REMARKS ON THE PARK TRANSFORMATION

A THESIS PRESENTED TO

THE FACULTY OF GRADUATE STUDIES

AND RESEARCH

THE UNIVERSITY OF MANITOBA

In partial fulfillment

of the requirements for the

degree Master of Science

in

Electrical Engineering

by

Albert Simeon

April 1963

ACKNOWLEDGMENTS

The author is grateful to W. Shepherd (Royal Military College, Shrivenham, England) for his suggestions on machine theory and to R. A. Johnson, (Associate Professor of Electrical Engineering, University of Manitoba) for his help in the mathematical formulation.

Table of Contents

ABSTRACT

Two new methods, based on the Park Transformation, are suggested for the derivation of the equations of equilibrium of three phase machines. The methods are demonstrated here by application to the induction motor.

The author considers the following points advantageous:

1. The mathematics is presented systematically by means of matrices and vectors, so that the complete derivation becomes reprintable.

2. Although the theory is based on idealization, all physical quantities can be measured in a simple way.

3. Machine theory is tied in closely in a rigorous manner with circuit theory and generalized mechanics.

INTRODUCTION

With the aid of the Park Transformation, a set of differential equations can be obtained for the induction motor in terms of the well known direct and quadrature-axis components[1,2,3]*: v_D, i_D and v_Q, i_Q. These equations are as compact as Park's equation for the synchronous machine.[4]

In reference (1) Brereton, Lewis and Young chose a velocity for the direct- and quadrature-axes in such a manner that the electrical quantities associated with these axes are related to the components in the standard phasor diagram of the induction motor. In particular, if, during a transient process, all quantities remain balanced, even if they do change slowly in amplitude, $\sqrt{i_D^2(t) + i_Q^2(t)}$ will describe the envelope of the individual phase currents. Such equations, then, have direct meaning to the electrical engineer.

This restriction of balance if applied only to the three input voltages is not unreasonable. Whenever the motor is connected to a standard three-phase supply the input voltages are balanced for all practical purposes. To arrive at Young's equations for the case of a balanced input is so much simpler than for the general case, (by means of the Park Transformation), that it is considered worth while to outline this method in the present thesis. It will be seen that due to the construction of the motor no further restriction has to be imposed for this simplified mathematical method.

* Superscripts indicate reference numbers in the Bibliography.

The resulting equations contain "speed-terms", and Kron[5] showed that such terms come about if the non-Riemannian form of the dynamical equations of Lagrange is employed. Kron discards the standard form to begin with because one is confronted by rotating reference frames. It will be shown that it is possible to relate the better known holonomic form for stationary reference frames to the result as well. This shows no advantage over Kron's rigorously established method, however, those not familiar with the finer logic of advanced dynamics may find it easier to use.

Chapter I

The Park Transformation for the Case of Balanced Quantities

Abstract:

It is shown in this section that balanced sinusoidal phase quantities, (such as currents, voltages, fluxes, etc.) if represented vectorially, add up to form a rotating vector. A projection of this vector onto a rotating plane is the basis of the Park-transformation. Balance implies that the three-phase quantities add to zero; hence an explicit statement of all three is redundant. This leads to a reduction of the transformation matrix.

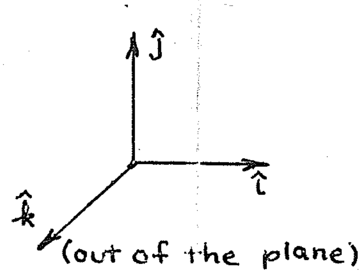
The three scalar quantities

$$p_a(t) = P_o(t) \cos \Omega t \qquad 1$$

$$p_b(t) = P_o(t) \cos(\Omega t - 2\pi/3) \qquad 2$$

and $$p_c(t) = P_o(t) \cos(\Omega t + 2\pi/3) \qquad 3$$

which could stand for currents in the stator of a synchronous machine for instance, may be represented along the axes of a three-sided symmetrical star (as shown in Fig. 1) and added vectorially, so that



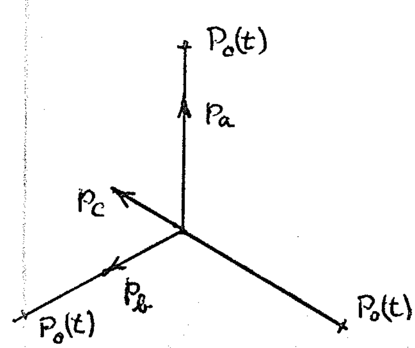


Fig. 1

the horizontal component becomes:

$$P_1(t) = P_o(t)\left[\cos\left(\Omega t + \tfrac{2\pi}{3}\right)\cos\left(-\tfrac{\pi}{6}\right) + \cos\left(\Omega t - \tfrac{2\pi}{3}\right)\cos\left(-\tfrac{5\pi}{6}\right)\right] \qquad 4$$

or $$P_1(t) = P_o(t)\left[\cos\left(\Omega t + \tfrac{2\pi}{3}\right)\tfrac{\sqrt{3}}{2} - \cos\left(\Omega t - \tfrac{2\pi}{3}\right)\tfrac{\sqrt{3}}{2}\right] \qquad 5$$

or $$P_1(t) = -\tfrac{3}{2} P_o(t) \sin \Omega t \qquad 6$$

and the vertical component:

$$P_2(t) = P_o(t)\left[\cos \Omega t + \cos\left(\Omega t - \tfrac{2\pi}{3}\right)\cos^2\tfrac{\pi}{3} + \cos\left(\Omega t + \tfrac{2\pi}{3}\right)\cos\left(-\tfrac{2\pi}{3}\right)\right] \qquad 7$$

or $$P_2(t) = \tfrac{3}{2} P_o(t) \cos \Omega t \qquad 8$$

Defining $\underline{P}(t)$ as $\hat{i}\,P_1(t) + \hat{j}\,P_2(t)$ in which $\hat{i}$ and $\hat{j}$ are horizontal and vertical unit vectors yields: 9

$$\underline{P}(t) = \tfrac{3}{2} P_o(t)\left[\hat{i}\,(-\sin \Omega t) + \hat{j}\,(\cos \Omega t)\right] \qquad 10$$

This vector can be represented in polar coordinates with the aid of the unit vectors $\hat{r}$ and $\hat{\delta}$. The magnitude of this vector is then

$$|\underline{P}| = \tfrac{3}{2} P_o(t) \qquad 11$$

and the associated angle δ becomes.

$$\delta = \tan^{-1}\left(\frac{\cos \Omega t}{-\sin \Omega t}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{-\sin \Omega t}{\cos \Omega t}\right) \qquad 12$$

or

$$\delta = \frac{\pi}{2} + \Omega(t) \qquad 13$$

The rate of change of which is $\dot{\delta} = \Omega$* 14

This implies that the magnitude of the vector $\underline{P}(t)$ is independent of rotation and that it turns counterclockwise at a constant angular velocity Ω.

This is the basis for the Park Transformation. The phase quantities are projected on a reference frame turning with a speed

* $(\dot{\ })$ means derivative re. t

equal to the angular frequency of the phase quantities when they are balanced and sinusoidal. The components along the rotating triad $\hat{i}'-\hat{j}'-\hat{k}$ are then called P_D along $\hat{i}'$, defining the "direct-axis", and P_Q along $\hat{j}'$ or the "quadrature-axis".

To make it easier for an inverse of the transformation to be taken, a third quantity P_3 is introduced which is proportional to the algebraic sum of p_a, p_b and p_c. The Park Transformation would then be of the form:

$$\begin{bmatrix} P_D \\ P_Q \\ P_3 \end{bmatrix} = \begin{bmatrix} \cos\delta & \cos\left(\delta - \frac{2\pi}{3}\right) & \cos\left(\delta + \frac{2\pi}{3}\right) \\ -\sin\delta & -\sin\left(\delta - \frac{2\pi}{3}\right) & -\sin\left(\delta + \frac{2\pi}{3}\right) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_a \\ p_b \\ p_c \end{bmatrix} \qquad 15$$

The rate of change δ is equal to Ω. Park actually chose different constants of proportionality (his D and Q quantities are 2/3 of the ones defined in eq. 15); furthermore, one could add any constant angle to the argument of the sinusoids, without affecting the result.

Now, under the assumption that p_a, p_b and p_c are balanced, by which is meant:

$$p_a + p_b + p_c = 0 \qquad 16$$

certainly, the differential equations of the particular device would be dependent. The quantities, p_a and p_b will be sufficient to describe the behaviour of the machine. P_3 will then always be zero and only a second order matrix is needed as the transformation. Substituting for p_c in (15), one obtains:

$$\begin{bmatrix} P_D \\ P_Q \end{bmatrix} = \begin{bmatrix} \cos\delta - \cos(\delta + \frac{2\pi}{3}) & \cos(\delta - \frac{2\pi}{3}) - \cos(\delta + \frac{2\pi}{3}) \\ -\sin\delta + \sin(\delta + \frac{2\pi}{3}) & -\sin(\delta - \frac{2\pi}{3}) + \sin(\delta + \frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} p_a \\ p_b \end{bmatrix} \qquad 17$$

$$\text{or} \quad \begin{bmatrix} P_D \\ P_Q \end{bmatrix} = \sqrt{3} \begin{bmatrix} -\sin(\delta - \frac{2\pi}{3}) & \sin\delta \\ -\cos(\delta - \frac{2\pi}{3}) & \cos\delta \end{bmatrix} \begin{bmatrix} p_a \\ p_b \end{bmatrix} \qquad 18$$

In shorter notation, this becomes:

$$\left[P(t)\right] = \left[A(\delta)\right]\left[p(t)\right] \qquad 19$$

Important in the subsequent development is the inverse of the transformation matrix $\left[A(\delta)\right]^{-1}$, for which Δ_A, the determinant of $\left[A(\delta)\right]$, is given by:

$$\Delta_A = -\sqrt{3}\sin(\delta - \tfrac{2\pi}{3})\sqrt{3}\cos\delta + \sqrt{3}\sin(\delta)\sqrt{3}\cos(\delta - \tfrac{2\pi}{3}) \qquad 20$$

or

$$\Delta_A = 3\left[\sin^2\delta \sin\tfrac{2\pi}{3} + \cos^2\delta \sin\tfrac{2\pi}{3}\right] \qquad 21$$

or finally,

$$\Delta_A = \frac{3\sqrt{3}}{2} \qquad 22$$

This results in the inverse:

$$\left[A(\delta)\right]^{-1} = \frac{2}{3}\begin{bmatrix} \cos\delta & -\sin\delta \\ \cos(\delta - \frac{2\pi}{3}) & -\sin(\delta - \frac{2\pi}{3}) \end{bmatrix} \qquad 23$$

A further important result, which will be utilized later, is that

$$\left\{\frac{d}{dt}\left[A(\delta)\right]^{-1}\right\}\left[P(t)\right] = \Omega\left[A(\delta)\right]^{-1}\begin{bmatrix} -P_Q \\ P_D \end{bmatrix} \qquad 24$$

which leads to the symbolism:

$$\left\{\frac{d}{dt}\left[A(\delta)\right]^{-1}\right\}\left[P(t)\right] = \Omega\left[A(\delta)\right]^{-1} \times \left[P(t)\right] \qquad 25$$

The proof of this will be found in Appendix A.

