Towards a Collective Understanding of Fraction as Quotient and a Deeper Mathematical Knowledge for Teaching

by

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Abstract

Research conducted in many countries has shown that teaching and learning fractions is a challenge for students and teachers (Cramer et al., 2002; Kieran, 1976; Lamon, 2007; Moss & Case, 1999; Tian & Siegler, 2018; Van Steenbrugge et al., 2014). One proposed cause of limited fractional understanding is an overemphasis on part-whole and an under emphasis on the four other interpretations of fractions in curriculum standards and learning activities (Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Lamon, 2007). Quotient, one of the five interpretations, is argued to be important as it is linked to multiplicative structures (Lamon, 2007), magnitude (Johanning & Mamer, 2014) and cardinality (Cooper et al., 2012). In this study, I used a methodology informed by hermeneutic phenomenology to analyse the K-8 mathematics curriculum standards and support documents in the province of Manitoba, Canada and found little explicit attention to the quotient interpretation. I also facilitated three interactive inquiry sessions with a small group of experienced Grades 4-8 teachers to explore their understanding of the quotient interpretation of fractions. Analysis of the transcripts and artifacts from the sessions revealed seven features which seemed to impact the development of mathematical knowledge for teaching (MKT) of fraction as quotient: problem-solving contexts, the quotient notation and generalization, re-defining quotient as more than the answer to a division question, difficulty describing fraction as quotient, lack of focus on fractions as relationships, working with indefinite amounts, and curriculum standards and support materials. Recommendations for schools, divisions, and Manitoba Education to support teachers with the concept of fraction as quotient are made and areas of future research are identified.

Keywords: quotient, fractions, mathematical knowledge for teaching, mathematics curriculum standards

Chapter 1: Introduction & Research Questions

Research conducted in many countries has shown that teaching and learning fractions is a challenge for both students and teachers (Cramer et al., 2002; Moss & Case, 1999; Tian & Siegler, 2018). Challenges with learning fractions have been linked to many causes such as difficulty with switching from additive thinking to multiplicative thinking (Cooper et al., 2012) and an overemphasis on a fraction as part of a whole, as compared with the other four interpretations of fractions (Behr et al., 1992). This study focuses on the interpretation of fractions known as 'fraction as quotient'; specifically, how teachers talk about their understanding of this interpretation and the ways in which it is evident in the Manitoba Education (2013) mathematics curriculum standards and in some support documents for teachers. This study addresses a gap I identified in the mathematics education literature by providing a close look at how practicing teachers understand the fraction as quotient interpretation and by describing how this interpretation is supported in the mathematics (2013) curriculum standards and support documents for the province of Manitoba, Canada.

Purpose and Research Questions

A deep understanding of fractions in students has been linked to later mathematical proficiency (Ezaki et al, 2023; Tian & Siegler, 2018; Salls, 2014). An understanding of fractions also supports the ability to reason proportionally, which Lamon (2005) argues is the foundation for larger, more complex topics in mathematics and science (p. 3). Several scholars highlight the need for teachers to explore the complex meaning of the quotient interpretation with students to develop a deep number and operation sense (Johanning & Mamer, 2014; Middleton et al., 2001;

Poon & Lewis, 2015). This study makes a key contribution to the mathematics education literature by examining how the concept of fraction as quotient is taken up in Manitoba, Canada.

Since curriculum standards are intended to direct what educators teach, this study provides an analysis of Manitoba's kindergarten to Grade 8 curriculum standards for mathematics (Manitoba Education, 2013) and some selected support documents for teachers. I analysed these documents in comparison with suggestions put forth by mathematics education researchers related to the effective teaching of fractions. Based on research regarding the importance of teaching all five interpretations of fractions, I also wanted to learn more about how educators, with at least five years teaching experience, understand fraction as quotient. Through dialogue with a small group of teachers, I hoped to better understand the complexity of this fraction interpretation and support our collective efforts to build our mathematics knowledge for teaching or MKT (Ball et al., 2008) about this aspect of mathematics teaching and learning.

My study was guided by the following research questions:

1. In what ways is fraction as quotient included in the K-8 Manitoba Curriculum Framework of Outcomes (2013) and in the Grades 3-8 Support Documents for Teachers? How does the approach to fraction as quotient in these documents relate to recommendations about this interpretation in the mathematics education literature?
2. What helps and hinders a group of experienced mathematics teachers working in Grades 4-8 to develop an understanding of fraction as quotient?
3. How does working collaboratively shift the ways that a group of teachers, including myself, think about teaching and learning fraction as quotient? Specifically, how do we

build our MKT of fraction as quotient and how does learning about fraction as quotient build our MKT?

Positionality

My experience with mathematics, specifically regarding fractions, both as a student and as a teacher affected how I conducted this study. As a student in elementary school, I had enjoyed math and had teachers in the early grades who made the mathematical concepts accessible to me through hands-on experiences. However, when I reached Grade 6, I have clear memories of being taught how to add and subtract fractions based on a memorized set of rules. This is when I became lost and could not keep up, instead allowing other, faster, and what I thought of as "smarter" students to answer questions. I was under the impression that I must be lacking a skill and that developing it was now unavailable to me. Years later in my beginning stages as a pre-service teacher, I discovered ways in which math could be accessible for all; through a better understanding of the curriculum standards and how the big ideas in math develop over time, as well as creatively designed lesson plans and a strong sense of pedagogy.

Now, after more than ten years as a generalist teacher of the early/middle grades in the province of Manitoba, I can confidently say that mathematics is one of my favourite subjects to teach. I have maintained a strong interest in how concepts are taught so that they can be accessible for students but I have also continued to grow in my MKT. I have taught Grades 3-6 and have been able to work with many Grade 3 classes as they journey into their first experiences with learning curricular outcomes focused on fractions. Due to the challenging nature of fractions for children (Cramer et al., 2002; Kieran, 1976; Tian & Siegler, 2018), I took this job seriously; striving to introduce ideas in conceptually appropriate ways for students, such as using a variety of representations beyond that of the circle, and delaying the introduction of numerical representations of fractions until the children displayed signs of readiness.

From my experience, fractions are a challenging concept for educators to teach. In professional development sessions, learning communities, and dialogue with colleagues, I have seen this trope often shared between the teachers of Grades 4-8. I have heard teachers report feeling unsure about the best way to teach certain concepts even as they struggle to accept a lack of student understanding with ideas such as equivalent fractions. Drawing on elements of Ball's theory of MKT (Ball et al., 2008) which I explore in Chapter 2, I believe that the struggle that teachers experience may relate to their missing content knowledge and need for a wider horizon view to better support students who demonstrate limited conceptual understanding.

In addition to my professional background and positionality, I also come from a place of curiosity and confusion. Although I have loved learning about mathematics concepts and teaching methods for the past ten years, I had never heard of the five interpretations of fractions until my graduate studies. When I took a closer look, I began to realize the influence and importance they could offer for a full fractional sense to develop for students and for teachers. And yet, I had never come across these interpretations explicitly identified in the Manitoba curriculum standards (Manitoba Education, 2013) or support documents for teachers, nor had I explored any interpretations of fractions other than part-whole and measure in my teaching. Therefore, I wanted to know more about how other teachers understand the five interpretations, with a specific look at quotient, and to examine the ways that the provincial curriculum standards and support documents support this way of thinking about fractions.

Overview of Thesis

In Chapter 2, I share the literature review that I conducted related to various aspects of teaching and learning fractions. I also summarize the three theoretical frameworks that I used to guide my study: the five interpretations of fractions, social constructivism, and mathematical knowledge for teaching. In Chapter 3, I describe my methodology which is rooted in hermeneutic phenomenology and explain the methods I used for data collection and analysis. In Chapter 4, I share my research findings from the document analysis and the interactive inquiry sessions in relation to the three research questions. Finally, in Chapter 5, I discuss the research findings in relation to existing literature, offer some recommendations, consider the significance and limitations of the study, and highlight areas for future research as well as next steps for myself as an educator.

Chapter 2: Literature Review & Theoretical Frameworks

In this chapter, I begin with a review of the literature that informed my research design as well as my data analysis, findings, and the recommendations I made based on the results of this study. I then explain some of the theoretical and empirical work that has been done around the five interpretations of fractions. In addition to this, since part of the study focuses on how a group of educators learn and understand fraction as quotient, I provide an overview of the theory of social constructivism which guided my planning, facilitation and interpretation of the interactive inquiry sessions. I also share an overview of Mathematical Knowledge for Teaching (MKT), a theory proposed by Ball et al. (2008), and explain how it connects to this study. Finally, I explain the commensurability of all three theoretical frameworks in relation to my study of the quotient interpretation of fractions.

Literature Review

I begin this section with a review of some key research insights into teaching fractions. I then share some literature on how mathematics curriculum standards influence the approaches that teachers take when teaching fractions. Next, I share an overview of literature that discusses how educators, both practicing and pre-service, understand conceptual models of fractions. As I conducted my literature search, I found that many researchers have written about quotient as the answer to a division question but not necessarily in relation to fractions and that little research has been done about how educators understand quotient as one of the five interpretations of fractions. Although quotient in the sense of the answer to a division problem is not the focus of my research, a point I explore in Chapter 4 in relation to my findings, I found the broader quotient literature to be a useful support for framing some of the difficulties teachers have with understanding the concept of fraction as quotient. For this reason, I have included some of this literature in my review. Finally, I share literature about how teachers, both practicing and pre-service, talk about their understanding of fractions. This literature was helpful as I analysed the data from the interactive inquiry sessions in my study.

Research Insights about Teaching Fractions

Challenges with learning fractions have been linked to many causes such as difficulty with switching from additive thinking to multiplicative thinking (Cooper et al., 2012), introducing fractional computation too early (Suh et al., 2012), and an overemphasis on a fraction as part of a whole as compared with the four other interpretations (Behr et al., 1992). Teachers must also be cognizant of the whole number bias that students often possess, which has been seen to impede their fraction understanding, especially when it comes to magnitude, density, and calculations with fractions (Tian & Siegler, 2018; Brown, 2015). For example, whole number bias might be at play if a child believes that 1/4 is smaller than 1/5 because 4 is smaller than 5, or that 1/4 + 1/5 is 2/9.

Mathematics curricula can support the learning of fractions by including a balance between conceptual thinking and procedural work (Tossavainen & Helenius, 2024), showcasing the many interpretations of fractions in a variety of contexts (Charalambous & Pitta-Pantazi, 2007), and providing many opportunities for students to generalize their understanding about number relationships (Brown, 2015; Cramer et al., 2002; Lamon, 2007). Unfortunately, some researchers have shared that the memorization of fraction rules as a method of learning often leads to an incorrect or inadequate understanding of fractions and that many teachers continue to struggle with teaching fractions (Fosnot & Dolk, 2002; Singh et al., 2020; Tossavainen & Helenius, 2024).

Research on Educators' Use of Curriculum Standards when Teaching about Fractions

Research done around how educators use curriculum standards documents when teaching fractions (Cramer et al., 2002; Behr et al., 1992; Middleton et al., 2001; Moss & Case, 1999; Norton et al., 2018) provides insights into the ways that mandated documents can influence classroom practice. Notably, the sequence of content in the Common Core State Standards (CCSS) adopted in some parts of the US, as described by Daro et al. (2011), is explained as an "approximation of the order in which students should learn the required content and skills. However, the current state standards are more prescriptive than they are descriptive" (p. 16). Daro et al. also identify a difference in pedagogy between grade levels in mathematics documents and maintain that elementary grades should be focused on cognitive development, high school should be focused on the logical development of mathematics, and the middle grades should be a bridge between these two (p. 41). This is relevant because Daro et al. highlight a need for mathematics curriculum standards to have continuity between concepts. In Chapters 4 and 5, I argue that this continuity is especially important for the challenging topic of fractions.

Other studies done in the US reveal similar insights into the ways that curriculum standards documents shape how teachers and students explore conceptions of fractions. Middleton et al. (2001) noted that in the early years, division of whole numbers and division of fractions, which are almost exclusively conveyed as part-whole constructs, are treated as separate entities in US curriculum standards, leaving it up to the middle grades to connect these two concepts (p. 2). Although this research is older and the CCSS have changed since then, Middleton et al. (2001) highlight a similar point to Daro et al. (2011), which is that there needs to be consideration for how mathematical concepts are conceptually linked in the curriculum standards. Similarly, research done by Siegler et al. (2012) highlighted that students who have difficulties with fractions also have a weak sense of whole number division, suggesting the two concepts could be linked. Norton et al. (2018) looked at the progression of fraction schemes in the CCSS and revealed that they begin with part-whole interpretations, as compared to other curriculum standards that begin with the measure interpretation of fractions in countries such as those in China (p. 210). In a US study done by Cramer et al. (2002), the authors argue that the trajectory for teaching fractions to students should begin with initial concepts of order and equivalence, the meaning of symbols, and an understanding of relationships before computational fluency, which should be saved for later grades (p. 112).

Although explicit reference to fraction concepts may begin in a specific grade (e.g. Grade 3 in Manitoba's curriculum standards), Cutting (2024) maintains that concepts of number that are linked to later fractional understanding should be introduced before this and in contextually appropriate ways for children. Research suggests that some of these ways might include measurement (Steffe, 2002), equivalence (Kieran, 1976), and partitioning (Charalambous & Pitta-Pantazi, 2007; Cutting, 2024; Kieran 1980; Lamon, 2007). Partitioning activities are particularly valuable due to their connection to equivalence and unit structures (Lamon, 2007). In addition, Kieran (1980) stated that partitioning may be as important to developing one's rational number system as counting is for one's system of whole numbers (p. 138). As I show in Chapter 4, partitioning in the form of 'sharing problems' can be helpful for teachers as they engage with the concept of fraction as quotient.

Researchers have identified other ways that curriculum standards can influence student understanding. In a US study, Cramer et al. (2002) argued that there is an overemphasis on part-whole shading tasks and not enough exposure to comparing different models for fractions (p. 112). Behr et al. (1992) warned against the treatment of fraction equivalence as an isolated topic, while a Canadian study revealed a curricular focus on teaching procedures as well as a lack of differentiation between fractions and whole numbers (Moss & Case, 1999). Each of these researchers have also noted that there are misconceptions children bring with them into middle grades fraction learning as well as pedagogical gaps that teachers miss, revealing areas of improvement needed in the implementation of curriculum standards. These include concepts such as an overuse of the circle model for fractions (Cramer et al., 2002), a belief that rational numbers are the same as whole numbers (Moss & Case, 1999), and an overreliance on fractional procures without meaning (Moss & Case, 1999). Many researchers also argue that fraction learning is best supported by curriculum standards documents where adequate time is given to the outcomes (Cramer et al., 2002; Daro et al., 2011) and where multiple models, relationships, and interpretations can be explored (Cramer et al., 2002; Moss & Case, 1999; Norton et al., 2018). These studies highlight the key distinction between the content of curriculum standards and how these standards may be enacted or implemented in classrooms. For this study, I analysed the Manitoba Education (2013) mathematics curriculum standards and some related support documents to look for connections with the literature on teaching fractions, particularly for middle years mathematics and in relation to quotient. However, I recognize that individual teachers may implement these curriculum standards in various ways.

Research on Teachers' Understandings of Conceptual Models of Fractions

Research from various contexts have revealed that many teachers have difficulty developing and demonstrating a conceptual understanding of fractions. In a US study about teachers' rational number knowledge, Suh et al. (2012) revealed that practicing teachers had difficulty with representing a conceptual understanding of fractions, as they were overly focused on rules and procedures without meaning (p. 476). Similar studies done in Sweden (Tossavainen & Helenius, 2024) and Slovenia and Kosovo (Kolar et al., 2018) showed that pre-service teachers had difficulty conceptualizing the meaning of fraction operations and models. Brown (2015), from a university in South Africa, argued that a relational understanding of rational numbers, with a specific example given to fractions, is key for educators as it allows for "more flexible and appropriate interpretations of children's responses" (p. 5). Several studies have also revealed that pre-service teachers struggled with selecting a fraction model to represent questions, beyond that of part-whole (Tossavainen & Helenius, 2024; Fuchs & Malone, 2020; Kolar et al., 2018; Brown, 2015).

To support teachers' understanding of fractions, these researchers made suggestions that echo those described in the previous section related to curriculum standards documents and student learning. Specifically, they recommend an increased focus on the connections between other interpretations of fractions and of rational numbers (Brown, 2015; Ezaki et al, 2023; Suh et al., 2012), greater emphasis on a conceptual understanding of fraction magnitude (Brown, 2015; Fuchs & Malone, 2020), more time for teachers to practice with fractional concepts (Fuchs & Malone, 2020), and greater opportunities for teachers to struggle through problems themselves, with a focus on mathematical reasoning (Suh et al., 2012). Given these findings, in this study I focused on working collaboratively with teachers to uncover and further develop their conceptual understanding of the quotient interpretation of fractions. As detailed in Chapter 3, I planned and facilitated three interactive inquiry sessions to provide time for teachers to deeply engage with this fraction interpretation, and as shown in Chapter 4, to struggle through problem-solving contexts that might support a deeper understanding.

Research on Teachers' Understanding of the Meaning of a Fraction

Understanding the meaning of "a fraction" (i.e. what is a fraction?) has been researched with practicing and pre-service teachers in countries such as Turkey (Doğan & Tertemiz, 2019), New Zealand (Getenet & Callingham, 2019), the US (Park et al., 2013; Reeder & Utley, 2017), and Indonesia (Purnomo et al., 2021). These studies have revealed that many teachers rely on a definition which includes the part-whole interpretation (Doğan & Tertemiz, 2019; Getenet & Callingham, 2019; Park et al., 2013; Purnomo et al., 2021; Reeder & Utley, 2017) and that some teachers struggle to see the connection to other possible meanings or other rational numbers, such as decimals, even after direct instruction and explanations are provided (Rathouz & Cengiz, 2013; Reeder & Utley, 2017). Researchers conveyed that some teachers also struggle with connecting the meaning of a fraction to prior knowledge, such as whole number division and that many teachers did not adequately explain that a fraction is a number itself (Park et al., 2013, p. 474). Doğan and Tertemiz (2019) clearly explain why it is important to understand what a fraction is by stating, "if the primary school teachers do not have sufficient knowledge about the fraction, teaching the fraction subject to the students will not be successful" (p. 67).

Researchers have made many recommendations for supporting this understanding, for both teachers and students. In a study done in Indonesia, Purnomo et al. (2021) suggest that teachers begin with fraction as quotient to build upon prior knowledge and that they should emphasize models such as partitioning and distributing (p. 203). Other recommendations are to use accurate language when discussing fractions (Getenet & Callingham, 2019; Park et al., 2013), to emphasize the connection between whole numbers and fractions (Park et al., 2013; Purnomo et al., 2021), to connect fraction meanings to each other in curriculum documents and materials (Doğan & Tertemiz, 2019), and to provide opportunities for teachers to discuss and reason with one another (Rathouz & Cengiz, 2013). These studies which frame how teachers, both experienced and pre-service, understand the meaning of a fraction, informed my analysis of the data from this study as discussed in Chapters 4 and 5. Notably in many of these studies, a broader sense of fraction meaning is explored whereas my study offers an in-depth exploration of the meaning of fraction as quotient.

Research on Teachers' Understanding of Division Involving Fractions

The understanding of division involving fractions, where the word *quotient* is used to indicate the answer to a division problem, has been researched with pre-service teachers (Alenazi, 2015; Kang, 2022; Van Steenbrugge et al., 2014). All three studies showed that pre-service teachers often relied on procedures and computation and lacked conceptual understanding. In a US study, Kang (2022) discovered that pre-service teachers would often make errors due to the use of incomplete algorithms or would use the wrong one altogether (p. 117), whereas Alenazi (2015) revealed that although pre-service teachers were able to solve quotient problems correctly, they were unable to provide any meaning, context, or alternative representation (p. 713).

Studies of division involving fractions have also been done with practicing teachers in the US (Copur-Gencturk, 2021; Lamburg & Wiest, 2014; Ma, 1999; Stohlmann et al., 2020). In these studies, it was revealed that teachers had difficulty representing this process with visual models (Copur-Gencturk, 2021; Lamburg & Wiest, 2014), working with a referent unit to solve questions (Stohlmann et al., 2020), and were unsure what to do with a remainder (Lamburg & Wiest, 2014). Similar to the research done with pre-service teachers, these studies also found that teachers often relied on memorized and algorithmic steps and procedures and had difficulty explaining what the procedures meant or how they were related to division (Copur-Gencturk, 2021; Lamburg & Wiest, 2014).

In interviews comparing how US and Chinese teachers understand division involving fractions, Ma (1999) revealed that teachers from the US used an algorithmic model, but that less than half of the participants reached a correct answer (p. 47). She compared this to the ways that Chinese teachers talked about their understanding, noting that teachers there mentioned the connection between fractions and whole numbers and were able to explain the meaning behind the procedures used (p. 50). Ma (1999) also shared that the models and representations used by teachers varied; US teachers mainly used food and money to represent the operation of division involving fractions whereas Chinese teachers were more diverse in their explanations, drawing from students' experiences such as what happens in a farm, factory, or family (p. 69). Finally, Ma (1999) found that most participants from the US seemed to only convey conceptual understanding for the partitive model of division involving fractions, giving it much more attention than the measurement or quotitive model for examining division as an operation (p. 71). This preference of one model over another may lead to an incomplete understanding of division for these teachers, since their connections within the topic have gaps (Ma, 1999).

Researchers have suggested ways that pre-service and practicing teachers can be supported in developing their conceptual understanding of division involving fractions. These include practices such as connecting the meaning of division to other mathematical representations and contexts (Alenazi, 2015; Copur-Gencturk, 2021; Lamburg & Wiest, 2014; Stohlmann et al., 2020), drawing upon the conceptual work of the measurement interpretation of fractions (Copur-Gencturk, 2021), and adequate time to develop an understanding of what it means to use the division operation with fractions (Copur-Gencturk, 2021; Lamburg & Wiest, 2014; Kang, 2022; Ma, 1999).

Although these studies provide insights into how educators understand division involving fractions, as noted earlier, this is not the same as understanding the quotient interpretation of fractions which centers around the idea that a fraction represents a division relationship, as put forward by Charalambous and Pitta-Pantazi (2007). Rather than supporting teachers to an understanding that $a \div b = a/b$ (i.e. the notation often used to convey fraction as quotient), which as I show in Chapter 4 is anything but straightforward, the majority of these studies work to support teachers with modelling division involving fractions using various representations. In contrast, my study examines how teachers understand and describe the quotient interpretation of fractions and how this interpretation is represented in Manitoba's curriculum standards and support documents. And yet, as I show in later chapters, the ways that teachers think about quotient both as an operation and as the answer to a division question seem to be related and may even interfere with how they develop their understanding of fraction as quotient.

Research on Talking with Teachers about their Understanding of Mathematics

As noted in my description of MKT in a later section within this chapter, knowledge for teaching mathematics requires more than just knowledge of mathematics (Hill et al., 2004), as there are also pedagogical and specialized content considerations that an educator must possess to facilitate students' understanding (Ball et al., 2008). Interviews conducted with teachers and pre-service teachers in the US shed light on how teachers talk about their practice and understanding, knowledge, pedagogy, and feelings about mathematics (Ball, 1990; Hill et al., 2005; Ma, 1999).

