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SUBLATTICES OF A FREE LATTICE

by

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INTRODUCTION

The study of free lattices was first initiated by P.M. Whitman. In his two famous papers ([26] and [27]) he exhibited the basic structure of the free lattice on n (unordered) generators; he showed that there exists a canonical form for the elements of such lattices, which enabled him to solve the word problem in a free lattice.

This concept was generalized in several different ways. R.P. Dilworth defined FL(P), the free lattice generated by a partially ordered set P as the "most general" lattice generated by P and preserving existing bounds in P. However, the solution of the word problem is lost in FL(P), except when P has a finite number of defining relations (cf. Trevor Evans [9]). Later, R.A. Dean developed another generalization of the concept of a free lattice on n generators ([6]). Dilworth called such a lattice the completely free lattice generated by a partially ordered set, and denoted it by CF(P). Here the word problem has a solution and this lattice has in fact many properties enjoyed by the free lattice on n generators. For instance, every element in CF(P) is either join-reducible or meet-reducible but not both, except for the generators which are doubly irreducible. In the event that P is unordered and $|P| \simeq n$, FL(P), CF(P) and FL(M) are isomorphic.

Another generalization of this concept, the relatively free lattice, was studied by B. Jónsson in [19] .

Free lattices are a basic tool in lattice theory. For instance, just like free algebras in an arbitrary equational class of algebras, free lattices are intimately connected with identities: if p and q are n-ary polynomials, then $\underset{\sim}{p} = \underset{\sim}{q}$ is valid in a variety $\underset{\sim}{K}$ of lattices if and only if $p\ (x_1, \ldots, x_n) = q\ (x_1, \ldots, x_n)$, where the x_i's are the free generators of the relatively free lattice on n generators in this variety (cf. [14]).

Moreover, many concepts coincide on free lattices. Modular sublattices of a free lattice are distributive ([7]). Every free lattice is projective ([15]). A finitely generated lattice is projective if and only if it is a sublattice of FL(3) (A.L. Kostinsky [21]). This is an extension of a result of R.N. McKenzie [23]. A distributive lattice is projective in the class of all lattices if and only if it is a sublattice of FL(3) (K.A. Baker and A.W. Hales [2]). A finite lattice has the property that if it is embeddable as a poset in a lattice L then it is embeddable as a sublattice in L if and only if it is a distributive sublattice of FL(3) (W. Poguntke and I. Rival [24]).

From this perspective it is clear that the concept of a sublattice of a free lattice is not only a natural one to consider, but that it is also a very useful one.

At the center of attention of this survey is the class $\underset{\sim}{S}$ of lattices satisfying the conditions (W), (SD), and (SD') (see section 2). B. Jónsson had proved in [18] that any sublattice of a free lattice satisfies (SD) and (SD'), whereas (W) is due to Whitman (see [26]). In [20], B. Jónsson and J.E. Kiefer conjectured that finite sublattices of FL(3) are characterised by these three conditions. The conjecture is still unsettled and has been so for the past thirteen years. In [10], F. Galvin and B. Jónsson showed that distributive sublattices of FL(3) are characterized by (W) (this is a special case of the conjecture since distributivity implies (SD) and (SD')). By considering sublattices of a free product, H. Lakser has proved that no non-trivial sublattice of FL(3) is simple ([22]).

Another class of interest is that of transferable lattices (see [11]). This concept was first introduced by G. Grätzer in a (possibly) more general form in [13]. A lattice is called transferable if it satisfies the following two conditions:

1) For any lattice K, L is a sublattice of I(K) (the ideal lattice of K) implies that L is a sublattice of K.
2) Let us denote the embedding of L into I(K) by φ, then there is an embedding ψ of L into K such that $\psi(x) \in \varphi(y)$ if and only if $x \leqslant y$.

A lattice that satisfies the first of the two conditions is called weakly transferable.

H.S. Gaskill and C.R. Platt have proved that, on the one hand, a lattice is transferable and satisfies (W) if and only if it is a sublattice of FL(3), and, on the other hand, that every transferable lattice satisfies (SD) and (SD') ([12]). Whether a lattice satisfying

(W), (SD) and (SD') is necessarily transferable or not is unknown. Recently, G. Grätzer supplied a proof for the claim that a weakly transferable lattice has no doubly reducible elements ([16]). In this connection, I. Rival and the author observed in [1] that in a lattice satisfying (SD), (SD'), the non-containment of a sublattice isomorphic to L_1 or L_2 (see figure 1) is equivalent to (W).

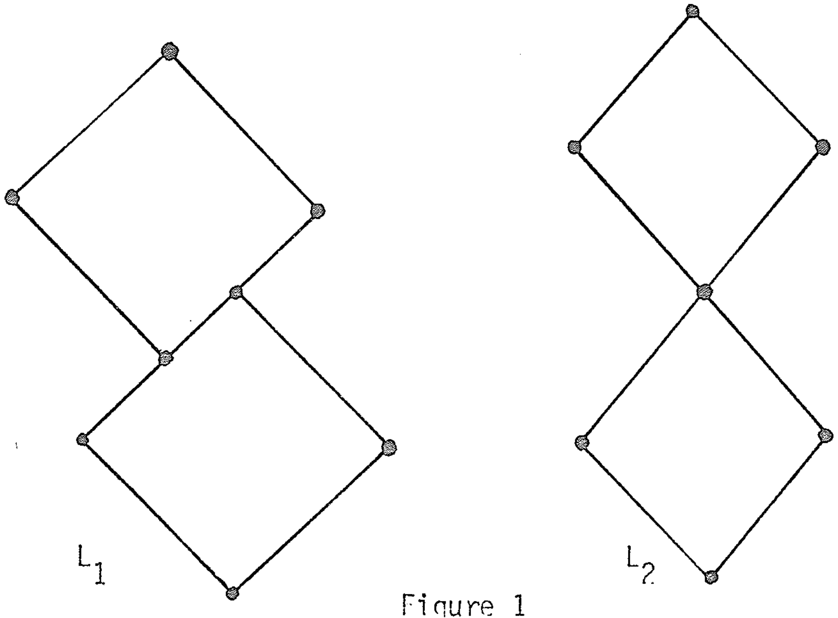


Figure 1

Therefore, proving that no transferable lattice can have L_1 as a sublattice would prove that every transferable lattice satisfies (W). This was done very recently by G. Grätzer and C.R. Platt in [17]. This result also proves that a sublattice of a transferable lattice is transferable, since it would also be a sublattice of FL(3) .

Finally, McKenzie ([23]) established the connection between bounded homomorphic images of FL(3) and sublattices of FL(3) by proving that a lattice is a bounded homomorphic image of FL(3) and it satisfies (W) if and only if it is a sublattice of FL(3) .

THE NOTATION:

In this survey, a lattice $\mathcal{L}$ and its underlying set L will be denoted by the same symbol L. The same applies for posets. The operations "join" and "meet" will be denoted by "+" and ".", respectively. Sometimes, a.b will simply be written ab.

If P is a subset of a lattice L, [P] will denote the sublattice of L generated by P. $\mathcal{M}_5$ will denote the 5-element modular nondistributive lattice, and $\mathcal{N}_5$ the 5-element non-modular lattice.

If a poset P is the disjoint union of k chains $c_1, \ldots, c_k$ of length $n_1, n_2, \ldots, n_k$ respectively, and no inclusion relations hold between elements of distinct chains, then P will be written $n_1 + n_2 + \ldots + n_k$ and will be called the sum of the disjoint chains $c_1, \ldots, c_k$.

For $a \leqslant b$ in a lattice L, the interval $[a, b]$ will sometimes be denoted by b/a (read: the <u>quotient</u> b over a). Also, the expression "$b \succ a$" means that "b <u>covers</u> a", i.e. there is no $x \in L$ such that $a < x < b$.

By a <u>canonical sum representation</u> for an element x in a lattice L, we mean a subset T_x such that 1) T_x is join-irredundant, 2) $\bigvee T_x = x$, 3) If H is a subset of L such that $\bigvee H = x$, than for every $y \in T_x$, there exists a $z \in H$ such that $y \leqslant z$. A canonical meet representation is defined dually.

1. BASIC DEFINITIONS AND THEOREMS:

1.1 DEFINITION: Let P be a poset and let $\underset{\sim}{K}$ be an equational class of lattices. A lattice $FK(P)$ is called a free lattice over $\underset{\sim}{K}$ generated by P if the following conditions are satisfied:

(1) $P \subseteq FK(P)$ and for $a,b,c \in P$, $\inf\{a,b\} = c$ in P if and only if $ab = c$ in $FK(P)$, and $\sup\{a,b\} = c$ in P if and only if $a + b = c$ in $FK(P)$;

(2) $[P] = FK(P)$;

(3) $FK(P) \in \underset{\sim}{K}$;

(4) Let $L \in \underset{\sim}{K}$ and let $\varphi : P \rightarrow L$ be a map with the property that if $a,b,c \in P$ and $\inf\{a,b\} = c$ in P, then $\varphi(a).\varphi(b) = \varphi(c)$ in L, and if $\sup\{a,b\} = c$ in P then $\varphi(a) + \varphi(b) = \varphi(c)$ in L. Then there exists a lattice homomorphism $\psi : FK(P) \longrightarrow L$ extending φ.

