Topics in the Notion of Amenability and its Generalizations for Banach Algebras

by

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Abstract

This thesis has two parts. The first part deals with some questions in amenability. We show that for a Banach algebra A with a bounded approximate identity, the amenability of $A\widehat{\otimes}A$, the amenability of $A\widehat{\otimes}A^{op}$ and the amenability of A are equivalent. Also if A is a closed ideal in a commutative Banach algebra B, then the (weak) amenability of $A\widehat{\otimes}B$ implies the (weak) amenability of A.
Finally, we show that if the Banach algebra A is amenable through multiplication π, then A is also amenable through any multiplication ρ such that $\|\rho - \pi\| < \frac{1}{11}$.

The second part deals with questions in generalized notions of amenability such as approximate amenability and bounded approximate amenability. First we prove some new results about approximately amenable Banach algebras. Then we state a characterization of approximately amenable Banach algebras and a characterization of boundedly approximately amenable Banach algebras.
Finally, we prove that $B(l^p(E))$ is not approximately amenable for Banach spaces E with certain properties. As a corollary of this part, we give a new proof that $B(l^2)$ is not approximately amenable.

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Introduction

The theory of amenable Banach algebras begins with B.E.Johnson's memoir in 1972 [12] and has proven to be of enormous importance in Banach algebra theory. The terminology comes from [12, Theorem 2.5]: a locally compact group G is amenable (in the usual sense) if and only if $L^1(G)$ is amenable.

For a Banach algebra A, X an A-bimodule is said to be a Banach A-bimodule if there exists $C > 0$ such that for any $a \in A$ and $x \in X$,

$$\|a.x\| \leq C\|a\|\|x\|;$$

$$\|x.a\| \leq C\|a\|\|x\|.$$

For a Banach A-bimodule X, X^* can be made into a Banach A-bimodule by the actions defined by

$$\langle a.f, x\rangle = \langle f, x.a\rangle,$$
$$\langle f.a, x\rangle = \langle f, a.x\rangle \quad (a \in A, f \in X^*, x \in X)$$

For a Banach algebra A, let X be a Banach A-bimodule. A linear mapping $D : A \longrightarrow X$ is said to be a derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in A).$$

The derivation D is said to be inner if there exists $x \in X$ such that

$$D(a) = a.x - x.a \qquad (a \in A).$$

The Banach algebra A is said to be amenable if every continuous derivation $D : A \longrightarrow X^*$, is inner for all Banach A-bimodules X.
For amenable Banach algebras A and B, we know that $A\widehat{\otimes}B$ is amenable. However the converse is not known to be true. Before us B.E.Johnson in [14] has proved that the amenability of $A\widehat{\otimes}B$ entails the amenability of A if the Banach algebra B has a certain property. However it is not known whether the answer is positive even for the case where $A = B$. So the question that comes to mind is:

Does amenability of $A\widehat{\otimes}A$ imply the amenability of A for a Banach algebra A?

We prove that if the Banach algebra A has a bounded approximate identity, then the answer is positive. Since having a bounded approximate identity is a necessary condition for an amenable Banach algebra, our proof is significant enough. Indeed we show that for a Banach algebra A with a bounded approximate identity the following are equivalent:

(i) A is amenable.

(ii) $A\widehat{\otimes}A$ is amenable.

(iii) $A\widehat{\otimes}A^{op}$ is amenable.

Then we prove that if A is a closed ideal in a commutative Banach algebra B, then the (weak) amenability of $A\widehat{\otimes}B$ entails (weak)amenability of A.

At the end of the section 2, we give a partly different proof for [13, Theorem 6.2] that leads us to a new result. We show that if a Banach algebra A is amenable with a multiplication π then A is also amenable with any multiplication ρ such that $\|\rho - \pi\| < \frac{1}{11}$. We have discovered that the constant $\frac{1}{11}$ is universal.

In section 3, we mostly concentrate on generalized notions of amenability such as approximate amenability and boundedly approximate amenability that were originally introduced in [7] and [8].

A Banach algebra A is approximately amenable if every continuous derivation $D : A \longrightarrow X^*$ is approximately inner for all Banach A-bimodules X. i.e. there exists a net $(\xi_i)_i \subseteq X^*$ such that

$$D(a) = \lim_i a.\xi_i - \xi_i.a \qquad (a \in A);$$

A is boundedly approximately amenable if the net $(\xi_u)_i$ can be found such that

$$\|a.\xi_i - \xi_i.a\| \leq M.\|a\| \qquad (a \in A),$$

for some $M > 0$.

First we prove some general results on approximate amenability. Then we find a characterization for approximate amenability and one for bounded approximate amenability, that are generalizations of [15, Theorem 1].

In the last part of section 3, we prove that certain classes of Banach algebras are not approximately amenable. This work is related to the work in [5],[17], [18] about non-amenability of certain classes of Banach algebras. We prove that if a Banach space E is such that for some $p \in [1, \infty)$, $l^p(E)$ satisfies the property $(*)$ (that we will define later) and $l^p(E) \oplus l^2 \cong l^2$, then the Banach algebra $B(l^p(E))$ is not approximately amenable. As an example of such a Banach algebra we can mention $B(l^2)$, and hence we find a new proof of Ozawa's result.

1 Preliminaries

1.1 Banach algebras

In this chapter we introduce some definitions and theorems that we will use in next chapters.

Throughout this chapter and the whole of this thesis, all of our spaces are linear spaces over the field of complex numbers $\mathbb{C}$.

For a normed space X, the dual space of X i.e. the space of all continuous linear functionals on X is denoted by X^*.

By an algebra A we mean a vector space with a multiplication that makes A together with sum of A into a ring and also satisfies the additional property

$$a(\alpha b) = (\alpha a)b = \alpha ab \qquad (\alpha \in \mathbb{C}, a, b \in A).$$

Definition 1.1. Suppose that A is an algebra with a norm. Then A is called normed algebra if the norm of A satisfies

$$\|ab\| \leq \|a\|\|b\| \qquad (a, b \in A).$$

A Banach algebra is a complete normed algebra. The Banach algebra A is unital if it has an identity e with $\|e\| = 1$.

Theorem 1.2. *Suppose that A is a Banach algebra with an identity e. If a is an element of A such that $\|a - e\| < 1$, then a is invertible.*

Proof: See [2, Lemma 2.1 on page 196]. $\square$

Definition 1.3. Let A to be an algebra. Then the unitization of A denoted by

$A^{\#}$, is the algebra $A \oplus \mathbb{C}$ with operations of addition, multiplication and scalar multiplication defined by

$$\begin{aligned}(a,\alpha)+(b,\beta) &= (a+b,\alpha+\beta),\\ \beta(a,\alpha) &= (\beta a,\beta\alpha),\\ (a,\alpha)(b,\beta) &= (ab+\beta a+\alpha b,\alpha\beta) \qquad (\alpha,\beta\in\mathbb{C}, a,b\in A).\end{aligned}$$

It can be easily seen that $A^{\#}$ is an algebra with the operations defined above. Also if A is a normed algebra, then $A^{\#}$ is a normed algebra with the norm

$$\|(a,\alpha)\| = \|a\| + |\alpha| \qquad (a\in A, \alpha\in\mathbb{C}).$$

The algebra $A^{\#}$ is a Banach algebra if A is and in this case $A^{\#}$ will ba unital Banach algebra with the unit $(0,1)$.
Throughout this chapter A is a Banach algebra unless otherwise indicated.

For $F \in A^{**}$ and $f \in A^{*}$, Ff is an element of A^{*} defined by

$$\langle Ff, a\rangle = \langle F, f.a\rangle \qquad (a\in A).$$

For $F, G \in A^{**}$, the first Arens product of F and G is defined by

$$\langle F\Box G, f\rangle = \langle F, Gf\rangle \qquad (f\in A^{*}).$$

From Goldstine's Theorem. it can be easily seen that if A^{**} has a right identity with respect to the first Arens product, then A has a right bounded approximate identity.

Definition 1.4. Let A to be a normed algebra. A bounded net $(e_i)_i \subset A$ is a bounded left approximate identity for A if

$$\|e_i a - a\| \longrightarrow_i 0 \qquad (a \in A).$$

In a similar way we can define the bounded right and two-sided approximate identities.

We note that if A has a bounded left approximate identity $(l_j)_j$ and a bounded right approximate identity $(r_i)_i$, then the net $(l_j + r_i - l_j r_i)$ gives us a two-sided bounded approximate identity for A.

Definition 1.5. Let X to be a bimodule over a Banach algebra A that is a Banach space itself. Then X is said to be a Banach A-bimodule if there exists a constant $C > 0$ such that for all $a \in A$, $x \in X$

$$\|a.x\| \leq C\|a\|\|x\|,$$

$$\|x.a\| \leq C\|a\|\|x\|.$$

In particular A is a Banach A-bimodule .

If X is a Banach A-bimodule, then X^* is a Banach A-bimodule by the actions defined by

$$\begin{aligned} &\langle a.f, x\rangle = \langle f, x.a\rangle \\ &\langle f.a, x\rangle = \langle f, a.x\rangle \quad (a \in A, f \in X^*, x \in X). \end{aligned}$$

The Banach A-bimodule X^* defined in this way is said to ba a dual Banach A-bimodule.

Definition 1.6. If X is a Banach A-bimodule, then a bounded (left) approximate identity for X is a bounded net $(e_i) \subset A$ such that

$$\|e_i x - x\| \longrightarrow_i 0 \qquad (x \in X).$$

Theorem 1.7. *Let A to be a Banach algebra and X be a Banach A-bimodule such that A has a bounded (left) approximate identity $(e_i)_i$ for X. Then for $y \in X$ and $\delta > 0$, there are elements $a \in A$ and $x \in X$ such that $y = a.x$ and $\|y - x\| < \delta$. Furthermore, we can find a such that $\|a\| \leq M$ where $M > 1$ is an upper bound for $(e_i)_i$.*

Proof: See [1, Proposition 11 on page 65] . □

Definition 1.8. A Banach A-bimodule X is said to be neo-unital if for every $x \in X$ there exists $y \in X$ and $a, b \in A$ such that $x = a.y.b$

Theorem 1.7 implies that every Banach A-bimodule X which has a bounded approximate identity in A is neo-unital.

Definition 1.9. Let X, Y and Z be normed spaces. A mapping $\phi : X \times Y \longrightarrow Z$ is said to be bilinear if

(i) for each $y \in Y$ the mapping $x \mapsto \phi(x, y)$ is linear;

(ii) for each $x \in X$ the mapping $y \mapsto \phi(x, y)$ is linear.

The bilinear map ϕ is said to be continuous (bounded) if there exists $M > 0$ such that

$$\|\phi(x, y)\| \leq M\|x\|\|y\| \qquad (x \in X, y \in Y). \qquad (\dagger)$$

And the norm of ϕ is defined to be the infimum of all M for which (†) is satisfied. The space of all continuous (bounded) bilinear functionals on $X \times Y$ is denoted by $BL(X, Y; \mathbb{C})$. It is a Banach space under the given norm.

Definition 1.10. Let X, Y be normed spaces. Then for $x \in X$ and $y \in Y$, $x \otimes y$ denotes an element of $BL(X^*, Y^*, \mathbb{C})$ that is defined by

$$(x \otimes y)(f, g) = f(x)g(y) \qquad (f \in X^*, g \in Y^*).$$

The Lin$\{x \otimes y : x \in X, y \in Y\}$ is denoted by $X \otimes Y$.

On $X \otimes Y$, one can define several norms; Injective and projective norms are defined as follows:

Definition 1.11. The injective norm of $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, denoted by $w(u)$, will be the norm that u inherits as an element of $BL(X^*, Y^*, \mathbb{C})$. So

$$w(u) = \sup\{|\sum_{i=1}^{n} f(x_i)g(y_i)| : f \in X^*, g \in Y^*, \|f\|, \|g\| \leq 1\}.$$

Definition 1.12. The projective tensor norm of $u \in X \otimes Y$, denoted by $p(u)$, is defined by

$$p(u) = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

By [1, Lemma 10 on page 233], we have

(i) $p(u) \geq w(u) \qquad (u \in X \otimes Y)$,

(ii) $p(x \otimes y) = w(x \otimes y) = \|x\|\|y\| \qquad (x \in X, y \in Y)$.

The completion of $X \otimes Y$ under the injective (projective) tensor norm is denoted by $X \check{\otimes} Y$ (respectively $X \hat{\otimes} Y$).

Now we state a proposition that we will use several times in this thesis.

Proposition 1.13. *The Banach space $X\hat{\otimes}Y$ can be represented as the linear subspace of $BL(X^*, Y^*, \mathbb{C})$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$ where $\sum_i \|x_i\|\|y_i\| < \infty$. Moreover $p(u)$ is the infimum of these sums.*

Proof: See [1, Proposition 12 on page 234]. □

Proposition 1.14. *Let A and B be Banach algebras. Then there exists a unique algebra product on $A\hat{\otimes}B$ such that for elementary tensors we have*

$$(a \otimes b)(c \otimes d) = ac \otimes bd \qquad (a, c \in A, b, d \in B).$$

Furthermore $A\hat{\otimes}B$ is a Banach algebra with respect to this product.

Proof: See [1, Proposition 17 (p.235) and Proposition 18 (p.236)]. □

Let A be a Banach algebra. Then $A\hat{\otimes}A$ is a Banach A-bimodule with module multiplications determined by

$$a.(b \otimes c) = ab \otimes c,$$
$$(b \otimes c).a = b \otimes ca \quad (a, b, c, d \in A).$$

It can be easily seen that if A has a bounded approximate identity $(e_i)_i$, then $(e_i)_i$ is also a bounded approximate identity for $A\hat{\otimes}A$ (as an A-bimodule) and hence $A\hat{\otimes}A$ will be a neo-unital A-bimodule.

Theorem 1.15. *Let A be a Banach algebra. Then there is a continuous linear mapping $\Psi : A^{**}\widehat{\otimes}A^{**} \longrightarrow (A\widehat{\otimes}A)^{**}$ such that for $a, b, c \in A$ and for $m \in A^{**}\widehat{\otimes}A^{**}$ the following hold:*

(i) $\Psi(a \otimes b) = a \otimes b$;

(ii) $a.\Psi(m) = \Psi(a.m)$;

(iii) $\Psi(m).a = \Psi(m.a)$;

(iv) $(\pi_A)^{**}(\Psi(m)) = \pi_{A^{**}}(m)$.

Proof: See [7, Lemma 1.7]. □

1.2 Amenable and weakly amenable Banach algebras

In this section, we give some basic definitions on the notion of amenability and weak amenability and some theorems that we will use later.

Definition 1.16. Let A be a Banach algebra and X a Banach A-bimodule. A linear mapping D from A into X is a derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in A).$$

We denote the set of all continuous derivations from A into X by $Z^1(A, X)$. Let X be a Banach A-bimodule and $x \in X$. Then the mapping $ad_x : A \longrightarrow X$ defined by $ad_x(a) = a.x - x.a$ is a continuous derivation. A derivation D is said to be inner if there exists $x \in X$ such that $D = ad_x$.

The question that comes to mind is that under which conditions a (continuous) derivations is inner. The following is one instance:
Let A be an algebra and X be an A-bimodule. Then A acts trivially on left (right) of X if $a.x = 0$ (respectively $x.a = 0$), for all $a \in A$, $x \in X$.

Theorem 1.17. *Let A be a Banach algebra with a two-sided bounded approximate identity and let X be a Banach A-bimodule such that A acts trivially on one side. Then every continuous derivation from A into X^* is inner.*

Proof: See [12, Proposition 1.5]. $\square$

Definition 1.18. A Banach algebra A is said to be amenable if for every Banach A-bimodule X , any continuous derivation from A into the dual Banach A-bimodule X^* is inner.

If our algebra has a (two-sided) bounded approximate identity, then for checking amenability we do not need to consider (continuous) derivations into all dual Banach modules X^*. In the next theorem we will see that we need only to consider continuous derivations into duals of neo-unital modules.

Theorem 1.19. *Let A to be a Banach algebra with a (two-sided) bounded approximate identity. Then A is amenable if and only if every continuous derivation from A into the dual of a neo-unital Banach A-bimodule is inner.*

Proof: See [12, Proposition 1.8]. $\square$

Theorem 1.20. *The Banach algebra A is amenable if and only if $A^{\#}$ is amenable.*

Proof: See [4, Proposition 2.8.58]. $\square$

Theorem 1.21. *Let A to be an amenable Banach algebra. Then A has a bounded approximate identity.*

Proof: See [19, Proposition 2.2.1].

Let $\pi : A\hat{\otimes}A \longrightarrow A$ be the so-called multiplication specified by $\pi(a \otimes b) = ab$.

Definition 1.22. A net $(m_i)_i$ in $A\hat{\otimes}A$ is an approximate diagonal if for all $a \in A$,

$$a.m_i - m_i.a \longrightarrow_i 0,$$

$$a\pi(m_i) \longrightarrow_i a.$$

Definition 1.23. An element $M \in (A\hat{\otimes}A)^{**}$ is said to be a virtual diagonal if for all $a \in A$,

$$a.M = M.a,$$

$$\pi^{**}(M)a = a.$$

In the next theorem , we will see a characterization of the amenability of A in terms of the A-bimodule $A\hat{\otimes}A$.

Theorem 1.24. *For a Banach algebra A the followings are equivalent:*

(i) A is amenable.

(ii) A has a bounded approximate diagonal.

(iii) A has a virtual diagonal.

Proof: See [19, Theorem 2.2.4 on page 45]. $\square$

Theorem 1.25. *Let A and B be amenable Banach algebras. Then $A\hat{\otimes}B$ is also amenable.*

Proof: See [12, Proposition 5.4]. $\square$

Definition 1.26. Let X to be a Banach A-bimodule. We define

$$Z_A(X^*) := \bigcap_{a\in A}\{f \in X^* : a.f = f.a\}.$$

Now we state another characterization of amenable Banach algebras.

Theorem 1.27. *For a Banach algebra A, the followings are equivalent:*

(i) A is amenable.

(ii) For any Banach A-bimodule X and any Banach A-submodule Y of X, each linear functional $f \in Z_A(Y^*)$ has an extension to a linear functional in $Z_A(X^*)$.

(iii) For any Banach A-bimodule X, there exists a projection P from X^* onto $Z_A(X^*)$ that commutes with any weak* continuous and bounded operator from X^* into X^* commuting with the action of A on X^*.

Proof: See [15, Theorem 1]. □

Let A be an algebra and suppose that X,Y and Z are left, right or two-sided A-modules and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are module morphisms. Then the sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is short exact if f is one-to-one, g is onto and im(f)=ker(g). A short exact sequence is said to be admissible if there is a bounded linear map $F : Y \longrightarrow X$ such that $Ff = Id_X$ and splits if additionally F can cab=n be chosen to be a module morphism.

We can relate amenability of A to the short exact sequences as follows:

Theorem 1.28. *Let A be an amenable Banach algebra, and let*

$$\Sigma : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an admissible short exact sequence of left or right Banach A-modules with X^* a *dual module. Then Σ splits.*

Proof: See [3, Theorem 2.3]. $\square$

Theorem 1.29. *Let A be a Banach algebra with a bounded approximate identity, X a left(or right) Banach A-module, f a left (or right) A-module morphism of A onto X with kernel J. Then the exact sequence*

$$\Sigma^* : 0 \longrightarrow X^* \xrightarrow{f^*} A^* \xrightarrow{\iota^*} J^* \longrightarrow 0$$

splits as a sequence of right (left) A-modules if and only if the left (right) ideal J has a bounded right (left) approximate identity.

Proof: See [3, Proposition 3.5]. $\square$

For the next theorem we let A^{op} be the same space as A but with reversed product and $\pi : A\hat{\otimes}A^{op} \longrightarrow A$ be the multiplication map specified by

$$\pi(a \otimes b) = ab \qquad (a, b \in A).$$

In this case ker π will be a closed subalgebra of $A\hat{\otimes}A^{op}$. Indeed kerπ will be a closed left ideal in $A\hat{\otimes}A^{op}$ and we have the following theorem.

Theorem 1.30. *The followings are equivalent:*

(i) *A is amenable.*

(ii) *A has a bounded approximate identity and* kerπ has a bounded right approximate identity.

Proof: See [3,Theorem 3.10]. $\square$

In the next Theorem, we see that the amenability is preserved by continuous homomorphisms with dense range.

