

Orderability, Generalized Torsion and Amalgams

by

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A thesis submitted to the Faculty of Graduate Studies of

The University of Manitoba

in partial fulfillment of the requirements of the degree of

Doctor of Philosophy

Department of Mathematics

University of Manitoba

Winnipeg

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Abstract

We study two related topics about orderable groups. First, we investigate a type of order-preserving isomorphism of free groups to study the bi-orderability of group extensions. We give a sufficient condition ensuring these isomorphisms are order-preserving. As an application, we answer a question of Kin and Rolfsen affirmatively. Second, we study generalized torsion elements in amalgams of groups. Our main theorem gives a sufficient condition ensuring an amalgam to be free of generalized torsion. As an application of the main theorem, we construct groups that are generalized torsion free and not left-orderable, solving problem 16.48 (on Page 98) of The Kourovka Notebook.

Acknowledgements

I am extremely grateful to my wife Ann and my son Danny; without their support, I even wouldn't have moved to Canada. I want to express my sincere gratitude to Adam for all his guidance and assistance. He is not only a great advisor, but also a cool friend and a good neighbor. Basically, this project wouldn't be possible without him.

I would also like to thank my friends Friedemann Brock, Fei Qi and Richard Stanley for writing recommendation letters in support of my application to this PhD program. I am especially grateful to Richard for his ongoing support of my research. I thank Michele Triestino and Yash Lodha for their insightful discussions on the generalized torsion problem, which contributed to my understanding of the topic.

感谢我的妹妹吴春，我为你的自信和优秀而骄傲。感谢我的父母，你们是勤劳，勇敢，正直和善良的人，我很幸运能出生在这个充满爱和力量的家。特别是我的妈妈，你自信的,自由不羁的天性真是太酷了。

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Chapter 1

Introduction

A **left-ordering** of a group G is a strict total order $<$ on G , such that it is invariant under left multiplication; i.e., $a < b$ implies that $ca < cb$ for all $a, b, c \in G$. Similarly, a **right-ordering** of a group is a total order invariant under right multiplication. If a left-ordering is also a right ordering, we call it a **bi-ordering**. A group is said to be **left-orderable** (respectively, **right-orderable**, **bi-orderable**) if it has a left-ordering (respectively, right-ordering, bi-ordering). For a total order $<$ of a group G , we define a total order $<'$ by declaring $x <' y \Leftrightarrow x^{-1} < y^{-1}$. We can then verify that $(<')'$ is $<$ and that $<$ is a left-ordering if and only if $<'$ is a right-ordering. Ergo, a group is left-orderable precisely when it's right-orderable. So we only consider left-orderings. We now move on to the contents of this thesis. We explain what each chapter is about, the corresponding background, and how our research is related to the works of others.

The goal of Chapter 2 is to prepare some known results about orderability, providing the foundation for the following chapters.

In Chapter 3, we study our first topic: the order-preserving automorphisms of free groups. In [23], the authors proved that complements of certain fibered knots in S^3 have bi-orderable fundamental groups. This initiated the research in this line. Let G be a group and $\varphi : G \rightarrow G$ be a group automorphism. If there is a bi-ordering $<$ of G such that $a < b$ if and only if $\varphi(a) < \varphi(b)$ for all $a, b \in G$, we say that φ is **order-preserving**, that $<$ is **invariant** under φ , and that φ **preserves**

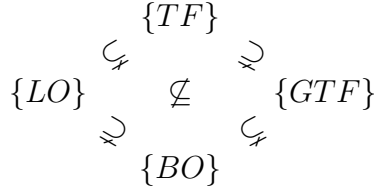


Figure 1.1: Containing Relations

<. For example, all the inner automorphisms of a bi-orderable group are order-preserving. In [18], a specific automorphism of the free group with three generators was considered; Kin and Rolfsen asked whether this automorphism is order-preserving, which is equivalent to whether the fundamental group of the so-called magic manifold is bi-orderable. We prove that the answer to the question of Kin and Rolfsen is positive (see Proposition 3.4.1), as an application of Theorem 3.3.7, which ensures a type of automorphism of free groups is order-preserving.

In Chapter 4, we study the second topic: generalized torsion in amalgam. A **torsion element** of a group G is a nontrivial element g in G whose order is finite; i.e., g is a torsion element if $g \neq 1$ and $g^n = 1$ for an integer $n \geq 2$. We write g^h for the conjugate $h^{-1}gh$. A **generalized torsion element** (also called a generalized torsion) of a group G is a nontrivial element g in G such that a product of conjugates of g is trivial; i.e., $g^{h_1}g^{h_2}\dots g^{h_n} = 1$ for some $h_1, \dots, h_n \in G$. A group without torsion elements (respectively, generalized torsion elements) is said to be **torsion-free** (respectively, **generalized torsion-free**). We use the short form LO, BO, TF and GTF for the words left-orderable, bi-orderable, torsion-free and generalized torsion-free, respectively. Denote by $\{BO\}$, $\{LO\}$, $\{TF\}$ and $\{GTF\}$ the set of groups which are BO, LO, TF and GTF respectively. We have containment relations given by Figure 1.1 (See Section 2.2) for proofs).

Clearly, we also know that $\{BO\} \subsetneq \{TF\}$ from the other known relations. The only unknown containment was whether $\{GTF\}$ is contained in $\{LO\}$. In fact, this is Problem 16.48 on Page 98 of The Kourovka Notebook [20]. This is also Problem 2.1 of [5]. We conjectured that there is a group which is GTF and non-LO and we tried to construct such an example. One idea is to construct a group from some “smaller” known groups, for example, using tools such as direct products or

exact sequences. However, we found that these tools don't work (see Remark 2.1.2). Thus we turn to the amalgams of groups and this is why we study generalized torsions in amalgams. As a result, we are able to give a sufficient condition ensuring an amalgam of two groups is GTF and this result—Theorem 4.4.6—is the main result of this chapter. As an application of this theorem, we give an analog—Corollary 4.6.3—of a result of George Bergman [3]. As another application of this theorem, Adam Clay proves in [6] that there exists a 3-manifold group which is GTF and not bi-orderable, resolving a conjecture of Motegi and Teragaito (Conjecture 1.1 of [21]). We want to remark that a lot of work has been done related to/towards this conjecture. For example, it was proved in [25] that the knot group of any negative twist knot has generalized torsion, and then in [26], the author further proved the existence of generalized torsion in some other manifold groups; generalized torsions of fundamental groups of hyperbolic manifolds are studied in [15].

The last chapter—Chapter 5—is to construct a group that is GTF and not left-orderable (see Theorem 5.1.3), thus solving the Kourovka Notebook problem. This gives yet another application of Theorem 4.4.6, thus can be seen as part of the second topic. We put it in a separate chapter since this part is bulky and the proof is quite combinatorial, which possesses a different taste from the contents in Chapter 4. We remark that there have been a number of investigations into generalized torsion in recent years, not necessarily directly related to manifold groups or group orderability. For example, a key result in [14] is that in a free product of torsion-free groups, every generalized torsion element is conjugate into a factor group. In [13] the author studied the groups whose generalized torsion elements are torsion elements. In [2], it is proved that all generalized torsion elements in FC groups (groups whose conjugate classes are finite) are torsion elements.

1.1 Contribution of authors

Finally, we want to point out that most of the contents of Chapter 3 are taken from [7] and most of the contents of Chapters 4 and 5 are from [6]. This thesis only contains the contents of which the author is the main contributor. The following are details of the contribution of Adam Clay, the

author's advisor.

In 2021, Clay told the author about this GTF and non-LO problem (Problem 16.48 on Page 98 of The Kourovka Notebook [20]) and his observation that the combination of the Chen-Heuer result [9, Theorem A] and the Ito-Motegi-Teragaito result [14, Theorem 2.4] can be applied to deal with the hyperbolic elements of the amalgams. Clay also thought that the results coming from studying amalgam of groups might be used to attack a conjecture of Motegi and Teragaito (Conjecture 1.1 of [21]). Clay had a lot of discussion with the author, which was especially helpful for the formulation of Theorem 4.4.6, one of the main results of this thesis.

About order-preserving isomorphisms of free groups, Clay had an idea about solving Kin-Rolfen's problem [18, Question 5.6]. He told the author about it, we did our first computation about it together, and it looked that his idea would work. The author did more study about it, gave some definitions and lemmas, and we had more discussions. Finally, the author formulized and proved Theorem 3.3.7, which is a generalization of a statement that we conjectured after our first discussion on this topic.

Chapter 2

Some known results in group theory

Let G be a group. A subsemigroup of G is a subset S of G which is closed under the multiplication of G ; i.e., $ab \in S$ for all $a, b \in S$. Note that the empty set is a subsemigroup of G . A subset S of G is normal if it's closed under conjugation; i.e., $s \in S, g \in G$ implies that $s^g \in S$. Therefore, a normal subsemigroup of G may be empty.

Let S be a nonempty set. Then the free group F_S generated by S is the unique group (up to isomorphism) containing S and satisfying the universal property: for every map f from the set S to a group G , there is a unique homomorphism $\phi : F_S \rightarrow G$, such that $\phi(s) = f(s)$ for all $s \in S$. We call the cardinality of S the rank of the free group F_S . It's well known that two free groups of the same rank are isomorphic. Therefore, for a positive integer n , the free groups of rank n are all isomorphic. Thus, for any set S whose cardinality is n , we will call F_S the free group of rank n .

The free group F_S can be constructed as follows. Let S^{-1} be the set $\{s^{-1} \mid s \in S\}$ (of symbols). We call $w = x_1 \cdots x_n$ with $x_i \in S \cup S^{-1}$ and $n \geq 0$ a **word** in S , where we write 1 for this word when $n = 0$. If $\{x_{i-1}, x_i\} = \{s, s^{-1}\}$ for an $s \in S$, we consider $w = x_1 \cdots x_{i-2} x_{i-1} x_i x_{i+1} \cdots x_n$ to be equal to $w' = x_1 \cdots x_{i-2} x_{i+1} \cdots x_n$, and say that we simplify the word w by omitting the pair $\{x_{i-1}, x_i\} = \{s, s^{-1}\}$. A word that cannot be simplified is called a **reduced** word. The set of F_S is formed by these words. (Since each word is equal to a reduced word, we can also say that F_S is formed by the reduced words.) The product $w_1 w_2$ of two words $w_1 = x_1 \cdots x_n$ and $w_2 = y_1 \cdots y_m$

is defined as their concatenation $x_1 \cdots x_n y_1 \cdots y_m$. It can be verified that this defines a group, S is a subset of F_S , s^{-1} is the inverse of s for every $s \in S$, and 1 is the identity element of the group. For example, let $S = \{a, b\}$. Then $\alpha = abb^{-1}ab$ and $\beta = b^{-1}a$ are two elements in F_S ; we have $\alpha = aab = a^2b$, and the product $\alpha\beta = a^2bb^{-1}a = a^3$.

Let R be a subset of F_S and N be the normal subgroup of F_S generated by R . We write $\langle S \mid R \rangle$ for the quotient group F_S/N and call it the group generated by S and subjected to the relations R . When a group G is isomorphic to $\langle S \mid R \rangle$, we say G has the presentation $\langle S \mid R \rangle$. We are mostly interested in the case when $S = \{a_1, a_2, \dots, a_n\}$ and $R = \{r_1, \dots, r_m\}$ are finite sets. In this case, we write $\langle a_1, a_2, \dots, a_n \mid r_1, \dots, r_m \rangle$ or $\langle a_1, a_2, \dots, a_n \mid r_1 = 1, \dots, r_n = 1 \rangle$ for the group F_S/N and call it the group generated by $a_1, a_2, \dots, a_n$ and subject to the relations $r_1 = 1, \dots, r_n = 1$. Abusing notation, we sometimes use the elements of F_S to represent the elements of the quotient group F_S/N .

Example 2.0.1. 1. $F_2 := \langle a, b \mid \rangle$ is the free group of rank two.

2. Let n be a positive integer. Then $H := \langle a, b \mid b, a^n \rangle$ is (isomorphic to) the cyclic group $\langle a \mid a^n = 1 \rangle$.

3. The group $G := \langle a, b \mid ba^{-1}ba \rangle$ is the well-known fundamental group of the Klein bottle. We have the identity $bb^a = 1$ in G , where we consider a, b as elements in G . (Recall that we write b^a for the conjugate $a^{-1}ba$.)

4. Let N be the normal subgroup of F_2 generated by $r := ba^{-1}ba$; i.e.,

$$N = \{a_1^{g_1} \cdots a_n^{g_n} \mid a_i \in \{r, r^{-1}\}, g_i \in F_2, n \geq 0\}.$$

Then the group $\langle a, b \mid ba^{-1}ba \rangle$ is isomorphic to the quotient group F_2/N .

It is well known that free groups are bi-orderable. (See [28] or its English translation [19].)

2.1 Left-orderable groups

For subsets S, T of a group G , we let $S^{-1} = \{g^{-1} \mid g \in S\}$ and $ST = \{gh \mid g \in S, h \in T\}$, called the inverse of S and the product of S and T , respectively. Let $<$ be a total order on a group G with the unit element denoted by 1. Let P be the set of elements g in G such that $1 < g$. If $<$ is a left-ordering (respectively, bi-ordering) of group G , then P satisfies the first two properties (respectively, all the three properties) of the following:

1. $PP \subset P$; i.e., P is a sub-semigroup of G ;
2. $G \setminus \{1\} = P \sqcup P^{-1}$; i.e., $G \setminus \{1\}$ is the disjoint union of P and P^{-1} ;
3. P is normal; i.e., $g^{-1}Pg \subseteq P$ for all $g \in G$.

We call P the **positive cone** of G determined by $<$. We say an element g and its sign is **positive** (respectively, **negative**) (with respect to $<$) if $g > 1$ (respectively, $g < 1$). On the other hand, if a group G admits the existence of a subset P satisfying the first two properties (respectively, all these three properties), then $g < h \Leftrightarrow g^{-1}h \in P$ defines a left-ordering (respectively, bi-ordering) $<$ of G , such that the positive cone of G determined by $<$ is P . To conclude, this correspondence gives a bijection between positive cones of G and orderings of G .

Now, we consider two ways of building groups from “smaller” groups.

Lemma 2.1.1. (1) Consider the direct product $A = A_1 \times A_2$ of two groups A_1 and A_2 . Then A is LO (respectively, GTF) if and only if both A_1 and A_2 are LO (respectively, GTF).

(2) Consider the exact sequence of groups

$$1 \rightarrow A_1 \xrightarrow{i} A \xrightarrow{\pi} A_2 \rightarrow 1. \quad (2.1)$$

Then A is LO if both A_1 and A_2 are LO. Also, it's possible that A is non-GTF when both A_1 and A_2 are GTF.

Proof. (1) If A is LO (respectively, GTF), then A_1 and A_2 are LO (respectively, GTF), as they are naturally subgroups of A . If A_1 and A_2 are both GTF, it's easy to verify that so is A .

Now if $<_1$ and $<_2$ are left-orderings of two groups A_1 and A_2 , respectively, then the lexicographic order of $A_1 \times A_2$ defined by declaring $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_1 a_2$ or $a_1 = a_2$ and $b_1 <_2 b_2$ is a left-ordering. This proves that A is LO if both A_1 and A_2 are LO.

(2) Assume that A_1, A_2 are left-orderable with positive cones P_1 and P_2 respectively. Then it can be verified that $P := i(P_1) \cup \pi^{-1}(P_2)$ is a positive cone of A and thus A is left-orderable.

Now we give an example of a group A that is non-GTF while both A_1 and A_2 are GTF. Let $A = \langle a, b \mid abab^{-1} \rangle$ be the group generated by a, b and subjected to the relation $abab^{-1} = 1$. (This is the well-known fundamental group of the Klein bottle.) Let A_1 be the subgroup of G generated by a , then A_1 is normal in A and $A_2 := A/A_1$ is isomorphic to the subgroup $\langle b \rangle$ of A generated by b . Then we have the exact sequence (2.1). Now A is not GTF as it has a generalized torsion element a and both the groups A_1 and A_2 are GTF as they are both infinite cyclic groups. (They are both infinite cyclic groups because A is TF, which is shown in (4) in the next subsection.) $\square$

Remark 2.1.2. Now we see that it's impossible to construct a group that is GTF and non-LO using the exact sequence (2.1) or direct product and “smaller groups” which are GTF and LO. Here the smaller groups are A_1, A_2 and A is the group constructed.

2.2 The known containment relations

In this section, we explain all the previously known relations given by Figure (1.1). As mentioned earlier, the only previously unknown relation—whether $\{GTF\}$ is contained in $\{LO\}$ —will be solved in Chapter 5 by a construction of a group which is GTF and non-LO.

(1) By definition, a BO group is LO and a GTF group is TF.

(2) If a group G is LO, we now show that it is TF.

Let $<$ be a left-ordering of G . Let $g \in G$ be nontrivial. Since $<$ is a total order of G , we have either $g > 1$ or $g < 1$. If $g > 1$, then we have $g^2 > g$ by left-multiplying with g and then similarly $g^3 > g^2, g^4 > g^3$ and so on. We thus see that for every $n \geq 1$, we have $g^n > 1$ by

the transitivity of $>$. This proves that if $g > 1$ then g is not a torsion element. Similarly, we can prove that g is not a torsion element if $g < 1$. Hence we proved that if G is LO, then G is TF.

- (3) If a group G is BO with a bi-ordering $<$, we show that it is GTF.

Let $g \in G$ be nontrivial. We want to prove that g is not a generalized torsion element. We can assume that $g > 1$ without loss of generality. Let $h_1, \dots, h_n \in G$. Then we have $h_1^{-1}gh_1 \cdots h_n^{-1}gh_n > 1$ because all these $h_i^{-1}gh_i$ are in the positive cone determined by $<$. Hence g is not a generalized torsion element of G . Thus, we showed that if G is BO, then G is GTF.

- (4) We now give examples of groups which are LO, non-BO, TF and non-GTF.

Let m be a nonzero integer. Consider the group G_m defined by the semi-direct product $G_m = \mathbb{Z}[1/m] \rtimes \mathbb{Z}$, where $\mathbb{Z}[1/m]$ is the underlying group of the subring of $\mathbb{R}$ generated by $\mathbb{Z} \cup \{\frac{1}{m}\}$, and the multiplication of G_m is given by

$$(\alpha, j)(\beta, l) = (\alpha + m^j\beta, j + l), \forall \alpha, \beta \in \mathbb{Z}[1/m], j, l \in \mathbb{Z}.$$

(This semidirect product comes from the homomorphism $\varphi : \mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}[1/m])$, where for each $j \in \mathbb{Z}$, the automorphism $\varphi(j)$ of $\mathbb{Z}[1/m]$ is defined by $\varphi(j) : \beta \mapsto m^j\beta$.)

Remark 2.2.1. It is well known that group G_m is isomorphic to the Baumslag–Solitar group $\langle a, b \mid bab^{-1} = a^m \rangle$, which is usually denoted as $BS(1, m)$; the isomorphism is given by $a \mapsto (1, 0), b \mapsto (0, 1)$. (See e.g. [27, Page 78].)

Define a total order $\prec$ on G_m by

$$(\alpha, j) \prec (\beta, l) \Leftrightarrow j < l \text{ or } j = l, \alpha < \beta,$$

where both $<$ on the right-hand side (RHS) are the usual order. We can then verify that $\prec$ is a right-ordering of G_m , thus G_m is LO (recalling that a group is left-orderable if and only

if it is right-orderable) and so G_m is TF by (2), and it is a bi-ordering if moreover $m > 0$. Letting $a = (1, 0), b = (0, 1)$, it can be verified that $bab^{-1}a^{-m} = 1$. Therefore, if $m < 0$, a is a generalized torsion element of G_m and thus G_m is not BO by (3).

Conclusion: G_m with $m < 0$ are LO, non-BO, TF and non-GTF.

- (5) A group which is TF but not LO can be found in Example 1.59 on Page 17 of [11].

Another TF and non-LO group is given by Example 4.1.2, where we constructed such a group as the amalgam of two free groups. We remark that the same idea is used to make the amalgam non-LO in the construction of a GTF and non-LO group in Section 5.1.

- (6) A group which is GTF but non-BO is given in Subsection 2.2.1.

2.2.1 Mura-Rhemtulla's GTF but non-BO group

In this subsection, we explain Mura-Rhemtulla's construction (see [22]) of a generalized torsion free and non-bi-orderable group. We remark here that we modified the construction a bit: we simply take B in the following construction to be a free group, while in Mura-Rhemtulla's original construction, B is a wreath product. This modification doesn't change the result that the group constructed is LO, GTF and non-BO, thus we still call this group Mura-Rhemtulla's group.

Mura-Rhemtulla's group G is a semidirect product $A \rtimes B$, where A is a subgroup of $\mathbb{R}$ and B is a free group. We first define A and B , the action of B on A , prove that G is GTF and finally show that G is not bi-orderable.

Let $\alpha = \frac{3+\sqrt{5}}{2}$, then $\alpha^{-1} = \frac{3-\sqrt{5}}{2}$. They are the roots of the polynomial $x^2 - 3x + 1$.

1. Group $A = \mathbb{Z}[1/2, \sqrt{2}, \alpha]$ is the underlying abelian group of the subring of $\mathbb{R}$ generated by $\{\frac{1}{2}, \sqrt{2}, \alpha\}$. Let $R = \mathbb{Z}[1/2]$, and view A as the free R -module generated by $1, \alpha, \sqrt{2}, \sqrt{2}\alpha$. Letting A_1 and A_2 be the sub-modules

$$A_1 = R1 + R\alpha, \quad A_2 = R\sqrt{2} + R\sqrt{2}\alpha,$$

we see that $A = A_1 \oplus A_2$. Since multiplication by 2 gives an automorphism of R , we know that multiplication by $\sqrt{2}$ gives an isomorphism from A_1 to A_2 and also one from A_2 to A_1 . Note that multiplication by α gives an automorphism of A_1 and one of A_2 .

2. The group B is the free group generated by x, y .¹
3. For $a = a_1 + a_2$ with $a_i \in A_i$, we define the action of x and the action of y by

$$a^x = a\sqrt{2} = a_2\sqrt{2} + a_1\sqrt{2}, \quad a^y = a_1\alpha + a_2\alpha^{-1}.$$

This defines the action of B on A . It can be verified directly that the action of $yx^{-1}yx$ is the identity, hence the normal subgroup N of B generated by $yx^{-1}yx$ acts on A trivially. Note that all elements in the quotient group $B/N = \langle x, y \mid yx^{-1}yx \rangle$ can be written in the form $y^i x^j$ with $i, j \in \mathbb{Z}$. (Note that again this is the fundamental group of the Klein bottle, and it is also the Baumslag-Solitar group G_{-1} using the notation of Remark 2.2.1.) Thus, for every element $g \in B$, there are $i, j \in \mathbb{Z}$ such that $a^g = a^{y^i x^j}$ for all $a \in A$.

4. We now prove that G is GTF. First, note that B is bi-orderable and that the projection $p_2 : A \rtimes B \rightarrow B$ sending (a, b) to b is a surjective homomorphism, so we only need to prove that no element in A is a generalized torsion element. Assume that $a = a_1 + a_2 \neq 0$, with $a_i \in A_i$, is a generalized torsion element of G . Then there are $b_1, \dots, b_n \in B$ ($n \geq 1$) such

¹In Mura-Rhemtulla's original construction, B is a wreath product

$$B = \langle y \rangle wr \langle x \rangle = Y \rtimes \langle x \rangle,$$

where $Y = \bigoplus_{i \in \mathbb{Z}} \langle y \rangle = \{f : \mathbb{Z} \rightarrow \langle y \rangle \mid f(k) \neq 1 \text{ only for finitely many } k\},$

and the action of $\langle x \rangle$ on Y is defined by $(x^i \cdot f)(k) = f(k + i)$. We identify y with the following f_0 in Y :

$$f_0(k) = \begin{cases} y & k = 0 \\ 1 & k \neq 0. \end{cases}$$

Then B is generated by x and y and the defining relations are $[y^{x^i}, y^{x^j}] = 1$ for $i, j \in \mathbb{Z}$.

that

$$\sum_{k=1}^n a^{b_k} = 0 \quad (2.2)$$

(Here we used that A is abelian.) We can (and do) assume that each b_t is of the form $y^i x^j$ by 3. Since we can apply a big enough power of x to (2.2), we can assume that all these j 's are positive. Since the action of x^2 is the multiplication by 2, we can assume that each b_k is of the form y^i or $y^i x$. (For example, we can write $a^{yx^3} = a^{yx} 2 = a^{yx} + a^{yx}$.) Considering these i may be negative or non-negative, we can write the left-hand side (LHS) of (2.2) as the sum of the following four terms:

$$S_1 = \sum_{p=1}^r a^{y^{i_p}}, \quad S_2 = \sum_{p=1}^s a^{y^{j_p}}, \quad T_1 = \sum_{p=1}^t a^{y^{k_p} x}, \quad T_2 = \sum_{p=1}^u a^{y^{l_p} x},$$

where i_p 's and k_p 's are non-negative integers, j_p 's and l_p 's are negative integers, and r, s, t, u are non-negative integers with $r + s + t + u = n$. Since $a^{y^i} = a_1 \alpha^i + a_2 \alpha^{-i}$, we have

$$S_1 = \sum_{p=1}^r (a_1 \alpha^{i_p} + a_2 \alpha^{-i_p}) = a_1 P_1(\alpha) + a_2 P_1(\alpha^{-1}),$$

where $P_1(x) = \sum_{p=1}^r x^{i_p}$ is an integer polynomial. Treating S_2, T_1, T_2 similarly, we find integer polynomials $P_1(x), P_2(x), Q_1(x)$ and $Q_2(x)$ with non-negative coefficients and at least one polynomial is nonzero, such that

$$\begin{aligned} & a_1 P_1(\alpha) + a_2 P_1(\alpha^{-1}) + a_1 P_2(\alpha^{-1}) + a_2 P_2(\alpha) \\ & + (a_1 Q_1(\alpha) + a_2 Q_1(\alpha^{-1}) + a_1 Q_2(\alpha^{-1}) + a_2 Q_2(\alpha)) \sqrt{2} = 0. \end{aligned}$$

Let $P(x) = P_1(x) + P_2(3-x)$. So $P(x)$ is an integer polynomial and $P(\alpha) = P_1(\alpha) + P_2(\alpha^{-1})$. Similarly, we let $Q(x) = Q_1(x) + Q_2(3-x)$. Now we have, considering the components in A_1

and A_2 ,

$$\begin{aligned} a_1 P(\alpha) + a_2 Q(\alpha^{-1})\sqrt{2} &= 0, \\ a_1 Q(\alpha)\sqrt{2} + a_2 P(\alpha^{-1}) &= 0. \end{aligned}$$

Since a_1, a_2 are not both zero, we have

$$P(\alpha)P(\alpha^{-1}) = 2Q(\alpha)Q(\alpha^{-1}). \quad (2.3)$$

Now, letting $P(\alpha) = a + b\alpha$ with $a, b \in \mathbb{Z}$, we have $P(\alpha^{-1}) = a + b\alpha^{-1}$. (One way to see this is that since α is a root of $P(x) - a - bx$, so is α^{-1} .) Thus we have

$$P(\alpha)P(\alpha^{-1}) = a^2 + 3ab + b^2.$$

If a, b are not both zero, then the largest m such that $2^m \mid P(\alpha)P(\alpha^{-1})$ is even. The same applies to $Q(\alpha^{-1})Q(\alpha)$. Therefore, by (2.3), we have $P(\alpha) = 0$ and $Q(\alpha) = 0$. But this is impossible, because $P(\alpha) = P_1(\alpha) + P_2(\alpha^{-1})$, $Q(\alpha) = Q_1(\alpha) + Q_2(\alpha^{-1})$, these P_i, Q_i have nonnegative coefficients, with at least one nonzero, and both α, α^{-1} are positive.

5. Now we turn to proving that G is not bi-orderable. Assume that G has a bi-ordering $\prec$. Then this bi-ordering restricted to A is invariant under the action of B . Note that $A = Rv_1 + Rv_2 + Rv_3 + Rv_4$ with $(v_1, v_2, v_3, v_4) = (1, \alpha, \sqrt{2}, \alpha\sqrt{2})$ and $R = \mathbb{Z}[\frac{1}{2}]$. Note that the isomorphisms given by x, y are R -linear maps on A given by

$$x \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \quad y \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}. \quad (2.4)$$

By Lemma 2.2.2, there is a nonzero vector (a_1, a_2, a_3, a_4) in $\mathbb{R}^4$, such that for all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$, $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 > 0$ implies that $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 \succ 0$. (The linear equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ defines a so-called hyperplane of $\mathbb{R}^4$.) Acting by x on

this, we have $x_1v_3 + x_2v_4 + x_3v_1 + x_4v_2 \succ 0$, which implies that $a_3x_1 + a_4x_2 + 2a_1x_3 + 2a_2x_4 \geq 0$. Therefore, $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 > 0$ implies that $a_3x_1 + a_4x_2 + 2a_1x_3 + 2a_2x_4 \geq 0$, for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Thus we have

$$(a_1, a_2, a_3, a_4) = t(a_3, a_4, 2a_1, 2a_2) \quad (2.5)$$

for some nonzero t in $\mathbb{R}$.² Similarly, acting y on the inequality $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 \succ 0$, we find

$$(a_1, a_2, a_3, a_4) = s(a_2, 3a_2 - a_1, 3a_3 - a_4, a_3) \quad (2.6)$$

for some nonzero s in $\mathbb{R}$.³ By (2.5), we have $a_1 = ta_3 = t(2ta_1) = 2t^2a_1$. Also, $a_1 \neq 0$, because $a_1 = 0$ leads to all $a_i = 0$ using (2.5) and (2.6), which can't happen. Hence $2t^2 = 1$. By (2.6), we have $a_1 = sa_2 = s^2(3a_2 - a_1) = s^2(3s^{-1}a_1 - a_1)$, leading to $1 = 3s - s^2$. Using (2.5) and (2.6), we have $a_1 = sa_2 = sta_4 = stsa_3 = sts2ta_1$, thus $2s^2t^2 = 1$. But $2t^2 = 1$ and $1 = 3s - s^2$, a contradiction.

We now turn to prove the following Lemma 2.2.2, which was used in the proof that the Mura-Rhemtulla group is non-BO. We remark that this lemma is a well-known result. For example, it appears in [10] implicitly. A hyperplane H in $\mathbb{R}^n$ is a $(n - 1)$ -dimensional subspace of $\mathbb{R}^n$. If H is a hyperplane in $\mathbb{R}^n$, then there are $a_1, \dots, a_n \in \mathbb{R}$, not all zero, such that

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \dots + a_nx_n = 0\}.$$

²To see this, note that if $a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0$ for a $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, then $\sum_{i=1}^4 a_i(y_i + \frac{a_i}{n}) > 0$ for all $n \geq 1$, hence $a_3(y_1 + \frac{a_1}{n}) + a_4(y_2 + \frac{a_2}{n}) + 2a_1(y_3 + \frac{a_3}{n}) + 2a_2(y_4 + \frac{a_4}{n}) \geq 0$ and then $a_3y_1 + a_4y_2 + 2a_1y_3 + 2a_2y_4 \geq 0$ by taking the limit $n \rightarrow \infty$. Then notice that $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 < 0$ implies that $a_3x_1 + a_4x_2 + 2a_1x_3 + 2a_2x_4 \leq 0$, for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Thus similarly, $a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0$ implies that $a_3y_1 + a_4y_2 + 2a_1y_3 + 2a_2y_4 \leq 0$ for all $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Therefore $a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0$ implies that $a_3y_1 + a_4y_2 + 2a_1y_3 + 2a_2y_4 = 0$. Hence the two hyperplanes must be equal.

³So, we proved that the vector (a_1, a_2, a_3, a_4) given by the bi-ordering is a common eigenvector of the two 4×4 matrices in (2.4), both corresponding to nonzero eigenvalues. (In fact, we can show that these eigenvalues are positive.) It is now an exercise in linear algebra to show that the two matrices don't have such common eigenvector. But here we give another argument to find a contradiction.

Writing $H(x) = a_1x_1 + \cdots + a_nx_n$, we also say that the hyperplane is given by $H(x)$.

Lemma 2.2.2. *Let $\prec$ be a bi-ordering on $G = \mathbb{Q}^n$, where n is a positive integer. Then there is a hyperplane given by $H(x) = a_1x_1 + \cdots + a_nx_n$ (with $a_i \in \mathbb{R}$ not all zero), such that for all $x \in \mathbb{Q}^n$, $H(x) > 0$ (respectively, $H(x) < 0$) implies $x \succ 0$ (respectively, $x \prec 0$).*

Proof. For $P \in \mathbb{R}^n$ and $r > 0$, we denote by $B(P, r)$ the open ball centered at P with radius r in $\mathbb{R}^n$. Let H be the set of points P in $\mathbb{R}^n$, such that for every $\epsilon > 0$, there are $x, y \in G \cap B(P, \epsilon)$, such that $x \prec 0$ and $y \succ 0$. Then by Lemma 3.3 of [10], H is a hyperplane in $\mathbb{R}^n$.