Ball (1990) stated that teachers must have a deep understanding of how to navigate connections between mathematics ideas and topics, as well as "be able to represent concepts in multiple ways" (p. 458). This moves beyond procedural ways of knowing. In interviews conducted with preservice teachers about their mathematical pedagogy, teachers shared some of their beliefs and feelings about the subject (Ball, 1990). Notably, both elementary and high school teachers shared that math is all about rules. Some teachers tried to explain concepts beyond the memorization techniques they teach to students, however, Ball stated that many of these teachers ended up explaining the rules and procedures rather than the underlying concepts. Her study revealed that even for those who wanted to teach conceptual understanding, they were unclear as to how this might look (p. 460). Along similar lines, Ma (1999) stated that to teach effectively, content and pedagogy must be well matched, and that having good pedagogy does not make up for lack of knowledge of a topic as teachers need to know the foundational pieces too (p. 61). She shared that teachers from the US who participated in her study seemed to have good intentions pedagogically; explaining how they represented fractions with different models such as circles and rectangles, as well as stories that involved food and recipes, but this could not make up for their lack of conceptual understanding (p. 61).

Some of these studies were conducted several years ago and teachers' conceptual understanding may have grown over the years, particularly since there has been an emphasis on the value of building teachers' mathematical knowledge in many contexts (Kilpatrick et al., 2001). Yet in my experience, many of these observations continue to be relevant for teachers that I interact with.

Teachers in various studies also revealed a common misconception that they shared; that elementary mathematics is simple and basic, as compared with mathematics in the older grades (Ball, 1990; Hill et al., 2004; Ma, 1999). This belief was aligned with the grade level stream the teachers entered: teachers who lacked confidence in their mathematical knowledge often came from those who taught younger grades, compared to higher levels of confidence seen in high school teachers (Ball, 1990). Ma (1999) counters this view by explaining that elementary mathematics is neither simple nor basic; it is instead foundational, requiring elementary teachers to have a profound understanding of fundamental mathematics (p. 106). As noted in Chapter 4, this view resonates with my observations which highlight the challenges for teachers in having breadth and depth in their MKT about fraction as quotient. My study offers a current perspective on how teachers' talk about their understanding of fraction as quotient and what their misconceptions are about this specialized content knowledge in a Canadian educational context.

Theoretical Frameworks

In this section, I provide an overview of the five interpretations of fractions as well as a deeper look at the quotient interpretation. I discuss social constructivism and MKT (Ball et al., 2008), as the two other theories that guided my study. Finally, I share the commensurability of all three theoretical frameworks and describe how they impacted this research.

Five Interpretations of Fractions: Definitions and Pedagogical Considerations

Fractions are generally understood to include five interpretations; part-whole, measure, ratio, operator, and quotient (Behr et al., 1983; Kieran, 1976). These interpretations intertwine to create a deep understanding of the rational number system (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). Each fraction can be thought of in relation to any of these interpretations. In Table 1, I provide a brief description and example of each interpretation, adapted from Purnomo et al. (2021) and influenced by my efforts to understand each interpretation. I also identify key sources for each interpretation. I provide more detail about these interpretations following the summary table.

Table 1

Explanation of the Five Interpretations of Fractions (adapted from Purnomo et al., 2021)

Interpretation	Brief Description	Example	Key Sources
Fractions as part-whole	Represents a situation in which a continuous quantity or a set of discrete objects are partitioned into equal parts.	⅔ is a part-whole because it means two parts out of three equal parts of one whole pizza.	Behr et al., 1983; Kieran, 1976; Lamon, 2007; Purnomo et al., 2021
Fractions as measure	Represents a measure of the quantity relative to one unit of that quantity.	⅔ is a measure because it means how far two sections of a ⅓ length are, in a distance between two places.	Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Purnomo et al., 2021
Fractions as ratio	Represents a comparison between two quantities, as they grow or shrink multiplicatively, in tandem.	⅔ is a ratio because it means the relationship between powder to water when making a pitcher of lemonade.	Behr et al., 1983; Lamon, 2007; Moss & Case, 1999
Fractions as operator	Represents a function which is applied to some other number, object, or set.	⅔ is an operator when applied to another thing, like ⅔ of 24 marbles.	Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Lamon, 2007
Fractions as quotient	Represents a divisive relationship where the numerator is the dividend	⅔ is a quotient because it means 2 granola bars shared between 3 friends.	Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007;

	and the denominator is the divisor.	Kieran, 1976; Lamon, 2007

Part-Whole Interpretation.

The part-whole interpretation is the most familiar and commonly taught view of fractions in early mathematics education (Lamon, 2007). It represents a situation where a whole—whether continuous, such as a pizza, or discrete, such as a set of objects—is divided into equal parts, and the fraction indicates how many of those parts are being considered. For example, $\frac{2}{3}$ means two parts out of three equal parts of one whole pizza. Some researchers posit that the part-whole interpretation is a fundamental concept in developing the other four interpretations (Behr, 1983; Kieran, 1976; Charalambous & Pitta-Pantazi, 2007). Other researchers warn against teaching part-whole as the only fraction interpretation, as this leads to an inadequate understanding of fractions (Lamon, 2007).

Measure Interpretation.

The measure interpretation positions fractions as numbers on a number line, emphasizing their role as measures of distance rather than pieces of a whole. In this view, a fraction represents the distance along the number line from zero to a point, relative to a unit interval. For instance, $\frac{2}{3}$ means the distance covered by two intervals of size $\frac{1}{3}$. This interpretation can support students to comprehend the density property of fractions (i.e. that there are an infinite number of fractions between two fractions) and reinforces the comparison of fractional quantities with order and equivalence (Charalambous & Pitta-Pantazi, 2007).

Ratio Interpretation.

The ratio interpretation views fractions as a comparison between two quantities, highlighting their multiplicative relationship (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). For example, ⅔ can represent the ratio of two cups of sugar to three cups of water in a lemonade recipe. This interpretation is essential for developing proportional reasoning and magnitude, which underpins many advanced mathematical concepts (Behr, 1983; Lamon, 2007). Students need to realize that there is a relationship between two quantities and understand the covariance-invariance property, which is that the two quantities in the ratio relationship change together (Charalambous & Pitta-Pantazi, 2007, p. 297). This distinguishes ratio from the part-whole interpretation.

Operator Interpretation.

The operator interpretation treats fractions as functions that scale or transform other quantities (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). In this sense, a fraction acts as an operator applied to a number, object, or set. For example, finding ⅔ of 24 marbles involves applying the operator ⅔ to 24, resulting in 16 marbles. This interpretation is particularly important for studying equivalence and comparisons of fractions as well as the operation of multiplication (Behr, 1983).

Fraction as Quotient Interpretation.

The quotient interpretation of fractions emphasizes the division relationship inherent in fractional notation, distinguishing it from other interpretations such as part-whole or operator. In this view, a fraction represents the result of dividing one quantity by another, where the numerator serves as the dividend and the denominator as the divisor (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). For example, the fraction ⅔ can be understood as the outcome of dividing two granola bars equally among three friends, meaning each person receives ⅔ of a bar. This interpretation moves beyond the idea of fractions as static parts of a whole and instead frames them as dynamic expressions of a division process.

Understanding fractions as quotients is critical because it connects to broader mathematical structures, particularly multiplicative reasoning and proportionality (Lamon, 2007; Moss & Case, 1999). When students recognize that ⅔ represents 2 ÷ 3, they begin to see fractions as numbers that express relationships rather than only as pieces of an object. This conceptual shift supports the development of rational number sense and prepares learners for advanced topics such as ratios, rates, and algebraic thinking. Furthermore, the quotient interpretation helps address common misconceptions, such as whole-number bias, where students incorrectly apply whole-number reasoning to fractions (Tian & Siegler, 2018). For instance, without understanding the division relationship, a student might assume that ⅕ is larger than ¼ because five is greater than four, rather than considering the relative size of the shares.

Researchers argue that teaching the quotient interpretation requires deliberate instructional strategies that highlight the meaning of division in fractional contexts (Johanning & Mamer, 2014; Middleton et al., 2001). This includes using real-world sharing scenarios, visual models such as partitioned bars or number lines, and explicit connections to division equations. By making these relationships visible, teachers can help students build a robust understanding of fractions as quotients, which in turn strengthens their mathematical knowledge for teaching (Ball et al., 2008). In this study, the quotient interpretation is positioned as specialized content knowledge within the MKT framework, underscoring its importance for educators working with fractions in the middle grades. Moreover, as demonstrated in the observations in Chapter 4 and discussed in Chapter 5, the concept of fraction as quotient is difficult to express and engage with in a conceptual manner, despite the relatively straightforward notation often used to express the interpretation (i.e. $a \div b = a/b$).

Pedagogical Considerations of the Five Interpretations.

Kieran (1976) explains that to fully grasp the meaning of fractions, we must understand that there are multiple interpretations of them. He argues that students must be given adequate experience with fractions, extending the work beyond procedures (p. 127). In addition to this, schools must be places where the various interpretations of fractions can be explored through different contexts, mechanisms to support this understanding can be practiced, and accurate fractional language is taught (p. 127). The understanding of all five interpretations is also seen as a prerequisite for solving problems that pertain to fractions (Charalambous & Pitta-Pantazi, 2007). Lamon (2007) notes that instruction does not necessarily need to include all the interpretations at once but rather over time with an additional focus on the multiplicative structure of rational number relationships to support students' fraction understanding (p. 658).

Teaching mathematics, including the five interpretations of fractions, in a connected and contextual way is also critical for unlocking the big ideas in number sense (Fosnot & Dolk, 2002; Lesh et al., 1979). It is important that teachers consider the context and situation in which they are exploring mathematical concepts with their students (Fosnot & Dolk, 2002), as this supports the development of students' mathematical literacy. Fosnot and Dolk (2002) suggest that if a mathematics problem is to be generalized by learners, they need to be given opportunities to explore relationships, put forth explanations and conjectures, and try to convince one another of their thinking. These authors refer to this process as "mathematizing" (p. 9) and they note that it is a powerful learning experience which is grounded in mathematical contexts. Fosnot and Dolk (2002) also specify that for a problem to be "mathematized" by learners, teachers need to be purposeful in their design and cognizant of three components: use of a familiar model over time and in different situations; a context that allows learners to access schema for the problem they are doing; and a situation that prompts questions and inquiry (p. 29). Therefore, educators need to be cognizant of how, when, and in what context the five interpretations are being introduced and modelled, so that students have the opportunity to develop generalizations about them. I return to this idea in Chapter 4 and show how the teacher participants and I begin to "mathematize" the quotient interpretation of fractions through problem-solving contexts.

Unfortunately, as noted earlier, research in many educational contexts has shown that there is an overemphasis on teaching the part-whole interpretation of fractions (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007) as well as a rush towards rote procedures and formulas (Cooper et al., 2012; Gabriel et al., 2023), both of which lead to a weakened understanding of fractions. This is relevant to my study because my preliminary review of the Manitoba Education (2013) mathematics curriculum standards suggested that minimal attention is given to the five interpretations, falling into the problematic pattern identified by researchers (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). I decided to more fully analyse the Manitoba Education (2013) mathematics curriculum standards and some specific support documents to gain insights into how teachers are being invited to interpret and teach fractions in the middle years. This is connected to my first research question: "In what ways is fraction as quotient included in the K-8 Manitoba Curriculum Framework of Outcomes (2013) and in the Grades 3-8 Support Documents for Teachers? How does the approach to fraction as quotient in these documents relate to recommendations about fraction as quotient in the mathematics education literature?" The five interpretations of fractions also underpin the interactive inquiry sessions I planned and facilitated with teachers in relation to my second research question; "What helps and hinders a group of teachers in Grades 4-8 to develop an understanding of fraction as quotient?"

Social Constructivism

Social constructivism is the view that learners construct meaning based on their social interactions (Vygotsky, 1962), that individuals become aware of their learning and the relationship to learning that they share with others (Adams, 2006), and that human development is socially situated (Izmirli, 2020). Learning is viewed as an active, ongoing process, in which meaning is closely connected to the environment in which it is created (Woolfolk, 1993). Success is not based on short-term outcomes, nor is it oriented toward target-driven tests and standards (Adams, 2006). Rather, this theory acknowledges the constant ebb and flow of learning, with a goal for all learners to deepen their understanding through a variety of opportunities. Vygotsky (1962) posited that educators can support learners' development of new concepts by introducing them as functional and comprehensive structures, rather than isolated situations, with learning recognized as a "live thinking process" (p. 105).

Social constructivism has been expressed in relation to mathematics education as a way of foregrounding that, "mathematics is a changing and evolving human product" (Izmirli, 2020, p. 3). School mathematics is a subject where processes such as questioning, listening to others, verbalizing, working together, and reasoning are key features which also reflect what it means to be a mathematician (Wedekind, 2011). Social constructivism positions the teacher as a facilitator, who is "expected to construct their own knowledge through vigorous and active investigation, possibly through group collaboration, and learn mathematics through problem solving, the true measure of proficiency in mathematics" (Izmirli, 2020, p. 3). In my experience, social interactions are not only important, but necessary for mathematical thinking to develop amongst students and teachers.

Social constructivism informed my study throughout each phase but particularly in the ways that I designed, facilitated and analysed the data from the interactive inquiry sessions. That is, I talked with teachers about how they understood the quotient interpretation of fractions over the course of the three inquiry sessions and then analysed the ideas shared in those interactions. This connects to my third research question: "How does working collaboratively shift the ways that a group of teachers, including myself, think about teaching and learning fraction as quotient? Specifically, how do we build our MKT of fraction as quotient and how does learning about fraction as quotient build our MKT?" As will be shown in Chapter 4, we had the opportunity to actively learn from one another, to unpack our current understandings and misconceptions, and to independently try things in our classrooms before returning for the next inquiry session. Since Vygotsky (1962) explained that learning is constructed through social interactions, I wanted to ensure that my research allowed for social learning to occur among the teachers who were participants in the study. Although no data was gathered from these teachers' classrooms, socially constructed learning may also have developed as these teachers worked alongside their students and saw fraction work unfolding in real-time.

Mathematical Knowledge for Teaching

Mathematical knowledge for teaching (MKT), put forth by Ball et al. (2008), is a theory that proposes that to teach math successfully one must have a level of mathematical knowledge that differs from having only content knowledge or pedagogical knowledge (Alsheri & Youssef, 2022). I draw on this theory in my study because I think that teachers need to have MKT, not just content knowledge or pedagogical knowledge, when teaching concepts that have proven to be challenging such as fractions (Cramer et al., 2002; Gomez et al., 2014; Lamon, 2007;

Tossavainen & Helenius, 2024). MKT is important because teacher knowledge, both of content and of pedagogy, is linked to the ways in which teachers conduct their instruction, which then influences the experiences of students (Ball, 1990; Hill et al., 2005; Ma, 1999).

Hill et al. (2008) give the example that MKT is the knowledge of "why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors students are likely to make with particular content" (p. 431). Within this framework, Ball et al. (2008) developed a model to convey the different aspects an educator with MKT would possess. The model includes two connected categories: subject matter knowledge and pedagogical content knowledge. Nested within the subject matter knowledge lies specific types of content knowledge: common, specialized, and horizon. Within the pedagogical content knowledge side of the model lies knowledge of content and teaching, knowledge of content and students, and knowledge of content and curriculum (p. 430).

Although a comprehensive review of MKT or other ways of thinking about teachers' mathematical knowledge is beyond the scope of this study, I see the specialized content knowledge and horizon content knowledge aspects of this model as particularly relevant for my research. This is because these aspects of MKT can support the work teachers do in "unpacking" mathematical knowledge for student understanding (Ball et al., 2008, p. 400). I am positioning the quotient interpretation of fractions as specialized content knowledge, in that it might not be needed all the time for all grades, but it is important for teachers to know about to make learning experiences meaningful for students. I am also drawing from Ball's horizon knowledge, as that is where I argue the five interpretations of fractions and their connection to deep fractional understanding can be located. In fraction research, specifically related to the quotient interpretation, teachers must make connections to prior learning experiences, such as equivalence (Kieran, 1976) and partitioning schemes (Charalambous & Pitta-Pantazi; Kieran, 1976; Kieran 1980; 2007; Lamon, 2007), as well as connections to the other interpretations of fractions (Kieran, 1976; Lamon, 2007), while keeping the big picture of mathematical ideas, such as how fractions are connected to proportionality (Lamon, 20007) in mind. Ball et al. (2008) describe this delicate dance that teachers must do to be successful mathematics educators, explaining, "[we] must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students" (p. 400). It is this 'unpacked' and 'visible' work that teachers do that is the focus of my research.

Commensurability of Theories

To help ensure the coherence of this study, I considered the commensurability of the three theoretical frameworks both in the design of the study and in the analysis and interpretation of the data. The five interpretations of fractions, as viewed through the lens of social constructivism, would need to be taught through contextually appropriate scenarios, collaboration, and exploration since, as warned by Izmirli (2020), "devoid of any context, mathematics becomes nothing more than a series of computations" (p. 5). In addition, Lamon (2007) pointed out that the five interpretations need to be taught over time and through connections to each other. I believe that these characteristics are consistent with the social constructivist view of learning as an active, ongoing process, in which understandings can shift (Woolfolk, 1993). As will be shown in Chapter 4, the teachers and I engaged in an active learning process about fraction as quotient while in the position of learners. These theories are also commensurable with the theory of MKT which emphasizes that an educator must have special knowledge for teaching mathematics (Ball et al., 2008). In this study, I consider how MKT can develop when teachers engage in mathematical discourse in collaboration with one another. As Ball et al. (2008) propose, to teach mathematics successfully, one must have a level of mathematical knowledge, which differs from having only content knowledge or pedagogical knowledge (Alsheri & Youssef, 2022). I argue that teachers need to have MKT to teach fraction concepts more effectively. Thus, the five interpretations of fractions, social constructivism, and MKT taken together provide a strong theoretical basis for understanding how to better support teachers with the challenging work of teaching fractions in middle years classrooms.

Chapter 3: Methodology and Research Methods

In this chapter, I outline my methodology and research methods, which includes the steps I took to gather and analyse my data and how I recruited teacher participants. I also explain how the research methods I used connect to my research questions and are appropriate for the theoretical frameworks on which the study is based.

Methodological Framework

The methodology I used was informed by my understanding of hermeneutic phenomenology. Hermeneutics acknowledges that both the context and content of a text need to be looked at closely to understand intended meanings (Merriam & Tisdell, 2016). Hermeneutic phenomenology, as developed by Gadamer (1990), is a way to investigate experiences as they are lived by people and to gain access to the meaning structures of lived experience by appropriating, clarifying, and reflectively making these meaning structures explicit (Bertomeu & Esteban, 2023). Text, such as interview transcriptions or documents such as curriculum standards, are considered data and are studied in a dynamic, iterative, and non-linear manner; the researcher moves between the whole of the text, its parts, and the contexts in which the whole and parts are embedded (Eatough & Smith, 2017) in what is known as the 'hermeneutic circle' (Eatough & Smith, 2017; Sloan & Bowe, 2013).

Reflexivity is recognized as an important aspect of hermeneutic phenomenology, where the researcher uses empathy or relevant prior experiences to support data analysis and the interpretation of meanings (Sloan & Bowe, 2013). In addition to this, the researcher uses "a flexible approach to questioning, which seeks access to pre-reflective knowledge and reflective perception of the phenomena" (Bertomeu & Esteban, 2023, p. 1455). Using hermeneutic phenomenology to inform my methodology meant that I was able to delve into the quotient interpretation of fractions through written and oral texts (as described below) and reflect on my experiences during the three interactive inquiry sessions to explore these teachers' understanding of fraction as quotient.

Notably, van Manen (as cited in Bertomeu & Esteban, 2023, p. 1454) said that hermeneutic phenomenological researchers should engage in six activities when conducting a study: focusing on a phenomenon that engages the researcher, inquiring into experiences of participants as they are lived, reflecting on essential themes, describing the phenomenon through writing and rewriting, maintaining a strong pedagogical orientation towards the phenomenon, and keeping a balance between the parts and the whole. I refer to these activities and describe how they guided my research methods in the following section.

Research Design

I conducted a document analysis of the K-8 Manitoba Curriculum Framework of Outcomes (2013) and the Grades 3-8 Support Documents for Teachers, as well as designed and facilitated three interactive inquiry sessions with a group of educators. My data includes my analysis of the curriculum documents as well as the transcriptions of the audio/video recordings from the sessions with teachers. I also draw on artifacts that were created during the interactive inquiry sessions which include participants' jot notes and writings on chart paper. These are the written and oral texts which I examined using hermeneutic phenomenology. In this section, I share my methods for data collection and how the data was analysed using hermeneutic phenomenology. I also discuss how trustworthiness was enhanced throughout the analysis process.

Document Analysis

I conducted a document analysis of the K-8 Manitoba Curriculum Framework of Outcomes (2013) and Grades 3-8 Support Documents for Teachers to unpack how the texts support the teaching and understanding of the quotient interpretation of fractions. This aspect of the study relates to my first research question: "In what ways is fraction as quotient included in the Manitoba K-8 mathematics curriculum standards and in the Grades 3-8 Support Documents for Teachers? How does the approach to fraction as quotient in these documents relate to recommendations about fraction as quotient in the mathematics education literature?"

I began with an analysis of the K-8 Manitoba Curriculum Framework of Outcomes (2013). This document, which provides the mandated curriculum for teachers in the province of Manitoba, is 168 pages in length and is organized into the front matter (Background, Introduction, Conceptual Framework, and Instructional Focus), General Learning Outcomes (GLOs), Specific Learning Outcomes (SLOs) which are grouped by strand (Number, Patterns and Relations, Shape and Space, and Statistics and Probability), and finally, the Achievement Indicators (AIs) which are related to the SLOs and organized by grade and strand. GLOs are defined as "overarching statements about what students are expected to learn in each strand/substrand", SLOs are defined as "statements that identify the specific skills, understanding, and knowledge students are required to attain by the end of a given grade", and AIs are described as "samples of how students may demonstrate their achievement of goals of a specific learning outcome" (Manitoba Education, 2013, p. 16). The curriculum standards state that the AIs should not be seen as the sole indicators of successful achievement of an SLO; they should be thought of as examples of how students might demonstrate their learning of that SLO (Manitoba Education, 2013, p. 16). The document also states that each section of the curriculum is not meant to stand alone, that learning should come from a problem-solving approach, and that connections between strands should be made to develop mathematical understanding (p. 16). The steps I followed to analyse this document are described later in this section.

After analysing the curriculum standards document, I analysed the Number strand of the Grades 3-8 Support Documents for Teachers. I share the details for this process later in this section. The support documents are meant to provide suggestions for instruction and assessment and are "intended to be used by teachers as they work with students in achieving outcomes and achievement indicators identified in Kindergarten to Grade 8 Mathematics: Manitoba Curricular Framework of Outcomes (2013)" (Manitoba Education, 2014a, p. 1). For each SLO in the curriculum standards, the support documents indicate an enduring understanding and an essential question to help frame the context and value of the learning goal. After the SLO and AIs are listed, the support document then describes the prior knowledge a child might bring with them to that outcome, background information and mathematical language the teacher might need for teaching, and a variety of suggested learning experiences and assessments. The support documents are thorough and are intended to provide a deeper understanding of each SLO and AI. Table 2 provides some information about the Grades 3-8 Support Documents, including the number of pages focused on the Number strand and the year of publication. To avoid confusion when discussing these documents which have similar publication dates, Table 2 also indicates the in-text citation I used in this thesis when I refer to each document.

Table 2

Information about the Grades 3-8 Support Documents

Grade	In-text Citation	# of pages in Number strand	Publication Year
Grade 3	Manitoba Education (2017a)	102	2017
Grade 4	Manitoba Education (2017b)	102	2017
Grade 5	Manitoba Education (2014a)	154	2014
Grade 6	Manitoba Education (2014b)	128	2014

Grade 7	Manitoba Education (2016)	154	2016
Grade 8	Manitoba Education (2015)	96	2015

My timeline for the document analysis phase of my research was not linear, nor did the themes I uncovered reflect an ascending trajectory, moving from the small to the larger parts of the text. Instead, the timeline was punctuated by days or even weeks of reflection, where I would pause the analysis and mentally sit with what I had come across so far. Being in the position of a teacher and using my reflexivity to my advantage, I was able to continue my work with students and teachers in my daily role and see the learning of fractions as I was in the midst of conducting this study. The benefit of this period of deep reflection was that I had time to consider quotient as it lies in the curriculum standards and support documents and I could take the time to think about questions I still had. I found myself returning to the document analysis with new or nagging questions, or a refreshed perspective each time. For this process, I considered three of van Manen's research activities (Bertomeu & Esteban, 2023): I focused on a phenomenon that engaged me as the researcher (i.e. fraction as quotient); I maintained a pedagogical orientation towards this phenomenon; and I was aware of keeping a balance between the parts and the whole as I moved through my analysis. I describe this process in more detail below.

