# A DIGITAL COMPUTER STUDY OF TIME AND FREQUENCY RESPONSE PARAMETER RELATIONSHIPS USING STRAIGHT LINE M CURVE APPROXIMATIONS



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## ABSTRACT

A digital computer is used to relate step response parameters of stable, linear, minimum phase control systems to several parameters of the closed loop frequency response magnitude diagram with a single resonant peak. Results are applicable over a wide range of variables and include new design charts and approximation equations for conversion between frequency and time domain parameters.

# PREFACE

The object of this thesis is to form a reliable method of relating time and frequency response parameters in order to facilitate the design of control systems. All pertinent literature published prior to the summer of 1959 was reviewed in an attempt to study the advantages and disadvantages of various methods of approach to the problem. As a result, the thesis contains many references to previous studies in this field. An extensive collection of references is included at the end in order to acquaint the reader with the previous papers.

Following a short introduction to the thesis subject, details of the methods used in correlating the time and frequency domain are given in Chapter 2. Computer programs which accomplish the conversion are described briefly in Chapter 3. In Chapters 4 to 7 the investigation results are presented and discussed. A discussion of the important error considerations is given in Chapter 8, together with several examples of the application of the conversion charts. As a supplement to the thesis the programmed conversion equations are explained in the Appendix.

The author wishes to thank the IBM Applied Science Representative in Winnipeg, Mr. D. Oakes, who energetically aided in the program preparation, and IBM itself for providing the necessary equipment. A helpful hand was also extended by Professor C. B.

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A special thanks is extended to the University of Ottawa Computing Centre which provided, without charge, computer time and an operator at their 650 installation in order that some of the valuable data for the conversion charts could be included in this thesis.

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# CHAPTER 1

# INTRODUCTION

## 1.1 THE SUBJECT

The design of control systems would be considerably simplified if it were possible to state, with confidence, the parameters of a unit step response which correspond to a given frequency response function for any system. There have been many attempts to find simple relationships between time and frequency domain parameters, but none of these have been wholly successful. (CH-B, DE-C, GR-C, JA-C, LU-A. WE-C, ZE-C)[1]. These investigations were prompted by the fact that, while definite relationships do exist between the two domains through Fourier transformations, the exact mathematical analysis is too complicated or even impossible to solve, in general, with the exception of a few simple cases. By necessity, then, the relationships are all founded on some approximation or are limited to a specific system or group of systems.

The object of this thesis is to attempt to investigate and clarify the relationship between well known parameters in the time domain

[1] Letters in parentheses refer to references. The letters before the dash give the first letters of the author's name and the symbol after the dash signifies the reference section.

(for example, the peak overshoot, rise time, and peak time of the unit step voltage response) and their equivalent closed loop frequency parameters (for example, the peak of the frequency response magnitude curve, and the various parameters describing the shape of this curve). Relationships which are most applicable in the field of control systems will be sought.

The majority of papers written on this subject are concerned with the open loop frequency characteristics (CH-B). However, some relationships are more easily investigated when dealing with the closed loop system response. This is particularly true of the presumed connection between the peak value of the closed loop frequency response magnitude curve and the peak overshoot to a step response. Most authors refer to this relationship in only vague terms which are unsatisfactory as a reliable design criterion. (CH-B, p. 408; BO-C, p. 211). This, among other things, will be examined and definite conclusions will be made with regard to the clarification of the exact connection and the establishment of the relationship as a reliable design method.

## 1.2 APPROXIMATION METHODS

One of the basic problems in conversion between the time and frequency domains becomes that of choosing an approximation method which best suits the problem at hand. A great deal of work has been done to investigate the various methods. They can be tabulated into two categories:

(a) methods designed to make an approximation to some time domain response for a given arbitrary input in order to obtain the approximate frequency characteristic (for references, see reference section A),

(b) methods designed to make an approximation to some frequency domain characteristic in order to obtain the approximate time response to a given input (for references, see reference section B).

In deciding from which category to choose an approximation method, the following desirable qualities are important:

(a) The method must be adaptable to large scale calculations, since it is desirable to consider as large a class of systems as possible. This factor eliminates methods based entirely on graphical manipulations.

(b) The method must not require the construction of large amounts of equipment, as time did not permit such work.

(c) The method should give reasonable accuracy in a minimum of time.

(d) The method must be particularly applicable to the design of control systems in which most of the calculations are done in the frequency domain.

This latter consideration suggests a frequency domain approximation which gives the corresponding step response as the result of a conversion procedure. The basic reference chosen is based on a Fourier conversion method (RU-~~A~~B) and will be discussed in Chapter 2.

# CHAPTER 2

# FREQUENCY TO
# STEP RESPONSE CONVERSION

## 2.1 THE CONVERSION METHOD

It was found that the method of frequency to step conversion as described by E. Rushton (RU-~~A~~B) offers the greatest advantages in regard to the problem at hand. The paper is based on an approximation of equation 2.1 which is basically a Fourier series summation for finding the unit step response. This integral is widely known and has been stated in one form or another in many papers.

$$\mathcal{J}(t) = \frac{M(0)}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{M(w)\ \sin(wt + \beta(w))}{w}\ dw \qquad (2.1)$$

where $t$ = response time in seconds,

$w$ = frequency in radians per second,

$\mathcal{J}(t)$ = unit step response,

$M(w)$ = magnitude of the frequency response at frequencies from zero to infinity,

$M(0)$ = magnitude of the frequency response at zero frequency,

$\beta(w)$ = phase of the frequency response at the frequencies from zero to infinity.

Having established this equation, the author proceeds to show that it may be restated in the following form

$$\Gamma(t) = \frac{2}{\pi} \int_0^{\infty} \frac{R(w)}{w} \sin(wt)\ dw \qquad (2.2)$$

where R(w) is the real part of the frequency response function.

The choice of a suitable "fundamental frequency", $w_f$, must now be made so that, when the integral is written as equation 2.3, the sum of the frequencies provide a true approximation to the actual integral.

$$\Gamma(t) = \frac{2}{\pi} \sum_{N=1,3,5,\ldots}^{\infty} \frac{R(N)\ \sin(Nw_f t)\ 2w_f}{Nw_f} \qquad (2.3)$$

where $dw \doteq \Delta w = 2w_f$

R(N) = value of the real part at the frequencies such that $w = Nw_f$.

Equation 2.3 then reduces to

$$\Gamma(t) = \frac{4}{\pi} \sum_{N = 1,3,5..}^{\infty} \frac{R(N)\ \sin(Nw_f t)}{N} \qquad (2.4)$$

which is, in effect, the equivalent of breaking up the real part curve, R(w), into a series of rectangles of height R(N) and width $\Delta w = 2w_f$.

In the derivation of this equation Rushton makes four limiting assumptions:

(a) The frequency response function, F(s), of the linear system which consists of a magnitude and phase, can be written as a ratio of two rational polynomials in s with real, constant coefficients.

(b) $\lim_{s \to \infty} F(s) = 0$

(c) F(s) is stable (has no poles in the right half of the s plane or on the imaginary axis).

(d) $w_f$ is not so large that any important portions of the R(w) curve at frequencies below $w_f$ are ignored. However, it must not be so small that the evaluation becomes impractical because it requires the inclusion of an excessive number of terms in the summation equation 2.4.[1]

Some of the advantages of this method, henceforth referred to as the RAE Method, are:

(a) Since it is easily programmed for a digital computer, it is possible to make the conversion completely automatic without the time consuming intermediate manipulation which is required by some methods (SO-B).

(b) The accuracy of the method may be controlled by the value of N.

(c) It applies to a wide range of systems limited only by the assumptions mentioned above.

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[1] In the investigation of a method similar to the RAE Method (WA2-B), it is stated that consideration of the responses up to the eleventh harmonic only can give transient response curves which are in error by less than two percent if R(w) has decayed sufficiently at the last harmonic taken.

## 2.2 THE FREQUENCY FUNCTION

The RAE method requires that the real part function be known. Since this function is usually not readily available to a designer, except through the calculation of the frequency response function, F(s), it is desirable to work from F(s). The starting point for this investigation is then F(jw), the magnitude of F(s) along the frequency axis. The curve of M(jw) versus frequency will henceforth be called the "M Curve", and will have logarithmic scales for ordinate and abscissa.

R(w) is calculated from M(jw) through

$$M(w) \cos \phi (w) = R(w) \tag{2.5}$$

This requires two major limiting assumptions in addition to those made previously.

(a) F(s) is a minimum phase function. Consequently, the phase can be found from the magnitude, M(w), since the amplitude response contains all the information necessary for obtaining such functions.

(b) M(w) can be adequately defined by some method which enables large scale calculation of its variations. This problem is solved by approximating the M curve using straight line segments. The various shapes of these M curves can then be simulated and the effect of the various parameter changes can be easily studied in relation to their effect on the step response parameters such as rise time, peak time, and overshoot.

It is interesting to note that such relationships would be of considerable use in synthesis problems in which frequency response curves of a prescribed shape could be the object of synthesis. In some problems this factor would eliminate the necessity of having to work in the time domain in which procedures may be more complicated. Also, in contrast with the charts presented by Chestnut and Mayer (CH-B), this method of analysis has an advantage in that it makes no distinction as to whether the control system is of the unity return ratio type or contains a feedback element.

All the types of basic M curves which were examined conform to equation 2.6.

$$\lim_{w \to 0} M(w) = 1 \qquad (2.6)$$

This equation implies, for a unity return ratio control system, that the loop gain increases without bound at low frequencies. Also, by the final value theorem

$$\lim_{t \to \infty} \mathcal{J}(t) = 1 \qquad (2.7)$$

Diagrams which show the variations of the M curve to be studied are given, along with the symbolism to be used, in Figure 2.1. It is emphasized that these curves are <u>not</u> asymptotic approximations to a frequency function but, rather, are straight line approximations to the actual M curve. These M curves are always characterized in the s plane by one pair of predominant complex poles and also, in some

## GENERAL M CURVE FORMS

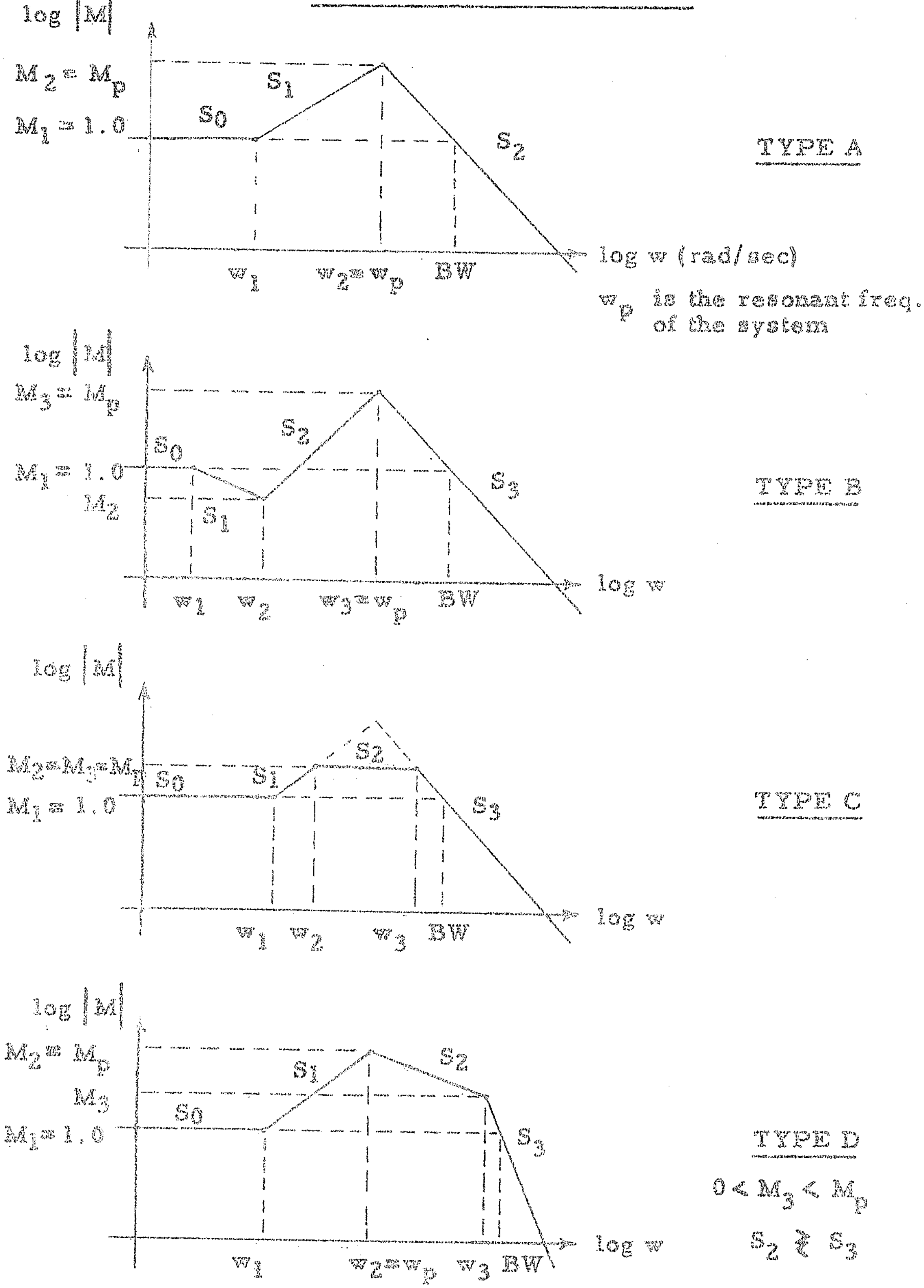


Figure 2.1 M curve types

cases, by a simple predominant pole or zero. Such M curves with moderate resonant peaks are regarded as desirable types of frequency response and are, therefore, the most common configurations.[2]

The segments of the M curves, when plotted on logarithmic paper, are straight lines. There are several valid reasons for this:

(a) It has been found through experience that the M curves representing the types of control systems considered are more easily approximated by straight line segments on this plane than on any other. In particular, since all systems ultimately approach infinite frequency at a rate of integral multiples of -6 db/octave (which, in itself, is a logarithmic scale), these curves will always approach a straight line on logarithmic paper at high frequency and, consequently, their high frequency attenuation may be approximated easily. This is important, since the high frequency portion of the M curve affects to a large degree the lower time values of a step response, and makes up that portion of the time response curve which is of greatest interest.

(b) As a result of the preceding statement, only integral values of the high frequency slopes require consideration.

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[2] See (CH-C, pp. 88-89) for a better visualization of the pole-zero effects on the magnitude and phase curves.

(c) As shown later, normalization is more easily accomplished because the shape of the curve is left unchanged by the frequency normalization procedures.

## 2.3 THE STEP RESPONSE

It is a fact that systems containing similar frequency characteristics must have transients of the same shape. The M curves of types A, C, and D are more or less of the same general shape, as they all possess some aspect of a dominant mode-type resonant frequency. They can, therefore, be expected to produce the step response associated with such a class of M curves. This is illustrated in Figure 2.2. M curves of type "B" are closely related to the response shown, but exhibit additional properties which will be discussed later.

The parameters of the step response are:

$t_1$ = time, in seconds, for the step response to rise to 10% of its final value,

$t_2$ = time, in seconds, for the step response to rise to 90% of its final value,

$t_r$ = 10% to 90% rise time (in seconds),

$t_p$ = time, in seconds, of the peak overshoot,

$\Theta_p$ = initial peak overshoot value expressed as the ratio of the controlled variable to the input at the initial peak.

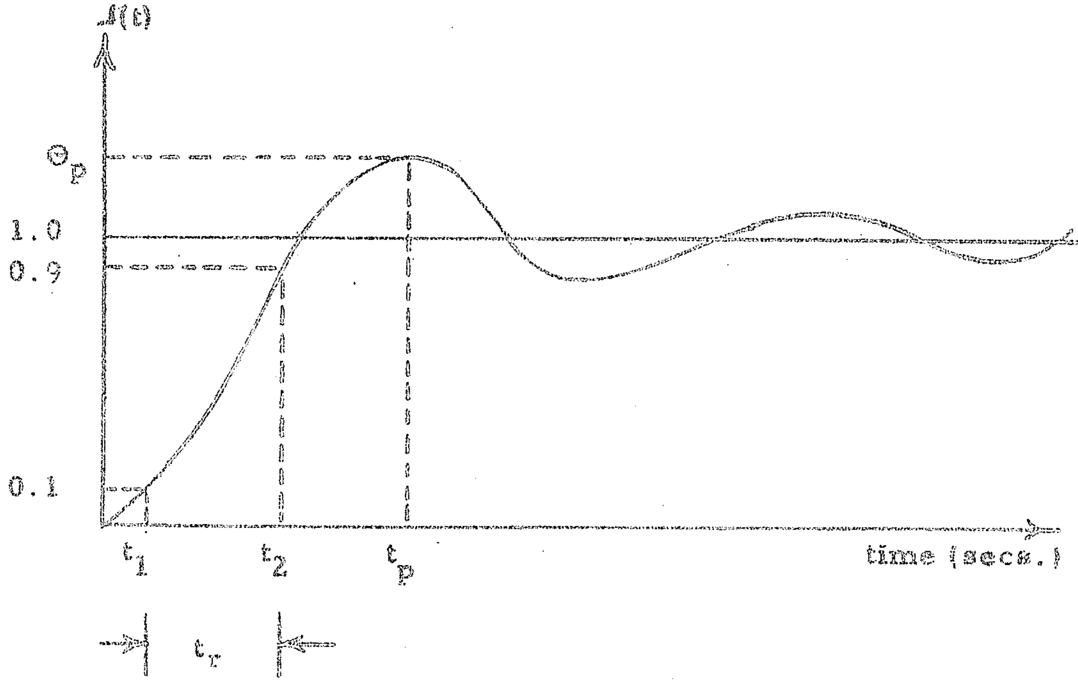



Figure 2.2 General form for the step response

If the peak overshoot is expressed as a percentage of the above response at its final value, then the symbol for overshoot will be $\mathcal{T}$.

## 2.4 NORMALIZATION OF DATA

In the analysis of parameter variations it is helpful to normalize the M curve along the frequency axis. This reduces by one the number of variables and aids in the construction of charts of results. However, when using these charts, it is necessary to establish a procedure which will account for the normalization that has taken place. A useful theorem is now developed for this purpose.

The following equation relates frequency and step response:

$$\Gamma(t') = \frac{2}{\pi} \int_0^\infty \frac{R(w') \sin(w't')}{w'} \, dw' \tag{2.8}$$

where

$\Gamma(t')$ = a step response function

$R(w')$ = the real part function of frequency

Now, changing the scale by substituting $t/n$ for $t'$, it follows that

$$\Gamma(t/n) = \frac{2}{\pi} \int_0^\infty \frac{R(w')}{w'} \sin\left(\frac{w't}{n}\right) dw' \tag{2.9}$$

Also, changing the variable $w'$ to $w$ by using the substitution

$$w = \frac{w'}{n} \tag{2.10}$$

it follows that

$$\Gamma(t/n) = \frac{2}{\pi} \int_0^\infty \frac{R(nw) \sin(wt)}{w} \, dw \tag{2.11}$$

The significance of this conversion is easily demonstrated. Given two real part curves, A and B, such that a point on B corresponds to a point on A shifted horizontally by a constant distance on the logarithmic paper, then

$$w_b/w_a = K \qquad \text{where } K \text{ is a constant} \tag{2.12}$$

An example of such an operation is shown in Figure 2.3.

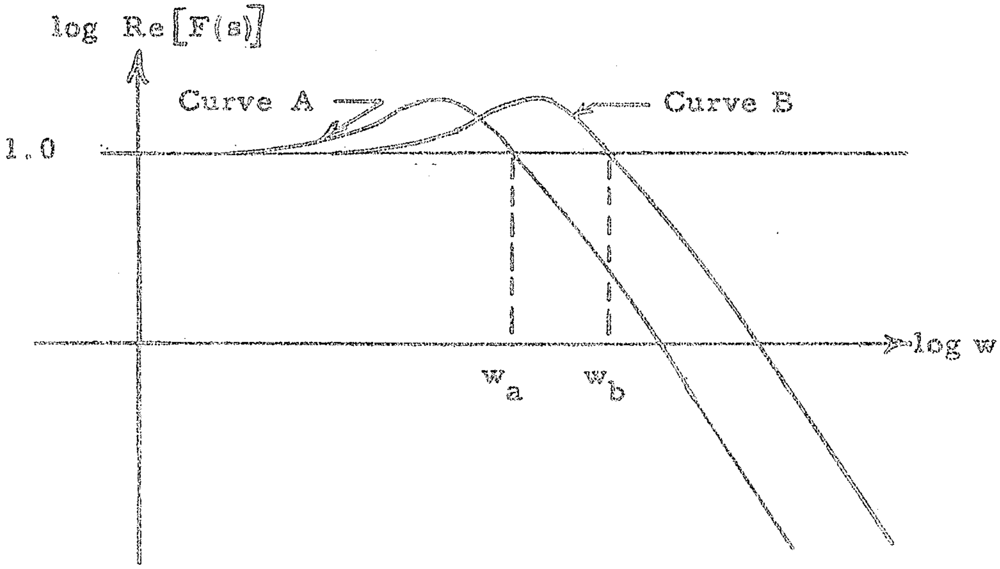



Figure 2.3 Frequency normalization of the real part curve.

Then, given the step response curve corresponding to curve A, (Figure 2.4), the equivalent response for curve B is as shown in Figure 2.5.

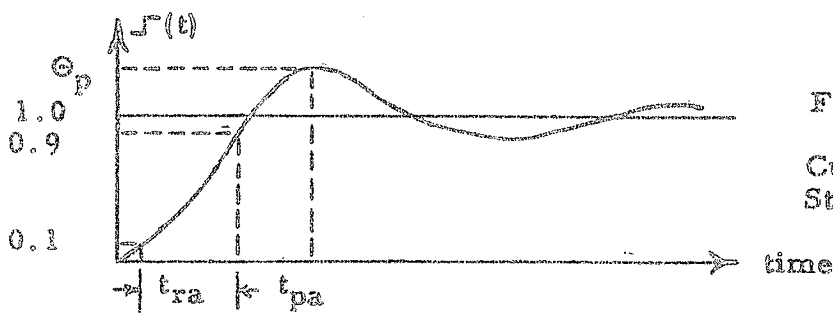



Figure 2.4

Curve A
Step Response

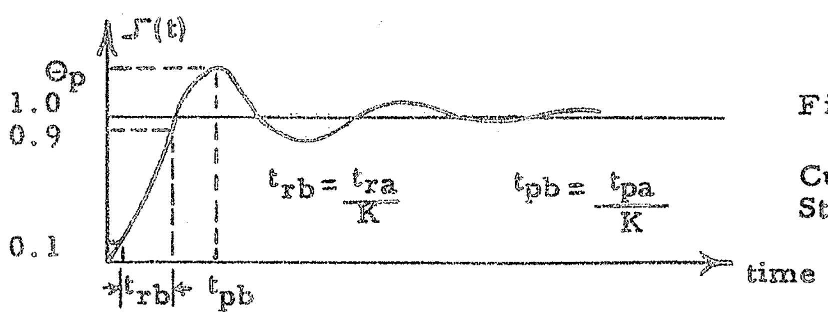



Figure 2.5

Curve B
Step Response

Furthermore, it is true that, under the conditions expressed in Chapter 2, the performance of the same operation on the M curve as was previously carried out for the real part curve results in identical changes in the step response curve. This can be deduced logically by referring to the Bode procedure for calculating phase using straight line Alpha diagrams (BO, pp. 69-77). According to Bode, the phase at any frequency, $\bar{w}$, can be found by adding terms which give the phase produced at the frequency, $\bar{w}$, due to each straight line segment. Each term is the product of

(a) a constant,

(b) the slope of the segment under consideration,

(c) a number which is dependent on the ratio of both ~~the farthest~~ break point frequenc~~y~~ies on the segment ~~to the frequency, $\bar{w}$~~.

Now it is obvious that the slope of the M curve does not change on the translation produced by frequency normalization, and that this translation does not change the ratio of its break point frequencies to $\bar{w}$. Consequently, each slope segment contributes the same amount to the phase at a frequency, $\bar{w}'$, at the equivalent point on the translated curve as at $\bar{w}$ on the original curve.

Therefore, the real part value at $\bar{w}'$ is the same as at $\bar{w}$. We can then conclude that a shift in the M curve by normalization results in an equivalent shift in the real part curve which does not involve an alteration in its shape.

This, in turn, leads to the conclusion that the shifting of the M curve by a factor, n, upward/downward with respect to frequency results in a shift of the step response curve downward/upward by the same factor, n.

In addition, it is obvious from the basic transformation equation (2.1) that the M curve amplitude also can be normalized and that such a normalization causes changes in the step response magnitude which are directly proportional to the ratio of change in the M curve.

## 2.5 INVESTIGATION METHODS

The aim of the investigations is twofold:

(a) to generate a set of curves whereby frequency functions having the basic forms shown in Figure 2.1 may be related to their respective time domain parameters for the purpose of the design of control systems,

(b) to investigate general relationships and trends existing between frequency parameters and time parameters if any simple connections do exist.

In order to accomplish these objectives it is necessary to set up a system of variation for the parameters of the M curve. A separate investigation into each of the four types of curves is given, as this method offers an easier and faster way of analysing and localizing the effects of the M curve parameters on the step response parameters. It should be noted, however, that limiting values in the variation of the parameters of one curve type may produce the outline of the curves of another type. For example, when the effect of the variation of $M_2$ is studied in curve type B, the limiting value of $M_2$ of 1.0 produces a curve of type A.

The initial investigation is concerned with the effect on the step response of the three parameters of curve type A. These parameters are chosen as the leading slope, $S_1$, the trailing slope, $S_2$, and the peak of the M curve, $M_p$. Normalizations are as follows:

Magnitude: $\lim_{w \to 0} |M| = 1.0$

Frequency: $w_2 = w_p = 1.05$ radians/second

The reason for the frequency normalization to 1.05 radians/second is related to programming problems and will be discussed in the next chapter.

A range of from five to seven values of each parameter is chosen which covers the range of stability, bandwidth, and M curve slopes of most control systems.

A similar method is followed for curve type D, although the addition of two extra trailing slope parameters necessitates the examination of a larger number of curves than was required for type A in order to deal with the same parameter ranges.

Two smaller investigations are carried out on curve types B and C where the effect of flattening the M peak and introducing an initial M curve dip is observed.

To accomplish these objectives, digital computer programs were set up for a Bendix G-15D computer as well as for an IBM 650 computer. Since several programming details affect the generation of the results, Chapter 3 is devoted to a brief explanation of the methods used to obtain the results.

# CHAPTER 3

# THE COMPUTER PROGRAM AND ITS EFFECT UPON THE ACCURACY OF RESULTS

## 3.1 GENERAL PROCEDURES

The intended object for including a chapter on the two computer programs is to provide a basis for estimating errors in the results and to give a reference for any further investigation subsequent to the findings of this thesis.

The program for accomplishing the conversion described in Chapter 2 was first written for the Bendix G - 15 Digital Computer. However, for reasons of increased accuracy of results and reduced computation time, the program was rewritten, with some changes, for the IBM 650. Nonetheless, the two programs were basically alike and the following description applies, in general, to both. Any distinctions will be pointed out in those instances where they are regarded as having affected the results to any appreciable degree.

The general procedure for obtaining the step response from the M curve is that of first obtaining the real part curve R(w) from equation 3.1.

$$R(w) = M(w) \cos \beta (w) \tag{3.1}$$

where M(w) is the magnitude and $\beta$ (w) the phase of the frequency response function. Equation 2.4 is then used to find $r(t)$.

An excellent phase approximation method which utilized Bode's Phase Integral (BO-C, pp. 69-74) was programmed for the IBM 650. However, as these results are only intermediate steps toward the final objectives, the procedure is explained briefly in the Appendix.

The value of R(w) is required for N points when the step response is calculated by equation 3.1. The fundamental frequency concept required the frequency of these points, C(N), to correspond to equation 3.2.

$$C(N) = w_f (2N - 1) \tag{3.2}$$

As was discussed briefly in the previous chapter, the adjustment of N and $w_f$ for best accuracy and optimum timing was the main problem which was encountered by the utilization of the RAE method. Consequently, a large portion of the total error in the results is attributed directly to this problem.

## 3.2 DETERMINATION OF THE FUNDAMENTAL FREQUENCY

The basic criterion (RU-B, p. 12) on which the value of N and $w_f$ is judged is the fact that negligible errors are incurred when the real part curve is "cut off" at a frequency, $w_c$, such that the frequency response has finally decayed to about 10% of its maximum value. The problem in this instance is that $w_c$ is not known until R(w) has been calculated, but R(w) cannot be calculated without knowing the correct value for $w_f$. As a result, it is necessary to guess $w_f$ and N for each curve before calculation begins.

One of the main reasons for such great concern regarding judgment of the value of $w_f$ is that inaccuracies which are caused by neglecting those frequencies beyond $w_c$ primarily affect the low time portion of the step response. It is just that portion which is of most significance in this thesis.

In order to make the program completely automatic, some system must be devised for predetermining N and $w_f$. Because of the wide range of curve types which were investigated, this was a troublesome problem. The most difficult curve types are those of type D where wide variations in $S_2$, $M_3$, $w_3$, and $S_3$ cause erratic variations of $w_c$.

With regard to curves of type D, the problem is solved by dividing the M plane trailing slope region into ten areas as shown in Figure 3.1. By a combination of calculation and experience, each region is associated with a certain value of $w_f$ and the various curves are classified into these regions according to the region in which their third break point falls and according, also, to the value of $S_3$. The circled points on the graph show the various positions of the third break point for type D curves. The effect of moving this point along the frequency axis and along the $|M|$ axis is studied in Chapter 7. The region number is included on the data input card for each curve

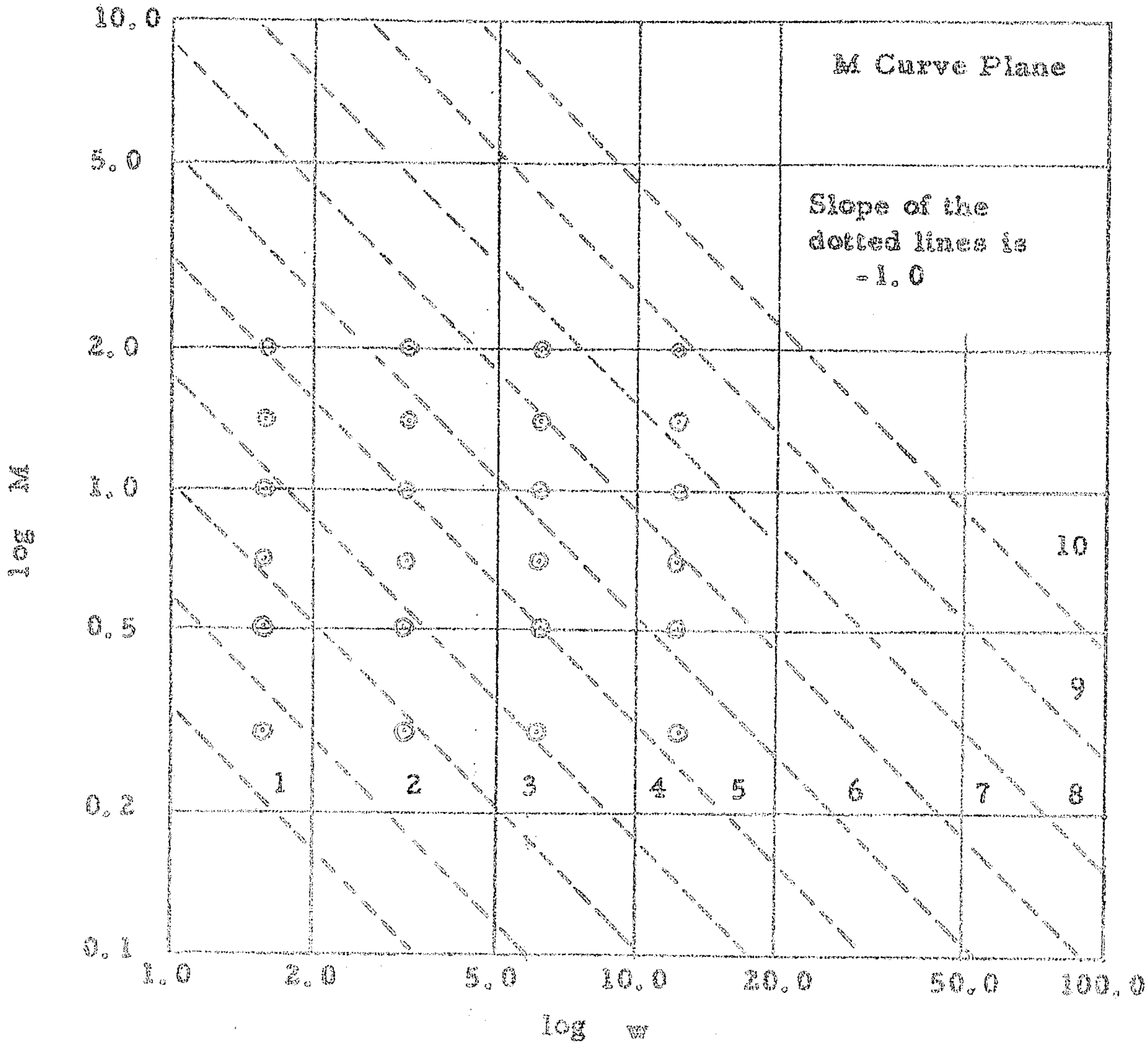



Figure 3.1 Procedure for estimating $w_f$ for curves with $S_3 = -1.0$

when running the calculations with the use of the IBM 650. The input cards are then electronically sorted on the correct column so that all curves of the same region (and thus of the same $w_f$) are computed together. This procedure eliminates the necessity of constantly changing $w_f$ and allows a certain amount of initial experimentation before $w_f$ values for each region are finally chosen.

It is found that values of N which are greater than 50 or 60 lead to excessive computing time and are therefore inefficient and impractical. Hence, most curves are computed with N between 45 and 55. However, it is obvious that, as the bandwidth increases, either a larger N or a larger $w_f$ is required. As increases in N above 50 or 60 cause relatively small decreases at the last point in R(w) due to the nature of the M curve, the solution to this difficulty is to increase $w_f$. A practical limit, however, for increasing $w_f$ is found when $w_f$ becomes so large that the initial C(N) values do not adequately describe the initial R(w) curve portions. It is this fact which produces excessive error in the procedure and causes the results to become relatively unreliable for those M curves which contain the largest bandwidths. As far as possible, such points are omitted from the charts.

## 3.3 PROGRAMMED NORMALIZATION

In all curves the resonant frequency of the system is normalised to

$$w_p \cong 1.05 \text{ radians/second}$$

The range of $w_f$ values which is practical in this case is $0.01 \leqslant w_f \leqslant 0.30$ Although an $w_p$ of 1.0 radians/second was initially chosen, it was found that a better R(w) approximation was made by the discrete points if one value for C(N) occurred at the resonant frequency. Investigation shows that there are more suitable, exact values for $w_f$

which satisfy this condition when $w_p$ is 1.05 radians/second than when the normalization is to 1.0 radian/second.

## 3.4 CHOICE OF TIME PARAMETERS FOR INVESTIGATION

The step response period of oscillation and decay time are not investigated in this thesis for several reasons:

(a) It was not economical (timewise) to carry the step response curve further than time $t_p$ because the number of points required to determine these parameters would drastically increase the time required and cut down the number of curve variations that could be investigated.

(b) It would increase the complexity of the program considerably because of the wide variation of the decay time with the stability of the control system.

The parameters

(a) peak time

(b) peak overshoot

(c) rise time

adequately describe the remaining portions of the step response curve and are somewhat easier to compute by the method used in this thesis. Another investigation could be made in future to determine the other parameters, but a different method of attacking the problem would probably be more efficient.

## 3.5 COMPARISON OF COMPUTER PROGRAMS

The calculation of the values for R(w) is considerably more accurate when conducted on the IBM 650, due to that machine's greater accumulator capacity and a more exact phase approximation (for details, see the Appendix). The IBM 650 ~~is~~ was used only for curves of type D, however, and in regard to these curves the errors caused by incorrect $w_f$ and N values overshadow all others.

It is interesting to note that the complete transformation of one M curve to its step response requires approximately seven minutes with the use of the more accurate IBM 650 program, while the smaller Bendix Computer, using a compiler, does slightly less calculation in the time of two hours!

The total combined running time for the results given in the next chapters is estimated at approximately 1000 hours.

