SELF-IMPEDANCE OF A LINEAR DIPOLE USING THE REACTION CONCEPT AND NUMERICAL DOUBLE INTEGRATION

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Stanley Stephen Hodge

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ABSTRACT

The Reaction Concept of V. H. Rumsey, as extended to antennas by Roger F. Harrington, is used to determine the self-resistance and self-reactance of a linear dipole. Values are given for antenna half-lengths varying from 0.05λ to 1.1 λ, the half-length to radius ratio being 74.2 (corresponding to an Ω of 10). The calculations are done using two different current assumptions, one due to C.T. Tai and one suggested by R. F. Harrington, and the results are compared with values from the literature. The double integrals involved in the equations are solved numerically on an IBM 360/model 50 digital computer, using a double application of the Trapezoidal Rule. A single calculation yielding self-resistance and self-reactance for one particular length dipole antenna requires about 15 seconds of computer time on the model 50.

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CHAPTER I

INTRODUCTION

This thesis is intended primarily as a contribution to a much larger overall project, that of analyzing certain antenna arrays theoretically. The intention of the overall project is to find expressions for array characteristics, such as input impedance and radiation pattern, and then place these expressions into a digital computer. Design work could proceed by changing various antenna element lengths and spacings and observing overall array changes.

The first antenna to be analyzed is the log-periodic dipole antenna which is composed of a special array of parallel dipoles. However, greater directivity from log-periodic arrays can be obtained by using V-shaped elements and eventually an attempt will be made to analyze the log-periodic V-antenna.

A study of antenna theory showed that Hallen's integral equation method for obtaining an integral equation for the current distribution on antennas was already well accepted, and C.T. Tai's[1*] variational solution of the

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* The numeral denotes reference number as listed in bibliography

equation was known to yield good results for the self-impedance of dipoles. Furthermore, Levis and Tai[2] expressed confidence in the variational approach for yielding good results for the impedance parameters of two parallel dipoles of arbitrary lengths as well as for an array of "n" parallel dipoles of arbitrary lengths. The Reaction Concept promised to combine both the integral equation method and the variational solution into one direct approach. Indeed, Levis and Tai[2] say with regard to their equations for the two dipoles: "Finally we may mention that (these results) can be obtained by applying V. H. Rumsey's[4] Reaction Principle to our problem. This method indeed leads to the admittance result more directly, but was not chosen here because the concepts are not as widely known".

Thus the Reaction Concept seemed to offer a very realistic and practical method on which to base eventual array analysis and design.

Now, both the variational method and the Reaction Concept yield double integral equations and to solve them requires that current distributions be assumed on the antennas. Answers will be good only if good choices are made for the current approximations. For the dipole case, C. T. Tai's[1] current approximation works extremely well and also is such that the impedance equation can be obtained in closed form through analytical integration. However, it is to be expected that when more general antenna configurations are attacked great difficulties will likely be encountered in finding suitable current distributions that give both expressions in closed form and also good results. Furthermore, for each new configuration several different current approximations would probably have to be tried before finding a suitable one for the problem.

Thus it was decided to investigate the feasibility of solving the double integrals numerically on a digital computer. This eliminates the need for worrying about distributions leading to closed form expressions, and also, since similar program logic for the integration would apply regardless of the distribution assumed, different approximations could be tried with relative ease.

Such, then, is the nature of this thesis - a study of the Reaction Concept coupled with numerical double integration as a practical method of finding antenna impedances. The scope of the thesis is quite limited, being essentially a "first-order" investigation. One particular case only is studied - that of finding the self-impedance of a linear dipole. This case is chosen as it has been done using the variational method, and a good current approximation is known. However, it is felt that self-impedance will be more difficult to determine than mutual impedance since the integrands have a much more singular nature in the self-impedance case. Also, no attempt is made to evolve a sophisticated method of digital integration. Instead, a straightforward double application of the Trapezoidal Rule is used. A similar approach (Simpson's Rule) has been used by Baker and La Grone[3] to find mutual impedance between thin dipoles using an assumed sinusoidal current approximation. However, Baker and La Grone only performed single integration by digital means and only needed a single application of Simpson's Rule.

To indicate how the success of the Reaction method depends on the current assumption, the self-impedance is determined using two different current approximations. It is seen that one approximation yields good results only for the short and intermediate length antennas investigated.

This thesis, of course, has implications for any problem involving the evaluation of similar double integrals.

CHAPTER II

THE REACTION CONCEPT

In this chapter, the Reaction Concept is introduced and the method of utilizing it to determine self-impedance of an antenna is explained.

I BRIEF HISTORY OF ANTENNA THEORY

In order to gain perspective regarding the position of the Reaction Concept in antenna theory, a brief history of antenna theory is included here. A much more comprehensive history can be found forming the introduction to the book by R. W. P. King[7].

Although many different approaches were attempted over the years to analyze antennas theoretically, the earliest treatments of the cylindrical center-driven antenna as a boundary-value problem are those of L. V. King[19] in 1937 and E. Hallén[21] in 1938. Using essentially the retarded-potential method of Pocklington[23], Hallén derived an integral equation for the current distribution along the antenna. Hallén solved his integral equation by a method of iteration in reciprocal powers of a parameter $\Omega = 2 \ln \frac{2h}{a}$ where h is the half-length of the antenna and a is its radius. Impedances are easily found once the current is known. (A good explanation of Hallén's method can be found in the book by Kraus[27]).

In 1941, Schelkunoff[24] presented a different treatment-his so-called non-uniform transmission line method. His starting point was the thin biconical antenna, which he solved as a boundary value problem concentrating on the fields rather than the current distributions. To apply the biconical antenna solution to the cylindrical case, Schelkunoff used a perturbation method. The conical boundary was considered perturbed into the cylindrical shape, and the perturbed wave functions calculated. In his article, Schelkunoff also discussed the shortcomings of previous works and since that time much of the interest in cylindrical antennas has been centered on the difference between Schelkunoff's and Hallén's method.

Hallén[25] finally showed, in 1948, that the first order impedance formula derived from Schelkunoff's theory can also be obtained by a so-called $\mu\Omega_2$- expansion from the general solution of the integral equation method.

In 1946, King and Middleton[26] used a different expansion parameter in the iteration of Hallén's integral equation in order to achieve more rapid convergence. This is the so-called King-Middleton ψ-expansion.

Searching for a more practical way of solving Hallen's integral equation, Storer[9] in 1950 developed a variational formulation. This method involved modifying the integral equation so that a specified quantity such as impedance is insensitive to errors in an assumed trial function for the current along the antenna. In the language of the calculus of variations, the first variation in impedance with respect to the current is zero, so that the impedance is stationary with respect to small changes in the current. More simply, the impedance is an integral equation in the current; with the integral stationary with respect to first-order changes in the current it follows that if the trial function of current is a good approximation to the true current, the approximate impedance will be a still better approximation to the true value of impedance. The more accurate the trial current, the better the value of impedance obtained. The trial current is chosen by considering it to be a linear combination of functions each multiplied by an adjustable constant or variational parameter. The parameters are adjusted by the Rayleigh-Ritz procedure which requires the partial derivation of the impedance with respect to each parameter be zero.

Tai[1], also in 1950, improved on Storer's results by using a better approximation for the current distribution in the variational method.

Then, in 1954, Rumsey presented his Reaction Concept. This method provided a more general and more straightforward means of obtaining stationary equations than did the variational method. Whereas variational techniques to obtain stationary equations varied for each problem, the reaction method determined stationary equations directly from Maxwell's equations. The reaction method also assumed a trial function that was a linear combination of functions each multiplied by an adjustable constant. A quantity called the reaction was defined. By forcing the assumed source to have the same reaction with certain "test" sources as the correct source would have, the constants were evaluated. This procedure will be explained in more detail presently.

Comments on the Driving Conditions

The theoretical analyses for the cylindrical antennas assumed the antennas to be driven by a slice generator which maintained a discontinuity in scalar potential at the center of the antenna. Since such a driving source is not physically realizable, it might appear that the theoretical results are quite unrelated to practical antennas. Indeed, experiments readily showed that the impedance an antenna presented to a transmission line was greatly dependent on the line used and the orientation of the line with respect to the antenna. In other words, the theoretical analyses did not account for the coupling effects and boundary conditions introduced by practical feed systems. However, the theoretical analyses still proved to be important and useful. Between 1947 and 1950, King and Wintermitz[14], King and Tomiyasu[13], and E. O. Hartig[15] bridged the gap between experiment and theory. They measured the apparent impedance of a dipole using many different transmission lines and orientations. For each case, the impedance was measured for various spacings of the conductors of the transmission line as the spacing between the conductors was decreased. It was found that, regardless of the line or orientation used, if the measured values were extrapolated to zero line spacing the values so obtained were in complete agreement with the theoretical results of the King-Middleton expansion or Tai's variational formulation.

Thus, the theoretical impedance of the antenna driven by a slice generator across an infinitesimal gap is independent of the feed line, and dependent only upon the length and radius of the antenna. The theoretical results are useful, since the researchers mentioned in this section found that lumped, terminal-zone networks could be designed to transform ideal theoretical values into measurable apparent values for different transmission lines and connections.

Though Tai obtained his variational solution using a Dirac delta function for the driving source field, he mentioned that the same equation results by assuming an impressed field distributed over a very small but finite gap at the center of the antenna, and assuming that the current is practically constant within this gap. (The derivation is similar to that presented by Albert and Synge[28]).

Harrington utilized this idea by assuming his antenna driven by a constant current source in the gap - that is, by a short filament of impressed electric current across the gap. Essentially, the antenna is assumed driven by a current source in the circuit sense, which can be used in the field problem because the input region - the gap - is of dimensions small compared to a wavelength.

The current source is particularly convenient for the reaction method, and was adopted as the method of exciting the dipole for this thesis.

The theoretical results of the King-Middleton expansion and Tai's variational formulation are the best theoretical results yet calculated for the dipole self-impedance. Results based on Schelkunoff's first order theory are also considered by Tai to be good. The reaction concept as applied in this thesis leads to results almost identical to the results of Tai.

II RECIPROCITY AND REACTION

In this section, the development leading up to the definition of Reaction follows for the most part that presented by Harrington[6]. Extensions to the theory can be found in Richmond[18] and Rumsey[29]. Good references on reciprocity theorems include Crowley[5], Richmond[18,30] and Carson[16].

Consider two monochromatic a.c. sources, namely the volume distributions of electric current $\underline{J}^a$ and $\underline{J}^b$. Consider them to exist in the same linear medium. The fields due to $\underline{J}^a$ acting alone we denote as $\underline{E}^a$, $\underline{H}^a$ and the fields due to $\underline{J}^b$ acting alone we denote as $\underline{E}^b$, $\underline{H}^b$. The field equations, written in complex form, are:

$$\nabla \times \underline{H}^a = y\,\underline{E}^a + \underline{J}^a \tag{2.1}$$

$$-\nabla \times \underline{E}^a = z\,\underline{H}^a \tag{2.2}$$

and

$$\nabla \times \underline{H}^b = y\,\underline{E}^b + \underline{J}^b \tag{2.3}$$

$$-\nabla \times \underline{E}^b = z\,\underline{H}^b \tag{2.4}$$

where: $y = \sigma + j\omega\epsilon \quad ; \quad z = j\omega\mu$

Of course, y and z are functions of position in the sense that they can have different values at different points in space if various types of material are present. The important point is that y and z are the same functions of position when $\underline{J}^a$ is acting alone as when $\underline{J}^b$ is acting alone.

Next, scalarly multiply equation (2.1) by $\underline{E}^b$ and equation (2.4) by $\underline{H}^a$ and add the equations to obtain:

$$\underline{E}^b \cdot (\nabla \times \underline{H}^a) - \underline{H}^a \cdot (\nabla \times \underline{E}^b) = y\,\underline{E}^b \cdot \underline{E}^a + \underline{E}^b \cdot \underline{J}^a + z\,\underline{H}^a \cdot \underline{H}^b \qquad (2.5)$$

Use vector identity $\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot \nabla \times \underline{A} - \underline{A} \cdot \nabla \times \underline{B}$
to simplify equation (2.5) to :

$$-\nabla \cdot (\underline{E}^b \times \underline{H}^a) = y\,\underline{E}^b \cdot \underline{E}^a + \underline{E}^b \cdot \underline{J}^a + z\,\underline{H}^a \cdot \underline{H}^b \qquad (2.6)$$

Similarly, scalarly multiply equation (2.3) by $\underline{E}^a$, equation (2.2) by $\underline{H}^b$, add the two equations and simplify to obtain:

$$-\nabla \cdot (\underline{E}^a \times \underline{H}^b) = y\,\underline{E}^a \cdot \underline{E}^b + \underline{E}^a \cdot \underline{J}^b + z\,\underline{H}^b \cdot \underline{H}^a \qquad (2.7)$$

Subtract (2.6) from (2.7)

$$-\nabla \cdot (\underline{E}^a \times \underline{H}^b - \underline{E}^b \times \underline{H}^a) = \underline{E}^a \cdot \underline{J}^b - \underline{E}^b \cdot \underline{J}^a \qquad (2.8)$$

Integrate both sides throughout the region containing the sources, and apply the divergence theorem to the left-hand side to obtain.

$$-\oiint (\underline{E}^a \times \underline{H}^b - \underline{E}^b \times \underline{H}^a) \cdot d\underline{S} = \iiint_V (\underline{E}^a \cdot \underline{J}^b - \underline{E}^b \cdot \underline{J}^a)\, dV \qquad (2.9)$$

Where volume V is the region containing the sources, and S is a closed surface surrounding that region.

Next postulate that all sources and matter are finite in extent. Distant from the sources and matter, using spherical co-ordinates r, θ, ϕ :

$$E_\theta = \eta H_\phi \qquad\qquad E_\phi = -\eta H_\theta \tag{2.10}$$

where $\eta = \sqrt{\dfrac{\mu}{\epsilon}}$

Integrating the left-hand side of equation (2.9) over a sphere of radius $r \to \infty$ the sources appear as point sources giving:

$$-\eta \oint\!\!\oint \left(H_\theta^a H_\theta^b + H_\phi^a H_\phi^b - H_\theta^b H_\theta^a - H_\phi^b H_\phi^a \right) dS = 0 \tag{2.11}$$

So equation (2.9) reduces to the following reciprocity equation

$$\iiint \underline{E}^a \cdot \underline{J}^b \, dV = \iiint \underline{E}^b \cdot \underline{J}^a \, dV \tag{2.12}$$

where the integrations extend over all space but, of course, will have values only over regions where $\underline{J}^b$ and $\underline{J}^a$ exist.