Theorem 1.31. *Let A be an amenable Banach algebra and φ be a continuous homomorphism from A into a Banach algebra B such that the range of φ is dense. Then B is also amenable.*

Proof: See [1, Proposition 11 on page 244]. $\square$

Definition 1.32. The Banach algebra A is said to be weakly amenable if every continuous derivation from A into the dual Banach A-bimodule A^* is inner.

Theorem 1.33. *Let A and B be commutative Banach algebras and φ be a continuous homomorphism from A into B with dense range. If A is weakly amenable, then B is also weakly amenable.*

Proof: See [11, Proposition 2.1]. $\square$

1.3 Generalized notions of amenability

In this section we introduce some generalized notions of amenability such as approximate amenability,bounded approximate amenability, pseudo amenability and approximate contractibility. Throughout this section X is a Banach A-bimodule.

Definition 1.34. A derivation $D : A \longrightarrow X$ is approximately inner if there exists a net $(\xi_i)_i$ in X such that for every $a \in A$, $D(a) = \lim_i a.\xi_i - \xi_i.a$ where the limit is taken in norm.

If X is a dual module and the above limit exists in the weak* topology, then we say that D is weak* approximately inner.

Definition 1.35. The Banach algebra A is approximately amenable if for every Banach A-bimodule X, every continuous derivation D from A into the dual Banach A-bimodule, X^* is approximately inner.

In a similar way, A is weak* approximately amenable if every continuous derivation from A to the dual Banach module X^* is weak* approximately inner. Now we state a characterization for approximate amenability.

Theorem 1.36. *The Banach algebra A is approximately amenable if and only if either of the following equivalent conditions hold:*

(i) There is net $(M_i) \subset (A^{\#}\hat{\otimes}A^{\#})^{**}$ such that for each $a \in A^{\#}$, $a.M_i - M_i.a \longrightarrow 0$ and $\pi^{**}(M_i) \longrightarrow 1$;

(ii) There is a net $(M_i) \subset (A^{\#}\hat{\otimes}A^{\#})^{**}$ such that for each $a \in A^{\#}$, $a.M_i - M_i.a \longrightarrow 0$ and $\pi^{**}(M_i) = 1$.

Proof: See [6, Theorem 2.1]. □

Definition 1.37. The Banach algebra A is approximately contractible if every continuous derivation from A into every Banach A-bimodule X is approximately inner.

In fact the three notions of approximate amenability, weak* approximate amenability and approximate contractibility are equivalent.

Theorem 1.38. *The following are equivalent:*

(i) A *is approximately contractible;*

(ii) A *is approximately amenable;*

(iii) A *is weak** *approximately amenable.*

Proof: See [8, Theorem 2.1]. □

Definition 1.39. The Banach algebra A is said to be boundedly approximately amenable if for every continuous derivation D from A into the dual of any Banach A-bimodule X, there exists $M > 0$ and a net $(\xi_i)_i$ such that for every $a \in A$,

$$D(a) = \lim_i a.\xi_i - \xi_i.a$$

$$\|a.\xi_i - \xi_i.a\| \leq M\|a\| \qquad (i \in I)$$

Proposition 1.40. *The Banach algebra* A *is boundedly approximately amenable if and only if there exists a constant* $M > 0$ *such that for any Banach* A*-bimodule* X*, and any continuous derivation* $D : A \longrightarrow X^*$ *there is a net* $(\xi_i)_i \subset X^*$ *such that*

(1) $\sup_i \|ad_{\xi_i}\| \leq M\|D\|$

(2) $D(a) = \lim_i ad_{\xi_i}(a) \quad (a \in A)$

Proof: See [8, Proposition 5.3]. □

Theorem 1.41. *Suppose that the Banach algebra A is boundedly approximately amenable. Then there is a net $(M_i)_i \subset (A^{\#}\hat{\otimes}A^{\#})^{**}$ and a constant $L > 0$ such that for each $a \in A^{\#}$, $a.M_i - M_i.a \longrightarrow 0$ and $\pi^{**}(M_i) \longrightarrow 1$ and $\|a.M_i - M_i.a\| \leq L\|a\|$. Conversely if A has the latter property and $(\pi^{**}(M_i))$ is bounded, then A is boundedly approximately amenable.*

Proof: See [8, Theorem 5.4]. $\square$

Definition 1.42. The Banach algebra A is said to be pseudo-amenable if it has an approximate diagonal.

Theorem 1.43. *The following are equivalent:*

(i) A has an approximate diagonal $(m_i)_i$ such that $(\pi(m_i))_i$ is bounded.

(ii) A is pseudo-amenable and has a bounded approximate identity.

(iii) A is approximately amenable and has a bounded approximate identity.

Proof: See [9, Proposition 3.2]. $\square$

Theorem 1.44. *The Banach algebra A is (boundedly) approximately amenable if and only if $A^{\#}$ is (boundedly) approximately amenable.*

Proof: See [6, Proposition 2.4 and 8, Lemma 5.9]. $\square$

1.4 Approximation Property

In this section, we introduce a Banach space property known as the approximation property.

Definition 1.45. A Banach space is said to have the approximation property if its identity operator can be uniformly approximated on every compact subset K by

finite-rank operators.(i.e., for every $\epsilon > 0$, there is a finite-rank operator T(depending on K and ϵ) such that $\|T(x) - x\| < \epsilon$ for every $x \in K$).
It is said to have bounded approximation property if there exists a number $C > 0$, independent of K and ϵ such that T can be chosen such that $\|T\| \leq C$. In this case we say that the Banach space has C-approximation Property.
A sequence $\{x_n\}$ in a Banach space X is called a Schauder basis of X if if for every $x \in X$, there is a unique sequence $(a_n)_n$ such that $x = \sum_n a_n x_n$. [16, definition 1.a.1].
Every Banach space with a (Schauder) basis has the bounded approximation property. [16, page 30].

For a Banach space E, let $K(E)$ denotes the space of compact operators on E , $F(E)$ denote the space of finite-rank operators on E and $A(E)$ denotes the closure of $F(E)$ in $B(E)$.

Theorem 1.46. *Suppose that E has the approximation property. Then $K(E)$ has a left approximate identity belonging to $F(E)$. In particular $K(E) = A(E)$.*

Proof: See [19, Proposition 3.1.1].

Theorem 1.47. *Let E to be a Banach space. Then E has the C-approximation property for some finite number $C > 1$ if and only if $K(E)$ has a bounded left approximate identity bounded by C belonging to $F(E)$.*

Proof: See [19, Theorem 3.1.2]. □

Theorem 1.48. *For a Banach space E the followings are equivalent:*

(i) E^* has the bounded approximation property.

(ii) $A(E)$ has a bounded approximate identity.

Proof: See [19, Corollary 3.1.5]. □

Theorem 1.49. *If E^* has the approximation property, then E has the approximation property.*

Proof: See [19, Corollary C.1.6]. □

Now suppose that E^* has the approximation property. Then by Theorem 1.49, E has the approximation property and therefore from Theorem 1.46, $A(E) = K(E)$. Then from Theorem 1.48, $K(E)$ has a bounded approximate identity. Hence we have the following corollary.

Corollary 1.50. *Let E to be a Banach space such that E^* has the bounded approximation property. Then $K(E)$ has a bounded approximate identity.*

2 Some notes on amenability

In this section we obtain some new results in amenability. In section 1 we introduced some definitions and known theorems in amenability.

2.1 The relation between the amenability of tensor products of Banach algebras with amenability of the original algebras

For amenable Banach algebras A and B, by Theorem 1.25, we know that $A\widehat{\otimes}B$ is amenable. In this section, we study the converse. We mainly concentrate on the special case where $A = B$. However, we will also obtain some results in the case where A is not necessarily equal to B. The following is earlier work due to B.E. Johnson:

Proposition 2.1. *Suppose that A is a Banach algebra and B is a Banach algebra such that there exists $b_0 \in B$ with $b_0 \notin \overline{\mathrm{Lin}\{bb_0 - b_0b : b \in B\}}$. If $A\widehat{\otimes}B$ is amenable then A is amenable.*

Proof: See [14, Proposition 3.5]. $\square$

The following theorem was brought to our attention by F.Ghahramani.

Theorem 2.2. *Suppose that A and B are Banach algebras and B has a non- zero character. If $A\widehat{\otimes}B$ is amenable, then A is also amenable.*

Proof: Let φ be a non- zero character in B and define the unique mapping $\theta : A\widehat{\otimes}B \longrightarrow A$ acting on elementary tensors by

$$\theta(a \otimes b) = \varphi(b)a \quad (a \in A, b \in B).$$

We show that θ is an algebra homomorphism (obviously θ is continuous). Since θ is linear, it is enough to check this for elementary tensors. To see this we have

$$\theta((a \otimes b)(c \otimes d)) = \theta((ac \otimes bd) = \varphi(bd)ac.$$

On the other hand

$$\theta(a \otimes b)\theta(c \otimes d) = \varphi(b)a\varphi(d)c = \varphi(bd)ac.$$

So

$$\theta((a \otimes b)(c \otimes d)) = \theta(a \otimes b)\theta(c \otimes d).$$

And since φ is non - zero, θ is surjective and hence from Theorem 1.29, A is amenable. $\square$

If A is a Banach algebra, A^{op} denotes the same space with the reversed product $a \times b = ba$. Throughout the following we let $\pi : A\widehat{\otimes}A^{op} \longrightarrow A$ be the mapping specified by acting on elementary tensors by $\pi(a \otimes b) = ab \quad (a, b \in A)$ and we let $K = \ker\pi$.

The Banach algebra A can be made into a left $A\widehat{\otimes}A^{op}$-module by the module multiplication defined by

$$(a \otimes b).c = acb \qquad (a, b, c \in A).$$

Theorem 2.3. *Suppose that $A\widehat{\otimes}A^{op}$ is amenable and A has a bounded approximate identity. Then A is amenable.*

Proof: Since A has a bounded approximate identity, the short exact sequence $(\prod^{op})^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\widehat{\otimes}A^{op})^* \xrightarrow{\imath^*} K^* \longrightarrow 0$ is an admissible short exact sequence of right $A\widehat{\otimes}A^{op}$-modules. ($\imath$ is the inclusion map).See [3, Lemma 1.2].

Since A^* is a dual $A\widehat{\otimes}A^{op}$-module, from Theorem 1.28, $(\prod^{op})^*$ splits and since $A\widehat{\otimes}A^{op}$ has a bounded approximate identity and π is onto (since A has a bounded approximate identity), then Theorem 1.29 implies that K has a bounded right approximate identity. Now since A has a bounded approximate identity, from Theorem 1.30, A is amenable. $\square$

Theorem 2.3 has been the motivation for us to consider the question of under which conditions on the tensor products, A has a bounded approximate identity. The following is one of them. Before going to next theorem, we need a lemma.

Lemma 2.4. *Let A to be a Banach algebra with a two-sided bounded approximate identity and X a Banach A-bimodule on which A acts trivially on one side. Then for every continuous derivation D from A into X, there exists a bounded net $(\zeta_i)_i$ in X such that $D(a) = \lim_i a.\zeta_i - \zeta_i.a \quad (a \in A)$.*

Proof: Since we can embed X into X^{**} through the canonical injection, we can consider D as a continuous derivation into the dual module X^{**}. Also since the action of A on one side of X is trivial, the action of A on other side of X^* is trivial. Therefore by Theorem 1.17, D is inner. Hence there exists $\xi \in X^{**}$ such that

$$D(a) = a.\xi - \xi.a \qquad (a \in A).$$

Now by Goldstine's Theorem, there is a bounded net $(\tau_j)_{j\in J}$ in X converging to ξ in weak* topology of X^{**}. Thus

$$D(a) = a.\xi - \xi.a = \mathrm{wk}^* - \lim_{\mathrm{j}} \mathrm{a}.\tau_{\mathrm{j}} - \tau_{\mathrm{j}}.\mathrm{a} \qquad (\mathrm{a} \in \mathrm{A}),$$

and hence

$$D(a) = \mathrm{wk} - \lim_{\mathrm{j}} a.\tau_j - \tau_j.a \qquad (a \in A).$$

Let $\Delta = \{a_1, a_2, ..., a_n\}$ be a finite subset of A. Then in $\bigoplus_{i=1}^{n} X$, we have

$$(D(a_1), ..., D(a_n)) \in weak - cl(co(\{(a_1.\tau_j - \tau_j.a_1, ..., a_n.\tau_j - \tau_j.a_n) : j \in J\}))$$

Therefore by Mazur's Theorem

$$(D(a_1), ..., D(a_n)) \in \mathrm{norm} - \mathrm{cl}(\mathrm{co}(\{(a_1.\tau_j - \tau_j.a_1, ..., a_n.\tau_j - \tau_j.a_n)) : j \in J\})$$

and hence for $\epsilon > 0$, there exists $\zeta_{\Delta,\epsilon} \in \mathrm{co}(\{\tau_j : h \in J\})$, such that

$$\|D(a_i) - (a_i.\zeta_{\Delta,\epsilon} - \zeta_{\Delta,\epsilon}.a_i)\| < \epsilon \qquad (a_i \in \Delta).$$

So by ordering the set of the finite subsets of A by inclusion and positive real numbers by decreasing order $(\zeta_{\Delta,\epsilon})$ is the desired net. $\square$

Theorem 2.5. *Suppose that $A\widehat{\otimes}A^{op}$ has a bounded approximate identity and each one of the topologies on A defined by the family of seminorms $\rho_a : b \mapsto \|ab\|$ and $\gamma_a : b \mapsto \|ba\|$ is stronger than weak topology on A. Then A has a (two-sided) bounded approximate identity.*

Proof: Suppose that $A\widehat{\otimes}A^{op}$ has a bounded approximate identity. We consider A as an $A\widehat{\otimes}A^{op}$-bimodule by actions specified by:

$$(a \otimes b) \bullet c = acb$$
$$c \bullet (a \otimes b) = 0 \quad (a, b, c \in A)$$

It can be easily seen that A is a Banach $A\widehat{\otimes}A^{op}$-bimodule with the actions above. Now we define a derivation $D : A\widehat{\otimes}A^{op} \longrightarrow A$ by acting on elementary tensors as $D(a \otimes b) = ab$. D is obviously continuous and also D is a derivation since

$$D((a \otimes b) \cdot (c \otimes d)) = D(ac \otimes db) = acdb$$

($\cdot$ is the product in $A\widehat{\otimes}A^{op}$). On the other hand:

$$(a \otimes b) \bullet D(c \otimes d) + D(a \otimes b) \bullet (c \otimes d) = (a \otimes b) \bullet cd = acdb$$

Therefore $D \in \mathbb{Z}^1(A\widehat{\otimes}A^{op}, A)$. Now since the right action of $A\widehat{\otimes}A^{op}$ on A is trivial and $A\widehat{\otimes}A^{op}$ has a bounded approximate identity, from Lemma 2.4, there exists a bounded net$(\zeta_i)_i$ in A such that $D(a \otimes b) = \lim_i ad_{\zeta_i}(a \otimes b)$.

So $ab = \lim_i a\zeta_i b$ and hence for all $a, b \in A$:

$$\lim_i a(b - \zeta_i b) = 0 \qquad \lim_i (b - b\zeta_i)a = 0 \tag{1}$$

If we denote the topology induced by the family of seminorms $\{\rho_a | a \in A\}$ by τ and the topology induced by the family of seminorms $\{\gamma_a | a \in A\}$ by ς, then from (1) we have:

$$a\zeta_i \longrightarrow a \quad (in\ \tau\ for\ \ all\ a \in A) \tag{2}$$

$$\zeta_i a \longrightarrow a \quad (in\ \varsigma\ for\ \ all\ a \in A) \tag{3}$$

We assumed both τ and ς to be stronger than the weak topology on A, so by (2) and (3) we have A has a weakly two-sided bounded approximate identity and hence A has a two-sided bounded approximate identity. $\square$

Remark Due to the proof of Theorem 2.5 we may impose weaker conditions and obtain weaker conclusions as below:

Theorem $\acute{2}$ *Suppose that $A\widehat{\otimes}A^{op}$ has a (two-sided) bounded approximate identity and the topology induced on A by the family of seminorms $\{\rho_a|a \in A\}$ where $\rho_a(b) = \|ab\|$ is stronger than the weak topology on A. Then A has a left bounded approximate identity.*

Theorem 2.6. *Suppose that $A\widehat{\otimes}A^{op}$ is amenable and that A has the property that each one of the topologies induced on A by the family of seminorms $\{\rho_a|a \in A\}$ where $\rho_a(b) = \|ab\|$ and $\{\gamma_a|a \in A\}$ where $\gamma_a(b) = \|ba\|$, are stronger than the weak topology on A. Then A is amenable.*

Proof: Firstly by the fact that $A\widehat{\otimes}A^{op}$ necessarily has a (two-sided) bounded approximate identity, from Theorem 2.5 we have that A has a (two-sided) bounded approximate identity and then from Theorem 2.3 we have A is amenable.$\square$

In the next Theorem we attempt to relate amenability of $A\widehat{\otimes}A$ (in the case that

A has a bounded approximate identity) to the amenability of $A\widehat{\otimes}A^{op}$ and then by using the preceding theorems, we attempt to prove the amenability of A when $A\widehat{\otimes}A$ is amenable. Before going to next theorem, we need a lemma.

Lemma 2.7. *Let A to be Banach algebra with a bounded approximate identity such that for any neo-unital Banach A-bimodule X and Y a closed submodule of X, every $f \in Z_A(Y^*)$ can be extended to a functional $\tilde{f} \in Z_A(X^*)$. Then A is amenable.*

Proof: As in the proof of [15, Theorem 1], for concluding the amenability of A, it is enough to have the property in the Lemma for the Banach A-bimodule $L = (A\widehat{\otimes}A)^*\widehat{\otimes}(A\widehat{\otimes}A)$ with the module actions

$$
\begin{aligned}
&a.(x^* \otimes x) = x^* \otimes a.x, \\
&(x^* \otimes x).a = x^* \otimes x.a \quad (a \in A, x \in (A\widehat{\otimes}A), x^* \in (A\widehat{\otimes}A)^*).
\end{aligned}
$$

Since A has a bounded approximate identity, $X = A\widehat{\otimes}A$ is neo-unital and hence by the above definition of the actions of A on L, L is also neo-unital. $\square$

Theorem 2.8. *Suppose that A is a Banach algebra with a bounded approximate identity such that $A\widehat{\otimes}A$ is amenable. Then $A\widehat{\otimes}A^{op}$ is also amenable.*

Proof: Suppose that X is a Banach neo-unital $A\widehat{\otimes}A^{op}$-bimodule and that $\bullet$ denotes the action of $A\widehat{\otimes}A^{op}$ on X. We define:

$$
\begin{aligned}
&(a \otimes b) \circ x = \lim_i (a \otimes e_i) \bullet x \bullet (e_i \otimes b), \\
&x \circ (a \otimes b) = \lim_i (e_i \otimes b) \bullet x \bullet (a \otimes e_i) \quad (x \in X \text{and } \ a, b \in A).
\end{aligned}
$$

First we note that the above limits exist because by the assumption that X is neo-unital we have:

If $x \in X$ then there exist $y \in X$ and $u, v \in A\widehat{\bigotimes}A^{op}$ such that $x = u \bullet y \bullet v$ and then we have

$$(a \otimes e_i) \bullet x \bullet (e_i \otimes b) = (a \otimes e_i) \bullet u \bullet y \bullet v \bullet (e_i \otimes b) = ((a \otimes e_i) \star u) \bullet y \bullet (v \star (e_i \otimes b)),$$

where $\star$ denotes the product in $A\widehat{\otimes}A^{op}$. Since $(e_i)_{i\in\Lambda}$ is a bounded approximate identity for A, it can be easily seen that $\lim_i (a \otimes e_i) \star u = a.u$ and $\lim_i v \star (e_i \otimes b) = v.b$, where $a.(e \otimes f) = ae \otimes f$ and $(e \otimes f).b = e \otimes bf$.
So $\lim_i (a \otimes e_i) \bullet x \bullet (e_i \otimes b)$ exists and we can similarly prove the existence of the second limit. Also $\circ$ makes X into a Banach $A\widehat{\bigotimes}A$-bimodule. To see the reason, by linearity, it is enough to check the module conditions for elementary tensors.

$$((a \otimes b)(c \otimes d)) \circ x = (ac \otimes bd) \circ x = \lim_i (ac \otimes e_i) \bullet x \bullet (e_i \otimes bd)$$