# CHAPTER 4

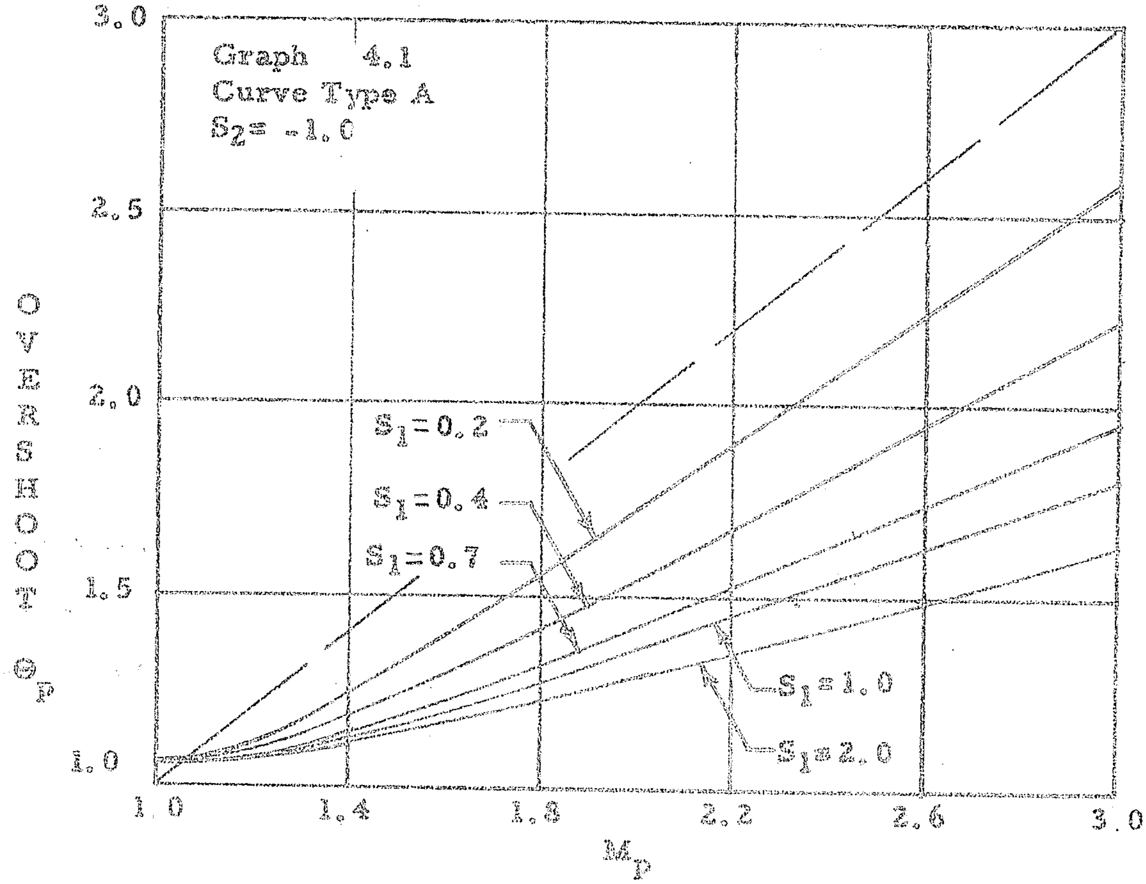
# PRESENTATION AND DISCUSSION OF RESULTS FOR CURVE TYPE "A"

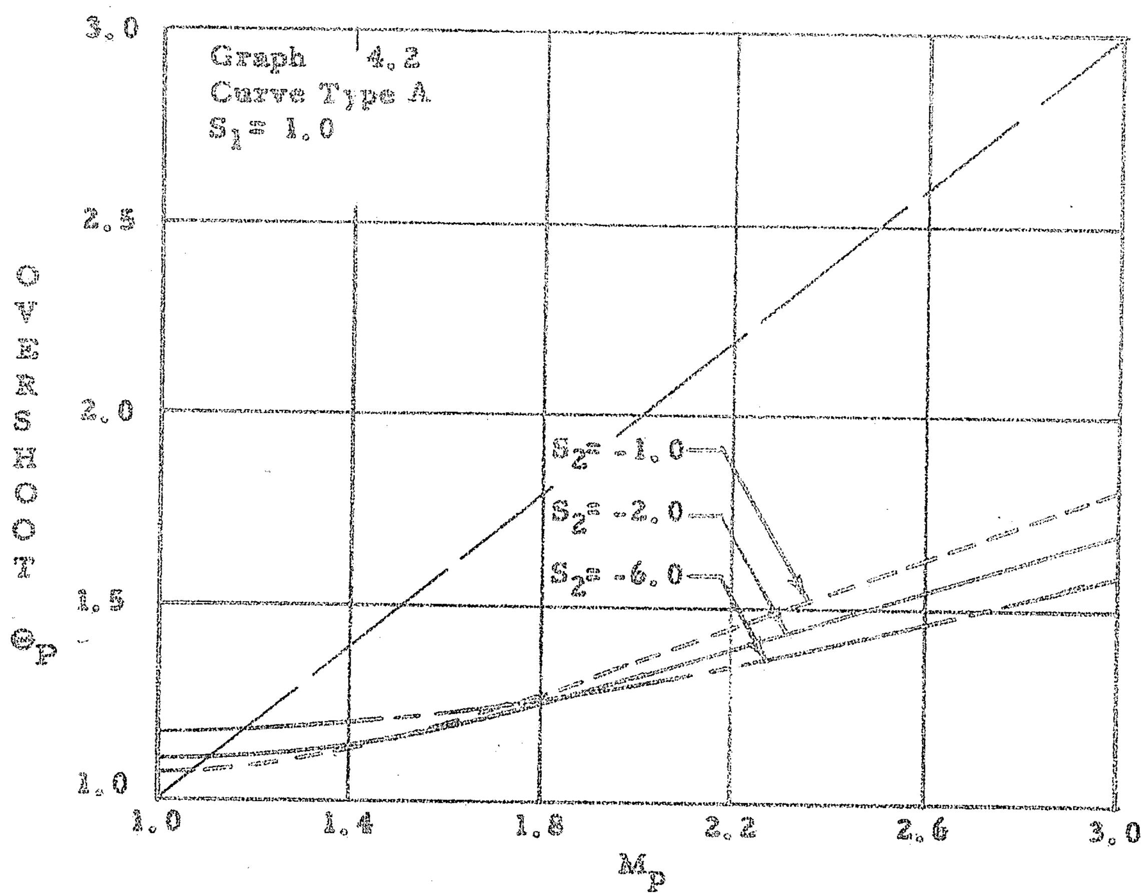
In order to serve the dual purpose of this thesis as was outlined in Chapter 2, results of the investigations are presented in the form of one chapter for each curve type. Chapters 4 and 7 contain conversion charts in addition to an abbreviated general discussion of results that occur in Chapters 4 to 7 inclusive.

## 4.1 OVERSHOOT

Graphs 4.1 and 4.2 were plotted in order to illustrate the effects of $M_p$, $S_1$, and $S_2$ on overshoot. The M curve variation which takes place for one curve of graph 4.1 is shown in Figure 4.1.

Several interesting criteria may be deduced from these curves. First, it is only at points above the diagonal line that the peak overshoot is greater than the M peak. The dividing line is in the area of $M_p$ equal to 1.1 to 1.2 and varies slightly with the value of $|S_2|$, being larger for larger $|S_2|$. This agrees closely with results published by Chestnut and Mayer (CH-B, p. 408) who worked from the open loop characteristics.




Graphs 4.1 and 4.2 Effect on overshoot of varying $M_p$ for several $S_1$ and $S_2$ values

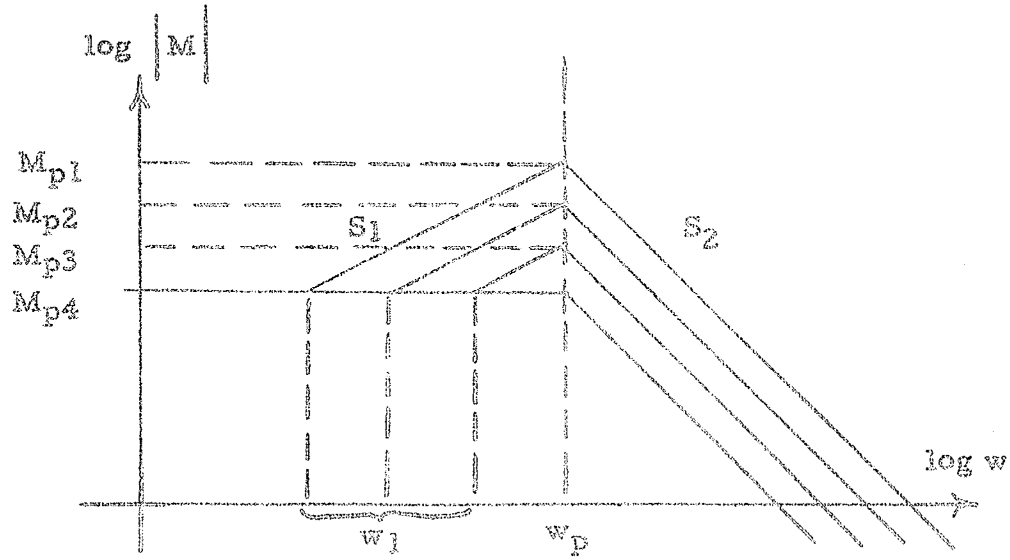



Figure 4.1 M curve variation with constant slopes

Also, there is, to some extent, a linear relation between M peak and overshoot for M peaks greater than approximately 1.5. The reason for such a relation can be traced to the method of M curve variation which is used to generate each curve. With reference to Figure 4.1, this method is such that each value of M for frequencies above $w_1$ is increased by a constant ratio of $M_{pi}/M_{p(i+1)}$. According to the amplitude normalization statement which was made in section 2.4, this is equivalent to increasing all step response values, including the overshoot value, by the same ratio if $w_1$ were close to zero. In our case this approximation appears to be applicable for smaller leading slopes, but it is not quite valid for the larger $S_1$ values because $w_1$ changes according to equation 4.1.

$$\lim_{S_1 \to \infty} w_1 = 1.05 \tag{4.1}$$

A check on the ratio of increase of overshoot for the curve of $S_1 = 0.2$ on graph 1 between $M_p = 2.5$ and $3.0$ gives the following result:

$$100 \left[ \frac{\Theta_p \text{ for } M_p = 3.0}{\Theta_p \text{ for } M_p = 2.5} \right] = \left[ \frac{2.58}{2.15} \right] 100 = 120\%$$

The ratio of amplitude change, in percent, on the M curve is

$$\left[ \frac{M_{p1}}{M_{p2}} \right] = 100 \left[ \frac{3.0}{2.5} \right] = 120\%$$

The overshoot thus increases by the same ratio as the increase of the M peak. Since, at $M_p$ equal to 3 on Graph 4.1, all $M_p$ values are less than their corresponding step overshoot, then as $M_p$ increases to higher values, the overshoot will always be more than the corresponding M peak value for the same system.

For an $S_1$ of 2.0, however, the percentage variation is

$$100 \left[ \frac{\Theta_p \text{ for } M_p = 3.0}{\Theta_p \text{ for } M_p = 2.5} \right] = \left[ \frac{1.63}{1.456} \right] 100 = 112\%$$

It is now apparent that here $w_1$ is not small enough for the amplitude approximation mentioned above to hold.

Third, the larger the value of $|S_2|$, the less the variation of overshoot with M peak changes.

Furthermore, if area (A) under the M curve above unity for type A curves is defined as equation (4.2),

$$A = \log\left[\frac{M_p}{M_0}\right]\log\left[\frac{BW}{w_1}\right]\Big/ 2 \qquad (4.2)$$

then overshoot seems, for the most part, to increase when the area above unity is increased. That is, for the same M peak, a sharper resonance peak does <u>not</u> give a larger overshoot as might be expected.

The variation of overshoot with changes in the initial slope shows that, in estimating the overshoot, only very limited accuracy is possible if the effect of the leading slope is ignored. A recent attempt (JA-C) to correlate overshoot with the shape of the frequency curve by means of "shape parameters" using equation 4.3 is considerably in error because the overshoot was related to the "form parameter", $F_p$, which is not a function of the leading slope. Jaworski's relationship is

$$\sigma = 58\, F_p - 39 \qquad (4.3)$$

where $F_p$ = form parameter

$$= \frac{B_3\, M_p}{B_6\, M_0}$$

and $B_3$ = bandwidth frequency at the 3 db down point

$B_6$ = bandwidth frequency at the 6 db down point

The method of relating parameters which is utilized in this thesis is a considerable improvement because the charts take this low frequency portion of the M curve into account.

Finally, it is apparent that the well known trend (JA-C, p. 398) of an increase in overshoot with a corresponding increase in the falling slope, holds only for the lower M peak portion of graph 4.2. However, the majority of control systems have M peaks in this region and therefore the generalization is valid within this class.

## 4.2 PEAK TIME

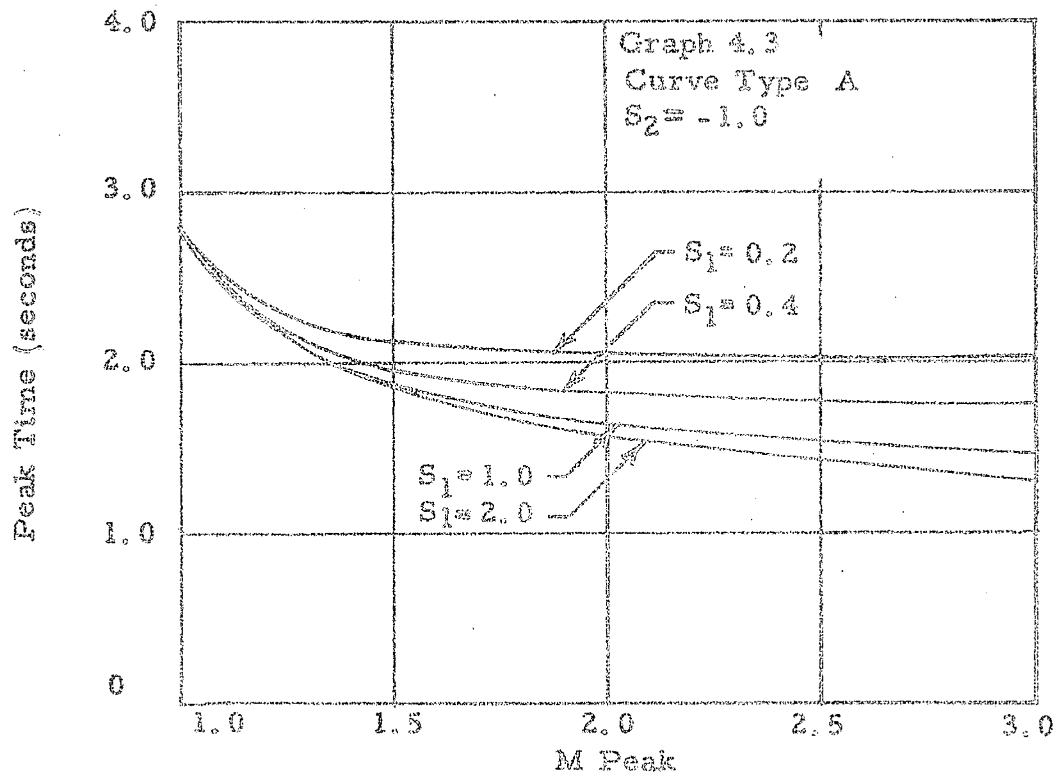
The variation of the peak time with the parameters of the M curve is shown in graphs 4.3 and 4.4. The effect of the various parameters on the time of the peak overshoot can be summarized as follows:

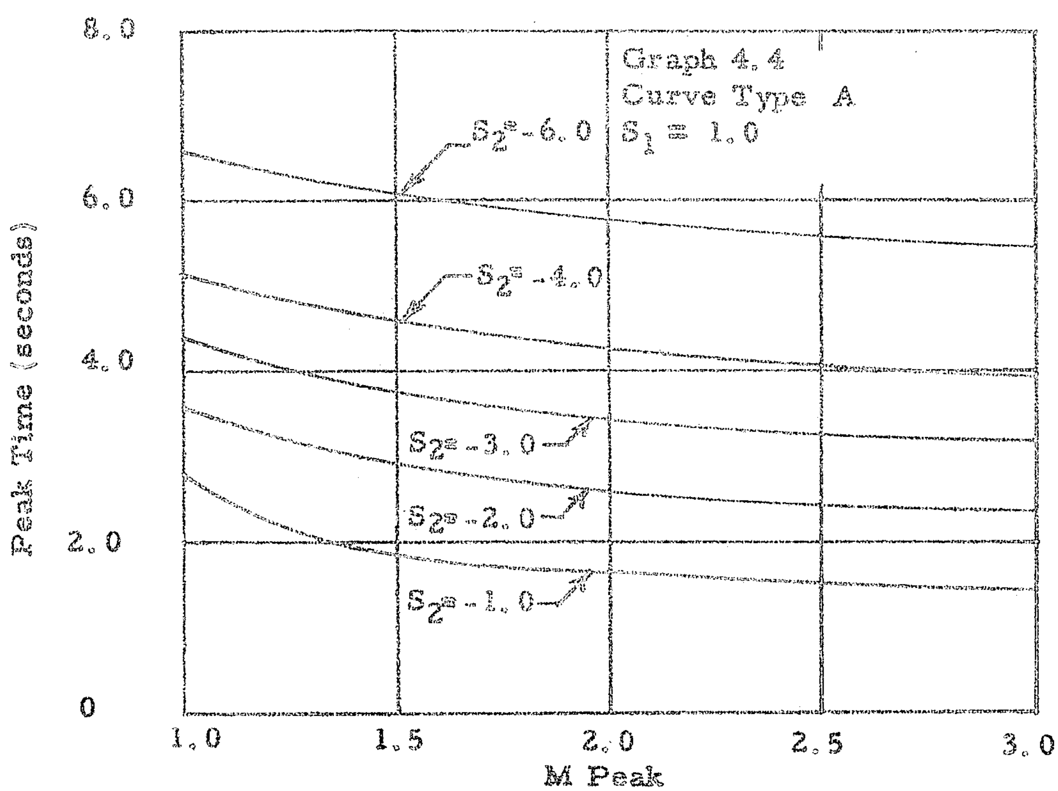
(a) increasing only the M peak decreases the peak time,

(b) increasing only the leading slope also decreases the peak time,

(c) increasing only the magnitude of the trailing slope increases the peak time.

Although empirical curve fitting procedures could be used on all data in order to generate equations relating the variables, it is obviously not practical to attempt this method when the relationships




**Graphs 4.3 and 4.4** The effect on peak time of varying the M peak for several slope values

are other than those which can be expressed as simple equations and are applicable over a wide range of the variables concerned. In any other case it is faster and more practical to use the conversion charts given. However, one case in which an empirical equation appears justifiable is presented at this time.

A study of the relationship of peak time to overshoot reveals that a conversion parameter, C, where C is given by equation 4.3A

$$C = \frac{|S_2|\sqrt{\theta_p}}{t_p} \qquad (4.3A)$$

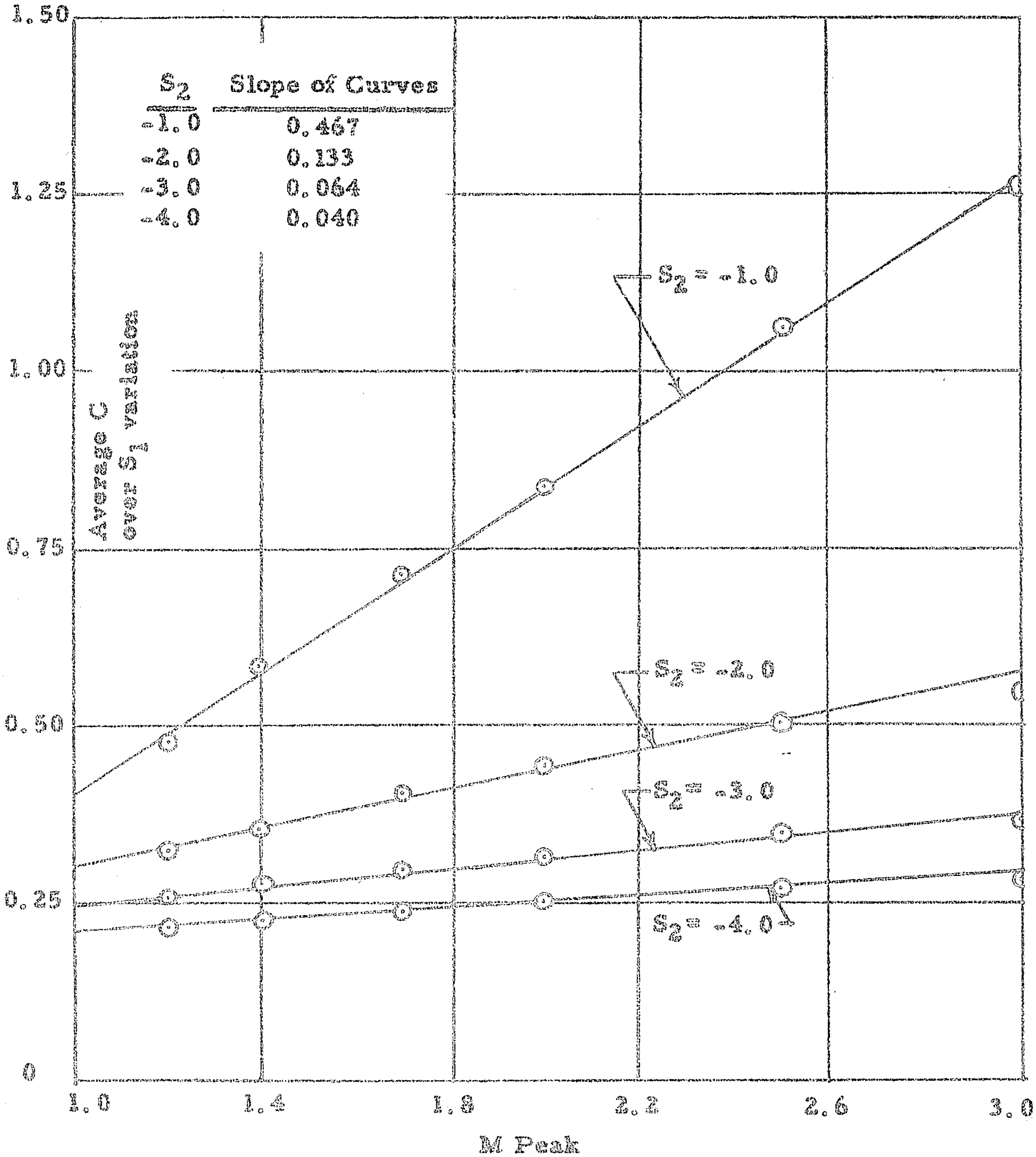
remains remarkably constant at all values of the leading slope for a given M peak and trailing slope. It also appears feasible to relate the parameters for the various M peaks and trailing slopes which are investigated.

A series of curves is presented in graph 4.5 which shows the variation of C with $M_p$ for various $S_2$. The relation is surprisingly linear and a study of the graph slopes and intercepts leads to equation 4.4.

$$C = (0.32 + 0.03\ S_2) + (\text{slope of C curve})\ (M_p - 0.74) \qquad \ldots.\ (4.4)$$

The various graph slopes are given on the same page. It is possible, by finite difference procedures, to express the graph slope as a function of $S_2$ and C can thus be rewritten as

$$C = (0.32 + 0.03\ S_2) + .037(M_p - 0.74)\ (S_2^{\,3} + 9.57\ S_2^{\,2} + 30.7\ S_2 + 34.7) \qquad \ldots.\ (4.5)$$




Graph 4.5 The average value of C for various $M_p$ and $S_2$

This states the relation

$$\Theta_p \approx f(t_p, S_2, M_p) \tag{4.6}$$

In the range of the variables considered for graph 4.1 the estimated average error for the approximation of a given C value for all $S_1$ at a specified $M_p$ and $S_2$ is 3 to 4 percent. This is illustrated in Table 4.1 in which the calculation of C is presented in tabular form. The deviation of the C value at any $S_1$ value from the corresponding average C value for all $S_1$ is observed to be small.

Since $S_2$ is used only as an integral value, it is helpful to restate equation 4.5 for the practical $S_2$ range. The conversion equations are given in Table 4.2.

## 4.3 RISE TIME

The graphs 4.6 and 4.7 show the variation of rise time for many combinations of leading and trailing slopes. As a summary, the lower rise times are caused by

(a) a larger M peak

(b) a smaller $|S_2|$

(c) a smaller $S_1$

In each of the above cases, it is assumed that the remaining variables are kept constant.

Points (a) and (b) comply with the well known fact that rise time decreases as bandwidth increases. However, it is seen from (c) that the rise time is also somewhat affected by the leading slope. The effect is smaller, however, at lower M peaks and increases

| | | $S_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S_2$ | $M_p$ | 0.2 | 0.4 | 0.7 | 1.0 | 1.3 | 1.7 | 2.0 | AVERAGE |
| -1.0 | 1.2 | 0.481 | 0.481 | 0.478 | 0.477 | 0.476 | 0.477 | 0.474 | 0.478 |
| | 1.4 | 0.579 | 0.589 | 0.583 | 0.584 | 0.584 | 0.579 | 0.577 | 0.582 |
| | 1.7 | 0.713 | 0.720 | 0.718 | 0.714 | 0.707 | 0.708 | 0.705 | 0.712 |
| | 2.0 | 0.843 | 0.851 | 0.847 | 0.839 | 0.830 | 0.826 | 0.825 | 0.837 |
| | 2.5 | 1.060 | 1.065 | 1.050 | 1.040 | 1.090 | 1.040 | 1.040 | 1.060 |
| | 3.0 | 1.270 | 1.270 | 1.260 | 1.250 | 1.260 | 1.260 | 1.260 | 1.260 |
| -2.0 | 1.2 | 0.326 | 0.324 | 0.322 | 0.322 | 0.320 | 0.320 | 0.319 | 0.322 |
| | 1.4 | 0.356 | 0.356 | 0.354 | 0.354 | 0.352 | 0.351 | 0.351 | 0.353 |
| | 1.7 | 0.396 | 0.404 | 0.407 | 0.407 | 0.405 | 0.405 | 0.405 | 0.404 |
| | 2.0 | 0.430 | 0.445 | 0.450 | 0.450 | 0.449 | 0.448 | 0.446 | 0.445 |
| | 2.5 | 0.482 | 0.504 | 0.509 | 0.506 | 0.506 | 0.503 | 0.504 | 0.502 |
| | 3.0 | 0.526 | 0.552 | 0.557 | 0.557 | 0.557 | 0.550 | 0.552 | 0.550 |
| -3.0 | 1.2 | 0.258 | 0.256 | 0.256 | 0.255 | 0.255 | 0.254 | 0.255 | 0.256 |
| | 1.4 | 0.276 | 0.275 | 0.274 | 0.273 | 0.272 | 0.271 | 0.271 | 0.273 |
| | 1.7 | 0.296 | 0.298 | 0.299 | 0.298 | 0.296 | 0.295 | 0.293 | 0.296 |
| | 2.0 | 0.312 | 0.317 | 0.318 | 0.317 | 0.317 | 0.314 | 0.313 | 0.315 |
| | 2.5 | 0.338 | 0.343 | 0.346 | 0.346 | 0.346 | 0.342 | 0.344 | 0.344 |
| | 3.0 | 0.359 | 0.367 | 0.371 | 0.370 | 0.371 | 0.371 | 0.370 | 0.368 |
| -4.0 | 1.2 | 0.215 | 0.214 | 0.214 | 0.213 | 0.213 | 0.212 | 0.212 | 0.213 |
| | 1.4 | 0.226 | 0.221 | 0.225 | 0.224 | 0.223 | 0.223 | 0.223 | 0.224 |
| | 1.7 | 0.238 | 0.240 | 0.240 | 0.240 | 0.238 | 0.237 | 0.238 | 0.239 |
| | 2.0 | 0.249 | 0.253 | 0.252 | 0.252 | 0.251 | 0.249 | 0.250 | 0.251 |
| | 2.5 | 0.264 | 0.269 | 0.270 | 0.270 | 0.270 | 0.269 | 0.269 | 0.269 |
| | 3.0 | 0.276 | 0.282 | 0.284 | 0.285 | 0.285 | 0.285 | 0.284 | 0.283 |

Table 4.1 Calculation of average C for all type "A" curves for all $S_1$ variations considered

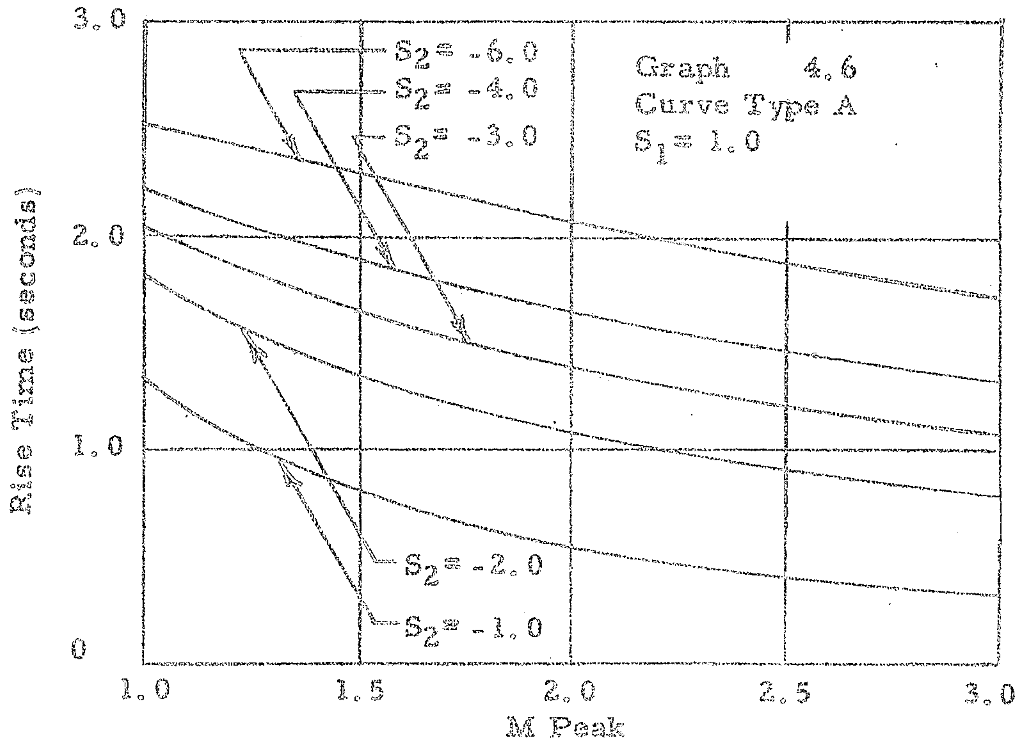
| $S_2$ | $t_p$ | |
|---|---|---|
| -1 | $t_p = \dfrac{\theta_p}{0.433\ M_p - 0.03}$ | (4.7) |
| -2 | $t_p = \dfrac{\sqrt{\theta_p}}{0.133\ M_p + 0.162}$ | (4.8) |
| -3 | $t_p = \dfrac{\sqrt[3]{\theta_p}}{0.064\ M_p + 0.183}$ | (4.9) |
| -4 | $t_p = \dfrac{\sqrt[4]{\theta_p}}{0.040\ M_p + 0.17}$ | (4.10) |

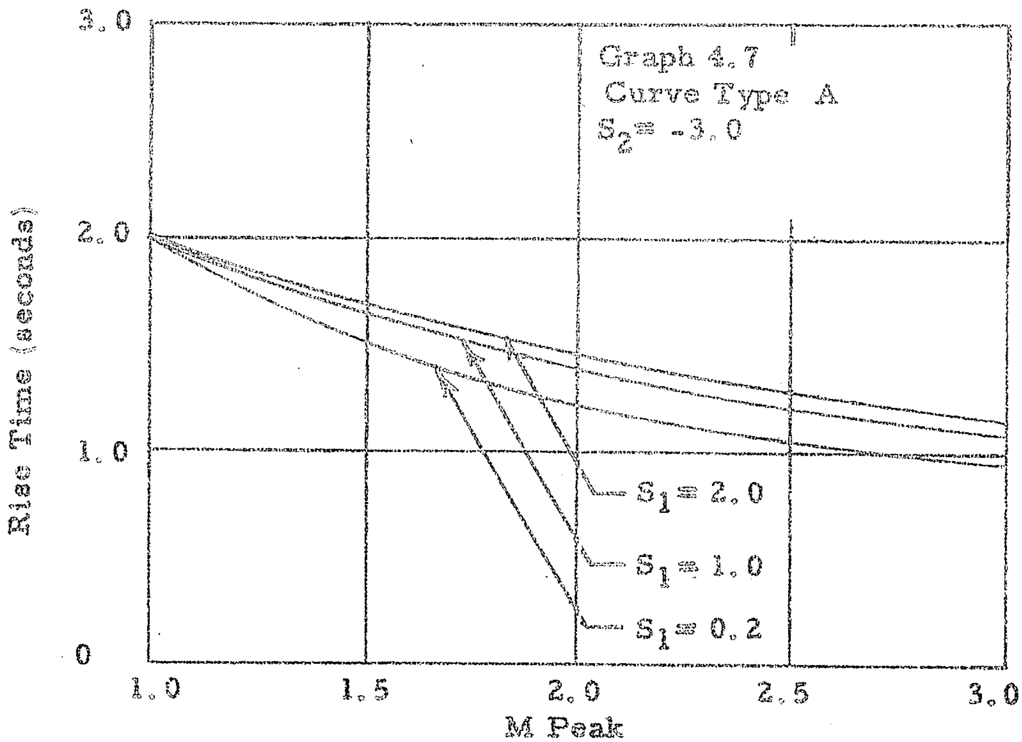
Table 4.2 Conversion equations for the peak time of Type A curves

considerably for higher M peaks.

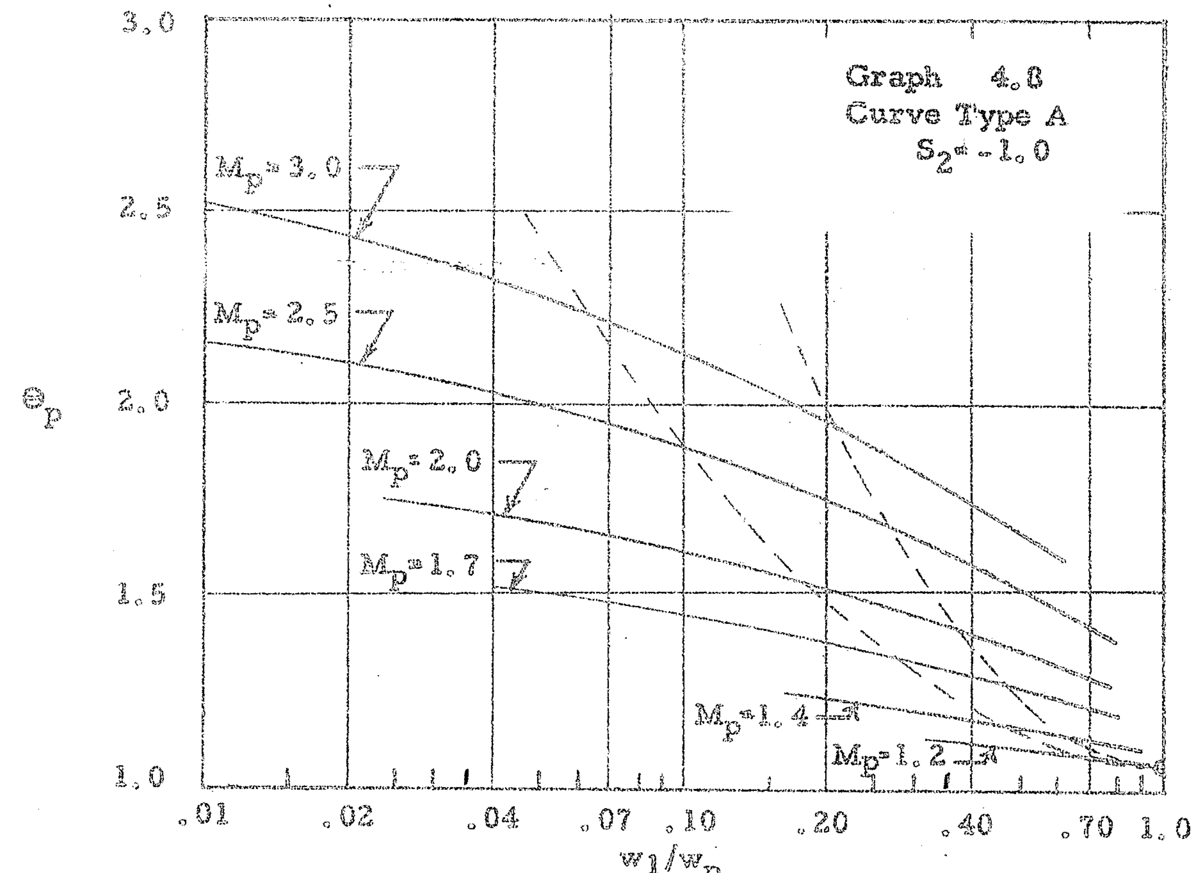
## 4.4 DESIGN CHARTS FOR TYPE "A" CURVES

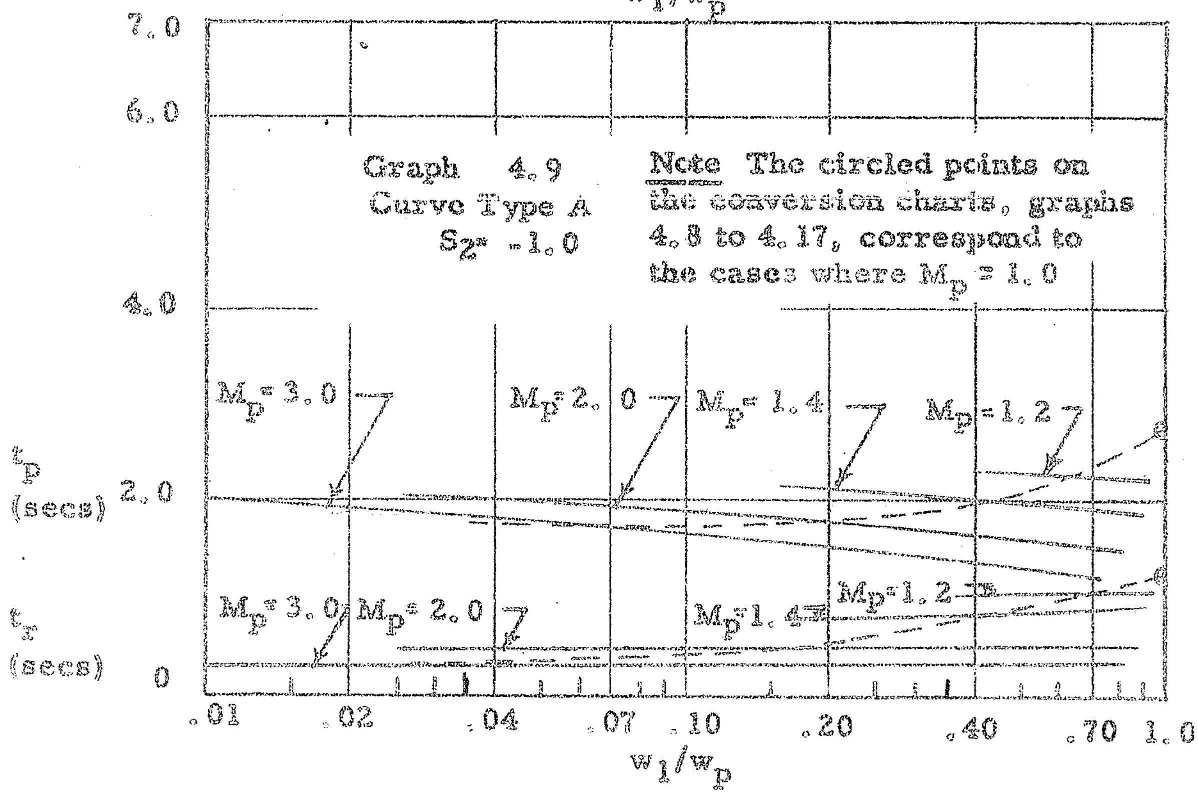
In order that a designer may make a good approximation to the overshoot, rise time, and peak time for M curves which resemble type A curves, a set of ten graphs is presented on the following pages. The abscissa for these graphs is carefully chosen as $w_1/w_p$



