Maximal Chains and Cells of Finite Planar Lattices

Kevin D. Doerksen

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

*****

COPYRIGHT PERMISSION

Maximal Chains and Cells of Finite Planar Lattices

BY

Kevin D. Doerksen

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirement of the degree

Of

MASTER OF SCIENCE

Abstract

Let L be a finite planar lattice with and let $\mathrm{Ch}\, L$ be the set of maximal chains on L. We show that each planar embedding $e(L)$ of L induces a left-right partial order on incomparable elements of L (D. Kelly and I. Rival [4]) and under this order, $\mathrm{Ch}\, L$ forms a distributive lattice. We further show that the poset of join-irreducible elements $\mathrm{J}(\mathrm{Ch}\, L)$ of $\mathrm{Ch}\, L$ forms a planar lattice after the addition of a maximal-element and a minimal-element and that this planar lattice (called $\mathrm{Ce}\, L$) can be obtained directly from the cells of L. For a given lattice L with planar embedding $e(L)$, the set of all lattices K such that $\mathrm{Ce}\, K \cong L$ is described. Finally it is shown that the cell lattice of the cell lattice of L, written $\mathrm{Ce}^2\, L$, is independent of the planar representation of L and can be extended to a lattice construction on all finite lattices.

Acknowledgements

I would like to thank all the participants of the universal algebra and lattice theory seminar. It was during this weekly seminar that I was first exposed to the research process. I would especially like to thank Professor Kelly for his insightful comments on planar lattices, and of course, Professor Grätzer for all his guidance and advice.

I would also like to thank the Department of Mathematics and the Faculty of Science for their financial support during my two years of study at the University of Manitoba.

Thanks also goes to my parents and brothers for their patience and support.

Finally, I would like to extend a warm thank-you to the coffee machine on the fourth floor. Without its daily contribution, I can honestly say that this thesis would never have been completed on time.

Contents

Introduction

Finite planar lattices have been studied extensively. Two key papers on the subject were produced by D. Kelly and I. Rival [4] and by C. R. Platt [**5**]. Both papers provide a complete description of the class of all finite planar lattices, and results from each paper will play prominent roles in this thesis.

The bulk of this thesis focuses on the relationship between the algebraic properties and the graph theoretic properties of finite planar lattices. In particular, it is shown that in a finite planar lattice L, each planar embedding induces a partial order on the set of maximal chains $\mathrm{Ch}\,L$ such that

- $\mathrm{Ch}\,L$ forms a distributive lattice, and
- The poset of join-irreducible elements of $\mathrm{Ch}\,L$ forms a planar lattice after the addition of a zero-element and a one-element.

Furthermore, a direct relationship is established between the join-irreducible elements of $\mathrm{Ch}\,L$ and the cells of L.

Chapter 1 develops the results using purely algebraic means. The partial order on the maximal chains of a planar lattice L is defined by using the left-right order defined in [4]. The cell lattice of L, denoted $\mathrm{Ce}\,L$, is defined and shown to be isomorphic to the join-irreducible elements of $\mathrm{Ch}\,L$ after removing the 0 and 1 elements from $\mathrm{Ce}\,L$.

Chapter 2 shows how the results from Chapter 1 can be obtained using graph theory. The main theorem from [**5**] plays a key role in this discussion. In the main result of this chapter, a complete description is given of all planar lattices K satisfying $\mathrm{Ce}\, K \cong L$, where L is a given planar lattice with planar embedding $e(L)$.

Finally, Chapter 3 uses work from the previous two chapters to develop applications which can be extended to all lattices.

CHAPTER 1

Maximal Chains of Planar Lattices

In this chapter we restrict our discussion to finite planar lattices. We use the left-right order, introduced by D. Kelly and I. Rival [**4**], to develop a partial order on the set of maximal chains of a planar lattice, L. We show that under this order, the set of maximal chains of L forms a distributive lattice.

In section 3, we show that the join-irreducible elements of the maximal chain lattice form a planar lattice, after the addition of a zero-element and a one-element. We further establish a connection between these join-irreducible maximal chains and the cells of our original planar lattice. We conclude this chapter by proving the converse statement: every distributive lattice whose poset of join-irreducibles is planar after adding a least element and a greatest element is the maximal chain lattice of a planar lattice.

1. Preliminaries

We begin with a brief introduction to lattice theory. For a complete treatment of the subject, see G. Grätzer [**2**].

DEFINITION 1.1. A partially ordered set (poset) is a set P together with a binary relation $\leq$ such that the following properties hold for all elements $a, b, c \in P$:

1. $a \leq a$ (Reflexivity)
2. $a \leq b$ and $b \leq a$ imply $a = b$ (Antisymmatry)
3. $a \leq b$ and $b \leq c$ imply $a \leq c$ (Transitivity)

Let P be a poset and let $a, b \in P$. Let $c \in P$ be an element such that c is an upper bound of a and b and if d is an upper bound of a and b then $c \leq d$. We say that c is a least upper bound of a and b and we write $c = a \vee b$ (called the *join* of a and b). We can similarly define the greatest lower bound (called the *meet*) of a and b and we write $a \wedge b$.

DEFINITION 1.2. A lattice L is a partially ordered set in which $a \vee b$ and $a \wedge b$ exist for all $a, b \in L$.

Let L be a lattice and let $a, b \in L$. If $a < b$ and for all $c \in L$, $a < c$ and $c \leq b$ imply $c = b$ then we say b *covers* a, or a is *covered* by b, written $a \prec b$. For $a, b \in L$, a and b are *incomparable* if $a \nleq b$ and $a \ngeq b$ (written $a \,||\, b$).

DEFINITION 1.3. An element $a \in L$ is *join-irreducible* (*meet-irreducible*) if $a = x \vee y$ ($a = x \wedge y$) implies either $x = a$ or $y = a$. If a is not join-irreducible (meet-irreducible) then a is *join-reducible* (*meet-reducible*).

If an element is both join-irreducible and meet-irreducible, then it is *doubly-irreducible*. If it is both join-reducible and meet-reducible, then it is *doubly-reducible*.

Every lattice L can be represented by a *Hasse diagram*, written $e(L)$. In a Hasse diagram of a lattice, L, every element is represented by a small circle, and if $a \prec b$ then b is set higher than a and is connected to a by a line. A *planar representation* of L is a Hasse diagram of L in which no two covering relations intersect, except possibly at their endpoints. L is *planar* if it has a planar representation.

2. A Partial Order on Maximal Chains of L

Let L be a planar lattice and let $x \in L$. Then each planar embedding $e(L)$ of L gives rise to a linear order on the set of lower (upper) covers of x. In particular, for lower covers y and z of x, we say y is to the left of z if the angle the $y \prec x$ line segment makes with the horizontal is less than the angle the $z \prec x$ line segment makes with the horizontal (from the left). In Figure 1.1, y is to the left of z.

Fix a planar representation $e(L)$ of L. We now define the left-right order on L (see [4]).

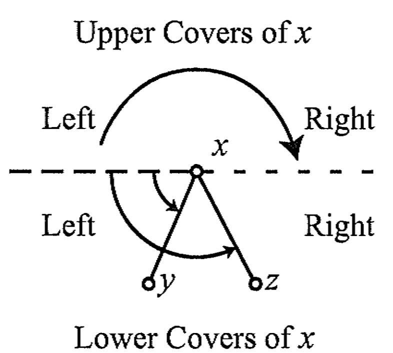

FIGURE 1.1. Linear order on lower covers of x in $e(L)$

DEFINITION 1.4. *x is to the left of y*, written $x \lambda y$, iff $x \,||\, y$ and there are lower covers x' and y' of $x \vee y$ such that $x \leq x'$ and $y \leq y'$ and x' is to the left of y' in $e(L)$.

A *maximal chain* is a sequence of elements

$$\{0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1\}$$

such that for each $i = 1$ to n, $x_{i-1} \prec x_i$. Recall that a planar representation is an embedding of L into the plane such that the second projection satisfies $\pi_2(x) < \pi_2(y)$ whenever $x < y$, and every covering relation $x \prec y$ is represented by a straight line segment from x to y in the plane. Therefore each maximal chain C represents a continuous function f_C in the second coordinate from the 0-element to the 1-element of L. So, for an an element $x \in L$ and a maximal chain $C \subset L$ with $x \notin C$, we can compare x to C by saying x is *to the left of* C whenever $\pi_2(x) \leq f_C(\pi_1(x))$ (pages 640 to 641 of [4] gives a detailed study of maximal chains of L as continuous functions in the plane). D. Kelly and I. Rival established the following result about maximal chains and the λ partial order in [4].

PROPOSITION 1.5 (D. Kelly and I. Rival, 1975). *If $x \lambda y$, then x is on the left of any maximal chain through y. If $x \,||\, y$ and x is on the left of some maximal chain through y, then $x \lambda y$*

From Proposition 1.5 we get $x \,||\, y$ iff x and y are comparable with respect to λ. We can use the left-right order to apply a partial order to maximal chains of L. Given two maximal chains A and B of L, $A \leq_\lambda B$ iff for all $a \in A$ and $b \in B$, $a \,||\, b$ implies $a \lambda b$. So $A <_\lambda B$ if there is at least one such pair. By Proposition 1.5, $A <_\lambda B$ iff A is to the left of B in $e(L)$.

Let $\mathrm{Ch}\, L$ represent the set of all maximal chains on L with leftmost chain being O_λ and rightmost chain being I_λ.

LEMMA 1.6. $\mathrm{Ch}\, L$ *forms a lattice under* $\leq_\lambda$.

PROOF. Let $A, B, C \in \mathrm{Ch}\, L$.

- *Reflexivity:* $A \leq_\lambda A$ since all elements of A are comparable.
- *Antisymmetry:* Assume $A \leq_\lambda B$ and $A \geq_\lambda B$. Then for all $a \in A$ and $b \in B$, $a \,||\, b$ implies $a \,\lambda\, b$ and $b \,\lambda\, a$ which is not possible. Therefore all elements of A are comparable to all elements of B and so $A = B$ since A and B are both maximal.
- *Transitivity:* Assume $A \leq_\lambda B$ and $B \leq_\lambda C$. Let $a \in A$ and $c \in C$ such that $a \,||\, c$. Then there is no maximal chain D such that $a, c \in D$. Therefore, for all $b \in B$, either $a \,||\, b$ or $b \,||\, c$ (or both). So either $a \,\lambda\, b$ or $b \,\lambda\, c$. If $a \,\lambda\, b$ then by Proposition 1.5, a is to the left of any maximal chain through b, and either C passes through b, or b is to the left of C. In either case, $a \,\lambda\, c$. Dually, if $b \,\lambda\, c$, then we conclude $a \,\lambda\, c$. Therefore $A \leq_\lambda C$.

Therefore, $\leq_\lambda$ forms a partial order on $\mathrm{Ch}\, L$.

Let $A, B \in \mathrm{Ch}\, L$. Define

$$\begin{aligned} A \wedge_\lambda B = &\{x \in A \cup B \mid x \in A \cap B\} \\ &\cup \{x \in A \cup B \mid \exists y \in A \cup B \text{ such that } x \,\lambda\, y\} \end{aligned}$$

So $A \wedge_\lambda B$ is the lefthand path in $e(L)$ wherever A and B diverge. As L is planar, if A and B ever cross, it is at points of the lattice, hence $A \wedge_\lambda B$ is in fact a maximal chain. Furthermore, $A \wedge_\lambda B \leq_\lambda A$ and $A \wedge_\lambda B \leq_\lambda B$. Finally, if $C \in \operatorname{Ch} L$ with $C \leq_\lambda A$ and $C \leq_\lambda B$ then for all $c \in C$ and all $a \in A \cup B$, $c \,||\, a$ implies $c \,\lambda\, a$. Therefore for all $b \in A \wedge B$, $c \,||\, b$ implies $c \,\lambda\, b$, since $A \wedge B \subset A \cup B$. Therefore $C <_\lambda A \wedge B$.

Dually, $A \vee_\lambda B$ is the righthand path in $e(L)$ wherever A and B diverge. Therefore $\operatorname{Ch} L$ forms a lattice under $\leq_\lambda$. □

When viewing our maximal chains as continuous functions in the plane, note that the partial order we defined on the maximal chains of L corresponds to the natural partial order given to continuous functions on a closed interval. That is, for f and g continuous on $[a, b]$ then $f \leq g$ iff $f(y) \leq g(y)$ for all $y \in [a, b]$. It comes as no surprise then, that $\operatorname{Ch} L$ is distributive (it is a sublattice of the distributive lattice of continuous functions on a closed interval). Nevertheless, we prove the distributivity of $\operatorname{Ch} L$ in the next section by a study of its join-irreducible elements, as the join-irreducible elements of $\operatorname{Ch} L$ play a key role in the remainder of our discussion.

We note that any maximal chain A divides L into two: all elements a of L on the left of A, written $a <_\lambda C$ and all elements on the right of A, written $a >_\lambda C$. Let $A \subset L$ and let C be a maximal chain in L. If a is on the left of C or $a \in C$ for all $a \in A$ then we will say $A \leq_\lambda C$. If $A \leq_\lambda C$ and there exists $a \in A$ such that a is on the left of C, then we say $A <_\lambda C$.

3. Elementary Cells and the Cell Lattice

A cell A in a lattice L is a sublattice of the form

$$\{0_A, x_1, \dots, x_m, y_1, \dots, y_n, 1_A\}$$

such that two maximal chains are formed:

$$A_L: \quad 0_A \prec x_1 \prec \cdots \prec x_m \prec 1_A$$

$$A_R: \quad 0_A \prec y_1 \prec \cdots \prec y_n \prec 1_A$$

where A_L is to the left of A_R, and such that for elements x_i and y_j in the cell, $x_i \vee y_j = 1_A$ and $x_i \wedge y_j = 0_A$.