The integrals of equation (2.12) are called reactions. By definition, then, reaction of field a on source b is:

$$\langle a, b \rangle = \iiint \underline{E}^{a} \cdot \underline{J}^{b} \, dV \tag{2.13}$$

In this notation, the reciprocity theorem expressed by equation (2.12) becomes:

$$\langle a, b \rangle = \langle b, a \rangle \tag{2.14}$$

That is, the reaction of field a on source b is equal to the reaction of field "b" on source a.

Because the field equations are linear, the following two identities hold true.

$$\langle a, b+c \rangle = \langle a, b \rangle + \langle a, c \rangle \tag{2.15}$$

$$\langle Aa, b \rangle = A \langle a, b \rangle \tag{2.16}$$

where A represents any scalar and Aa represents the source a increased in strength by the factor A.

The term self-reaction denotes the reaction of a field on its own sources, that is, $\langle a,a \rangle$.

If the source $\underline{J}^b$ is a surface distribution of current over a closed surface S, then by taking the limiting case of equation (2.13) as the volume distributions approach surface distributions the form of the reaction for surface currents becomes:

$$\langle a, b\rangle = \oiint \underline{E}^a \cdot \underline{J}^b \, dS \tag{2.17}$$

where $\underline{J}^b$ is now a surface distribution of current.

If the source b is a circuit current source, that is, is a short filament of impressed current I^b of constant value extending over an incremental length $d\ell$ then the reaction <a,b> of a field with a current source is :

$$\langle a, b\rangle = \int_0^{d\ell} \underline{E}^a \cdot I^b \underline{d\ell} = I^b \int_0^{d\ell} \underline{E}^a \cdot \underline{d\ell}$$

$$\text{or} \quad \langle a, b\rangle = - V^a I^b \tag{2.18}$$

where V^a is the voltage across the b source due to some, not yet specified, a source.

Some Comments on Reaction

The reaction <a,b> is of course just a scalar quantity. However, it should be noted that the definition brings together quantities from different situations. For instance, field $\underline{E}^a$ exists at an entirely different time than does $\underline{J}^b$, since $\underline{E}^a$ was produced by $\underline{J}^a$ acting alone, and $\underline{J}^b$ also acted by itself. Nevertheless the definition proves to be an extremely practical and useful one, since many parameters and measurable quantities of interest in electromagnetism can be expressed in terms of reactions. For instance, if source b is a unit current generator across the terminals of some antenna then reaction <a,b> is seen to be equal to the open circuit voltage generated at the antenna's terminals by some source a. As another example, the impedance parameters of a multiport network can be shown to be proportional to reactions.

Because the reaction definition is in terms of an integral, mathematical advantages are gained when dealing with singular fields and singular source functions as pointed out by Rumsey[29].

Perhaps the most important feature of the reaction concept is that it leads to a general procedure for establishing stationary formulas, as explained in the next section.

Stationary Nature of $\langle a,b \rangle$.

Consider that some quantity of interest can be determined if the reaction $\langle c_a, c_b \rangle$ can be found, where the symbol c indicates that correct or true source distributions and fields for the problem are involved. However, in many problems the correct fields and sources are unknown, and approximate sources (or fields) and their corresponding fields (or sources) must be assumed to obtain a solution. Thus, a and b are assumed as approximations for c_a and c_b respectively and reaction $\langle a,b \rangle$ then determined as an approximation to reaction $\langle c_a, c_b \rangle$.

It will now be shown that if the following conditions can be enforced, namely

$$\langle a,b \rangle = \langle c_a, b \rangle = \langle a, c_b \rangle \tag{2.19}$$

that the reaction $\langle a,b \rangle$ is stationary for small variations of a and b about c_a and c_b. Again, the proof follows that given by Harrington. The theory of the calculus of variations can be found in books such as Pipes[31] and Davis[32].

To proceed, let

$$a = c_a + p_a e_a \qquad\qquad b = c_b + p_b e_b \tag{2.20}$$

where e_a and e_b are functions of the variables of integration but must be zero at the end points of the range of integration. The functions e_a and e_b are "error" functions that account for the difference between a and c_a, and b and c_b. Also, p_a and p_b are constants such that when they are small, a and b closely approximate c_a and c_b. If $\langle a,b\rangle$ is stationary, first order changes in p_a and p_b, when p_a and p_b are extremely small to begin with, should cause only a second order change in $\langle a,b\rangle$. Mathematically, $\langle a,b\rangle$ stationary means:

$$\left.\frac{\partial\langle a,b\rangle}{\partial p_a}\right|_{p_a=p_b=0} = \left.\frac{\partial\langle a,b\rangle}{\partial p_b}\right|_{p_a=p_b=0} = 0 \tag{2.21}$$

To show this, start with equation (2.20) to obtain

$$\langle a,b\rangle = \langle c_a + p_a e_a, c_b + p_b e_b\rangle \tag{2.22}$$

Using the identities of equation (2.15) and (2.16):

$$\langle a,b\rangle = \langle c_a, c_b\rangle + p_a\langle e_a, c_b\rangle + p_b\langle c_a, e_b\rangle + p_a p_b\langle e_a, e_b\rangle \tag{2.23}$$

From the equalities of equation (2.19):

$$\begin{aligned}\langle a, b\rangle &= \langle c_a, b\rangle = \langle c_a, c_b + p_b e_b\rangle \\ &= \langle c_a, c_b\rangle + p_b\langle c_a, e_b\rangle\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\langle a,b\rangle &= \langle a, c_b\rangle = \langle c_a + p_a e_a, c_b\rangle \\ &= \langle c_a, c_b\rangle + p_a\langle e_a, c_b\rangle\end{aligned} \tag{2.25}$$

Substituting equations (2.24) and (2.25) into (2.23)

$$\langle a,b\rangle = \langle c_a, c_b\rangle - p_a p_b\langle e_a, e_b\rangle \tag{2.26}$$

Thus $$\left.\frac{\partial\langle a,b\rangle}{\partial p_a}\right|_{p_b=0} = -p_b\langle e_a, e_b\rangle\Big|_{p_b=0} = 0 \tag{2.27}$$

Similarly: $$\left.\frac{\partial\langle a,b\rangle}{\partial p_b}\right|_{p_a=0} = 0 \tag{2.28}$$

so that:

$$\left.\frac{\partial\langle a,b\rangle}{\partial P_a}\right|_{P_a=P_b=0} = \left.\frac{\partial\langle a,b\rangle}{\partial P_b}\right|_{P_a=P_b=0} = 0 \qquad (2.29)$$

and $\langle a,b\rangle$ is stationary.

Now, the stationary nature of $\langle a,b\rangle$ is not by itself necessarily of any help in obtaining good answers if the approximate distributions differ greatly from the correct distributions. However, the conditions of equation (2.19) can be thought of as more than just a means of obtaining stationary equations. They are really the conditions expected to do the most to force the approximate reaction to be the best approximation for the problem at hand. That is, suppose $\langle a,b\rangle$ is to approximate $\langle c_a, c_b\rangle$. Then the condition $\langle a,b\rangle = \langle c_a, b\rangle$ can be thought of as meaning that b is being used as a test source to test that a acts the same as the correct source as far as its reaction with the test source b is concerned. Similarily, in $\langle a,b\rangle = \langle a, c_b\rangle$ a is really testing b to make sure that b is equivalent to c_b for the reaction with test source a.

All this is still no guarantee of course that $\langle a,b\rangle$ will closely approximate $\langle c_a, c_b\rangle$ but all possible constraints inherent in the problem have been applied. That is, sources involved in the problem have been used as tests for other sources involved in the problem, and the tests applied have been directed as far as possible to making $\langle a,b\rangle$ equal $\langle c_a, c_b\rangle$. Essentially, the Reaction Method applies necessary conditions for $\langle a,b\rangle$ to equal $\langle c_a, c_b\rangle$

but cannot apply sufficient conditions (unless-as is not practical-a complete orthogonal set of trial functions could be assumed for the approximate sources, as explained in Rumsey[4]).

As a point of terminology, equation (2.19) is thought of as saying that all trial sources look the same to themselves as they look to the correct sources.

III DIPOLE SELF-IMPEDANCE IN TERMS OF REACTIONS

The Reaction Method will now be used to obtain an expression for the input or self-impedance of an isolated cylindrical dipole driven by a current source at its terminal gap,as in Figure 2.1.

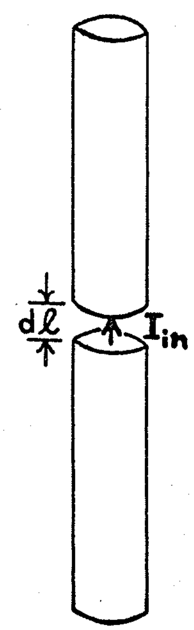


Figure 2.1 A DIPOLE EXCITED BY A CURRENT SOURCE

Here the source of the fields is the current along the antenna as well as in the gap. The antenna structure is assumed to be perfectly conducting. Thus, the correct current distribution $\underline{J}^c$ will distribute itself as a surface current along the antenna. Also, $\underline{J}^c$ must be distributed such that the tangential component of its total electric field, $\underline{E}^c$, will vanish on the conductor surface. Hence, with the antenna terminals close together, the reaction of $\underline{E}^c$ with $\underline{J}^c$ is just the reaction of the field with the current source at the terminals and is of the form $-V^c_{in} I_{in}$. That is:

$$\langle c, c \rangle = -V^c_{in}\, I_{in} \qquad (2.30)$$

where V^c_{in} is the correct input voltage to the dipole when I_{in} is the input current. Equation (2.30) can be rearranged to give:

$$\frac{\langle c, c \rangle}{I_{in}^2} = \frac{-V^c_{in}}{I_{in}} \qquad (2.31)$$

But the ratio of $\frac{V^c_{in}}{I_{in}}$ is by definition the input impedance of the dipole, so:

$$Z^c_{in} = \frac{-\langle c, c \rangle}{I_{in}^2} \qquad (2.32)$$

Next, just for convenience, consider the current source to be an equivalent surface current of constant value across the gap and uniformly distributed around the gap as in Figure 2.2.

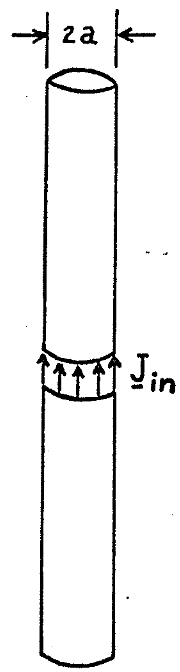


Figure 2.2 SURFACE CURRENT EQUIVALENT OF THE FILAMENTARY INPUT CURRENT

This avoids breaking the integration into two parts- one across the gap and the other around the antenna structure. The result is identical providing:

$$2\pi a\, J_{in} = I_{in} \qquad (2.33)$$

where a is the antenna radius. However, now the gap can be included as part of the antenna surface and one surface integral can be written for the reactions that follow. (In Chapter III, the surface currents will all be replaced by equivalent filamentary currents for calculating fields).

To continue, since the correct current distribution cannot be found without knowing the correct electromagnetic field, and the correct field cannot be found without knowing the correct current distribution, $\langle c,c \rangle$ cannot be calculated. Thus, the Reaction Method seeks an approximation to $\langle c,c \rangle$. A trial surface-current distribution $\underline{J}^a$ is assumed on the antenna, the electric field corresponding to $\underline{J}^a$ being $\underline{E}^a$. The approximate input impedance can now be expressed in terms of the self-reaction $\langle a,a \rangle$.

$$Z_{in}^{a} = \frac{-\langle a,a \rangle}{I_{in}^{2}} = -\frac{1}{I_{in}^{2}} \oiint \underline{E}^{a} \cdot \underline{J}^{a} \, dS \qquad (2.34)$$

where S is the antenna surface. Also, since neither Z_{in}^{a} or Z_{in}^{c} are dependent on the magnitude and phase of I_{in}, the same input current source is considered when talking about either the correct or approximate distribution.

Next, to force the approximation to be good the following equality can be imposed:

$$\langle a,a \rangle = \langle c,a \rangle \qquad (2.35)$$

Since $\langle c,a \rangle = \langle a,c \rangle$ by reciprocity the constraints of equation (2.19) have been met and Z_{in}^{a} is stationary about the true current.

To proceed, the Reaction Method assumes the trial source to be a linear combination of functions , $a = Uu + Vv + \cdots$, where $U, V, \cdots$, are adjustable constants to be determined. Choosing the current on the antenna to be represented by two functions, the trial current is then a surface current of the form:

$$\underline{J}^{a} = U\underline{J}^{u} + V\underline{J}^{v} \tag{2.36}$$

Equation (2.35) then becomes:

$$\langle a,a\rangle = \langle c,a\rangle = \langle c, Uu + Vv\rangle$$

or:

$$\langle a,a\rangle = U\langle c,u\rangle + V\langle c,v\rangle \tag{2.37}$$

U and V must be adjusted to suit the problem,and this is accomplished by introducing test sources.The approximate source is forced to look the same to the test sources as does the correct source as far as reaction is concerned. Now, even though many test sources could be invented for testing purposes, with only two adjustable parameters in the approximate source only two test sources can be used. Otherwise, inconsistent equations will most likely result since there will be more equations than unknowns. Hence,consider introducing two test sources x and y and adjust source a using the following test equations:

$$\langle a,x\rangle = \langle c,x\rangle \tag{2.38}$$

$$\langle a,y\rangle = \langle c,y\rangle \tag{2.39}$$

The question is, which test sources x and y to use? Obviously, a test source is really only available for testing if its reactions of equations (2.38) and (2.39) can be calculated. Also, to make the tests pertinent to the problem under consideration, the general guide is to use as test sources, sources involved in the problem. For the dipole case, the only sources available and inherent in the problem are the sources u and v. Thus, test as follows:

$$\langle a,u\rangle = \langle c,u\rangle \tag{2.40}$$

$$\langle a,v\rangle = \langle c,v\rangle \tag{2.41}$$

Both tests, it must be noted, are consistent with the aim of imposing equation (2.35).