Let $\underset{\sim}{L}$ be the class of all lattices. Since every poset can be embedded in a lattice (see [15], p.44), the existence of $FL(P)$ is guaranteed by the following:

1.2 THEOREM: [15] Let P be a poset and let $\underset{\sim}{K}$ be an equational class of lattices. Then $FK(P)$ exists if and only if the following condition is satisfied: There exists a lattice L in $\underset{\sim}{K}$ such that $P \subseteq L$ and for a, b, $c \in P$, $\inf\{a,b\} = c$ in P if and only if $a.b = c$ in L and $\sup\{a,b\} = c$ in P if and only if $a + b = c$ in L.

If P is unordered and $|P| = m$, we write $FK(m)$ for $FK(P)$ and call it the free lattice on m generators over $\underset{\sim}{K}$.

1.3 COROLLARY: For any non-trivial equational class $\underset{\sim}{K}$, and for any cardinal m, a free lattice over $\underset{\sim}{K}$ with m generators, $FK(m)$ exists.

In the case where $\underset{\sim}{K} = \underset{\sim}{L}$ (the class of all lattices), it is possible to solve the "word problem", i.e. given two elements a, b of $FL(m)$, it is possible to decide whether $a \leq b$ or not.

1.4 THEOREM: [26] In the free lattice generated by a set of elements x_i,

(1) $x_i \leq x_j$ if and only if $i = j$

(2) recursively $a \leq b$ holds if and only if one or more of the following holds:

(a) $a \equiv a_1 + a_2$, where $a_1 \leq b$ and $a_2 \leq b$

(b) $a \equiv a_1 \cdot a_2$ where $a_1 \leq b$ or $a_2 \leq b$

(c) $b \equiv b_1 + b_2$ where $a \leq b_1$ or $a \leq b_2$

(d) $b \equiv b_1 \cdot b_2$ where $a \leq b_1$ and $a \leq b_2$

1.5 COROLLARY: In a free lattice, the following condition holds:

(W) If $ab \leq c + d$ then either $ab \leq c$ or $ab \leq d$ or $a \leq c + d$ or $b \leq c + d$.

It is possible to prove that (W) holds in $FL(3)$ without reference to the solution to the word problem (see Alan Day [36]).

The canonical form of an element in a free lattice is its representation as a polynomial of minimal length in the generators. Whitman proved that such a representation is unique up to commutativity and associativity.

In a free lattice on m generators, a subset $\{U_i \mid i \in R\}$ with $|R| = n$ is called a free set if the sublattice of FL(m) that it generates is isomorphic to FL(n) . Necessary and sufficient conditions for a subset of FL(n) to be free were given by Whitman [27] .

1.6 THEOREM: [27] A subset $\{U_i\}$ of the elements of FL(n) is free if and only if $U_j \leqslant \sum_{i \in S} U_i$ and its dual (where S is a finite set of indices) each imply $j \in S$.

In FL(3) , Whitman found a free set with four elements. This implies that FL(4) is isomorphically embedded in FL(3), and repeating the process on three of the generators of FL(4), he concluded that FL(3) contains FL(n) as a sublattice, for any finite or countable n. Since any three generators of FL(n) form a free set, we have

1.7 THEOREM: A lattice L is a sublattice of FL(3) if and only if it is a sublattice of FL(n), for any finite or countable n.

The solution of the word problem is not valid for FL(P), and it may be useful to consider another type of lattice generated by P, first introduced by Dilworth [8], and investigated by Dean [6].

1.8 DEFINITION: Let P be a poset. A lattice $CF(P)$ is called a completely free lattice generated by P if the following conditions are satisfied:

(1) $P \subseteq CF(P)$ and for $a,b \in P$, $a \leqslant b$ in P implies $ab = a$ in $CF(P)$.

(2) $[P] = CF(P)$.

(3) Let L be a lattice and let $\varphi : P \longrightarrow L$ be an isotone map with the property that $a \leqslant b$ in P implies $\varphi(a) \cdot \varphi(b) = \varphi(a)$ in L. Then there exists a lattice homomorphism ψ : $CF(P) \longrightarrow L$ extending φ.

Whereas $FL(P)$ preserves the least upper and greatest lower bounds of pairs of elements of P, $CF(P)$ does not. However, if no such bounds exist in P except for pairs of comparable elements, $CF(P)$ and $FL(P)$ are the same. Hence, if P is a sum of disjoint chains $c_1, c_2, \ldots c_k$, having $n_1, n_2, \ldots, n_k$, elements respectively we have $CF(n_1 + n_2 + \ldots + n_k) \cong FL(n_1 + n_2 + \ldots + n_k)$. This fact is important in view of the following:

1.9 THEOREM: (R.A. Dean, [7]) For any countable partially ordered set P, $CF(P)$ is a sublattice of $FL(3)$.

1.10 COROLLARY: (H. Rolf, [25]) $FL(n_1 + n_2 + \ldots + n_k)$ is a sublattice of $FL(3)$.

Hence, any sublattice of $FL(n_1 + n_2 + \ldots + n_k)$ is a sublattice of $FL(3)$. The converse of Corollary 1.10 is true if P is a sum of three or more disjoint chains, but it is not always true when P is the sum of two chains: neither $CF(2 + 2)$ nor $CF(4 + 1)$ contains $FL(3)$, however, both $FL(3 + 2)$ and $FL(5 + 1)$ do. (H. Rolf, [25]) (For the diagrams of $CF(2 + 2)$ and $CF(4 + 1)$ see appendix two). We finally need to recall one result of B. Jónsson on freely generated lattices.

1.11 THEOREM: [20] A lattice generated by a set X is freely generated by X if and only if it satisfies (W) and every finite nonempty subset of X is additively and multiplicatively irredundant.

2. ESSENTIAL PROPERTIES OF FL(3)

2.1 LEMMA: (Jonsson [18] and Whitman [26]). Let F be a free lattice on three generators and let $a,b,c,d \in F$. Then the following hold:

(W) If $ab \leqslant c + d$ then either $ab \leqslant c$ or $ab \leqslant d$ or $a \leqslant c + d$ or $b \leqslant c + d$.

(SD) If $d = a + b = a + c$ then $d = a + bc$.

(SD') If $d = ab = ac$ then $d = a(b + c)$.

Proof: (W) See Corollary 1.5.

(SD) Let $d = a + b = a + c$. The condition is trivially true if any two of a,b,c,d, are equal. So, we can assume that d is join reducible and has $\sum_{i \in S} U_i$ as its canonical representation. It follows from [26], that for every U_i , we have:

$$(U_i \leqslant a \text{ or } U_i \leqslant b) \text{ and } (U_i \leqslant a \text{ or } U_i \leqslant c)$$

If $U_i \leqslant a$, then $U_i \leqslant a + bc$. If $U_i \nleqslant a$, then we must have $U_i \leqslant b$ and $U_i \leqslant c$; hence $U_i \leqslant bc$. In both cases we get $U_i \leqslant a + bc$ for all $i \in S$, hence $U \leqslant a + bc$. The inverse inclusion is trivial.

(SD') This is simply the dual of (SD) , hence it follows from the above considerations since FL(3) is self dual.

Let us denote by $\underset{\sim}{S}$ the class of lattices that satisfy (W), (SD) and (SD'). Lemma 2.1 says that the class of sublattices of FL(3) is a subclass of $\underset{\sim}{S}$.

2.2 COROLLARY: For any lattice L in S , no element can be both join and meet reducible.

2.3 THEOREM: [18] A lattice L in $\underset{\sim}{S}$ which has finite length is finite.

Proof: This in fact is a corollary of (SD) alone (or (SD') alone, of course). The proof goes by induction on n, the length of the sublattice A, assuming it is true for lattices of length n-1. Let M be the set of atoms of A. We prove first that M is finite. Indeed, let $a \in M$, and $N = M - \{a\}$. Then $ab = o$ for every b in N. From (SD') and the finiteness of the length of A, it follows that $a \cdot (\sum_{b \in N} b) = o$ hence $\sum_{b \in N} b = c \neq 1$. Hence the lattice $[o,c]$ is of length at most $n - 1$, hence it is finite. It follows that N and M are finite. Now, for every atom $a \in M$, the interval $[a,1]$ is of length at most $n - 1$, hence it is finite. Since the lattice A is contained in $(\bigcup_{a \in M} [a,1]) \cup \{o\}$ and there is only a finite number of such intervals, it follows that A is finite.