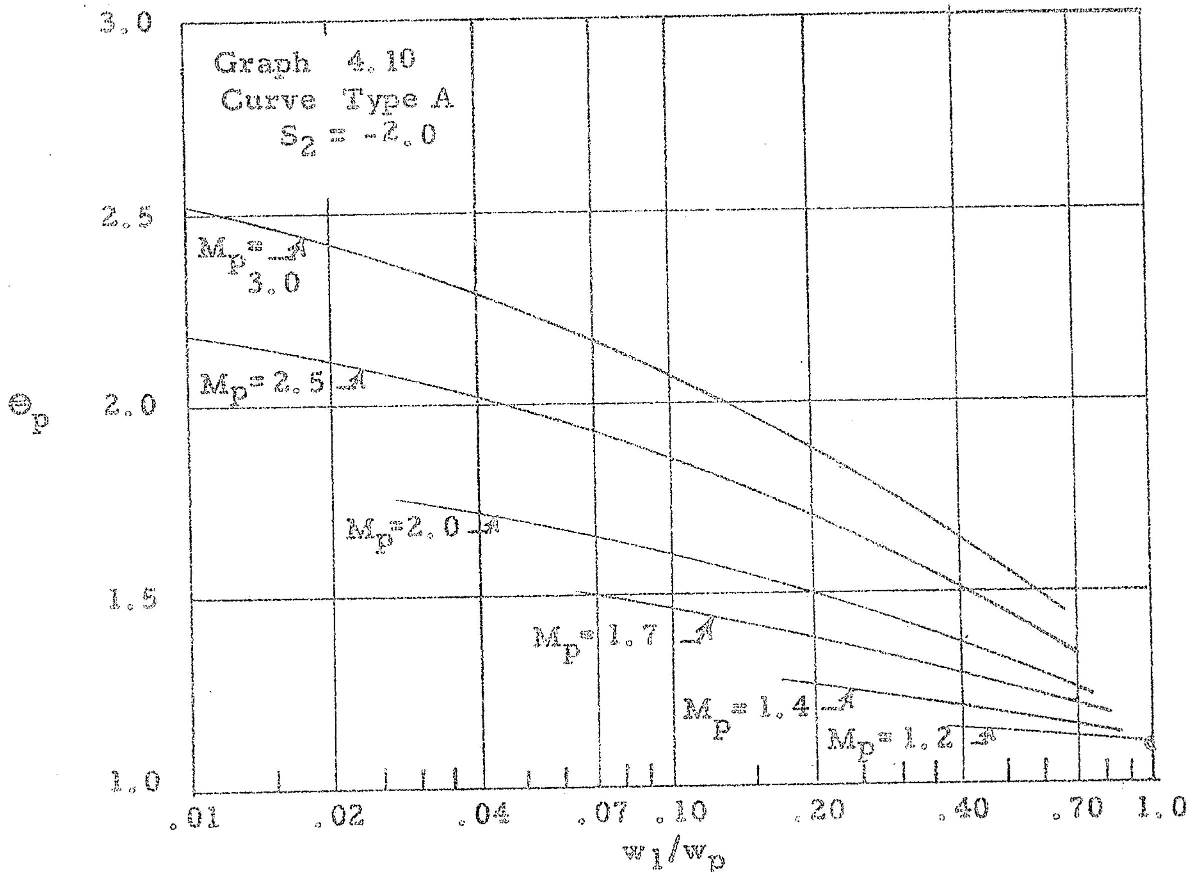



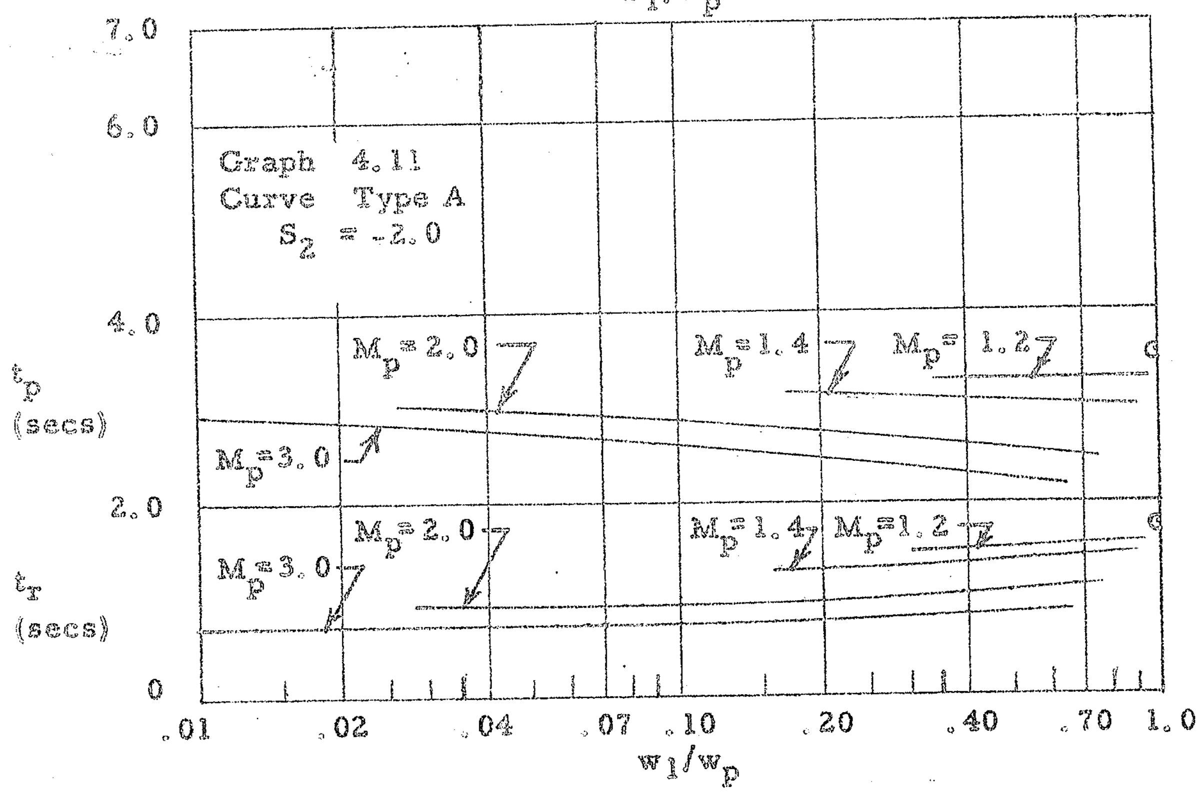

Graphs 4.6 and 4.7 Variation of rise time with M peak for several leading and trailing slopes



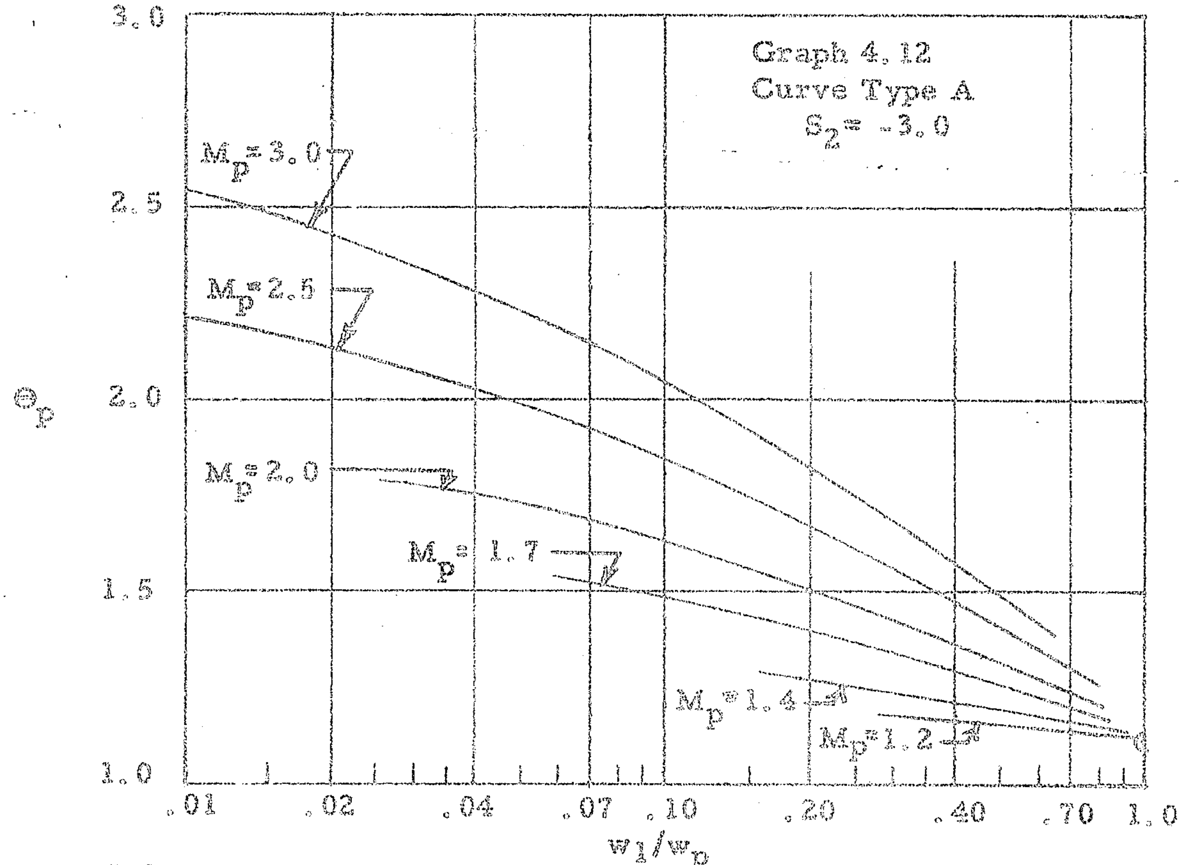



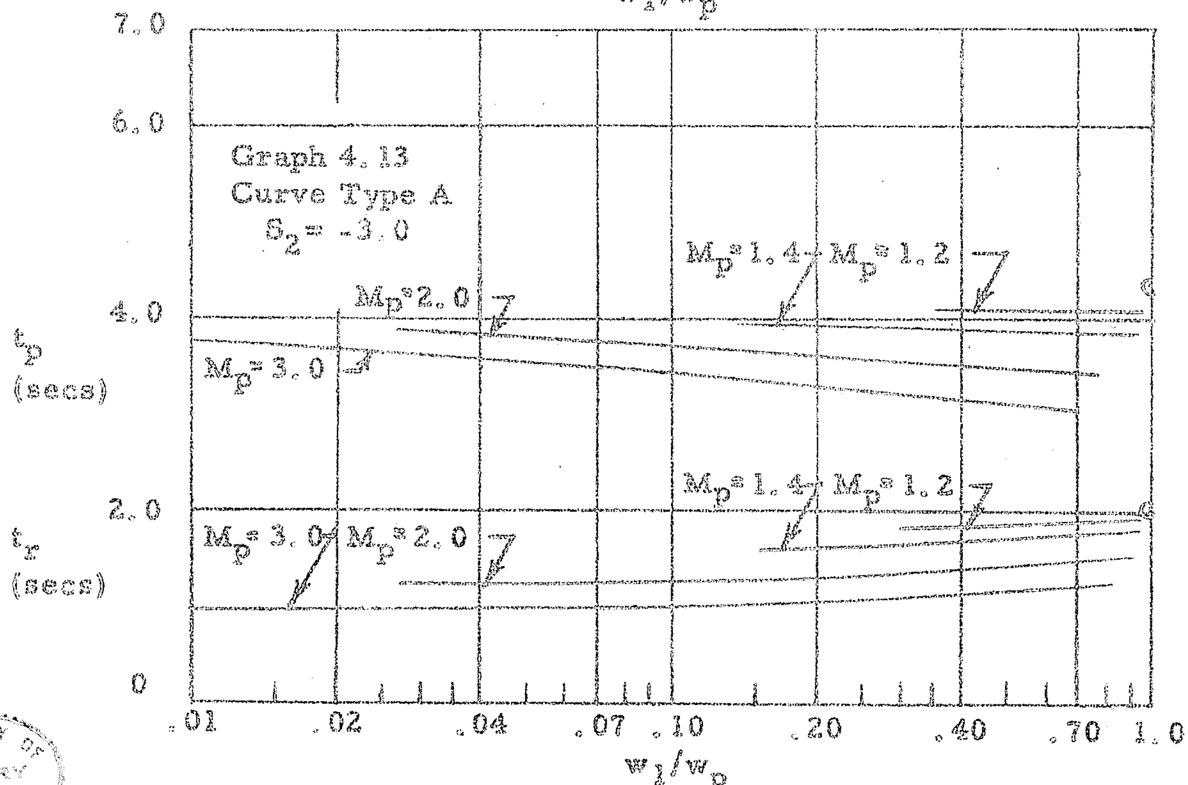

Graphs 4.8 and 4.9 Design conversion chart for $S_2$ = -1.0



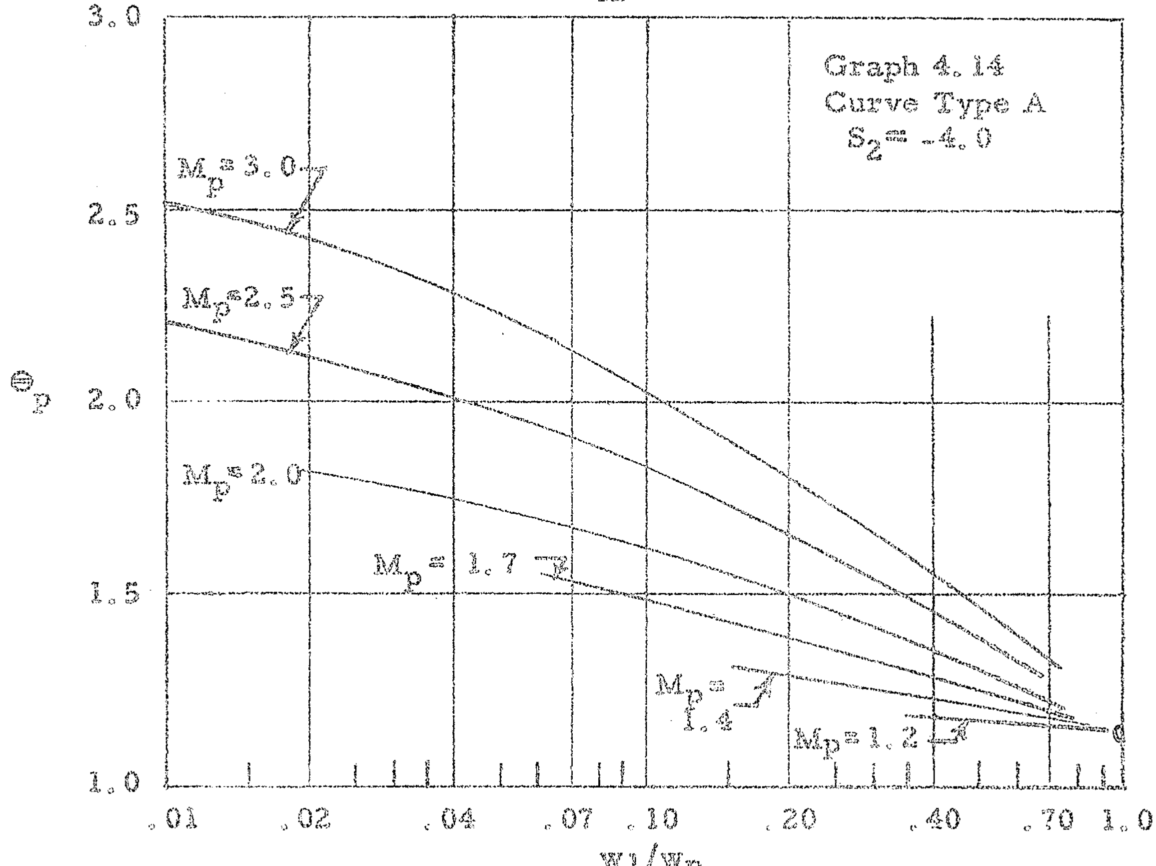

Graphs 4.10 and 4.11 Design conversion chart for $S_2$ = -2.0

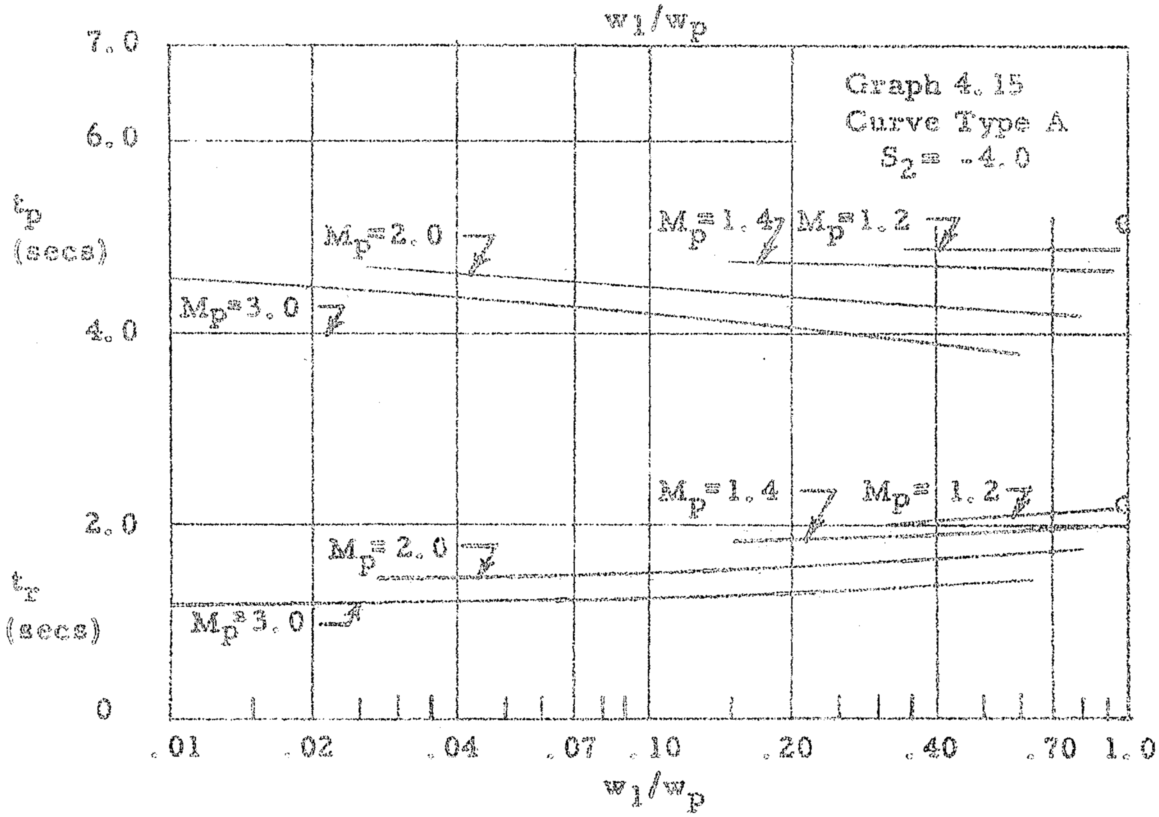


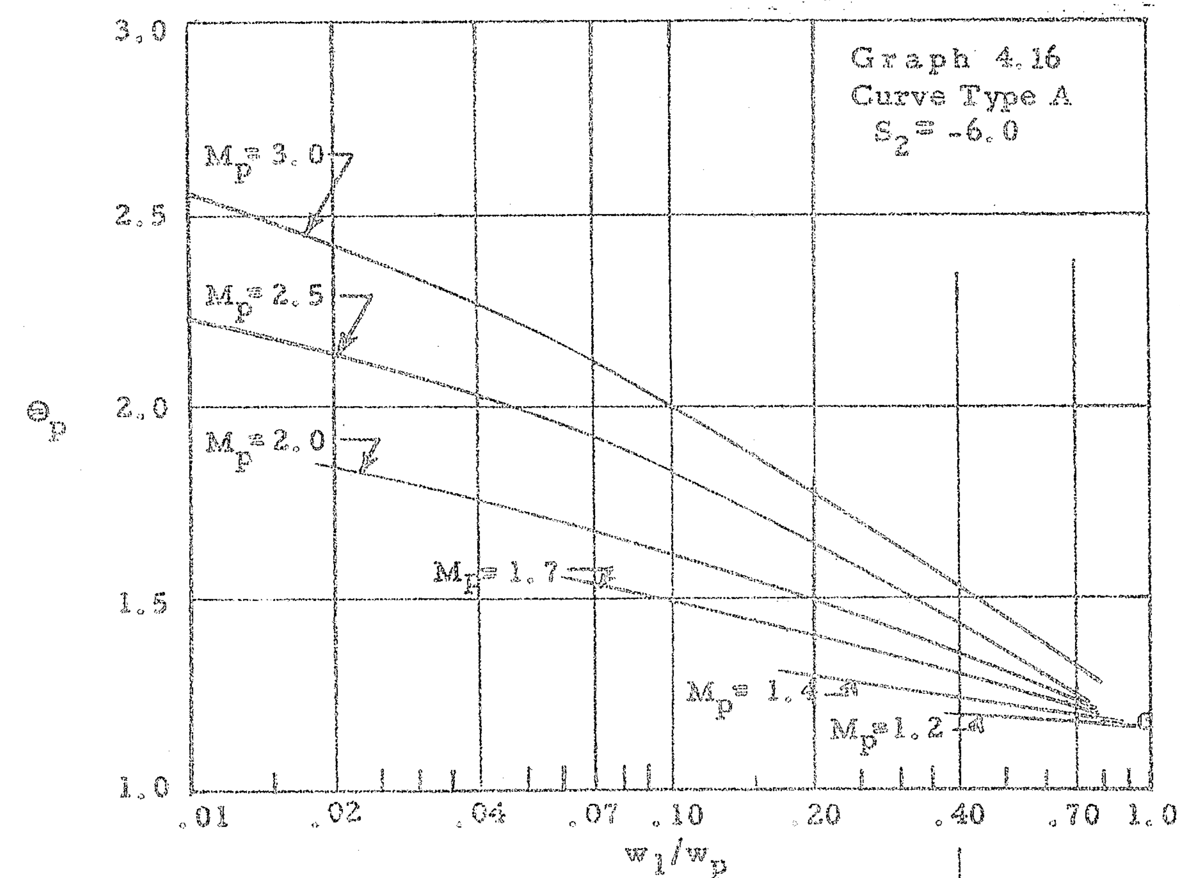


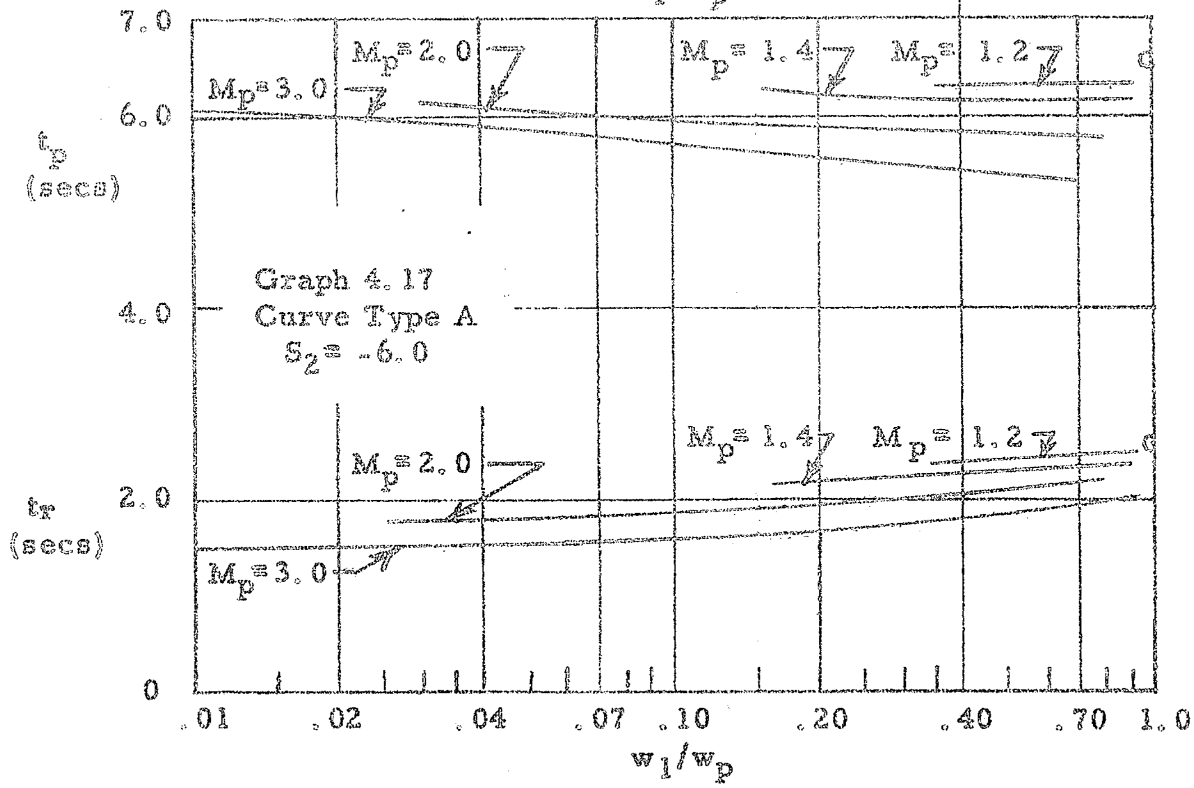

Graphs 4.12 and 4.13 Design conversion chart for $S_2$ = -3.0




Graphs 4.14 and 4.15 Design conversion chart for $S_2$ = -4.0




Graphs 4.16 and 4.17 Design conversion chart for $S_2 = -6.0$

in order that the most difficult variable for interpolation be one of the parameters. Since all curves ultimately have an integral slope to infinite frequency, the constant parameter of each graph is $S_2$. The only other variable, $M_p$, is quite easily interpolated. An example of the application of these graphs is given in Chapter 8 which deals with design considerations.

With $w_1/w_p$ as the abscissa of these curves, several interesting features are evident. First, as this ratio approaches zero, with constant M peak and $S_2$, the overshoot approaches $M_p$ times the overshoot associated with the M curve of unity M peak and having the same $S_2$. This fact follows from the amplitude normalization rule. The above rule greatly helps in extrapolating the curves to lower abscissa ratios than those given in the graphs. By observing the overshoot graphs at the top of each page, it is seen that this rule establishes the fact that all curves will exhibit a "saturation effect" at lower abscissa ratios, even though those curves for the lower M peak values may not show this definitely as far as the curves were taken.

Second, by means of the frequency normalization rule, we see that for all $M_p$

$$\lim_{w_1/w_p \to 0} t_p = \left(\text{peak time for [redacted] } M_p = 1.0 \text{ at the same } S_2\right) \qquad (4.11)$$

This also establishes, within narrow limits, the value of $t_p$ for systems falling outside the range of the given curves (see top

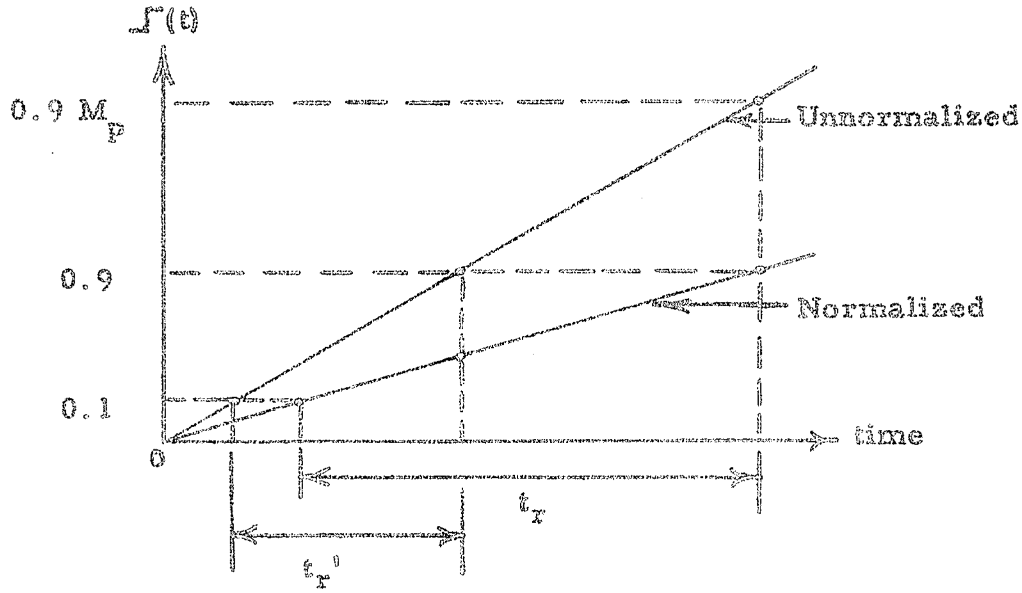



Figure 4.2 The effect of normalization on the rise time

section of all bottom graphs).

Furthermore, if we approximate the initial part of $\Gamma(t)$ by a straight line, then it can be seen from Figure 4.2 that

$$t_{r'} \cong \frac{t_r}{M_p} \tag{4.12}$$

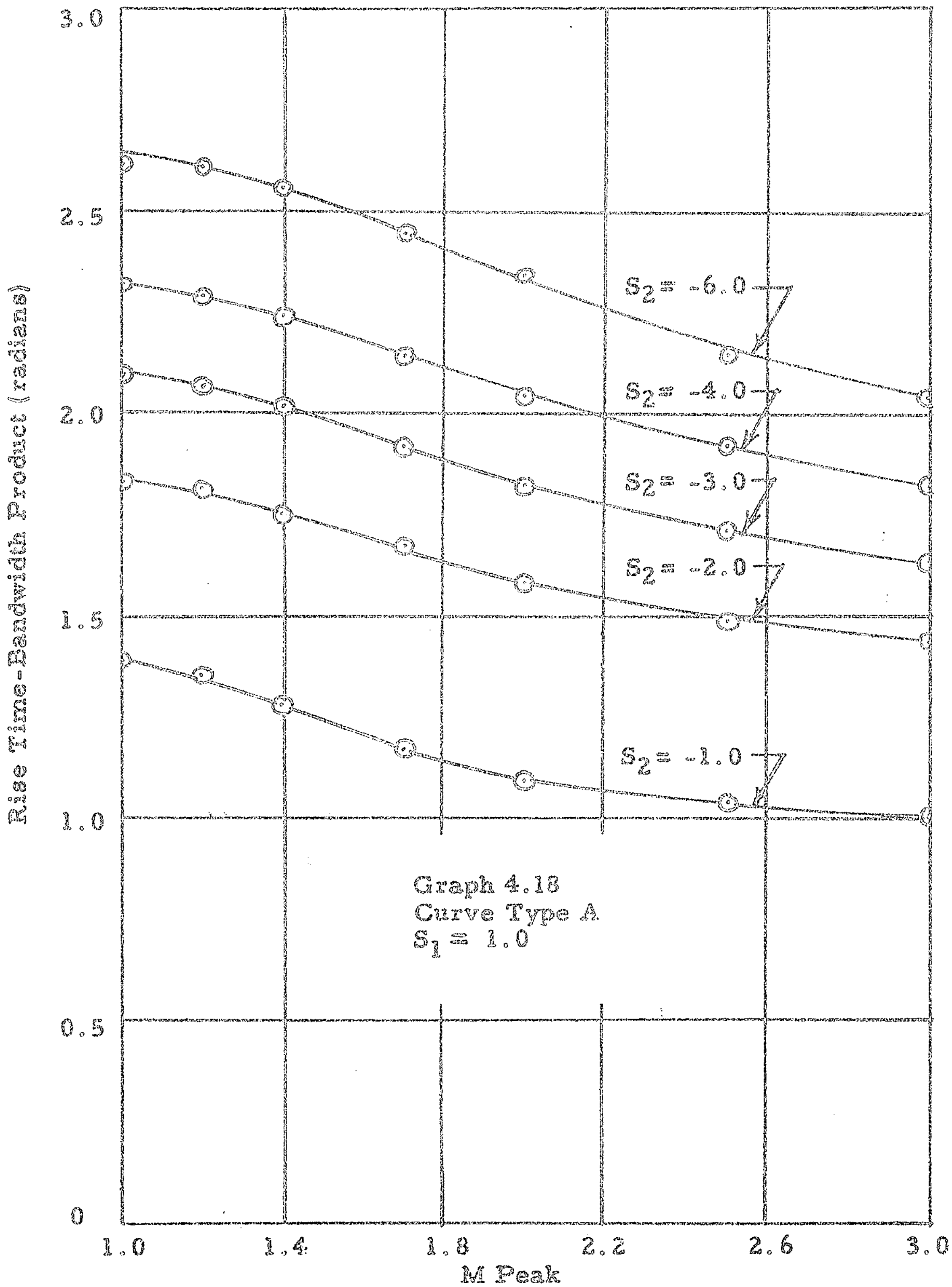
However, since the actual step function is not really linear as shown, the approximation turns out to be inaccurate and serves only to show the order by which $t_{r'}$ is different than $t_r$.

Note that the dotted lines on the first graph sheet indicate the locus of constant $S_1$.

Since each graph is drawn for constant $S_2$ and each curve is drawn for constant M peak, then, to other scales, the curves show the variation of the ordinate parameters with M peak and $S_1$ changes at constant bandwidth. Note that the rise time for these curve sets remains almost constant for any given M peak and $S_2$ when $S_1$ is varied. This is reasonable, since the initial part of a system time response is determined primarily by the high frequency pole-zeros of the frequency response and is determined very little by the low frequency pole-zeros. The latter affect primarily the final portion of the time response. Since the variation here is strictly on the low frequency portion of the M curve, there is a more pronounced effect on the peak time and overshoot than on the rise time, since the former occur at larger time values than those which determine the rise time.

Since it is widely recognized that rise time is approximately inversely proportional to bandwidth, it is interesting to plot the rise time-bandwidth product against M peak for various $S_2$ values. The result indicates the extent of the effect on the relationship of variations in the M curve. It is obvious that this product is not constant, but changes noticeably under the specified variations, according to graph 4.18. However, since most stable systems have M peaks of less than 1.5, the variation is not large when it covers this limited range.

Having investigated the most basic type of M curve, it is now advantageous to extend the range of application of the method by considering, in the next chapters, several variations of the basic type A curve.




**Graph 4.18** **M peak variation effect on the rise time-bandwidth product**

# CHAPTER 5

# PRESENTATION AND DISCUSSION OF RESULTS FOR CURVE TYPE "B"

## 5.1 INITIAL M CURVE DIPS

In order to investigate the effect of an initial dip of the M curve on time response parameters, a set of M curve variations, as shown in Figure 5.1, are considered.

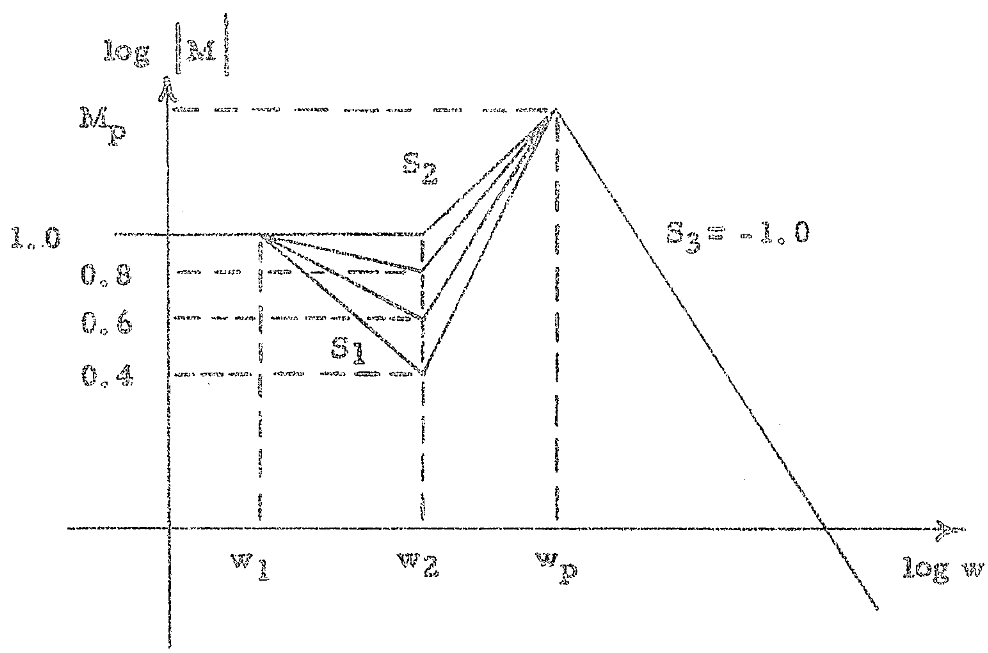



Figure 5.1 Basic M curve variation for type "B" curves

This type of M curve is characterized by the existence of a predominant simple pole in addition to a pair of predominant complex poles of the closed loop frequency response function. That is, the S plane may appear similar to Figure 5.2 or Figure 5.3.

An S plane configuration like that shown in Figure 5.2 infers that there is a long tail in the transient response, because the simple pole closest to the imaginary axis of the S plane gives the longest settling time (CH-C).

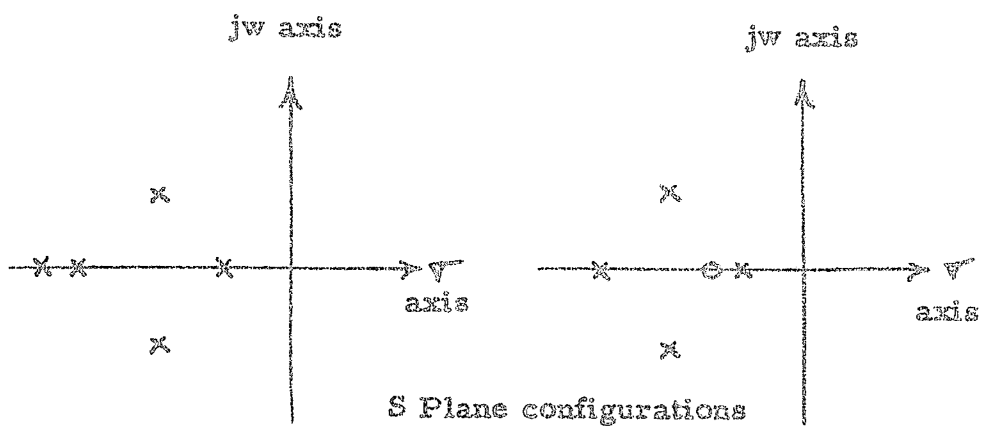



Figure 5.2 Figure 5.3

M curves which are characterized by the shape shown in Figure 5.1, then, are not as stable as the other forms investigated. Consequently, the form of the step response deviates from that shown in Figure 2.2. The initial peak in the response may not be the highest peak because of the existence of the long tail produced by the predominant critical points on the $\sigma$ axis. This is more applicable to the cases in which the M peak is low and the initial dip in the M curve is more pronounced. It must be remembered, however, that this effect occurs in the type "B" curves and that the peak overshoot and peak time to which reference is made in the results pertain to the first transient peak only.

