ANALYSIS OF INVENTORY LOT SIZE PROBLEM

BY

HARPREET GREWAL

A Thesis
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ABSTRACT

Lot size inventory control problem with ambiguous variable demand is important in industry as inventories can be a major commitment of monetary resources and affect virtually every aspect of daily operations. Inventories can be an important competitive weapon and are a major control problem in many companies, and improper lot sizes can affect the inventory levels and the costs associated with them.

In the present thesis, we formulate and analyze such a problem and its variations under both crisp and fuzzy environments with a variety of assumptions. Specifically, we model such a problem with imprecise variable demand and imprecise total cost that includes setup, backordering and holding costs, as zero-one integer linear program. Also, the problem is modeled as piece wise integer linear program by incorporating quantity discounts. Furthermore, we consider such a problem, with ambiguous variable demand to be satisfied as closely as possible. Therefore we use goal-programming approach to solve such a problem. The present thesis seeks to provide an alternate, easy to understand and hopefully improved means of obtaining optimal lot sizing solution.

The thesis demonstrates the adaptability of the approaches used, for decision making in a variety of applications, particularly scheduling purchases and production. The methods presented in this thesis are computationally effective and beneficial for determining the optimal solution for inventory lot sizing problems.

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Chapter 1

INTRODUCTION

The architecture of any in-house production system, built up from several production cells, may be implemented in different fashions (flow lines or work centers for instance). This macro-structure further refines as each production cell provides the capability to perform a group of operations. Raw materials and component parts float concurrently through the complex system in order to be processed and assembled, until a final product comes out ready for delivery.

Production planning and scheduling is one of the most challenging subjects for the management. It appears to be a hierarchical process ranging from long to medium to short term decisions. The Master Production Schedule (MPS) defines the external demand. The goal is to find a feasible production plan that meets the requirements and provides release dates and amounts for all products including component parts. For economical reasons, just finding feasible plan is not sufficient. Production plans can be evaluated by means of an objective function (e.g. a function that measures the setup and holding costs). Then, the aim of production planning is to find a feasible production plan with optimum or close to optimum objective function value.

Lot sizing is a significant aspect of Materials Requirement Planning (MRP) production planning process. Although its perceived importance has declined as a result of system development, wide spread Just in Time (JIT) orientations and development of satisfactory heuristics, lot sizing is still a major component of a balanced MRP operation. Optimizing routines for this multi-level lot-sizing problem have been shown to be all too demanding form a computing standpoint in both practical as well as research environments. The inability to efficiently solve even medium sized research problems, and use of unfamiliar or tedious problem reduction or solution methods, lead practitioners to use simpler heuristics, and researchers and developers of lot sizing heuristics to forego optimality comparisons.

The present thesis seeks to provide an alternate, easy to understand and hopefully improved means of obtaining optimal lot sizing solution. We consider inventory problem with variable demand rate and allowing back orders, and model the problem using integer linear programming approach, both under crisp and fuzzy environments, with a finite-planning horizon. Also, an attempt is made to model the same problem of replenishing inventory, incorporating quantity discounts. Furthermore, the inventory lot size problem is presented using goal programming approach coupled with integer linear programming.

In the present chapter we give a brief introduction to these problems.

1.1 Basic Inventory Concepts

1 Inventory

An inventory is the stock or the store of an item or a resource used by an organization. Good inventory management is important to all firms, whether manufacturing or service. Four reasons for its importance are:

- Inventories can be a major commitment of monetary resources.

- Inventories affect virtually every aspect of daily operations.
- Inventories can be a major competitive weapon.
- Inventories are the major control problem in many companies.

2 Independent Demand Items

These are shipped as end items to customers and may be finished goods or spare/repair parts. Demand is market-based, and is independent of the demand for other items.

3 Dependent Demand Items

These are used in the production of a finished product. Such items may be raw materials, component parts or subassemblies. Demand is based on the number needed in each higher-level (parent) item where the part is used. Dependent demand items are frequently managed by certain inventory replenishment systems such as MRP or JIT systems.

1.2 Methods for Measurement of Inventory

There are three accounting categories, or types, of inventories:

- Raw materials
- Work in process
- Intermediate storage
- Finished goods

There are at least three methods for measuring inventory

1. **Aggregate inventory value** (average or maximum) gives answer to the question of HOW MUCH is the stock of inventory?
2. **Weeks of supply** (or other time unit) provides answer to the question of HOW LONG will inventory last?
3. **Inventory turnover or turns** (ratio of sales to inventory) gives answer to the question of HOW MANY times inventory is sold?

1.3 The Objectives of Inventory Management

There are two major objectives of inventory control, which commonly are in conflict. The operations manager's problem is in striking a balance between the two. These objectives are

- maximize the level of customer service, and
- minimizing the cost of providing an adequate level of customer service, promoting efficiency in production or purchasing.

The organization's Inventory Management System must carry out objectives set by upper management and must perform in such a way that the organization's profit or performance is enhanced. The objectives set by management will frequently fall into either of two categories:

- customer service objectives, and
- inventory investment objectives.

The first category includes such concepts as service level and stock-out rate, and the second category includes such items as number of inventory turnovers per time period. Generally, the achievement of higher levels of customer service, however defined, is accomplished with larger amounts of inventory, and is subject to diminishing returns. The achievement of higher levels of the investment objectives is generally met with smaller inventories. Thus, we see the basic conflict of inventory management: some objectives call for economizing on inventory levels, while other objectives call for increasing inventories. These objectives may create conflict along departmental lines: finance wants smaller sums tied up in inventory, while marketing wants larger amounts so that customer orders can be more promptly satisfied.

1.4 Inventory Decisions

Inventory problems are encountered in many phases of the production process. For example

- A manufacturer must determine the amount of raw materials to order.
- A manufacturer must determine the quantity of in-process inventory to 'ship' to another stage of production. The processing speed at one department or workstation may differ from the speed of another. An in-process inventory will allow each station to operate at its own optimal rate. The inventory therefore de-couples the two production rates.
- A distributor or manufacturer must determine the quantity of finished goods to ship to a customer.

- A retailer or wholesaler must determine the quantity of goods to order for resale.
- An individual or a corporation must determine the quantity of funds to be transferred from one account to another.
- A manufacturer may need to protect himself from supplier shortages or disruptions. A buffer stock or safety stock of raw materials can provide this protection. Likewise, a retailer may need to keep safety stocks of products in order to manage ambiguity/impreciseness in the level of demand for those products.

1.5 Types of Inventories

Six types of inventory, based on the organization's motivation for holding it, are

1 De-coupling Inventory

This buffer stock maintains independence of operations, allows two stages in the manufacturing process to operate at individually appropriate speeds. This type of inventory also serves to allow a uniform rate manufacturing process to be protected against variations in the demand for the product.

2 Lot Sizing Inventory

Lot sizing is the purchasing or producing items in large enough lots to take advantage of cost efficiencies, quantity discounts, learning curves, scale economies, etc.

3 Fluctuation Inventories

Fluctuation inventories arise from keeping sufficient added inventory to protect against imprecise/random, recurring variations in demand, usage or arrival rates, which can result in stockouts when demand exceeds the level of inventory. These would be safety stock inventories.

4 Anticipation Inventories

Anticipation inventories are specific planned changes in inventory position in anticipation of infrequent events, such as strikes, shipment delays, vacations, special sales and promotions, etc. A special type of anticipation inventory is the hedge inventory, which occurs in anticipation of a price change. Anticipation inventories may be increases or decreases. A firm might increase its inventory in anticipation of a supplier's price increase, but decrease its inventory in anticipation of a price decrease.

5 Transportation Inventories

Transportation inventories also referred to as pipeline inventory are items in transit from one production stage to another, from warehouse to retailer, etc. This is a type of inventory often neglected. Such inventories are important since they do reflect moneys tied up for periods of time, and hence do incur inventory-holding costs.

6 Speculation Inventory

Speculative inventory exists when objects are held for resale, not for direct use in a manufacturing process.

1.6 Single and Multi-period Inventory Models

1 Single-Period Inventory Models

This class of problem referred to as single-period inventory problem is based only on one decision i.e. HOW MUCH to stock for a single time period? Unlike most other classes of inventory problems, there is no decision regarding WHEN replenishment takes place.

2 Multi-Period Inventory Models

Since the single-period models are built around a quantity decision only, most inventory models are constructed around a pair of decisions. The essence of such inventory decisions is two-fold, and resides in the determination of QUANTITY and TIMING. How much should be ordered, shipped, etc.; and how (or when) are replenishment orders triggered? How much to order? A few very large orders or many very small orders? In the first case, ordering costs are held down, but carrying costs soar. In the second case, carrying costs are small because average inventory is small, but ordering costs are exorbitant.

1.7 Elements of Inventory Costs

There are several costs that influence the inventory decision.

1. The cost to place replenishment order. In the simplest models this is treated as fixed regardless of the size or amount of the order.
2. The cost to hold inventory. This may be a fixed sum per unit per time period, or it may be a fixed percent of value per time period. That is, holding costs may consist of both physical storage and capital costs (foregone earnings). Among the relevant costs are

warehouse rental (implicit or explicit) clerical costs of counting inventory, insurance, security, taxes, obsolescence, damage, theft (burglars and employees), reduced item life, spoilage, and the value associated with funds tied up in inventory. This cost of capital may be the actual cost of funds borrowed to purchase inventory, the interest that could be saved if that money were used to retire debt or an internal rate of return, representing gains made from using the same funds on, for example, a plant expansion.

3. The costs of managing shortages or backorders. A firm that runs out of a product may initiate a special order, or take additional special steps to satisfy the customer. The use of rainchecks at retail is a case of extra cost, labor and paperwork associated with backorders.
4. The costs of acquiring the items themselves. This is relevant in any quantity discount model.

The overall total cost can be deterministic, stochastic or imprecise. The emphasis in the present thesis is on imprecise total cost.

1.8 Classification of Inventory Models

There are several ways of classifying the inventory models. Some of the attributes useful in distinguishing between various inventory models are given in this section. (Gill, 1992).

1 Number of Items

- Single Item – This type of model recognizes one type of product at a time. If the demand rate changes from period to period, then the problem becomes that of a dynamic lot-sizing problem.

- Multi Item – This type of model considers a number of products simultaneously. These products must have at least one interrelating or binding factor such as budget or capacity constraint or a common setup.

2 Stocking Points

- Single Echelon Models – Only one stocking location is considered.
- Multi Echelon Models – More than one interconnected stocking locations are considered.

3 Frequency of Review

This is the frequency of assessment of the current stock position of the system and the implementation of the ordering decision.

- Periodic – Placement of orders is done at discrete points in time, with a given periodicity.
- Continuous – Order placement can occur at any time.

4 Order Quantity

- Fixed – Order quantity is fixed to the same amount each time.
- Variable – Order quantity can be variable.

5 Planning Horizon

- Finite – Demands are recognized over a limited number of periods.
- Infinite – Demands are recognized over an unlimited number of periods.

6 Demand

- Deterministic – Demands are known with certainty over the planning horizon.

  a) Static – Demand rate is constant over every period.

  b) Dynamic – Demand rate is not necessarily constant.

- Stochastic (Probabilistic) – Demand is unknown, and must be estimated. The demand probability distribution may be known or unknown.

  The emphasis in the present thesis is to deal with an inventory lot size problem with ambiguous or imprecisely known demand.

7 Lead Time

- Zero – No time elapses between placement and receipt of orders.
- Non Zero – Significant time elapses between the placement and receipt of orders. This time may be constant or random.

8 Capacity

- Capacitated – There are capacity restrictions on the amount produced or ordered.
- Un-capacitated – Capacity is assumed to be unlimited.

9 Unsatisfied Demand

- Not allowed – In this case, all demand is met and no shortages are allowed.
- Allowed – Demand not satisfied in a particular period may be retained and satisfied in a future period (backlogging), partially retained and partially lost or completely lost (no backlogging).

1.9 Lot Size Inventory Problem

Two basic questions to be answered in most of the inventory situations are;

- when to order (the reorder point) and,
- how much to order (the lot size).

When the demand rate is constant over time, the associated problem of planning is rather simple because the use of the Classical Economic Order Quantity Model (EOQ Model) gives us optimal results. But when the demand rate varies over time, i.e. not necessarily constant from one period to another, the associated problem of planning is a bit more challenging and is said to be dynamic in nature. The problem considered for this study is uncapacitated single item lot sizing problem with dynamic demand. This problem was first addressed by Wagner and Whitin (1958) under the assumption of deterministic demand. The uncapacitated assumption can be justified to some extent in an MRP (Material Requirement Planning) environment on the condition that a good master production schedule exists which takes these capacity restrictions into consideration. This master schedule is aimed at smoothing the production load and can make use of fine tuning devices such as adjusting the lead times, subcontracting, overtime, alternate routings etc. However, in certain situations it may be difficult to ignore the capacity restrictions in actual lot size decisions since this would lead to an infeasible master schedule and subsequently to more frequent re planning. Further, we would add that inventory problems are universal and intricate in nature, so no particular model can represent all the inventory situations.

1.10 Solution Approaches

Wagner and Whitin (1958) suggested a dynamic programming algorithm to deal with the uncapacitated inventory control problem. Though the approach gives optimal results, the complex nature of dynamic programming makes it difficult to understand and therefore makes it practically useless. There are numerous heuristic methods available in the literature that will be discussed in the literature survey in the next chapter. These heuristics are easy to use but not necessarily optimal. In certain practical situations such as for dedicated production lines, group technology and FMS, it is impossible to ignore setup costs or setup times. Each time a setup is done, a cost is incurred. This suggests an integer linear programming approach with some binary variables representing the setups. We use such integer linear programming models in the following chapters. The main underlying assumption in most of the models is that "demand is deterministically known". But demand is always forecasted, and most of the times the forecasts do not turn out to be precisely correct. Furthermore, in practice most of the companies are limited by budget restrictions. Setting targets or goals on cost figures is a very common practice in the industrial and business world, and in some situations these restrictions have some elasticity/ambiguity. This suggests the possibility of applying fuzzy logic to some industrial problems.

In some cases, the decision-maker might not really want to actually maximize or minimize the objective function, but rather may want to reach some "aspiration level" which might not even be crisply defined. In real world problems, this can happen because sometimes it is simply not possible to obtain precise data, or the cost of obtaining precise data is too high. This imprecision in data arises because of complex nature of real world problems. So the problem becomes that of modeling with imprecise data. We will analyze our problems by means of a fuzzy logic approach when some sort of ambiguity in available budget and demands is involved. Fuzzy set theory is a tool that gives reasonable analysis of complex systems without making the process of analysis too complex. Also, there might be situations in which a decision-maker needs to consider multiple criteria in arriving at the overall best decision. Goal programming is such a technique that handles multi-criteria situations within the general framework of linear programming. We will use goal programming to obtain overall best decision when goals are associated with different priorities.

In the following lines, we give a brief introduction to linear programming, fuzzy set theory and goal programming.

1.11 Linear Programming

It is a mathematical method of allocating scarce resources to achieve an objective, such as maximizing profit (Lee et. al., 1981). Linear programming involves the description of a real world decision situation as a mathematical model that consists of a linear objective function and linear resource constraints. Once the problem has been identified, the goals of management established, and the applicability of the linear programming determined, the next step in solving an unstructured, real world problem is the formulation of a mathematical model. This entails three major steps:

- Identification of solution variables (the quantity of the activity in question).
- The development of an objective function that is a linear relationship of the solution variables, and
- The determination of system constraints, which are also linear relationships of the decision variables, which reflect the limited resources of the problem.

1.11.1 The Generalized Linear Programming Model

Decision Variables

In each problem, decision variables, which denote a level of activity or quantity produced, are defined. For a general model, n decision variables are defined as

x_j = quantity of activity j, where j = 1, 2, ..., n.

Objective Function

The objective function represents the sum total of the contribution of each decision variable in the model towards an objective. It is represented as

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_jx_j + \ldots + c_nx_n$

where

Z = the total value of the objective function

c_j = the contribution per unit of activity j (j = 1, 2, ..., n)

System Constraints

The constraints of a linear programming model represent the limited availability of resources in the problem. Let the amount of each of m resources available be defined as b_i (for i = 1, 2, ..., m). we also define a_{ij} as the amount of resource i consumed per unit of activity j (j = 1, 2, ..., n). Thus, the constraint equations can be defined as

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mj}x_n \ (\leq, =, \geq)\ b_m$

$x_1, x_2, \ldots, x_n \geq 0.$

1.12 Fuzzy Set Theory

Theory of fuzzy sets is basically a theory of graded concepts (Zimmerman, 1991). A central concept of fuzzy set theory is that it is permissible for an element to belong partly to a fuzzy set.

Let X be a space of points or objects, with a generic element of X denoted by x. Thus X={x}.

1.12.1 Fuzzy Set

Let x ε X. A fuzzy set A in X is characterized by a membership function (M.F) $\mu_A(x)$ which associates with each point in X, a real number in the interval [0,1], with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A. Thus, nearer the value of $\mu_A(x)$ to 1, higher the grade of belongingness of x in A.

In conventional crisp set theory, $\mu_A(x)$ can take only two values 1 or 0 depending on whether the element belongs or does not belong to the set A. Therefore, if X = {x} is a collection of objects denoted generically by x, then a fuzzy set in X is a set of ordered pairs, $A = \{(x, \mu_A(x))/x \varepsilon X\}$, where $\mu_A(x)$ maps X to the membership space [0,1].

1.12.2 Intersection of Fuzzy Sets

Intersection of two fuzzy sets A and B with respective M.F.'s $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set D whose M.F is $\mu_D(x) = \min[\mu_A(x), \mu_B(x)], \quad x \varepsilon X$.

1.13 Goal Programming

Any goal-programming model will be formulated using the following guidelines (Lee et. al., 1981):

1. Following two types of variables will be a part of any formulation

- decision variables (the x's) and the
- deviational variables (the d^+s and d^-s).

2. Two classes of constraints can exist in a given goal programming model i.e.

- structural constraints, which are generally considered environmental constraints and are not directly related to goals
- goal constraints which are directly related to goals.

3. Finally, while in most cases a goal constraint will contain both an underachievement (d^-) and an over achievement (d^+) deviational variable, even when both do not appear in the objective function, it is not mandatory that both be included. Omission of either type of deviational variable in the goal constraint, however, bounds the goal in the direction of omission. That is, omission of d^+ places an upper bound in the goal, while omission of d^- forces a lower bound on the goal.

Assuming that there are m goals, p structural constraints, n decision variables, and K priority levels, the general model can be expressed as follows:

$$\text{Minimize} \qquad Z = \sum_{k=1}^{K} P_k \sum_{i=1}^{m} \left(w_{i,k}^{+} d_i^{+} + w_{i,k}^{-} d_i^{-} \right)$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i \qquad i = 1,\ldots, m$$

$$\sum_{j=1}^{n} a_{ij}x_{ij} \quad (\leq, =, \geq) \quad b_i \qquad i = m+1,\ldots, m+p$$

$$x_j, d_i^+, d_i^- \geq 0, \qquad j = 1,\ldots,n; \; i = 1,\ldots, m.$$

where P_k = the priority coefficient for the k th priority.

$w_{i,k}^+$ = the relative weight of the d_i^+ variable in the k th priority level.

$w_{i,k}^-$ = the relative weight of the d_I^- variable in the k th priority level.

It is important to decide the priorities for the goals. First of all, the goal with the highest priority is exclusively considered in the objective function and the problem is solved minimizing the deviation of this goal. Then, the next step involves adding the value of this deviation of highest priority goal as a constraint in the problem, and solving the problem, minimizing the deviation of next priority goal. Again this deviation is added as a constraint in the problem. Similarly proceeding in the same manner, the problem is solved minimizing the deviation of the goal with the least priority.

1.14 Organization of the Thesis

In the present thesis, an important problem in the field of industrial engineering i.e. lot size inventory control problem (addressed by Wagner and Whitin, 1958) has been revisited. The inventory lot-sizing problem has been modeled as zero one integer linear program with variable demand, setup, backordering, and carrying costs, under both crisp and fuzzy environments. Also, the problem is modeled as piece wise integer linear program by incorporating quantity discounts. Besides, goal-programming approach is applied to this problem. Also, a fuzzy logic approach to deal with inventory lot sizing problem, when the data known is imprecise, is proposed.