Now let H be given by $H(x) = a_1x_1 + \cdots + a_nx_n$, with $a_i \in \mathbb{R}$ and not all zero. Fix a $P_0 \in \mathbb{R}^n \setminus H$. Then there is an $\epsilon > 0$ such that elements in $\mathbb{Q}^n \cap B(P_0, \epsilon)$ are all $\succ 0$ or all $\prec 0$. We can assume that they are all $\succ 0$ as we can replace P_0 with $-P_0$ if necessary. Clearly, $B(P_0, \epsilon)$ doesn't intersect H and thus we can assume that all $P \in B(P_0, \epsilon)$ satisfy $H(P) > 0$. (Because if all $H(P) < 0$, we can replace $H(x)$ with $-H(x) = -a_1x_1 - \cdots - a_nx_n$.)

Now assume $Q \in \mathbb{Q}^n$ satisfies $H(Q) > 0$, we want to prove that $Q \succ 0$. There is an $\epsilon > 0$ such that points in $B(Q, \epsilon) \cap \mathbb{Q}^n$ are all $\succ 0$ or all $\prec 0$. We are done if it's the former. If it's the latter, then we consider that for all x in the line segment P_0Q , there is an $\epsilon(x) > 0$ such that elements in $V_x := B(x, \epsilon(x)) \cap \mathbb{Q}^n$ are all $\succ 0$ or $\prec 0$. (This is because $H(x) > 0$ for all x in P_0Q , by linearity of H and that $H(Q) > 0, H(P_0) > 0$.) For convenience, we call V_x positive (respectively, negative) if all its elements are $\succ 0$ (respectively, $\prec 0$). By the compactness of the line segment, there are finitely many points $x_1, \dots, x_k$ in P_0Q , such that these V_{x_i} cover P_0Q . We can then find a subset $\{y_1, \dots, y_l\}$ of $\{x_1, \dots, x_k\}$ such that V_{y_1} is positive, each $V_{y_i} \cap V_{y_{i+1}}$ is nonempty and V_{y_l} is negative. Letting t be maximal such that $V_{y_1}, V_{y_2}, \dots, V_{y_t}$ are positive, we have $1 \leq t \leq l-1$ and $V_{y_{t+1}}$ is negative. But $V_{y_t} \cap V_{y_{t+1}} \neq \emptyset$, this is a contradiction.

Similarly, if $Q \in \mathbb{Q}^n$ satisfies $H(Q) < 0$, we have $Q \prec 0$. □

Chapter 3

Order preserving automorphisms of free groups

Now we turn to study order-preserving automorphisms of free groups. The main purpose of this chapter is to answer a question of Kin and Rolfsen in [18]. Another purpose of this chapter is to develop some tools which may be useful in the (desired) construction of GTF but non-LO groups.

Our study is based on the Artin action [1] of the braid groups B_n upon the free group F_n of rank n , as in [18]. In the course of our investigation, we introduce a new method that may be applied more generally to establish that an automorphism of a free group preserves some bi-ordering.

The remainder of this chapter is organized as following. In Section 3.1 we define order-preserving automorphisms and introduce the main result of this chapter. In Section 3.2, we introduce generalizations of the “positive eigenvalue” condition of [23], and use it to show that certain automorphisms of infinitely generated abelian groups preserve a bi-ordering. In Section 3.3, we bootstrap this result to prove our main theorem. In Section 3.4, we apply our main theorem to analyze the Artin action on F_n and show that the fundamental group of the magic manifold is bi-orderable.

3.1 Introduction

First, some definitions are in order. Let $\varphi : G \rightarrow G$ be an automorphism. If there is a bi-ordering $<$ of G such that $f < g$ if and only if $\varphi(f) < \varphi(g)$ for all $f, g \in G$, we say that φ is **order-preserving**, that $<$ is **invariant** under φ , and that φ **preserves** $<$.

Our main theorem for showing that an automorphism of a free group $\varphi : F \rightarrow F$ is order-preserving is the following. We say that an orbit $\mathcal{O}$ of a bijection $\sigma : I \rightarrow I$ is finite (respectively, infinite) if the cardinality $|\mathcal{O}|$ of $\mathcal{O}$ is finite (respectively, infinite).

Theorem 3.3.7. Let F be the free group generated by $\{x_i \mid i \in I\}$ where I is a nonempty set. Let $\varphi : F \rightarrow F$ be an automorphism satisfying

$$\varphi(x_i) = w_i x_{\sigma(i)} w_i^{-1}, \forall i \in I$$

where $w_i \in F$ and σ is a bijection on I . Assume that $\sigma(i_0) = i_0$ for some $i_0 \in I$ and let $h : F \rightarrow \mathbb{Z}$ be the homomorphism defined by $h(x_{i_0}) = 1$ and $h(x_j) = 0$ for all $j \neq i_0$. Assume that for each finite orbit $\mathcal{O}$ of σ we have

$$\gcd\left(|\mathcal{O}|, \sum_{i \in \mathcal{O}} h(w_i)\right) = 1.$$

Then there exists a bi-ordering of F invariant under φ .

One source of automorphisms of F_n having a formula as in Theorem 3.3.7 is the Artin action of B_n on F_n [1]. The braid group B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$, subject to the relations:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } 1 \leq i \leq n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ for } |i-j| > 1. \end{aligned}$$

There is an embedding of B_n into the automorphism group of the free group F_n on generators $\{x_1, \dots, x_n\}$, which yields the Artin action of B_n on F_n . The embedding sends each generator σ_i

of B_n to the automorphism which acts on each generator x_j of F_n according to

$$x_j^{\sigma_i} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i \\ x_i & \text{if } j = i + 1 \\ x_j & \text{if } j \neq i, i + 1. \end{cases}$$

An unanswered question in [18] was whether the braid $\beta = \sigma_1^2 \sigma_2^{-1} \in B_3$ yields an automorphism that preserves a bi-ordering of F_3 . This is equivalent to whether the fundamental group of the so-called “magic manifold” is bi-orderable; the magic manifold is a 3-cusped hyperbolic 3-manifold conjectured to be of minimal volume. We are able to answer this question in the affirmative using Theorem 3.3.7 (see Proposition 3.4.1). We also generate some new examples of order-preserving braids.

3.2 Ordered bases, modules, and order-preserving homomorphisms

Let $(S, <)$ be a set with a strict total ordering and $f : S \rightarrow S$ be a function. We say that f preserves the ordering $<$ if $a < b$ implies that $f(a) < f(b)$ for all $a, b \in S$, and in this case we say that $<$ is invariant under f , and that the function f is order-preserving. Note that our definition of order-preserving is therefore context-sensitive: a bijection $f : S \rightarrow S$ from a set S to itself is order-preserving if there is a strict total ordering $<$ of S preserved by f , whereas an automorphism $\varphi : G \rightarrow G$ of a group is order-preserving if there is a bi-ordering $<$ of G preserved by φ .

In what follows, we let R be the ring $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, each equipped with its usual ordering. Given a free R -module M , an **ordered basis** for M is a pair $(\{v_i \mid i \in I\}, \prec)$ where $\{v_i \mid i \in I\}$ is a basis for the R -module M , and $\prec$ is a strict total ordering of the index set I .

Definition 3.2.1. *Given a free R -module M and an ordered basis $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$, the **lexicographic ordering of M associated with $\mathcal{B}$** is the bi-ordering $<_{\mathcal{B}}$ of the abelian group M*

defined as follows. Given $\alpha = \sum_i a_i v_i \in M \setminus \{0\}$, set $i(\alpha) = \min_{\prec} \{j \in I \mid a_j \neq 0\}$ and define $0 <_{\mathcal{B}} \alpha$ if only if $0 < a_{i(\alpha)}$.

Definition 3.2.2. Let M be a free R -module, $f : M \rightarrow M$ an R -module homomorphism, and $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ an ordered basis for M . We say that f is **positively triangular with respect to $\mathcal{B}$** if there is an order-preserving map $\eta : I \rightarrow I$ with respect to $\prec$ such that for each $i \in I$,

$$f(v_i) = \lambda_i v_{\eta(i)} + \sum_{j \succ \eta(i)} c_{i,j} v_j,$$

where $\lambda_i, c_{i,j} \in R$ and $\lambda_i > 0$. We say that f is **positively triangular** if there exists an ordered basis $\mathcal{B}$ such that f is positively triangular with respect to $\mathcal{B}$.

Remark 3.2.3. Note that if M is a finite rank R -module, say with ordered basis $\mathcal{B} = \{v_1, \dots, v_n\}$ where the index set $\{1, \dots, n\}$ is equipped with its natural ordering, then $f : M \rightarrow M$ is positively triangular with respect to $\mathcal{B}$ if and only if f is represented relative to $\mathcal{B}$ by an upper triangular matrix with positive entries on the diagonal.

Lemma 3.2.4. Let M be a free R -module and $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ be an ordered basis for M . If $f : M \rightarrow M$ is an R -module homomorphism that is positively triangular with respect to $\mathcal{B}$, then f preserves the bi-ordering $<_{\mathcal{B}}$.

Proof. Suppose f is positively triangular with respect to $\mathcal{B}$ and let $\prec, \eta, \lambda_i$ and $c_{i,j}$ be as given in Definition 3.2.2. Recall that if $\alpha = \sum_i a_i v_i \in M \setminus \{0\}$ then $i(\alpha) := \min_{\prec} \{j \in I \mid a_j \neq 0\}$. To check that $<_{\mathcal{B}}$ is invariant under f , assume that $\alpha = \sum_i a_i v_i >_{\mathcal{B}} 0$. We want to prove that $f(\alpha) >_{\mathcal{B}} 0$.

We have:

$$\begin{aligned}
f(\alpha) &= a_{i(\alpha)} f(v_{i(\alpha)}) + \sum_{j \succ i(\alpha)} a_j f(v_j) \\
&= a_{i(\alpha)} \left(\lambda_{i(\alpha)} v_{\eta(i(\alpha))} + \sum_{j \succ \eta(i(\alpha))} c_{i(\alpha),j} v_j \right) \\
&\quad + \sum_{j \succ i(\alpha)} a_j \left(\lambda_j v_{\eta(j)} + \sum_{k \succ \eta(j)} c_{j,k} v_k \right) \\
&= a_{i(\alpha)} \lambda_{i(\alpha)} v_{\eta(i(\alpha))} + \sum_{k \succ \eta(i(\alpha))} c'_{\alpha,k} v_k.
\end{aligned}$$

To reach the last equality, we have regrouped terms to arrive at new coefficients $c'_{\alpha,k}$ for $k \succ \eta(i(\alpha))$, where we used the fact that $k \succeq \eta(j)$ and $j \succ i(\alpha)$ implies $k \succ \eta(i(\alpha))$, as $\succ$ is invariant under η .

Since both $a_{i(\alpha)}$ and $\lambda_{i(\alpha)}$ are positive, we conclude that $f(\alpha) >_{\mathcal{B}} 0$, as desired. $\square$

Definition 3.2.5. Let $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ and $\mathcal{C} = (\{w_j \mid j \in J\}, \triangleleft)$ be ordered bases of R -modules of M and N respectively. We define the **tensor** $\mathcal{B} \otimes \mathcal{C}$ to be the ordered basis $(\{v_i \otimes w_j \mid (i, j) \in I \times J\}, <)$ of the R -module $M \otimes N$, where the total ordering $<$ on $I \times J$ is defined lexicographically: $(i, j) < (s, t)$ if $i \prec s$, or $i = s$ and $j \triangleleft t$.

Lemma 3.2.6. Let $f : M \rightarrow M$ and $g : N \rightarrow N$ be R -module homomorphisms. Assume that f and g are positively triangular with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Then the R -module homomorphism $f \otimes g : M \otimes N \rightarrow M \otimes N$ is positively triangular with respect to $\mathcal{B} \otimes \mathcal{C}$.

Proof. Let $\mathcal{B} = (\{v_i \mid i \in I\}, \prec)$ and $\mathcal{C} = (\{w_j \mid j \in J\}, \triangleleft)$. For f , let $\eta : I \rightarrow I$ be as in Definition 3.2.2, and for g , we will use the function $\xi : J \rightarrow J$. Define the map $\sigma : I \times J \rightarrow I \times J$ by $\sigma(i, j) = (\eta(i), \xi(j))$, which clearly preserves the lexicographic ordering $<$ of $I \times J$. Writing $u_{(i,j)}$

for $v_i \otimes w_j$, we have

$$\begin{aligned}
(f \otimes g)(u_{(i,j)}) &= (f \otimes g)(v_i \otimes w_j) = f(v_i) \otimes g(w_j) \\
&= \left(\lambda_i v_{\eta(i)} + \sum_{s \succ \eta(i)} a_{i,s} v_s \right) \otimes \left(\mu_j w_{\xi(j)} + \sum_{t \triangleright \xi(j)} b_{j,t} w_t \right) \\
&= \lambda_i \mu_j u_{(\eta(i), \xi(j))} + \sum_{t \triangleright \xi(j)} \lambda_i b_{j,t} u_{(\eta(i), t)} \\
&\quad + \sum_{s \succ \eta(i)} a_{i,s} \mu_j u_{(s, \xi(j))} \\
&\quad + \sum_{s \succ \eta(i), t \triangleright \xi(j)} a_{i,s} b_{j,t} u_{(s,t)} \\
&= \lambda_i \mu_j u_{\sigma(i,j)} + \sum_{(s,t) \succ \sigma(i,j)} c_{i,j,s,t} u_{(s,t)},
\end{aligned}$$

for some $c_{i,j,s,t} \in R$. As $\lambda_i \mu_j > 0$, $f \otimes g$ is thus positively triangular with respect to $\mathcal{B} \otimes \mathcal{C}$. $\square$

3.3 Order-preserving automorphisms of free groups

We now aim to build upon the results of the last section, bootstrapping to yield order-preserving automorphisms of free groups of arbitrary rank. Our first lemma is a generalization of [23, Lemma 4.5], wherein we confirm that the free group F in that proof need not be finitely generated. The proof remains largely unchanged, with two exceptions: (1) the summations appearing below over the variable j are sums of terms drawn from an infinite set (though each summation is still finite), and (2) equation (3.3) is demonstrated directly without referring to the Jacobian matrix as in [23]. Despite the changes being minor, we present the proof here for the sake of completeness, and so that the reader can confirm for themselves that the conclusion of the lemma is independent of the cardinality of the generating set of F .

We will use a tool from [12], so let's recall it. Let G be a group and $\mathbb{Z}G$ be the group ring of G over the integers. Let $\epsilon : \mathbb{Z}G \rightarrow \mathbb{Z}$ be the ring homomorphism defined by $\epsilon(\sum_{i=1}^m k_i g_i) = \sum_{i=1}^m k_i$.

Then a **derivation** in $\mathbb{Z}G$ is a map $D : \mathbb{Z}G \rightarrow \mathbb{Z}G$, such that for all $u, v \in \mathbb{Z}G$

$$D(u + v) = D(u) + D(v), \quad D(u \cdot v) = Du \cdot \epsilon(v) + u \cdot D(v).$$

To give the setup of the lemma, we consider G to be a free group F generated by $\{z_i \mid i \in J\}$, where J is not necessarily finite (unlike [23, Lemma 4.5]). By the theorem in Page 550 of [12], for each $i \in J$, there is a derivation, denoted as $D_i = \frac{\partial}{\partial z_i}$, in $\mathbb{Z}F$ such that $D_i(z_j) = \delta_{i,j}$ for all $j \in J$. (Here $\delta_{i,j}$ is the Kronecker delta function.) Let F_k be the k -th term of the lower central series of F , defined by $F_1 = F$ and $F_{k+1} = [F_k, F]$. Let H be the abelianization $F/[F, F]$ of F , written additively. Set $I = \ker(\epsilon)$, which is an ideal of the ring $\mathbb{Z}F$. For $g \in F$ and $k \in \{1, 2, \dots\}$, we have $g \in F_k$ if and only if $g - 1 \in I^k$, by the second sentence of the first paragraph on page 557 of [12]. Therefore, there is an injective homomorphism $\psi_k : F_k/F_{k+1} \rightarrow I^k/I^{k+1}$, sending $[g]$ to $[g - 1]$, where the square brackets are to be interpreted as equivalence classes in the appropriate quotient.

Lemma 3.3.1. *A basis of I^k/I^{k+1} is*

$$\left\{ [(z_{j_1} - 1) \cdots (z_{j_k} - 1)] \mid j_1, \dots, j_k \in J \right\}.$$

Proof. This generates the group because of the Taylor expansion formula (3.3) in [12]. To prove that it is $\mathbb{Z}$ -independent, we use the fact that $D_{i_1} \cdots D_{i_k}((z_{j_1} - 1) \cdots (z_{j_k} - 1)) \neq 0$ if and only if $(i_1, \dots, i_k) = (j_1, \dots, j_k)$, which can easily be proved by induction. $\square$

Hence, we can define a $\mathbb{Z}$ -module isomorphism $i_k : I^k/I^{k+1} \rightarrow H^{\otimes k}$ by

$$i_k([(z_{j_1} - 1) \cdots (z_{j_k} - 1)]) = a_{j_1} \otimes \cdots \otimes a_{j_k},$$

where $a_j := [z_j]$ is the image of z_j in H . Moreover, there is an injective $\mathbb{Z}$ -module homomorphism j_k from $H^{\otimes k}$ to $(H \otimes \mathbb{R})^k$, given by

$$j_k(a_1 \otimes \cdots \otimes a_k) = (a_1 \otimes 1) \otimes \cdots \otimes (a_k \otimes 1), \forall a_1, \dots, a_k \in H.$$

Lemma 3.3.2. *For every automorphism $\varphi : F \rightarrow F$ and every positive integer k , the following diagram commutes:*

$$\begin{array}{ccccccc}
F_k/F_{k+1} & \xrightarrow{\psi_k} & I^k/I^{k+1} & \xrightarrow{i_k} & H^{\otimes k} & \xrightarrow{j_k} & (H \otimes \mathbb{R})^{\otimes k} \\
\downarrow \varphi_k & & \downarrow \varphi'_k & & \downarrow \varphi_{ab}^{\otimes k} & & \downarrow (\varphi_{ab} \otimes Id)^{\otimes k} \\
F_k/F_{k+1} & \xrightarrow{\psi_k} & I^k/I^{k+1} & \xrightarrow{i_k} & H^{\otimes k} & \xrightarrow{j_k} & (H \otimes \mathbb{R})^{\otimes k}
\end{array}$$

where the vertical maps are induced from φ .

Proof. It is easy to check that the first rectangle and the last rectangle are commutative, because all the maps involved are canonical. Therefore, we focus on the commutativity of the middle rectangle. Given $\alpha := [(z_{j_1} - 1) \cdots (z_{j_k} - 1)] \in I^k/I^{k+1}$, it is enough to prove that $i_k(\varphi'_k(\alpha)) = \varphi_{ab}^{\otimes k}(i_k(\alpha))$ for all such α .

First, $\varphi'_k(\alpha) = [(\varphi(z_{j_1}) - 1) \cdots (\varphi(z_{j_k}) - 1)]$. To simplify the notation, we write $D_j^o(w)$ for $\epsilon(D_j(w))$ ($w \in \mathbb{Z}F$). Using the Taylor expansion formula (3.3) in [12] again, we have

$$\varphi(z_{j_s}) - 1 = \sum_j D_j^o(\varphi(z_{j_s}))(z_j - 1) + O(2),$$

where $O(2)$ is a term in I^2 . We compute

$$\begin{aligned}
i_k(\varphi'_k(\alpha)) &= i_k \left(\left[\left(\sum_j D_j^o(\varphi(z_{j_1}))(z_j - 1) \right) \cdots \left(\sum_j D_j^o(\varphi(z_{j_k}))(z_j - 1) \right) \right] \right) \\
&= \left(\sum_j D_j^o(\varphi(z_{j_1}))a_j \right) \otimes \cdots \otimes \left(\sum_j D_j^o(\varphi(z_{j_k}))a_j \right) \\
&= \left[\sum_j D_j^o(\varphi(z_{j_1}))z_j \right] \otimes \cdots \otimes \left[\sum_j D_j^o(\varphi(z_{j_k}))z_j \right].
\end{aligned} \tag{3.1}$$

Second, we have

$$\begin{aligned}
\varphi_{ab}^{\otimes k}(i_k(\alpha)) &= \varphi_{ab}^{\otimes k}(a_{j_1} \otimes \cdots \otimes a_{j_k}) \\
&= \varphi_{ab}(a_{j_1}) \otimes \cdots \otimes \varphi_{ab}(a_{j_k}) \\
&= [\varphi(z_{j_1})] \otimes \cdots \otimes [\varphi(z_{j_k})].
\end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we only need to prove that for $w \in F$, we have

$$[w] = \left[\sum_j D_j^o(w) z_j \right] \tag{3.3}$$

in H . But this is true, because $D_j^o(w)$ is the sum of the exponents of z_j that appear in w , and recall that we write the operation in H additively. $\square$

We are now ready to prove the main result of this section, which by Remark 3.2.3, is a generalization of [23, Theorem 2.6] from finitely generated free groups to free groups of arbitrary rank.

Proposition 3.3.3. *Let F be a free group, $\varphi : F \rightarrow F$ be an automorphism, set $H := F/[F, F]$, and let $\varphi_{ab} : H \rightarrow H$ be the automorphism induced by φ . If $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R}$ is positively triangular, then there is a bi-ordering $<$ on F which is invariant under φ .*

Proof. Let k be an arbitrary positive integer. Let F_k be the k -th term in the lower central series of F and $\varphi_k : F_k/F_{k+1} \rightarrow F_k/F_{k+1}$ be the automorphism induced by φ . By Lemma 3.2.6, $(\varphi_{ab} \otimes Id)^{\otimes k} : (H \otimes \mathbb{R})^{\otimes k} \rightarrow (H \otimes \mathbb{R})^{\otimes k}$ is positively triangular. By Lemma 3.2.4, there is a bi-ordering $<'_k$ of $(H \otimes \mathbb{R})^{\otimes k}$ which is invariant under $(\varphi_{ab} \otimes Id)^{\otimes k}$.

By Lemma 3.3.2, the ordering $<'_k$ can be used to define a bi-ordering $<_k$ of F_k/F_{k+1} that is invariant under φ_k . The ordering is defined by declaring that for $a, b \in F_k/F_{k+1}$, we have $a <_k b$ if and only if $j_k(i_k(\psi_k(a))) <'_k j_k(i_k(\psi_k(b)))$.

Finally, we use these orderings $<_k$ ($k = 1, 2, \dots$) to define a bi-ordering $<$ on F . It is known that $\cap_{k=1}^{\infty} F_k = \{1\}$, thus for $a \neq 1$ in F there is the largest k , denoted as $k(a)$, such that $a \in F_k$.

Then we define $a > 1$ if and only if $[a] >_{k(a)} 1$ in $F_{k(a)}/F_{k(a)+1}$, where $[a]$ indicates the equivalence class in the quotient $F_{k(a)}/F_{k(a)+1}$ which contains a . This defines a bi-ordering of F and it is invariant under φ by construction. $\square$

Proposition 3.3.3 tells us that for automorphisms φ of a free group F , a sufficient condition for φ to be order-preserving is that $\varphi_{ab} \otimes Id$ is positively triangular. However, this condition is not a necessary condition for φ to be order-preserving. We will see in Example 3.3.9 that there exist homomorphisms φ such that $\varphi_{ab} \otimes Id$ is not positively triangular, but φ is still order-preserving.

Remark 3.3.4. Let φ be an automorphism of a free group F and set $H = F/[F, F]$. If $\varphi_{ab} : H \rightarrow H$ is positively triangular, then $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R}$ is positively triangular. However, the reverse is not true, and it is for this reason that we insist $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R}$ be positively triangular in Proposition 3.3.3.

As an example of this behavior, let F be the free group generated by x_1, x_2 . Let φ be the automorphism of F given by

$$\varphi(x_1) = x_1^2 x_2 \text{ and } \varphi(x_2) = x_1^2 x_2 x_1 x_2.$$

Writing the abelian group $H = F/[F, F]$ additively and the cosets of x_1 and x_2 as $\overline{x_1}$ and $\overline{x_2}$ respectively, φ_{ab} sends $\overline{x_1}$ to $2\overline{x_1} + \overline{x_2}$ and $\overline{x_2}$ to $3\overline{x_1} + 2\overline{x_2}$ respectively. The matrix of $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R}$ is therefore $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ with eigenvalues $2 \pm \sqrt{3}$. Hence, $\varphi_{ab} : H \rightarrow H$ is not positively triangular, whereas $\varphi_{ab} \otimes Id : H \otimes \mathbb{R} \rightarrow H \otimes \mathbb{R}$ is positively triangular because the matrix A is diagonalizable over $\mathbb{R}$. Note that the map φ is thus order-preserving by Proposition 3.3.3; we can explicitly define a bi-ordering on $H \otimes \mathbb{R}$ which is invariant under $\varphi_{ab} \otimes Id$: $\overline{x_1} \otimes a + \overline{x_2} \otimes b > 0$ if and only if $a + b\sqrt{3} > 0$ for $a, b \in \mathbb{R}$.

We can check that the construction of $<$ in the proof of Proposition 3.3.3 doesn't depend on the homomorphism φ , and instead depends only on the underlying choice of ordered basis. In fact, the ordering is constructed as in our next definition, where (3.4) involves Definition 3.2.1, Definition 3.2.5 and the diagram in Lemma 3.3.2.

Definition 3.3.5. Let F be a free group and $\mathcal{B}$ be an ordered basis of $(F/[F, F]) \otimes \mathbb{R}$. Let $<_k$ denote the ordering of F_k/F_{k+1} defined by $a <_k b$ if and only if

$$j_k(i_k(\psi_k(a))) <_{\mathcal{B}^{\otimes k}} j_k(i_k(\psi_k(b))). \quad (3.4)$$

Let $<$ be the bi-ordering of F induced by the orderings $<_k$ as in the proof of Proposition 3.3.3. We call $<$ the **bi-ordering of F associated with $\mathcal{B}$** and denote it by $<_{\mathcal{B}}$.

We can thus, upon reviewing the proof, give a version of Proposition 3.3.3 that makes explicit the invariant bi-ordering.

Theorem 3.3.6. Let F be a free group and $\mathcal{B}$ be an ordered basis of $(F/[F, F]) \otimes \mathbb{R}$. Let $<_{\mathcal{B}}$ be the bi-ordering of F associated with $\mathcal{B}$. If $\varphi : F \rightarrow F$ is an automorphism such that

$$\varphi_{ab} \otimes Id : (F/[F, F]) \otimes \mathbb{R} \rightarrow (F/[F, F]) \otimes \mathbb{R}$$

is positively triangular with respect to $\mathcal{B}$, then $<_{\mathcal{B}}$ is invariant under φ .

As an application of Theorem 3.3.6, we are now ready to prove Theorem 3.3.7, our main result from the introduction.

Theorem 3.3.7. Let F be the free group generated by $\{x_i \mid i \in I\}$ where I is a nonempty set. Suppose that $\varphi : F \rightarrow F$ is an automorphism satisfying

$$\varphi(x_i) = w_i x_{\sigma(i)} w_i^{-1}, \forall i \in I,$$

where $w_i \in F$ and σ is a bijection on I . Assume that σ fixes $i_0 \in I$. Define the homomorphism $h : F \rightarrow \mathbb{Z}$ by $h(x_{i_0}) = 1$ and $h(x_j) = 0$ for all $j \neq i_0$. Assume that for each finite orbit $\mathcal{O}$ of σ , we have

$$\gcd \left(|\mathcal{O}|, \sum_{i \in \mathcal{O}} h(w_i) \right) = 1.$$

Then there exists a bi-ordering of F invariant under φ .

Proof. Let K be the kernel of h . We see that $\varphi(K) = K$ and thus the restriction $\varphi|_K$ is an automorphism of K . Let $\psi : F \rightarrow F$ be the conjugation $\alpha \rightarrow x_{i_0} \alpha x_{i_0}^{-1}$. Similarly, we can verify that $\psi|_K$ is an automorphism on K . We now proceed in two steps.

First, we assume that there is a bi-ordering $<_K$ on K which is invariant under both $\varphi|_K$ and $\psi|_K$. Under this assumption, we now show that the lexicographic left-ordering on F coming from the short exact sequence

$$1 \rightarrow K \xrightarrow{i} F \xrightarrow{h} \{0\}$$

is in fact a bi-ordering invariant under φ . Here, i is the inclusion of K in F . The positive cone of F is defined to be

$$P_F := i(P_K) \cup h^{-1}(\mathbb{Z}_{>0}) = P_K \cup \{\alpha \in F_n \mid h(\alpha) > 0\},$$

where P_K is the positive cone of K corresponding to $<_K$. To verify that it gives a bi-ordering, we only need to verify that $x_{i_0}^k P_K x_{i_0}^{-k} = P_K$ for all $k \in \mathbb{Z}$. This is true because $x_{i_0} P_K x_{i_0}^{-1} = \psi|_K(P_K) = P_K$ as $<_K$ is invariant under $\psi|_K$. To verify that it is invariant under φ , we want to show that $\varphi(P_F) \subseteq P_F$. This is true, because $\varphi(P_K) = \varphi|_K(P_K) = P_K$ using that $<_K$ is invariant under $\varphi|_K$; and if $\alpha \in F$ satisfies $h(\alpha) > 0$, then we have $h(\varphi(\alpha)) = h(\alpha) > 0$.

So, to prove the theorem it suffices to prove that there is a bi-ordering on K which is invariant under both $\varphi|_K$ and $\psi|_K$, which is our second step in the proof.

It can be shown that K has a free basis:

$$\left\{ x_{i_0}^j x_i x_{i_0}^{-j} \mid i \in I \setminus \{i_0\}, j \in \mathbb{Z} \right\}.$$

For $\alpha \in K$, we write $\bar{\alpha}$ for the image of α in the quotient $H := K/[K, K]$, which is the abelianization of K . Then H is a free abelian group and

$$\mathfrak{B} := \{A_{i,j} \mid i \in I \setminus \{i_0\}, j \in \mathbb{Z}\} \text{ with } A_{i,j} := \overline{x_{i_0}^j x_i x_{i_0}^{-j}} \quad (3.5)$$

is a basis of H . By Theorem 3.3.6 and the second sentence of Remark 3.3.4, we only need to prove that there is an ordered basis $\mathcal{B}$ of H , such that $(\varphi|_K)_{ab}$ and $(\psi|_K)_{ab}$ are positively triangular with respect to $\mathcal{B}$. We proceed with the following sub-steps:

1. Claim: For the abelianization $(\varphi|_K)_{ab}$ of $\varphi|_K$ on H , $i \in I$ and $j \in \mathbb{Z}$, we have

$$(\varphi|_K)_{ab}(A_{i,j}) = A_{\sigma(i),j+h(w_i)}. \quad (3.6)$$

Proof of claim: Note that we have $\overline{x_{i_0}^r} A_{i,j} \overline{x_{i_0}^{-r}} = A_{i,j+r}$, and that for $k \neq i_0$ we have $\overline{x_k} = A_{k,0}$ and therefore $\overline{x_k^r} A_{i,j} \overline{x_k^{-r}} = A_{i,j}$ since H is abelian. Therefore, for $\alpha \in F$, we have $\overline{\alpha} A_{i,j} \overline{\alpha^{-1}} = A_{i,j+h(\alpha)}$. Now for the abelianization $(\varphi|_K)_{ab}$ of $\varphi|_K$ on H , $i \in I \setminus \{i_0\}$ and $j \in \mathbb{Z}$, we have

$$\begin{aligned} (\varphi|_K)_{ab}(A_{i,j}) &= \overline{\varphi(x_{i_0}^j x_i x_{i_0}^{-j})} \\ &= \overline{(w_{i_0} x_{i_0} w_{i_0}^{-1})^j w_i A_{\sigma(i),0} w_i^{-1} (w_{i_0} x_{i_0} w_{i_0}^{-1})^{-j}} \\ &= A_{\sigma(i),j+h(w_i)}. \end{aligned}$$

□

2. For each infinite orbit $\mathcal{O}$ of σ , we define $V_{\mathcal{O},t} \in H$ for $t \in \mathcal{O} \times \mathbb{Z}$ as the following:

$$V_{\mathcal{O},t} = V_{\mathcal{O},(i,j)} = A_{i,j} \text{ for } t = (i,j) \in \mathcal{O} \times \mathbb{Z}.$$

We have, for $t = (i,j) \in \mathcal{O} \times \mathbb{Z}$,

$$(\psi|_K)_{ab}(V_{\mathcal{O},t}) = (\psi|_K)_{ab}(A_{i,j}) = A_{i,j+1} = V_{\mathcal{O},(i,j+1)}, \quad (3.7)$$

$$(\varphi|_K)_{ab}(V_{\mathcal{O},t}) = (\varphi|_K)_{ab}(A_{i,j}) = A_{\sigma(i),j+h(w_i)} = V_{\mathcal{O},(\sigma(i),j+h(w_i))}. \quad (3.8)$$

3. For each finite orbit $\mathcal{O}$ which is not equal to the orbit $\{i_0\}$, we define $V_{\mathcal{O},t} \in H$ for $t \in \mathbb{Z}$ as follows:

We fix a tuple $(k_1, \dots, k_r)$ such that $\mathcal{O} = \{k_1, \dots, k_r\}$ with $r = |\mathcal{O}|$, $\sigma(k_i) = k_{i+1}$ for $1 \leq i \leq r-1$ and $\sigma(k_r) = k_1$. Let $h_{\mathcal{O}} = \sum_{i=1}^r h(w_{k_i})$. Note that $\gcd(h_{\mathcal{O}}, r) = 1$ by the assumption. Let $y_i = \sum_{1 \leq j \leq i-1} -h(w_{k_j})$ for $i = 1, 2, \dots, r$. (Then we have, for example, $y_1 = 0$, $y_r - h(w_{k_r}) = -h_{\mathcal{O}}$.) Since $\gcd(h_{\mathcal{O}}, r) = 1$, given $t \in \mathbb{Z}$, there is a unique $i = i(t) \in \{1, 2, \dots, r\}$ and a unique $j = j(t) \in \mathbb{Z}$ such that $t = r(y_i + j) + ih_{\mathcal{O}}$. Define a map $k : \mathbb{Z} \rightarrow \mathcal{O}$ by $k(t) = k_{i(t)}$. Now, let

$$V_{\mathcal{O},t} = V_{\mathcal{O},r(y_i+j)+ih_{\mathcal{O}}} := A_{k_i,j} = A_{k_{i(t)},j(t)} = A_{k(t),j(t)} \in H. \quad (3.9)$$

Note that the map $E_{\mathcal{O}} : \mathbb{Z} \rightarrow \mathcal{O} \times \mathbb{Z}$ given by $t \mapsto (k(t), j(t))$ is a bijection.