Analysis of the K-8 Manitoba Curriculum Standards

My analysis for the K-8 Manitoba Curriculum Framework of Outcomes (2013) occurred over four steps. I used both a physical copy and digital copy of the document in this process. I provide an overview of the steps taken in Table 3.

Table 3

Steps Taken in the Analysis of the K-8 Manitoba Curriculum Framework of Outcomes (2013)

Step	Document Section	Analysis	Process
1	GLOs and SLOs of the K-8 Number strand	Looked for words and phrases that were connected to the quotient interpretation	Physical copy
2	SLOs and AIs of K-8 Number strand	Same as Step 1	Physical copy
3	Curriculum front matter	Looked for how the curriculum writers positioned the learning of fractions	Physical copy
4	SLOs and AIs across all strands	Looked for conceptual development to support the meaning of a fraction and quotient as an interpretation instead of the answer to a division question	Digital copy

I began this analysis by looking at the General Learning Outcomes (GLOs) and Specific Learning Outcomes (SLOs) for the kindergarten to Grade 8 Number strand, using a physical copy of the Manitoba curriculum standards. I started with this step because I believed that it would inform the rest of my analysis, given that my previous experiences with the curriculum standards showed that fraction outcomes are located in the Number strand. I was looking for words and phrases that connected to my prior understanding of fraction as quotient, as informed by my literature review. I highlighted examples in the SLOs such as *part*, *group*, *partial*, *divide*, *division*, and *quotient*. I also highlighted any mention or concept that related to the five interpretations of fractions.

In the second step, I focused solely on the kindergarten to Grade 8 SLOs and AIs for the Number strand. I wanted to see if quotient was mentioned, explicitly or implicitly, in the parts of the document that I use most often as a teacher.

In the third step and keeping in mind the relationship that exists between the parts and the whole of a text, I moved from the GLOs and SLOs back to the front matter of the curriculum standards. As I read and re-read these parts of the document using a physical copy, I highlighted phrases or paragraphs that positioned how the curriculum writers characterized what it means to learn mathematics. I used my reflexivity and understanding as informed by my literature review to interpret this writing through the lens of learning about fractions, specifically fraction as quotient. I noted various words that I interpreted as places where the learning of fractions could be situated.

Finally for the fourth step, I decided to move to a digital copy of the curriculum standards so that I could use the 'search and find' tool to look for key words within the GLOs, SLOs, and AIs. This tool helped me to move back and forth within parts of the document more easily. I focused my attention on outcomes where the meaning of fractions is developed instead of on outcomes that involve students conducting operations with fractions. I wanted to see if quotient, as a fraction interpretation and a way to understand the relationship that exists within a fraction, was evident in the document, rather than how to calculate the answer to a division problem involving fractions. I did a 'search and find' for key words that relate to ideas that researchers argue are relevant to learning about fractions which are connected with quotient. These words included: quotient (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieran, 1976; Kieran 1980) partition (Charalambous & Pitta-Pantazi, 2007; Kieran, 1976; Kieran 1980; Lamon, 2007), sharing (Fosnot & Dolk, 2002), proportional reasoning (Lamon, 2005), and unit fraction (Fosnot & Dolk, 2002), and variations of these words, such as unit, parts, share, and unitize. I looked for these words in the curriculum document across all strands: Number, Patterns and Relations, Space and Shape, and Statistics and Probability. This led me to look more broadly at Patterns and

Relations, Space and Shape, and Statistics and Probability for concepts of quotient as well as the other four interpretations of fractions. Here again, I moved from the Number strand to other parts of the document to gain greater insights. Although this part of my analysis revealed many things about how the K-8 Manitoba curriculum standards document positions learning about fractions, in this thesis I report primarily on places where fraction as quotient was most evident.

Throughout this process, I kept notes about what I was seeing and made lists of the SLOs, AIs, or specific words I felt were related to fraction as quotient. Each time I looked at the K-8 Manitoba curriculum standards document for another round of analysis, I read my notes from the previous session and kept these in mind as I worked. In keeping with hermeneutic phenomenology, this process of reading through my notes helped me to see the "layers" of themes that were appearing (Eatough & Smith, 2017, p. 13). Using this process, I identified nine distinct places, within SLOs and/or AIs, where the quotient interpretation of fractions was evident. In Chapter 4, I identify these passages and discuss how fraction as quotient might be seen through the lens of these nine SLOs and AIs.

Analysis of the Grades 3-8 Support Documents for Teachers.

In the second phase of the document analysis, I turned my attention to the Grades 3-8 Support Documents for Teachers. I was initially going to look at all four curricular strands within these support documents but based on my analyses of the curriculum standards document, I decided to focus only on the Number strand support documents for Grades 3-8. Table 4 provides an overview of the steps taken for this part of the analysis.

Table 4

Steps Taken for the Document Analysis of the Grades 3-8 Support Documents for Teachers

Step	Document Section	Analysis	Process
1	Front matter of Number strand for Grades 3-8 support documents	Looked for how the curriculum writers positioned teaching and learning, compared to the K-8 curriculum	Digital copy
2	Fraction SLOs and AIs in the Number strand for Grades 3-8 support documents	Looked for how the quotient interpretation might be interpreted	Digital copy
3	All SLOs and AIs in the Number strand for Grades 3-8 support documents	Looked for explicit mentions of "quotient"	Digital copy

For the first step, I read the front matter of the Number strand support document for each grade. I compared this text to ideas in the front matter of the Manitoba Education (2013) curriculum standards, with respect to perspectives on effective mathematics teaching and learning. In the second step, I looked specifically at the support document materials for each of the fraction SLOs and for any other SLO where I thought that fraction as quotient could fit, according to the first phase of my document analyses. I highlighted and made notes about these, with the intent of bringing these pieces to the interactive inquiry sessions where I hoped to gain participating teachers' perspectives on how these passages relate to fraction as quotient. Finally in the third step, I used the 'search and find' tool for the word *quotient* within each of the Grades 3-8 support documents in the Number strand section. I took note of how often it was mentioned, where it was mentioned, and in what capacity or context.

Throughout this process, I made notes and created a list of what I was seeing, identifying places where the concept of fraction as quotient was implicitly or explicitly supported. My notes included the page number and SLO, as well as definitions, essential questions, enduring understandings, background information, and potential learning activities. Each time I looked at a new grade level support document, I read my notes and lists from the previous session and kept these in mind as I worked, once again uncovering "layers" where fraction as quotient was appearing (Eatough & Smith, 2017, p.13). At the end of each session, I reviewed the notes I collected during that session and compared them to the previous list of places where I felt fraction as quotient was evident. I then created an updated list based on the new information I had gathered. Through this process, I identified five places within the support documents where the quotient interpretation was most evident. The five places that I identified are described in Chapter 4.

Three Interactive Inquiry Sessions

In this section, I begin with how I planned and recruited participants for the three interactive inquiry sessions. I then share details about how the sessions were conducted. Finally, I go into detail about my analysis process for the transcripts and artifacts from the three sessions and share some codes that I identified during this part of my study.

Planning and Recruiting Participants.

I began planning for the interactive inquiry sessions while I was in the process of completing the document analysis and then conducted the sessions in May and June of 2025. I used four criteria to invite teacher participants for the interactive inquiry sessions: grade level taught; familiarity; teaching experience; and reflectiveness. I chose to work with teachers with experience teaching in Grades 4 to 8 because these were the grades where I had previously observed many teachers expressing a challenge with teaching fractions. I wanted to work with teachers who were familiar to me and with whom I had a strong positive professional relationship because I believed they would be more inclined to share their teaching experiences and understanding of fractions as compared with teachers I did not know. I hoped this would provide access to rich and vulnerable conversations, where the teachers would express their concerns, questions, and challenges with teaching fractions and with the concept of fraction as quotient. I wanted participants to have at least five years of teaching experience because I wanted them to have a clearer sense of their pedagogy and to have had ample experience with teaching fractions. I also thought their years of teaching experience might mean they were more open to the five interpretations of fractions, specifically quotient, and that they might be more able to build on their existing MKT during the three sessions than teachers who had less than five years of experience. Based on my experience, I anticipated that they would likely not yet be familiar with the five interpretations of fractions. As a fourth criteria, I invited teachers that had demonstrated a high degree of reflectiveness about mathematics teaching and learning in prior interactions with me. I used this as an inclusion criterion because I wanted participants to reflect on and be willing to share their experiences and consider their shifting understanding of fraction as quotient over the course of the three sessions.

After I obtained approval from the Research Ethics Board at the University of Manitoba (see Appendix A), school division approval to conduct the study, and permission from the school principal to use a space in the building to meet with teachers, I reached out to colleagues by e-mail. Although these teachers were familiar to me, I held no power or professional influence over them in any capacity and this was made clear in the invitation that I sent them. In the e-mail, I gave a brief explanation of my research topic, goals for the interactive sessions, and the timeline for participation. I sent the invitation to five teachers I had worked with and all five agreed to participate. Informed consent forms were e-mailed out, which included a section on how the teachers would like to be referred to in my dissemination of the study. Three teachers chose to be referred to by their teaching role (i.e. "a Grade 6 teacher") while two teachers indicated that they preferred to be referred to by a pseudonym, chosen by me. I chose "Julie" and "John" as pseudonyms for these two participants. Both Julie and John were teaching Grade 7/8 multi-age classes at the time of the study. Each participant received a $40 gift card to a local bookstore as a token of my appreciation for the time and thought they devoted to the study. Once consent forms were signed and collected, gift cards were e-mailed out, and dates were chosen for the interactive inquiry sessions. Table 5 provides some background information about each participant.

Table 5

Demographic Information about the Teacher Participants

Participants	**Background information**
Grade 4/5 Teacher	15 years teaching experience, previously taught younger grades
Grade 6 Teacher	7 years teaching experience, recently returned from parental leave and began teaching Grade 6, previously taught younger grades
Grade 7/8 Teacher	15 years teaching experience including 3 years teaching high school mathematics
John (teaches Grades 7/8)	10 years teaching experience, used to work in collaboration with Julie
Julie (teaches Grades 7/8)	14 years teaching experience, used to work in collaboration with John as a learning support teacher; currently shares math teaching with another teacher at her school (Julie teaches Grade 8 and the other educator teaches Grade 7)

Conducting the Interactive Inquiry Sessions.

I conducted three interactive inquiry sessions with the five teachers described in Table 5. All of the teachers had been teaching for a minimum of five years, with four of them teaching for at least ten years. Four of the five teachers also knew one another prior to attending the sessions. The purpose of the sessions was to gain an awareness of how the teachers understand fractions, to position fraction as quotient as an avenue to deeper understanding and develop this concept through collective work and productive struggle (Dingman et al., 2019), and to unpack how the teachers utilize and navigate the Manitoba curriculum standards documents in relation to fractions. I also wanted to be a learner in these sessions, rather than an instructor. My stance as a learner connects to my belief about mathematics and social constructivism (Vygotsky, 1962) and is the reason that I refer to these sessions as interactive inquiries rather than focus group sessions.

The interactive inquiry sessions targeted my second and third research questions: "What helps and hinders a group of teachers in Grades 4-8 to develop an understanding of fraction as quotient?" and "How does working collaboratively shift the ways that a group of teachers, including myself, think about teaching and learning fraction as quotient? Specifically, how do we build our MKT of fraction as quotient and how does learning about fraction as quotient build our MKT?" The interactive inquiry sessions were guided by questions that I asked (see Appendix B), as well as several problem-solving contexts that we explored (see Appendix D). I provide an overview of the three sessions in Table 6 and provide more detail about each session below the table.

Table 6

Overview of the Three Interactive Inquiry Sessions

Session	Date	Duration	Focus	Activities	Data Collected
1	May 2025	45 min	Introduce the study & five interpretations	Virtual discussion, brief presentation	Audio/video recording
2	June 2025	90 min	Explore interpretations & quotient	Rational number line task, sub sandwich problem, co-construct quotient problem	Audio/video recording, chart paper artifacts
3	June 2025	90 min	Curriculum analysis & reflection	Review curriculum excerpts, discuss quotient	Audio/video recording, chart paper artifacts,

	interpretation, and share shifts in thinking	participant notes

The first meeting was conducted virtually and was approximately 45 minutes in length. The purpose of this session was to discuss the research project, get a sense of participants' current experiences with teaching fractions, and introduce the five interpretations of fractions. I used Microsoft Teams to make a video and audio recording of the session. Participants were able to join virtually from their schools. All five chose to keep their camera on during the session. As noted in Appendix B, I asked questions such as, "What do you think of when you consider 'quotient' in your teaching?" and "How do you explain the meaning of a fraction to students?" I also did a brief presentation about the five interpretations of fractions. The slides that I shared with participants in this presentation are shown in Appendix C. Two of the participants shared that they took notes during the presentation. I describe my observations and analysis of this session in Chapter 4.

Two weeks later, we came back together for an in-person interactive inquiry session. The session lasted approximately 90 minutes and was held after school in a meeting room, in a school convenient to the teachers. At this session we reviewed the five interpretations of fractions and shared our current thinking about these interpretations. We worked through a rational number line task and the sub sandwich problem (see Appendix D for a brief description of each task). Both tasks were adapted from Fosnot and Dolk (2002). As we worked on these tasks, we explained our thinking, described how we imagined students would react to the problems, and began to unpack the quotient interpretation of fractions. During this session, I had available resources such as chart paper, markers, fraction tiles, cuisenaire rods, and handouts detailing the five interpretations (see Appendix E, Figure 4). After working on the two tasks from Fosnot and

Dolk, we worked on creating our own problem where fraction as quotient was centered and we discussed the notation, a/b = a÷b, which is one way of expressing fraction as quotient (Charalambous & Pitta-Pantazi, 2007). We ended the session by sharing success stories and challenges when teaching fractions to students. The session was audio-video recorded using Microsoft Teams, with my computer positioned on the conference table so that all participants were visible. I photographed all artifacts that we created as we worked through the tasks (i.e. chart paper with written annotations and diagrams) so that I could retain them as data files in digital format. I also kept the chart paper that we had written on so that we could refer to it in Session 3.

The following week, we came back together at the same location for one more, 90-minute interactive inquiry session. At this session, we began by revisiting the quotient problem that we had co-constructed in Session 2 and discussed the meaning and intent of quotient as an interpretation of fractions that students might use. Some of the teachers shared their recent teaching observations, as some had engaged with fraction activities such as Fosnot and Dolk's (2002) rational number line task and the sub sandwich problem with students in the days since we met for Session 2. Next, we turned our attention to the curriculum standards and support documents. Teachers shared how they used these documents, their challenges with navigation through the documents, and suggestions for ways they might be improved. I shared the relevant SLOs and AIs that I had identified from my document analysis, and the teachers shared their thoughts about these points. They also read through some of the front matter of the curriculum standards document including what the province of Manitoba writes about the "Nature of Mathematics" (Manitoba Education, 2013, p. 16). The teachers described how they might interpret these ideas with regard to the quotient interpretation of fractions. Finally, they reflected upon their understanding of the meaning of a fraction and shared if and how their thinking had shifted since Session 1, along with how they imagined using their new learning moving forward. This session was also audio-video recorded using Microsoft Teams and chart paper artifacts that were created were photographed and retained as part of the data.

Analysis of the Three Interactive Inquiry Sessions.

As previously noted, all three sessions were audio-video recorded. The data from these sessions were multi-modal as they included chart paper where participants made visual representations and notes, in addition to the audio and video recording. I began the analysis of these sessions by transcribing each one using Otter AI. After the initial transcription was completed, I read through the text of the transcript while listening and viewing the video recording. This process enabled me to add expressions, pauses, and describe moments of silence where teachers were working and thinking to the transcripts. Reading through the transcripts, while listening and viewing the video also helped me to become more familiar with the data and allowed me to begin to identify initial themes that emerged during each session.

Once the transcripts were edited, I began to read them again as text only in order to more clearly identify initial codes for themes in the data. Some examples of the codes that were identified at this stage included the teachers' understanding of fractions, experiences teaching in multi-age classrooms, and conceptual versus procedural teaching. Table 7 provides sample codes for this stage.

Table 7

Initial Sample Codes

Initial Code	**Description**	**Example Excerpt**
Teachers' understanding of fractions	How teachers describe the meaning of a fraction	"A fraction is partitioning of the whole…half isn't just in the middle, it's half of something, and how that

		works with different models." (Grade 7/8 teacher, Session 1, Lines 136-137)
Experiences teaching in multi-age classrooms	Teachers' experiences with multi-age classrooms and navigating mathematics teaching and curriculum	"Because you do need that much time to work with them [fractions] and to understand them. And we get that in multi-age, if we're lucky to be there both years, and if they're lucky enough to be with us for two years." (Grade 7/8 teacher, Session 2, Lines 990-992)
Conceptual versus procedural teaching	Teachers' struggle to find the balance between conceptual understanding and procedural fluency	"I try to go with reasoning. 'Okay, this is a formula. But what does this formula mean?' So, we play with what's happening and say, 'Okay, play with this'." (John, Session 3, Lines 575-577)

Once the initial coding was completed for the three transcripts, I put the codes from each transcript together into one document. I printed these so that I could read through each of them as a whole and identify changes over the three sessions. Some of the larger themes that emerged were references to and reflections about the five interpretations of fractions; reflections about fraction as quotient; references to the curriculum; and misconceptions and struggles shared by teachers. I then re-read each transcript to see if the themes I had identified within their parts (i.e. passages within each transcript) still felt coherent in relation to the ideas that developed when I looked at each session as a whole. Table 8 provides sample codes with examples for these larger themes.

Table 8

Sample Codes for Larger Themes

Larger Themes	Description	Example Excerpt
Five interpretations of fractions	Conveys an understanding or reflection on one of the five interpretations of fractions	"I found myself as a teacher going back to part-whole." (Grade 6 teacher, Session 2, Line 76)

Fraction as quotient	Reasoning about what fraction as quotient means	"I feel like it's a cool trick." (Grade 6 teacher, Session 2, Line 1414)
Curriculum standards navigation	Challenges using the standards document	"If you're a new teacher and a new teacher to multi-age [classroom], especially math, this is so overwhelming!" (Julie, Session 3, Lines 1138-1139)
Misconceptions and struggles	Incorrect reasoning or confusion about learning fractions	"I find in the early years, the emphasis on the symbolic [representation] really trips them up." (Grade 4/5 teacher, Session 2, Line 1993)

Throughout this iterative and non-linear process, I kept van Manen's practices of the hermeneutic researcher (Bertomeu & Esteban, 2023) in mind, specifically those of inquiring into experiences of participants as they are lived, reflecting on essential themes, describing the phenomenon through writing and rewriting, and keeping a balance between the parts and the whole. The hermeneutic circle (Eatough & Smith, 2017; Sloan & Bowe, 2013) was evident during this process, as the words spoken by teachers existed in the context of their own sentences but also in the conversation as a whole within each session and ultimately across the three sessions. The words spoken by teachers were also influenced by what others said and by what I said as we interacted. I was cognizant of the relationship that existed between the words that were spoken and I grouped chunks of phrases together during my analysis since meaning would have been lost if each sentence was looked at in isolation. I was also aware of the reflexivity I brought as a researcher to this analysis, as the way in which I interpreted what was being said and the themes that I identified were influenced by my experiences, beliefs, and understanding of teaching and learning fractions.

Finally, I compared the themes to one another and to the photographs of the artifacts created by teachers in the sessions to help determine findings for each research question. I did this by taking each common codes document, which was sorted by session number, and made notes about what seemed to be the most prevalent theme and/or trend in each session. I compared each session's most prevalent theme/trend to the other sessions at the end of the codes document to identify an over-arching takeaway from my analysis related to that particular code. Once this was complete, I was able to read through these statements to identify the topics I wanted to focus on in sharing my results. I chose topics that in my experience and based on my reading of the literature provide worthwhile insights about the process of coming to understand fraction as quotient. This analysis process resulted in my identifying seven features which seemed to help and/or hinder teachers' understanding of fraction as quotient. I share these results in Chapter 4.

Trustworthiness of the Methodology and Research Methods

I engaged in in three practices as a researcher to enhance the trustworthiness of this research: grounding all phases of the research in mathematics education literature; engaging in on-going conversations between myself and my academic advisor; and providing transparency about the research process and participant thinking by including quotations and excerpts from the data. I briefly describe each practice in this section.

I used the existing mathematics literature as a lens to guide my analysis, therefore enhancing the credibility to the research. For example, many researchers share that part-whole is a commonly used and over-emphasized fraction interpretation (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). When I analysed the curriculum standards documents, I looked for evidence of emphasis of this interpretation over others. The credibility of this research is also increased because I drew heavily upon research described in my literature review when I planned and conducted the interactive inquiry sessions. For example, some of the research I read highlighted teachers' struggles with describing the meaning of a fraction to students (Doğan & Tertemiz, 2019; Getenet & Callingham, 2019; Park et al., 2013). Therefore, I intentionally asked the teacher participants how they understand the meaning of a fraction and flagged their descriptions. As another example, I planned to engage in collaborative problem-solving during Session 2 because both the MKT and social constructivist literature identifies this as an effective way to facilitate learning.

I engaged in on-going and extensive dialogue with my academic advisor throughout my analysis of the interactive inquiry sessions, which included discussing the coding process, initial findings, and overall themes for both the document analysis and the analysis of the interactive inquiry sessions. This adds a level of trustworthiness to my research, as my interpretations of references to fraction as quotient in the document analysis were not completed in isolation. The conversations with my academic advisor throughout this process also helped to ensure that I kept the mathematics education literature in mind and both challenged and supported me to identify what was impactful for the teacher participants as they worked with the concept of fraction as quotient.

Finally, I have endeavored to be as transparent as possible about the research process and about what the teacher participants said and shared, by using many direct quotations where I share the findings in Chapter 4. These quotations illustrate what I identified as helpful and as a hinderance for developing an understanding of fraction as quotient. Transparency is also added to this study through my positionality, as I made sure to be open and honest with the teacher participants about my own learning throughout the interactive inquiry sessions and to briefly summarize my positionality earlier in this thesis.

Chapter 4: Research Findings

In this chapter I begin with my observations and findings from the analysis of the K-8 Manitoba Curriculum Framework of Outcomes (2013) and the Grades 3-8 Support Documents for Teachers. I identify the SLOs and AIs where I see evidence of fraction as quotient and explain the reasoning for my interpretation of each instance. Following this, I share some of my observations and analysis of the interactive inquiry sessions. Specifically, I describe seven features that seemed to impact our collective efforts to further develop our understanding of fraction as quotient, sometimes helping and sometimes hindering our learning. I provide a more extensive discussion of these features in relation to the research literature in Chapter 5. Taking a step back, I conclude this chapter with a discussion of how the social constructivist approach used in the inquiry sessions seemed to support the development of our collective MKT.