Chapter 1 provided an introduction to the concepts and problems considered in this thesis. Chapter 2 deals with the literature review of the problems considered with an objective to recognize the work done by other researchers. The inventory problem with variable demand rate under both crisp and fuzzy environments with a planning horizon of N periods and allowing backorders is considered in Chapter 3. Chapter 4 deals with modeling the inventory problem with variable demand, setup, backordering, and carrying costs, incorporating quantity discounts, under both crisp and fuzzy environments, as a piecewise integer linear programming model. Chapter 5 presents the inventory lot size problem using goal-programming approach coupled with integer linear programming. Finally, the conclusion and the discussion on the contribution made by the thesis, along with some recommendations for further research are given in Chapter 6.

Chapter 2

LITERATURE SURVEY

This chapter provides a survey of the literature dealing with inventory lot sizing problems and other concepts considered in this thesis. The purpose of this chapter is to review the developments, and to identify the status of existing literature in this area.

2.1 Review of Literature on Inventory Lot Size Problem

Conceivably the first reported work on inventory control was by Harris (1915). He derived the classic EOQ formula. Then, Wilson (1934) contributed a statistical approach to find order points, thereby popularizing the EOQ formula in practice. The basic formula for economic order lot sizing is as follows:

$$EOQ = \sqrt{\frac{2.A.D}{v.r}}$$

where,

A = Fixed cost for the replenishment of an order,

D = demand rate of the item (normally annual usage rate),

v = unit variable cost,

r = cost of one dollar of item tied up in inventory for a unit of time.

Note that D and r should have same unit time basis (i.e. if annual demand is considered, then r must be considered for one year, not one month).

This method determines a single point or quantity and assumes a constant demand. But, when the demand rate varies from period to period, the results from the EOQ formula may be deceptive.

The approaches found little realistic recognition for at least few decades. This was due to the fact that the 1930's and 1940's were periods of great depression for the industrial and business world. The question before many companies was that of survival, not optimization. During World War II, different companies were mainly concerned about meeting the wartime needs and a backlog for civilian demands started appearing. This restrained demand for civilian goods provided a market for every item that could be produced. Once the postwar backlog was satisfied, firms started reasoning in terms of optimization, because the problem became that of over-production. Inventory control models received their real boost from operation research techniques developed during World War II.

The technique which performs optimally in a situation with variable demand was first suggested by Wagner and Whitin (1958) in their well known paper. They used dynamic programming to solve the problem, perhaps forced by the recursive nature of the computations. Their work was based on some important theorems established in their paper. These theorems were themselves based upon the assumption that initial inventory is zero ($I_0 = 0$). Before stating their algorithm, we shall briefly state these theorems.

Theorem 1

There always exists an optimal policy such that

$$I_{t-1} . X_t = 0 \quad \text{for } t = 1, 2, \ldots, N.$$

Where I_{t-1} is the inventory entering a period t, X_t is the amount produced in period t and N is the length of planning horizon. This means that replenishment can be made only when the inventory level becomes zero, i.e. having positive inventory and producing at the same time never leads to optimality.

Theorem 2

There exists an optimal policy such that for all t

$X_t = 0$ or

$$X_t = \sum_{j=t}^{k} d_j$$

for some k, $t \leq k \leq N$,

where X_t is the amount produced in period t and d_j is the demand in period j. This means that for any given period, production is either zero or is sum of subsequent demands for some number of periods in the future.

Theorem 3

There exists an optimal policy such that if demand d_t^* in a period t^* is satisfied by some amount X_t^{**} produced in period t^{**}, $t^{**} < t^*$, then d_t, $(t = t^{**} + 1, \ldots, t^* - 1)$ is also satisfied by X_t^{**}.

Theorem 4

Given that $I = 0$ for period t, it is optimal to consider periods 1 through $t - 1$ by themselves.

Planning Horizon Theorem

The planning horizon theorem states in part that if it is optimal to incur a setup cost in period t^* when periods 1 through t^* are considered by themselves, then we may let $X_{t^*} > 0$ in the N period model without foregoing optimality. By theorems 1 and 4 it follows further that we adopt an optimal program for periods 1 through $t^* - 1$ considered separately.

The Algorithm

According to Wagner and Whitin (1958), the algorithm at period t^*, $t^* = 1, 2, \ldots, N$, may be generally stated as:

1. Consider the policies of ordering at period t^{**}, $t^{**} = 1, 2, \ldots, t^*$ and filling demands d_t, $t = t^{**}, t^{**} + 1, \ldots, t^*$, by this order.
2. Determine the total cost of these t^* different policies by adding the ordering and holding costs associated with placing an order at period t^{**}, and the cost of acting optimally for periods 1 through t^{**}-1 considered separately. The latter cost is computed previously in computations for periods $t = 1, 2, \ldots, t^*$-1.
3. From these t^* alternatives, select the minimum cost policy for periods 1 through t^* considered independently.
4. Proceed to period t^*+1 (or stop if $t^* = N$).

Since this time, however, the literature has tended to either ignore or, at least, to minimize the significance of the contribution of the Wagner-Whitin solution to this class of the problem due to its complicated nature and enormous computational efforts required. Frequently, excuses such as "the high computational burden and the near impossibility of explaining it to the average MRP user" or "the complexity of the procedure inhibits its understanding by the layman, and acts as an obstacle to its adoption process" were used to justify other approximate (not optimal) alternatives to the Wagner-Whitin procedure (Fordyce and Webster, 1984).

Triggered by Wagner and Whitin's work, a number of papers appeared in various journals. Some of them tried to improve on Wagner-Whitin algorithm, others gave some heuristics which were fundamentally off shoots of Wagner-Whitin algorithm with the emphasis on making computation scheme more easy, though may not be optimal.

Wagner (1960) refined his approach further taking into account time varying manufacturing costs. Eppen et. al. (1969) and Zabel (1964) evolved theorems that decrease the computational effort required to find optimal policies and established the existence of planning horizons. Zangwill (1969) included backordering costs in the problem and gave a network representation to the problem. Blackburn and Kunreuther (1974) also considered a backlogging case.

Fordyce and Webster (1984) presented the Wagner-Whitin algorithm in a simple and straightforward computational style in a tabular form, without using any mathematical notation or formulas. Carrying on with their research, Fordyce and Webster (1985) demonstrated the ability of Wagner-Whitin Algorithm to be modified to situations in which unit cost price is not constant over the planning horizon and included quantity discounts. Tersine and Toelle (1985) examined various methods for determining lot sizes in presence of either all units or incremental discounts. Aucump (1985) contributed another discrete lot sizing strategy, which he claims beats all others when the time horizon is large enough to include about two EOQ cycles of firm demand. Sumichrast (1986) compared the Fordyce –

Webster algorithm with other heuristics for determining order policy when quantity discounts are possible. Bahl and Zoints (1986) formulated the problem as a fixed charge problem and made lot sizing decisions by comparing the minimum savings of having a setup in each period to the maximum savings of having an order in that period. Saydam and McKnew (1987) developed a microcomputer program for finding out optimal policy using Wagner-Whitin algorithm in TURBO PASCAL. Jacobs and Khumawala (1987) presented a simple graphic, branch and bound optimal procedure, which is computationally equivalent to Wagner-Whitin algorithm (1958).

2.2 Classification of Literature

Zoller and Robrade (1988) suggested a convenient classification scheme for the existing literature by categorizing it into following three categories:

1. Optimizing techniques
2. Stop rules (heuristics) and
3. Heuristic algorithms.

Using the above classification scheme, we discuss the literature as follows:

2.2.1 Optimizing Techniques

The optimization techniques like EOQ and Wagner-Whitin algorithm have been discussed above.

2.2.2 Stop Rules

Stop rules (Zoller and Robrade, 1988) increase the cycle length and stop as soon as some transformation of the controllable cost is reached. Controllable cost C(t), is normally the sum of ordering and holding cost and given as

$$C(t) = S + H.\sum_{h=1}^{t}(h-1).d_h$$

where d_h is the demand quantity in period h, H is the holding cost per period per unit, and S is the fixed cost of each replenishment.

- **Least Unit Cost Rule (LUC)**

This is probably the earliest heuristic, the exact origin of which hasn't been traced out. Gorham (1968) compares the LUC and least total cost (LTC) methods and concludes that LUC method is erratic. Although it performs well on one set of data, it fairs poorly on another set of data. LUC (Wemmerlov, 1981) divides the total cost by the demand quantities to find out the cost per unit U(t) as follows:

$$U(t) = \frac{C(t)}{\sum_{h=1}^{t} d_h}$$

and stops as soon as $U(t+1) \geq U(t)$.

- **Part Period Rule**

The part period rule was developed for IBM's software package due to its simplicity in programming. It was introduced by DeMatteis (1968) and Mendoza (1968) and is basically the same as the Least Total Cost (LTC) rule (Gorham, 1968). The basic criterion in these rules is that the requirements for the successive periods can be added to the same lot so long as the cumulative carrying cost does not exceed the ordering cost, i.e.

$$H.\sum_{h=1}^{t}(h-1).d_h \leq S$$

and stops as soon as

$$H.\sum_{h=1}^{t+1}(h-1).d_h > S.$$

- **The Silver and Meal Rule (SMR)**

This is perhaps the most famous heuristic method (Silver and Meal, 1973). Silver-Meal rule is identical to Least Unit Cost rule except that the total cost is divided by the number of periods included in the lot instead of by the sum of demand quantities. It computes the cost per period P(t) as follows:

P(t) = C(t)/t, and stops as soon as

P(t+1) > P(t).

- **Groff's Rule**

Groff (1979) introduced a policy under which the demand for a period is added to the lot if the marginal savings in ordering cost are greater than the marginal increase in carrying cost. In mathematical terms,

Marginal savings in ordering cost = (S/t) – (S/t+1) = S/(t. (t+1))

Marginal increase in holding cost = $(1/2).H.d_{t+1}$

Groff's rule adds the demand for the period to the lot if $S/(t.(t+1)) > (1/2).H.d_{t+1}$ and stops as soon as

$$(1/2).H.d_{t+1} \geq S/(t.(t+1))$$

- **Incremental Order Quantity (IOQ)**

Boe and Yilmax (1983), Freeland and Colley (1982) suggested that cycle length should be increased so long as incremental carrying costs $H.t.d_{t+1}$ does not exceed S and it stops as soon as

$$H.t.d_{t+1} \geq S$$

- **Period Order Quantity (POQ)**

Period Order Quantity is an EOQ based technique. If there are considerable variations in the demand pattern, then the results from the simple EOQ formula can be misleading. Better results can be obtained by adopting a slightly different approach (Brown, 1977). The EOQ quantity is divided by the average demand during one period to obtain the number of periods whose requirements are to be covered by the lot size (rounded to the nearest positive integer).

T_{POQ} = EOQ / (Average demand during one period)

If D_{avg} is the average demand for one period, then

$$EOQ = \sqrt{\frac{2.A.D_{avg}}{v.r}}$$

$$T_{POQ} = \sqrt{\frac{2.A}{v.r.D_{avg}}}$$

Thus in POQ method. the time between orders remains fixed. but lot size changes.

2.2.3 Heuristic Algorithms

In the previous section we discussed some rules which were basically single pass stop rules. The stop rules terminate when some transformation of controllable cost is reached, while the algorithms further look ahead or back and compare different alternatives to improve the overall decision.

- **IOQ Algorithm**

Trux (1972) proposed to use the IOQ rule to find a safe maximum and then examines if the corresponding lot can be split into two lots. Gaither (1983) determined two subsequent lengths and examined if shifting a demand from first lot to second lot is more profitable or not. In fact. Gaither (1983) is an improved version of his previous algorithm Gaither (1981). after the comments from Silver (1983) and Wemmerlov (1983).

- **Part Period Algorithm**

To improve the performance of PPR many attempts have been made. DeMatteis (1968) suggested that the cycle length determined by the PPR should be subjected to look ahead or look back to determine if the periods of large demand exist. Blackburn and Millen (1980) proposed that the cycle length determined by PPR could be increased if a closer balance of ordering and carrying costs can be maintained. Karni (1981) proposed that pairs of lots should be combined into a single order through an iterative procedure with a maximum gain in terms of net cost reduction.

- **Silver Meal Algorithm**

Silver and Meal (1973) made an observation that cost per period is not necessarily convex and may hence have many local minima, however, SMR identifies only the first minima. Blackburn and Millen (1980) suggested that the absolute minima should be found by exhaustive enumeration of C(t) over the entire planning horizon.

2.3 Other Approaches

Trigeiro (1987) examined a mathematical programming method of accounting for capacity costs for deterministic, multi item, single operation lot sizing problem. Webster (1989) further extended his research and contributed a backorder version of the Wagner-Whitin discrete demand algorithm using dynamic programming. Shtub (1990) presented a model of cellular production system and a heuristic lot sizing procedure which is based on tradeoff between setup cost and inventory carrying cost for MRP systems. Golany et. al. (1991) successfully applied a goal programming inventory control model at a large chemical plant in Israel.

In addition to the approaches discussed above, there are numerous other ones. Detailed description of these heuristic techniques can be found in Plossl (1985), Silver and Peterson (1985), Zoller and Robrade (1988), Nydick and Weiss (1989). The relative performance of different heuristic methods is compared in Karni (1986), Nydick and Weiss (1989), Zoller and Robrade (1988) and Drexl and Kimms (1997). Zoller and Robrade (1988) provide an extensive comparative study of different methodologies implemented in commercial softwares. Ptak (1988) provides an excellent comparison of various inventory models and carrying costs.

Prentis and Khumawala (1989) developed two heuristics based on branch and bound method to solve closed loop MRP lot sizing problems. Trigeiro (1989) also developed an algorithm for capacitated lot sizing problem, single machine lot sizing problem, with non-stationary costs, demands and setup times.

Shtub (1990) presented a model of cellular production system and a heuristic lot sizing procedure which is based on tradeoff between setup cost and inventory carrying cost for MRP systems. McKnew et. al. (1991) presented a zero one linear formulation of the multilevel lot-sizing problem for MRP systems without capacity constraints. Tersine and Barman (1991) derived optimum lot sizing algorithm for dual discount situations by structuring quantity and freight discounts into order size decision in deterministic EOQ system. Bretthauer et. al. (1994) formulated a resource constrained production and inventory management model as a nonlinear integer program.

Dellart and Melo (1996) presented two heuristic procedures for determining production strategies under the conditions of constant capacity. Stadtler (1996) formulated a mixed integer programming model for calculating lot sizes and performing sensitivity analysis by varying end of period inventory levels. Kimms (1996) also considered the same problem under the assumption of starting with initial inventory. Voros (1995) determined the range of feasibility of setup costs. Gurnani (1996) considered the lot-sizing problem under random demand and quantity discounts being offered after the orders have been placed. Martel (1998) formulated the problem with holding cost as a function of purchasing price.

Moncer and Ben-Daya (1999) developed stochastic inventory models as continuous and periodic review models with mixture of backorders, lost sales and the base stock model.

Given the magnitude of the published efforts on MRP and its various components, and the importance of the lot sizing function in real systems, the need is recognized for an improved algorithm that would address the two primary issues of efficiency and near optimality. Such a technique would represent not only an aid to MRP researchers and a foundation for future efforts to confront more complex cases, but also as a tool in study of JIT management. Lot sizing approaches are necessary to help both practitioners and researchers identify the point at which setup proficiency and process flexibility have reached JIT standards. Until that point is reached, lot-sizing techniques will also continue to be a practical necessity.

2.4 Motivation and Objectives of the Research

The present research was largely motivated by the benefits of optimal lot sizing. The selection of research problems and the objectives in this thesis depend on the following factors discussed below.

- Throughout industrial establishments worldwide, it is commonly experienced that the demand for the products manufactured by them varies periodically. Therefore, in order to satisfy the demand, the raw materials need to be acquired and processed so as to align the production as per demand. Due to changing demand, the lot sizes in which the lower level items need to be processed, also vary. Consequently, a careful decision is required as far as lot sizing is concerned, and this decision depends upon numerous financial factors such as setup or acquiring costs, holding and backordering costs etc. In addition, certain factors such as space and capacity restrictions, quantity discounts etc. also influence the decision.

- The available data associated with the variables of lot sizing problem, is commonly imprecise. For example, the exact number of units demanded, the exact holding and backordering cost, setup costs etc are rarely valid. Due to this ambiguity in the data available, the problem model cannot yield an optimal solution. Therefore, some tool is required to account for this impreciseness in data.
- Many times the objectives have different priorities. For example, holding cost may be less important as compared to setup cost, whereby one can conclude that the model will suggest larger lot sizes to be run less frequently, thereby saving setup costs. Therefore, some sort of mathematical mechanism is required to do this sort of analysis.

To deal with these issues mentioned above, we have used mathematical tools like integer linear programming, piecewise integer linear programming, fuzzy programming and goal programming to arrive at satisfactory solutions.

Chapter 3

LOT SIZING INVENTORY PROBLEM WITH VARIABLE DEMAND RATE AND BACKORDERS ALLOWED UNDER CRISP AND FUZZY ENVIRONMENTS

In the present chapter, we consider variable demand rate inventory problem with back orders allowed, both under crisp and fuzzy environments with a finite-planning horizon. To obtain a solution to such a problem, under crisp environment we formulate a linear programming model, and under fuzzy environment we solve a max-min linear program with a finite number of integer and 0-1 variables. Furthermore, it is observed that our results obtained by using linear programming under crisp environment are similar to the results obtained by other researchers using the dynamic programming approach.

3.1 Introduction

Lot sizing is a significant aspect in production planning process of Materials Requirement Planning (MRP). In variable demand rate inventory problems, the management is compelled to furnish accurate data. Nevertheless, in practice, management always wants some sort of flexibility or room to accommodate for the imprecision in the available data. Since the variable demand is forecasted, it is rarely known exactly as the forecasts do not always turn out to be crisply accurate. Thus, in practice there is always a component of vagueness (fuzziness) in available data. Also, it is a common practice, that the management specifies a budget limit and asks the production planner to stay possibly below the budget limit. It is this "possibly below" circumstance that adds to the element of fuzziness in the problem. Under condition of fuzzy demand and when the budget allocated is not a precise number, the methods developed for the crisp problem may not work well to furnish an optimal solution to the problem. In the present chapter, we take advantage of the fuzzy set theory by Zadeh (1965), Bellman and Zadeh (1970), and use the approach developed by Zimmermann (1991), and model such a problem as a symmetric fuzzy linear program (in which both the objective function and some constraints are fuzzy). Thereby, creating the fuzzy regions around the forecasted demand by using the idea of tolerance and subsequently introducing appropriate membership functions, we obtain a max-min linear programming problem (Zadeh (1965), Bellman and Zadeh (1970), Zimmermann (1991)), with a finite number of integer and 0-1 variables.

Bector et. al. (1992) presented a method in which a linear programming model is formulated with number of zero one variables exactly equal to number of periods in planning horizon under both crisp and fuzzy environments. However, they did not include in the problem the alternative of backordering, and therefore the costs associated with backordering were not included in the overall costs. The present chapter is an extension of their work incorporating the concept of backordering. This chapter also reveals the simplicity of the linear programming method as compared to the backorder version of Wagner-Whitin algorithm (1958) given by Webster (1989) using dynamic programming. The numerical example used by Webster (1989) is solved, using linear programming approach developed in the present chapter, and it is observed that the results obtained are identical.

3.2 Integer Linear Programming Formulation under Crisp Environment

3.2.1 Assumptions

For this model, the following assumptions are made (Bector et. al. (1992)):

1. The demand varies from one time period to another and is assumed to be known.
2. The units needed to satisfy demand in a particular period can be acquired at any time including the backorders.
3. Acquisition costs (setup costs of production run or ordering or follow-up costs for purchased parts) are fixed relative to the quantity acquired, but may vary from one time period to another.
4. Holding cost in a particular time period represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.
5. Holding cost may vary from one time period to another.
6. Backorder cost represents the cost of backordering one unit of product from the period in which it was needed to the next time period. Units backordered more than one time period would be charged the accumulated backordering cost of all the periods from the time period of demand till the time period of acquisition.
7. Backorder cost may vary from one time period to another.
8. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.

9. The unit variable cost does not depend upon replenishment quantity i.e., no quantity discounts are permitted.

10. The product is treated entirely independently of the other products i.e., benefits from joint replenishment do not exist or are ignored.