To adjust U and V using equations (2.40) and (2,41) the procedure is:

$$\langle a,u\rangle = \langle c,u\rangle$$

$$\langle Uu+Vv,u\rangle = \langle c,u\rangle$$

$$U\langle u,u\rangle + V\langle v,u\rangle = \langle c,u\rangle \tag{2.42}$$

and

$$\langle a,v\rangle = \langle c,v\rangle$$

$$\langle Uu+Vv,v\rangle = \langle c,v\rangle$$

$$U\langle u,v\rangle + V\langle v,v\rangle = \langle c,v\rangle \tag{2.43}$$

Solving for U and V from equations (2.42) and (2.43), the result in matrix notation is:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \langle u,u\rangle & \langle v,u\rangle \\ \langle u,v\rangle & \langle v,v\rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle c,u\rangle \\ \langle c,v\rangle \end{bmatrix} \tag{2.44}$$

Substituting equation (2.44) into equation (2.37) :

$$\langle a,a\rangle = \begin{bmatrix} \langle c,u\rangle & \langle c,v\rangle \end{bmatrix} \begin{bmatrix} \langle u,u\rangle & \langle v,u\rangle \\ \langle u,v\rangle & \langle v,v\rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle c,u\rangle \\ \langle c,v\rangle \end{bmatrix} \tag{2.45}$$

By reciprocity $\langle u,v\rangle = \langle v,u\rangle$.

Expanding equation (2.45) gives:

$$\langle a,a\rangle = \frac{\langle c,u\rangle^2\langle v,v\rangle - 2\langle c,u\rangle\langle c,v\rangle\langle u,v\rangle + \langle c,v\rangle^2\langle u,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2} \tag{2.46}$$

Since $\underline{E}^c$ vanishes everywhere along the antenna except at the feed:

$$\langle c,u\rangle = -V_{in}^c\, I^u(0) \tag{2.47}$$

$$\langle c,v\rangle = -V_{in}^c\, I^v(0) \tag{2.48}$$

where $I^u(0)$ and $I^v(0)$ are the values of the trial currents u and v respectively at the input. Thus:

$$\langle a,a\rangle = -(V_{in}^c)^2\left[\frac{I^u(0)^2\langle v,v\rangle - 2I^u(0)I^v(0)\langle u,v\rangle + I^v(0)^2\langle u,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\right] \quad (2.49)$$

Dividing both sides by I_{in}^2 :

$$\frac{\langle a,a\rangle}{I_{in}^2} = -\left(\frac{V_{in}^c}{I_{in}}\right)^2\left[\frac{I^u(0)^2\langle v,v\rangle - 2I^u(0)I^v(0)\langle u,v\rangle + I^v(0)^2\langle u,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\right] \quad (2.50)$$

Hence:

$$-Z_{in}^a = (Z_{in}^c)^2\left[\frac{I^u(0)^2\langle v,v\rangle - 2I^u(0)I^v(0)\langle u,v\rangle + I^v(0)^2\langle u,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}\right] \quad (2.51)$$

If Z_{in}^a is to closely approximate Z_{in}^c , Z_{in}^a must be found from:

$$Z_{in}^a = \frac{\langle u,v\rangle^2 - \langle u,u\rangle\langle v,v\rangle}{I^v(0)^2\langle u,u\rangle - 2I^u(0)I^v(0)\langle u,v\rangle + I^u(0)^2\langle v,v\rangle} \quad (2.52)$$

(This equation corresponds to equation (7.99), page 353 of Harrington[6], if an error in sign is corrected in Harrington's equation)

Chapter III explains how the reactions of equation (2.52) are evaluated.

CHAPTER III

EQUATIONS FOR THE REACTIONS

In this chapter, an equation for the reaction $\langle u,v\rangle$ in terms of assumed filamentary currents I^u and I^v is derived. A similar derivation would apply to determine $\langle u,u\rangle$ and $\langle v, v\rangle$. The form of the current assumptions to be used are discussed, and then the final equations for each reaction are given.

I REACTION $\langle u,v\rangle$

From the definition of reaction for surface currents:

$$\langle u,v\rangle = \oiint \underline{E}^u . \underline{J}^v dS \qquad (3.1)$$

where surface S is the antenna surface. $\underline{J}^u$ and $\underline{J}^v$ are assumed antenna surface current distributions, and $\underline{E}^u$ is the electric field on the surface of the antenna due to surface current $\underline{J}^u$.

Attention will first be concentrated on determining $\underline{E}^u$. The dipole is considered oriented along the z axis as in Figure 3.1 .

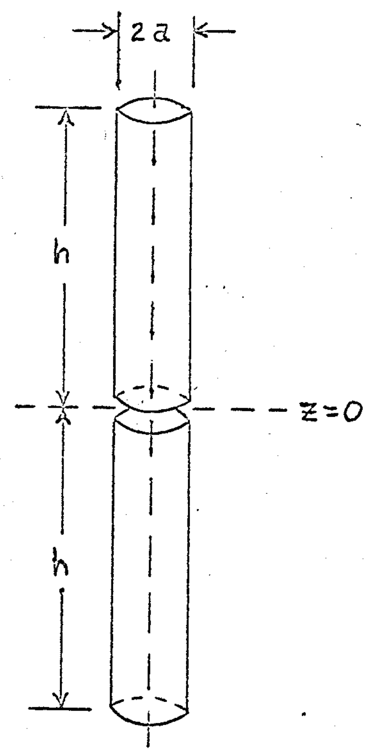


Figure 3.1 THE CYLINDRICAL ANTENNA

King and Harrison[8] prove that for thin antennas the magnetic vector potential $\underline{A}$, at all points outside a cylindrical conductor including its surface (except those points within distances of an end face comparable with its radius) is given to a good approximation by assuming that the current is filamentary and distributed along the center of the antenna. C. T. Tai[1,2] and R. F. Harrington[6] consider this an excellent approximation, and it is utilized in this section to find $\underline{E}^u$ from $\underline{J}^u$.

Accordingly, the symbol I^u is used to represent the filamentary equivalent current of $\underline{J}^u$ where the two are related by:

$$I^u = \frac{J^u}{2\pi a} \tag{3.2}$$

with a being the radius of the dipole.

With the dipole along the z axis, the current is z directed, and is some function of z. Vertical distances along the center of of the axis shall be symbolized z' to distinguish them from vertical distances along the antenna surface which will be symbolized z. Thus the filamentary current can be written as $I^u(z')$.

An expanded view of the antenna is shown as Figure 3.2

Denote the magnetic potential vector on the antenna surface due to I^u by the symbol $\underline{A}^u$. Since I^u is z-directed, $\underline{A}^u$ must have only a z - component - that is:

$$\underline{A}^u = A_z^u \hat{k} \tag{3.3}$$

where $\hat{k}$ is a unit vector in the z direction. Also, since I^u is along the center of the antenna, symmetry demands that $\underline{A}^u$ and $\underline{E}^u$ be independent of ϕ. Thus along the antenna surface $\underline{A}^u$ and $\underline{E}^u$ are functions only of z, namely $\underline{A}^u(z)$ and $\underline{E}^u(z)$.

At any point z' along the center of the antenna, the current $I^u(z')$ extending over an incremental length dz' forms a current element or electric dipole of moment $I^u(z')dz'$ which is z-directed, and shown in Figure 3.2. At any point P (a,ϕ,z) on the antenna

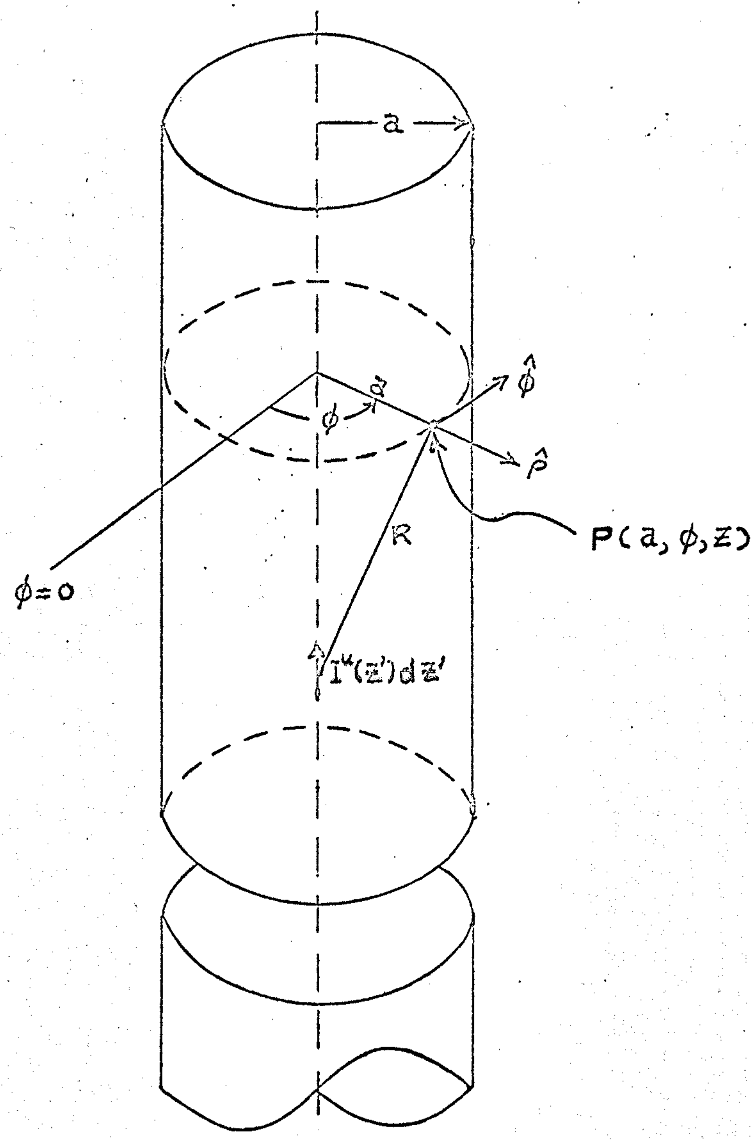


Figure 3.2 EXPANDED VIEW OF THE DIPOLE

surface the contribution $dA_z^u(z)$ to $A_z^u(z)$ due to electric dipole $I^u(z')dz'$ is given by:

$$dA_z^u(z) = \frac{I^u(z')\, e^{-j\beta\sqrt{(Z-Z')^2 + a^2}}}{4\pi\sqrt{(Z-Z')^2 + a^2}}\, dz' \qquad (3.4)$$

A_z^u is found, of course, as the summation of all such dA_z^u,

$$A_z^u(z) = \int_{-h}^{h} \frac{I^u(z')\, e^{-j\beta\sqrt{(Z-Z')^2 + a^2}}}{4\pi\sqrt{(Z-Z')^2 + a^2}}\, dz' \qquad (3.5)$$

where h is the dipole half-length.

To find $\underline{E}^u$ from $\underline{A}^u$ the following equation, true for time-harmonic waves with the antenna radiating into free space, is used:

$$\underline{E}^u = -j\omega\mu_0 \underline{A}^u + \frac{1}{j\omega\epsilon_0} \nabla(\nabla\cdot\underline{A}^u) \qquad (3.6)$$

With $\underline{A}^u$ only in the $\hat{k}$ direction, this reduces to

$$\underline{E}^u(z) = -j\omega\mu_0 A_z^u(z)\hat{k} + \frac{1}{j\omega\epsilon_0} \nabla\left(\frac{\partial}{\partial Z} A_z^u(z)\right) \qquad (3.7)$$

Furthermore, since eventually the dot product $\underline{E}^u \cdot \underline{J}^v$ must be taken, and since $\underline{J}^v$ is entirely in the z direction, only the $\hat{k}$ component of $\underline{E}^u(z)$ will enter the final expression for <u,v> . Thus, only the $\hat{k}$ component of $\nabla(\frac{\partial}{\partial Z} A_z^u(Z))$ is retained to obtain:

$$E_z^u(z)\hat{k} = \left(-j\omega\mu_0 A_z^u(z) + \frac{1}{j\omega\epsilon_0}\frac{\partial^2 A_z^u(z)}{\partial z^2}\right)\hat{k} \tag{3.8}$$

Substituting equation (3.5) into equation (3.8) gives:

$$E_z^u(z) = \frac{1}{j\omega\epsilon_0}\left(\omega^2\mu_0\epsilon_0 + \frac{\partial^2}{\partial z^2}\right)\int_{-h}^{h}\frac{I^u(z')\, e^{-j\beta\sqrt{(z-z')^2+a^2}}}{4\pi\sqrt{(z-z')^2+a^2}}\,dz' \tag{3.9}$$

The integration is with respect to z' and the differentiation is with respect to z, so the differentiation can be performed within the integral sign. This differentiation is straightforward. The following equations are introduced at this point

$$\omega^2\mu_0\epsilon_0 = \left(\frac{2\pi}{\lambda}\right)^2 \tag{3.10}$$

$$\beta = \frac{2\pi}{\lambda} \tag{3.11}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \doteq 120\,\pi \text{ ohms} \tag{3.12}$$

Performing the differentiation along with some simple manipulations the following expressions result for the real and imaginary parts of $E_z^u(z)$ (I^u and I^v will be seen later to be real):

$$\mathrm{Re}\, E_z^u(z) = \frac{\eta_0}{8\pi^2\lambda}\int_{-h}^{h} I^u(z')\, F_1(z,z')\, dz' \tag{3.13}$$

$$\mathrm{Im}\, E_z^u(z) = \frac{\eta_0}{8\pi^2\lambda}\int_{-h}^{h} I^u(z')\; F_2(z,z')\,dz' \tag{3.14}$$

For equations (3.13) and (3.14), the functions F_1 and F_2 are defined by

$$F_1(z,z') \equiv \cos 2\pi\sqrt{(z-z')^2+a^2}\left[\frac{-2\pi}{(\sqrt{(z-z')^2+a^2})^2} + \frac{6\pi(z-z')^2}{(\sqrt{(z-z')^2+a^2})^4}\right]$$

$$+ \sin 2\pi\sqrt{(z-z')^2+a^2}\left[\frac{-4\pi^2}{\sqrt{(z-z')^2+a^2}} + \frac{4\pi^2(z-z')^2+1}{(\sqrt{(z-z')^2+a^2})^3} - \frac{3(z-z')^2}{(\sqrt{(z-z')^2+a^2})^5}\right] \tag{3.15}$$

$$F_2(z,z') \equiv -\sin 2\pi\sqrt{(z-z')^2+a^2}\left[\frac{-2\pi}{(\sqrt{(z-z')^2+a^2})^2} + \frac{6\pi(z-z')^2}{(\sqrt{(z-z')^2+a^2})^4}\right]$$

$$+ \cos 2\pi\sqrt{(z-z')^2+a^2}\left[\frac{-4\pi^2}{\sqrt{(z-z')^2+a^2}} + \frac{4\pi^2(z-z')^2+1}{(\sqrt{(z-z')^2+a^2})^3} - \frac{3(z-z')^2}{(\sqrt{(z-z')^2+a^2})^5}\right] \tag{3.16}$$

In equations (3.13), (3.14), (3.15), and (3.16) all distances - z', z, h, and a - are now expressed as a fraction of wavelength λ.