In this investigation the variables are stipulated as:

(a) $M_p$

(b) $M_2$

(c) $w_1/w_2$

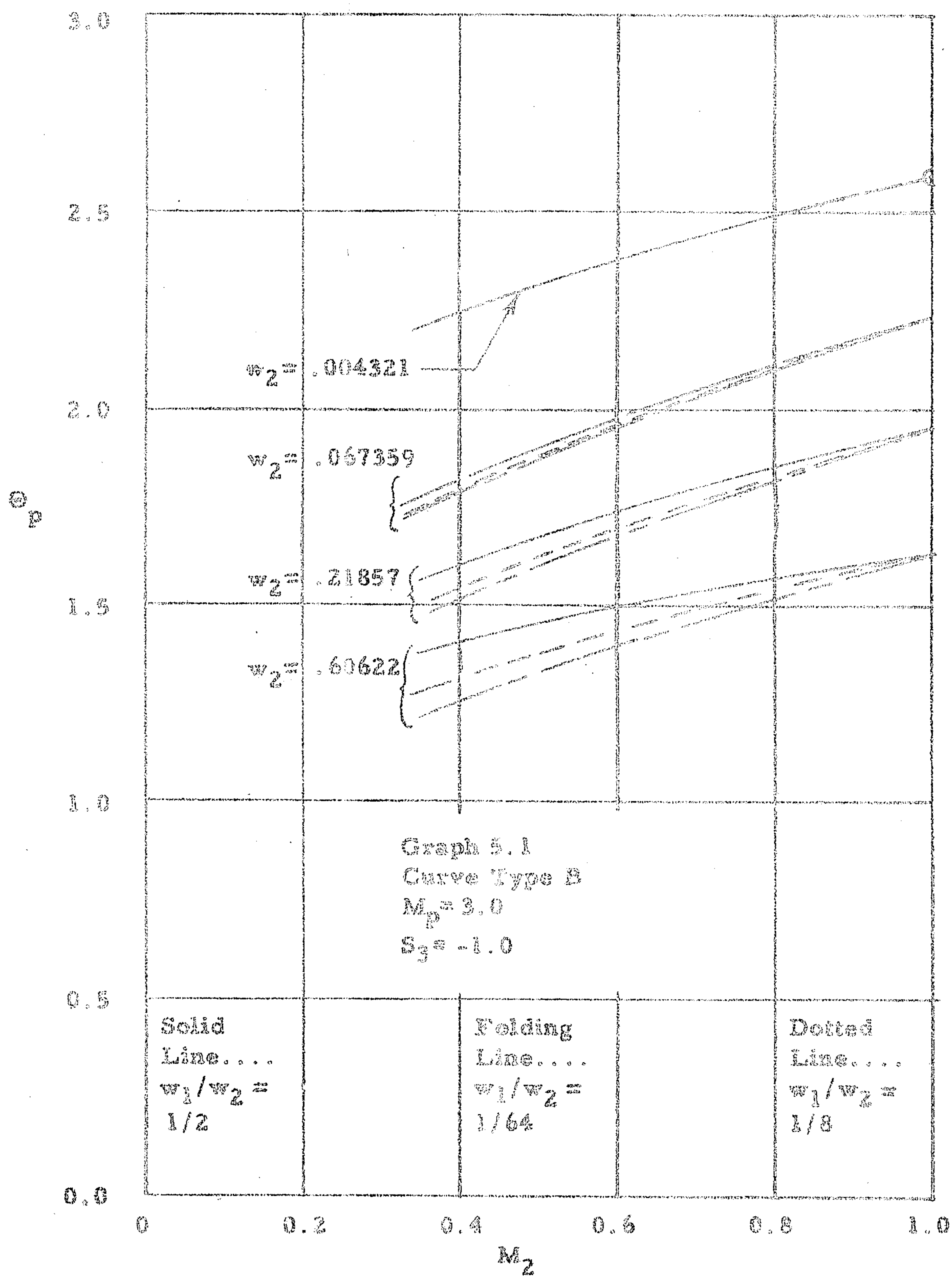
(d) $w_2$

The slope of the final M curve segment is at all times equal to -1.0. The results, however, are intended only as a guide for corrections to curves of type "A", and not as a complete reference of all possible variations.

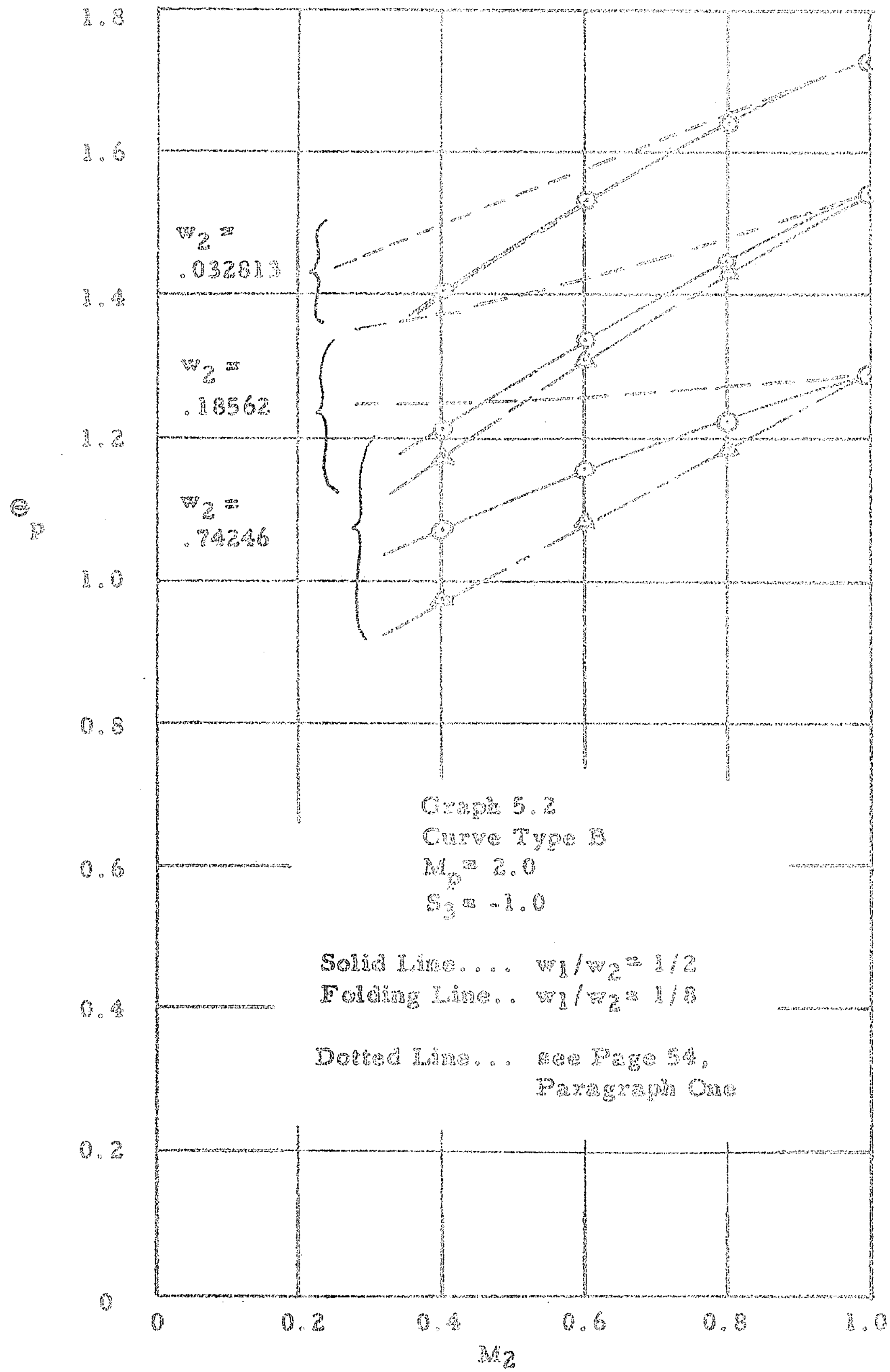
## 5.2 OVERSHOOT

A variation of the type described in section 5.1 produces a decrease of overshoot with a corresponding decrease in $M_2$. This rate of decrease varies slightly in accordance with changes in $w_2$, as shown in graph 5.1 and graph 5.2. It varies more radically, however, with changes in the ratio $w_1/w_2$. As seen in these graphs, the rate of decrease of overshoot is larger for M curves with smaller $w_1/w_2$ ratios. These findings would seem to indicate that those areas which are below the unity M curve line constitute a controlling force on the decrease of overshoot.

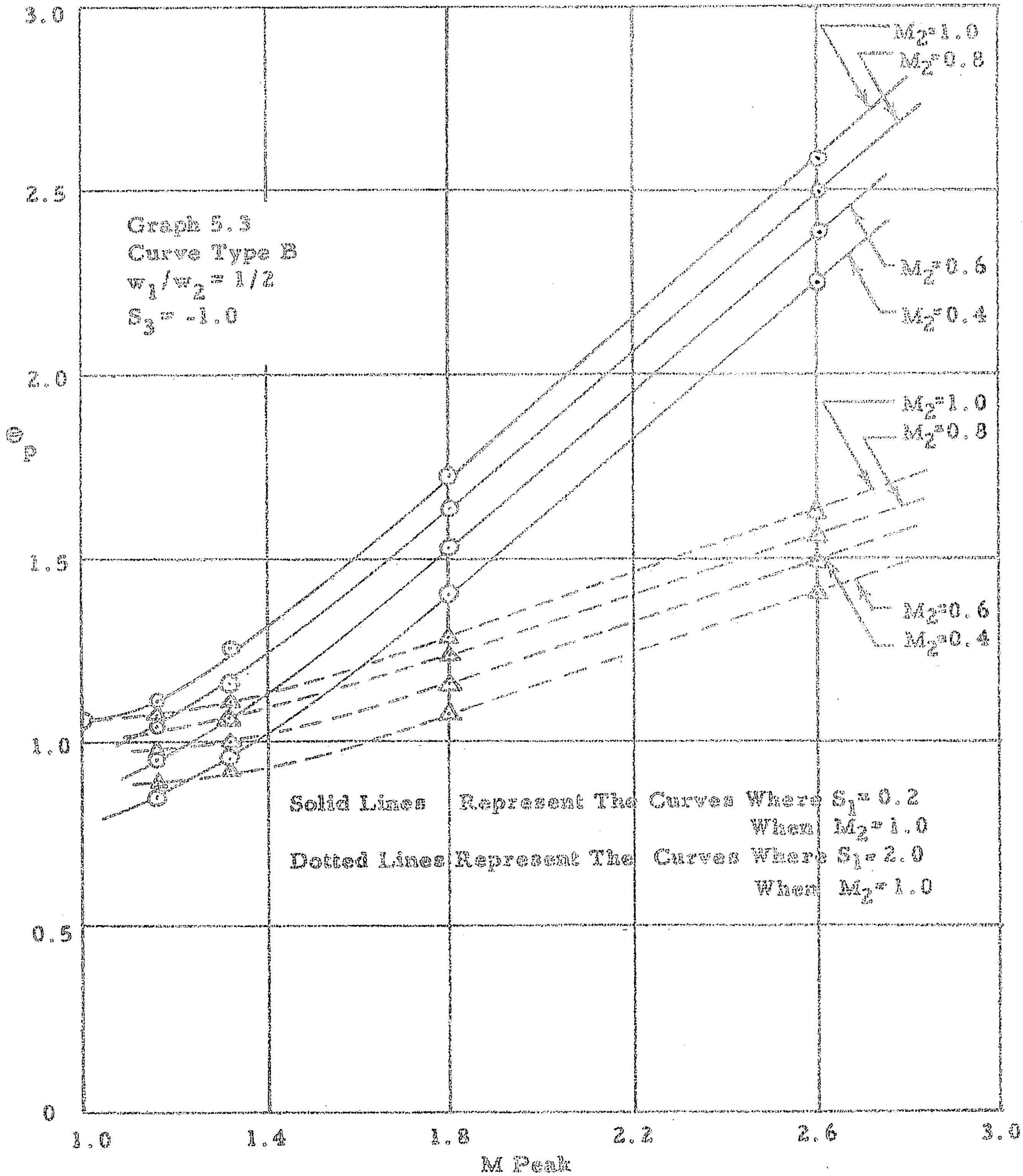
A significant relation shown by graph 5.3 is that the overshoot is decreased by a constant amount at all M peak values by an initial




Graph 5.1 Overshoot variation for curves with an initial dip




Graph 5.2 Overshoot variation for curves with an initial dip




Graph 5.3 Overshoot variation for Type B curves showing the effect of lowering the initial dip

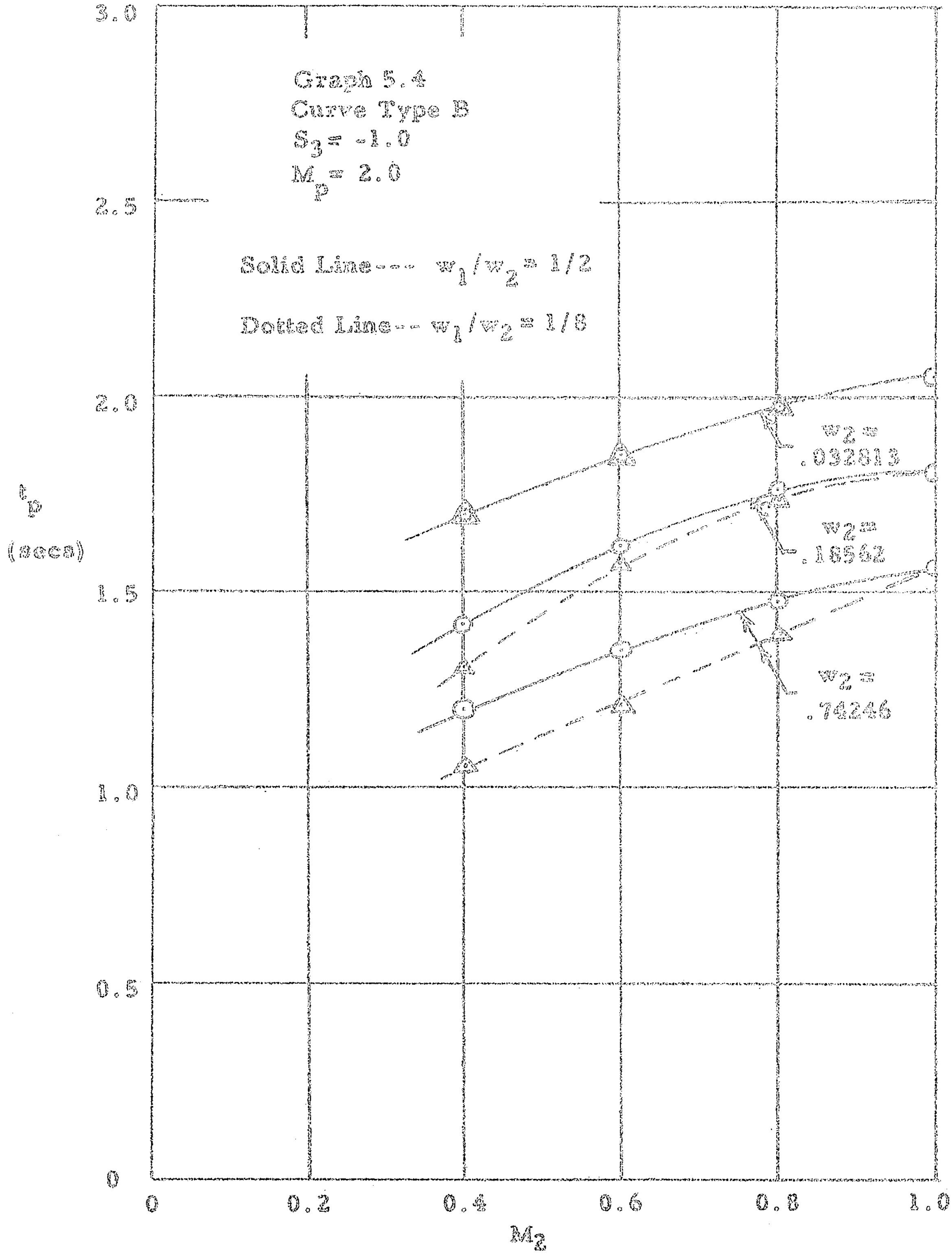
dip of a given $M_2$ value. This seems to maintain validity regardless of the value of $w_2$.

It is interesting to observe what portion of the decrease of overshoot with $M_2$ changes is accounted for by the loss of area above unity on the M curve. With reference to graph 5.2, the short dotted lines represent the amount of overshoot decrease caused solely by the decrease of area above unity on the M curve. The relative decreases of each term indicate that the loss of area above and below the unity M line is a predominant factor in determining the amount of overshoot decrease. However, there seems to be no obvious relationship here that could be stated quantitatively.

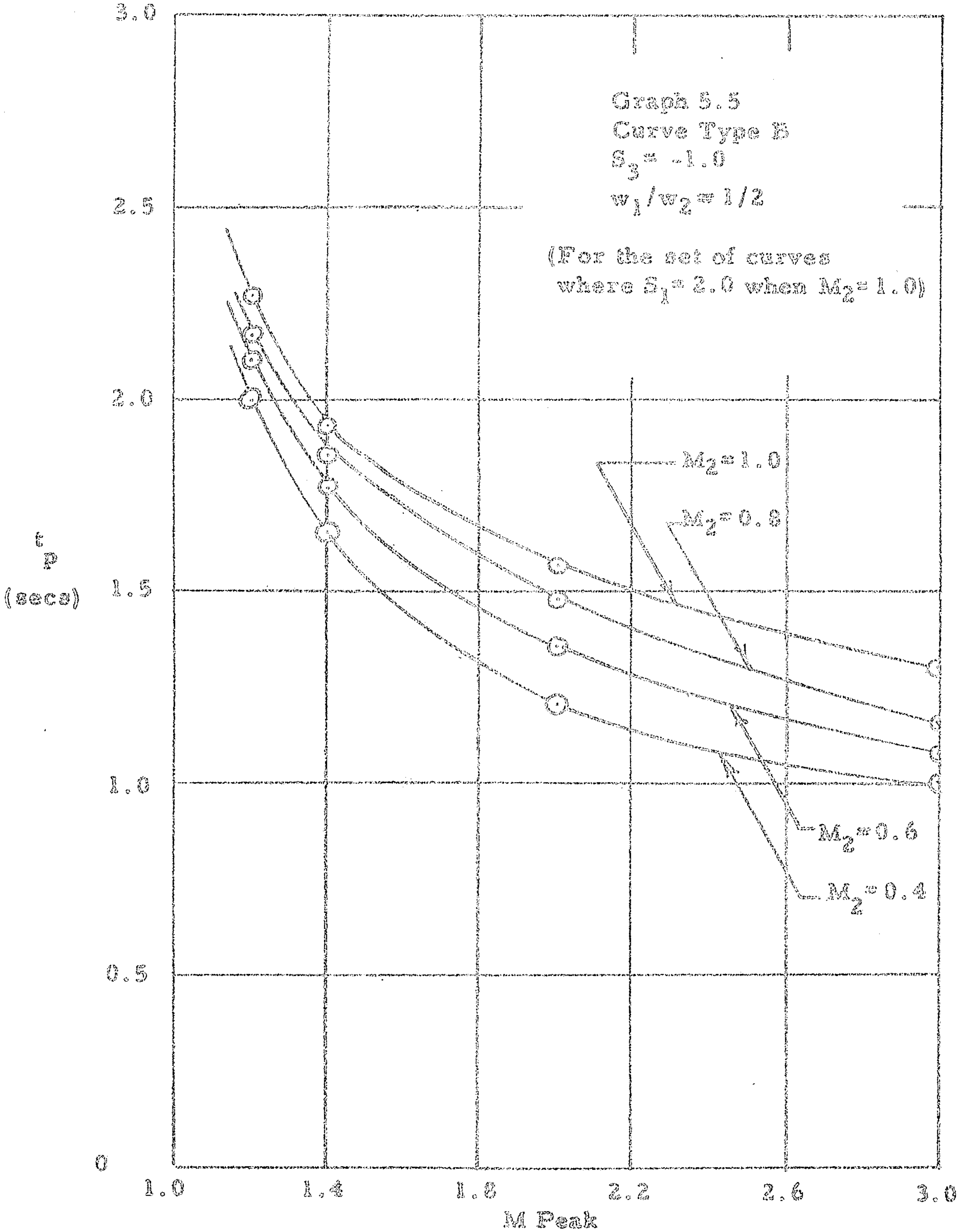
## 5.3 PEAK TIME

A graphical picture of the variation of peak time with ~~lower~~ $M_2$ ~~curve~~ values at the second finite break point for several $w_2$ and $w_1/w_2$ is given in graph 5.4. The most prominant aspect of these series of curves is that the trends are very similar to those of the overshoot curves. Indeed, it can be shown that the equations previously given (equations 4.5 to 4.10) for finding the peak time from overshoot for curve type "A" are equally valid for curves of type "B" over the range considered. However, since time limited the investigation of these types of curves for the final slope values other than -1.0, no definite statement can be made as to whether or not this relationship remains valid for the other $S_2$ values. It does, however, seem likely that it would.

A sample of the variation trends in the peak time over the range of M peaks is shown in graph 5.5.




Graph 5.4 Peak time variation for Type "B" curves with various initial dips




Graph 5.5 Peak time variation over a range of M peaks for Type B curves

In summary, then, the variation of peak time is to

(a) decrease for decrease in $M_2$,

(b) decrease for decrease of $w_1/w_2$ ratio,

(c) decrease for increase in M peak,

(d) decrease for increase in $w_2$.

It is assumed in each case that the variable mentioned is the only one which is not kept constant.

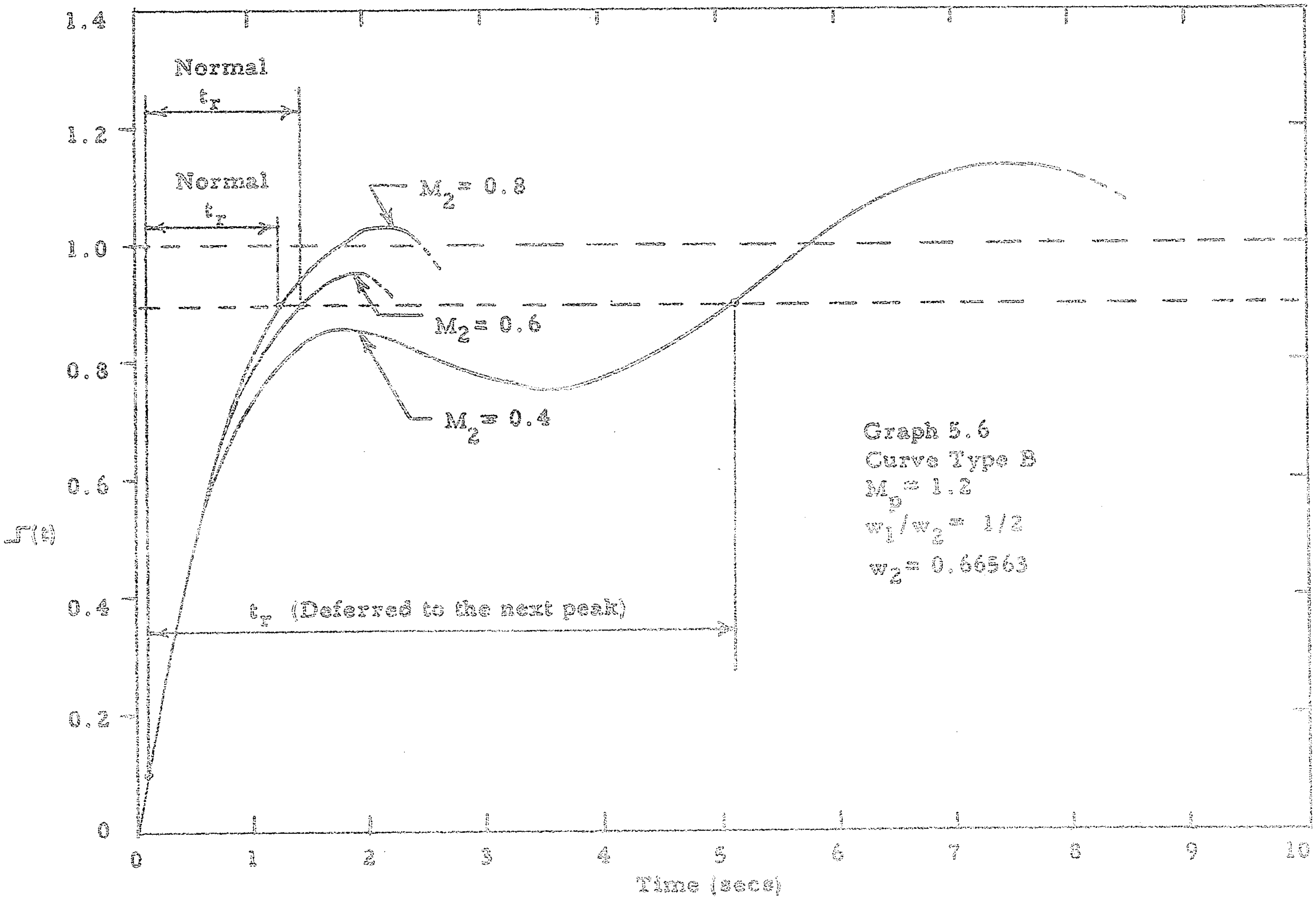
## 5.4 RISE TIME

It is rather difficult to specify any concrete basis as an aid in the estimation of rise time for a type "B" curve. This is due to the fact that, for M curves with a large initial dip or those with small M peaks, the step response curve takes on another form, as described at the beginning of this section. This type of response is shown in graph 5.6, which is plotted for $M_p$ equal to 1.2 with a $w_1/w_2$ ratio of 1/2 and $w_2$ equal to .66563. It is obvious that there will be a point at which the first peak becomes less than 0.9 and, consequently, the rise time will then be deferred to the following peaks until such time as a succeeding peak reaches the 90% level. In this investigation, then, the 10% to 90% rise time is not as satisfactory a definition of rise time as the other well known meanings.

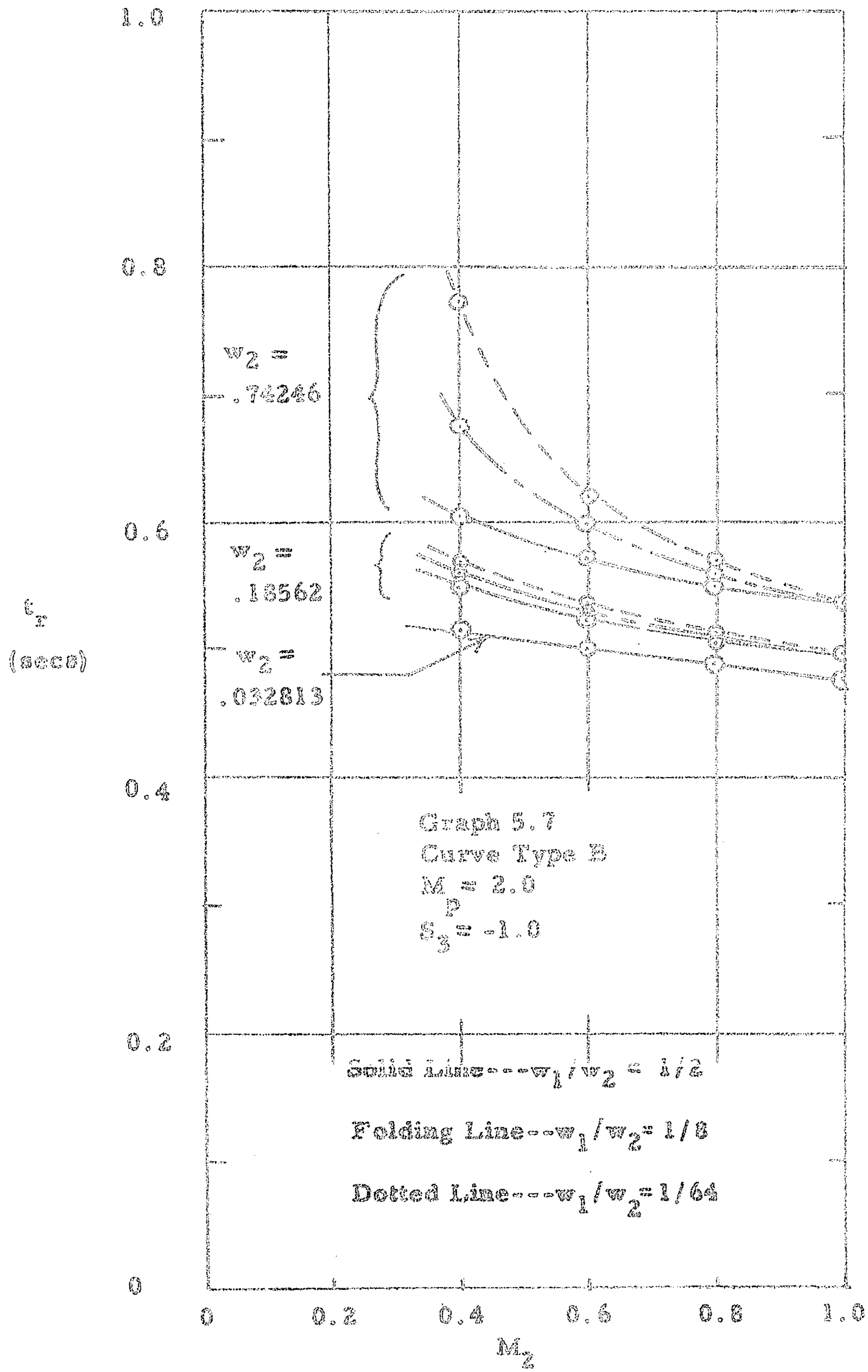
The trends in the data are shown in graphs 5.7 and 5.8. It is seen that rise time

(a) increases as $M_2$ decreases,

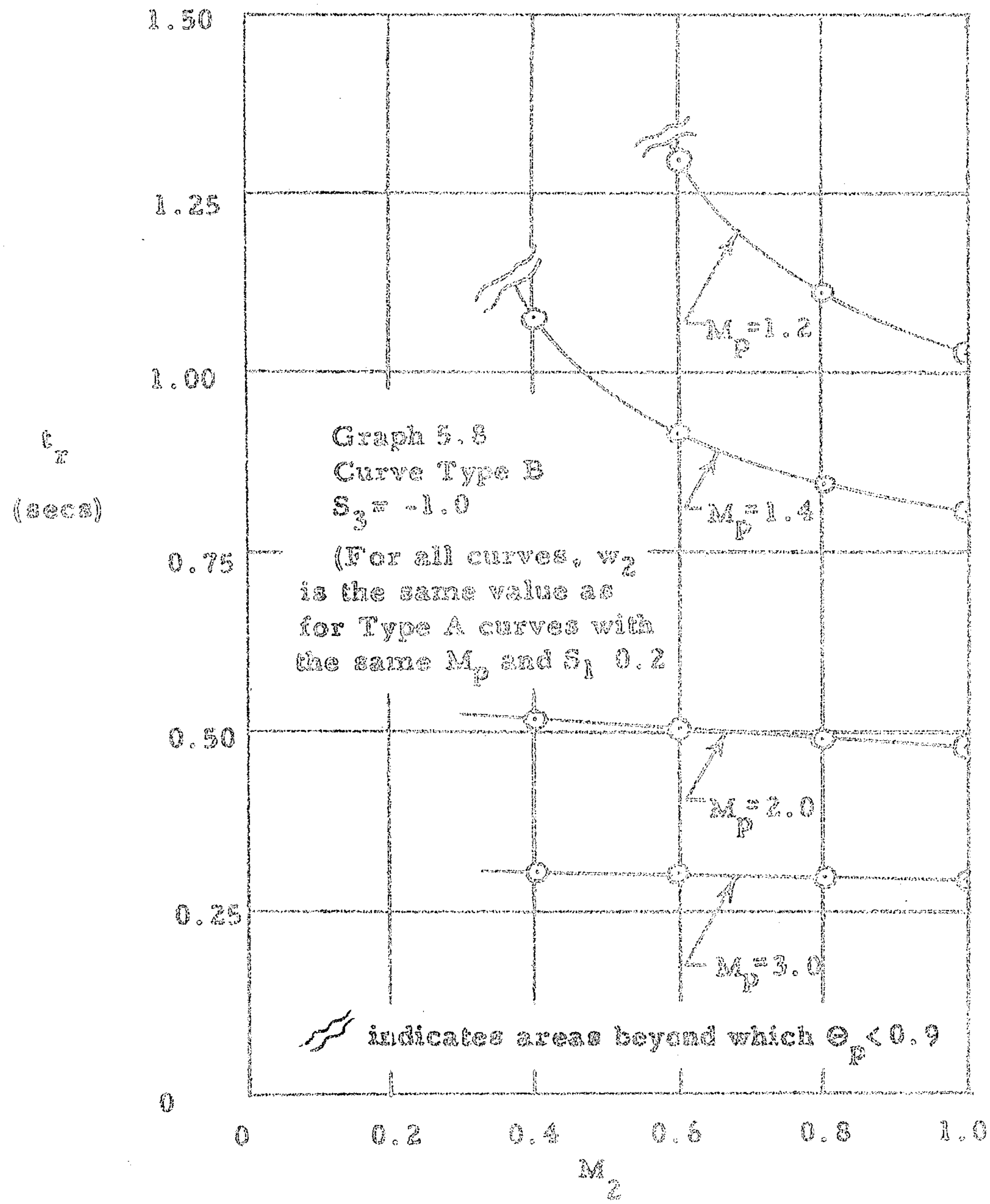
(b) increases as $M_p$ decreases,




Graph 5.6 Example of the deferred rise time for the Type B curves




Graph 5.7 Rise time variation at different $M_2$ values




Graph 5.8 Effect of changing the M peak on the rise time for Type B curves at various $M_2$ values

(c) increases as $w_2$ increases,

(d) increases as $w_1/w_2$ decreases.

Any further discussion regarding this matter seems to be unwarranted, although further investigation might be made with another rise time definition.

Type "C" curves will now be investigated in a similar manner in Chapter 6.

# CHAPTER 6

# PRESENTATION AND DISCUSSION OF RESULTS FOR CURVE TYPE "C"

## 6.1 FLAT PEAKED M CURVES

With regard to a realistic system, no M curve actually has an M peak with a first derivative discontinuity at the resonant frequency. Instead, all such curves have somewhat rounded peaks and many more systems have a flat response for a range of frequencies about $w_p$. An important consideration for the future utility of curves which are presented in Chapter 4, then, is that of the observation of the effect on the step response of a flattening of the resonant peaks and, further, that suitable criteria be stated whereby the applicability of the conversion curves may be broadened in order to take into account such cases.

Similar to the type of investigation carried out in Chapter 5, this treatment will not be exhaustive. However, it is hoped that the few examples offered by way of illustration in this chapter are sufficient to enable a reliable approximation to be made of the "flattening effect".

Step response parameter variations to changes in M curves in the manner of Figure 6.1 are to be examined. That is, the case of the flattened M peak will be accounted for, first by approximating

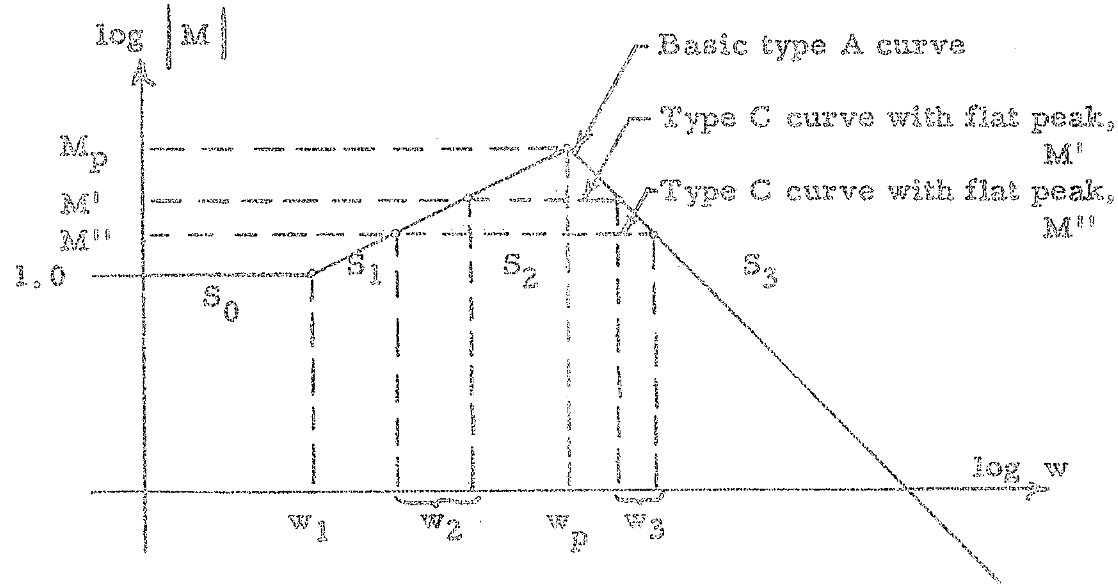



Figure 6.1 M curve variation in type "C" curves

the M curve with a type "A" curve and then by approximating the flattened portion by means of a zero slope segment which intersects the slopes previously called $S_1$ and $S_2$.

It is immediately apparent that the effect on the step response of using the highest $M_2$ values (where $M_2 = M_3 = M_p$) and lowest $M_2$ values possible (where $M_2 = M_3 = 1.0$) can be found from previous results of the type "A" curves discussed in Chapter 4 (graphs 4.8 to 4.17). It remains only to study the effects in the intermediate stages where

$$1.0 < M_2 = M_3 < M_p \tag{6.1}$$

A sample of four basic curves is taken from the many cases investigated in Chapter 4. These are chosen so that a wide variation of the basic parameters is included. The curve parameters are given in Table 6.1.

| CURVE | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $M_p$ | 3.0 | 3.0 | 1.7 | 1.4 |
| $S_1$ | 1.0 | 0.4 | 1.0 | 0.4 |
| $S_2$ | -1.0 | -3.0 | -1.0 | -1.0 |

Table 6.1 Sample curve parameters

Three intermediate $M_2$ points are chosen for each curve and the corresponding step response is calculated.