4. Claim: For a finite orbit $\mathcal{O}$ we have, for $t \in \mathbb{Z}$,

$$(\psi|_K)_{ab}(V_{\mathcal{O},t}) = V_{\mathcal{O},t+|\mathcal{O}|}, \quad (3.10)$$

$$(\varphi|_K)_{ab}(V_{\mathcal{O},t}) = V_{\mathcal{O},t+h_{\mathcal{O}}}, \quad (3.11)$$

where $h_{\mathcal{O}} = \sum_{i \in \mathcal{O}} h(w_i)$.

Proof of claim. Fix an arbitrary $t \in \mathbb{Z}$. We have $t = r(y_i + j) + ih_{\mathcal{O}}$ where y_i , $i = i(t)$ and $j = j(t)$ were defined in (3). By (3.9), we have $V_{\mathcal{O},t} = A_{k_i,j}$.

Thus we find

$$(\psi|_K)_{ab}(V_{\mathcal{O},t}) = \psi(A_{k_i,j}) = A_{k_i,j+1} = V_{\mathcal{O},t+r} = V_{\mathcal{O},t+|\mathcal{O}|},$$

proving (3.10).

To prove (3.11), we note that

$$(\varphi|_K)_{ab}(V_{\mathcal{O},t}) = (\varphi|_K)_{ab}(A_{k_i,j}). \quad (3.12)$$

We have the following, by (3.6) and recalling the definition of y_i :

(a) If $1 \leq i(t) \leq r-1$, (3.12) is equal to

$$A_{k_{i+1}, j+h(w_{k_i})} = V_{\mathcal{O}, r(j+h(w_{k_i})+y_{i+1})+(i+1)h_{\mathcal{O}}} = V_{\mathcal{O}, t+h_{\mathcal{O}}}$$

as desired.

(b) If $i(t) = r$, (3.12) is equal to

$$A_{k_1, j+h(w_{k_r})} = V_{\mathcal{O}, r(j+h(w_{k_r})+y_1)+h_{\mathcal{O}}} = V_{\mathcal{O}, t+h_{\mathcal{O}}}$$

as desired.

□

5. Define an ordered basis of H .

Let $\mathbb{O}(\sigma)$ be the set of orbits of σ which are not equal to $\{i_0\}$. For a finite orbit $\mathcal{O}$, we let $\mu(\mathcal{O}) = \mathbb{Z}$. For an infinite orbit $\mathcal{O}$, we let $\mu(\mathcal{O}) = \mathcal{O} \times \mathbb{Z}$. Then

$$\mathfrak{B}' := \{V_{\mathcal{O}, t} \mid \mathcal{O} \in \mathbb{O}(\sigma), t \in \mu(\mathcal{O})\}$$

is a basis of H . To see this, let us check that $\mathfrak{B}' = \mathfrak{B}$, the latter being the basis of H given by (3.5). First, it is easy to check that $\mathfrak{B}' \subseteq \mathfrak{B}$ by definition. Now, let $A_{i,j} \in \mathfrak{B}$ and we want to show $A_{i,j} \in \mathfrak{B}'$. Consider two cases. For the first case, suppose i is contained in an infinite orbit $\mathcal{O}$. Then $(i, j) \in \mu(\mathcal{O}) = \mathcal{O} \times \mathbb{Z}$ and thus $A_{i,j} = V_{\mathcal{O}, (i,j)}$ is in $\mathfrak{B}'$. For the second case, suppose i is contained in a finite orbit $\mathcal{O}$ distinct from $\{i_0\}$. Recall that $E_{\mathcal{O}}(t) = (k(t), j(t))$ is a bijection from $\mathbb{Z}$ to $\mathcal{O} \times \mathbb{Z}$. Hence, there is a unique $t \in \mathbb{Z}$, such that $k(t) = i$ and $j(t) = j$. Now we have $V_{\mathcal{O}, t} = A_{k(t), j(t)} = A_{i,j}$, thus $A_{i,j}$ is contained in $\mathfrak{B}'$.

We now define a total order $\prec$ on the index set

$$\mathcal{I} := \{(\mathcal{O}, t) \mid \mathcal{O} \in \mathbb{O}(\sigma), t \in \mu(\mathcal{O})\}.$$

First, let $<'$ be an arbitrary total order of $\mathbb{O}(\sigma)$. Second, we define an ordering $<$ on each $\mu(\mathcal{O})$. If $\mathcal{O}$ is finite, $<$ is the usual order on $\mu(\mathcal{O}) = \mathbb{Z}$. When $\mathcal{O}$ is infinite, let $(i, j), (i', j') \in \mu(\mathcal{O}) = \mathcal{O} \times \mathbb{Z}$. We write $(i, j) < (i', j')$ if and only if $i' \in \{\sigma^k(i) \mid k \geq 1\}$ or $(i', j') \in \{(i, j + k) \mid k \geq 1\}$.

The order $\prec$ is defined lexicographically: $(\mathcal{O}, t) \prec (\mathcal{O}', t')$ if and only if $\mathcal{O} <' \mathcal{O}'$ or $\mathcal{O} = \mathcal{O}'$ and $t < t'$ in $\mu(\mathcal{O})$. Now set $\mathcal{B} = (\mathfrak{B}', \prec)$, which is an ordered basis of H .

6. Last, we check that both $(\varphi|_K)_{ab}$ and $(\psi|_K)_{ab}$ are positively triangular with respect to $\mathcal{B}$.

This essentially follows from (3.7), (3.8), (3.10) and (3.11). The details are as follows.

First, define $\eta : \mathcal{I} \rightarrow \mathcal{I}$ by

$$\eta(\mathcal{O}, t) = \begin{cases} (\mathcal{O}, (\sigma(i), j + h(w_i))) & \text{if } |\mathcal{O}| = \infty \text{ and } t = (i, j) \in \mathcal{O} \times \mathbb{Z}; \\ (\mathcal{O}, t + h_{\mathcal{O}}) & \text{if } |\mathcal{O}| < \infty \text{ and } t \in \mathbb{Z}. \end{cases}$$

We can then verify that η preserves the order $\prec$. Therefore, $(\varphi|_K)_{ab}$ is positively triangular with respect to $\mathcal{B}$ by (3.8) and (3.11).

Second, define $\zeta : \mathcal{I} \rightarrow \mathcal{I}$ by

$$\zeta(\mathcal{O}, t) = \begin{cases} (\mathcal{O}, (i, j + 1)) & \text{if } |\mathcal{O}| = \infty \text{ and } t = (i, j) \in \mathcal{O} \times \mathbb{Z}; \\ (\mathcal{O}, t + |\mathcal{O}|) & \text{if } |\mathcal{O}| < \infty \text{ and } t \in \mathbb{Z}. \end{cases}$$

We can verify that ζ preserves the order $\prec$. Therefore, $(\psi|_K)_{ab}$ is positively triangular with respect to $\mathcal{B}$ by (3.7) and (3.10).

□

Theorem 3.3.7 says that if an automorphism as in the statement of the theorem satisfies the coprime condition: $\gcd(h_{\mathcal{O}}, |\mathcal{O}|) = 1$ for each finite orbit $\mathcal{O}$ of σ , then the automorphism preserves a bi-ordering of the free group. If we replace this coprime condition with the (weaker) non-vanishing

condition: $h_{\mathcal{O}} \neq 0$ for each finite orbit of σ with $|\mathcal{O}| \geq 2$, we can still prove that the automorphism preserves a left-ordering.

The proof of this fact remains almost the same; the major change is the definition of $V_{\mathcal{O},t} \in H$ for $t \in \mathbb{Z}$ and each finite orbit $\mathcal{O}$ of σ , which is given by the following. Assume that $\mathcal{O} = \{k_1, \dots, k_r\}$ with $r = |\mathcal{O}|$, $\sigma(k_i) = k_{i+1}$ for $1 \leq i \leq r-1$. If $r = 1$, we just define $V_{\mathcal{O},t} = A_{k_1,t} \in H$, where $t \in \mathbb{Z}$. Now consider $r \geq 2$. For each $t \in \mathbb{Z}$, there are unique integers $i = i(t), j = j(t), m = m(t)$ (by the Euclidean algorithm) such that

$$t = (mr + j)h_{\mathcal{O}} + i \text{ with } 0 \leq i \leq |h_{\mathcal{O}}| - 1, 1 \leq j \leq r, m \in \mathbb{Z}.$$

Then we define

$$V_{\mathcal{O},t} = A_{k_{j(t)}, m(t)h_{\mathcal{O}} + h_{j(t)} + i(t)}, \quad \forall t \in \mathbb{Z},$$

where $h_j := \sum_{1 \leq p \leq j-1} h(w_{k_p})$ for $j = 1, 2, \dots, r$. We then verify that the constructed bi-ordering on K is still invariant under ϕ and that the left-ordering from the exact sequence is invariant under φ .

Example 3.3.8. Consider the free group F with basis $\{x_1, x_2, x_3\}$ and $\varphi : F \rightarrow F$ given by $x_1 \mapsto x_3x_2x_3^{-1}, x_2 \mapsto x_3x_1x_3^{-1}, x_3 \mapsto x_3$. With $i_0 = 3$ we see $h_{\mathcal{O}} = 2$ for the orbit $\mathcal{O} = \{1, 2\}$, and so φ preserves a left-ordering, although it clearly does not preserve a bi-ordering.

Example 3.3.9. Theorem 3.3.7 gives us an example of an automorphism φ on a free group F which is order-preserving but $\varphi_{ab} \otimes Id$ is not positively triangular on $F/[F, F] \otimes \mathbb{R}$. For instance, let F be the free group with basis $\{x_1, x_2, x_3\}$ and let φ be the automorphism of F defined by

$$x_1 \mapsto x_3x_2x_3^{-1}, x_2 \mapsto x_1, x_3 \mapsto x_3.$$

Then φ is order-preserving by Theorem 3.3.7. However, $\varphi_{ab} \otimes Id$ is not positively triangular on $F/[F, F] \otimes \mathbb{R}$.

To see this, suppose that $\varphi_{ab} \otimes Id : F/[F, F] \otimes \mathbb{R} \rightarrow F/[F, F] \otimes \mathbb{R}$ is positively triangular, then

by Lemma 3.2.4 $\varphi_{ab} \otimes Id$ preserves a bi-ordering on $F/[F, F] \otimes \mathbb{R}$. But this is impossible as $\varphi_{ab} \otimes Id$ swaps the two elements $\overline{x_1} \otimes 1$ and $\overline{x_2} \otimes 1$.

3.4 The fundamental group of the magic manifold is bi-orderable

Recall from the introduction that the braid group B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$, and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$. Also recall that the Artin action of B_n on F_n is determined by the following action of the generators σ_i of B_n on the generators x_j of F_n :

$$x_j^{\sigma_i} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i \\ x_i & \text{if } j = i+1 \\ x_j & \text{if } j \neq i, i+1. \end{cases}$$

We will call a braid $\beta \in B_n$ order-preserving if its action on F_n induces an order-preserving automorphism $F \rightarrow F$.

3.4.1 Minimal volume 3-manifolds

In [18], the authors initiated a study of bi-orderability of the fundamental groups of minimal volume cusped hyperbolic 3-manifolds. For such manifolds, they determined whether or not the fundamental group is bi-orderable for all examples with five or fewer cusps, with one exception. The fundamental group of a three-cusped manifold, called the “magic manifold”, could not be addressed by the techniques of [18].

Proposition 3.4.1. *The fundamental group of the magic manifold is bi-orderable.*

Proof. By [18, Section 5.5], the fundamental group of the magic manifold is bi-orderable if and only if the braid $\sigma_1^2 \sigma_2^{-1}$ is order-preserving. Let F_3 denote the free group on generators x_1, x_2, x_3 .

One can verify that $\sigma_1^2\sigma_2^{-1}$ induces a map $\varphi : F_3 \rightarrow F_3$ given by

$$\varphi(x_1) = x_1x_3x_1x_3^{-1}x_1^{-1}, \varphi(x_2) = x_1x_3x_1^{-1}, \varphi(x_3) = x_3^{-1}x_2x_3,$$

the underlying permutation σ is the transposition $(2\ 3)$.

In order to apply Theorem 3.3.7, we define $h : F_3 \rightarrow \mathbb{Z}$ by $h(x_1) = 1, h(x_2) = 0, h(x_3) = 0$. Observe that σ has a single orbit $\mathcal{O} = \{2, 3\}$ that does not contain 1, and that $h_{\mathcal{O}} = h(x_1) + h(x_3^{-1}) = 1$. Since $\gcd(2, 1) = 1$, the braid $\sigma_1^2\sigma_2^{-1}$ is order-preserving. $\square$

This resolves [18, Question 5.6] in the affirmative. Moreover, from the proof, we see that the braid $\beta = \sigma_1^2\sigma_2^{-1}$ is order-preserving. Note that multiplication of this by σ_2^2 does not change the underlying permutation, and also does not alter the value of $h_{\mathcal{O}}$ in the above discussion. The same is true of multiplication by σ_2^{-2} . Therefore, the braid $\sigma_1^2\sigma_2^{2n-1}$ is also order-preserving. On the other hand, $\sigma_1^2\sigma_2^{2n}$, being a pure braid, is also order-preserving. Therefore, we have the following.

Theorem 3.4.2. *For every integer n , the braid $\sigma_1^2\sigma_2^n$ is order-preserving.*

We note that it was shown by Johnson, Scherich and Turner that the braids $\sigma_1\sigma_2^{2k+1}$ are not order-preserving for any integer k [17]. Such 3-braids have underlying permutation a single cycle. On the other hand, Jonathan and Le recently proved that there exist order-preserving braids whose permutation is a single cycle [16].

Chapter 4

Generalized Torsions in Amalgams

The goal of the remainder of this thesis is to solve Problem 16.48 on Page 98 of The Kourovka Notebook [20]: Does there exist a group which is generalized torsion free (GTF) but not left-orderable? We conjectured that the answer is positive and we will construct such a group in Chapter 5. Our example uses the amalgam $G = A *_C B$ of two groups A and B , amalgamated by a common subgroup C . In this chapter, we study the generalized torsions in amalgams. The main result of this chapter is Theorem 4.4.6, which gives a sufficient condition ensuring an amalgam be GTF. This theorem will be used in Chapter 5 in our construction of the amalgam which is GTF and non-LO.

4.1 Introduction to amalgams

We refer to Serre's book [24] for definitions and notations about amalgams of groups. Let G_i ($i \in I$) be a collection of groups, C be a group and for each $j \in I$, and $i_j : C \rightarrow G_j$ be an injective homomorphism. By Proposition 1 on Page 1 of [24], there is a unique group G ¹ and homomorphisms $f : C \rightarrow G$, $f_j : G_j \rightarrow G$ with $f = f_j \circ i_j$ for all $j \in I$ such that the universal property is satisfied: if there is a group H , homomorphisms $h : C \rightarrow H$ and $h_j : G_j \rightarrow H$ with

¹up to isomorphism, of course

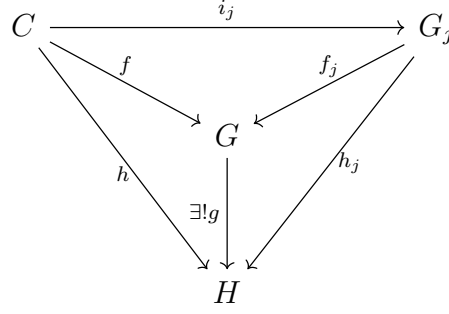


Figure 4.1: Commutative Diagram For Amalgams

$h = h_j \circ i_j$ for all $j \in I$, then there is a unique homomorphism $g : G \rightarrow H$ such that $g \circ f = h$ and $g \circ f_j = h_j$ for all $j \in I$ (see the commutative diagram in Figure 4.1). Furthermore, the homomorphisms f and f_j ($j \in I$) are injective, by Theorem 1 on Page 3 of [24]. Therefore, we can (and do) view C as a common subgroup of G_j ($j \in I$) and each G_j as a subgroup of G . We denote by $G = *_C G_i$ this unique group associated with $(C, \{G_j, i_j | j \in I\})$, and call it the **free product** of G_i ($i \in I$) **amalgamated** by (their common subgroup) C . When there is no risk of confusion, we also call $G = *_C G_i$ the **amalgam** of G_i ($i \in I$), without specifying the common subgroup amalgamated.

In the next chapter, we will focus on the amalgam of two groups; i.e., we will mostly consider $I = \{1, 2\}$ (and C a common subgroup of G_1 and G_2). In this case, we write the amalgam as $G = G_1 *_C G_2$. If C is the trivial subgroup $\{1\}$, we denote this as $G_1 * G_2$ and call it the free product of G_1 and G_2 . For example, the free group $G = \langle a, b \rangle$ is the free product $\langle a \rangle * \langle b \rangle$. Another scenario we consider quite often is as follows. Let C and D be subgroups of G_1 and G_2 , respectively. Assume that $\phi : C \rightarrow D$ is an isomorphism. Let i_1 be the inclusion of C into G_1 . Let $i_2 = i_D \circ \phi$, where i_D is the inclusion of D into G_2 . Then i_1, i_2 are injective homomorphisms from C to G_1 and G_2 respectively. Now we write the amalgam associated with $(C, \{G_j, i_j | j \in \{1, 2\}\})$ as $G = G_1 *_\phi G_2$, and call it the free product of G_1 and G_2 amalgamated by ϕ . By the proof of Proposition 1 on Page 1 of [24], we know that if G_i has presentation $G_i = \langle S_i | R_i \rangle$, then the

amalgam has presentation

$$G_1 *_\phi G_2 = \langle S_1 \sqcup S_2 | R_1 \sqcup R_2 \sqcup \{c^{-1}\phi(c) | c \in C\} \rangle.$$

Note that the unions are disjoint unions. We give an example to further explain this.

Example 4.1.1. If $G_1 = G_2 = A$, then the amalgam is written as $G = A *_C A$. Now let's say A is the free group $\langle a, b \mid \rangle$ and C is the subgroup of A generated by $\{ab, a^2b^{-2}\}$. To write $A *_C A$ using generators and relations, we let $B = \langle c, d \mid \rangle$ and D be the subgroup of B generated by $\{cd, c^2d^{-2}\}$. Note that D and B are “copies” of C and A , respectively, in the sense that the isomorphism $\Phi : A \rightarrow B$ defined by $\Phi(a) = c, \Phi(b) = d$ restricted to C gives an isomorphism $\phi := \Phi|_C$ from C to D . Now $A *_C A$ can be written as follows:

$$A *_\phi B = \langle a, b, c, d \mid ab = cd, a^2b^{-2} = c^2d^{-2} \rangle.$$

This amalgam will be mentioned again (see Example 4.6.4).

Let us return to the definition of amalgams. Recall that we view C and G_i as subgroups of G . By Theorem 1 on Page 3 of [24], every element $g \in G$ can be written as $g = cg_1g_2 \cdots g_n$ with $n \geq 0$, $c \in C$ and $g_j \in G_{i_j} \setminus C$ satisfying $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, n-1$, and the **index vector** $(i_1, \dots, i_n)$ is uniquely determined by g . (The index vector is empty $()$ when g is in C .) We say $g = cg_1g_2 \cdots g_n$ is an **alternating product**, each g_i is a **component** of g and that the j th component of g is from G_{i_j} (and from $G_{i_j} \setminus C$). We define the **length** of g to be $l(g) = n$. For example, $l(g) = 0$ (respectively, $l(g) = 1$) if and only if $g \in C$ (respectively, $g \in \cup_{i \in I} G_i \setminus C$). Note that when $l(g) \geq 1$, we can require the “ c ” in the alternating product $g = cg_1g_2 \cdots g_n$ to be the unit element 1 and thus write g in the form $g = g_1g_2 \cdots g_n$. Also note that by the uniqueness of index vectors, we have $G_i \cap G_j = C$ for all $i, j \in I$ with $i \neq j$.

Given a tuple $(g_1, \dots, g_n)$ with $g_i \in G$, we say that $(g_1, \dots, g_n)$ is **reduced** if $l(g_1 \cdots g_n) = l(g_1) + \cdots + l(g_n)$. When no confusion will arise from doing so, we will simply say that $g_1 \cdots g_n$ is a reduced product (of n terms) and make no reference to the underlying tuple. If g_1g_2 is a reduced

product, we say that g_1 (respectively, g_2) is a **left factor** (respectively, **right factor**) of g_1g_2 . Note that in the case that $l(g_i) \geq 1$ for all $i = 2, \dots, n-1$, the product $g_1 \cdots g_n$ is reduced if and only if $l(g_i g_{i+1}) = l(g_i) + l(g_{i+1})$ for all i with $1 \leq i \leq n-1$.

For $g, h \in G$, write g as an alternating product $g = cx_m \cdots x_1$ with $c \in C$ and write $h = y_1 \cdots y_n d$, where $y_1 \cdots y_n$ is an alternating product (with $l(y_i) = 1$) and $d \in C$. Denote by $K(g, h)$ the maximum $k \geq 0$ such that $x_k \cdots x_1 y_1 \cdots y_k \in C$; we call $K(g, h)$ the **cancellation number** of the product gh . Note that if g is written as another alternating product $g = c'x'_k \cdots x'_1$ (with $c' \in C$), then by the uniqueness of index vectors we have $m = k$ and that for every i with $1 \leq i \leq m$, there is an $c_i \in C$ such that $x_i \cdots x_1 = c_i x'_i \cdots x'_1$. Similarly, if we write h in another form $h = y'_1 \cdots y'_l d'$ where $y'_1 \cdots y'_l$ is an alternating product (with $l(y'_i) = 1$) and $d' \in C$, we have $l = n$ and for each i with $1 \leq i \leq n$, there is an $d_i \in C$ such that $y_1 \cdots y_i = y'_1 \cdots y'_i d_i$. Therefore, the cancellation number $K(g, h)$ depends only on g and h , and not how the alternating products are written.

To end this section, let's see how to use amalgams to construct groups which are torsion-free and non-LO. We remark that we use the same idea to construct a non-LO group in Chapter 5. The only difference is that we want the group there to be GTF as well, not just TF, and that's why the construction there is more complicated than the next example.

Example 4.1.2. Consider the free groups $A = \langle a, b \mid \rangle$ and $B = \langle c, d \mid \rangle$. Let C be the subgroup of A generated by $\{\alpha_1, \dots, \alpha_8\}$, where $\alpha_1, \dots, \alpha_8$ are $ab, a^2b^2, a^3b^3, a^4b^4, a^5b^{-5}, a^6b^{-6}, a^7b^{-7}, a^8b^{-8}$, respectively. Similarly, let D be the subgroup of B generated by $\{\beta_1, \beta_2, \dots, \beta_8\}$, where $\beta_1, \dots, \beta_8$ are $cd, c^2d^{-2}, c^{-3}d^3, c^{-4}d^{-4}, c^5d^5, c^6d^{-6}, c^{-7}d^7, c^{-8}d^{-8}$ respectively. Then $\phi : \alpha_i \mapsto \beta_i$ ($i = 1, 2, \dots, 8$) defines an isomorphism from C to D .

Claim: The amalgam $G = A *_\phi B$ is torsion-free and non-LO.

Proof. Since it's well-known that free groups are torsion-free (and in fact generalized torsion free), G is torsion-free by Corollary 2 on Page 6 of [24].²

To show that it's non-LO, we assume that $<$ is a left-ordering of G . Recall that we say an

²It is also easy to prove this from scratch, only using the uniqueness of the index vector of elements in amalgams.

element g and its sign are positive (respectively, negative) if $g > 1$ (respectively, $g < 1$). Note that a, b (respectively, c, d) are nontrivial in G as they are nontrivial in A (respectively, B). Thus, each of a, b, c, d is either positive or negative. There are two cases to consider:

1. If a, b have the same sign, then $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ have the same sign.³ Note that there is at least one negative and one positive among $\beta_1, \beta_2, \beta_3, \beta_4$, no matter what the signs of c, d are. This contradicts that $\alpha_i = \beta_i$ for $i = 1, 2, 3, 4$ in G .
2. If a, b have different signs, then $\alpha_5, \alpha_6, \alpha_7, \alpha_8$ have the same sign. Note that there is at least one negative and one positive among $\beta_5, \beta_6, \beta_7, \beta_8$. This contradicts that $\alpha_i = \beta_i$ for $i = 5, \dots, 8$ in G .

□

4.2 Some results about products in amalgams

In this section, we prepare some results which will be used in the next section. Let $G = *_C G_i$ ($i \in I$) be an amalgam of groups. If $g \in G$ has index vector $(i_1, \dots, i_n)$ and $n \geq 1$, then we call i_1 (respectively, i_n) the left-end (respectively, right-end) index of g , denoted as $LEI(g)$ (respectively, $REI(g)$). Note that $LEI(g)$ and $REI(g)$ are undefined if $g \in C$.

Given a tuple $(g_1, \dots, g_n)$ where $g_k \in G$ for $k = 1, \dots, n$ with at least one $l(g_k) \geq 1$, let i (respectively, j) be the minimal (respectively, maximal) index such that $l(g_i) \geq 1$ (respectively, $l(g_j) \geq 1$). If $LEI(g_1 \cdots g_n) = LEI(g_i)$ (respectively, $REI(g_1 \cdots g_n) = REI(g_j)$), we say the tuple $(g_1, \dots, g_n)$ is **left end-preserving** (respectively, **right end-preserving**). We say the tuple is **end-preserving** if it is both left end-preserving and right end-preserving. As in the case of reduced products, we will sometimes make no reference to the underlying tuple, and instead write that $g_1 \cdots g_n$ is left or right end-preserving, when no confusion arises from doing so. Note that,

³This is because in LO groups, products of positive (respectively, negative) elements are positive (respectively, negative).

by definition, $l(g_1 \cdots g_n) \geq 1$ if the tuple $(g_1, \dots, g_n)$ is either left end-preserving or right end-preserving.

For the definitions of reduced products and end-preserving products, we will often bracket a product to indicate the underlying tuple. As an example, if we say the product $g_1 \cdots g_n$ is reduced, this means that the tuple $(g_1, \dots, g_n)$ is reduced; whereas if we say that the product $(g_1 g_2) g_3 \cdots g_n$ is reduced, we mean that the tuple $(g_1 g_2, g_3, \dots, g_n)$ is reduced. We now give an example to illustrate the necessity of doing this. Consider the amalgam $\langle a \mid \rangle * \langle b \mid \rangle$. Let $g_1 = a, g_2 = a^{-1}b, g_3 = b^{-1}a$. Then $g_1 g_2 g_3$ and $g_1(g_2 g_3)$ are end-preserving but $(g_1 g_2) g_3$ is not left end-preserving.

Lemma 4.2.1. *Let $\alpha, \beta \in G$ and $k = K(\alpha, \beta)$.*

- (1) *If $l(\alpha) > 0$, then the product $\alpha\beta$ is left end-preserving if and only if $k < l(\alpha)$.*
- (2) *Assume that $\alpha\beta$ is left end-preserving and β_1 is a left factor of β . Also assume that $l(\alpha) > 0$ or $l(\beta_1) > 0$. Then $\alpha\beta_1$ is left end-preserving.*

Proof. First, we prove (1). As it is clearly true if $l(\beta) = 0$, we assume that $l(\beta) > 0$. Write $\alpha = x_m \cdots x_1$ and $\beta = y_1 \cdots y_n$ as alternating products, so $k \leq m$ and $k \leq n$. Also, we have $c_i := x_i \cdots x_1 y_1 \cdots y_i \in C$ for all i with $1 \leq i \leq k$. Consider two cases.

- (i) $k \geq l(\alpha)$. We then have $k = m \leq n$. If $k = m = n$, then $\alpha\beta = c_m \in C$ and thus $\alpha\beta$ is not left end-preserving. On the other hand, if $k = m < n$, then $\alpha\beta = c_m y_{m+1} \cdots y_n$. Because y_m and y_{m+1} have different indices and x_m and y_m have the same index (as $c_{m-1}, c_m = x_m c_{m-1} y_m \in C$), we have $LEI(\alpha\beta) \neq LEI(\alpha)$, and thus the product $\alpha\beta$ is not left end-preserving.
- (ii) $k < l(\alpha)$. Then $\alpha\beta$ is equal to one of the following alternating products

$$x_m \cdots x_{k+2} (x_{k+1} c_k y_{k+1}) y_{k+2} \cdots y_n \text{ or } x_m \cdots x_{k+2} (x_{k+1} c_k) y_{k+1} \cdots y_n,$$

where the brackets are used to indicate a single component of the alternating product. This shows that the product $\alpha\beta$ is left end-preserving.

Now we prove (2). If $l(\alpha) = 0$, then $l(\beta_1) > 0$ and $\alpha\beta_1$ is (left) end-preserving by definition. Otherwise, the claim follows from (1) as we have $K(\alpha, \beta_1) \leq K(\alpha, \beta)$. $\square$

We remark there is an analog to Lemma 4.2.1 concerning right end-preserving, cancellation numbers and right factors. We only state one of the two versions of this result, as the other version can be deduced easily from the one provided and/or proved similarly. Note that this situation happens several times later, and this occurs quite frequently in Chapter 5.

The next result, which follows from Lemma 4.2.1, will (only) be used in the proof of Theorem 5.2.13.

Corollary 4.2.2. *Let $\alpha_i, g_i \in G$, $n \geq 1$. Consider the following product*

$$T = g_0\alpha_1g_1\alpha_2g_2 \cdots \alpha_ng_n.$$

Assume that for each $i \in \{1, 2, \dots, n-1\}$, we have

$$K(g'_{i-1,2}\alpha_i, g_i) + K(g_i, \alpha_{i+1}g'_{i+1,1}) \leq l(g_i) - 2 \quad (4.1)$$

for every right factor $g'_{i-1,2}$ of g_{i-1} and every left factor $g'_{i+1,1}$ of g_{i+1} . Then $T \neq 1$.