The following matrix definitions will be needed:

$$\sqrt{3}\begin{bmatrix} \sin(\theta + \frac{2\pi}{3}) & -\sin\theta \\ \sin\theta & -\sin(\theta - \frac{2\pi}{3}) \end{bmatrix}\Big[A(\alpha)\Big]^{-1} \equiv \Big[B(\theta)\Big]\Big[A(\alpha)\Big]^{-1} \qquad 26$$

and
$$\sqrt{3}\begin{bmatrix} -\sin(\theta - \frac{2\pi}{3}) & \sin\theta \\ -\sin\theta & \sin(\theta + \frac{2\pi}{3}) \end{bmatrix}\Big[A(\alpha)\Big]^{-1} \equiv \Big[B(-\theta)\Big]\Big[A(\alpha)\Big]^{-1} \qquad 27$$

It will be shown in Appendix A that the products (26) and (27) simplify to the following useful results:

$$\Big[B(\theta)\Big]\Big[A(\alpha)\Big]^{-1} = \frac{3}{2}\Big[A(\alpha+\theta)\Big]^{-1} \qquad 28$$

and

$$\Big[B(-\theta)\Big]\Big[A(\alpha)\Big]^{-1} = \frac{3}{2}\Big[A(\alpha-\theta)\Big]^{-1} \qquad 29$$

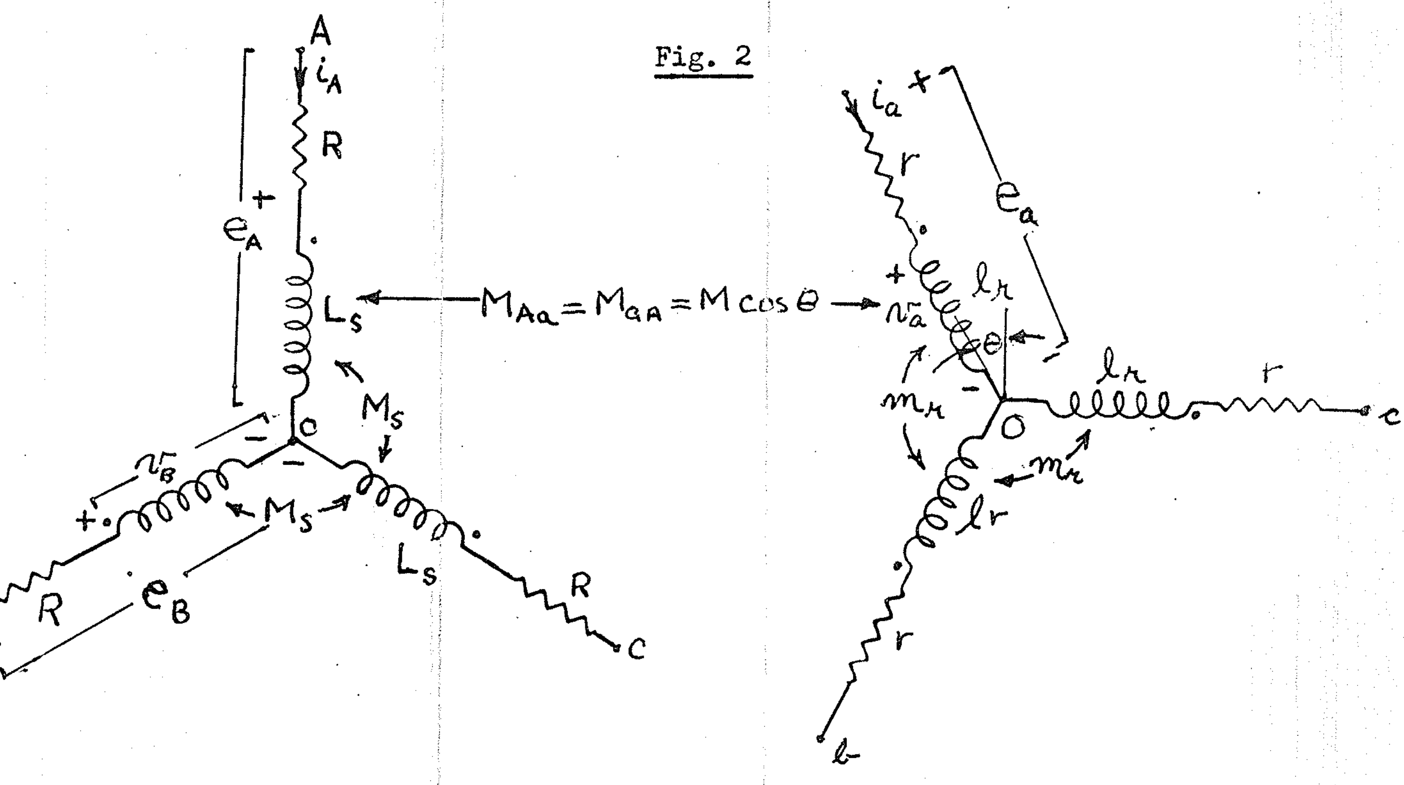
Chapter II

The Induction Motor

Abstract: Stanley's equations[3] of a linearized and simplified induction motor are put into a simpler form.

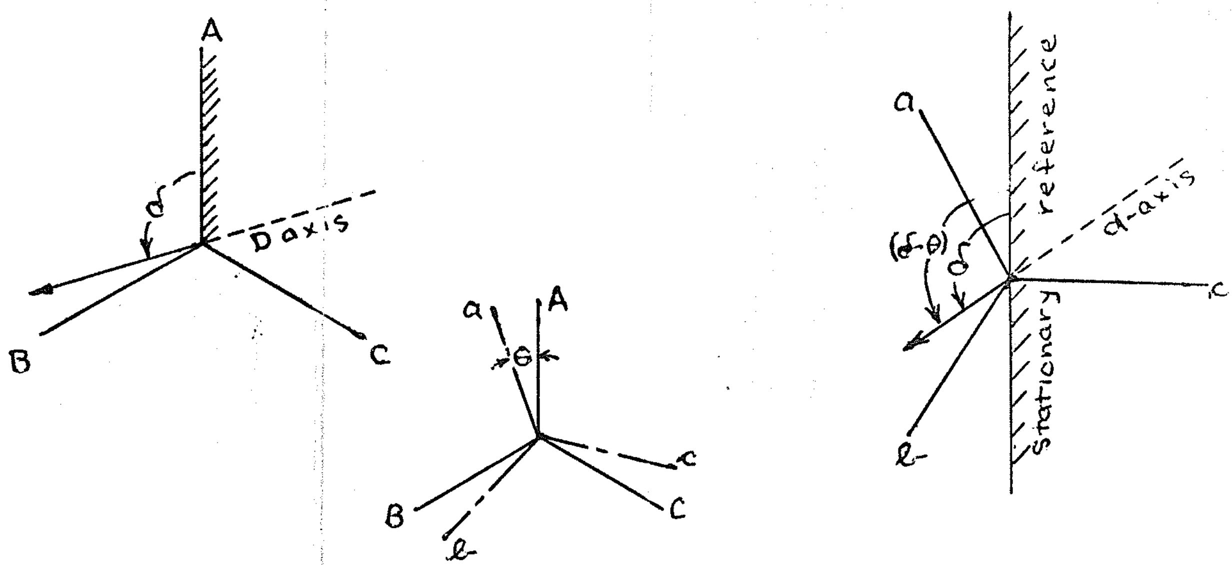
In order to keep the physical principles as clear as possible, consider an idealized machine with perfectly linear behaviour and only one coil per phase. In both stator and rotor the phases are Y-connected. This guarantees that all currents add to zero and if, furthermore, no external voltages are applied to the rotor, only the applied stator voltages need obey the restriction mentioned in the introduction. Furthermore, let the stator and rotor have perfect radial symmetry. It is known that, under these conditions, sinusoidal currents are induced in the rotor with angular frequency equal to the slip frequency $\Omega-\omega$, where Ω is the angular frequency imposed on the stator and ω is the angular velocity of the rotor. This implies that a projection is required of the rotor phase-quantities on axes d and q that rotate with speed $\Omega-\omega$ with respect to the rotor. Stanley[3] made his projections on axes fixed in the stator. If this is done, however, the direct- and quadrature-axis quantities are no longer stationary in the steady-state, this being the case in Park's development for the synchronous machine and in Young's equations for the induction motor. Nevertheless, since Young does not develop his result, but merely states what is by no means obvious, Stanley's equations (before the transformation is applied) will be used as starting point, with the exception that i_c will be set equal to $-i_B - i_A$ for the stator and $i_c = -i_a - i_b$ for the rotor. Stanley's assumption of sinusoidal variation of the mutual inductance between stator and rotor phases will also be employed here, since it is justified from the point-of-view that the actual variation with angle may be expressed in a Fourier Series of which the fundamental may be considered a reasonable approximation.

Fig. 2



M_{Aa} is the mutual inductance between phase A of the stator and phase a of the rotor. The other mutual inductances may be found by inspection of Fig. 3, which is a symbolic representation of Fig. 2.

Fig. 3



Stanley's differential equations, written in matrix form, are:

$$
\begin{bmatrix} e_A \\ e_B \\ e_C \\ \hline e_a \\ e_b \\ e_c \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
R & 0 & 0 & 0 & 0 & 0 \\
0 & R & 0 & 0 & 0 & 0 \\
0 & 0 & R & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & r & 0 & 0 \\
0 & 0 & 0 & 0 & r & 0 \\
0 & 0 & 0 & 0 & 0 & r
\end{array}\right]
\begin{bmatrix} i_A \\ i_B \\ i_C \\ \hline i_a \\ i_b \\ i_c \end{bmatrix}
+
$$

$$
\frac{d}{dt}\left\{
\left[\begin{array}{ccc|ccc}
L_s & M_s & M_s & M\cos\theta & M\cos(\theta+\frac{2\pi}{3}) & M\cos(\theta-\frac{2\pi}{3}) \\
M_s & L_s & M_s & M\cos(\theta-\frac{2\pi}{3}) & M\cos\theta & M\cos(\theta+\frac{2\pi}{3}) \\
M_s & M_s & L_s & M\cos(\theta+\frac{2\pi}{3}) & M\cos(\theta-\frac{2\pi}{3}) & M\cos\theta \\ \hline
M\cos\theta & M\cos(\theta-\frac{2\pi}{3}) & M\cos(\theta+\frac{2\pi}{3}) & l_r & m_r & m_r \\
M\cos(\theta+\frac{2\pi}{3}) & M\cos\theta & M\cos(\theta-\frac{2\pi}{3}) & m_r & l_r & m_r \\
M\cos(\theta-\frac{2\pi}{3}) & M\cos(\theta+\frac{2\pi}{3}) & M\cos\theta & m_r & m_r & l_r
\end{array}\right]
\begin{bmatrix} i_A \\ i_B \\ i_C \\ \hline i_a \\ i_b \\ i_c \end{bmatrix}
\right.
$$

The matrices are now partitioned along the broken lines. Leaving out the third and sixth equation one obtains the following statements:

$$\begin{bmatrix} e_A \\ e_B \end{bmatrix} = R \begin{bmatrix} i_A \\ i_B \end{bmatrix} + \frac{d}{dt}\left\{ \left[\begin{array}{cc|c} L_s & M_s & M_s \\ M_s & L_s & M_s \end{array}\right] \left[\begin{array}{c} i_A \\ i_B \\ \hline -i_A - i_B \end{array}\right] + \left[\begin{array}{cc|c} M\cos\theta & M\cos(\theta+\frac{2\pi}{3}) & M\cos(\theta-\frac{2\pi}{3}) \\ M\cos(\theta-\frac{2\pi}{3}) & M\cos\theta & M\cos(\theta+\frac{2\pi}{3}) \end{array}\right] \left[\begin{array}{c} i_a \\ i_b \\ \hline -i_a - i_b \end{array}\right] \right\} \quad 31$$

$$\begin{bmatrix} e_a \\ e_b \end{bmatrix} = r \begin{bmatrix} i_a \\ i_b \end{bmatrix} + \frac{d}{dt} \left[\begin{array}{cc|c} \ell_r & m_r & m_r \\ m_r & \ell_r & m_r \end{array}\right] \left[\begin{array}{c} i_a \\ i_b \\ \hline -i_a - i_b \end{array}\right] + \left[\begin{array}{cc|c} M\cos\theta & M\cos(\theta-\frac{2\pi}{3}) & M\cos(\theta+\frac{2\pi}{3}) \\ M\cos(\theta+\frac{2\pi}{3}) & M\cos\theta & M\cos(\theta-\frac{2\pi}{3}) \end{array}\right] \left[\begin{array}{c} i_A \\ i_B \\ \hline -i_A - i_B \end{array}\right] \Bigg\} \quad 32$$

A further partitioning is indicated in both (31) and (32). Subsequently the matrices are partly recombined according to the following example:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline -x-y \end{array}\right] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} -x-y \end{bmatrix} \quad 33$$

$$\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} -x-y \end{bmatrix} = - \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} (x+y) = \begin{bmatrix} -a_{13} & -a_{13} \\ -a_{23} & -a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad 34$$

Equation (34) is inserted into (33). The column matrix is factored and the square matrices are added to produce:

$$\begin{bmatrix} a_{11} & a_{1} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = \begin{bmatrix} a_{11}-a_{13} & a_{12}-a_{13} \\ a_{21}-a_{23} & a_{22}-a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad 35$$

These operations are performed on (31) and (32), resulting in (36) and (38). The symbols L and ℓ stand for $L_s - M_s$ and

$l_r - m_r$ respectively. From now on it will be considered self-understood that the operator $\frac{d}{dt}$ operates on all matrices that follow it in a product. Square brackets will be reserved for matrices, round brackets for scalars.

$$\begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} R+L\frac{d}{dt} & 0 \\ 0 & R+L\frac{d}{dt} \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} + \sqrt{3}M\frac{d}{dt} \begin{bmatrix} \sin(\theta+\frac{2\pi}{3}) & -\sin\theta \\ \sin\theta & -\sin(\theta-\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} \qquad 36$$

or symbolically:

$$\left[\mathcal{E}_s\right] = \left(R + L\frac{d}{dt}\right)\left[\mathcal{I}_s\right] + M\frac{d}{dt}\left[B(\theta)\right]\left[\mathcal{I}_R\right] \qquad 37$$

Matrix $[B]$ is that of equation (26). The other matrices in (37) are defined by (36).