On the other hand:

$$\begin{aligned}
(a \otimes b) \circ ((c \otimes d) \circ x) &= (a \otimes b) \circ (\lim_j (c \otimes e_j) \bullet x \bullet (e_j \otimes d)) \\
&= \lim_i (a \otimes e_i) \bullet (\lim_j (c \otimes e_j) \bullet x \bullet (e_j \otimes d)) \bullet (e_i \otimes b) \\
&= \mathrm{lim_i lim_j} (ac \otimes e_j e_i) \bullet x \bullet (e_j e_i \otimes bd) \\
&= \mathrm{lim_i} (ac \otimes e_i) \bullet x \bullet (e_i \otimes bd).
\end{aligned}$$

Hence

$$((a \otimes b)(c \otimes d)) \circ x = (ac \otimes bd) \circ x = (a \otimes b) \circ ((c \otimes d) \circ x).$$

In a similar way we can show that

$$x \circ ((a \otimes b)(c \otimes d)) = (x \circ (a \otimes b))(c \otimes d).$$

Also we have:

$$\begin{aligned}((a\otimes b)\circ x)\circ(c\otimes d) &= \mathrm{lim_i}(e_i\otimes d)\bullet(\mathrm{lim_j}(a\otimes e_j)\bullet x\bullet(e_j\otimes b))\bullet(c\otimes e_i)\\ &= \mathrm{lim_i}\mathrm{lim_j}((e_i\otimes d)\star(a\otimes e_j))\bullet x\bullet((e_j\otimes b)\star(c\otimes e_i))\\ &= \mathrm{lim_i}\mathrm{lim_j}(e_ia\otimes e_jd)\bullet x\bullet(e_jc\otimes e_ib)\\ &= (a\otimes d)\bullet x\bullet(c\otimes b).\end{aligned}$$

On the other hand:

$$\begin{aligned}(a\otimes b)\circ(x\circ(c\otimes d)) &= \mathrm{lim_i}\mathrm{lim_j}(a\otimes e_i)\bullet((e_j\otimes d)\bullet x\bullet(c\otimes e_j))\bullet(e_i\otimes b)\\ &= \mathrm{lim_i}\mathrm{lim_j}((ae_j\otimes de_i)\bullet x\bullet(ce_i\otimes be_j)\\ &= (a\otimes d)\bullet x\bullet(c\otimes b).\end{aligned}$$

Hence

$$((a\otimes b)\circ x)\circ(c\otimes d)=(a\otimes b)\circ(x\circ(c\otimes d)).$$

So X is an $A\widehat{\otimes}A$-bimodule for the action $\circ$. Also since the net (e_i) is bounded, it can be easily seen that X is indeed a Banach $A\widehat{\otimes}A$-bimodule for $\circ$. For a Banach $A\widehat{\otimes}A^{op}$-bimodule X, $X_\dagger$ denotes X as an $A\widehat{\otimes}A$-bimodule (via the action $\circ$).
Now if Y is a closed submodule of X and $f\in Z_{A\widehat{\otimes}A^{op}}(Y^*)$, we show that $f\in Z_{A\widehat{\otimes}A}(Y_\dagger^*)$. To prove the above statement we have

$$\begin{aligned}(a\otimes b)\circ f &= \mathrm{wk}^*-\lim_{\mathrm{i}}(a\otimes e_i)\bullet f\bullet(e_i\otimes b)\\ &= \mathrm{wk}^*-\lim_{\mathrm{i}}\mathrm{f}\bullet(a\otimes e_i)\bullet(e_i\otimes b)\\ &= \mathrm{wk}^*-\lim_{\mathrm{i}}\mathrm{f}\bullet(ae_i\otimes be_i)\\ &= f\bullet(a\otimes b).\end{aligned}$$

Similarly

$$f \circ (a \otimes b) = (a \otimes b) \bullet f.$$

Thus

$$f \in Z_{A\widehat{\otimes}A}(Y_{\dagger}^{*}).$$

Now from Theorem 1.27, f has an extension to an $\tilde{f} \in Z_{A\widehat{\otimes}A}(X_{\dagger}^{*})$. We show that $\tilde{f} \in Z_{A\widehat{\otimes}A^{op}}(X^{*})$ For this purpose we have:

$$\begin{aligned}
(a \otimes b) \bullet \tilde{f} &= \mathrm{wk}^* - \lim_{\mathrm{i}} -\mathrm{wk}^* - \lim_{\mathrm{j}} ((a \otimes e_i)(e_j \otimes b)) \bullet \tilde{f} \bullet (e_i \otimes e_j) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} (a \otimes e_i)(\mathrm{wk}^* - \lim_{\mathrm{j}} (\mathrm{e_j} \otimes \mathrm{b}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_i} \otimes \mathrm{e_j})) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} (a \otimes e_i) \bullet (\tilde{f} \circ (e_i \otimes b)) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} (a \otimes e_i) \bullet ((e_i \otimes b) \circ \tilde{f}) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} (a \otimes e_i) \bullet (\mathrm{wk}^* - \lim_{\mathrm{j}} (\mathrm{e_i} \otimes \mathrm{e_j}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_j} \otimes \mathrm{b})) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} \mathrm{wk}^* - \lim_{\mathrm{j}} (ae_i \otimes e_j e_i) \bullet \tilde{f} \bullet (e_j \otimes b) \\
&= \mathrm{wk}^* - \lim_{\mathrm{i}} (a \otimes e_i) \bullet \tilde{f} \bullet (e_i \otimes b) \\
&= (a \otimes b) \circ \tilde{f}.
\end{aligned}$$

similarly we have $\tilde{f} \bullet (a \otimes b) = \tilde{f} \circ (a \otimes b)$ and since $\tilde{f} \in Z_{A\widehat{\otimes}A}(X_{\dagger}^{*})$, we have $(a \otimes b) \bullet \tilde{f} = \tilde{f} \bullet (a \otimes b)$and hence

$$\tilde{f} \in Z_{A\widehat{\otimes}A^{op}}(X^{*}).$$

Since Y was an arbitrary closed submodule of X and f was arbitrary in $Z_{A\widehat{\otimes}A^{op}}(Y^{*})$, from Lemma 2.7, we have that $A\widehat{\otimes}A^{op}$ is amenable. $\square$

Theorem 2.9. *Suppose that $A\widehat{\otimes}A$ is amenable and A has a bounded approximate identity. Then A is amenable.*

Proof: By the preceding Theorem we have that $A\widehat{\otimes}A^{op}$ is amenable. Since A has a bounded approximate identity, from Theorem 2.3, A is amenable. $\square$

Since having a bounded approximate identity is a necessary condition for an algebra to be amenable, Theorem 2.9 has the minimum conditions. If we can prove that amenability of $A\widehat{\otimes}A$ implies that A has a bounded approximate identity, then we can even drop the condition in Theorem 2.9 that A has a bounded approximate identity.

2.2 Some results in commutative Banach algebras

Now we go to the case where our algebra A is commutative. First we prove the following general result.
For the Banach algebra A, we define

$$A^2 = \mathrm{Lin}\{ab : a, b \in A\}.$$

Theorem 2.10. *Suppose that B is a Banach algebra and A is a closed subalgebra of B such that $A\widehat{\otimes}B$ is weakly amenable. Then $(A^2)^- = A$*

Proof: *Suppose* that $A\widehat{\otimes}B$ is weakly amenable and $(A^2)^- \neq A$. Then from the Hahn-Banach Theorem there exists a $\lambda \in A^*$ such that $\lambda|_{A^2} = 0$ and $\lambda \neq 0$. So there exists an $a_0 \in A$ such that $\lambda(a_0) = 1$. We denote a Hahn-Banach extension of λ on B by $\tilde{\lambda}$. So $\tilde{\lambda} \in B^*$ and we define:
$D : A\widehat{\otimes}B \longrightarrow (A\widehat{\otimes}B)^*$ by $D(a \otimes b) = \tilde{\lambda}(a)\tilde{\lambda}(b)(\tilde{\lambda} \otimes \tilde{\lambda})$
where $(\tilde{\lambda} \otimes \tilde{\lambda})(c \otimes d) = \tilde{\lambda}(c)\tilde{\lambda}(d)$.

Then D is a continuous derivation since

$$D((a \otimes b)(c \otimes d)) = D(ac \otimes bd) = \tilde{\lambda}(ac)\tilde{\lambda}(bd)(\tilde{\lambda} \otimes \tilde{\lambda}) = 0$$

On the other hand for $a, c, x \in A$ and $b, d, y \in B$ we have:

$$\langle (a \otimes b).D(c \otimes d), x \otimes y \rangle = \langle D(c \otimes d), xa \otimes yb \rangle = \tilde{\lambda}(c)\tilde{\lambda}(d)\tilde{\lambda}(xa)\tilde{\lambda}(yb) = 0$$

and similarly

$$\langle D(a \otimes b).(c \otimes d), x \otimes y \rangle = \langle D(a \otimes b), cx \otimes dy \rangle = \tilde{\lambda}(a)\tilde{\lambda}(b)\tilde{\lambda}(cx)\tilde{\lambda}(dy) = 0.$$

So $D \in Z^1(A\widehat{\otimes}B, (A\widehat{\otimes}B)^*)$ and hence from weak amenability of $(A\widehat{\otimes}B)$ it follows that $D = ad(\xi)$ for some $\xi \in (A\widehat{\otimes}B)^*$.
So

$$\begin{aligned} \langle D(a_0 \otimes a_0), (a_0 \otimes a_0) \rangle &= \langle (a_0 \otimes a_0).\xi - \xi.(a_0 \otimes a_0), a_0 \otimes a_0 \rangle \\ &= \langle \xi, (a_0^2 \otimes a_0^2) - (a_0^2 \otimes a_0^2) \rangle \\ &= 0 \end{aligned}$$

But we have:

$$\langle D(a_0 \otimes a_0), (a_0 \otimes a_0) \rangle = \tilde{\lambda}(a_0)\tilde{\lambda}(a_0)(\tilde{\lambda} \otimes \tilde{\lambda})(a_0 \otimes a_0) = (\tilde{\lambda}(a_0))^4 = 1,$$

which is a contradiction and hence $(A^2)^- = A$ $\square$

Theorem 2.11. *Suppose that B is a commutative Banach algebra and A is a closed ideal in B such that $A\widehat{\otimes}B$ is(weakly) amenable. Then A is (weakly) amenable.*

Proof Suppose that $A\widehat{\otimes}B$ is (weakly) amenable. Then we define $\varphi : A\widehat{\otimes}B \longrightarrow A$ by $\varphi(a \otimes b) = ab$. First it can be easily seen that φ is continuous and is an algebra homomorphism. Also by Theorem 2.10 we have $\varphi(A\hat{\otimes}B)^- = A$. So by Theorem 1.33, for weakly amenable case, and Theorem 1.31, for amenable case, A is (weakly) amenable. $\square$

2.3 Perturbation of Banach algebras and amenability

In this section we will see that if a Banach algebra is amenable with a given product, then it is amenable for any other multiplication close enough to the first one. This result has first been proven by B.E.Johnson in [13, Theorem 6.2]. Here we will give a proof for the main Theorem that is different from original proof in its second half. The advantage of our proof is that the neighborhood we find for multiplications does not depend on the algebra. Indeed we will show that if the difference between the multiplications (in norm), is less that $\frac{1}{11}$, then amenability will be preserved.

For two closed subspaces Y and Z of a Banach space X,their Hausdorff distance is defined by

$$d(Y, Z) = \max\{\sup\{d(y, Z) : \|y\| \leq 1\}, \sup\{d(z, Y) : \|z\| \leq 1\}\}$$

Lemma 2.12. *Let Y and Z be closed subspaces of a Banach space X. Suppose that there is a projection P of X onto Y with $\|P\| < d(Y, Z)^{-1} - 1$. Then P maps Z one* *to one onto* Y *and the inverse* α *of* $P|_Z$*satisfies* $(d = d(Y, Z))$

$$\|\alpha\| \leq (1 + d)(1 - \|P\|d)^{-1}$$

$$\|\alpha(y) - y\| \leq ((1 + d)(1 - \|P\|d)^{-1} - 1)\|y\|$$

$$\|P(z) - z\| \leq d(1 + \|P\|)\|z\|$$

Proof: See [13, Lemma 5.2].

Lemma 2.13. *Let* X_1 *and* X_2 *be Banach spaces and* $S, T \in B(X_1, X_2)$ *and let* S *be onto. Suppose that there exists* $K > 0$ *such that for all* $y \in X_2$*, there is* $x \in X_1$ *with* $\|x\| \leq K\|y\|$ *and* $S(x) = y$*. If* $K\|S - T\| < 1$*, then* T *will also be onto and for each* $y \in X_2$*, there exists* $x \in X_1$ *such that* $\|x\| \leq K(1 - K\epsilon)^{-1}\|y\|$ *and* $T(x) = y$*, where* $\epsilon = \|S - T\|$*.*

Proof: It is a special case of [13, Lemma 6.1].

Note: Suppose that π and ρ are two multiplications on a Banach algebra A. Then we have the following:

(i) Both π and ρ can be considered as continuous linear functionals from $A\widehat{\otimes}A^{op}$ into A that take the elementary tensors $a \otimes b$ into the product of a and b. $a_\pi b$ and $a_\rho b$ respectively denoted by $\pi(a \otimes b)$ and $\rho(a \otimes b)$.

(ii) If $\pi^\#$ and $\rho^\#$ are the multiplications respectively induced by π and ρ on $A^\#$ then we have

$$\|(\pi^\# - \rho^\#)((a, \alpha) \otimes (b, \beta))\| = \|a_\pi b - a_\rho b\| \leq \|\pi - \rho\|\|a\|\|b\| \quad (a, b \in A).$$

And hence

$$\|(\pi^{\#} - \rho^{\#})((a,\alpha) \otimes (b,\beta))\| \leq \|\pi - \rho\|\|(a,\alpha)\|\|(b,\beta)\|$$

Thus we have

$$\|\pi^{\#} - \rho^{\#}\| \leq \|\pi - \rho\|.$$

Theorem 2.14. *Suppose that (A,π) is an amenable Banach algebra. Then there exists $\epsilon > 0$ such that if ρ is another multiplication on A with $\|\rho - \pi\| < \epsilon$, then (A,ρ) is also amenable.*

Proof: By the note above, we can assume that A has and identity 1 for both multiplications π and ρ. Let $j : A \longrightarrow A\widehat{\bigotimes}A$ be defined by $j(a) = a \otimes 1$.
Then $\|j\| \leq 1$ and $\pi j = Id_A$. So $\pi^{**}j^{**} = Id_{A^{**}}$. It can be easily checked that $P = \mathrm{Id}_{(A\widehat{\bigotimes}A)^{**}} - j^{**}\pi^{**}$ is a projection onto $\ker\pi^{**}$ with norm at most 2.
By Lemma 2.13, and letting $X_1 = (A\widehat{\bigotimes}A)^{**}$and $X_2 = A^{**}$, $S_1 = \pi^{**}$,$T_1 = \rho^{**}$,by $K = 1$ (since$\|\mathrm{j}^{**}\| \leq 1$), we get that for $\|S_1 - T_1\| = \epsilon < 1$, ρ^{**} will be onto and for every $F \in \ker\pi^{**}$, there is $B \in (A\widehat{\bigotimes}A)^{**}$ such that $\rho^{**}(B) = \rho^{**}(F)$ and

$$\|B\| \leq (1-\epsilon)^{-1}\|\rho^{**}(F)\| = (1-\epsilon)^{-1}\|\rho^{**}(F) - \pi^{**}(F)\| \leq (1-\epsilon)^{-1}\epsilon\|F\|$$

So $F - B \in \ker\rho^{**}$ and $\|F - (F-B)\| = \|B\| \leq \epsilon(1-\epsilon)^{-1}\|F\|$. So that

$$\sup\{d(F, \ker\rho^{**}) : \mathrm{F} \in \ker\pi^{**}\text{and } \|\mathrm{F}\| \leq 1\} \leq \epsilon(1-\epsilon)^{-1}.$$

And similarly by changing the role of S_1 and T_1, we will obtain

$$\sup\{d(F, ker\pi^{**}) : F \in \ker\rho^{**}\text{and } \|\mathrm{F}\| \leq 1\} \leq \epsilon(1-\epsilon)^{-1}$$

Hence

$$d := d(\ker\pi^{**}, \ker\rho^{**}) \leq \epsilon(1-\epsilon)^{-1}.$$

So if $\epsilon < \frac{1}{4}$, then

$$\|P\| \leq 2 < (\epsilon(1-\epsilon)^{-1})^{-1} - 1 \leq d(\ker\pi^{**}, \ker\rho^{**})^{-1} - 1.$$

And hence by Lemma 2.12, there exists a linear homeomorphism α from $\ker\pi^{**}$ onto $\ker\rho^{**}$ such that

$$\|\alpha\| \leq (1-3\epsilon)^{-1}, \|\alpha^{-1}\| \leq \|P\| \leq 2$$

$$\|F - \alpha(F)\| \leq 3\epsilon(1-3\epsilon)^{-1}\|F\| \quad (F \in \ker\pi^{**})$$

$$\|F - \alpha^{-1}(F)\| \leq 3\epsilon(1-\epsilon)^{-1}\|F\| \quad (F \in \ker\rho^{**}).$$

Suppose that $F \in (A\widehat{\otimes}A)$ is an elementary tensor say $b \otimes c$ for $b, c \in A$. Then for $a \in A$, we have

$$\begin{aligned}\|a._\pi F - a._\rho F\| &= \|a._\pi(b\otimes c) - a._\rho(b \otimes c)\| \\ &= \|a_\pi b \otimes c - a_\rho b \otimes c\| = \|(a_\rho b - a_\pi b)\|\|c\| \\ &\leq \|\rho - \pi\|\|a \otimes b\|\|c\| \\ &\leq \epsilon\|a\|\|b\|\|c\| = \epsilon\|a\|\|F\|.\end{aligned}$$

So that

$$\|a._\pi F - a._\rho F\| \leq \epsilon\|F\|\|a\| \quad (a \in A, F \in A\widehat{\otimes}A).$$

And by using Goldsteine's Theorem, we have

$$\|a._\pi F - a._\rho F\| \leq \epsilon\|F\|\|a\| \quad (F \in (A\widehat{\otimes}A)^{**}) \qquad (\dagger)$$

Similarly

$$\|F._\pi a - F._\rho a\| \leq \epsilon\|a\|\|F\| \quad (a \in A, F \in (A\widehat{\otimes}A)^{**}).$$

Our proof will be different from the original proof from this part.

Now consider the derivation $D : A \longrightarrow \ker\pi^{**}(\cong (\ker\pi)^{**})$ by $D(a) = a \otimes 1 - 1 \otimes a$. Then amenability of (A, π) implies the existence of an element $\xi \in \ker\pi^{**}$ such that

$$a \otimes 1 - 1 \otimes a = a._\pi\xi - \xi._\pi a \quad (a \in A).$$

Let $\delta = \alpha(\xi) \in \ker\rho^{**}$. Then we have

$$\begin{aligned}\|a._\pi\xi - a._\rho\delta\| &= \|a._\pi\xi - a._\rho(\alpha(\xi))\| \\ &\leq \|a._\pi\xi - a._\pi(\alpha(\xi))\| + \|a._\pi(\alpha(\xi)) - a._\rho(\alpha(\xi))\| \\ &\leq 3\epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\| + \epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\|. \quad \text{(By properties of } \alpha \text{ and } (\dagger))\end{aligned}$$

And similarly

$$\|\xi._\pi a - \delta._\rho a\| \leq 4\epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\|.$$

so that

$$\begin{aligned}\|a \otimes 1 - 1 \otimes a - (a._\rho\delta - \delta._\rho a)\| &= \|a._\pi\xi - \xi._\pi a - (a._\rho\delta - \delta._\rho a)\| \\ &\leq \|a._\pi\xi - a._\rho\delta\| + \|\xi._\pi a - \delta._\rho a\| \\ &\leq 8\epsilon(1-3\epsilon)^{-1}\|a\|.\end{aligned}$$

So

$$\|a \otimes 1 - 1 \otimes a - (a._\rho\delta - \delta._\rho a)\| \leq O(\epsilon)\|a\| \qquad (a \in A). \qquad (\ddagger)$$

where $O(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$.