Proof. For each $i \in \{1, 2, \dots, n-1\}$, let l_i (respectively, r_i) be the maximum of $K(g'_{i-1,2}\alpha_i, g_i)$ (respectively, $K(g_i, \alpha_{i+1}g'_{i+1,1})$) for all right factors $g'_{i-1,2}$ of g_{i-1} (respectively, left factors $g'_{i+1,1}$ of g_{i+1}). Note that we have $l_i + r_i \leq l(g_i) - 2$ by (4.1). Now, for each $i \in \{1, 2, \dots, n-1\}$, we write $g_i = g_{i,1}g_{i,2}$ as a reduced product such that $l(g_{i,1}) = l_i + 1$, so $l(g_{i,2}) \geq r_i + 1$. Also, let $g_{0,2} = g_0$ and $g_{n,1} = g_n$. Now by Lemma 4.2.1, $g_{i-1,2}\alpha_i g_{i,1}$ (respectively, $g_{i,2}\alpha_{i+1}g_{i+1,1}$) is right (respectively, left) end-preserving for all $i \in \{1, 2, \dots, n-1\}$. Then the following product is reduced and thus nontrivial

$$T := (g_0\alpha_1g_{1,1})(g_{1,2}\alpha_2g_{2,1}) \cdots (g_{n-2,2}\alpha_{n-1}g_{n-1,1})(g_{n-1,2}\alpha_ng_n).$$

$\square$

The next result is about the relation between cancellation numbers and right factors. It will (only) be used in the proof of Lemma 4.3.4, which assists in proving Proposition 4.3.8, the main result of Section 4.3.

Lemma 4.2.3. *Assume that $K(\alpha, \beta) = l(\beta)$ and γ is a right factor of $\alpha\beta$. Then there is a right factor α_2 of α such that $\gamma = \alpha_2\beta$.*

Proof. First, there is a reduced product $\alpha = \alpha'_1\alpha'_2$ such that $\alpha'_2\beta = 1$. Hence $\alpha\beta = \alpha'_1$. Now γ is a right factor of α'_1 , thus we can write α'_1 as a reduced product $\alpha'_1 = \gamma_1\gamma$. Letting $\alpha_2 = \gamma\alpha'_2$, we have $\alpha = \alpha'_1\alpha'_2 = \gamma_1\gamma\alpha'_2 = \gamma_1\alpha_2$, all being reduced products. Thus, α_2 is a right factor of α and we have $\gamma = \gamma_1 = \gamma\alpha'_2\beta = \alpha_2\beta$, as desired. $\square$

4.3 Taming products of conjugates

To study generalized torsion, we need to investigate products of conjugates, which may involve lots of cancellations and be hard to control. In this section, we introduce the techniques which are essential in taming the product of conjugates.

The setup of this section is as follows. We let $G = *_C G_i$ be an amalgam of groups and consider the tuple $v = ((t_1, g_1), \dots, (t_n, g_n))$, where $n \geq 1$, $t_i, g_i \in G$, and the associated product of conjugates $T = C_1 C_2 \cdots C_n$, where $C_i = t_i^{g_i}$. For convenience of notation, we set $t_0 = t_{n+1} = g_0 = g_{n+1} = 1$.

We first make the following definitions to aid in our analysis of products of conjugates.

Definition 4.3.1. *Given i with $1 \leq i \leq n$, we say that t_i is **cancellable** if at least one of the following is true:*

1. t_i is cancellable from LHS, meaning that there exists a right factor R_{i-1} of g_{i-1} such that $R_{i-1}g_i^{-1}t_i \in C$;
2. t_i is cancellable from RHS, meaning that there exists a left factor L_{i+1} of g_{i+1}^{-1} such that $t_i g_i L_{i+1} \in C$;

3. t_i is cancellable two-sidedly, meaning that there exists a right factor R_{i-1} of g_{i-1} and a left factor L_{i+1} of g_{i+1}^{-1} such that $R_{i-1}t_i^{g_i}L_{i+1} \in C$.

Lemma 4.3.2. *Fix an i with $1 \leq i \leq n$. Assume that $l(t_i) \geq 1$ and the product $g_i^{-1}t_i g_i$ is reduced. If t_i is cancellable from LHS (respectively, RHS), then $l(g_{i-1}C_i) < l(g_{i-1})$ (respectively, $l(g_{i+1}C_i^{-1}) < l(g_{i+1})$).*

Proof. Assume that t_i is cancellable from LHS. Then we can write $g_{i-1} = L_{i-1}R_{i-1}$ as a reduced product, such that $c_i := R_{i-1}g_i^{-1}t_i \in C$. With $R_{i-1}^{-1}c_i = g_i^{-1}t_i$, we have $l(R_{i-1}) = l(g_i^{-1}) + l(t_i) > l(g_i)$, using that $g_i^{-1}t_i g_i$ is reduced and $l(t_i) > 0$. Now we have, as desired,

$$\begin{aligned} l(g_{i-1}C_i) &= l(L_{i-1}R_{i-1}g_i^{-1}t_i g_i) = l(L_{i-1}c_i g_i) \\ &\leq l(L_{i-1}) + 0 + l(g_i) < l(L_{i-1}) + l(R_{i-1}) = l(g_{i-1}). \end{aligned}$$

The case when t_i is cancellable from RHS can be treated similarly. We write $g_{i+1}^{-1} = L_{i+1}R_{i+1}$ as a reduced product such that $d_i := t_i g_i L_{i+1} \in C$. Then $l(L_{i+1}) > l(g_i)$ and

$$l(g_{i+1}C_i^{-1}) = l(C_i g_{i+1}^{-1}) = l(g_i^{-1}t_i g_i L_{i+1} R_{i+1}) = l(g_i^{-1}d_i R_{i+1}) < l(L_{i+1}) + l(R_{i+1}) = l(g_{i+1}).$$

□

One trick we use to tame the products of conjugates is to notice that $C_{i-1}C_i = C_i C'_{i-1}$, where $C'_{i-1} = C_{i-1}^{C_i} = t_{i-1}^{g_{i-1}C_i}$. This gives a glimpse into the role of the next result, which is to assist in the proof of Theorem 4.4.1, a key result of Section 4.4.

Proposition 4.3.3. *Fix an i with $1 \leq i \leq n$. Let $g'_{i+1} = g_{i+1}C_i^{-1}$ and $g'_{i-1} = g_{i-1}C_i$. Assume that the product $g_i^{-1}t_i g_i$ is reduced and $l(t_i) \geq 1$. Also assume that t_i is cancellable, $l(g'_{i+1}) \geq l(g_{i+1})$ and $l(g'_{i-1}) \geq l(g_{i-1})$. Then $l(g'_{i-1}) = l(g_{i-1}) > l(g_i)$ and $l(g'_{i+1}) = l(g_{i+1}) > l(g_i)$.*

Proof. By Lemma 4.3.2, we know that t_i is not cancellable from LHS or RHS, hence it is cancellable two-sidedly; i.e., we can write $g_{i-1} = L_{i-1}R_{i-1}$ and $g_{i+1}^{-1} = L_{i+1}R_{i+1}$ as reduced products, such

that $c_i := R_{i-1}C_iL_{i+1} \in C$. Now we have

$$l(R_{i-1}) = l(C_iL_{i+1}), \quad l(L_{i+1}) = l(R_{i-1}C_i), \quad (4.2)$$

$$l(C_i) \leq l(R_{i-1}) + l(L_{i+1}). \quad (4.3)$$

Here, the inequality follows by considering the length of $C_i = R_{i-1}^{-1}c_iL_{i+1}^{-1}$.

Next, we proceed as follows:

- (1) We prove that $l(C_iL_{i+1}) = l(L_{i+1})$ and $l(R_{i-1}C_i) = l(R_{i-1})$.

We only prove the first equality, as the second one follows from the first one and (4.2). First, suppose that $l(C_iL_{i+1}) < l(L_{i+1})$. Then we have

$$l(g'_{i+1}^{-1}) = l(C_iL_{i+1}R_{i+1}) \leq l(C_iL_{i+1}) + l(R_{i+1}) < l(L_{i+1}) + l(R_{i+1}) = l(g_{i+1}),$$

contradicting the assumptions of the lemma. Second, if $l(C_iL_{i+1}) > l(L_{i+1})$, then we deduce $l(R_{i-1}) > l(R_{i-1}C_i)$ from (4.2). Adding $l(L_{i-1})$ to both sides, we find

$$l(g_{i-1}) = l(L_{i-1}) + l(R_{i-1}) > l(L_{i-1}) + l(R_{i-1}C_i) \geq l(L_{i-1}R_{i-1}C_i) = l(g'_{i-1}),$$

again a contradiction. This finishes the proof that $l(C_iL_{i+1}) = l(L_{i+1})$.

- (2) We have, using (1),

$$l(g'_{i-1}) = l(L_{i-1}R_{i-1}C_i) \leq l(L_{i-1}) + l(R_{i-1}C_i) = l(L_{i-1}) + l(R_{i-1}) = l(g_{i-1}).$$

From this and our assumptions, it follows that $l(g'_{i-1}) = l(g_{i-1})$. Similarly, we have $l(g'_{i+1}) = l(g_{i+1})$.

- (3) Combining (1), (4.2) and (4.3), we find that $l(R_{i-1}) = l(L_{i+1}) \geq \frac{1}{2}l(C_i)$.

(4) Note that $l(C_i) = 2l(g_i) + l(t_i)$. Applying the result of (3), we have

$$l(g_{i-1}) \geq l(R_{i-1}) \geq \frac{1}{2}l(C_i) = l(g_i) + \frac{1}{2}l(t_i) > l(g_i),$$

as $l(t_i) \geq 1$. Similarly, $l(g_{i+1}) > l(g_i)$.

□

Our key result is that the length of the product $t_1^{g_1} \cdots t_n^{g_n}$ is bounded below by n when $((t_1, g_1), \dots, (t_n, g_n))$ is tamed (see Definition 4.3.6 and Proposition 4.3.8). We first provide two lemmas which build towards this result.

For each i with $1 \leq i \leq n$, we set $r_i = l(g_{i-1}g_i^{-1})$. When $r_i \geq 1$, we write $g_{i-1}g_i^{-1}$ as an alternating product

$$g_{i-1}g_i^{-1} = e_{i,1} \cdots e_{i,r_i}.$$

Lemma 4.3.4. *Fix an i with $1 \leq i \leq n$. Assume that $g_i^{-1}t_i g_i$ is reduced, $l(t_i) = 1$ and t_i is not cancellable. Let L_i be a left factor of $g_i g_{i+1}^{-1}$.*

(1) *If $l(L_i) \geq 1$ and $l(t_i e_{i+1,1}) = 1$, then the product $t_i L_i$ is not reduced.*

(2) *Assume that $t_i L_i$ is not reduced. Then $l(g_i g_{i+1}^{-1}) \geq 1$, $l(t_i e_{i+1,1}) = 1$, and there is a left factor L_{i+1} of g_{i+1}^{-1} such that $e_{i+1,1} = g_i L_{i+1}$.*

(3) *The product $t_i L_i$ is end-preserving.*

Proof. (1) This is clear, as $e_{i+1,1}$ is a left factor of L_i .

(2) First, we have $l(g_i g_{i+1}^{-1}) \geq 1$, as otherwise, $L_i \in C$ and thus the product $t_i L_i$ is reduced.

Next, since the product $t_i g_i$ is reduced, we know that either $g_i \in C$ or $g_i \notin C$ and the product $g_i g_{i+1}^{-1}$ is not left end-preserving and thus $K(g_i, g_{i+1}^{-1}) = l(g_i)$ by Lemma 4.2.1. By Lemma 4.2.3, there is a left factor L_{i+1} of g_{i+1}^{-1} such that $e_{i+1,1} = g_i L_{i+1}$.

Last, let's show that $l(t_i e_{i+1,1}) = 1$. We only need to rule out two other cases. (i) If $l(t_i e_{i+1,1}) = 0$, then $t_i g_i L_{i+1} = t_i e_{i+1,1} \in C$, and thus t_i is cancellable (from RHS), contradicting the assumption. (ii) If $l(t_i e_{i+1,1}) = 2$, then $t_i (g_i g_{i+1}^{-1})$ is reduced and thus so is $t_i L_i$, contradicting the assumption.

- (3) If the product $t_i L_i$ is reduced, it is end-preserving. So we assume that $t_i L_i$ is not reduced. Then $l(t_i e_{i+1,1}) = 1$ by (2) and $l(L_i) \geq 1$. Since L_i is a left factor of $g_i g_{i+1}^{-1} = e_{i+1,1} \cdots e_{i+1,r_{i+1}}$, the right-hand side (RHS) of which is an alternating product, we know that $L_i = e_{i+1,1} \cdots e_{i+1,j} c$ for some j with $1 \leq j \leq r_{i+1}$ and $c \in C$. Therefore,

$$t_i L_i = (t_i e_{i+1,1}) e_{i+1,2} \cdots e_{i+1,j} c,$$

and thus $t_i L_i$ is end-preserving, as $(t_i e_{i+1,1}) e_{i+1,2} \cdots e_{i+1,j}$ is an alternating product. □

Note that the parallel version of Lemma 4.3.4 considers a right factor R_{i-1} of $g_{i-1} g_i^{-1}$, with the other conditions unchanged, and the conclusions are the following.

- (1) If $l(R_{i-1}) \geq 1$ and $l(e_{i,r_i} t_i) = 1$, then the product $R_{i-1} t_i$ is not reduced.
- (2) If $R_{i-1} t_i$ is not reduced, then $l(g_{i-1} g_i^{-1}) \geq 1$, $l(e_{i,r_i} t_i) = 1$, and there is a right factor R'_{i-1} of g_{i-1} such that $e_{i,r_i} = R'_{i-1} g_i^{-1}$.
- (3) The product $R_{i-1} t_i$ is end-preserving.

Lemma 4.3.5. *Fix an i with $1 \leq i \leq n$. Suppose $g_i^{-1} t_i g_i$ is a reduced product, $l(t_i) = 1$ and t_i is not cancellable. Let L_i be a left factor of $g_i g_{i+1}^{-1}$ and R_{i-1} a right factor of $g_{i-1} g_i^{-1}$. Then we have:*

- (1) $t_i L_i, R_{i-1} t_i, R_{i-1} t_i L_i, (R_{i-1} t_i) L_i, R_{i-1} (t_i L_i)$ are end-preserving;
- (2) If $l(g_i g_{i+1}^{-1}) \geq 1$ and $l(t_i e_{i+1,1}) = 1$, then $l(R_{i-1} t_i e_{i+1,1}) = l(R_{i-1} t_i)$.

Proof. (1) We use Lemma 4.3.4 (and its “right version”) and proceed in 3 steps.

- (a) First, $t_i L_i$ and $R_{i-1} t_i$ are end-preserving by Lemma 4.3.4(3).
- (b) We prove that $R_{i-1}(t_i L_i)$ is right end-preserving. By Lemma 4.2.1, it suffices to prove that $K(g_{i-1} g_i^{-1}, t_i L_i) = 0$. Assuming otherwise, then depending on whether the product $t_i L_i$ is reduced or not,

$$e_{i,r_i} t_i \in C \text{ or } e_{i,r_i} t_i e_{i+1,1} \in C, \quad (4.4)$$

respectively, where $e_{i+1,1} = g_i L_{i+1}$ for some left factor L_{i+1} of g_{i+1}^{-1} by Lemma 4.3.4. Note that $t_i L_i$ is (left) end-preserving and $(g_{i-1} g_i^{-1})(t_i L_i)$ is not reduced, thus the product $(g_{i-1} g_i^{-1}) t_i$ is not reduced either. Hence, by Lemma 4.3.4, there is a right factor R'_{i-1} of g_{i-1} such that $e_{i,r_i} = R'_{i-1} g_i^{-1}$. Plugging the expressions for $e_{i+1,1}$ and e_{i,r_i} into (4.4), we find $R'_{i-1} g_i^{-1} t_i \in C$ or $R'_{i-1} g_i^{-1} t_i g_i L_{i+1} \in C$, implying that t_i is cancellable, contradicting the assumption. This finishes proving that $R_{i-1}(t_i L_i)$ is right end-preserving.

- (c) We now conclude the proof of (1). It follows similarly to (b) that the product $(R_{i-1} t_i) L_i$ is left end-preserving. We have $R_{i-1} t_i L_i$ is right end-preserving because both $R_{i-1}(t_i L_i)$ and $t_i L_i$ are so. Similarly, $R_{i-1} t_i L_i$ is left end-preserving. So $R_{i-1} t_i L_i$ is end-preserving, from which it follows that $(R_{i-1} t_i) L_i$ is right end-preserving and $R_{i-1}(t_i L_i)$ is left end-preserving.

- (2) If $l(R_{i-1}) = 0$, the equality is clear. Otherwise, since $e_{i,r_i} t_i e_{i+1,1}, t_i e_{i+1,1} \notin C$ by (1), $R_{i-1} t_i e_{i+1,1}$ can be written as one of the following alternating products

$$c e_{i,j} \cdots e_{i,r_i-1} (e_{i,r_i} t_i e_{i+1,1}), \quad c e_{i,j} \cdots e_{i,r_i-1} e_{i,r_i} (t_i e_{i+1,1}),$$

where $c \in C$ and $1 \leq j \leq r_i$. We can write $R_{i-1} t_i$ similarly, with the only difference being the omission of $e_{i+1,1}$. We see that if $l(e_{i,r_i} t_i) = 1$ (respectively, $l(e_{i,r_i} t_i) = 2$), then both $l(R_{i-1} t_i e_{i+1,1})$ and $l(R_{i-1} t_i)$ are equal to $r_i - j + 1$ (respectively, $r_i - j + 2$).

□

The following definition will allow us to control products of conjugates in an amalgam.

Definition 4.3.6. We say the tuple $v = ((t_1, g_1), \dots, (t_n, g_n))$ is **tamed** if the following are satisfied:

1. for $i = 1, \dots, n$, $l(t_i) = 1$ and $t_i^{g_i} = g_i^{-1}t_i g_i$ is a reduced product;
2. for $i = 1, \dots, n-1$, if $l(g_i g_{i+1}^{-1}) = 0$, then $l(t_i t_{i+1}) = 2$;
3. no t_i is cancellable.

Lemma 4.3.7. Assume that $((t_1, g_1), \dots, (t_n, g_n))$ is tamed. Let $T_i = t_1^{g_1} \cdots t_i^{g_i}$ for $1 \leq i \leq n$ and $T_0 = 1$. (Also recall that $t_0 = g_0 = 1$.) Then for each i with $1 \leq i \leq n$,

$$T_i = (T_{i-1} g_{i-1}^{-1} \delta_i) (\delta_i^{-1} g_{i-1} g_i^{-1} t_i) g_i \quad (4.5)$$

is a reduced product (of three terms), where

$$\delta_i := \begin{cases} e_{i,1} & \text{if } l(g_{i-1} g_i^{-1}) \geq 1 \text{ and } l(t_{i-1} e_{i,1}) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We use Lemma 4.3.5 and make induction on i . When $i = 1$, $T_1 = \delta_1 (\delta_1^{-1} g_1^{-1} t_1) g_1$ is clearly a reduced product. Now assume that for i with $1 \leq i < n$, (4.5) is a reduced product; we will show that T_{i+1} is also a reduced product of three terms as in (4.5). Consider

$$\begin{aligned} T_{i+1} &= T_i g_{i+1}^{-1} t_{i+1} g_{i+1} \\ &= (T_{i-1} g_{i-1}^{-1} \delta_i) (\delta_i^{-1} g_{i-1} g_i^{-1} t_i) g_i g_{i+1}^{-1} t_{i+1} g_{i+1} \\ &= (T_{i-1} g_{i-1}^{-1} \delta_i) (\delta_i^{-1} g_{i-1} g_i^{-1} t_i \delta_{i+1}) (\delta_{i+1}^{-1} g_i g_{i+1}^{-1} t_{i+1}) g_{i+1}. \end{aligned}$$

We will prove that the last expression is a reduced product, then by combining the first two terms,

$T_{i+1} = (T_i g_i^{-1} \delta_{i+1}) (\delta_{i+1}^{-1} g_i g_{i+1}^{-1} t_{i+1}) g_{i+1}$ is a reduced product, as desired.

For ease of exposition, write $T_{i+1} = S_1 S_2 S_3 S_4$ and $T_i = S_1 S'_2 g_i$ with

$$S_1 = T_{i-1} g_{i-1}^{-1} \delta_i, S_2 = \delta_i^{-1} g_{i-1} g_i^{-1} t_i \delta_{i+1}, S_3 = \delta_{i+1}^{-1} g_i g_{i+1}^{-1} t_{i+1}, S_4 = g_{i+1}, S'_2 = \delta_i^{-1} g_{i-1} g_i^{-1} t_i.$$

Since $(\delta_i^{-1} g_{i-1} g_i^{-1}) t_i \delta_{i+1}$ and $(\delta_{i+1}^{-1} g_i g_{i+1}^{-1}) t_{i+1}$ are end-preserving by 4.3.5(1), we have $l(S_2) \geq 1$ and $l(S_3) \geq 1$, and thus we only need to show that $S_k S_{k+1}$ is reduced for $k = 1, 2, 3$. Since $S_1 S'_2$ is reduced, $S_2 = S'_2 \delta_{i+1}$ is (left) end-preserving and $l(S'_2) \geq 1$ by Lemma 4.3.5(1), $S_1 S_2$ is reduced. Since $t_{i+1} g_{i+1}$ is reduced and $(\delta_{i+1}^{-1} g_i g_{i+1}^{-1}) t_{i+1}$ is (right) end-preserving by Lemma 4.3.5(1), $S_3 S_4$ is reduced.

Now we only need to show that $S_2 S_3$ is reduced. Since the product $S_2 = (\delta_i^{-1} g_{i-1} g_i^{-1}) t_i \delta_{i+1}$ is (right) end-preserving and $S_3 = (\delta_{i+1}^{-1} g_i g_{i+1}^{-1}) t_{i+1}$ is (left) end-preserving by Lemma 4.3.5(1), we only need to show that the product of the rightmost term of S_2 which is not in C , and the leftmost term of S_3 which is not in C , is reduced. This is given by analyzing the following possible cases.

1. Assume that $l(g_i g_{i+1}^{-1}) = 0$. In this case, $\delta_{i+1} = 1$, $g_i g_{i+1}^{-1} \in C$ and $l(t_i t_{i+1}) = 2$ by tameness. So $t_i t_{i+1}$ is reduced, as desired.
2. Assume that $l(g_i g_{i+1}^{-1}) \geq 1$ and $l(t_i e_{i+1,1}) = 2$. Then $\delta_{i+1} = 1$ and $t_i (g_i g_{i+1}^{-1})$ is reduced, as desired.
3. Assume that $l(g_i g_{i+1}^{-1}) \geq 1$ and $l(t_i e_{i+1,1}) = 1$. In this case, $\delta_{i+1} = e_{i+1,1}$. Now consider two possible cases. (i) If $l(g_i g_{i+1}^{-1}) \geq 2$, then $\delta_{i+1} (e_{i+1,1}^{-1} g_i g_{i+1}^{-1})$ is reduced, as desired. (ii) If $l(g_i g_{i+1}^{-1}) = 1$, then $e_{i+1,1}^{-1} g_i g_{i+1}^{-1} = 1$ and we must show that $e_{i+1,1} t_{i+1}$ is reduced. Since $g_i g_{i+1}^{-1} = e_{i+1,1}$, $e_{i+1,1}$ is a right factor of g_{i+1}^{-1} or a left factor of g_i . But $e_{i+1,1}$ can't be a left factor of g_i , because $t_i g_i$ is reduced and $t_i e_{i+1,1}$ is not. Now since $g_{i+1}^{-1} t_{i+1}$ is reduced, so is $e_{i+1,1} t_{i+1}$, as desired.

□

The next result plays an indispensable role in the proof of Theorem 4.4.1 of Section 4.4.

Proposition 4.3.8. *If $((t_1, g_1), \dots, (t_n, g_n))$ is tamed, then $l(t_1^{g_1} \cdots t_n^{g_n}) \geq l(g_1) + n + l(g_n)$.*

Proof. Let $T_i = t_1^{g_1} \cdots t_i^{g_i}$ for $1 \leq i \leq n$. We will show by induction that

$$l(T_i) \geq l(g_1) + i + l(g_i) \text{ for all } i \in \{1, 2, \dots, n\}. \quad (4.6)$$

We proceed with two steps, using the definition of δ_i from the statement of Lemma 4.3.7.

(1) We prove that $l(T_i g_i^{-1} \delta_{i+1}) = l(T_i g_i^{-1})$ for $1 \leq i \leq n-1$.

This is true if $\delta_{i+1} = 1$. Thus, we consider when $\delta_{i+1} = e_{i+1,1}$, meaning that $l(t_i e_{i+1,1}) = 1$ and $l(g_i g_{i+1}^{-1}) \geq 1$. Consider the products

$$T_i g_i^{-1} = (T_{i-1} g_{i-1}^{-1} \delta_i)(\delta_i^{-1} g_{i-1} g_i^{-1} t_i), \quad T_i g_i^{-1} \delta_{i+1} = (T_{i-1} g_{i-1}^{-1} \delta_i)(\delta_i^{-1} g_{i-1} g_i^{-1} t_i e_{i+1,1}).$$

The first product is reduced by Lemma 4.3.7, and so is the second product because $(\delta_i^{-1} g_{i-1} g_i^{-1} t_i) e_{i+1,1}$ is (left) end-preserving and $l(\delta_i^{-1} g_{i-1} g_i^{-1} t_i) \geq 1$ by Lemma 4.3.5(1). Thus it suffices to prove that $l(\delta_i^{-1} g_{i-1} g_i^{-1} t_i) = l(\delta_i^{-1} g_{i-1} g_i^{-1} t_i e_{i+1,1})$, which holds by Lemma 4.3.5(2).

(2) Now we prove (4.6) by induction. It is true when $i = 1$ since $t_1^{g_1}$ is reduced. Assume that it is true for T_{i-1} and consider T_i . We have:

$$\begin{aligned} l(T_i) &= l(T_{i-1} g_{i-1}^{-1} \delta_i) + l(\delta_i^{-1} g_{i-1} g_i^{-1} t_i) + l(g_i) \quad (\text{by Lemma 4.3.7}) \\ &\geq l(T_{i-1} g_{i-1}^{-1}) + 1 + l(g_i) \quad (\text{by (1) and Lemma 4.3.5(1)}) \\ &\geq l(T_{i-1}) - l(g_{i-1}) + 1 + l(g_i) \\ &\geq (l(g_1) + (i-1) + l(g_{i-1})) - l(g_{i-1}) + 1 + l(g_i) \quad (\text{by the inductive hypothesis}) \\ &= l(g_1) + i + l(g_i). \end{aligned}$$

□

4.4 Generalized torsion in amalgams

The main goal of this section is to derive Theorem 4.4.6, which gives a sufficient condition ensuring that an amalgam of two groups is GTF.

For a subset S of a group G , we denote by $\text{NSS}_G(S)$ the normal subsemigroup of G generated by S ; i.e., $\text{NSS}_G(S)$ is formed by elements of the form $s_1^{g_1} \cdots s_n^{g_n}$ with $n \geq 1$, $s_i \in S$ and $g_i \in G$. Recall that we allow a (normal) subsemigroup to be empty.

Theorem 4.4.1. *Consider the amalgam $G = A *_C B$ of groups A and B , where C is a common subgroup of A and B . Let P and Q be normal subsemigroups of A and B , respectively. The following are equivalent:*

- (1) $P \cap C = Q \cap C$;
- (2) $A \cap \text{NSS}_G(P \cup Q) = P$ and $B \cap \text{NSS}_G(P \cup Q) = Q$;
- (3) $C \cap \text{NSS}_G(P \cup Q) = P \cap Q$.

Proof. Note that (2) implies (3) by taking the intersection of terms appearing on LHS of the equalities in (2), and recalling that $A \cap B = C$. To see (3) implies (1), note that with (3) we have $C \cap P \subseteq C \cap \text{NSS}_G(P \cup Q) = P \cap Q$, thus $C \cap P \subseteq P \cap Q \cap C \subseteq Q \cap C$ and then similarly $Q \cap C \subseteq P \cap C$, leading to $P \cap C = Q \cap C$.

Thus, we only need to prove that $P \cap C = Q \cap C$ implies $A \cap \text{NSS}_G(P \cup Q) = P$, as the proof of $B \cap \text{NSS}_G(P \cup Q) = Q$ is nearly identical. First, the inclusion $A \cap \text{NSS}_G(P \cup Q) \supseteq P$ is clear. So all that remains is to prove that $A \cap \text{NSS}_G(P \cup Q) \subseteq P$.

To this end, suppose (1) holds and there exists $e_1^{f_1} \cdots e_m^{f_m} \in A$ with $e_i \in P \cup Q$, $f_i \in G$ and $m \geq 1$, and assume that $e_1^{f_1} \cdots e_m^{f_m} \notin P$ in order to find a contradiction. Let $\mathcal{S}$ be the set of tuples $((s_1, h_1), \dots, (s_k, h_k))$ such that $k \geq 1$, $s_i \in P \cup Q$, $h_i \in G$ and $s_1^{h_1} \cdots s_k^{h_k} \in A \setminus P$. Then $\mathcal{S}$ is nonempty as it contains the tuple $((e_1, f_1), \dots, (e_m, f_m))$. For $w = ((s_1, h_1), \dots, (s_k, h_k))$ in $\mathcal{S}$, we associate three non-negative integers to w : $NC(w) = k$, $L(w) = \sum_{i=1}^k l(h_i)$ and $N(w)$, which is the number of pairs (i, j) with $i, j \in \{1, \dots, k\}$ such that $i < j$ and $l(h_i) < l(h_j)$.

Choose $v_0 = ((t_1, g_1), \dots, (t_n, g_n)) \in \mathcal{S}$ such that $(NC(v), L(v), N(v))$ is minimal, where the set $\mathbb{N}^3$ is given the lexicographic order: $(a_1, a_2, a_3) < (b_1, b_2, b_3)$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$ or $a_1 = b_1, a_2 = b_2$ and $a_3 < b_3$. We will finish the proof by showing that v_0 must be tamed. Then by Proposition 4.3.8, $l(t_1^{g_1} \cdots t_n^{g_n}) \geq n$ and thus $n = 1$ as $t_1^{g_1} \cdots t_n^{g_n} \in A$. But then the product $t_1^{g_1} \cdots t_n^{g_n}$ reduces to $t_1^{g_1}$ with $l(t_1^{g_1}) = 1$. As (t_1, g_1) is tamed, the product $t_1^{g_1}$ is reduced and $l(t_1) = 1$. Hence we have $t_1 \in A \setminus C$ and $g_1 \in C$, recalling that $t_1^{g_1} \in A$. But this leads to $t_1 \in (A \setminus C) \cap (P \cup Q) = P \setminus C$ and then $t_1^{g_1} \in P$, a contradiction.

We use the minimality of v_0 and that $P \cap C = Q \cap C$ to check that v_0 is tamed as follows.

- (1) We first check that $l(t_i) = 1$ for all $i = 1, \dots, n$. Note that in order to check this, we need only check that $t_i \notin C$ for all i , since $t_i \in P \cup Q$ already implies $l(t_i) \leq 1$.

Assume that t_i is in C . Then t_i is in $P \cap C = Q \cap C$ by the assumption. Consider two possible cases.

- (a) Assume $l(g_i) \geq 1$. In this case, we can write $g_i = c_i x_1 \cdots x_m$ as an alternating product with $m \geq 1$. Set $t'_i = t_i^{c_i x_1}$ and $g'_i = x_2 \cdots x_m$. Observe that $t'_i \in P \cup Q$, $g'_i \in G$ and

$$t_1^{g_1} \cdots t_n^{g_n} = t_1^{g_1} \cdots t_{i-1}^{g_{i-1}} (t'_i)^{g'_i} t_{i+1}^{g_{i+1}} \cdots t_n^{g_n} \in A \setminus P,$$

so we have

$$w := ((t_1, g_1), \dots, (t_{i-1}, g_{i-1}), (t'_i, g'_i), (t_{i+1}, g_{i+1}), \dots, (t_n, g_n)) \in \mathcal{S}.$$

But w satisfies $NC(w) = NC(v_0)$ and $L(w) < L(v_0)$, contradicting the minimality of v_0 .

- (b) Assume $l(g_i) = 0$. This means $g_i \in C$ and so we have $c_i := t_i^{g_i} \in P \cap C$, as $t_i \in P \cap C$.

Thus we have $n \geq 2$. Observe that

$$t_1^{g_1} \cdots t_n^{g_n} = t_1^{g_1} \cdots t_{i-1}^{g_{i-1}} t_{i+1}^{g'_{i+1}} \cdots t_n^{g'_n} c_i$$

where $g'_j = g_j c_i^{-1}$ for $j = i+1, \dots, n$. Now, $c_i \in C \subseteq A$ and by assumption $t_1^{g_1} \dots t_n^{g_n} \in A$, it thus follows that $t_1^{g_1} \dots t_{i-1}^{g_{i-1}} t_{i+1}^{g'_{i+1}} \dots t_n^{g'_n} \in A$. Moreover, $t_1^{g_1} \dots t_{i-1}^{g_{i-1}} t_{i+1}^{g'_{i+1}} \dots t_n^{g'_n} \notin P$, because otherwise, since $c_i \in P$, we have $t_1^{g_1} \dots t_n^{g_n} \in P$, a contradiction. We conclude that

$$w := ((t_1, g_1), \dots, (t_{i-1}, g_{i-1}), (t_{i+1}, g'_{i+1}), \dots, (t_n, g'_n)) \in \mathcal{S},$$

a contradiction since $NC(w) < NC(v_0)$.