## 6.2 INTERMEDIATE EFFECTS AND APPROXIMATION RULES

The effect of intermediate $M_2$ values on the step response parameters is very similar for all four curves, as might be expected. A typical example of the intermediate variation is that for curve 2 which is shown in graph 6.1 and 6.2. The overshoot rises linearly with M peak up to a point and then saturates as $M_2$ approaches $M_p$. To a good approximation, the main variation of rise time and peak time takes place when

$$M_2 < \left[\frac{M_p + 2}{3}\right] \tag{6.2}$$

A general rule for the approximation of intermediate rise time and peak time values is

(a) for $\dfrac{(M_p + 2)}{3} \leq M_2 < M_p$

$$\text{use} \quad t_{ri} = t_{rp} \tag{6.3}$$

$$\text{and} \quad t_{pi} = t_{pp} \tag{6.4}$$

(b) for $1.0 < M_2 < \dfrac{(M_p + 2)}{3}$

$$\text{use} \quad t_{pi} = \left[\frac{3(M_{pi} - 1.0)(t_{pp} - t_{po})}{(M_{pp} - 1)}\right] + t_{po} \tag{6.5}$$

$$\text{and} \quad t_{ri} = \left[\frac{3(M_{pi} - 1.0)(t_{rp} - t_{ro})}{(M_{pp} - 1)}\right] + t_{ro} \tag{6.6}$$

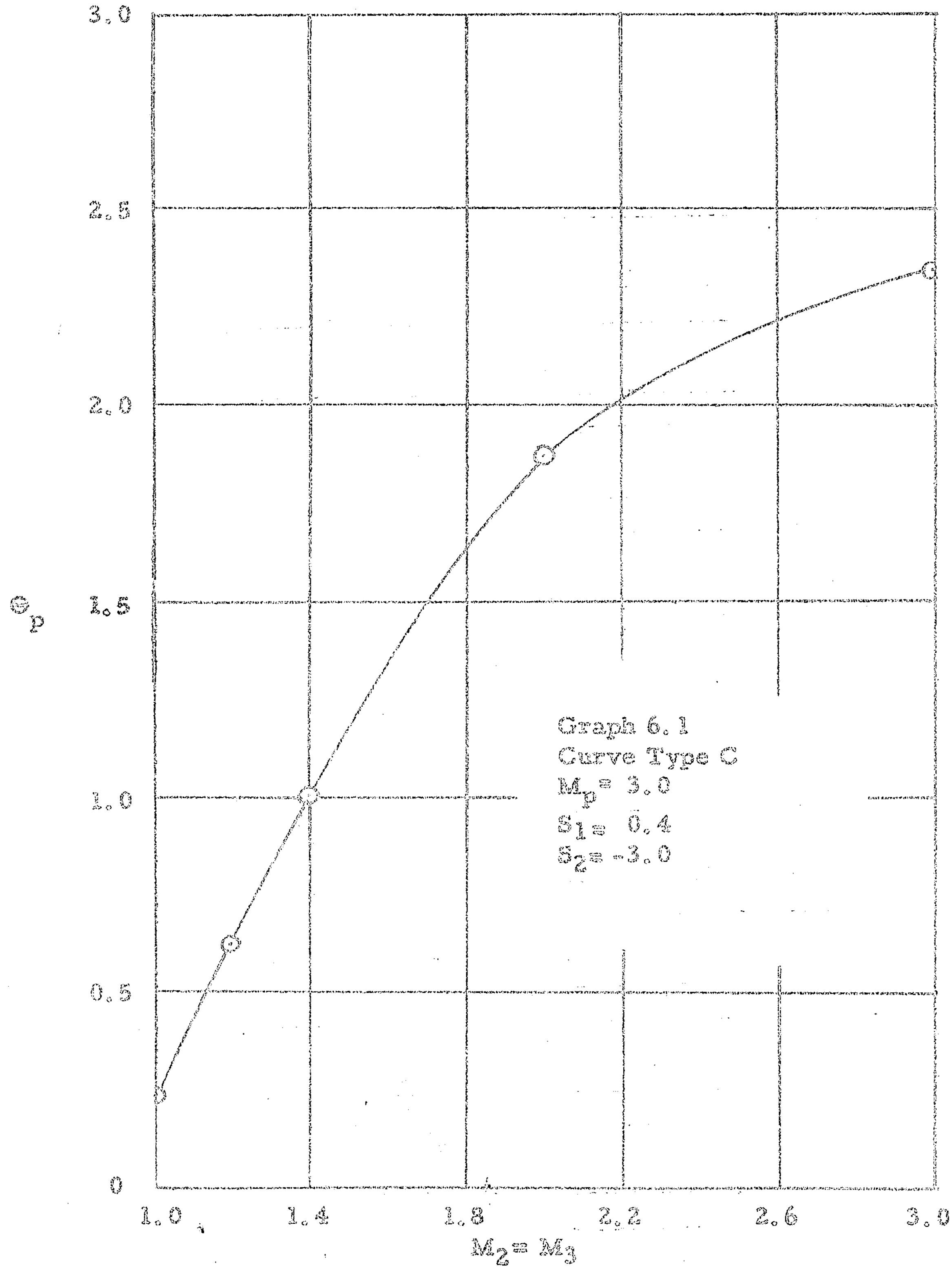
In the second subscript

i refers to intermediate parameters

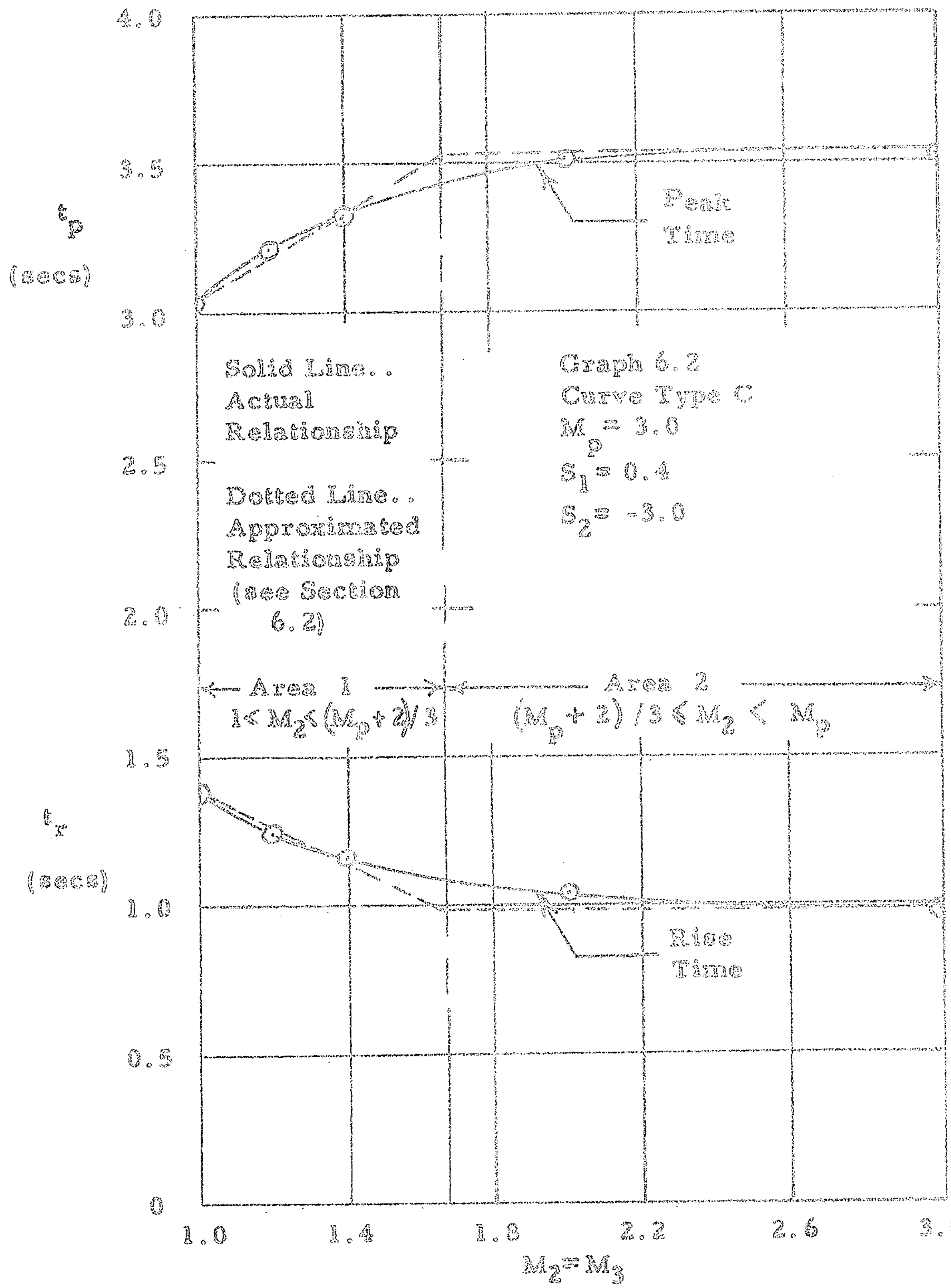
p refers to value of parameter when $M_2 = M_p$

o refers to value of parameter when $M_2 = 1.0$

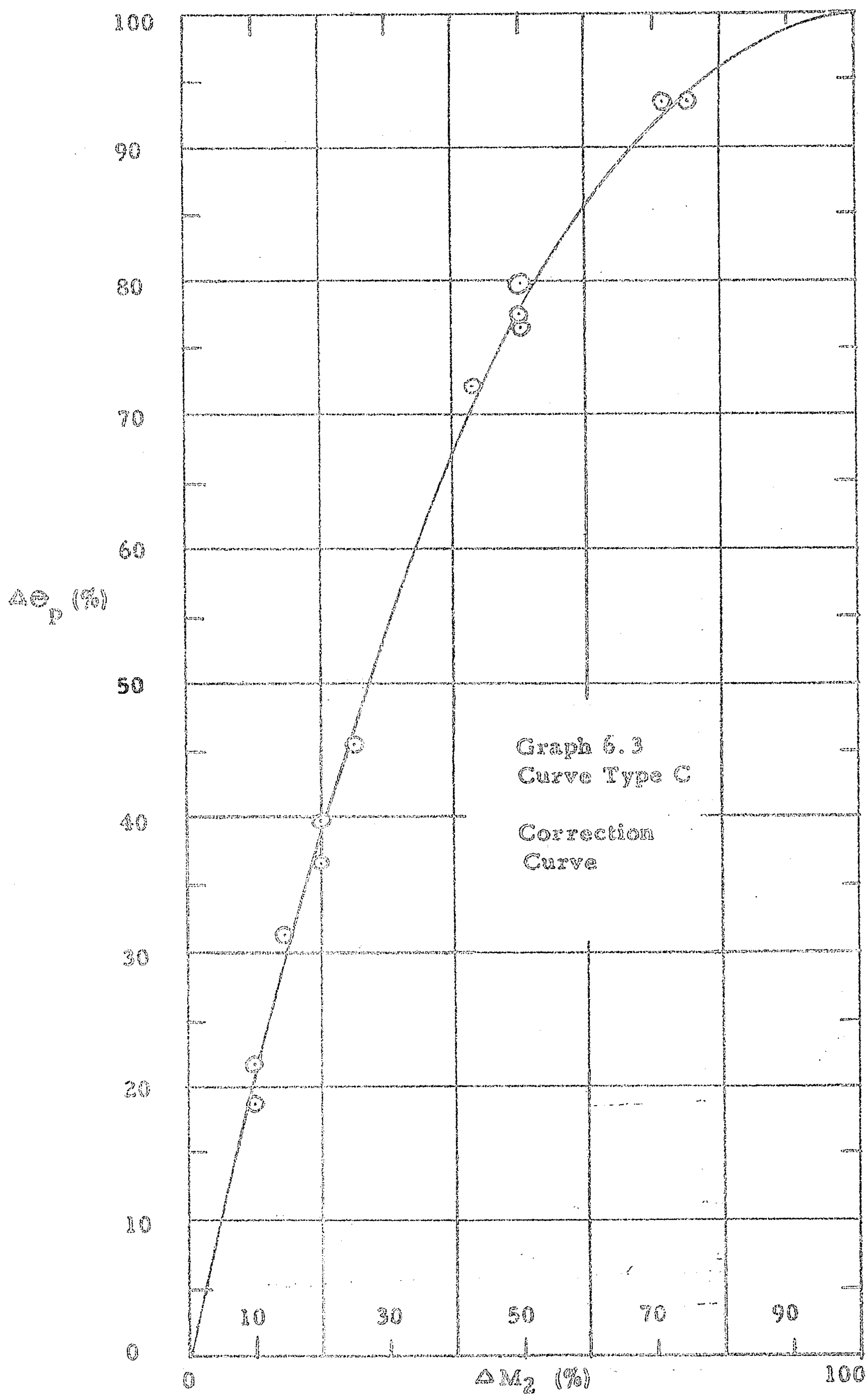
A convenient method for estimating intermediate overshoots is shown on graph 6.3, where the ordinate and abscissa are expressed in percent of the total variation in overshoot and $M_2$ respectively. All intermediate points which were investigated are shown and an approximate curve is drawn through them. Such a graph appears practical because all curves investigated have very similar patterns of variation.




Graph 6.1 Typical intermediate variation of overshoot for Type "C" curves




Graph 6.2 Typical intermediate variation of rise time and peak time for Type "C" curves




Graph 6.3 Correction curve used to estimate $\Theta_{pi}$ for Type "C" curves

Another aspect of the variation of overshoot with $M_2$ is evident when the overshoot is plotted against the area of the M curve above unity for various $M_2$ values. Graph 6.4 shows the results for curves 1 and 2 and graph 6.5 shows the results for curves 3 and 4. The aspect of the relationship which is immediately obvious is that the relationships are nearly linear for all curves when $M_2 < 1.5$. Also, for curves with $M_p < 1.5$, the relationship over the entire variation is a linear one. This, then, is a good approximation method for low M peak curves.

Quantitatively, it may be stated, for low M peak curves, that

$$\Theta_{pi} = \Theta_{po} + \left[ \frac{(\Theta_{pp} - \Theta_{po})\, A_i}{A_p} \right] \qquad (6.7)$$

where $A_i$ = area above unity for the intermediate curve,

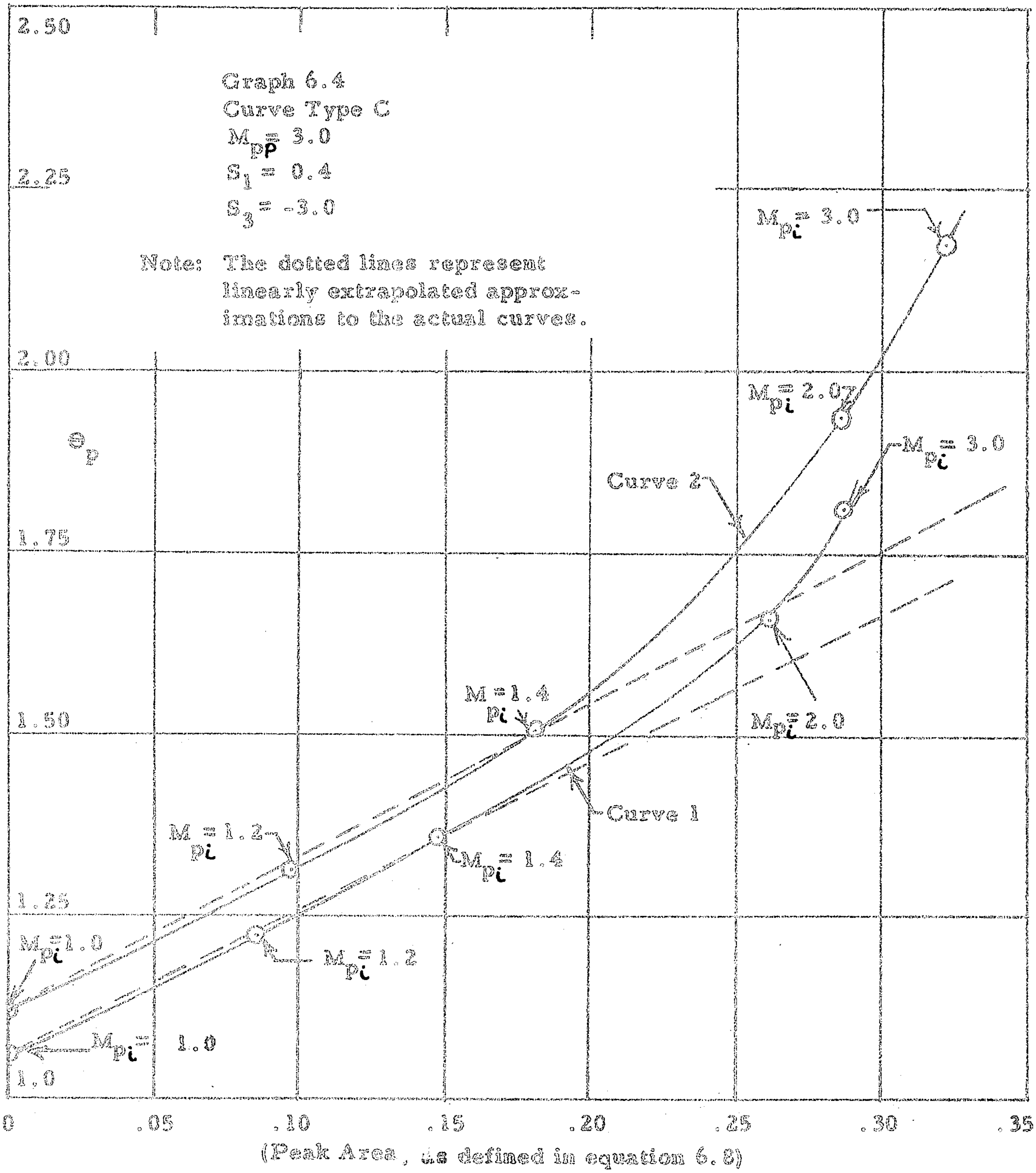
$A_p$ = area above unity for the M curve where $M_2 = M_p$.

Now, since

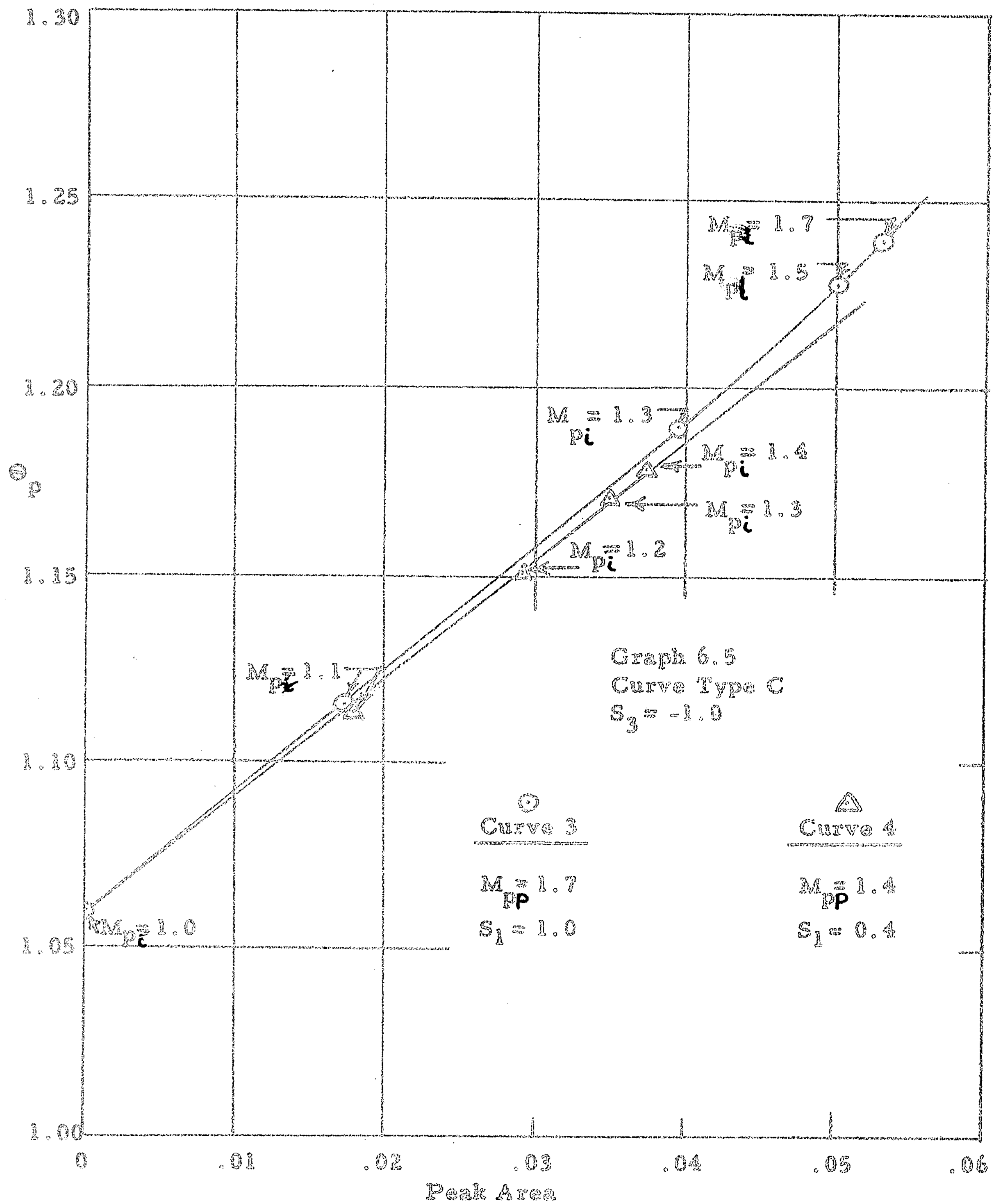
$$A = \frac{\ln M_p \ln \left( w_3 / w_2 \right)}{2} \qquad (6.8)$$

then

$$\Theta_{pi} = \Theta_{po} + (\Theta_{pp} - \Theta_{po})\left(1 - \left[ \frac{\ln\left(\frac{M_{pp}}{M_{2i}}\right) \ln\left(\frac{w_{3i}}{w_{2i}}\right)}{\ln (M_{pp}) \ln\left(\frac{w_{3p}}{w_{2p}}\right)} \right]\right) \qquad (6.9)$$




Graph 6.4 Relationship of overshoot to area of the M curve above unity for Type "C" curve $M_2$ variations




Graph 6.5 Relationship of overshoot to area of the M curve above unity for Type "C" curves with lower M peaks than those in Graph 6.4

However, due to the nature of the curve type "C" variation, the following equation is valid:

$$\ln\left[\frac{w_{3p}}{w_{2p}}\right] \Big/ \ln\left[\frac{w_{3i}}{w_{2i}}\right] = \frac{\ln\left(M_{pp}\right)}{\ln\left(\frac{M_{pp}}{M_{2i}}\right)} \tag{6.10}$$

Then, if equation 6.10 is substituted into equation 6.9, the result becomes

$$\Theta_{pi} = \Theta_{pp} - (\Theta_{pp} - \Theta_{po})\left[\frac{\ln\left(\frac{M_p}{M_2}\right)}{\ln M_p}\right]^2 \tag{6.11}$$

This equation approximates intermediate overshoots for M curves with resonant peaks up to approximately 1.5.

The foregoing has shown how to account for flatter resonance peaks than those which were dealt with in Chapter 4. However, only a relatively small sample was studied, and consequently, only further practical use will ultimately decide the reliability of the method.

# CHAPTER 7

# PRESENTATION AND DISCUSSION OF RESULTS FOR CURVE TYPE "D"

## 7.1 M CURVES WITH 3 BREAK POINTS

It is advantageous to consider M curves with 3 break points on the finite M plane in order to increase the versatility of the approximation method. In this chapter a set of 18 conversion charts and 9 tables are presented. These charts and tables present the variations of the three step response parameters to changes in the basic M curve parameters as follows:

| | | |
|---|---|---|
| (a) | M Peak | $1.0 \leqslant M_p \leqslant 3.0$ |
| (b) | $M_3$ | $0.3 \leqslant M_3 \leqslant 2.0$ |
| (c) | $S_1$ | $S_1 = 0.4,\ 1.0,$ and $2.0$ |
| (d) | $S_3$ | $S_3 = -1.0,\ -2.0,$ and $-4.$ |
| (e) | $w_3$ | $w_3 = 1.5,\ 3.0,$ and $6.0$ |

In addition, the inequality given in 7.1 must be valid.

$$M_3 \leqslant M_p \qquad (7.1)$$

That is, the highest M curve point must fall at $w_2 = 1.05$ radians/second on normalization and the third break point must be at a frequency higher than $w_p$. This stipulation, in effect, eliminates from consideration in this thesis, those curves with a break point, other than $w_1$, which occurs on the leading portion of the curve.

The 18 charts previously mentioned are concerned with peak overshoot and rise time variations. Tables are used to present the peak time data because it has been found that those M curves with

(a) M peaks near 1.0 or

(b) $S_2$ close to zero with $w_3/w_2 \gtrsim 3.0$

have a tendency to produce a step response similar to that described in Section 5.4 where the first peak is not necessarily the highest peak. In these cases, the peak time shows erratic variations caused by the movement of the highest step response point from one peak to another. Furthermore, this movement is, in some cases, critically dependent on the accuracy of the computer program which computes the real part curve. In many cases the same M curve which was run first on the IBM 650 computer, and then on the Bendix G-15D computer gave radically different results and these discrepancies were caused entirely by this phenomenon. Consequently, the data ~~is~~ are difficult to plot and interpolation becomes more complicated and unreliable. However, much of the data ~~is~~ are of considerable importance. It is in this interest that the data for peak time ~~is~~ are listed in tabular form.

In deciding upon which region a particular M curve falls, it is generally true that those points in the tables where the approximation is close to producing a Type A curve (for example, where $S_2 \doteq S_3$ and $|S_2| \gtrsim 1.0$) are the most reliable.

The rise time and peak overshoot data ~~is~~ are relatively unaffected by this phenomenon and the data ~~is~~ are therefore considerably more reliable.

The overshoot conversion charts appear on pages 77 to 85 inclusive. In these graphs it is important to note that

(a) the solid line represents $S_3 = -1.0$

(b) the dotted line represents $S_3 = -2.0$

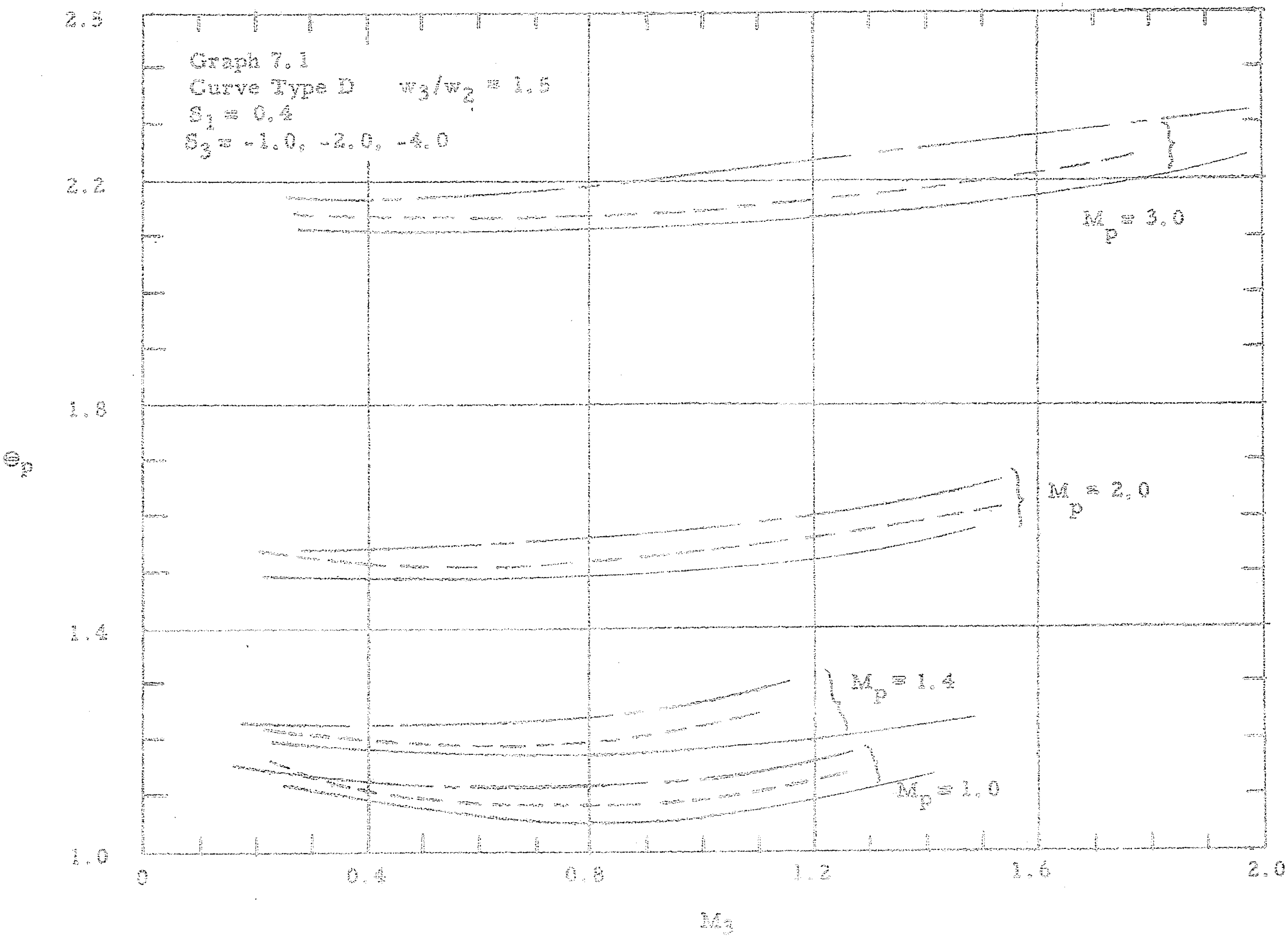
(c) the centre line represents $S_3 = -4.0$

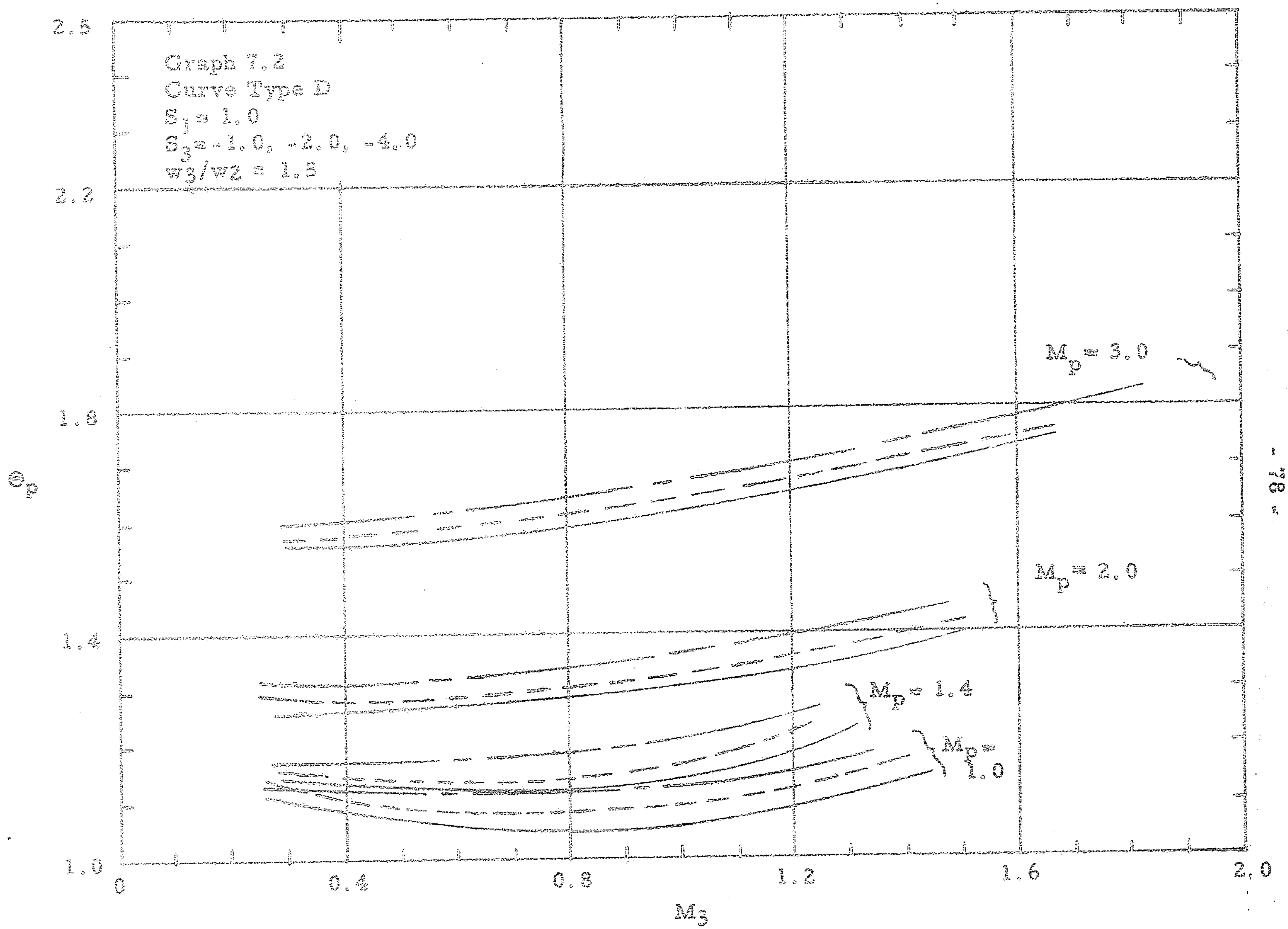
These graphs are followed by a set of 9 more graphs which are concerned with the rise time and are given on pages 86 to 94 inclusive.

PEAK OVERSHOOT CONVERSION CHARTS

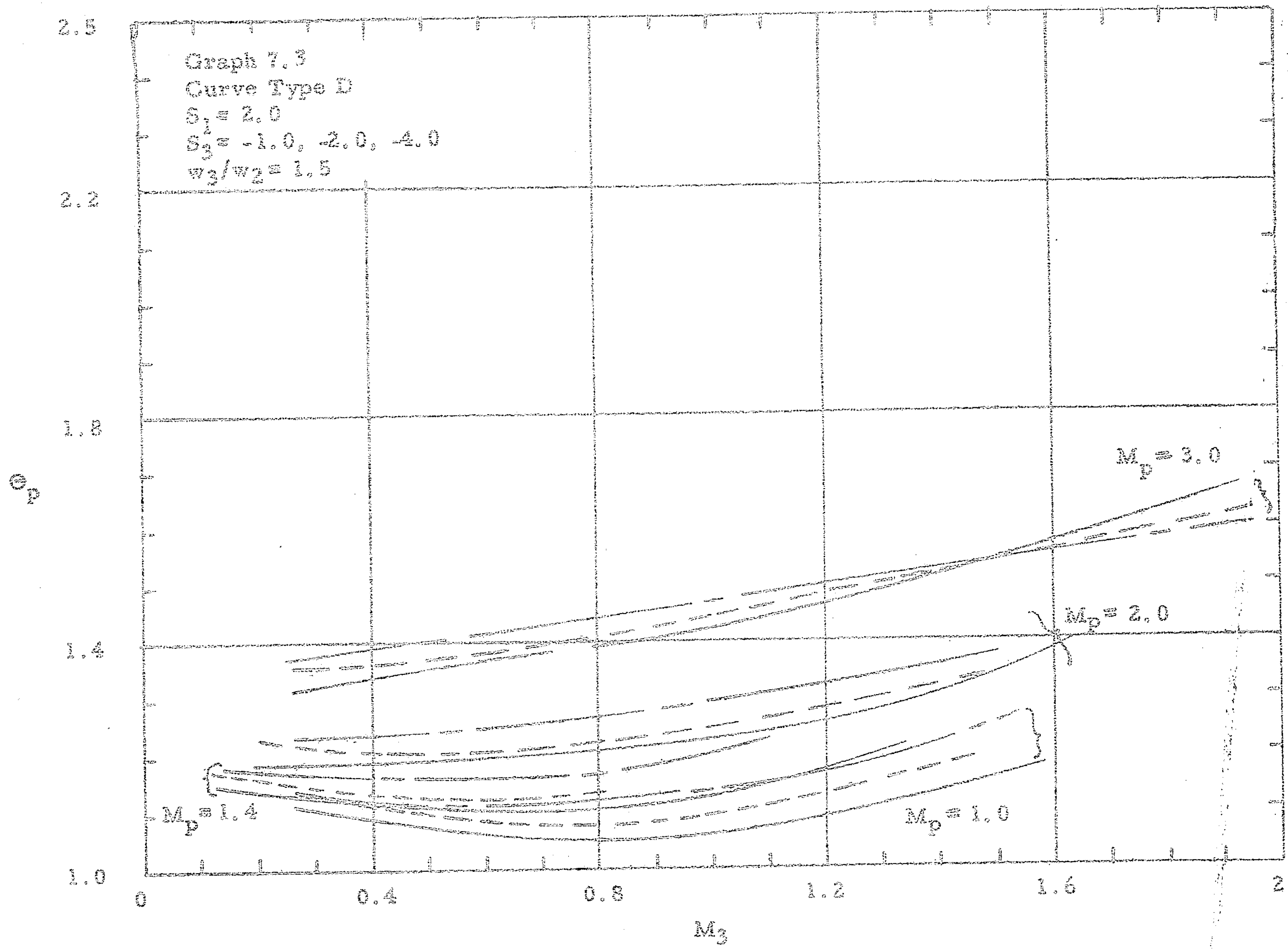


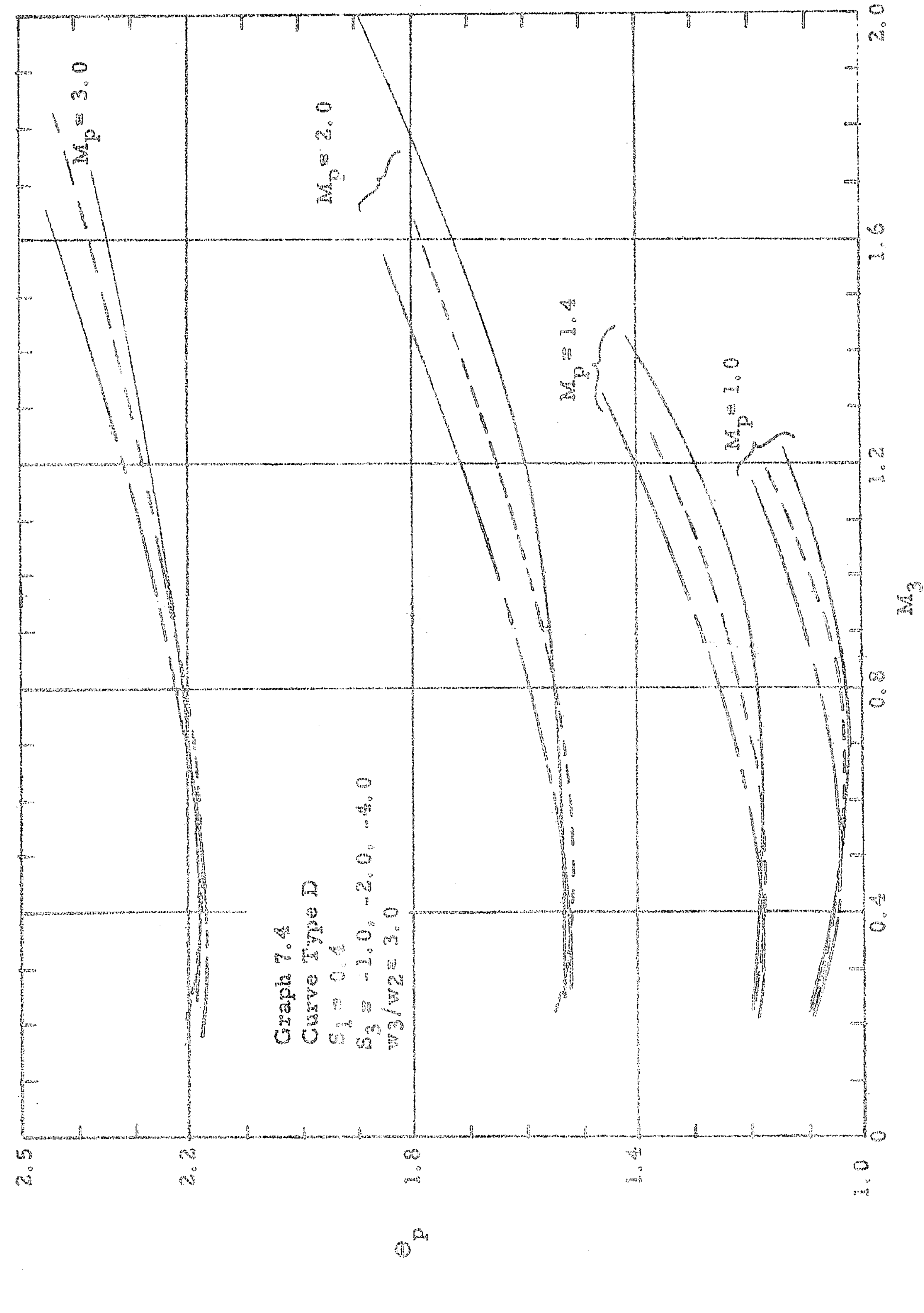
RISE TIME CONVERSION CHARTS

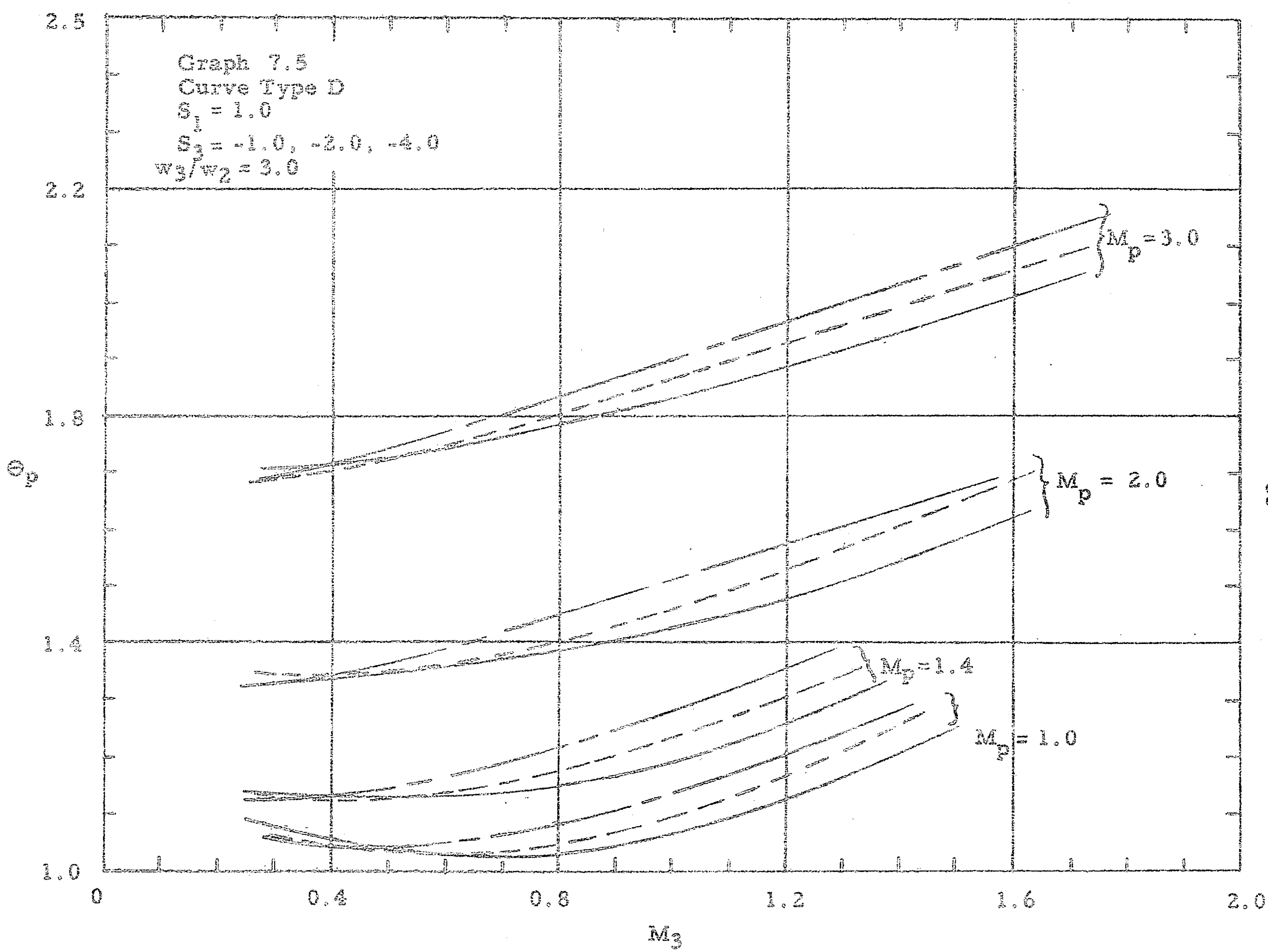

Graph 7.1
Curve Type D    $w_3/w_2 = 1.5$
$S_1 = 0.4$
$S_3 = -1.0, -2.0, -4.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
$\Theta_p$
2.5
2.2
1.8
1.4
1.0
0
0.4
0.8
1.2
1.6
2.0
$M_3$