Points a, b and c are connected together; the rotor equations become then:

$$\begin{bmatrix} e_a \\ e_b \end{bmatrix} = 0 = \begin{bmatrix} r+l\frac{d}{dt} & 0 \\ 0 & r+l\frac{d}{dt} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} + \sqrt{3}M\frac{d}{dt} \begin{bmatrix} \sin(\theta+\frac{2\pi}{3}) & \sin\theta \\ -\sin\theta & \sin(\theta+\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} \qquad 38$$

or

$$\left[\mathcal{E}_R\right] = 0 = \left(r + l\frac{d}{dt}\right)\left[\mathcal{I}_R\right] + M\frac{d}{dt}\left[B(-\theta)\right]\left[\mathcal{I}_s\right] \qquad 39$$

Chapter III

Application of the Transformation

Abstract: The results of Chapter I are applied to the simplified form of Stanley's equations. The resulting differential equations correspond to those proposed by Young[1].

In Chapter I the direct and quadrature quantities have been related to the phase-quantities by the transformations $[A]$ and $[A]^{-1}$

$$[P_s]=[A(\delta)][\mathcal{P}_s] \quad , \quad [\mathcal{P}_s]=[A(\delta)]^{-1}[P_s] \qquad 40$$

for the stator, and for the rotor:

$$[P_R]=[A(\delta-\theta)][\mathcal{P}_R] \quad , \quad [\mathcal{P}_R]=[A(\delta-\theta)]^{-1}[P_R] \qquad 41$$

The need for the choice of the angle $\delta-\theta$, the time derivative of which is $\Omega-\omega$, has been explained in Chapter II.

Expressing equations (37) and (39) in terms of (40) and (41) produces:

$$[A(\delta)]^{-1}[E_s]=(R+L\tfrac{d}{dt})[A(\delta)]^{-1}[I_s]+M\tfrac{d}{dt}[B(\theta)][A(\delta-\theta)]^{-1}[I_R] \qquad 42$$

and

$$[A(\delta-\theta)]^{-1}[E_R]=0=(r+\ell\tfrac{d}{dt})[A(\delta-\theta)]^{-1}[I_R]+M\tfrac{d}{dt}[B(-\theta)][A(\delta)]^{-1}[I_s] \qquad 43$$

As pointed out before, the transformations $[A]$ and $[A]^{-1}$ are applicable only if the given phase quantities are balanced, which is always the case for the currents, but for the voltages this is not necessarily so. In this sense equation (42) is restricted, but not equation (43).

In order to get the equations in standard form (42) will be multiplied by $[A(\delta)]$ and (43) by $[A(\delta-\theta)]$ yielding:

$$[E_s]=[A(\delta)](R+L\tfrac{d}{dt})[A(\delta)]^{-1}[I_s]+[A(\delta)](M\tfrac{d}{dt})[B(\theta)][A(\delta-\theta)]^{-1}[I_R] \qquad 44$$

and

$$0=\left[A(\delta-\theta)\right]\left(r+\ell\frac{d}{dt}\right)\left[A(\delta-\theta)\right]^{-1}\left[I_R\right]+\left[A(\delta-\theta)\right]\left(M\frac{d}{dt}\right)\left[B(-\theta)\right]\left[A(\delta)\right]^{-1}\left[I_s\right]=\left[E_R\right] \qquad 45$$

By (28) and (29) respectively (44) and (45) become:

$$\left[E_s\right]=\left[A(\delta)\right]\left(R+L\frac{d}{dt}\right)\left[A(\delta)\right]^{-1}\left[I_s\right]+\frac{3}{2}\left[A(\delta)\right]\left(M\frac{d}{dt}\right)\left[A(\delta)\right]^{-1}\left[I_R\right] \qquad 46$$

$$\left[E_R\right]=0=\left[A(\delta-\theta)\right]\left(r+\ell\frac{d}{dt}\right)\left[A(\delta-\theta)\right]^{-1}\left[I_R\right]+\frac{3}{2}\left[A(\delta-\theta)\right]\left(M\frac{d}{dt}\right)\left[A(\delta-\theta)\right]^{-1}\left[I_s\right] \qquad 47$$

By the product rule for differentiation and the fact that AA^{-1} is equal to the unit matrix equation (46) becomes:

$$\left[E_s\right]=\left(R+L\frac{d}{dt}\right)\left[I_s\right]+\left[A(\delta)\right]\left\{L\frac{d}{dt}\left[A(\delta)\right]^{-1}\right\}\left[I_s\right]+\frac{3}{2}M\frac{d}{dt}\left[I_R\right]+\frac{3}{2}\left[A(\delta)\right]\left\{M\frac{d}{dt}\left[A(\delta)\right]\right\}\left[I_R\right] \qquad 48$$

Finally relationship (25) is applied to produce:

$$\left[E_s\right]=\left(R+L\frac{d}{dt}\right)\left[I_s\right]+\left[A(\delta)\right](L\Omega)\left[A(\delta)\right]^{-1\times}\left[I_s\right]+\frac{3}{2}M\frac{d}{dt}\left[I_R\right]+\frac{3}{2}\left[A(\delta)\right](M\Omega)\left[A(\delta)\right]^{-1\times}\left[I_R\right] \qquad 49$$

Again the scalars $(L\Omega)$ and $(M\Omega)$ may be taken out of the position they occupy in (49), leading to the standard result, which has been obtained here by a very compact and easily applicable method, viz:

$$\left[E_s\right]=\left(R+L\frac{d}{dt}\right)\left[I_s\right]+\Omega L^{\times}\left[I_s\right]+\frac{3}{2}M\frac{d}{dt}\left[I_R\right]+\frac{3}{2}M\,\Omega^{\times}\left[I_R\right] \qquad 50$$

In exactly the same fashion (47) becomes

$$\left[E_R\right]=0=\left(r+\ell\frac{d}{dt}\right)\left[I_R\right]+\ell(\Omega-\omega)^{\times}\left[I_R\right]+\frac{3}{2}M\frac{d}{dt}\left[I_s\right]+\frac{3}{2}M(\Omega-\omega)^{\times}\left[I_s\right] \qquad 51$$

where $\Omega-\omega$ equals $\frac{d}{dt}(\delta-\theta)$ 52

In expanded form (50) and (51) read:

$$e_D=\left(R+L\frac{d}{dt}\right)i_D-\Omega L i_Q+\frac{3}{2}M\frac{d}{dt}\,i_d-\frac{3}{2}M\Omega\,i_q \qquad 53$$

$$e_Q=\left(R+L\frac{d}{dt}\right)i_Q+\Omega L\,i_D+\frac{3}{2}M\frac{d}{dt}\,i_q+\frac{3}{2}M\Omega\,i_d \qquad 54$$

and

$$e_d=0=\left(r+\ell\frac{d}{dt}\right)i_d-(\Omega-\omega)\ell i_q+\frac{3}{2}M\frac{d}{dt}\,i_D-\frac{3}{2}M(\Omega-\omega)\,i_Q \qquad 55$$

$$e_q=0=\left(r+\ell\frac{d}{dt}\right)i_q+(\Omega-\omega)\ell i_q+\frac{3}{2}M\frac{d}{dt}\,i_Q-\frac{3}{2}M(\Omega-\omega)\,i_D \qquad 56$$

This result is identical to the one stated by Young in reference (1). According to the outline of Young's procedure (given in the 1961 edition of Fitzgerald and Kingsley[11]) it is likely that Young applied the unrestricted Park Transformation. The present author performed independently the procedure (utilizing the unrestricted transformation) and produced eventually equations (53) to (56); however, the length and complexity of the manipulations are so forbidding that they cannot be reprinted here; in fact, for that reason, they are not found anywhere in the literature. On the other hand, the method that has been carried out in this chapter is compact and easily applied. But, it must be kept in mind, that even if these equations (53 to 56) are identical to Young's equations, they can only be used, if the motor is fed from a balanced supply, for it has not been proved to the reader that they hold true in a more general way, as well. The restriction leaves a certain degree of dissatisfaction. Yet, to accept Young's equations takes an act of faith, which is totally unsatisfactory in the conventional spirit of science.

A further, rather subtle, justification for the restriction of balance is that the solution to a set of differential equations in terms of direct- and quadrature-components can be interpreted (and this is generally done for the synchronous machine) as an expression for the envelope of the sinusoidal variations in each phase. This is shown by equations (10) to (14), which can be rewritten, according to the discussion following equation (14), in a manner given by

$$\underline{P}(t) = \imath' P_D(t) + \jmath' P_Q(t) \tag{57}$$

$$\left| \underline{P}(t) \right| = \sqrt{P_D^2(t) + P_Q^2(t)} \tag{58}$$

which by equation (11) equals $\frac{3}{2} P_o(t)$. $P_o(t)$ is the common envelope of the phase-quantities.

$$P_o(t) = \frac{2}{3}\sqrt{P_D^2(t) + P_Q^2(t)} \tag{59}$$

If the phase-quantities are not balanced they cannot have a common envelope, and the terms P_D and P_Q have no longer an obvious physical interpretation. Having performed the Park Transformation under such conditions one has transformed a set of differential equations (30) with direct physical meaning into another set of differential equations with obscure physical meaning, although there is still the advantage that all trigonometric terms have been eliminated.

A further remark, regarding the result, is that all constants occurring in the equations can be measured by two open-circuit tests, in which the rotor of the induction motor is held stationary in any arbitrary position and a constant, forward-sequence, 3-phase emf is applied to the stator and rotor in turn. For these tests equations (55) and (56) should be set equal to $e_d \neq 0$ and $e_q \neq 0$ respectively. For both tests ω and all $\frac{d()}{dt}$ are zero. In the first test the rotor currents are zero, for which equations (53) to (56) result in:

$$e_D = R i_D - \Omega L i_Q \tag{60}$$

$$e_Q = R i_Q + \Omega L i_D \tag{61}$$

for the stator, and

$$e_d = \frac{-3}{2} M \Omega i_Q \qquad (62)$$

$$e_q = \frac{3}{2} M \Omega i_D \qquad (63)$$

for the rotor. All quantities in (60) to (63) are directly proportional to their respective rms-values. Equation (61) is multiplied through by j and added to (60) to produce:

$$e_D + j e_Q = R(i_D + j i_Q) + \Omega L (j j i_Q + i_D) \qquad (64)$$

or

$$E_S = (R + j \Omega L) I_S \qquad (65)$$

and in the same way (62) and (63) become:

$$E_R = j \Omega \frac{3}{2} M \; I_S \qquad (66)$$

The terms R, L and $\frac{3}{2}M$ are found with the aid of a voltmeter, ammeter and a wattmeter, and r and ℓ are found when a similar test is applied to the rotor.

Chapter IV

The Result in Compact Form.

Abstract: The differential equations are expressed in phasor and vector form. The vector formulation permits a further contraction with the aid of a "gradient"-operator.