The evaluation of $\langle u,v\rangle$ can now proceed. Recall equation (3.1), namely

$$\langle u,v\rangle = \oiint \underline{E}^{u}\cdot\underline{J}^{v}\,dS \tag{3.1}$$

Some simplifications can be introduced here. Firstly, $\underline{J}^{v}$ will be assumed such that $\underline{J}^{v}=0$ at the ends of the antenna, so that the integration need not be taken over the ends. Furthermore, $\underline{J}^{v}$ is assumed uniformly distributed around the antenna so that it is not a function of ϕ but only of z. Lastly, since $\underline{J}^{v}$ is z directed we can replace $\underline{E}^{u}$ by E_z^{u} and hence eliminate the dot product. Mathematically these simplifications mean:

$$\langle u,v\rangle = \int_{-h}^{h}\int_{0}^{2\pi} \lambda E_z^{u}(z)\, J^{v}(z)\, a\, d\phi\, dz \tag{3.17}$$

The λ factor is included so that dz is in terms of wavelength. With $\underline{E}^{u}$ and $\underline{J}^{v}$ independent of ϕ, this reduces to

$$\langle u,v\rangle = 2\pi a \int_{-h}^{h} \lambda E_z^u(z)\, J^v(z)\, dz$$

$$= \int_{-h}^{h} \lambda E_z^u(z)\, 2\pi a\, J^v(z)\, dz \qquad (3.18)$$

From what has been discussed previously, $2\pi a\ J^v(z)$ just represents an assumed equivalent filamentary current $I^v(z)$ Hence

$$\langle u,v\rangle = \int_{-h}^{h} \lambda E_z^u(z)\, I^v(z)\, dz \qquad (3.19)$$

Because the dipole is symmetrical about $z=0$, the currents I^u and I^v are assumed to be symmetrical about $z=0$. Since I^u is symmetrical about $z=0$, $\underline{A}^u$ and $\underline{E}^u$ are also symmetrical about $z=0$. Thus the product $E_z{}^u(z)I^v(z)$ is an even function of z and equation (3.19) can be written

$$\langle u,v\rangle = 2\int_{0}^{h} \lambda E_z^u(z)\, I^v(z)\, dz \qquad (3.20)$$

Substituting equations (3.13) and (3.14) into equation (3.20) the following expressions result for the real and imaginary parts of $\langle u,v\rangle$

$$\mathrm{Re}\,\langle u,v\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^u(z')\,I^v(z)\,F_1(z,z')\,dz'\,dz \tag{3.21}$$

$$\mathrm{Im}\,\langle u,v\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^u(z')\,I^v(z)\,F_2(z,z')\,dz'\,dz \tag{3.22}$$

An almost identical derivation applies for <u,u> and <v,v>, yielding:

$$\mathrm{Re}\,\langle u,u\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^u(z')\,I^u(z)\,F_1(z,z')\,dz'\,dz \tag{3.23}$$

$$\mathrm{Im}\,\langle u,u\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^u(z')\,I^u(z)\,F_2(z,z')\,dz'\,dz \tag{3.24}$$

$$\mathrm{Re}\,\langle v,v\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^v(z')\,I^v(z)\,F_1(z,z')\,dz'\,dz \tag{3.25}$$

$$\mathrm{Im}\,\langle v,v\rangle = \frac{\eta_o}{4\pi^2}\int_0^h\int_{-h}^h I^v(z')\,I^v(z)\,F_2(z,z')\,dz'\,dz \tag{3.26}$$

II CURRENT ASSUMPTIONS

Storer[9] used the following trial functions to obtain the first order (two trial functions) variational solution for input impedance:

$$I^u = \sin 2\pi (h - |z|) \tag{3.27}$$

$$I^v = 1 - \cos 2\pi (h - |z|)$$

Thus the total current I_T at any point on the antenna would be given by

$$I_T(z) = U \sin 2\pi(h-|z|) + V\left[1 - \cos 2\pi(h-|z|)\right] \quad |z| \leq h \tag{3.28}$$

where, again, h and z are expressed in terms of wavelength.

Since $I_T(0)$ - the input current - is zero when h=1,2,3..., Storer's analysis does not provide valid answers in these cases. Tai [1,2] uses

$$I^u = \sin 2\pi (h - |z|)$$

$$I^v = 2\pi (h-|z|) \cos 2\pi (h-|z|) \tag{3.29}$$

so that

$$I_T(z) = U \sin 2\pi (h-|z|) + V 2\pi(h-|z|)\cos 2\pi (h-|z|) \tag{3.30}$$

$$|z| \leq h$$

Here, I_T does not vanish at z=0 for any value of h, and thus first order solutions based on these trial functions are finite for antennas of any length.

Harrington[6] suggests the following trial functions

$$\begin{aligned} I^u &= \sin 2\pi(h-|z|) \\ I^v &= h-|z| \end{aligned} \tag{3.31}$$

giving

$$I_T(z) = U \sin 2\pi(h-|z|) + V(h-|z|) \qquad |z| \leq h \tag{3.32}$$

which is finite at z=0 for all h > 0.

In each case, U and V are in general complex constants that are to be adjusted using the Reaction Concept as explained in Chapter II.

Notice that each current approximation involves the sine term. It can be shown [10] that the current in any infinitely thin perfectly conducting antenna is exactly sinusoidal. Additional terms are added in an attempt to account for the finite thickness of practical antennas.

This thesis solves for the dipole impedance using both Tai's approximation and Harrington's suggested approximation.

III FINAL REACTION EQUATIONS

It is only necessary now to substitute the trial functions of equation (3.29) or equation (3.31) into equations (3.21) to (3.26) to arrive at the final expressions for the real and imaginary parts of the reactions <u,v> , <u,u> , and <v,v> .

Reactions Using Tai's Current Distribution

$$\operatorname{Re}\langle u,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\, 2\pi(h-|z|)\cos 2\pi(h-|z|)F_1(z,z')dz'dz \tag{3.33}$$

$$\operatorname{Im}\langle u,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\, 2\pi(h-|z|)\cos 2\pi(h-|z|)F_2(z,z')dz'dz \tag{3.34}$$

$$\operatorname{Re}\langle u,u\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\sin 2\pi(h-|z|)\, F_1(z,z')dz'dz \tag{3.35}$$

$$\operatorname{Im}\langle u,u\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\sin 2\pi(h-|z|)\, F_2(z,z')dz'dz \tag{3.36}$$

$$\operatorname{Re}\langle v,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h 2\pi(h-|z'|)\cos 2\pi(h-|z'|)\, 2\pi(h-|z|)\cos 2\pi(h-|z|)F_1(z,z')dz'dz \tag{3.37}$$

$$\operatorname{Im}\langle v,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h 2\pi(h-|z'|)\cos 2\pi(h-|z'|)\, 2\pi(h-|z|)\cos 2\pi(h-|z|)F_2(z,z')dz'dz \tag{3.38}$$

Reactions Using Harrington's Suggested Distribution

$$\mathrm{Re}\langle u,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\,(h-|z|)\;F_1(z,z')\,dz'\,dz \qquad (3.39)$$

$$\mathrm{Im}\langle u,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\,(h-|z|)\;F_2(z,z')\,dz'\,dz \qquad (3.40)$$

$$\mathrm{Re}\langle u,u\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\sin 2\pi(h-|z|)\,F_1(z,z')\,dz'\,dz \qquad (3.41)$$

$$\mathrm{Im}\langle u,u\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h \sin 2\pi(h-|z'|)\sin 2\pi(h-|z|)\,F_2(z,z')\,dz'\,dz \qquad (3.42)$$

$$\mathrm{Re}\langle v,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h (h-|z'|)\,(h-|z|)\,F_1(z,z')\,dz'\,dz \qquad (3.43)$$

$$\mathrm{Im}\langle v,v\rangle = \frac{\eta_0}{4\pi^2}\int_0^h\int_{-h}^h (h-|z'|)(h-|z|)\,F_2(z,z')\,dz'\,dz \qquad (3.44)$$

For equations (3.33) to (3.44), F_1 and F_2 are defined by equations (3.15) and (3.16) respectively.

CHAPTER IV

SELF-RESISTANCE AND SELF-REACTANCE EQUATIONS

Equation (2.52) which gives the input impedance of the isolated dipole in terms of the reactions is repeated here for convenience as equation (4.1):

$$Z_{in} = \frac{\langle u,v\rangle^2 - \langle u,u\rangle\langle v,v\rangle}{I^v(0)^2\langle u,u\rangle - 2\,I^u(0)\,I^v(0)\langle u,v\rangle + I^u(0)^2\langle v,v\rangle} \tag{4.1}$$

The real and imaginary parts of the reactions have been derived in chapter III. The next step is to separate Z_{in} into its real and imaginary parts, which will be in terms of the real and imaginary parts of the reactions. The real part of Z_{in} can then be identified, of course, as the input or self-resistance and the imaginary part of Z_{in} as the input or self-reactance of the isolated dipole.

Representing the complex number Z_{in} as the ratio of a complex numerator and a complex denominator:

$$Z_{in} = \frac{A + jB}{C + jD} \tag{4.2}$$

Then self-resistance R_{in} is given by:

$$R_{in} = \frac{AC + BD}{C^2 + D^2} \tag{4.3}$$

and self-reactance X_{in} is given by

$$X_{in} = \frac{BC - AD}{C^2 + D^2} \tag{4.4}$$

where, in terms of reactions:

$$A \equiv (\mathrm{Re}\langle u,v\rangle)^2 - (\mathrm{Im}\langle u,v\rangle)^2 - \mathrm{Re}\langle u,u\rangle \mathrm{Re}\langle v,v\rangle + \mathrm{Im}\langle u,u\rangle \mathrm{Im}\langle v,v\rangle \tag{4.5}$$

$$B \equiv 2\,\mathrm{Re}\langle u,v\rangle \mathrm{Im}\langle u,v\rangle - \mathrm{Re}\langle u,u\rangle \mathrm{Im}\langle v,v\rangle - \mathrm{Re}\langle v,v\rangle \mathrm{Im}\langle u,u\rangle \tag{4.6}$$

$$C \equiv I^v(0)^2\, \mathrm{Re}\langle u,u\rangle - 2\, I^u(0) I^v(0)\, \mathrm{Re}\langle u,v\rangle + I^u(0)^2\, \mathrm{Re}\langle v,v\rangle \tag{4.7}$$

$$D \equiv I^v(0)^2\, \mathrm{Im}\langle u,u\rangle - 2\, I^u(0) I^v(0)\, \mathrm{Im}\langle u,v\rangle + I^u(0)^2\, \mathrm{Im}\langle v,v\rangle \tag{4.8}$$

It remains but to perform the integrations indicated in Chapter III to obtain the reactions, and then to substitute the results into equations (4.3) and (4.4). The purpose of this thesis is to do the integrations numerically on a digital computer. The method is explained in Chapter V.

CHAPTER V

THE COMPUTER PROGRAM

In this chapter, the major aspects of the computer program are discussed. The actual program as written in Fortran IV for the IBM 360 model 50 digital computer is given in Appendix A.

I DOUBLE APPLICATION OF THE TRAPEZOIDAL RULE

The Trapezoidal Rule for the approximate integration of single definite integrals can be found in many introductory calculus books. A simple extension of this Rule makes it applicable for approximate integration of double definite integrals. The method is described briefly in this section.

Consider finding an approximation to the following double integral

$$V = \int_0^b \int_{-a}^{a} g(x,y)\, dx\, dy \qquad (5.1)$$

This integral of course gives the volume enclosed between the x-y plane and the g(x,y) curve. Figure 5.1 is an attempt to depict the three-dimensional nature of a representative g(x,y) plotted in the x,y,g coordinate system.