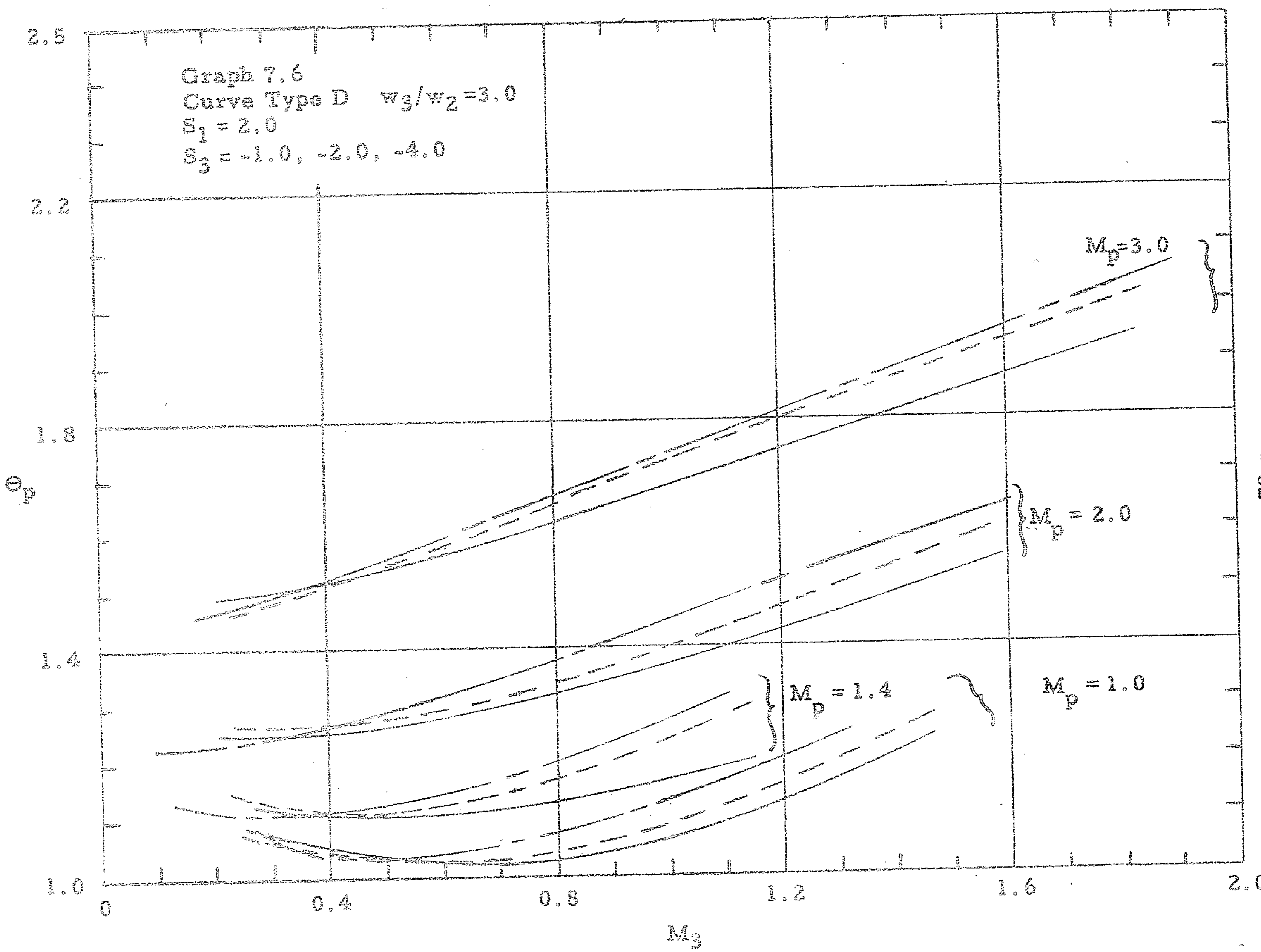
Graph 7.2
Curve Type D
$S_1 = 1.0$
$S_3 = -1.0, -2.0, -4.0$
$w_3/w_2 = 1.5$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
2.5
2.2
1.8
1.4
1.0
$\Theta_p$
0
0.4
0.8
1.2
1.6
2.0
$M_3$

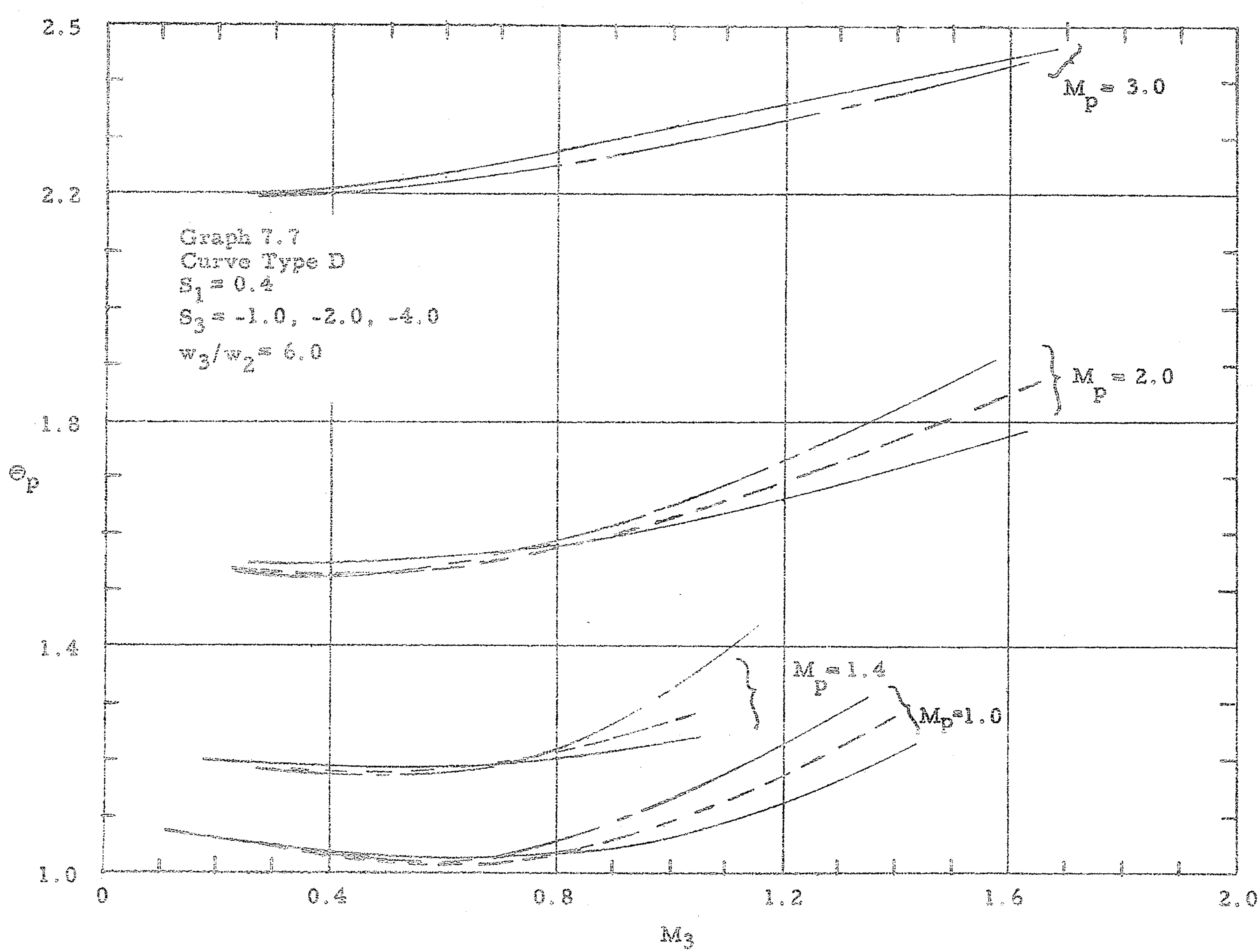

Graph 7.3
Curve Type D
$S_1 = 2.0$
$S_3 = -1.0, -2.0, -4.0$
$w_3/w_2 = 1.5$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.0$
$M_p = 1.4$
$\Theta_p$
$M_3$
2.5
2.2
1.8
1.4
1.0
0
0.4
0.8
1.2
1.6
2.0


Graph 7.4
Curve Type D
$S_1 = 0.4$
$S_3 = -1.0, -2.0, -4.0$
$w_3/w_2 = 3.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
$M_3$
$\theta_p$
0
0.4
0.8
1.2
1.6
2.0
1.0
1.4
1.8
2.2
2.5

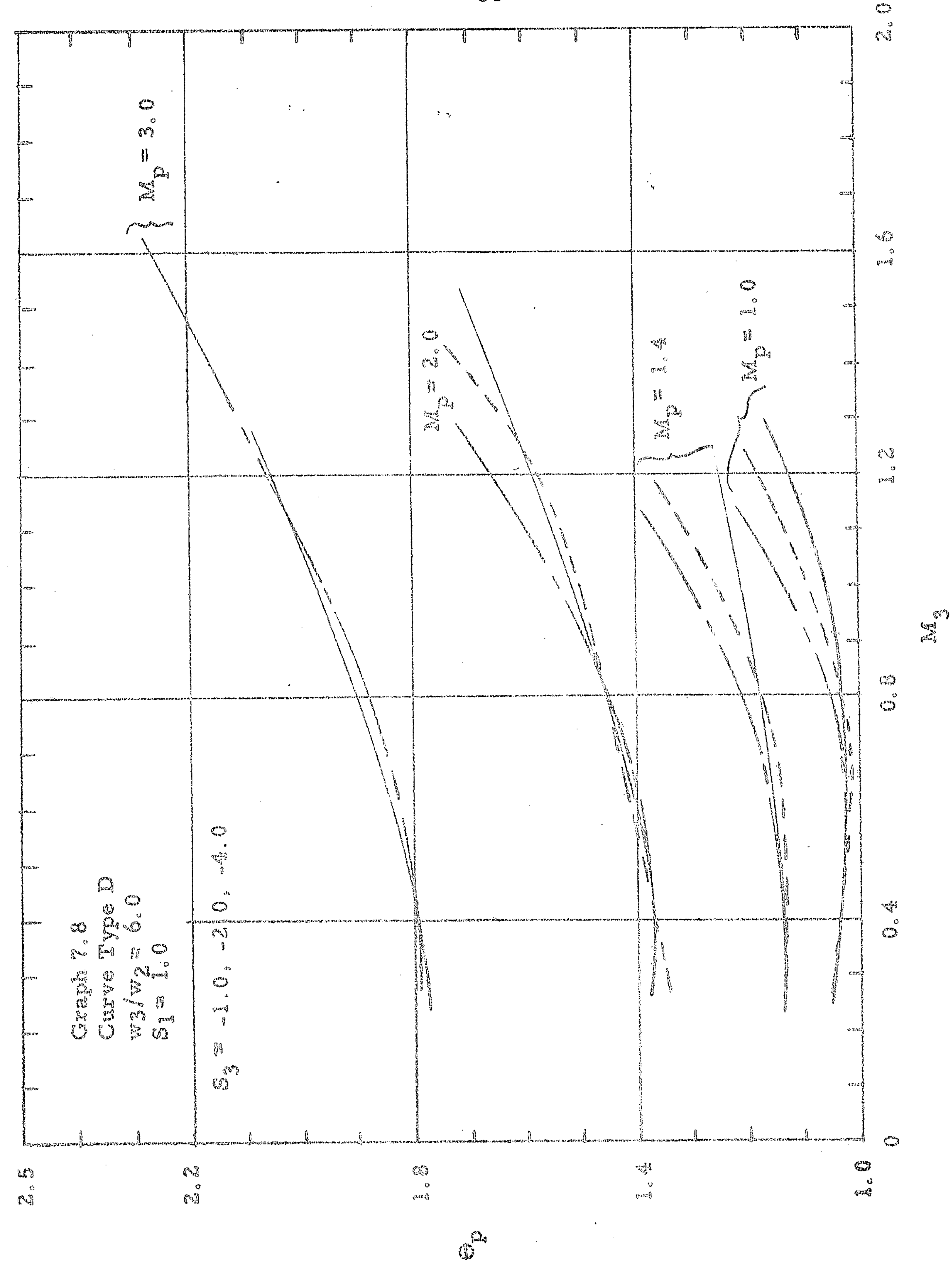


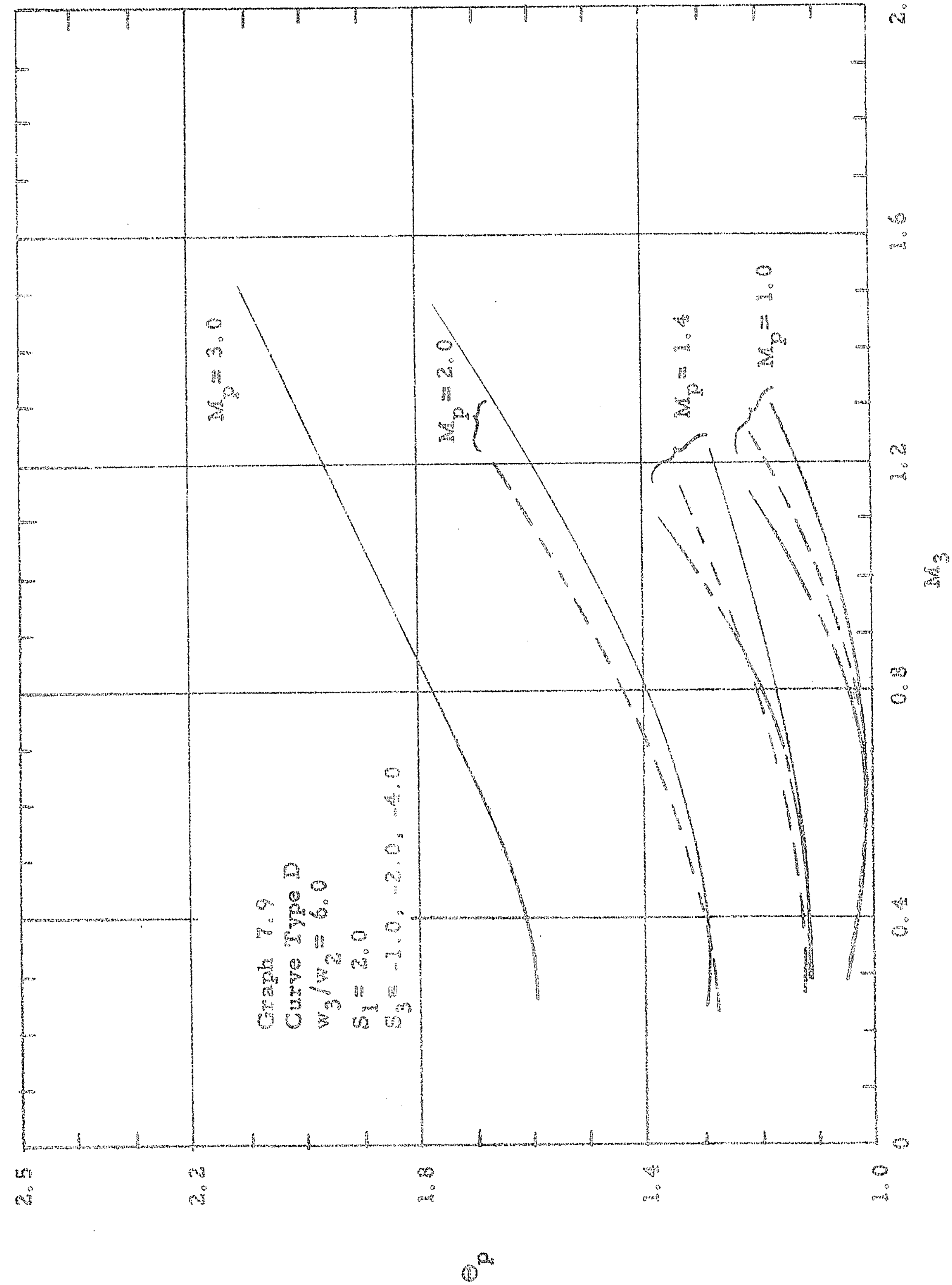
Graph 7.5
Curve Type D
$S_1 = 1.0$
$S_3 = -1.0, -2.0, -4.0$
$w_3/w_2 = 3.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
2.5
2.2
1.8
1.4
1.0
$\Theta_p$
0
0.4
0.8
1.2
1.6
2.0
$M_3$

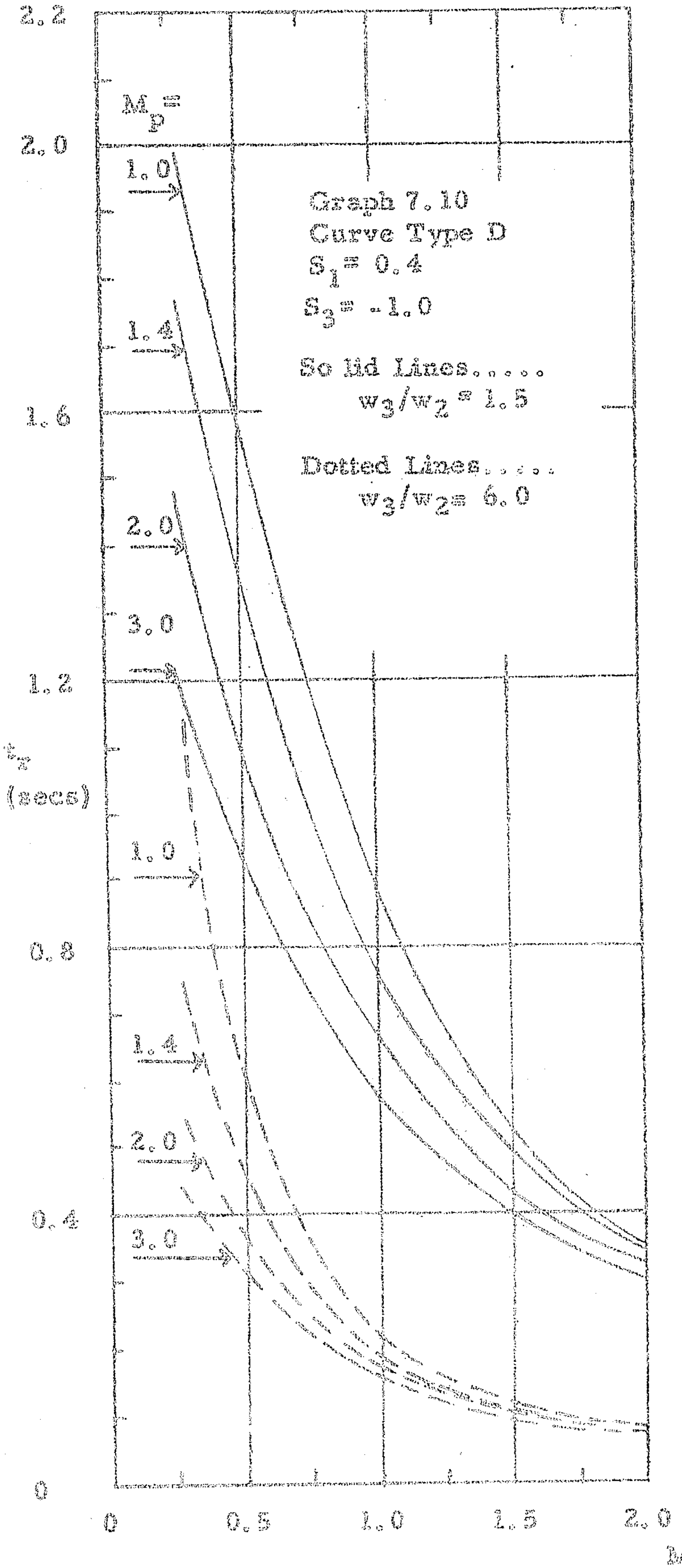

Graph 7.6
Curve Type D $w_3/w_2 = 3.0$
$S_1 = 2.0$
$S_3 = -1.0, -2.0, -4.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
$\Theta_p$
2.5
2.2
1.8
1.4
1.0
0
0.4
0.8
1.2
1.6
2.0
$M_3$



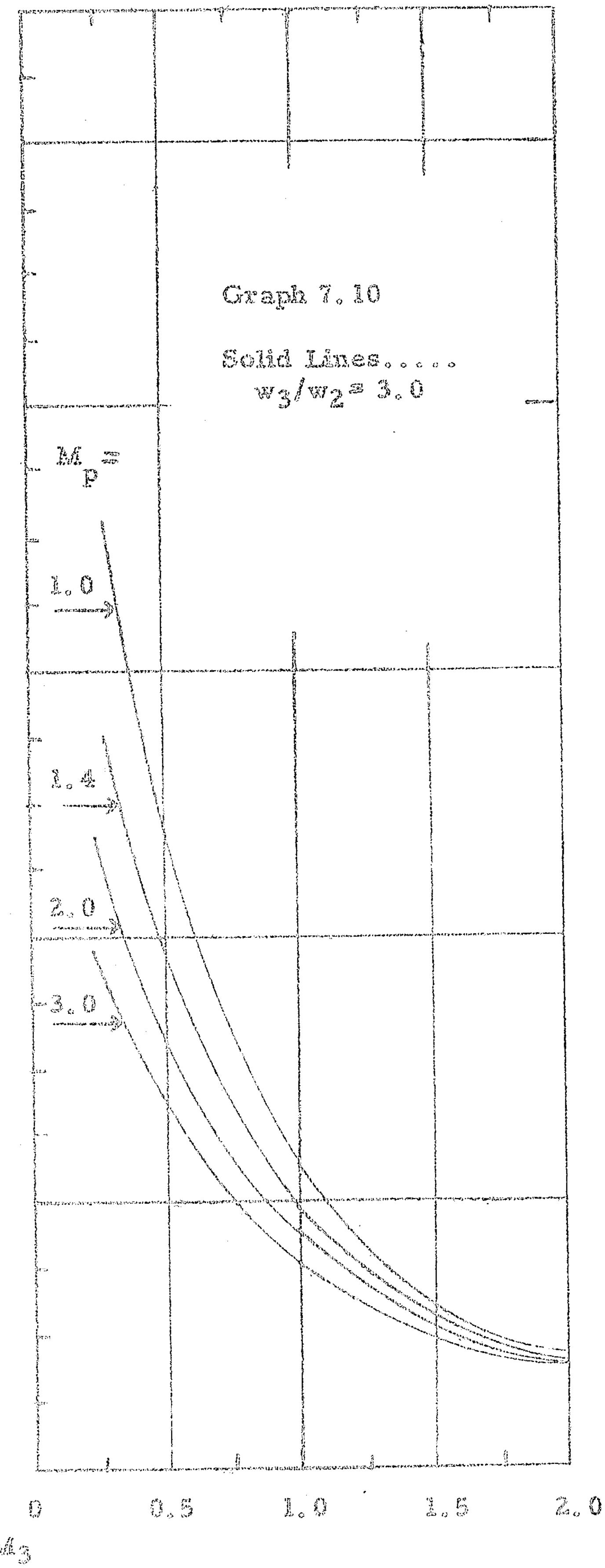
Graph 7.7
Curve Type D
$S_1 = 0.4$
$S_3 = -1.0,\ -2.0,\ -4.0$
$w_3/w_2 = 6.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
2.5
2.2
1.8
1.4
1.0
$\Theta_p$
0
0.4
0.8
1.2
1.6
2.0
$M_3$


Graph 7.8
Curve Type D
w3/w2 = 6.0
$S_1$ = 1.0
$S_3$ = -1.0, -2.0, -4.0
$M_p$ = 3.0
$M_p$ = 2.0
$M_p$ = 1.4
$M_p$ = 1.0
$M_3$
$\theta_p$
0
0.4
0.8
1.2
1.6
2.0
1.0
1.4
1.8
2.2
2.5

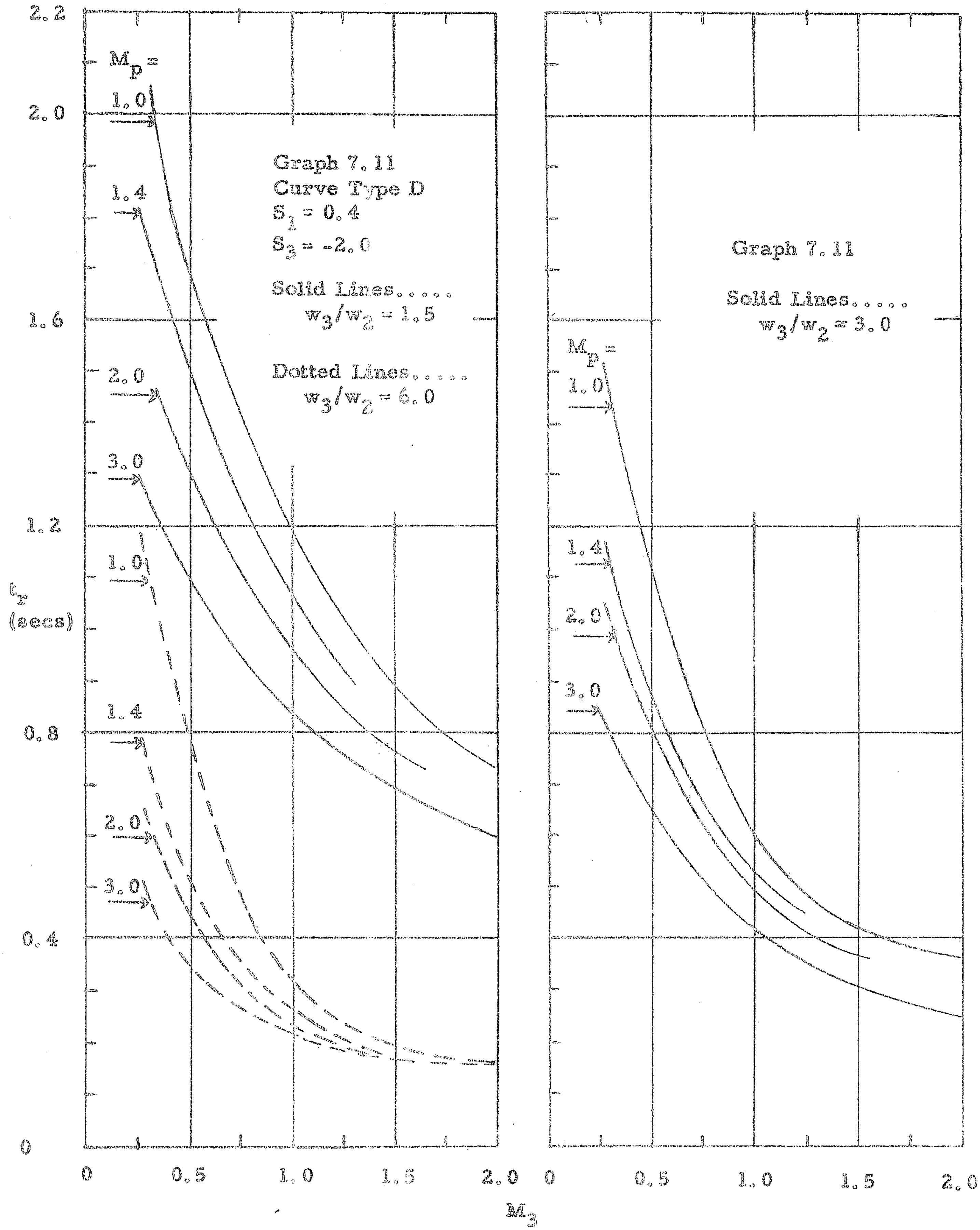

Graph 7.9
Curve Type D
$w_3/w_2 = 6.0$
$S_1 = 2.0$
$S_3 = -1.0, -2.0, -4.0$
$M_p = 3.0$
$M_p = 2.0$
$M_p = 1.4$
$M_p = 1.0$
$M_3$
$\theta_p$
0
0.4
0.8
1.2
1.6
2.0
1.0
1.4
1.8
2.2
2.5

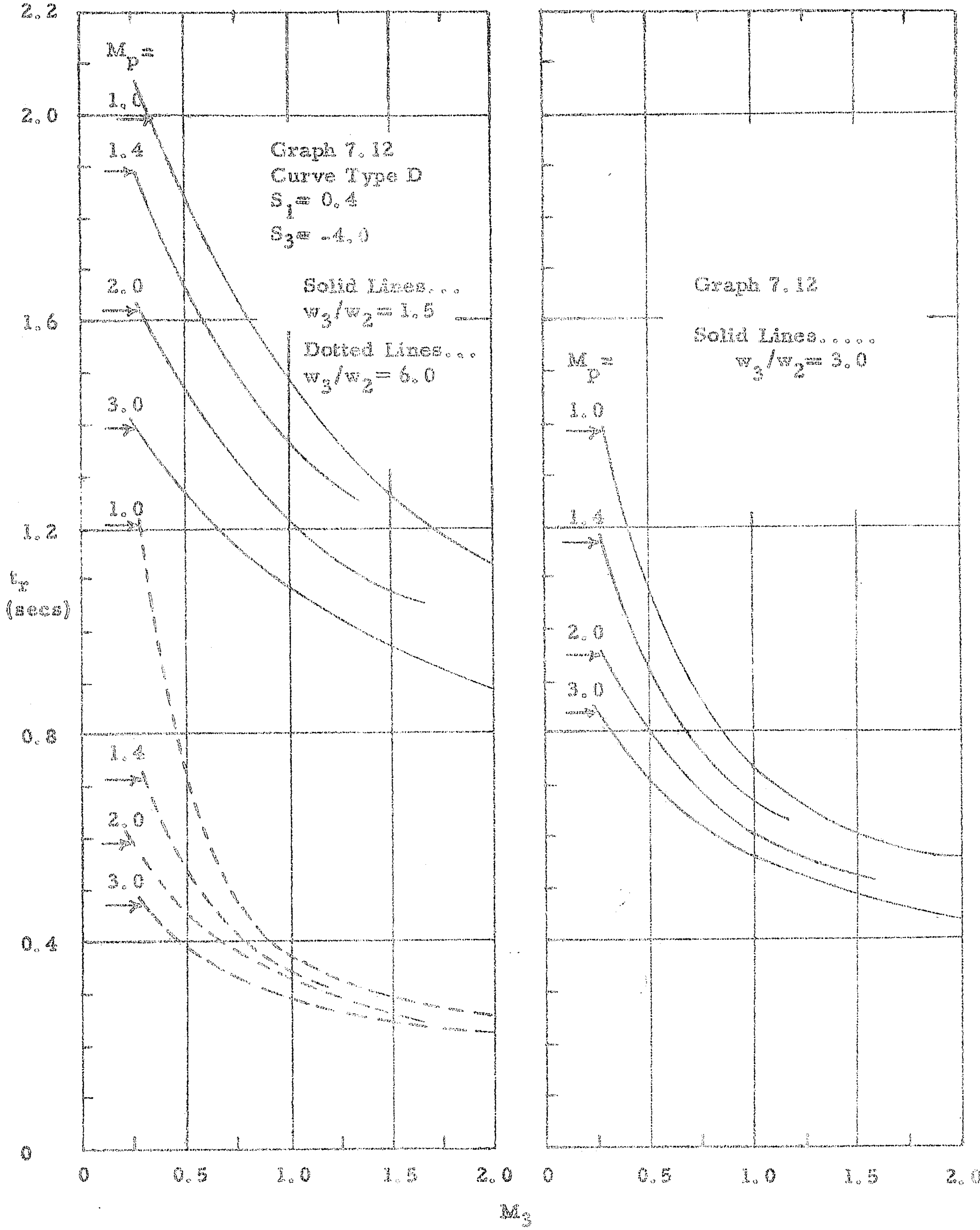


Graph 7.10
Curve Type D
$S_1$ = 0.4
$S_3$ = -1.0
Solid Lines..... $w_3/w_2$ = 1.5
Dotted Lines..... $w_3/w_2$ = 6.0
$M_p$ =
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
$t_x$ (secs)
0
0.4
0.8
1.2
1.6
2.0
2.2
0
0.5
1.0
1.5
2.0