From now on all the multiplications we consider are respect to the multiplication ρ on A. We denote the multiplication in $A\widehat{\otimes}A^{op}$ by $\star_\rho$. Also we indicate the Arens product on $(A\widehat{\otimes}A^{op})^{**}$ with the same notation. So for elementary tensors,

$$(a \otimes b) \star_\rho (c \otimes d) = ac \otimes db$$

For $R = \sum_i a_i \otimes b_i \in \ker\rho$ we have

$$\begin{aligned} R \star_\rho \delta - R &= \sum_i (a_i \otimes b_i) \star_\rho \delta - \delta \sum_i a_i b_i - \sum_i a_i \otimes b_i + 1 \otimes \sum_i a_i b_i \\ &= \sum_i (a_i \cdot_\rho \delta - \delta \cdot_\rho a_i - a_i \otimes 1 + 1 \otimes a_i) \cdot_\rho b_i. \end{aligned}$$

So

$$\begin{aligned} \|R \star_\rho \delta - R\| &= \| \sum_i (a_i \cdot_\rho \delta - \delta \cdot_\rho a_i - a_i \otimes 1 + 1 \otimes a_i) \cdot_\rho b_i \| \\ &\leq \sum_i \| \frac{a_i}{\|a_i\|} \cdot_\rho \delta - \delta \cdot_\rho \frac{a_i}{\|a_i\|} + \frac{a_i}{\|a_i\|} \otimes 1 + 1 \otimes \frac{a_i}{\|a_i\|} \| \|a_i\| \|b_i\| \\ &\leq \|R\| \sup_{a \in A_1} \|a \cdot_\rho \delta - \delta \cdot_\rho a - a \otimes 1 + 1 \otimes a\|. \end{aligned}$$

Now if $R \in (\ker\rho)^{**}$, then by Goldstine's Theorem, there exists a net $(r_i)_i$ with $\|r_i\| \leq \|R\|$, in $\ker\pi$ such that $r_i \longrightarrow_i R \quad$ wk*. Note that since $\ker\rho^{**} \cong (\ker\rho)^{**}$, isometrically, for notational convenience, we do not disguise between δ as an element in $\ker\rho^{**}$ and its image as an element of $(\ker\rho)^{**}$.

Thus

$$r_i \cdot_\rho \delta - r_i \longrightarrow_i R \cdot_\rho \delta - R \quad \text{wk}^*.$$

And hence $\|R._\rho\delta - R\| \leq \sup_i \|r_i\cdot_\rho\delta - r_i\|$. So we have

$$\|R \star_\rho \delta - R\| \leq \|R\| \sup_{a \in A_1} \|a._\rho\delta - \delta._\rho a - a \otimes 1 + 1 \otimes a\| \quad (R \in (\ker\rho)^{**}).$$

And hence by (‡), we obtain

$$\|R \star_\rho \delta - R\| \leq O(\epsilon)\|R\| \qquad (R \in (\ker\rho)^{**}).$$

If we define $\lambda : (\ker\rho)^{**} \longrightarrow (\ker\rho)^{**}$ by $\lambda(S) = S \star_\rho \delta$, then for ϵ sufficiently small such that $O(\epsilon) < 1$, $\|\lambda - Id_{(\ker\rho)^{**}}\| < 1$ and thus λ will be invertible.
Since λ is surjective, there exists $x \in (\ker\rho)^{**}$ such that $\lambda(x) = \delta$. So $x \star_\rho \delta = \delta$ and hence for every $y \in (\ker\rho)^{**}$, we have $(y \star_\rho x - y) \star_\rho \delta = 0$. But this means that

$$\lambda(y \star_\rho x - y) = 0 \qquad (y \in (\ker\rho)^{**}).$$

Now by injectivity of λ, we have

$$y \star_\rho x = y \qquad (y \in (\ker\rho)^{**}).$$

Hence x will be a right identity for $(\ker\rho)^{**}$ and hence $\ker\rho$ has a bounded right approximate identity. So from Theorem 1.28, (A, ρ) is amenable. $\square$

Corollary 2.15. *Let (A, π) be an amenable Banach algebra. If ρ is another multiplication on A such that $\|\pi - \rho\| < \frac{1}{11}$, then (A, ρ) is also amenable.*

Proof: From the argument of the proof the Theorem above, (A, ρ) will be amenable if for $\epsilon = \|\pi - \rho\|$, $O(\epsilon) = 8\epsilon(1 - 3\epsilon)^{-1} < 1$ and this condition is satisfied under the assumption $\epsilon < \frac{1}{11}$. $\square$

3 A note on approximate amenability

3.1 Some General results about approximately amenable Banach algebras

In this section we will mention some results about approximately amenable Banach algebras and we will use them to prove some results about some specific Banach algebras in other sections. In the Preliminaries section, we saw some known results in approximate amenability that we will make use of to prove some of the results in this section.

Theorem 3.1. *Suppose that A is a Banach algebra with identity. Then the conditions below are equivalent:*
(i) A is approximately amenable.
(ii) The short exact sequence $\prod^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\widehat{\bigotimes}A)^* \xrightarrow{\imath^*} K^* \longrightarrow 0$ of A-bimodules approximately splits. i.e. there exists a net $(F_i)_i$, $F_i : (A\widehat{\bigotimes}A)^* \longrightarrow A^*$ of left inverses of π^* such that for all $a \in A$ and all $f \in (A\widehat{\bigotimes}A)^*$:
(1) $F_i(a.f) - a.F_i(f) \longrightarrow 0$;
(2) $F_i(f.a) - F_i(f).a \longrightarrow 0$.

Proof: Suppose that A is approximately amenable. Since A has an identity, A has an approximate diagonal $(m_i)_i \subset A\widehat{\bigotimes}A$ such that $\pi(m_i) = 1$, where we denote the identity of A by 1.
Now we define $F_i : (A\widehat{\bigotimes}A)^* \longrightarrow A^*$ by:

$$\langle F_i(f), a\rangle = \langle f, m_i.a\rangle \quad (f \in (A\widehat{\otimes}A)^*,\ a \in A).$$

So for every $f \in A^*$ we have:

$$\langle F_i(\pi^*(f)), a\rangle = \langle \pi^*(f), m_i.a\rangle = \langle f, \pi(m_i).a\rangle = \langle f, a\rangle.$$

So that

$$F_i\pi^* = \mathrm{Id}_{\mathrm{A}^*}$$

Also we have

$$\langle F_i(b.f) - b.F_i(f), a\rangle = \langle b.f, m_i.a\rangle - \langle f, m_i.ab\rangle = 0$$

and

$$\langle F_i(f.b) - F_i(f).b, a\rangle = \langle f.b, m_i.a\rangle - \langle f, m_i.ba\rangle = \langle a.f, b.m_i - m_i.b\rangle.$$

So if $\| a \| \leq 1$, then by choosing i such that $\| b.m_i - m_i.b \| < \epsilon$, we have

$$\| F_i(f.b) - F_i(f).b \| < \epsilon$$

which gives us the result. Conversely suppose that such a net $(F_i)_i$ exists. Then let $M_i = F_i^*(\hat{1}) \in (A\widehat{\bigotimes}A)^{**}$. So for $f \in (A\widehat{\bigotimes}A)^*$, we have

$$\langle a.M_i, f\rangle = \langle F_i(f.a), 1\rangle$$

and

$$\langle M_i.a, f\rangle = \langle F_i(a.f), 1\rangle.$$

For given $\epsilon > 0$, there exists i_0 such that for $i > i_0$,

$$|\langle F_i(a.f) - a.F_i(f), 1\rangle|, |\langle F_i(f.a) - F_i(f).a, 1\rangle| < \epsilon/2.$$

So

$$\begin{aligned} |\langle a.M_i - M_i.a, f\rangle| &= |\langle F_i(f.a) - F_i(a.f), 1\rangle| \\ &\leq |\langle F_i(a.f) - a.F_i(f), 1\rangle| + |\langle F_i(f.a) - F_i(f).a, 1\rangle| \\ &< \epsilon. \end{aligned}$$

Also for $g \in A^*$ we have:

$$\langle \pi^{**}(M_i), g\rangle = \langle M_i, \pi^*(g)\rangle = \langle F_i^*(\hat{1}), \pi^*(g)\rangle = \langle \hat{1}, F_i(\pi^*(g))\rangle = \langle \hat{1}, g\rangle = \langle g, 1\rangle.$$

Now if $\Phi \subset (A\widehat{\bigotimes}A)^*$, $\Omega \subset A^*$ and $\Lambda \subset A$ are finite sets, then by Goldstine's Theorem we have $m_{i_0}(i_0(\Phi, \Omega, \Lambda, \epsilon)) \in A\widehat{\bigotimes}A$ such that

$$|\langle a.m_{i_0} - m_{i_0}.a, f\rangle| < \epsilon, \;\; |\langle \pi(m_{i_0}) - 1, g\rangle| < \epsilon \quad (f \in \Phi, g \in \Omega, a \in \Lambda).$$

So there exists $(m_\lambda)_{\lambda \in I} \in A\widehat{\bigotimes}A$ such that

$$\mathrm{wk} - \lim_\lambda a.m_\lambda - m_\lambda.a = 0,$$

$$\mathrm{wk} - \lim_\lambda \pi(\mathrm{m}_\lambda) = 1.$$

Now for $F = \{a_1, a_2, ..., a_n\}$, a finite subset of A, we have in $(A\widehat{\bigotimes}A) \bigoplus ... \bigoplus (A\widehat{\bigotimes}A) \bigoplus A$:

$$(0, ..., 0, 1) \in \mathrm{wk} - \mathrm{cl}(\mathrm{co}\{(a_1.m_\lambda - m_\lambda.a_1, ..., a_n.m_\lambda - m_\lambda.a_n, \pi(m_\lambda))|\; \lambda \in I\}).$$

So by Mazur's Theorem, we have

$$(0, ..., 0, 1) \in \parallel . \parallel -\mathrm{cl}(\mathrm{co}\{(a_1.m_\lambda - m_\lambda.a_1, ..., a_n.m_\lambda - m_\lambda.a_n, \pi(m_\lambda))| \ \lambda \in I\}).$$

And so for given $\epsilon > 0$, there exists $n_{F,\epsilon} \in \mathrm{co}\{\mathrm{m}_\lambda | \lambda \in \mathrm{I}\}$ such that for $1 \leq i \leq n$,

$$\parallel a_i.n_{F,\epsilon} - n_{F,\epsilon}.a_i \parallel < \epsilon. \tag{2}$$

$$\parallel \pi(n_{F,\epsilon}) - 1 \parallel < \epsilon. \tag{3}$$

Hence there exists a net $(n_j)_j \subset A\widehat{\otimes}A$ such that for all $a \in A$:

$$\lim_j a.n_j - n_j.a = 0. \tag{4}$$

$$\lim_j \pi(n_j) = 1. \tag{5}$$

Hence A has an approximate diagonal which together with A having an identity, implies that A is approximately amenable. $\square$

Remark 3.2. In the proof of Theorem 3.1, if A is approximately amenable with identity, then the i_0 such that: $\parallel F_i(f.b) - F_i(f).b \parallel < \epsilon$ for $i > i_0$, was independent of f for $\parallel f \parallel \leq 1$, so that the convergences in Theorem 3.1 are uniform for $\parallel f \parallel \leq 1$.

So we have

Theorem 3.3. *Suppose that A is a Banach algebra with identity. Then these two conditions are equivalent:*
(i) A is approximately amenable.
(ii) The short exact sequence $\prod^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\widehat{\bigotimes}A)^* \xrightarrow{1^*} K^* \longrightarrow 0$ *of A-bimodules uniformly approximately splits. i.e. there exists a net* $(F_i)_i$, F_i :

$(A\widehat{\bigotimes}A)^* \longrightarrow A^*$ *of left inverses of* π^* *such that for all* $a \in A$ *and all* $f \in (A\widehat{\bigotimes}A)^*$ *:*

(1) $F_i(a.f) - a.F_i(f) \longrightarrow 0$*;*

(2) $F_i(f.a) - F_i(f).a \longrightarrow 0$*;*

where the convergences above are uniform for $\| f \| \leq 1$*.*

Proof: It is clear by Theorem 3.1 and Remark 3.2. $\square$

Corollary 3.4. *Suppose that* A *is a Banach algebra with identity. Then the short exact sequence* $\prod^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\widehat{\bigotimes}A)^* \xrightarrow{\imath^*} K^* \longrightarrow 0$ *of* A*-bimodules uniformly approximately splits if and only if approximately splits.*

Proof: *If* $\prod^*$ uniformly approximately splits, then it obviously approximately splits. Conversely if $\prod^*$ approximately splits, then by Theorem 3.1 , A is approximately amenable. Now by Theorem 3.3, $\prod^*$ uniformly approximately splits. $\square$

Proposition 3.5. *Let* A *to be a Banach algebra with identity. Also suppose that there exists a net* $(M_i) \subset (A\widehat{\bigotimes}A)^{**}$ *such that* $\mathrm{wk}^* - \lim(\mathrm{a.M_i} - \mathrm{M_i.a}) = 0$ *and* $\pi^{**}(M_i) = 1$ *for all* i*, where* 1 *is the identity of* A*. Then* A *is approximately amenable.*

Proof: Suppose that there is such a net, X is a unit-linked Banach A-bimodule and $D : A \longrightarrow X^*$ is a continuous derivation. Then we define $\mu_x \in (A\widehat{\bigotimes}A)^*$ by $\mu_x(a \otimes b) = \langle a.D(b), x\rangle$. If we define $f_i \in X^*$ by $f_i(x) = M_i(\mu_x)$, then we have:

$$\begin{aligned}\langle a.f_i - f_i.a, x\rangle &= \langle a.M_i - M_i.a, \mu_x\rangle + \langle \pi^{**}(M_i).D(a), x\rangle \\ &= \langle a.M_i - M_i.a, \mu_x\rangle + \langle D(a), x\rangle.\end{aligned}$$

So we have

$$\langle a.f_i - f_i.a - D(a), x\rangle = \langle a.M_i - M_i.a, \mu_x\rangle.$$

And hence

$$\lim_i \langle a.f_i - f_i.a - D(a), x\rangle = \lim_i \langle a.M_i - M_i.a, \mu_x\rangle = 0 \quad (x \in X)$$

which shows that D is wk* approximately inner and hence A is weak* approximately amenable. Therefore by Theorem 1.38, A is approximately amenable. □

In the next Theorem π denotes the multiplication map from $A^{\#}\widehat{\bigotimes}(A^{\#})^{op}$ into $A^{\#}$ and K denotes Kerπ.

Theorem 3.6. *For a Banach algebra A, the following statements are equivalent:*
(i) A is approximately amenable.
(ii) There exists a net $(u_i)_i \subseteq K$ such that $ku_i - k \longrightarrow_i 0$ for all $k \in K \cap (A \bigotimes A)$

Proof: Suppose that A is approximately amenable. Then from Theorem 1.38, A is approximately contractible. Suppose that 1 denotes the identity of $A^{\#}$. Then $D : A \longrightarrow K$ defined by $D(a) = a \otimes 1 - 1 \otimes a$ is a continuous derivation and hence is approximately inner. So there exists a net $(v_j)_j \subseteq K$ such that $D(a) = \lim_i (a.v_i - v_i.a)$. Now if $\Omega \subseteq K \cap (A \bigotimes A)$ is the finite set $\{k_1, k_2, ..., k_m\}$, for $k_r \in \Omega$, we have $k_r = \sum_{q=1}^{n} a_{q,r} \otimes b_{q,r}$ such that $\sum_{q=1}^{n} a_{q,r} b_{q,r} = 0$.

Then we have

$$k_r v_j - k_r = \textstyle\sum_{q=1}^{n} a_{q,r}.v_j.b_{q,r} - \sum_{q=1}^{n} a_{q,r} \otimes b_{q,r}$$

$$= \sum_{q=1}^{n} a_{q,r}.v_j.b_{q,r} - \sum_{q=1}^{n} v_j.a_{q,r}b_{q,r} - \sum_{q=1}^{n} a_{q,r} \otimes b_{q,r} + \sum_{q=1}^{n} 1 \otimes a_{q,r}b_{q,r}$$

So we have

$$\begin{aligned} k_r.v_j - k_r &= \sum_{q=1}^{n}(a_{q,r}.v_j - v_j.a_{q,r} - (a_{q,r} \otimes 1 - 1 \otimes a_{q,r})).b_{q,r} \\ &= \sum_{q=1}^{n}(a_{q,r}.v_j - v_j.a_{q,r} - D(a_{q,r})).b_{q,r}. \end{aligned}$$

Since $a_{q,r}v_j - v_j.a_{q,r} \longrightarrow_j D(a_{q,r})$ $(1 \leq q \leq n, 1 \leq r \leq m)$, for given $\epsilon > 0$, we can choose v_{j_0} such that

$$\| kv_{j_0} - k \| < \epsilon/2 \quad (k \in \Omega).$$

So we can find a net $(u_i)_i \subseteq K$ such that

$$ku_i - k \longrightarrow_i 0 \quad (k \in K \cap (A \otimes A)).$$

Conversely, suppose that there is such a net $(u_i)_i$. For every $a \in A^{\#}$, we have $a \otimes 1 - 1 \otimes a \in K \cap (A \bigotimes A)$. So

$$a \otimes 1 - 1 \otimes a = \lim_i (a \otimes 1 - 1 \otimes a)u_i.$$

So that:

$$a \otimes 1 - 1 \otimes a = \lim_i (a.u_i - u_i.a).$$

So by letting $v_i = 1 \otimes 1 - u_i \in (A^{\#} \widehat{\bigotimes} A^{\#})$, for $a \in A$ we have

$$\lim_i a.v_i - v_i.a = 0.$$

Also we have $\pi(v_i) = 1$ for all i. Hence $A^{\#}$ is pseudo amenable and since $A^{\#}$ has identity , from Theorem 1.43, $A^{\#}$ is approximately amenable. Hence A is approximately amenable. $\square$

Note: If A has a bounded approximate identity, then by the Cohen factorization Theorem, π will be surjective and hence by [4, Theorem A.3.48], we have

$$(\ker\pi)^{**} \cong \ker\pi^{**} = \mathrm{wk}^* - \mathrm{cl}(\ker\pi).$$

Also we note that if a net $(k_i)_i$ converges to L in $\sigma((\ker\pi)^{**}, (\ker\pi)^*)$ then $(k_i)_i$ also converges to $\theta(L)$ in $\sigma((A\widehat{\bigotimes}A)^{**}, (A\widehat{\bigotimes}A)^*)$ where θ denotes the isometric isomorphism from $(\ker\pi)^{**}$ onto $\ker\pi^{**}$ and also we have the converse statement. So whenever we talk about the convergence of a net $(k_i)_i \subseteq \ker\pi$ in the wk* topology, it doesn't matter that we take wk* topology as $\sigma((\ker\pi)^{**}, (\ker\pi)^*)$ or $\sigma((A\widehat{\bigotimes}A)^{**}, (A\widehat{\bigotimes}A)^*)$.
In the following Lemma and Theorem, K denotes the $\ker\pi$ where $\pi : A\widehat{\bigotimes}A^{op} \longrightarrow A$ is the multiplication map.

Lemma 3.7. *Suppose that A is a Banach algebra with a bounded approximate identity. Then every element L of K^{**} is wk* limit of a sequence*
$(k_i)_i \subseteq K \cap (A \bigotimes A)$.

Proof: *Take $L \in K^{**}$.* Then by Goldstine's Theorem, There exists a net $(k_g)_g \subseteq K$ such that

$$\mathrm{wk}^* - \lim_{\mathrm{g}} k_g = L \qquad (1)$$

Take $\epsilon > 0$. If $k_g = \sum_k a_k^g \otimes b_k^g$, then there exists $n_g \in \mathbb{N}$ such that

$$\sum_{k=n_g}^{\infty} \| a_k^g \| \| b_k^g \| < \epsilon.$$

Let $k_{g,\epsilon} = \sum_{k=1}^{n_g} a_k^g \otimes b_k^g$. Then we have

$$|\langle k_{g,\epsilon} - L, \phi\rangle| \leq |\langle k_{g,\epsilon} - k_g, \phi\rangle| + |\langle k_g - L, \phi\rangle| \leq \epsilon \parallel \phi \parallel + |\langle k_g - L, \phi\rangle| \quad (\phi \in (A\widehat{\otimes}A)^*).$$

Therefore for a finite subset Δ of $(A\widehat{\bigotimes}A)^*$, from (1) and the above inequality, there exists $k_{g_\Delta,\epsilon} \in K \cap (A \otimes A)$ such that

$$|\langle k_{g_\Delta,\epsilon} - L, \lambda\rangle| < \epsilon \quad (\lambda \in \Delta).$$

So if we order $\mathbb{R}^+$ by decreasing order and the finite subsets of $(A\widehat{\bigotimes}A)^*$ by inclusion, the net $(k_{g_\Delta,\epsilon}) \subseteq K \cap (A \bigotimes A)$ converges to L in the wk* topology. $\square$

Theorem 3.8. *Suppose that A is an approximately amenable Banach algebra with a bounded approximate identity. Then there exists a net $(u_i)_i \subseteq K$ such that*
$ku_i \longrightarrow_i k \quad (k \in K \cap (A \bigotimes A))$.