A cell A in L will be called an *elementary cell* if it has empty interior, that is, if there is no maximal chain C in the sublattice $[0_A, 1_A]$ such that $A_L <_\lambda C <_\lambda A_R$. For example, in $\mathcal{M}_3$, there are three cells, $A = \{0, a, b, 1\}$, $B = \{0, a, c, 1\}$, and $C = \{0, b, c, 1\}$, but only A and C are elementary cells (see figure 1.2). Clearly the left-right order determines the elementary cells of L. For a fixed a Hasse diagram of

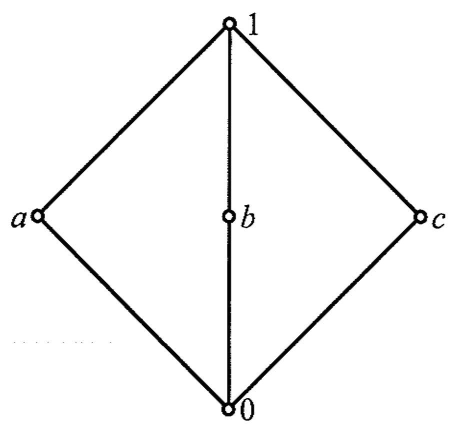

FIGURE 1.2. $\mathcal{M}_3$

L we will let $C(L)$ correspond to the set of all elementary cells in that diagram.

LEMMA 1.7. *Let A be an elementary cell and let C be a maximal chain. Then either $A <_\lambda C$ or $A >_\lambda C$.*

PROOF. For the sake of contradiction, assume there are $a, b \in A$ such that $a <_\lambda C$ and $b >_\lambda C$. As A_L is to the left of A_R, we can assume $a \in A_L$ and $b \in A_R$. Now A_L and A_R share endpoints, so a portion C must lie between A_L and A_R, call it C_A. That is, in the sublattice $[0_A, 1_A]$, $A_L <_\lambda C_A <_\lambda A_R$. As L is planar, C must cross A_L and A_R at elements a_l and a_r of L respectively. Without loss of generality, we can assume $a_l < a_r$. But this creates a maximal chain:

$$C'_A = \overbrace{0_A \prec a_0 \prec \cdots \prec}^{\text{in } A_L} a_l \overbrace{\prec c_i \prec \cdots \prec}^{C_A} a_r \overbrace{\prec a_{r+1} \prec \cdots \prec 1_A}^{\text{in } A_R}$$

in $[0_A, 1_A]$ such that $A_L <_\lambda C'_A <_\lambda A_R$ contradicting the irreducibility of A. $\square$

Fix a planar representation of L and let A be an elementary cell. We can extend A_L and A_R into maximal chains on L in several useful ways. Let A_{L_0} and A_{L_1} correspond to the leftmost paths in $[0, 0_A]$ and $[1_A, 1]$ respectively. Similarly define A_{R_0} and A_{R_1} to be the rightmost paths. We then let

- $A^- = A_{L_0} \cup A_L \cup A_{L_1}$
- $A^+ = A_{R_0} \cup A_R \cup A_{R_1}$
- $A^\vee = A_{L_0} \cup A_R \cup A_{L_1}$
- $A^\wedge = A_{R_0} \cup A_L \cup A_{R_0}$

We call A^- (A^+) the left (right) maximal chain extension of A_L (A_R). We later show that $A^\vee$ ($A^\wedge$) are the join-irreducible (meet-irreducible) elements of $\mathrm{Ch}\, L$. In fact, we show that for every join-irreducible (meet-irreducible) element C of $\mathrm{Ch}\, L$, there is a cell A such that $C = A^\vee (= A^\wedge)$.

With A^- and A^+ acting as boundaries, we obtain four regions oriented with respect to A (see figure 1.3):

(1) $\mathbf{T}(A) = \{x \in L \mid x \geq 1_A\}$

(2) $\mathbf{B}(A) = \{x \in L \mid x \leq 0_A\}$

(3) $\mathbf{L}(A) = \{x \in L \mid x \leq_\lambda A^-\}$

(4) $\mathbf{R}(A) = \{x \in L \mid x \geq_\lambda A^+\}$

LEMMA 1.8. $\boldsymbol{B}(A) \cup \boldsymbol{T}(A) \cup \boldsymbol{L}(A) \cup \boldsymbol{R}(A) = L$

PROOF. Let $x \in L$ and assume $x >_\lambda A^-$ and $x <_\lambda A^+$. Then x is to the right of A^- and to the left of A^+. So $x \notin A$ and as A is an elementary cell, either $x \in [0, 0_A]$ or $x \in [1_A, 1]$. In either case, we are done. □

As the borders between the four regions are maximal chains, by Lemma 1.7, every other elementary cell of L is contained in exactly one of the regions. This gives us a means of comparing elementary cells in L.

LEMMA 1.9. *Let L be a finite planar lattice and fix a planar representation of L. For elementary cells A and B, we have:*

(1) $A \in \boldsymbol{B}(B)$ *iff* $B \in \boldsymbol{T}(A)$

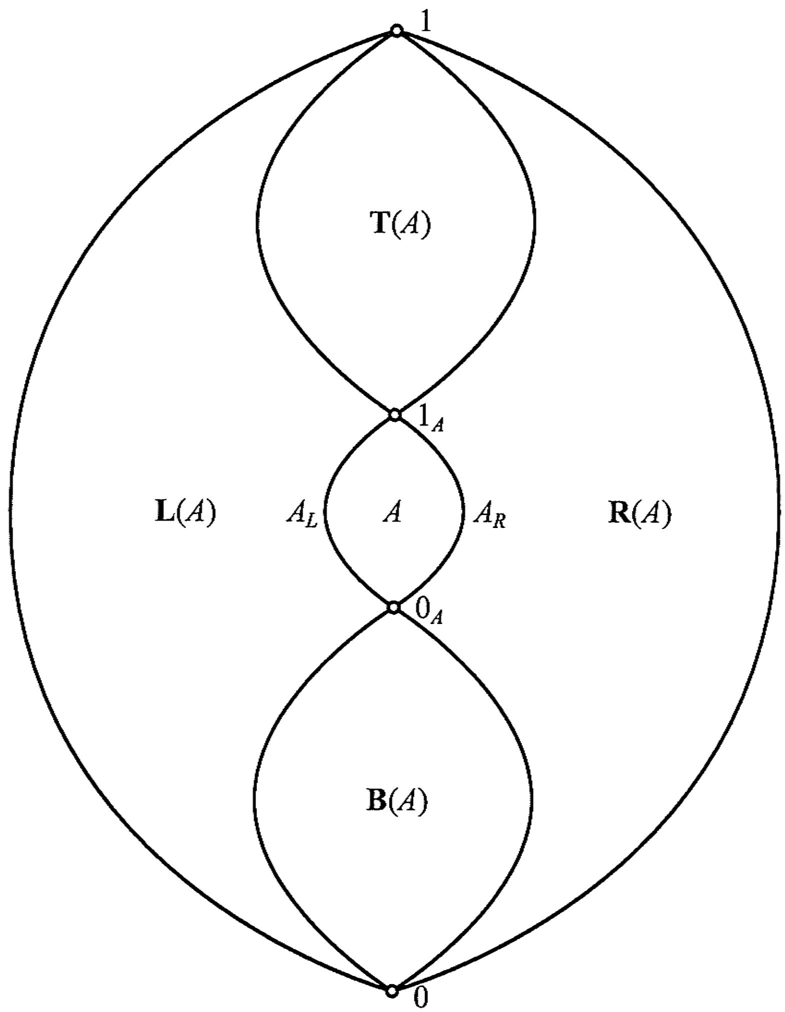

FIGURE 1.3. The four regions of a planar lattice L with respect to an elementary cell A.

(2) $A \in \boldsymbol{L}(B)$ *iff* $B \in \boldsymbol{R}(A)$

PROOF. Assume $A \in \mathbf{B}(B)$. Then for every $a \in A$, $a \leq 0_B$. In particular, $1_A \leq 0_B$. So for every $b \in B$, $1_A \leq 0_B \leq b$. Therefore $B \in \mathbf{T}(A)$.

Now assume $A \in \mathbf{L}(B)$. Then for every $a \in A$, $a \leq_\lambda B^-$. So in particular, $A_R \leq_\lambda B^-$. Therefore $A^\vee \leq_\lambda B^-$. If $B \in \mathbf{L}(A)$, then

$B^{\vee} \leq_{\lambda} A^{-}$. But then $B^{\vee} \leq_{\lambda} A^{-} < A^{\vee} \leq_{\lambda} B^{-} < B^{\vee}$, a contradiction. Therefore $B \notin \mathbf{L}(A)$, and by (1), $B \notin \mathbf{B}(A)$ and $B \notin \mathbf{T}(A)$. Therefore, $B \in \mathbf{R}(A)$. □

Using Lemma 1.9, we can impose a partial order on the elementary cells of L. For elementary cells A and B, let $A <_C B$ iff $A \in \mathbf{L}(B)$. So $A \leq_C B$ iff $A <_{\lambda} B^{\vee}$.

LEMMA 1.10. *Let A and B be elementary cells in L. Then $A \leq_C B$ iff there exists $a \in A$ and $b \in B$ such that $a \,\lambda\, b$.*

PROOF. Assume $A \leq_C B$. Then $A <_{\lambda} B^{\vee}$. Therefore there exists $a \in A$ and $c \in B^{\vee}$ such that $a \,\lambda\, c$. But $B^{\vee}$ is the leftmost maximal chain containing B_R. Therefore there exists $b \in B_R$ such that $a \,\lambda\, b$.

For the converse, assume A and B are incomparable cells. Then either $A \in \mathbf{B}(B)$ or $A \in \mathbf{T}(B)$. So either $1_A \leq 0_B$ or $0_A \geq 1_B$. In either case, every $a \in A$ is comparable to every $b \in B$. Therefore, there is no $a \in A$ and $b \in B$ such that $a \,\lambda\, b$ or $b \,\lambda\, a$. □

This order relation gives us a convenient means of determining the structure of $\mathrm{Ch}\, L$. If we let $\mathrm{J}(\mathrm{Ch}\, L)$ correspond to the poset of join-irreducible elements of $\mathrm{Ch}\, L$ then we have the following result.

THEOREM 1.11. *Define the function $\varphi : C(L) \longrightarrow \mathrm{J}(\mathrm{Ch}\, L)$ by $A\varphi = A^{\vee}$. Then φ is a bijection. Furthermore, if $C(L)$ is given the partial order $<_C$, then φ is an isomorphism between posets.*

PROOF. Let A be an elementary cell. If B is an elementary cell such that $B \leq_{\lambda} A^{\vee}$ then $B \leq_C A$. So $\sup\{B \in C(L) \mid B \leq_{\lambda} A^{\vee}\} = A$.

Therefore if $C \in \operatorname{Ch} L$ can be transformed into $A^{\vee}$ by exactly one (elementary) cell transformation to the right, then $C = A^{-}$ and the cell used in the transformation is A. So $A^{-} \prec_{\lambda} A^{\vee}$ and A^{-} is the only maximal chain covered by $A^{\vee}$ in $\operatorname{Ch} L$. Therefore $A^{\vee} \in \mathrm{J}(\operatorname{Ch} L)$ and φ is well-defined.

Let A and B be elementary cells such that $A\varphi = B\varphi$. Then $A^{\vee} = B^{\vee}$. So $A <_{\lambda} B^{\vee}$ implies $A \leq_{C} B$. But $B <_{\lambda} A^{\vee}$ implies $B \leq_{C} A$. Therefore $A = B$ and φ is one-to-one.

Let $C \in \mathrm{J}(\operatorname{Ch} L)$. Then there is exactly one $D \in \operatorname{Ch} L$ such that $D \prec_{\lambda} C$. As $D <_{\lambda} C$, there exist subchains C_1 and D_1 of C and D respectively such that C_1 and D_1 share endpoints a and b and such that $d \,\lambda\, c$ for all $d \in D_1 \setminus \{a, b\}$ and $c \in C_1 \setminus \{a, b\}$. Since $D \prec_{\lambda} C$, there is no maximal chain B such that $D <_{\lambda} B <_{\lambda} C$, so there is only one set of such subchains, and $D_1 \cup C_1$ forms an elementary cell in L, call it A. Therefore there is a (unique) cell transformation A corresponding to the covering relation $D \prec_{\lambda} C$. So $A^{\vee} = A\varphi = C$ and as D is the unique lower cover of C, A is the unique cell such that $A\varphi = C$. Therefore, φ is onto.

Finally, let A and B be elementary cells. Then

$$\begin{aligned}
A \leq_{C} B &\text{ iff } A <_{\lambda} B^{\vee} \\
&\text{ iff } A_R \leq_{\lambda} B^{\vee} \\
&\text{ iff } \exists C \in \operatorname{Ch} L \text{ such that } A_R \subset C \text{ and } C \leq_{\lambda} B^{\vee} \\
&\text{ iff } A^{\vee} \leq_{\lambda} B^{\vee} \qquad (\text{since } A^{\vee} = \min{}_{\lambda}\{C \in \operatorname{Ch} L \mid A_R \in C\})
\end{aligned}$$

Therefore φ is a poset isomorphism. $\square$

Dually, $A^\wedge$ corresponds isomorphically to the meet-irreducible elements of $\operatorname{Ch} L$.

LEMMA 1.12. $\operatorname{Ch} L$ *is distributive.*

PROOF. Let $C \in \operatorname{Ch} L$ and let

$$r(C) = \{A \in \mathrm{J}(\operatorname{Ch} L) \mid A \leq_\lambda C\}$$

So $r(C)$ is the set of all join-irreducible elements of $\operatorname{Ch} L$ less than C. If $A \in r(C)$ and $B \in \mathrm{J}(\operatorname{Ch} L)$ with $B \leq_\lambda A$ then $B \leq_\lambda A \leq_\lambda C$, so $B \in r(C)$. Therefore, $r(C)$ is *hereditary*, that is, if $A \in r(C)$ and $B \leq A$ then $B \in r(C)$. Define the map from $\operatorname{Ch} L$ to the set of all hereditary subsets on $\mathrm{J}(\operatorname{Ch} L)$ by:

$$\varphi : C \mapsto r(C)$$

Now since $\operatorname{Ch} L$ is finite, every element is the join of nonzero join-irreducible elements. Thus $C = \bigvee r(C)$, showing that φ is one-to-one. By Lemma 1.11, we know that the join irreducible elements below C correspond to the elementary cells to the left of C. Therefore for $C, D \in \operatorname{Ch} L$, $r(C \vee D) = r(C) \cup r(D)$ and $r(C \wedge D) = r(C) \cap r(D)$. So we get $(C \vee D)\varphi = C\varphi \cup D\varphi$ and $(C \wedge D)\varphi = C\varphi \cap D\varphi$.