A first induction shows that A can have at most n atoms: note that the atoms of $[o,c]$ are exactly those $b \in N$. A second induction shows that $|A| \leqslant 2(n!)$.

2.4 THEOREM: [18] A lattice L in $\underset{\sim}{S}$ that satisfies the double chain condition is finite.

Proof: Let us first assume that every quotient properly contained in A is finite. For $x \in A$, $x \notin \{o,1\}$, x must contain some atom $a \in A$ (since every descending chain is finite).

Let B be the set of atoms different from a. We claim that this set is finite. If not there should be a denumerable subset of atoms $\{b_1,\ldots, b_m,\ldots\} \subseteq B$. But the chain $\{a, a + b_1, a + b_1 + b_2,\ldots, a + b_1 +\ldots + b_m,\ldots\}$ must be finite, i.e. there exists an element $c = b_1 + \ldots + b_n$ and by (SD') $ac = o$. Hence $c \neq 1$, and $(c]$ is finite. Hence we can only have finitely many atoms, and for every atom a, the interval $[a)$ is finite, hence A is finite. Now let A properly contain some infinite quotient $[a,b]$, and let $[a,b]$ properly contain some infinite quotient $[a_1,b_1]$ and so on. We get a descending chain $\{b,b_1,b_2,\ldots\}$ and an ascending chain $\{a,a_1,a_2,\ldots\}$. Since both must be finite, there exists some n such that $[a_n,b_n]$ is infinite but every quotient properly contained in $[a_n,b_n]$ is finite. From the first part of the proof we conclude that $[a_n,b_n]$ must be finite, contradicting the assumption that A contains some infinite quotient. This completes the proof of the theorem.

2.5 THEOREM: (Dean [7]) A lattice L in $\underset{\sim}{S}$ which is generated by three distinct elements a,b,c such that $ab = ac = bc$ is given by the following diagram, or one obtained by identifying one or more of the following pairs : (a, a_1), (b, b_1), (c, c_1).

Proof: The condition (SD') implies that $a + b$, $a + c$, $c + d$ are distinct and incomparable. Hence, they generate an eight-element Boolean lattice. Let us denote its atoms by a_1,b_1,c_1. It remains

to prove that $a_1b_1 = ab = o$
and $a_1c_1 = b_1c_1 = o$.
Now: $c(a_1b_1) \leqslant c(a + b) = o$.
Also $b(a_1b_1) \leqslant b(a_1) \leqslant b(a + c) = o$.
Hence, by (SD') we get
$(c + b) \,.\, (a_1b_1) = o$, but
$(c + b) \geqslant b_1 \geqslant a_1b_1$. It follows
that $a_1b_1 = o$.

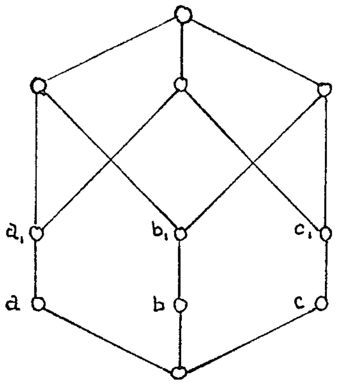


Figure 2

2.6 COROLLARY : $\mathfrak{M}_5$ does not belong to the class $\underset{\sim}{S}$. In particular, every modular sublattice of FL(3) is distributive.

Note that the lattice of 2.5 does occur in FL(3), with none of the pairs (a,a_1), (b,b_1), (c,c_1) identified: it is the sublattice generated by $\{x + (x + y)(x + z)(y + z),\ y + (x + z)(x + y)(y + z),\ z + (x + y)(x + z)(y + z)\}$, where x,y,z are the generators of FL(3).

The lattice generated by a four element join irredundant set

To get the main result concerning this situation we need a few lemmas which are due to B. Jónsson and J.E. Kiefer. These results are not only valid for sublattices of a free lattice, but for any lattice L in the class $\underset{\sim}{S}$.

2.7 LEMMA: [20] If a lattice A satisfies (W) and $a_1, a_2, a_3, v \in A$ are such that:

(i) $a_1 \nleq a_2 + a_3 + v$ and cyclically

(ii) $v \nleq a_i$ for $i = 1, 2, 3$

(iii) v is multiplicatively irreducible,

then A contains a free lattice with three generators as a sublattice.

Proof: The condition (W) together with (i), (ii), (iii) will ensure that the set $\{b_1, b_2, b_3\}$ where $b_1 = a_1 + (a_2 + v)(a_3 + v)$ and cyclically is a free set (see Theorem 1.6). By Theorem 1.11, the result follows.

2.8 LEMMA: [20] The following are equivalent:

(i) For all $a, b, c \in A$, $u = a + b = a + c$ implies $u = a + bc$

(ii) If u is an element of a lattice A, for any positive integers m, n and for all $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$, $u = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ implies $u = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j$

(iii) If u is an element of a lattice A, any two sum representations of u have a common refinement. If u has canonical sum-representation then (i) - (iii) hold.

If A is finite and satisfies (i) – (iii), then u has a canonical sum representation.

Proof: (i) implies (ii): We prove the stronger statement, that (i) implies the following statement denoted by $P(n,m)$: $u = v + \sum_{i=1}^{m} a_i = v + \sum_{j=1}^{n} b_j$ implies $u = v + \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j$. $P(1,1)$ is just (SD), so we proceed to prove $P(1,n)$ by induction on n. Let $u = v + a_1 = v + \sum_{j=1}^{n} b_j$. Let $b_n' = b_1 + \ldots + b_{n-1}$. Then

$u = (v + b_n') + a = (v + b_n') + b_n$ (since $b_n \leqslant u$). Therefore, by (i), $u = (v + b_n') + a \, . \, b_n$. But also $u = v + a_1b_n + a$, (since $a_1b_n \leqslant u$). So $P(1,n-1)$ together with the two previous lines imply $u = (v + a_1b_n) + \sum_{j=1}^{n-1} a_1b_j$, i.e. $u = v + \sum_{j=1}^{n} a_1b_j$. To prove $P(M,N)$, (where $m > 1$), let $a_m' = a_1 + \ldots + a_{m-1}$, and let us have $u = v + a_m' + a_m = v + \sum_{j=1}^{n} b_j$. Since $a_m' \leqslant u$, we can rewrite the last equality, $u = v + a_m' + a_m = v + a_m' + \sum_{j=1}^{n} b_j$. Now, $P(1,n)$ implies $u = v + a_m' + \sum_{j=1}^{n} a_m b_j$. Therefore $u = v + (\sum_{j=1}^{n} a_mb_j) + \sum_{i=1}^{m-1} a_i = (v + \sum_{j=1}^{n} a_mb_j) + \sum_{j=1}^{n} b_j$ and by $P(n-1, m)$, we get $u = (v + \sum_{j=1}^{n} a_mb_j) + \sum_{i=1}^{n-1}\sum_{j=1}^{m} a_ib_j = v + \sum_{i=1}^{m}\sum_{j=1}^{n} a_ib_j$

The implication of (iii) by (ii) is obvious. To prove that (iii) implies (i), let us have $u = a + b = a + c$, and let us have a common refinement $u = \sum_{i=1}^{n} d_i$. As in the proof of lemma 2.1, it follows that every d_i is either contained in a or contained in both b and c, and the result follows in the same way. The last two statements of the Theorem are obvious.

2.9 COROLLARY: [20] In a finite lattice A that satisfies (SD), every element has a canonical sum representation.

2.10 LEMMA: [16] If a finite lattice A satisfies (W) and (SD') and if the elements $a_1, a_2, a_3, v \in A$ are such that:

(i) $a_1 \nleqslant a_2 + a_3 + v$ and cyclically

(ii) $v \nleqslant a_i$ for $i = 1, 2, 3$

then $w = (a_2 + a_3 + v)\ (a_3 + a + v)\ (a_1 + a_2 + v)$.

Proof: The proof will be completed by applying (2.7) to the elements $b_i = a_i + v$ $(i = 1, 2, 3)$ and $w = w_1 + w_2 + w_3$ where $w_1 = (a_1 + v)\ (a_2 + a_3 + v)$ and cyclically. Since A is finite, one of the three conditions (i) – (iii) in (2.7) must fail. It is easy to check that $b_1 \nleq b_2 + b_3 + w$ and cyclically (by expanding the right hand side and applying (W)). Now one of the two remaining conditions of (2.7) must fail, since A is finite. So, either one of the b's contain w, or else w is multiplicatively reducible hence additively irreducible. If $w \leqslant b_1$, i.e. $b_1 \geqslant w_2$ and $b_1 \geqslant w_3$, then $(a_2 + v)\ (a_3 + a_1 + v) \leqslant a_1$ and $(a_3 + v)\ (a_1 + a_2 + v) \leqslant a_1 + v$. Now applying (W), we get that

$$v = (a_2 + v)\ (a_3 + a_1 + v) = (a_3 + v)\ (a_1 + a_2 + v)$$

and by (2.8) (ii) we get the desired conclusion. If w is additively irreducible, then $w = w_i$ for some i, say $i = 1$, which means that $w \leqslant b_1$, and the conclusion follows as above.