Proof: *Suppose* that A is approximately amenable with bounded approximate identity $(e_\alpha)_\alpha$. Then $(e_\alpha)_\alpha$ has a wk*-cluster point E in A^{**}. Without loss of generality, we can assume that wk$^* - \lim e_\alpha = E$. Thus E is a right identity for $(A^{**}, \square)$ and $a.E = E.a = a \quad \forall a \in A$.
Now we define the map $D : A \longrightarrow \ker(\pi^{**}) (\cong \mathrm{K}^{**})$ by $D(a) = \Psi(a \otimes E - E \otimes a)$ where $\Psi : A^{**}\widehat{\otimes}A^{**} \longrightarrow (A\widehat{\otimes}A)^{**}$ is as in the Theorem 1.15. Obviously D maps A into $\ker\pi^{**}$ since $\pi^{**}(\Psi(a\otimes E - E\otimes a)) = \pi_{A^{**}}(a\otimes E - E\otimes a) = aE - Ea = 0 \qquad (a \in A)$.
Obviously D defines a continuous derivation from A into K^{**} and hence approximate amenability of A implies the existence of a net $(v_j)_{j\in J} \subseteq K^{**}$ such that

$$D(a) = \lim_j (a.v_j - v_j.a) \quad (a \in A)$$

So we have

$$\Psi(a \otimes E - E \otimes a) = \lim_j (a.v_j - v_j.a).$$

Take $k = \sum_{t=1}^n a_t \otimes b_t \in K \cap A \otimes A$. So we have $\sum_{t=1}^n a_t b_t = 0$ and hence we have

$$kv_j - k = \sum_{t=1}^n a_t v_j b_t - \sum_{t=1}^n v_j.a_t b_t - \Psi(\sum_{t=1}^n a_t \otimes b_t - \sum_{t=1}^n E \otimes a_t b_t).$$

Thus we have

$$kv_j - k = \sum_{t=1}^n (a_t.v_j - v_j.a_t - \Psi(a_t \otimes E - E \otimes a_t)).b_t.$$

Since $a_t.v_j - v_j.a_t - \Psi(a_t \otimes E - E \otimes a_t) \longrightarrow_j 0 \ \ (1 \leq t \leq n)$, we have

$$\lim_j k.v_j - k = 0 \qquad (k \in K \cap (A \otimes A)). \qquad (\dagger)$$

Now for $\epsilon > 0$, a finite subset Φ of K^* and a finite subset $\Omega = \{k_1, k_2, ..., k_m\}$ of K and for any $j \in J$, by using Goldstine's Theorem, we can choose $w_{j,\Phi,\Omega} \in K$ such that

$$|\langle w_{j,\Phi,\Omega} - v_j, \phi.k\rangle| < \epsilon/2 \qquad (k \in \Omega, \phi \in \Phi). \qquad (\ddagger)$$

So if $k \in \Omega$ and $\phi \in \Phi$,then by ($\ddagger$), we have

$$\begin{aligned} |\langle kw_{j,\Phi,\Omega} - k, \phi\rangle| &\leq |\langle kw_{j,\Phi,\Omega} - kv_j, \phi\rangle| + |\langle kv_j - k, \phi\rangle| \\ &= |\langle w_{j,\Phi,\Omega} - v_j, \phi.k\rangle| + |\langle kv_j - k, \phi\rangle| \\ &\leq \epsilon/2 + |\langle kv_j - k, \phi\rangle| \end{aligned}$$

and hence by (†), we can choose $j_0 \in J$ such that:

$$|\langle kw_{j,\Phi,\Omega} - k, \phi\rangle| \leq \epsilon \quad j \geq j_0.$$

And hence there exists a net $(w_\lambda)_{\lambda\in\Lambda}$ in K such that

$$wk - \lim_\lambda (kw_\lambda - k) = 0 \quad (k \in K \cap (A \bigotimes A)).$$

Now in $\bigoplus_{s=1}^m K$, we have

$$(0, 0, ..., 0) \in \mathrm{wk} - \mathrm{cl}(\mathrm{co}(\{(k_1 w_\lambda - k_1, ..., k_m w_\lambda - k_m)|\lambda \in \Lambda\})).$$

And hence by Mazur's Theorem,

$$(0, 0, ..., 0) \in \mathrm{norm} - \mathrm{cl}(\mathrm{co}(\{(k_1 w_\lambda - k_1, ..., k_m w_\lambda - k_m)|\lambda \in \Lambda\})).$$

So for $\epsilon > 0$, we can choose $u = u_{\Omega,\epsilon} \in \mathrm{co}(\{w_\lambda|\lambda \in \Lambda\})$ such that

$$\| ku - k \| < \epsilon \quad (k \in \Omega).$$

So we can find a net $(u_i)_i \subseteq K$ such that

$$\lim_i ku_i - k = 0 \quad (k \in K \cap (A \otimes A)). \qquad \square$$

Before going to Corollary 3.10, we need the following proposition.

Proposition 3.9. *Let A be a Banach algebra with a bounded approximate identity and K to be the kernel of the multiplication map $\pi : A\widehat{\bigotimes}A^{op} \longrightarrow A$. Then for a given $\epsilon > 0$, every element $k \in K$ can be written as the sum of k_1,k_2 where $k_1, k_2 \in K$ and* *also* $k_1 \in A \otimes A$ *and* $\| k_2 \| < \epsilon$.

Proof: Suppose that A has a bounded approximate identity bounded by $M > 1$. Take $\epsilon > 0$ and $k \in K$. Then we can find $t_1 \in A \bigotimes A$ and $t_2 \in A \widehat{\otimes} A$ such that

$$k = t_1 + t_2 \quad and \quad \| t_2 \|_p < \epsilon/4M$$

where $\| \, . \, \|_p$ denotes the projective tensor norm.
Now by Theorem 1.7, there are $a, b \in A$ such that $\| a \| \leq M$ and $\| \pi(t_2) - b \| < \epsilon/4M$ and $\pi(t_2) = ab$.

We let

$$k_1 = t_1 + a \otimes b \ , \ \ k_2 = t_2 - a \otimes b.$$

So we have

$$\pi(k_1) = \pi(t_1) + ab = \pi(t_1) + \pi(t_2) = \pi(k) = 0.$$

Obviously we have $\pi(k_2) = 0$. Also we have

$$\|k_2\| \leq \|t_2\| + \|a\|\|b\|. \qquad (\dagger)$$

But we have $\|b\| \leq \epsilon/4M + \|\pi(t_2)\| < \epsilon/2M$. Hence by $(\dagger)$ we have:

$$\|k_2\| \leq \epsilon/2M + M\epsilon/2M < \epsilon.$$

And hence k_1, k_2 have the required properties. $\square$

Corollary 3.10. *Let A be an approximately amenable Banach algebra with a bounded approximate identity and e be an idempotent in A. Then for a finite subset F of* *eAe and given* $\epsilon > 0$*, there are* $a_1, b_1, ..., a_r, b_r \in A$ *such that*

$$\sum_{k=1}^{r} a_k b_k = e$$

and

$$\| \sum_{k=1}^{r} x a_k \otimes b_k - a_k \otimes b_k x \| < \epsilon \quad (x \in F).$$

Proof: Take $\epsilon > 0$. Since e is an idempotent, then for every $x \in F$, we have $xe - ex = 0$. So that the set $\Omega = \{x \otimes e - e \otimes x | x \in F\}$ is contained in $\ker\pi \cap (A \bigotimes A)$. Now by using Theorem 3.8, we easily see that there exists $k \in \ker\pi$ such that

$$\| (x \otimes e - e \otimes x) \circ k - (x \otimes e - e \otimes x) \| < \epsilon \quad (x \in F). \tag{1}$$

where $\circ$ is the multiplication in $A \widehat{\bigotimes} A^{op}$.
Then $k = k_1 + k_2$ where k_1 and k_2 are as in the Proposition 3.9 with $\epsilon' = \epsilon/4K$, where $K \geq 1$ and $K \geq Max\{\| x \| : x \in F\}$. Then by letting $d = e \otimes e - (e \otimes e) \circ k_1$ we have $d \in A \bigotimes A$ and $\pi(d) = e$.
Also we have:

$$\begin{aligned} x.d - d.x &= x.(e \otimes e - (e \otimes e) \circ (k + (k_1 - k)) - (e \otimes e - (e \otimes e) \circ (k + (k_1 - k)).x \\ &= x \otimes e - e \otimes x - (x \otimes e - e \otimes x) \circ k + (x \otimes e - e \otimes x) \circ (-k_2). \end{aligned}$$

By (1) and using the fact that $\| k_2 \| < \epsilon'$, we obtain:

$$\| x.d - d.x \| < \epsilon' + 2K\epsilon' < \epsilon \quad (x \in F). \tag{2}$$

So if $d = \sum_{i=1}^{r} a_i \otimes b_i$, we have

$$\sum_{i=1}^{r} a_i b_i = \pi(d) = e,$$

and by (2) we have

$$\| \sum_{i=1}^{r} x a_i \otimes b_i - a_i \otimes b_i x \| < \epsilon \qquad (x \in F).$$

So $a_1, b_1, ..., a_r, b_r$ have the desired property. $\square$

3.2 Characterization of approximately amenable and boundedly approximately amenable Banach algebras

Now we state a characterization for approximate amenability and one for bounded approximate amenability.

Theorem 3.11. *For a Banach algebra A the following conditions are equivalent:*

(i) A is approximately amenable.

(ii) For a Banach A-bimodule X and a Banach submodule Y of X, if $f \in Z_A(Y^*)$, then there exists a net $(g_i)_{i \in I} \subset X^*$ of extensions of f such that $\lim_i (a.g_i - g_i.a) = 0$.

(iii) For any Banach A-bimodule X, there exists a net $(P_i)_{i \in I}$, $P_i : X^* \longrightarrow X^*$ each P_i is a continuous operator and $P_i|_{Z_A(X^*)} = Id_{Z_A(X^*)}$. Also P_i commutes with every $\mathrm{wk}^* - \mathrm{wk}^*$ *continuous bounded operator from X^* into X^* commuting with the action of A on X^* and* $\lim_i (a.P_i(f) - P_i(f).a) = 0 \quad \forall f \in X^*$.

Proof: First since approximate amenability of A is equivalent to the approximate amenability of $A^{\#}$, and by using the fact that $Z_{A^{\#}}(X^*) = Z_A(X^*)$, it is enough to prove the Theorem for the case that A has an identity.

$(i) \Longrightarrow (ii)$ Suppose that $\tilde{f} \in X^*$ is an extension of f. Then $\delta(a) := a.\tilde{f} - \tilde{f}.a$ is a continuous derivation from A into $Y^{\perp}(\cong (\frac{X}{Y})^*)$. So approximate amenability of A results in the existence of a net $(h_i)_i \subset Y^{\perp}$ such that

$$\lim_i (a.h_i - h_i.a) = a.\tilde{f} - \tilde{f}.a.$$

Now let $g_i = \tilde{f} - h_i$. So we have

$$\lim_i (a.g_i - g_i.a) = a.\tilde{f} - \tilde{f}.a - \lim_i (a.h_i - h_i.a) = 0.$$

It is obvious that each g_i is an extension of f.

$(ii) \Longrightarrow (iii)$ Let $L = X^* \hat{\bigotimes} X$ and make L an A-bimodule by:

$$a.(f \otimes x) = f \otimes a.x;$$
$$(f \otimes x).a = f \otimes x.a \ \ (a \in A, f \in X^*, x \in X).$$

Now let H and K to be the closed linear span of the sets below, respectively,

$$\{T^*(f) \otimes x - f \otimes T(x) | f \in X^*, T \in \Omega, x \in X\}$$

$$\{f \otimes x | f \in Z_A(X^*), x \in X\},$$

where Ω is the set of all continuous operators on X that commute with the action of A on X.

Obviously H, K are Banach submodules L. So is the closed linear span of H, K say Y. Hence Y/H is a Banach A-submodule of L/H.

Now define $\varphi \in L^*$ by $\varphi(f \otimes x) = \langle f, x \rangle$ for $f \in X^*$ and $x \in X$. Then we have $\varphi(T^*(f) \otimes x - f \otimes T(x)) = \langle T^*(f), x \rangle - \langle f, T(x) \rangle = 0$.

So we have $\varphi \in H^{\perp}$ and hence we can define $\lambda \in (Y/H)^*$ by $\lambda(\bar{y}) = \phi(y)$.

Now if $f \in Z_A(X^*), x \in X$, we have

$$\begin{aligned}
\langle a.\lambda, \overline{f \otimes x} \rangle &= \langle \varphi, f \otimes x.a \rangle = \langle f, x.a \rangle = \langle f, a.x \rangle \\
&= \langle \varphi, f \otimes a.x \rangle = \langle \lambda, \overline{f \otimes a.x} \rangle = \langle \lambda, a.\overline{f \otimes x} \rangle \\
&= \langle \lambda.a, \overline{f \otimes x} \rangle.
\end{aligned}$$

We observe that if $\bar{y} \in Y/H$, then $\bar{y} = \bar{k}$ for some $k \in K$ and since every element $k \in K$ can be approximated by a linear combination of $f_i \otimes x_i$ where$f_i \in Z_A(X^*)$ and $x \in X$, we have $\lambda \in Z_A((Y/H)^*)$. Now by (ii), there is a net $(\tilde{\lambda_i})_i \subset (L/H)^*$ of extensions of λ such that

$$\lim_i (a.\tilde{\lambda_i} - \tilde{\lambda_i}.a) = 0.$$

Now we define $P_i : X^* \longrightarrow X^*$ by:

$$\langle P_i(f), x \rangle = \langle \tilde{\lambda_i}, \overline{f \otimes x} \rangle \qquad (f \in X^*, x \in X).$$

If $f \in Z_A(X^*)$, then we have

$$\langle P_i(f), x \rangle = \langle \tilde{\lambda_i}, \overline{f \otimes x} \rangle = \langle \tilde{\lambda}, \overline{f \otimes x} \rangle = \langle f, x \rangle.$$

So $P_i|_{Z_A(X^*)} = \mathrm{Id}_{\mathrm{Z_A(X^*)}}$.

Now suppose that $T : X^* \longrightarrow X^*$, is a $wk^* - wk^*$ continuous and bounded operator which commutes with the action of A on X^*. Then, since T is bounded $wk^* - wk^*$ continuous operator we have $T = S^*$ for some $S \in B(X)$. Also S has to be in Ω, because if $f \in X^*, a \in A$ and $x \in X$, we have

$$\begin{aligned}\langle f, S(a.x)\rangle &= \langle S^*(f), a.x\rangle = \langle T(f), a.x\rangle \\ &= \langle T(f).a, x\rangle = \langle T(f.a), x\rangle \\ &= \langle S^*(f.a), x\rangle = \langle f.a, S(x)\rangle \\ &= \langle f, a.S(x)\rangle.\end{aligned}$$

So

$$S(a.x) = a.S(x) \qquad (x \in X, a \in A).$$

Similarly we have $S(x.a) = S(x).a$. So $S \in \Omega$ and hence we have:

$$\langle P_i(T(f)), x\rangle = \langle \tilde{\lambda}_i, \overline{T(f) \otimes x}\rangle = \langle \tilde{\lambda}_i, \overline{S^*(f) \otimes x}\rangle.$$

But since $S \in \Omega$ we have $S^*(f) \otimes x - f \otimes S(x) \in H$ and hence

$$\langle \tilde{\lambda}_i, \overline{S^*(f) \otimes x}\rangle = \langle \tilde{\lambda}_i, \overline{f \otimes S(x)}\rangle = \langle P_i(f), S(x)\rangle = \langle S^*(P_i(f)), x\rangle = \langle T(P_i(f)), x\rangle.$$

So for all i,

$$P_i T = T P_i.$$

Also we have

$$\langle a.P_i(f) - P_i(f).a, x\rangle = \langle P_i(f), a.x - x.a\rangle = \langle \tilde{\lambda}_i, a.\overline{f \otimes (a.x - x.a)}\rangle$$
$$= \langle \tilde{\lambda}_i, a.\overline{f \otimes x} - \overline{f \otimes x}.a\rangle = \langle \tilde{\lambda}_i.a - a.\tilde{\lambda}_i, \overline{f \otimes x}\rangle.$$

So

$$|\langle a.P_i(f) - P_i(f).a, x\rangle| \leq \| \tilde{\lambda}_i.a - a.\tilde{\lambda}_i \| \|f\|\|x\|.$$

Hence

$$\|a.P_i(f) - P_i.(f)a\| \leq \|\tilde{\lambda}_i.a - a.\tilde{\lambda}_i\|\|f\|.$$

Now since $\lim_i(\tilde{\lambda}_i.a - a.\tilde{\lambda}_i) = 0$,

$$\lim_i(a.P_i(f) - P_i(f).a) = 0.$$

So $(P_i)_i$ has the required properties in (iii).

$(iii) \Longrightarrow (i)$ In part (iii), let $X = A\widehat{\bigotimes}A$. Then X is a Banach A-bimodule by the usual actions of A on $A\widehat{\bigotimes}A$. i.e.

$$a.(b \otimes c) = ab \otimes c \ , \ (b \otimes c).a = b \otimes ca \quad (a, b, c \in A).$$

Also suppose that the net $(P_i)_i$, $P_i : X^* \longrightarrow X^*$ has the properties mentioned in part (iii). Now we define $q : X^* \longrightarrow X^*$ by

$$\langle q(f), a \otimes b\rangle = \langle f, b \otimes a\rangle \quad (f \in X^*, a, b \in A).$$

Now let

$$M_i = q^*(P_i^*(1 \otimes 1)).$$

Then for $\phi \in (A\widehat{\bigotimes}A)^*$, we have

$$\langle M_i.a, \phi\rangle = \langle M_i, a.\phi\rangle = \langle q^*(P_i^*(1 \otimes 1)), a.\phi\rangle = \langle P_i^*(1 \otimes 1), q(a.\phi)\rangle.$$

But we have

$$\langle q(a.\phi), b \otimes c\rangle = \langle a.\phi, c \otimes b\rangle = \langle \phi, c \otimes ba\rangle = \langle q(\phi), ba \otimes c\rangle = \langle q(\phi), R_a(b \otimes c)\rangle,$$

where $R_a : A\widehat{\otimes}A \longrightarrow A\widehat{\otimes}A$ is defined by

$$R_a(b \otimes c) = ba \otimes c \quad (a, b, c \in A).$$

So we have

$$\langle q(a.\phi), b \otimes c\rangle = \langle R_a^*(q(\phi), b \otimes c\rangle.$$

And hence

$$q(a.\phi) = R_a^*(q(\phi) \quad (a \in A \text{ and } \phi \in (A\widehat{\otimes}A)^*).$$

Obviously R_a commutes with the action of A on $A\widehat{\otimes}A$. So by the assumptions of (iii), we have

$$P_i R_a^* = R_a^* P_i \quad (a \in A, i \in I).$$

Hence we have

$$\begin{aligned}\langle M_i.a,\phi\rangle &= \langle M_i, a.\phi\rangle = \langle P_i^*(1\otimes 1), q(a.\phi)\rangle = \langle 1\otimes 1, P_i(q(a.\phi)\rangle \\ &= \langle 1\otimes 1, P_i(R_a^*(q(\phi)))\rangle = \langle 1\otimes 1, R_a^*(P_i(q(\phi)))\rangle \\ &= \langle P_i(q(\phi)), a\otimes 1\rangle = \langle P_i(q(\phi)).a, 1\otimes 1\rangle.\end{aligned}$$

Similarly we have

$$\langle a.M_i,\phi\rangle = \langle a.P_i(q(\phi)), 1\otimes 1\rangle.$$

So

$$\lim_i \langle a.M_i - M_i.a,\phi\rangle = \lim_i \langle a.P_i(q(\phi)) - P_i(q(\phi)).a, 1\otimes 1\rangle = 0 \ \ \text{(by assumption)}$$

and hence

$$\mathrm{wk}^* - \lim_{\mathrm{i}}\,(a.M_i - M_i.a) = \mathit{0}.$$

Also for $f \in A^*$, we have:

$$\langle \pi^{**}(M_i), f\rangle = \langle P_i^*(1\otimes 1), q(\pi^*(f))\rangle.$$

Also we have

$$\begin{aligned}\langle a.q(\pi^*(f)), b\otimes c\rangle &= \langle q(\pi^*(f)), b\otimes ca\rangle = \langle \pi^*(f), ca\otimes b\rangle \\ &= \langle f, cab\rangle = \langle \pi^*(f), c\otimes ab\rangle \\ &= \langle q(\pi^*(f)), ab\otimes c\rangle = \langle q(\pi^*(f)).a, b\otimes c\rangle.\end{aligned}$$

So we have

$$q(\pi^*(f)) \in Z_A(X^*).$$

And hence we have

$$\langle \pi^{**}(M_i), f\rangle = \langle 1\otimes 1, P_i(q(\pi^*(f)))\rangle = \langle 1\otimes 1, q(\pi^*(f))\rangle = \langle \pi^*(f), 1\otimes 1\rangle = \langle f, 1\rangle.$$

Hence

$$\pi^{**}(M_i) = 1 \quad (i \in I).$$

Now by Proposition 3.5, A is approximately amenable. $\square$

Remark 3.12. Due to the proof of Theorem 3.11, we can replace (iii) by $(iii)'$ as follows:

$(iii)'$ For any Banach A-bimodule X, there exists a net $(P_i)_{i\in I}$, $P_i : X^* \longrightarrow X^*$ such that each P_i is a continuous operator and $P_i|_{Z_A(X^*)} = Id_{Z_A(X^*)}$. Also P_i commutes with every $\text{wk}^* - \text{wk}^*$ continuous bounded operator from X^* into X^* commuting with the action of A on X^* and $\lim_i(a.P_i(f) - P_i(f).a) = 0$ uniformly for $\| f \| \leq 1$.