3.2.2 Notation

Let,

x_{ij} = the number of units acquired (manufactured or purchased) in period i for demand in period j (associated with x_{ij} we use, the holding cost when $i < j$, and backordering cost when $i > j$),

h_i = holding cost (in dollars) per unit during period i,

$\sum_{i=j}^{k-1} h_i$ = holding cost (in dollars) per unit from period j to period k, $k > j$,

s_i = cost of replenishment in dollars in period i (in manufacturing the units, it is setup cost incurred each time the units are produced, and for purchasing the units, it is the ordering cost). This cost is irrespective of the number of units acquired,

N = length of planning horizon, i.e., the number of periods in planning horizon,

b_i = backorder cost (in dollars) per unit for period i,

$\sum_{i=j}^{k-1} b_i$ = backordering cost for items ordered in period j received in period k, $k > j$,

d_j = number of units demanded in period j,

$$y_i = \begin{cases} 0 & \text{if } x_{ij} = 0, \text{ i.e., if no replenishment is made in period i, for all j,} \\ 1 & \text{if } x_{ij} > 0, \text{ i.e., if replenishment is made in period i, for any i,} \end{cases}$$

G = a large number $\geq d_1 + d_2 + \ldots + d_N$.

3.2.3 Objective

Our objective in this problem is to minimize the total cost of acquiring the units x_{ij}'s (i, j = 1, 2, ..., N), such that certain variable demand constraints are satisfied, where the total cost of acquiring the units is the sum of total carrying cost, total backordering cost and the total setup cost.

Minimize total cost = Total carrying cost + Total backorder cost + Total setup cost.

3.2.4 General Formulation

Thus, under the crisp environment, we have the following problem (3C).

(3C)
$$\text{Min}\left\{ \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} + \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} b_i x_{kj} + \sum_{i=1}^{N} s_i y_i \right\}$$

subject to

$$\sum_{j=1}^{N} x_{ij} \leq G\, y_i \qquad i = 1, 2, \ldots, N,$$

$$\sum_{i=1}^{N} x_{ij} = d_j \qquad j = 1, 2, \ldots, N,$$

$$x_{ij} \geq 0 \qquad i, j = 1, 2, \ldots, N,$$

$$y_i = 0, 1 \qquad i = 1, 2, \ldots, N.$$

The structure of the constraints of (3C) is such that it will always generate an integer solution. Therefore, we need not impose the condition of integrality on x_{ij}'s.

3.2.5 Numerical Example under crisp demand

We illustrate our method through the following numerical example (represented in Table 3.1) taken from Webster (1989). We assume, as in Webster (1989), that setup cost, holding cost and backorder cost remain constant throughout the planning horizon.

Let, for $i = 1, 2, \ldots, 6$, $s_i = \$100$, $h_i = \$1.00$ per unit per time period, and $b_i = \$0.50$ per unit per time period, and let

Table 3.1 Representation of the Problem

Period	1	2	3	4	5	6
1	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
2	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}
3	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}
4	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}
5	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}
6	x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}
Demand	20	50	10	10	50	20

Selecting $G = 200 > 20 + 50 + 10 + 10 + 50 + 20 = 160$, we have the following formulation

Minimize $1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 0.5x_{21} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{31} + 0.5x_{32} + 1x_{34} + 2x_{35} + 3x_{36} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 1x_{45} + 2x_{46} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} + 1x_{56} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 + 100y_6$

subject to

$$x_{11}+ x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 20$$

$$x_{12}+ x_{22} + x_{32} + x_{42} + x_{52} + x_{62} = 50$$

$$x_{13}+ x_{23} + x_{33} + x_{43} + x_{53} + x_{63} = 10$$

$$x_{14}+ x_{24} + x_{34} + x_{44} + x_{54} + x_{64} = 10$$

$$x_{15}+ x_{25} + x_{35} + x_{45} + x_{55} + x_{65} = 50$$

$$x_{16}+ x_{26} + x_{36} + x_{46} + x_{56} + x_{66} = 20$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - 200y_1 \leq 0$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - 200y_2 \leq 0$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - 200y_3 \leq 0$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - 200y_4 \leq 0$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - 200y_5 \leq 0$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - 200y_6 \leq 0$$

$$x_{ij} \geq 0 \qquad i, j = 1,2,.....6$$

$$y_i = 0,1. \qquad i = 1,2,.....6.$$

3.2.6 Results

Solving the above problem using WinQSB (Chang, 1998), we obtain the following Table 3.2. The replenishment schedule obtained is given in Table 3.2.1.

Table 3.2 Results of crisp problem with backorders allowed.

Variable	Value	Variable	Value	Variable	Value
x_{21}	20	x_{54}	10	y_2	1
x_{22}	50	x_{55}	50	y_5	1
x_{23}	10	x_{56}	20		

Table 3.2.1 Replenishment Schedule of crisp problem with backorders allowed.

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	20	0	0	20
2	50	80	10	0
3	10	0	0	0
4	10	0	0	10
5	50	80	20	0
6	20	0	0	0

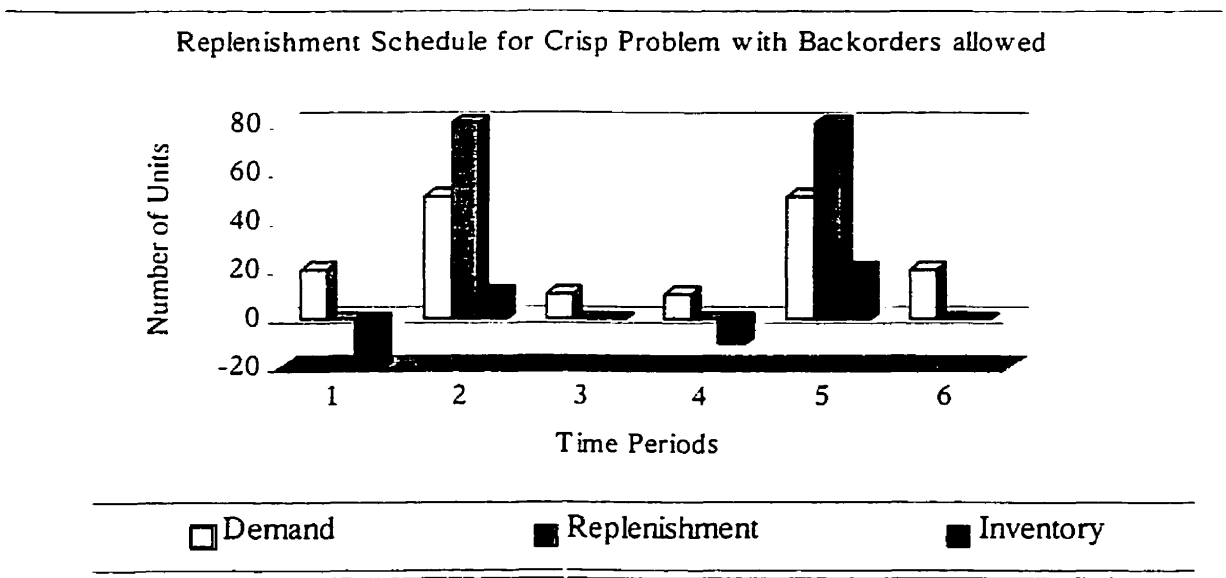


Figure 3.1 Replenishment Schedule for Crisp Problem with Backorders allowed

3.2.7 Interpretation of the Results

Since x_{ij} is the quantity produced in Period i for demand in Period j, therefore, from Tables 3.2 and 3.2.1 and Figure 3.1, we have $x_{21} + x_{22} + x_{23} = 20 + 50 + 10 = 80$ units should be acquired in Period 2. Out of these 80 units, 20 were backordered in Period 1 and acquired in Period 2, 50 units would be acquired in Period 2 for Period 2, and 10 units are carried over to Period 3. Again, the interpretation for $x_{54} + x_{55} + x_{56} = 10 + 50 + 20 = 80$ units acquired in Period 5 is that 10 units are used to fill the backorder from Period 4, 50 units are used to fill the demand in Period 5 itself, and the remaining 20 are carried over to Period 6. The values $y_2 = 1$ and $y_5 = 1$ indicate that the acquisition of the units for satisfying total demand is done in Period 2 and Period 5 only. Therefore, the setup cost is incurred in Periods 2 and 5. The total minimum cost or the value of the objective function is $245. It may be observed that the above results are as in Webster (1989).

3.3 Formulation under Fuzzy Environment

As already mentioned, problems of imprecise demand or data can be handled effectively by taking advantage of fuzzy set theory (Zadeh (1965), Zimmermann (1991), and Bellman and Zadeh (1970)).

The characteristics of inventory problem being considered that require it to be formulated in fuzzy environment are:

1. **Imprecise total cost limit levels**. The management provides an upper bound of the estimation of the total cost of acquisition z_0 over the entire planning horizon. The actual

costs would be likely to stay below this upper bound. A tolerance that defines the dispersion of the total cost may be given in the form of fraction of z_0.

2. **Imprecise dynamic demand.** Since demand is always forecasted and forecasts are rarely accurate to the exact number of units, the management can provide a tolerance level in form of a fraction of imprecisely known demand, that provides a range above and below the forecasted demand in which the actual demand is likely to occur.

We now formulate the problem under the following additional assumptions.

3.3.1 Additional Assumptions

(i) The total cost over the entire planning horizon of N periods stays possibly below a given limit.

(ii) The demand that varies from period to period is known imprecisely.

3.3.2 Objective

The objective of the model is to stay below an imprecisely stated upper bound on the total cost keeping in view the imprecise demand for a finite number of periods.

3.3.3 Additional Notation

Let,

z_0 = imprecisely known total cost limit specified by the management,

p_0 = tolerance level associated with the imprecisely known total cost z_0,

p_j = tolerance level associated with imprecisely known demand d_j, for all j,

μ_{OF} = membership function associated with imprecisely known total cost z_0,

μ_{jL} = membership function corresponding to lower side of the constraint associated with imprecisely known demand d_j, for all j,

μ_{jU} = membership function corresponding to upper side of the constraint associated with imprecisely known demand d_j, for all j,

All other variables and symbols have the same meaning as in crisp formulation.

3.3.4 Formulation under Fuzzy Environments

Using Zimmerman's notation (Zimmermann, 1991), the following crisp constraints

$$\sum_{i=1}^{N} x_{ij} = d_j \qquad j = 1, 2, \ldots, N$$

can be rewritten in fuzzy environment as

$$\sum_{i=1}^{N} x_{ij} \tilde{\leq} d_j \qquad j = 1, 2, \ldots, N$$

with μ_{jL} as the corresponding membership function, and

$$\sum_{i=1}^{N} x_{ij} \tilde{\geq} d_j \qquad j = 1, 2, \ldots, N$$

with μ_{jU} as the corresponding membership function, where '$\tilde{\leq} d_j$' (or '$\tilde{\geq} d_j$', respectively) means that the corresponding fuzzy constraint is 'essentially $\leq d_j$ '(or essentially $\geq d_j$', respectively), for all j (Zimmermann,1991),

Then, under fuzzy environments, our linear programming problem becomes the following problem (3FP).

(3FP) Find x_{ij}'s , $i, j = 1, 2, \ldots, N$, that satisfy the following.

For the objective function, we have,

$$\sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} + \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} b_i x_{kj} + \sum_{i=1}^{N} s_i y_i \tilde{\leq} z_0 \qquad (1)$$

With μ_{OF} as the corresponding membership function for the objective function.

The fuzzy demand constraints with corresponding membership functions μ_{jL} and μ_{jU} are,

$$\sum_{i=1}^{N} x_{ij} \tilde{\leq} d_j \qquad j = 1, 2, \ldots, N \qquad (2)$$

$$\sum_{i=1}^{N} x_{ij} \tilde{\geq} d_j \qquad j = 1, 2, \ldots, N \qquad (3)$$

and the crisp constraints are written as

$$\sum_{j=1}^{N} x_{ij} \leq G\, y_i \qquad i = 1, 2, \ldots, N, \qquad (4)$$

$$x_{ij} \geq 0 \quad \text{and integer} \qquad i, j = 1, 2, \ldots, N. \qquad (5)$$

$$y_i = 0,1 \qquad i = 1, 2, \ldots, N. \qquad (6)$$

$$0 \leq \lambda \leq 1 \qquad (7)$$

3.3.4.1 Membership Functions

Following Zimmermann (1991), below we define the membership functions for the fuzzy objective and fuzzy constraints.

Objective Function

Let us denote our objective function by $f_0(x)$. Figure 3.2 represents the graphical representation of this function.

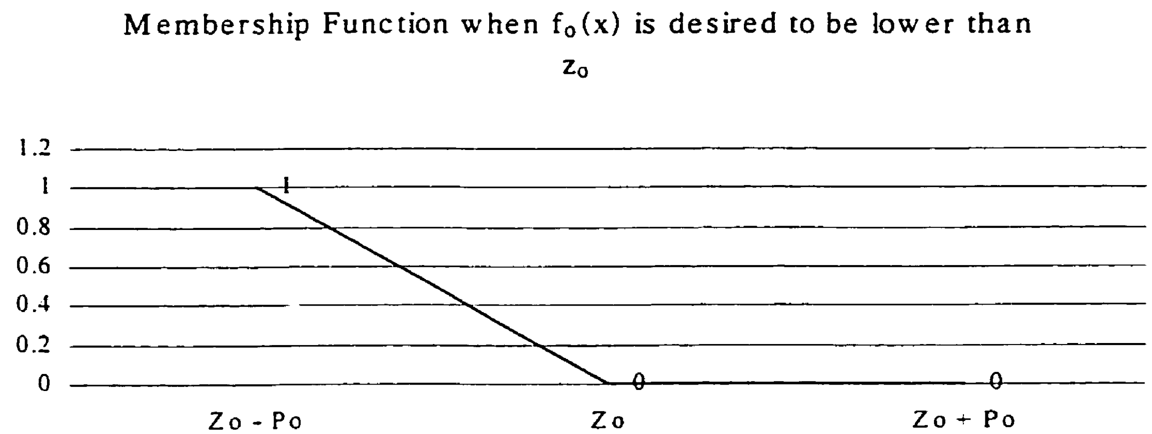


Figure 3.2 Membership function for Objective Function

Then, if

$f_0(x)$ is desired to be possibly lower than z_0 , the membership function μ_{OF} for objective function is written as

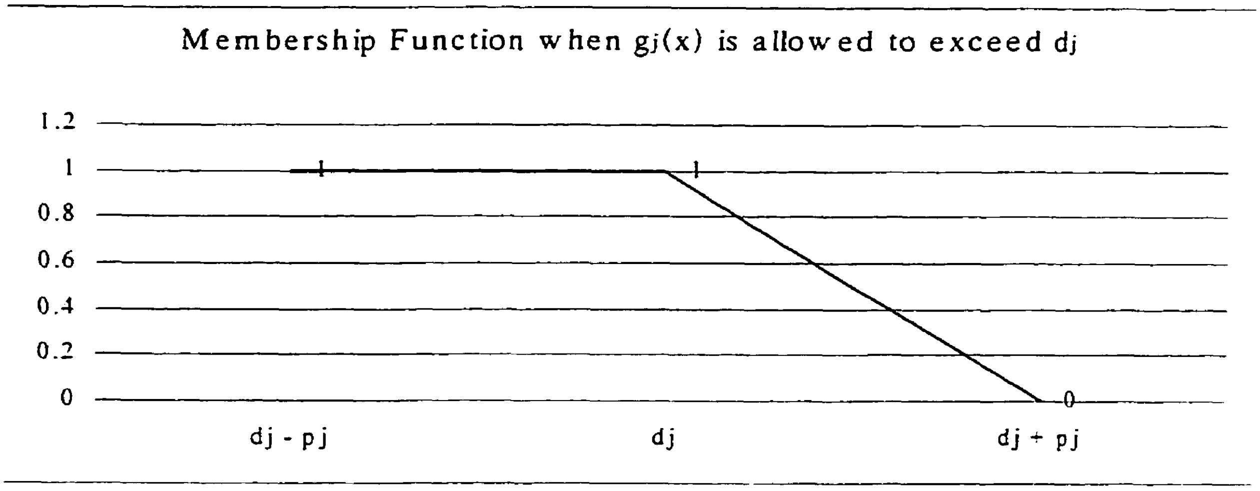
$$\mu_{OF} = \begin{cases} 1 & \text{if } f_0(x) \le z_0 - p_0 \\ 1 - \dfrac{[f_0(x) - (z_0 - p_0)]}{p_0} & \text{if } z_0 - p_0 < f_0(x) \le z_0 \\ 0 & \text{if } z_0 < f_0(x) \end{cases}$$

Demand Constraints

The membership functions for fuzzy constraints are obtained below.

We denote our fuzzy constraint functions by $g_j(x)$, $j = 1, 2, \ldots, N$.

Figure 3.3 Membership function for the upper side of the fuzzy region of the fuzzy constraint



Then,

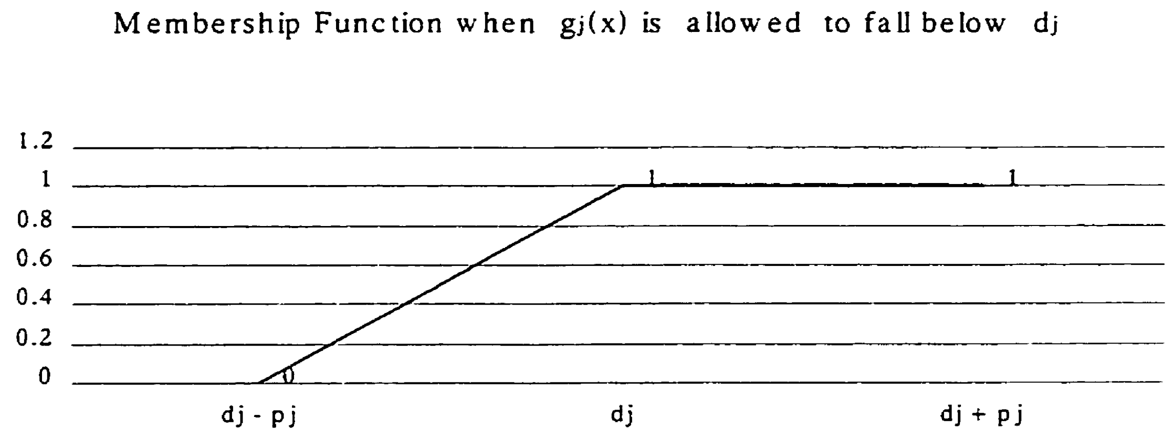
μ_{jU} = membership function for the upper side of the fuzzy region of the fuzzy constraint corresponding to period j, is taken as (represented in Figure 3.3)

$$\mu_{jU} = \begin{cases} 1 & \text{if} \quad g_j(x) \leq d_j \\ 1 - \dfrac{[g_j(x) - d_j]}{p_j} & \text{if} \quad d_j < g_j(x) \leq d_j + p_j \\ 0 & \text{if} \quad d_j + p_j < g_j(x) \end{cases}$$

Also,

μ_{jL} = membership function for the lower side of the fuzzy region of the fuzzy constraint corresponding to period j, is taken as (represented in Figure 3.4)

Figure 3.4 Membership function for the lower side of the fuzzy region of the fuzzy constraint



$$\mu_{jL} = \begin{cases} 1 & \text{if} \quad g_j(x) \geq d_j \\ 1 - \dfrac{[d_j - g_j(x)]}{p_j} & \text{if} \quad d_j - p_j < g_j(x) \leq d_j \\ 0 & \text{if} \quad d_j - p_j > g_j(x) \end{cases}$$

Once the membership functions are known, then the intersection of these fuzzy sets given by (1), (2), and (3) is to be found out to get a decision. Let $\mu_D(x)$ be the membership function of the fuzzy set 'decision' of the model. Thus $\mu_D(x)$

$$\mu_D(x) = \min(\mu_{OF}, \mu_{1L}, \mu_{2L}, \mu_{3L}, \ldots, \mu_{NL}, \mu_{1U}, \mu_{2U}, \mu_{3U}, \ldots, \mu_{NU})$$

Since, we are interested in large value of $\mu_D(x)$ over (4), (5), (6) and (7), therefore, following Zimmermann (1991), we obtain

maximize $\mu_D(x) = \min(\mu_{OF}, \mu_{1L}, \mu_{2L}, \mu_{3L}, \ldots, \mu_{NL}, \mu_{1U}, \mu_{2U}, \mu_{3U}, \ldots, \mu_{NU})$

subject to the constraints (4)-(7).