Findings from the Curriculum Standards and Support Documents Analysis

For my first research question, I analysed each section of the Manitoba Education (2013) curriculum standards document to identify how it supports the teaching and learning of fraction as quotient. As noted in Chapter 3, my analysis included the front matter, the GLOs, the SLOs, and the AIs. I also analysed the Grades 3 to 8 Number strand support documents for teachers which are "intended to be used by teachers as they work with students in achieving outcomes and achievement indicators identified in [the curriculum standards]" (Manitoba Education, 2014a, p. 1). In this section, I begin by sharing the results from the nine SLOs and AIs that I identified where fraction as quotient is evident and/or that could support teaching fraction as quotient. I then describe the five places where fraction as quotient is evident in the Grades 3-8 support documents. I also share my rationale for why I identified each of these passages as related to teaching and learning about fraction as quotient.

Results from the Specific Learning Outcomes and Achievement Indicators

In this section, I share the results from the document analysis by grade level. I refer to each finding as an "interpretation" to reflect my experience of moving through the process of hermeneutic phenomenology which guided my analysis (Eatough & Smith, 2017, p. 12). The first connection to fraction as quotient that I identified appears in Grade 1 of the curriculum standards and the final one I identified appears in Grade 8. I did not review any curriculum standards beyond Grade 8, so it is possible that there are SLOs or AIs in higher grades that relate to fraction as quotient. As noted in Chapter 3, for each SLO the curriculum standards also provide examples of what this understanding might look like in the form of Achievement Indicators (AIs). Therefore, for some of the SLOs I identify, I also include one or more AIs whose language reflects the fraction as quotient interpretation, as an additional place where the quotient interpretation was represented. Table 9 provides an overview of the nine features and a brief description as to where these are located in the curriculum standards. A more detailed explanation for each example, where I explain my reasoning as to how each passage connects to an understanding of fraction as quotient is provided after the table.

Table 9

Nine Interpretations (SLOs and AIs) Where Fraction as Quotient Appears

	Grade	SLO or AI	Connection to Quotient
1	1	AI: Partition any quantity up to 20 into two parts	Partitioning supports early divisive thinking
2	3	SLO: Demonstrate an understanding of division using equal sharing	Equal sharing contexts reflect quotient situations
3	3	SLO: Describe situations in which fractions are used	Allows for inclusion of quotient contexts
4	3	AI: Model and explain the meaning of numerator and denominator	Encourages unpacking the fraction structure and could allow for understanding fractions as relationships

5	5	AI: Express remainders as fractions (e.g. five apples shared by two people)	Direct link to sharing and quotient
6	6	SLO: Represent generalizations using equations	Opens door for quotient generalization
7	8	SLO: Solve problems involving rates, ratios, and proportional reasoning	Related to quotient interpretation
8	8	AI: Explain meaning of a/b within a context	Explicit reference to fraction meaning
9	8	AI: Provide a context where a/b represents quotient	Most direct link to quotient interpretation

Interpretation 1) *Partition* appears as an AI for one of the Grade 1 SLOs. The SLO states, "Represent and describe numbers to 20, concretely, pictorially, and symbolically" (p. 58). The relevant AI that I identified related to this SLO says, "Partition any quantity up to 20 into two parts and identify the number of objects in each part" (p. 58). In the mathematics education literature, partitioning is defined as fair-sharing (Lamon, 2007) and described as "the ability to divide an object or objects into a given number of like parts" (Kieran, 1976, p. 121). Although this AI refers to whole numbers, I see it as connected to the development of fraction as quotient because researchers have shared the importance of partitioning for later fractional understanding (Charalambous & Pitta-Pantazi, 2007; Cutting, 2024; Kieran, 1976; Kieran 1980; Lamon, 2007). Thus, even though the SLO and related AI do not mention fractions, I see this as a relevant passage in the curriculum that could support teaching and learning about fraction as quotient.

Interpretation 2) *Sharing* appears in a Grade 3 Number strand SLO which states, "Demonstrate an understanding of division by: representing and explaining division using equal sharing and equal grouping; creating and solving problems in context[s] that involve equal sharing and equal grouping; modelling equal sharing and equal grouping using concrete and visual representations, and recording the process symbolically" (p. 80). This SLO deals with equal sharing when modelling the concept of division, which relates to partitioning through fair-sharing (Lamon, 2007) and is connected to fraction as quotient. I also believe this SLO is connected to fraction as quotient because at the heart of this interpretation is the idea that any fraction can be seen as an expression of a division situation (Charalambous & Pitta-Pantazi, 2007, p. 299) and helping students more fully understand the concept of division could help them to understand fraction as quotient. Here again, although the SLO does not mention fractions explicitly, the SLO could be achieved through the use of equal sharing problems that involve fractions.

Interpretation 3) The conceptual development of the meaning of fractions is most explicitly stated in the Grade 3 SLO, "Demonstrate an understanding of fractions by describing situations in which fractions are used" (p. 81). This SLO could align with researchers' recommendations that in teaching the meaning of fractions, educators should emphasize a range of models (Purnomo et al., 2021), the connection between whole numbers and fractions (Park et al., 2013; Purnomo et al., 2021) and use contexts that allow learners access to schema and inquiry (Fosnot & Dolk, 2002). Thus, although the SLO does not refer to fraction as quotient, teachers could include situations where fractions are used to show a divisive relationship and thereby relate to the quotient interpretation.

Interpretation 4) Fraction as quotient can also be interpreted in one of the AIs that is provided for teachers to use for the SLO described in the previous item. The AI states "Model and explain the meaning of numerator and denominator" (p. 81). This AI implies that there is an important element to what the bipartite fraction notation represents. This is relevant because the quotient interpretation focuses on the idea that the fraction itself is a division relationship, where $a/b = a \div b$ (Behr et al., 1983), which involves unpacking what both numbers within a fraction represent, as well as how they relate to one another. That is, in the fraction as quotient interpretation, the numerator of a fraction is the dividend while the denominator of a fraction is the divisor. Notably, “model” is used in this AI instead of wording such as ‘explain’ which suggests that teachers need to do more than provide a definition of numerator and denominator.

Interpretation 5) *Share*, as a mathematics concept, appears again in a Grade 5 AI for a division SLO. The SLO states, “Demonstrate an understanding of division (1- and 2-digit divisors and up to 4-digit dividends), concretely, pictorially, and symbolically, and interpret remainders” and the relevant AI states “express remainders as fractions (e.g. five apples shared by two people)” (p. 102). This AI is connected to students’ development of the concept of fraction as quotient through fair-sharing and partitioning (Lamon (2007). Notably, fractions are only mentioned in relation to remainders. However, I see this as an opportunity for the quotient interpretation to be emphasized because a teacher could choose to explore the meaning of fractions through problem-solving contexts, such as “five apples shared by two people” by using the generalization of fraction as quotient or the notation, $a/b = a \div b$ (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). I refer to the quotient generalization as the general case in which learners see that the numerator is what is being shared, the denominator is how it is being shared, and the fraction represented is the relationship as well as the partitioned amount.

Interpretation 6) *Generalization* appears in a Grade 6 SLO within the Patterns and Relations strand (p. 118); “Represent generalizations arising from number relationships using equations with letter variables.” This outcome addresses relationships as well as generalizations, which is what the concept of fraction as quotient represents (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). Teachers could use this SLO as an opportunity to connect to the quotient interpretation of fractions since relationships between the numerator and denominator are key in conceptually understanding the meaning of a fraction. For generalizations, teachers could explore the quotient generalization, as a way of meeting this SLO. As noted in Chapter 5, SLOs such as this one open the door for teaching about fraction as quotient even though the current versions of the standards and support documents do not make this connection explicit.

Interpretation 7) *Proportional reasoning* is mentioned in a Grade 8 SLO in the Number strand: “Solve problems that involve rates, ratios, and proportional reasoning” (p.140). I see this SLO as connected to fraction as quotient because a solid and connected understanding of all five fraction interpretations, including the quotient interpretation, has been shown to contribute to the ability to reason proportionally (Lamon, 2007). In addition to this, proportional reasoning is “critical to mathematical and scientific thinking” (Lamon, 2007, p. 637). Teachers might more effectively achieve this SLO by including fraction as quotient in teaching and learning activities.

Interpretations 8 & 9) For the previous SLO, there are two AIs which teachers can use that I see as connected to fraction as quotient. The first one is “Explain the meaning of a/b within a context” and the second is, “Provide a context in which a/b represents a fraction, rate, ratio, quotient, or probability” (p. 140). These two AIs are the most explicitly connected to fraction as quotient of all the passages that I identified because they unpack the meaning of a fraction as being more than part of a whole. These AIs also include some of the notation used to explain fraction as quotient and the second one uses the term *quotient* in a relational context rather than as a way to denote the solution to a division question. These two AIs provide evidence that fraction as quotient is most explicit in the AIs for one SLO in Grade 8 even though the quotient interpretation of fractions is not directly stated in any of the SLOs in the curriculum standards.

Results from the Grades 3-8 Number Strand Support Documents

Fraction as quotient is evident in five places in the support documents, specifically in Grades 3, 4, 6, and 8. Table 10 provides an overview of these five passages and a brief description as to where each one is located. I identify these passages in order of grade level, referring to them again as interpretations. I share my reasoning behind identifying each passage as related to fraction as quotient below.

Table 10

Five Interpretations of Fraction as Quotient in the Grades 3-8 Support Documents

	Grade	Location	Connection to Quotient
1	3	Definition includes “a quotient in the form a/b”	Explicit mention
2	4	Same definition as Grade 3	Explicit mention
3	4	Performance task: Sharing cheese slices	Implicit quotient context
4	6	Same definition as Grades 3 & 4	Explicit mention
5	8	Checklist: “Explain when a/b represents a quotient”	Explicit mention

Interpretations 1, 2, & 4) In Grades 3, 4, and 6, a definition for a fraction is provided that connects to the quotient interpretation. Within these three grades, “fraction” is defined as, “a number that represents part of a whole, part of a set, or a quotient in the form *a*/*b*, which can be read as *a* divided by *b*” (Manitoba Education, 2014b, p. 40; Manitoba Education, 2017a, p. 93; Manitoba Education, 2017b, p. 71). Notably, fraction as quotient is explicitly referenced in this definition in the support documents even though this definition is not included anywhere in the curriculum standards document. The Grade 3 support document which includes this fraction definition is connected to the relevant Grade 3 SLO and AI that I mentioned for interpretations 3 and 4 in Table 9. However, there is no information provided in the later sections of the Grade 3, 4, and 6 support documents about learning experiences that might help students explore the concept of fraction as quotient. Instead, the learning experiences focus on fractions as part of a whole and part of a set (Manitoba Education, 2014b; Manitoba Education, 2017a; Manitoba Education, 2017b).

Interpretation 3) I located a place where fraction as quotient was implicit in a "performance task" suggested for Grade 4 (Manitoba Education, 2017b). The task describes a sharing activity: "Show how 4 people can share 3 cheese slices; Show how 3 people can share 2 cheese slices; Show how 12 people can share 6 cheese slices; Show how 6 people can share 4 cheese slices" (p. 77). This activity is similar to a sharing activity that the participating teachers engaged in during one of the interactive inquiry sessions, as discussed later. However, in the Grade 4 support document, teachers are not helped to see how this activity would connect to fraction as quotient and students could work on the task in ways that do not explicitly engage with this fraction interpretation. In addition, the performance task is suggested as an assessment of understanding fractions as part of a set rather than being related to the quotient interpretation.

Interpretation 5) Fraction as quotient appears explicitly in the support documents as part of the Grade 8 observation checklist for an SLO focused on ratio in the curriculum standards. This SLO along with two of the AIs are places that I identified as related to fraction as quotient in the section on the curriculum standards in Table 9 (see Interpretations 7, 8, & 9). The support document includes, "Explain when *a/b* represents a quotient" as one of eleven checklist indicators (Manitoba Education, 2015, p. 51). Teachers are provided with one example of a learning experience that connects to this checklist, where fraction as quotient is somewhat evident, "Explain that five spaces in the room are labelled *fraction, rate, ratio, quotient*, and *probability*. Present students with one scenario from BLM 8.N.4.2: Meaning of *a/b* ?" (p. 50). Here the emphasis is on the meaning of a/b. However, there is no support given to what a quotient or fraction as quotient could mean in this learning experience, nor in any other sections of this support document.

Summary of Document Analysis Results

My analysis shows that there are SLOs and AIs where fraction as quotient can be implicitly interpreted, as well as more explicit references to fraction as quotient in some of the support documents for teachers, as described above. Notably, teachers are required to follow the curriculum standards document whereas use of the support documents is optional. Although these examples acknowledge fraction as quotient in definitions for some grade levels, the curriculum standards documents fail to operationalize this definition through any tasks, examples, or teacher prompts. This omission is very likely to create a gap between conceptual intent and the way the curriculum is enacted in classrooms. In addition to this, outcomes which support the development of quotient, such as partitioning (Charalambous & Pitta-Pantazi, 2007; Kieran, 1976; Lamon, 2007) and sharing (Fosnot & Dolk, 2002), although evident in the Manitoba Education (2013) mathematics curriculum standards, are not positioned as precursors to fractional understanding because these outcomes do not include direct references to fractions. Notably, none of the support documents convey the progression of fraction schemes across the grades. Finally, although some relational focus can be interpreted in one Grade 3 AI, as described above (Table 9, Interpretation 3), there is little emphasis on conceptual understanding of fractions as relationships in the SLOs or AIs. In Chapter 5, I compare these findings to research and offer recommendations.

Findings from the Interactive Inquiry Sessions

The interactive inquiry sessions with teachers were intended to provide insights about my second and third research questions. The data was very rich and many features of the interactions seemed to help nudge each person toward a deeper understanding of fraction as quotient. At the same time, analysis of the interactions revealed some factors that seemed to be barriers to getting to a deeper understanding. In this section, I share seven features that seemed to impact our collective understanding of quotient and convey how each feature was helpful, a hinderance, or in some cases, both. The seven features are: problem-solving contexts, the quotient notation and generalization, re-defining quotient as more than the answer to a division question, difficulties with describing fraction as quotient, focusing on the relationship dimension of fractions, working with indefinite amounts, and the mathematics curriculum standards and support documents. After discussing each feature, I describe through the lens of social constructivism evidence that working collaboratively supported us in deepening our understanding of quotient and contributed to our MKT. I then reflect on how my thinking about fraction as quotient shifted and how I developed my own MKT through preparing, facilitating, and analysing these sessions.

Features Which Impacted Teachers' Understanding of Quotient

Table 11 provides an overview of the seven features that seemed to impact how the teachers and I developed our understanding of quotient. For each feature, I identify the observed role and whether the feature was helpful and/or a hinderance. Following the table, I explain each feature and include some dialogue from the transcripts to show how the feature seemed to impact our understanding of fraction as quotient. The excerpts I share are examples of each feature which, in most cases, was also evident in other interactions in the sessions. That is, the examples that are shared do not include all occasions where a feature was evident in the data.

Table 11

Seven Features which Impacted Teachers' Understanding of Quotient

Feature	Observed Role	Impact
Problem-Solving Contexts	Teachers engaged with sharing problems and created their own quotient-based problem	Helpful
Quotient Notation and Generalization	Teachers discussed and worked with a ÷ b = a/b notation	Helpful/Hinderance
Re-defining Quotient	Teachers revisited their prior definition of quotient as the answer to a division question	Hinderance

Describing Fraction as Quotient	Teachers attempted to articulate the meaning of a fraction as a quotient	Hinderance
Relationship Dimension	Teachers explored the relationship between the numerator and the denominator and compared quotient to ratio	Helpful
Indefinite Amounts	Teachers worked with ambiguous contexts (i.e. "bags of chips")	Helpful
Curriculum and Support Documents	Teachers examined how these documents position fraction learning	Helpful/Hinderance

Problem-Solving Contexts.

Problem solving contexts seemed to be supportive of teachers' understanding of fraction as quotient and helped us to "mathematize" this new learning (Fosnot & Dolk, 2002, p. 29). As explained in Chapter 2, Fosnot and Dolk say that for a problem to be "mathematized" by learners, teachers need to be purposeful in their design and cognizant of three components: use of a familiar model over time and in different situations; a context that allows learners to access schema for the problem they are doing; and a situation that prompts questions and inquiry. The problems we explored together seemed to provide this opportunity. I first discuss evidence that Fosnot and Dolk's (2002) "Sub Sandwich Problem" (see Appendix D) supported teachers' beliefs about teaching conceptual understanding through problems and helped us recognize the need for more contextual problems. I also show that using manipulatives, such as graph paper, during problem solving tasks helped us to model our thinking of fraction as quotient. Finally, I share how developing our own problem-solving context centered around fraction as quotient helped our collective understanding to deepen.

Working with the "Sub Sandwich Problem."

I had anticipated that problem solving would be helpful for developing an understanding of fraction as quotient. This was primarily based on my experience as a mathematics teacher where I had seen problem solving as an effective avenue for learners to identify or generalize patterns. Although we didn't engage in any problem solving in Session 1, I designed Session 2 around the sub sandwich problem because I hoped that this problem would help everyone visualize a situation where fraction as quotient was evident.

We began by looking at a poster that showed five students sharing three sub sandwiches (see Appendix D). Julie began to partition three subs that I had drawn on the chart paper, dividing each sub initially in half, so that each of the five students received ½ of a sub (see Appendix E, Figure 2). She then partitioned the remaining half into fifths, with one of each of these pieces also going to each student. She stated that these pieces would be the same as tenths. When asked how much each student would get, or what ½ plus 1/10 is, she said, "So altogether it's like, [pauses and squints eyes] six-tenths?" (Session 2, Line 1262). Then, moments later, she elaborated, "Like, right away I thought, like '[O]K, three subs shared with… over five people, [writes down 3/5 on chart paper] so I guess, three-fifths [squints a little]" (Session 2, Lines 1282-1283). This moment is interesting because it seems like Julie does not recognize that the situation of three subs shared between five students is the fraction as quotient, 3/5, even though she says she thought of three "over five" right away. She also does not seem to immediately recognize that 6/10 and 3/5 are equivalent fractions. I think this quotation reveals the complexity of the quotient interpretation of fractions, which is highlighted by the problem we were solving.

When asked to explain her thinking, Julie writes that ½ is the same as 5/10 on the chart paper (Appendix E, Figure 2) and says that therefore, 5/10 + 1/10 = 6/10, which is also equivalent to 3/5. She adds, "Like this [gesturing to her written fraction addition on the chart paper] is just the easiest representation of like, three [fifths], but to make sense of this [gesturing again to the chart paper] as the <u>answer</u> to this [gesturing to the sub sandwich poster] is hard" (Session 2, Lines 1323-1325). In the same conversation, the Grade 6 teacher also said, "And that's that missing piece of not being taught fractions as quotient" (Session 2, Line 1319). In this interchange, it seems as though the problem context is helping these two teachers get closer to understanding how fraction as quotient can be helpful in this sharing problem.

A few moments later, the Grade 7/8 teacher shared how she had done this problem with her class earlier in the school year and had been surprised by the reveal of fraction as quotient within it. She said, "See, when my kids did that [referring to Julie's three "over five" explanation], I was like, it can't be that simple... And I was like, hold on. Maybe it is that simple [laughing with Grade 6 teacher]" (Session 2, Lines 1307-1312). By simple, I think she meant that when working with a sharing problem, the portion that any person gets is always a fraction, with the number of objects that are being shared in the numerator (i.e. sub sandwiches in this problem) and the number of people sharing the objects in the denominator. I look more closely at this generalization in the related section below. Observing Julie partition the sub sandwiches and share her thinking highlighted how the teachers viewed and experienced problem-solving contexts as important opportunities for building mathematics concepts, specifically fraction as quotient.

Discussion of the sub sandwich problem continued in Session 3. The Grade 6 teacher shared how she had tried this problem with her students, since exploring it with us in Session 2. She described how her students seemed to have been able to recognize the generalization of fraction as quotient within the problem, but then when she gave them another contextual problem in the following lesson, it was clear that their understanding was not yet solid:

> I was like, 'Okay, so we have five bags of chips and three kids. Who can tell me…?' And like dead silence. And so, I was so stoked [referring to the previous lesson she had done with them with the sub sandwiches], but I think they didn't totally have the conceptual

understanding, or it's not ingrained yet, or whatever… And they were doing problem after problem, like got it, got it, got it [referring to recognizing the fraction as quotient generalization in the sub sandwich problem]. And then today, just like, staring at me blankly [laughs]" (Lines 642-646).

It is evident that the Grade 6 teacher saw the importance of problem-solving contexts for developing a conceptual understanding of the quotient interpretation, as she independently tried to create more problems for her learners to unpack.

Similarly, Julie voiced the need for more problems to highlight the quotient interpretation of fractions, which provides further evidence that problem-solving contexts were helpful for her understanding of quotient. During Session 3, she stated,

> I wish there were more problems that helped me wrap my head around the scenarios. Like '[Okay], the sub one or the chip bags' [referring to the sub sandwich problem and the potato chip problem that was shared by the Grade 6 teacher]…I need to figure out the opportunities to teach that, I guess, because I'm not as familiar with it. So, I need to do a little bit more work on how I can get there. Like my takeaway from this is it's important [fraction as quotient] and I never thought about it really before this … but my work then is how? What are the problems I need to find to make it explicit? (Lines 2014-2017).

These observations suggest that including contextual problems which highlight the quotient interpretation of fractions, either in curriculum standards documents, support documents, or other mathematics teaching resources would be helpful for teachers as they "mathematize" the quotient interpretation of fractions (Fosnot & Dolk, 2002, p. 29) and would support their students in doing so as well.

Use of Manipulatives when Problem Solving.

The use of manipulatives while working on the problem contexts also helped to deepen our understanding. When I was planning Session 2, I knew I wanted to bring chart paper as a way for people to document their collaborative mathematical thinking. I ended up bringing chart paper sized graph paper to the session rather than plain or lined chart paper, as that was what was available to me. Unexpectedly, this tool helped to deepen understanding during the sub sandwich problem. For instance, some teachers remarked, “I just think it make[s] a lot more sense if we have the graph [paper] (John, Session 2, Line 1434) and “It’s cool ‘cause it kinda gives you the space” [referring to the graph paper] (Grade 7/8 teacher, Session 2, Line 1447). The Grade 6 teacher demonstrated the value of the graph paper as we explored the scenario where three subs are shared by five students. She drew out the sandwiches and divided them up using the graph paper squares (see Appendix E, Figure 1) and explained, “I was like, ‘Oh, well, yeah, three-fifths.’ But then I was like, ‘What does that look like? How do you cut that? (Session 2, Lines 1415-1416). The Grade 6 teacher then wondered about how to represent the sandwiches for the next scenario in the problem and whether or not changing the size of the wholes affected the result (Session 2, Lines 1483-1485). The Grade 6 teacher was “mathematizing” the quotient interpretation here by drawing out the sub sandwich model many times and in different situations (Fosnot & Dolk, 2002, p. 29). The graph paper tool seemed to support the Grade 6 teacher’s ability to reason with fraction as quotient, while also supporting all of the teachers’ ability to draw, partition, and compare subs as a group.

Co-Creating Our Own Problem.

The evidence of problem-solving contexts as supports for the participants’ understanding of quotient was also made clear when we worked collaboratively to create our own problem involving fraction as quotient. In Session 2, I asked that we try to come up with a new question that targets the a÷b= a/b notation, where fraction as quotient was the central fraction interpretation instead of part-whole or ratio (Session 2, Lines 1782-1789). There was a short pause and then the Grade 6 teacher asked for some clarification; “So maybe I'm mis-conceptualizing this, but then it would have to be something where you're splitting something, right? Like it would have to be like a division problem essentially” (Session 2, Lines 1792-1793). In this quotation, it seemed to me that the teacher was both realizing herself but also helping others to see the meaning of fraction as quotient as a relationship that is a divisive situation. After some discussion, we arrived at the following problem: *There are five classes walking to the mall for a field trip. Two of the teachers are driving and can’t walk with their classrooms, leaving the three remaining teachers to supervise all the students on the walk. How many classes do each of the teachers have to supervise?* (Session 2, Lines 1844-1859). Later, I discuss some specific features of this problem that seemed to support a deeper understanding of fraction as quotient including that the number of students is not specified, only the number of classes. The teachers conveyed their understanding of this problem in relation to fraction as quotient, as follows:

> Grade 7/8 teacher: So, there's five classes and there were five teachers. Now there's three, only three teachers.
>
> Grade 6 teacher: So, each teacher gets five-thirds of a class [expressed with a sense of discovery]
>
> (Session 2, Lines 1856-59)

Through this, the teachers again were “mathematizing”, as they created a problem-solving situation that prompted questions and inquiry (Fosnot & Dolk, 2002, p. 29).