- (2) We now show that each $t_i^{g_i}$ is a reduced product. Fix i , and write $g_i = c_i x_1 \dots x_m$ as an alternating product, and observe that there are two possible ways in which the product $t_i^{g_i}$ might fail to be reduced: either $t_i, x_1 \in A \setminus C$ or $t_i, x_1 \in B \setminus C$.

In either of these situations, we set $t'_i = t_i^{c_i x_1}$ and $g'_i = x_2 \dots x_m$ and argue as in the previous case (1)a that

$$w := ((t_1, g_1), \dots, (t_{i-1}, g_{i-1}), (t'_i, g'_i), (t_{i+1}, g_{i+1}), \dots, (t_n, g_n)) \in \mathcal{S},$$

which contradicts the minimality of v_0 .

- (3) Next, we show that if $g_i g_{i+1}^{-1} \in C$, then $l(t_i t_{i+1}) = 2$.

Assume that $c_i := g_i g_{i+1}^{-1} \in C$. Note that

$$t_i^{g_i} t_{i+1}^{g_{i+1}} = t_i^{g_i} t_{i+1}^{c_i^{-1} g_i} = (t_i t_{i+1}^{c_i^{-1}})^{g_i} = t_i'^{g_i} \text{ with } t'_i = t_i t_{i+1}^{c_i^{-1}}.$$

Now if $l(t_i t_{i+1}) \leq 1$, then t_i and t_{i+1} are in the same set $P \setminus C$ or $Q \setminus C$. In this case $t'_i \in P \cup Q$, which leads to the existence of a tuple $w \in \mathcal{S}$ with $NC(w) < NC(v_0)$, a contradiction. Hence $l(t_i t_{i+1}) = 2$ as desired.

- (4) Last, we show that no t_i is cancellable. Assume that t_i is cancellable for a fixed i with $1 \leq i \leq n$. We will find a contradiction using Proposition 4.3.3 and the notations there. Letting $C'_j = t_j^{g'_j}$ for $j = i-1, i+1$, we have $C_{i-1} C_i = C_i C'_{i-1}$ and $C_i C_{i+1} = C'_{i+1} C_i$. Using

these identities, we rewrite the product $C_1 \cdots C_n$ and consider the corresponding tuples

$$\begin{aligned} u_i &:= ((t_1, g_1), \dots, (t_i, g_i), (t_{i-1}, g'_{i-1}), (t_{i+1}, g_{i+1}), \dots, (t_n, g_n)), \\ w_i &:= ((t_1, g_1), \dots, (t_{i-1}, g_{i-1}), (t_{i+1}, g'_{i+1}), (t_i, g_i), \dots, (t_n, g_n)). \end{aligned}$$

If $l(g'_{i-1}) < l(g_{i-1})$ (respectively, $l(g'_{i+1}) < l(g_{i+1})$), then $i > 1$, $u_i \in \mathcal{S}$ and $L(u_i) < L(v_0)$ (respectively, $i < n$, $w_i \in \mathcal{S}$ and $L(w_i) < L(v_0)$), contradicting the minimality of v_0 . Therefore $l(g'_{i-1}) \geq l(g_{i-1})$ and $l(g'_{i+1}) \geq l(g_{i+1})$. Then $i < n$, since otherwise t_n is cancelable from LHS (as $g_{n+1} = 1$), implying that $l(g'_{i-1}) < l(g_{i-1})$ by Lemma 4.3.2, a contradiction. Now $w_i \in \mathcal{S}$, and $l(g'_{i+1}) = l(g_{i+1}) > l(g_i)$ by Proposition 4.3.3, implying that $L(w_i) = L(v_0)$ while $N(w_i) < N(v_0)$, a contradiction.

□

Proposition 4.4.2. *Consider the amalgam $G = A *_C B$. Given any subset $R \subseteq A$, there exist normal subsemigroups P and Q of A and B , respectively, satisfying*

$$(1) R \subseteq P, \quad (2) P \cap C = Q \cap C, \quad (3) P \subseteq \text{NSS}_G(R).$$

Proof. Let R be a subset of A . We first define sets $S_A^{(i)}, S_B^{(i)}$ for $i \in \{0, 1, 2, \dots\}$. Let $S_A^{(0)} = \text{NSS}_A(R)$ and then iteratively define

$$S_B^{(i)} = \text{NSS}_B(S_A^{(i)} \cap C), \quad S_A^{(i+1)} = \text{NSS}_A(R \cup (S_B^{(i)} \cap C)) \text{ for } i = 0, 1, 2, \dots$$

Now take $P = \cup_{i \geq 0} S_A^{(i)}$ and $Q = \cup_{i \geq 0} S_B^{(i)}$. We claim that these sets have the required properties.

First, suppose that $g \in S_A^{(i)} \cap C$, which is the generating set for $S_B^{(i)}$. Therefore $g \in S_B^{(i)} \cap C$, meaning $g \in S_A^{(i+1)}$. We already know $g \in C$, so together this gives $g \in S_A^{(i+1)} \cap C$. Now we have inclusions $S_A^{(i)} \cap C \subseteq S_A^{(i+1)} \cap C$ for $i \geq 0$, and correspondingly, $S_B^{(i)} \subseteq S_B^{(i+1)}$ for all $i \geq 0$. It follows that $S_A^{(i)} \subseteq S_A^{(i+1)}$ for all $i \geq 0$. From these inclusions, it follows that P and Q are normal subsemigroups (of A and B , respectively).

Now, $R \subseteq P$ by construction, so (1) holds.

We now check that $P \cap C = Q \cap C$. Let $g \in P \cap C$. Then $g \in S_A^{(i)} \cap C$ for some $i \geq 0$, and therefore $g \in S_B^{(i)} \cap C$ so that $g \in Q \cap C$. Thus, we have $P \cap C \subseteq Q \cap C$. For the other inclusion, suppose $g \in Q \cap C$, so that $g \in S_B^{(i)} \cap C$ for some $i \geq 0$. Then clearly $g \in S_A^{(i+1)} \cap C$, so that $g \in P \cap C$. This shows the reverse inclusion so that (2) holds.

Last, observe that $S_A^{(0)} \subseteq \text{NSS}_G(R)$, which serves as the base case for an induction. Suppose that $S_A^{(i)} \subseteq \text{NSS}_G(R)$, then $S_A^{(i)} \cap C \subseteq \text{NSS}_G(R)$, so that $S_B^{(i)} \subseteq \text{NSS}_G(R)$. But then $R \cup (S_B^{(i)} \cap C) \subseteq \text{NSS}_G(R)$, implying $S_A^{(i+1)} \subseteq \text{NSS}_G(R)$. Upon taking the union of all $S_A^{(i)}$, we conclude $P \subseteq \text{NSS}_G(R)$ and so (3) holds. $\square$

The next result follows immediately by combining Theorem 4.4.1 and Proposition 4.4.2. It gives necessary and sufficient conditions which guarantee that an element of $A \subseteq A *_C B$ is not a generalized torsion element of $G = A *_C B$.

Theorem 4.4.3. *Consider an amalgam $G = A *_C B$. Let R be a subset of $A \setminus \{1\}$. Then $1 \notin \text{NSS}_G(R)$ if and only if there exist normal subsemigroups P and Q of A and B , respectively, such that $R \subseteq P$, $1 \notin P$, and $P \cap C = Q \cap C$.*

Proof. Suppose there are such P and Q , then by Theorem 4.4.1, we have $A \cap \text{NSS}_G(P \cup Q) = P$. Since $1 \in A$ and $1 \notin P$, we have $1 \notin \text{NSS}_G(P \cup Q)$. Therefore $1 \notin \text{NSS}_G(R)$, as $R \subseteq P$.

On the other hand, assume that $1 \notin \text{NSS}_G(R)$. Then apply Proposition 4.4.2, and note that $P \subseteq \text{NSS}_G(R)$ guarantees that $1 \notin P$. $\square$

While the previous theorem allows us to rule out generalized torsion in G arising from elements of $A \cup B$, we need the following result to deal with other elements in the amalgam.

Definition 4.4.4. ([9]) *Let G be a group, and H a subgroup. The subgroup H is called **relatively torsion-free**, or **RTF** for short, if for all $g \in G \setminus H$ and all $h_1, \dots, h_k \in H$ ($k \geq 1$), we have*

$$gh_1gh_2 \cdots gh_k \neq 1.$$

In the proof below, we use $\text{scl}(g)$ to denote the stable commutator length of g . (See [8] for background on stable commutator length.)

Proposition 4.4.5. *Suppose that $G = A *_C B$ and suppose that $g \in G$ is not conjugate into either A or B . If C is RTF in both A and B , then g is not a generalized torsion element.*

Proof. The usual construction of the Bass-Serre tree associated with the amalgam $A *_C B$ provides a tree Γ such that the vertex stabilizers of the G -action on Γ are precisely the conjugate subgroups $h^{-1}Ah$, $h^{-1}Bh$, $h \in G$. Thus, if $g \in G$ is not conjugate into either A or B , then g stabilizes no vertices of Γ . Consequently, if C is RTF in A and B then $\text{scl}(g) \geq \frac{1}{2}$ by [9, Theorem A]. On the other hand, generalized torsion elements of G have scl strictly less than $\frac{1}{2}$ by [14, Theorem 2.4], so g cannot be generalized torsion. $\square$

As a consequence of Theorem 4.4.3, we can give necessary and sufficient conditions that an amalgam contains no generalized torsion elements that are conjugate into one of the factors; and sufficient conditions that the amalgam contain no generalized torsion whatsoever. The reader should note the similarity between the conditions required in the theorem below, and the conditions required in [4, Theorem A].

Theorem 4.4.6. *Let $G = A *_C B$ be the free product of groups A and B , amalgamated by a common subgroup C . Then no element conjugate to an element of $A \cup B$ is a generalized torsion element of G if and only if there is a family P_i ($i \in I$) of normal subsemigroups of A and a family Q_j ($j \in J$) of normal subsemigroups of B satisfying the following:*

1. $A \setminus \{1\} = \cup_{i \in I} P_i$ and $B \setminus \{1\} = \cup_{j \in J} Q_j$;
2. For every $i \in I$ (respectively, $j \in J$) there is an $j \in J$ (respectively, $i \in I$) such that $P_i \cap C = Q_j \cap C$.

If moreover C is RTF in both A and B , respectively, then G is GTF.

Proof. First, assume the existence of the two families of normal subsemigroups, and note that it suffices to prove that no element in $A \cup B$ is a generalized torsion element. Now for every $a \in A \setminus \{1\}$,

there exists $i \in I$ such that $a \in P_i$. Then there exists $j \in J$ such that $P_i \cap C = Q_j \cap C$. Applying Theorem 4.4.3 with $R = \{a\}$ we get that $1 \notin \text{NSS}_G(\{a\})$, which says that a is not a generalized torsion element in G . Thus, no element in A is a generalized torsion element of G . Similarly, we can prove that B contains no generalized torsion of G .

For the converse, we assume that $A \cup B$ contains no generalized torsion elements of G . Then for every $a \in A \setminus \{1\}$, we have $1 \notin \text{NSS}_G(\{a\})$. Thus, by Theorem 4.4.3, there are normal subsemigroups P_a and Q_a of A and B , respectively, such that $a \in P_a$, $1 \notin P_a$ and $P_a \cap C = Q_a \cap C$. Similarly, for every $b \in B \setminus \{1\}$, there are normal subsemigroups Q'_b and P'_b of B and A respectively, such that $b \in Q'_b$, $1 \notin Q'_b$ and $P'_b \cap C = Q'_b \cap C$.

Now the families $\{P_a \mid a \in A \setminus \{1\}\} \cup \{P'_b \mid b \in B \setminus \{1\}\}$ and $\{Q_a \mid a \in A \setminus \{1\}\} \cup \{Q'_b \mid b \in B \setminus \{1\}\}$ satisfy the required properties.

The last claim of the theorem now follows from Proposition 4.4.5. $\square$

4.5 Some corollaries of the main theorems

Let A be a group, and recall that a subgroup C of A is malnormal in A if $a^{-1}Ca \cap C$ is the trivial subgroup $\{1\}$ for all $a \in A \setminus C$. Put it another way, C is malnormal in A if and only if for every $c \in C \setminus \{1\}$ and $a \in A \setminus C$, we have that $c^a \in A \setminus C$. The first three results in this section, Lemma 4.5.1, Lemma 4.5.3 and Corollary 4.5.4 are all easily proved from definitions, not requiring the assistance of theorems from the previous section. The last two results, Corollary 4.5.5 and 4.5.6 are consequences of the main theorems of the last section, of which the first one is analog of Lemma 4.5.1 and the second one is the basis of our construction of GTF and non-LO groups in Chapter 5.

We first state a fact about malnormality in amalgams, which is well-known and easy to prove.

Lemma 4.5.1. *Let $G = A *_C B$ where C is a common subgroup of A and B . Then A is malnormal in G if and only if C is malnormal in B .*

Proof. Assume that A is malnormal in G . If C is not malnormal in B , then there is a $c \in C \setminus \{1\}$ and $b \in B \setminus C$ such that $c^b \in C$. Note that in G , we have $A \cap B = C$. Hence, we have $c \in A \setminus \{1\}$

and $b \in G \setminus A$ and $c^b \in A$, which contradicts the malnormality of A in G .

Now assume that C is malnormal in B . Let $a \in A \setminus \{1\}$ and $g \in G \setminus A$. We want to show that a^g is not in A . Write g as a reduced product $g = dg_1$, where $d \in (A \setminus C) \cup (B \setminus C)$ and $g_1 \in G$. Consider two possible cases.

1. Assume $d \in B \setminus C$. We consider two subcases.

- (a) If $a \in A \setminus C$, then $a^g = g_1^{-1}d^{-1}adg_1$ is a reduced product of 5 terms and thus $a^g \notin A$.
- (b) If $a \in C$, then $a^g = g_1^{-1}(a^d)g_1$ is a reduced product of 3 terms, with $a^d \in B \setminus C$, and thus $a^g \notin A$.

2. If $d \in A \setminus C$, then a^d is either in $A \setminus C$ or in C . Moreover, $g_1 \notin A$ as $g = dg_1 \notin A$, and thus g_1 can be written as a reduced product $g_1 = bg_2$ for some $b \in B \setminus C$ and $g_2 \in G$, recalling that dg_1 is reduced. We consider two subcases.

- (a) If $a^d \in A \setminus C$, then $a^g = g_1^{-1}(a^d)g_1$ is a reduced product of 3 terms. Hence $a^g \notin A$ as $g_1 \notin A$.
- (b) If $a^d \in C$, then $a^g = g_2^{-1}((a^d)^b)g_2$ is a reduced product of 3 terms, with $(a^d)^b \in B \setminus C$, hence $a^g \notin A$ follows as well.

□

We now define some analogs of malnormality. They share some similar properties as malnormality and/or will be useful in Chapter 5 for the construction of GTF and non-LO groups.

Definition 4.5.2. Let C be a subgroup of a group A .

(1) We say that C is **multi-malnormal** in A if for every normal subsemigroup C' of C with $1 \notin C'$, we have

$$c_1^{a_1} \cdots c_n^{a_n} \in A \setminus C, \text{ for all } c_i \in C', a_i \in A \setminus C, n \geq 1.$$

(2) We say C is **strong malnormal** in A if for every $c \in C \setminus \{1\}$, we have

$$c^{a_1} \cdots c^{a_n} \in A \setminus C, \text{ for all } a_i \in A \setminus C, n \geq 1.$$

(3) A subgroup C of group A is said to be **weakly multi-malnormal** in A if for any normal subsemigroup C' of C with $1 \notin C'$, we have $\text{NSS}_A(C') \cap C = C'$.

The next result tells us that multi-malnormality implies weakly multi-malnormality, and for subgroups of a GTF group, multi-malnormality implies strong malnormality which implies malnormality.

Lemma 4.5.3. *Let C be a subgroup of group A .*

- (1) *If C is strong malnormal in A , then it is malnormal in A .*
- (2) *If C is GTF and multi-malnormal in A , then C is strong malnormal in A .*
- (3) *C is multi-malnormal in A if and only if for any normal subsemigroup C' of C with $1 \notin C'$, we have*

$$c_1^{a_1} \cdots c_n^{a_n} \in A \setminus C, \text{ for all } c_i \in C', a_i \in A \text{ with at least one } a_i \in A \setminus C.$$

- (4) *C is strong malnormal in A if and only if for every $c \in C \setminus \{1\}$, we have*

$$c^{a_1} \cdots c^{a_n} \in A \setminus C, \text{ for all } a_i \in A \text{ with at least one } a_i \in A \setminus C.$$

- (5) *If C is multi-malnormal in A , then C is weakly multi-malnormal in A .*

Proof. (1) This is obvious.

- (2) This is clear noticing that for $c \in C \setminus \{1\}$, the normal subsemigroup C' of C generated by c doesn't contain 1 when C is GTF.

- (3) First, the “if” is obvious.

To prove the “only if”, we assume that C is multi-malnormal in A . We assume that $c_1, \dots, c_n \in C'$ and $a_1, \dots, a_n \in A$ and that there is an $a_j \in A \setminus C$, such that $T := c_1^{a_1} \cdots c_n^{a_n} \in$

C . We want to find a contradiction. We can also assume that such n is minimal. By multi-malnormality, there exists $a_i \in C$. Note that we can rewrite:

$$T = c_i^{a_i} (c_1^{a_1} \dots c_{i-1}^{a_{i-1}})^{c_i^{a_i}} c_{i+1}^{a_{i+1}} \dots c_n^{a_n} = c_i^{a_i} c_1^{a'_1} \dots c_{i-1}^{a'_{i-1}} c_{i+1}^{a_{i+1}} \dots c_n^{a_n} = c_i^{a_i} T',$$

where $a'_k = a_k c_i^{a_i}$ and $T' = c_1^{a'_1} \dots c_{i-1}^{a'_{i-1}} c_{i+1}^{a_{i+1}} \dots c_n^{a_n}$. But now $T' \notin C$ by minimality of n , a contradiction since $T' = (c_i^{-1})^{a_i} T \in C$.

(4) This is proved similarly to (3) .

(5) Assume that C is multi-malnormal in A . Let C' be a normal subsemigroup of C with $1 \notin C'$.

We want to prove that $\text{NSS}_A(C') \cap C = C'$. First, the RHS is clearly contained in the LHS. To prove the other direction, assume that $T := c_1^{a_1} \dots c_n^{a_n}$ with $a_i \in A$ and $c_i \in C'$ is contained in the LHS. Then all $a_i \in C$, as otherwise T is not in C by (3) . Now T is in C' by the normality of C' in C , as desired.

□

Combining (2) and (4) of Lemma 4.5.3, we have the following result.

Corollary 4.5.4. *Assume that a subgroup C of a group A is strong malnormal or multi-malnormal in A . Then C doesn't contain a generalized torsion element of A if and only if C is GTF.*

The next result is an analog of one direction of Lemma 4.5.1; the proof uses Theorem 4.4.1.

Corollary 4.5.5. *Assume that in the amalgam $G = A *_C B$, C is weakly multi-malnormal in B . Then A is weakly multi-malnormal in G .*

Proof. Let A' be a normal subsemigroup of A with $1 \notin A'$. We want to prove that

$$A \cap \text{NSS}_G(A') = A'.$$

Let $P = A', C' = P \cap C$ and $Q = \text{NSS}_B(C')$. Then C' is a normal subsemigroup of C with $1 \notin C'$.

As C is weakly multi-malnormal in B , we have

$$Q \cap C = C' = P \cap C.$$

Therefore, by Theorem 4.4.1, we have $A \cap \text{NSS}_G(P \cup Q) = P$. Now we have

$$A \cap \text{NSS}_G(A') = A \cap \text{NSS}_G(P) \subseteq A \cap \text{NSS}_G(P \cup Q) = P = A',$$

from which the desired equality follows as the other direction is clear.

□

Note that the trivial group $\{1\}$ is malnormal and weakly multi-malnormal in every group. Therefore, in every free product $A * B$ of two groups, the factor group A is malnormal and weakly multi-malnormal.

Our construction of GTF and non-LO groups in Chapter 5 is based on the next result. Note that by (5) of Lemma 4.5.3, if we replace the weakly multi-malnormal condition by the stronger multi-malnormal condition in Corollary 4.5.6, its conclusion still holds.

Corollary 4.5.6. *Consider the amalgam $G = A *_C B$, where both A and B are GTF groups. If C is RTF and weakly multi-malnormal in both A and B , then G is GTF.*

Proof. We only need to find two families of normal subsemigroups satisfying the conditions in Theorem 4.4.6. First, for $a \in A \setminus \{1\}$, let $N_a = \text{NSS}_A(\{a\})$ and $M_a = \text{NSS}_B(N_a \cap C)$. Note that $1 \notin N_a$ as A is GTF, thus 1 is not in $C' := N_a \cap C$ which is a normal subsemigroup of C . Hence we have

$$M_a \cap C = \text{NSS}_B(C') \cap C = C' = N_a \cap C,$$

and thus $1 \notin M_a$ as $1 \notin N_a \cap C$ and $1 \in C$.

Second, for every $b \in B \setminus \{1\}$, let $N'_b = \text{NSS}_B(\{b\})$ and $M'_b = \text{NSS}_A(N'_b \cap C)$. Then similarly, $1 \notin N'_b$, $M'_b \cap C = N'_b \cap C$ and $1 \notin M'_b$.

Last, we define the families of normal subsemigroups of A and B

$$\mathcal{F}_A = \{N_a \mid a \in A \setminus \{1\}\} \cup \{M'_b \mid b \in B \setminus \{1\}\},$$

$$\mathcal{F}_B = \{M_a \mid a \in A \setminus \{1\}\} \cup \{N'_b \mid b \in B \setminus \{1\}\}.$$

These two families satisfy the two conditions in Theorem 4.4.6. □

4.6 Examples, and an analog of a theorem of Bergman

Using Theorem 4.4.6, it is easy to create many examples of generalized torsion-free amalgams.

Corollary 4.6.1. *Let $G = A *_C B$. Assume that C is RTF in A and B respectively. If there are bi-orderings of A and B with positive cones P and Q , respectively, such that $P \cap C = Q \cap C$, then G is GTF.*

Proof. The families $\{P, P^{-1}\}$ and $\{Q, Q^{-1}\}$ satisfy the conditions in Theorem 4.4.6. □

Remark 4.6.2. The amalgam of two groups being GTF does not imply that the amalgamated subgroup is RTF in even one of the two factor groups. For example, consider $G = \langle a, b, c, d \mid a^2 = c, b = d^2 \rangle$. This is isomorphic to the free group $G' = \langle a, d \mid \rangle$ because the relations $a^2 = c$ and $d^2 = b$ are used to eliminate the generators c and b . It is also isomorphic to the free product of the two free groups $A = \langle a, b \mid \rangle, B = \langle c, d \mid \rangle$, amalgamated by $\phi : C \rightarrow D$, where C is the subgroup of A generated by $\{a^2, b\}$, D is the subgroup of B generated by $\{c, d^2\}$ and ϕ is given by $\phi(a^2) = c, \phi(b) = d^2$. Clearly, C (respectively, D) is not RTF in A (respectively, B).

Our main result also allows for a GTF-analog of an orderability theorem of George Bergman [3, Theorems 21 and 28]. In particular, this analog shows that despite the previous examples, there are circumstances where the RTF condition is necessary and sufficient that the amalgam be GTF.

Corollary 4.6.3. *Let A be a group and $C \subseteq A$ a subgroup. Then $G := A *_C A$ is GTF if and only if A is GTF and C is RTF in A .*

Proof. Suppose A is GTF and C is RTF in A . For $a \in A$, let $N_a = \text{NSS}_A(\{a\})$. Then taking one copy of the family $\{N_a \mid a \in A \setminus \{1\}\}$ for each factor yields two families of normal subsemigroups that satisfy the conditions in Theorem 4.4.6.

To prove the other direction, we assume that G is GTF. Then clearly A is GTF, so we only need to show that C is RTF in A . We prepare some notation. The group G is an amalgam of two copies of A , call them A_1 and A_2 . For each $a \in A_1$, we will write $a' \in A_2$ to denote the image of a under the identity map $A_1 \rightarrow A_2$. Note that for $g \in A$, we have $g = g'$ in G if and only if $g \in C$.

Assume C is not RTF in A and we need to find a contradiction. Then, there exists $a \in A \setminus C$ and $c_1, \dots, c_n \in C$ such that

$$T := ac_1 \cdots ac_{n-1}ac_n \in C.$$

In particular $a'a^{-1} \neq 1$ and $T = T'$ in G , and the second equality can be rewritten as

$$c_n^{-1}a^{-1}c_{n-1}^{-1}a^{-1} \cdots c_1^{-1}a^{-1} \cdot a'c_1 \cdots a'c_{n-1}a'c_n = 1.$$

We claim that the product

$$P_n := c_n^{-1}a^{-1}c_{n-1}^{-1}a^{-1} \cdots c_1^{-1}a^{-1} \cdot a'c_1 \cdots a'c_{n-1}a'c_n$$

is a product of n conjugates of $a'a^{-1}$. Note that $P_1 = c_1^{-1}a^{-1}a'c_1 = (a'a^{-1})^{ac_1}$ so that the base case for an induction holds, and for $n > 1$ we can write $P_n = (P_{n-1}a'a^{-1})^{ac_n}$, proving the claim. As $P_n = 1$, the group G contains $a'a^{-1}$ as a generalized torsion element, a desired contradiction. $\square$

Example 4.6.4. As an example of Corollary 4.6.3, we construct an explicit amalgam that is not GTF when the RTF condition is not satisfied. This group is actually given in Example 4.1.1. Recall that $G = A *_C A = A *_\phi B$, where $A = \langle a, b \mid \rangle$, $B = \langle c, d \mid \rangle$ are free groups of rank two. Also recall that C and D are the subgroups of A and B generated by $\{ab, a^2b^{-2}\}$ and $\{cd, c^2d^{-2}\}$, respectively, $\Phi : A \rightarrow B$ is the isomorphism given by $\Phi(a) = c, \Phi(b) = d$ and ϕ is the restriction $\Phi|_C$. We have the following properties.

1. The isomorphism ϕ is order-preserving. This is because it is the restriction of the isomorphism $\Phi : A \rightarrow B$. Thus, if P is the positive cone of any bi-ordering of A , the positive cones P and $\Phi(P)$ satisfy the condition $\phi(P \cap C) = \Phi(P) \cap D$, which is the intersection condition of Corollary 4.6.1.
2. In G , the element $a^{-1}c$ is generalized torsion. To see this, we note that in G , $ab = cd$ and thus $a^{-1}c = bd^{-1}$. Second, we note that $b^{-2}d^2 = a^{-2}c^2 = a^{-1}c(a^{-1}c)^c$. Therefore

$$(a^{-1}c)(a^{-1}c)^{d^{-1}}[(a^{-1}c)(a^{-1}c)^c]^{d^{-2}} = bd^{-1}(bd^{-1})^{d^{-1}}(b^{-2}d^2)^{d^{-2}} = 1.$$

3. The subgroup C (respectively, D) is not RTF in A (respectively, B). We check the case of C in A . First, note that $a^{-1} \notin C$, which we can see by considering the homomorphism $\psi : A \rightarrow \mathbb{Z}$ defined by $\psi(a) = \psi(b) = 1$, and noting that $\psi(C) \subseteq 2\mathbb{Z}$ and $\psi(a^{-1}) = -1$. On the other hand, $ab, a^2b^{-2} \in C$ and yet $(a^{-1} \cdot ab)^2 \cdot a^{-2} \cdot a^2b^{-2} = 1$.

Remark 4.6.5. By Corollary 4.6.3, the conclusion of Theorem 4.4.6 does not hold in general if the RTF condition is dropped. In fact, the conclusion does not hold in general even if we replace the RTF condition with the weaker condition that C is RTF in at least one of A and B . To see this, consider the amalgam $G = A *_\phi B$, where $A = \langle a \mid \rangle$, $B = \langle b, c \mid bc = cb \rangle$, C and D are (cyclic) subgroups of A and B generated by a^m for a fixed $m \geq 2$ and c , respectively, and the isomorphism $\phi : C \rightarrow D$ is given by $\phi(a^m) = c$. Clearly, C is not RTF in A and D is RTF in B . Consider element $\alpha = [a, b]$ of G , where we write $[x, y]$ for the commutator $xyx^{-1}y^{-1}$. Since $[a^m, b] = [c, b] = 1$, iteratively applying the well-known commutator formula

$$[xy, z] = [y, z]^{x^{-1}}[x, z]$$

to $[a^m, b]$, we find a product of m conjugates of $\alpha = [a, b]$ that is trivial. Hence α is a generalized torsion element (as $m \geq 2$).⁴

⁴Note that this amalgam is in fact the Baumslag-Solitar group $BS(m, m) = \langle a, b \mid ba^mb^{-1} = a^m \rangle$.

Chapter 5

A group which is GTF and non-LO

As an application of our main result Theorem 4.4.6, in this chapter we construct an amalgam which is generalized torsion free and not left-orderable. Our construction $G = A *_\phi B$ is given by Definition 5.1.1 and Theorem 5.1.3.

In Section 5.1 we define an amalgam $G = A *_\phi B$ where $\phi : C \rightarrow D$ is an isomorphism between subgroups $C \subseteq A$ and $D \subseteq B$. We prove that G is not left-orderable, and prove that it is GTF assuming the results in the subsequent subsections. In Section 5.2, we prove that C is RTF in A . We introduce the left-first product in Section 5.3 and study some of its properties, which are applied in Section 5.4 to prove that C is multi-malnormal in A .

5.1 The construction of a GTF and non-LO group

In this section, we define the amalgam $G = A *_\phi B$ which we will prove is GTF and non-LO. In the definition below, $\text{sign}(x) = \frac{x}{|x|}$ is the usual sign function of real numbers.

Definition 5.1.1. *Let C be the subgroup of the free group $A = \langle a, b \mid \rangle$ generated by $S = \{\alpha_i \mid i = 1, 2, \dots, m\}$, with $\alpha_i = a^{k_{i1}^{(1)}} b^{k_{i1}^{(2)}} a^{k_{i2}^{(1)}} b^{k_{i2}^{(2)}} \dots a^{k_{is}^{(1)}} b^{k_{is}^{(2)}}$, $m \geq 8$ and $s \geq 10$, where $k_{ij}^{(t)}$ are nonzero integers satisfying the following conditions.*

(A) The cardinalities of the following two sets are both $sm + \binom{m}{2}$:

$$\begin{aligned} & \{|k_{ij}^{(1)}| \mid 1 \leq i \leq m, 1 \leq j \leq s\} \cup \{|k_{i1}^{(1)} - k_{i'1}^{(1)}| \mid 1 \leq i < i' \leq m\}, \\ & \{|k_{ij}^{(2)}| \mid 1 \leq i \leq m, 1 \leq j \leq s\} \cup \{|k_{is}^{(2)} - k_{i's}^{(2)}| \mid 1 \leq i < i' \leq m\}. \end{aligned}$$

(B) For $i \in \{1, 2, \dots, 8\}$ and $t \in \{1, 2\}$, $\epsilon_{it} := \text{sign}(k_{i1}^{(t)}) = \text{sign}(k_{i2}^{(t)}) = \dots = \text{sign}(k_{is}^{(t)})$.

(C) For $i \in \{1, 2, 3, 4\}$, $\epsilon_{i1} = \epsilon_{i2} = 1$ and for $i \in \{5, 6, 7, 8\}$, $\epsilon_{i1} = -\epsilon_{i2} = 1$.

Condition (A) says that the $sm + \binom{m}{2}$ numbers $k_{ij}^{(1)}$ for all i, j and $k_{i1}^{(t)} - k_{i'1}^{(t)}$ for all $i < i'$ have distinct absolute values and the $sm + \binom{m}{2}$ numbers $k_{ij}^{(2)}$ for all i, j and $k_{is}^{(t)} - k_{i's}^{(t)}$ for all $i < i'$ have distinct absolute values. Conditions (B) and (C) together say that all exponents of a and b in $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are positive, and for $\alpha_5, \alpha_6, \alpha_7, \alpha_8$, the exponents of a are positive and those of b are negative. Condition (A) and the restriction that $s \geq 10$ will be needed to show that C is RTF and multi-malnormal in A (see Theorem 5.2.13 and Theorem 5.4.7), while conditions (B) and (C) together with $m \geq 8$ are used to show the non-left-orderability of the amalgam in Theorem 5.1.3.