Replacing $\mu_D(x)$ by λ , we have the following problem (3P) along the lines of Zimmermann (1991) ;

(3P) max λ

subject to

$$\mu_{OF} \geq \lambda$$

$$\mu_{jL} \geq \lambda \qquad j = 1, 2, \ldots, N,$$

$$\mu_{jU} \geq \lambda \qquad j = 1, 2, \ldots, N,$$

and crisp constraints (4)-(7)

It is observed that (3P) is a crisp linear program whose optimal solution provides a solution to (3FP).

3.3.5 Numerical Example under Fuzzy Environments

Below we write a fuzzified format (3FC) of (3C). In this example we assume a tolerance level of approximately 30% in demands and 0.25% in total cost. Therefore z_0 is \$245 and p_0 is \$0.6125. For the demand constraints, the tolerances are $p_1 = 6$, $p_2 = 15$, $p_3 = 3$, $p_4 = 3$, $p_5 = 15$, $p_6 = 6$, where as the rest of the data is same as in (3C). Figure 3.5 represents the variation in demand in each time period. Choosing G = 200 >160, we have

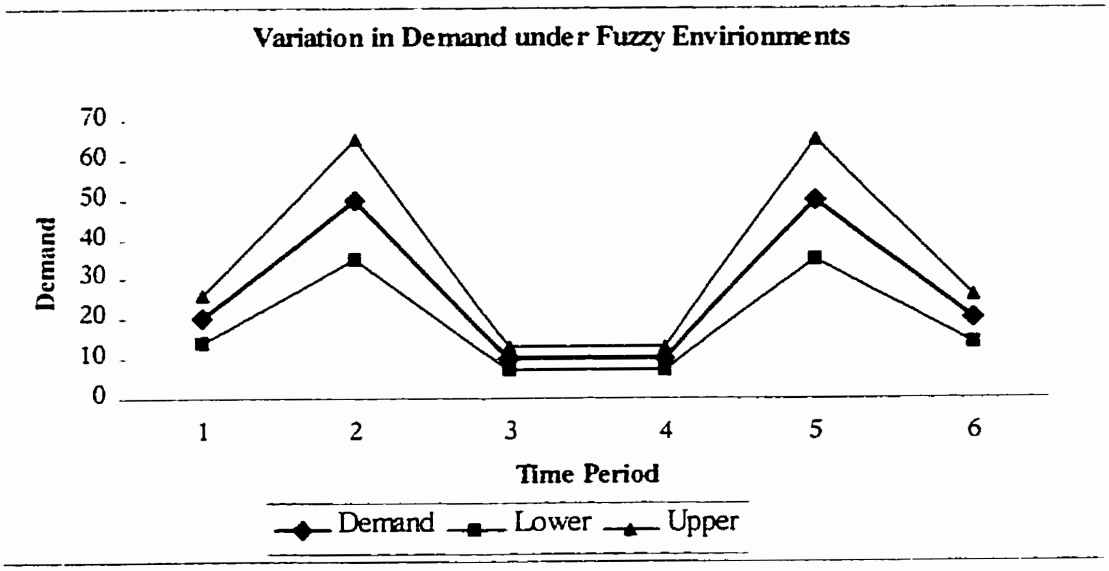


Figure 3.5 Variation in Demand under Fuzzy Environments

(3FC) Maximize λ

subject to

$$1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 0.5x_{21} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{31} + 0.5x_{32} + 1x_{34} + 2x_{35} + 3x_{36} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 1x_{45} + 2x_{46} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} + 1x_{56} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 + 100y_6 + 0.6125\lambda \leq 245 \quad (1)$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} - 6\lambda \geq 14 \quad (2)$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} + 6\lambda \leq 26 \quad (3)$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} - 15\lambda \geq 35 \quad (4)$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} + 15\lambda \leq 65 \quad (5)$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} - 3\lambda \geq 7 \quad (6)$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} + 3\lambda \leq 13 \quad (7)$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} - 3\lambda \geq 7 \quad (8)$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} + 3\lambda \leq 13 \quad (9)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} - 15\lambda \geq 35 \quad (10)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} + 15\lambda \leq 65 \quad (11)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} - 6\lambda \geq 14 \quad (12)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} + 6\lambda \leq 26 \quad (13)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - 200y_1 \leq 0 \quad (14)$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - 200y_2 \leq 0 \quad (15)$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - 200y_3 \leq 0 \quad (16)$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - 200y_4 \leq 0 \quad (17)$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - 200y_5 \leq 0 \quad (18)$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - 200y_6 \leq 0 \quad (19)$$

$$x_{ij} \geq 0 \text{ and integer} \qquad i, j = 1, 2, \ldots, 6.$$

$$y_i = 0, 1. \qquad i = 1, 2, \ldots, 6.$$

$$0 \leq \lambda \leq 1$$

3.3.6 Results.

The optimal solution to (3FC) is as in Table 3.3. The replenishment schedule obtained from Table 3.3 is given in Table 3.3.1.

Table 3.3 Results under fuzzy environments allowing backorders.

Variable	Value	Variable	Value
x_{21}	20	x_{55}	49
x_{22}	50	x_{56}	19
x_{23}	10	y_2	1
x_{54}	10	y_5	1
λ	0.833		

Table 3.3.1 Replenishment Schedule under fuzzy environments allowing backorders

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	20	0	0	20
2	50	80	10	0
3	10	0	0	0
4	10	0	0	10
5	50	78	19	0
6	20	0	0	0

Values of the membership functions for the above solution are provided in the last row of Table 3.5.

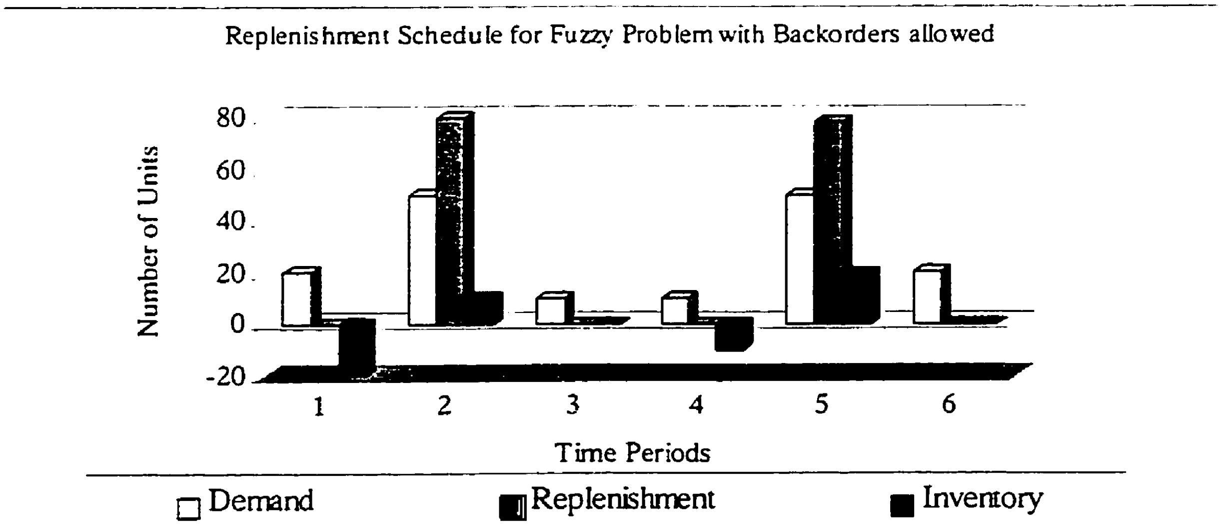


Figure 3.6 Replenishment schedule for Fuzzy Problem with Backorders allowed

3.3.7 Interpretation of the Results

Since x_{ij} is the quantity produced in Period i for demand in Period j, therefore from Tables 3.3 and 3.3.1and Figure 3.6, we have $x_{21} + x_{22} + x_{23} = 20 + 50 + 10 = 80$ units should be acquired in Period 2. Out of these 80 units, 20 were backordered in Period 1 and acquired in Period 2. 50 units would be acquired in Period 2 for Period 2, and 10 units are carried over to Period 3. Again, the interpretation for $x_{54} + x_{55} + x_{56} = 10 + 49 + 19 = 78$ units acquired in Period 5, is that 10 units are used to fill the backorder from Period 4, 49 units are used to fill the demand in Period 5 itself, and the remaining 19 are carried over to Period 6. The values $y_2 = 1$ and $y_5 = 1$ indicate that the acquisition of the units for satisfying total demand is done in Period 2 and Period 5 only. Therefore, the setup cost is incurred in Periods 2 and 5. The total minimum cost of the objective function is $244. The values of membership functions and the optimal value of λ are given in the Table 3.6.

The following Table 3.4 and Figure 3.7 shows the behavior of the value of λ corresponding to changes in tolerance levels, of 10, 20, 30, 40 and 50 percent for imprecisely known demand, and of 0.25, 0.5, 1, 2, 3, 4, and 5 percent for imprecisely known total cost.

Table 3.4 – Value of λ corresponding to Demand Tolerance and Total Cost Tolerance

Demand Tolerance	Total Cost Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	0.500	0.500	0.500	0.400	0.340	0.250	0.200
15%	0.667	0.667	0.612	0.510	0.466	0.408	0.330
20%	0.750	0.750	0.700	0.600	0.544	0.500	0.500
25%	0.800	0.800	0.680	0.600	0.600	0.600	0.600
30%	0.833	0.833	0.800	0.733	0.667	0.667	0.667

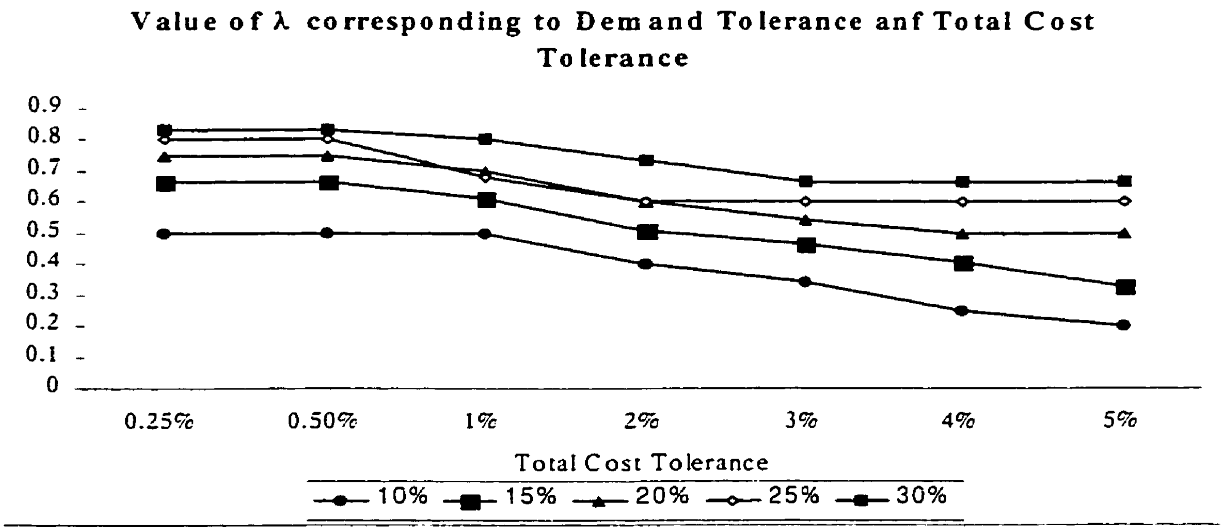


Figure 3.7 Value of λ corresponding to Demand Tolerance and Total Cost Tolerance

Table 3.5 –Total Cost corresponding to Demand Tolerance and Total Cost Tolerance

Demand Tolerance	Total Cost Tolerance						
	0.25%	0.5%	1%	2%	3%	4%	5%
10%	$244	$244	$243.5	$242.5	$242.5	$242.5	$242.5
15%	$244	$244	$243.5	$242.5	$241	$241	$240.5
20%	$244	$244	$242.5	$241	$241	$240	$238.5
25%	$244	$244	$242.5	$241	$240.5	$238.5	$237
30%	$244	$243.5	$242.5	$241	$240	$237	$235

Also, Table 3.6 shows the values of membership functions calculated under the conditions of 0.25% total cost tolerance and variable demand tolerance of 10%, 15%, 20%, 25% and 30%. The value of membership function corresponding to upper side of each constraint associated with imprecisely known demand is equal to 1 for this particular case. Therefore. these values are not listed in the table.

Table 3.6 Values of membership functions under various tolerance levels for demand at 0.25% tolerance level for total cost.

Demand Tolerance	λ	μ_{OF}	μ_{1L}	μ_{2L}	μ_{3L}	μ_{4L}	μ_{5L}	μ_{6L}
10%	0.50	1	1	1	1	1	0.80	0.50
15%	0.66	1	1	1	1	1	0.87	0.66
20%	0.75	1	1	1	1	1	0.90	0.75
25%	0.80	1	1	1	1	1	0.92	0.80
30%	0.83	1	1	1	1	1	0.93	0.83

3.3.8 Discussion of the solution in view of Table 3.4, 3.5 and 3.6

Table 3.4 and Figure 3.7 shows different values of λ for various tolerance levels for the imprecisely known total cost and imprecisely known demand. Also, Table 3.5 shows different values of Total Cost for various tolerance levels for the imprecisely known total cost and imprecisely known demand. Note that in this formulation the membership function λ is used to express the degree of certainty of the solution with respect to fuzzy parameters, total cost and imprecisely known demands (Zimmerman, 1991). From Table 3.4, it is observed that with the increase in the tolerance level for imprecisely known total cost, the value of λ decreases. This shows that the smaller the value of membership grade λ, the smaller is the support for the solution and hence, lower the degree of certainty of solution. On the other hand, it is observed that with increase in tolerance limits for imprecisely known demand, the value of λ increases. This shows that the larger the value of membership grade λ, the larger is the support for the solution. It can therefore be concluded that fuzzy programming does not provide just another crisp solution, instead it produces the optimum solution corresponding to the pre-specified tolerance levels of constraints.

Table 3.6 shows that the membership functions corresponding to lower side of the constraint associated with imprecisely known demand for Periods 5 and 6, i.e. constraints (10) and (12), are not completely satisfied all the times. It is observed that demand in Period 5 is completely satisfied only 80% of the time when the demand tolerance is 10%, and increases to 93% satisfaction upon increasing the demand tolerance to 30%. Similarly, in Period 6, demand satisfaction increases from 50% to 83% under the same conditions. That explains the difference in values of x_{55} and x_{56} in Tables 3.2 when compared to Table 3.3.

In the above examination, the relationship between total cost and choices of membership grades, and the range of demand for each period are investigated for a range of possible values of membership grade between 0 and 1. Such an examination is required in order to provide the MRP planner with sufficient information on the implication of the choice of the membership grade prior to the final choice determined by him. Another advantage of fuzzy programming is that it admits imprecise data, especially the data that are not statistically defined (Saturdi et. al., 1995). This feature is particularly useful for the situation when the management in an organization is not able to specify precisely the total cost limit, but is rather able to provide lower and upper bounds, with a prespecified tolerance interval above or below these bounds, taken as representing imprecision in setting of such bounds. As already stated, fuzzy set theory permits the partial belonging of an element to a fuzzy set characterized by a membership function that takes values in the interval [0,1]. Thus, fuzzy programming produces most satisfactory solution within a pre specified interval, whereas a conventional crisp set theory constraint only permits an element either to belong (membership grade 1) or not to belong (membership grade 0) to the set.

Chapter 4

INVENTORY REPLENISHMENT PROBLEM WITH VARIABLE DEMAND AND QUANTITY DISCOUNTS: AN INTEGER LINEAR PROGRAMMING APPROACH

In the present chapter, we consider variable demand rate inventory problem with quantity discounts permitted, both under crisp and fuzzy environments, with a finite planning horizon. The problem is formulated under two different scenarios; one not allowing backorders, and the other allowing backorders. To obtain a solution to such problems, we formulate a piecewise integer linear programming model, with a finite number of integer and 0-1 variables. Furthermore, it is observed that our results obtained by using piecewise integer linear programming model under crisp environment are similar to the results obtained by other researchers using the dynamic programming approach.

4.1 Introduction

Price discounts are often offered by suppliers to encourage larger orders. Benefits for purchaser from bigger orders include reduction in unit price, lower shipping and handling costs, and reduction in ordering costs owing to fewer orders. These benefits have to be measured against the incremental increase in carrying costs. As the order size increases, more space must be provided for storage and the costs of holding the larger inventory level correspondingly increase. Another appropriate consideration, though difficult to quantify, is the risk of obsolescence or functional depreciation. Larger inventories magnify the loss that would result if design or demand changes make the stored supplies less valuable. The buyer must be able to select the most advantageous discount opportunities given total storage availability and financial capacity. Any savings allowed by ordering larger quantities should be evaluated against the risks incurred from maintaining higher stock levels. Risks are gauged by stability of past demand, resale value of stock, and market trends.

4.2 General Quantity Discount Model

Let q be the quantity ordered each time an order is placed, and let $b_1, b_2, \ldots, b_R$ be those points where a price change (or a price break) occurs. Then, the general quantity discount model (Winston, 1994) may be described as follows:

If $q < b_1$, each item costs p_1 dollars,

If $b_1 \leq q < b_2$, each item costs p_2 dollars,

. . . .

If $b_{R-2} \leq q < b_{R-1}$, each item costs p_{R-1} dollars,

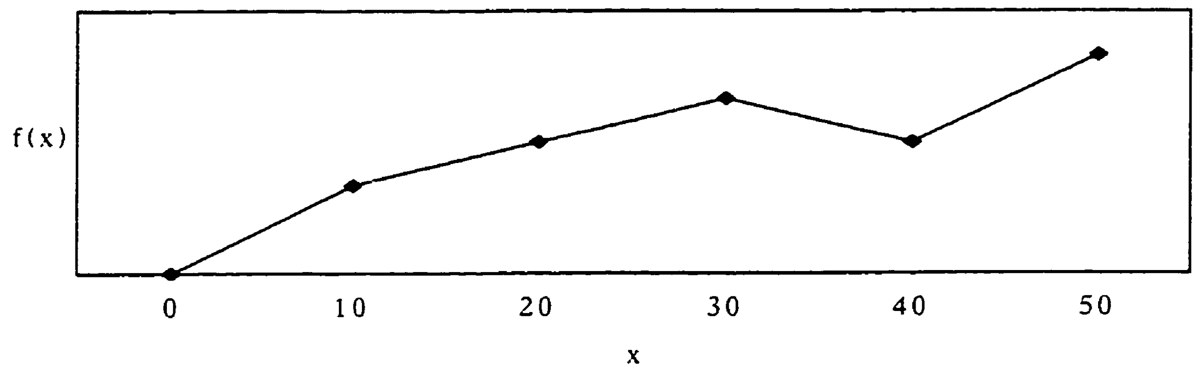
If $b_{R-1} \leq q < b_R = \infty$, each item costs p_R dollars.

Since larger order quantities should be associated with lower prices, therefore, without any loss of generality we take $p_R < p_{R-1} < \ldots < p_2 < p_1$.

4.3 Generalized Integer Programming Model using Quantity Discounts

Integer programming makes use of 0-1 variables to model optimization problems involving piecewise linear functions (Winston, 1994). A piecewise linear function consists of several straight-line segments. For example, the piecewise linear function in Figure 4.1 is made up of four straight-line segments. The points where the slope of the piecewise linear function changes are called break points of the function. Thus, 0, 10, 30, 40, and 50 are break points of the function pictured in Figure 4.1.

Figure 4.1 Piecewise Linear Function showing Break Points.



Since piecewise linear function is not a linear function, it can be concluded that linear programming cannot be used to solve optimization problems involving these functions. However, by using the concept of convex combination and the 0-1 variables(Winston,1994), piecewise linear functions can be represented in a linear form. To do so, we assume that a piecewise linear function f(x) has break points $b_1, b_2, \ldots, b_R$, and for some k (k = 1, 2, ..., R-1),

$b_k \le x \le b_{k+1}$. Then, for some z_k, x may be written as

$$x = z_k b_k + (1 - z_k) b_{k+1} \quad , \text{ where, } 0 \le z_k \le 1.$$

Since f(x) is linear for $b_k \le x \le b_{k+1}$, we may write f(x) also as a convex combination of $f(b_k)$ and $f(b_{k+1})$ as follows.