Finally, let H be a hereditary subset of $\mathrm{J}(\operatorname{Ch} L)$. So H is a set of elementary cells of L such that if $A \in H$ then $B \in H$ for all $B \in \mathbf{L}(A)$. Let $M \subset H$ be the set of all maximal cells in H. Then for all $A, B \in H$, $A \in \mathbf{B}(B)$ (or $B \in \mathbf{B}(A)$), so for all $a \in A_R$ and $b \in B_R$, $a < b$ ($b < a$). Let C be the leftmost maximal chain such that $A_R \subset C$ for all $A \in M$. Then $B \in r(C)$ iff $B \leq_C A$ for some $A \in M$. Therefore $r(C) = H$ and φ is onto.

Therefore $\mathrm{Ch}\, L$ is isomorphic to a ring of sets, and so $\mathrm{Ch}\, L$ is distributive (G. Birkhoff [1] and M. H. Stone [7]). $\square$

During the remainder of the discussion, it will be useful to add a zero and a one element to $\mathrm{J}(\mathrm{Ch}\, L)$. We will write $\mathrm{J}_0(\mathrm{Ch}\, L)$ for $\mathrm{J}(\mathrm{Ch}\, L) \cup \{0, 1\}$. To maintain the isomorphic relation between the join-irreducible chains and cells of L, we add a zero cell O_C and a one cell I_C to $C(L)$. The leftmost chain O_λ of L will be considered the right chain $(O_C)_R$ of O_C and O_C will not have a left chain. Similarly, $(I_C)_L = I_\lambda$ and $(I_C)_R = \emptyset$.

DEFINITION 1.13. Let L be a finite planar lattice. Define the *cell lattice* or *cell structure* of L to be $\mathrm{Ce}\, L = C(L) \cup \{O_C, I_C\}$, partially ordered by $<_C$.

For any given cell $A \subset L$, we will let

$$A_r = A \setminus A_L \qquad \text{and} \qquad A_l = A \setminus A_R.$$

Notice that for $A \neq O_C$ and $A \neq I_C$, $A_r = A_R \setminus \{0_A, 1_A\}$, and $A_l = A_L \setminus \{0_A, 1_A\}$.

LEMMA 1.14. *For every element $x \in L$, there are unique cells A and B such that $x \in A_r$ and $x \in B_l$. Similarly, for every covering relation $x \prec y \subset L$, there are unique cells C and D such that $x \prec y \subset C_R$ and $x \prec y \subset D_L$.*

PROOF. Let $x \in L \setminus \{0, 1\}$ and let X_L be the leftmost chain in $[0, x]$ and in $[x, 1]$. If $x \in O_\lambda$ then $X_L \cong O_\lambda$ and x is uniquely on $(O_C)_R$.

Otherwise there are cells to the left of X_L. Let

$$\mathcal{A} = \{A \in \mathrm{Ce}\, L \mid A <_\lambda X_L\}$$

and let $A \in \mathcal{A}$. If A is a maximal element of $\mathcal{A}$ then $A_R \subset X_L$. But X_L is the leftmost path in $(0, x)$ and in $(x, 1)$. Therefore $0_A < x < 1_A$, so $x \in A_l$. Also, as $0_A, 1_A \in X_L$, we have $A^\vee = X_L$. Therefore A is the unique maximal element of $\mathcal{A}$ and so it is the unique cell such that $x \in A_r$.

Dually, using the rightmost maximal chain containing x, we can show there is a unique cell B such that $x \in B_l$.

Now let $X = x \prec y$ be an edge in L. Then there are unique cells A and B such that $x \in A_r$ and $y \in B_r$. If $A = B$ then $X \subset A_R$ and we are done. Otherwise, either $x = 0_B$ or $y = 1_A$. So either $X \subset B_R$ or $X \subset B_R$ (but not both). In either case, we are done.

Dually, there is a unique cell C such that $x \prec y \subset C_L$.

If $x = 0$ or $x = 1$ then trivially, O_C is the unique cell such that $x \in (O_C)_r$ and I_C is the unique cell such that $x \in (I_C)_l$ (because $(O_C)_L = \emptyset$ and $(I_C)_R = \emptyset$ respectively). $\square$

THEOREM 1.15. $\mathrm{Ce}\, L$ *is a planar lattice.*

PROOF. Let $A, B \in \mathrm{Ce}\, L$ and define the following two linear orders on $\mathrm{Ce}\, L$:

(1) $A <_0 B$ iff $A \in \mathbf{L}(B)$ or $A \in \mathbf{B}(B)$.
(2) $A <_1 B$ iff $A \in \mathbf{L}(B)$ or $A \in \mathbf{T}(B)$.

Lemma 1.9 guarantees that these are both linear orders. Also $A <_C B$ iff $A \in \mathbf{L}(B)$ iff $A <_0 B$ and $A <_1 B$. Therefore $\mathrm{Ce}\, L$ is of dimension no greater than 2, and has dimension 2 iff there exist cells A and B such that $A \in \mathbf{B}(B)$.

Let $A, B \in \operatorname{Ce} L$. If A and B are comparable, then $A \vee_C B$ and $A \wedge_C B$ are trivially $\max_C\{A, B\}$ and $\min_C\{A, B\}$ respectively.

So assume A and B are incomparable. Then A^- and B^- are incomparable in $\operatorname{Ch} L$. Therefore, there exists an $x \in A^- \cap B^-$ such that $x \neq 0$ and $x \neq 1$. Then $A^- \wedge_\lambda B^-$ is the leftmost path in $(0, x)$ and in $(x, 1)$. By Lemma 1.14, there is a unique cell C such that $x \in C_r$. Since $A^- \wedge_\lambda B^-$ is the leftmost path in $(0, x)$ and in $(x, 1)$, C must be the unique elementary cell such that $C_R \subset A^- \wedge_\lambda B^-$. But this implies C is the unique cell such that $C^\vee = A^- \wedge_\lambda B^-$.

Now let $D \in \operatorname{Ce} L$ such that $D <_C A$ and $D <_C B$. Then $D <_\lambda A^-$ and $D <_\lambda B^-$. Therefore $D <_\lambda \left(A^- \wedge_\lambda B^-\right) = C^\vee$. So $D \leq_C C$ and $C = A \wedge_C B$.

Dually, $A \vee_C B$ exists and therefore $\operatorname{Ce} L$ forms a planar lattice. $\square$

As a planar lattice, $\operatorname{Ce} L$ may have a variety of planar representations and corresponding left-right orders. We can use the two linear orders from the above proof to describe one such representation by using the partial order on L. For elements $a, b \in \operatorname{Ce} L$ with corresponding cells $A, B \subset L$, we know

$$
\begin{aligned}
a \,||\, b \text{ iff either } & a <_0 b \text{ and } a >_1 b, \text{ or } a >_0 b \text{ and } a <_1 b \\
& \text{iff either } A \in \mathbf{T}(B) \text{ or } A \in \mathbf{B}(B) \\
& \text{iff } 1_A \leq 0_B \text{ or } 1_B \leq 0_A \\
& \text{iff } x \leq y \;\forall x \in A \text{ and } y \in B \text{ or } x \geq y \;\forall x \in A \text{ and } y \in B.
\end{aligned}
$$

We can therefore set $a \lambda b$ in $\mathrm{Ce}\, L$ iff $1_A \leq 0_B$. We will refer to this left-right order on $\mathrm{Ce}\, L$ as the left-right order of $\mathrm{Ce}\, L$ *induced by* L.

The remaining results in this section about the cell lattice will be useful tools for the remainder of the paper. Let L be a planar lattice.

LEMMA 1.16. *Let $a, b \in \mathrm{Ce}\, L$ and let A and B be the corresponding elementary cells in L. Then the following are equivalent:*

(1) $a \prec b$
(2) $B_L \subset A^+$
(3) $A_R \subset B^-$

PROOF. 1 implies 2: We will prove the contrapositive. Assume $a < b$ but there exists a $x \in B_L$ such that $x \notin A^+$. Then x is to the right of A^+ since $B \in \mathbf{R}(A)$. As A^+ is the rightmost maximal chain containing A_R, and hence the rightmost maximal chain containing both 0_A and 1_A, we know that for any $y \in A_l$, we have $y \lambda x$.

Now, by Lemma 1.15, there is a unique cell C such that $x \in C_r$. As $x \notin A_r$, we know $C \neq A$. So $A <_C C$ since there exist $y \in A$ such that $y \lambda x$. But $C <_C B$ since for any $z \in C_l$, we have $z \lambda x$. Therefore, there exists $c \in \mathrm{Ce}\, L$ such that $a < c < b$, so $a \not\prec b$.

2 implies 3: Assume $B_L \subset A^+$. Then, in particular, 0_B and 1_B both lie on A^+. As $B \in \mathbf{R}(A)$, we know $0_B < 1_A$ and $1_B > 0_A$. Therefore either (at least) one of 1_B and 0_B lies on A_r, or $A_R \subset B_L$. Obviously in the latter case, we would have $A_R \subset B^-$. On the other hand, if $0_B \in A_r$ but $1_B \geq 1_A$ then the portion of A_R which lies between 0_A and 0_B would be on B^- because of the irreducibility of A. The

portion of A_R which lies between 0_B and 1_A lies on B_L since $B_L \subset A^+$. Therefore $A_R \subset B^-$. The other cases where $1_B \in A_r$ and where both 0_B and 1_B lie on A_r are proved similarly.

3 implies 1: Assume $A_R \subset B^-$. Then $1_A > 0_B$ and $0_A < 1_B$ as otherwise B^- would not be the leftmost maximal chain in $[0, 0_B]$ or $[1_B, 1]$ respectively. So there exists $x \in A_r$ such that $x \in B$. Therefore, for $y \in A_l$ we have $x \, \lambda \, y$, so $A <_C B$.

Now let C be an elementary cell in L such that $A \leq_C C$ and $C <_C B$. Then $C \in \mathbf{L}(B)$, so $C <_\lambda B^-$. But $A_R \subset B^-$, so either there exist $x \in A$ and $y \in C$ such that $y \, \lambda \, x$ or every $y \in C$ is comparable to every $x \in A_R$. In the latter case, we would conclude that either $1_C \leq 0_A$ or $0_C \geq 1_A$, so A and C are incomparable cells, which contradicts $A \leq_C C$. Therefore, there exist $x \in A$ and $y \in C$ such that $y \, \lambda \, x$. So $C \leq_C A$, and as $A \leq_C C$, we conclude $A = C$ as desired. Therefore $a \prec b$ in $\mathrm{Ce}\, L$. $\square$

COROLLARY 1.17. *Let $a, b \in \mathrm{Ce}\, L$ and let A and B be the corresponding elementary cells in L. If $a \prec b$ then A_R and B_L share (at least) one edge.*

PROOF. Let $a \prec b$. By Lemma 1.16, we know $A_R \subset B^-$, and as A and B are comparable cells, we know that $1_A \not\leq 0_B$ and $1_B \not\leq 0_A$. Therefore, A and B must have at least one edge in common. $\square$

4. Planar Lattices as Cell Lattices

Every planar lattice has a planar cell lattice. In this section, we show that for every planar lattice L, there is a planar lattice K such that $L = \operatorname{Ce} K$. We prove the following theorem:

THEOREM 1.18. *Let L be a finite planar lattice. Then* $\operatorname{Ch} L$ *forms a distributive lattice whose poset of join-irreducibles* $\mathrm{J}(\operatorname{Ch} L)$ *forms a planar lattice when adjoining a 0 and a 1 element.*

Conversely, let D be a finite distributive lattice. If the poset of join-irreducible elements $J(D)$ forms a planar lattice after adjoining a 0 and a 1 then there is a planar lattice L such that $\operatorname{Ch} L \cong D$.

To create K, a direct relation between the cells and edges of L and the elements of K will be established. Similarly, a direct relation will be established between the cells of K and the elements of L. To avoid confusion, lowercase letters at the end of the alphabet $x, y, z, \ldots$ will be used to refer to elements of K, while the uppercase letters $X, Y, Z, \ldots$ will refer to the edge or cell of L which corresponds to that element. Similarly, elements of L will be denoted by lowercase letters at the beginning of the alphabet $a, b, c, \ldots$, while the corresponding cells of K will be denoted by the respective uppercase letters $A, B, C, \ldots$. The zero and one element of L will be denoted 0_L and 1_L, and similarly, the zero and one element of K will have a 'K' as a subscript for distinction.

Let $E(L)$ be the set of all edges (covering pairs) on L. Define the set $K = E(L) \cup \operatorname{Ce} L$. In order to standardize notation between cells and edges, when X is an edge of L, we will define $X_R = X_L = X$. In this way, we can extend the definitions of X^+, X^-, $X^\vee$, and $X^\wedge$ to include the case when X is an edge. In this case, we note $X^+ = X^\wedge$ and $X^- = X^\vee$.

Now define an order relation on K: for $x, y \in K$, $x < y$ iff $x \neq y$ and there exists $a \in X$ and $b \in Y$ such that $a \,\lambda\, b$.

Now X is either an elementary cell or an edge in L. In either case, for any maximal chain $C \subset L$, either $X \leq_\lambda C$ or $X \geq_\lambda C$ (see Lemma 1.7). Therefore if $x < y$ in K then there exist $a \in X$ and $b \in Y$ such that $a \,\lambda\, b$. So a is on the left of any maximal chain through b (Proposition 1.5). Therefore X is on the left of any maximal chain through b, so $X <_\lambda Y^\vee$. Similarly, $Y >_\lambda X^\wedge$. Conversely, if $X <_\lambda Y^\vee$ then there exists $a \in X$ such that a is on the left of $Y^\vee$, and as $Y^\vee$ is the leftmost maximal chain passing through Y_R, this implies there is $b \in Y_R$ such that $a \,\lambda\, b$. Therefore, we can redefine the order relation as follows: for $x, y \in K$, $x < y$ iff $x \neq y$ and $X <_\lambda Y^\vee$ (or equivalently $Y >_\lambda X^\wedge$).