2.11 THEOREM: [20] If a lattice A is generated by an additively irredundant four element set $\{P_1, \ldots, P_4\}$ such that $(P_2 + P_3 + P_4)(P_3 + P_4 + P_1)(P_4 + P_1 + P_2) = P_4$ and cyclically, then the order of A is 22 and A is isomorphic to the lattice B_4 generated by the atoms in a free lattice on four generators. (see figure 3)

Proof: Define $z = p_1 p_2 p_3 p_4$, $u = p_1 + p_2 + p_3 + p_4$, $q_1 = p_2 + p_3 + p_4$ and cyclically, $a_{ij} = p_i + p_j$, $b_{ij} = q_i q_j$

$(i,j = 1, 2, 3, 4$ and $i \neq j)$. The proof is done by checking the addition and multiplication table, looking at all possible combinations.

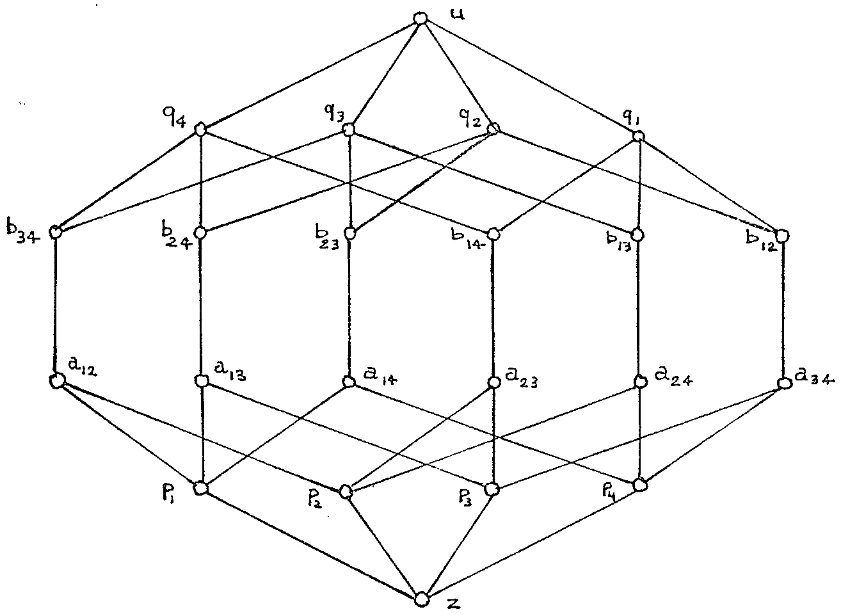


Figure 3 : B_4

The next result follows at once:

<u>2.12 THEOREM</u>: [20] If A is a finite lattice that satisfies (W) and (SD'), and if A contains a four element subset that is additively irredundant, than A contains a sublattice isomorphic to B_4.

<u>Proof</u>: Apply (2.10) and (2.11).

We have now determined all the lattices in the class $\underset{\sim}{S}$ which are generated by their atoms when the number of atoms is less than or equal to four. The next results show that this is the maximum number of atoms a finite lattice in $\underset{\sim}{S}$ can have.

2.13 LEMMA: [20] If A is a finite lattice that satisfies (W), then every representation of an element as a sum or a product of more than four elements is redundant.

Proof: Let $u = a_1 + a_2 + \ldots + a_n$ where $n \geqslant 5$, and assume this representation to be irredundant. Apply (2.7) with $v = a_4 + \ldots + a_n$; Conditions (i) and (ii) are obviously satisfied, and (iii) is satisfied because v is join-reducible hence meet-irreducible. Therefore A would contain FL(3), but A is finite. Hence one of (i)–(iii) must fail and the representation is not irredundant.

2.14 THEOREM: [20] A finite lattice A that satisfies (W) and (SD') contains at most four atoms.

Proof: (SD') implies that the atoms form an additively irredundant set. Hence by (2.13) we can have at most four atoms.

3. FORMING NEW SUBLATTICES OF FL(3)

3.1 LEMMA: (B. Jónsson [18]) Let m be an infinite cardinal. For any $a,b \in FL(m)$ such that $a < b$, the interval $[a,b]$ contains $FL(m)$ as a sublattice.

Proof: (J. Berman [4]) Let X be the generating set of $FL(m)$, and let $S \subseteq X$ be such that $a,b \in [S]$, the sublattice of $FL(m)$ generated by S. Let $T = X-S$ and let $U = \{(a + x) \, . \, b \mid x \in T\}$. It is obvious that the sublattice generated by U is contained in $[a,b]$. We now prove that $[U] = FL(U)$, and since S can be chosen to be finite, the result will follow. This means that we have to prove that U is a free set (as defined in section 1). Let us have $U = \{u_i \mid i \in I\}$, and $u_i \leq \sum_{j \in J} u_j$. We'll prove that $i \in J$.

Indeed, let us define the mapping $f{:}X \to FL(S)$ by

$$f(x) = \begin{cases} x & \text{if } x \in S \\ \bar{a} & \text{if } x \notin S \text{ and } (a+x) \cdot b = u_j \text{ for some } j \in J \\ \bar{b} & \text{otherwise} \end{cases}$$

(where $\bar{a}, \bar{b}$ correspond to a and b in the isomorphism $[S] \cong FL(S)$). Extend this mapping to a homomorphism $h{:}FL(X) \to FL(S)$. Under this homomorphism every u_j $(j \in J)$ is mapped onto $\bar{a}$. If $i \notin J$, u_i would be mapped to $\bar{b}$, and we would get $\bar{b} \leq \bar{a}$, a contradiction. Hence $i \in J$. The dual statement is proved similarly.

3.2 COROLLARY: (Jónsson [18]) If the lattice A is the union of a denumerable chain $\mathcal{A}$ of sublattices each of which is isomorphic to a sublattice of a free lattice on three generators, then A is isomorphic to a sublattice of a free lattice on three generators.

Proof: It is enough to recall that $FL(3)$ contains an infinite chain isomorphic to $\mathcal{A}$.

3.3 COROLLARY: (Jónsson [18]) Let L be a lattice, A and B two sublattices of L such that $A \cap B = \emptyset$, $A \cup B = L - \{0, 1\}$ and $a \cdot b = o$, $a + b = 1$ for every $a \in A$, $b \in B$. Then: if A and B are sublattices of $FL(3)$, so is L.

Proof: It is enough to notice that the lattice $\mathcal{N}_6$ of figure 4 is a sublattice of the lattice of figure 2 (Theorem 2.5). Hence, $\mathcal{N}_6$ is a sublattice of $FL(\aleph_0)$, and so are A and B. The interval $[x, y]$ in $FL(\aleph_0)$ contains an isomorphic copy $\bar{A}$ of A, and the interval $[x_1, y_1]$ contains an isomorphic copy $\bar{B}$ of B. Therefore the lattice $\bar{B} \cup \bar{A} \cup \{0, 1\}$ is a sublattice of $FL(\aleph_0)$ hence of $FL(3)$.

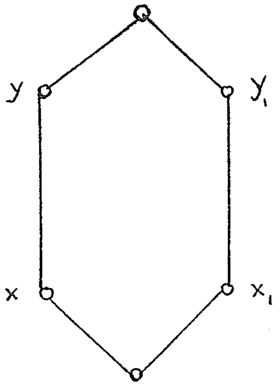


Figure 4

3.4 COROLLARY: Size of a sublattice of $FL(3)$ of finite length (Jónsson and Kiefer [20]). Upper and lower bounds can be got for the maximum size f_n of such a lattice of length n. Let A be a lattice attaining such a maximum (it exists, since an upper bound was already found in (2.3)). Forming the lattice of 3.3 with $B = A$, we get a lattice having $2f_n + 2$ elements and length $n + 2$. Therefore: $f_{n+2} > 2f_n$. Since $f_0 = 1$ and $f_1 = 2$, we get $f_n > (\sqrt{2})^n$. Also, if A is a lattice of length n and p is an atom, then $\{x \mid x \geqslant p\}$ and $\{x \mid x \in A \text{ and } px = o\}$ are sublattices of A of length at most $n - 1$. Hence $f_n \leqslant 2f_{n-1}$; therefore $f_n \leqslant 2^n$.

A sharper upper bound was mentioned by Jónsson and Kiefer, but it requires much more involved computations. For instance, it is known that $\lim_{n\to\infty} \sqrt[n]{f_n} \leqslant R$, where R is the positive root of the equation $R^5 - R^4 - R^2 - 1 = 0$, which is less than 1.571. The conjecture is that $\lim_{n\to\infty} \sqrt[n]{f_n} = \sqrt{2}$.