$f(x) = z_k f(b_k) + (1 - z_k) f(b_{k+1})$

Now, in order to express a piecewise linear function via linear constraints and 0-1 variables, we follow the following two steps:

Step 1.

Wherever f(x) occurs in the optimization problem, replace x and f(x) by their convex combinations

$$x = z_1 b_1 + z_2 b_2 + \ldots + z_R b_R$$

and $$f(x) = z_1 f(b_1) + z_2 f(b_2) + \ldots + z_R f(b_R),$$

where, $z_1 + z_2 + \ldots + z_R = 1$, and $z_i \geq 0$, ($i = 1, 2, \ldots, R$.).

Step 2.

Add the following constraints to the problem,

$$z_1 \leq m_1,$$

$$z_2 \leq m_1 + m_2,$$

$$z_3 \leq m_2 + m_3$$

.

.

$$z_{R-1} \leq m_{R-2} + m_{R-1},$$

and $$z_R \leq m_{R-1}$$

where, $$m_1 + m_2 + \ldots + m_{R-1} = 1$$

and $$m_i = 0 \text{ or } 1 \quad (i = 1, 2, \ldots, R-1.)$$

4.4 Piecewise Integer Linear Programming Formulation under Crisp Environments with no Backorders allowed

4.4.1 Assumptions

For this model, in addition to the above break points and the related prices, we make the following additional assumptions as in Bector et. al. (1992).

1. The demand varies from one time period to another and is assumed to be known.
2. The units needed to satisfy demand in a particular period must be purchased during a previous period or at the beginning of a particular period during which it is required.
3. Acquisition costs (ordering or follow-up costs for purchased parts) are fixed relative to the quantity purchased, but may vary from one time period to another.
4. Holding cost in a particular time period represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.
5. Holding cost may vary from one time period to another.
6. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.
7. The unit variable cost depends upon replenishment quantity i.e. quantity discounts are permitted.
8. The product is treated entirely independently of the other products i.e. benefits from joint replenishment do not exist or are ignored.

4.4.2 Notation

In addition to the notation used in section 3.2.2 of Chapter 3, let

R = number of quantity discount break points,

P_i = total quantity purchased in period i,

z_{is} = a continuous non negative variable corresponding to period i at price break point s,

m_{is} = a binary variable corresponding to period i at price break point s,

q_s = quantity required to be purchased at break point s,

$f(q_s)$ = function that calculates the purchasing cost associated with q_s.

4.4.3 Objective

Our objective in this problem is to minimize the total cost of acquiring the units x_{ij}'s (i, j = 1, 2, ..., N), such that certain time varying demand constraints are satisfied, where the total cost of acquiring the units is the sum of total carrying cost, the total setup cost and total purchasing cost.

Minimize total cost Z = Total carrying cost + Total setup cost + Total Purchasing Cost.

4.4.4 General Formulation

Then, under the crisp environment, we have the following problem (4C).

$$\text{(4C)} \qquad \text{Min}\left\{ \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} + \sum_{i=1}^{N} s_i y_i + \sum_{s=1}^{R} \sum_{i=1}^{N} f(q_s)\, z_{is} \right\}$$

subject to

$$\sum_{j=1}^{N} x_{ij} \le G y_i \qquad i = 1, 2, \ldots, N,$$

$$\sum_{i=1}^{j} x_{ij} = d_j \qquad j = 1, 2, \ldots, N,$$

$$\sum_{j=1}^{N} x_{ij} - P_i = 0 \qquad i = 1, 2, \ldots, N,$$

$$P_i - \sum_{s=1}^{R} q_s z_{is} = 0 \qquad i = 1, 2, \ldots, N$$

$$z_{i1} \le m_{i1} \qquad i = 1, 2, \ldots, N,$$

$$z_{i2} \le m_{i1} + m_{i2} \qquad i = 1, 2, \ldots, N,$$

$$z_{i3} \le m_{i2} + m_{i3} \qquad i = 1, 2, \ldots, N,$$

.

.

$$z_{i\,R-1} \le m_{i\,R-2} + m_{i\,R-1} \qquad i = 1, 2, \ldots, N,$$

$$z_{i\,R} \le m_{i\,R-1} \qquad i = 1, 2, \ldots, N,$$

$$\sum_{s=1}^{R} z_{is} = 1 \qquad i = 1, 2, \ldots, N$$

$$\sum_{s=1}^{R-1} m_{is} = 1 \qquad i = 1, 2, \ldots, N$$

$$x_{ij} \ge 0 \text{ and integer} \qquad i, j = 1, 2, \ldots, N, \text{ and } i \le j.$$

$$y_i = 0,1. \qquad i = 1, 2, \ldots, N.$$

$$z_{is} \ge 0 \qquad i = 1, 2, \ldots, N, \text{ and } s = 1, 2, \ldots, R.$$

$$m_{is} = 0,1. \qquad i = 1, 2, \ldots, N, \text{ and } s = 1, 2, \ldots, R\text{-}1.$$

4.4.5 Numerical Example

We illustrate our method through the following numerical example taken from (Fordyce, 1985). We assume, as in Fordyce (1985), that the setup cost and holding cost remain constant throughout the planning horizon.

Let, for $i = 1, 2,\ldots, 6$, and $R = 1, 2,\ldots, 4$,

$s_i = \$100$, $h_i = \$1.00$ per unit per time period.

The specific quantity discount in effect is a price of \$10 per unit for orders of less than 100 units, \$8 per unit from 100 units to 150 units, and \$7 per unit for orders in excess of 150 units. Therefore $q_1 = 0$, $q_2 = 100$, $q_3 = 150$ and $q_4 = 160$. The corresponding $f(q_s)$'s are calculated below.

$f(q_1) = \$0$

$f(q_2) = 100*10 = \$1000$

$f(q_3) = 150*8 = \$1200$

$f(q_4) = 160*7 = \$1120$.

and let ,

Table 4.1 Demand Data for each Time Period.

Period	1	2	3	4	5	6
Demand	20	50	10	10	50	20

Selecting $G = 200 > 20 + 50 + 10 + 10 + 50 + 20 = 160$, we have the following formulation

Minimize

$$1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{34} + 2x_{35} + 3x_{36} + 1x_{45} + 2x_{46} + 1x_{56} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 + 100y_6 + 0z_{11} + 1000z_{12} + 1200z_{13} + 1120z_{14} + 0z_{21} + 1000z_{22} \div 1200z_{23} + 1120z_{24} + 0z_{31} + 1000z_{32} + 1200z_{33} + 1120z_{34} + 0z_{41} + 1000z_{42} + 1200z_{43} + 1120z_{44} + 0z_{51} + 1000z_{52} + 1200z_{53} + 1120z_{54} + 0z_{61} + 1000z_{62} + 1200z_{63} + 1120z_{64}$$

subject to

$$x_{11} = 20 \quad (1)$$

$$x_{12} + x_{22} = 50 \quad (2)$$

$$x_{13} + x_{23} + x_{33} = 10 \quad (3)$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 10 \quad (4)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} = 0 \quad (5)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} = 20 \quad (6)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - 200y_1 \leq 0 \quad (7)$$

$$x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - 200y_2 \leq 0 \quad (8)$$

$$x_{33} + x_{34} + x_{35} + x_{36} - 200y_3 \leq 0 \quad (9)$$

$$x_{44} + x_{45} + x_{46} - 200y_4 \leq 0 \quad (10)$$

$$x_{55} + x_{56} - 200y_5 \leq 0 \quad (11)$$

$$x_{66} - 200y_6 \leq 0 \quad (12)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - P_1 = 0 \quad (13)$$

$$x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - P_2 = 0 \quad (14)$$

$$x_{33} + x_{34} + x_{35} + x_{36} - P_3 = 0 \quad (15)$$

$$x_{44} + x_{45} + x_{46} - P_4 = 0 \quad (16)$$

$$x_{55} + x_{56} - P_5 = 0 \quad (17)$$

$$x_{66} - P_6 = 0 \quad (18)$$

$$P_1 - 0z_{11} - 100z_{12} - 150z_{13} - 160z_{14} = 0 \quad (19)$$

$$P_2 - 0z_{21} - 100z_{22} - 150z_{23} - 160z_{24} = 0 \quad (20)$$

$$P_3 - 0z_{31} - 100z_{32} - 150z_{33} - 160z_{34} = 0 \quad (21)$$

$$P_4 - 0z_{41} - 100z_{42} - 150z_{43} - 160z_{44} = 0 \quad (22)$$

$$P_5 - 0z_{51} - 100z_{52} - 150z_{53} - 160z_{54} = 0 \quad (23)$$

$$P_6 - 0z_{61} - 100z_{62} - 150z_{63} - 160z_{64} = 0 \quad (24)$$

$$1z_{11} - 1m_{11} \leq 0 \quad (25)$$

$$1z_{21} - 1m_{21} \leq 0 \quad (26)$$

$$1z_{31} - 1m_{31} \leq 0 \quad (27)$$

$$1z_{41} - 1m_{41} \leq 0 \quad (28)$$

$$1z_{51} - 1m_{51} \leq 0 \quad (29)$$

$$1z_{61} - 1m_{61} \leq 0 \quad (30)$$

$$1z_{12} - 1m_{11} - 1m_{12} \leq 0 \quad (31)$$

$$1z_{22} - 1m_{21} - 1m_{22} \leq 0 \quad (32)$$

$$1z_{32} - 1m_{31} - 1m_{32} \leq 0 \quad (33)$$

$$1z_{42} - 1m_{41} - 1m_{42} \leq 0 \quad (34)$$

$$1z_{52} - 1m_{51} - 1m_{52} \leq 0 \quad (35)$$

$1z_{62} - 1m_{61} - 1m_{62} \leq 0$ (36)

$1z_{13} - 1m_{12} - 1m_{13} \leq 0$ (37)

$1z_{23} - 1m_{22} - 1m_{23} \leq 0$ (38)

$1z_{33} - 1m_{32} - 1m_{33} \leq 0$ (39)

$1z_{43} - 1m_{42} - 1m_{43} \leq 0$ (40)

$1z_{53} - 1m_{52} - 1m_{53} \leq 0$ (41)

$1z_{63} - 1m_{62} - 1m_{63} \leq 0$ (42)

$1z_{14} - 1m_{13} \leq 0$ (43)

$1z_{24} - 1m_{23} \leq 0$ (44)

$1z_{34} - 1m_{33} \leq 0$ (45)

$1z_{44} - 1m_{43} \leq 0$ (46)

$1z_{54} - 1m_{53} \leq 0$ (47)

$1z_{64} - 1m_{63} \leq 0$ (48)

$1z_{11} + 1z_{12} + 1z_{13} + 1z_{14} = 1$ (49)

$1z_{21} + 1z_{22} + 1z_{23} + 1z_{24} = 1$ (50)

$1z_{31} + 1z_{32} + 1z_{33} + 1z_{34} = 1$ (51)

$1z_{41} + 1z_{42} + 1z_{43} + 1z_{44} = 1$ (52)

$1z_{51} + 1z_{52} + 1z_{53} + 1z_{54} = 1$ (53)

$1z_{61} + 1z_{62} + 1z_{63} + 1z_{64} = 1$ (54)

$1m_{11} + 1m_{12} + 1m_{13} = 1$ (55)

$1m_{21} + 1m_{22} + 1m_{23} = 1$ (56)

$$1m_{31} + 1m_{32} + 1m_{33} \qquad = 1 \qquad (57)$$

$$1m_{41} + 1m_{42} + 1m_{43} \qquad = 1 \qquad (58)$$

$$1m_{51} + 1m_{52} + 1m_{53} \qquad = 1 \qquad (59)$$

$$1m_{61} + 1m_{62} + 1m_{63} \qquad = 1 \qquad (60)$$

$x_{ij} \geq 0$, and integer $\quad i, j = 1, 2, \ldots, 6$, and $i \leq j$.

$y_i = 0, 1. \quad i = 1, 2, \ldots, 6.$

$z_{is} \geq 0 \quad i = 1, 2, \ldots, 6$, and $s = 1, 2, \ldots, 4$.

$m_{is} = 0, 1. \quad i = 1, 2, \ldots, 6$, and $s = 1, 2, \ldots, 3$.

4.4.6 Results

Solving the above problem, we obtain the following Table 4.2.

Table 4.2. Results of the Crisp Problem with Quantity Discounts with no backorders.

Variable	Value	Variable	Value	Variable	Value
x_{11}	20	P_1	160	m_{13}	1
x_{12}	50	z_{14}	1	m_{21}	1
x_{13}	10	z_{21}	1	m_{31}	1
x_{14}	10	z_{31}	1	m_{41}	1
x_{15}	50	z_{41}	1	m_{51}	1
x_{16}	20	z_{51}	1	m_{61}	1
y_1	1	z_{61}	1	Z optimal	1620

Table 4.2.1 Replenishment Schedule for the crisp problem with Quantity Discounts with no backorders

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	20	160	140	0
2	50	0	90	0
3	10	0	80	0
4	10	0	70	0
5	50	0	20	0
6	20	0	0	0

4.4.7 Interpretation of the Results

Since x_{ij} is the quantity purchased in Period i for demand in Period j, therefore from Tables 4.2 and 4.2.1 and Figure 4.2, we have $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} = 20 + 50 + 10 +10 + 50 + 20 = 160$ units should be acquired in Period 1. The value of $P_1 = 160$ indicates that entire purchase of 160 units to be made in Period 1 at a rate of $7 per unit. The values of z_{14}, z_{21}, z_{31}, z_{41}, z_{51} and z_{61} are equal to 1, thereby satisfying constraints (49) to (54). Similarly, the values of m_{13}, m_{21}, m_{31}, m_{41}, m_{51} and m_{61} are equal to 1, thereby satisfying constraints (55) to (60). The value of $y_1 = 1$ indicates that the acquisition of the units for satisfying total demand is done in Period 1 only. Therefore, the setup cost is incurred in Period 1. The total minimum cost or the value of the objective function is $1620. It may be observed that the above results are as in Fordyce (1985)

4.5 Piecewise Integer Linear Programming Formulation under Crisp Environments with Backorders allowed

We now consider the above problem with option of backordering. It is important to point out here, that backordering with quantity discounts is not considered by Fordyce-Webster (1985).

4.5.1 Assumptions

For this model, the following additional assumptions are made (Bector et. al (1992)):

1. The units needed to satisfy demand in a particular period can be acquired at any time including the backorders.
2. Backorder cost represents the cost of backordering one unit of product from the period in which it was needed to the next time period. Units backordered more than one time period would be charged to the accumulated backordering cost of all the periods from the time period of demand till the time period of acquisition.
3. Backorder cost may vary from one time period to another.

4.5.2 Objective

Our objective in this problem is to minimize the total cost of acquiring the units x_{ij}'s (i, j = 1, 2, ..., N), such that certain variable demand constraints are satisfied, where the total cost of acquiring the units is the sum of total carrying cost, total backordering cost , the total setup cost and total purchasing cost.

Minimize total cost Z = Total carrying cost + Total Backordering Cost + Total setup cost + Total Purchasing Cost.

4.5.3 Generalized Piecewise Integer Linear Programming Model

Under the crisp environment, we have the following problem (4D).

$$\text{(4D)} \qquad \text{Min}\left\{ \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} + \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} b_i x_{kj} + \sum_{i=1}^{N} s_i y_i + \sum_{s=1}^{R} \sum_{i=1}^{N} f(q_s) z_{is} \right\}$$

subject to

$$\sum_{j=1}^{N} x_{ij} \le G\, y_i \qquad i = 1, 2, \ldots, N,$$

$$\sum_{i=1}^{N} x_{ij} = d_j \qquad j = 1, 2, \ldots, N,$$

$$\sum_{j=1}^{N} x_{ij} - P_i = 0 \qquad i = 1, 2, \ldots, N,$$

$$P_i - \sum_{s=1}^{R} q_s z_{is} = 0 \qquad i = 1, 2, \ldots, N$$

$$z_{i1} \le m_{i1} \qquad i = 1, 2, \ldots, N,$$

$$z_{i2} \le m_{i1} + m_{i2} \qquad i = 1, 2, \ldots, N,$$

$$z_{i3} \le m_{i2} + m_{i3} \qquad i = 1, 2, \ldots, N,$$

.

.

$$z_{i\,R-1} \le m_{i\,R-2} + m_{i\,R-1} \qquad i = 1, 2, \ldots, N,$$

$$z_{iR} \le m_{i\,R-1} \qquad i = 1, 2, \ldots, N,$$

$$\sum_{s=1}^{R} z_{is} = 1 \qquad i = 1, 2, \ldots, N$$

$$\sum_{s=1}^{R-1} m_{is} = 1 \qquad i = 1, 2, \ldots, N$$

$x_{ij} \geq 0$ and integer $i, j = 1, 2, \ldots, N,$

$y_i = 0, 1.$ $i = 1, 2, \ldots, N.$

$z_{is} \geq 0$ $i = 1, 2, \ldots, N,$ and $s = 1, 2, \ldots, R.$

$m_{is} = 0, 1.$ $i = 1, 2, \ldots, N,$ and $s = 1, 2, \ldots, R-1.$

4.5.4 Numerical Example

We illustrate our method through the example taken above, including a new parameter i.e. the backordering cost. b_i = \$0.50 per unit per time period backordered. b_i is assumed to be the same in all the time periods. Then we have the following problem formulation.

Minimize $1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 0.5x_{21} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{31} + 0.5x_{32} + 1x_{34} + 2x_{35} + 3x_{36} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 1x_{45} + 2x_{46} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} + 1x_{56} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 + 100y_6 + 0z_{11} + 1000z_{12} + 1200z_{13} + 1120z_{14} + 0z_{21} + 1000z_{22} + 1200z_{23} + 1120z_{24} + 0z_{31} + 1000z_{32} + 1200z_{33} + 1120z_{34} + 0z_{41} + 1000z_{42} + 1200z_{43} + 1120z_{44} + 0z_{51} + 1000z_{52} + 1200z_{53} + 1120z_{54} + 0z_{61} + 1000z_{62} + 1200z_{63} + 1120z_{64}$

subject to

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 20 \quad (1)$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} = 50 \quad (2)$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} = 10 \quad (3)$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} = 10 \quad (4)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} = 50 \quad (5)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} = 20 \quad (6)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - 200y_1 \leq 0 \quad (7)$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - 200y_2 \leq 0 \quad (8)$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - 200y_3 \leq 0 \quad (9)$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - 200y_4 \leq 0 \quad (10)$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - 200y_5 \leq 0 \quad (11)$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - 200y_6 \leq 0 \quad (12)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - P_1 = 0 \quad (13)$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - P_2 = 0 \quad (14)$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - P_3 = 0 \quad (15)$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - P_4 = 0 \quad (16)$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - P_5 = 0 \quad (17)$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - P_6 = 0 \quad (18)$$

and the rest of the constraints (19) to (60) as in previous formulation 4.4.5.

4.5.5 Results

Solving the above problem. we obtain the following Table 4.3.

Table 4.3 Results of Crisp Problem with Quantity Discounts and Backorders allowed.