As discussed earlier, the direct- and quadrature-axis quantities are components of vectors, defined in rotating planes. In two dimensions a complex number can also be utilized for the representation of a vector. Let the complex number notation defining a directed quantity carry no special symbolism but, the corresponding vector notation carry an underlining bar. To carry this through, equation (54) is multiplied by and added to (53) to obtain:

$$(e_D + j e_Q) - R(i_D + j i_Q) = L\frac{d}{dt}(i_D + j i_Q) + j\Omega L(i_D + j i_Q) + \frac{3}{2}M\frac{d}{dt}(i_d + j i_q) + j\Omega\frac{3}{2}M(i_d + j i_q) \qquad 67$$

It has tacitly assumed that $\frac{d}{dt}(j) = 0$. This is a constraint on the operator $\frac{d}{dt}$. If, for instance, equation (67) is to be regarded as a statement holding true within a rotating plane in which j is fixed, then the constraint on $\frac{d}{dt}$ demands that this operator has to stand for the rate of change of the particular vector it operates on as viewed from an observer turning with the rotating plane. The constraint relationship

$$\frac{d}{dt}(j) = 0 \qquad 68$$

for the operator $\frac{d}{dt}$ must be observed whether the rate of turning of the plane in which j is fixed is physically meaningful or not. It is conventional to give such operators different symbols such as

$\left(\frac{d}{dt}\right)_1$ or $\left(\frac{\delta}{\delta t}\right)$. However, the rate of change of a scalar is the same viewed from all reference frames, barring relativistic principles. For example the quantities $\dot{i}_D$ and $\dot{i}_Q$ are two such scalars.

Now, it is seen in figure 3 that, due to the particular projections employed, the D-axis and the d-axis always coincide. A simple definition for a complex plane would be to choose a phasor $\dot{j}$ such that it always lies along both the Q-axis and the q-axis. Let such a $\dot{j}$ be denoted by j_{Q-q}, which then, turns with angular velocity Ω with respect to the stator windings and $\Omega-\omega$ with respect to the rotor windings. By the constraint relationship (68) on the operator $\frac{d}{dt}$, this operator denotes the rate of change observed by an observer turning with the rotating plane. Let this $\frac{d}{dt}$ be denoted by $\left(\frac{d}{dt}\right)_{Q-q}$. (68) is then in particular:

$$\left(\frac{d}{dt}\right)_{Q-q}\left(j_{Q-q}\right)= 0 \tag{69}$$

Equation (67) defines the following phasor-symbols:

$$e_S - R\,i_S = v_S = \left(\frac{d}{dt}\right)_{Q-q}\left(L i_S + \frac{3}{2} M i_R\right) + j_{Q-q}\,\Omega\left(L i_S + \frac{3}{2} M i_R\right) \tag{70}$$

For the rotor one would obtain similarly:

$$v_R = \left(\frac{d}{dt}\right)_{Q-q}\left(\ell i_R + \frac{3}{2} M i_S\right) + j_{Q-q}\left(\Omega-\omega\right)\left(\ell i_R + \frac{3}{2} M i_S\right) \tag{71}$$

In view of the meaning of the operator $\left(\frac{d}{dt}\right)_{Q-q}$ and in the light of the discussion at the beginning of Chapter I, (70) and (71) degenerate in the steady state i.e. $\left(\frac{d}{dt}\right)_{Q-q} \simeq 0$ to the well known equations which lead to the equivalent circuit of the induction motor[6].

In two-dimensional problems complex numbers and vectors can serve the same purpose. The notation will be different, but the physical interpretation will be unchanged. In order to write (70) and (71) in vector notation it is necessary to "translate" the meaning of the operator $\dot{j}$: $\dot{j}$ rotates a phasor by an angle of $\frac{\pi}{2}$ in the counterclockwise direction. Since all vectors in (70) and (71) are coplanar the cross-product $\hat{k} \times \underline{\dot{i}}_S$ will produce the same effect as $\dot{j}\, \dot{i}_S$.

$\hat{k}$ is defined by:

$$|\hat{k}| = 1 \tag{72}$$

$$\hat{k} \cdot \underline{\dot{i}}_{S,R} = 0 \qquad \text{all } \underline{\dot{i}}_{S,R} \tag{73}$$

With the vector notation the following two statements are identical to (70) and (71):

$$\underline{v}_S = \left(\frac{d}{dt}\right)_{Q-q} \left(L\,\underline{\dot{i}}_S + \frac{3}{2} M\,\underline{\dot{i}}_R\right) + \Omega\,\hat{k} \times \left(L\,\underline{\dot{i}}_S + \frac{3}{2} M\,\underline{\dot{i}}_R\right) \tag{74}$$

$$\underline{v}_R = \left(\frac{d}{dt}\right)_{Q-q} \left(l\,\underline{\dot{i}}_R + \frac{3}{2} M\,\underline{\dot{i}}_S\right) + (\Omega - \omega)\,\hat{k} \times \left(l\,\underline{\dot{i}}_R + \frac{3}{2} M\,\underline{\dot{i}}_S\right) \tag{75}$$

It is not necessary to define the vectors v and $\underline{\dot{i}}_{S,R}$ any further than that they are in fact the same "arrows" occurring in (70) and (71).

It is customary[8] to define angular velocity axial vectors. The direction of rotation is defined by the cork-screw rule. In this case then the angular velocity of the rotating field is

$$\underline{\Omega} = \hat{k}\,\Omega \tag{76}$$

and that of the rotor

$$\underline{\omega} = \hat{k}\,\omega \tag{77}$$

and that of the field with respect to the rotor windings:

$$\underline{\Omega} - \underline{\omega} = \hat{k}(\Omega - \omega) \qquad 78$$

According to the discussion on page 133 in Goldstein[8] which results in his formula (4-102)

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \underline{\omega} \times \qquad 79$$

it is possible to generalize this relationship to changing the operator from one rotating coordinate system turning with velocity $\underline{\Omega}$ to another rotating with ω.

Let (79) hold true for the relation between the rotor windings and a Newtonian reference frame. ω is the angular velocity of the rotor with respect to the stator or the Newtonian reference frame. Let the following notation represent equation (79):

$$\left(\frac{d}{dt}\right)_S = \left(\frac{d}{dt}\right)_R + \underline{\omega} \times \qquad 80$$

Similarly:

$$\left(\frac{d}{dt}\right)_S = \left(\frac{d}{dt}\right)_{Q-q} + \underline{\Omega} \times \qquad 81$$

Subtracting (80) from (81) one obtains:

$$\left(\frac{d}{dt}\right)_R = \left(\frac{d}{dt}\right)_{Q-q} + (\underline{\Omega} - \underline{\omega}) \times \qquad 82$$

Direct application of equations (81) and (82) allows a representation of equations (74) and (75) in the form:

$$\underline{v}_S = \left(\frac{d}{dt}\right)_S \left(L \underline{i}_S + \frac{3}{2} M \underline{i}_R\right) \qquad 83$$

and

$$\underline{v}_R = \left(\frac{d}{dt}\right)_R \left(l \underline{i}_R + \frac{3}{2} M \underline{i}_S\right) \qquad 84$$

These equations represent four statements of Faraday's law in vector form. They would have been directly obtained if in chapter III $\frac{d}{dt}\left[A(\delta_1)\right]^{-1}$ and $\frac{d}{dt}\left[A(\delta_2-\theta)\right]^{-1}$ had been taken as zero, in other words if the projections had been performed onto stationary axes from stationary coils for the stator, and $\delta_2-\theta$ held constant for all θ for the rotor. Such a procedure would have been equivalent to a simple introduction of generalized coordinates (see Goldstein pp 10 - 12). It must be pointed out that the constraint relationships

$$i_A + i_B + i_C = 0 \longrightarrow q_A + q_B + q_C - C_1 = 0 \qquad 85$$

and

$$i_a + i_b + i_c = 0 \longrightarrow q_a + q_b + q_c - C_2 = 0 \qquad 86$$

are holonomic (Goldstein: equation 1-35). For the reader not acquainted with mechanics it is worth noting that the branch to mesh transformation in circuit theory is a change to a particular form of generalized coordinates and velocities: the mesh currents.

To the reader acquainted with circuit theory and classical mechanics it may be of interest that because the mesh method is equivalent to generalized mechanics it can be independently verified that the transpose of the mesh transformation is the matrix premultiplying the branch-voltage matrix in Kirchoff's voltage law[9]:

$$\left[i_b\right] = \left[B\right]'\left[i_m\right] \quad \text{or} \quad i_b = \sum_m \frac{\partial i_b}{\partial i_m} i_m \quad \text{for all } b \qquad 87$$

and

$$\left[B\right]\left[v_b\right] = 0 \qquad 88$$

The general element q_{bm} of the matrix $\left[B\right]'$ is defined by:

$$q_{bm} = \left.\frac{\partial i_b}{\partial i_m}\right| \text{with all other } q \text{ held constant.} \qquad 89$$

According to Goldstein (1-46) page 17 a generalized force is found by:

$$Q_i = \sum_j F_j \frac{\partial r_j}{\partial q_i}\bigg|_{\text{all other } q \text{ held constant}} \qquad 90$$

The "cancellation of the dots" (ref. 10, p 413, 15.107) states:

$$\frac{\partial r_j}{\partial q_i} = \frac{\partial \dot{r}_j}{\partial \dot{q}_i}\bigg|_{\text{all other } \dot{q} \text{ and } q \text{ held constant.}} \qquad 91$$

so that (90) may be rewritten in terms of (91)

$$Q_i = \sum_j F_j \frac{\partial \dot{r}_j}{\partial \dot{q}_i} \qquad 92$$

This becomes after translation into the ideas of total emf around a mesh m, emf sources in a branch l, mesh currents and branch currents:

$$e_m = \sum_l e_l \frac{\partial i_m}{\partial i_l}\bigg|_{\text{all other } i_m \text{ constant}} \equiv \sum_l g_{ml} e_l \qquad 93$$

This may be written out in matrix form for all the meshes defined by equation (87);

$$[e_m] = \left[\frac{\partial i_l}{\partial i_m}\right][e_l] \equiv [g_{ml}][e_l] \qquad 94$$

Now, the matrix $[g_{ml}]$ is evidently the transpose of the matrix $[B]' \equiv [g_{lm}]$. Therefore (94) can be rewritten

$$[e_m] = [B][e_l] \qquad 95$$

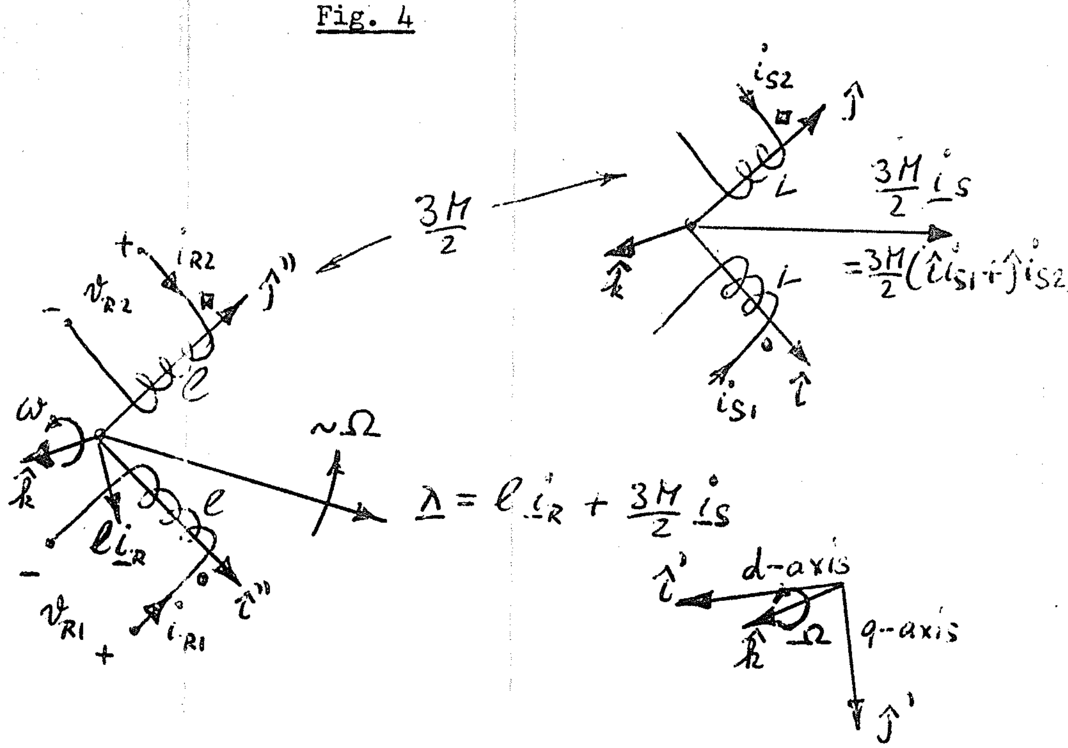
Now Kirchhoff's voltage law (equation 88) states that (95) is exactly the way the mesh emf must be found:

$$0 = [B][v_l] \equiv [B][z_l(p)\, i_l] + [B][e_l] \qquad 96$$

All this has no direct bearing on equations (83) and (84) other than that those equations in component form represent also, though less familiar generally, a way of choosing generalized coordinates. They form the basis of Kron's "primitive machine"[7], although Kron states immediately equations (74) and (75) (ref 7, pp 118 to 122).