4. DISTRIBUTIVE SUBLATTICES OF FL(3):

All denumerable distributive sublattices of a free lattice were characterized by F. Galvin and B. Jónsson [10]. These are all the distributive lattices that are countable and satisfy (W). ((SD) and (SD') follow from distributivety anyway).

4.1 DEFINITION: A lattice A will be called linearly indecomposable if A cannot be written as $B \cup C$ where B and C are sublattices of A and $b < c$ for all $b \in B$, $c \in C$.

4.2 LEMMA: [10]. If D is a distributive lattice satisfying (W) and x, y, z are such that no two of them are comparable, then they generate the eight-element boolean lattice.

Proof: We prove the lemma by proving that x, y, z satisfy either $xy = xz = yz$ or $x + y = x + z = y + z$ and in this case the lemma follows from (2.5). Since D is distributive, then

$$(x + y)(x + z)(y + z) = xy + xz + yz \qquad (1)$$

But D satisfies (W), hence if the left hand side is join-irreducible, we must have $xy \geqslant xz + yz$ (say); Again, xy is meet-reducible, hence join-irreducible, so we have, say: $xy = xz$ (2)

Now $y \geqslant xy$, $z \geqslant xz = xy$, therefore: $yz \geqslant xy$ (3)

Also: $x + (yz) = (x + y)(x + z)$ (4)

If the left hand side in (4) is join-reducible, we conclude that either $x + y = x + (yz)$ or $x + z = x + (yz)$. The first

possibility implies that $\{x, y + z, y, yz, xy\}$ forms a lattice isomorphic to $\mathcal{N}_5$, the second possibility implies that $\{x, x + z, z, yz, xy\}$ forms an $\mathcal{N}_5$. Hence the left hand side in (4) must be join-irreducible and $x + yz = x$ (for $x + yz = yz$ would imply that x and y are comparable). Now $yz \leqslant x$, hence $yz \leqslant xy$. From (3), we conclude that $xy = yz = xz$. If the left hand side of (1) is meet-irreducible, we get the dual conclusion.
This completes the proof of the lemma.

4.3 LEMMA: [10] Let D be a linearly indecomposable distributive lattice having (W). Then the width of D is at most 3. If the width of D is 3, D is a Boolean lattice with eight elements.
Proof: If the width of D is 3 or more, then D contains 3 mutually incomparable elements, hence generating B, the Boolean lattice with eight elements. Let u and z be the unit and zero of B, respectively, and let p, q, r be the atoms of B.

No element of D can lie between z and any of the atoms. For if x was such an element, lying between z and say p, then the sublattice generated by x and B would not satisfy (W), for $q + x = (q + p)(p + x + r)$. Now let $d \in D - B$; let $p' = z + pd$, $q' = z + qd$, $r' = z + rd$. Since p is additively irreducible, either $p' = z$ or $p = pd \leqslant d$ and similarly for q and r. All different possibilities yield $d > u$ or $d < z$. Now if $D - B$ is not empty, it is a sublattice of D: For if $a, b < z$, then $a + b < z$ since z is join-irreducible; and similarly for u, i.e. D is linearly decomposable, contrary to our assumption. Hence $D - B$ must be empty.

4.4 LEMMA: [10] Let D be a linearly indecomposable denumerable distributive lattice satisfying (W). If the width of D is 2, then D is isomorphic with a direct product of a chain C and the two-element chain.

Proof: Let a_1 be any element of D. Then there exists some a_2 which is incomparable with a_1 (otherwise D would be decomposable). We will construct the sublattice of D whose underlying set is $\{x \mid x \geqslant a_1 \cdot a_2\}$. Every element of D is comparable either to a_1 or to a_2 or to both. Write the set $D_1 = \{x \mid x \geqslant a_1 \text{ or } x \geqslant a_2\}$ as a disjoint union $D_1 = D_{11} \cup D_{12} \cup D_{13}$ where $D_{11} = \{x \mid x > a_1 \text{ and } x \not\geqslant a_2\}$, $D_{12} = \{x \mid x > a_2 \text{ and } x \not\geqslant a_1\}$ $D_{13} = \{x \mid x \geqslant a_1 + a_2\}$. D_{11} and D_{12} are chains, obviously.

Claim 1: $a_1 + a_2$ covers at least one of a_1 or a_2.

If not, then there exist b_1, b_2 such that $a_1 < b_1 < a_1 + a_2$ and $a_2 < b_2 < a_1 + a_2$. Then $b_1 \cdot b_2$ must be comparable to at least one of a_1 or a_2, since the width of D is two. But

$b_1 b_2 \geqslant a_1$ implies $b_2 \geqslant a_1 + a_2$ (contradiction);

$b_1 b_2 \geqslant a_2$ implies $b_1 \geqslant a_1 + a_2$ (contradiction);

$b_1 b_2 \leqslant a_1$ (or a_2) implies that $\mathcal{N}_5$ is a sublattice of D.

Hence no such pair can exist, and the claim is proved. Without loss of generality, we can assume that $a_1 + a_2 \succ a_2$.

Claim 2: $[a_1 a_2, a_2] \cong [a_1, a + a_2]$.

This is trivially true because of distributivity, and in fact $[a_1a_2, a_1 + a_2] \cong C_1 \times \underset{\sim}{2}$, where C_1 is a chain.

Claim 3: $D_{12} = \emptyset$ implies $D_{13} = \emptyset$ (i.e. $a_1 + a_2$ is the unit of D). If D_{13} is not empty, it becomes a decomposable sublattice of D, contrary to the hypothesis. In this case, $[a_1a_2) \cong C_1 \times \underset{\sim}{2}$. So assume $D_{12} \neq \emptyset$.

Claim 4: $x \in D_{11}$ implies $x \leq a_1 + a_2$.

If not, then for any $y \in D_{12}$ the elements x, $a_1 + a_2$, y would be mutually incomparable.

Claim 5: For every $x \in D_{12}$, $x(a_1 + a_2) = a_2$.

This is a direct corollary of claim 1.

Claim 6: For $x_i, x_j \in D_{12}$ such that $x_i > x_j$, let $y_n = x_n + (a_1 + a_2)$ $(n = i, j)$. Then $y_i > y_j$.

$y_i = y_j$ would contradict distributivity.

Claim 7: For every $y \in D_{13}$, $(y \neq a_1 + a_2)$ y is of the form $x + (a_1 + a_2)$ for some $x \in D_{12}$.

Assume that for no $x \in D_{12}$, $y = x + (a_1 + a_2)$. First, $y < x$ for no $x \in D_{12}$, by the definition of this set. Also, if $y > x$ for all $x \in D_{12}$, then D is decomposable. Therefore there must exist $x \in D_{12}$ such that x and y are incomparable. But then the sublattice $\{x, x \cdot y, (x \cdot y) + (a_1+a_2), y, x+y\}$ is isomorphic to $\mathcal{N}_5$. Hence every $y \in D_{13}$ is of the form $x + (a_1 + a_2)$ for some $x \in D_{12}$, and it follows that $[a_1a_2) \cong C_2 \times \underset{\sim}{2}$ where C_2 is the linear sum of C_1 and $[a_1, a_1 + a_2]$. By duality the sublattice $(a_2]$ has the dual structure of $[a_1)$, since $a_2 \succ a_1a_2$, and

the lattice D is isomorphic to the product $C \times \underset{\sim}{2}$ where C is the linear sum of C_1, C_2 and $\{x \mid x < a_1, x \nless a_2\}$.
This completes the proof of the lemma.

4.5 LEMMA: 10 Every simply ordered subset of a free lattice is denumerable.

Proof: Let F be freely generated by X. If X is denumerable, the result is trivial. So, let us assume that X in non-denumerable. Let X_0 be a denumerable infinite subset of X, and let F_0 be the sublattice of F generated by X_0. Define the following relation on F: $a \equiv b$ if and only if there is an automorphism f of F such that $f(a) = b$. This is obviously an equivalence relation. Every equivalence class contains an element of F_0: given $a \in F$, there exists a finite subset Y of X such that a belongs to the sublattice generated by Y. There also exists a permutation that maps Y into X_0, and it can be extended to an automorphism f under which $a \equiv f(a) \in F_0$. Hence, the number of equivalence classes is denumerable. Now given a simply ordered subset of F, we will show that no two of its members can belong to the same class. Indeed assume that $f(a) = b$, that $a \leqslant b$, and that Y is a finite subset of X such that a belongs to the sublattice $[Y]$ generated by Y. Define $g: F \rightarrow F$ in the following way: $g(x) = f(x)$ whenever $x \in Y$, $g(x) = x$ whenever $x \in X - Y$, and extend it to an automorphism. So $g(x) = f(x)$ on Y. Also, g is of finite order, since it permutes a finite number of generators. So we get:

$b = g(a) \leqslant g^2(a) \leqslant \ldots \leqslant g^n(a) = a$ from which it follows that $a = b$.