To see this observe that when we made P_i in (ii), we had

$$\|a.P_i(f) - P_i(f).a\| \leq \|\tilde{\lambda}_i.a - a.\tilde{\lambda}_i\|\|f\|.$$

And hence

$$\lim_i(a.P_i(f) - P_i(f).a) = 0 \quad \text{uniformly for } \|f\| leq1.$$

Corollary 3.13. *For a Banach algebra A , these conditions are equivalent:*

(i) There exists a net $(P_i)_i$, $P_i : (A\widehat{\otimes}A)^* \longrightarrow (A\widehat{\otimes}A)^*$ such that each P_i is a continuous operator and $P_i|_{Z_A((A\widehat{\otimes}A)^*)} = Id_{Z_A((A\widehat{\otimes}A)^*)}$. Also P_i commutes with every $\text{wk}^* - \text{wk}^*$ *continuous bounded operator from $(A\widehat{\otimes}A)^*$ into $(A\widehat{\otimes}A)^*$ commuting*

with the action of A *on* $(A\widehat{\otimes}A)^*$ *and* $\lim_i(a.P_i(f) - P_i(f).a) = 0 \quad (f \in (A\widehat{\otimes}A)^*)$.

(ii) There exists a net $(P_i)_i$, $P_i : (A\widehat{\otimes}A)^* \longrightarrow (A\widehat{\otimes}A)^*$ *such that each* P_i *is a continuous operator and* $P_i|_{Z_A((A\widehat{\otimes}A)^*)} = Id_{Z_A((A\widehat{\otimes}A)^*)}$. *Also* P_i *commutes with every* $\mathrm{wk}^* - \mathrm{wk}^*$ *continuous bounded operator from* $(A\widehat{\otimes}A)^*$ *into* $(A\widehat{\otimes}A)^*$ *commuting with the action of* A *on* $(A\widehat{\otimes}A)^*$ *and* $\lim_i(a.P_i(f) - P_i(f).a) = 0$ *uniformly for* $f \in (A\widehat{\otimes}A)^*$ *with* $\|f\| \leq 1$.

Proof: $(i) \Longrightarrow (ii)$ By using the argument of Theorem 3.11, we see that A is approximately amenable and hence by the preceding Remark, (ii) holds.
$(ii) \Longrightarrow (i)$ Obvious. $\square$

Now we give a characterization for bounded approximate amenability.

Theorem 3.14. *For a Banach algebra* A *, the following are equivalent:*

(i) A *is boundedly approximately amenable;*

(ii) there exists $M > 0$ *such that for every Banach* A*-bimodule* X *and any Banach submodule* Y *of* X*, if* $f \in Z_A(Y^*)$*, then there exists a net* $(g_i)_{i\in I} \subset X^*$ *of extensions of* f *such that* $\lim_i(a.g_i - g_i.a) = 0$ *and* $\|a.g_i - g_i.a\| \leq 2M\|f\|\|a\| \quad (a \in A)$.

(iii) There exists $M > 0$ *such that for any Banach* A*-bimodule* X*, there exists a net* $(P_i)_{i\in I}$, $P_i : X^* \longrightarrow X^*$ *such that each* P_i *is a continuous operator and* $P_i|_{Z_A(X^*)} = Id_{Z_A(X^*)}$. *Also* P_i *commutes with every* $wk^* - wk^*$ *continuous bounded operator from*

X^* into X^* commuting with the action of A on X^*, and $\lim_i(a.P_i(f) - P_i(f).a) = 0$ *(uniformly for $\|f\| \leq 1$) and* $\|a.P_i(f) - P_i(f).a\| \leq 2M\|a\|\|f\| \quad (f \in X^*, a \in A)$.

Proof: Again, it is enough to prove the theorem for the case that A has an identity.

$(i) \Rightarrow (ii)$ From Proposition 1.40 and by the notations we used in the proof of Theorem 3.11, there exists $K > 0$ and a net $(h_i)_i \subseteq Y^\perp$ such that for $a \in A$,

$$\|a.h_i - h_i.a\| \leq K\|\delta\|\|a\|,$$

and

$$a.\tilde{f} - \tilde{f}.a = \lim_i (a.h_i - h_i.a).$$

But $\|\delta\| \leq 2\|\tilde{f}\| = 2\|f\|$. Hence

$$\|a.h_i - h_i.a\| \leq 2K\|f\|\|a\| \quad (a \in A).$$

For $a \in A$, by letting $g_i = \tilde{f} - h_i$ we have:

$$\|a.g_i - g_i.a\| \leq 2(K+2)\|f\| \; |a\|,$$

$$\lim_i a.g_i - g_i.a = 0.$$

So it is enough to let $M = K + 2$. Since K does not depend on the derivation δ and the module X, M is also independent of X.

$(ii) \Longrightarrow (iii)$ The proof of this part is exactly the same as the proof of the corresponding part in Theorem 3.11. Additionally when we find the net $(\lambda_i)_i \subseteq (L/H)^*$

of extensions of λ we can assume that

$$\|a.\tilde{\lambda} - \tilde{\lambda}_i.a\| \leq 2M\|\lambda\|\|a\| \quad (i \in I, a \in A).$$

Since $\|\lambda\| \leq 1$, we have

$$\|a.\tilde{\lambda} - \tilde{\lambda}_i.a\| \leq 2M\|a\|,$$

where M comes from part (ii) and hence does not depend on the Banach $A-$module X. So if we define $P_i : X^* \longrightarrow X^*$ by

$$\langle P_i(f), x\rangle = \langle \tilde{\lambda}_i, \overline{f \otimes x}\rangle \quad (f \in X^*, x \in X),$$

then by the argument in the proof of Theorem 3.11, P_i satisfies all of the properties given in the statement of the Theorem. To see the last property, due to the proof of Theorem 3.11, we have

$$\| a.P_i(f) - P_i(f).a \| \leq \| a.\tilde{\lambda} - \tilde{\lambda}_i.a \| \| f \| \leq 2M \| a \| \| f \| \; (f \in X^*, a \in A).$$

$(iii) \Longrightarrow (i)$ Suppose that there is such a net $(P_i)_i$. By the argument of the proof of Theorem 3.11 and defining M_i in a similar way, we have

$$|\langle a.M_i - M_i.a, \phi\rangle| = |\langle a.P_i(q(\phi)) - P_i(q(\phi)).a, 1 \otimes 1\rangle| \quad (a \in A, \phi \in (A\widehat{\otimes}A)^*).$$

So by part (iii),

$$|\langle a.P_i(q(\phi)) - P_i(q(\phi)).a, 1 \otimes 1\rangle| \leq 2M\|q(\phi)\|\|a\| \quad (a \in A, \phi \in (A\widehat{\otimes}A)^*).$$

Since $\|q\| \leq 1$, then we have

$$|\langle a.M_i - M_i.a, \phi\rangle| \leq 2M\|\phi\|\|a\| \quad (a \in A, \phi \in (A\widehat{\otimes}A)^*).$$

Hence we have

$$\|a.M_i - M_i.a\| \leq 2M\|a\| \quad (a \in A),$$

and

$$\lim_i \langle a.M_i - M_i.a, \phi\rangle = \lim_i \langle a.P_i(q(\phi)) - P_i(q(\phi)).a, 1\otimes 1\rangle = 0 \quad \text{(uniformly for } \|\phi\| \leq 1).$$

Hence

$$\lim_i a.M_i - M_i.a = 0 \quad (a \in A).$$

Also similar to the proof of Theorem 3.11, we have that $\pi^{**}(M_i) = 1$ for all i.
Hence by Theorem 1.41, A is boundedly approximately amenable. $\square$

3.3 Non-approximate amenability of certain classes of Banach algebras

In this section, we introduce a class of Banach algebras that are not approximately amenable. First we need some preliminaries.
The following result is based on a result of V.Runde and M.Daws in [5] for amenable Banach algebras. Here we extend that result to the case where our Banach algebra is approximately amenable.

Theorem 3.15. *Let E be a Banach space. Then the Banach algebra $B(l^p(E))$ is approximately amenable if and only if $l^\infty(B(l^p(E)))$ is approximately amenable.*

Proof: First we make a useful isometric isomorphism between $l^p(E)$ and $l^p(l^p(E))$. We define $\varphi : l^p(l^p(E)) \longrightarrow l^p(\mathbb{N}^2, E)$ by:

$$\varphi(a)(m,n) = (a(n))(m) \;\; for\ all\ a \in l^p(l^p(E)).$$

φ is obviously onto and linear and for a in $l^p(l^p(E))$ we have

$$\|\varphi(a)\|_p^p = \sum_{n,m} \|\varphi(a)(m,n)\|^p = \sum_n \sum_m \|(a(n))(m)\|^p = \sum_n \|a(n)\|_p^p = \|a\|_p^p.$$

So $\| \varphi \| = 1$. Now since $\mathbb{N}^2 \cong \mathbb{N}$, we have $l^p(l^p(E)) \cong l^p(E)$.
For notational convenience we denote the isometric isomorphism between $l^p(l^p(E))$and $l^p(E)$ by φ ($\varphi : l^p(l^p(E)) \longrightarrow l^p(E)$)
For each n we let $P_n : l^p(l^p(E)) \longrightarrow l^p(l^p(E))$ to be the projection onto the nth coordinate and we define $Q : B(l^p(l^p(E)) \longrightarrow B(l^p(l^p(E))$ by:

$$Q(T) = \sum_n P_n T P_n \quad\;\; (T \in B(l^p(l^p(E))).$$

Firstly we have the above sum is convergent in the strong operator topology (S.O.T).

Since for $X \in l^p(l^p(E))$ we have

$$\sum_{k=1}^{n} \|P_k T P_k(X)\|_p^p = \sum_{k=1}^{n} \|P_k(T(X_k))\|_p^p \leq \sum_{k=1}^{n} \|P_k\|^p \|T\|^p \|X_k\|_p^p,$$

where $X_k \in l^p(l^p(E))$ is defined by $(X_k)(n) = X(k)$ for $n = k$ and $(X_k)(n) = 0$ for $n \neq k$.

Since $\|P_k\| \leq 1$ for all k, we have

$$\sum_{k=1}^{n} \|P_k T P_k(X)\|_p^p \leq \|T\|^p \sum_{k=1}^{n} \|X_k\|_p^p \leq \|T\|^p \|X\|_p^p$$

and hence

$$\sum_{n=1}^{\infty} \|P_k T P_k(X)\|_p \leq \|T\| \|X\|_p.$$

Let $A = l^{\infty}(B(l^p(E))$. Then we can embed A into $B(l^p(l^p(E)))$ by identifying each element of A with a block diagonal matrix in $B(l^p(l^p(E)))$. So we have

$$(T(X))(n) = (T(n))(X(n)) \quad (T \in A \text{ and } \mathrm{X} \in \mathrm{l^p(l^p(E))}).$$

By definition of Q , it is obvious that Q maps $B(l^p(l^p(E)))$ onto A. In this case we have $(Q(T))(n) = P_n T_n$ where $T_n : l^p(E) \longrightarrow l^p(l^p(E))$ is defined by $T_n(x) = T(X)$ where $X \in l^p(l^p(E))$ is the vector with x in its nth coordinate and 0 in other coordinates. Also Q is a projection onto A since if $T \in A$ and $X \in l^p(l^p(E))$ we have

$$P_n T_n P_n(X) = P_n T(n)(X_n) = T(n)(X_n) \qquad (n \in \mathbb{N}).$$

Hence for all $X \in l^p(l^p(E))$ we have

$$Q(T)(X) = T(X)$$

Now we define $U_n, V_n \in B(l^p(l^p(E)))$ by

$$U_n(x) = \varphi^{-1}(x(n)) \quad (x \in l^p(l^p(E)).$$

$$V_n(x)(k) = \begin{cases} \varphi(x) & k = n \\ 0 & k \neq n \end{cases} \qquad (x \in l^p(l^p(E))).$$

So for $x \in l^p(l^p(E))$, we have

$$(U_n V_n)(x) = \varphi^{-1}(\varphi(x)) = x.$$

Hence

$$U_n V_n = \mathrm{Id}_{l^p(l^p(E))}.$$

Also we have $(V_n U_n)(x) = V_n(\varphi^{-1}(x(n)))$. So

$$((V_n U_n)(x))(n) = \varphi(\varphi^{-1}(x(n))) = x(n),$$

and $((V_n U_n)(x))(k) = 0$ for $k \neq n$. Hence $V_n U_n = P_n$.

So we have

$$V_n = P_n V_n \ \ , \ \ U_n = U_n P_n. \tag{6}$$

For $n \neq m$ we have

$$(U_n V_m)(x) = \varphi^{-1}(P_n(V_m(x))) = 0.$$

So

$$U_n V_m = 0 \qquad n \neq m \tag{7}$$

Now we define:

$$Q_L : B(l^p(l^p(E))) \longrightarrow B(l^p(l^p(E))) \qquad T \longmapsto \sum_n P_n T U_n;$$

$$Q_R : B(l^p(l^p(E))) \longrightarrow B(l^p(l^p(E))) \qquad T \longmapsto \sum_n V_n T P_n.$$

Similar to what we proved for Q, the series above are convergent in S.O.T . Since elements of A are identified by Block diagonal matrices in $B(l^p(l^p(E)))$, it is clear that Q_L is a left A-module morphism and Q_R is a right A-module morphism.
From (6), it is clear that, Q_L and Q_R attain their values in A.

For $S, T \in B(l^p(l^p(E)))$ we have

$$S_N = \sum_{k=1}^{N} P_k S U_k \longrightarrow_N Q_L(S);$$

$$T_N = \sum_{k=1}^{N} V_k T P_k \longrightarrow_N Q_R(T),$$

where the limits are taken in S.O.T .
Also noting that

$$\begin{aligned}
\|S_N(X)\|_P^P &\leq \sum_{k=1}^{N} \|(P_k S U_k)(X)\|_p^p \\
&= \sum_{k=1}^{N} \|P_k S(\varphi^{-1}(X(k)))\|_p^p \\
&\leq \sum_{k=1}^{N} \|P_k\|^p \|S\|^p \|X(k)\|_p^p \\
&\leq \|S\|^p \|X\|_p^p \quad (X \in l^p(l^p(E)))
\end{aligned}$$

we have $\|S_N\| \leq \|S\|$.

Similarly we have

$$\|T_N\| \leq \|T\| \qquad (N \in \mathbb{N}).$$

So

$$\begin{aligned}\|(S_N T_N)(X) - (Q_L(S)Q_R(T))(X)\| &\leq \|(S_N T_N - S_N Q_R(T))(X) + (S_N Q_R(T) - Q_L(S)Q_R(T))(X)\| \\ &\leq \|S_N\|\|(T_N - Q_R(T))(X)\| + \|(S_N - Q_L(S))(Q_R(T))(X)\|.\end{aligned}$$

Hence boundedness of $(\|S_N\|)_N$ implies that

$$S_N T_N \longrightarrow_N Q_L(S)Q_R(T),$$

where the limit is taken in S.O.T.

So we have

$$\begin{aligned}(Q_L(S))(Q_R(T)) &= \text{S.O.T} - \lim_N (\sum_{k=1}^{N} P_k S U_k)(\sum_{k=1}^{N} V_k T P_k) \\ &= \text{S.O.T} - \sum_{k,t=1}^{N} P_k S U_k V_t T P_t \\ &= \text{S.O.T} - \lim_N \sum_{k=1}^{N} P_k S U_k V_k T P_k \qquad (\text{By}(7)) \\ &= \text{S.O.T} - \lim_N \sum_{k=1}^{N} P_k S T P_k = Q(ST).\end{aligned}$$

So

$$\pi \circ (Q_L \otimes Q_R) = Q \circ \pi. \tag{8}$$

Now suppose that $B(l^p(E))$ is approximately amenable. Since $B(l^p(E))$ has identity, by Theorem 1.43 it has an approximate diagonal $(d_\alpha)_{\alpha\in\Lambda}$.

Also we can assume that $\pi(d_\alpha) = \mathrm{Id}_{B(l^p(E))} \quad (\alpha \in \Lambda)$. Since if not, we can consider:

$$d_\alpha - \pi(d_\alpha)\otimes 1 + 1\otimes 1 \quad (1 = \mathrm{Id}_{l^p(E)}).$$

Now we take the net $((Q_L \otimes Q_R)(d_\alpha))_{\alpha\in\Lambda}$ and we prove that this net gives us an approximate diagonal for $A\widehat{\otimes}A$ .Since $l^p(l^p(E)) \cong l^p(E)$, $B(l^p(l^p(E))) \cong B(l^p(E))$. Also since $Q_L \otimes Q_R$ is an A-module morphism and attains its values in $A\widehat{\otimes}A$, we have for $(a \in A)$,

$$a.((Q_L \otimes Q_R)(d_\alpha)) - ((Q_L \otimes Q_R)(d_\alpha)).a = (Q_L \otimes Q_R)(d_\alpha.a - a.d_\alpha) \longrightarrow_\alpha 0.$$

Also by (8), we have

$$\pi((Q_L \otimes Q_R)(d_\alpha)) = Q(\pi(d_\alpha)) = Q(\mathrm{Id}_{l^p(l^p(E))}) = \mathrm{Id}_{l^p(l^p(E))}$$

The latter can be considered as Id_{A}. So A has an approximate diagonal and since $A = l^\infty(B(l^p(E)))$ has an identity, by Theorem 1.43, A is approximately amenable. The converse is obvious since $B(l^p(E))$ can be considered as a closed ideal in $l^\infty(B(l^p(E)))$ that has an identity and hence is approximately amenable. $\square$

We have even the more general case:

Theorem 3.16. *Let E to be a Banach space. Then for $1 \leq p < \infty$ and $r \in \mathbb{N}$, the Banach algebra $B(\bigoplus_{k=1}^r l^p(E))$ is approximately amenable if and only if $l^\infty(B(\bigoplus_{k=1}^r l^p(E))$ is approximately amenable.*

Proof: Similar to what we had in the proof of Theorem 3.15, we can identify $\bigoplus_{k=1}^{r} l^p(l^p(E))$ isometrically with $\bigoplus_{k=1}^{r} l^p(E)$ through a mapping φ. If we let $A = l^\infty(B(\bigoplus_{k=1}^{r} l^p(E)))$, then we can embed A into $B(\bigoplus_{k=1}^{r} l^p(l^p(E)))$ as block diagonal matrices.