To understand equation (84) consider the following arrangement for the rotor:

Fig. 4



For the sake of simplicity, the fixed triad $\hat{\imath}-\hat{\jmath}-\hat{k}$ was chosen arbitrarily to coincide instantaneously with $\hat{\imath}''-\hat{\jmath}''-\hat{k}$. There are two fictitious coils fixed in the rotor. A flux-linkage vector $\underline{\lambda}$ sweeps through these coils with angular velocity $\Omega-\omega$ in the steady state, and induces in them according to equation (84) the voltages

$$v_{R1}=\frac{d}{dt}\left(\ell i_{R1}+\frac{3}{2}M i_{S1}\right)$$

and

$$v_{R2} = \frac{d}{dt}\left(l\, i_{R2} + \frac{3}{2} M\, i_{s2} \right) \qquad 98$$

At any other instant the fixed triad may either be redefined, or i_{s1} and i_{s2} may be replaced by $\underline{i}_s \cdot \hat{\imath}''$ and $\underline{i}_s \cdot \hat{\jmath}''$ in (97) and (98). Since (97) and (98) are scalar equations it is no longer necessary to write the differential operator as $\left(\frac{d}{dt}\right)_R$. All the scalar quantities vary sinusoidally in the steady state as in equation 10. A further remark concerning equation (84) is this: The equation might be written as either

$$\underline{v}_R = \hat{\imath}' v_d + \hat{\jmath}' v_q = \left(\frac{d}{dt}\right)_R \left(l[\hat{\imath}' i_d + \hat{\jmath}' i_q] + \frac{3}{2} M [\hat{\imath}\, i_D + \hat{\jmath}\, i_Q] \right) \qquad 99$$

or

$$\underline{v}_R = \hat{\imath}'' v_{R1} + \hat{\jmath}'' v_{R2} = \left(\frac{d}{dt}\right)_R \left(l[\hat{\imath}'' i_{R1} + \hat{\jmath}'' i_{R2}] + \frac{3}{2} M [\hat{\imath}'' \underline{i}_s \cdot \hat{\imath}'' + \hat{\jmath}'' \underline{i}_s \cdot \hat{\jmath}] \right) \qquad 100$$

(Using $\underline{i}_s \cdot \hat{\imath}''$ as a definition for i_{s1}) or any other choice of axes for that matter. However, only equation (100) can be decomposed into component form directly, since only $\left(\frac{d}{dt}\right)_R \hat{\imath}''$ and $\left(\frac{d}{dt}\right)_R \hat{\jmath}''$ are zero. Which way, however, $\hat{\imath}''$ and $\hat{\jmath}''$ are fixed momentarily is arbitrary, so that there are still infinitely many possible ways of interpreting equation (100).

Finally let the operator

$$\hat{\imath}' \frac{\partial}{\partial i_d}\bigg|_{i_q, i_D, i_Q} + \hat{\jmath}' \frac{\partial}{\partial i_q}\bigg|_{i_d, i_D, i_Q} = \hat{\imath}'' \frac{\partial}{\partial i_{R1}}\bigg|_{i_{R2}, i_{s1}, i_{s2}} + \hat{\jmath}'' \frac{\partial}{\partial i_{R2}}\bigg|_{i_{R1}, i_{s1}, i_{s2}} = \text{etc.} \qquad 101$$

be defined by the symbol, ∇_R . This, obviously, has the properties of a "gradient".

Similarly, define:

$$\hat{i}' \frac{\partial}{\partial i_D} + \hat{j}' \frac{\partial}{\partial i_Q} = \hat{i} \frac{\partial}{\partial i_{s_1}} + \hat{j} \frac{\partial}{\partial i_{s_2}} = \text{etc.} = \nabla_s \qquad 102$$

and now, consider the function φ, (whatever its physical significance may be) given by:

$$\varphi = \frac{1}{2} L \, \underline{i}_s \cdot \underline{i}_s + \frac{3}{2} M \, \underline{i}_s \cdot \underline{i}_R + \frac{1}{2} \ell \, \underline{i}_R \cdot \underline{i}_R \qquad 103$$

Then evidently (83) and (84) can be written as

$$\underline{v}_s = \left(\frac{d}{dt}\right)_s \left(\nabla_s \varphi\right) \qquad 104$$

$$\underline{v}_R = \left(\frac{d}{dt}\right)_R \left(\nabla_R \varphi\right) \qquad 105$$

Equation (104) and (105) are a statement of equations 53 to 56. These equations hold true for the vector quantities occurring in (103) and it is immaterial how one chooses the relative positions of the triads along which one wishes to express the components of each vector.

Chapter V

An Alternate Method

Abstract: The differential equations (104) and (105) will be derived from energy considerations.*

In Appendix B it is shown that the magnetic energy stored in the induction-motor is equal to

$$T_e = \frac{2}{3}\varphi$$

where φ is given by equation (103).

There exists a more or less intuitive way of obtaining equation (106) quickly. For this consider equations (70) and (71) in the steady state:

$$e_s - R i_s = j_{Q-q}\,\Omega\left(L i_s + \frac{3}{2} M i_R\right) \qquad 107$$

and

$$-r i_R = j_{Q-q}\,(\Omega-\omega)\left(\ell i_R + \frac{3}{2} M i_s\right) \qquad 108$$

Multiply (108) through by $\frac{1}{s} = \frac{\Omega}{\Omega-\omega}$ to obtain:

$$-\frac{r}{s} i_R = j_{Q-q}\,\Omega\cdot\left(\ell i_R + \frac{3}{2} M i_s\right) \qquad 109$$

(107) and (109) may be represented by an equivalent circuit:

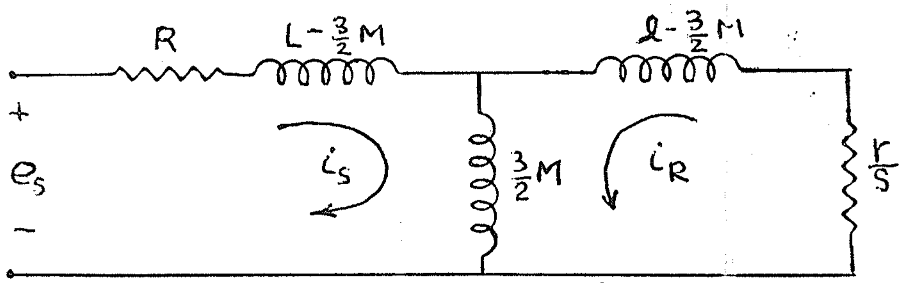


Fig.5

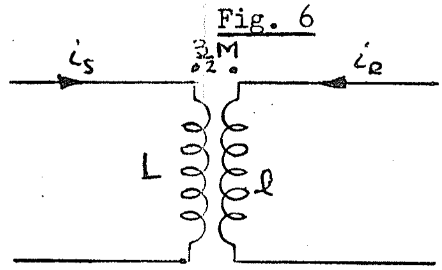
(angular frequency Ω)

*In order to understand this chapter the reader will have to study either reference 8 or 10. It is beyond the scope of this thesis to develop this background here.

Anyone familiar with machine theory recognizes this circuit immediately. It is an equivalent circuit for the induction motor in the steady state. To the machines engineer the inductors designated by the symbols $L-\frac{3M}{2}$ etc. have meaning as such, and they form the starting point for many theoretical studies. The preceeding symbolism will be retained, with the assumption that $L-\frac{3M}{2}$, etc. are well known, even if generally a different symbolism is employed.

The T-network of the three inductors shown in Fig. 5 is in turn equivalent to the transformer shown in Fig. 6.

Fig. 6



The average magnetic energy stored in this network may be written in the form[9]

$$T_m = \frac{1}{2}\begin{bmatrix} i_S & i_R \end{bmatrix}\begin{bmatrix} L & \frac{3}{2}M \\ \frac{3}{2}M & \ell \end{bmatrix}\begin{bmatrix} i_S^* \\ i_R^* \end{bmatrix} \qquad 110$$

The network of Fig. 5 is usually employed to find the stored energy[11]. But it has to be remembered that the resulting expression always has to be multiplied by some constant of proportionality. Let the total magnetic energy stored in the machine be T_e. Then the relationship between T_e and T_m is

$$T_e = k\,T_m \qquad 111$$

The immediate task is now to find the constant of proportionality.

It is known that if a balanced three-phase system is represented as a single-phase system then the currents and voltages in the single phase system must have $\sqrt{3}$ times the actual value in order to give numerically correct values for energy and power.

Equation (11) states

$$|i_s| = \frac{3}{2} i_{so}$$

i_{so} in turn equals $\sqrt{2}$ times the effective value per phase, i.e. $\sqrt{\frac{2}{3}}$ times the effective value for the 3-phase system. Hence one can state:

$$|i_s| = \frac{3}{2}\sqrt{\frac{2}{3}}\,|i_{s_{eff.}}| = \sqrt{\frac{3}{2}}\,|i_{s_{eff.}}| \qquad 112$$

But T_e will be given by some functional relationship, F, involving the effective current values:

$$T_e \equiv F(i_{s_{eff.}}, i_{R_{eff}}) \qquad 113$$

By (112) this becomes

$$T_e \equiv F\left(\sqrt{\tfrac{2}{3}}\, i_s, \sqrt{\tfrac{2}{3}}\, i_R\right) \qquad 114$$

Equation (11) is given as:

$$T_m \equiv T_m(i_s, i_R) \qquad 115$$

or by (111):

$$T_e \equiv k\,T_m(i_s, i_R) \qquad 116$$

Equation (110) is quadratic in i_s and i_R, so that (116) may be rewritten as

$$T_e \equiv T_m(\sqrt{k}\, i_s, \sqrt{k}\, i_R) \qquad 117$$

Equations (114) and (117) imply that the functional relationships are the same in both relations and that the constant of proportionality

k is equal to $\frac{2}{3}$.

$$k = \frac{2}{3} \tag{118}$$

If equation (110) is expanded one obtains finally:

$$T_m = \frac{1}{2}\left\{ L i_s i_s^* + \frac{3}{2} M (i_s i_R^* + i_R i_s^*) + \ell i_R i_R^* \right\} \tag{119}$$

This in turn becomes:

$$T_m = \frac{1}{2} L (i_D^2 + i_Q^2) + \frac{3}{2} M (i_D i_d + i_Q i_q) + \frac{1}{2} \ell (i_d^2 + i_q^2) \tag{120}$$

and in vector notation may be stated in the form

$$T_m = \frac{1}{2} L \underline{i}_s \cdot \underline{i}_s + \frac{3}{2} M \underline{i}_s \cdot \underline{i}_R + \frac{1}{2} \ell \underline{i}_R \cdot \underline{i}_R \tag{121}$$

Hence, it follows by equation (103) that

$$T_m = \varphi \quad \text{and} \quad T_e = \frac{2}{3} \varphi \tag{122}$$

This is the total average magnetic energy stored in the machine in the steady state. That equation (122) holds under transient conditions and instantaneously as well, is shown by the rigorous proof in Appendix B.