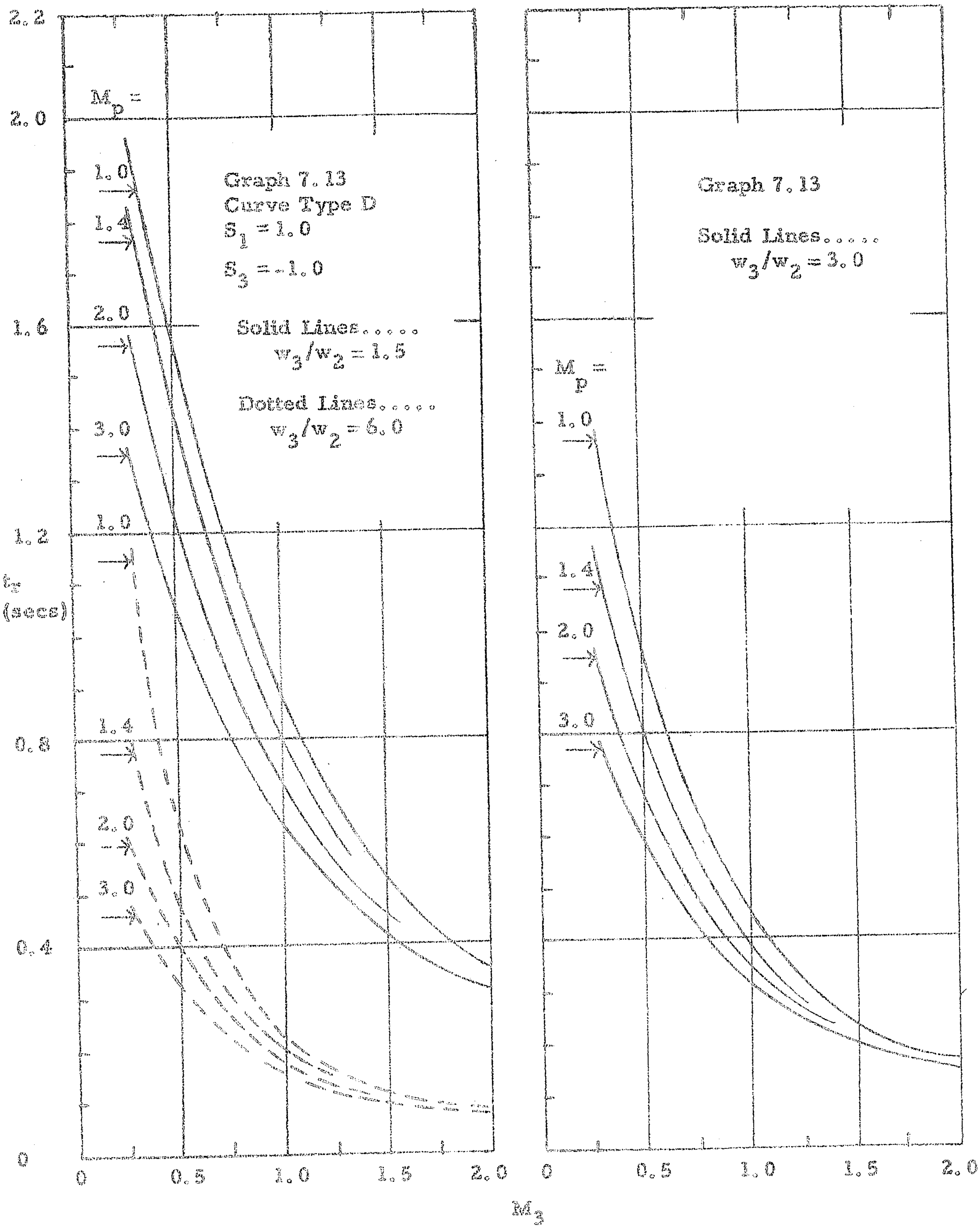



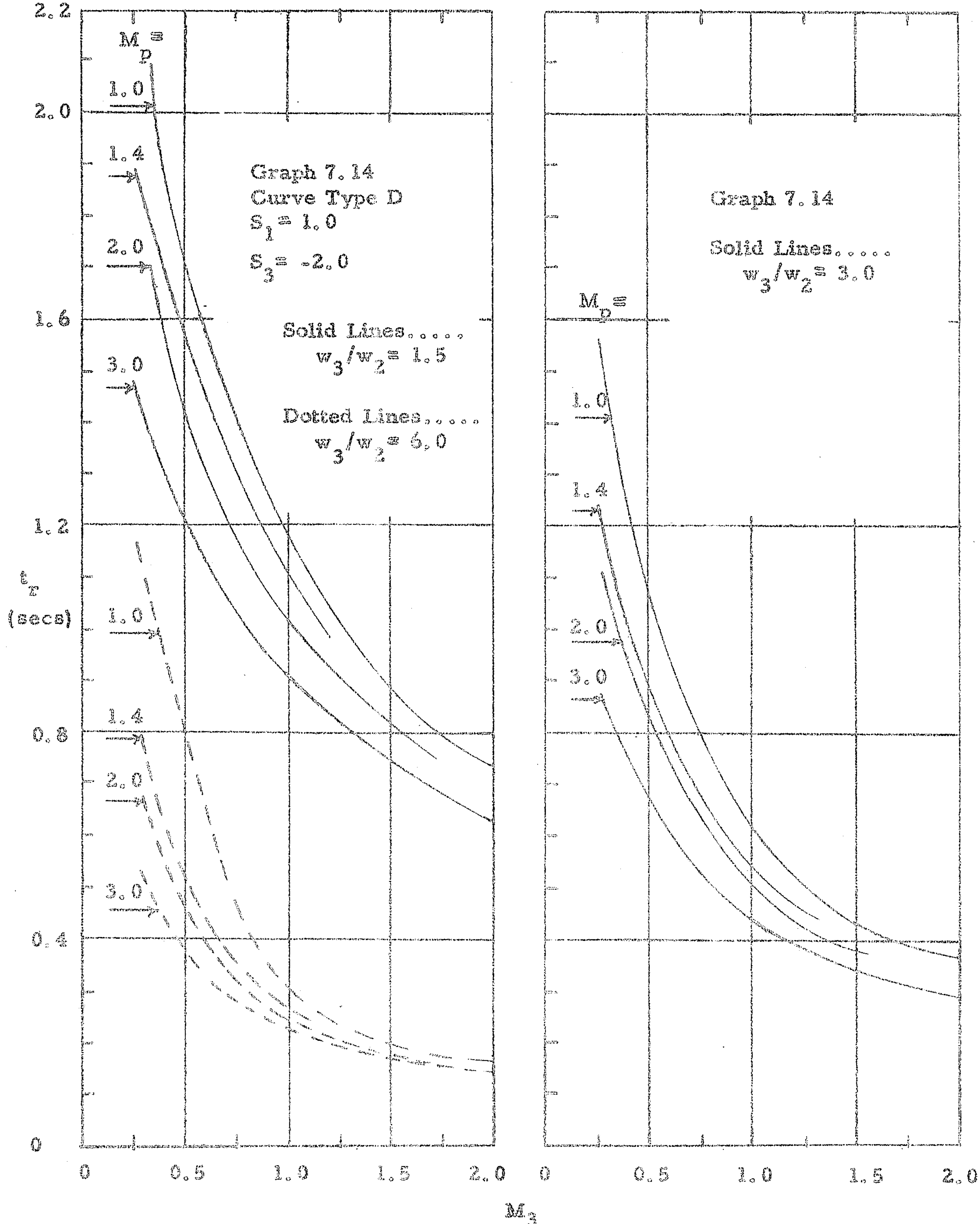
Graph 7.10
Solid Lines..... $w_3/w_2$ = 3.0
$M_p$ =
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
$M_3$

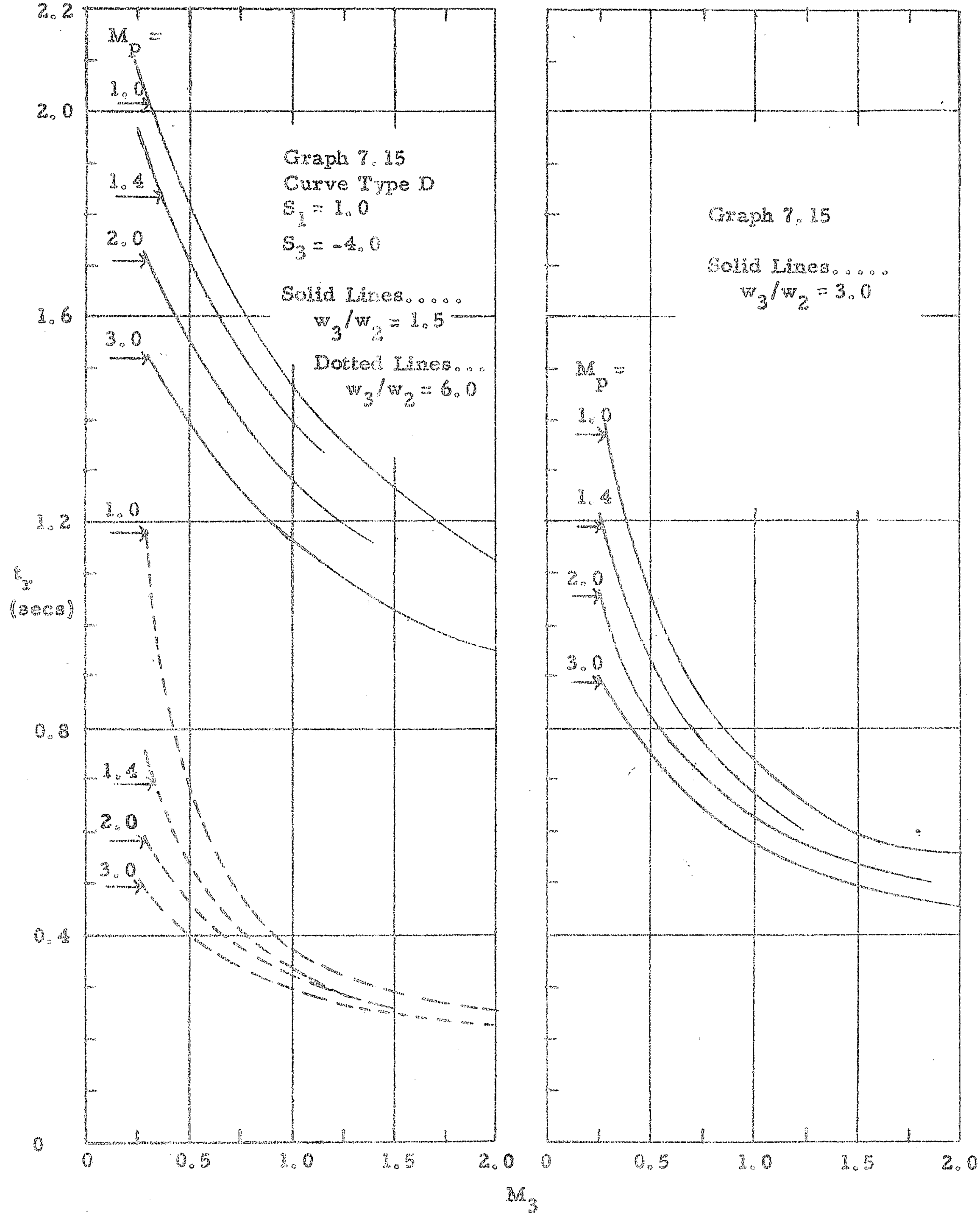

2.2
2.0
1.6
1.2
0.8
0.4
0
$t_r$ (secs)
$M_p$ =
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
Graph 7.11
Curve Type D
$S_1 = 0.4$
$S_3 = -2.0$
Solid Lines.....
$w_3/w_2 = 1.5$
Dotted Lines.....
$w_3/w_2 = 6.0$
Graph 7.11
Solid Lines.....
$w_3/w_2 = 3.0$
$M_p$ =
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
0
0.5
1.0
1.5
2.0
$M_3$

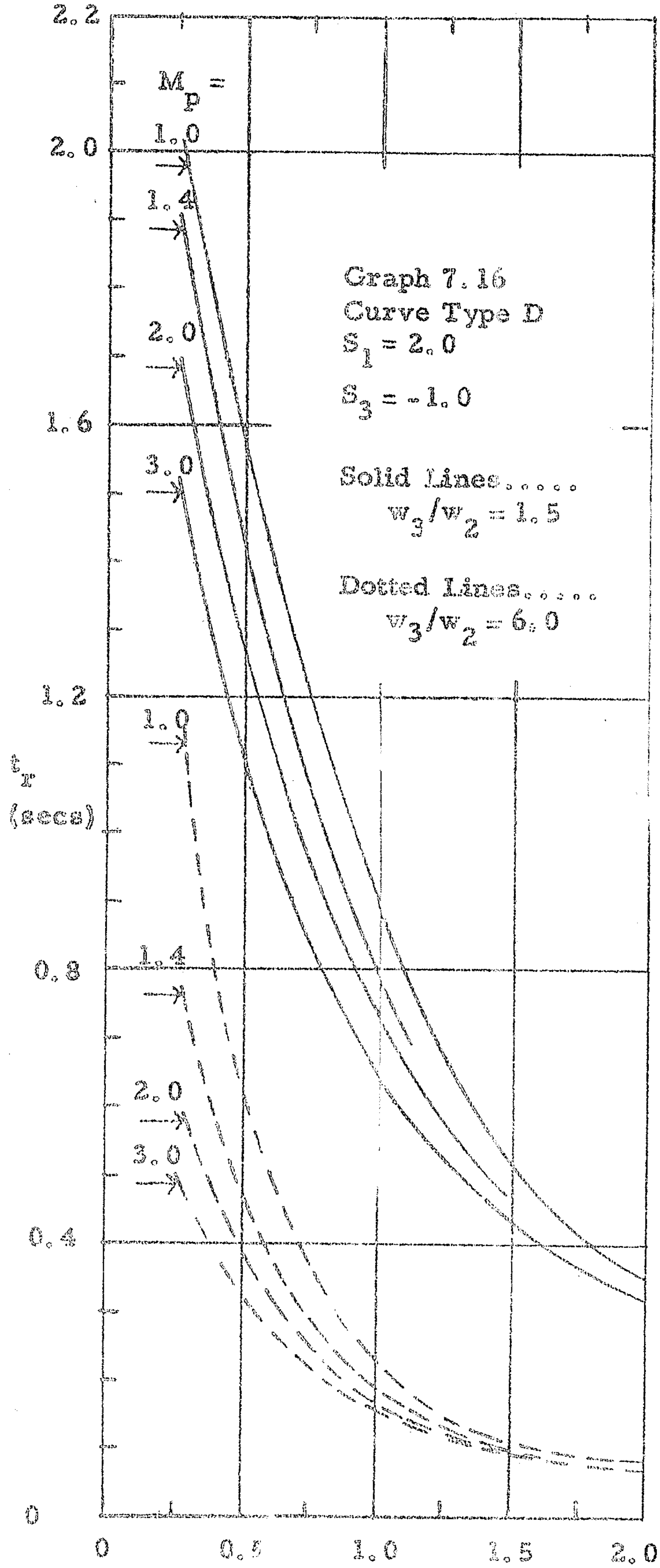


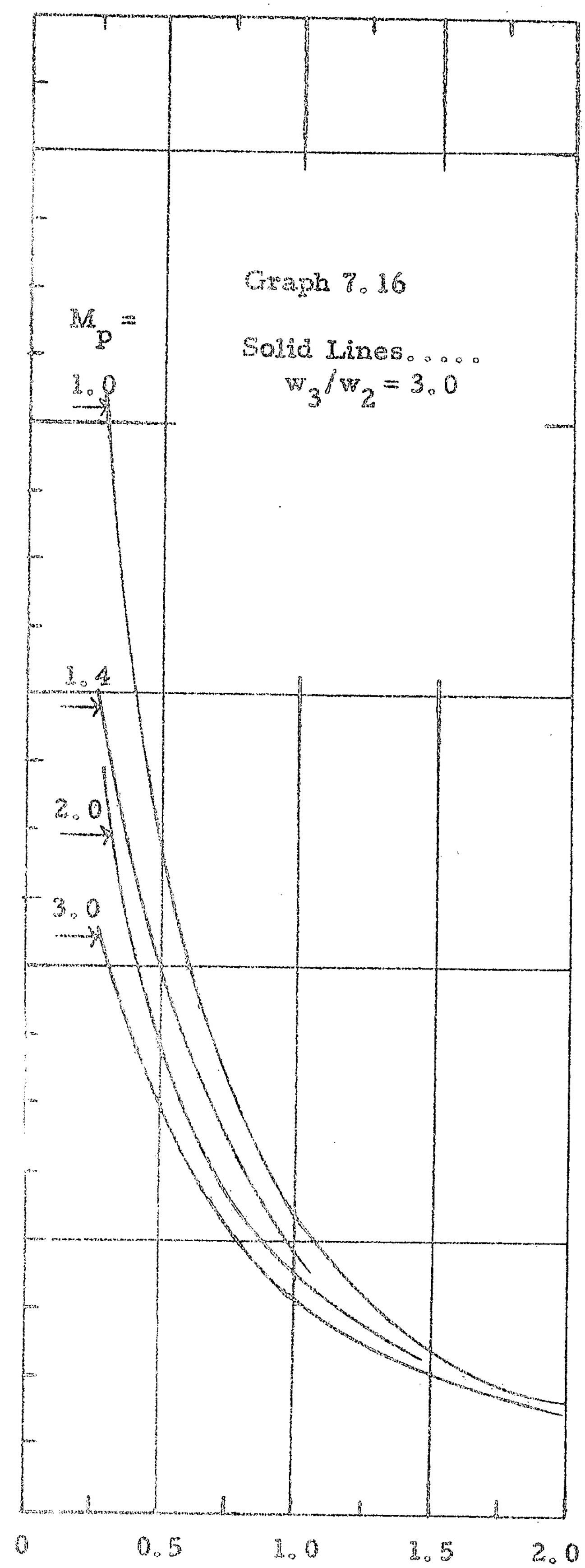
2.2
2.0
1.6
1.2
0.8
0.4
0
$t_r$ (secs)
$M_p$ =
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
Graph 7.12
Curve Type D
$S_1$ = 0.4
$S_3$ = -4.0
Solid Lines...
$w_3/w_2$ = 1.5
Dotted Lines...
$w_3/w_2$ = 6.0
Graph 7.12
Solid Lines.....
$w_3/w_2$ = 3.0
$M_p$ =
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
0
0.5
1.0
1.5
2.0
$M_3$


Graph 7.13
Curve Type D
$S_1 = 1.0$
$S_3 = -1.0$
Solid Lines.....
$w_3/w_2 = 1.5$
Dotted Lines.....
$w_3/w_2 = 6.0$
$M_p =$
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
$t_r$
(secs)
2.2
2.0
1.6
1.2
0.8
0.4
0
0
0.5
1.0
1.5
2.0
Graph 7.13
Solid Lines.....
$w_3/w_2 = 3.0$
$M_p =$
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
$M_3$

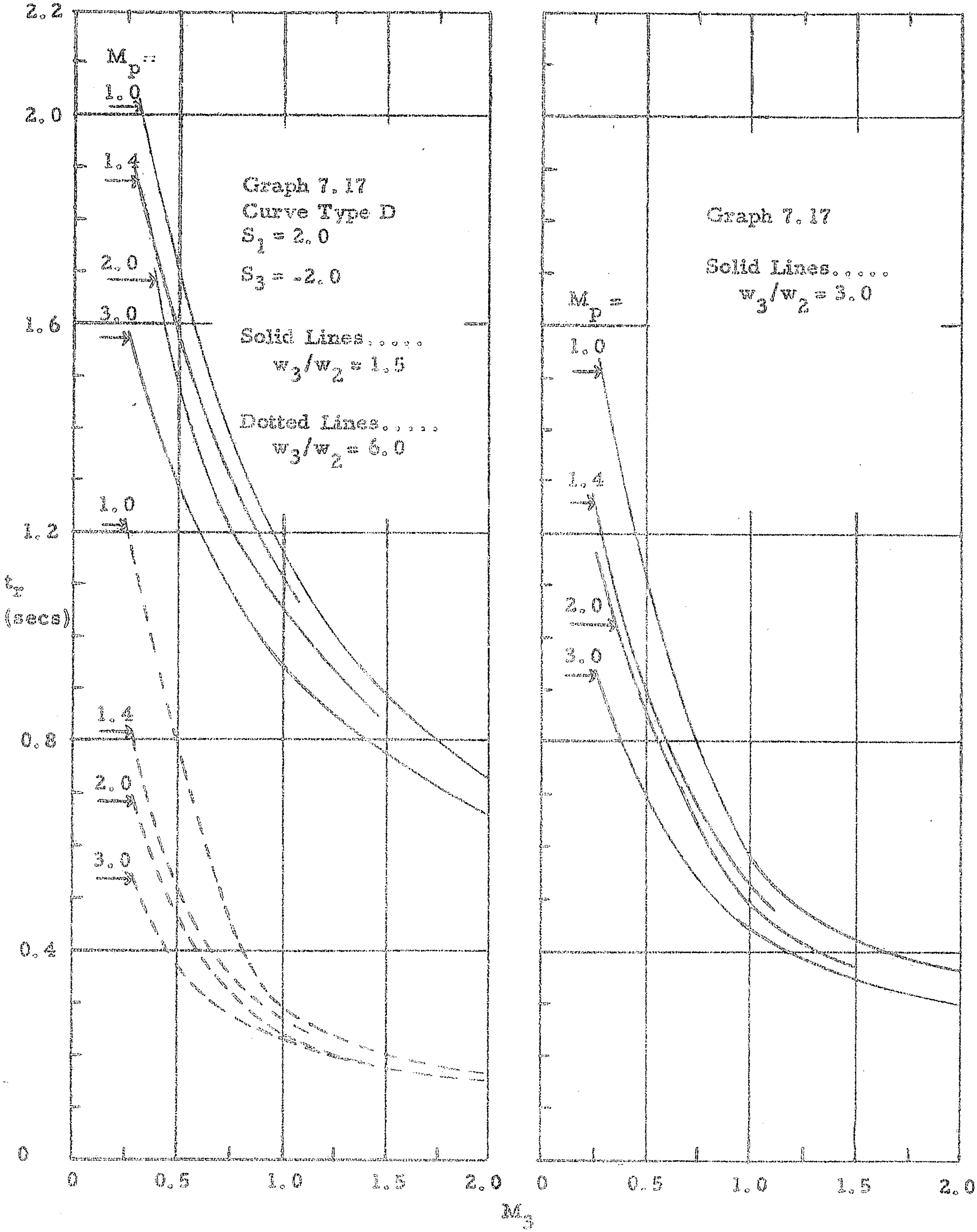

Graph 7.14
Curve Type D
$S_1 = 1.0$
$S_3 = -2.0$
Solid Lines.....
$w_3/w_2 = 1.5$
Dotted Lines.....
$w_3/w_2 = 6.0$
$M_p =$
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
$t_r$
(secs)
0
0.4
0.8
1.2
1.6
2.0
2.2
0
0.5
1.0
1.5
2.0
Graph 7.14
Solid Lines.....
$w_3/w_2 = 3.0$
$M_p =$
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
$M_3$

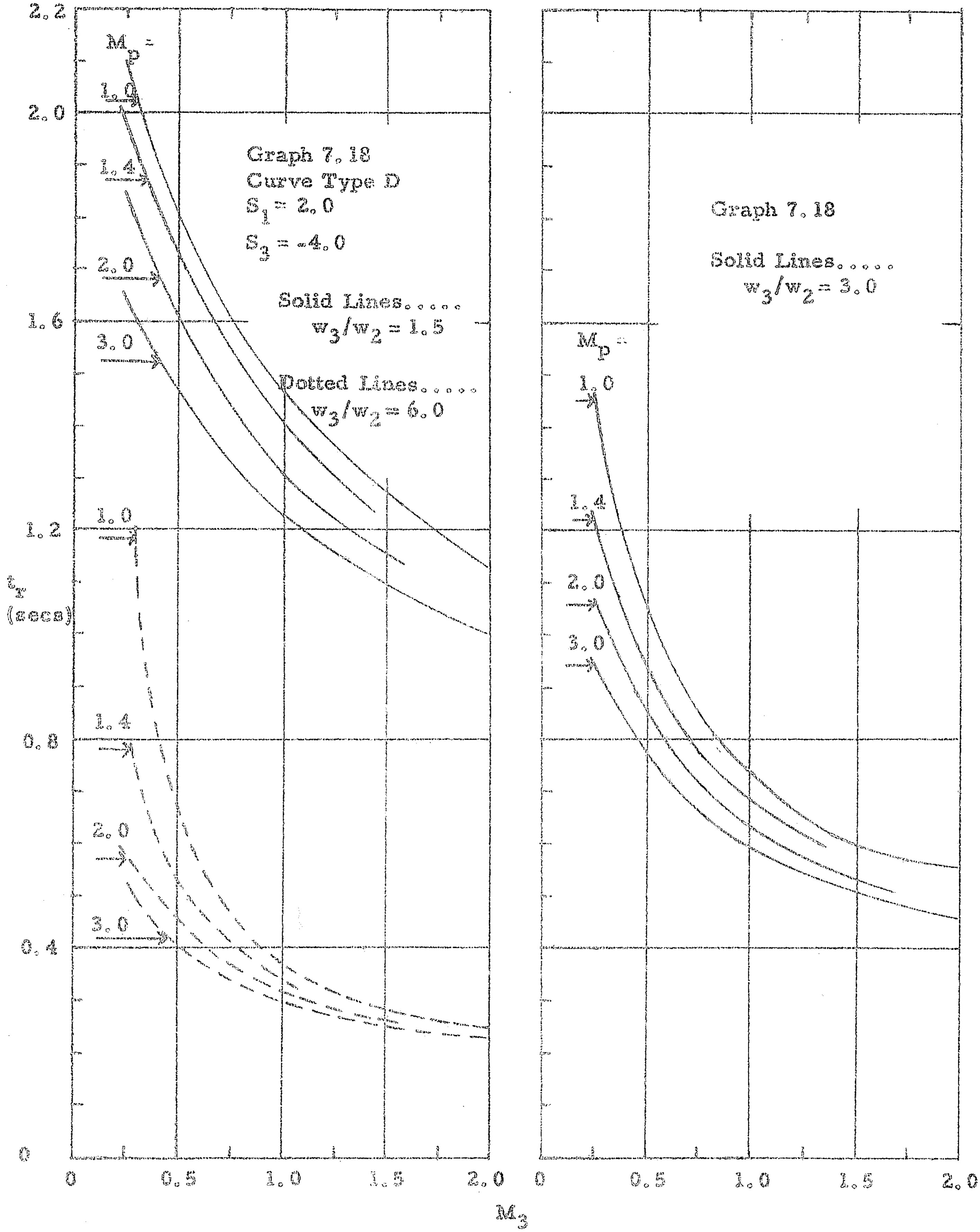

2.2
2.0
1.6
1.2
0.8
0.4
0
$t_r$ (secs)
$M_p$ =
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
Graph 7.15
Curve Type D
$S_1$ = 1.0
$S_3$ = -4.0
Solid Lines.....
$w_3/w_2$ = 1.5
Dotted Lines...
$w_3/w_2$ = 6.0
Graph 7.15
Solid Lines.....
$w_3/w_2$ = 3.0
$M_p$ =
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
0
0.5
1.0
1.5
2.0
$M_3$


2.2
2.0
1.6
1.2
0.8
0.4
0
$t_r$ (secs)
$M_p$ =
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
Graph 7.16
Curve Type D
$S_1 = 2.0$
$S_3 = -1.0$
Solid Lines......
$w_3/w_2 = 1.5$
Dotted Lines......
$w_3/w_2 = 6.0$
0
0.5
1.0
1.5
2.0


Graph 7.16
$M_p$ =
Solid Lines......
$w_3/w_2 = 3.0$
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0

$M_3$


Graph 7.17
Curve Type D
$S_1 = 2.0$
$S_3 = -2.0$
Solid Lines.....
$w_3/w_2 = 1.5$
Dotted Lines.....
$w_3/w_2 = 6.0$
$M_p =$
1.0
1.4
2.0
3.0
1.0
1.4
2.0
3.0
$t_r$
(secs)
2.2
2.0
1.6
1.2
0.8
0.4
0
0
0.5
1.0
1.5
2.0
Graph 7.17
Solid Lines.....
$w_3/w_2 = 3.0$
$M_p =$
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
$M_3$



2.2
2.0
1.6
1.2
0.8
0.4
0
$t_r$ (secs)
$M_p$ =
1.0
1.4
2.0
3.0
Graph 7.18
Curve Type D
$S_1 = 2.0$
$S_3 = -4.0$
Solid Lines..... $w_3/w_2 = 1.5$
Dotted Lines..... $w_3/w_2 = 6.0$
1.0
1.4
2.0
3.0
Graph 7.18
Solid Lines..... $w_3/w_2 = 3.0$
$M_p$ =
1.0
1.4
2.0
3.0
0
0.5
1.0
1.5
2.0
0
0.5
1.0
1.5
2.0
$M_3$

PEAK TIME CONVERSION TABLES

FOR TYPE "D" CURVES......PAGES 96 TO 104

TABLE 7.1

$S_1 = 0.4$

$S_3 = -1.0$

| $M_p$ | $M_3$: 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | | | $w_3/w_2 = 1.5$ | | | |
| 1.0 | 3.69 | 3.16 | 2.55 | 1.86 | | |
| 1.4 | 3.47 | 2.91 | 2.44 | 1.98 | | |
| 2.0 | 3.49 | 2.96 | 2.62 | 2.21 | 1.78 | |
| 3.0 | 3.69 | 3.20 | 2.93 | 2.55 | 2.18 | 1.74 |
| | | | $w_3/w_2 = 3.0$ | | | |
| 1.0 | 2.86 | 2.92 | 1.95 | 0.93 | | |
| 1.4 | 2.39 | 2.05 | 1.72 | 1.08 | | |
| 2.0 | 2.37 | 2.05 | 1.78 | 1.40 | 1.01 | |
| 3.0 | 2.53 | 2.24 | 1.99 | 1.74 | 1.42 | 1.02 |
| | | | $w_3/w_2 = 6.0$ | | | |
| 1.0 | 2.88 | 1.88 | 2.18 | 0.47 | | |
| 1.4 | 1.87 | 1.75 | 1.37 | 0.59 | | |
| 2.0 | 1.85 | 1.69 | 1.39 | 1.17 | 0.58 | |
| 3.0 | 2.01 | 1.77 | 1.52 | 1.35 | 1.20 | 0.58 |

TABLE 7.2

$S_1 = 0.4$

$S_3 = -2.0$

| $M_p$ | $M_3$ 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | $w_3/w_2 = 1.5$ | | | | | |
| 1.0 | 4.14 | 3.60 | 2.94 | 2.38 | | |
| 1.4 | 3.81 | 3.33 | 2.83 | 2.43 | | |
| 2.0 | 3.91 | 3.42 | 3.01 | 2.69 | 2.31 | |
| 3.0 | 4.03 | 3.63 | 3.30 | 2.98 | 2.60 | 2.28 |
| | $w_3/w_2 = 3.0$ | | | | | |
| 1.0 | 3.36 | 2.90 | 1.42 | 1.19 | | |
| 1.4 | 2.67 | 2.00 | 1.68 | 1.29 | | |
| 2.0 | 2.47 | 2.24 | 1.99 | 1.53 | 1.24 | |
| 3.0 | 2.78 | 2.44 | 2.17 | 1.86 | 1.55 | 1.26 |
| | $w_3/w_2 = 6.0$ | | | | | |
| 1.0 | 2.50 | 2.16 | 1.68 | 0.60 | | |
| 1.4 | 1.84 | 1.61 | 1.55 | 0.66 | | |
| 2.0 | 2.09 | 1.63 | 1.55 | 1.37 | 0.67 | |
| 3.0 | 2.03 | 1.82 | 1.62 | 1.47 | 1.40 | 0.65 |

TABLE 7.3

$S_1 = 0.4$

$S_3 = -4.0$

| $M_p$ | $M_3$ 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | | | $w_3/w_2 = 1.5$ | | | |
| 1.0 | 4.73 | 4.18 | 3.79 | 3.41 | | |
| 1.4 | 4.56 | 4.09 | 3.75 | 3.43 | | |
| 2.0 | 4.63 | 4.21 | 3.89 | 3.56 | 3.28 | |
| 3.0 | 4.82 | 4.43 | 4.16 | 3.86 | 3.56 | 3.26 |
| | | | $w_3/w_2 = 3.0$ | | | |
| 1.0 | 3.57 | 3.58 | 1.91 | 1.71 | | |
| 1.4 | 2.96 | 2.30 | 1.98 | 1.77 | | |
| 2.0 | 2.98 | 2.50 | 2.16 | 1.92 | 1.72 | |
| 3.0 | 3.17 | 2.80 | 2.48 | 2.15 | 1.93 | 1.74 |
| | | | $w_3/w_2 = 6.0$ | | | |
| 1.0 | 2.78 | 2.73 | 0.94 | 0.85 | | |
| 1.4 | 2.05 | 1.84 | 1.01 | 0.91 | | |
| 2.0 | 2.11 | 1.88 | 1.79 | 0.98 | 0.90 | |
| 3.0 | 2.28 | 1.99 | 1.84 | 1.77 | 1.01 | 0.89 |

TABLE 7.4

$S_1 = 1.0$

$S_3 = -1.0$

| $M_p$ | $M_3$: 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | | | $w_3/w_2 = 1.5$ | | | |
| 1.0 | 3.69 | 3.16 | 2.55 | 1.86 | | |
| 1.4 | 3.47 | 2.88 | 2.38 | 1.93 | | |
| 2.0 | 3.29 | 2.76 | 2.38 | 1.99 | 1.61 | |
| 3.0 | 3.29 | 2.80 | 2.48 | 2.15 | 1.79 | 1.40 |
| | | | $w_3/w_2 = 3.0$ | | | |
| 1.0 | 2.86 | 2.92 | 1.95 | 0.93 | | |
| 1.4 | 2.33 | 1.98 | 1.45 | 1.12 | | |
| 2.0 | 2.18 | 1.87 | 1.61 | 1.21 | 0.92 | |
| 3.0 | 2.21 | 1.91 | 1.69 | 1.40 | 1.12 | 0.87 |
| | | | $w_3/w_2 = 6.0$ | | | |
| 1.0 | 2.88 | 1.88 | 2.18 | 0.47 | | |
| 1.4 | 1.85 | 1.39 | 1.29 | 0.54 | | |
| 2.0 | 1.70 | 1.50 | 1.17 | 0.92 | 0.53 | |
| 3.0 | 1.66 | 1.44 | 1.23 | 1.03 | 0.72 | 0.48 |

TABLE 7.5

$S_1 = 1.0$

$S_3 = -2.0$

| $M_p$ | $M_3$ 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | $w_3/w_2 = 1.5$ | | | | | |
| 1.0 | 4.14 | 3.60 | 2.94 | 2.38 | | |
| 1.4 | 3.82 | 3.31 | 2.78 | 2.39 | | |
| 2.0 | 3.78 | 3.18 | 2.83 | 2.49 | 2.15 | |
| 3.0 | 3.66 | 3.21 | 2.92 | 2.58 | 2.30 | 1.97 |
| | $w_3/w_2 = 3.0$ | | | | | |
| 1.0 | 3.36 | 2.90 | 1.42 | 1.19 | | |
| 1.4 | 2.64 | 1.90 | 1.65 | 1.25 | | |
| 2.0 | 2.28 | 1.87 | 1.58 | 1.39 | 1.19 | |
| 3.0 | 2.43 | 2.03 | 1.80 | 1.58 | 1.31 | 1.15 |
| | $w_3/w_2 = 6.0$ | | | | | |
| 1.0 | 2.50 | 2.16 | 1.68 | 0.60 | | |
| 1.4 | 1.81 | 1.60 | 1.50 | 0.64 | | |
| 2.0 | 1.72 | 1.38 | 1.28 | 0.80 | 0.65 | |
| 3.0 | 1.74 | 1.53 | 1.32 | 1.15 | 0.77 | 0.60 |

TABLE 7.6

$S_1 = 1.0$

$S_3 = -4.0$

| $M_p$ | $M_3$: 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | | | $w_3/w_2 = 1.5$ | | | |
| 1.0 | 4.73 | 4.18 | 3.79 | 3.41 | | |
| 1.4 | 4.57 | 4.06 | 3.72 | 3.39 | | |
| 2.0 | 4.46 | 4.02 | 3.72 | 3.42 | 3.16 | |
| 3.0 | 4.46 | 4.08 | 3.80 | 3.52 | 3.26 | 3.01 |
| | | | $w_3/w_2 = 3.0$ | | | |
| 1.0 | 3.57 | 3.58 | 1.91 | 1.71 | | |
| 1.4 | 2.90 | 2.22 | 1.94 | 1.74 | | |
| 2.0 | 2.73 | 2.28 | 2.03 | 1.82 | 1.67 | |
| 3.0 | 2.79 | 2.42 | 2.18 | 1.95 | 1.79 | 1.63 |
| | | | $w_3/w_2 = 6.0$ | | | |
| 1.0 | 2.78 | 2.73 | 0.94 | 0.85 | | |
| 1.4 | 2.00 | 1.83 | 1.01 | 0.89 | | |
| 2.0 | 1.93 | 1.78 | 1.08 | 0.97 | 0.87 | |
| 3.0 | 1.94 | 1.77 | 1.67 | 1.09 | 0.94 | 0.87 |

TABLE 7.7

$S_1 = 2.0$

$S_3 = -1.0$

| $M_p$ | $M_3$: 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | $w_3/w_2 = 1.5$ | | | | | |
| 1.0 | 3.69 | 3.16 | 2.55 | 1.86 | | |
| 1.4 | 3.48 | 2.91 | 2.38 | 1.90 | | |
| 2.0 | 3.28 | 2.71 | 2.31 | 1.93 | 1.52 | |
| 3.0 | 3.16 | 2.64 | 2.30 | 1.99 | 1.63 | 1.28 |
| | $w_3/w_2 = 3.0$ | | | | | |
| 1.0 | 2.86 | 2.92 | 1.95 | 0.93 | | |
| 1.4 | 2.32 | 1.96 | 1.61 | 1.10 | | |
| 2.0 | 2.12 | 1.79 | 1.52 | 1.12 | 0.87 | |
| 3.0 | 2.06 | 1.76 | 1.56 | 1.28 | 1.03 | 0.80 |
| | $w_3/w_2 = 6.0$ | | | | | |
| 1.0 | 2.88 | 1.88 | 2.18 | 0.47 | | |
| 1.4 | 1.81 | 1.72 | 1.29 | 0.54 | | |
| 2.0 | 1.62 | 1.39 | 1.07 | 0.82 | 0.51 | |
| 3.0 | 1.53 | 1.27 | 1.11 | 0.91 | 0.70 | 0.48 |

TABLE 7.8

$S_1 = 2.0$

$S_3 = -2.0$

| $M_p$ | $M_3$ 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | $w_3/w_2 = 1.5$ | | | | | |
| 1.0 | 4.14 | 3.60 | 2.94 | 2.38 | | |
| 1.4 | 3.83 | 3.33 | 2.77 | 2.38 | | |
| 2.0 | 3.79 | 3.15 | 2.79 | 2.43 | 2.10 | |
| 3.0 | 3.55 | 3.08 | 2.80 | 2.44 | 2.18 | 1.84 |
| | $w_3/w_2 = 3.0$ | | | | | |
| 1.0 | 3.36 | 2.90 | 1.42 | 1.19 | | |
| 1.4 | 2.63 | 1.89 | 1.62 | 1.24 | | |
| 2.0 | 2.25 | 1.78 | 1.52 | 1.32 | 1.15 | |
| 3.0 | 2.26 | 1.90 | 1.69 | 1.47 | 1.24 | 1.10 |
| | $w_3/w_2 = 6.0$ | | | | | |
| 1.0 | 2.50 | 2.16 | 1.68 | 0.60 | | |
| 1.4 | 1.79 | 1.59 | 1.50 | 0.64 | | |
| 2.0 | 1.59 | 1.31 | 1.21 | 0.79 | 0.64 | |
| 3.0 | 1.66 | 1.38 | 1.22 | 0.90 | 0.70 | 0.61 |

TABLE 7.9

$S_1 = 2.0$

$S_3 = -4.0$

| $M_p$ | $M_3$ 0.3 | 0.5 | 0.7 | 1.0 | 1.4 | 2.0 |
|---|---|---|---|---|---|---|
| | | | $w_3/w_2 = 1.5$ | | | |
| 1.0 | 4.73 | 4.18 | 3.79 | 3.41 | | |
| 1.4 | 4.57 | 4.06 | 3.72 | 3.38 | | |
| 2.0 | 4.43 | 3.98 | 3.68 | 3.37 | 3.10 | |
| 3.0 | 4.36 | 3.96 | 3.68 | 3.41 | 3.16 | 2.91 |
| | | | $w_3/w_2 = 3.0$ | | | |
| 1.0 | 3.57 | 3.58 | 1.91 | 1.71 | | |
| 1.4 | 2.89 | 2.21 | 1.95 | 1.74 | | |
| 2.0 | 2.68 | 2.22 | 1.98 | 1.79 | 1.65 | |
| 3.0 | 2.62 | 2.27 | 2.06 | 1.88 | 1.72 | 1.59 |
| | | | $w_3/w_2 = 6.0$ | | | |
| 1.0 | 2.78 | 2.73 | 0.94 | 0.85 | | |
| 1.4 | 1.99 | 1.81 | 0.99 | 0.90 | | |
| 2.0 | 1.90 | 1.74 | 1.09 | 0.94 | 0.88 | |
| 3.0 | 1.83 | 1.63 | 1.19 | 1.03 | 0.94 | 0.84 |

# CHAPTER 8

# ERROR ANALYSIS USING AN EXAMPLE SYSTEM

## 8.1 THE SAMPLE SYSTEM

The method which will be used to analyse the errors which occur in both setting up of the conversion charts and using the charts in design is that of studying specific cases of such procedures and drawing conclusions from the experience. At this point, it is hardly necessary to emphasize that there are a multitude of errors associated with the method and, due to the complicated nature of the conversion equations, it is out of the question to analyse such errors in a general manner. The object here is only to develop an idea of the magnitude of the various errors and to draw conclusions regarding the possible methods of reducing them.