4.6 THEOREM: [10] For any distributive lattice D, the following are equivalent:

(1) D is isomorphic to a sublattice of $FL(3)$.

(2) D is isomorphic to a sublattice of a free lattice.

(3) D is the union of a denumerable linearly ordered family $\mathscr{C}$ of sublattices where each member of $\mathscr{C}$ is either a one-element lattice or an eight-element Boolean lattice, or a direct product of a two-element chain and a denumberable chain.

(4) D is denumerable and satisfies (W).

Proof: Clearly (1) implies (2).

(2) implies (3): This follows from (4.3), (4.4) and (4.5).

(3) implies (4): Obvious, from (4.3), (4.4), (4.5).

(3) implies (1): In the free lattice on 3 generators, the atoms generate the boolean lattice with eight elements. Also, if C is a denumerable chain, it has an isomorphic copy in the free lattice on 3 generators under some isomorphism f.

Now the map: $g: \{0,1\} \times C \to FL(5)$ defined by

$g: \langle 1, c\rangle \mapsto x_4 + (x_5 \cdot f(c))$; $g: \langle o, c\rangle \mapsto (x_4 + (x_5 \cdot f(c))) \cdot x_4$

(where $x_1, \ldots, x_5$ are the free generators of $FL(5)$, and f maps $\mathscr{C}$ into $FL(x_1, x_2, x_3)$) is an isomorphism. The rest of the statement is proved by applying 3.2.

For finite distributive lattices, the following characterization is due to W. Poguntke and I. Rival [24].

4.7 DEFINITION: [24] The property $\underset{\sim}{\Lambda}(L)$ is said to hold for a lattice L if every lattice which contains an order-isomorphic copy of L also contains a lattice-isomorphic copy of L.

4.8 THEOREM: [24] For a finite lattice L the following conditions are equivalent:

(i) $\underset{\sim}{\Lambda}(L)$ holds;

(ii) L is a distributive sublattice of $FL(3)$.

Proof: We first prove that (i) implies that L is distributive and has no doubly reducible element.

Indeed, since every poset can be embedded in a distributive lattice, namely the lattice of its order ideals, it follows that L must be distributive. Also, if d is a doubly reducible element in L, and if d_1, d_2 are distinct elements not in L, then the set $K := (L - \{d\}) \cup \{d_1, d_2\}$ with the partial ordering $\leq'$ defined by:

for $x, y \in L - \{d\}$, $x \leq' y$ whenever $x \leq y$; $d_1 \leq' d_2$;

for $x \in L - \{d\}$, $i = 1, 2$, $x <' d_i$ $(x >' d_i)$ whenever $x < d$ $(x > d)$

is a lattice. $(K; \leq')$ contains an order-isomorphic copy of L, but no lattice-isomorphic copy of it.

Now, by Theorem 4.6, it is enough to prove that if L is the union of a finite linearly ordered family of sublattices each of which is either a one-element lattice or an eight-element Boolean lattice or a direct product of a two-element chain and a finite chain, then $\underset{\sim}{\Lambda}(L)$ holds.

If L is an eight-element Boolean algebra, let the lattice K contain an order-isomorphic copy of L, with the atoms of L mapped to a_1, a_2, a_3. Then it is easy to verify that the set $\{a_1 + a_2, a_1 + a_3, a_2 + a_3\}$ generates a lattice-isomorphic copy of L.

Now let the lattice K contain $C \times \underset{\sim}{2}$ as a sub-poset, where the elements of C are $c_1 < c_2 < \dots < c_n$. Denote $\langle c_i, 0 \rangle$ by a_i, and $\langle c_i, 1 \rangle$ by b_i. Then we can verify that the set $\{d_1, d_2, \dots d_n, e_1, e_2, \dots, e_n\}$ is a lattice-isomorphic copy of $C \times \underset{\sim}{2}$, where $d_i = a_n \cdot b_i$, and $e_i = b_i + a_i$.

Finally, let L be the union of a linearly ordered family of sublattices L_i, $(i \in I)$, where each L_i is as specified above, and let L be contained in K as a sub-poset. Now, by taking the universal bounds of each of the L_i's in K, it is possible to find, for each $i \in I$, and interval $[0_i, 1_i]$ in K containing L_i as a sub-poset. Hence $[0_i, 1_i]$ contains a lattice-isomorphic copy L_i' of L_i, and the lattice $L' = \bigcup L_i$ is isomorphic to L.

This completes the proof of the theorem.

5. THE DECOMPOSITION THEOREM:

Under the hypothesis of 2.12, B. Jónsson and J.E. Kiefer proved that the lattice A can be expressed as a union of certain sublattices, and that A is isomorphic to a sublattice of a free lattice if and only if each of the summands is isomorphic to a sublattice of a free lattice.

5.1 THEOREM: [20] Suppose A is a finite lattice in the class $\underset{\sim}{S}$ and assume that the lattice B_4 (Fig. 3) is a sublattice of A. Let $A' = A - \{x \mid x \in A \text{ and } z < x < u\}$ $\quad C_{ij} = \{x \mid x \in A \text{ and } a_{ij} < x < b_{kl}\}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$ Then A' is a sublattice of A, each C_{ij} is a sublattice of A and

$$A = A' \cup B_4 \cup C_{12} \cup C_{13} \cup C_{14} \cup C_{23} \cup C_{24} \cup C_{34}$$

Furthermore: If $\{i, j, k, l\} = \{1, 2, 3, 4\}$ then:

(i) $x \in C_{ij}$ and $y \in C_{ik}$ implies that $x + y = q_l$ and $xy = p_i$; $x \in C_{ij}$ and $y \in C_{kl}$ implies that $x + y = u$ and $xy = z$.

(ii) If $x \in A'$ and $z \leqslant y \leqslant u$ then: $x \nleqslant z$ implies that $x + y = x + u$; $x \ngeqslant u$ implies that $xy = xz$.

Proof: We start by proving that the set A is the union of A', B_4 and the six sets C_{ij}. Let $v \in A - B_4$, and $z < v < u$. We show that $v \in C_{ij}$ for some i and j. But first, we prove that p_i covers u, and that a_{ij} covers p_i; it will follow by duality that u covers q_i and that q_i covers b_{ij}.
Indeed let p_i cover some element $d \geqslant z$. The condition (W) applied

on a_{ik}, a_{il}, d and p_{ij} implies that $p_i \leqslant p_j + d$. Hence $p_i(p_j + d) = d$ and the same is true if j is replaced by k or l. Now conditions (i) and (ii) of 2.7 are satisfied with $\{a_1, a_2, a_3\} = \{p_j, p_k, p_l\}$ and $v = d + p_j$. But since A is finite, then v must be join irreducible i.e. $d \leqslant p_j$. It follows that $d = z$. Also, let $p_i \leqslant d < a_{ij}$. Then $p_j\, d = z$ and $p_j q_j = z$. So, by (SD'), we have $p_j(d + q_j) = z$. i.e. $d < q_j$. Hence $d \leqslant a_{ij} q_j = p_i$, i.e. $d = p_j$.

Now, if $z < v < u$, v must contain some atom, say p_i, since u/z cannot have more than four atoms. If v does not contain any other atom, then $v \not\geqslant a_{ij}$ and $v \,.\, a_{ij} = p_i$ (and similarly with j replaced by k or l). Now $p_i = va_{ij} = va_{ik} = va_{il}$ hence $p_i = v(a_{ij} + a_{ik} + a_{il}) = vu = v$ i.e. $p_i = v$, which contradicts the condition that $v \notin B_4$. So, we must also have $v \geqslant p_i$; Repeating the same type of arguments we conclude that either $v \in B_4$ or $v \in C_{ij}$ and only the last possibility is compatible with our assumption. This proves the claim that

$$A = A' \cup B_4 \cup C_{12} \cup C_{13} \cup C_{14} \cup C_{23} \cup C_{24} \cup C_{34}.$$

It is obvious that the sets C_{ij} are sublattices of A. It is also clear that (i) holds, since $a_{ij} + a_{ik} = q_l$, $b_{kl} \cdot b_{jl} = p_i$, etc... To prove (ii), we prove that if $x \in A'$ and $x \nleqslant z$ then $u \leqslant x + z$. For if $u \nleqslant x + z$, we would have $u(x + z) < q_i$ for some i. Hence $p_i(x + z) = p_i$, $u.(x + z) \leqslant p_i q_i = z$, therefore $p_i(x + z + q) = p_i(x + q_i) = z$. This obviously implies that $x = q_i \geqslant u$. But since u is the only element covering q_i, we conclude that

$x \leqslant q_i$. Hence $z \leqslant z + x \leqslant q_i$, and it follows that one of the atoms of u/z is contained in $z + x$, say $p_j \leqslant z + x$. Now: $u = q_j + p_j = q_j + x$, hence $u = q_j + p_j x$, hence $p_j x \geqslant p_j$, i.e. $x \notin A$. This proves that $u \leqslant x + z$, from which the first part of (ii) follows immediately. The second part follows by duality.