Whether the vectors in (122) are expressed in terms of rotating or stationary coordinates does not matter, since vectors are invariant under transformation. However, when (122) is used for the "kinetic-energy" function in the Lagrangian formulation (Ref. 8, Chapter 1 or Ref. 10, pp 418, 419) care has to be taken with regard to the operator $\left(\frac{d}{dt}\right)$.

Since the present energy function only depends on currents, i.e. the $\dot{q}$'s, the term $\frac{\partial T}{\partial q_\rho}$ in equation 15.144 of reference 10 may be ommitted. Therefore, the equation simplifies to:

$$\frac{d}{dt}\left(\frac{\partial T_e(\dot{q})}{\partial \dot{q}_\rho}\right) = \sum \frac{\partial \underline{r}_i}{\partial q_\rho} \cdot \underline{p}_i \tag{123}$$

By the rule of "the cancellation of the dots"

$$\frac{\partial r_i}{\partial q_p} \equiv \frac{\partial \dot{r}_i}{\partial \dot{q}_p} \equiv g_{ip} \qquad 124$$

one may rewrite (123) in those terms. Let all three rotor currents i_a, i_b and i_c form a vector $\underline{i}$ such as the column matrix on the right hand side of equation (15), and all three rotor flux linkages a vector $\underline{\lambda}$. The left-hand-side of equation (15) contains the direct- and quadrature-axis quantities, which we understand to be defined along rotating axes, but only by virtue of the statement directly beneath equation (15). The same verbal statement would have to be given along with the inverse of the Park Transformation. Equation (124) defines the elements of the inverse of the Park Transformation g_{ip} together with the verbal statement that the rate of change of $\delta-\theta$ is $\Omega-\omega$. If the statement is not given and the angle $\delta-\theta$ is taken to be stationary, then the Park Transformation for the rotor defines the quantities shown in Fig. 4. In the development of equation (123) in ref. 8 or 10 it is pointed out that when the various partial derivatives are taken time is also to be held constant. For this reason, then, we choose the quantities i_{R1}, i_{R2} for the rotor and similarly i_{s1} and i_{s2} for the stator as generalized velocities. The elements of the inverse transformation for the rotor are given by $\frac{\partial \underline{i}}{\partial i_{R1,2}}$ which is the inverse of the Park Transformation with the angle $\delta-\theta$ held constant. Then, when applying (123) to the rotor only, the summation sign drops out $\left[\frac{\partial i}{\partial i_{s1,2}}\right]$ and (123) becomes;

$$\frac{d}{dt}\left(\frac{\partial T_e(\dot{q})}{\partial i_{R1,2}}\right) = \frac{\partial \underline{i}}{\partial i_{R1,2}} \cdot \dot{\underline{\lambda}} = \frac{\partial \underline{i}}{\partial i_{R1,2}} \cdot \frac{d}{dt}\underline{\lambda} \qquad 125$$

$\frac{\partial \lambda}{\partial t}$ cannot, off hand, be identified with a vector form of Faraday's law for the actual rotor-coils. The reference system in which the operator $\frac{d}{dt}$ is a measure of the rate of change must be specified. Obviously, Faraday's law only holds if $\frac{d}{dt}$ signifies the rate of change as seen from the rotor coils. Hence, if (125) is written as

$$\left(\frac{d}{dt}\right)_R \left(\frac{\partial\, T_e(\dot{q})}{\partial \dot{i}_{R1,2}}\right) = \frac{\partial \underline{i}}{\partial i_{R1,2}} \cdot \left(\frac{d}{dt}\right)_R \underline{\lambda} \qquad 126$$

only then can the term $\left(\frac{d}{dt}\right)_R \underline{\lambda}$ be identified with $\mathcal{V}$ the voltage vector for the rotor:

$$\underline{\mathcal{V}} \equiv \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \qquad 127$$

Now, according to the discussion following equation (100), $\left(\frac{d}{dt}\right)_R \hat{i}$ and $\left(\frac{d}{dt}\right)_R \hat{j}$ are zero. So, equation 126 may be written explicitly, after having multiplied through by $\hat{i}''$ and $\hat{j}''$, i.e. having converted the "vector" $\left(\frac{d}{dt}\right)_R \left(\frac{\partial T_e(\dot{q})}{\partial i_{R1,2}}\right)$ from a column-matrix into the more conventional form,

$$\left(\frac{d}{dt}\right)_R \left(\hat{i}'' \frac{\partial T_e}{\partial i_{R1}}\right) = \hat{i}'' \frac{\partial \underline{i}}{\partial i_{R1}} \cdot \underline{\mathcal{V}} \equiv \hat{i}'' Q_1 \qquad 128$$

and

$$\left(\frac{d}{dt}\right)_R \left(\hat{j}'' \frac{\partial T_e}{\partial i_{R2}}\right) = \hat{j}'' \frac{\partial \underline{i}}{\partial i_{R2}} \cdot \underline{\mathcal{V}} \equiv \hat{j}'' Q_2 \qquad 129$$

The Q's are the so-called generalized forces.

Addition of (128) and (129) results in:

$$\left(\frac{d}{dt}\right)_R \left(\nabla_R T_e\right) = \hat{i}'' Q_1 + \hat{j}'' Q_2 = \underline{Q}_R \qquad 130$$

One would similarly obtain:

$$\left(\frac{d}{dt}\right)_S \left(\nabla T_e\right) = \underline{Q}_S \qquad 131$$

Equations (130) and (131) show that by means of the operators ∇_R

and ∇_s one can use the vector quantities in T_e directly as quasi-generalized velocities. This idea can be extended to the general case when $\frac{\partial T}{\partial q_\ell} \neq 0$ by defining a gradient for the q's, in a similar manner to definitiions (101) and (102). So far, it is known what T_e stands for in terms of known machine-constants. The Q's can be determined in two ways: by the principle of virtual work or by their definition (equations (128) and (129)). The first method is simpler, but not particularly rigorous and general. The second method sheds some additional light on the Park Transformation, and is therefore also carried out in appendix B.

$\underline{Q}_S$ and $\underline{Q}_R$ are related to the virtual work[8] by the equations:

$$d\,T_e = \underline{Q}_S \cdot d\underline{q}_S + \underline{Q}_R \cdot d\underline{q}_R \qquad 132$$

In this equation the $d\underline{q}$'s are defined by:

$$d\,T_e = \underline{Q}_S \cdot \underline{i}_S\, dt + \underline{Q}_R \cdot \underline{i}_R\, dt \qquad 133$$

Equation (133) may be rewritten in terms of the electric power flowing into the system of coils

$$p = \frac{d\,T_e}{dt} = \underline{Q}_S \cdot \underline{i}_S + \underline{Q}_R \cdot \underline{i}_R \qquad 134$$

or by equation (112):

$$p = \sqrt{\tfrac{3}{2}}\; \underline{Q}_S \cdot \underline{i}_{S\,eff} + \sqrt{\tfrac{3}{2}}\; \underline{Q}_R \cdot \underline{i}_{R\,eff} \qquad 135$$

which implies:

$$\sqrt{\tfrac{3}{2}}\; \underline{Q}_S = \underline{v}_{S\,eff} \qquad 136$$

But, it is known that in a balanced 3-phase system

$$\left| v_{s_{eff}} \right| = \sqrt{3} \left| \left(v_{s_{eff.}} / phase \right) \right| \qquad 137$$

holds true. And in general, for a sinusoid one has

$$v_{s_{eff.}} / phase = \frac{v_o}{\sqrt{2}} \qquad 138$$

where v_o is the amplitude, which in turn is related to v_s by the inverse Park Transformation or equation (11)

$$v_o = \frac{2}{3} \left| \underline{v}_s \right| \qquad 139$$

We obtain finally:

$$\sqrt{\frac{3}{2}}\ \underline{Q}_s = (\sqrt{3}) \left(\frac{1}{\sqrt{2}} \right) \frac{2}{3}\ \underline{v}_s \qquad 140$$

or

$$\underline{Q}_s = \frac{2}{3}\ \underline{v}_s \qquad 141$$

and similarly:

$$\underline{Q}_R = \frac{2}{3}\ \underline{v}_R \qquad 142$$

Substituting (141) and (142) into (130) and (131), and keeping in mind equation (122) we get:

$$\left(\frac{d}{dt} \right)_S (\nabla_s \varphi) = \underline{v}_s \qquad 143$$

$$\left(\frac{d}{dt} \right)_R (\nabla_R \varphi) = \underline{v}_R \qquad 144$$

This puts equations (104) and (105) on a physical basis, and therefore, working backwards it is possible to arrive at Young's equations without trigonometric manipulations starting from the well known equivalent steady-state circuit of the induction motor. This inherently demands that all transients be slow enough so that the electrical quantities may be always considered approximately sinusoidal. This restriction is used in setting up equations (110) and (141). It will be shown in appendix B that this restriction is not necessary. It should furthermore be noted that the ordinary form of Lagrange's equations for "stationary" reference frames was used. Evidently, the stator is stationary but not the rotor. In reference 8 and 10 the equations are derived by establishing first a general kinematical relationship, equation (123), which holds whether the reference frame is stationary or not. Subsequent identification with Newton's second law demands that the reference frame be stationary, this is not necessarily the case for identification with Faraday's law, which has been pointed out in detail in the discussion following equation (125).

Chapter VI

The Torque Relationship

Abstract: It is necessary to introduce a different set of generalized coordinates, with the aid of which the torque relationship is deduced.

For the determination of the torque it is necessary to augment the relationship for the magnetic energy stored in the system, T_e, by the mechanical energy stored in the inertia, J. The total "kinetic" energy then becomes:

$$T = T_e + \frac{1}{2} J \omega^2 \qquad 145$$

ω will be a generalized velocity and the generalized coordinate associated with ω is θ, the angular displacement of the rotor. For the remaining velocities, i_d and i_q cannot be chosen, since the contain implicitly, through the Park Transformation, the angle θ.

Let us, therefore, choose instead for the rotor two mesh-currents; $i_a(t)$ and $i_b(t)$ are a particularly convenient choice. It is shown in appendix B, that the expression for T_e holds no matter what the time relationship of the currents is: i_a and i_b do not have to be sinusoidal!

For all practical purposes we can leave T_e in the old form and simply think of i_d and i_q to be functions of θ, i_a and i_b, the functional relationship being given by:

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \left[A(\delta - \theta) \right] \begin{bmatrix} i_a \\ i_b \end{bmatrix}^{\sim} \qquad 146$$

In particular equation (145) becomes:

$$T(i_D, i_Q, i_a, i_b, \omega, \theta, t) = \frac{1}{3} L (i_D^2 + i_q^2) + M[i_D i_d(i_a, i_b, \theta) + $$
$$+ i_Q i_q(i_a, i_b, \theta) + \frac{1}{3} \ell [i_d^2(i_a, i_b, \theta) + i_q^2(i_a i_b \theta)] + \frac{1}{2} J\omega^2 \qquad 147$$

The electrical relations have already been established in Chapter V. Hence only the following equation has to be evaluated:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right) - \frac{\partial T}{\partial \theta} = Q_{\dot{\theta}} \qquad 148$$

is the generalized force, in this case evidently the externally applied torque in the direction of ω less any possible friction.

The partial derivatives $\frac{\partial}{\partial \omega}$ and $\frac{\partial}{\partial \theta}$ imply that all quantities appearing in the bracket of $T(i_D, i_Q, i_a, i_b, \omega, \theta, t)$ except of course, ω and θ respectively, must be held constant, and no others.

Evidentally $\frac{\partial T}{\partial \omega}$ becomes simply $J\omega$, the angular momentum.