Lemma 5.1.2. *The subgroup C in Definition 5.1.1 does exist; i.e., there exist nonzero integers $k_{ij}^{(t)}$ satisfying the conditions (A), (B) and (C).*

Proof. Fix a $t \in \{1, 2\}$. We want to give nonzero values to $k_{ij}^{(t)}$ such that they satisfy conditions (A), (B) and (C). We only need to show that we can fix the values of these $k_{ij}^{(t)}$'s one by one, such that every time we fix the value of one more $k_{ij}^{(t)}$, the $k_{ij}^{(t)}$'s with values given don't violate the conditions.

But this is clear by induction. Indeed, assume that N of the $k_{ij}^{(t)}$'s are given values already and they satisfy the conditions. Now for any new $k_{ij}^{(t)}$ which hasn't been assigned a value, there are only finitely many choices of values for this new $k_{ij}^{(t)}$ such that the conditions will be violated. So, we can choose a value for this new $k_{ij}^{(t)}$'s such that these $N + 1$ valued $k_{ij}^{(t)}$ satisfy the conditions.

□

That S is a free generating set of C is proved in Lemma 5.2.1(4), but we implicitly use this fact in the statement of the next theorem, which is the main result of this section. The details of the proof constitute the remainder of the manuscript.

Theorem 5.1.3. *Assume the setup in Definition 5.1.1. Let $B = \langle c, d \mid \rangle$, and isomorphism $\Phi : A \rightarrow B$ be given by $\Phi(a) = c$, $\Phi(b) = d$ and let $D = \Phi(C)$. Let $\phi : C \rightarrow D$ be an isomorphism such that it sends $\alpha_1, \dots, \alpha_8$ to $\beta_1, \beta_2^{-1}, \beta_5, \beta_6^{-1}, \beta_3, \beta_4^{-1}, \beta_7, \beta_8^{-1}$, respectively, where $\beta_i = \Phi(\alpha_i)$.*

*Then the amalgam $G = A *_\phi B$ is generalized torsion free and not left-orderable. Consequently, there exists a group which is generalized torsion free and not left-orderable.*

Proof. The second statement follows from the first by Lemma 5.1.2. Now we prove the first statement.

By Theorem 5.2.13, C is RTF in A , and thus D is RTF in B . By Theorem 5.4.7, C is multi-malnornal in A and thus D is multi-malnornal in B . Together with the fact that free groups are GTF, these results imply that G is GTF by Corollary 4.5.6.

The amalgam G is not left-orderable because of the same reason as given in Example 4.1.2. But here we give it a proof anyway. To prove that G is not left-orderable, first note that none of a, b, c, d is equal to the identity. Thus by Theorem 1.48 in [11] we only need to prove that 1 is contained in the subsemigroup of G generated by $\{a^{\epsilon_1}, b^{\epsilon_2}, c^{\epsilon_3}, d^{\epsilon_4}\}$ for all the choices $\epsilon_i = \pm 1$. We can assume $\epsilon_1 = 1$ for all choices under consideration, by taking inverses. However, by the conditions on the exponents given in Definition 5.1.1 parts (B) and (C), the element $1 = \alpha_i \phi(\alpha_i)^{-1}$ for $i = 1, 2, \dots, 8$ belongs to the subsemigroup of G generated by the following sets respectively:

$$\begin{aligned} &\{a, b, c^{-1}, d^{-1}\}, \{a, b, c, d\}, \{a, b, c^{-1}, d\}, \{a, b, c, d^{-1}\}, \\ &\{a, b^{-1}, c^{-1}, d^{-1}\}, \{a, b^{-1}, c, d\}, \{a, b^{-1}, c^{-1}, d\}, \{a, b^{-1}, c, d^{-1}\}. \end{aligned}$$

□

Now it remains to prove that C is RTF and multi-malnornal in A . This is the task of the next three sections.

5.2 The RTF-ness of C in A

Our goal in this subsection is to prove that C in Definition 5.1.1 is RTF in A . We view A as the free product $\langle a; \rangle * \langle b; \rangle$ of infinite cyclic groups $\langle a; \rangle$ and $\langle b; \rangle$. Hence, the length $l(g)$ of $g \in A$ is defined. We remark that $l(g)$ is *not* the usual length with respect to the generating set $\{a, b\}$. Assume that $l(g) = n$. We can then write g as an alternating product $g = x_1 \cdots x_n$, where $x_i := c_i^{k_i}$ with $k_i \in \mathbb{Z} \setminus \{0\}$, $c_i \in \{a, b\}$ and $c_i \neq c_{i+1}$. In this case, the alternating product is uniquely determined by g and we also call it the **reduced form** of g . We call $x_i = c_i^{k_i}$ the **i th component** of g and denote it as $B_i(g)$. For convenience, we also call x_i the **$(n - i + 1)$ th component from RHS** and denoted this as $x_i = RB_{n-i+1}(g)$. For $i \leq n$, we define the **left i th factor** and the **right i th factor** of g to be

$$L_i(g) = x_1 x_2 \cdots x_i, \text{ and } R_i(g) = x_{n-(i-1)} \cdots x_{n-1} x_n.$$

For convenience, we define $L_0(g) = 1$ and $R_0(g) = 1$. For $1 \leq i \leq j \leq n$, we call $x_i \cdots x_j$ a **factor** of g and denote it as $B_{[i,j]}(g)$. Hence, for example, the left i th factor $L_i(g)$ is $B_{[1,i]}(g)$, and the i th component $B_i(g)$ is $B_{[i,i]}(g)$. Note that $B_{[i,j]}(g)$ is only defined when $1 \leq i \leq j \leq l(g)$.

We also define the sets $L_i(C) = \{L_i(c) \mid c \in C, l(c) \geq i\}$ and $R_i(C) = \{R_i(c) \mid c \in C, l(c) \geq i\}$. Recall that if $l(g_1 \cdots g_n) = \sum_{i=1}^n l(g_i)$ for $g_i \in A$, we say the product $g_1 \cdots g_n$ is reduced. When $g_1, \dots, g_n \in A$ with all $l(g_i) \geq 2$ for $2 \leq i \leq n - 1$, we say that the product $g_1 \cdots g_n$ is **almost reduced** if $l(g_i g_{i+1}) \geq l(g_i) + l(g_{i+1}) - 1$ for all i with $1 \leq i \leq n - 1$.

We need some results about the structure of the elements in C . Recall that $S = \{\alpha_i \mid i = 1, 2, \dots, m\}$ is the generating set of C .

Lemma 5.2.1. *Let $u_1, \dots, u_k \in S \cup S^{-1}$ with $k \geq 1$, such that $u_i u_{i+1} \neq 1$ for all i with $1 \leq i \leq k - 1$. Then we have :*

- (1) *the product $u_1 \cdots u_m$ is almost reduced;*
- (2) *$L_{2s-1}(u_1 \cdots u_k) = L_{2s-1}(u_1)$ and $R_{2s-1}(u_1 \cdots u_k) = R_{2s-1}(u_k)$;*
- (3) *$l(u_1 \cdots u_k) \geq 2ks - (k - 1)$.*

(4) S is a free generating set of C and $l(g) \geq 2s$ for all $g \in C \setminus \{1\}$;

(5) If the alternating product $A_1 \cdots A_{i+2}$ is in $L_{i+2}(C)$ with $i \geq 1$, then $A_{i+2} \neq A_i^{-1}$.

Proof. We first show that for $u, v \in S \cup S^{-1}$ with $uv \neq 1$, we have $K(u, v) = 0$ and $l(uv) \geq 4s - 1$. This is because the numbers $k_{i1}^{(1)}$ for $i = 1, \dots, m$ are distinct and the number $k_{is}^{(2)}$ for $i = 1, \dots, m$ are distinct. To elaborate on this, recall that $\alpha_i = a^{k_{i1}^{(1)}} b^{k_{i1}^{(2)}} \cdots a^{k_{is}^{(1)}} b^{k_{is}^{(2)}}$. If u and v are both from S , then the product uv is reduced, because $R_1(u)$ is a power of b and $L_1(v)$ is a power of a . If u and v are both from S^{-1} , then similarly uv is reduced. If $u \in S$ and $v \in S^{-1}$, we can assume that $u = \alpha_i$ and $v = \alpha_j^{-1}$ with $i \neq j$. Then we have

$$\begin{aligned} uv &= a^{k_{i1}^{(1)}} b^{k_{i1}^{(2)}} \cdots a^{k_{is}^{(1)}} b^{k_{is}^{(2)}} \cdot b^{-k_{js}^{(2)}} a^{-k_{js}^{(1)}} \cdots b^{-k_{j1}^{(2)}} a^{-k_{j1}^{(1)}} \\ &= a^{k_{i1}^{(1)}} b^{k_{i1}^{(2)}} \cdots a^{k_{is}^{(1)}} b^{k_{is}^{(2)} - k_{js}^{(2)}} a^{-k_{js}^{(1)}} \cdots b^{-k_{j1}^{(2)}} a^{-k_{j1}^{(1)}}, \end{aligned}$$

where the last expression is an alternating product of $4s - 1$ components, as $k_{is}^{(2)} - k_{js}^{(2)} \neq 0$. The case that $u \in S^{-1}$ and $v \in S$ can be treated similarly. In all these cases we find that $K(u, v) = 0$ and $l(uv) \geq 4s - 1$.

With this result in hand, (1) follows immediately, (2) and (3) follow by making induction on k and (4) follows from (3).

To prove (5), we assume that $A_{i+2} = A_i^{-1}$. Then $h := B_{[i, i+2]}(c) = A_i A_{i+1} A_i^{-1}$ for a $c \in C$. Now c can be written as an almost reduced product $v_1 \cdots v_k$, with each $v_j \in S \cup S^{-1}$ and each $v_i v_{i+1} \neq 1$. Since $l(v_i) = 2s \geq 4$, we know that h is a factor of some $v_i v_{i+1}$. However by Condition (A), there is not a pair of components of $v_i v_{i+1}$ such that they are the inverse of each other. This is a contradiction. $\square$

Let $Sc(C)$ be the set of components of elements in C ; i.e.,

$$Sc(C) = \{E \mid E \text{ is a component of some } c \in C\} \subset A.$$

Definition 5.2.2. Define a map $p : Sc(C) \rightarrow A$, called the **prefix**, as follows. Given $E \in Sc(C)$, there are two possibilities:

1. There exists a unique $A_1 \cdots A_{2s} \in S \cup S^{-1}$ with $E = A_i$ for some i with $1 \leq i \leq 2s$. Set

$$p(E) = p(A_i) = A_1 \cdots A_{i-1} \text{ for } 1 \leq i \leq 2s, (\text{with } p(A_1) = 1).$$

2. There exists a unique pair $B_1 \cdots B_{2s}, A_1 \cdots A_{2s} \in S \cup S^{-1}$ with $E = B_{2s}A_1$. Set

$$p(E) = p(B_{2s}A_1) = B_1 \cdots B_{2s-1}.$$

Note that the uniqueness claims in Definition 5.2.2 follow from condition (A). We can verify that

$$p(E)E(p(E^{-1}))^{-1} \in C \text{ for all } E \in Sc(C). \quad (5.1)$$

Lemma 5.2.3. Assume that an alternating product $D_1 \cdots D_i$ (respectively, $E_i \cdots E_1$) with $1 \leq i$ is in $L_i(C)$ (respectively, $R_i(C)$). Then $D_1 \cdots D_{i-1}(p(D_i))^{-1}$ (respectively, $p(E_i)E_iE_{i-1} \cdots E_1$) is in C .

Proof. We only prove the case involving $D_1 \cdots D_i$, as the other case can be proved similarly.

Let $D_1 \cdots D_i = L_i(c)$ for some $c \in C$. There are two possible cases to consider, where we write the component D_i in bold and blue (so D_i is A_k or $B_{2s}A_1$, respectively):

$$c = g_1ug_2 = g_1A_1 \cdots \mathbf{A_k} \cdots A_{2s}g_2, \text{ or } c = g_1vug_2 = g_1B_1 \cdots B_{2s-1}(\mathbf{B_{2s}A_1})A_2 \cdots A_{2s}g_2,$$

where $g_1, g_2 \in C$, both g_1ug_2 and g_1vug_2 are almost reduced products, $u = A_1 \cdots A_{2s}$ and $v = B_1 \cdots B_{2s}$ are alternating products with $u, v \in S \cup S^{-1}$, $l(B_{2s}A_1) = 1$ and the product g_1A_1

(respectively, $A_{2s}g_2$) is reduced if $k = 1$ (respectively, $k = 2s$). Now we have for the first case

$$D_1 \cdots D_{i-1}(p(D_i))^{-1} = g_1 A_1 \cdots A_{k-1}(p(A_k))^{-1} = g_1 A_1 \cdots A_{k-1} \cdot (A_1 \cdots A_{k-1})^{-1} = g_1 \in C,$$

as desired. In the second case, we similarly find $D_1 \cdots D_{i-1}(p(D_i))^{-1} = g_1 \in C$. $\square$

Definition 5.2.4. Let $g = A_p \cdots A_1$ and $h = B_1 \cdots B_q$ be elements of A in reduced form, and $k = K(g, h)$.

1. We say that the components A_i for $1 \leq i \leq k$ of g and the components B_j for $1 \leq j \leq k$ of h are **canceled** in the product gh . For each i with $1 \leq i \leq k$, we say the components A_i and B_i cancel (in the product gh).
2. We say a component A_i (of g) is **unaltered** (in the product gh) if (i) $p \geq i \geq k + 2$ or (ii) $p \geq i = k + 1 = q + 1$ or (iii) $p, q \geq i = k + 1$ and $l(A_{k+1}B_{k+1}) = 2$ ¹. Similarly, a component B_j (of h) is **unaltered** (in the product gh) if (i) $q \geq j \geq k + 2$ or (ii) $q \geq j = k + 1 = p + 1$ or (iii) $p, q \geq j = k + 1$ and $l(A_{k+1}B_{k+1}) = 2$.
3. We call the component $A_{k+1}B_{k+1}$ of gh the **merge** of A_{k+1} and B_{k+1} if $p, q \geq k + 1$ and $l(A_{k+1}B_{k+1}) = 1$ ².
4. We say that a factor $B_{[i,j]}(g)$ is **unaltered** (respectively, **canceled**) if all $B_t(g)$ for $i \leq t \leq j$ are **unaltered** (respectively, **canceled**). Similarly, a factor $B_{[i,j]}(h)$ is **unaltered** (respectively, **canceled**) if all $B_t(h)$ for $i \leq t \leq j$ are **unaltered** (respectively, **canceled**). A factor g' of g and a factor h' of h is said to **cancel** if $l(g') = l(h')$ and for each i with $1 \leq i \leq l(g')$, $RB_i(g')$ and $B_i(h')$ cancel in gh . A factor of g or h is said to be **altered** in the product gh if it's not **unaltered**.

Note that sometimes we use the notation $P(g, h)$ for the product gh . For example, $P(g_1g_2, g_3g_4)$

¹Since we are considering the free group of rank two, this can only happen when $k = 0$.

²Note that if $p, q \geq k + 1 \geq 2$, then $l(A_{k+1}B_{k+1}) = 1$, because we are working in the free group of rank 2; this doesn't follow if the rank is greater than 2.

refers to the product gh , where $g = g_1g_2$ and $h = g_3g_4$. We may also just write it as the product $(g_1g_2)(g_3g_4)$ when there is no risk of confusion.

Remark 5.2.5. We remark that a factor f of h is unaltered in $P(g, h)$ if and only if we can write $h = h'fh''$ as a reduced product such that the product $P(gh', fh'')$ is reduced. Moreover, h is a right factor of gh if and only if $P(g, h)$ is reduced. (The “if” is clear. Also, if h is a right factor of gh , then we can write gh as a reduced product $g'h$, but then $g = g'$, so $P(g, h)$ is reduced.)

Note an altered component is either one that is canceled or one that is “combined with” another component to form a merge of the product, while an unaltered component “remains” a component of the product.

Let $\gamma \in A$ and $1 \leq i \leq l(\gamma)$. If $L_i(\gamma) \in L_i(C)$ (respectively, $L_i(\gamma) \notin L_i(C)$), we say that γ is left i -**compatible** (respectively, **left i -incompatible**) with C . We say that γ is **left compatible** (respectively, **left incompatible**) with C if it is left $l(\gamma)$ -compatible (respectively, left $l(\gamma)$ -incompatible) with C . We similarly define right i -compatibility, right i -incompatibility and right compatibility. For $\alpha \in A$, let $\Lambda(\alpha)$ (respectively, $P(\alpha)$) be the largest integer i (with $i \leq l(\alpha)$) such that $L_i(\alpha) \in L_i(C)$ (respectively, $R_i(\alpha) \in R_i(C)$). For example, we have $\Lambda(c) = l(c) = P(c)$ for all $c \in C$. We also have

$$\Lambda(\beta) = \max\{K(g, \beta) \mid g \in C\} \text{ and } P(\beta) = \max\{K(\beta, g) \mid g \in C\}, \text{ for all } \beta \in A. \quad (5.2)$$

Note that $\Lambda(\alpha^{-1}) = P(\alpha)$ for all $\alpha \in A$, because $(L_i(C))^{-1} = R_i(C)$. Note that if γ is left or right i -incompatible with C , then $\gamma \notin C$, an observation used in Subsection 5.4.3.

An element $\alpha \in A$ is called **left C -simplified** if either (1) $\Lambda(\alpha) < s$ or (2) $\alpha \in L_s(C)$. (Note that $\alpha \in L_s(C)$ implies that $\Lambda(\alpha) = s$.) We define right C -simplified similarly. We say that α is **C -simplified** if it is both left and right C -simplified.

Lemma 5.2.6. *Let $\alpha \in A \setminus C$.*

- (1) *There exists $c \in C$ and a left (respectively, right) C -simplified α' such that $\alpha = c\alpha'$ (respectively, $\alpha = \alpha'c$). Moreover, $l(\alpha') \leq l(\alpha)$, and $l(\alpha') < l(\alpha)$ if α is not C -simplified.*

(2) There exist $c_1, c_2 \in C$ and α' C -simplified such that $\alpha = c_1 \alpha' c_2$ with $l(\alpha') \leq l(\alpha)$.

Proof. We prove by induction on $l(\alpha)$. Both statements (1) and (2) are true if $l(\alpha) < s$, as we can take $c = c_1 = c_2 = 1$.

(1) We only prove the statement that is not in parentheses. It is clearly true if α is left C -simplified. Now assume that α is not left C -simplified. Then $\Lambda(\alpha) \geq s$ and $\alpha \notin L_s(C)$. Thus $l(\alpha) > s$ and there is a $c_1 \in C$ such that $L_s(\alpha) = L_s(c_1)$. By (2) of Lemma 5.2.1, there is a $u \in S \cup S^{-1}$ such that $L_s(c_1) = L_s(u)$, and thus $L_s(\alpha) = L_s(u)$. Letting $\alpha_1 = u^{-1}\alpha$, we have $\alpha = u\alpha_1$ and we show in the next paragraph that $l(\alpha_1) < l(\alpha)$. Thus, by induction we can write $\alpha_1 = c'\alpha'$ with $c' \in C$ and α' left C -simplified and $l(\alpha') \leq l(\alpha_1)$, leading to the desired product $\alpha = (uc')\alpha'$ with $uc' \in C$ and α' left C -simplified and $l(\alpha') < l(\alpha)$.

Let $k = K(u^{-1}, \alpha)$. Then $k \geq s$ as $L_s(\alpha) = L_s(u)$. If $k > s$, then we clearly have $l(\alpha_1) < l(\alpha)$ as $l(u) = 2s$. If $k = s$, then there is a merge of components after the cancellations, as $l(u^{-1}), l(\alpha) > s$. Thus, we have $l(\alpha_1) = l(u^{-1}\alpha) < l(\alpha)$ again.

(2) By (1), there is a $c'_1 \in C$ and an α_1 which is left C -simplified such that $\alpha = c'_1 \alpha_1$ and $l(\alpha_1) \leq l(\alpha)$. If α_1 is right C -simplified, then we are done. If not, then there is a $c'_2 \in C$ and an α_2 right C -simplified such that $\alpha_1 = \alpha_2 c'_2$ and $l(\alpha_2) < l(\alpha_1)$. Now we have $\alpha = c'_1 \alpha_2 c'_2$ and $l(\alpha_2) < l(\alpha)$. Using induction, there are $c'_1, c'_2 \in C$ and α' C -simplified such that $\alpha_2 = c'_1 \alpha' c'_2$ with $l(\alpha') \leq l(\alpha_2)$. Now we have $\alpha = (c'_1 c'_1) \alpha' (c'_2 c'_2)$ where $c'_1 c'_1$ and $c'_2 c'_2$ are in C , α' C -simplified and $l(\alpha') \leq l(\alpha)$.

□

The next result follows from Lemma 5.2.3.

Corollary 5.2.7. *Let $\beta \in A \setminus C$, h be a left factor of an element in C , $k = K(\beta, h)$ and $n = l(\beta)$. Then βh is not left $(n - k + 1)$ -compatible with C . If moreover $k \leq l(h) - 2$, then βh is left $(n - k + 1)$ -incompatible with C .*

Proof. The second statement follows from the first, as $l(\beta h) \geq n' := n - k + 1$ when $k \leq l(h) - 2$. So we assume that βh is n' -compatible with C and need to find a contradiction. By Lemma 5.2.3, letting $B = B_{n'}(\beta h)$, we have

$$d := L_{n-k}(\beta h)(p(B))^{-1} \in C, \text{ and } c_i := L_{i-1}(h)p(B_i(h))^{-1} \in C \text{ for } 1 \leq i \leq l(h).$$

Also, we can write $\beta = \beta' \alpha^{-1}$, $h = \alpha h'$ as reduced products, with $l(\alpha) = k$. So, $\beta h = \beta' h'$, $l(\beta') = n - k$, $L_k(h) = \alpha$, and $K(\beta', h') = 0$. We consider two cases.

1. If $\beta' h'$ is reduced, then $L_{n-k}(\beta h) = \beta'$, $B = B_{n'}(\beta' h') = B_{k+1}(h)$, and so $(p(B))^{-1} = (L_k(h))^{-1} c_{k+1}$. Therefore, $d = \beta' (L_k(h))^{-1} c_{k+1} = \beta c_{k+1}$, leading to $\beta \in C$, a contradiction.
2. If $\beta' h'$ is not reduced, then $B = B_{k+2}(h)$ and $L_{n-k}(\beta' h') = \beta' B_{k+1}(h)$. So $d = \beta' B_{k+1}(h) \cdot (L_{k+1}(h))^{-1} c_{k+2} = \beta' (L_k(h))^{-1} c_{k+2} = \beta c_{k+2}$, leading to $\beta \in C$, a contradiction again.

□

Lemma 5.2.8. *Let $\beta \in A \setminus C$, $g \in C$ and h be a left factor of an element in C . Then we have*

$$K(g, \beta h) \leq K(g, \beta) + 1.$$

Proof. We argue by contradiction, Let $i = K(g, \beta)$ and assume that $i' := K(g, \beta h) \geq i + 2$. Note that i (respectively, i') is maximal such that $L_i(\beta) = L_i(g^{-1})$ (respectively, $L_{i'}(\beta h) = L_{i'}(g^{-1})$); in particular, βh is left $(i+2)$ -compatible with g . However, letting $n = l(\beta)$ and $k = K(\beta, h)$, βh is not left $(n-k+1)$ -compatible with C by Corollary 5.2.7. Hence $n - k + 1 \geq i + 3$, leading to $k \leq n - i - 2$ and thus $L_{i+1}(\beta)$ is unaltered in $P(\beta, h)$. Therefore, $L_{i+1}(\beta) = L_{i+1}(\beta h) = L_{i+1}(g^{-1})$, contradicting the maximality of i . □

Recall that a subgroup H of group G is **isolated** in G if $g \in G \setminus H$, $g^n \in H$ with $n \geq 2$ implies that $g^n = 1$. When G is a group without a torsion element, this is equivalent to $g^n \in G \setminus H$ for all $g \in G \setminus H$ and $n \geq 2$. By the next result, C is isolated in A .

Lemma 5.2.9. *Let $\alpha \in A \setminus C$, $g \in C$ and $n \geq 2$. Then we have $\alpha^n \in A \setminus C$ and $K(g, \alpha^n) \leq K(g, \alpha) + 1$.*

Proof. Let $k = K(g, \alpha)$. So k is maximal such that $L_k(\alpha) = L_k(g^{-1})$. Write α as an alternating product

$$\alpha = A_1^{-1} \cdots A_r^{-1} B_1 \cdots B_t A_r \cdots A_1,$$

where $r \geq 0$, $t \geq 1$ and $B_1 B_t \neq 1$. (Also, $l(B_1 B_t) = 1$ if $r > 0$.) Then we can write α^n as one of the following alternating products:

$$A_1^{-1} \cdots A_r^{-1} (B_1^n) A_r \cdots A_1 \quad (\text{when } t = 1),$$

$$B_1 \cdots B_t B_1 \cdots B_t \cdots B_1 \cdots B_t \quad (\text{when } t \geq 2, l(B_1 B_t) = 2),$$

$$A_1^{-1} \cdots A_r^{-1} B_1 B_2 \cdots B_{t-1} (B_t B_1) B_2 \cdots (B_t B_1) B_2 \cdots B_t A_r \cdots A_1 \quad (\text{when } t \geq 2, l(B_1 B_t) = 1).$$

Suppose α^n is the first product. If $r = 0$, then $l(\alpha^n) = 1$ and thus $K(g, \alpha^n) \leq 1 \leq k + 1$, and $\alpha^n \notin C$ by (4) of Lemma 5.2.1. If $r \geq 1$, then by (5) of Lemma 5.2.1, we know that $\alpha^n \notin C$ and $A_1^{-1} \cdots A_r^{-1} (B_1^n) A_r$ is not $L_{r+2}(g^{-1})$. Hence $K(g, \alpha^n) \leq r + 1$. So, $K(g, \alpha^n) \leq k + 1$ if $k \geq r$. If $k < r$, then $K(g, \alpha^n) = k$ as $L_r(\alpha^n) = L_r(\alpha)$. So we always have $K(g, \alpha^n) \leq k + 1$, as desired.

From now on we only consider the other two cases. We first show by contradiction that $L_{r+t+1}(\alpha^n) \notin L_{r+t+1}(C)$. Suppose that $L_{r+t+1}(\alpha^n) \in L_{r+t+1}(C)$.

1. Suppose α^n is the second product. Then $r = 0$ and $B_1 \cdots B_t B_1 \in L_{t+1}(C)$. Hence $B_1 \in L_1(C)$, and by Lemma 5.2.3, $\alpha = B_1 \cdots B_t = B_1 \cdots B_t (p(B_1))^{-1} \in C$, a contradiction.
2. Suppose α^n is the third product. Recalling that $l(B_t B_1) = 1$, by Lemma 5.2.3 we have

$$h_1 := A_1^{-1} \cdots A_r^{-1} B_1 \cdots B_{t-1} B_t B_1 (p(B_2))^{-1} \in C, \quad h_2 := A_1^{-1} \cdots A_r^{-1} B_1 (p(B_2))^{-1} \in C,$$

from which we find that $h_1 h_2^{-1} = \alpha$ is in C , again a contradiction.

It now follows that $\alpha^n \notin C$, as $l(\alpha^n) \geq r + t + 1$. Suppose $K(g, \alpha^n) \geq k + 2$. Then as

$L_{r+t-1}(\alpha) = L_{r+t-1}(\alpha^n)$, this forces $k \geq r+t-1$, so $K(g, \alpha^n) \geq r+t+1$. But then $L_{r+t+1}(\alpha^n) = L_{r+t+1}(g^{-1}) \in L_{r+t+1}(C)$, a contradiction. Thus $K(g, \alpha^n) \leq k+1$, as desired. $\square$

Combining Lemma 5.2.8, Lemma 5.2.9 and (5.2), we have the next result.

Corollary 5.2.10. *If $\alpha \in A \setminus C$, $g \in C$, $n \geq 1$, and h is a left factor of an element in C , then*

$$K(g, \alpha^n h) \leq K(g, \alpha) + 2 \leq \Lambda(\alpha) + 2.$$

Lemma 5.2.11. *Let $\alpha \in A \setminus C$ be C -simplified, h_2 (respectively, h_1) be a right (respectively, left) factor of an element in C , $g \in S \cup S^{-1}$ and $m \geq 1$ be an integer. Write $g = A_1 \cdots A_{2s}$ as an alternating product and assume that $i := K(\alpha, g)$ and $j := K(g, \alpha)$ are both nonzero. Then we have:*

(1) $l(\alpha)$ is even, the product α^m is reduced, $R_{l(\alpha)}(\alpha^m) = \alpha$, $K(h_2, \alpha) \leq j$ and $i, j \leq s-1$;

(2) if $K(h_2 \alpha^m, g) > i$ or $K(g, \alpha^m h_1) > j$, then $m = 1$ and α is the alternating product

$$A_{2s}^{-1} \cdots A_{2s-j+1}^{-1} (A_{2s-j}^{-1} A_{i+1}^{-1}) A_i^{-1} \cdots A_1^{-1}; \quad (5.3)$$

(3) $K(h_2 \alpha^m, g) \leq i+1$ and $K(g, \alpha^m h_1) \leq j+1$.

Proof. (1) Since i, j are nonzero and $l(A_1 A_{2s}) = 2$, we conclude $l(B_1(\alpha) R_1(\alpha)) = 2$ and so $l(\alpha)$ is even. Hence, the product $\alpha^m = \alpha \cdots \alpha$ is reduced and thus $R_{l(\alpha)}(\alpha^m) = \alpha$.

Note that $i, j \leq s$ as α is C -simplified. If $i = s$, then $\alpha = (A_1 \cdots A_s)^{-1}$ as α is C -simplified, and thus $R_1(g) L_1(\alpha) = A_{2s} A_s^{-1} \neq 1$ by condition (A), contradicting $j > 0$. Hence $i \leq s-1$, and similarly $j \leq s-1$. We also have $\Lambda(\alpha) = j$ and hence $K(h_2, \alpha) \leq j$.

(2) Since $A_1, \dots, A_{2s}$ are distinct, $L_j(\alpha) = (R_j(g))^{-1}$ and $R_i(\alpha) = (L_i(g))^{-1}$ don't have common components, hence α can be written as alternating product

$$\alpha = A_{2s}^{-1} \cdots A_{2s-j+1}^{-1} B_1 \cdots B_t A_i^{-1} \cdots A_1^{-1}, \text{ with } t \geq 0.$$

Assume that $K(h_2\alpha^m, g) \geq i+1$. Then $RB_{i+1}(\alpha^m) = RB_{i+1}(\alpha)$ is altered in $P(h_2, \alpha^m)$. But this can't happen if $m \geq 2$ as $K(h_2, \alpha^m) = K(h_2, \alpha) \leq j$. Hence $m = 1$ and $R_{i+1}(h_2\alpha) = A_{i+1}^{-1} \cdots A_1^{-1}$. Note that h_2 is a right factor of an element in C and the component A_{2s}^{-1} of α is canceled in $P(h_2, \alpha)$. Therefore, letting $j' = \min\{l(h_2), 2s-1\}$, we have $R_{j'}(h_2) = A_{2s-j'+1} \cdots A_{2s}$. Now, if $t = 0$ or $t \geq 2$, we can't have $R_{i+1}(h_2\alpha) = A_{i+1}^{-1} \cdots A_1^{-1}$. So $t = 1$ and we also need $2s - j' + 1 \leq 2s - j$ and $A_{2s-j}B_1 = A_{i+1}^{-1}$, from which $B_1 = A_{2s-j}^{-1}A_{i+1}^{-1}$ follows. This proves (2) when $K(h_2\alpha^m, g) \geq i+1$. We treat the case $K(g, \alpha^m h_1) \geq j+1$ similarly and reach the same conclusion.

- (3) If $K(h_2\alpha^m, g) \geq i+2$, then we still have $m = 1$ by the argument of (2). But $K(h_2\alpha, g) \leq K(\alpha, g) + 1 = i+1$ by (the other version of) Lemma 5.2.8, a contradiction. This proves the first inequality; the second is proved similarly.

□

The next result helps handle a special case in the proof of the RTF-ness of C in A .