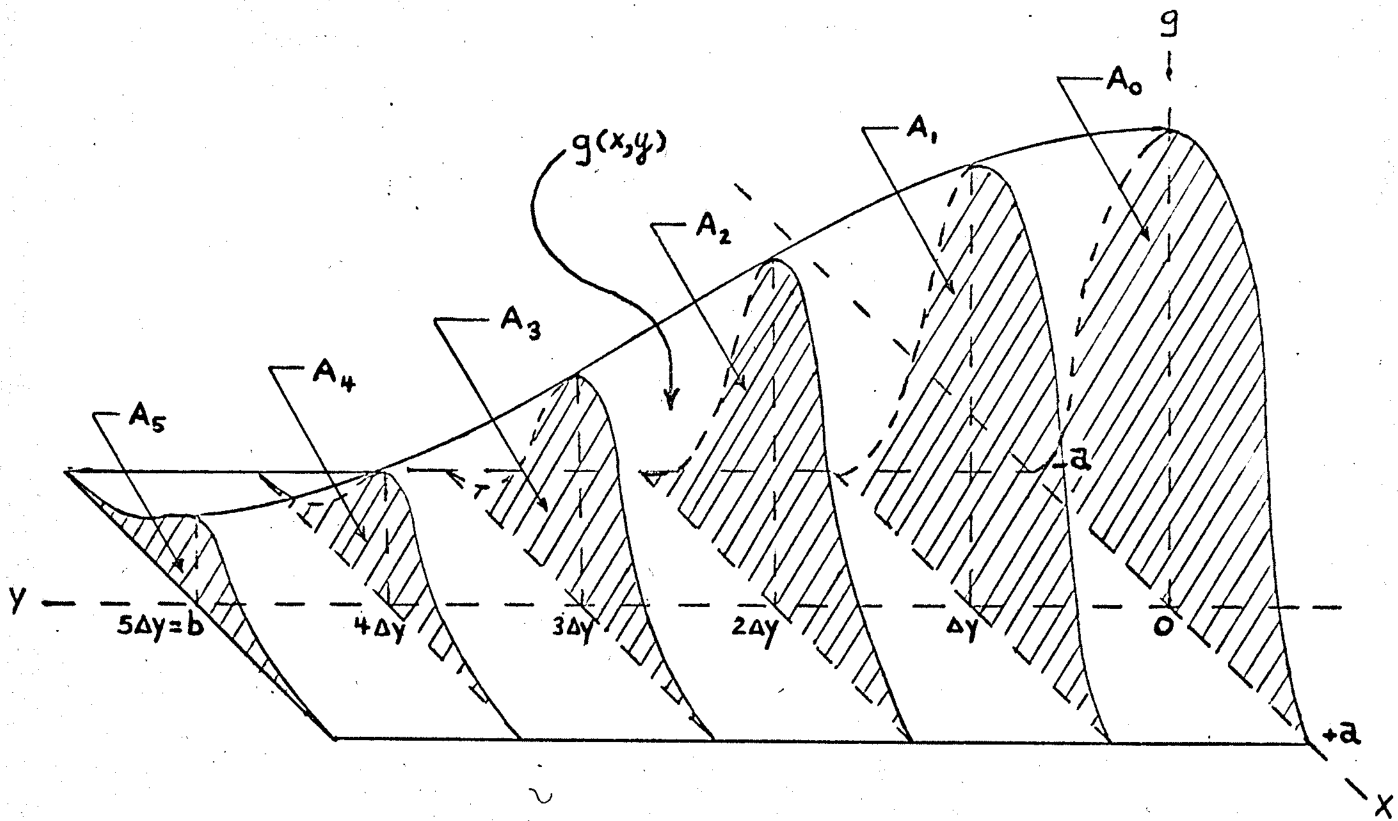


Figure 5.1   THREE-DIMENSIONAL SKETCH OF A POSSIBLE g(x,y)

The shaded portions marked A_o, A_1, ...,A_5 are cross sectional areas perpendicular to the y-axis.

The y-axis is subdividedinto equal increments Δy, where:

$$\Delta y = \frac{b}{N_b} \tag{5.2}$$

N_b is an integer large enough to give a good approximate integration. In Figure 5.1, $N_b = 5$

To approximate the volume under the g(x,y) curve, it is assumed that the cross sectional area varies linearly with y. Thus:

$$V \doteq \frac{(A_0+A_1)}{2}\Delta y + \frac{(A_1+A_2)}{2}\Delta y + \frac{(A_2+A_3)}{2}\Delta y + \frac{(A_3+A_4)}{2}\Delta y + \frac{(A_4+A_5)}{2}\Delta y \qquad (5.3)$$

$$\text{or: } \quad V \doteq \Delta y\left[\frac{A_0}{2} + A_1 + A_2 + A_3 + A_4 + \frac{A_5}{2}\right] \qquad (5.4)$$

Of course, the A's are just the single integrals with respect to x at the various y values, that is:

$$V \doteq \Delta y\left[\int_{-a}^{a}\frac{g(x,0)}{2}\,dx + \int_{-a}^{a} g(x,\Delta y)\,dx + \int_{-a}^{a} g(x,2\Delta y)\,dx + \cdots + \int_{-a}^{a}\frac{g(x,5\Delta y)}{2}\,dx\right] \qquad (5.5)$$

Now, these cross sectional areas can also be approximated using the Trapezoidal Rule. For example, consider $A_1 = \int_{-a}^{a} g(x,\Delta y)\,dx$

Subdivide the x axis into equal segments Δx where:

$$\Delta x = \frac{a}{N_a} \qquad (5.6)$$

N_a is an integer large enough to give a good approximate integration. For the sake of illustration, let $N_a = 5$.

To approximate A_1, the area under the $g(x,\Delta y)$ curve, it is assumed that $g(x,\Delta y)$ varies linearly with x. Then the Trapezoidal Rule gives the following approximation to A_1:

$$A_1 \doteq \Delta x \left[\frac{g(-a,\Delta y)}{2} + g(-a+\Delta x,\Delta y) + \cdots + g(-a+9\Delta x,\Delta y) + \frac{g(-a+10\Delta x,\Delta y)}{2} \right] \tag{5.7}$$

However, if g(x,y) is such as in Figure 5.1, where:

$$g(-a,y) = g(+a,y) = 0 \tag{5.8}$$

then equation (5.7) simplifies to

$$A_1 \doteq \Delta x \left[g(-a+\Delta x,\Delta y) + \cdots + g(-a+9\Delta x,\Delta y) \right] \tag{5.9}$$

Similar expressions hold for A_o, A_2, A_3, A_4, and A_5 and V becomes:

$$\begin{aligned} V \doteq \Delta y \Delta x \Big\{ & \tfrac{1}{2}\left[g(-a+\Delta x,0) + \cdots + g(-a+9\Delta x,0) \right] \\ & + \left[g(-a+\Delta x,\Delta y) + \cdots + g(-a+9\Delta x,\Delta y) \right] \\ & + \cdots + \tfrac{1}{2}\left[g(-a+\Delta x,5\Delta y) + \cdots + g(-a+9\Delta x,5\Delta y) \right] \Big\} \end{aligned} \tag{5.10}$$

II PROGRAM FEATURES AND CONSIDERATIONS

The evaluation of the reactions require solving double integrals of the form

$$\int_{0}^{h}\int_{-h}^{h} f(z,z')\,dz'\,dz$$

Because I^u and I^v are zero at $z'=h$ and $z'=-h$, $f(z,z')$ is zero at $z'=h$ and $z'=-h$. Hence the simpler form of the Trapezoidal Rule, namely equation (5.9), can be used. Also, the last cross-sectional area, at $z=h$, is zero. These facts simplify the program logic somewhat by eliminating three program branches otherwise necessary to account for the factor of one-half.

The integration with respect to z' was found to be somewhat more critical than the integration with respect to z. Satisfactory results were obtained with $\Delta z' = \frac{h}{200}$ and $\Delta z = \frac{h}{100}$. Now, for each value of z, the inner integration with respect to z' involves evaluation of $f(z,z')$ at about 200 values of z'. Thus with z divided into 100 divisions calculations involved in evaluating the inner integral could have to be repeated up to 20,000 times. Hence care must be exercised to avoid unnecessary calculation steps, and whenever possible, to store numbers that are used repeatedly. For example $F_1(z,z')$ and $F_2(z,z')$ (equation 3.15 and 3.16) have many terms in common. These terms are evaluated just once and then used in the expressions for both F_1 and F_2. Similarly, F_1 and F_2 are common to all the reactions (see equations 3.21 to 3.26). Thus, in a manner to be explained next, F_1 and F_2 are calculated once, stored, and then used to evaluate all the reactions.

A Weighting Function Technique

The most important feature of the program is the method by which F_1 and F_2 are used as types of weighting functions in calculating reactions.

To explain the procedure, consider as a specific example determining:

$$\int_0^h I^v(z) \left(\int_{-h}^h I^u(z') F_1(z,z') dz' \right) dz$$

From what has been said about the Trapezoidal method, approximating this double integral involves a summation of terms of the form

$$I^v(z_1) \int_{-h}^h I^u(z') F_1(z_1, z') dz'$$

Here, of course, z_1 is a particular value of z between z=0 and z=h, and is some multiple of Δz.

Now, some observations must be made regarding $F_1(z,z')$. Similar observations can be seen to apply also to $F_2(z,z')$

Consider equation 3.15 defining $F_1(z,z')$. It is to be noted that z and z' always appear in the form $(z - z')^2$. Thus F_1 can be considered as a function of $(z - z')$ - denoted $F_1(z - z')$. Using this fact, recalculation of F_1 for each different value of z_1 can be avoided.

First, $F_1(z - z')$ is calculated for values of $(z - z')$ from 0 to $-2h$ in increments of $\Delta z'$ and each value stored in an array in the computer. Values of F_1 for $(z - z')$ from 0 to $+2h$ can be inferred from the values calculated because $F_1(z - z')$ must be symmetrical about $(z - z') = 0$. This symmetry follows because $z - z'$ always appears as $(z - z')^2$ in $F_1(z - z')$. Figure 5.2 illustrates this calculation of F_1, though the shape is not necessarily representative of the actual shape of F_1. The dotted portion of the curve of Figure 5.2 is obtained by symmetry. The small circles represent values of F_1 calculated and stored.

Next, regardless of the values of Δz and $\Delta z'$, Δz is chosen to be a multiple of $\Delta z'$.

$I^u(z')$ is then calculated from $z' = 0$ to $z' = h$ in increments of $\Delta z'$, and each value obtained is stored in an array. Values for $I^u(z')$ for $z'=0$ to $z'=-h$ can be inferred from the calculated values since I^u is symmetrical about $z'=0$. In Figure 5.3, $I^u(z')$ is shown assuming $I^u(z') = \sin 2\pi(h - |z'|)$ for a dipole with $\frac{\lambda}{4} < h < \frac{\lambda}{2}$.

The evaluation of $\int_{-h}^{h} I^u(z') F_1(z_1, z') dz'$ involves taking the product $I^u(z')F_1(z_1,z')$ at each value of z' with z' varying from $-h$ to h in increments of $\Delta z'$, then taking the summation of the product terms.

The $F_1(z_1,z')$ curve is obtained merely by placing the origin of the $F_1(z - z')$ curve at $z'=z_1$. This is shown in Figure 5.3. Since Δz is a multiple of $\Delta z'$, z_1, being a multiple of Δz, is a multiple of $\Delta z'$. Hence F_1 and I^u have both been calculated for $z'=z_1$, as well as at integer increments of $\Delta z'$ on both sides of $z'=z_1$.

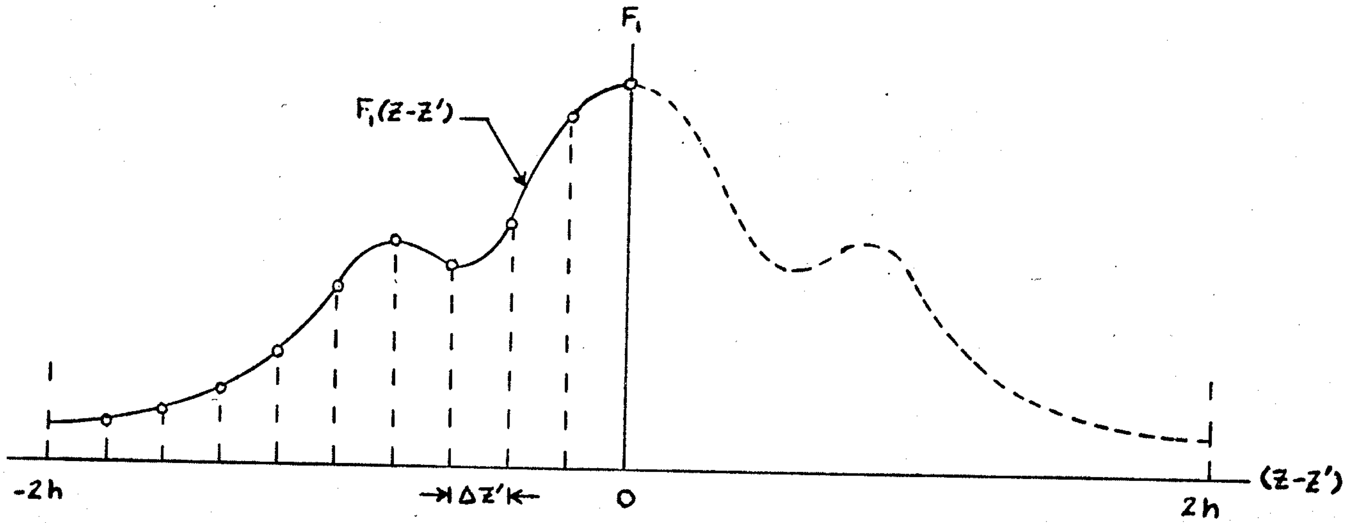


Figure 5.2 AN ILLUSTRATIVE $F_1(z-z')$

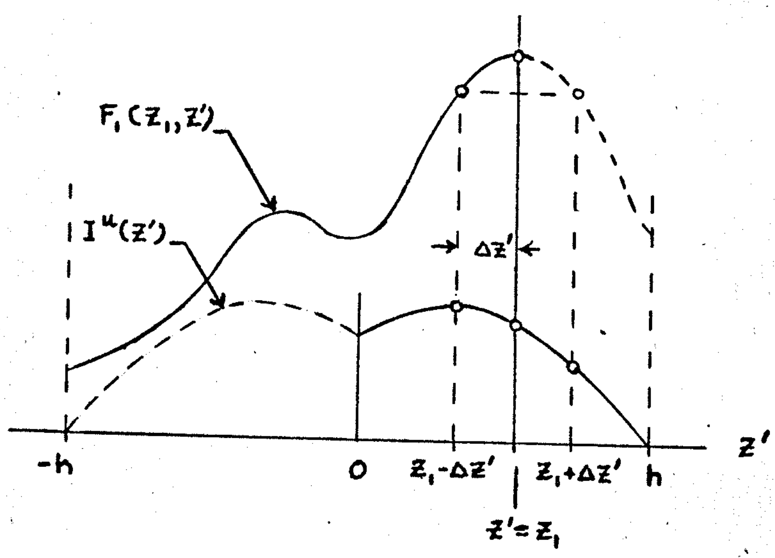


Figure 5.3 INTEGRATING $I^u(z')F_1(z_1,z')$

Thus, $\int_{-h}^{h} I^u(z')F_1(z_1,z')dz'$ can be found using the summation of $I^u(z')F_1(z - z')$ products in the following manner:

$$\int_{-h}^{h} I^u(z')F_1(z_1,z')dz' \doteq \Delta z'\Big[I^u(z_1)F_1(0) + I^u(z_1+\Delta z')F_1(\Delta z')$$
$$+ I^u(z_1-\Delta z')F_1(-\Delta z') + I^u(z_1+2\Delta z')F_1(2\Delta z') + I^u(z_1-2\Delta z')F_1(-2\Delta z')$$
$$+ \cdots \Big]$$

Since $F_1(z - z') = F_1(z' - z)$ by symmetry, this simplifies to

$$\int_{-h}^{h} I^u(z')F_1(z_1,z')dz' \doteq \Delta z' \Big\{ I^u(z_1)F(0) + \big[I^u(z_1+\Delta z')$$
$$+ I^u(z_1-\Delta z')\big] F_1(-\Delta z') + \big[I^u(z_1+2\Delta z') + I^u(z_1-2\Delta z')\big] F_1(-2\Delta z')$$
$$+ \cdots \Big\} \tag{5.11}$$

Because $I^u(z')$ and $F_1(z - z')$ are already stored as linear arrays, performing the operations indicated by equation (5.11) is a simple matter of multiplying the two arrays together term by term.