In order to analyse techniques of approximating M curves by straight line segments and obtaining from the conversion charts the system time parameters, an example system having the loop gain function, A, of equation 8.1 is chosen.

$$A = \frac{K(1+0.2S)^2}{S^3(1+0.0125S)^2} \tag{8.1}$$

In order to obtain reasonable stability in the system, the gain constant, K, is chosen so that the open loop crossover frequency is at the centre of the -1 slope of the open loop Bode diagram.

A short calculation will show that K = 500. Using this value for K and assuming that the system has a unity return ratio, then it follows that

$$M(S) = \frac{A(S)}{1 + A(S)} \tag{8.2}$$

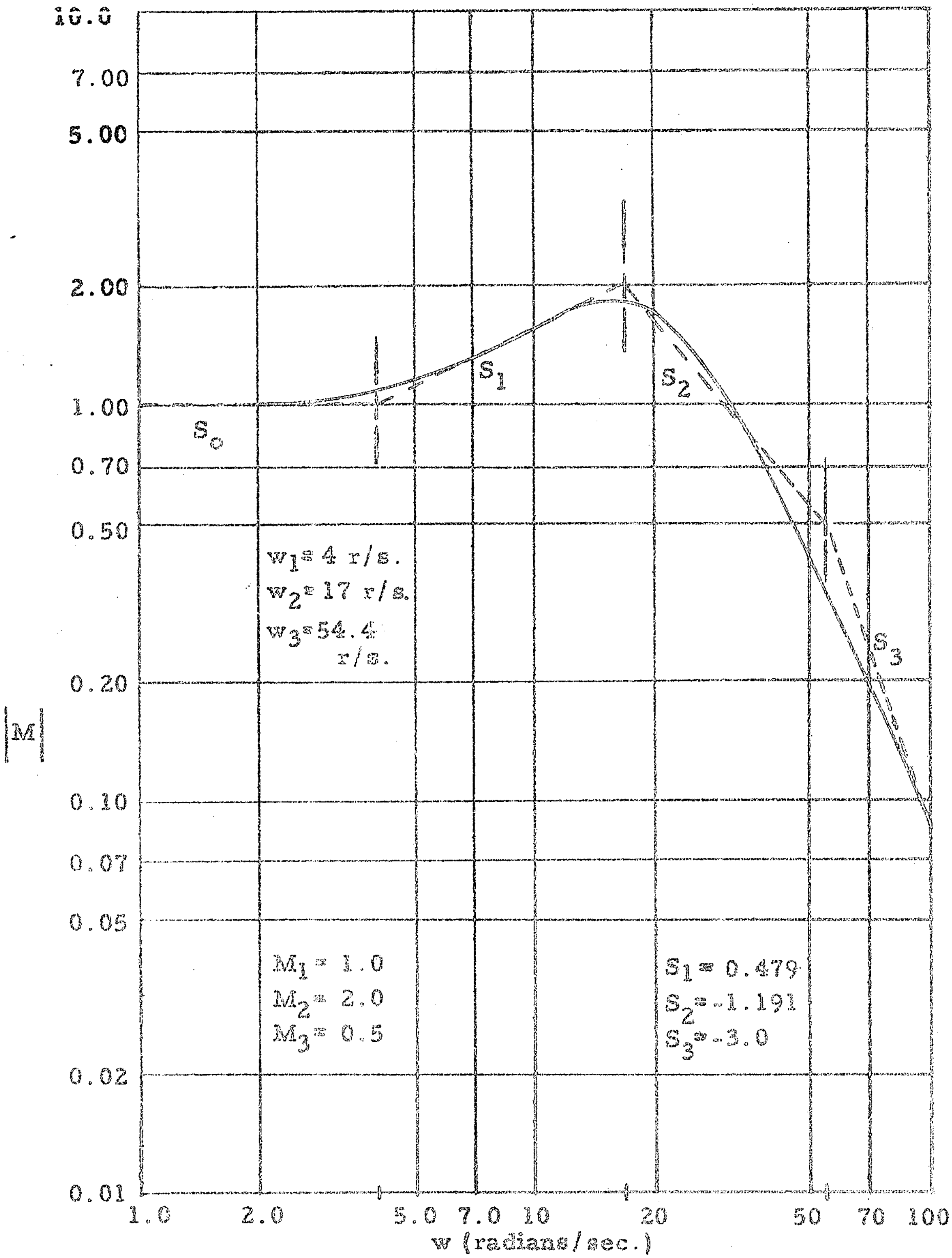
or

$$= \frac{500\,(1 + 0.2\,S)^2}{500\,(1 + 0.2\,S)^2 + S^3\,(1 + 0.0125\,S)^2} \tag{8.3}$$

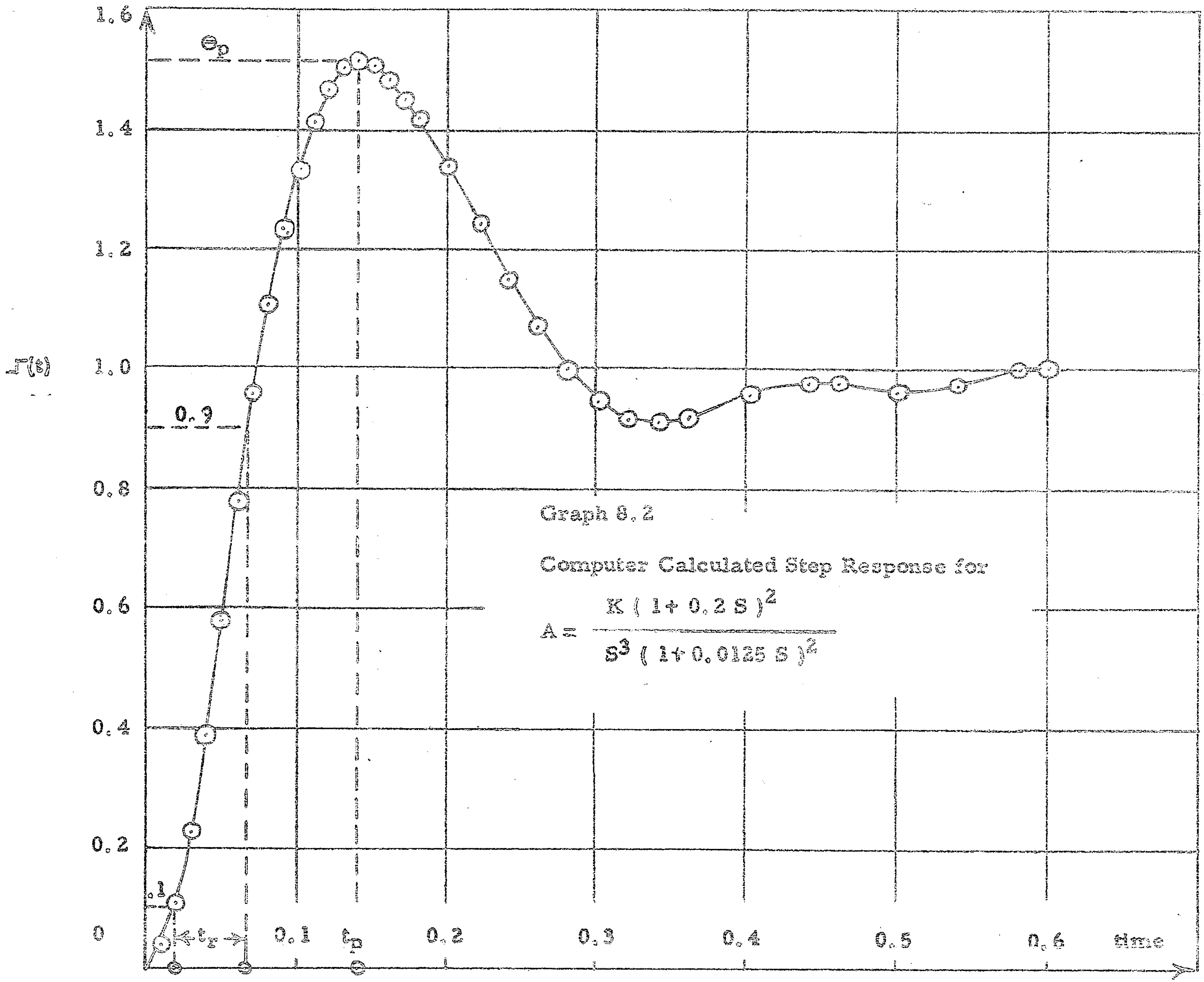
The associated M curve is then the magnitude of this function along the imaginary axis, M(jw), and this is plotted in Graph 8.1 as the solid line.

From equation 8.3, the Bendix G-15 Digital Computer can be programmed to calculate the real part of M(jw), R(jw), at 93 points. The first point, which is $w_f$ in the succeeding step response calculation, is w = 0.5 radians/second. All other frequencies follow at $\Delta w = 2\,w_f$. It may be shown that at the last frequency (where w = 92.5 radians/second) the real part has decayed to a magnitude of 0.098 and it may then be concluded that the cut-off error here is negligible (see Section 3.2).

The resultant step response, as calculated by computer from equation 2.4, is shown in Graph 8.2. The correct time parameters,




Graph 8.1 M curve for A(S) and three break point approximation




Graph 8.2

Computer Calculated Step Response for

$$A = \frac{K\,(1 + 0.2\,S)^2}{S^3\,(1 + 0.0125\,S)^2}$$

taken from the digitalized output data, are then

$$\Theta_p = 1.512$$

$$t_p = 0.141 \text{ seconds}$$

$$t_r = 0.0486 \text{ seconds}$$

These parameter values are very close to the exact values since an abnormally large number of points has been used in their calculation. The error in determining the parameters from the discrete data of the step response is also negligible.

Various tests will now be performed to determine errors associated with the less accurate approximations.

## 8.2 STEP PARAMETER DETERMINATION USING TYPE "D" CONVERSION CURVES

There is no formal procedure for obtaining the step parameters using the conversion curves. This is due to the variety of M curves which may be analysed by the general method and to various ordinates and abscissae which are used to display the conversion curves. There are, however, many helpful points which have been found by experience to simplify the technique.

When approximating an M curve by straight line segments, there is a certain margin for choice of the segment positions. Referring to curves of type "D", it has been found that chart conversion is facilitated if at least one (and preferably more) of the M curve parameters fall on one of the given values of $M_p$, $M_3$, $w_3/w_2$, or $S_1$ that were used in preparing the charts. Otherwise, interpolation becomes a major problem in some cases where other than a very rough time parameter approximation is desired. The frequency parameters which should be given preference in the segment fitting procedure are $w_3/w_2$, and $M_3$, and $S_3$.

In general, errors incurred in plotting the data make up the largest inaccuracies of the charts and, consequently, no more accuracy than a few percent should be expected from the charts. It is obvious, therefore, that two values for the parameter which are to be used in interpolation should differ by this order of magnitude or more.

As an example of the techniques involved, the M curve of Graph 8.1 is approximated with a type "D" M curve as shown by the dotted lines. The data for the segment approximations ~~is~~ are also listed on the graph. ~~Note that the frequency of the third finite break point~~ $|M_3|$ was chosen as 0.5 ~~radians/second~~, one of the values of $M_3$ used ~~for~~ to plot the charts of Chapter 7.

Also, the ratio of $w_3/w_2$ was

$$w_3/w_2 = 54.4/17 = 3.2$$

which is close to the ratio, 3.0, used in the charts.

The M peak was set at 2.0. It is noted, however, that the results of Chapter 6 indicate that a 10% flattening of this peak to more closely conform to the actual M curve makes only a fraction of 1% difference to the overshoot and effectively no difference to the peak time or rise time.

There are, then, only two parameters for which interpolation may be required. They are $S_1$ and $S_3$. A calculation of $S_1$ follows.

$$S_1 = \frac{\log(M_p/M_1)}{\log(w_p/w_1)} = \frac{\log 2}{\log(17/4)} = 0.479 \qquad (8.4)$$

Since, for the case of this example, $S_3 = -3.0$, then it is necessary to use the results from the $S_3 = -2.0$ and $-4.0$ curves. Performing this interpolation on Graph 7.4, the peak overshoot for a leading slope of 0.4 is

$$\Theta_p\,(S_1 = 0.4) = 1.53$$

and from Graph 7.5, the peak overshoot for a leading slope of 1.0 is

$$\Theta_p\,(S_1 = 1.0) = 1.35$$

A check on the variations of overshoot with the initial slope, $S_1$, confirms that a linear interpolation here is justified. Therefore, the $S_1$ interpolation between

$$0.4 \leqslant S_1 \leqslant 1.0$$

gives an approximate peak overshoot of 1.50. This is less than 1%

from the accurately calculated value (given in Section 8.1) of

$$\Theta_p \text{ (actual)} = 1.512$$

The procedure has been shown to involve little work and a simple M curve approximation. Experience actually reveals that approximating this M curve by more segments is not only impractical, but rarely decreases the error significantly. It is conceded, however, that the average error in using these techniques is likely to be slightly higher since not all M curves are approximated as easily or as closely as the example curve.

The rise time is obtained even more easily since, from Graphs 7.11 and 7.12, it is seen that in either case the normalized rise time ($t_{rn}$) for $S_1 = 0.4$ is

$$t_{rn} = 0.83 \text{ secs.}$$

while the value for $S_1 = 1.0$ is 0.80 secs. To the accuracy of the graphs, then, linear interpolation gives

$$t_{rn} \text{ (interpolated)} = 0.81 \text{ secs.}$$

The effect of frequency normalization is shown in equation 8.5.

$$t_r = \frac{t_{rn} \times 1.05}{17} = \frac{0.81 \times 1.05}{17}$$

$$= 0.0503 \text{ secs.} \qquad (8.5)$$

It has been found that the rise time is within 3 1/2 % of the accurate value given in Section 8.1.

To find the peak time, the case is slightly different than that of the rise time since the peak times at $S_3 = -2.0$ and $S_3 = -4.0$ are not too close. Interpolating between Tables 7.2 and 7.5, and between Tables 7.3 and 7.6, the peak times are found to be:

$$t_{pn} ( S_3 = -2.0 ) = 2.15 \text{ secs.}$$

$$t_{pn} ( S_3 = -4.0 ) = 2.40 \text{ secs.}$$

$$t_{pn} ( S_3 = -3.0 ) = 2.27 \text{ secs.}$$

Taking into account the effect of the conversion curve normalization, it is seen that

$$t_p = \frac{2.27 \times 1.05}{17} = 0.140 \text{ secs.}$$

There is only a fraction of 1% error here.

It may be concluded, therefore, that linear interpolation is valid for most cases since enough curves have been calculated that there is little error in interpolating linearly ~~even though~~ most curves are not closely linear.

## 8.3 STEP PARAMETER DETERMINATION USING TYPE "A" CONVERSION CURVES

It is useful to show the effect on the time parameters of approximating the same sample system M curve by a Type A curve.

A good approximation of the Type A form is shown in Figure 8.1. When this approximation is superimposed on Graph 8.1, it is seen that, although the final M curve slope is -2.0, the trailing slope of the Type A

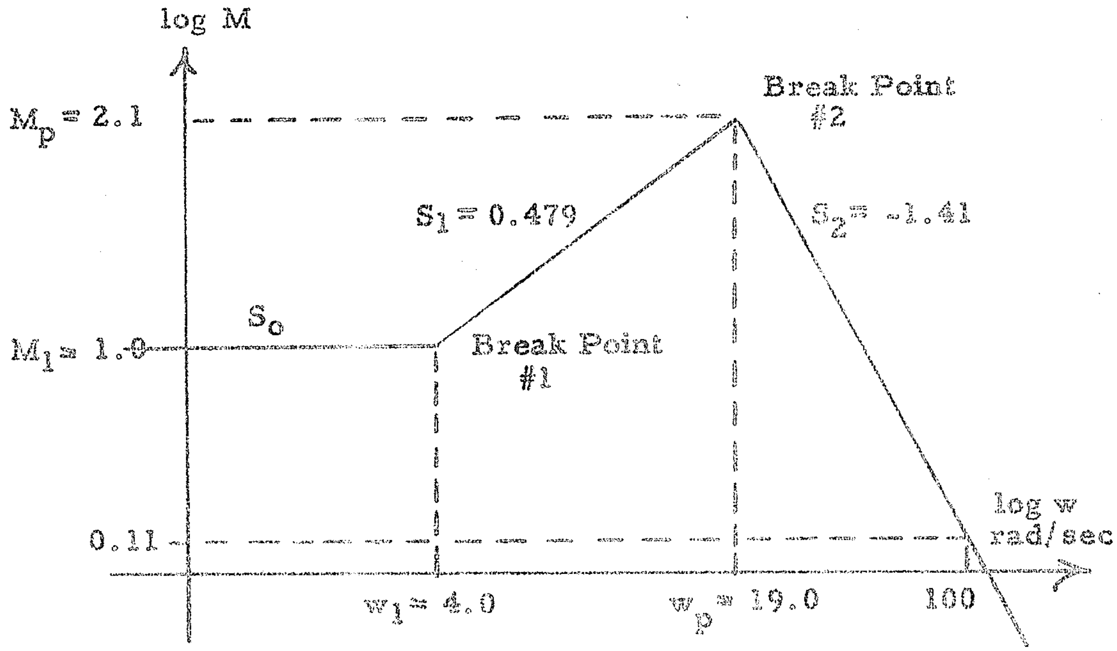



Figure 8.1 Type A curve approximation

approximation was made equal to -1.41 in order to more evenly distribute the positive and negative errors between the Type A approximation and the actual curve in the trailing slope region. These errors will, henceforth, be referred to, in general, as the "M errors".

Referring to the appropriate design conversion charts for Type A curves, Graphs 4.8 to 4.11, the underlined data in Tables 8.1 to 8.3 below is obtained. The remaining data may be found by linear interpolation. Taking the centre figure in each table and accounting for curve normalization, the resultant time parameters may then be found as below (errors are shown in brackets).

(a) $\Theta_p = 1.52$ (0.53%)

(b) $t_r = \dfrac{0.66 \times 1.05}{19} = 0.0365$ secs. (-25%)

TABLE 8.1 **OVERSHOOT INTERPOLATION**

| | | $M_p$ | | |
|---|---|---|---|---|
| | | 2.0 | 2.1 | 2.5 |
| $S_2$ | -1.0 | 1.51 | 1.53 | 1.74 |
| | -1.41 | 1.50 | 1.52 | 1.72 |
| | -2.0 | 1.48 | 1.50 | 1.68 |

TABLE 8.2 **RISE TIME INTERPOLATION**

| | | $M_p$ | | |
|---|---|---|---|---|
| | | 2.0 | 2.1 | 3.0 |
| $S_2$ | -1.0 | 0.50 | 0.48 | 0.30 |
| | -1.41 | 0.68 | 0.66 | 0.48 |
| | -2.0 | 0.95 | 0.93 | 0.75 |

TABLE 8.3 **PEAK TIME INTERPOLATION**

| | | $M_p$ | | |
|---|---|---|---|---|
| | | 2.0 | 2.1 | 3.0 |
| $S_2$ | -1.0 | 1.75 | 1.72 | 1.50 |
| | -1.41 | 2.16 | 2.13 | 1.89 |
| | -2.0 | 2.75 | 2.72 | 2.45 |

(c) $$t_p = \frac{2.13 \times 1.05}{19} = 0.118 \text{ secs.} \quad (-16.3\%)$$

These results show that the overshoot was approximated well, but the rise time and peak time are significantly lower than the actual values. The errors may be explained by the following arguments.

First, it is interesting to note that both the rise time and peak time errors may be cut to less than 7% by using a trailing slope of -2.0 instead of $S_2 = -1.41$ (leaving all other M curve parameters unchanged). This is surprising because it introduces much larger M errors in the $S_2$ region. However, this situation reveals an important aspect of the problem which must be emphasized. <u>It is not sufficient to attempt to reduce M errors at frequencies below cut-off when approximating an M curve.</u> The approximation attempted in this section failed because it assumed that the phase associated with the system frequency response as frequency approaches infinity is not -180 degrees (as it would have been if $S_2$ were -2.0), but is equal to

$$-1.41 \times 90^{\circ} = -127^{\circ}$$

(Of course, this is physically impossible in actual systems, but may be used analytically as an approximation tool).

The difference in phase at high frequencies causes the shape of the real part curve to change considerably and, as a consequence, large errors occur in the step response, especially at low time values.

The rise time and peak time are considerably in error, then, because the M curve approximation minimized M errors, but failed to make R(w) a close approximation to the actual real part curve.

On the other hand, the peak overshoot had little error in this instance because the peak overshoots associated with the cases where $S_2 = -1.0$ and $S_2 = -2.0$ are approximately the same value.

The most important aspect of this discussion is that it establishes the importance of the Type "D" approximation. Although this curve type is slightly more cumbersome to handle, it is much more valuable because it allows the designer more latitude to adjust the phase as well as the magnitude of the frequency response at frequencies between $w_p$ and $w_c$. In general, it is wise to approximate the final segment with a slope equal to that determined by the slope of the actual transfer function at high frequency. The slope, $S_2$, can then be used mainly for the minimization of M errors at the frequencies above $w_p$.

# CHAPTER 9

# FINAL THOUGHTS

## 9.1 SUMMARY

In the interests of studying the overall effect of the method and suggesting improvements and shortcomings, a short resume is attempted here, the main purpose of which is to stimulate thought on topics which were brought up in the course of discussion, and were not fully answered due to lack of time.

It is believed that the basic method of attacking the problem is sound and is of sufficient generality that it provides answers to the problem on a sufficiently large scale. Chapter 4 covers the Type A curves quite adequately and Chapter 6 provides the groundwork for increasing the applicability of the method to include flat peaked M curves. The conversion curves of Chapter 7 are probably the most useful portions of the thesis.

More work could be done on Chapter 5 where the discussion does not go far enough to show how curves of Type B may be handled quantitatively.

There are many places where relationships between variables appear probable, but have not been shown conclusively. The following may be stated as examples of this above statement and other unanswered questions:

(a) The correction curve for estimating $\Theta_{pi}$ (Graph 6.3) appears to be a very useful one. Can it be shown that it holds over a wider range of M curve samples than those taken here?

(b) Equation 4.3A has been shown to have applicability to curves of Type A and B. How much wider applicability does it have ? Can it be improved to provide a relationship between peak time and overshoot for other curve types?

(c) Peak area calculations show promise of enabling an easy transformation from M curve shape to obtain the step response overshoot (as Graph 6.4, 6.5). Is there really any relationship that is of practical importance between the decrease of peak area for Type B curves and the overshoot as is suggested at the end of Section 5.2? During the calculation of the results of curve Type D, the peak area for each curve was also listed. However, time did not permit large scale investigation of the results.

(d) In Chapter 6, Curves 1 and 2 of the sample tabulated in table 6.1 have the ~~same M peaks~~. Graph 6.4 shows the relationship of peak area to overshoot for intermediate curves. The curves maintain a parallel configuration . Is this always true and, if so, why?

(e) In the Type B cases depicted on Graph 5.3, why are the curves parallel? Can this be developed into ~~some~~ design equations or graphs?

(f) This method can be applied to systems with dead time. Is it possible to expand the technique to other non-minimum phase systems in a more general manner?

(g) Can other techniques be set up to study the other step response parameters which are not covered in this thesis (as suggested in Section 3.4)?

(h) An attempt to correlate the locus of pole and zero position shifts with the changes in the shape of the segment approximation of the M curve ended in failure because the type of pole-zero pattern chosen to represent the Type A curve variations was not sufficiently versatile to adequately describe the response under the entire M curve parameter variations. In particular, where $S_1$ was constant at 0.2 and the M peak was increased from 1.0 to 3.0, those M curves with peaks of 2.0 or over were not adequately described by two zeros and three poles. Can other pole-zero patterns be shown to be practical in approximating such curves and, if so, can the locus of such pole-zero position shifts supply the answers to clarify those relationships which are not yet clear?

As with the results indicated in this thesis, the answers to each of these questions above will require much practical usage before the ultimate reliability and usefulness is decided.

# APPENDIX

# DESCRIPTION OF
# THE MATHEMATICAL METHODS
# IN THE PROGRAMMED CONVERSION ROUTINE

## A.1 BLOCK DIAGRAM

A simplified block diagram of the various calculations involved in the conversion routine is given in Figure A.1. (For a detailed block diagram of each section see the University of Manitoba 650 program notes for this conversion routine.) Some of the more important sections will now be examined more closely.

## A.2 $|M|$ CALCULATION

The method used to calculate $|M|$ curve points is straightforward. For the xth straight line segment of the M curve, equation A.1 is valid (See Figure A.2).

$$\ln \left[\frac{M(x')}{M(x)}\right] = S_x \ln \left[\frac{w(x')}{w(x)}\right] \tag{A.1}$$

Therefore, $|M|$ at any point, $w_x'$, on the M curve can be expressed as equation A.2.

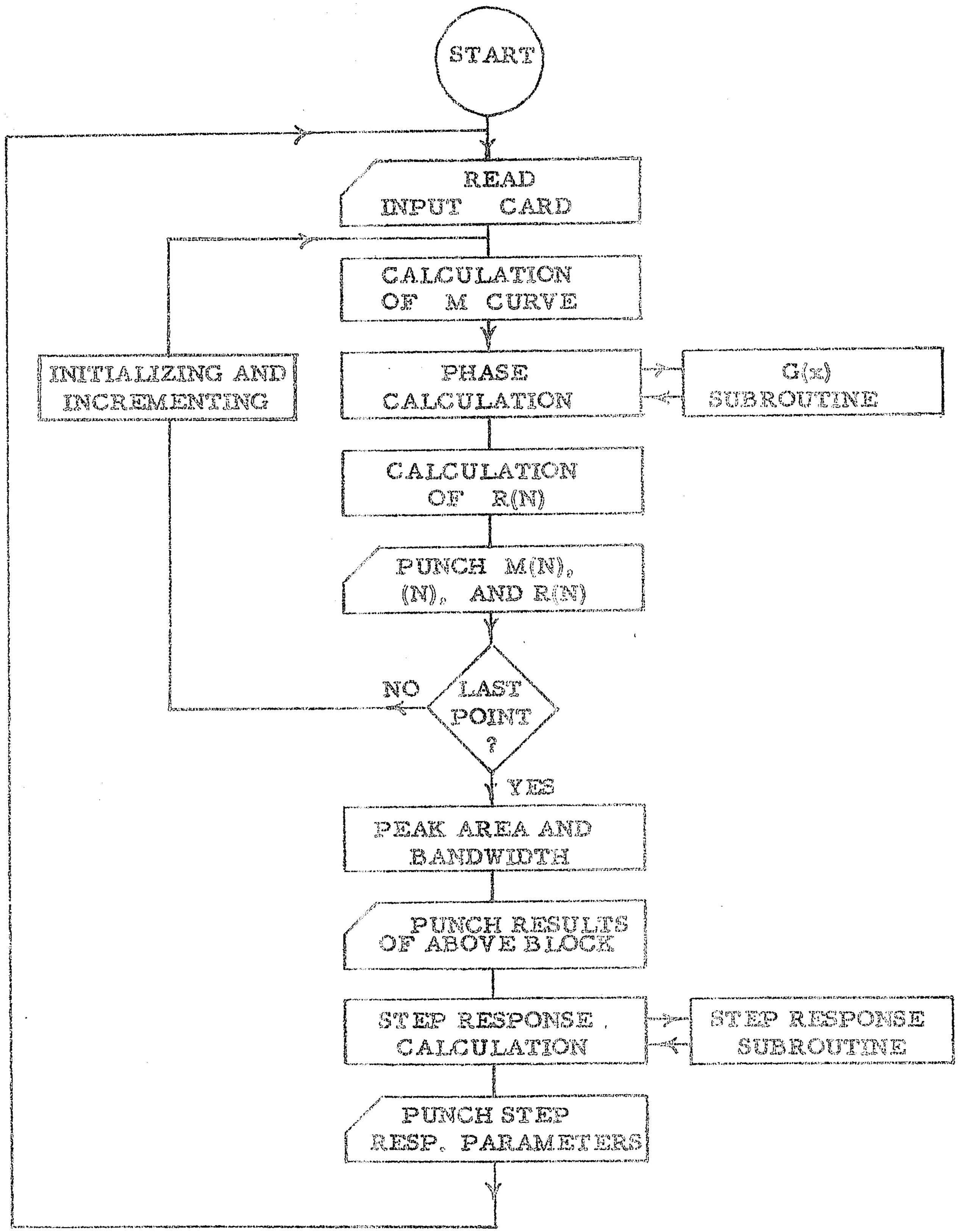



Figure A. 1 Basic block diagram of the conversion routine

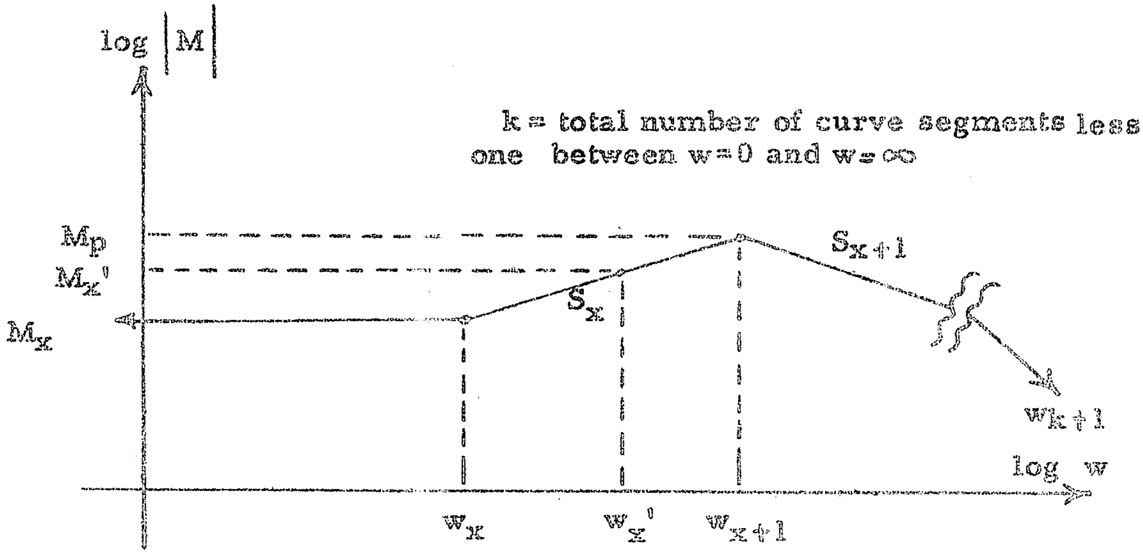



Figure A.2 Generalized M curve

$$M(x') = \exp\left[ S_x \ln\left(\frac{w_x'}{w_x}\right) + \ln\left(M_x\right)\right] \tag{A.2}$$

If the points, $C(N)$, taken along the curve at frequencies of

$$C(N) = (2N - 1)\, w_f \tag{A.3}$$

define $x$ such that, ~~when~~

$$w_x \leq C(N) < w_{x+1} \tag{A.4}$$

then let

$$x = i \tag{A.4a}$$

Therefore, $|M|$ at the N points, $M(N)$, can be written as

$$M(N) = \exp\left[ S_i \ln\left(\frac{C(N)}{w_i}\right) + \ln\left(M_i\right)\right] \tag{A.5}$$

for $1 \leq i \leq k$ and when $k = 0$, then $M(N) = 1$

## A.3 PHASE CALCULATION

A compact, efficient, and fast method for calculating the phase was developed and warrants special attention here. The method is based on the Bode Integral (BO-C) which is shown in equation A.6.

$$\beta_c = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha}{du} \ln \coth \left| \frac{u}{2} \right| du \qquad (A.6)$$

where $u = \ln (w/w_c)$

$w_c$ = frequency of interest (rad/sec)

$\beta_c$ = phase at the frequency, $w_c$

This integral, in effect, states that the phase at any frequency is determined by the slope of the M curve at all frequencies. However, the slopes nearest the frequency of interest, $w_c$, are most dominant in determining the phase, $\beta_c$.

Now, if the M curve is approximated by straight line segments, as has been done here, then, for a particular segment, $d\alpha/du$ is a constant and equals the slope of that portion of the M curve. Now, Bower and Schultheiss have shown that, for a segment of slope S, the percentage contribution, $\beta_p$, to the total phase shift at $w_c$ contributed by this segment is as shown in equation A.7.

$$\beta_p = \left[ \frac{\pi^2}{8} - \sum_{\alpha=1}^{\infty} \frac{x^{(2\alpha-1)}}{(2\alpha-1)^2} \right] \frac{400}{\pi^2} = \frac{400\, g(x)}{\pi^2} \qquad (A.7)$$

where $x = w_c/w_0$

and $w_0$ = end point of the frequency range under consideration

It is seen that the series will converge rapidly except where $w_o$ falls close to $w_c$, that is, where x approaches unity. This presents a major difficulty in the digital computer programming since points where x approaches unity occur often in the calculation of $\beta_c$. To speed calculation for such values of x, another series was developed which converged much faster in these regions.

## PHASE SERIES DERIVATION

It is known from elementary mathematics that

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \qquad (A.8)$$

$$\operatorname{arctanh}(x) = \sum_{k=1}^{\infty} \frac{x^{(2k-1)}}{2k-1} \qquad (A.9)$$

$$\operatorname{arctanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad (A.10)$$

$$\int x\, e^{ax} dx = \frac{(ax-1)}{a^2}\, e^{ax} + C \qquad (\text{for } a \neq 0) \qquad (A.11)$$

Now, from the previous definition of g(x) equation A.12 follows.

$$\frac{d\, g(x)}{dx} = -\frac{1}{x} \sum_{k=1}^{\infty} \frac{x^{(2k-1)}}{2k-1} \qquad (A.12)$$

$$= -\frac{\operatorname{arctanh}(x)}{x} = \frac{1}{2x} \ln\left(\frac{1-x}{1\ \ x}\right) \qquad (A.13)$$

If the substitution, A.14, is made

$$y = -2 \operatorname{arctanh}(x) = \ln\left(\frac{1 - x}{1 + x}\right) \qquad (A.14)$$

then, it is found that

$$x = \frac{1 - e^{y}}{1 + e^{y}} \qquad (A.15)$$

and that

$$dx = \frac{-2\, e^{y}}{(1 + e^{y})^{2}}\, dy \qquad (A.16)$$

Hence,

$$g(x) = C + \int \frac{1}{2x} \ln\left(\frac{1 - x}{1 + x}\right) dx = C + \int \frac{-y\, e^{y}\, dy}{1 - e^{2y}} \qquad (A.17)$$

Expanding $\left(\frac{1}{1 - e^{2y}}\right)$ as a geometric series, the result is

$$g(x) = -\int \sum_{k=1}^{\infty} y\, e^{2ky - y}\, dy \qquad (A.18)$$

$$= -\int \sum_{k=1}^{\infty} y\, e^{(2k - 1)y}\, dy \qquad (A.19)$$

and applying equation A.11, it follows that

$$g(x) = C - \sum_{k=1}^{\infty}\left(\frac{(2k-1)y - 1}{(2k-1)^2}\right)e^{(2k-1)y} \qquad (A.20)$$

and from equation A.14, the final result is

$$g(x) = C + \sum_{k=1}^{\infty} \frac{1-(2k-1)\ln\left[(1-x)/(1+x)\right]}{(2k-1)^2}\left(\frac{1-x}{1+x}\right)^{2k-1} \qquad (A.21)$$

A comparison of the rapidity of convergence for various values of x has been made to show the relative merits of equation A.21 and the former g(x) definition (see Table A.1). This was done by finding the number of terms required so that the last term computed is less than $10^{-9}$. This does not necessarily ensure 9 significant figures in the sum, but when the number of terms required is small, it may be expected to be within $\pm 1$ in the 9th decimal place. If the number of terms required is very large, the error may well exceed $\pm 1$ in the 8th decimal place. This is another reason for using a series which required only a few terms to converge.

Even though equation A.21 requires the computation of a logarithm, it appears from the table given on the next page that equation A.21 will be a faster series to compute than A.7 whenever x is more than 0.5. for x less than 0.5, it is simpler and probably faster to use equation A.7. For x equal to zero or unity, the choice of the wrong series results in the computation of about 15,000 terms, and since values near unity for x will frequently be encountered, the value of equation A.21 becomes obvious.

| x | Number of terms required for convergence: Equation A.7 | Equation A.21 |
|---|---|---|
| 0.01 | 2 | 267 |
| 0.05 | 4 | 69 |
| 0.20 | 6 | 21 |
| 0.40 | 9 | 11 |
| 0.45 | 10 | 10 |
| 0.50 | 12 | 9 |
| 0.70 | 20 | 6 |
| 0.90 | 55 | 4 |
| 0.95 | 100 | 4 |

Table A.1 Convergence comparison

## USE OF g(x) TO CALCULATE PHASE

For each $n = 1\ (1)\ N$ and for each $i = 1\ (1)\ k$, define $f(i,n)$ such that equations A.22 through A.24 hold.

(a) $$f(i,n) = -g\left[\frac{C(N)}{w_i}\right] \quad \text{if} \quad C(N) \leqslant w_i \qquad (A.22)$$

(b) $$f(i,n) = g\left[\frac{w_i}{C(N)}\right] \quad \text{if} \quad C(N) > w_i \qquad (A.23)$$

(c) $$f(k+1,\ n) \quad \frac{-\pi^2}{8} \qquad (A.24)$$

The following summation then yields $\beta_c$

$$\beta_c(N) = \frac{360}{\pi^2} \sum_{i=1}^{k} s_i \left[ f(i,n) - f(i+1,\ n) \right] \qquad (A.25)$$

Using the equations here to calculate phase, it is interesting to note that the time required by the IBM 650 to calculate the phase at one frequency is about 2 seconds.

The real part was easily found from the previous results by

$$R(N) = M(N) \cos \beta(N) \qquad (2.5)$$

The approximate time required to compute N points of R(N) was found to be given by

$$\text{Time (mins)} \doteq \frac{N}{25} \qquad (A.25a)$$

## A.4 STEP RESPONSE CALCULATION

As shown previously, the step response, $\Gamma(t)$, can be calculated from

(a) the time, t,

(b) the fundamental frequency, $w_f$, and

(c) the real part curve, R(N), at N points

using equation A.26.

$$\Gamma(t) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{R(N) \sin (2N - 1)\, w_f t}{(2N - 1)} \qquad (A.26)$$

for $t \geqslant 0$.

The problem is then to find

$t_p$ = time of peak overshoot

$\Theta_p$ = peak overshoot

$t_r$ = 10% to 90% rise time

The major difficulty encountered here is that the previous equation is transcendental and cannot be solved explicitly for time in terms of the other parameters. A trial and error solution is therefore imperative.

A linear interpolation between trial points was programmed to approximate the above quantities. The resultant calculation time was comparatively long, being approximately as given in equation A.27 for the IBM 650.

$$\text{Time (mins)} \approx \frac{N}{10} \text{ mins.} \qquad \text{(A.27)}$$

The Bendix program, which calculated all curves except Type "D", requires a running time of approximately 15 times that which the IBM 650 requires. The details of programming are not quite so sophisticated as those described above, but the poorer accuracy resulting from this cause is, for the greater part, undetectable to the scale of the graphs.

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