Proposition 5.2.12. *Let $\alpha \in A \setminus C$ be C -simplified, $g \in C \setminus \{1\}$, $m, n \geq 1$ and h_2 (respectively, h_1) be a right (respectively, left) factor of an element in C . Assume that*

$$K(h_2\alpha^m, g) + K(g, \alpha^n h_1) > l(g) - 2. \quad (5.4)$$

Then $g \in S \cup S^{-1}$ and $R_s(g)\alpha L_s(g)$ is the component $B_s(g)$ or the component $B_{s+1}(g)$.

Proof. First $\Lambda(\alpha), P(\alpha) \leq s$ as α is C -simplified. So $i, j \leq s$ by (5.2), where $i := K(\alpha, g)$, $j := K(g, \alpha)$. Then we proceed to prove the following claims.

Claim (1) We have $g \in S \cup S^{-1}$ and so the RHS of (5.4) is $2s - 2$.

To show this, suppose $g \notin S \cup S^{-1}$. Then $l(g) \geq 4s - 1$ by (4) and (3) of Lemma 5.2.1. By Corollary 5.2.10 (and its “right-sided” version), we get

$$K(h_2\alpha^m, g) + K(g, \alpha^n h_1) \leq P(\alpha) + 2 + \Lambda(\alpha) + 2 \leq 2s + 4.$$

Now (5.4) gives $2s + 4 > 4s - 3$ or $2s < 7$, a contradiction since $s \geq 10$.

Claim (2) Both i and j are nonzero. (Recall that $i := K(\alpha, g)$ $j := K(g, \alpha)$.)

Because if one of them is zero, then the LHS of (5.4) is at most $s + 4$ by Corollary 5.2.10, the RHS is $2s - 2$ by Claim (1), leading to $s < 6$, again a contradiction to $s \geq 10$.

Claim (3) We have $l(\alpha)$ is even and $i, j \leq s - 1$.

This now follows from Lemma 5.2.11(1).

Claim (4) We have $i, j \geq s - 2$.

To prove this, suppose $i \leq s - 3$. Then by Lemma 5.2.11(3) and Claim (3), we have

$$K(h_2\alpha^m, g) + K(g, \alpha^n h_1) \leq i + 1 + j + 1 \leq (s - 3) + 1 + (s - 1) + 1 = 2s - 2,$$

contradicting (5.4) by Claim (1). Similarly, we show that $j \leq s - 3$ is not possible.

Claim (5) We have $K(h_2\alpha^m, g) > i$ or $K(g, \alpha^n h_1) > j$.

Because otherwise we have, using Claim (3) and Claim (1),

$$K(h_2\alpha^m, g) + K(g, \alpha^n h_1) \leq s - 1 + s - 1 = 2s - 2 = l(g) - 2,$$

contradicting (5.4).

We now apply Lemma 5.2.11(2) to conclude that α is the alternating product

$$A_{2s}^{-1} \cdots A_{2s-j+1}^{-1} (A_{2s-j}^{-1} A_{i+1}^{-1}) A_i^{-1} \cdots A_1^{-1},$$

where $A_1 \cdots A_{2s}$ is the reduced form of g . In particular, we have $l(\alpha) = i + j + 1$. Since $l(\alpha)$ is even and $i, j \in \{s - 2, s - 1\}$, we have $\{i, j\} = \{s - 1, s - 2\}$. First, consider $i = s - 1$. Then $\alpha = A_{2s}^{-1} \cdots A_{s+3}^{-1} (A_{s+2}^{-1} A_s^{-1}) A_{s-1}^{-1} \cdots A_1^{-1}$, from which we have $R_s(g) \alpha L_s(g) = A_{s+1}$, as desired. Similarly, we find $R_s(g) \alpha L_s(g) = A_s$ when $i = s - 2$. $\square$

We are now ready to prove the main result of this subsection.

Theorem 5.2.13. *The subgroup C of A given by Definition 5.1.1 is RTF in A .³*

Proof. Let $\beta \in A \setminus C$. We need to prove that $T := h_0 \beta h_1 \beta h_2 \cdots \beta h_k \neq 1$ for all $h_i \in C$ and $k \geq 1$. By Lemma 5.2.6, we can write $\beta = c_1 \alpha c_2$, where $c_i \in C$ and α is C -simplified. Plugging this into the expression of T , we can rewrite T in the following form

$$T = g_0 \alpha^{m_1} g_1 \alpha^{m_2} g_2 \cdots \alpha^{m_{n-1}} g_{n-1} \alpha^{m_n} g_n \quad (5.5)$$

where $g_i \in C$, $m_i \geq 1$, $n \geq 1$ and $g_1, g_2, \dots, g_{n-1}$ are nontrivial. Applying Corollary 4.2.2 to (5.5) and considering α_i in the corollary to be α^{m_i} , we know that $T \neq 1$ if the conditions of the corollary are satisfied. Therefore, we only need to consider the case when there is an i with $1 \leq i \leq n-1$, a right factor $g'_{i-1,2}$ of g_{i-1} and a left factor $g'_{i+1,1}$ of g_{i+1} , such that

$$K(g'_{i-1,2} \alpha^{m_i}, g_i) + K(g_i, \alpha^{m_{i+1}} g'_{i+1,1}) > l(g_i) - 2. \quad (5.6)$$

Then by Proposition 5.2.12, we have $g_i \in S \cup S^{-1}$ and $\gamma := R_s(g_i) \alpha L_s(g_i)$ is $B_s(g_i)$ or $B_{s+1}(g_i)$.

Let $\tilde{C}$ be the subgroup of A generated by $\tilde{S} := (S \setminus \{g_i, g_i^{-1}\}) \cup \{L_s(g_i), R_s(g_i)\}$. Then C is a subgroup of $\tilde{C}$. Plugging $\alpha = (R_s(g_i))^{-1} \gamma (L_s(g_i))^{-1}$ into the expression (5.5), we can write T as

$$T = h_0 \gamma^{n_1} h_1 \gamma^{n_2} h_2 \cdots \gamma^{n_r} h_r \quad (5.7)$$

where $h_i \in \tilde{C}$, $n_i \geq 1$, $r \geq 1$ and $h_1, \dots, h_{r-1}$ are nontrivial. We want to prove that $T \neq 1$. This can be derived from another application of Corollary 4.2.2, using that $l(\gamma) = 1$. We first prove some claims.

Claim (1) We have $l(g) \geq s$ for all $g \in \tilde{C} \setminus \{1\}$.

³In fact, here we only need condition (A) on the exponents $k_{ij}^{(t)}$; conditions (B), (C) and that $m \geq 8$ are not required for this proof.

This is because all elements in the generating set $\tilde{S}$ have length at least s and that $K(u, v) = 0$ for all $u, v \in \tilde{S} \cup \tilde{S}^{-1}$ with $uv \neq 1$.

Claim (2) If the alternating product $B_i \cdots B_1$ (respectively, $D_1 \cdots D_j$) is a right (respectively, left) factor of an element in $\tilde{C}$, with $i, j \leq s - 1$ and $B_i D_j = 1$, then $i = j$ and $B_k D_k = 1$ for all $k \leq i$.

To see this, note that $i, j \leq s - 1$ implies $B_i \cdots B_1 = R_i(u)$ and $D_1 \cdots D_j = L_j(v)$ for some $u, v \in \tilde{S} \cup \tilde{S}^{-1}$. Hence $B_1^{-1} \cdots B_i^{-1} = L_i(u^{-1})$. Since $B_i^{-1} = D_j$, by Definition 5.1.1(A), we have $u^{-1} = v$ and also $i = j$. Then $B_k^{-1} = D_k$ for all $k \leq i$ follows, as desired.

Claim (3) For any right (respectively, left) factor h of an element in $\tilde{C}$, for every $g \in \tilde{C} \setminus \{1\}$, and every positive integer m , we have $K(h\gamma^m, g) \leq 1$ (respectively, $K(g, \gamma^m h) \leq 1$).

We only prove the case when h is a right factor. Assume that $K(h\gamma^m, g) \geq 2$ and we will find a contradiction. Let $h = B_r \cdots B_1$ and $g = D_1 \cdots D_t$ be alternating products. Then $l(h\gamma^m), t \geq 2$ and $h\gamma^m$ is one of the following alternating products:

$$B_r \cdots B_1(\gamma^m), B_r \cdots B_2(B_1\gamma^m), \text{ or } B_r \cdots B_2,$$

which leads to $B_1 D_2 = 1$, $B_1 \gamma^m D_1 = 1$ and $B_2 D_2 = 1$, and $B_2 D_1 = 1$, respectively. But none of these is possible by Claim (2).

Now, for each i with $1 \leq i \leq r - 1$, every right factor $h_{i-1,2}$ of h_{i-1} and every left factor $h_{i+1,1}$ of h_{i+1} , using Claim (3), $s \geq 4$ and Claim (1), we have

$$K(h_{i-1,2}\gamma^{n_i}, h_i) + K(h_i, \gamma^{n_{i+1}}h_{i+1,1}) \leq 2 \leq s - 2 \leq l(h_i) - 2.$$

With this, it follows from Corollary 4.2.2 that T in (5.7) is nontrivial. □

5.3 Left-first product and origin of components

Having completed the proof of the RTF-ness of C in A , we now want to deal with the multi-malnormality of C in A . In this section, we provide some tools and results, which will be applied in the next section in the proof that C is multi-malnormal in A .

Consider a product $c_1^{g_1} \cdots c_n^{g_n}$ with $c_i \in C$ and $g_i \in A \setminus C$. Our primary technique will be to express $c_1^{g_1} \cdots c_n^{g_n}$ as an alternating product, and then use the exponent conditions of Definition 5.1.1 to limit cancellation. However, in order to use these exponent conditions, we must keep track of which components “come from” the c_i ’s, which will be accomplished in Subsection 5.4 using the tools of this section.

Let us first look at an example to illustrate the subtlety of this task. Consider a product $g_1 g_2 g_3$ with $g_1 = a, g_2 = a^{-1}, g_3 = a$, so $g_1 g_2 g_3 = a$. Where the component a “comes from” is ambiguous: it “comes from” g_1 if we first compute $g_2 g_3$ and then $g_1(g_2 g_3)$; it “comes from” g_3 if we compute $g_1 g_2$ first and then calculate $(g_1 g_2)g_3$. Therefore, the order of the multiplication matters when considering the origin of the components appearing in the final alternating product (even though the product is independent of the order in which we carried out the multiplications).

Recall that in Definition 5.2.4, we defined the unalteredness and cancellation of components and factors in a product gh of two elements g and h . We now generalize these to a product $g_1 \cdots g_n$. Note that if a component A_i of g is unaltered in gh , then we can view A_i as a component of gh , and doing so allows us to talk about whether A_i is unaltered in $(gh)f$ for $f \in A$. This iterative approach to unaltered components is key to the definition below.

Definition 5.3.1. Let $g_1, \dots, g_n \in A$ and $P_t = g_1 g_2 \cdots g_t$ for $1 \leq t \leq n$. We call the tuple $(P_1, P_2, \dots, P_n)$ the **left-first product** of $(g_1, \dots, g_n)$ and denote it as $LFP(g_1, \dots, g_n)$.

1. Let g'_i be a factor of g_i with $1 \leq i \leq n$. We say that g'_i is **unaltered** in $LFP(g_1, \dots, g_n)$ if it's unaltered in $P_{j-1}g_j$, for $j = i, i+1, \dots, n$. (Here, we use that when g'_i is unaltered in $P_{j-1}g_j$ we may view it as a factor of P_j .) We say that g'_i is **altered** in $LFP(g_1, \dots, g_n)$ if it's not unaltered in $LFP(g_1, \dots, g_n)$.

2. Let g'_i (respectively, g'_j) be a factor of g_i (respectively, g_j) with $1 \leq i < j \leq n$. We say g'_i and g'_j **cancel** in $LFP(g_1, \dots, g_n)$ if g'_i is unaltered in $LFP(g_1, \dots, g_{j-1})$ and g'_i and g'_j cancel in $P_{j-1}g_j$.

Similarly, we define the right-first product $RFP(g_n, \dots, g_1)$ of the tuple $(g_n, \dots, g_1)$ to be $(P_1, P_2, \dots, P_n)$, where $P_i = g_i \cdots g_1$; a factor g'_i of g_i is unaltered in $RFP(g_n, \dots, g_1)$ if it's unaltered in $g_j P_{j-1}$ for $j = i, i+1, \dots, n$; and a factor g'_j of g_j and a factor g'_i of g_i with $n \geq j > i \geq 1$ cancel in $RFP(g_n, \dots, g_1)$ if g'_i is unaltered in $RFP(g_{j-1}, \dots, g_1)$ and g'_j, g'_i cancel in $g_j P_{j-1}$. Moreover, for convenience, we say that g'_i is **canceled by** g''_j in $LFP(g_1, \dots, g_n)$ (respectively, $RFP(g_n, \dots, g_1)$) if g'_i and a factor g'_j of g''_j cancel in $LFP(g_1, \dots, g_n)$ (respectively, $RFP(g_n, \dots, g_1)$).

The rest of this section presents several results concerning left-first products and right-first products. Note that for each result involving the left-first product, there is an analogous result involving the right-first product and vice versa. For brevity, we usually only state and prove one of the two versions for each pair of results.

For two factors $f_1 := B_{[i_1, j_1]}(g)$, $f_2 := B_{[i_2, j_2]}(g)$ of g with $1 \leq i_1 \leq j_1 < i_2 \leq j_2 \leq l(g)$, we say that f_2 is **to the RHS** of f_1 in g . The next result is clear and follows by induction.

Lemma 5.3.2. *Consider the product $g = g_1 g_2 \cdots g_n$. Let f_i, f_j be factors of g_i, g_j , respectively with $1 \leq i < j \leq n$. Suppose f_i, f_j are unaltered in $LFP(g_1, \dots, g_n)$. Then f_j is to the RHS of f_i in g .*

Lemma 5.3.3. *Let $g_1, \dots, g_n \in A$ with $n \geq 2$. Assume that the component $RB_r(g_1)$ of g_1 and the component $B_t(g_n)$ of g_n cancel in $LFP(g_1, \dots, g_n)$, then we have $RB_r(g_1)B_t(g_n) = 1$ and $R_r(g_1)g_2 \cdots g_{n-1}L_t(g_n) = 1$.*

Proof. We use induction on n . For $n = 2$ it is true by definition, so consider $n \geq 3$. Note that $RB_r(g_1)$ is unaltered in $g_1 g_2$ as it is unaltered in $LFP(g_1, \dots, g_{n-1})$, so $RB_r(g_1) = RB_{r'}(g_1 g_2)$ for a positive integer r' with $R_{r'}(g_1 g_2) = R_r(g_1)g_2$. Now $RB_{r'}(g_1 g_2)$ and $B_t(g_n)$ cancel in $LFP(g_1 g_2, g_3, \dots, g_n)$. Thus by the induction assumption, we have $RB_{r'}(g_1 g_2)B_t(g_n) = 1$ and $R_{r'}(g_1 g_2)g_3 \cdots g_{n-1}L_t(g_n) = 1$, from which desired equalities follow by replacement. $\square$

Lemma 5.3.4. *Let $g, h \in A$ and $1 \leq i \leq l(h)$. Then $R_i(gh) = R_i(h)$ if and only if $R_i(h)$ is unaltered in the product gh .*

Proof. The direction of “if” is clear. Now assume that $R_i(gh) = R_i(h)$. Writing $h = h_1 R_i(h)$ as reduced product, $R_i(h)$ is a right factor of $gh = (gh_1)R_i(h)$. Now by Remark 5.2.5, $P(gh_1, R_i(h))$ is reduced, and thus $R_i(h)$ is unaltered in $P(g, h)$, as desired. $\square$

Corollary 5.3.5. *Let $g_1, \dots, g_n \in A$, $1 \leq i \leq l(g_1)$ and $1 \leq j \leq n$. If $L_i(g_1)$ is unaltered in $LFP(g_1, \dots, g_n)$, then $L_i(g_1)$ is unaltered in $P(g_1 \cdots g_j, g_{j+1} \cdots g_n)$.*

Proof. As $L_i(g_1)$ is unaltered in $LFP(g_1, \dots, g_n)$, we have $L_i(g_1) = L_i(g_1 \cdots g_j) = L_i(g_1 \cdots g_n)$. So $L_i(g_1)$ is unaltered in the product $P(g_1 \cdots g_j, g_{j+1} \cdots g_n)$ by (the left version of) Lemma 5.3.4. $\square$

The next result is about the multiplicativity of the unalteredness of left-first product.

Lemma 5.3.6. *Let $g_1, \dots, g_n \in A$, $1 < k < n$ and B_t be a component of g_k . Then B_t is unaltered in $LFP(g_1, \dots, g_n)$ if and only if B_t is unaltered in both $LFP(g_1, \dots, g_k)$ and $LFP(g_k, \dots, g_n)$.*

Proof. Note that B_t is unaltered in $LFP(g_1, \dots, g_k)$ if B_t is unaltered in $LFP(g_1, \dots, g_n)$. Now we assume that B_t is unaltered in $LFP(g_1, \dots, g_k)$ and we will prove that B_t is unaltered in $LFP(g_1, \dots, g_n)$ if and only if B_t is unaltered in $LFP(g_k, \dots, g_n)$. This finishes the proof.

First, we write $g_k = g'_k B_t g''_k$ as a reduced product and let $\pi_{k-1} = g_1 \cdots g_{k-1}$. Since B_t is unaltered in the product $P(\pi_{k-1}, g_k)$, the last component of $\pi_{k-1} g'_k B_t$ is B_t . Note that the last component of $g'_k B_t$ is B_t as well. Hence for every $h \in A$, $P(\pi_{k-1} g'_k B_t, h)$ is reduced if and only if $P(g'_k B_t, h)$ is reduced.

Now, B_t is unaltered in $P(g_1, \dots, g_n)$ if and only if for each j with $k+1 \leq j \leq n$, the product $P(\pi_{k-1} g'_k B_t, g''_k g_{k+1} \cdots g_j)$ is reduced, which is equivalent to $P(g'_k B_t, g''_k g_{k+1} \cdots g_j)$ being reduced. Hence B_t is unaltered in $P(g_1, \dots, g_n)$ if and only if it is unaltered in $P(g_k, \dots, g_n)$, as desired. $\square$

Corollary 5.3.7. *Let $g_1, \dots, g_n \in A$ and $1 < k < n$. Assume that components B_1 of g_1 and B_k of g_k are unaltered in $LFP(g_1, \dots, g_k)$ and B_1 is altered in $LFP(g_1, \dots, g_n)$. Then the component B_k of g_k is altered in $LFP(g_k, \dots, g_n)$.*

Proof. First B_k is to the RHS of B_1 in $g_1 \cdots g_k$ by Lemma 5.3.2. Since B_1 is altered in the left first product $LFP(g_1 \cdots g_k, g_{k+1}, \dots, g_n)$, so is B_k . Thus B_k is altered in $LFP(g_1, \dots, g_n)$ as well. But B_k is unaltered in $LFP(g_1, \dots, g_k)$, so B_k is altered in $LFP(g_k, \dots, g_n)$ by Lemma 5.3.6. $\square$

Lemma 5.3.8. *Let $g_1, \dots, g_n \in A$, S be a factor of g_i , and T be a factor of g_j with $1 \leq i < j \leq n$. Assume that S and T cancel in $LFP(g_1, \dots, g_n)$. Then S and T cancel in $LFP(g_i, \dots, g_j)$.*

Proof. Since S is unaltered in $LFP(g_1, \dots, g_{j-1})$, it is unaltered in $LFP(g_i, \dots, g_{j-1})$ by Lemma 5.3.6. So we only need to prove that S and T cancel in $P(g_i \cdots g_{j-1}, g_j)$.

We write $g_i = g'_i S g''_i$ as a reduced product. Since S is unaltered in $LFP(g_1, \dots, g_{j-1})$ (respectively, $LFP(g_i, \dots, g_{j-1})$), the product $P(g_1 \cdots g_{i-1} g'_i S, g''_i g_{i+1} \cdots g_{j-1})$ (respectively, the product $P(g'_i S, g''_i g_{i+1} \cdots g_{j-1})$) is reduced. Since S and T cancel in $P(g_1 \cdots g_{j-1}, g_j)$, writing $g_j = g'_j T g''_j$ as a reduced product, we have that $(g''_i g_{i+1} \cdots g_{j-1}) g'_j = 1$ and $ST = 1$. Now we see that S and T cancel in $P(g_i \cdots g_{j-1}, g_j)$. $\square$

The next result is about the associativity of cancellation. It will only be used at the very end of this chapter.

Lemma 5.3.9. *Let α, β, γ be in A . Assume that a factor S of α is unaltered in $P(\alpha, \beta)$ and a factor T of γ is unaltered in $P(\beta, \gamma)$. Then S and T cancel in $P(\alpha, \beta\gamma)$ if and only if they cancel in $P(\alpha\beta, \gamma)$.*

Proof. Let $\alpha = \alpha' S \alpha''$ and $\gamma = \gamma' T \gamma''$ be reduced products. Then the products $(\alpha' S)(\alpha'' \beta)$ and $(\beta \gamma')(T \gamma'')$ are reduced. We now have that S and T cancel in $P(\alpha, \beta\gamma)$ (respectively, $P(\alpha\beta, \gamma)$) if and only if $\alpha''(\beta \gamma') = 1$ (respectively, $(\alpha'' \beta) \gamma' = 1$) and $ST = 1$, which implies the desired equivalence. $\square$

5.4 Multi-malnormality of C in A

Let C' be a normal subsemigroup of C with $1 \notin C'$. We are interested in products of conjugates of the form $T := C_1 C_2 \cdots C_n$, where $C_i = c_i^{g_i}$ with $c_i \in C'$ and $g_i \in A \setminus C$. In this section, we

first examine a single conjugate, then study the product of conjugates, and finally prove that C is multi-malnormal in A (see Theorem 5.4.7).

5.4.1 Almost reduced form and standard form of conjugates

Before defining almost reduced form and standard form, we first look at some examples to see the ideas. Note that we are considering elements of the form c^g , with $c \in C$ and $g \in A \setminus C$, where C is given in Definition 5.1.1, where we require $s \geq 10$. Since an element of length $2s$ with $s \geq 10$ is too long to write down explicitly and such length is not essential for us to see the idea, we will just consider elements of shorter length in another subgroup C_1 of A in the next example.

Example 5.4.1. Let's say $\beta_1 = aba^2b^2$ and $\beta_2 = a^3b^3a^4b^4$ are two elements in the generating set S_1 of a subgroup C_1 of the group $A = \langle a, b \mid \rangle$.

1. Consider c^g with $c = \beta_1$ and $g = ab^3$. Note that there is a cancellation in the multiplication $g^{-1}c$. We write $g = a\gamma$ with $\gamma = b^3$, so we have

$$c^g = \gamma^{-1}a^{-1}(aba^2b^2)a\gamma = \gamma^{-1}(ba^2b^2a)\gamma = \chi^\gamma,$$

where $\chi = ba^2b^2a$ is a cyclic conjugate of c . We call χ^γ the almost reduced form of c^g . Now there is a merge in $\gamma^{-1}\chi$ and $\chi\gamma$ is a reduced product. We continue to write c^g in the following form

$$c^g = \gamma^{-1}(ba^2b^2a)\gamma = b^{-3}(ba^2b^2a)b^3 = b^{-2}(a^2b^2a)b^3 = \lambda\mu\rho,$$

where $\lambda = b^{-2}$ is $\gamma^{-1}L_1(\chi)$, $\mu = (L_1(\chi))^{-1}\chi$ and $\rho = \gamma$. We call $\lambda\mu\rho$ the standard form of c^g . We see that $\gamma^{-1}\chi\gamma$ is an almost reduced product and $\lambda\mu\rho$ is a reduced product.

2. Consider c^g with $c = aba^2b^2$ and $g = ab$. We have

$$c^g = (ab)^{-1}(aba^2b^2)ab = a^2b^2ab = \chi^\gamma = \lambda\mu\rho,$$

where $\chi = a^2b^2ab$ is a cyclic conjugate of c , $\gamma = 1$, $\mu = \chi$ and $\lambda = \rho = 1$. Thus, in this case, the almost reduced form and the standard form look the same.

3. Let $c = \beta_1\beta_2^{-1} = aba^2b^{-2}a^{-4}b^{-3}a^{-3}$, $g = a^2ba$. We have

$$\begin{aligned} c^g &= g^{-1}cg = a^{-1}b^{-1}a^{-2}(aba^2b^{-2}a^{-4}b^{-3}a^{-3})a^2ba = \gamma^{-1}\chi\gamma \\ &= a^{-1}b^{-1}a^{-1}(ba^2b^{-2}a^{-4}b^{-3})a^{-1}ba \\ &= \lambda\mu\rho, \end{aligned}$$

where $\chi = c$, $\gamma = g$, $\lambda = g^{-1}L_1(\chi)$, $\mu = L_1(\chi)^{-1}\chi R_1(\chi)^{-1}$ and $\rho = R_1(\chi)g$. In this case, the almost reduced form χ^γ appears the same as its original form c^g .

With these examples in mind, let's now define the almost reduced form and standard form formally. Recall the subgroup C of A given in Definition 5.1.1.

Definition 5.4.2. Consider c^g with $c \in C \setminus \{1\}$ and $g \in A \setminus C$. We define the **almost reduced form** χ^γ of c^g by

$$\chi = c^{L_k(g)}, \quad \gamma = (L_k(g))^{-1}g,$$

where $k := \max\{K(g^{-1}, c), K(c, g)\}$. The **standard form** $\lambda\mu\rho$ of c^g is defined by

$$\lambda = \gamma^{-1}\xi_1, \quad \mu = \xi_1^{-1}\chi\xi_2^{-1}, \quad \rho = \xi_2\gamma,$$

where ξ_1 (respectively, ξ_2) is 1 if the product $\gamma^{-1}\chi$ (respectively, $\chi\gamma$) is reduced and it is $L_1(\chi)$ (respectively, $R_1(\chi)$) otherwise.

We remark that we will only need to consider the case that g is left C -simplified, in which case $K(\gamma^{-1}, \chi)$ and $K(\chi, \gamma)$ are both zero, and thus the products $\gamma^{-1}\chi$ and $\chi\gamma$ are reduced or have a merge. Thus the definition of the standard form says that if there is a merge in $\gamma^{-1}\chi$ (respectively, $\chi\gamma$), then we let the first (respectively, last) component of χ “merge into” γ^{-1} (respectively, γ) to form λ (respectively, ρ). We next record several essential properties of these definitions.

Lemma 5.4.3. *Consider c^g with $c \in C \setminus \{1\}$ and $g \in A \setminus C$ left C -simplified. Let χ^γ and $\lambda\mu\rho$ be the almost reduced form and standard form of c^g , respectively. Let $i = K(g^{-1}, c)$, $j = K(c, g)$ and $k = \max\{i, j\}$. Then we have the following:*

(1) *The product $\gamma^{-1}\chi\gamma$ is almost reduced and the product $\lambda\mu\rho$ is reduced.*

(2) *$l(\lambda) = l(\gamma) = l(\rho)$. Moreover, if $l(\gamma) > 0$, we have*

$$L_{l(\gamma)-1}(\lambda) = L_{l(\gamma)-1}(\gamma^{-1}), \quad R_{l(\gamma)-1}(\rho) = R_{l(\gamma)-1}(\gamma).$$

(3) *$l(\chi) \geq 2s$, $l(\mu) \geq 2s - 1$, $i + j \leq l(c) - s$, $i + j \leq l(\mu) - s + 1$ and $l(c^g) \geq 2s - 1 + 2l(\gamma)$.*

(4) *c^g is left $(l(\gamma) + 2)$ -incompatible and right $(l(\gamma) + 2)$ -incompatible with C .*

Proof. (1) These claims are direct consequences of the definitions.

(2) By definition, $\lambda = \gamma^{-1}$ or $\lambda = \gamma^{-1}L_1(\chi)$ when the product $\gamma^{-1}\chi$ has a merge. Therefore, we have $l(\lambda) = l(\gamma)$ and γ^{-1} and λ differ at most at the last component, implying that $L_{l(\gamma)-1}(\lambda) = L_{l(\gamma)-1}(\gamma^{-1})$ when $l(\gamma) > 0$. Similarly, we prove the other equalities.

(3) First $i, j \leq s$ as g is left C -simplified. As $\lambda\mu\rho$ is reduced, the last inequality follows from the second using (2). Now we prove the first four inequalities considering two cases.

(a) Assume that $i \neq 0$ and $j \neq 0$. Then write $c = u_1 \cdots u_p$ as a reduced word using the free generating set S , it must satisfy $u_1 = u_p^{-1}$ and $p \geq 3$ because of Definition 5.1.1(A), therefore $i = j \leq s$ and $l(c) \geq 6s - 2$, from which the first four inequalities follow.

(b) Assume that $i = 0$ or $j = 0$. Then $l(\chi) \geq l(c) - 1$, as one of the products $(L_k(g))^{-1}c$ and $cL_k(g)$ is almost reduced. Write $c = u_1 \cdots u_p$ as a reduced word using the free generating set S . If $p \geq 2$, we clearly have all the first four inequalities, using $l(c) \geq 4s - 1$. If $p = 1$, then $l(c) = 2s$, and $l(\chi) = 2s$ as one of the products $(L_k(g))^{-1}c$ and $cL_k(g)$ is reduced. Also, since $\gamma^{-1}\chi$ or $\chi\gamma$ is reduced, ξ_1 or ξ_2 appearing in Definition 5.4.2 is the identity, hence $l(\mu) \geq 2s - 1$, from which the fourth inequality follows.

- (4) Let $l' = l(\gamma) + 2 = n - k + 2$, where $n := l(g)$. We only prove that c^g is left l' -incompatible with C , as the other statement can be done similarly. We have the following.

Claim (a) $L_{l'}(c^g) = L_{l'}(g^{-1}c)$. To see this, write $c = c_1c_2$ as a reduced product, such that $l(c_2) = j + 2$, then $l(c_1) = l(c) - j - 2 \geq i + s - 2 \geq i + 8$, thus $K(g^{-1}, c_1) = i$. We now see that $R_1(c_1)$ (respectively, $L_1(c_2)$) is unaltered in $g^{-1}c_1$ (respectively, c_2g), and thus the products $(g^{-1}c_1)(c_2g)$ and $(g^{-1}c_1)c_2$ are reduced. Recalling $l(c) \geq i + j + s$ and $i \leq k$, we have

$$\begin{aligned} l(g^{-1}c_1) &\geq l(g^{-1}) + l(c_1) - 2i - 1 = n + l(c) - j - 2 - 2i - 1 \\ &\geq n - i - 3 + s \geq n - k - 3 + s = l' - 5 + s \geq l' + 5. \end{aligned}$$

Therefore $L_{l'}(c^g) = L_{l'}(g^{-1}c_1c_2g) = L_{l'}(g^{-1}c_1) = L_{l'}(g^{-1}c_1c_2) = L_{l'}(g^{-1}c)$.

Claim (b) If $k = i$, then c^g is left l' -incompatible with C . This is because of $L_{l'}(c^g) = L_{l'}(g^{-1}c)$ and that $g^{-1}c$ is $(n - k + 1)$ -incompatible with C by Corollary 5.2.7.

Claim (c) If $k > i$, then c^g is l' -incompatible with C . To prove this, we first note that $k = j$, and then $i = 0$ by the discussion in (3)(a), and also $R_k(c)L_k(g) = 1$. Now consider two subcases.

- i. Assume $k = 1$. In this case, we want to prove that $L_{n+1}(g^{-1}c) \notin L_{n+1}(C)$. But this is true by Corollary 5.2.7 (as $i = 0$).
- ii. Assume $k \geq 2$. In this case, we have $L_{n-k+1}(g^{-1}) = L_{n-k+1}(g^{-1}c)$ (as $i = 0$). Thus we only need to show that $L_{n-k+1}(g^{-1}) \notin L_{n-k+1}(C)$. Otherwise, $R_{n-k+1}(g) \in R_{n-k+1}(C)$ and so $g' := p(B_k(g))R_{n-k+1}(g) \in C$ by Lemma 5.2.3. But $p(B_k(g)) = L_{k-1}(g)$ because $L_k(g) = (R_k(c))^{-1} = L_k(c^{-1}) = L_k(u)$ for some $u \in S \cup S^{-1}$. Hence $C \ni g' = L_{k-1}(g)R_{n-k+1}(g) = g$, a contradiction.

□

The next result will assist us in proving Corollary 5.4.5, which together with Lemma 5.3.3

tells us that if a component from μ_i and a component from μ_j cancel, then the partial product $g_i C_{i+1} \cdots C_{j-1} g_j^{-1}$ is in C , and therefore we can reduce the number of conjugates in the product $T = C_1 \cdots C_n$ using Lemma 5.4.6.