Program logic is included to avoid calculations of products for $z' \geqslant h$ and calculation stops when $z' = -h+\Delta z'$. At this point, all the product terms have been summed and this number is multiplied by $I^v(z_1)$. z_1 is incremented by Δz and the procedure repeated until z covers the range z=0 to $z=h-\Delta z$. Conditioning the z=0 term by a factor of one-half, summing all terms for z from 0 to $h-\Delta z$, and then multiplying by $\Delta z' \Delta z$ results in the approximation to

$$\int_0^h \int_{-h}^h I^v(z)\, I^u(z')\, F_1(z,z')\, dz'\, dz \,.$$

Similar logic follows for the other double integrals.

The remainder of the computer program is relatively straightforward.

CHAPTER VI

RESULTS

In this chapter, the results of the computer calculations are presented in graphical form. Figures 6.1, 6.2, 6.3, and 6.4 were obtained using Tai's trial currents, while Figures 6.5 and 6.6 were obtained using Harrington's trial currents. Calculations involving Tai's approximation were done for dipole half-lengths from 0.05λ to 1.1λ at .05λ intervals. Calculations involving Harrington's approximation were done for dipole half-lengths from 0.05λ to 1.4λ at .05λ intervals. For all calculations, antenna half-length to radius ratio was taken as 74.2. Various curves from the literature[1] are plotted for the sake of comparisons.

All computer calculations were done in single precision.

A Note On Antenna Thickness

The half-length to radius ratio used in all the calculations is 74.2. For those familiar with the Hallen integral - equation method, this ratio corresponds to Ω= 10, where Ω is defined by:

$$\Omega = 2 \ln \left(\frac{2h}{a} \right)$$

Most VHF and UHF antennas that are self-supporting will be about this thickness or likely somewhat thicker.

To analyze thinner antennas, exactly the same procedure as outlined in this thesis can be used. However, it is to be expected that as the antennas become thinner F_1 and F_2 will become more singular in nature, thus requiring the integrations to be performed with smaller values of dz' and dz.