LEMMA 1.19. *$<$ is a partial order on K.*

PROOF. Trivially $<$ is antireflexive.

Let $x, y \in K$ such that $x < y$. Then there exists $a \in X$ and $b \in Y$ such that $a \,\lambda\, b$. So $X \leq_\lambda Y^\vee$ and therefore there cannot be $c \in X$ and $d \in Y$ such that $d \,\lambda\, c$. Therefore $y \not< x$ and $<$ is antisymmetric.

Let $x, y, z \in K$ such that $x < y$ and $y < z$. Then $X <_\lambda Y^\vee$ and $Y <_\lambda Z^\vee$. Therefore $Y^\vee \leq_\lambda Z^\vee$ so $X <_\lambda Z^\vee$. If $Y^\vee <_\lambda Z^\vee$ then $a <_\lambda Z^\vee$ for all $a \in X$, so $X \neq Z$ and therefore $x < z$ as desired. So assume $Y^\vee = Z^\vee$. As $y < z$, there exist $b \in Y$ and $c \in Z$ such that $b \lambda c$, therefore Y is a cell and Z is an edge such that $Z \subset Y_R$. As $X <_\lambda Y^\vee = Z^\vee$, there exist $a \in X$ such that $a <_\lambda Z^\vee$. Therefore $X \neq Z$ and so $x < z$. Therefore $<$ is transitive. $\square$

THEOREM 1.20. *$(K, <)$ is a planar lattice.*

PROOF. Let $x, y \in K$ and let $z \in K$ be an upper bound of x and y. Then there exist $a \in X$, $b \in Z$ such that $a \lambda b$ and $c \in Y$, $d \in Z$ such that $c \lambda d$. If Z is an edge of L then there is a unique cell $W \subset L$ such that $Z \subset W_R$ (by Lemma 1.14). So w is also an upper bound of x and y (since $b, d \in W$) and $w < z$ (since $Z \subset W_R$ and $W_L <_\lambda W_R$). Therefore, if the least upper bound of x and y exists in K, it must correspond to a cell in L, and so we will restrict our discussion to cells of L.

For $x \in K$, the set of all elements $z \in K$ such that $z > x$ is the set of all cells Z of L such that $Z >_\lambda X^\wedge$. Therefore z is an upper bound of x and y in K iff $Z >_\lambda X^\wedge$ and $Z >_\lambda Y^\wedge$; that is, iff $Z >_\lambda X^\wedge \vee_\lambda Y^\wedge$. So $x \vee y$ will exist in K iff there is a unique cell Z such that $Z_L \subset X^\wedge \vee_\lambda Y^\wedge$. But, either X is an elementary cell, or it is an edge, in which case there is a unique cell X' such that $X \subset X'_L$, so $X^\wedge = X'^\wedge$. In either case, $X^\wedge$ corresponds to a meet-irreducible element of $\operatorname{Ch} L$. Similarly, $Y^\wedge$ is a meet-irreducible element of $\operatorname{Ch} L$. But the meet-irreducible elements of $\operatorname{Ch} L$ correspond isomorphically to the cells of L, call them X' and Y' (dual of Theorem 1.11. So $x \vee y$ will exist iff $X' \vee_C Y'$ exists. But $\operatorname{Ce} L$ is a lattice, so $X' \vee_C Y'$ exists. So if $Z = X' \vee_C Y'$ then $z = x \vee y$ in K.

Dually, using $X^\vee \wedge_\lambda Y^\vee$, we can obtain $x \wedge y$. Therefore, K is a lattice.

We can easily show that K has dimension no greater than 2. Let $x, y \in K$ with $x \neq y$. If $x \,||\, y$, then every element of X must be comparable to every element of Y. Therefore, either $1_Y \leq 0_X$ or $1_X \leq 0_Y$. As these two conditions are mutually exclusive, we can now define two total orders $<_0$ and $<_1$ on K whose intersection forms the partial order of K:

- $x <_0 y$ iff $x < y$ or $1_X \leq 0_Y$ in L
- $x <_1 y$ iff $x < y$ or $0_X \geq 1_Y$ in L.

Therefore, K is a planar lattice. $\square$

So, for $x, y \in K$, $x \,||\, y$ iff either $1_X \leq 0_Y$ or $0_X \geq 1_Y$. That is, x and y are incomparable iff either $a \leq b$ for every $a \in X$ and $b \in Y$ or $a \geq b$ for every $a \in X$ and $b \in Y$. So the left-right order of the lattice K is determined by the partial order of L, where $x \,\lambda\, y$ iff for every $a \in X$ and $b \in Y$, $a \leq b$.

In order to analyze the cell structure of K, it is useful to first describe the covering relations in K.

LEMMA 1.21. *Let $x, y \in K$. Then $x \prec y$ iff one of the following occurs:*

(1) *X is a cell and Y is an edge such that $Y \subset X_R$,*

(2) *Y is a cell and X is an edge such that $X \subset Y_L$.*

PROOF. Assume X is a cell and Y is an edge such that $Y \subset X_R$. Then for $a \in X_l$ and $b \in Y$ we have $a \,\lambda\, b$ so $x < y$.

Now let $z \in K$ such that $z < y$ and $z \geq x$. Then $X^{\vee} \leq Z^{\vee}$ and $Z^{\vee} \leq Y^{\vee}$. But as $Y \subset X_R$, we have $X^{\vee} = Y^{\vee}$. So $Z^{\vee} = X^{\vee} = Y^{\vee}$. Now $z < y$, so there exists $b \in Y$ and $c \in Z$ such that $c \lambda b$. But $Z^{\vee} = Y^{\vee}$, so Z must be an elementary cell such that $Z_R \subset Y^{\vee}$ (for, if Z were an edge then $Z^{-} = Z^{\vee}$ so every element of Z would be comparable to every element of Y). But $Z^{\vee} = X^{\vee}$ so X is the unique cell such that $X_R \subset Y^{\vee}$. Therefore $X = Z$ and therefore $x \prec y$.

By using a dual argument of the above, we can similarly show that if Y is a cell and X is an edge such that $X \subset Y_L$ then $x \prec y$.

To prove the converse, we show that if $x < y$ but x and y do not satisfy either above condition then there exists a $z \in K$ such that $x < z < y$. So assume $x < y$. Then $x \neq y$ and there exist $a \in X$ and $b \in Y$ such that $a \lambda b$. Now assume X is a cell in L.

Claim. We can choose $a \in X$ and $b \in Y$ such that $a \in X_R$ and $a \lambda b$

Proof of Claim. Assume this is not the case. Then every $b \in Y$ is on some maximal chain C such that $X_R \subset C$. But Y is not an edge on X_R by assumption. Therefore either $b > 1_X$ (or $b < 0_X$) for every $b \in Y$. But then $a < b$ ($a > b$) for every $a \in X$ contradicting $x < y$. ∎

Let $Z \subset X_R$ be an edge on X_R such that $a \in Z$. Then $z < y$ and by our above work, $x < z$.

Now assume that X is an edge in L. By Lemma 1.14, there is a unique cell Z such that $X \subset Z_L$. Furthermore, $Z \neq Y$ as otherwise x and y would satisfy condition 2. But $X \subset Z$, so there exist $a \in Z$

and $b \in Y$ such that $z \lambda b$. Therefore $z < y$ and by our above work, $x < z$. □

Let $a \in L$ and define $k(a) = \{x \in K \mid a \in X\}$.

LEMMA 1.22. *$k(a)$ is an elementary cell in K. Conversely, for every elementary cell $A \subset K$, there is $a \in L$ such that $A = k(a)$.*

PROOF. For $0_L \in L$, $k(0_L)$ consists of all the edges between 0_L and atoms of L and all the elementary cells to which these edges belong. Therefore, by Lemma 1.21, this is a covering chain in K. As 0_L is an element of the zero cell and the one cell of L, $k(0_L)$ is a maximal chain of K. Furthermore, $0 \in X$ for every $x \in k(0_L)$ and $0_L \leq a$ for every $a \in L$. Therefore, for every $x \in k(0_L)$ and $y \in K$, if $x \| y$ then $x \lambda y$. Therefore, $k(0_L)$ is the leftmost maximal chain in K, and therefore it is the zero cell of K (since $(O_C)_R = O_\lambda$ and $(O_C)_L = \emptyset$). Therefore $k(0_L) = O_C$ in K.

Dually, for $1_L \in L$, $k(1_L)$ is the one cell of K.

Now let $a \in L \setminus \{0, 1\}$. Then there exist elementary cells X and Y of L such that $a \in X_r$ and $a \in Y_l$.

CLAIM. For all $z \in k(a)$, $x \leq z$ and $z \leq y$.

Proof of Claim. If $Y = I_C$ in L, then $y = 1_K$ so trivially $z \leq y$ for all $z \in k(a)$. Otherwise, there exists $b \in Y_r$, so $a \lambda b$ in L (since $a \in Y_l$). Let $z \in k(a)$. Then $a \in Z$ and $b \in Y$ with $a \lambda b$, so $Z \leq Y$. So $y = \bigvee k(a)$ as required. Dually, $x = \bigwedge k(a)$. ■

Now let $z \in k(a) \setminus \{x, y\}$. As X and Y are the unique cells such that $a \in X_r$ and $a \in Y_l$, we know that either $a = 1_Z$ or $a = 0_Z$.

Therefore, we can divide $k(a) \setminus \{x, y\}$ into two subsets:, call them A_l and A_r respectively:

- $A_l = \{z \in k(a) \mid a = 1_Z\}$
- $A_r = \{z \in k(a) \mid a = 0_Z\}$

We know that both sets are non-empty, since $a \neq 0_L$ and $a \neq 1_L$ imply the existence of edges W and Z such that $a = 1_W$ and $a = 0_Z$. Furthermore, if $z \in A_r$ then $a = 0_Z$ so Z is in the convex sublattice $[a, 1_L]$ of L. Therefore, we know by the first half of the proof, that A_l is a covering chain in K, and if we let $A_L = A_l \cup \{x, y\}$ then A_L is a covering chain in K with maximal element y and minimal element x.

Similarly, if we define $A_R = A_r \cup \{x, y\}$ then A_R is a covering chain in K. Let $w \in A_l$ and $z \in A_r$. Then $a = 1_W$ and $a = 0_Z$. Therefore $1_W = 0_Z$, so $w \,\lambda\, z$ in K. Therefore $k(a)$ is a cell in K. Furthermore, since $1_W = 0_Z$ for every $w \in A_l$ and $z \in A_r$ there can no edge or cell V in K such that $1_W \leq 0_V < 1_V \leq 0_Z$. Therefore $k(a)$ is an elementary cell in K.

Conversely, assume $A \in \operatorname{Ce} K$ with maximal element $y = 1_A$ and minimal element $x = 0_A$. Then for every $w \in A_l$ and every $z \in A_r$, $x < w < y$ and $x < z < y$, but $w \wedge z = x$ and $w \vee z = y$. Therefore X and Y are elementary cells in L. Also, $w \,\lambda\, z$ in K, so $1_W \leq 0_Z$ in L.

Claim. For every $w \in A_l$ and $z \in A_r$, $1_W = 0_Z$.

Proof of Claim. Assume, for the sake of contradiction, that there exist $w \in A_l$ and $z \in A_r$ such that $1_W \neq 0_Z$. Then $1_W < 0_Z$, so there is a covering chain from 1_W to 0_Z. Therefore there is an edge V such that $1_W \leq 0_V < 1_V \leq 0_Z$. But this would imply $v \in K$ such that $w \,\lambda\, v$

and $v \lambda z$. Also if $0_V > 1_X$ then $0_Z > 1_X$, contradicting $z > x$ in K. Similarly, $0_V \not< 0_X$. Therefore there exists $b \in X$ such that $0_V \,||\, b$. Now $0_V \leq c$ for every $c \in Z$, and there exists a $c \in Z$ and $b \in X$ such that $b \lambda c$. Therefore as a corollary to Proposition 1.5, $b \lambda 0_V$. So $v > x$ in K. Similarly, $v < y$. But this contradicts the irreducibility of A. ■

Let $a \in L$ be the unique element such that $1_W = 0_Z = a$, for every $w \in A_l$ and $z \in A_r$. We need only show that $a \in X$ and $a \in Y$. But there exists a $w \in A$ such that $x \prec w$ and $a = 1_W$. Therefore, by Lemma 1.21, $W \subset X_R$ so $a \in X_R$. Similarly, there exists $z \in A$ such that $z \prec y$ and $a = 0_Z$. Therefore $a \in Y_L$. So $a \in L$ is the unique element such that $a \in X$ for every $x \in A$. Therefore $A \subset k(a)$. But from the first half of this proof, we know that $k(a)$ is an elementary cell in K. Therefore $k(a) = A$, and we are done. □

THEOREM 1.23. *For every finite planar lattice L, there is a finite planar lattice K such that* $\mathrm{Ce}\, K \cong L$.

PROOF. We will use the K lattice that we have constructed in this section. We already know by Lemma 1.22 that the k mapping is an bijection between L and $\mathrm{Ce}\, K$. We need only show that the order is preserved. So let $a, b \in L$ such that $a < b$. If $a = 0_L$ ($b = 1_L$) then we know $k(a) = O_C$ ($k(b) = I_C$) in K, so $k(a) < k(b)$ as desired. So assume $a > 0_L$ and $b < 1_L$. Then there exist edges X and Y in L such that $a = 1_X$ and $b = 0_Y$. But then $1_X < 0_Y$ so $x \lambda y$ in K. As $x \in k(a)$ and $y \in k(b)$, by Lemma 1.10, we are done.

Conversely, let A and B be elementary cells of K such that $A \leq_\lambda B$ and let a and b be the corresponding elements of L such that $k(a) = A$ and $k(b) = B$. Then there exist $x \in A$ and $y \in B$ such that $x \,\lambda\, y$. So $a \leq 1_X \leq 0_Y \leq b$ as desired. Therefore $\mathrm{Ce}\, K \cong L$. □

We can now prove Theorem 1.18.