Lemma 5.4.4. *Let $c \in C \setminus \{1\}$, and let $g \in A \setminus C$ be left C -simplified. Let $c^g = D_1 \cdots D_t \cdots D_n$ be in reduced form. Let $\lambda\mu\rho$ be the standard form of c^g . If D_t is a component of μ , then we have*

$$gD_1 \cdots D_{t-1}(p(D_t))^{-1} \in C \quad \text{and} \quad p(D_t)D_t \cdots D_n g^{-1} \in C. \quad (5.8)$$

Proof. Note that each memberships in (5.8) implies the other, as the product of the terms considered is equal to $c \in C$. Thus, we only need to show that for each component D_t of μ , one of the memberships holds. We use the notation of Definition 5.4.2. Let $i = K(g^{-1}, c), j = K(c, g)$, so $k = \max\{i, j\}$. Let $h_1 = L_k(g)$, so $\chi = h_1^{-1}ch_1, g = h_1\gamma$ is a reduced product. Also $i + j \leq l(c) - s$ by Lemma 5.4.3((3)), which implies $R_1(h_1^{-1}ch_1) = R_1(ch_1)$. We first prove two claims.

Claim (1) We have $g\lambda\mu$ (respectively, $\mu\rho g^{-1}$) is left (respectively, right) compatible with C .

We prove the statement about $g\lambda\mu$. The other can be done similarly. Note that

$$g\lambda\mu = h_1\gamma \cdot \gamma^{-1}\xi_1 \cdot \xi_1^{-1}h_1^{-1}ch_1\xi_2^{-1} = ch_1\xi_2^{-1}$$

where ξ_2 is 1 of $R_1(\chi) = R_1(h_1^{-1}ch_1) = R_1(ch_1)$. Hence $g\lambda\mu$ is a left factor of ch_1 . Thus, we only need to prove that ch_1 is left compatible with C . If $k = j$, then $R_k(c)h_1 = 1$ and then ch_1 is a left factor of c , so ch_1 is left compatible with C . If $k > j$, then $k = i$ and then $h_1^{-1}L_k(c) = 1$, thus $ch_1 = cL_k(c)$ is a left factor of c^2 , hence again ch_1 is left compatible with C .

Claim (2) We have $R_{l(\mu)-j-1}(\mu)$ (respectively, $L_{l(\mu)-i-1}(\mu)$) is unaltered in $P(g, \lambda\mu)$ (respectively, $P(\mu\rho, g^{-1})$).

Again, we only present the proof of the statement about $P(g, \lambda\mu)$. This is clearly true if $L_1(\mu)$ is not canceled by g in $P(g, \lambda\mu)$. Now we assume that $L_1(\mu)$ is canceled by g . Recalling that $g = h_1\gamma$ and $\lambda\mu$ are reduced products, and $l(\gamma) = l(\lambda)$, we know then $\gamma\lambda = 1$, thus $\xi_1 = 1$ as $\lambda = \gamma^{-1}\xi_1$.

Hence $P(g, \lambda\mu) = P(h_1\gamma, \gamma^{-1}\mu)$ with $\mu = h_1^{-1}ch_1\xi_2^{-1}$ and we only need to prove that $K(h_1, \mu) \leq j$. This is true if $k = j$, as $l(h_1) = k$. If $k > j$, then $k = i$ and c can be written as a reduced product h_1c' , now $\mu = c'h_1\xi_2^{-1}$, and $L_1(\mu) = L_1(c')$ as $l(c) \geq i + j + s$ by Lemma 5.4.3((3)), thus $h_1\mu$ is reduced, implying $K(h_1, \mu) = 0 \leq j$, as desired.

By Lemma 5.2.3, Claim (1) and Claim (2) together imply that if D_t is a component of $R_{l(\mu)-j-1}(\mu)$ (respectively, $L_{l(\mu)-i-1}(\mu)$), we have the first (respectively, second) membership of (5.8). But $i + j \leq l(\mu) - s - 1$ by Lemma 5.4.3((3)), so all components of μ are components of $R_{l(\mu)-j-1}(\mu)$ or components of $L_{l(\mu)-i-1}(\mu)$. So for each component D_t of μ , one of the containments holds, as desired. $\square$

5.4.2 Products of conjugates

We now turn to the study of a product of conjugates. The next result follows from Lemma 5.4.4.

Corollary 5.4.5. *Consider a product $T = C_1 \cdots C_n$, where for each i with $1 \leq i \leq n$, $C_i = c_i^{g_i}$, $c_i \in C \setminus \{1\}$, $g_i \in A \setminus C$ is left C -simplified and $C_i = \lambda_i \mu_i \rho_i$ is the standard form of C_i . Consider the multiplication $C_r C_{r+1} \cdots C_t$ with $1 \leq r < t \leq n$. Let $C_r = B_1 \cdots B_p \cdots B_k$ and $C_t = D_1 \cdots D_q \cdots D_l$ be in reduced form, where B_p is a component of μ_r and D_q is a component of μ_t . Assume that $B_p D_q = 1$ and $B_p \cdots B_k C_{r+1} \cdots C_{t-1} D_1 \cdots D_q = 1$. We have $c_{rt} := g_r C_{r+1} \cdots C_{t-1} g_t^{-1} \in C$.*

Proof. By Lemma 5.4.4, we have

$$\eta := p(B_p)B_p \cdots B_k g_r^{-1} \in C, \quad \eta' := g_t D_1 \cdots D_{q-1} (p(D_q))^{-1} \in C.$$

Since $\delta := p(D_q)D_q(p(D_q^{-1}))^{-1} \in C$ by (5.1) and $B_p D_q = 1$, we find

$$C \ni \theta := \eta' \delta = g_t D_1 \cdots D_q (p(D_q^{-1}))^{-1} = g_t D_1 \cdots D_q (p(B_p))^{-1}.$$

Now we obtain $c_{rt} = g_r C_{r+1} \cdots C_{t-1} g_t^{-1} = \eta^{-1} \theta^{-1} \in C$ by the computation

$$1 = B_p \cdots B_k C_{r+1} \cdots C_{t-1} D_1 \cdots D_q = (p(B_p))^{-1} \eta g_r C_{r+1} \cdots C_{t-1} g_t^{-1} \theta p(B_p).$$

□

So, this result tells us that if a certain type of cancellation happens in the product T , then the partial product c_{rt} is in C . The next result says that if $c_{rt} \in C$, then we can rewrite T as a product of fewer conjugates than the original expression for T .

Lemma 5.4.6. *Let C' be a normal subsemigroup of C , $T = C_1 \cdots C_n$ with $C_i = c_i^{g_i}$, $c_i \in C'$ and $g_i \in A$ for all i . Assume that $c_{rt} := g_r C_{r+1} \cdots C_{t-1} g_t^{-1} \in C$ with $1 \leq r < t \leq n$. We have*

$$C_r C_{r+1} \cdots C_{t-1} C_t = C_r' C_{r+1} \cdots C_{t-1}, \text{ where } C_r' = (c_r')^{g_r} \text{ for some } c_r' \in C'.$$

Proof. We have the computation

$$\begin{aligned} C_r C_{r+1} \cdots C_{t-1} C_t &= g_r^{-1} c_r g_r C_{r+1} \cdots C_{t-1} g_t^{-1} c_t g_t = g_r^{-1} c_r c_{rt} c_t g_t \\ &= g_r^{-1} c_r c_{rt} c_t c_{rt}^{-1} g_r C_{r+1} \cdots C_{t-1} = C_r' C_{r+1} \cdots C_{t-1}, \end{aligned}$$

where $C_r' = c_r'^{g_r}$ with $c_r' = c_r c_t^{-1} \in C'$, as C' is a normal subsemigroup of C . □

5.4.3 Proof of multi-malnormality of C

We now prove the main result of this subsection.

Theorem 5.4.7. *The subgroup C given by Definition 5.1.1 is multi-malnormal in A .*

Proof. Let C' be a normal subsemigroup of C with $1 \notin C'$. Let $\mathcal{S}$ be the set of tuples $((t_1, a_1), \dots, (t_k, a_k))$ such that $t_i \in C'$, $a_i \in A \setminus C$, $k \geq 1$ and $t_1^{a_1} \cdots t_k^{a_k} \in C$. We want to prove that $\mathcal{S}$ is empty. We argue by contradiction, and assume that $\mathcal{S}$ is nonempty.

For a tuple $V := ((t_1, a_1), \dots, (t_k, a_k)) \in \mathcal{S}$, we associate two numbers with V : the number of conjugates $NC(V) = k$ and the sum of conjugate lengths $SLC(V) = \sum_{i=1}^k l(t_i^{a_i})$. These define a map $\phi : \mathcal{S} \rightarrow \mathbb{N}^2$, where $\phi(V) = (NC(V), SLC(V))$. Lexicographically order $\mathbb{N}^2$ according to the rule $(m_1, n_1) < (m_2, n_2)$ if $m_1 < m_2$, or if $m_1 = m_2$ and $n_1 < n_2$. Choose $V_0 \in \mathcal{S}$ satisfying $\phi(V_0) \leq \phi(V)$ for all $V \in \mathcal{S}$.

Suppose $V_0 = ((c_1, g_1), \dots, (c_n, g_n))$, then $n \geq 1$, $c_i \in C'$, $g_i \in A \setminus C$ and $\Pi := c_1^{g_1} \dots c_n^{g_n} \in C$. Moreover, by using Lemma 5.2.6(1) we may assume that these g_i are left C -simplified. For each $t \in \{1, \dots, n\}$, let $C_t = c_t^{g_t} = \chi_t^{\gamma_t} = \lambda_t \mu_t \rho_t$, where the last two expressions are the almost reduced form and the standard form of C_t . To simplify notation, for $1 \leq r \leq t \leq n$, we let $\Pi_{r,t} = C_r C_{r+1} \dots C_t$ (and for convenience, define $\Pi_{r,t} = 1$ when $r > t$). We also write $L_{i,j}$ for the left-first product $LFP(C_i, \dots, C_j)$, and $R_{i,j}$ for the right-first product $RFP(C_i, \dots, C_j)$. Let x be a fixed integer satisfying $3 \leq x \leq s - 6$. We now proceed by two steps to reach a contradiction.

Our first step is to establish three preparatory claims using the minimality of V_0 .

Claim A : Let $1 \leq r < t \leq n$. We have $l(\gamma_r \Pi_{r+1,t}) \geq l(\gamma_r) - 1$ and $l(\Pi_{r,t-1} \gamma_t^{-1}) \geq l(\gamma_t^{-1}) - 1$.

Proof of **Claim A**: We only prove the first inequality as the second one can be done similarly. For contradiction, assume $l(\gamma_r \Pi_{r+1,t}) < l(\gamma_r) - 1$. Letting $C'_r = C_r^{\Pi_{r+1,t}}$, we have

$$\Pi_{r,t} = C_r \Pi_{r+1,t} = \Pi_{r+1,t} \Pi_{r+1,t}^{-1} C_r \Pi_{r+1,t} = \Pi_{r+1,t} C'_r,$$

and hence we find

$$\begin{aligned} \Pi &= \Pi_{1,n} = \Pi_{1,r-1} \Pi_{r,t} \Pi_{t+1,n} = \Pi_{1,r-1} \Pi_{r+1,t} C'_r \Pi_{t+1,n} \\ &= \Pi_{1,r-1} \Pi_{r+1,t} C'_r \Pi_{t+1,n} C_r'^{-1} C'_r \\ &= \Pi_{1,r-1} \Pi_{r+1,t} \Pi_{t+1,n}^{C_r'^{-1}} C'_r. \end{aligned}$$

If $C'_r \in C$, $\Pi' := \Pi_{1,r-1} \Pi_{r+1,t} \Pi_{t+1,n}^{C_r'^{-1}}$ is in C as $\Pi \in C$, and Π' is a product of $n - 1$ conjugates of the form c^g with $c \in C'$ and $g \in A \setminus C$, contradicting the minimality of V_0 . Therefore, we have

$C'_r \in A \setminus C$. Since $C'_r = C_r^{\Pi_{r+1,t}} = c_r^{g_r \Pi_{r+1,t}} = c_r^{g'_r}$, where $g'_r := g_r \Pi_{r+1,t}$, we have $g'_r \in A \setminus C$ as $c_r \in C$. Also, $C'_r = (\chi_r^{\gamma_r})^{\Pi_{r+1,t}} = \chi_r^{\gamma_r \Pi_{r+1,t}}$, and thus

$$l(C'_r) \leq l(\chi_r) + 2l(\gamma_r \Pi_{r+1,t}) < l(\chi_r) + 2(l(\gamma_r) - 1) = l(\chi_r) + 2l(\gamma_r) - 2 \leq l(C_r)$$

where the last inequality is because the product $C_r = \gamma_r^{-1} \chi_r \gamma_r$ is almost reduced. Considering the product $\Pi = \Pi_{1,r-1} \Pi_{r+1,t} C'_r \Pi_{t+1,n}$, we obtain a tuple $V_1 \in \mathcal{S}$, such that $NC(V_1) = NC(V_0)$ but $SLC(V_1) < SLC(V_0)$, contradicting the minimality of V_0 . This completes the proof of **Claim A**. $\square$

Claim B : Let $1 \leq i < j \leq n$ and assume that $l(\Pi_{i,j-1}) \geq l(C_i) - 6$. Then $l(\Pi_{i,j-1}) \geq l(\gamma_i) + x + 1$ and $B_{l(\gamma_i)+x+1}(\Pi_{i,j-1})$ is not canceled by the factor λ_j of C_j in $P(\Pi_{i,j-1}, C_j)$.

Proof of **Claim B**: To prove the inequality, we only need $l(C_i) - 6 \geq l(\gamma_i) + x + 1$, thus by the last equality of Lemma 5.4.3((3)), we only need $2s - 1 - 6 \geq x + 1$, which follows from $3 \leq x \leq s - 6$.

Now we prove the second statement by contradiction, assuming it is canceled by λ_j . Then the number of components of $\Pi_{i,j-1}$ canceled by λ_j in $P(\Pi_{i,j-1}, \lambda_j)$ is at least $\kappa := l(\Pi_{i,j-1}) - l(\gamma_i) - x$, so $K(\Pi_{i,j-1}, \lambda_j) \geq \kappa$. Since γ_j^{-1} and λ_j are of the same length, and at most differ in their right-most components (by Lemma 5.4.3)((2)), thus $K(\Pi_{i,j-1}, \gamma_j^{-1}) \geq \kappa - 1$. It now follows that

$$\begin{aligned} l(\Pi_{i,j-1} \gamma_j^{-1}) &\leq l(\Pi_{i,j-1}) + l(\gamma_j^{-1}) - 2K(\Pi_{i,j-1}, \gamma_j^{-1}) \\ &\leq l(\Pi_{i,j-1}) + l(\gamma_j^{-1}) - 2(\kappa - 1) \\ &= l(\gamma_j^{-1}) - l(\Pi_{i,j-1}) + 2l(\gamma_i) + 2x + 2 \\ &\leq l(\gamma_j^{-1}) - l(C_i) + 6 + 2l(\gamma_i) + 2x + 2 \\ &\leq l(\gamma_j^{-1}) - 2s + 2x + 9 \leq l(\gamma_j^{-1}) - 3, \end{aligned}$$

where the penultimate inequality is obtained using the last inequality of Lemma 5.4.3((3)) and the last inequality is because $x \leq s - 6$. This contradicts **Claim A** and thus finishes proving **Claim B**. $\square$

Claim C : Let $1 \leq i \leq r < t \leq j \leq n$. Let B and D be components of μ_r and μ_t , respectively. Then B and D do not cancel in $L_{i,j}$. (Recall that we write $L_{i,j}$ for $LFP(C_i, \dots, C_j)$.)

Proof of **Claim C**: We prove by contradiction, supposing that the components B and D cancel in $L_{i,j}$. By Lemma 5.3.8, B and D cancel in $L_{r,t}$ too. Letting $B = RB_p(C_r)$, $D = B_q(C_t)$, we have $BD = 1$ and $R_p(C_r)C_{r+1} \cdots C_{t-1}L_q(C_t) = 1$ by Lemma 5.3.3. By Corollary 5.4.5, $g_r C_{r+1} \cdots C_{t-1} g_t^{-1} \in C$. Applying Lemma 5.4.6, we have $\Pi_{r,t} = C'_r C_{r+1} \cdots C_{t-1}$, where $C'_r = c'^{g_r}_r$ with $c'_r \in C'$. Now $\Pi = \Pi_{1,r-1} C'_r C_{r+1} \cdots C_{t-1} \Pi_{t+1,n} \in C$ is a product of $n-1$ terms of the form c^g with $c \in C'$ and $g \in A \setminus C$. This contradicts the minimality of V_0 , and thus completes the proof of **Claim C**. $\square$

Our second step is to prove the following claim to finish the proof of the theorem. To ease notation, we write $\mathcal{L}_t$ in place of $L_{l(\gamma_t)+x}(C_t)$ and $\mathcal{R}_t$ in place of $R_{l(\gamma_t)+x}(C_t)$.

Claim 5.4.8. If i, j are integers satisfying $1 \leq i \leq j \leq n$, then $\mathcal{L}_i$ is unaltered in $L_{i,j}$ and $\mathcal{R}_j$ is unaltered in $R_{i,j}$.

Let us see how the claim contradicts the nonemptiness of $\mathcal{S}$. Assuming Claim 5.4.8, we know $\mathcal{L}_1$ is unaltered in $L_{1,n}$ and so $L_{l(\gamma_1)+x}(C_1) = L_{l(\gamma_1)+x}(\Pi)$. But C_1 is $(l(\gamma_1) + 2)$ -incompatible with C by Lemma 5.4.3((4)), so that Π is also $(l(\gamma_1) + 2)$ -incompatible with C , in particular, $\Pi \notin C$. Recalling that the assumption that $\mathcal{S}$ is nonempty leads to the existence of V_0 and that $\Pi \in C$, this gives a desired contradiction.

The remainder of this chapter is therefore devoted to proving Claim 5.4.8. We induct on the number $j - i + 1$ of conjugates in the product $\Pi_{i,j}$. Note that if $j - i + 1 = 1$, then the conclusion of Claim 5.4.8 is trivially true. So, let $K \geq 2$ and assume that the conclusion of Claim 5.4.8 holds for all i, j with $j - i + 1 < K$. Assuming that $j - i + 1 = K$, we want to prove that the conclusion of Claim 5.4.8 is true in this case. We proceed with two substeps to achieve this goal. Our first substep is to prove the following **Claim D** and **Claim E** based on the induction assumption.

Claim D : Let i' be an integer satisfying $i \leq i' < j$. Assume that a component $B_{i'}$ of $C_{i'}$ is unaltered in $L_{i',j-1}$ but altered in $L_{i',j}$. Then the factor $RB_{l(\gamma_{i'})+2}(C_{i'})$ of $C_{i'}$ is unaltered in

$L_{i',j-1}$.

Proof of **Claim D**: We prove this with two sub-substeps. Note that the claim is trivially true if $i' = j - 1$, and so we focus on the case that $i' \leq j - 2$.

- (a) Let k be a fixed integer satisfying $i' + 1 \leq k \leq j - 1$. We prove that at least one component of C_k is altered in $P(\Pi_{i',k-1}, C_k)$, and if t is maximal such that $B_t(C_k)$ is altered, then $l(\gamma_k) + x \leq t \leq l(C_k) - (l(\gamma_k) + x)$.

To prove this, we rule out two other possible cases for the product $P(\Pi_{i',k-1}, C_k)$.

- i. No component of C_k is altered or the rightmost component of C_k altered is $B_t(C_k)$ with $t < l(\gamma_k) + x$. In this case, $B_{l(\gamma_k)+x}(C_k)$ is unaltered in $L_{i',k}$, and by assumption, the component $B_{i'}$ of $C_{i'}$ is unaltered in $L_{i',j-1}$ (thus also unaltered in $L_{i',k}$) but altered in $L_{i',j}$. Therefore by Corollary 5.3.7, $B_{l(\gamma_k)+x}(C_k)$ is altered in $L_{k,j}$. But this means $\mathcal{L}_k$ is altered in $L_{k,j}$, contradicting the induction assumption as $j - k + 1 < K$.
 - ii. The rightmost component of C_k altered is $B_t(C_k)$ with $t > l(C_k) - (l(\gamma_k) + x)$. Then $B_t(C_k)$ is a component of $\mathcal{R}_k$, thus $\mathcal{R}_k$ is altered in $P(\Pi_{i',k-1}, C_k)$. But $\mathcal{R}_k$ is unaltered in $R_{i',k}$ by the induction assumption (as $k - i' + 1 < K$), thus $\mathcal{R}_k$ is unaltered in $P(\Pi_{i',k-1}, C_k)$ by (the right version of) Corollary 5.3.5, a contradiction.
- (b) Second, we prove the following Subclaim. Let's first see how this Subclaim finishes the proof of **Claim D**. By this Subclaim, for each k with $i' + 1 \leq k \leq j - 1$, $\eta_{i'} := \lambda_{i'} \mu_{i'}(R_1(\mu_{i'}))^{-1}$ is a left factor of $\Pi_{i',k}$, and thus $\eta_{i'}$ is unaltered in $P(\Pi_{i',k-1}, C_k)$ by (the left version of) Lemma 5.3.4. Therefore $\eta_{i'}$ is unaltered in $L_{i',j-1}$ and thus so is its right-most component $RB_{l(\gamma_{i'})+2}(C_{i'})$, as desired by **Claim D**.

Subclaim: For every k with $i' \leq k \leq j - 1$, $\Pi_{i',k}$ can be written as an almost reduced product

$$\Pi_{i',k} = \lambda_{i'} \zeta_{j_0} \zeta_{j_1} \cdots \zeta_{j_{r_k}} h_k, \quad (5.9)$$

where $\zeta_{j_0} = \mu_{i'}$, $r_k \geq 0$, $h_k \in A$, $i' = j_0 < j_1 < \cdots < j_{r_k} \leq k$, the products

$\lambda_{i'}\zeta_{j_0}$ and $\zeta_{j_{r_k}}h_k$ are reduced, each ζ_t is a right factor of μ_t with $l(\zeta_t) \geq x+1$, each $\zeta'_t := (L_1(\zeta_t))^{-1}\zeta_t(R_1(\zeta_t))^{-1}$ is unaltered in $L_{i',k}$, and $\zeta''_{j_0} := \zeta_{j_0}(R_1(\zeta_{j_0}))^{-1}$ and $\zeta''_{j_{r_k}} := (L_1(\zeta_{j_{r_k}}))^{-1}\zeta_{j_{r_k}}$ are unaltered in $L_{i',k}$.

To prove this subclaim, we induct on k . It is clearly true when $k = i'$, as we can take $r_k = 0$ and $h_{i'} = \rho_{i'}$. Now assume that $\Pi_{i',k}$ with $i' \leq k \leq j-2$ can be written as the desired almost reduced product (5.9). We want to prove that $\Pi_{i',k+1}$ can too. We have

$$\Pi_{i',k+1} = \Pi_{i',k}C_{k+1} = \lambda_{i'}\zeta_{j_0}\zeta_{j_1}\cdots\zeta_{j_{r_k}}h_k\lambda_{k+1}A_1\cdots A_{m_{k+1}}\rho_{k+1},$$

where we write μ_{k+1} in reduced form $A_1\cdots A_{m_{k+1}}$. By **Claim D(a)**, the rightmost component of C_{k+1} being altered in $P(\Pi_{i',k}, C_{k+1})$ is A_r with $x \leq r \leq m_{k+1} - x$. Thus A_{r-1} is canceled in $P(\Pi_{i',k}, C_{k+1})$. Let $\delta_k = \zeta_{j_0}\zeta_{j_1}\cdots\zeta_{j_{r_k}}$. Then $\Pi_{i',k} = \lambda_{i'}\delta_k h_k$ is a reduced product. We want to show that A_{r-1} is canceled by h_k in $P(\Pi_{i',k}, C_{k+1})$. First, it cannot be canceled by $\lambda_{i'}$, as that will lead to $\mathcal{L}_{i'}$ being altered in $L_{i',k+1}$, contradicting the induction assumption (as $k+1-i'+1 < K$). Second, note that each ζ'_t for $t = j_0, j_1, \dots, j_{r_k}$ is a factor of μ_t unaltered in $L_{i',k}$, so ζ'_t cannot cancel A_{r-1} by **Claim C** as A_{r-1} is a component μ_{k+1} . Similarly, ζ''_{j_0} or $\zeta''_{j_{r_k}}$ cannot cancel A_{r-1} either. Therefore, if A_{r-1} is canceled by δ_k , then it can only be canceled by $R_1(\zeta_{j_{t-1}})L_1(\zeta_{j_t})$ for some $t \in \{1, 2, \dots, r_k\}$, but then A_{r-2} (note $r \geq x \geq 3$) is canceled by a component of ζ'_{j_t} , contradicting **Claim C**. Therefore A_{r-1} can only be canceled by a component of h_k , call this component D . We consider the following two possibilities.

- i. Suppose D is not the left-most component of h_k . In this case, letting $h_{k+1} = h_k C_{k+1}$, we have

$$\Pi_{i',k+1} = \lambda_{i'}\zeta_{j_0}\zeta_{j_1}\cdots\zeta_{j_{r_k}}h_{k+1},$$

which is the desired almost reduced product.

- ii. Suppose D is the left-most component of h_k . In this case, we have $h_k\lambda_{k+1}A_1\cdots A_{r-1} = 1$

and thus

$$\Pi_{i',k+1} = \lambda_{i'} \zeta_{j_0} \zeta_{j_1} \cdots \zeta_{j_{r_k}} A_r \cdots A_{m_{k+1}} \rho_{k+1} = \lambda_{i'} \zeta_{j_0} \zeta_{j_1} \cdots \zeta_{j_{r_k}} \zeta_{j_{r_{k+1}}} \rho_{k+1},$$

where $\zeta_{j_{r_{k+1}}} := A_r \cdots A_{m_{k+1}}$ is a right factor of μ_{k+1} with $l(\zeta_{j_{r_{k+1}}}) = m_{k+1} - r + 1 \geq x + 1$, $K(\zeta_{j_{r_k}}, \zeta_{j_{r_{k+1}}}) = 0$ and the product $\zeta_{j_{r_{k+1}}} \rho_{k+1}$ is reduced. This gives the desired almost reduced product for $\Pi_{i',k+1}$, proving the Subclaim and thus **Claim D**. $\square$

Claim E : Assume that $\mathcal{L}_i$ is altered in $L_{i,j}$. Then $l(\Pi_{i,j-1}) \geq l(C_i) - 6$.

Proof of **Claim E**: We prove this by contradiction, supposing that $l(\Pi_{i,j-1}) < l(C_i) - 6$. First we have $i \neq j - 1$ and thus $i < j - 1$ as $i < j$. Letting $r = K(C_i, \Pi_{i+1,j-1})$, we have

$$l(C_i) - 6 > l(\Pi_{i,j-1}) \geq l(C_i) + l(\Pi_{i+1,j-1}) - 2r - 1,$$

from which we find $l(\Pi_{i+1,j-1}) < 2r - 5$. Since $\mathcal{L}_i$ is unaltered in $L_{i,j-1}$ by the induction assumption, and it is altered in $L_{i,j}$, we know that by **Claim D**, $RB_2(\mu_i) = RB_{l(\gamma_i)+2}(C_i)$ is unaltered in $L_{i,j-1}$, hence unaltered in $P(C_i, \Pi_{i+1,j-1})$ by Corollary 5.3.5. Now as exactly r components of C_i are canceled in $P(C_i, \Pi_{i+1,j-1}) = P(\lambda_i \mu_i \rho_i, \Pi_{i+1,j-1})$, we have $K(\rho_i, \Pi_{i+1,j-1}) \geq r - 1$. It then follows that $K(\gamma_i, \Pi_{i+1,j-1}) \geq r - 2$ by Lemma 5.4.3((2)). Using this, the inequality $l(\Pi_{i+1,j-1}) < 2r - 5$ and **Claim A**, we reach a contradiction:

$$l(\gamma_i) - 1 \leq l(\gamma_i \Pi_{i+1,j-1}) \leq l(\gamma_i) + l(\Pi_{i+1,j-1}) - 2(r - 2) < l(\gamma_i) - 1.$$

This completes the proof of **Claim E**. $\square$

Our second substep is to complete the proof of Claim 5.4.8, using **Claim B**, **Claim C**, **Claim D** and **Claim E**. We only prove $\mathcal{L}_i$ is unaltered in $L_{i,j}$, as the “right version” is proved similarly.

Suppose that $\mathcal{L}_i$ is altered in $L_{i,j}$. By the induction assumption $\mathcal{L}_i$ is unaltered in $L_{i,j-1}$, so the component $RB_{l(\gamma_i)+2}(C_i)$ is unaltered in $L_{i,j-1}$ by **Claim D**. It follows that $B_i := B_{l(\gamma_i)+x+1}(C_i)$ is unaltered in $L_{i,j-1}$ as well, because $l(C_i) \geq 2l(\gamma_i) + 2s - 1$ by Lemma 5.4.3(3) and $2s - 1 \geq x + 3$.

Now, since $\mathcal{L}_i$ is unaltered in $L_{i,j-1}$ but altered in $L_{i,j}$, it is therefore altered in $P(\Pi_{i,j-1}, C_j)$, implying that B_i is canceled in $P(\Pi_{i,j-1}, C_j)$. But since B_i is a component from μ_i , it cannot be canceled by μ_j in $P(\Pi_{i,j-1}, C_j)$ by **Claim C**. Combining **Claim E** and **Claim B**, we know that B_i is not canceled by λ_j either. Therefore B_i is canceled by ρ_j in $P(\Pi_{i,j-1}, C_j)$.

Because B_i is canceled by ρ_j in $P(\Pi_{i,j-1}, C_j)$, we know $\mathcal{R}_j$ is altered in $P(\Pi_{i,j-1}, C_j)$. So $\mathcal{R}_j$ is altered in $R_{i,j}$ by (the right version of) Corollary 5.3.5. Starting from that $\mathcal{R}_j$ is altered in $R_{i,j}$, we make an argument similar to the previous two paragraphs but using the right version of the claims whenever necessary, to conclude that $D_j := RB_{l(\gamma_j)+x+1}(C_j)$ is unaltered in $R_{i+1,j}$ and D_j is canceled by λ_i in the product $P(C_i, \Pi_{i+1,j})$.

Now, as D_j is unaltered in $R_{i+1,j}$, it is unaltered in $P(\Pi_{i+1,j-1}, C_j)$ by (the right version of) Corollary 5.3.5. Moreover, the factor λ_i of C_i is unaltered in $P(C_i, \Pi_{i+1,j-1})$ by the induction assumption and Corollary 5.3.5. We can therefore apply Lemma 5.3.9, concluding that since D_j is canceled by λ_i in the product $P(C_i, \Pi_{i+1,j}) = P(C_i, \Pi_{i+1,j-1}C_j)$, D_j is canceled by λ_i in $P(C_i\Pi_{i+1,j-1}, C_j) = P(\Pi_{i,j-1}, C_j)$.

Hence in the product $P(\Pi_{i,j-1}, C_j)$, B_i (respectively, D_j) is canceled by ρ_j (respectively, λ_i). We now show this leads to a contradiction. Since $B_i = B_{l(\gamma_i)+x+1}(C_i)$ is unaltered in $L_{i,j-1}$, λ_i is a left factor of C_i and $l(\lambda_i) = l(\gamma_i)$, we can write $\Pi_{i,j-1} = \lambda_i\sigma_1 B_i\tau_1$ as a reduced product, where $\sigma_1, \tau_1 \in A$. Similarly, as $D_j = RB_{l(\gamma_j)+x+1}(C_j)$ and ρ_j is a right factor of C_j with $l(\rho_j) = l(\gamma_j)$, we can write C_j as a reduced product $\sigma_2 D_j\tau_2\rho_j$, where $\sigma_2, \tau_2 \in A$. Then the cancellation of B_i by ρ_j in the product $\Pi_{i,j-1}C_j = (\lambda_i\sigma_1 B_i\tau_1)(\sigma_2 D_j\tau_2\rho_j)$ implies that D_j is canceled by τ_1 , contradicting that D_j is canceled by λ_i . $\square$

Chapter 6

Conclusion

Theorem 3.3.7 ensures that a type of automorphism of free groups is order-preserving. As an application, we proved that the answer to a question of Kin and Rolfsen in [18] is positive.

We presented Theorem 4.4.6, which gives a sufficient condition ensuring an amalgam of two groups is GTF. As an application, we prove an analog—Corollary 4.6.3—of a result of George Bergman [3]. As another application, we prove that there exists a group which is GTF and non-LO, solving problem 16.48 (on Page 98) of The Kourovka Notebook.

To prove that the group is GTF and non-LO, we introduce left-first product and use it to prove that certain subgroups of a free group of rank two are multi-malnormal. This tool—left-first product—may play a role in the study of other related problems.

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