Variable	Value	Variable	Value	Variable	Value
x_{51}	20	P_5	160	m_{11}	1
x_{52}	50	z_{11}	1	m_{21}	1
x_{53}	10	z_{21}	1	m_{31}	1
x_{54}	10	z_{31}	1	m_{41}	1
x_{55}	50	z_{41}	1	m_{53}	1
x_{56}	20	z_{54}	1	m_{61}	1
y_5	1	z_{61}	1	Z optimal	1370

Table 4.3.1 Replenishment Schedule for crisp problem with Quantity Discounts and Backorders allowed

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	20	0	0	20
2	50	0	0	70
3	10	0	0	80
4	10	0	0	90
5	50	160	20	0
6	20	0	0	0

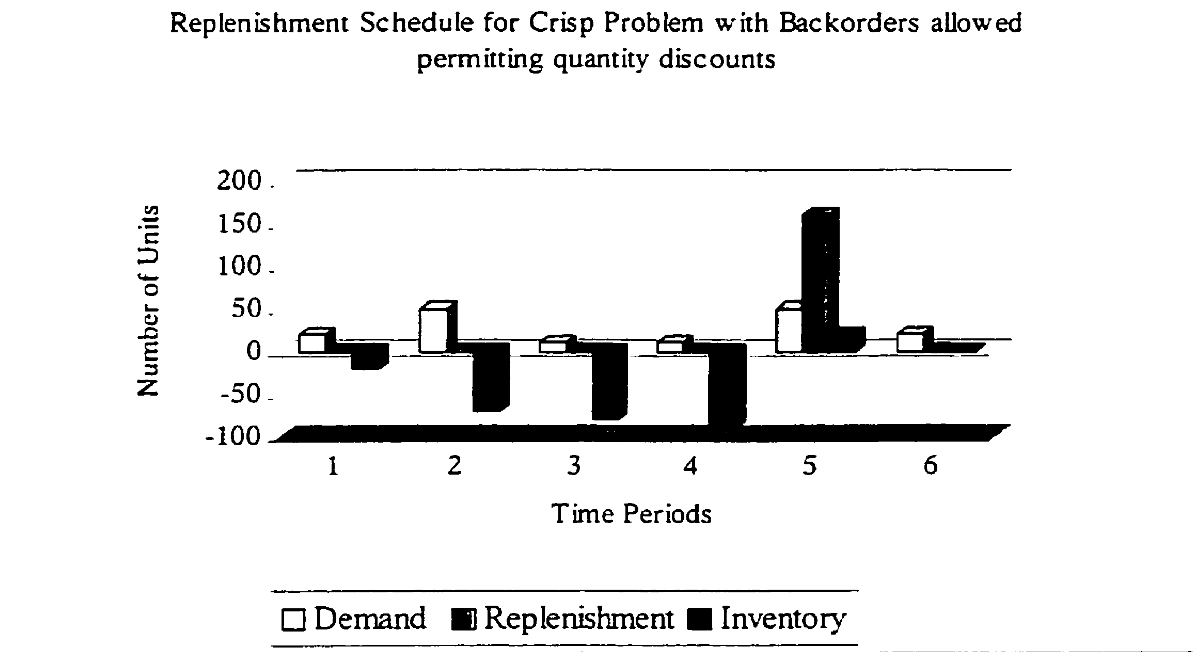


Figure 4.2 Replenishment Schedule for Crisp problem with Backorders allowed permitting quantity discounts

4.5.6 Interpretation of the Results

Since x_{ij} is the quantity purchased in Period i for demand in Period j, therefore from Tables 4.3 and 4.3.1 and Figure 4.2, we have $x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} = 20 + 50 + 10 + 10 + 50 + 20 = 160$ units should be acquired in Period 5. The value of $P_5 = 160$ indicates that entire purchase of 160 units to be made in Period 5 at a rate of $7 per unit. The values of z_{11}, z_{21}, z_{31}, z_{41}, z_{54} and z_{61} are equal to 1, thereby satisfying constraints (49) to (54). Similarly, the values of m_{11}, m_{21}, m_{31}, m_{41}, m_{53} and m_{61} are equal to 1, thereby satisfying constraints (55) to (60). The value of $y_5 = 1$ indicates that the acquisition of the units for satisfying total demand is done in Period 5 only. Therefore, the setup cost is incurred in Period 5. The total minimum cost or the value of the objective function is $1370.

4.6 Piecewise Integer Linear Programming under Fuzzy Environments with Backorders allowed

We formulate the problem under fuzzy environments under the following additional assumptions.

4.6.1 Additional Assumptions

1. The total cost over the entire planning horizon of N periods stays possibly below a given limit.
2. The demand that varies from period to period is known imprecisely.

4.6.2 Objective

The objective of the model is to stay below an imprecisely stated upper bound on the total cost keeping in view the imprecise demand for a finite number of periods.

4.6.3 Generalized Fuzzy Formulation

Using Zimmerman's notation (1991), the following crisp constraints

$$\sum_{i=1}^{N} x_{ij} = d_j \qquad j = 1, 2, \ldots, N$$

can be rewritten in fuzzy environment as

$$\sum_{i=1}^{N} x_{ij} \tilde{\lesssim} d_j \qquad j = 1, 2, \ldots, N$$

with μ_{jL} as the corresponding membership function, and

$$\sum_{i=1}^{N} x_{ij} \tilde{\geq} d_j \qquad j = 1, 2, \ldots, N$$

with μ_{jU} as the corresponding membership function, where '$\tilde{\leq} d_j$' (or '$\tilde{\geq} d_j$', respectively) means that the corresponding fuzzy constraint is 'essentially $\leq d_j$' (or essentially $\geq d_j$', respectively), for all j (Zimmerman (1991)).

Then, under fuzzy environments, our linear programming problem becomes the following problem (4FP).

(4FP) Find x_{ij}'s, $i, j = 1, 2, \ldots, N$, that satisfy the following.

For the objective function, we have,

$$\text{Min}\left\{\sum_{j=1}^{N}\sum_{k=2}^{N}\sum_{i=j}^{k-1} h_i x_{jk} + \sum_{j=1}^{N}\sum_{k=2}^{N}\sum_{i=j}^{k-1} b_i x_{kj} + \sum_{i=1}^{N} s_i y_i + \sum_{s=1}^{R}\sum_{i=1}^{N} f(q_s) z_{is}\right\} \tilde{\leq} z_0 \quad (1)$$

with μ_{OF} as the corresponding membership function for the objective function.

The fuzzy demand constraints with corresponding membership functions μ_{jL} and μ_{jU} are,

$$\sum_{i=1}^{N} x_{ij} \tilde{\leq} d_j \qquad j = 1, 2, \ldots, N \qquad (2)$$

$$\sum_{i=1}^{N} x_{ij} \tilde{\geq} d_j \qquad j = 1, 2, \ldots, N \qquad (3)$$

and the crisp constraints are written as

$$\sum_{j=1}^{N} x_{ij} \leq G\, y_i \qquad i = 1, 2, \ldots, N, \qquad (4)$$

$$\sum_{j=1}^{N} x_{ij} - P_i = 0 \qquad i = 1, 2, \ldots, N. \qquad (5)$$

$$P_i - \sum_{s=1}^{R} q_s z_{is} = 0 \qquad i = 1, 2, \ldots, N \qquad (6)$$

$$z_{i1} \le m_{i1} \qquad i = 1, 2, \ldots, N, \qquad (7)$$

$$z_{i2} \le m_{i1} + m_{i2} \qquad i = 1, 2, \ldots, N, \qquad (8)$$

$$z_{i3} \le m_{i2} + m_{i3} \qquad i = 1, 2, \ldots, N, \qquad (9)$$

.

.

$$z_{i\,R-1} \le m_{i\,R-2} + m_{i\,R-1} \qquad i = 1, 2, \ldots, N, \qquad (10)$$

$$z_{i\,R} \le m_{i\,R-1} \qquad i = 1, 2, \ldots, N, \qquad (11)$$

$$\sum_{s=1}^{R} z_{is} = 1 \qquad i = 1, 2, \ldots, N \qquad (12)$$

$$\sum_{s=1}^{R-1} m_{is} = 1 \qquad i = 1, 2, \ldots, N \qquad (13)$$

$$x_{ij} \ge 0 \text{ and integer} \qquad i, j = 1, 2, \ldots, N, \qquad (14)$$

$$y_i = 0, 1. \qquad i = 1, 2, \ldots, N. \qquad (15)$$

$$z_{is} \ge 0 \qquad i = 1, 2, \ldots, N, \text{ and } s = 1, 2, \ldots, R \qquad (16)$$

$$m_{is} = 0, 1. \qquad i = 1, 2, \ldots, N, \text{ and } s = 1, 2, \ldots, R-1 \qquad (17)$$

$$0 \le \lambda \le 1 \qquad (18)$$

Membership Functions

The membership functions for the fuzzy objective and fuzzy constraints are defined as in section 3.3.4.1 of Chapter 3.

Once the membership functions are known, then the intersection of these fuzzy sets is to be found out to get a decision. Let $\mu_D(x)$ be the membership function of the fuzzy set 'decision' of the model. Thus $\mu_D(x)$

$$\mu_D(x) = \min\,(\mu_{OF}, \mu_{1L}, \mu_{2L}, \mu_{3L}, \ldots, \mu_{NL}, \mu_{1U}, \mu_{2U}, \mu_{3U}, \ldots, \mu_{NU})$$

Since, we are interested in large value of $\mu_D(x)$ over (4) through (18), therefore, following Zimmermann (1991) we obtain

maximize $\mu_D(x) = \min\,(\mu_{OF}, \mu_{1L}, \mu_{2L}, \mu_{3L}, \ldots, \mu_{NL}, \mu_{1U}, \mu_{2U}, \mu_{3U}, \ldots, \mu_{NU})$

subject to the constraints (4)-(18).

Replacing $\mu_D(x)$ by λ , we have the following problem (4P) along the lines of Zimmermann (1991):

(4P) max λ

subject to

$\mu_{OF} \geq \lambda$

$\mu_{jL} \geq \lambda$ $\quad j = 1, 2, \ldots, N,$

$\mu_{jU} \geq \lambda$ $\quad j = 1, 2, \ldots, N,$

and crisp constraints (4)-(18)

It is observed that (4P) is a crisp linear program whose optimal solution provides a solution to (4FP).

4.6.4 Numerical Example under Fuzzy Environments

Below we write a fuzzified format (4FD) of (4D). In this example we assume a tolerance level of approximately 5% in demands and 0.25% in total cost. Therefore z_0 equal to \$1370 and p_0 is \$3.42. For the demand constraints, the tolerances are $p_1 = 1$, $p_2 = 2.5$, $p_3 = 0.5$, $p_4 = 0.5$, $p_5 = 2.5$, $p_6 = 1$, where as the rest of the data is same as in (4D). The variation in demand under fuzzy environments is shown in Figure 4.3. Choosing G = 200 >160, we have

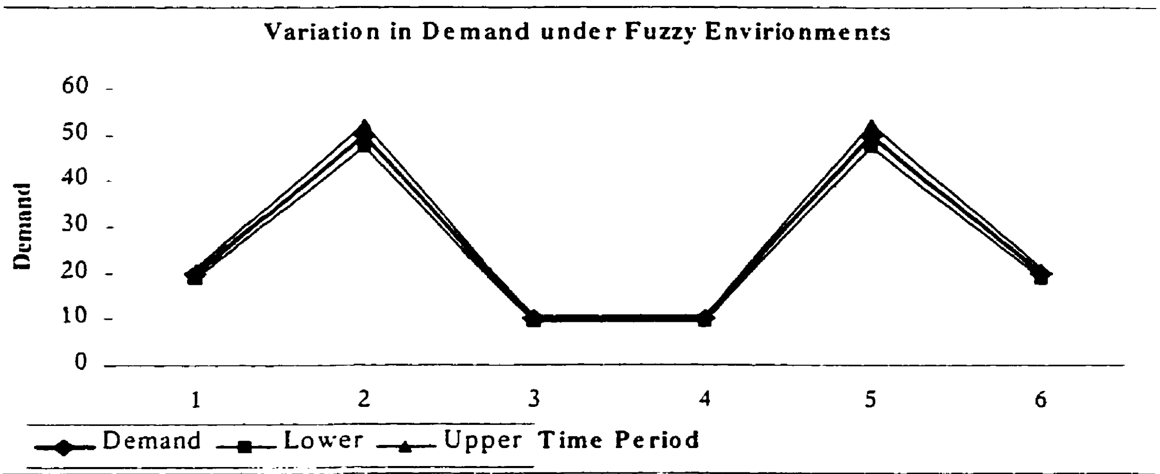


Figure 4.3 Variation in demand under Fuzzy environments

(4FD)

Maximize λ

subject to

$1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 0.5x_{21} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{31} + 0.5x_{32} +$

$1x_{34} + 2x_{35} + 3x_{36} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 1x_{45} + 2x_{46} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} +$

$1x_{56} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 +$

$$100y_6 + 0z_{11} + 1000z_{12} + 1200z_{13} + 1120z_{14} + 0z_{21} + 1000z_{22} + 1200z_{23} + 1120z_{24} + 0z_{31} + 1000z_{32} + 1200z_{33} + 1120z_{34} + 0z_{41} + 1000z_{42} + 1200z_{43} + 1120z_{44} + 0z_{51} + 1000z_{52} + 1200z_{53} + 1120z_{54} + 0z_{61} + 1000z_{62} + 1200z_{63} + 1120z_{64} + 3.42\lambda \leq 1370 \quad (1)$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} - 1\lambda \geq 19 \quad (2)$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} + 1\lambda \leq 21 \quad (3)$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} - 2.5\lambda \geq 47.5 \quad (4)$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} + 2.5\lambda \leq 52.5 \quad (5)$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} - 0.5\lambda \geq 9.5 \quad (6)$$

$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} + 0.5\lambda \leq 10.5 \quad (7)$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} - 0.5\lambda \geq 9.5 \quad (8)$$

$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} + 0.5\lambda \leq 10.5 \quad (9)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} - 2.5\lambda \geq 47.5 \quad (10)$$

$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} + 2.5\lambda \leq 52.5 \quad (11)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} - 1\lambda \geq 19 \quad (12)$$

$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} + 1\lambda \leq 21 \quad (13)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - 200y_1 \leq 0 \quad (14)$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - 200y_2 \leq 0 \quad (15)$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - 200y_3 \leq 0 \quad (16)$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - 200y_4 \leq 0 \quad (17)$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - 200y_5 \leq 0 \quad (18)$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - 200y_6 \leq 0 \quad (19)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} - P_1 = 0 \quad (20)$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} - P_2 = 0 \quad (21)$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} - P_3 = 0 \quad (22)$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} - P_4 = 0 \quad (23)$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} - P_5 = 0 \quad (24)$$

$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} - P_6 = 0 \quad (25)$$

$$P_1 - 0z_{11} - 100z_{12} - 150z_{13} - 160z_{14} = 0 \quad (26)$$

$$P_2 - 0z_{21} - 100z_{22} - 150z_{23} - 160z_{24} = 0 \quad (27)$$

$$P_3 - 0z_{31} - 100z_{32} - 150z_{33} - 160z_{34} = 0 \quad (28)$$

$$P_4 - 0z_{41} - 100z_{42} - 150z_{43} - 160z_{44} = 0 \quad (29)$$

$$P_5 - 0z_{51} - 100z_{52} - 150z_{53} - 160z_{54} = 0 \quad (30)$$

$$P_6 - 0z_{61} - 100z_{62} - 150z_{63} - 160z_{64} = 0 \quad (31)$$

$$1z_{11} - 1m_{11} \leq 0 \quad (32)$$

$$1z_{21} - 1m_{21} \leq 0 \quad (33)$$

$$1z_{31} - 1m_{31} \leq 0 \quad (34)$$

$$1z_{41} - 1m_{41} \leq 0 \quad (35)$$

$$1z_{51} - 1m_{51} \leq 0 \quad (36)$$

$$1z_{61} - 1m_{61} \leq 0 \quad (37)$$

$$1z_{12} - 1m_{11} - 1m_{12} \leq 0 \quad (38)$$

$$1z_{22} - 1m_{21} - 1m_{22} \leq 0 \quad (39)$$

$$1z_{32} - 1m_{31} - 1m_{32} \leq 0 \quad (40)$$

$$1z_{42} - 1m_{41} - 1m_{42} \leq 0 \quad (41)$$

$$1z_{52} - 1m_{51} - 1m_{52} \leq 0 \quad (42)$$

$$1z_{62} - 1m_{61} - 1m_{62} \leq 0 \quad (43)$$

$$1z_{13} - 1m_{12} - 1m_{13} \leq 0 \quad (44)$$

$$1z_{23} - 1m_{22} - 1m_{23} \leq 0 \quad (45)$$

$$1z_{33} - 1m_{32} - 1m_{33} \leq 0 \quad (46)$$

$$1z_{43} - 1m_{42} - 1m_{43} \leq 0 \quad (47)$$

$$1z_{53} - 1m_{52} - 1m_{53} \leq 0 \quad (48)$$

$$1z_{63} - 1m_{62} - 1m_{63} \leq 0 \quad (49)$$

$$1z_{14} - 1m_{13} \leq 0 \quad (50)$$

$$1z_{24} - 1m_{23} \leq 0 \quad (51)$$

$$1z_{34} - 1m_{33} \leq 0 \quad (52)$$

$$1z_{44} - 1m_{43} \leq 0 \quad (53)$$

$$1z_{54} - 1m_{53} \leq 0 \quad (54)$$

$$1z_{64} - 1m_{63} \leq 0 \quad (55)$$

$$1z_{11} + 1z_{12} + 1z_{13} + 1z_{14} = 1 \quad (56)$$

$$1z_{21} + 1z_{22} + 1z_{23} + 1z_{24} = 1 \quad (57)$$

$$1z_{31} + 1z_{32} + 1z_{33} + 1z_{34} = 1 \quad (58)$$

$$1z_{41} + 1z_{42} + 1z_{43} + 1z_{44} = 1 \quad (59)$$

$$1z_{51} + 1z_{52} + 1z_{53} + 1z_{54} = 1 \quad (60)$$

$$1z_{61} + 1z_{62} + 1z_{63} + 1z_{64} = 1 \quad (61)$$

$$1m_{11} + 1m_{12} + 1m_{13} = 1 \quad (62)$$

$$1m_{21} + 1m_{22} + 1m_{23} = 1 \quad (63)$$

$$1m_{31} + 1m_{32} + 1m_{33} = 1 \quad (64)$$

$$1m_{41} + 1m_{42} + 1m_{43} = 1 \quad (65)$$

$$1m_{51} + 1m_{52} + 1m_{53} = 1 \quad (66)$$

$$1m_{61} + 1m_{62} + 1m_{63} = 1 \quad (67)$$

$x_{ij} \geq 0$ and integer $i, j = 1, 2, \ldots, 6,$

$y_i = 0, 1.$ $i = 1, 2, \ldots, 6.$

$z_{is} \geq 0$ $i = 1, 2, \ldots, 6,$ and $s = 1, 2, \ldots, 4.$

$m_{is} = 0, 1.$ $i = 1, 2, \ldots, 6,$ and $s = 1, 2, 3.$

$0 \leq \lambda \leq 1$

4.6.5 Results

The optimal solution to (4FD) is as in Table 4.4.