$\frac{\partial T}{\partial \theta}$ is evaluated by the well known rule for partial derivatives:

$$\frac{\partial T}{\partial \theta} = \left[\frac{\partial T(i_D, i_Q, i_d, i_q, \omega, t)}{\partial i_d}\right]\left[\frac{\partial i_d}{\partial \theta}\bigg|_{t,\, i_a,\, i_b = const.}\right] + \frac{\partial T}{\partial i_q} \cdot \frac{\partial i_q}{\partial \theta} \qquad 149$$

or $$\frac{\partial T}{\partial \theta} = M\left[i_D \frac{\partial i_d}{\partial \theta} + i_Q \frac{\partial i_q}{\partial \theta}\right] + \frac{2}{3} \ell \left[i_d \frac{\partial i_d}{\partial \theta} + i_q \frac{\partial i_q}{\partial \theta}\right] \qquad 150$$

where $\frac{\partial i_d}{\partial \theta}$ and $\frac{\partial i_Q}{\partial \theta}$ are found by the definition of i_d & i_q, viz:

$$\frac{\partial}{\partial \theta}\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \left\{\frac{\partial}{\partial \theta}[A(\delta - \theta)]\right\}\begin{bmatrix} i_a \\ i_b \end{bmatrix} \qquad 151$$

By inspection of equation (18), with δ replaced by $\delta - \theta$, it is seen that when $\frac{\partial}{\partial \theta}$ of every term in $A(\delta - \theta)$ is taken a new four element matrix results. Comparison with $A(\delta-\theta)$ reveals that if the order of the equations in (151) is reversed and $\frac{\partial i_d}{\partial \theta}$ is multiplied through by -1, then the right hand side of (18) is reproduced. This is similar to the proof of (25) in Appendix A and yields:

$$\frac{\partial}{\partial \theta}\begin{bmatrix} i_q \\ -i_d \end{bmatrix} = \left[A(\delta-\theta)\right]\begin{bmatrix} i_a \\ i_b \end{bmatrix} \equiv \begin{bmatrix} i_d \\ i_q \end{bmatrix} \tag{152}$$

Hence, (152) results in:

$$\frac{\partial i_d}{\partial \theta} = -i_q \tag{153}$$

and

$$\frac{\partial i_q}{\partial \theta} = i_d \tag{154}$$

so that, equation (150) becomes

$$\frac{\partial T}{\partial \theta} = M\left[i_Q\, i_d - i_D\, i_q\right] \tag{155}$$

This result can also be produced by a more or less intuitive argument:

Consider T in the form of dot-products of vectors

$$T = \frac{2}{3}\left[\frac{1}{2} L\, \underline{i}_s \cdot \underline{i}_s + \frac{3}{2} M\, \underline{i}_s \cdot \underline{i}_R + \frac{1}{2}\, l\, \underline{i}_R \cdot \underline{i}_2\right] + \frac{1}{2} J \omega^2 \tag{156}$$

The vector i_R is defined in terms of the rotor position θ.

Then

$$\frac{\partial \underline{i}_R}{\partial \theta} \tag{157}$$

may be thought of in terms of a virtual displacement given to the vector $\underline{i}_R$ as shown in Fig. 7.

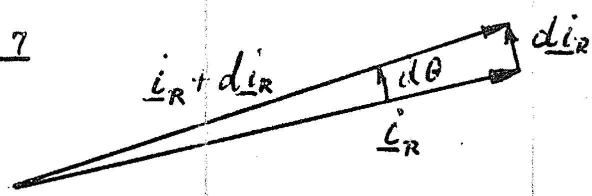


Fig. 7

The length of the vector $d\,\underline{i}_R$ is given by

$$|d\,\underline{i}_R| = |\underline{i}_R|\,|d\theta|$$

and the direction is given by that of the cross-product:

$$\measuredangle\, d\underline{i}_R = \frac{\hat{k} \times \underline{i}_R}{|\underline{i}_R|} = \hat{r} \qquad 158$$

$\hat{r}$ is a unit vector having the direction of $d\underline{i}_r$

$d\underline{i}_R$ may be then expressed as $\hat{r}\,|d\underline{i}_R|$ and

$$\frac{d\underline{i}_r}{d\theta} = \frac{\hat{r}\,|d\underline{i}_R|}{d\theta} = \frac{\hat{r}\,|\underline{i}_R||d\theta|}{d\theta} = \hat{r}\,|\underline{i}_r| = \frac{\hat{k} \times \underline{i}_R}{|\underline{i}_R|}\,|\underline{i}_R| = \hat{k} \times \underline{i}_r \qquad 159$$

Assuming that $\frac{d\underline{i}_r}{\theta} = \frac{\partial \underline{i}_R}{\partial \theta}$ is correct

we obtain $$\frac{\partial \underline{i}_R}{\partial \theta} = \hat{k} \times \underline{i}_R \qquad 160$$

and $\frac{\partial T}{\partial \theta}$ becomes:

$$\frac{\partial T}{\partial \theta} = M\underline{i}_S \cdot \frac{\partial \underline{i}_R}{\partial \theta} + \frac{l}{3}\,\underline{i}_R \cdot \frac{\partial \underline{i}_R}{\partial \theta} = M\underline{i}_S \cdot \hat{k} \times \underline{i}_R + \frac{l}{3}(\underline{i}_R \cdot \hat{k} \times \underline{i}_R) \qquad 161$$

or

$$\frac{\partial T}{\partial \theta} = M\hat{k} \cdot \underline{i}_R \times \underline{i}_S = M\begin{vmatrix} 0 & 0 & 1 \\ i_d & i_q & 0 \\ i_D & i_q & 0 \end{vmatrix} = M(i_Q\, i_d - i_D\, i_q) \qquad 162$$

or, alternatively,

$$M\hat{k} \cdot \underline{i}_R \times \underline{i}_S = M\,|\underline{i}_R||\underline{i}_S|\sin\alpha \qquad 163$$

where α is the torque angle, the angle by which the phasor $\underline{i}_R$ lags the phasor $\underline{i}_S$. This is the most conventional way of stating internally developed torque. Equation (148) may now be written explicitly as: $$\frac{d}{dt}(J\omega) - M\hat{k} \cdot \underline{i}_R \times \underline{i}_S = Q_\theta \qquad 164$$

Q_θ = externally applied torque - (friction + windage)

$$\text{i.e.} \quad Q_\theta = \tau - [K\,\mathrm{sgn}(\omega) + B\omega] \qquad 165$$

This results then in the fifth differential equation governing the behaviour of the motor:

$$J\frac{d\omega}{dt} + B\omega + k\,\mathrm{sgn}(\omega) = M\left[i_Q i_d - i_D i_q \right] + \tau \qquad 166$$

The other four equations are equations (53) through (56).

The solution of these five equations cannot be carried out in general terms, because they are all non-linear and, therefore, the solution to one particular input cannot be related to the solution to any other input, even one proportional to the first input.

The author believes that the derivation of these equations gains its value mainly from the fact that some aspects of the intimate relationship between the disciplines of electrical machines, circuit theory and classical mechanics have been demonstrated.

Appendix A

A.1 PROOF OF EQUATION (25)

Differentiating equation (23) one obtains:

$$\frac{d}{d\delta}\left[A(\delta)\right]^{-1} = \frac{2}{3}\begin{bmatrix} -\sin\delta & -\cos\delta \\ -\cos\left(\delta-\frac{2\pi}{3}\right) & -\sin\left(\delta-\frac{2\pi}{3}\right) \end{bmatrix} \equiv \left[D(\delta)\right] \qquad 167$$

or

$$\left\{\frac{d}{dt}\left[A(\delta)\right]^{-1}\right\}\left[P(t)\right] = \frac{d\delta}{dt}\left\{\frac{d}{d\delta}\left[A(\delta)\right]^{-1}\right\}\left[P(t)\right] = \Omega\left[D(\delta)\right]\left[P(t)\right] \qquad 168$$

When $[P]$ is changed to ${}^{x}[P]$, defined by equations (24) and (25), the the columns in $[D]$ must be interchanged and the resulting first column in $[D]$ must be multiplied by -1, if the statement (168) is to remain unchanged. Comparison of (167) with (23) shows that this manipulation within the matrix $[D]$ has reproduced the matrix $[A]^{-1}$, which proves equation (25).

A.2 PROOF OF EQUATION (28)

Consider first $\left\{\left[B(\theta)\right]+\left[B(-\theta)\right]\right\}\left[A(\alpha)\right]^{-1}$

$$\left[B(\theta)\right] + \left[B(-\theta)\right] = \sqrt{3}\left\{\sin\left(\theta+\frac{2\pi}{3}\right) - \sin\left(\theta-\frac{2\pi}{3}\right)\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2\sqrt{3}\,\frac{\sqrt{3}}{2}\cos\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 169$$

so that the first statement becomes:

$$\left\{\left[B(\theta)\right] + \left[B(-\theta)\right]\right\}\left[A(\alpha)\right]^{-1} = 2\begin{bmatrix} \cos\alpha\cos\theta & -\sin\alpha\cos\theta \\ \cos\left(\alpha-\frac{2\pi}{3}\right)\cos\theta & -\sin\left(\alpha-\frac{2\pi}{3}\right)\cos\theta \end{bmatrix} \qquad 170$$

Now use of the trigonometric identities

$$2\cos\alpha\cos\theta = \cos(\alpha-\theta) + \cos(\alpha+\theta) \qquad 171$$

and
$$2\sin\alpha\cos\theta = \sin(\alpha-\theta) + \sin(\alpha+\theta)$$

and forming two matrices out of (170) results in

$$\begin{bmatrix} \cos(\alpha-\theta) & -\sin(\alpha-\theta) \\ \cos(\alpha-\theta-\frac{2\pi}{3}) & -\sin(\alpha-\theta-\frac{2\pi}{3}) \end{bmatrix} + \begin{bmatrix} \cos(\alpha+\theta) & -\sin(\alpha+\theta) \\ \cos(\alpha+\theta-\frac{2\pi}{3}) & -\sin(\alpha+\theta-\frac{2\pi}{3}) \end{bmatrix}$$

$$= \frac{3}{2} A^{-1}(\alpha-\theta) + \frac{3}{2} A^{-1}(\alpha+\theta) \qquad 172$$

By a similar manipulation the reader may satisfy himself that also the following statement also holds true:

$$\left\{\left[B(\theta)\right] - \left[B(-\theta)\right]\right\} \left[A(\alpha)\right]^{-1} = \frac{3}{2} A^{-1}(\alpha+\theta) - \frac{3}{2} A^{-1}(\alpha-\theta) \qquad 173$$

Addition of (172) and (173) results in:

$$\left[B(\theta)\right] \left[A(\alpha)\right] = \frac{3}{2} A^{-1}(\alpha+\theta) \qquad 174$$

Appendix B

B.1 PROOF THAT EQUATION (106) HOLDS ALSO FOR NON-SINUSOIDAL PHASE-CURRENTS.

Consider equations (30). It may be represented as

$$\begin{bmatrix} \mathcal{E}_S \\ \mathcal{E}_R \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \mathcal{I}_S \\ \mathcal{I}_R \end{bmatrix} + \frac{d}{dt} \left\{ \begin{bmatrix} \mathcal{L}_{SS} & \mathcal{L}_{SR} \\ \mathcal{L}_{RS} = \mathcal{L}_{SR} & \mathcal{L}_{RR} \end{bmatrix} \begin{bmatrix} \mathcal{I}_S \\ \mathcal{I}_R \end{bmatrix} \right\} \qquad 175$$

The curly bracket contains a statement of the flux linkages. Premultiplications by $\frac{1}{2}\left[\mathcal{I}_S \,\vdots\, \mathcal{I}_R'\right]$ of this braket gives the magnetic energy stored in the system, whether the currents are sinusoidal or not, provided the system is linear so that:

$$T_e = \frac{1}{2}\left[\mathcal{I}_S' \,\vdots\, \mathcal{I}_R'\right] \left[\begin{array}{c|c} \mathcal{L}_{SS} & \mathcal{L}_{SR} \\ \hline \mathcal{L}_{SR} & \mathcal{L}_{RR} \end{array}\right] \left[\begin{array}{c} \mathcal{I}_S \\ \hline \mathcal{I}_R \end{array}\right] \qquad 176$$

This matrix product is now partitioned to produce:

$$2\,T_e = \mathcal{I}_S' \mathcal{L}_{SS} \mathcal{I}_S + \mathcal{I}_S' \mathcal{L}_{SR} \mathcal{I}_R + \mathcal{I}_R' \mathcal{L}_{SR}' \mathcal{I}_S + \mathcal{I}_R' \mathcal{L}_{RR} \mathcal{I}_R \qquad 177$$