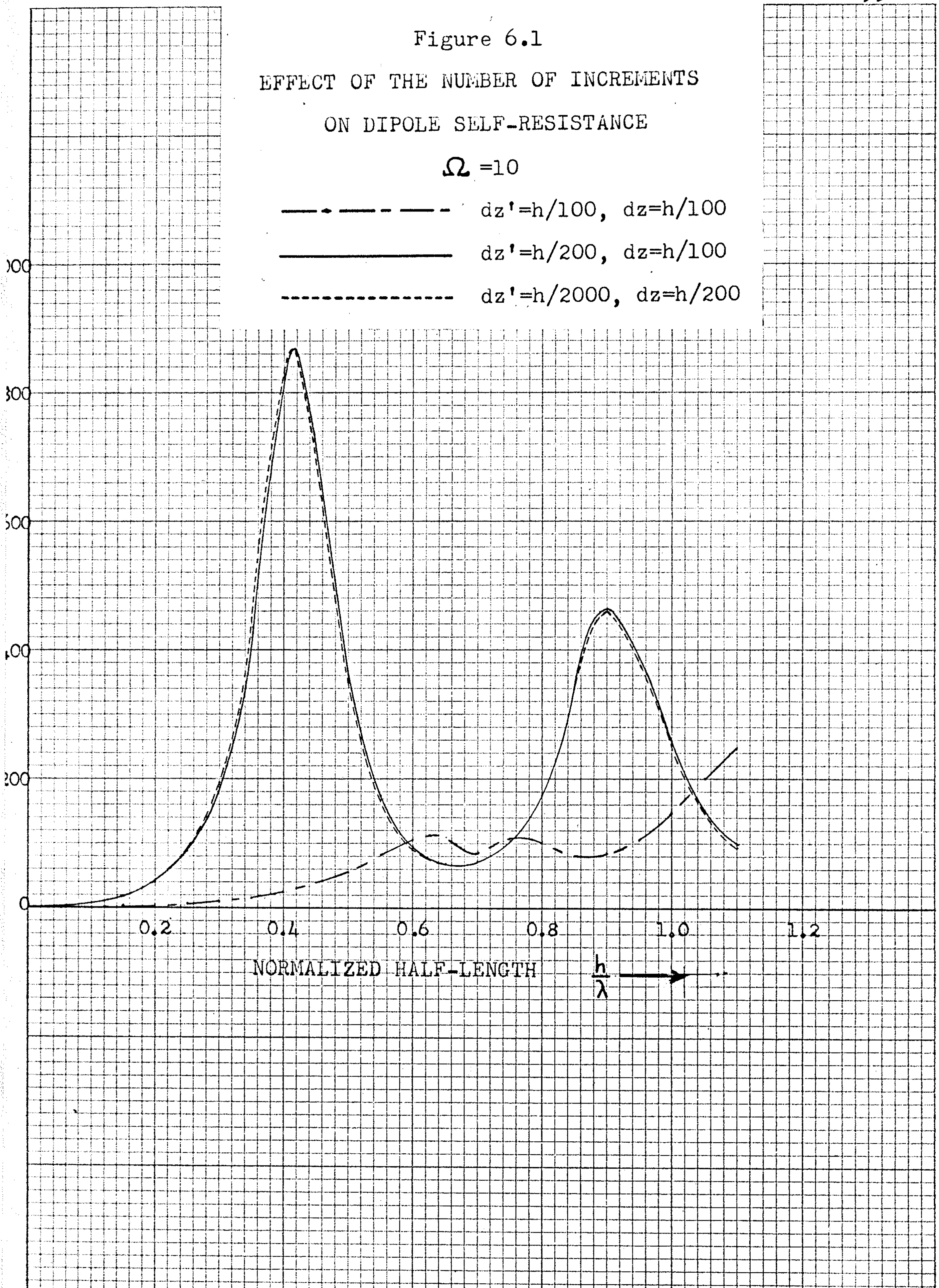


Figure 6.1

EFFECT OF THE NUMBER OF INCREMENTS ON DIPOLE SELF-RESISTANCE

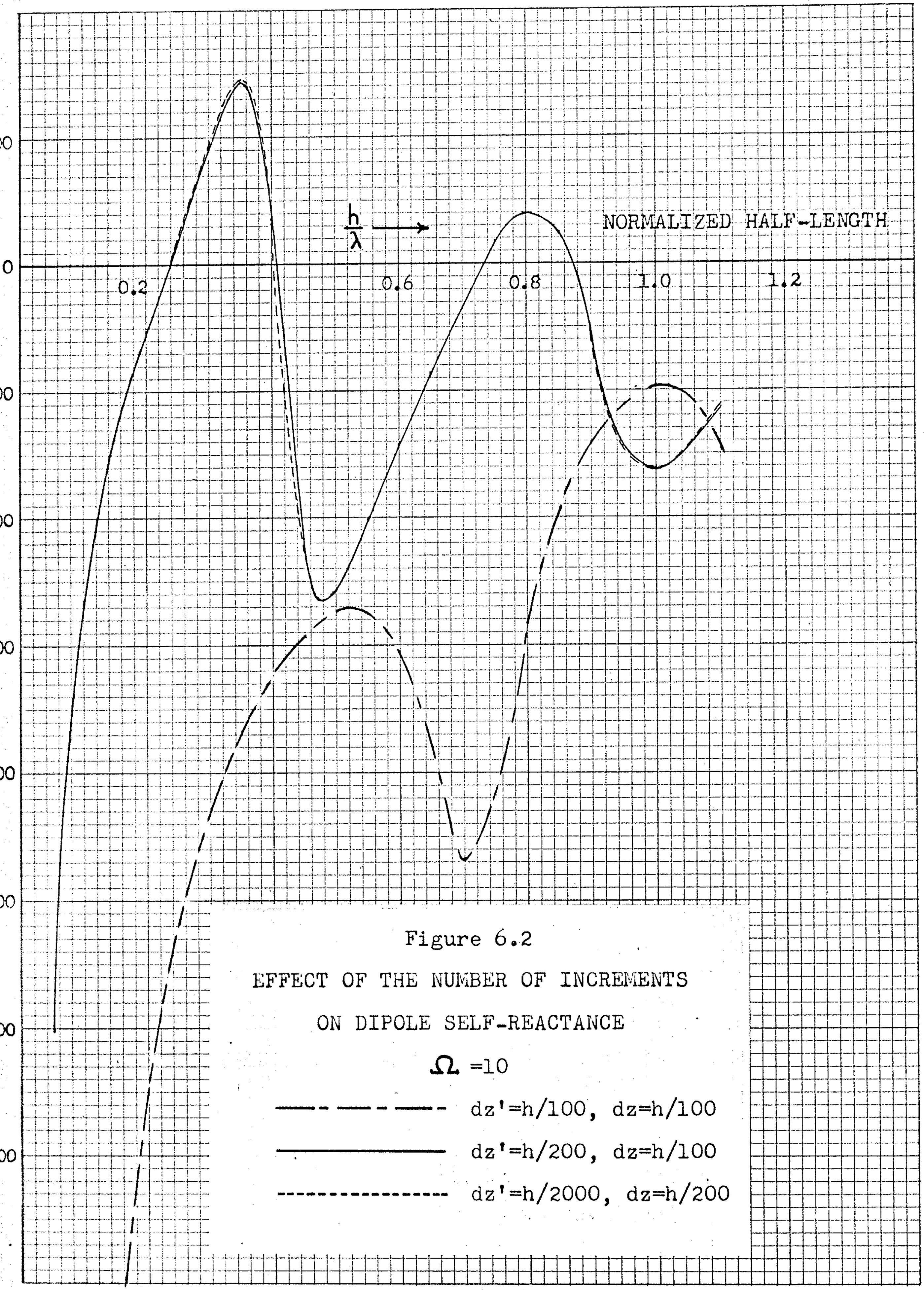


Figure 6.2

EFFECT OF THE NUMBER OF INCREMENTS

ON DIPOLE SELF-REACTANCE

Ω =10

— - — - — dz'=h/100, dz=h/100

——— dz'=h/200, dz=h/100

- - - - - - dz'=h/2000, dz=h/200

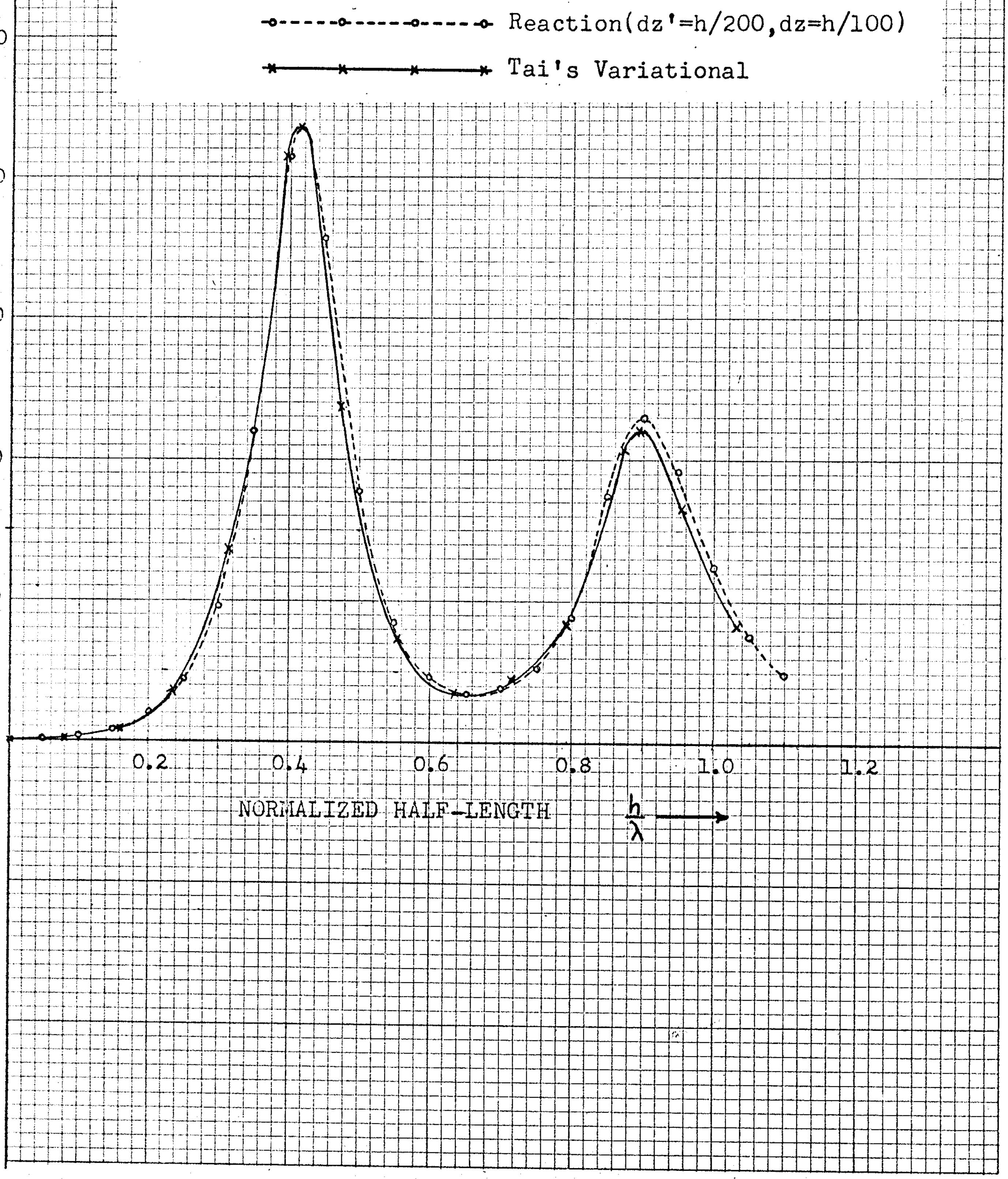
Figure 6.3

DIPOLE SELF-RESISTANCE

REACTION CONCEPT USING TAI'S CURRENT APPROXIMATION

COMPARED WITH TAI'S VARIATIONAL SOLUTION

$\Omega = 10$



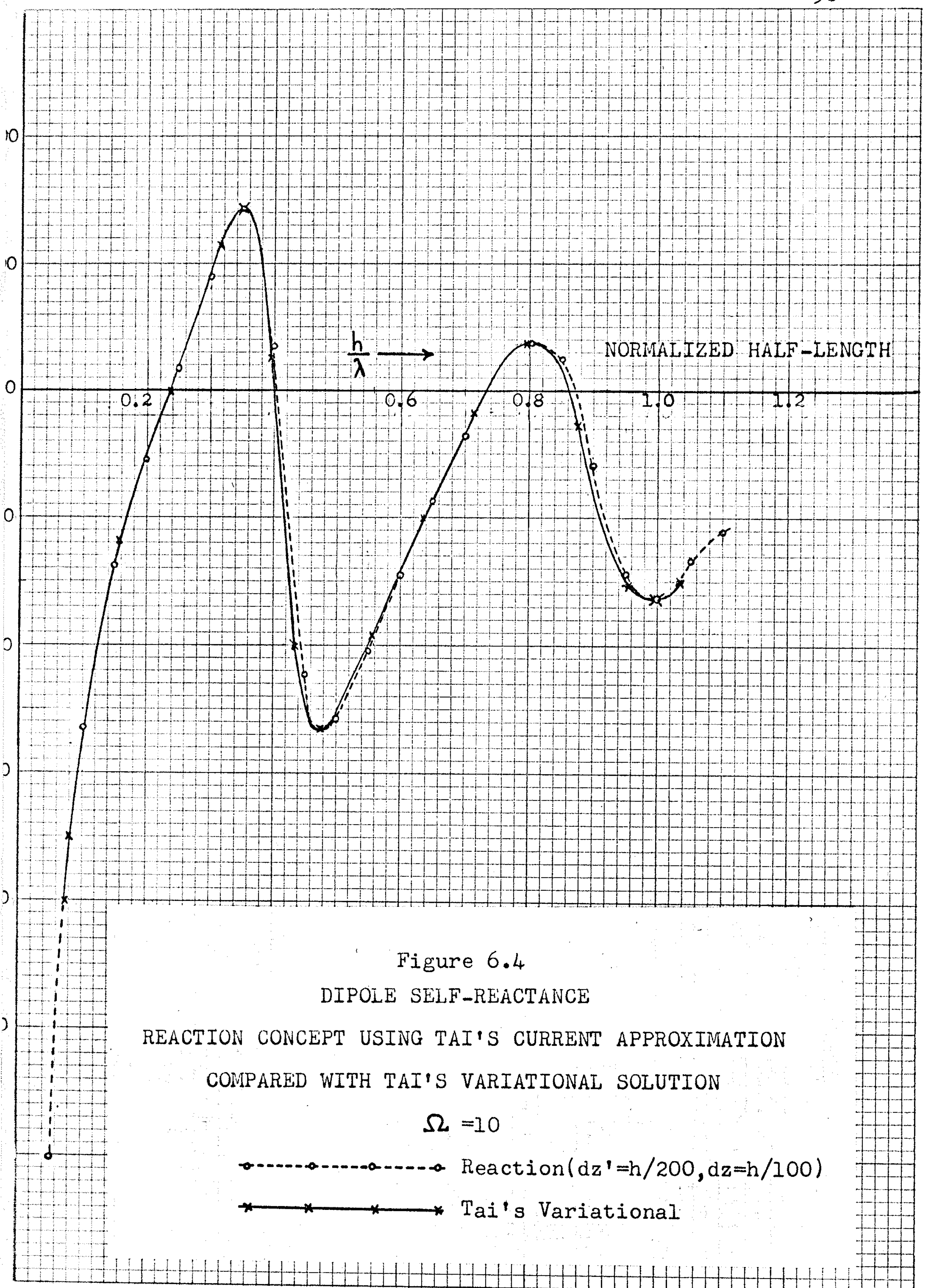


Figure 6.4

DIPOLE SELF-REACTANCE

REACTION CONCEPT USING TAI'S CURRENT APPROXIMATION

COMPARED WITH TAI'S VARIATIONAL SOLUTION

$\Omega = 10$

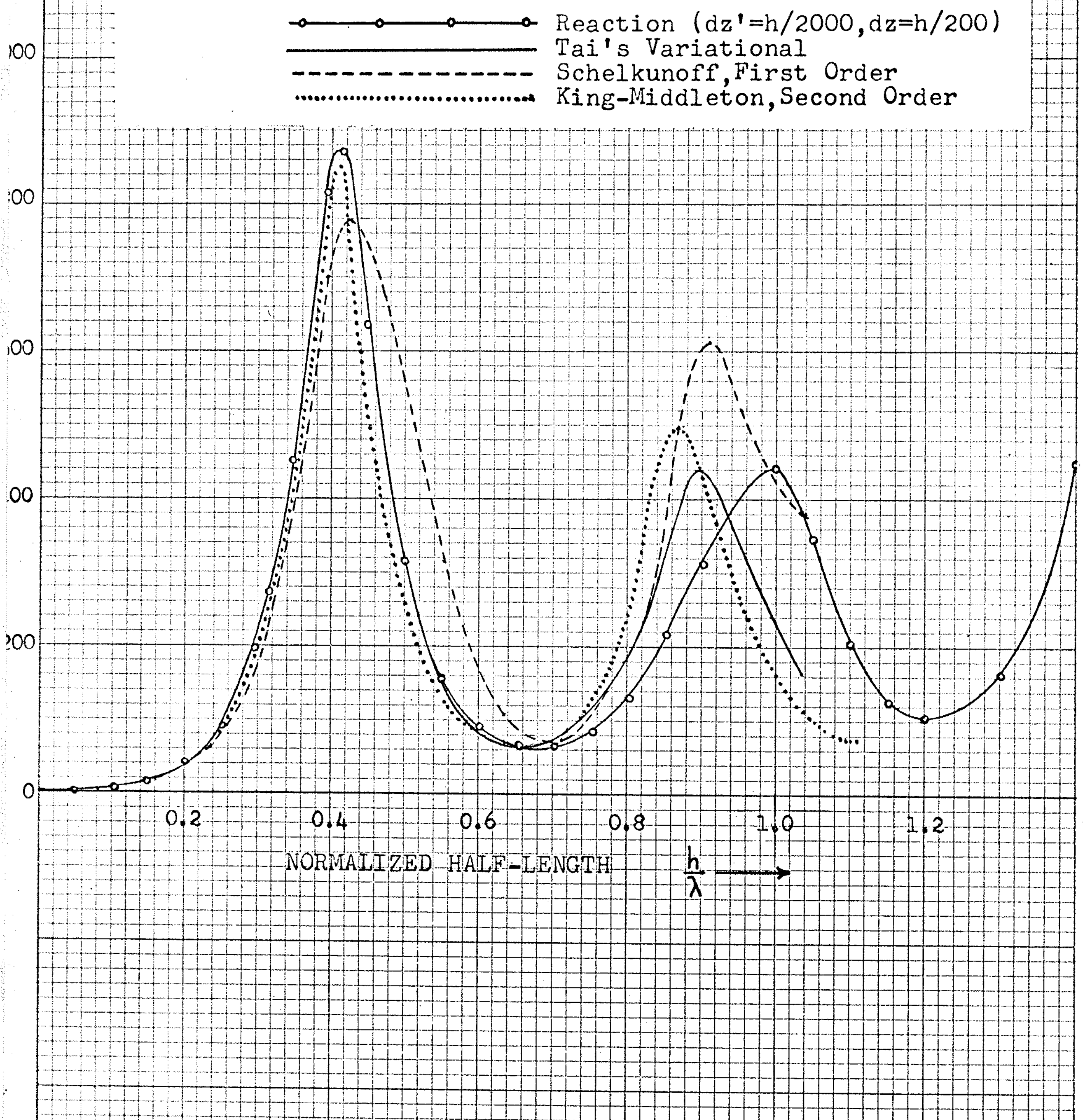
Figure 6.5

DIPOLE SELF-RESISTANCE

REACTION CONCEPT USING HARRINGTON'S CURRENT APPROXIMATION

COMPARED WITH DIFFLRENT METHODS

$\Omega = 10$



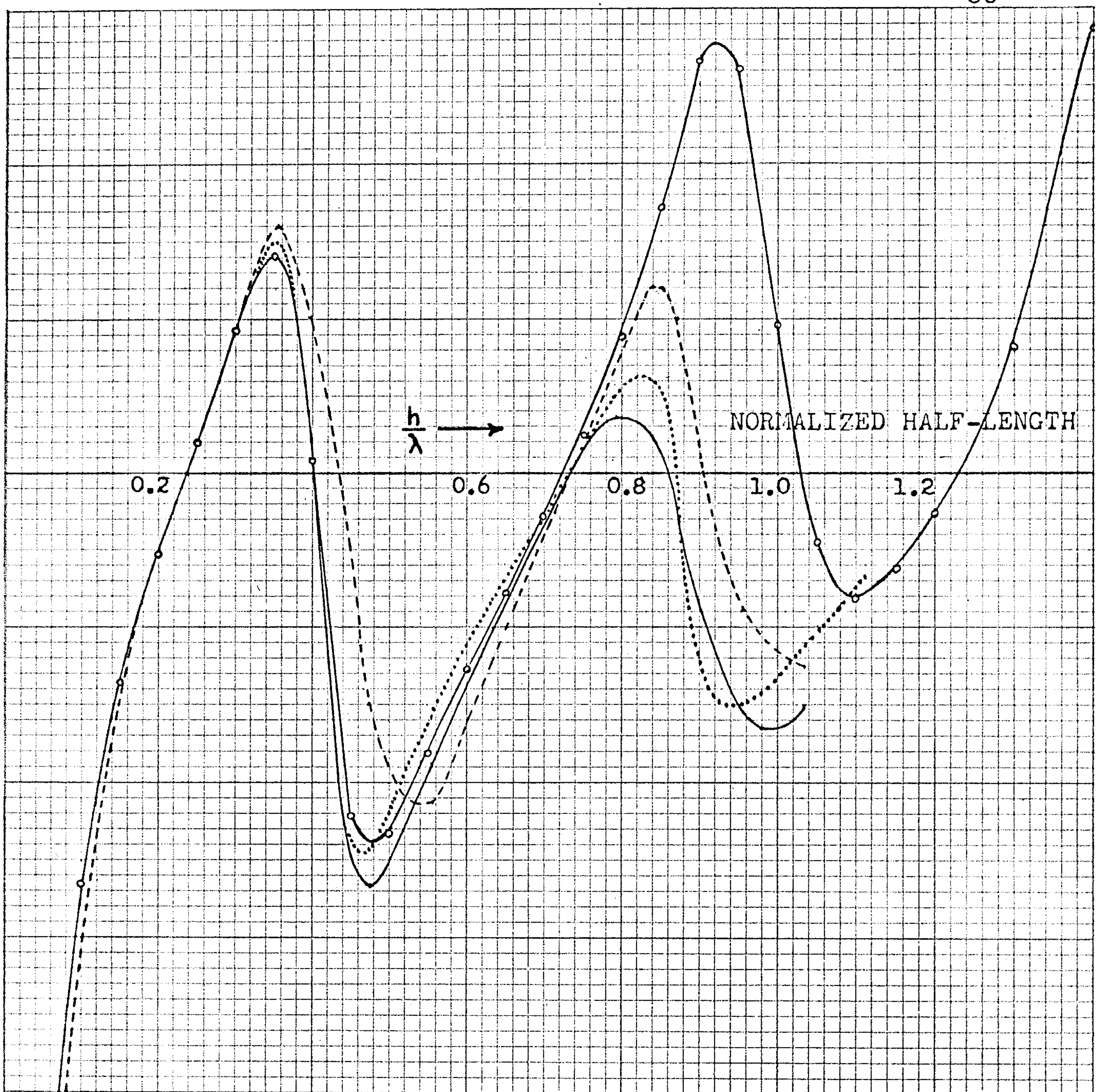


Figure 6.6

DIPOLE SELF-REACTANCE

REACTION CONCEPT USING HARRINGTON'S CURRENT APPROXIMATION

COMPARED WITH DIFFERENT METHODS

$\Omega = 10$

Reaction ($dz' = h/2000$, $dz = h/200$)
Tai's Variational
Schelkunoff, First Order
King-Middleton, Second Order

VII

DISCUSSION AND CONCLUSIONS

The Effect Of The Number Of Increments

Figures 6.1 and 6.2 strikingly illustrate the importance of using small enough values of dz' and dz. The results for $dz' = \frac{h}{100}$ and $dz = \frac{h}{100}$ show almost no resemblance to the results obtained by merely reducing dz' to $\frac{h}{200}$. Furthermore, the convergence of the answers is quite abrupt. The results for $dz' = \frac{h}{2000}$, $dz = \frac{h}{200}$ differ only very slightly from the results with dz' a factor of 10 and dz a factor of 2 times larger.

Though not plotted, results were obtained for many other choices of dz' and dz, namely:

$$dz' = \frac{h}{400} \qquad dz = \frac{h}{100}$$

$$dz' = \frac{h}{500} \qquad dz = \frac{h}{100}$$

$$dz' = \frac{h}{1000} \qquad dz = \frac{h}{100}$$

$$dz' = \frac{h}{2000} \qquad dz = \frac{h}{100}$$

In all cases the results fell between the results for $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$, and $dz' = \frac{h}{2000}$, $dz = \frac{h}{200}$.