Table 4.4 Results under Fuzzy Environments with Quantity Discounts and Backorders

Variable	Value	Variable	Value	Variable	Value
x_{51}	20	z_{54}	1	μ_{2L}	0.6
x_{52}	49	z_{61}	1	μ_{2U}	1
x_{53}	10	m_{11}	1	μ_{3L}	1
x_{54}	10	m_{21}	1	μ_{3U}	1
x_{55}	51	m_{31}	1	μ_{4L}	1
x_{56}	20	m_{41}	1	μ_{4U}	1
y_5	1	m_{53}	1	μ_{5L}	1
P_5	160	m_{61}	1	μ_{5U}	0.6
z_{11}	1	λ	0.44	μ_{6L}	1
z_{21}	1	μ_{OF}	0.44	μ_{6U}	1
z_{31}	1	μ_{1L}	1	Z optimal	1368.5
z_{41}	1	μ_{1U}	1		

Table 4.4.1 Replenishment Schedule as per Table 4.4

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	20	0	0	20
2	50	0	0	69
3	10	0	0	79
4	10	0	0	89
5	50	160	20	0
6	20	0	0	0

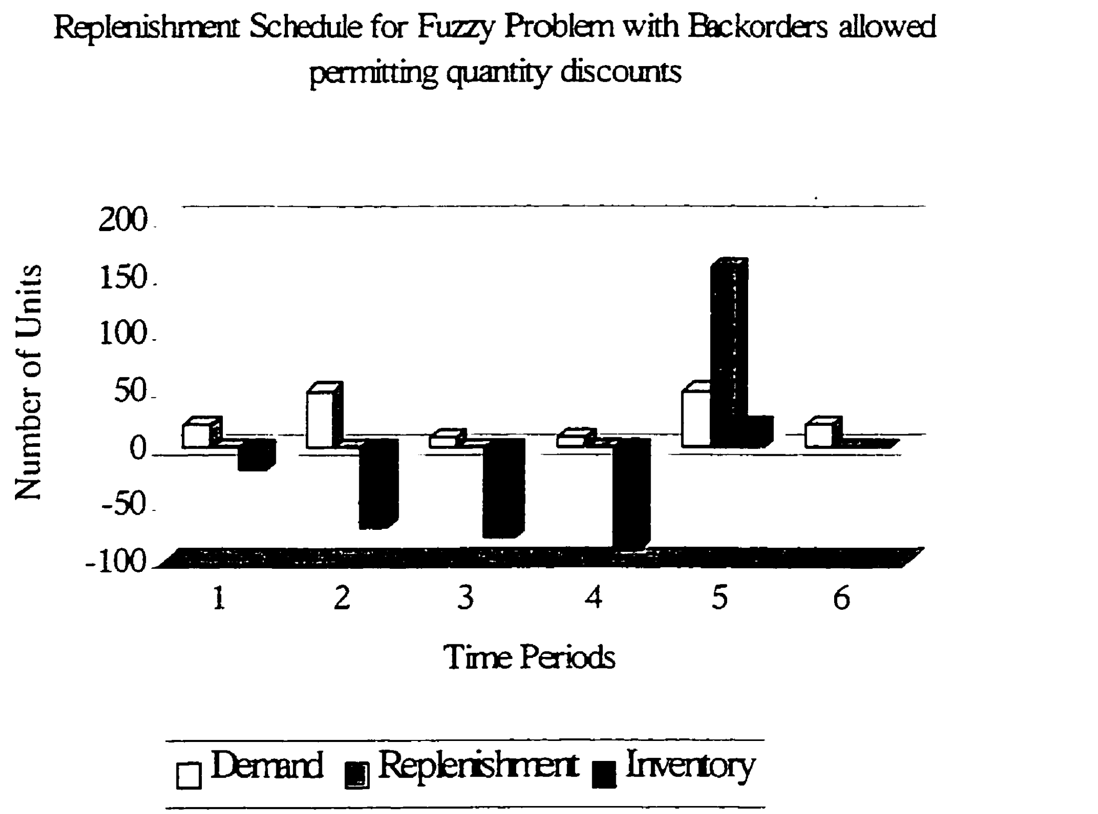


Figure 4.4 Replenishment Schedule for Fuzzy problem with backorders allowed permitting quantity discounts.

4.6.6 Interpretation of the Results

Since x_{ij} is the quantity purchased in Period i for demand in Period j, therefore from Tables 4.4 and 4.4.1 and Figure 4.4, we have $x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} = 20 + 49 + 10 + 10 + 51 + 20 = 160$ units should be acquired in Period 5. The value of $P_5 = 160$ indicates that entire purchase of 160 units to be made in Period 5 at a rate of $7 per unit. The values of z_{11}, z_{21}, z_{31}, z_{41}, z_{54} and z_{61} are equal to 1, thereby satisfying constraints (56) to (61). Similarly, the values of m_{11}, m_{21}, m_{31}, m_{41}, m_{53} and m_{61} are equal to 1, thereby satisfying constraints (62) to (67). The value of $y_5 = 1$ indicates that the acquisition of the units for satisfying total demand is done in Period 5 only. Therefore, the setup cost is incurred in Period 5. The total minimum cost or the value of the objective function is $1368.50. It is observed that constraints (4) and (11) are satisfied at 60% levels which explain the values of x_{52} and x_{55} being different in Table 4.4 from Table 4.3. The membership function λ is used to express the degree of certainty of the solution with respect to fuzzy parameters, total cost and imprecisely known demands and in this case is 44%.

Table 4.5 shows the values of membership functions calculated under the conditions of 3%, 4% and 5% tolerance limits for imprecisely known demands, and 0.05%, 0.10%, 0.25%, 0.5%, 1%, 3%, and 5% tolerances in the total cost.

Table 4.5 Values of Membership Functions under different Tolerance Limits for Demands and Total Cost.

	Demand Tolerance 3%						
				Total Cost Tolerance			
Variable	5%	3%	1%	0.5%	0.25%	0.01%	0.05%
λ	0.02	0.04	0.11	0.22	0.33	0.33	0.33
μ_{OF}	0.02	0.04	0.11	0.22	0.44	1	1
μ_{2L}	0.33	0.33	0.33	0.33	0.33	0.33	0.33
μ_{5U}	0.33	0.33	0.33	0.33	0.33	0.33	0.33
Total Cost	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1368.50**
	Demand Tolerance 4%						
Variable	5%	3%	1%	0.5%	0.25%	0.01%	0.05%
λ	0.02	0.04	0.11	0.22	0.44	0.50	0.00
μ_{OF}	0.02	0.04	0.11	0.22	0.44	1	0.00
μ_{2L}	0.50	0.50	0.50	0.50	0.50	0.50	1.00
μ_{5U}	0.50	0.50	0.50	0.50	0.50	0.50	1.00
Total Cost	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1368.50**	**1370.00**
	Demand Tolerance 5%						
Variable	5%	3%	1%	0.5%	**0.25%**	0.01%	0.05%
λ	0.04	0.07	0.11	0.22	**0.44**	0.00	0.00
μ_{OF}	0.04	0.07	0.11	0.22	**0.44**	0.73	1
μ_{2L}	0.20	0.20	0.60	0.60	**0.60**	1.00	1.00
μ_{5U}	0.20	0.20	0.60	0.60	**0.60**	0.60	0.60
Total Cost	**1367.00**	**1367.00**	**1368.50**	**1368.50**	**1368.50**	**1369.00**	**1369.00**

The values of the rest of membership functions, not shown in the table, are 1.

4.6.7 Discussion of the solution in view of Table 4.5

Table 4.5 shows different values of membership functions for various tolerance levels for the imprecisely known total cost and imprecisely known demand. Note that in this formulation the membership function λ is used to express the degree of certainty of the solution with respect to fuzzy parameters, total cost and imprecisely known demands (Zimmerman, (1991)).

Table 4.5 shows that the membership functions corresponding to lower side of the constraint associated with imprecisely known demand for Period 2, and upper side of the constraint associated with imprecisely known demand for Period 5 are not completely satisfied all the times. It can be seen that the corresponding constraints (4) and (11) are never fully satisfied unlike the other constraints where the value of corresponding membership functions reaches 1. Therefore, these two constraints influence the overall solution of the problem and should be paid more attention to. It is observed that the values of the membership functions corresponding to these two constraints increase as the tolerance in demands and total cost increases i.e. as the demand and total cost constraints are relaxed. It is evident from the fact that demand in Period 2 and 5 is completely satisfied only 33% of the time when the demand tolerance is 3%, and increases to 60% satisfaction upon increasing the demand tolerance to 5%, under 0.25% tolerance level for total cost. That explains the difference in values of x_{52} and x_{55} in Tables 4.4 when compared to Table 4.3.

Chapter 5

GOAL PROGRAMMING APPROACH FOR INVENTORY LOT SIZING PROBLEM

In this chapter, the inventory lot size problem is presented using goal programming approach coupled with integer linear programming. The problem of inventory lot size with variable demand is considered under circumstances allowing backorders and sensitivity analysis performed by alternating priorities assigned to the goals. It is observed that the results obtained are similar to the ones achieved by applying linear programming technique in the previous chapters, and the results obtained by other researchers using dynamic programming.

5.1 Introduction

Goal programming is primarily a variation of linear programming (Davis, 1984). A key factor that differentiates goal programming from linear programming is the structure and the use of objective function. In linear programming, only one goal is incorporated into the objective function, while in goal programming, multiple goals are incorporated. This is accomplished by expressing the goal in the form of a constraint, including a deviational variable in the constraint to reflect the extent to which the goal is or not achieved, and incorporating the deviational variable in the objective function. In linear programming, the objective is either maximized or minimized, while in goal programming the objective is to minimize the deviations from the specified goals.

Since the deviations from the set of goals are minimized, a goal-programming model can handle multiple goals with different dimensions and units of measures. If multiple goals exist, priority can be specified, and the goal programming solution process will operate in such a manner that the goal with the highest priority is satisfied as closely as possible before considering goals of lower priority. Whereas linear programming seeks to identify the optimal solution from a set of feasible solutions, goal programming identifies the point that best satisfies a problem's set of goals.

One of the key advantages of goal programming is that it can provide information beyond that provided by linear programming, and thus is more useful as an aid to management in its decision making process. For example, assume that the management of a firm has employed linear programming to maximize the profits in production of its products. Assume further that the company is committed to an order of N units for a specific product and that the company's goal is to meet that commitment. In a linear programming formulation, the goal would be specified as a constraint, and if sufficient production capacity existed, the linear programming model would provide a solution. If insufficient capacity exists, however, an infeasible solution will result. The goal programming treatment of the problem would provide a solution regardless. If sufficient capacity exists, the goal programming solution would be the same as linear programming solution; but where insufficient capacity exists, the goal programming solution would depend upon the order of priorities, the cost of overtime, and profits, but a solution would be provided. From the goal programming solution, the management would be able to determine the required overtime and profits foregone to meet the goal of customer satisfaction.

5.2 Goal Programming Formulation for Inventory Replenishment Problem with Backorders allowed

5.2.1 Assumptions

This model is formulated under the same assumptions as in section 3.2.1 of Chapter 3.

5.2.2 Objective Function

Our objective (in order of priority) will be to

Minimize

1 over achievement in total cost (Priority 1)

2 over achievement in total holding cost (Priority 2)

3 over achievement in total backordering cost (Priority 3)

4 under achievement in fulfilling demand in each period (Priority 4)

5 over achievement in fulfilling demand in each period (Priority 4)

For Priorities 4 and 5, the under achievements and over achievements in terms of number of units, in each period, have to be multiplied by the backordering costs ($0.5 per unit per period) and holding costs ($1 per unit per period), respectively, in order to have the same unit of measurement in the objective function, i.e. in terms of cost.

5.2.3 Notation

In addition to the notation used in section 3.2.2 of Chapter 3, let,

O_{OF} = over achievement in the total cost

O_j = over achievement in fulfilling the demand in period j

U_j = under achievement in fulfilling the demand in period j

H = over achievement in total holding cost

B = over achievement in total backordering cost.

5.2.4 Formulation

We have the following goal programming problem (5C).

Minimize $$P_1 O_{OF} + P_2 H + P_3 B + P_4 (0.5 \sum_{j=1}^{N} U_j + 1 \sum_{j=1}^{N} O_j)$$

Subject to

$$\sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} + \sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} b_i x_{kj} + \sum_{i=1}^{N} s_i y_i - O_{OF} = 0$$

$$\sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} h_i x_{jk} - H = 0$$

$$\sum_{j=1}^{N} \sum_{k=2}^{N} \sum_{i=j}^{k-1} b_i x_{kj} - B = 0$$

$$\sum_{i=1}^{N} x_{ij} = d_j \qquad j = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N} x_{ij} \leq G\, y_i \qquad i = 1, 2, \ldots, N.$$

$$\sum_{j=1}^{N} x_{ij} + U_j - O_j = d_j \qquad i, j = 1, 2, \ldots, N$$

$$x_{ij} \geq 0 \text{ and integer } i, j = 1, 2, \ldots, N$$

$$y_i = 0, 1 \qquad i = 1, 2, \ldots, N.$$

5.2.5 Numerical Example

We illustrate our method through the following numerical example (3C) taken from section 3.2.5 of Chapter 3. Therefore, we have the following goal programming model.

Minimize $P_1(O_{OF}) + P_2(H) + P_3(B) + P_4(0.5U_1 + O_1 + 0.5U_2 + O_2 + 0.5U_3 + O_3 + 0.5U_4 + O_4 + 0.5U_5 + O_5 + 0.5U_6 + O_6)$

subject to

$$1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 0.5x_{21} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{31} + 0.5x_{32} + 1x_{34} + 2x_{35} + 3x_{36} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 1x_{45} + 2x_{46} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} + 1x_{56} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} + 100y_1 + 100y_2 + 100y_3 + 100y_4 + 100y_5 + 100y_6 - O_{OF} = 0 \quad (1)$$

$$1x_{12} + 2x_{13} + 3x_{14} + 4x_{15} + 5x_{16} + 1x_{23} + 2x_{24} + 3x_{25} + 4x_{26} + 1x_{34} + 2x_{35} + 3x_{36} + 1x_{45} + 2x_{46} + 1x_{56} - H = 0 \quad (2)$$

$$0.5x_{21} + 1x_{31} + 0.5x_{32} + 1.5x_{41} + 1x_{42} + 0.5x_{43} + 2x_{51} + 1.5x_{52} + 1x_{53} + 0.5x_{54} + 2.5x_{61} + 2x_{62} + 1.5x_{63} + 1x_{64} + 0.5x_{65} - B = 0 \quad (3)$$

$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16}+U_1-O_1 = 20 \quad (4)$$

$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25}+x_{26}+U_2-O_2 = 50 \quad (5)$$

$$x_{31}+x_{32}+x_{33}+x_{34}+x_{35}+x_{36}+U_3-O_3 = 10 \quad (6)$$

$$x_{41}+x_{42}+x_{43}+x_{44}+x_{45}+x_{46}+U_4-O_4 = 10 \quad (7)$$

$$x_{51}+x_{52}+x_{53}+x_{54}+x_{55}+x_{56}+U_5-O_5 = 50 \quad (8)$$

$$x_{61}+x_{62}+x_{63}+x_{64}+x_{65}+x_{66}+U_6-O_6 = 20 \quad (9)$$

$$x_{11}+x_{21}+x_{31}+x_{41}+x_{51}+x_{61} = 20 \quad (10)$$

$$x_{12}+x_{22}+x_{32}+x_{42}+x_{52}+x_{62} = 50 \quad (11)$$

$$x_{13}+x_{23}+x_{33}+x_{43}+x_{53}+x_{63} = 10 \quad (12)$$

$$x_{14}+x_{24}+x_{34}+x_{44}+x_{54}+x_{64} = 10 \quad (13)$$

$$x_{15}+x_{25}+x_{35}+x_{45}+x_{55}+x_{65} = 50 \quad (14)$$

$$x_{16}+x_{26}+x_{36}+x_{46}+x_{56}+x_{66} = 20 \quad (15)$$

$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16}-200y_1 \leq 0 \quad (16)$$

$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25}+x_{26}-200y_2 \leq 0 \quad (17)$$

$$x_{31}+x_{32}+x_{33}+x_{34}+x_{35}+x_{36}-200y_3 \leq 0 \quad (18)$$

$$x_{41}+x_{42}+x_{43}+x_{44}+x_{45}+x_{46}-200y_4 \leq 0 \quad (19)$$

$$x_{51}+x_{52}+x_{53}+x_{54}+x_{55}+x_{56}-200y_5 \leq 0 \quad (20)$$

$$x_{61}+x_{62}+x_{63}+x_{64}+x_{65}+x_{66}-200y_6 \leq 0 \quad (21)$$

$x_{ij} \geq 0$ and integer $\quad i, j = 1,2,\ldots..6.$

$y_i = 0,1. \quad i = 1,2,\ldots\ldots.6.$

5.2.6 Solution to the Problem

The problem is solved in the following steps.

1. The first or the most important priority is to minimize deviation in the total cost i.e. O_{OF}. Therefore, the problem becomes,

   Minimize O_{OF}

   subject to constraints (1) through (21).

2. Upon solving the problem, we obtain the value of O_{OF} equal to \$245.
3. Add $O_{OF} = 245$ as a constraint (22) in the problem.
4. Next priority is to minimize the deviation in total holding cost H. Therefore the problem now becomes,

   Minimize H

   subject to constraints (1) through (22)

5. Upon solving the problem, the value of H is obtained as \$20.
6. Add H = 20 as constraint (23) in the problem.
7. Third priority is to minimize deviation in total backorder cost B. Therefore the problem now becomes

   Minimize B

   subject to constraints (1) through (23)

8. Solving the problem, the value of B is obtained as \$25.
9. Add B = 25 as constraint (24) in the problem.

10. Our last priority is to minimize the inventory shortages or excess inventory during any period. Therefore our goal is to minimize the deviations in demand in each period. The problem now becomes

Minimize $0.5U_1 + O_1 + 0.5U_2 + O_2 + 0.5U_3 + O_3 + 0.5U_4 + O_4 + 0.5U_5 + O_5 + 0.5U_6 + O_6$

subject to constraints (1) through (24).

5.2.7 Results

The following Table 5.1 lists the result obtained upon solving the problem as per steps listed above.

Table 5.1 Results of Goal Programming problem

Variable	Value	Variable	Value
x_{21}	20	O_{OF}	245
x_{22}	50	U_1	20
x_{53}	10	U_3	10
x_{54}	10	U_4	10
x_{55}	50	U_6	20
x_{56}	20	O_2	20
y_2	1	O_5	40
y_5	1	H	20
All other variables	0	B	25

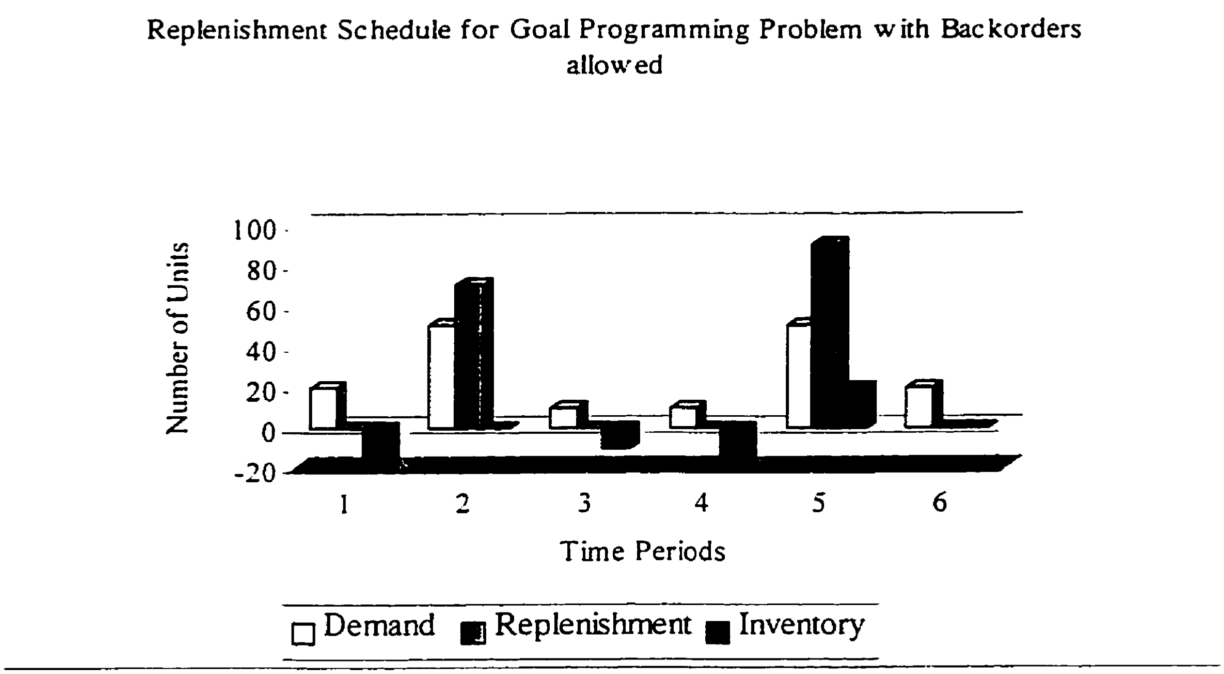


Figure 5.1 Replenishment Schedule for Goal Programming Problem with backorders allowed.

5.2.8 Interpretation of the Results

Since x_{ij} is the quantity acquired in Period i for demand in Period j, therefore from Table 5.1 and Figure 5.1, we have $x_{21} + x_{22} = 20 + 50 = 70$ units should be acquired in Period 2. Then $x_{53} + x_{54} + x_{55} + x_{56} = 10 + 10 + 50 + 20 = 90$ units should be acquired in Period 5. The values of y_2 and $y_5 = 1$ indicate that the acquisition of the units for satisfying total demand is done in Periods 2 and 5. Therefore, the setup cost is incurred in Period 2 and 5. The value of O_{OF} = \$245 indicates that the minimum over achievement in total cost is \$245. Since the demand for first period was backordered, therefore there is an underachievement in demand of 20 units which is reflected in the value of U_1. Since nothing was acquired in Period 3, 4 and 6, therefore the values of U_3, U_4 and U_6 reflect this underachievement. Also as the entire acquisitions are made in Period 2 and 5 only, there is an over achievement of 20 and 40 units in these time periods which is indicated by values of O_2 and O_5. The total minimum cost is $245. The minimum total holding cost and total backordering cost is $20 and $25 respectively.