Each product is a scalar so that its transpose is equal to itself. The transpose of the 3rd term is equal to the second term since we have

$$\left[\mathcal{I}_R' \mathcal{L}_{SR}' \mathcal{I}_S\right]' = \left[\mathcal{I}_S\right]' \left[\mathcal{I}_R' \mathcal{L}_{SR}\right]' = \left[\mathcal{I}_S\right]' \left[\mathcal{L}_{SR}\right] \left[\mathcal{I}_R\right] \qquad 178$$

(177) may then be rewritten in the form

$$T_e = \frac{1}{2} \mathcal{I}_S' \mathcal{L}_{SS} \mathcal{I}_S + \mathcal{I}_S' \mathcal{L}_{SR} \mathcal{I}_R + \frac{1}{2} \mathcal{I}_R' \mathcal{L}_{RR} \mathcal{I}_R \qquad 179$$

By the inverse of equation (15) the phase quantities may be expressed in the direct- and quadrature-axis quantities to give:

$$T_e = \frac{1}{2} I_s' P^{-1}(\delta)' \mathcal{L}_{ss} \; P^{-1}(\delta) I_s + I_s' P^{-1}(\delta)' \mathcal{L}_{sR} P^{-1}(\delta-\theta) I_R$$
$$+ \frac{1}{2} I_R' P^{-1}(\delta-\theta) \mathcal{L}_{RR} \; P^{-1}(\delta-\theta) \qquad 180$$

Let this define the quantities:

$$T_e = T_1 + T_2 + T_3 \qquad 181$$

The inverse of equation (15), $P^{-1}(\delta)$, becomes:

$$P^{-1}(\delta) = \frac{2}{3} \begin{bmatrix} \cos\delta & -\sin\delta & \frac{1}{2} \\ \cos(\delta - \frac{2\pi}{3}) & -\sin(\delta - \frac{2\pi}{3}) & \frac{1}{2} \\ \cos(\delta + \frac{2\pi}{3}) & -\sin(\delta + \frac{2\pi}{3}) & \frac{1}{2} \end{bmatrix} \qquad 182$$

It may be noted that the following relationship exists between the Park Transformation and its inverse by comparison of equations (15) and (182):

$$P^{-1}(\delta)' = \frac{2}{3} P(\delta) - \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad 183$$

Because of the constraint given by Kirchoff's current law the zero-sequence components are missing:

$$I_s' = [\, i_D, i_Q, 0 \,] \qquad 184$$

and since in the expression for T_1 I_s' premultiples equation (183) all terms involving the second matrix of (183) are zero so that we get

$$2T_1 = \frac{2}{3} [\, i_D, i_Q \;\; 0 \,][P(\delta)] \begin{bmatrix} L_s & M_s & M_s \\ M_s & L_s & M_s \\ M_s & M_s & L_s \end{bmatrix} [\, P(\delta)^{-1}] \begin{bmatrix} i_D \\ i_Q \\ 0 \end{bmatrix} \qquad 185$$

The product $P(\delta) \mathcal{L}_{ss} P^{-1}(\delta)$ may be changed into:

$$P(\delta) \left\{ \begin{bmatrix} L_s - M_s & 0 & 0 \\ 0 & L_s - M_s & 0 \\ 0 & 0 & L_s - M_s \end{bmatrix} + M_s \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} P(\delta)^{-1} \qquad 186$$

or

$$(L_S - M_S)\, P(\delta)\, \mathcal{U}\, P(\delta)^{-1} + M_S\, P(\delta)\, \mathcal{V}\, P(\delta)^{-1} \qquad 187$$

where $\mathcal{U}$ is the unit matrix and $\mathcal{V}$ is defined on comparison of (187) with equation (186). Since three sinusoids of equal magnitude spaced $\frac{2\pi}{3}$ radians apart add to zero $\mathcal{V}P(\delta)^{-1}$ becomes

$$\mathcal{V}\, P(\delta)^{-1} = \frac{2}{3}\begin{bmatrix} 0 & 0 & 3/2 \\ 0 & 0 & 3/2 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \equiv \mathcal{W} \qquad 188$$

and similarly

$$P(\delta)\, \mathcal{W} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{Z} \end{bmatrix} \equiv \mathcal{Z}$$

$\mathcal{Z} I_S$ results in zero.

By definition of $\mathcal{U}$, the unit matrix:

$$P(\delta)\, \mathcal{U}\, P(\delta)^{-1} = P(\delta)\, P(\delta)^{-1} = \mathcal{U} \qquad 189$$

It follows that $2T_1$, simplified to

$$2T_1 = \frac{2}{3}(L_S - M_S)\, i_D^2 + \frac{2}{3}(L_S - M_S)\, i_Q^2 \qquad 190$$

or

$$\frac{3}{2}T_1 = \frac{1}{2} L\, i_D^2 + \frac{1}{2} L\, i_Q^2 \qquad 191$$

quite similarly we would obtain:

$$\frac{3}{2}T_3 = \frac{1}{2} \ell\, i_d^2 + \frac{1}{2} \ell\, i_q^2 \qquad 192$$

T_2 may be written as:

$$T_2 = [\, i_D \;\; i_Q \;\; 0 \,] \left(\frac{2}{3}\right) P(\delta)\, \mathcal{L}_{SR}\, P(\delta-\theta)^{-1} \begin{bmatrix} i_d \\ i_q \\ 0 \end{bmatrix} \qquad 193$$

The rows and columns of $P(\delta)$ and $P(\delta-\theta)^{-1}$ that contribute nothing are omitted to give:

$$T_2=[i_D\ i_Q]\left(\tfrac{2}{3}\right)\begin{bmatrix}\cos\delta & \cos(\delta-\frac{2\pi}{3}) & \cos(\delta+\frac{2\pi}{3})\\ -\sin\delta & -\sin(\delta-\frac{2\pi}{3}) & -\sin(\delta+\frac{2\pi}{3})\end{bmatrix}\left[\mathcal{L}_{SR}\right]\begin{bmatrix}\frac{2}{3}\cos(\delta-\theta) & -\frac{2}{3}\sin(\delta-\theta)\\ \frac{2}{3}\cos(\delta-\theta-\frac{2\pi}{3}) & -\frac{2}{3}\sin(\delta-\theta-\frac{2\pi}{3})\\ \frac{2}{3}\cos(\delta-\theta+\frac{2\pi}{3}) & -\frac{2}{3}\sin(\delta-\theta+\frac{2\pi}{3})\end{bmatrix}\begin{bmatrix}i_d\\ i_q\end{bmatrix} \quad 194$$

Let the following symbolism be defined by (194):

$$T_2 = \underline{I}_s'\left(\tfrac{2}{3}\right)\underline{P(\delta)}\,\mathcal{L}_{SR}\,P(\delta-\theta)^{-1}\,I_R \quad 195$$

When the product $P(\delta)\,\mathcal{L}_{SR}$ is evaluated it is useful to keep in mind the double angle relationships (171). $P(\delta)\,\mathcal{L}_{SR}$ becomes by the definition of $\mathcal{L}_{SR}$ in equation (30) after some relatively simple manipulation:

$$\underline{P(\delta)}\,\mathcal{L}_{SR} = \tfrac{3}{2}M\,\underline{P(\delta-\theta)} \quad 196$$

Now $\underline{P(\delta-\theta)}$ and $\underline{P(\delta-\theta)^{-1}}$ are still inverse matrices, although the originals have been deleted by a row and a column respectively:

$$P(\delta-\theta)P(\delta-\theta)^{-1} = \mathcal{U} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad 197$$

However, this particular order of multiplication is necessary.

T_2 becomes then

$$T_2 = \tfrac{2}{3}\,\underline{I}_s'\,\tfrac{3}{2}M\,\mathcal{U}\,\underline{I}_R \quad 198$$

or

$$\tfrac{3}{2}T_2 = \tfrac{3}{2}M\left(i_D i_d + i_Q i_q\right) \quad 199$$

It follows from equations (191), (192) and (199) that equation (106) is verified, viz:

$$\tfrac{3}{2}\,T_e = \varphi \quad 200$$

B.2 PROOF OF EQUATIONS (141) AND (142).

By equations (128) and (129) the for the rotor are given by:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial i_a}{\partial i_{R_1}} & \frac{\partial i_b}{\partial i_{R_1}} & \frac{\partial i_c}{\partial i_{R_1}} \\ \frac{\partial i_a}{\partial i_{R_2}} & \frac{\partial i_b}{\partial i_{R_2}} & \frac{\partial i_c}{\partial i_{R_2}} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \qquad 201$$

The relationship between $\underline{i}$ and $i_{R_{1,2}}$ is given by the inverse Park Transformation with $\delta - \theta$ to be held stationary.

$$\underline{i} = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \left[P(\delta-\theta)\right]^{-1} \begin{bmatrix} i_{R_1} \\ i_{R_2} \\ 0 \end{bmatrix} = \underline{P^{-1}(\delta-\theta)} \begin{bmatrix} i_{R_1} \\ i_{R_2} \end{bmatrix} \qquad 202$$

The elements of $\underline{P^{-1}(\delta-\theta)}$ are given by the definition of $\underline{P^{-1}(\delta-\theta)}$, viz. equation (194) or alternatively by:

$$\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \begin{bmatrix} \frac{\partial i_a}{\partial i_{R_1}} & \frac{\partial i_a}{\partial i_{R_2}} \\ \frac{\partial i_b}{\partial i_{R_1}} & \frac{\partial i_b}{\partial i_{R_2}} \\ \frac{\partial i_c}{\partial i_{R_1}} & \frac{\partial i_c}{\partial i_{R_2}} \end{bmatrix} \begin{bmatrix} i_{R_1} \\ i_{R_2} \end{bmatrix} \qquad 203$$

It follows that 201 may be written as

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \left[P^{-1}(\delta-\theta)\right]' \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \qquad 204$$

By equation (183) we have

$$\left[\underline{P^{-1}(\delta-\theta)}\right]' = \frac{2}{3}\,\underline{P(\delta-\theta)} \qquad 205$$

By the definition of $P(\delta-\theta)$ which is the Park Transformation without a statement about any possible zero-sequency one obtains

$$\left[\underline{P(\delta-\theta)}\right]\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} v_{R1} \\ v_{R2} \end{bmatrix} \qquad 206$$

The voltages do not have to be sinusoidal or balanced. It follows that (204) may be written as

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} v_{R1} \\ v_{R2} \end{bmatrix} \qquad 207$$

On multiplication of the first row by $\hat{i}''$ the second row by $\hat{j}''$ and subsequent addition the final result is

$$\underline{Q}_R = \frac{2}{3}\,\underline{v}_R \qquad 208$$

Bibliography

1. D. S. Brereton, D. G. Lewis, and C. C. Young, "Representation of Induction Motor Loads During Power-System Stability Studies", Trans. AIEE, Vol. 76, Pt. III, pp 451-460, 1957.

2. K. G. Black and R. J. Noorda, "Analog Computer Study of Wind-Tunnel Drives", Trans. AIEE, Vol. 76, Pt. I, pp. 745-750, 1957.

3. H. C. Stanley, "An Analysis of the Induction Machine", Trans. AIEE, Vol. 57, pp. 751-755, 1938.

4. R. H. Park, "Two-Reaction Theory of Synchronous Machinery," Transactions of the AIEE, Vol. 48, pp. 716-727, July 1929.

5. G. Kron, "Non-Riemannian Dynamics of Rotating Electrical Machinery", Journal of Mathematics and Physics, 1934, pp. 103-194.

6. G. Kron, "Equivalent Circuits of Electric Machinery", Wiley and Sons, New York, 1951, (p. 77).

7. G. Kron, "Tensors for Circuits", Dover, New York, 1959, (Ch. 31).

8. H. Goldstein, "Classical Mechanics", Addison-Wesley Pub. Co., Reading, Mass., 1960.

9. Seshu and Balabanian, "Network Analysis", Wiley & Sons, New York, 1959, (Ch. 4, p. 80).

10. Synge and Griffith, "Principles of Mechanics", McGraw-Hill Book Co. Inc., New York, Toronto, London, 1959, Ch. 15.

11. Fitzgerald and Kingsley, "Electric Machinery", McGraw-Hill Book Co. Inc., Now York, Toronto, London, 1961.