PROOF. Let L be a finite planar lattice. Then by Lemma 1.12, $\mathrm{Ch}\, L$ is a finite distributive lattice whose poset of join-irreducibles is isomorphic to $C(L)$ (Theorem 1.11). But when we adjoin a zero and a one, we get $\mathrm{Ce}\, L$, which is a finite planar lattice by Theorem 1.15.

Conversely, let D be a distributive lattice whose poset of join-irreducible elements $\mathrm{J}(D)$ forms a planar lattice after adjoining a zero and a one. Let L be $\mathrm{J}(D) \cup \{0, 1\}$. Then by Theorem 1.23, there is a finite planar lattice K such that $\mathrm{Ce}\, K \cong L$. But $\mathrm{Ce}\, K = C(K) \cup \{0, 1\}$ so $C(K) \cong \mathrm{J}(D)$. By Theorem 1.11, $C(K) \cong \mathrm{J}(\mathrm{Ch}\, K)$, therefore $D \cong \mathrm{Ch}\, K$. □

So for every planar lattice L, there exists a planar lattice K such that $L \cong \mathrm{Ce}\, K$. However K is by no means unique. For example, any finite chain, C_n, will have the two element chain C_2 as its cell lattice.

CHAPTER 2

Cell Lattices and Dual Graphs

In this chapter we use graph theory to obtain an alternate description of cell lattices of planar lattices. We show that by using dual graphs, we can obtain the covering graph of the cell lattice $\operatorname{Ce} L$ of L. We conclude this chapter by showing that this process is reversible, thereby allowing us to obtain a complete description of all lattices K such that $\operatorname{Ce} K = L$.

1. Introduction

In [**5**], C. R. Platt showed that every finite lattice L is planar if and only if the graph, obtained from the covering graph of L by adding an edge between its least and greatest elements, is a planar graph. By using his main theorem, we show that the cell lattice can be directly obtained from the *dual graph* of the extended covering graph of L.

We will begin with some relevant graph theory.

2. Background Graph Theory

DEFINITION 2.1. A *directed multigraph* (or *digraph*) G is a vertex set $V(G)$ together with an edge set $E(G)$ where each edge $E \in E(G)$ is a ordered pair of vertices. A *simple digraph* is a digraph with no repeated edges (multiple edges) and no edges which begin and end at the same vertex (loops).

In this paper, multiple edges will be allowed in a general graph G, so the terms 'graph' and 'multigraph' will be interchangeable. If a graph G cannot have multiple edges, then it will be explicitly called a 'simple graph' to avoid confusion. A graph is planar if it can be drawn in the plane with no intersecting edges.

DEFINITION 2.2. Let G and H be graphs such that $V(G) \subset V(H)$ and $E(G) \subset E(H)$. Then G is a *subgraph* of H and H is a *supergraph* of G, written $G \subseteq H$.

For a planar graph G, its *geometric dual graph*, G^*, is constructed by placing a vertex in each face of G and if two faces have an edge E in common, joining the corresponding vertices by an edge E^* crossing only at E (Ref: Diag). We note that in this definition, we must allow for the existence of multiple edges and loops. Clearly G^* has a loop if and only if G has a vertex incident with only one edge. Similarly, G^* has multiple edges if and only if G has two faces which sharing more than one edge. The dual graph G^* of a planar graph G is itself a planar graph.

3. Cell Lattices as Graphs

We can now resume our discussion. In [**5**], C. R. Platt proved that a finite lattice L is planar if and only if the graph obtained from its Hasse diagram by adding an edge between the 0 and 1 of L is a planar graph. This planar graph will be called the *extended covering graph*, denoted $g(L)$ and the added edge will be called the *distinguished edge*, written D.

As a directed graph (digraph), $g(L)$ will have the orientation which respects the covering relation on L. That is, every edge $E = x \prec y$ from the covering graph of the lattice is directed from the lesser element x to the greater element y. The distinguished edge is oriented from 1 to 0, so we can think of 0 as the upper cover of 1 in the extended graph. In this way, $g(L)$ will be a digraph such that every cycle contains the distinguished edge.

C. R. Platt gave the following necessary and sufficient conditions in order for a digraph to be the covering graph of a planar lattice (see [**5**]).

THEOREM 2.3 (C. R. Platt, 1976). *Let G be a simple digraph with at least 2 elements. Then G is the (oriented) covering graph of a planar lattice if and only if there exist elements 0 and 1 in G such that the following hold:*

(1) $G \cup \langle 1, 0 \rangle$ *is planar*
(2) G *contains no cycles*
(3) *If $x \in G$ then there is a path from 0 to x and a path from x to 1 in G*
(4) G *is* strongly antitransitive *(that is, if there is a driected path from x to y in G of length ≥ 2 then there is no edge from x to y in G).*

From Theorem 2.3, we obtain the following necessary and sufficient conditions for a digraph to be the extended covering graph of a planar lattice.

COROLLARY 2.4. *Let G be a simple digraph with at least one edge. Then G is the extended covering graph of a planar lattice if and only if there exists an edge D in G such that the following hold:*

(1) *G is planar*

(2) *Every cycle in G contains the edge D*

(3) *If $x \in G$ then there exists a cycle $\mathcal{C}$ such that $x \in \mathcal{C}$*

(4) *G is strongly antitransitive.*

Now let L be a finite planar lattice with $|L| \geq 2$ and fix a planar representation $e(L)$ of L. Then, by adding the distinguished edge, we obtain the associated planar graph $g(L)$. With the addition of this edge, we note that the exterior, unbounded region of the lattice is divided into two new regions. One region has the leftmost maximal chain, O_λ together with the distinguished edge as the boundary; the other region has the rightmost maximal chain I_λ together with the distinguished edge as the boundary. We consider the distinguished edge to form both the left chain of the zero cell, O_C, and the right chain of the one cell, I_C. As such, the two exterior regions of the lattice described above are respectively the cells O_C and I_C. Every other cell of the lattice represents a unique region of the corresponding graph. In this manner, we have a bijective relation between the elementary cells of L and the regions of $g(L)$.

Let $g^*(L)$ be the dual graph of $g(L)$, and let D^* be the corresponding edge in $g^*(L)$ of the distinguished edge D in $g(L)$. By Lemma 1.14, we know that for every edge E, there are unique cells C and D such that $E \subset C_R$ and $E \subset D_L$. We therefore apply the following orientation to $g^*(L)$. Every dual edge E^* of $g^*(L)$ is oriented towards the cell of $g(L)$ which contains E on its left chain (see figure 2.1).

We now discuss some properties of $g^*(L)$.

LEMMA 2.5. *$g^*(L)$ contains no loops.*

PROOF. As the distinguished edge D represents the covering relation $1 \prec 0$, every element of L (including 0 and 1) has both an upper and a lower cover in the extended covering graph. Therefore every element is incident with at least two edges, one directed *to* that element and one directed *from* that element. So $g^*(L)$ has no loops. $\square$

LEMMA 2.6. *Every cycle in $g^*(L)$ contains the edge D^*.*

PROOF. Let $\mathcal{C}$ be a cycle in $g^*(L)$. So

$$\mathcal{C} = \quad c_0^* \to c_1^* \to \cdots \to c_{n-1}^* \to c_0^*.$$

As each element of $g^*(L)$ represents a region in $g(L)$ and hence an elementary cell of L, we obtain a sequence of cells of L:

$$C_0, C_1, \ldots, C_{n-1}, C_n = C_0$$

such that for all $i = 0$ to n, the right chain of C_i shares an edge E_i with the left chain of C_{i+1}. If E_i is the distinguished edge, then $C_i = I_C$ and $C_{i+1} = O_C$; otherwise, $C_{i-1} < C_i$. Therefore, we know one of the edges used in the cycle must be the distinguished edge, as otherwise

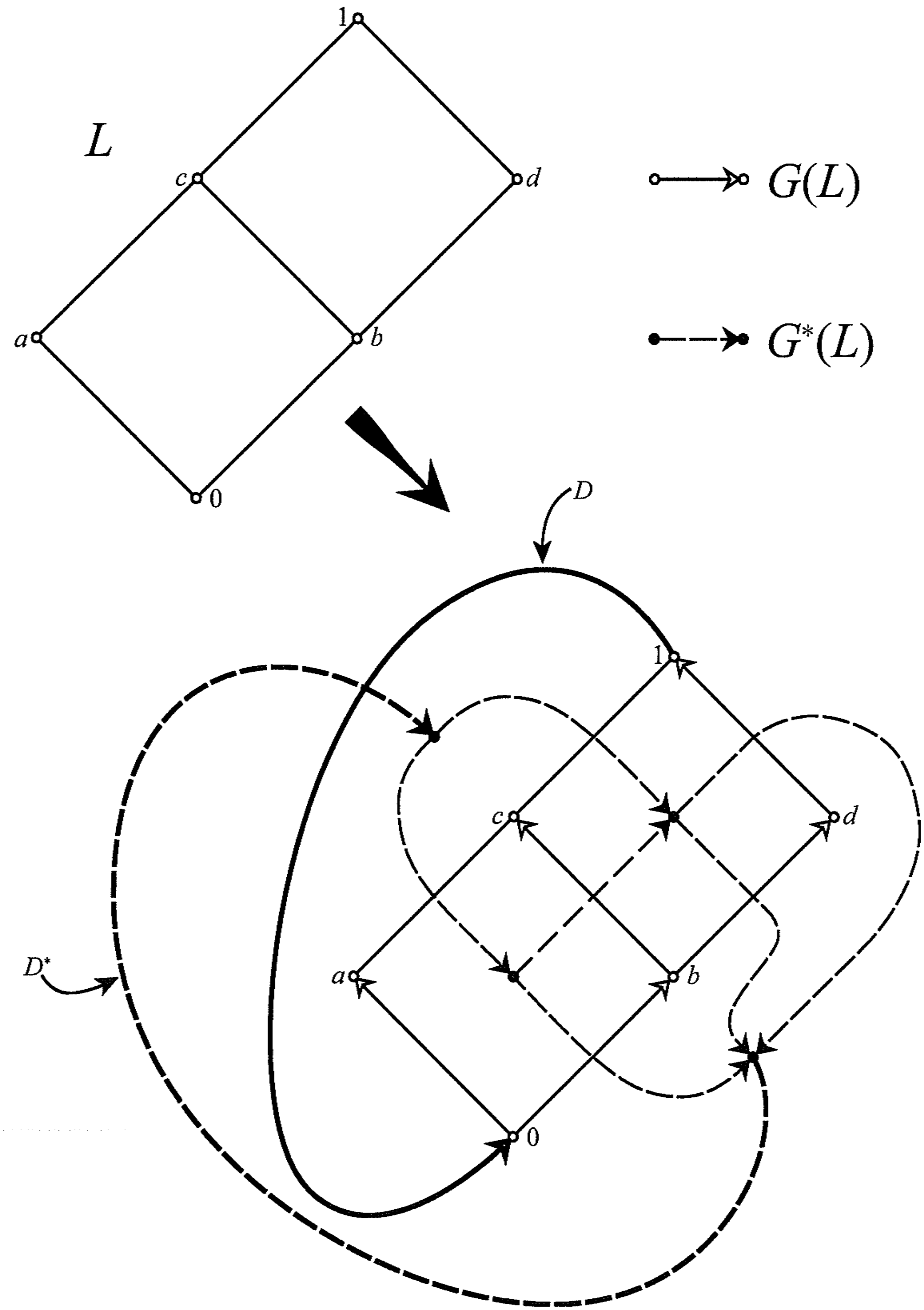

FIGURE 2.1. The extended covering graph of a lattice and the associated dual graph.

the partial order of $\mathrm{Ce}\,L$ would contain a cycle. So every cycle in $g^*(L)$ must use the dual distinguished edge, or equivalently, $g^*(L) \setminus \{D^*\}$ contains no cycles. □

LEMMA 2.7. *Let $x \in g^*(L)$. Then there exists a cycle $\mathcal{C}$ such that $x \in \mathcal{C}$.*

PROOF. Let A and B be two elementary cells such that $A \prec_C B$. By Corollary 1.17, we know A_R and B_L must have at least one edge in common. So the corresponding elements a^* and b^* are connected by an edge from a^* to b^* in $g^*(L)$.

Let $\mathcal{C}$ be a maximal chain in $\mathrm{Ce}\,L$. So

$$\mathcal{C} = O_C \prec_C C_1 \prec_C \cdots \prec_C C_{n-1} \prec_C I_C$$

As each covering relation represents an edge in $g^*(L)$, we obtain a path:

$$O_C \to C_1 \to \cdots \to C_{n-1} \to I_C.$$

With the addition of the distinguished edge, we obtain a cycle in $g^*(L)$. As every element of $\mathrm{Ce}\,L$ is on a maximal chain, we know that every element of $g^*(L)$ is on a cycle. □

So $g^*(L)$ is an planar digraph which satisfies (1), (2), and (3) of Corollary 2.4. However, $g^*(L)$ is not a simple graph and it may not be strongly antitransitive either. Clearly, $g^*(L)$ contains multiple edges if and only if there are two regions of $g(L)$ which share a border of more than one edge. But $g(L)$ was obtained by a planar representation of the lattice L, and every finite planar lattice is dismantlable (see [**6**]). Therefore L contains a doubly-irreducible element x which implies the cells to either side of x share both edges connected to x. Therefore for every lattice L with greater than one element, $g^*(L)$ will have multiple edges.

Although $g^*(L)$ does not satisfy Corollary 2.4 and hence is not the extended covering graph of a planar lattice, we can obtain planar lattices from it in two different ways. We can either find the largest subgraph of $g^*(L)$ which satisfies Corollary 2.4, or we can "add" new elements onto the middle of the problem edges (those edges which are either multiple edges, or which disrupt the antitransivity of $g^*(L)$) to obtain a larger graph which satisfies Corollary 2.4.

The first way is to use the acyclic digraph $g^*(L) \setminus D^*$ to impose a partial order on elements of $g^*(L)$. For $x, y \in g^*(L)$, we set $x < y$ iff there is a *directed walk* from x to y in $g^*(L) \setminus D^*$ (that is, if and only if there is a sequence of vertices $x = a_0, a_1, \ldots, a_n = y$ such that for each i, a_i and a_{i+1} are connected by an edge from a_i to a_{i+1}). This technique to induce a partial order from an acyclic digraph will be called the *directed walk partial order* of a digraph. In this case, the poset we obtained will be called L^*.