We continued to reflect on the problem we created the following week, at the beginning of Session 3. Having contextual examples for fractions as quotient remained important in grounding our understanding as summarized by Julie when she said, “It [the quotient interpretation of fractions] needs to live in problems” (Session 3, Line 173). Towards the end of Session 3, Julie reiterated this once more by redefining fraction as quotient; “I think just saying that it represents a quantity, or quantities, in the variety of scenarios and contexts, matters. I think that's an important piece” (Session 3, Lines 2121-2122), to which the Grade 4/5, Grade 6, and Grade 7/8 teacher nodded in agreement, with the Grade 7/8 teacher also gesturing at Julie to convey that this statement resonated with her.

The examples we worked through and created collaboratively not only helped us begin to “mathematize” the meaning of fraction as quotient, but they also highlighted the need for more problems and contexts for this fraction interpretation to build our MKT (Fosnot & Dolk, 2002). Although the teachers displayed growth in their understanding of quotient and MKT through this feature, as discussed in relation to things that seemed to hinder learning, I also saw evidence that their understanding was still new and fragile.

The Quotient Notation and Generalization.

Unpacking the notation of fraction as quotient, a÷b= a/b, as well as the generalized use of fraction as quotient with the teachers seemed to be both helpful as well as a hinderance for these teachers’ understanding of the interpretation. In this section, I discuss instances where some teachers did not see the notation and/or generalization as relevant to the quotient interpretation and other times where explicitly talking about it seemed to help.

The quotient notation and generalization came up for the first time in Session 1 when I briefly stated that fraction as quotient targets a÷b= a/b and gave the example that 6/4 is a fraction as quotient that can be understood as six brownies shared between four friends (see slides in Appendix C). I then asked what the teachers thought the answer to 3÷5 was, with Julie responding, "six-tenths" (Session 1, Line 2874). Note that this discussion took place in Session 1, prior to the work we did using these fractions in relation to the sub sandwich problem in Session 2. When asked to explain her thinking, Julie shared that she thought of an equivalent fraction, out of one hundred: "I used a fraction to solve it, but thought about my answer as a decimal, if that makes sense" (Session 1, Lines 282-283). Although I did not redirect Julie to think about 3÷5 as 3/5, which might have helped to unpack the quotient generalization, it is interesting to note that Julie was not applying the quotient generalization here. In other words, she did not think about the question 3÷5 as being the same thing as the fraction three fifths or 3/5. These observations suggest that at this point, the quotient generalization did not yet hold meaning for Julie, even though I had just stated it. Some participants did come to realize the idea that 'fractions are division' but this understanding was not yet evident in Session 1.

In Session 2, after working through the sub sandwich problem from Fosnot and Dolk (2002), we unpacked the a÷b= a/b notation and generalization more explicitly. In the context of the sub sandwich problem or other sharing problems, the quotient generalization helps the learner to realize that the portion or fraction that each person would receive is always the numerator (the number of items available to share) divided by the denominator (the number of people sharing). For example, if there are three subs shared by five people, each person would receive 3/5 of one sub. If there are nine subs shared by seven people, the portion each person would receive would be 9/7 of one sub. Later in the same session, we discussed the potential value of exploring this generalization and notation with students and for helping students develop an understanding of fraction as quotient. This exchange prompted Julie to wonder if teaching the notation to students would take away from the productive struggle of discovering it themselves, and said, “Like I guess once they come to this realization, does all of this great struggle kind of go out the window?” (Session 2, Lines 1685-1686). The Grade 6 teacher replied, **“**But isn’t that kind of the point? Like, we struggle to get to a place that’s you know, like…we struggle through addition so that we can get proficient at it?” (Session 2, Lines 1689-1690). Through this interaction, it seemed as if Julie’s understanding or appreciation of fraction as quotient was hindered by focusing on the quotient notation, which might be applied in a more procedural manner and undermine conceptual understanding, whereas the Grade 6 teacher’s understanding was supported or reinforced by using the generalization.

The other teachers then shared their thoughts, conveying a clearer understanding of quotient and its place in mathematics instruction: “[I] never put that there [referring to the a÷b= a/b notation], that this is what it looked like. That's interesting” (John, Session 2, Line 1732); “I feel like it would make so much sense for kids, especially moving into like, improper fractions and looking at it as quotient” (Grade 4/5 teacher, Session 2, Lines 1736-1737). In this instance, although the Grade 4/5 teacher expressed that the quotient generalization was helpful, John’s takeaway about the notation as “interesting” was harder to interpret.

In Session 3, the quotient notation, a÷b=a/b, also impacted the ways in which teachers seemed to understand the meaning of this fraction interpretation, with its connection to division. After some discussion of this generalization, I wrote ‘6÷3=6/3’ on the chart paper (see Appendix D, Figure 4). After I wrote this, the Grade 6 teacher responded,

> And they [referring to her students] did find this [gesturing to 6÷3=6/3 on the chart paper]. But like, getting from this to this was hard [gesturing to 6÷3=6/3 on the chart

paper]. It was just like, reaffirming that these are the same [both sides of the equation]… but yeah, I agree that connection was a struggle (Session 3, Lines 275-280).

I think the struggle the Grade 6 teacher is referring to in this passage is in connecting the quotient notation to the generalized understanding of its meaning. To this, the Grade 4/5 teacher later added, "But now I'm thinking of it, and I'm like, why would I not teach fraction as quotient when I'm teaching division?" (Session 3, Line 491), revealing that she was seeing the value and meaning of the generalization beyond the notation.

At the end of the session, when asked if their understanding of the meaning of fractions had shifted since Session 1, the Grade 6 teacher said, "I had my kids chanting 'fractions are division' today [people laugh], trying to get that point across…So this has kind of helped flesh that out" (Session 3, Lines 2115-2116). Highlighting the divisive relationship in terms of the generalization within fraction as quotient seemed to support the Grade 4/5 and Grade 6 teachers' understanding of fraction as quotient. However, it was less clear if it was helpful for the other three participants.

Re-defining Quotient as More than the Answer to a Division Question.

The participants previously held understandings of the term quotient seemed to pose a challenge as they worked to develop an understanding of fraction as quotient. In Session 1, John and Julie revealed a pre-existing definition of quotient in their minds, sharing that quotient means the answer to a division problem, whereas the Grade 7/8 teacher shared that quotient means division involving fractions (Session 1, Lines 18-41). In retrospect, I can see how these limited understandings may have impacted the participants' ability to think about fraction as quotient as a relationship, as it meant re-defining the concept of quotient. If I had been aware of this in Session 1, I might have responded in a different way when the teachers described how they understood the meaning of the word quotient.

In Session 2, when we looked at the definition of fraction as quotient more deeply, I shared that fraction as quotient means more than the answer to a division problem:

> Jillian: The research that I've done will say that quotient as fraction is not the answer to a division problem or division with fractions. And I went down a big rabbit hole thinking 'Oh it's the same thing'. But it's actually not, because the fraction itself is a quotient. [Long pause, people are quiet] [But] the vocabulary of quotient teaches us that that's the answer to a division problem.
>
> John: Can you repeat what you said?
>
> Grade 7/8: Yeah, no, I'm trying to follow you and I'm like 'Uhhhh' [laughs]
>
> (Session 2, Lines 672-687)

In this interaction, it is evident to me that these teachers' existing definition of quotient impacted their ability to see quotient in a different manner or to understand quotient as a relationship in addition to quotient as the answer to a division question.

In Session 3, we once again revisited the definition of quotient as more than the answer to a division question. When we looked back on the problem we co-created in Session 2, we talked about how five classes shared between three teachers is 5/3, and this is fraction as quotient because the fraction represents the divisive relationship that exists between the classes and the teachers. Julie and the Grade 7/8 teacher added,

> Julie: I think kids feel like math ought to be harder. And if we say it's, you know, 12 markers split amongst 10 people. You can't split a marker. Actually, it's a bad example,

but, like, [bags of] chips or whatever, it almost seems too good to be true, that the answer is just those numbers, right?

Grade 7/8: [It's] Like, you have to do something with them.

Julie: Yeah. Like it's a trick [to just leave the numbers as a fraction].

Grade 7/8: It shouldn't look the same as what you started with

(Session 3, Lines 212-224)

This interaction highlights the struggle the teachers were engaging in as they tried to re-define quotient as more than the answer to a division problem. It also conveys the challenge they felt with the fraction as quotient definition which does not require that the operation be performed but is simply expressing the relationship that exists. In contrast with the more common use of the word quotient, simply stating the fraction seemed too simple or a sort of math trick or shortcut. Thus, our previously held definition of quotient seemed to hinder our ability to gain a deeper understanding of the fraction as quotient interpretation. The persistent way that the word quotient is used in mathematics resources and curriculum standards documents suggests to teachers that an operation is needed to achieve a quotient which seems to create a barrier in coming to understand the concept of fraction as quotient.

Difficulties with Describing Fraction as Quotient.

There was some confusion when it came to describing fraction as quotient using words, beyond explaining that it was not the answer to a division question. In Session 1, after a brief introduction to the five interpretations of fractions (see Appendix C), I asked the teachers to reflect on the new information and share how things felt so far. The Grade 7/8 teacher revealed that her new way of describing quotient was still developing, "I guess I still wonder, like between fraction as operator and fraction as quotient...They feel very similar. And I guess, you know, I'm starting to wonder, how different are they? Are they actually different interpretations?" (Session 1, Lines 588-591). I agreed with her statement and explained that my own understanding of the five interpretations of fractions was still developing, even after working with these concepts for over two years.

In Session 2, I shared that a fraction can be any one of the interpretations. To this, the Grade 7/8 teacher expressed, "So, they can easily flip to each other" (Line 699), to which Julie nodded in agreement. This moment reveals that describing fractions as various interpretations, depending on the context, was supportive of their understanding of quotient. However, moments later, when I reiterated how a fraction as quotient is different than the answer to a division problem, Julie and the Grade 7/8 teacher looked for further explanation, expressing confusion over the fact that the Purnomo et al. (2021) definition that I had provided on the handout described quotient as, "the result of a division situation." Julie asked, "Isn't the result an answer?" (Session 2, Line 736). I was trying to convey that "result", in the case of quotient, meant that the fraction is already a quotient because of the divisive situation it conveys; that the word result did not mean the result of a computation, but I did not seem to be able to express this idea very effectively. This interaction, following the one immediately prior, conveys the challenges but also the importance of clear and precise language when describing the quotient interpretation.

This confusion was expressed again in Session 3, when Julie asked for a clearer description that would distinguish fraction as quotient from quotient as the answer to a division problem:

> I think I'm struggling with the fact that the word quotient can have two different meanings. And I appreciate all the different [fraction] interpretations, but sometimes I

> wonder about the vocabulary with math. And I get it’s important for kids. It's important for us to understand. But when you get down this road of quotient is this and this, especially using the word fraction with it too... [I] just wonder, what is the purpose? Like, what's the reasoning to know that? (Session 3, Lines 107-112).

Later in Session 3, when looking at the Manitoba Education (2013) curriculum standards document, the teachers began to notice that quotient was a term used often, but not as a relationship or in the fractional sense; only as a solution to a division question. Through this, the Grade 4/5 teacher conveyed a deeper understanding of fraction as quotient. She agreed with the idea that one’s prior knowledge of what quotient means, in an isolated sense, may contribute to confusion among learners:

> If they're [the curriculum writers] introducing division in that sense, that makes me think it should be getting introduced in those multiple ways, right? Because then it does become so finite. Then they're going to grade, whatever, and then, like, ‘No, no, this is quotient. This is division. This is how we only look at it (Session 3, Lines 1334-1337).

It was clear that the muddiness of fraction as quotient, in the ways it was defined in the curriculum standards document and in how it was described during the sessions, tended to hinder most of these teachers’ understanding. In addition to this, for myself as an educator and researcher, the ways in which I describe both quotient and fraction as quotient have also shifted. I expand more upon my own MKT in this regard in a later section below.

Focusing on the Relationship Dimension of Fractions.

Viewing fraction as quotient through the lens of a relationship seemed to be helpful for supporting teachers’ understanding of this interpretation. Focusing on relationship occurred in two different, yet related ways. One of the ways is through a consideration of the relationship between the numerator and denominator within a fraction, while the other looks at the relationship between the quotient and ratio interpretations of fractions. Both aspects of the relationship dimension of fractions seemed to impact the understanding teachers had about the quotient interpretation.

During Session 1, I saw little evidence of a shift in MKT through focusing on the relationship dimension, although John did share how he tries to connect some of the rational number representations to each other, from a relational perspective; "I try to do that when I when I talk about division [use of mathematical language], I start talking about those vocabularies and from there, I connect to fractions, and right after fractions is ratios, so that they can see the relationship" (Session 1, Lines 537-539).

The relationship dimension of fractions was also not mentioned in Session 2 as something helpful for teachers' understanding of the quotient interpretation. However, several interactions during Session 3 highlighted how thinking about fractions as relationships deepened for participants. I will focus on the thinking of four participants to illustrate this observation (Julie, John, the Grade 7/8 teacher, and the Grade 4/5 teacher). I present the examples in the order that they unfolded during the session since the earlier interactions may have paved the way for participants' later realizations. The importance of seeing fractions as relationships for understanding the fraction as quotient interpretation was initially made clear by Julie when she shared,

> When I look at the two differences here for fraction as part-whole [referring to the handout of the five fraction interpretations, see Appendix E, Figure 4], it says 'needs to be a continuous quantity of one thing, like one pizza, one chocolate bar', one whatever, or discrete objects…of the same thing. Whereas fraction as quotient, you're looking at

> two things. You've got granola bars and friends, teachers and kids (Session 3, Lines 173-188).

A few moments later, John shared how the notion of relationships between or within numbers was new to him as an explicit way of thinking about fractions, even though he had been using this idea as he worked on ratio with students. He openly shared his developing understanding, which contributed to the other teachers' development of MKT as well:

> It's interesting when we talk about fractions, right? Last time we specifically talked about [how] it's basically the relationship between them, right? But I don't recall where I used that [term of] relationship when I was talking about fractions. I did when I started looking at ratios. That is where we specifically said, again, 'ratios are just like fractions', but vocabulary? I didn't think of that when we were talking about fractions (Session 3, Lines 554-589).

Shortly after this, the Grade 7/8 teacher responded, "So, then I wonder, is fractions as a ratio and fractions as quotient… are they the same? Except ratio talks about those implications to bigger quantities?" (Session 3, Lines 595-596). A connection was made explicit between the relationship that exists within a fraction as quotient and the ratio interpretation of fractions. This shift was important for us during the interactive inquiry sessions and helped us begin to re-define fraction as quotient, using our new understanding of the interpretation and our developing MKT. Notably, we still had partial and fragile understandings and struggled to see the difference between fraction as quotient and fraction as ratio (which adds an inherent multiplicative dimension) but working with the idea of relationship seemed to help to move us to new understandings.

At the end of Session 3, the Grade 4/5 teacher expressed her understanding of the relationship within fraction as quotient and conveyed how she believed it would also be helpful for other learners. She said, “I definitely touch upon that. Like we talk about the relationship between the numerator and the denominator, but not specifically in the form of quotient, which I think [would] make so much more sense for kids” (Session 3, Lines 1375-1377). The Grade 7/8 teacher also conveyed the impact that the relationship dimension of fractions had on her understanding by saying, “The idea of like, you can look at parts of a set, or partitioning a set, or a whole or whatever, but then the relationship of like two things together...I think that's like a new understanding for me” (Session 3, Lines 2101-2103). These quotations provide compelling evidence that the Grade 4/5 and Grade 7/8 teachers were supported by thinking about the relational aspect of fraction as quotient and found it to be helpful for their developing understanding of this fraction interpretation.

Working with Indefinite Amounts.

Working with indefinite amounts rather than discrete quantities, also seemed to support the teachers’ understanding of fraction as quotient. Although I did not see evidence of this feature in Session 1, I began to recognize how this was helping in Session 2 when we co-created our own problem where fraction as quotient was centered. The teachers talked about the intentional use of the word “class” within the problem in contrast with a specific number of students:

Grade 7/8 teacher: Yeah. I mean, you should take a bit of a class.

Julie: And I used ‘class’ because there's not a number [of items].

Grade 6 teacher: And that's good. I like that you said that because then we can't rely on the [dividing or reducing the] number, right? That's our final answer. You can't, I don't

> know, because we're not reducing it to [another fraction]. The five-thirds is the only answer that makes sense (Session 2, Lines 1871-1891).

The teachers came to this realization that using an indefinite amount could be helpful through their dialogue; this was not an idea that I initiated. They all agreed that working with indefinite amounts aligned with fraction as quotient without highlighting or creating confusion between fraction as quotient and some of the other fraction interpretations (i.e. ratio and operator). The idea that an indefinite or more ambiguous quantity was helpful for understanding fraction as quotient was revisited again in Session 3. The Grade 6 teacher shared an example from her recent teaching with this in mind, saying,

> I think, I've been thinking about it a bit, and I think we talked about this last time, but to me, the key is that it can't be something measurable. When I was doing it [fraction a quotient] with my kids today, I was talking about bags of chips. How much of a bag of chips? . . . You know, like, I think it has to be something that's a little bit ambiguous (Session 3, Lines 23-27).

I think by measurable, the Grade 6 teacher was getting at the idea that when an indefinite amount is used, it is easier to think about the relationship dimension of fraction as quotient within a divisive situation. This seemed to help the Grade 7/8 teacher's understanding because she added, "Because I guess it has to be a non-countable set" (Session 3, Line 47), to which the Grade 6 teacher responded, "[Because] then [if it is a countable set] there's no reason to represent it as quotient" (Session 3, Line 53). Here, I think the point was that when you have numbers that can be easily reduced or divided, it is less necessary to retain the relationship idea in your mind and learners may revert to part-whole thinking or simply solve the division question. This idea also resonated with Julie who later referred to the indefinite "bag of chips" idea at the end of Session

3 when she said, “I wish there were more problems that helped me wrap my head around the scenarios, like the sub [sandwich] one or the chip bags” (Session 3, Lines 2012-2013). Julie’s observation, and the dialogue among these teachers, suggests that problem contexts with indefinite or ambiguous amounts rather than precise quantities may be helpful for supporting teachers to think about fraction as quotient because indefinite amounts make it easier to retain the idea that a fraction is a relationship.

The Mathematics Curriculum Standards and Support Documents.

The Manitoba Education (2013) mathematics curriculum standards document and related support documents seems to have hindered our collective understanding of fraction as quotient, although it is my position that these documents have the potential to be supportive of fraction as quotient, as I explain in this section. During Session 1, in response to my question asking how the teachers explain what the meaning of a fraction is to students, four participants (the Grade 4/5 teacher, Julie, John, and the Grade 7/8 teacher) shared that fractions are parts of a whole (Session 1, Lines 86-152). I think that this initial limited understanding of fractions likely reflects the prevalent representation of part-whole in the Manitoba Education (2013) curriculum standards document.

In Session 2, the teachers began to share how they felt the Manitoba Education (2013) curriculum standards hinder fraction teaching and learning because the SLOs in each grade level seem to be isolated from meaningful fraction concepts. For example, Julie, John, and the Grade 7/8 teacher all shared that students are not always ready for the SLO focused on fractions included in their grade level and that further conceptual understanding is needed, related to SLOs at earlier grade levels (Session 2, Lines 1005-1023). The Grade 4/5 teacher also reflected on the progression of learning fractions that is supported by a teacher’s MKT (Ball et al., 2008), saying,

> What do they need to know before they can even get to this thing that the Grade 8 or 7 [outcome] says? Because there's some [outcomes] in Grade 1 and Grade 2 and Grade 3, right? They don't explicitly talk about fractions, but there is an understanding that then led them to this sort of understanding [of] fractional number (Session 2, Lines 2217-2221).

It was clear that the teachers used the curriculum standards document as a guide, rather than a prescriptive set of SLOs. This sems to be related to their own MKT and autonomy as teachers but also seems to be their response to the limitations they see with the standards document. In addition to this, the Grade 4/5 teacher stated that she felt there are outcomes connected to the understanding of fractions that occur before the first fraction SLO in Grade 3 (Manitoba Education, 2013, p. 81). She wondered how these are taught since the curriculum standards do not make this connection explicit (Session 2, Lines 2215-2221).

We discussed the Manitoba Education (2013) curriculum standards documents and the related support documents more explicitly during Session 3 as we collaboratively analysed passages related to the quotient interpretation. The teachers noticed that quotient was not mentioned in any of the fraction SLOs, nor was there emphasis placed on the conceptual meaning of fractions; something we had been unpacking throughout the three sessions. Instead, the teachers shared that they believed the curriculum standards document is large and too focused on procedures involving fractions. For instance, Julie said, "I think sometimes there's a race to cover content, to do it operationally. And ordering [and] comparing fractions is so difficult because unless you convert them to decimals, it involves that conceptual understanding of space, right?" (Session 3, Lines 540-542). In Chapter 5, I argue that fraction as quotient can support learners' conceptual understanding of fractions and should therefore be positioned as such in the standards document.

All five of the teachers also shared that they believed the Manitoba Education (2013) mathematics curriculum includes too many learning outcomes. The teachers expressed a need for a more concise and continuous document to convey the development of fractional learning, with the Grade 7/8 teacher wondering, "Why are we hitting our heads against the wall with the document that should be helping us?" (Session 3, Lines 1841-1842). The Grade 6 teacher shared similar thoughts by saying, "I just wonder… how do you do that in a way that's accessible to teachers… because this all feels [like] there's a lot in the curriculum already. How do you add quotient in?" (Session 3, Lines 1503-1506). Although the teachers had shared that they use the curriculum standards document as a guide, they also revealed a desire for a supportive document that they could use and trust. At the same time, most participants stated that they had not used the Grades 3 to 8 support documents (i.e. the documents that I analysed in this study) very frequently. In addition to this, based on the passages from the support documents that I shared with them, they commented that even if they had used these documents more frequently, these passages would not be sufficient to assist them with understanding or teaching students about the fraction as quotient interpretation.

Although the teachers expressed challenges with the curriculum standards and support documents, they also described some of the features what would make these documents more helpful for teaching fractions. Thus, I argue that the Manitoba Education (2013) curriculum standards and related support documents have the potential to be helpful documents for supporting an understanding of fraction as quotient. In Chapter 5, I provide my recommendations and suggestions for improving these documents.

Social Constructivism Supports Teachers' Development of MKT

It was clear to me from the first interactive inquiry session that this was a group of teachers with characteristics that might not be typical of most teachers. The participants immediately appeared open and honest about their understanding of fractions in ways that I believe were genuine. For example, one teacher shared that the phrasing of fraction as quotient was new to them (Grade 4/5 teacher, Session 1, Line 52), while another openly wondered about the overlap between the quotient and operator interpretations of fractions (Grade 7/8 teacher, Session 1, Line 589). Throughout Session 2, the participants continued to be vulnerable about their understanding of fractions, such as when the Grade 7/8 teacher said, "I think the ratio one [ratio interpretation of fractions] I find hard. Because it can just so easily become one of the other ones" (Session 2, Lines 629-636), or when John said, "It's interesting, we don't focus on this [gesturing to the quotient interpretation on the handout]" (Session 2, Lines 562-653). It was evident that the teachers felt comfortable to share their thinking, including things that they found challenging. In my experience, many mathematics teachers are uncomfortable revealing things that they do not fully understand and for this reason I think the vulnerability revealed by this group of teachers was not typical of most teachers but was very helpful for our collaborative learning process.