Let $P_n : \bigoplus_{k=1}^{r} l^p(l^p(E)) \longrightarrow \bigoplus_{k=1}^{r} l^p(l^p(E))$ be the projection onto the nth coordinate. We define $Q : B(\bigoplus_{k=1}^{r} l^p(l^p(E))) \longrightarrow B(\bigoplus_{k=1}^{r} l^p(l^p(E)))$ by

$$Q(T) = \sum_n P_n T P_n \qquad (T \in B(\bigoplus_{k=1}^{r} l^p(l^p(E))).$$

Again the series above is convergent in strong operator topology, since for $X_k = (x_{n,k})_n$ in $l^p(l^p(E))$ and $X = \bigoplus_{k=1}^{r} X_k$ in $\bigoplus_{k=1}^{r} l^p(l^p(E))$, we have

$$\begin{aligned}(\sum_n \|P_n T P_n(X)\|)^p &\leq \sum_n \|T\|^p (\sum_{k=1}^{r} \|x_{n,k}\|_p)^p \\ &\leq \|T\|^p r^{p-1} \sum_n \sum_{k=1}^{r} \|x_{n,k}\|_p^p \\ &= \|T\|^p r^{p-1} (\sum_{k=1}^{r} \|X_k\|^p) \\ &\leq \|T\|^p \|\bigoplus_{k=1}^{r} X_k\|^p.\end{aligned}$$

Hence we have

$$\|Q(T)\| \leq r^{\frac{p-1}{p}} \|T\| \qquad (T \in B(\bigoplus_{k=1}^{r} l^p(l^p(E))).$$

For each $n \in \mathbb{N}$ and every $X = \bigoplus_{k=1}^{r} X_k$ in $\bigoplus_{k=1}^{r} l^p(l^p(E))$,we define $U_n, V_n \in B(\bigoplus_{k=1}^{r} l^p(l^p(E)))$ by

$$U_n(\bigoplus_{k=1}^{r} X_k) = \varphi^{-1}(\bigoplus_{k=1}^{r} X_k(n)).$$

$$V_n(\bigoplus_{k=1}^{r} X_k)(i) = \begin{cases} \varphi(\bigoplus_{k=1}^{r} X_k) & i = n \\ 0 & i \neq n \end{cases}$$

In a similar way to the proof of Theorem 3.15, for $n \in \mathbb{N}$, we have

$$U_n V_n = \mathrm{Id}_{\bigoplus_{\mathrm{k=1}}^{\mathrm{r}} l^p(l^p(E))};$$

$$V_n U_n = P_n.$$

Therefore for $n \in \mathbb{N}$,

$$V_n = P_n V_n \quad , \quad U_n = U_n P_n \tag{9}$$

Also we have

$$U_n V_m = 0 \qquad m \neq n \tag{10}$$

Now we define $Q_L, Q_R : B(\bigoplus_{k=1}^{r} l^p(l^p(E))) \longrightarrow B(\bigoplus_{k=1}^{r} l^p(l^p(E)))$ by

$$Q_L : T \mapsto \sum_n P_n T U_n;$$

$$Q_R : T \mapsto \sum_n V_n T P_n.$$

By the same way as we proved for Q, the above series are convergent in the strong operator topology and

$$\|Q_L(T)\| \leq r^{\frac{p-1}{p}} \|T\|; \tag{11}$$

$$\|Q_R(T)\| \leq r^{\frac{p-1}{p}} \|T\|. \tag{12}$$

Similar to what we did in the proof of Theorem 3.15, and by using (10) , we have

$$Q_L(S) Q_R(T) = Q(ST) \qquad (S, T \in B(\bigoplus_{k=1}^{r} l^p(l^p(E)))).$$

Therefore

$$\pi \circ (Q_L \otimes Q_R) = Q \circ \pi. \tag{13}$$

Also by (9), both Q_L and Q_R attain their values in A and from identifying A with block diagonal matrices, Q_L is a left and Q_R is a right A-module morphism. Now suppose that $B(\bigoplus_{k=1}^{r} l^p(l^p(E)))$ is approximately amenable. So by Theorem 1.43 it has an approximate diagonal say $(d_\alpha)_\alpha$. Also this approximate diagonal can be chosen such that $\pi(d_\alpha) = \mathrm{Id}_{\bigoplus_{\mathrm{k}=1}^{\mathrm{r}} l^p(l^p(E))}$. In a similar way to the proof of Theorem 3.15, $((Q_L \otimes Q_R)(d_\alpha))_\alpha$ will be an approximate diagonal for A with $\pi((Q_L \otimes Q_R)(d_\alpha)) = \mathrm{Id}_{\mathrm{A}}$. Hence again by Theorem 1.43, A will be approximately amenable. The converse is obvious. $\square$

Lemma 3.17. *Suppose that A is a Banach algebra with a bounded approximate identity. Then also the Banach algebra $l^\infty(A)$ has a bounded approximate identity.*

Proof: *Suppose* that $(e_\alpha)_{\alpha\in\Lambda}$ is a bounded approximate identity for A. Let $D = \{e_\alpha | \alpha \in \Lambda\}$. Then we define the set E by:

$$E = \{f | f : \mathbb{N} \longrightarrow D, f \text{ is a function}\}$$

Obviously E is a bounded subset of $l^\infty(A)$. We denote an element $f \in E$ by $(f_n)_{n\in\mathbb{N}}$. Then for $(a_n)_n \in l^\infty(A)$ we have:

For every $\epsilon > 0$ and for all $n \in \mathbb{N}$, there exists $e_{n,\epsilon} \in D$ such that

$$\|a_n e_{n,\epsilon} - a_n\| < \epsilon, \|e_{n,\epsilon} a_n - a_n\| < \epsilon.$$

So if we consider $e = (e_{n,\epsilon})_n$, then we have

$$\|ae - e\|_\infty = \|(a_n e_{n,\epsilon} - a_n)_n\| = \sup_n \|a_n e_{n,\epsilon} - a_n\| \leq \epsilon,$$

and similarly $\| ea - a \|_\infty < \epsilon$.

Hence for each $\epsilon > 0$ and each element $a \in l^\infty(A)$, we found an element $e_{a,\epsilon} \in E$ such that $\|ea - a\|_\infty < \epsilon$ and $\|ae - a\|_\infty < \epsilon$. Now boundedness of E implies that $l^\infty(A)$ has a bounded approximate identity. $\square$

Corollary 3.18. *Let $p \in (1, \infty)$ be such that $B(l^p)$ is approximately amenable. Then $l^\infty(K(l^p))$ is approximately amenable.*

Proof: By letting $E = \mathbb{C}$ in preceding Theorem, $l^\infty(B(l^p))$ is approximately amenable. $K(l^p)$ has a bounded approximate identity (since l^p has the approximation property). Hence by the preceding Lemma, $l^\infty(K(l^p))$ has a bounded approximate identity. So $l^\infty(K(l^p))$ is a closed ideal in $l^\infty(B(l^p))$ with bounded approximate identity. Hence $l^\infty(K(l^p))$ is approximately amenable. $\square$

Lemma 3.19. *Let X be a Banach space. Then for $1 \leq p < \infty$, $l^p(X)$ can be identified with the completion of the algebraic tensor product of $l^p(\mathbb{N}) \otimes X$ (as a subalgebra of $l^p(X)$).*

Proof We define a mapping $\varphi : l^p(\mathbb{N}) \otimes X \longrightarrow l^p(X)$ by

$$\varphi((a_n)_n \otimes x) = (a_n x)_n \quad ((a_n)_n \in l^p(\mathbb{N}), x \in X)$$

Obviously φ is one to one and also $Ran(\varphi)$ contains all sequences with finite support in $l^p(X)$. Since such sequences are dense in $l^p(X)$, we have the result. $\square$

Lemma 3.20. *Let E be a Banach space and define $Tr : F(E) \longrightarrow \mathbb{C}$ by*

$$Tr(x \otimes x^*) = \langle x^*, x\rangle \qquad (x \in E, x^* \in E^*)$$

Then we have the following:

(i) $Tr(ST) = Tr(TS) \quad (S \in F(E), T \in B(E))$

(ii) If for another Banach space G, $T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in F(E \oplus G)$ *, then*

$$Tr(T) = Tr(T_{1,1}) + Tr(T_{2,2})$$

(iii)If $(x_n)_n$ is a basis for E, then

$$Tr(T) = \sum_{n=1}^{\infty} \langle x_n^*, T(x_n)\rangle \quad (T \in F(E))$$

Proof: First since Tr is linear, it is enough to prove the statements for the elementary tensors $x \otimes x^*$. For (i), if $S = x \otimes x^*$, we have $TS = T(x) \otimes x^*$ and $ST = x \otimes T^*(x^*)$ and hence

$$Tr(TS) = \langle x^*, T(x)\rangle = \langle T^*(x^*), x\rangle = Tr(ST).$$

For (ii), if $T_{1,1} = x_1 \otimes x_1^*$, $T_{1,2} = y_1 \otimes x_2^*$, $T_{2,1} = x_2 \otimes y_1^*$, $T_{2,2} = y_2 \otimes y_2^*$, for $x_1, x_2 \in E, x_1^*, x_2^* \in E^*$ and $y_1, y_2 \in G, y_1^*, y_2^* \in G^*$ we have

$$\begin{aligned} Tr(T) &= \langle y_1^*, x_2\rangle + \langle x_2^*, y_1\rangle + \langle y_2^*, y_2\rangle \\ &= \langle x_1^*, x\rangle + \langle y_2^*, y_2\rangle = Tr(T_{1,1}) + Tr(T_{2,2}). \end{aligned}$$

For part (iii), let $T = x \otimes x^*$. Since $x \in E$ has a representation $x = \sum_n \lambda_n x_n$, where the λ_n 's are determined uniquely, we have

$$Tr(T) = \langle x^*, x\rangle = \sum_n \lambda_n \langle x^*, x_n\rangle = \langle x_n^*, x\rangle\langle x^*, x_n\rangle = \sum_n \langle x_n^*, T(x_n)\rangle. \qquad \square$$

Let $1 \leq p \leq \infty$ and let q be such that $\frac{1}{p}+\frac{1}{q}$=1. We denote by $\{e_n\}$ (respectively $\{e_n^*\}$) the standard basis for l^p (respectively l^q).

Lemma 3.21. *Let $N \in \mathbb{N}$. For $x \in B(l_p, l_p^N)$ and $y \in B(l_q, l_q^N)$, we have*

$$\sum_n \|x(e_n)\|\|y(e_n^*)\| \leq N\|x\|\|y\|$$

Proof: See [17, Lemma 2.1]. $\square$

In the next Lemma and the theorem following it, we let $\mathbb{P}$ denote the set of prime numbers. For $p \in \mathbb{P}$, Λ_P denotes the projective plane on the field $\mathbb{Z}_p$ i.e. the set of all triples (a, b, c) where $a, b, c \in \mathbb{Z}_p$ and we identify two such triples if one is multiple of another. $SL(3, \mathbb{Z})$ denotes all 3×3 matrices with entries in $\mathbb{Z}$ that have determinant equal to 1. Then $SL(3, \mathbb{Z})$ acts on Λ_p through matrix multiplication and therefore this action induces a representation π_p from $SL(3, \mathbb{Z})$ on $l^2(\Lambda_p)$. Also it can be easily seen that π_p is a unitary representation. This representation is determined by

$$\pi_p(g)(a) = g.a \qquad (g \in SL(3, \mathbb{Z}), a \in l^2(\Lambda_p)),$$

where . denotes the action of $SL(3, \mathbb{Z})$ on $l^2(\Lambda_p)$.

Note that $SL(3, \mathbb{Z})$ is clearly a group (under standard matrix multiplication) and also this group is finitely generated.

Lemma 3.22. *(Ozawa) It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with this property: for each $p \in \mathbb{P}$, there are $\xi_{1,p}, \eta_{1,p}, ..., \xi_{r,p}, \eta_{r,p} \in l^2(\Lambda_p)$ such that*

$\sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \neq 0$ *and*

$$\| \sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} - (\pi_p(g_j) \otimes \pi_p(g_j))(\xi_{k,p} \otimes \eta_{k,p}) \|_{l^2(\Lambda_p)\hat{\otimes} l^2(\Lambda_p)}$$

$$\leq \epsilon \| \sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \|_{l^2(\Lambda_p)\hat{\otimes} l^2(\Lambda_p)}.$$

Proof: See [18, page 4]. $\square$

V.Runde in [18] proved the following result for the amenable case. Here, we prove the more general result by adding the condition that E^* has the bounded approximation property. Also we give all the details of the original proof.

Theorem 3.23. *Let E be a Banach such that E^* has the bounded approximation property. Furthermore suppose that E has a basis $(x_n)_n$ such that there is $C > 0$ with*

$$\sum_{n=1} \|Sx_n\| \|Tx_n^*\| \leq CN\|S\|\|T\| \quad (S \in B(E, l_N^2), T \in B(E^*, l_N^2), N \in \mathbb{N}).$$

Then $l^\infty(K(l^2 \oplus E))$ is not approximately amenable.

Proof: Firstly we can identify $K(l^2 \oplus E)$ with the matrix

$$\begin{pmatrix} K(l^2) & K(E, l^2) \\ K(l^2, E) & K(E) \end{pmatrix}.$$

Also we can easily embed $B(l^2(\Lambda_P))$ in the upper left corner of the matrix above. We let $A = l^\infty(\mathbb{P}, K(l^2 \oplus E)) \cong l^\infty(K(l^2 \oplus E))$. Since $(l^2 \oplus E)^* = l^2 \oplus E^*$ has the bounded approximation property , by Corollary 1.50, $K(l^2 \oplus E)$ has a bounded approximate identity and hence $A = l^{\infty}(\mathbb{P}, K(l^2 \oplus E))$, has a bounded approximate identity.

$l^{\infty} - \bigoplus_{p \in \mathbb{P}} B(l^2(\Lambda_P))$ can be considered as the closed subalgebra of A. In particular $F := \{(\pi_p(g_j))_{p \in \mathbb{P}} | j = 1, 2, ..., m+1\}$ can be identified with a finite subset of A.

Also we can assume that A acts on

$$l^2(\mathbb{P}, l^2 \oplus E) \cong l^2(\mathbb{P}, l^2) \oplus l^2(\mathbb{P}, E),$$

through matrix multiplication. So if $a = (a_p) \in A$ and $t = (t_p) \in l^2(\mathbb{P}, l^2 \oplus E)$, we have

$$(at)(p) = a_p(t_p).$$

By Lemma 3.19 we may identify $l^2(\mathbb{P}, l^2)$ and $l^2(\mathbb{P}, E)$ by the completions of $l^2(\mathbb{P}) \otimes l^2$ and $l^2(\mathbb{P}) \otimes E$.

Now for each prime number p, we define the operator P_p on $l^2(\mathbb{P}, l^2) \oplus l^2(\mathbb{P}, E)$ to be the projection onto the first $|\Lambda_p|$ coordinates of the p^{th} l^2-summand of $l^2(\mathbb{P}, l^2) \oplus l^2(\mathbb{P}, E)$.

If $a = (a_p)_p \in A$ and $a^* = (a_p^*)_p \in A$, then for $q \in \mathbb{P}$ and $n \in \mathbb{N}$, we have

$$(a(e_q \otimes e_n))(p) = a_p((e_q \otimes e_n)(p)) = \begin{cases} a_p(e_n) & p = q \\ 0 & p \neq q. \end{cases} \tag{1}$$

Also since $e_p^* = e_p$ and $e_n^* = e_n$, we have

$$(a^*(e_q^* \otimes e_n^*))(p) = a_p^*((e_q^* \otimes e_n^*)(p)) = \begin{cases} a_p^*(e_n^*) & p = q \\ 0 & p \neq q. \end{cases} \tag{2}$$

Similarly we have

$$(a(e_q \otimes x_n))(p) = a_p((e_q \otimes x_n)(p)) = \begin{cases} a_p(x_n) & p = q \\ 0 & p \neq q. \end{cases} \tag{3}$$

And

$$(a^*(e_q^* \otimes x_n^*))(p) = a_p^*((e_q^* \otimes x_n^*)(p)) = \begin{cases} a_p^*(x_n^*) & p = q \\ 0 & p \neq q. \end{cases} \tag{4}$$

Let $a = (a_p)_{p\in\mathbb{P}}$ and $b = (b_p)_{p\in\mathbb{P}}$ be elements of A. Then if $(s_n)_n \in l^2$, $a_p((s_n)_n)$ denotes the element of $l^2(\mathbb{P}, l^2 \oplus E)$ that is 0 in all coordinates except p and is equal to $a_p(e_n)$ in the *pth* coordinate, and the similar definition for $b_p^*(e_n^*)$, then by letting $x((s_n)_n) = P_p(a_p((s_n)_n))$ and $y((s_n)_n) = P_p(b_p^*((s_n)_n))$, and by exploiting Lemma 3.21, we obtain

$$\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} \|P_p(a(e_q \otimes e_n))\| \|P_p^*(b^*(e_q^* \otimes e_n^*)\| = \sum_{n=1}^{\infty} \|P_p(a_p(e_n))\| \|P_p(b_p^*(e_n^*))\|$$

$$\leq |\Lambda_p| \|a_p\| \|b_p\| \leq |\Lambda_p| \|a\| \|b\|. \quad (\dagger)$$

If we define $T \in B(E, l^2_{|\Lambda_p|})$ and $S \in B(E^*, l^2_{|\Lambda_p|})$ by

$$T(x) = P_p(a_p(x)) \ \ and \ \ S(x^*) = P_p(b_p^*(x^*)) \qquad (x \in E, x^* \in E^*),$$

then by the assumption of the theorem we have

$$\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} \|P_p(a(e_q \otimes x_n))\| \|P_p^*(b^*(e_q^* \otimes x_n^*)\| = \sum_{n=1}^{\infty} \|P_p(a_p(x_n))\| \|P_p(b_p^*(x_n^*))\|$$

$$\leq C|\Lambda_p| \parallel a_p\| \|b_p\| \leq C|\Lambda_p| \|a\| \|b\|. \quad (\ddagger)$$

Set $e = (P_p)_{p\in\mathbb{P}}$ and let $a = (a_p)_{p\in\mathbb{P}} \in A$. Since each P_p can be considered as an element of $K(l^2)$, e is an idempotent in A. Also since $r \in l^2(\Lambda_p)$ can be considered as an element of l^2 that takes 0 on the coordinates after $|\Lambda_p|$, $(P_p a_p P_p)(r)$ is an element of $l^2(\Lambda_p)$ that is the first $|\Lambda_P|$ coordinates of $a_p(r)$. So we have $\| (P_p a_p P_p)(r) \| \leq \| a_p \| \| r \|$ and hence

$$eae = (P_p a_p P_p) \in l^\infty - \bigoplus_{p\in\mathbb{P}} B(l^2(\Lambda_p)).$$

On the other hand for $(T_p) \in l^\infty - \bigoplus_{p\in\mathbb{P}} B(l^2(\Lambda_p))$, $(P_p T_p P_p)(r) = T_p(r)$. So $eTe = (P_p T_p P_p) = (T_p) = T$ and hence we have

$$eAe = l^\infty - \bigoplus_{p\in\mathbb{P}} B(l^2(\Lambda_p)).$$

Now if $A = l^\infty(\mathbb{P}, l^2 \oplus E)$ is approximately amenable, then given $\epsilon > 0$, by corollary 3.10, there are $a_{1,1}, ..., a_r, b_1, b_2, ..., b_r \in A$ such that

$$\sum_{k=1}^{r} a_k b_k = e$$

and

$$\sum_{k=1}^{r} \|x a_k \otimes b_k - a_k \otimes b_k x\| < \frac{\epsilon}{(C+1)(m+1)} \qquad (x \in F). \tag{5}$$

For each $p \in \mathbb{P}$, we define $\varphi_{p,q,n} : (A \widehat{\bigotimes} A) \longrightarrow l^2(\Lambda_p) \widehat{\bigotimes} l^2(\Lambda_p)$ by acting on elementary tensors by

$$\varphi_{p,q,n}(a \otimes b) = P_p(a(e_q \otimes e_n)) \otimes P_p^*(b^*(e_q^* \otimes e_n^*)) + P_p(a(e_q \otimes x_n)) \otimes P_p^*(b^*(e_q^* \otimes x_n^*)).$$

By (†) and (‡), we have

$$\sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty} \| \varphi_{p,q,n}(a\otimes b) \| \leq (C+1)|\Lambda_p| \| a \| \| b \| \qquad (a,b\in A).$$

Let $m=\sum_i a_i\otimes b_i \in A\widehat{\otimes}A$. Then

$$\sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty} \|\varphi_{p,q,n}(m)\| \leq \sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty}\sum_i \|\varphi_{p,q,n}(a_i\otimes b_i)\| \leq (C+1)|\Lambda_p|\sum_i \|a_i\|\|b_i\|.$$

Hence

$$\sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty} \|\varphi_{p,q,n}(m)\| \leq (C+1)|\Lambda_p|\|m\| \qquad (p\in\mathbb{P}). \qquad (\spadesuit)$$