Let $x, y \in L^*$ such that $x \prec y$. Then $x < y$ and there is no z such that $x < z < y$. So in $g^*(L)$, there is a directed walk from x to y, and no such walk can pass through any other element along the way. Therefore there is an edge from x to y and so the covering graph of L^* is a subgraph of $g^*(L)$.

LEMMA 2.8. $L^* \cong \mathrm{Ce}\, L$

PROOF. Let $a, b \in \operatorname{Ce} L$ such that $a \prec b$. By Corollary 1.17, there exists an edge $E = x \prec y$ in L such that $E \subset A_R \cap B_L$. Therefore, in the graph $g(L)$, the regions A and B share the edge E, which lies on the left chain of B. So in $g^*(L)$, the edge E^* points from a^* to b^*, so there is a directed walk from a^* to b^* in $g^*(L)$. Therefore, $a^* < b^*$ in L^*.

Conversely, let $a^*, b^* \in L^*$ such that $a^* \prec b^*$. Then there is an edge E^* from a^* to b^* in $g^*(L)$. So the cells A and B share the edge E in L, and since E^* points from a^* to b^*, we know $E \subset A_R \cap B_L$. Therefore $a < b$ in $\operatorname{Ce} L$.

Therefore $\operatorname{Ce} L \cong L^*$. □

So the directed walk partial order of $g(L) \setminus D^*$ is the cell lattice of L. Or equivalently, the extended covering graph of $\operatorname{Ce} L$ is a maximal subgraph of $g^*(L)$ which spans $g^*(L)$ and is strongly antitransitive (maximal in the sense that the addition of any edge of $g^*(L)$ not already in $\operatorname{Ce} L$ will either create a multiple edge or will disrupt the strong anritransitivity of $\operatorname{Ce} L$).

The second approach is to "add" elements onto the middle of some edges of $g^*(L)$ to obtain a graph which satisfies Corollary 2.4 (and hence is the extended covering graph of a poset). By adding elements to an edge, we mean replacing an edge $p \to q$ with a chain of elements

$$p(= r_0) \to r_1 \to \cdots \to r_{n-1} \to q(= r_n)$$

where each r_i is adjacent only with the two edges $\langle r_{i-1}, r_i \rangle$ and $\langle r_i, r_{i+1} \rangle$ in the graph, for $i = 1$ to $n - 1$. Clearly, we obtain an infinite family of possible posets (dependant on the number of elements added to each edge). Adding a single element to an edge is called a subdivision of that edge. Two graphs are *homeomorphic* if they can be obtained from a common graph by a sequence of subdivision by lines (see, for example [**3**], page 107). Therefore, this technique will give us an infinite family of posets whose extended covering graphs are homeomorphic to $g^*(L)$.

The set of all planar lattices K whose extended covering graphs $g(K)$ are homeomorphic to $g^*(L)$ will be denoted $\mathcal{G}^*(L)$. So we are working with all graphs which are homeomorphic to $g^*(L)$ and which satisfy Corollary 2.4.

THEOREM 2.9. *If $K \in \mathcal{G}^*(L)$ then K is a planar lattice such that* $\operatorname{Ce} K \cong L$.

We begin by proving a Lemma:

LEMMA 2.10. *Let H and G be two planar graphs. Then the following are equivalent:*

(1) *G and H are homeomorphic*
(2) *The simple graphs obtained from G^* and H^* by identifying all multiple edges between pairs of vertices with single edges are isomorphic.*

PROOF. Trivial. We simply observe that, given a graph G and an edge E of G, the subdivision of E into E_0, E_1 merely replaces the edge E^* in G^* by the two edges E_0^* and E_1^*, both of which connect the same two elements as did E. □

With this Lemma, we can prove Theorem 2.9.

PROOF. Let $K \in \mathcal{G}^*(L)$. Then K is a planar lattice whose extended covering graph $g(K)$ is homeomorphic to $g^*(L)$. Let D_K be the distinguished edge in K. If there are more than one possible edges eligible to be the distinguished edge, we choose one of the edges in K which correspond to the dual distinguished edge D^* in $g^*(L)$ by a series of subdivision of lines.

By Lemma 2.10, the simple graphs obtained by $g^*(K)$ and $g(L)$ by identifying all multiple edges between pairs of vertices with single edges are isomorphic. In particular, the underlying set of elements is invariant between $g^*(K)$ and $g(L)$. So for two elements x and y in the underlying set, there exists (at least) one edge from x to y in $g^*(K)$ if and only if there exists (at least) one edge from x to y in $g(L)$. So the directed walk poset of $g(L) \setminus D$ is isomorphic to the poset induced by the directed edges of $g^*(K) \setminus D_K^*$. In other words, $L \cong \operatorname{Ce} K$, as desired. $\square$

The easiest way to construct a homeomorphic image of $g^*(L)$ which satisfies Corollary 2.4 is to subdivide each edge in $g^*(L) \setminus D^*$ exactly once. Doing this, we obtain the graph of a lattice with elements corresponding to the cells and edges of $e(L)$. That is, we obtain the lattice K, constructed in section 4.

Notice that although any edge in $g^*(L)$ *can* be subdivided, there are certain edges which *must* be subdivided in order to create a graph which satisfies Corollary 2.4 and hence represents a lattice in $\mathcal{G}^*(L)$. We know $g^*(L)$ is not a simple graph and also may not be strongly antitransitive. So there are two types of edges which must be subdivided in $g^*(L)$:

- An edge E which disrupts the strong antitransitivity of $g^*(L)$ – that is, an edge $E = \langle a, b\rangle$ for which there exists a path of length greater than two from a to b in $g^*(L) \setminus \{E\}$
- A multiple edge – that is, an edge $E = \langle a, b\rangle$ for which there exists a path of length two from a to b in $g^*(L) \setminus \{E\}$

Combining the above two cases, an edge $E = \langle a, b\rangle$ in $g^*(L)$ must be subdivided iff there exists a path from a to b in $g^*(L) \setminus \{E\}$.

We can partially order the lattices in $\mathcal{G}^*(L)$ by the number of elements in each lattice. With this partial order, $\mathcal{G}^*(L)$ will have a least member $\mathrm{C}^* L = \bigwedge \mathcal{G}^*(L)$ whose extended covering graph is obtained from $g^*(L)$ by only subdividing the edges which must be subdivided, doing so exactly once in each case. Working backwards now: $E^* = \langle A, B\rangle$ is an edge which is not subdivided in transforming $g^*(L)$ to $\mathrm{C}^* L$ iff there is no path from A to B in $g^*(L) \setminus \{E^*\}$. That is, if and only if $E = x \prec y$ is an edge in $g(L)$ such that $\{E\} = A_R \cap B_L$ where A and B are elementary cells in L such that $A \prec_C B$. By Lemma 1.16, we know $A_R \subset B^-$ and $B_L \subset A^+$. Also, since A and B are comparable, we have $1_A \nleq 0_B$ and $1_B \nleq 0_A$. Therefore $1_A > 0_B$ and $1_B > 0_A$. As A and B share only one edge, there are only two possibilities for E: either $E = 0_A \prec 1_B$ or $E = 0_B \prec 1_A$.

So E^* is not subdivided in obtaining $\mathrm{C}^* L$ iff $E = 0_A \prec 1_B$ for two arbitrary cells A and B of L. Equivalently, this occurs if $E = a \prec b$ and there exist elements $c, d \in L$ where $c \neq b$ and $d \neq b$ such that $a \prec c$ and $d \prec b$.

DEFINITION 2.11. An edge $E = a \prec b$ in L is called a *prunable edge* iff $L \setminus \{E\}$ is a lattice.

LEMMA 2.12. *Let L be a planar lattice and let $a \prec b \subset L$. Then $a \prec b$ is prunable iff there exist $c, d \in L$ where $c \neq b$ and $d \neq b$ such that $a \prec c$ and $d \prec b$.*

PROOF. Assume there exist $c, d \in L$ such that $a \prec c$ and $d \prec b$. Let $K = L \setminus \{a \prec b\}$. Now $a \prec c$ and $c \neq b$, so $b \neq 1$. As $c > a$, the covering relation $a \prec b$ does not occur in any covering chain from c to 1. Therefore, for all $x \in L$ such that $x \neq 1$, either $a > x$ or not. In either case, there is a covering chain from x to 1. Therefore for all $x \in K$, $x \leq 1$. Similarly, since $a \neq 0$, we can show that for all $x \in K$, $x \geq 0$.

As L is planar, there is a left-right order λ on L. K was obtained from L by removing an edge, so there are more incomparable elements in K than in L, namely those elements $x < y$ in L where every covering chain from x to y must use the edge $a \prec b$. We can extend λ to include these elements by setting $x \lambda y$ iff $a \lambda d$. Therefore K is planar, and as it has a zero and a one element, K is a lattice.

Conversely, assume d does not exist. Then a is a maximal element in K. But $a \prec b$ in L implies that $a \neq 1$. Therefore $a \vee 1$ does not exist in K. Similarly, if c does not exist then $b \wedge 0$ does not exist. In either case, K is not a lattice. $\square$

Therefore, given a lattice L, we can construct $\mathrm{C}^* L$ by subdividing the dual edges of all non-prunable edges of L. Using the terminology in

Section 4, $\mathrm{C}^* L$ is the lattice of all cells and non-prunable edges, with the partial order: for $x, y \in \mathrm{C}^* L$, $x \leq y$ if there exist $a \in X$ and $b \in Y$ such that $a \lambda b$. Although $\mathrm{C}^* L$ is the least member of $\mathcal{G}^*(L)$, we have not established that this lattice is a lower bound for all lattices having L as a cell lattice.

For a given lattice L with a fixed planar representation $e(L)$, let $\mathcal{C}^*(L)$ be the set of all lattices K with a planar representation $e(K)$ such that $L \cong \mathrm{Ce}\, K$. By Theorem 2.9, we know that $\mathcal{G}^*(L) \subset \mathcal{C}^*(L)$. However, it is not necessarily true that $\mathcal{G}^*(L) = \mathcal{C}^*(L)$. Given a poset P, there may be non-homeomorphic simple acyclic digraphs $G_0, G_1, \ldots$ with the directed walk partial order P (see figure 2.2). Let $e(P)$ denote the covering digraph of P and define the *partial order digraph* of P to be $G(P)$, where $G(P)$ has the elements of P as elements, and for $x, y \in P$ there is an edge from x to y iff $x < y$. The following proposition will allow us to characterize $\mathcal{C}^*(L)$.

PROPOSITION 2.13. *Let G be an acyclic digraph and P be a poset. Let G' be the simple graph obtained from G by identifying all multiple edges between pairs of vertices with single edges. Then the following are equivalent:*

(1) *P is the directed walk partial order of G*
(2) $e(P) \subseteq G' \subseteq G(P)$.

PROOF. Let Q directed walk poset of G. We note that G' and G have the same directed walk partial order, so without loss of generality,

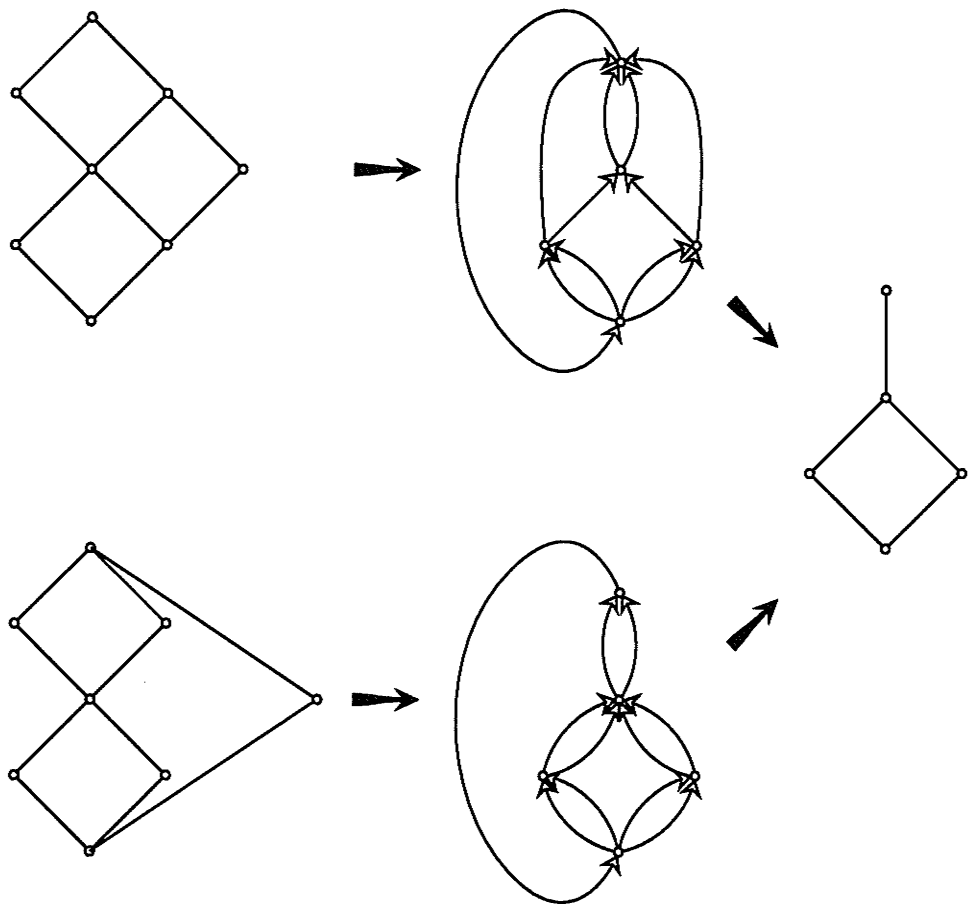

FIGURE 2.2. Two lattices whose extended covering graphs are non-homeomorphic but have the same cell lattice.

assume G is a simple graph (so $G = G'$). We also note that since $V(e(P)) = V(G(P))$, we have $V(G) = V(e(P))$.