These participants' willingness to be open and honest continued into Session 3, when they continued to be vulnerable about their current understanding of fraction as quotient. For instance, the Grade 4/5 teacher said, "Some of the other ones are a little muddy for me [referring to the five interpretations of fractions]" (Session 3, Line 2005). I believe this openness was related to the fact that these were experienced teachers, that they all knew me, and that most of them knew one another. I think that these circumstances, which I intentionally included in my research design, helped to create a socially constructed learning setting where we were all able to deepen our understanding (Vygotsky, 1962). Given the openness of these teachers as we worked to construct the meaning of fraction as quotient, many ideas were co-created. That is, one teacher might share a thought and it was then elaborated on or further developed by other participants. I see this as an example of what Vygotsky (1962) referred to as a “live-thinking process” (p. 105). The following interaction from Session 3 is an example how this process unfolded many times during the three sessions:

> Grade 7/8: So, then I wonder, then, is fractions as a ratio and fractions as quotient like, are they the same, except ratio talks about, like, you know, those implications to bigger quantities?
>
> Jillian (PI): Yeah? And like, the proportional aspect. The ratio does have two things, like the quotient.
>
> Grade 7/8: Yeah, because they’re comparing the same thing.
>
> John: And the quotient [motions to interpretations handout]. That’s what we say now.
>
> Grade 7/8: Because it goes back to your point [talking to the Grade 6 teacher] that, when you're saying it's two different things compared.
>
> Jillian (PI): Yeah.
>
> John: [A] fraction can be one [thing] or it can be two?
>
> Julie: Or three or four [everyone laughs]
>
> (Session 3, Lines 595-627)

It is also important to note that I was honest and vocal about my own learning process around fraction as quotient, which included my misunderstandings and confusion. Just as I learn alongside students that I teach, I felt it was important to do this work honestly and to learn alongside the teachers who participated in the study. Therefore, my own vulnerability may have positively impacted the teachers' ability to be open about their vulnerability as well.

Finally, it was clear that all the teachers had a shift in their MKT regarding fraction as quotient after participating in the three interactive inquiry sessions. This was evident when I compared comments they made in the first session with comments they made in the third session. When teachers were asked about their understanding of the meaning of fractions in Session 1, four teachers explained that fractions are 'part of a whole', with the fifth teacher revealing that they relate it to the measure interpretation by getting students to count fraction parts (i.e. one-fifth plus another one-fifth is two-fifths). The teachers were asked again, at the end of Session 3, to describe their understanding of the meaning of fractions. This time, four teachers shared the revelations they had experienced; "Mine [understanding] is exponentially more" (Grade 4/5 teacher, Session 3, Line 2085); "The relationship of two things together, I think that's a new understanding for me" (Grade 7/8 teacher, Session 3, Line 2102); "I think just saying that it represents a quantity, or quantities, in the variety of scenarios and contexts, matters. I think that's an important piece" (Julie, Session 3, Lines 2122-2123); and,

> I had my kids chanting "fractions are division" today, trying to get that point across. So, I mean, that's added to whatever we said the first time. I mean again, like obviously, that's not the only thing fractions are. It's all of these things (Grade 6 teacher, Session 3, Lines 2115-2117).

At the same time, John expressed some uncertainty about where the new learning would fit into the rest of his teaching, saying, "I'm still trying to [wrap] my mind around all of this" (Session 3, Line 2131). He expressed new learning and a growth in his MKT at other times during the sessions but also shared his uncertainty. For example, in Session 2 he expressed a deeper understanding of the concept of quotient when he stated:

> We were looking at equations [referring to his classroom teaching]. And when we talk about the equation, we are saying it is balancing two sides. So, if I'm saying two as my quotient [gestures to 10/5 on the chart paper], [I] never thought, 'Why this side?' If this is a quotient, quotient is equal to quotient. So, this side is quotient too, right?" (Session 2, Lines 922-925).

The development of MKT was also evident when four of the teachers shared how they imagined they would use quotient and some of the other fraction interpretations in their teaching. For example, the Grade 4/5 teacher expressed that she believed this would impact her future teaching and said, "I just feel like it's so relevant to fractions…When we've just done so much work with division, I feel for them to look at it in that sense would click so much quicker" (Session 3, Lines 1988-1994). The Grade 4/5 teacher also jotted down ideas on her five interpretations of fractions handout noting where she could weave these interpretations into other areas of her mathematics instruction (see Appendix E, Figure 4). Another teacher shared, "I'm excited to try all these things [referring to the five interpretations]. Yesterday, I made a game that was fraction as measure. And so, just thinking about how can I get all of these in?" (Grade 6 teacher, Session 3, Lines 2045-2047). Therefore, I concluded that the process of socially constructed meaning making during the interactive inquiry sessions, helped these teachers further develop their MKT for teaching fractions, with a specific deepening around their specialized content knowledge for fraction as quotient (Ball et al., 2008).

My Understanding of Quotient and MKT

As the researcher, teacher, and someone who has also been working to deepen my understanding of the five interpretations of fractions for more than two years, I further developed my understanding of quotient because of the interactive inquiry sessions. Listening and learning alongside the teacher participants to what was helpful and what hindered their understanding of fraction as quotient supported my understanding. These interactions provided opportunities for me to re-evaluate and re-define quotient out loud and to receive feedback in real time. As a classroom teacher when I facilitate a mathematics lesson with students, I make adjustments in the moment using my MKT and this impacts the learning experiences of those in front of me. As I planned for and facilitated each of these sessions, my MKT around fraction as quotient and the implications of this interpretation for teaching were continually adjusted and tuned.

One example of this process was a shift in how I defined quotient. In Session 1, I said it was the "the result of a division situation". However, this way of expressing the interpretation implied that fraction as quotient was more about the solution than about the relationship. Julie later pointed out that "result" felt like 'answer' and this made differentiating the quotient interpretation of fractions from quotient as the solution to a division question more difficult. Based on my learning through the interactive inquiry sessions, I began to recognize how impactful the relational notion of fractions was to teachers, specifically John and the Grade 4/5 teacher. This influenced the way I later described fraction as quotient. For instance, in Session 3, I said,

> That relationship piece between the numerator and the denominator is so important, for all of these fraction interpretations. But for quotient especially because you need to know… what's the thing that you're partitioning, or what's the thing that you're sharing? And to know how those, the numerator and the denominator, sort of interact with each

other, right? It's not one out of three pieces or whatever. It's like, ‘there's five classes that are split between three teachers’ [referring to the problem we had co-created in Session 2]. That feels very interconnected [Session 3, Lines 84-90].

Reflecting on these sessions, I now think of fraction as quotient as a dynamic, divisive relationship between two things. I believe it is important to continue to recognize my learning in this process in relation to the five interpretations of fractions and as part of social constructivism. Of course, my learning is also intertwined with my positionality as I continue to build my MKT.

Summary of Results from the Interactive Inquiry Sessions

Analysis of the interactive inquiry sessions revealed that four of the five teachers were not familiar with the concept of fraction as quotient prior to the first session. The fifth teacher participant revealed that although she was familiar with fraction as quotient prior to the inquiry sessions, she had found it challenging and “tricky” to unpack with students (Grade 6 teacher, Session 1, Lines 25-30). My analysis revealed that all of the experienced teachers struggled in some way to deepen their understanding of this concept and that seven key features shaped their engagement with the concept. The transcripts and artifacts also show that teachers collaborated actively, as they shared ideas, designed problems, and revisited their understanding of fraction meaning throughout the sessions. By the third session, all participants reported noticeable shifts in how they described fractions, indicating meaningful progress in their conceptual understanding. In Chapter 5, I discuss what the results from the document analysis and the interactive inquiry sessions suggest with regard to research, as well as the insights for furthering MKT for teachers.

Chapter 5: Discussion, Recommendations, and Next Steps

The purpose of this study was to examine how fraction as quotient is represented in Manitoba's K–8 mathematics curriculum and support documents, and to explore how teachers understand and develop mathematical knowledge for teaching (MKT) of this interpretation. In this chapter, I discuss the findings for the three research questions in relation to the mathematics education literature: (1) In what ways is fraction as quotient included in the K-8 Manitoba Curriculum Framework of Outcomes (2013) and in the Grades 3-8 Support Documents for Teachers? How does the approach to fraction as quotient in these documents relate to recommendations about this interpretation in the mathematics education literature? (2) What helps and hinders a group of experienced mathematics teachers working in Grades 4-8 to develop an understanding of fraction as quotient? (3) How does working collaboratively shift the ways that a group of teachers, including myself, think about teaching and learning fraction as quotient? Specifically, how do we build our MKT of fraction as quotient and how does learning about fraction as quotient build our MKT? In this chapter, I also identify some limitations of the study, describe the significance of the work, and offer some recommendations for better supporting teachers to deepen their understanding of fraction as quotient.

Discussion of Findings

In this section, I take a step back and look at each area of my study to identify my conclusions and make connections to the literature.

Discussion of RQ1: Fraction as Quotient in Curriculum and Support Documents

In response to RQ1, the findings indicate that fraction as quotient appears in nine places in the curriculum standards and five places in the support documents, mostly in definitions and as isolated achievement indicators. These mentions are limited and lack continuity across grades, indicating minimal emphasis on this fraction interpretation as compared with the part-whole interpretation. These findings suggest that Manitoba's curriculum does not adequately support the conceptual development of quotient. This contrasts with research recommending early and explicit integration of quotient to strengthen proportional reasoning for later mathematical proficiency (Lamon, 2007). My analysis also revealed that fraction as part-whole is the most prominent fraction interpretation in the Manitoba Education (2013) curriculum standards. The emphasis on part-whole is a challenge that several researchers have identified with fractional teaching and understanding, and a factor that may lead to an incomplete conceptual understanding of fractions (Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Lamon, 2007, Norton et al., 2018; Tossavainen & Helenius, 2024). Curriculum standards researchers warn against beginning fraction outcomes with the part-whole interpretation and suggest that beginning with measure (Norton et al., 2018) or quotient (Purnomo et al., 2021) may support students to develop a deeper understanding. Accordingly, even though there are statements about fraction as quotient in some of the support documents, these are likely to be overlooked both because use of the support documents is optional and because so much attention is placed on the part-whole interpretation.

In addition to the emphasis placed on fractions as part-whole, the word quotient appears often in the Manitoba Education (2013) curriculum standards but is used to denote the answer to a division problem, rather than the divisional relationship at the heart of the fraction as quotient interpretation. The repeated definition of quotient as the solution to a division problem limits the new meanings that teachers can ascribe to the word quotient, including fraction as quotient and the idea that a quotient is a relationship between two elements. As described in Chapter 4, this limitation was evident among the participating teachers during Session 1 when I asked them about their understanding of quotient.

In terms of the connection between fractions and division outcomes in the curriculum, researchers recommend that these should be linked, especially in the earlier grades (Middleton et al., 2001), and that a stronger conceptual understanding of these outcomes is a predictor of success in high school (Siegler et al., 2012). In the Manitoba Education (2013) curriculum standards, the SLOs for fractions and division are located in close proximity to one another in Grades 3 to 8, with the division SLOs occurring right before the fraction SLOs in Grades 3 to 6, and a couple of SLOs apart in Grades 7 and 8. However, conceptual connections between the SLOs in the document is non-existent. That is, there is no reference or outcome stating that division should be related to fractions. Thus, they are located in close proximity to one another but the connections between them are not indicated. In contrast, this connection is explicitly stated for the relationship between decimals and fractions in Grades 4 (p. 93) and 5 (p. 103), with language such as “Relate decimals to fractions (to hundredths)” (p. 93). It is puzzling and unfortunate that there is no explicit connection made between division and the meaning of a fraction in the curriculum standards document.

Finally, the findings from the curriculum standards analysis suggest a serious limitation for describing fractions as relationships; something that supports the development of conceptual understanding of fractions (Getenet & Callingham, 2019; Park et al., 2013). According to my analysis, there is only one interpretation where unpacking the relationship within a fraction might possibly be explored (Manitoba Education, 2013, p. 81). However, the language in this achievement indicator is vague and does not include an explicit connection. Therefore, the opportunity to explore fractions as relationships with students is likely to be missed. It certainly is not something the experienced teachers in this study were exploring with their students. In addition to this, the other fraction outcomes identified in this study focus on procedures and part-whole tasks, something which researchers have warned against (Cramer et al., 2002).

Overall, the findings for RQ1 highlight a gap between curriculum design and research-based recommendations for teaching fraction as quotient. I share specific recommendations to target these challenges in a later section.

Discussion of RQ2: Factors Impacting Teachers' Understanding

For RQ2, the interactive inquiry sessions demonstrated that supporting teachers to deepen their understanding of the quotient interpretation of fractions is complex and challenging. With that being said, each of the teachers shared a new understanding of fractions as relationships, recognized the importance of teaching the five interpretations of fractions beyond the part-whole interpretation, and saw connections between fraction as quotient and other concepts in their mathematics teaching. There were seven features which seemed to impact teachers' understanding. In this discussion I focus on five of the seven features which I think are important contributions to the mathematics education literature: problem-solving contexts, re-defining quotient as more than the answer to a division question, difficulties with describing fraction as quotient, focusing on the relationship dimension of fractions, and working with indefinite amounts.

The findings suggest that teachers need opportunities to explore quotient through contextual problems and collaborative reasoning. This aligns with research emphasizing the need for extended time and conceptual focus in teacher learning (Fuchs & Malone, 2020) and with research showing that teachers need opportunities to struggle through problems themselves, with a focus on mathematical reasoning (Suh et al., 2012). Problem solving is widely acknowledged as an important avenue for constructing meaning with fractional concepts (Cramer et al., 2010; Fosnot & Dolk, 2002; Johanning & Mamer, 2014; Kang, 2022; Poon & Lewis, 2015; Salls, 2014) and this aligns with the findings of my study. In Salls' (2014) study with Grade 7 students, the complexity of supporting learners to see and use the quotient interpretation to solve problems was conveyed. Salls' findings align with my analysis of these teachers' experience which revealed that although the quotient interpretation is a challenge to understand, problem-solving contexts can support the construction of this understanding in meaningful ways.

The importance of re-defining quotient as more than the answer to a division question in order to develop an understanding of fraction as quotient is not something that I found in the literature. In fact, most of the literature I reviewed for this study and discussed in Chapter 2 for challenges regarding division involving fractions also refers to quotient as the answer to a division problem (Alenazi, 2015; Copur-Gencturk, 2021; Kang, 2022; Ma, 1999). As I showed in Chapter 4, this definition of quotient was a hinderance for the teachers during the interactive inquiry sessions, as it impeded their efforts to develop a new understanding of fraction as quotient. Simply stated, these teachers struggled to understand fraction as quotient as a relationship because they primarily understood quotient to be the answer to a division question rather than a relationship between the dividend and the divisor.

Difficulties with finding clear and concise language to describe the concept of fraction as quotient is another aspect of working with this fraction interpretation that that I was unable to locate in the literature. The literature which I reviewed provided their descriptions of fraction as quotient, often including terminology such as partitive and quotitive constructs (Behr, 1992; Charalambous & Pitta-Pantazi, 2007), the quotient generalization (Behr, 1983; Charalambous & Pitta-Pantazi, 2007; Salls, 2014), or describing classroom activities to model the interpretation

(Lamon, 2007; Salls, 2014). However, most of these sources do not discuss the challenges of articulating this concept, a feature which hindered the teacher participants in this study from deepening their understanding. Purnomo et al. (2021) describes a fraction as quotient by stating, “any fraction can be seen as the result of a division situation” (p. 185). However, as I showed in my analysis of the interactive inquiry sessions, this description actually hindered some of the participating teachers’ developing understanding of the quotient interpretation because they understood “result” to mean “answer,” rather than a relationship. In a study done by Middleton et al. (2001), there is a brief reference to challenges with describing quotient when they say, “students' resistance to consider a fraction as a quotient was due to the fact that children had such a strong understanding of the Part-Whole subconstruct” (p. 269), but this, along with the other studies referenced, does not highlight the challenges that educators can have with describing the fraction as a quotient interpretation, and the ways that specific language choices may or may not support the development of this understanding. Foregrounding the challenge of expressing this fraction interpretation is a key contribution of my study.

The importance of focusing on the relationship dimension of fractions as a way of helping people more fully understand the quotient interpretation of fractions is also not evident in the literature that I located. According to the five interpretations of fractions, I found that the relationship which exists within a fraction is referred to as part of the ratio interpretation (Behr et al., 1983, p. 95) but less explicitly in discussions of the quotient interpretation. The relational aspect of the quotient interpretation and/or comparing the relational aspects between quotient and ratio is briefly touched on by Lamon (2007), who stated, “Quotients shared natural connections with ratios and rates, and there was a great deal of power in being able to interchange quotients and ratios in sharing and comparing contexts” (p. 658). This quotation suggests the close relationship between the quotient and ratio interpretations of fractions and that understanding the relationship aspect of ratio may help with understanding the relationship aspect of fraction as quotient. In fact, this overlap or blurring of the ratio and quotient interpretations of fractions emerged during several interchanges between participants in the interactive inquiry sessions in my study. Research reveals that focusing on the relationship dimension of fractions is helpful for learners' conceptual development of fractions in general (Brown, 2015; Castro-Rodríguez et al., 2015; Champion & Wheeler, 2014). The findings from my study suggest that focusing on the relationship dimension of fractions may be particularly powerful for facilitating teachers' understanding of fraction as quotient as well as supporting them to develop a deeper conceptual understanding of fractions in general.

Working with indefinite amounts as an avenue to understand fraction as quotient is another aspect of teaching and learning about fraction as quotient that I was not able to locate in existing literature. Studies which discuss the five interpretations of fractions seem to refer to discrete or continuous quantities (Behr et al., 1983; Behr, 1992; Charalambous & Pitta-Pantazi, 2007; Kieran, 1976). In the study done by Salls (2014), the word "object" is used to define fraction as quotient ("when objects are shared among a number of groups" p. 370) which is a little more ambiguous than a discrete quantity. However, the example that follows uses "biscuits" and involves working with a specific quantity ("8 biscuits in a can of refrigerated biscuits among 5 people") (p. 370). Charalambous and Pitta-Pantazi (2007) specify that there is no constraint regarding the size of the fraction: "the numerator can be smaller, equal to or bigger than the denominator, and subsequently, the quantity that results from the fair-sharing activity can be less than, equal to or more than the unit" (p. 299). Although this might be helpful in furthering one's understanding and description of fraction as quotient, this, and the other studies referenced here, do not suggest that working with an indefinite amount might support the development of fraction as quotient thinking and might help to foreground the idea of a fraction as a relationship, as the teachers in my study discovered.

Overall, the findings for RQ2 underscore the importance of problem-solving contexts, focusing on the relationship dimension of fractions, and working with indefinite amounts as supportive for teacher learning. I believe that in the context where I teach there is a need for professional development opportunities for teachers and that these experiences should reflect the features identified in this study. I offer specific recommendations for schools and districts related to these features and suggest areas of further research in a later section.

Discussion of RQ3: Collaborative Learning and MKT Development

Regarding RQ3, collaborative inquiry fostered noticeable shifts in teachers' descriptions of fractions and their mathematical knowledge for teaching (MKT). By Session 3, all participants articulated a broader view of fractions beyond part-whole. As I shared in Chapter 4, I argue that this provides evidence of our growing MKT about this aspect of fractions, as well as conveyed that learning about fraction as quotient supported our MKT. These findings suggest that social constructivist approaches can enhance teachers' specialized content knowledge. This aligns with Vygotsky's (1962) theory of learning as socially constructed and supports research on collaborative professional development as a means of developing MKT (Ball et al., 2008). In my study, we unpacked the specialized content knowledge of fraction as quotient during the sessions, the teachers were the learners, and they developed new comprehensive structures for fractional understanding in collaboration with one another. In keeping with social constructivism, I think that this experience was more meaningful and impactful for all of us than if the learning process had been done by each teacher working in isolation.

Overall, the findings for RQ3 highlight the value of collaborative inquiry for building MKT and conceptual understanding of fractions. This study suggests that this type of in-depth work to develop MKT takes time but is important. Professional development should incorporate structured collaborative sessions, focused on unpacking complex fraction interpretations and linking them to classroom practice. I expand on this as an area for future research, as well as in my discussion of the recommendations, significance and limitations of this study.

Recommendations for Schools, Divisions and Manitoba Education

Based on my understanding of the mathematics education research and the insights that came from my analyses, I have three main recommendations for schools, divisions and Manitoba Education in relation to more effective approaches to teaching and learning fractions. I consider these recommendations as important for schools, divisions and Manitoba Education as they endeavour to better support teachers and students in developing their conceptual understanding of fractions including the interpretation of fraction as quotient.

More Fraction as Quotient in the Curriculum Standards and Support Documents

I recommend that there should be more explicit reference to fraction as quotient in the Manitoba Education (2013) mathematics curriculum standards. Specifically, explicit reference in the Grades 3-5 fraction SLOs and/or related AIs is needed, since these are where the first formal fraction outcomes occur in Manitoba. Explicit reference to fraction as quotient would target the meaning of a fraction as a relationship of division in a deeper sense than the current curriculum standards. For example, in the Grade 3 AI that states, “Model and explain the meaning of numerator and denominator” (p. 81), the standards document could be revised to say, “Model and explain the meaning of the numerator and denominator in a fraction, as a relationship between the two parts, as a partitioning and/or divisive relationship.” Another alternative would be to use the definition for "fraction", found in the Grades 3, 4, and 6 support documents, and to place this definition in the curriculum standards for every grade. The definition should be placed directly in the fraction SLOs; "A number that represents part of a whole, part of a set, or a quotient in the form *a*/*b*, which can be read as *a* divided by *b*" (Manitoba Education, 2014b, p. 40; Manitoba Education, 2017a, p. 93; Manitoba Education, 2017b, p. 71). Placing greater and more explicit emphasis on the quotient interpretation of fractions in the mandatory curriculum standards document, not just the optional support documents, could more effectively support students and teachers to develop a deeper understanding of fractions.

Researchers also recommend that there needs to be explicit connections developed between the outcomes that deal with division of whole numbers and the outcomes related to fractions, and not merely placing the outcomes close to one another in the document (Middleton et al., 2001). I argue that this would also help the conceptual development of the meaning of fractions. I recommend that language such as "Relate fractions to division" be added to the Grades 3-5 SLOs, which would mirror the connection that is already stated between fractions and decimals in Grades 4 and 5 (Manitoba Education, 2013, p. 93 & 103).

Once fraction as quotient is explicitly referenced in the curriculum standards document and the conceptual understanding of fractions is more supported, the support documents could then provide examples of problem-solving contexts that would assist with developing the concept of fraction as quotient. This is something that the teacher participants identified as an essential feature for understanding fraction as quotient. Although the teacher participants expressed limited use of the support documents, placing problem-solving examples into these documents could be beneficial for other educators. These problem-solving examples could include helpful features such as contexts that would help students to discover the quotient generalization, recommendations for the use of graph paper or other manipulatives, and some examples that include indefinite amounts. Notably, these revisions to the support documents would be even more effective if professional development focused specifically on the support documents was offered to teachers. When designing problems, the support document writers should keep in mind the three components of how learners “mathematize” during problem solving: the use of a familiar model over time and in different situations; a context that allows learners to access schema for the problem they are doing; and a situation that prompts questions and inquiry (Fosnot & Dolk, 2002, p. 29). These features helped us to better understand the quotient interpretation of fractions during the interactive inquiry sessions and should be considered for future problems.