We still have to prove that A' is a sublattice of A. We first show that $A' - \{u,z\}$ is a sublattice of A, that is it is closed under joins and meets. Indeed, let $z \leqslant c + d \leqslant u$ then we show that $z \leqslant c$ or $z \leqslant d$. Assume this fails, then $q_1 q_2 q_3 q_4 = z \leqslant c + d$, hence $q_i \leqslant c + d$ for some i, (by (W)), and it follows that $c + d = u$ or $c + d = q_i$. If $c + d = u$, then c and d cannot both be contained in q_i, and we may assume that $c \nleqslant q_i$. Therefore $q_i + c = u$ and, since $q_i + p_i = u$ it follows by (SD) that $q_i + p_i c = u$. Since by hypothesis $z \nleqslant c$, we have $p_i \nleqslant c$ so that $p_i c < p_i$. Therefore $p_i c \leqslant z \leqslant q_i$ which is a contradiction. If $c + d = q_i$, we derive a contradiction again by noticing that b_{ij} cannot contain both c and d. Assume $c \nleqslant b_{ij}$, it follows that $b_{ij} + c = q_i$, and since $b_{ij} + p_j = q_i$, we get $b_{ij} + p_i c = q_i$. And as before, $p_i c \leqslant z$, and $b_{ij} = q_i$, a contradiction.

Now (ii) implies that A' is closed under the meet of any of its elements with u or z, and this completes the proof of the theorem.

5.2 THEOREM: (B. Jónsson and J.E. Keifer [20]) Under the hypothesis of (5.1), if A' and all the lattices C_{ij} are isomorphic to sublattices of free lattices, then so is A.

Proof: Let f be an isomorphism of B_4 into $FL(\aleph_0)$. C_{ij} can be mapped into $f(b_{kl})/f(a_{ij})$ by an isomorphism f_{ij}, since this quotient contains an isomorphic copy of $FL(\aleph_0)$ (see 3.1). Also, A' can be mapped into $FL(\aleph_0)$ by an isomorphism h, and $h(u)/h(z)$ contains an isomorphic copy of u/z. Let k be the map that agrees with h on $A' - \{u,z\}$, with f on B_4 and with f_{ij} on C_{ij}. Then k is seen to be an isomorphism.

This completes the proof of the theorem.

6. CONCLUSION:

It may be convenient to explicitly formulate various problems that have tacitly appeared in this survey, as well as to mention some others which are indirectly connected with them.

Problem 1: [Jónsson and Kiefer] Is every lattice satisfying (SD), (SD') and (W) a sublattice of FL(3)? Equivalently, is there a lattice which satisfies (W), (SD) and (SD') but which is not transferable?

The solution to this problem would be made easier if there was a way of describing lattices satisfying (W), (SD) and (SD'). In view of the result of [1], it is reasonable to formulate the question as follows:

Problem 2: Can the conditions (W), (SD) and (SD') be characterized by the non-containment of a finite list of finite lattices?

Three classes coincided in the class $\underset{\sim}{L}$ of all lattices: the class of finite sublattices of a free lattice, that of finite transferable lattices and that of finite projective lattices. The distributive lattices of these classes are exactly the lattices having the property $\underset{\sim}{\Lambda}$ (see 4.7). Therefore, it is reasonable to ask whether the same relationship holds between these concepts when generalised to an arbitrary class $\underset{\sim}{K}$ of lattices. Unfortunately, this is not the case; for example, every finite distributive lattice is a sublattice of some free lattice in the class $\underset{\sim}{D}$ of distributive lattices, however not every distributive lattice is projective in D. Therefore, we should recall a problem first raised in [24]:

Problem 3: Characterize those finite lattices L for which $\underset{\sim}{\Lambda}_{\underset{\sim}{D}}(L)$ holds. Are these, for example, just the finite lattices projective in $\underset{\sim}{D}$?

We should also mention that in [24], it was shown that every lattice satisfying $\underset{\sim}{\Lambda}_{\underset{\sim}{K}}(L)$ is transferable in the class $\underset{\sim}{K}$.

APPENDIX ONE

PROPOSITION: [1] In a finite lattice L satisfying (SD) and (SD'), the condition (W) fails if and only if L contains a sublattice isomorphic to L_1 or L_2 (see figure 1).

Proof: If L has a doubly reducible element, then L obviously has a sublattice isomorphic to L_2.

Now let us assume that (W) fails, and in fact let a_1, a_2, b_1, b_2 be such that $a_1 \cdot a_2 \leqslant b_1 + b_2$ and

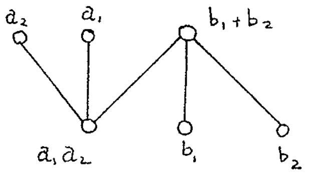
$a_i \nleq b_1 + b_2$, $a_1 \cdot a_2 \nleq b_i$ $(i = 1, 2)$.

Since L is finite, we can select a_1, a_2 to be minimal with respect to the failure of (W), that is

$a_i \succ a_i \cdot (b_1 + b_2)$ $(i = 1, 2)$

and dually:

$b_i \prec b_i + (a_1 \cdot a_2)$ $(i = 1, 2)$.



Now, we claim that there exists $m \in L$ such that at least one of the following holds: $b_1 + b_2 < m < a_1 + (b_1 + b_2)$, or

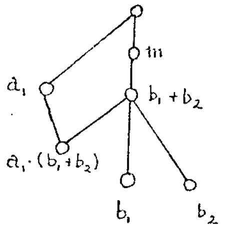
$b_1 + b_2 < m < a_2 + (b_1 + b_2)$.

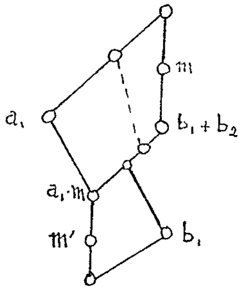
Indeed, if both $a_1 + (b_1 + b_2)$ and $a_2 + (b_1 + b_2)$ covered $(b_1 + b_2)$, we would have $a_1 + (b_1 + b_2) = a_2 + (b_1 + b_2)$ (since $b_1 + b_2$ is meet-irreducible) and $a_1 \cdot a_2 < b_1 + b_2$, contradicting the assumption that (SD) holds in L. Therefore, without loss of generality, we may assume that

$b_1 + b_2 < m < a_1 + (b_1 + b_2)$.

It follows from the minimality of a_1 that $a_1 \cdot m = a_1 \cdot (b_1 + b_2)$.

Replacing a_2 by m and dualizing the above argument, we may conclude that there exists $m' \in L$ such that $b_1 \cdot (a_1 \cdot m) < m' < a_1 \cdot m$. We also have: $b_1 + m' = b_1 + a_1 \cdot m$. Now, the set $\{(a_1 + b_1 + m')(b_1 + b_2), a_1, a_1 m, (a_1 + b_1 + m'), m', b_1, m'b_1\}$ forms a sublattice of L isomorphic to L_1. This completes the proof of the proposition.





APPENDIX TWO

Diagram of $FL(\check{2} + 2)$. [Rolf, 25]