Assume $e(P) \subseteq G \subseteq G(P)$. For $a, b \in V(G)$, if there is a directed walk from a to b in $e(P)$ then there is a directed walk from a to b in G (because $e(P) \subseteq G$). Therefore the directed walk poset of $e(P)$ is a subposet of Q. Similarly Q is a subposet of the directed walk poset of $G(P)$. But the directed walk poset of $e(P)$ and $G(P)$ is P in both cases. So $P \subset Q$ and $Q \subset P$, therefore $P = Q$ as desired.

Now assume $e(P)$ is not a subgraph of G. Then there exists an edge $\langle a, b\rangle \in E(e(P))$ which does not occur in G. So $a \prec b$ in P. However, in G, either there is no path from a to b, in which case $a \not< b$ in Q, or there is a path from a to b in G of length greater than 2, so there exists an element $c \in Q$ such that $a < c < b$. In either case, we know $Q \ncong P$.

If G is not a subgraph of $G(P)$ then there exists an edge $\langle a, b\rangle \in G$ which is not an edge in $G(P)$. Since $\langle a, b\rangle$ is not an edge in $G(P)$, we know $a \not< b$ in P, but $a < b$ in Q. Therefore $Q \ncong P$. $\square$

We can now characterize all lattices K which are members of $\mathcal{C}^*(L)$. For a planar lattice K, let $g_\wedge(K)$ be the homeomorphic graph of $g(K)$ obtained by removing all doubly-irreducible elements of K (that is, of all graphs homeomorphic to $g(K)$, $g_\wedge(K)$ is the graph with the smallest vertex set).

THEOREM 2.14. *Let L be a finite planar lattice. For a planar lattice K, the following are equivalent:*

(1) $K \in \mathcal{C}^*(L)$
(2) $g(L) \subseteq g^*_\wedge(K) \subseteq \big(G(L) \cup \langle 1, 0\rangle\big)$.

PROOF. Combine Lemma 2.10 with Proposition 2.13. $\square$

CHAPTER 3

A New Lattice Construction

1. Limitations of the Cell Lattice

Although useful for describing the structure of the maximal chains in a planar lattice, the cell lattice has its limitations. For a given lattice L, different planar representations may give rise to different left-right orders and hence to different cell structures. Therefore there can be several lattices $L_0, L_1, \ldots, L_n$ such that $K \in \mathcal{C}^*(L_i)$ for $i = 0$ to n. Furthermore, we can find lattices K_0 and K_1 such that $K_0 \in \mathcal{C}^*(L_0)$ and $K_1 \in \mathcal{C}^*(L_1)$ but such that $K_0 \notin \mathcal{C}^*(K_1)$ and $K_1 \notin \mathcal{C}^*(K_0)$ (see figure 3.1). Therefore, the relation $K \equiv L[\Theta]$ iff $\operatorname{Ce} K = \operatorname{Ce} L$ is not an equivalence relation on the class, **Pl**, of all finite planar lattices.

Furthermore, the cell lattice is only defined for finite planar lattices; there is no analogous lattice construction for non-planar lattices.

In this chapter, we obtain a lattice construction related to the cell lattice which addresses all these limitations.

2. An Alternate Description of $\operatorname{Ce}^2 L$

Let L be a finite planar lattice. We will write $\operatorname{Ce}^2 L$ for the cell lattice of the cell lattice of L, where $\operatorname{Ce} L$ has the left-right order induced by L (the left-right order induced by L is defined on page 17). That is, we let $\operatorname{Ce}^2 L = \operatorname{Ce}(\operatorname{Ce} L)$. In this section, we study $\operatorname{Ce}^2 L$.

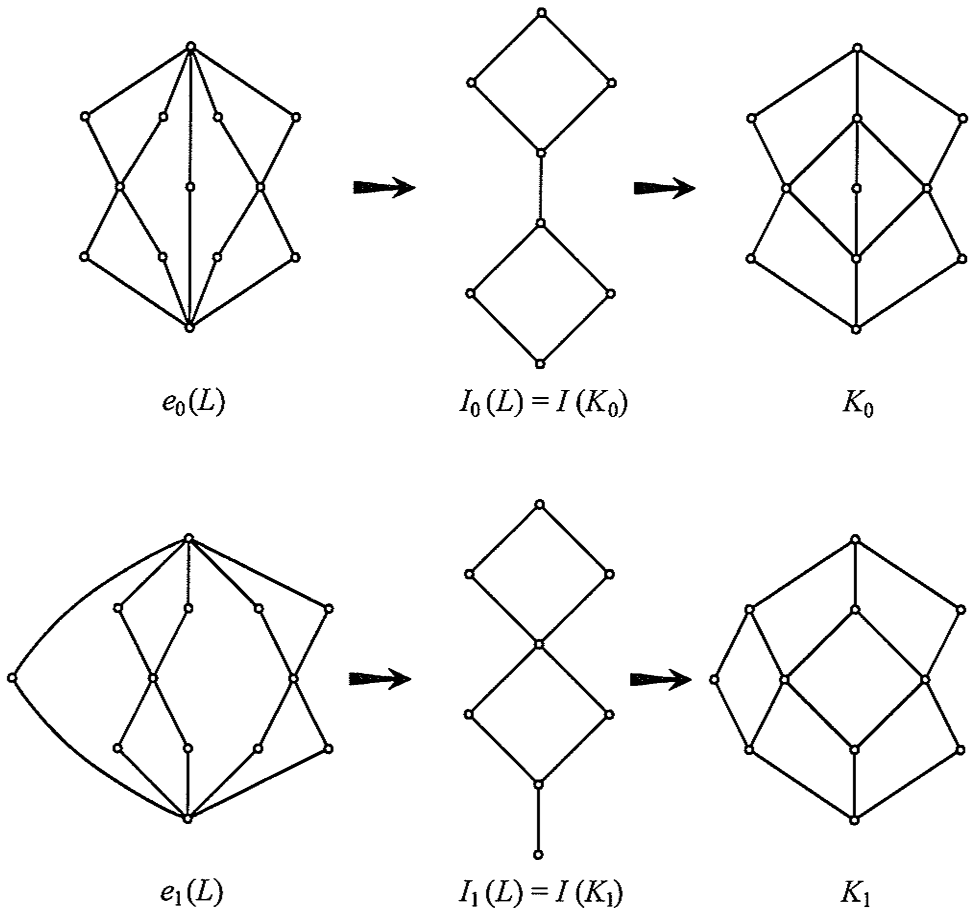

FIGURE 3.1. Different planar representations of a lattice giving rise to two non-isomorphic cell structures

We begin by defining a congruence relation, Φ on L. For $a, b \in L$ with $a \prec b$:

(3.1) $\quad a \equiv b\ [\Phi]$ iff a is meet-irreducible and b is join-irreducible.

It is useful to observe that Φ does not depend on the planarity of L.

The following proposition proves that Φ does in fact describe a congruence relation on L. For $a, b \in L$, the *principle* congruence relation $\Theta(a,b)$ of a and b is the least congruence relation on L such that $a \equiv b\ [\Theta(a,b)]$.

PROPOSITION 3.1. *Let $a, b \in L$ with $a \prec b$. Then the following are equivalent:*

(1) *a is meet-irreducible and b is join-irreducible,*

(2) *$a \prec b$ is the unique edge of L which is collapsed under $\Theta(a,b)$,*

(3) *For $C \in \mathrm{Ce}\, L$*
   - *$a \in C_l$ if and only $b \in C_l$ and*
   - *$a \in C_r$ if and only $b \in C_r$.*

PROOF. 1 implies 2: Let $a \in \mathrm{J}(L)$ and $b \in \mathrm{M}(L)$ and let $x \in L$ with $x \neq a$ and $x \neq b$. Then $x \wedge b < b$ so $x \wedge b \leq a$. But $x \neq a$, therefore $x \wedge b < a$, and hence $x \wedge b = x \wedge a$. Dually, $x \vee a = x \vee b$.

2 implies 1: Assume b is join-reducible. Then there exists $c \in L$ such that $c \prec b$ and $c \neq a$. So $c \wedge b = c$ but $c \wedge a \neq c$. Therefore $c \wedge a \equiv c\ [\Theta(a,b)]$.

Similarly, if a is meet-reducible then we can find an edge $x \prec y$ such that $x \equiv y\ [\Theta(a,b)]$.

1 implies 3: Let a be meet-irreducible and let b be join-irreducible. Let $C, D \in \mathrm{Ce}\, L$ be the unique cells such that $a \prec b \subset C_R$ and $a \prec b \subset D_L$ (by Lemma 1.14). Since a is meet-irreducible, a cannot be the zero element of a cell. Similarly, b cannot be the one-element of a

cell. Therefore $a \prec b \subset C_r$ and so both $a \in C_r$ and $b \in C_r$. Similarly, $a \prec b \subset D_l$.

3 implies 1: Trivial. $\square$

The congruence relation, Φ will be called the *cell congruence relation* on L because of part 3 of Proposition 3.1. By using the results from Proposition 3.1, we can rewrite equation 3.1 as:

$$\begin{aligned} \Phi &= \bigvee \left\{\Theta(a,b) \mid a \prec b, a \text{ meet-irreducible}, b \text{ join-irreducible}\right\} \\ &= \bigvee \left\{\Theta(a,b) \mid a \prec b \text{ is the unique edge collapsed by } \Theta(a,b)\right\} \qquad (3.2) \end{aligned}$$

So every congruence class of Φ consists of either isolated elements:

$$[x]\Phi = \{x\},$$

or of a chain of elements:

$$[x]\Phi = \{x_0 \prec \cdots \prec x_n \mid n \geq 1\}$$

where each x_i is meet-irreducible for $0 \leq i \leq n-1$ and each x_j is join-irreducible for $1 \leq j \leq n$.

LEMMA 3.2. *Let $x \in L$ and let $[x]\Phi = \{x_0 \prec \cdots \prec x_n \mid n \geq 0\}$. Then we have:*

- *$[x]\Phi$ is join-reducible in L/Φ iff x_0 is join-reducible in L,*
- *$[x]\Phi$ is meet-reducible in L/Φ iff x_n is meet-reducible in L.*

PROOF. Trivial. If x_0 is join-reducible, then there exist $a, b \in L$ with $a \neq b$ such that $a \prec x_0$ and $b \prec x_0$. Also, since x_0 is join-reducible, we know $[a]\Phi \neq [x]\Phi$ and $[b]\Phi \neq [x]\Phi$. But $a \,||\, b$, so $[a]\Phi \neq [b]\Phi$. Finally $a \vee b = x_0$ implies $[a]\Phi \vee [b]\Phi = [x_0]\Phi = [x]\Phi$, so $[x]\Phi$ is join-reducible.

Conversely, let $[x]\Phi$ be join-reducible. Then there exist $a, b \in L$ such that $[a]\Phi \neq [x]\Phi$ and $[b]\Phi \neq [x]\Phi$ but $[a]\Phi \vee [b]\Phi = [x]\Phi$. Therefore $a \vee b \in [x]\Phi$, and as x_i is join-irreducible for $1 \leq i \leq n$, we conclude $a \vee b = x_0$. Therefore x_0 is join-reducible.

The case for meet-irreducibility follows by duality. $\square$

We can now describe $\mathrm{Ce}^2 L$.

THEOREM 3.3. *Let P be the poset of doubly-reducible elements of L/Φ and let $P_0^1 = P \cup \{0_P, 1_P\}$ where $0_P, 1_P \notin P$ are added least and greatest elements respectively. Then $P_0^1 \cong \mathrm{Ce}^2 L$.*

PROOF. $\mathrm{Ce}^2 L$ is the cell lattice of $\mathrm{Ce}\, L$, where $\mathrm{Ce}\, L$ has the natural left-right order induced by L. Therefore, for every $x \in \mathrm{Ce}^2 L$ there is a corresponding cell X in $\mathrm{Ce}\, L$. We will restrict our attention to $\mathrm{Ce}^2 L \setminus \{0, 1\}$. So

$$X = \{0_X, a_0, \ldots, a_m, b_0, \ldots, b_n, 1_X\},$$

with the following left and right chains:

$$X_L = 0_X \prec a_0 \prec \cdots \prec a_m \prec 1_X$$

$$X_R = 0_X \prec b_0 \prec \cdots \prec b_n \prec 1_X.$$

The elements of X are elements of $\mathrm{Ce}\, L$, hence are elementary cells of L. Let A_i, B_j, I_X and O_X be the cells in L corresponding to a_i, b_j, 1_X and 0_X respectively (for $0 \leq i \leq m$ and $0 \leq j \leq n$).

By Corollary 1.17, A_i and A_{i+1} share an edge for each i, as do B_j and B_{j+1} for each j. Also, by Lemma 1.16,

- $(A_0)_L \subset (O_X)^+$ and $(B_0)_L \subset (O_X)^+$,

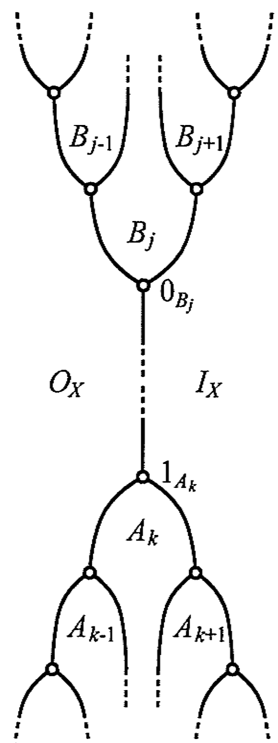

FIGURE 3.2. The cells of L which form the elements of X in $\mathrm{Ce}\, L$

- $(A_m)_R \subset (I_X)^-$ and $(B_n)_R \subset (I_X)^-$.

CLAIM. There is a $k \leq m$ such that, for all $i \leq m$, $1_{A_i} \leq 1_{A_k}$, that is, 1_{A_k} is maximal in $\mathcal{A}(X) = \{1_A \mid a \in X_l\}$. Similarly, $\mathcal{B}(X) = \{0_B \mid b \in X_r\}$ has a minimal element; call it 0_{B_j} (see figure 3.2).