As stated above, my analysis of the Manitoba Education (2013) standards document also suggests that there is too much emphasis on the part-whole interpretation and not enough on the other interpretations, which includes quotient. It seems likely that the over-emphasis on part-whole in the Manitoba Education (2013) curriculum standards document is the reason why four of the teacher participants in this study initially explained that fractions are parts of a whole (Session 1, Lines 86-152). The teacher participants noted the prevalence of this interpretation in Session 3, when they shared their thoughts about the curriculum standards document. The Manitoba Education (2013) mathematics curriculum standards should include explicit reference to all five interpretations, over the course of the Grades 3-8 fraction SLOs, and consider the mathematics education literature when designing the order and prominence of the interpretations in the revised curriculum standards. Including a balance between all five interpretations with a clear focus on the multiplicative relationships within them would support a deeper conceptual understanding of fractions for students (Lamon, 2007).

In addition to this, the teachers shared that the layout and design of the curriculum standards document is too large and disconnected, with an emphasis on procedures instead of developing conceptual meaning. Their observations align with the findings from my curriculum analysis, as even when fraction as quotient was explicitly referenced (Manitoba Education, 2017a, p. 93), it is done so with the quotient notation rather than by providing conceptually considered meanings. The lack of conceptual understanding of what fractions are and specifically fractions as quotients in the curriculum standards is an area that needs to be addressed when the document is revised. Teachers need to feel supported as they enact the curriculum standards they are required to teach (Ball & Cohen, 1996), therefore careful considerations are needed to better support the teaching of fractions conceptually. Manitoba Education should work in collaboration with mathematics education researchers to further examine these shortcomings and determine necessary revisions to the curriculum standards and support documents.

Focusing on the Relationship Dimension of Fractions and Using Indefinite Amounts

The work with the teacher participants revealed that focusing on the relationship dimension of fractions, such as comparing fraction as quotient to fraction as ratio and understanding the relationship between the numerator and denominator, was helpful for understanding the quotient interpretation. In addition to this, a focus on the quotient interpretation through indefinite or ambiguous amounts in a problem-solving context was even more helpful for moving our understanding along and supported the development of our MKT. I see these two features as connected. Focusing on the relationship dimension of fractions and using indefinite amounts when thinking about the relationships seemed to be helpful because when we can't give a precise or calculated answer, we are more comfortable expressing the fraction as a relationship. Therefore, I recommend that Manitoba curriculum standards consider these two features during document revisions by including explicit language in the SLOs (i.e. "relationship") and AI examples (i.e. "Consider posing a fraction as quotient problem using indefinite amounts: e.g. 'If three chip bags are shared amongst four friends, how much of a chip bag does each friend get?'"). I also recommend that professional development be provided for teachers on how to explore these features as learners themselves, such as designing their own problems and how to make these features available for students.

Redefining Quotient and the Connection Between Fractions and Division

One of the hinderances to developing an understanding of fraction as quotient for the teachers I worked with was their previously held definitions of quotient. Therefore, I recommend that the language of a 'fraction as a quotient' be placed in the meaning-making SLOs and AIs, alongside the part-whole language that already exists. Specifically, redefining quotient through the lens of fractions should appear in Grades 3 to 5 in the Manitoba Education (2013) mathematics curriculum standards. This would be one step in the direction of re-branding quotient as more than the answer to a division question and could help educators to have a deeper understanding of the concept of quotient as a relationship. Social constructivism foregrounds that prior knowledge influences a learner's new knowledge (Amineh & Asl, 2015). In the case of this study, the prior knowledge of the meaning of the word quotient seemed to interfere with thinking about fraction as quotient.

From a professional development perspective, this is similar to research related to re-defining the meaning of the equal sign (Lee & Pang, 2021; Prediger, 2009), as there was a pedagogical shift in mathematical learning from using the equal sign in a procedural manner to flag the 'answer' to a problem, to a conceptual understanding of the meaning of the equal sign as a ‘balance between two sides’. For instance, understanding the equal sign in this way was included as one of the key concepts in the Manitoba Grade 3 numeracy assessment. In that assessment, “Sees the equal symbol as meaning only ‘give me the answer’ to a number sentence”, is considered a basic understanding, whereas “Understands and can explain the relationship between two different expressions” is considered to meet expectations (Manitoba Education, 2014c). Helping teachers to more fully understand the meaning of the word quotient, less as a procedural outcome (the answer to a division question) and more as a relationship, could also be done by discussing some quotients that are referred to in other contexts (i.e. intelligence quotient or IQ, social quotient, and quotients in biology, chemistry and physics).

The findings of this study also suggest that professional development around the meaning of division is needed. As stated above, explicit connections between fraction and division SLOs should be included in the Manitoba Education (2013) mathematics curriculum standards. As researchers say that moving beyond additive thinking towards multiplicative thinking is paramount to fractional understanding and proportional reasoning (Kieran, 1976; Lamon, 2007; Purnomo et al., 2021), I have yet to come across how implications of division as only subtractive thinking, versus quotitive or partitive may have impacted fractional reasoning. However, the ways that participants discussed division in this study suggest to me that more work around this may be needed. That is, the teacher participants revealed some fragile understandings about the meaning of division (Session 1, Lines 280-428). More professional development that targets the meaning of division in a conceptual way and how this too relates to fractions would be very beneficial.

Significance of the Study for Mathematics Education

The findings of this study reveal that fraction as quotient is a difficult concept, even for teachers with ten to fifteen years of teaching experience. I see this as signaling that more work around the five interpretations of fractions, coupled with curricular support, is needed in order to enable teachers to teach conceptual understandings of fractions more effectively. This study provided some insights into how this work might be done.

My research offers new insights into understanding the quotient interpretation of fractions with teachers, which are not evident in the literature. First, my study reveals that problem-solving contexts, as an avenue to support experienced teachers' understanding of the quotient interpretation, was helpful. My study reveals that working with indefinite amounts rather than discrete quantities, as well as a focus on the relationship dimension of fraction as quotient was also helpful. To view a fraction as a quotient, rather than just through the lens of the quotient notation and generalization, targets a deeper meaning of what a fraction is. Working with indefinite quantities and focusing on the relational dimension of fractions placed an emphasis on the dynamic relationship at play between the bipartite numbers and the context in which the fraction came from, rather than the division procedure. This supported our collective MKT for fraction as quotient and a deeper conceptual understanding of fractions. These features are not evident in the literature and my study offers clear evidence that they were particularly impactful for developing an understanding of quotient.

Other features that came through in my study, such as the challenges to use and unpack the quotient notation and generalization, re-defining quotient as more than the answer to a division problem, and difficulties describing quotient, are also not evident in the literature.

Therefore, my study offers key insights into how to support teachers to understand the concept of fraction as quotient.

In addition to these areas of significance, my study reveals that the Manitoba Education (2013) mathematics curriculum standards are under-supportive of fraction as quotient, do not offer a balance between all five fraction interpretations, nor are they aligned with some of the things that researchers suggest are important for teaching fractions in general (Alenazi, 2015; Copur-Gencturk, 2021; Lamburg & Wiest, 2014; Norton et al., 2018; Stohlmann et al., 2020; Tossavainen & Helenius, 2024). As stated earlier in this chapter, my study highlights the need for redesigning the Manitoba curriculum standards in ways that will support both teachers and learners and provides specific recommendations for how that might be acheived.

Limitations

As with all research, there are limitations to my study. My data reflects the voices and perspectives of these five experienced individuals who were comfortable exploring mathematics concepts together. The impact of three sessions on teachers' MKT about fraction as quotient might unfold quite differently with educators who were less experienced or who were not comfortable sharing their mathematical thinking and their vulnerability as teachers and learners.

Although one of the recruitment criteria for the participants was experience with teaching within Grades 4 to 8, three of the five teachers were working in a Grade 7/8 multi-age classroom. The participants might have brought other aspects about fraction as quotient forward if there were more Grades 4 to 6 teachers represented.

Another feature of the research design which limited the insights I could gain about the research questions is that I only met with teachers on three occasions. This may have limited the amount of learning the teachers were able to demonstrate. In addition, if there were more sessions, I might have gained a clearer insight into the features which supported the teachers' understanding of the quotient interpretation, such as working with indefinite amounts or focusing on the relationship dimension of fractions.

The sessions were scheduled at the end of the school year and this could have been a challenge for the capacity the teachers were able to offer in terms of participation. That is, they may have felt burnt out from the school year and end of term report card writing, therefore offering less feedback during the sessions than they may have at a different point in the school year. As the sessions were also near the end of the school year, many teachers were also no longer actively teaching fractions in their classrooms. This may have reduced the amount of reflection that teachers could engage in regarding the value of these concepts for their classroom practices.

Finally, the teachers and I only designed one problem where fraction as quotient was centered. Although this problem reinforced the conclusions I drew about the difficulty of quotient as an interpretation, designing and exploring multiple problems would likely have provided additional valuable insights into some features that were revealed in the study, such as the quotient notation and generalization and our ability to describe quotient. Designing more problems could also have revealed features that were not identified in the study.

Areas for Further Research

The quotient interpretation proved to be complex to understand, describe, and use for this group of experienced and open-minded teachers. The findings of this study suggest several areas of further research that might help to support teachers' understanding of this important yet challenging fraction interpretation. Four areas of research seem particularly valuable to me and I share them in this section.

First, I think that working with indefinite amounts, as opposed to a specific quantity, when developing problem contexts related to fraction as quotient is an area that needs to be further explored. This was not a feature that I anticipated when I planned the interactive inquiry sessions but was one that seemed powerful for these teachers given that it was referred to throughout our time together in Sessions 2 and 3. Research around this topic, including if it is in fact helpful and why that is, should be done.

Second, I think that the relationship dimension of fractions, specifically comparing fraction as quotient to fraction as ratio, should be explored. The central idea that in both interpretations two things are being compared seemed to be helpful for the teachers' understanding of quotient and should be further researched. At the same time, finding effective ways to distinguish between these two fraction interpretations, if they are separate interpretations, would be valuable.

Third, research needs to be done around Manitoba students' conceptual understanding of fractions through the lens of the quotient interpretation. My study revealed that the Manitoba Education (2013) mathematics curriculum standards has areas of weakness. As I recommended above, the curriculum standards should include more fraction as quotient, specifically in Grades 3-5, to support the development of conceptual understanding in students. Therefore, more research needs to be done to see if fraction as quotient is a more successful avenue to understanding fractions, as compared with the emphasis on part-whole that many Manitoba students experience.

The fourth area of future research that I see as valuable would be a deeper investigation into how other teachers, including those with fewer years of teaching experience or who are less comfortable revealing their misconceptions, build their MKT around the five interpretations of fractions. My study revealed that we were able to build our MKT of fraction as quotient and that learning this interpretation also supported our collective MKT. My study illuminated that this type of work takes time but is valuable for teachers' professional learning. With that in mind, more research about socially constructed learning through a focus on one or all five interpretations of fractions is needed and would be a valuable insight into how other educators understand, describe, and utilize the interpretations in their math classrooms.

Next Steps for My Learning and My Classroom Practice

As a teacher, my next steps are to teach fractions through the lens of quotient, alongside that of the part-whole interpretation, and to link this to concepts of division. I want to unpack the relationship between the numerator and denominator in a fraction with students and work through this in a variety of problem-solving contexts, so that they can "mathematize" the meaning of fraction as quotient for themselves (Fosnot & Dolk, 2002). I also want to see if the quotient notation and generalization is more or less impactful for students than it was for the teacher participants, when it is something constructed themselves through situations that reveal the generalized case of fraction as quotient, rather than told to them explicitly through the notation. I anticipate achieving this by, for example, working through more sharing problems with students so that they can construct the quotient generalization.

As a researcher, my next step would be to conduct a similar study but with a different group of experienced teachers, to see if MKT was still developed and if the same features about what was impactful for an understanding of fraction as quotient came through. Finally, as an educator working in the province of Manitoba, another next step for me would be to offer professional development for practicing teachers within my school and division about the importance of fraction as quotient as part of building a conceptual understanding of fractions and expanding our collective MKT.

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Appendix A

University of Manitoba | Research Ethics and Compliance

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PROTOCOL APPROVAL

Effective: May 16, 2025 Expiry: May 15, 2026

Principal Investigator:	Jillian Dempsey
Advisor(s):	Martha Koch
Protocol Number:	HE2025-0102
Protocol Title:	*Towards a Collective Understanding of Fractions as Quotient and a Deeper Mathematical Knowledge for Teaching*

Merissa Daborn, Acting Chair, REB2

Research Ethics Board 2 has reviewed and approved the above research. The Office of Human Research Ethics (OHRE) is constituted and operates in accordance with the current *Tri-Council Policy Statement: Ethical Conduct for Research Involving Humans- TCPS 2 (2022)*.

Please note the following important information about your protocol approval:

i. Approval is granted for the research and purposes described in the protocol only.

ii. Any changes to the protocol or research materials must be approved by the OHRE **before implementation**.

iii. Any **deviations** to the research or **adverse events** must be reported to the OHRE immediately through an **REB Event**.

iv. This approval is valid **for one year only.** A Renewal Request must be submitted and approved prior to the above expiry date.

v. A **Protocol Closure** must be submitted to the OHRE when the research is complete or if the research is terminated.

vi. The University of Manitoba may request to audit your research documentation to confirm compliance with this approved protocol, and with the UM *Ethics of Research Involving Humans* policies and procedures.

Appendix B

First Interactive Inquiry Session: Guiding Questions

1. Tell me about your understanding of fractions as “quotient”? What do you think of when you consider “quotient” in your teaching?
2. What is a fraction? What is your understanding of this? How do you explain what a fraction is to students?
3. Consider: what is 3÷5?
4. Three bars of chocolate are shared equally between five children. How much does each child get?
5. Consider this: what is 48÷7?
6. How does 3÷5 feel compared to 48÷7? In the classroom, how do you think these would be approached by students?
7. 5 interpretations at a glance: How are they sitting? What surprised you? Which of these feel familiar or unfamiliar to you? What questions do you still have?
8. Revisit question 1: Tell me about your understanding of fractions as quotient. What do you think of when you consider “quotient” in your teaching?

Second Interactive Inquiry Session: Guiding Questions

1. Go over the 5 interpretations of fractions once again
2. Then let’s jump into a problem:

As a group, ask learners to place these cards in cardinal order on an open number line and to explain their thinking as they do so.

2.6 75% ⅕ 1/10 0.75 5.05 50% 1 and 1/4 ⅜ 4/7 0.8 0.5714

5 and ½ ⅘ ⅓ ⅙ 0.333… 33 and ⅓% 1.5 5%

3. Does anyone want to share any insights about this rational number line problem?
4. What is something you learned or noticed about your own mathematical thinking during this task?
5. Let's reflect on the five interpretations of fractions…can you give an example as to times you lean into some interpretations over others? I.e., during the rational number line problem, in your teaching, or in your own mathematical thinking?
6. In which interpretations do you feel less confident? Can you reflect as to why this might be?
7. Let's work through another problem together; the Sub Sandwiches problem (Fosnot & Dolk, 2002) as a group.
8. How does quotient reveal itself here? Do you feel as though other interpretations are present during this problem-solving task?
9. What does a/b = a÷b mean? Is this always true? Can we prove this using a visual representation? What did you notice about quotient during these representations we just made? Do you feel as though other interpretations are present here too?
10. Let's work together to create some other problems, in which the quotient interpretation of fractions is centered.
11. Tell me about a time or process when teaching fractions to students was challenging?
12. Tell me about a time or process when teaching fractions to students was successful?
13. Finally: In your classrooms, sometime between now and the next time we meet for our in-person workshop, you may wish to try a rational number line problem, inspired by Fosnot and Dolk (2002, p. 149). As you guide students through this problem, think about

the strategies they used, what was easy or challenging, and what you noticed about your own mathematical thinking.

Third Interactive Inquiry Session: Guiding Questions

1. Revisit quotient: The problem we made last week: There are 5 classes walking to the mall. 2 of their teachers are driving, which only leaves 3 teachers to supervise the classes. How many 'classes' does each teacher get? 5/3 of a class. 5/3 is fraction as quotient: Why? How do you understand this? Relationship between numerator and denominator?
2. Some people shared how this seems too simple, and yet also shared how $5 \div 3 = 5/3$ has never been explicitly explored. Some wondered if exploring this takes the opportunity for struggle away, whereas others wondered if that isn't the point of learning all math generalizations? What are your thoughts?
3. What's another math "generalization" that kids use to make things simpler?
4. Does fraction as quotient seem valuable to explore, explicitly, with students?
5. Some of you shared that some of the interpretations don't come up in the younger grades. But do you think it's valuable in the future?
6. Share and reflect upon the Fosnot rational number line task, if you chose to do this with students. Does anyone want to share any insights about the rational number line problem?
7. What is something you learned or noticed about your own mathematical thinking while observing this task? Student challenges and successes?
8. Let's look at the curriculum standards: I want to know about your experience using the Manitoba math curriculum standards and support documents for teaching fractions. Can you tell me what this looks like?

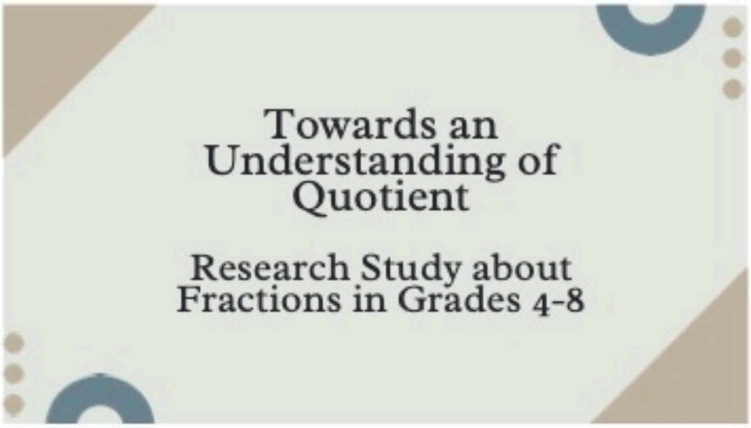
9. Tell me about how you might use the Manitoba math curriculum standards and support documents to teach the meaning of a fraction, i.e., quotient? Either previous experience or moving forward?
10. Let's look at the front matter of the document: Which mathematical processes connect to fraction as quotient? Can we look at visualization? (p. 14)
11. Look at support docs/SLOs for outcomes and AIs. What do you think of these? Which interpretations do you see? Do you see an opportunity anywhere for quotient?
12. If you could offer some feedback to the Manitoba math curriculum standards developers about how fractions are represented, what would it be?
13. Are there aspects of 'fraction as quotient' that you are still finding hard to hold onto?
14. How do you believe your understanding of quotient will impact your teaching, going forward? Or the 5 interpretations?
15. How do you believe your understanding of quotient will impact the way you navigate curriculum, going forward? Or the 5 interpretations?
16. At the first session, you said:
    a) "I talk about fraction as being part or parts of a whole."
    b) "I would agree. I also try to emphasize that you could have part of a whole or part of a set." "And also means visuals."
    c) "It can be part of one thing, or it can be part of multiple things also." And connecting to place value.
    d) "Shifting more towards partition of a whole versus just parts has really helped them with their operations."

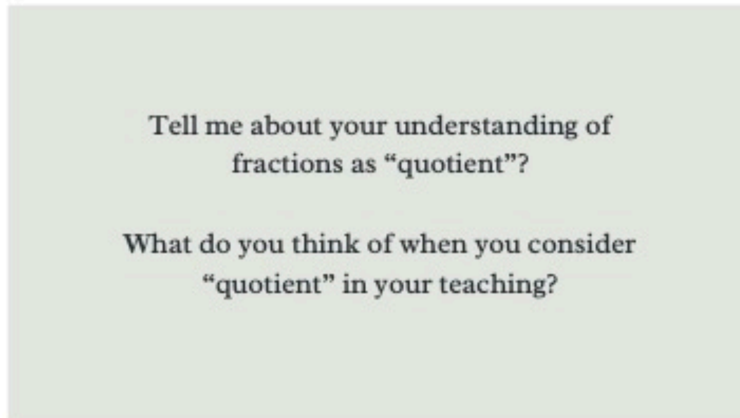
e) Connection to place value and counting fraction parts, i.e. one fifth, now you have two of them.

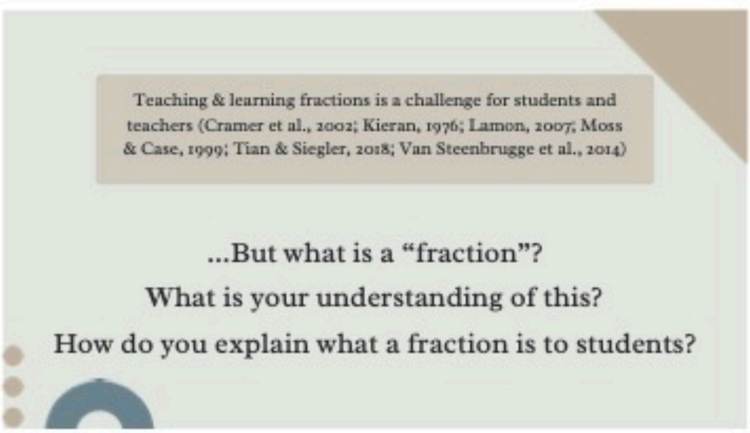
17. Has your understanding of the meaning of fractions changed since then? If so, how would you explain the meaning of a fraction to students now?
18. Is there anything else you have learned from these sessions or that you would apply to your teaching?