5.3 Effect Of Changing Priorities of Goals on the Solution.

Let us examine the effect upon changing certain priorities. We interchange the second and third priorities in the above problem. Instead of having the second priority as minimizing the deviation in total holding cost (H), our second priority now is to minimize the deviation in the total backordering cost (B) and the third priority as minimizing the deviation in total holding cost (H). Now the problem becomes

Minimize $P_1(O_{OF}) + P_2(B) + P_3(H) + P_4(0.5U_1 + O_1 + 0.5U_2 + O_2 + 0.5U_3 + O_3 + 0.5U_4 + O_4 + 0.5U_5 + O_5 + 0.5U_6 + O_6)$

subject to constraints (1) through (21) mentioned above in 5.2.5.

5.3.1 Solution to the Goal Programming problem with priorities changed

We solve the problem through the following steps.

1. The first or the most important priority is to minimize deviation in the total cost i.e. O_{OF}. Therefore, the problem becomes,

Minimize O_{OF}

subject to constraints (1) through (21).

2. Upon solving the problem, we obtain the value of O_{OF} equal to \$245.
3. Add $O_{OF} = 245$ as a constraint (22) in the problem.
4. Next priority is to minimize the deviation in total backordering cost B. Therefore the problem now becomes,

   Minimize B

   subject to constraints (1) through (22)
5. Upon solving the problem, the value of B is obtained as \$15.
6. Add B = 15 as constraint (23) in the problem.
7. Third priority is to minimize deviation in total holding cost H. Therefore the problem now becomes

   Minimize H

   subject to constraints (1) through (23)
8. Solving the problem, the value of H is obtained as \$30.
9. Add H = 30 as constraint (24) in the problem.
10. Our last priority is to minimize the inventory shortages or excess inventory during any period. Therefore our goal is to minimize the deviations in demand in each period. The problem now becomes

    Minimize $0.5U_1 + O_1 + 0.5U_2 + O_2 + 0.5U_3 + O_3 + 0.5U_4 + O_4 + 0.5U_5 + O_5 + 0.5U_6 + O_6$

    subject to constraints (1) through (24).

5.3.2 Results

The following Table 5.2 lists the result obtained upon solving the problem as per steps listed above.

Table 5.2 Results of the Goal programming problem with priorities interchanged

Variable	Value	Variable	Value
x_{21}	20	O_{OF}	245
x_{22}	50	U_1	20
x_{23}	10	U_3	10
x_{54}	10	U_4	10
x_{55}	50	U_6	20
x_{56}	20	O_2	30
y_2	1	O_5	30
y_5	1	H	30
All other variables	0	B	15

5.3.3 Interpretation of the Results

Since x_{ij} is the quantity acquired in Period i for demand in Period j, therefore from Table 5.2, we have $x_{21} + x_{22} + x_{23} = 20 + 50 + 10 = 80$ units should be acquired in Period 2. Then $x_{54} + x_{55} + x_{56} = 10 + 50 + 20 = 80$ units should be acquired in Period 5. The values of y_2 and $y_5 = 1$ indicate that the acquisition of the units for satisfying total demand is done in Periods 2 and 5. Therefore, the setup cost is incurred in Periods 2 and 5. The value of $O_{OF} =$ \$245 indicates that the minimum over achievement in total cost is \$245. Since the demand for first period was backordered therefore there is an underachievement in demand of 20 units which is reflected in the value of U_1. Also, as nothing was acquired in period 3, 4 and 6, therefore the values of U_3, U_4 and U_6 show the under achievements. Also as the entire acquisitions are made in Periods 2 and 5 only, there is an over achievement of 30 units in each of these time periods which is indicated by values of O_2 and O_5. The total minimum cost is $245. The minimum holding cost and backordering cost is $30 and $15 respectively.

5.4 Comparison of Results

On comparing the results of both the cases, it is seen that the overall total cost remains the same as it had the highest priority in both the cases. However, we observe that upon interchanging the priorities of holding and backordering costs, the solution changes. In the first case, holding cost was important than backordering cost whereas in the second case backordering cost is more important than the holding cost. The holding cost in previous case was $20 and when the priority was lowered it was calculated to be $30. The backordering cost in the previous case was $25 and upon increasing the priority, it was calculated to be $15. There is a trade off between holding cost and backordering cost in both the cases as their sum remains the same. This explains the change in pattern for acquiring the units.

Chapter 6

CONCLUSION, CONTRIBUTION AND RECOMMENDATIONS

In the present chapter, we state the contributions and conclusions of this dissertation. Finally, we give some recommendations for further research on the problems considered in this dissertation.

6.1 Conclusion and Contribution

In the present dissertation, an important problem in the field of industrial engineering i.e. lot size inventory control problem (addressed by Wagner and Whitin, 1958) has been revisited. We have modeled inventory lot sizing problem as zero one integer linear program with variable demand, setup, backordering, and carrying costs, under both crisp and fuzzy environments. Also, the problem is modeled as piece wise integer linear program by incorporating quantity discounts. Besides, goal-programming approach is applied to this problem. Furthermore, we propose a fuzzy logic approach to deal with inventory lot sizing problem when the data known is imprecise. A new interpretation for backordering variables (x_{ij}. when i>j, Chapter 3) is a contribution of this thesis. Earlier this type of interpretation was used in transportation type linear programming problems.

The inventory problem with variable demand rate under both crisp and fuzzy environments with a planning horizon of N periods and allowing backorders is considered in Chapter 3. Under crisp environments, we formulate the problem as an integer linear programming problem with exactly N variables restricted to zero one value. However, one underlying assumption in the above model, and most of the models in the literature is that demand is deterministically known. But demand is forecasted and forecasts rarely-if-ever turn out to be crisply correct. Therefore, the models based on precise knowledge of demand have little practical applications. We deal with such a problem through fuzzy logic approach. Under fuzzy environments, the problem is formulated as a max-min linear program, with exactly N variables restricted to zero one value

In Chapter 4 we have modeled an inventory problem with variable demand, setup, backordering, and carrying costs, incorporating quantity discounts under both crisp and fuzzy environments, as a piecewise integer linear programming model. It is observed that the results obtained in the crisp problem without incorporating backorders are the same as obtained by other researchers using dynamic programming. Also, using fuzzy logic, we are able to identify the constraints that affect the overall solution considerably through the values of membership functions.

Chapter 5 presents the inventory lot size problem is using goal-programming approach coupled with integer linear programming. The problem of inventory lot size with variable demand is considered under circumstances allowing backorders and sensitivity analysis performed by alternating priorities assigned to the goals. It is observed that the results obtained are similar to the ones achieved by applying linear programming technique in the previous chapters, and the results obtained by other researchers using dynamic programming. A goal-programming model can handle multiple goals with different dimensions and units of measures. Linear programming assigns equal priority to all the goals whereas goal programming treats the problem differently as desired by the researcher. It is seen that whereas linear programming seeks to identify the optimal solution from a set of feasible solutions, goal programming identifies the point that best satisfies a problem's set of goals. One of the key advantages of goal programming is that it can provide information beyond that provided by linear programming, and thus is more useful as an aid to management in its decision making process. Goal programming will always provide a solution in terms of deviations from the goal, whereas linear programming might render an infeasible solution if the constraints are not satisfied.

It is suggested that the methods presented in this dissertation are computationally effective and useful for determining the optimal solution to inventory lot size problems. Although most real productions systems still exceed the limits of the proposed model, its potential impact on both practical and theoretical decision making is apparent.

6.2 Practical Applications and Recommendations for Future Research

The application of quantitative techniques, such as integer linear program presented in this dissertation, in the field of supply chain or logistics, has to be considered one of the most successful marriages of academia and industry. Due to incredible growth in desktop computing power and introduction of improved algorithms, it is now possible to solve very large problems in matter of minutes.

The effective management of product supply chain has grown in importance with the realization that it represents a major opportunity for organizations to improve operational performance and overall margins. A typical supply chain network can be described very accurately by means of linear objective/cost functions and variable number of linear constraints and relationships. The linear cost function incorporates "straight line" financial components such as variable manufacturing costs, overtime cost, direct and transfer freight, and revenue. The bulk of operational and business constraints can also be represented in a linear, straight line format:

- Resource utilization
- Demand fulfillment
- Production capacity
- Inventory storage capacity
- Minimum inventory requirements
- Fleet utilization
- Product formulations and yields
- Throughput capacity.

The integer linear program presented in this dissertation can be modified to take into consideration above-mentioned resources available in order to optimize supply chain. This will involve adding additional constraints representing the resource availability. Therefore, a true supply chain optimization system will typically incorporate three components:

- An optimizer (the mathematical programming algorithm)
- A mathematical model/representation of the supply chain
- A front-end, menu driven database management and reporting system.

The front-end database management and reporting component serves as the interface between the user, the optimization model and the other corporate sources of data. It should allow the user to first define the specifics of the scenario to be analyzed and then review the proposed supply/distribution plan via graphs, charts or management reports.

Conceptually, the supply chain optimization model will establish longer-term production allocation, inventory and supply patterns that will then serve as input and drive the shorter-term scheduling model.

In a nutshell, in any type of industry, the basic goal remains the same: to identify the most cost effective or profitable way of getting the right product to the right place at the right time, given a host of working and business constraints and parameters. Therefore, for an organization with multiple locations, production processes, products, modes and customers, where instinct and experience are not able to cope up with the size and complexity of the operation, this is the type of planning aid to be examined.

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Appendix

Case Study – Application of Integer Linear Programming Technique under Crisp and Fuzzy Environments at ABC Industries Ltd.

ABC Industries Ltd., Winnipeg is a leading producer of hydraulic cylinders in North America. The various models of hydraulic cylinders manufactured at ABC vary from 2" to 5" bore, and 6" to 48" stroke length with all the combinations between bore size and stroke length. The cylinders with 3" bore size are the most popular size. The 3" rod cap used in 3" bore size cylinder is common for all the cylinders with stroke length from 6" to 48". This specific 3" rod cap is replenished on a JIT system with help of a pull tag. Whenever the assembly department requires this rod cap, they move the pull tag to the machine shop. The machine shop gets the castings from the foundry and machine the parts as per the quantity mentioned on the tag. Each pull tag replenishes 600 parts. The requirements for this rod cap for the next six weeks (as obtained from the MRP system) are given below.

Demand Data for 3" rod cap for 6 weeks.

Week	1	2	3	4	5	6
Demand	500	900	700	900	800	500

Each time the machine shop sets up for making this rod cap, it takes one hour to change the fixtures and tools and reprogram the CNC lathe and the Vertical Machining Center in the Rod-Cell. The estimated cost of the setup, which includes the costs of operator and machine idle time, amounts to $300 per setup. The company standards indicate that the cost to hold a 3" rod cap in inventory is approximately $1 per unit per week. If the machine shop is unable to provide the parts within the replenishment lead time of one week, the production of 3" bore size cylinders gets backlogged. It is estimated that every time a 3" bore size cylinder gets backordered, the backlogging cost broken down comes out to be $0.50 per week for every 3" rod cap.

The aim of the production planner is to find out an optimal replenishment schedule such that the requirements are met at the minimum possible expense. It is assumed that there is no inventory at the beginning.

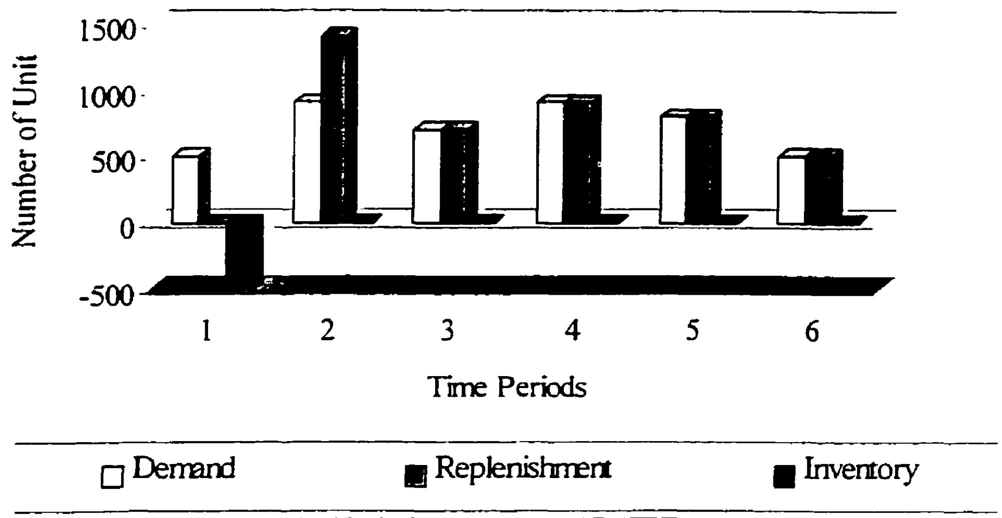
Implementing the Integer Linear Programming Technique discussed in the thesis

Using the information available above, the production planner formulates the integer linear program as described in Chapter 3. Upon solving the problem, the following replenishment schedule is obtained.

Replenishment Schedule using Linear Programming Technique

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	500	0	0	500
2	900	1400	0	0
3	700	700	0	0
4	900	900	0	0
5	800	800	0	0
6	500	500	0	0
Costs		$1500	$0	$250

Case Study - Replenishment Schedule for Crisp Problem with Backorders allowed

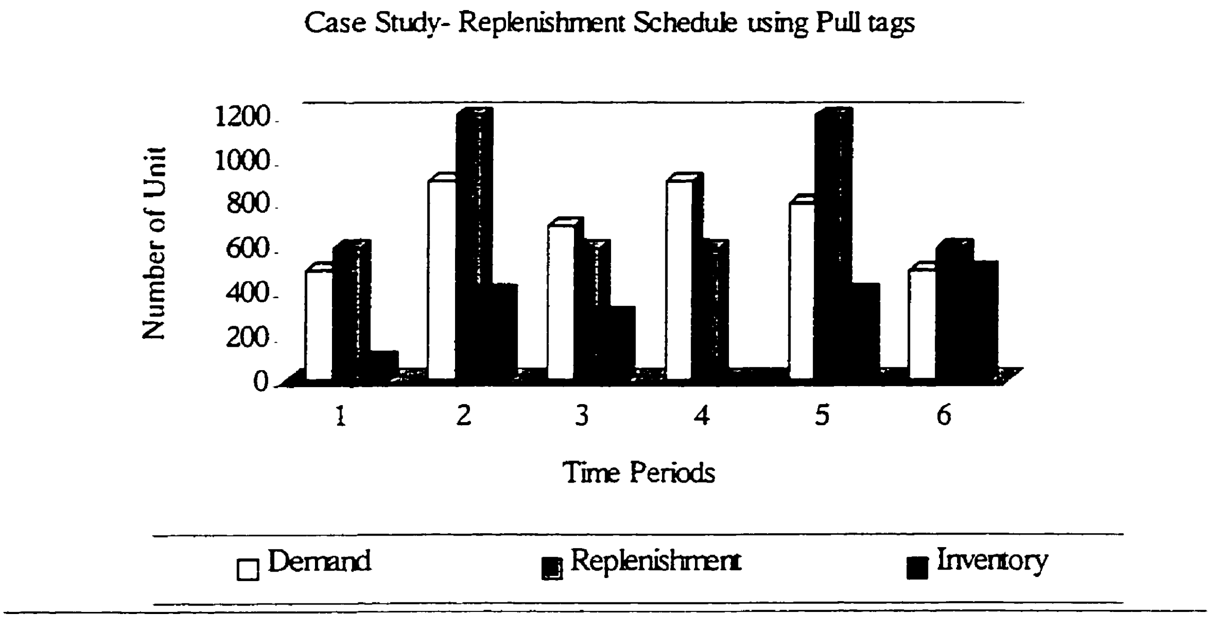


The total cost of this replenishment schedule is $1750, which includes costs for five setups in Periods 2, 3, 4, 5 and 6 at the rate of $300 per setup, and cost of backordering 500 units in Period 1 at rate of $0.50 per unit.

The replenishment schedule using the pull tags being currently used at ABC would yield the following replenishment schedule.

Replenishment Schedule using Pull Tags

Period	Demand	Replenishment Schedule	Units Carried	Units Back ordered
1	500	600	100	0
2	900	1200	400	0
3	700	600	300	0
4	900	600	0	0
5	800	1200	400	0
6	500	600	500	0
Costs		$2400	$1700	$0



The total cost of replenishment using pull tags is $4100, which includes setup cost of eight setups at $300 per setup, and holding cost of $1700.

Application of fuzzy theory in case of Uncertain Demand and Imprecise Total Cost Levels

The demand data provided above, for the 3" rod cap, is not always accurate. The production planner examines some old reports for the past demands and compares them to the requirements generated by the MRP system, and finds out that the demand varies between 15% of the demand numbers provided by the MRP system. Based upon the outcome of the linear program under crisp environments, the management asks the production planner find a replenishment schedule staying below the total cost of $1750. Therefore, the production planner has now been given a flexibility of 15% in demand and an upper budget limit of $1750 to work with.

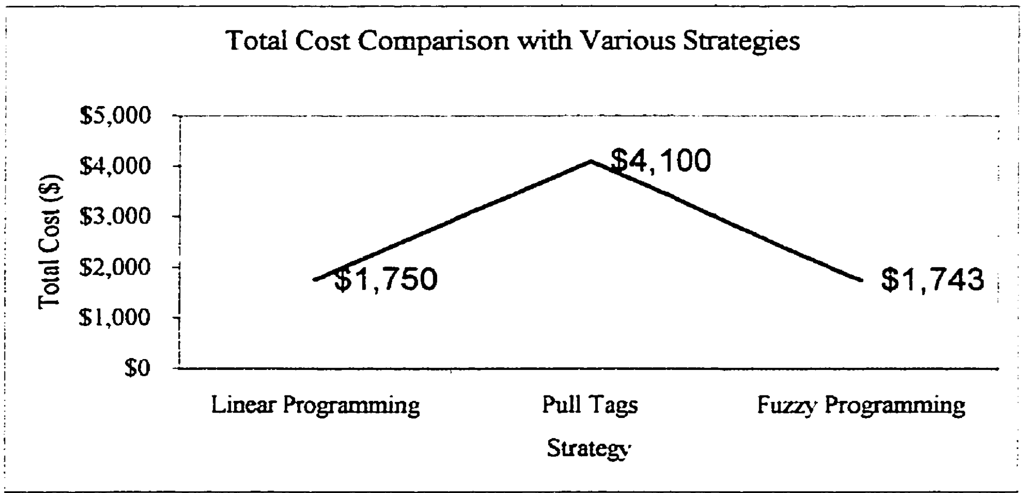
Using the technique of fuzzy set theory, discussed in the Chapter 3 of this thesis, the production planner formulates the max-min integer linear program and solves the problem.

Replenishment Schedule under 15% demand tolerance and 0.05% Total Cost Tolerance

Period	Demand	Replenishment Schedule	Units allocated	Units Acquired less than demand	Units Carried	Units Back ordered
1	500	0	486	14	0	486
2	900	1359	873	27	0	0
3	700	679	679	21	0	0
4	900	873	873	27	0	0
5	800	776	776	24	0	0
6	500	485	485	15	0	0
	4300	$1500	4172	128	$0	$243

The results indicate that the replenishment would be made for a total of 4172 units as compared to the demand figures of 4300 units provided by MRP system. The total cost for the replenishment was $1743, which constitutes 5 setups for $1500 and backordering cost of 486 units for the first time period for one time period at the rate of $0.50 per unit per time period i.e. $243. This yields a reduction of $7 in total cost as compared to the solution under crisp environments. Also, the value of λ is 0.80, which suggests that the degree of certainty of the solution is 80%.

The following graph compares the total cost incurred for replenishment using all the three strategies, i.e. using integer linear programming, using the pull tags with fixed quantity and fuzzy integer linear programming.



Total Cost Comparison with various Strategies for ABC Industries Ltd.

From the above comparison, it is evident that using integer linear programming technique under fuzzy environments gives a better solution as compared to the fixed lot sizing technique by pull tags being currently used at ABC Industries Ltd. This approach can help the production planner to plan the production more economically.