The numerical integration was found most sensitive for the inner integral, the one with respect to z'. Thus dz' must be smaller than dz. Results were calculated using $dz = \frac{h}{50}$, $dz' = \frac{h}{500}$ that differed by less than 10% from corresponding values obtained with $dz = \frac{h}{100}$, $dz' = \frac{h}{200}$.

Since results can be dependent on the size of dz' and dz, calculations using this numerical method may have to be repeated several times for various values of dz' and dz until increment values are found such that further decreases in size do not appreciably influence the results.

Results Using Tai's Approximation

Figure 6.3 and 6.4 compare the results obtained by the Reaction Concept with the results obtained by Tai using the Variational Method. The agreement for both resistance and reactance is excellent.

The Reaction Concept results plotted were obtained using $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$. The results using $dz' = \frac{h}{2000}$, $dz = \frac{h}{200}$ fall even closer to Tai's results. Such close agreement indicates that both methods lead to the same equation for impedance. Numerical integration likely accounts for calculation differences.

Results Using Harrington's Current Approximation

Figures 6.5 and 6.6 illustrate that the success of the Reaction Concept is quite dependent on the current approximation used. These figures compare the results obtained using the Reaction Concept with Harrington's trial currents and the results obtained by Tai using the variational solution. Harrington's approximation is seen to yield very good agreement with Tai's results for dipole half-lengths less than about 0.7λ. However, for antennas longer than this, the Reaction Concept using Harrington's approximation yields results that rapidly diverge from Tai's results. This is particularly true for the case of reactance where the error becomes extremely large, although the curves still retain similar shapes.

Just as a point of interest, the results using the King-Middleton Method and Schelkunoff's Method are also plotted on Figures 6.5 and 6.6. Even though an appreciable difference is seen to exist between the results of Schelkunoff and those of King-Middleton and Tai, still Tai considers Schelkunoff's First Order Theory to yield fairly good results. For antenna half-length less than 0.7λ, the Reaction Concept using Harrington's current approximation provides results that are much more in line with Tai and King-Middleton than are the results of Schelkunoff.

Ease Of Re-Programming For Different Current Approximation

It might be expected that using $I^V = h - |z|$ in Harrington's approximation, instead of $I^V = 2\pi(h-|z|)\cos 2\pi(h-|z|)$ as used in Tai's approximation, would make analytical integration easier. However, this is not so, and the integration leads to extremely complicated expressions like those given in the final footnote of an article by King and Harrison Jr.[12].

On the other hand, when doing the integrations numerically only four computer cards had to be changed to change the program from Tai's approximation to Harrington's approximation. The changes required are pointed out in Appendix A.

Calculation Precision:

The results using Tai's approximation and $dz' = \frac{h}{2000}$, $dz = \frac{h}{200}$ were calculated using both double and single precision on the computer. The final answers by the two different methods varied by no more than ± 1 in the fourth significant digit. Hence, single precision was considered adequate.

Storage Requirements:

Appendix A shows the computer program to be quite short, so it poses no storage problem in itself. The majority of the storage is used for the F_1, F_2, I^u, and I^v arrays. The number of values stored depends of course on the number of increments used in the integration with respect to z'. Table 7.1 shows the storage requirements of the arrays for three different values of dz'.

Table 7.1 Array Storage Requirements For Various dz' Values

dz'	F_1	F_2	I^u	I^v	Total Number Of Stored Values
$\frac{h}{200}$	400	400	200	200	1200
$\frac{h}{1000}$	2000	2000	1000	1000	6000
$\frac{h}{2000}$	4000	4000	2000	2000	12,000

The particular compiler used would not allow dimensioning arrays to store more than 12,000 values total in the central processing unit. For values of dz' smaller than $\frac{h}{2000}$, disc storage would have to be utilized.

Time Considerations:

Of prime inportance is the time required to calculate the results on the computer. With Tai's current approximation, the calculations with $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$ offer a good compromise between accuracy of results on one hand and calculation time on the other. For $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$ calculation time for resistance and reactance corresponding to a particular value of h is 15 seconds, using Fortran IV, single precision, on the IBM 360/ Model 50.

When dz' is divided by 10 to $dz' = \frac{h}{2000}$ and dz is divided by 2 to $dz = \frac{h}{200}$, then computation time is closely 10 X 2 = 20 times as long as for $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$. That is, computation time for $dz' = \frac{h}{2000}$, $dz = \frac{h}{200}$ is about 300 seconds for each value of h.

However, the IBM 360/ Model 65 will perform the same calculations 3 to 4 times faster than the Model 50. Thus, using the model 65 results for $dz' = \frac{h}{200}$, $dz = \frac{h}{100}$ could be obtained in less than 5 seconds for each value of h.

In all cases, compilation time on the Model 50 was about 16 seconds.

Conclusions

(1) The Reaction Concept coupled with numerical double integration provides a very good method for solving for the input impedance of an isolated dipole. It leads by a straightforward procedure to excellent results within an acceptable computing time.

(2) The success of the Reaction Concept is dependent upon the choice of the current approximation. Using numerical integration, different current approximations can be tried with relative ease.

(3) Unless integration increments are small enough, the results obtained by the method of this thesis are meaningless. On the other hand, using more increments than necessary results in unnecessary computer computation time.

(4) The present state of computer technology is such as to make numerical double integration practical for solving integrals similar to those encountered in this thesis.

(5) On the basis of its success in solving the dipole problem, the Reaction Concept utilizing numerical double integration merits consideration as a method for solving for impedances of other antenna configurations.

(6) The Reaction Method appears to yield the same equation for impedance as does the Variational Method.

APPENDIX A

SAMPLE COMPUTER PROGRAM

The computer program presented in this section is one to calculate self-resistance and self-reactance using Tai's current approximation with $dz' = \frac{h}{500}$, $dz = \frac{h}{50}$ and h varying from 0.05 λ to 1.1 λ. The card changes necessary to convert the program to Harrington's approximation are placed in parenthesis alongside the corresponding card in the sample program.

Some comments are important regarding the program. Firstly, it is to be noted that the program is written as if two different antenna lengths h_1 and h_2 and four trial currents I^u, I^v, I^m, and I^n are involved. The reason for this is that the program used for finding self-impedance originally formed a portion of a program the author was preparing to find mutual impedance between two parallel V - antennas of different lengths h_1 and h_2. It was observed that by letting $h_1=h_2=h, I^u=I^m$, and $I^v=I^n$, and letting the distance of separation of the two antennas equal the radius of the single dipole, that one portion of the mutual impedance equation reduced to the self-impedance equation for the dipole. Harrington[6] makes similar observations to obtain the first order (one trial function) variational solution for input impedance of a dipole from his first order dipole mutual impedance solution.

By retaining the mutual impedance features of the program, many calculations become redundant. For example, the real and imaginary parts of both $\langle u,n \rangle$ and $\langle v,m \rangle$ are calculated even though they are the same quantities for the self-impedance case. However, this is felt to be a desirable feature for this thesis as the program combines somewhat the worst aspects of impedance calculations - namely, the greater number of calculations that would be involved for mutual impedance and the more singular nature of the F_1 and F_2 functions in the self-impedance case. Thus, computation times using this program will be slightly on the conservative side.

```
      SELF IMPEDANCE
      RESISTANCE AND REACTANCE

    1 FORMAT(3F20.0,I10)
    3 FORMAT(1H0,3HH1=F18.8,2X3HH2=F18.8,2X6HRATIO=F18.8)
    4 FORMAT(1H ,11HRESISTANCE=F18.8,5X10HREACTANCE=F18.8)
   90 FORMAT(1H ,4HRUM=F25.0,4HCUM=F25.0,4HRUN=F25.0,4HCUN=F25.0)
   91 FORMAT(1H ,4HRVM=F25.0,4HCVM=F25.0,4HRVN=F25.0,4HCVN=F25.0)
      DIMENSION RF(2001),CF(2001),SN(1001),CS(1001)

    9 READ (1,1) TOPI,H1,RATIO,NO
)0    DO 80 N=1,22
      H1=H1+0.05
      H2=H1
      D=H1/RATIO
      DZ2=0.02*H2
      DZ1=0.002*H1
      DZDZ=DZ1*DZ2
      TOPI2=TOPI**2
      D2=D**2
20    DO 82 I=1,500
      M=I-1
      XM=M
      Z1=XM*DZ1
      XZ1=TOPI*(H1-Z1)
      SN(I)=SIN(XZ1)
      CS(I)=XZ1*COS(XZ1)                    (CS(I)=H1-Z1)
      ZMZ2=Z1**2
      DIST2=ZMZ2+D2
      DIST=SQRT(DIST2)
      A1=1.000000/DIST2
      A2=ZMZ2*A1
      A3=3.000000*A2
      A4=A3/DIST2
      A5=TOPI2*A2
      TERM2=(-TOPI2+A5+A1-A4)/DIST
      TERM1=TOPI*A1*(-1.000000+A3)
      TOPIS=TOPI*DIST
      COSR=COS(TOPIS)
      SINR=SIN(TOPIS)
      RF(I)=COSR*TERM1+SINR*TERM2
      CF(I)=COSR*TERM2-SINR*TERM1
2     CONTINUE
30    DO 83 I=501,1000
      M=I-1
      XM=M
      ZMZ2=(XM*DZ1)**2
      DIST2=ZMZ2+D2
      DIST=SQRT(DIST2)
      A1=1.000000/DIST2
      A2=ZMZ2*A1
      A3=3.000000*A2
      A4=A3/DIST2
      A5=TOPI2*A2
      TERM2=(-TOPI2+A5+A1-A4)/DIST
      TERM1=TOPI*A1*(-1.000000+A3)
      TOPIS=TOPI*DIST
```

```
      COSR=COS(TOPIS)
      SINR=SIN(TOPIS)
      RF(I)=COSR*TERM1+SINR*TERM2
      CF(I)=COSR*TERM2-SINR*TERM1
      CONTINUE
      RUM=0.0
      CUM=0.0
      RUN=0.0
      CUN=0.0
      RVM=0.0
      CVM=0.0
      RVN=0.0
      CVN=0.0
      DO 5 K=1,50
      J=K-1
      XJ=J
      Z2=DZ2*XJ
      XZ2=TOPI*(H2-Z2)
      XMMULT=SIN(XZ2)
      XNMULT=XZ2*COS(XZ2)               (XNMULT=H2-Z2)
      RUMN=0.0
      CUMN=0.0
      RVMN=0.0
      CVMN=0.0
      J101=J*10+1
      L=-1
      L=L+1
      LL=L+1
      IP=J101+L
      IM=J101-L
      IF(L) 13,18,13
   13 IF(IM) 14,14,15
   14 IM=-IM+2
      IF(IM-500) 15,15,16
   16 GO TO 57
   15 IF(IP-500) 17,17,18
   17 UX=SN(IP)+SN(IM)
      VX=CS(IP)+CS(IM)
      GO TO 19
   18 UX=SN(IM)
      VX=CS(IM)
   19 RUX=RF(LL)*UX
      CUX=CF(LL)*UX
      RVX=RF(LL)*VX
      CVX=CF(LL)*VX
      RUMN=RUMN+RUX
      CUMN=CUMN+CUX
      RVMN=RVMN+RVX
      CVMN=CVMN+CVX
      GO TO 84
      IF(K-2) 7,8,8
    7 RUMN5=RUMN*0.5000000
      CUMN5=CUMN*0.5000000
      RVMN5=RVMN*0.5000000
      CVMN5=CVMN*0.5000000
      RUM=RUM+XMMULT*RUMN5
      CUM=CUM+XMMULT*CUMN5
      RUN=RUN+XNMULT*RUMN5
      CUN=CUN+XNMULT*CUMN5
      RVM=RVM+XMMULT*RVMN5
```

```
      CVM=CVM+XMMULT*CVMN5
      RVN=RVN+XNMULT*RVMN5
      CVN=CVN+XNMULT*CVMN5
      GO TO 5
    8 RUM=RUM+XMMULT*RUMN
      CUM=CUM+XMMULT*CUMN
      RUN=RUN+XNMULT*RUMN
      CUN=CUN+XNMULT*CUMN
      RVM=RVM+XMMULT*RVMN
      CVM=CVM+XMMULT*CVMN
      RVN=RVN+XNMULT*RVMN
      CVN=CVN+XNMULT*CVMN
      CONTINUE
   72 ARG1=TOPI*H1
      ARG2=TOPI*H2
      FIM=SIN(ARG2)
      FIN=ARG2*COS(ARG2)              (FIN=H1)
      FIU=SIN(ARG1)
      FIV=ARG1*COS(ARG1)              (FIV=H1)
      FIVIM=FIV*FIM
      FIUIM=FIU*FIM
      FIVIN=FIV*FIN
      FIUIN=FIU*FIN
      RNUM1=RUM*RVN-CUM*CVN
      CNUM1=RUM*CVN+RVN*CUM
      RNUM2=RUN*RVM-CUN*CVM
      CNUM2=RUN*CVM+RVM*CUN
      RNUM=RNUM1-RNUM2
      CNUM=CNUM1-CNUM2
      RDEN=FIVIM*RUN-FIUIM*RVN-FIVIN*RUM+FIUIN*RVM
      CDEN=FIVIM*CUN-FIUIM*CVN-FIVIN*CUM+FIUIN*CVM
      RDEN2=RDEN**2
      CDEN2=CDEN**2
      DENOM=RDEN2+CDEN2
      B1=RNUM*RDEN+CNUM*CDEN
      RRRR=B1/DENOM
      B2=RDEN*CNUM-RNUM*CDEN
      COMPL=B2/DENOM
      FACTOR=60.00000*DZDZ/TOPI
      RESIST=FACTOR*RRRR
      REACT=FACTOR*COMPL
      WRITE (3,3) H1,H2,RATIO
      WRITE (3,4) RESIST,REACT
      WRITE (3,90) RUM,CUM,RUN,CUN
      WRITE (3,91) RVM,CVM,RVN,CVN
      CONTINUE
      IF(NO) 9,9,101
  101 CALL EXIT
      END
ATA
283185          0.0000000          74.20000          10000
```

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