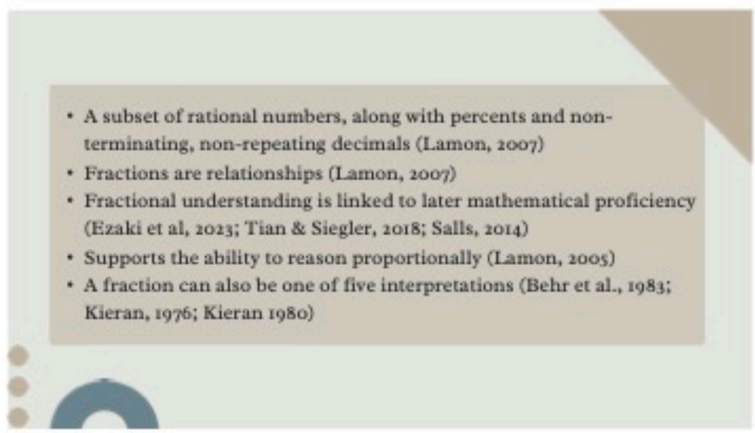
Appendix C

Canva presentation from first session

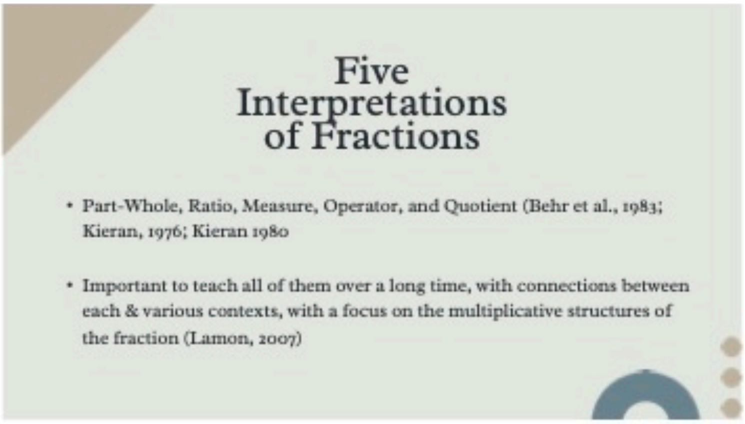


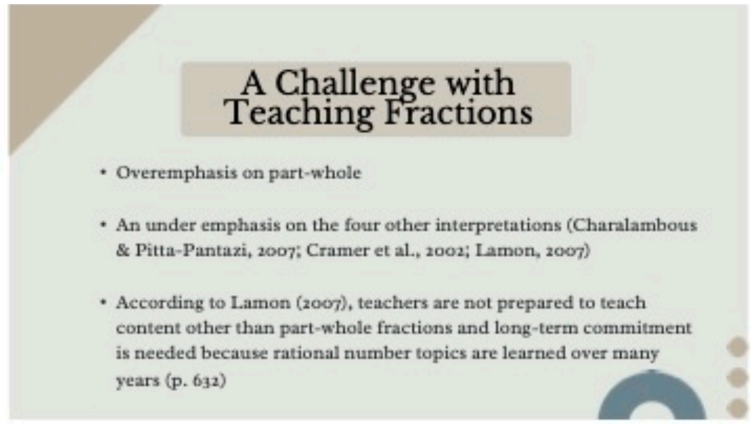


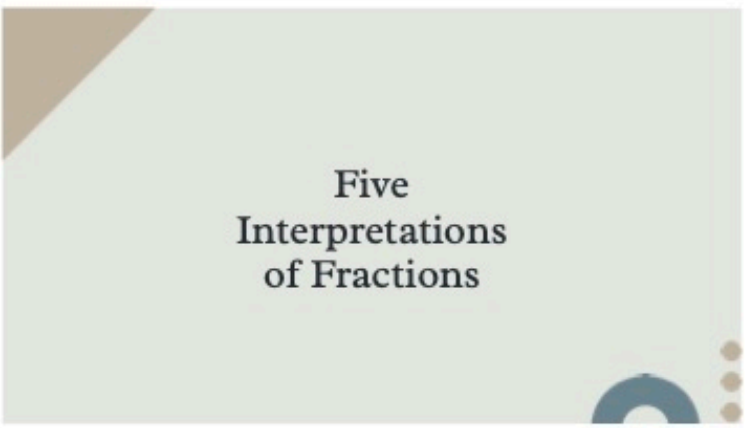


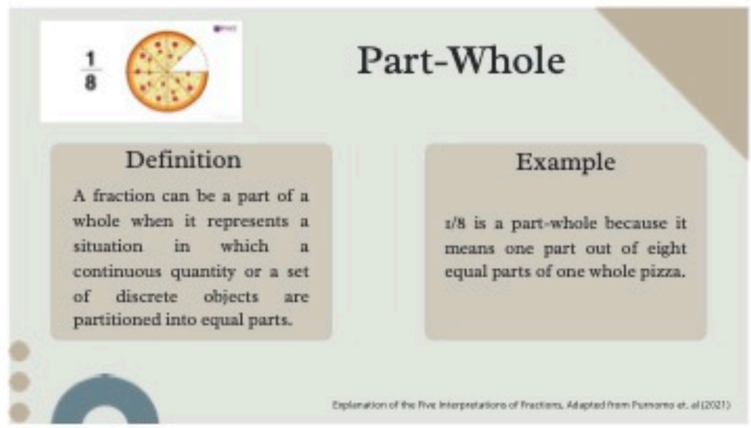


Slides 1 to 4

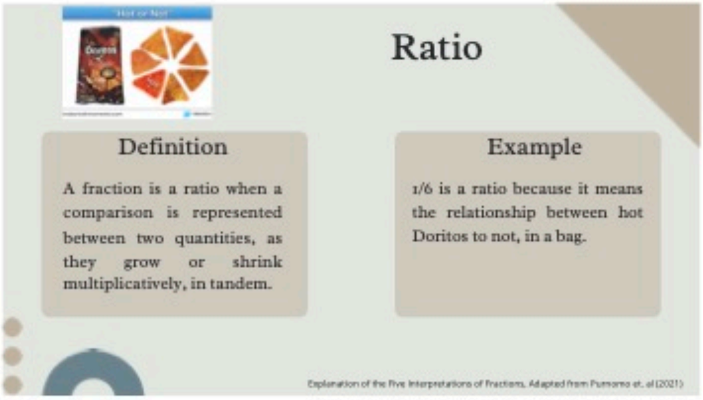

Five Interpretations of Fractions
• Part-Whole, Ratio, Measure, Operator, and Quotient (Behr et al., 1983; Kieran, 1976; Kieran 1980
• Important to teach all of them over a long time, with connections between each & various contexts, with a focus on the multiplicative structures of the fraction (Lamon, 2007)

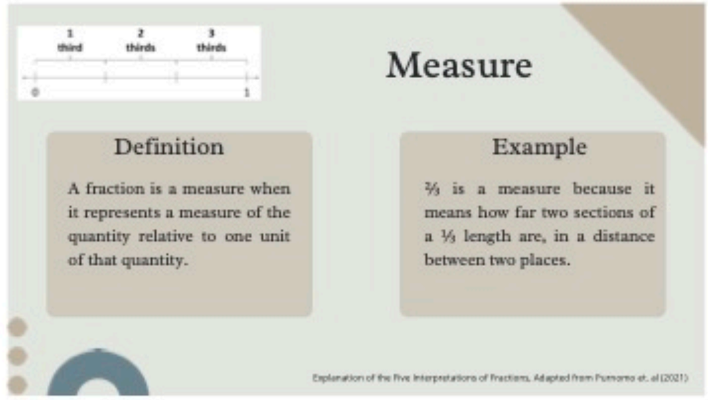

A Challenge with Teaching Fractions
• Overemphasis on part-whole
• An under emphasis on the four other interpretations (Charalambous & Pitta-Pantazi, 2007; Cramer et al., 2002; Lamon, 2007)
• According to Lamon (2007), teachers are not prepared to teach content other than part-whole fractions and long-term commitment is needed because rational number topics are learned over many years (p. 632)

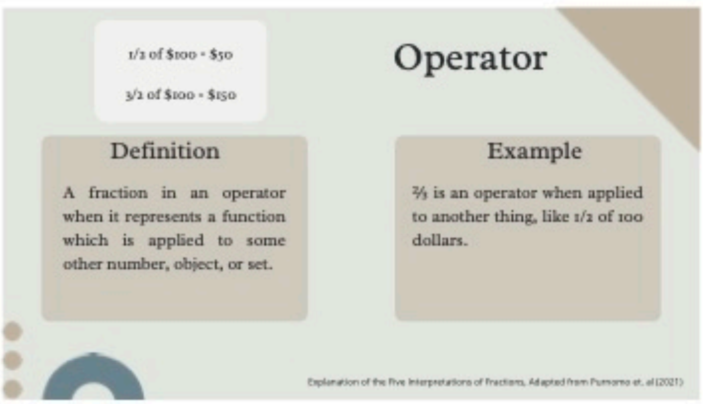

Five Interpretations of Fractions

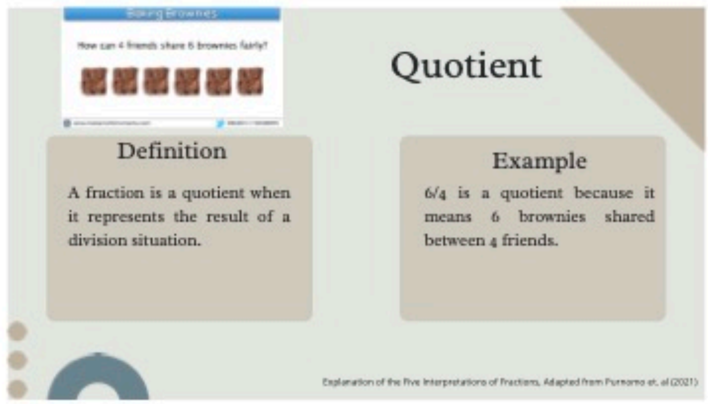

1/8
Part-Whole
Definition
A fraction can be a part of a whole when it represents a situation in which a continuous quantity or a set of discrete objects are partitioned into equal parts.
Example
1/8 is a part-whole because it means one part out of eight equal parts of one whole pizza.
Explanation of the Five Interpretations of Fractions. Adapted from Purnomo et. al (2021)

Slides 5 to 8





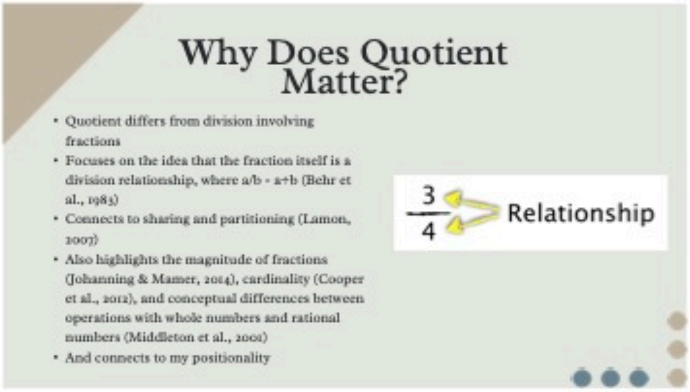


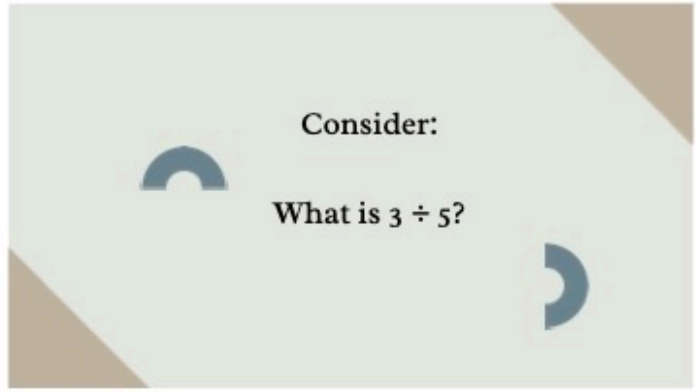


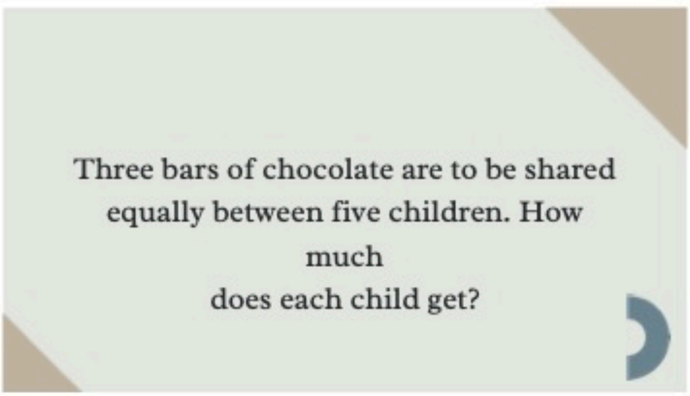
*

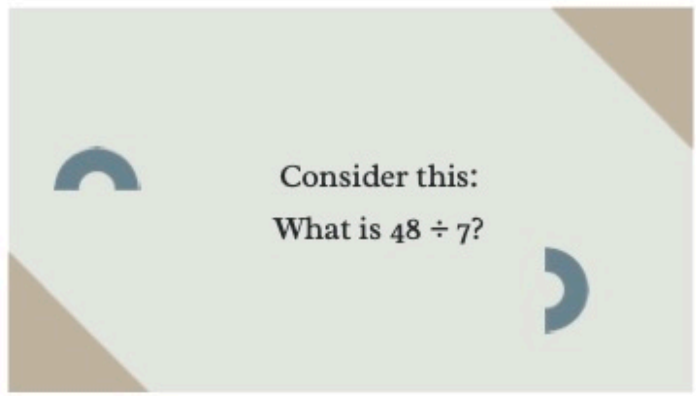
Slides 9 to 12

*Correction on "Operator" slide that I explained during the presentation: "A fraction is an operator when it represents a function which is applied to some other number, object, or set. Example: ½ is an operator when applied to another thing, like ½ of 100 dollars."

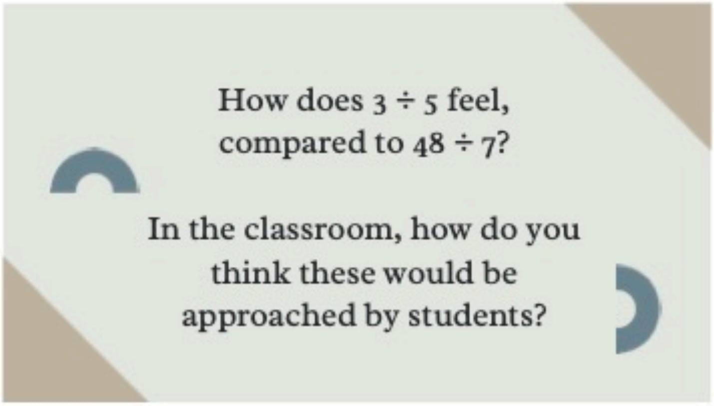


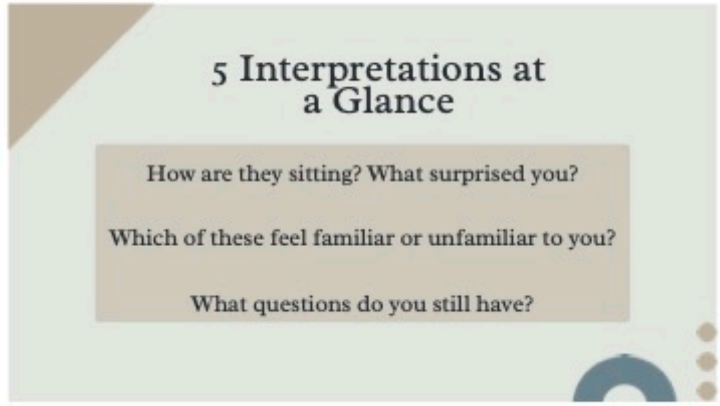


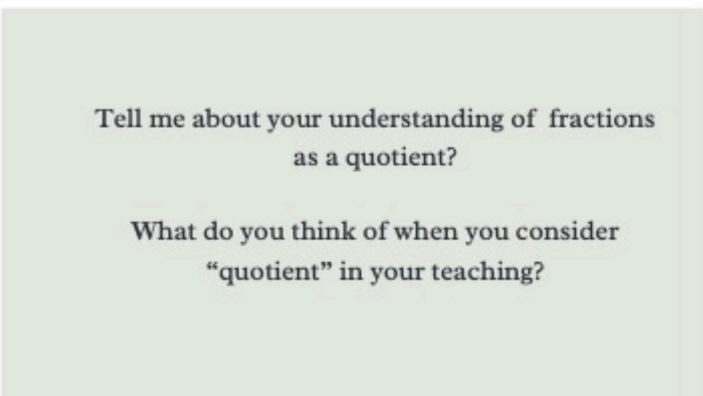


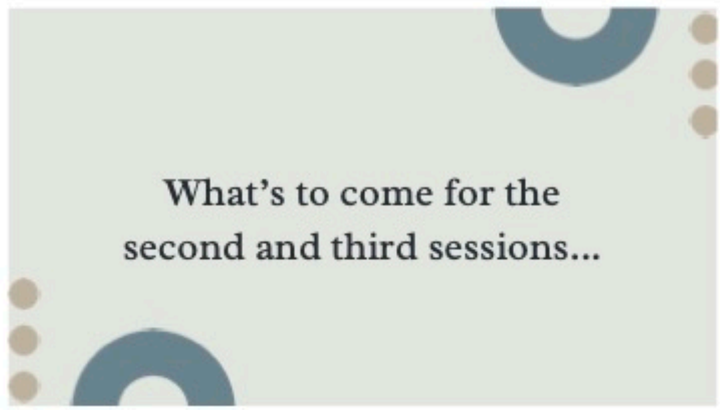


Slides 13 to 16









Slides 17 to 20

Slide 21

Appendix D

Description of the Number Line Task

This task was adapted from Fosnot and Dolk (2002). Learners are asked to place the following rational numbers in order on an open number line:

75%   ⅕   1/10   0.75   5.05   50%   1 and 1/4   ⅜   4/7   0.8   0.5714

5 and ½   ⅘   ⅓   ⅙   0.333…   33 and ⅓%   1.5   5%   2.6

"The quantities chosen are related in interesting ways…Also, the fractions, decimals, and percents are being treated as rational numbers: investigating them as quantities and placing them on the number line- actually building a "number space"- *is* the context" (Fosnot & Dolk, p. 148).

Description of the "Sub Sandwich Problem"

This task was also adapted from Fosnot and Dolk (2002). In this problem, children are on a field trip and are taking a break for lunch. The children are put into groups to share submarine sandwiches. Each group has a different quantity of children and a different quantity of sandwiches. The learners are asked to figure out how much submarine sandwich each child, in each group gets, and determine whether or not it is fair.

Fosnot, C., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals and percents.* Heinemann.

Appendix E

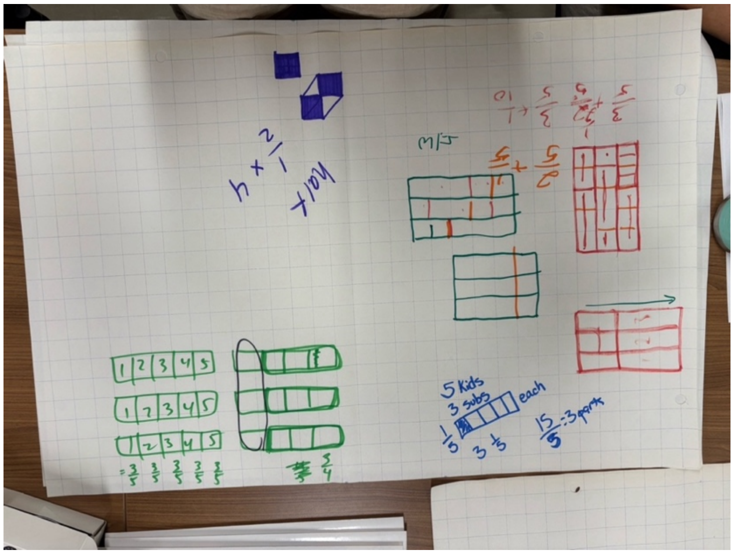


Figure 1: Collaborative chart paper from second session.

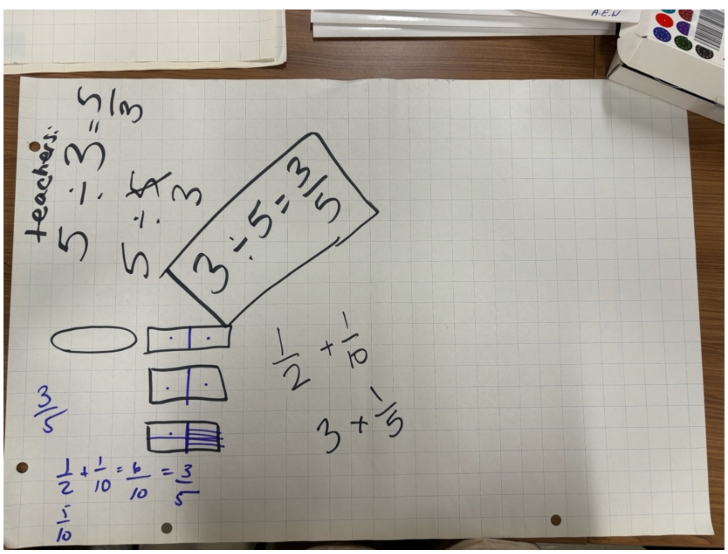


Figure 2: Another collaborative chart paper from second session.

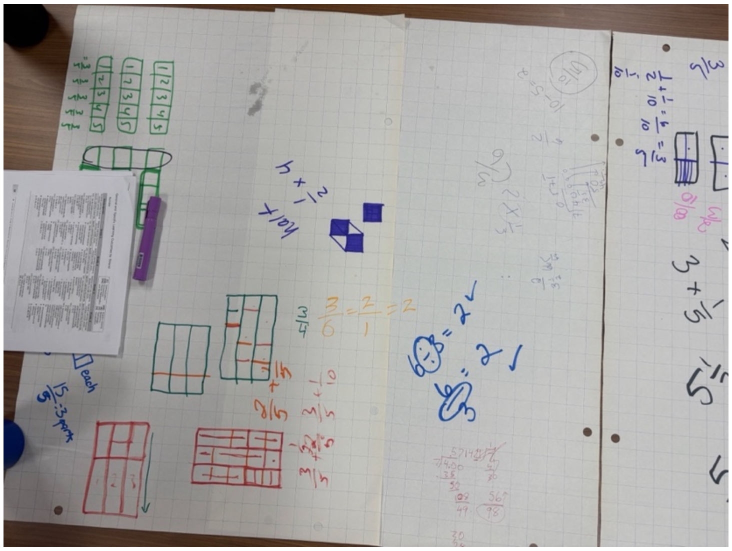


Figure 3: Collaborative chart paper, continued in third session.

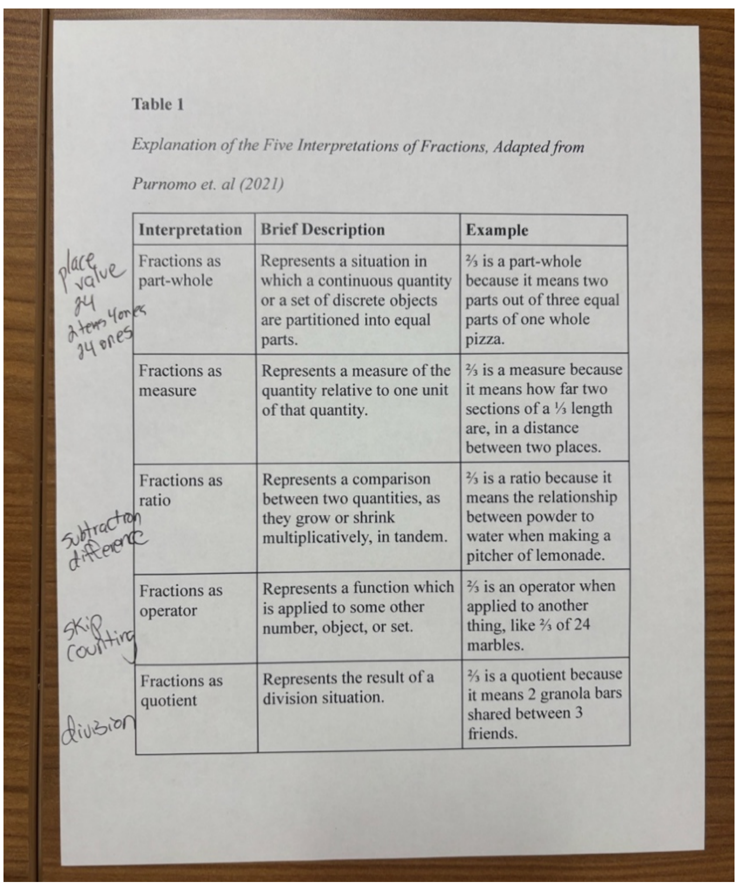


Table 1

Explanation of the Five Interpretations of Fractions, Adapted from Purnomo et. al (2021)

Interpretation	Brief Description	Example
Fractions as part-whole	Represents a situation in which a continuous quantity or a set of discrete objects are partitioned into equal parts.	⅔ is a part-whole because it means two parts out of three equal parts of one whole pizza.
Fractions as measure	Represents a measure of the quantity relative to one unit of that quantity.	⅔ is a measure because it means how far two sections of a ⅓ length are, in a distance between two places.
Fractions as ratio	Represents a comparison between two quantities, as they grow or shrink multiplicatively, in tandem.	⅔ is a ratio because it means the relationship between powder to water when making a pitcher of lemonade.
Fractions as operator	Represents a function which is applied to some other number, object, or set.	⅔ is an operator when applied to another thing, like ⅔ of 24 marbles.
Fractions as quotient	Represents the result of a division situation.	⅔ is a quotient because it means 2 granola bars shared between 3 friends.

Figure 4: The Five Interpretations of Fractions handout and the Grade 4/5 teacher's notes about where she could include each.