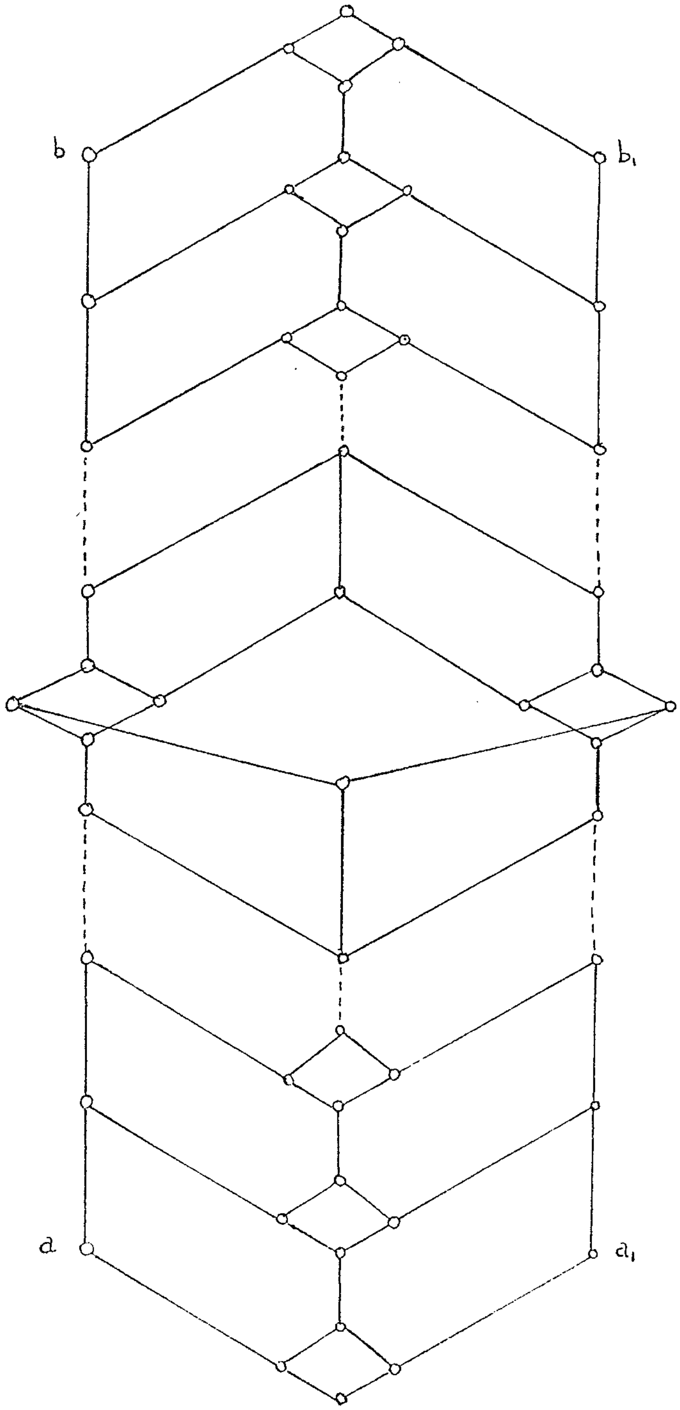
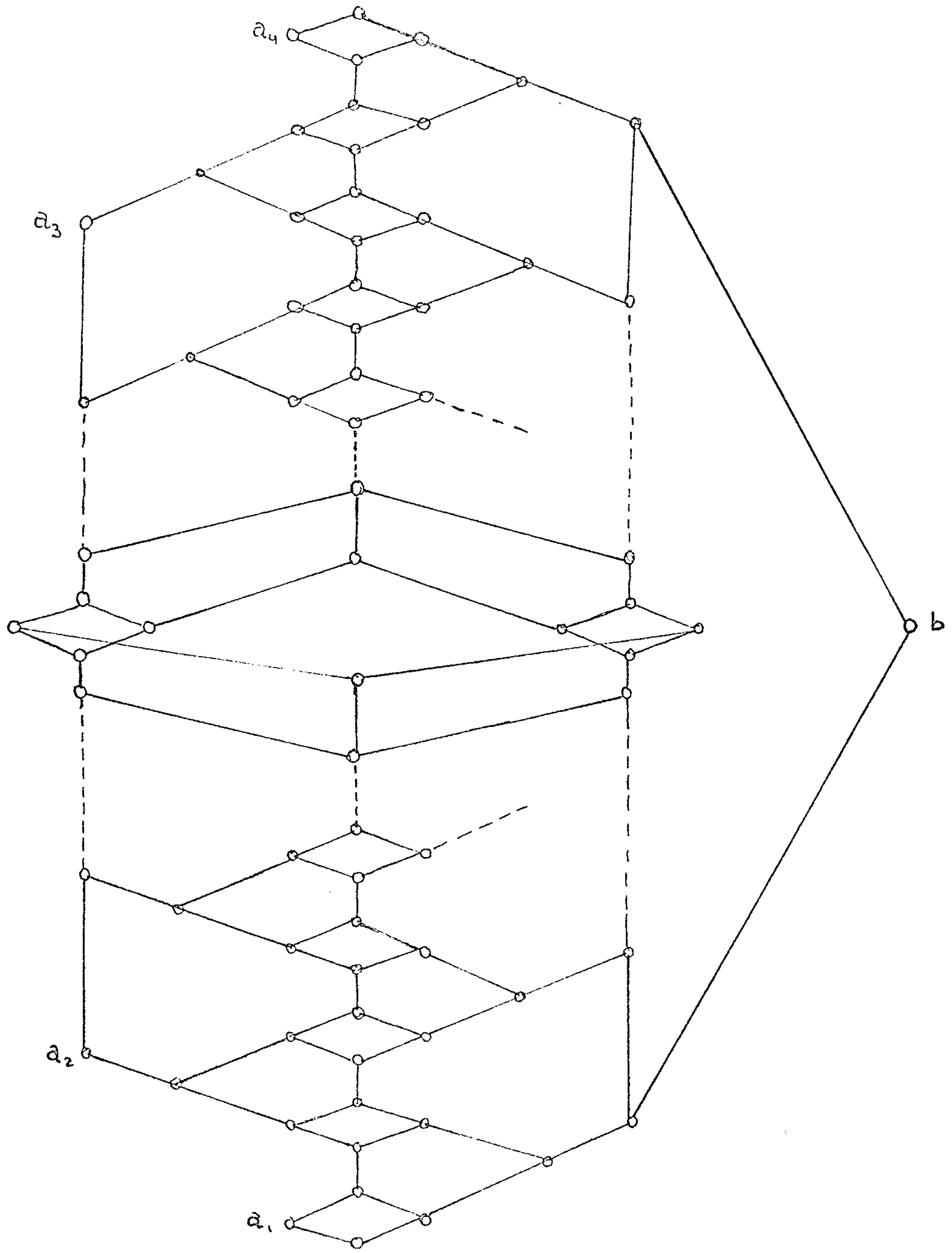
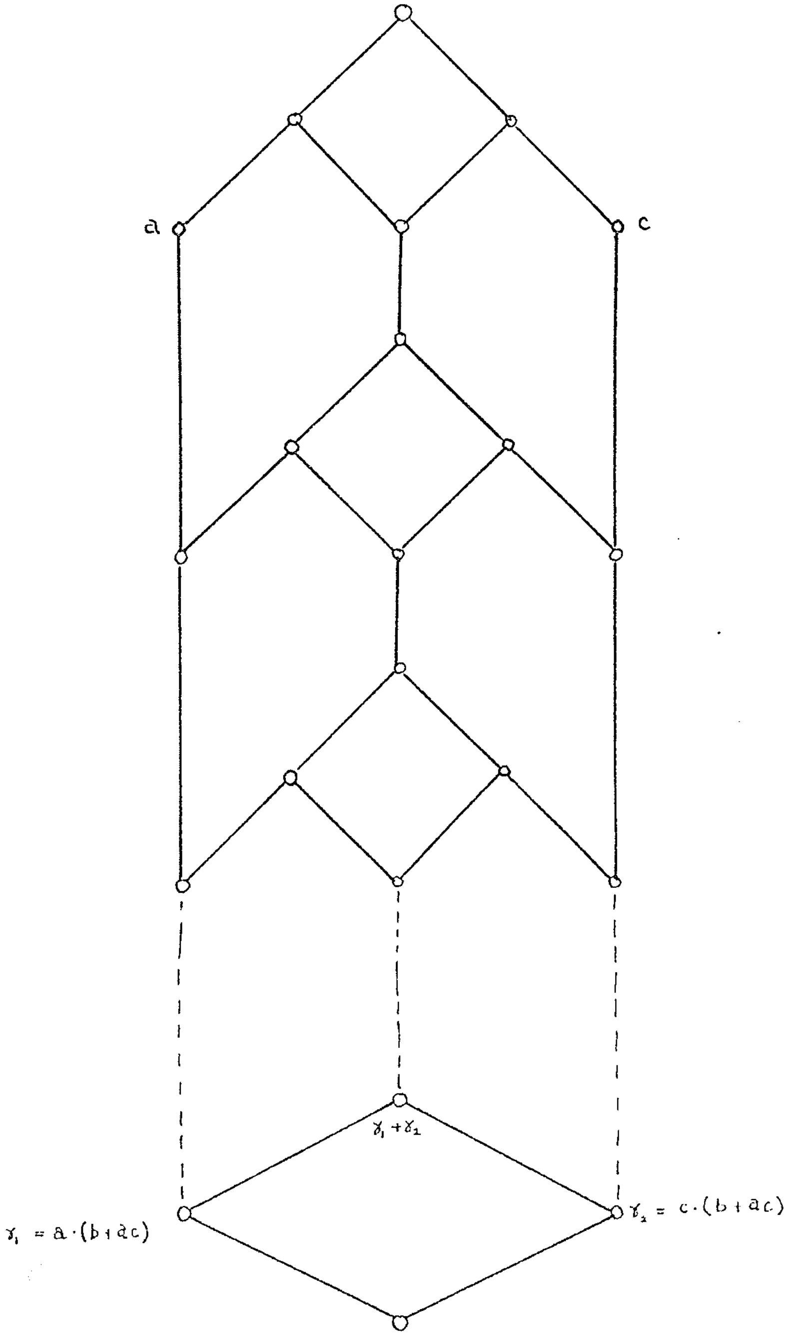


Diagram of FL(4 + 1). [Rolf, 25]



The sublattice of FL(3) generated by $\{a,(a(b+ac))+(c(b+ac)),\ c\}$ where a,b,c are the generators of FL(3). [Dean,7]



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