Proof of Claim. We will prove that $\max \mathcal{A}(X)$ exists; the existence of $\min \mathcal{B}(X)$ is assured by duality. So assume $\max \mathcal{A}(X)$ does not exist.

Then there exist $1 \leq s < t \leq m$ such that:

$$1_{A_s} \nleq 1_{A_i} \text{ and } 1_{A_t} \nleq 1_{A_i} \text{ for all } i \leq n. \tag{3.3}$$

In particular, $1_{A_s} \nleq 1_{A_t}$ and $1_{A_t} \nleq 1_{A_s}$, so $1_{A_s} \,||\, 1_{A_t}$. Since $s < t$, we know $1_{A_s} \lambda\, 1_{A_t}$. By Lemma 1.14, there is a unique cell $C \subset L$ such that $1_{A_s} \in C_l$. Let c be the corresponding element in $\mathrm{Ce}\, L$. We claim that c lies in the interior of X, contradicting the irreducibility of X.

- c is to the right of X_L.

  Proof. We know $1_{A_s} \in C_l$, so $a_s < c$ in $\mathrm{Ce}\, L$. Also $1_{A_s} \lambda\, 1_{A_t}$ implies $c < a_t$. But $1_{A_s} < 1_C$, so by 3.3, $c \notin X_L$. As $1_C > 1_{A_s}$ and X_L is a covering chain, we can conclude there is an i with $s < i < t$ such that $a_i \lambda\, c$ in $\mathrm{Ce}\, L$.

- $0_X < c < 1_X$.

  Proof. We have already shown that $a_s < c < a_t$.

- c is to the left of X_R.

  Proof. X is a cell in $\mathrm{Ce}\, L$, so $a_i \vee b_j = 1_X$ and $a_i \vee b_j = 1_X$ for all $i \leq m$ and $j \leq n$. But $a_s < c < a_t$ implies $b_j \wedge c = 0_X$ and $b_j \vee c = 1_X$ for all j. Therefore $c \,||\, b_j$, and as $1_{A_s} \in C$ and $a_s \lambda\, b_j$ for all $j \leq n$, we conclude $c \lambda\, b_j$.

Thus X is not an elementary cell – a contradiction. So $\max \mathcal{A}(X)$ exists. ■

Let $1_A = \max \mathcal{A}(X)$ and $0_B = \min \mathcal{B}(X)$. Then $1_A \leq 0_B$, and as X is an elementary cell in $\mathrm{Ce}\, L$, the convex sublattice $[1_A, 0_B] \subset L$

must be a chain (possibly of length 0, if $1_A = 0_B$). Therefore $[1_A, 0_B]$ is collapsed to a single element under Φ, and as 1_A is join-reducible and 0_B is meet-reducible,

$$[1_A]\Phi = [0_B]\Phi = [1_A, 0_B].$$

Define the map $\psi : \mathrm{Ce}^2 L \setminus \{0,1\} \longrightarrow P$ by

$$x\psi = [1_A]\Phi$$

where $x \in \mathrm{Ce}^2 L$ corresponds to the elementary cell $X \in \mathrm{Ce}\, L$ and $1_A = \max \mathcal{A}(X)$. We already know ψ is well-defined function by the first part of this proof.

Let $x \in P$. Then $x = [z]\Phi$ for some $z \in L$. Since $x \in P$, $[z]\Phi$ is doubly-reducible in L/Φ. Therefore $[z]\Phi = \{z_0 \prec \ldots \prec z_n \mid n \geq 0\}$ where z_0 is join-reducible and z_n is meet-reducible (Lemma 3.2). So there exist cells A and B such that $z_0 = 1_A$ and $z_n = 0_B$. If $n > 0$ then as z_i is join-irreducible for $1 \leq i \leq n$, there is no cell C such that $z_i = 1_C$. Similarly, there is no cell C such that $z_j = 0_C$ for all $0 \leq j \leq n-1$. Therefore there exist cells O and I such that $[z_0, z_n] \subset O_r$ and $[z_0, z_n] \subset I_l$. If we let $a, b, o, i \in \mathrm{Ce}\, L$ correspond to A, B, O, and I respectively, then we have shown that $o < a < i$, $o < b < i$, and $a \,\lambda\, b$. In fact, we know $a \wedge b = o$, since $1_A \in O_r$ and $0_B \in O_r$. Similarly, $a \vee b = i$. Finally, let $c \in \mathrm{Ce}\, L$ such that $a \,\lambda\, c$ and $c \,||\, b$. Then, for the corresponding cell $C \subset L$, $0_C \geq 1_A (= z_0)$. Since $c \,||\, b$, we know either $1_C \leq 0_B$ or $0_C \geq 1_B$. But $0_C \geq 1_A$ and $[1_A, 0_B]$ is a chain (hence contains no cells). Therefore $0_C \geq 1_B$, and hence $b \,\lambda\, c$. Therefore, there exists a unique cell $Z \subset \mathrm{Ce}\, L$ such that $\{0_Z, c, d, 1_Z\} \subset Z$, so ψ is onto.

Let $x, y \in \mathrm{Ce}^2 L$ such that $x\psi = y\psi$. Let X, Y be the cells in $\mathrm{Ce}\, L$ which correspond to x and y respectively and let $1_A = \max \mathcal{A}(X)$ and $1_B = \max \mathcal{A}(Y)$. So A and B are cells of L such that, for the corresponding elements $a, b \in \mathrm{Ce}\, L$, we have $a \in X_l$ and $b \in Y_l$ (by the definition of $\mathcal{A}(X)$). Now $x\psi = y\psi$ implies $[1_A]\Phi = [1_B]\Phi$. In particular, $1_A \in [1_B]\Phi$. But 1_A and 1_B are both join-reducible, so 1_A must be the least element in $[1_B\Phi$, that is, $1_A = 1_B$. Therefore $a \in Y_l$, and by Lemma 1.14, $Y = X$. Therefore $x = y$ and ψ is one-to-one and hence is a bijection.

Finally, let $x, y \in \mathrm{Ce}^2 L$ with corresponding cells X and Y of $\mathrm{Ce}\, L$, and let

$$x\psi = [1_A]\Phi = [1_A, 0_B]$$
$$y\psi = [1_C]\Phi = [1_C, 0_D],$$

where A, B, C, and D are cells of L. Then, for the corresponding elements a, b, c, d of $\mathrm{Ce}\, L$, we have $a \in X_l$, $b \in X_r$, $c \in Y_l$, and $d \in Y_r$.

Let $x < y$. Then there exist elements $e, f \in \mathrm{Ce}\, L$ such that $e \in X$, $f \in Y$, and $e \lambda f$ (Lemma 1.10). Therefore, for the corresponding cells $E, F \subset L$, $1_E \leq 0_F$. Furthermore, since $x \neq y$ we can assume that $e \in X_R$ and $f \in Y_L$. Therefore, either $e \in X_r$ of $e = 0_X$ or $e = 1_X$. In any case, $1_E > 0_B$. Similarly, $0_F < 1_C$. Therefore $0_B < 1_C$, so $x\psi < y\psi$.

Conversely, let $x\psi < y\psi$. Then $[1_A]\Phi < [1_C]\Phi$, and so $0_B < 1_C$. In particular, $1_A < 0_D$. Therefore $a \lambda d$, so by Lemma 1.10, $x \leq y$. But

$x\psi \neq y\psi$ and ψ is one-to-one, so $x < y$. Therefore ψ is an isomorphism from $\mathrm{Ce}^2 L \setminus \{0,1\}$ onto P.

If we expand ψ to map zero-element to zero-element and one-element to one-element, then we have an isomorphism from $\mathrm{Ce}^2 L$ onto P_0^1, completing the proof. $\square$

COROLLARY 3.4. $\mathrm{Ce}^2 L$ *is independent of planar representation* $e(L)$ *of* L.

3. Applications to General Lattices

In Section 2, we showed $\mathrm{Ce}^2 L$ can be obtained from L by the doubly-reducible elements of L/Φ, and hence can be obtained without any reference to the planarity of L. We can therefore expand the definition of $\mathrm{Ce}^2 L$ to include the class of all finite lattices.

DEFINITION 3.5. Let L be a finite lattice and let P be the poset of doubly-reducible elements of L/Φ, where Φ is the congruence relation on L described in equation 3.2. We define the *lattice of doubly reducible cell classes* of L to be $\mathrm{Re}\, L = P \cup \{0,1\}$ where $0, 1 \notin P$ and $0 < p < 1$ for all $p \in P$.

By Theorem 3.3, we know $\mathrm{Re}\, L = \mathrm{Ce}^2 L$ for all finite planar lattices L, and hence know that $\mathrm{Re}\, L$ is a lattice whenever L is planar. We must show that $\mathrm{Re}\, L$ is a lattice for all finite lattices.

THEOREM 3.6. $\mathrm{Re}\, L$ *is a lattice.*

PROOF. Let $a, b \in \mathrm{Re}\, L$. We must show that $a \vee b$ exists in $\mathrm{Re}\, L$; the existence of $a \wedge b$ follows by duality. Then $a = [x]\Phi$ and $b = [y]\Phi$

for some $x, y \in L$ and $[x]\Phi$ and $[y]\Phi$ are both doubly-reducible in L/Φ. From our work on Φ in Section 2, we have:

$$a = [x]\Phi = \{x_0 \prec \cdots \prec x_m \mid m \geq 0\}$$

and

$$b = [y]\Phi = \{y_0 \prec \cdots \prec y_n \mid n \geq 0\}$$

where $x_i, y_j \in L$ for $0 \leq i \leq m$, $0 \leq j \leq n$, and such that x_0 and y_0 are both join-reducible and x_m and y_n are both meet-reducible (Lemma 3.2). If $m > 0$ ($n > 0$), then we also have x_i (y_j) being join-irreducible for $1 \leq i \leq m$ ($1 \leq j \leq n$) and x_i (y_j) being meet-irreducible for $0 \leq i \leq m-1$ ($0 \leq j \leq n-1$).

Clearly, if $a \vee b$ exists in $\operatorname{Re} L$ then $a \vee b \geq [a_m \vee b_n]\Phi$. So look at $a_m \vee b_n$.If there are no meet-reducible elements x such that $x \geq a_m \vee b_n$, then by Lemma 3.2, there is no doubly-irreducible element $y \in L/\Phi$ such that $y \geq [A_m \vee b_n]\Phi$. So a and b have no common upper bound in P. Therefore $a \vee b = 1$ in $\operatorname{Re} L$.

On the other hand assume there exists at least one meet-reducible element greater than or equal to $a_m \vee b_n$ in L. Let

$$\mathcal{M} = \{x \geq a_m \vee b_n \mid x \text{ is meet-reducible}\}$$

As L is finite, $\mathcal{M}$ has a greatest lower bound, $d = \wedge\mathcal{M}$.

CLAIM. $d \in \mathcal{M}$ (Hence d is meet-reducible.)

Proof of Claim. If $d \notin \mathcal{M}$, then d is meet-irreducible. So d has a unique upper cover x. Since $d = \wedge\mathcal{M}$ and $d < x$, we know there exists a $y \in \mathcal{M}$ such that $d \leq y < x$. Therefore, $d = y$, a contradiction. ■

If $a_m \vee b_n \neq d$ then for all $x \in L$ such that $a_m \vee b_n \leq x < c$, x is meet-irreducible. Therefore $[a_m \vee b_n, d]$ forms a chain (possibly of length zero if $d = a_m \vee b_n$). Set $\mathcal{C} = [a_m \vee b_n, d]$ and let

$$\mathcal{J} = \{x \in \mathcal{C} \mid x \text{ is join-reducible in } L\}.$$

Then $\mathcal{J} \neq \emptyset$ because $a_m \vee b_n \in \mathcal{J}$. Also, $\mathcal{J} \subset \mathcal{C}$ implies $\mathcal{J}$ forms a chain of one or more elements. Since L is finite, $\mathcal{J}$ has a least upper bound, $c = \vee\mathcal{J}$. By a dual argument to that used for $\wedge\mathcal{M}$, we know $c \in \mathcal{J}$, and hence is join-reducible.

So we have $[c, d]$ forming a chain of one or more elements such that c is join-reducible and d is meet-reducible. Furthermore, if $c \neq d$ then we know every $x \in [c, d]$ with $x > c$ is join-irreducible, and every $y \in [c, d]$ with $y < d$ is meet-irreducible. Therefore $[c]\Phi = [c, d]$ and by Lemma 3.2, $[c]\Phi \in P$.

We already know that $[c]\Phi$ is an upper bound of a and b; we will show that $a \vee b = [c]\Phi$. Let $x \in \operatorname{Re} L$ such that $x > a$ and $x > b$. If $x = 1$, then $x > [c]\Phi$ and we are done. Otherwise, $x = \{x_0 \prec \cdots \prec x_k\}$ with $x_i \in L$ for $i = 0$ to k. As $x \in P$, we know x is doubly-reducible in L/Φ. Therefore x_k is meet-reducible. Furthermore, $x > a$ and $x > b$ together imply that $x_i \geq a_m \vee b_n$ for all i. In particular, $x_k \geq a_m \vee b_n$. Therefore $x_k \in \mathcal{M}$, so $d \leq x_k$ and we conclude $[c]\Phi \leq x$.

By duality, $a \wedge b$ exists. Therefore, $\operatorname{Re} L$ is a lattice. $\square$

Bibliography

1. G. Birkoff, *On the combination of subalgebras*, Proc. Cambridge Philos. Soc. **29** (1933), 441–464.
2. George Grätzer, *General lattice theory*, second ed., Birkhäuser Verlag, Basel, 1998, New appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille. MR **2000b:**06001
3. Frank Harary, *Graph theory*, Addison-Wesley Publishing Co., Inc., Philippines, 1969.
4. David Kelly and Ivan Rival, *Planar lattices*, Canad. J. Math. **27** (1975), no. 3, 636–665. MR 52 #2974
5. C. R. Platt, *Planar lattices and planar graphs*, J. Combinatorial Theory Ser. B **21** (1976), no. 1, 30–39. MR 55 #2683
6. Ivan Rival, *Lattices with doubly irreducible elements*, Canad. Math. Bull. **17** (1974), 91–95. MR 50 #12837
7. M. H. Stone, *The theory of representations for boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), 37–111.