Fix $1\leq j\leq m+1$ and $p_0\in\mathbb{P}$. By the way that we embed $\bigoplus_{p\in\mathbb{P}} B(l^2(\Lambda_p))$ in A and by the definition of P_{p_0}, we have

$$(((\pi_p(g_j))_{p\in\mathbb{P}}a)(e_q\otimes e_n))(p) = \begin{cases} (\pi_q(g_j)a_q)(e_n) & p=q \\ 0 & p\neq q\ , \end{cases}$$

so that

$$P_{p_0}(((\pi_p(g_j))_{p\in\mathbb{P}}a)(e_q\otimes e_n)) = \pi_{p_0}(g_j)(a_{p_0}(e_n)) = (\pi_p(g_j))_{p\in\mathbb{P}}P_{p_0}(a(e_q\otimes e_n)).$$

And similarly

$$P_{p_0}(((\pi_p(g_j))_{p\in\mathbb{P}}a)(e_q\otimes x_n)) = (\pi_p(g_j))_{p\in\mathbb{P}}P_{p_0}(a(e_q\otimes x_n)).$$

Also we have

$$(b(\pi_p(g_j))_{p\in\mathbb{P}})^*(e_q^*\otimes e_n^*))(p) = ((\pi_p(g_j)^* b_p^*)_{p\in\mathbb{P}}(e_q^*\otimes e_n^*))(p) = \begin{cases} (\pi_q(g_j)^*(b_q^*(e_n^*)) & p = q \\ 0 & p \neq q\ . \end{cases}$$

By using the fact that $\pi_p(g_j)^* = \pi_p(g_j)^{-1} \quad (p\in\mathbb{P})$, we obtain

$$P_{p_0}^*((b(\pi_p(g_j))_{p\in\mathbb{P}}^*)(e_q^*\otimes e_n^*)) = \pi_{p_0}^{-1}(g_j)(b_{p_0}^*(e_n^*)) = (\pi_p(g_j))_{p\in\mathbb{P}}^{-1} P_{p_0}^*(b^*(e_q^*\otimes e_n^*)).$$

And similarly,

$$P_{p_0}^*((b(\pi_p(g_j))_{p\in\mathbb{P}}^*)(e_q^*\otimes x_n^*)) = (\pi_p(g_j))_{p\in\mathbb{P}}^{-1} P_{p_0}^*(b^*(e_q^*\otimes x_n^*)).$$

By the above equalities, we obtain

$$\begin{aligned}\varphi_{p,q,n}(\sum_{k=1}^r (\pi_p(g_j))_{p\in\mathbb{P}} a_k\otimes b_k - a_k\otimes b_k(\pi_p(g_j))_{p\in\mathbb{P}}) = &\sum_{k=1}^r (\pi_p(g_j))_{p\in\mathbb{P}} P_{p_0}(a_k(e_q\otimes e_n))\otimes P_{p_0}^*(b_k^*(e_q^*\otimes e_n^*)) \\ &+ (\pi_p(g_j))_{p\in\mathbb{P}} P_{p_0}(a_k(e_q\otimes x_n))\otimes P_{p_0}^*(b_k^*(e_q^*\otimes x_n^*)) \\ &- P_{p_0}(a_k(e_q\otimes e_n)\otimes (\pi_p(g_j))_{p\in\mathbb{P}}^{-1} P_{p_0}^*(e_q^*\otimes e_n^*) \\ &- P_{p_0}(a_k(e_q\otimes x_n))\otimes (\pi_p(g_j))_{p\in\mathbb{P}}^{-1} P_{p_0}^*(b_k^*(e_q^*\otimes x_n^*)).\end{aligned}$$

For $q, p\in\mathbb{P}$ and $n\in\mathbb{N}$, we define $T_p(q,n)\in l^2(\Lambda_p)\widehat{\bigotimes} l^2(\Lambda_p)$ by

$$T_p(q,n) := \sum_{k=1}^r P_p(a_k(e_q\otimes e_n))\otimes P_p^*(b_k^*(e_q^*\otimes e_n^*)) + P_p(a_k(e_q\otimes x_n))\otimes P_p^*(b_k^*(e_q^*\otimes x_n^*)).$$

It can be easily seen that

$$T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}}\otimes(\pi_p(g_j))_{p\in\mathbb{P}}) T_p(q,n) =$$

$$= -(\mathrm{Id} \otimes (\pi_{\mathrm{p}}(\mathrm{g_j}))_{\mathrm{p}\in\mathbb{P}})(\varphi_{\mathrm{p,q,n}}(\sum_{\mathrm{k=1}}^{\mathrm{r}}(\pi_{\mathrm{p}}(\mathrm{g_j}))_{\mathrm{p}\in\mathbb{P}}\mathrm{a_k} \otimes \mathrm{b_k} - \mathrm{a_k} \otimes \mathrm{b_k}(\pi_{\mathrm{p}}(\mathrm{g_j}))_{\mathrm{p}\in\mathbb{P}})).$$

Since $\|(\pi_p(g_j))_{p\in\mathbb{P}}\| \leq 1$,

$$\| T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}})T_p(q,n) \| \leq$$
$$\| \varphi_{p,q,n}(\sum_{k=1}^{r}(\pi_p(g_j))_{p\in\mathbb{P}}a_k \otimes b_k - a_k \otimes b_k(\pi_p(g_j))_{p\in\mathbb{P}})) \|,$$

and hence by (♠) and (5), we obtain

$$\textstyle\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}})T_p(q,n) \| \leq$$

$$\leq (C+1)|\Lambda_p| \, \| \sum_{k=1}^{r}(\pi_p(g_j))_{p\in\mathbb{P}}a_k \otimes b_k - a_k \otimes b_k(\pi_p(g_j))_{p\in\mathbb{P}})) \| \leq \frac{\epsilon}{(m+1)}|\Lambda_p|.$$

Thus

$$\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{m+1} \| T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}})T_p(q,n) \| \leq \epsilon|\Lambda_p|. \tag{6}$$

For $k = 1, 2, ..., r$, write $a_k = (a_{k,q})_{q\in\mathbb{P}}$. So by equalities $(1), (2), (3), (4)$, we have

$$\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} T_p(q,n) = \sum_{n=1}^{\infty} \sum_{k=1}^{r} P_p(a_{k,p}(e_n)) \otimes P_p^*(b_{k,p}^*(e_n^*)) + P_p(a_{k,p}(x_n) \otimes P_p^*(b_{k,p}^*(x_n^*)). \tag{7}$$

We specify $\phi \in ((l^2(\Lambda_p)\widehat{\bigotimes} l^2(\Lambda_p))^*$ by $\phi(x \otimes y) = \langle y, x\rangle$, where $\langle y, x\rangle$ indicates the action of y as an element of $(l^2(\Lambda_p)^*)$ on x. We have

$$|\phi(\sum_{i=1}^{\infty} x_i \otimes y_i)| = |\sum_{i=1}^{\infty}\langle y_i, x_i\rangle| \leq \sum_{i=1}^{\infty} \|y_i\|_2 \|x_i\|_2,$$

and hence

$$\|\phi\| \leq 1. \tag{14}$$

By (7), (8)we have

$$\begin{aligned}
\sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty} \parallel T_p(q,n) \parallel &= \sum_{n=1}^{\infty} \parallel \sum_{k=1}^{r} P_p(a_{k,p}(e_n)) \otimes P_p^*(b_{k,p}^*(e_n^*)) + P_p(a_{k,p}(x_n) \otimes P_p^*(b_{k,p}^*(x_n^*)) \parallel \\
&\geq \sum_{n=1}^{\infty} | \sum_{k=1}^{r} \langle P_p^*(b_{k,p}^*(e_n^*)), P_p(a_{k,p}(e_n))\rangle + \langle P_p^*(b_{k,p}^*(x_n^*)), P_p(a_{k,p}(x_n))\rangle | \\
&\geq | \sum_{n=1}^{\infty}\sum_{k=1}^{r} \langle P_p^*(b_{k,p}^*(e_n^*)), P_p(a_{k,p}(e_n))\rangle + \langle P_p^*(b_{k,p}^*(x_n^*)), P_p(a_{k,p}(x_n))\rangle | \\
&= | \sum_{n=1}^{\infty}\sum_{k=1}^{r} \langle e_n^*, b_{k,p}(P_p(a_{k,p}(e_n))\rangle + \langle x_n^*, b_{k,p}P_p(a_{k,p}(x_n))\rangle | \quad (\text{Since } P_p^2 = P_p) \\
&= |Tr(\sum_{k=1}^{r} b_{k,p}P_p a_{k,p})| \quad (\text{By Lemma 3.20 parts(ii), (iii)}) \\
&= |Tr(\sum_{k=1}^{r} P_p a_{k,p} b_{k,p})| \quad (\text{By Lemma 3.20, part(i)}) \\
&= |Tr(P_p)| \quad (\text{Since} \sum_{k=1}^{r} a_{k,p} b_{k,p} = P_p)
\end{aligned}$$

We also have

$$|Tr(P_p)| = \sum_{n} \langle e_n^*, P_p(e_n)\rangle = \sum_{n=1}^{|\Lambda_p|} \langle e_n^*, e_n\rangle = |\Lambda_p|.$$

And hence we obtain

$$\sum_{q\in\mathbb{P}}\sum_{n=1}^{\infty} \|T_p(q,n)\| \geq |\Lambda_p|. \tag{15}$$

Equations (6) and (9) together imply the existence of $q \in \mathbb{P}$ and $n \in \mathbb{N}$ such that $T_P(q,n) \neq 0$ and

$$\sum_{j=1}^{m+1} \parallel T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}}) T_p(q,n) \parallel \leq \epsilon \parallel T_p(q,n) \parallel .$$

Thus for $1 \leq j \leq m+1$, we get

$$\| T_p(q,n) - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}})T_p(q,n) \| \leq \epsilon \| T_p(q,n) \| . \tag{16}$$

By definition of $T_p(q,n)$, there are $\xi_{1,p}, \eta_{1,p}, ..., \xi_{2r,p}, \eta_{2r,p} \in l^2(\Lambda_P)$ such that

$$T_p(q,n) = \sum_{k=1}^{2r} \xi_{k,p} \otimes \eta_{k,p}.$$

So $\sum_{k=1}^{2r} \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and by (10), we have

$$\| \sum_{k=1}^{2r} \xi_{k,p} \otimes \eta_{k,p} - ((\pi_p(g_j))_{p\in\mathbb{P}} \otimes (\pi_p(g_j))_{p\in\mathbb{P}})(\xi_{k,p} \otimes \eta_{k,p}) \| \leq \epsilon \| \sum_{k=1}^{2r} \xi_{k,p} \otimes \eta_{k,p} \| .$$

Since $p \in \mathbb{P}$ was arbitrary, the above statement contradicts Ozawa's Lemma. $\square$

Definition 3.24. We say a Banach space E has property $(*)$ if E^* has the bounded approximation property and also E has a basis $(x_n)_n$ such that there is a $C > 0$ with

$$\sum_{n=1} \| Sx_n \| \| Tx_n^* \| \leq CN \| S \| \| T \| \qquad (S \in B(E, l_N^2), T \in B(E^*, l_N^2), N \in \mathbb{N}).$$

Theorem 3.25. *Let $p \in [1,\infty)$ and suppose that E is a Banach space such that $l^p(E)$ has the property $(*)$ and also $l^p(E)$ is isometrically isomorphic to $l^p(E) \oplus l^2$. Then $B(l^p(E))$ is not approximately amenable.*

Proof: Suppose that $B(l^p(E))$ is approximately amenable. So by Theorem 3.15, $l^\infty(B(l^p(E))$ is also approximately amenable. Since $l^p(E)$ has the property $(*)$, in particular $l^p(E)^*$ has the bounded approximation property. So from Corollary 1.50, $K(l^p(E))$ has a bounded approximate identity. Hence by Lemma 3.18, $l^\infty(K(l^p(E))$ also has a bounded approximate identity and since $l^\infty(K(l^p(E))$ is a closed ideal in

$l^\infty(B(l^p(E))$, $l^\infty(K(l^p(E)))$ is approximately amenable. But since $l^\infty(K(l^p(E))) \cong l^\infty(K(l^p(E) \oplus l^2))$, $l^\infty(K(l^p(E) \oplus l^2))$ is approximately amenable which is impossible by Theorem 3.23. So $B(l^p(E))$ is not approximately amenable. $\square$

By Theorem 3.25, we can see that property $(*)$ plays an important role in non-approximate amenability of certain classes of Banach algebras. Now we try to find some classes of Banach algebras that satisfy the property $(*)$. In the next theorem,we will see that for every $p \in (1, \infty)$, l^p satisfies the property $(*)$. Before going to the next Theorem, we need some preliminaries and a Lemma.

For $1 \leq p < \infty$, the operator $T : E \longrightarrow F$ is called p-summing if the operator $\mathrm{Id}_{l^p} \otimes T : l^p \otimes E \longrightarrow l^p \otimes F$ can be extended to a bounded linear operator from $l^p \check{\otimes} E \longrightarrow l^p(F)$ where $\check{\otimes}$ denotes the injective tensor product.
In this case the operator norm of $\mathrm{Id}_{l^p} \otimes T : l^p \check{\otimes} E \longrightarrow l^p(F)$ is called the p-summing norm of T denoted by $\pi_p(T)$.

We have introduced a proof for the following as we were not able to find it in the literature.

Lemma 3.26. *For $1 < p < \infty$, $B(l^p, l^2_N)$ is isometrically isomorphic to $l^q \check{\otimes} l^2_N$ where $\frac{1}{p} + \frac{1}{q}$ =1.*

Proof: First we can algebraically identify $B(l^p, l^2_N)$ by $l^q \otimes l^2_N$ by using the fact that every operator in $B(l^p, l^2_N)$ is a finite rank operator. It can be easily seen that the isomorphism $\varphi : B(l^p, l^2_N) \longrightarrow l^q \otimes l^2_N$ acts on $T \in B(l^p, l^2_N)$ as below

$$\varphi(T) = \sum_{i=1}^{N} x_i^* \otimes e_i.$$

Where e_i is the element of l^2_N that has 1 in its *ith* coordinate and 0 in its other coordinates and $x_i^* \in l^q$ is defined by

$$\langle x_i^*, x\rangle = (f(x))(i) \qquad i = 1, 2, ..., N.$$

Now we show that φ indeed defines an isometric isomorphism between the spaces $B(l^p, l_N^2)$ and $l^q \check{\otimes} l_N^2$.

For this we have

$$\begin{aligned}
\|\varphi(T)\|_w = \|\sum_{i=1}^N x_i^* \otimes e_i\|_w &= \sup\{|\sum_{i=1}^N \langle x_i^*, x\rangle\langle b, e_i\rangle| : x \in (l^p)_1, b = (b_i)_{i=1}^N \in (l_N^2)_1\} \\
&= \sup\{|\sum_{i=1}^N \langle x_i^*, x\rangle b_i| : x \in (l^p)_1, \sum_{i=1}^N |b_i|^2 \leq 1\} \\
&\leq \sup\{\sum_{i=1}^N |\langle x_i^*, x\rangle|^2 \sum_{i=1}^N |b_i|^2 : x \in (l^p)_1, \sum_{i=1}^N |b_i|^2 \leq 1\} \\
&\leq \sup\{\sum_{i=1}^N |\langle x_i^*, x\rangle|^2 : x \in (l^p)_1\} = \|T\|_{op}.
\end{aligned}$$

On the other hand for given $\epsilon > 0$, suppose that $x \in (l^p)_1$ is such that $\|T(x)\|_{l_N^2} \geq \|T\| - \epsilon$. For $1 \leq i \leq n$, we let $b_i = \frac{1}{\|T\|}\overline{\langle x_i^*, x\rangle}$ and consider $b = (b_i)_{i=1}^N \in l_N^2$. We have

$$\|b\|_{l_N^2} = \sum_{i=1}^N |b_i|^2 \leq \sum_{i=1}^N \frac{1}{\|T\|^2}|\langle x_i^*, x\rangle|^2 = \frac{1}{\|T\|^2}\|T(x)\|^2 \leq \frac{\|T\|^2}{\|T\|^2} = 1.$$

So we have

$$
\begin{aligned}
\|\varphi(T)\|_w = \|\sum_{i=1}^N x_i^* \otimes e_i\|_w &= \sup\{|\sum_{i=1}^N \langle x_i^*, x\rangle\langle b, e_i\rangle| : x \in (l^p)_1, b = (b_i)_{i=1}^N \in (l_N^2)_1\} \\
&\geq |\sum_{i=1}^N \langle x_i^*, x\rangle \frac{1}{\|T\|}\overline{\langle x^*, x\rangle}| = \frac{1}{\|T\|}\sum_{i=1}^N |\langle x^*, x\rangle|^2 \\
&= \frac{1}{\|T\|}\|T(x)\|^2 \geq \frac{(\|T\| - \epsilon)^2}{\|T\|}.
\end{aligned}
$$

And the last term tends to $\|T\|$ as $\epsilon \longrightarrow 0$, so that $\|\varphi(T)\|_w \geq \|T\|_{op}$.

Therefore

$$\|T\|_{op} = \|\varphi(T)\|_w.$$

Hence φ is an isometry. $\square$

Theorem 3.27. *For $p \in (1, \infty)$, l^p has the property $(*)$.*

Proof: Since $l_p^* = l^q$ has a Schauder basis, it has the bounded approximation property. Let $N \in \mathbb{N}$ and $\{\delta_n : n \in \mathbb{N}\}$ be the regular basis for l^p. By the argument of the preceding lemma, for every element $S = \sum_{i=1}^N x_i^* \otimes e_i \in B(l^p, l_N^2)$ where $x_i^* \in l^q$, $\phi(S) = (S(\delta_n))_n$ belongs to $l^q(l_N^2)$. Since

$$
\begin{aligned}
\sum_n \|S(e_n)\|_{l_N^2}^q = \sum_n \|(x_i^*(n))_{i=1}^N\|^q &= \sum_n (\sum_i \|x_i^*(n)\|^2)^{\frac{q}{2}} \\
&\leq K(q, N)\sum_{i=1}^N \sum_n |x_i^*(n)|^q = K(q, N)\sum_{i=1}^N \|x_i^*\|_q^q < \infty,
\end{aligned}
$$

where $K(q, N)$ is a finite number depending only on q, N. So for $S \in B(l^p, l_N^2)$ and $T \in B(l^q, l_N^2)$, by Holder's inequality, we have

$$\sum_{n=1}^{\infty} \|S(\delta_n)\|\|T(\delta_n^*)\| \leq \|\phi(S)\|_{l^q(l_N^2)}\|\phi(T)\|_{l^p(l_N^2)}. \tag{17}$$

From [10, Theorem 5], $\mathrm{Id}_{l^2_N}$ is p-summing and the p-summing norm satisfies:

$$\pi_p(\mathrm{Id}_{l^2_N}) \sim \sqrt{N} \sim \pi_q(\mathrm{Id}_{l^2_N}).$$

So there are $C_p, C_q > 0$ such that

$$\pi_p(\mathrm{Id}_{l^2_N}) \leq C_p\sqrt{N} \text{ and } \mathrm{\pi_q(id_{l^2_N}) \leq C_q\sqrt{N} \quad (N \in \mathbb{N})}.$$

Now since for $S \in B(l^p, l^2_N)$, $\phi(S) = id_{l^p} \otimes \mathrm{Id}_{l^2_N}$ and since $B(l^p, l^2_N)$ is isometrically isomorphic to $l^q \check{\otimes} l^2_N$, we have:

$$\|\phi(S)\|_{l^q(l^2_N)} \leq \pi_q(id_{l^2_N})\|S\|_{op}.$$

And similarly

$$\|\phi(T)\|_{l^p(l^2_N)} \leq \pi_p(id_{l^2_N})\|T\|_{op} \quad (T \in B(l^q, l^2_N)).$$

So by (11), we have

$$\sum_{n=1}^{\infty} \|S(\delta_n)\|\|T(\delta_n^*)\| \leq C_pC_qN\|S\|_{op}\|T\|_{op} \quad (N \in \mathbb{N}, S \in B(l^p, l^2_N), T \in B(l^q, l^2_N)).$$

So it is enough to let $C = C_pC_q$ to get the desired result. $\square$

Corollary 3.28. $B(l^2)$*is not approximately amenable.*

Proof: From Theorem 3.27, $l^2(\mathbb{C}) = l^2$ has the property $(*)$. Also $l^2(\mathbb{C}) \oplus l^2 = l^2 \oplus l^2 \cong l^2$. So in Theorem 3.25, by letting $E = \mathbb{C}$, $B(l^2)$ is not approximately amenable. $\square$

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