# PARTIAL LATTICES

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# ABSTRACT

If we consider a subset of a lattice and the lattice operations restricted to that subset, the resulting algebraic structure is not necessarily a lattice, but instead a partial algebra which is called a partial lattice. Partial lattices are of interest because their study solves certain problems in lattice theory.

This thesis began as a review of three papers on partial lattices. The first was an attempt by Yu. I. Sorkin, Dokl. Akad. Nauk SSSR 95 (1954), 931 to establish a system of identities that would characterize partial lattices. Using the paper of N. Funayama (3) we realize that Sorkin's result is in error. Chapter one is an extension of Funayama's results which gives a minimal system of identities to characterize partial lattices.

The second paper reviewed is "On the problem of isomorphism of lattices" by M. M. Gluhov (5) in which Gluhov relied heavily on the incorrect identities given in Sorkin's paper. Chapter two characterizes the free extension of a partial lattice using the identities of Chapter One. Unfortunately, the final result that a partial lattice has a unique basis could not be proven although it is believed to be true.

The third paper reviewed, "On a lattice-theoretical theorem of a kind similar to Grusko's theorem" by M. M. Gluhov (4) studies the free product of lattices. The main result

is that if a lattice is a free product of k lattices, each with a finite number of generators, then the minimum number of generators of the lattice is equal to the sum of the corresponding number of generators of the k free factors.

TABLE OF CONTENTS

## CHAPTER ONE

Let $\mathcal{L}$ be a lattice. We may consider subsets of L and the lattice operations restricted to this subset. If we have chosen a sublattice, then the resulting algebraic structure is a lattice. But, in general, when we consider a subset of the lattice, the restricted operations are merely partial operations and this algebraic structure is called a <u>partial lattice</u>.

Funayama (3) characterized partial lattices but did not present a minimal system of defining equations for partial lattices. The object of Chapter One is to extend Funayama's results.

<u>Definition</u>. Let $\mathcal{L} = \langle L; \underset{L}{\vee}, \underset{L}{\wedge} \rangle$ be a lattice. We consider P, a subset of L, and define two partial binary operations, $\vee$ and $\wedge$ on P as follows:

if $a, b \in P$

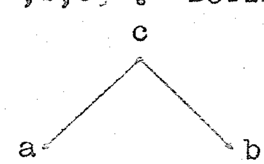
1) $a \vee b$ exists if and only if $a \underset{L}{\vee} b \in P$ and then

   $a \vee b = a \underset{L}{\vee} b$

2) $a \wedge b$ exists if and only if $a \underset{L}{\wedge} b \in P$ and then

   $a \wedge b = a \underset{L}{\wedge} b$ then $\mathcal{P} = \langle P; \vee, \wedge \rangle$ is called a <u>partial lattice</u>.

<u>Example</u>. Let P be the three element set $\{a, b, c\}$. Define $a \vee b = c$. Do not define $a \wedge b$.



We shall now define two partial orderings on P.

Define $a \leq_J b$ if and only if $a \vee b$ exists and $a \vee b = b$.

$a \leq_M b$ if and only if $a \wedge b$ exists and $a \wedge b = a$.

We shall now prove that these two partial orderings are equivalent.

Lemma. $a \leq_J b$ if and only if $a \leq_M b$.

Proof. Identify $\leq_J$ and $\leq_M$ with the existing partial order in the lattice L.

Define lub $\{a,b\} = c$ if and only if $a \leq c$, $b \leq c$, and if there exists d such that $a \leq d$, $b \leq d$, then $c \leq d$.

In a similar manner we can define glb $\{a,b\} = c$ if and only if $c \leq a$, $c \leq b$, and if there exists d such that $d \leq a$, $d \leq b$, then $d \leq c$.

## Imbedding Theorem for Partial Lattices

It is known that any partially ordered set can be embedded in a complete lattice preserving the inclusion relation and all glbs and lubs.

Consider an algebraic structure with two binary partial operations $\vee$ and $\wedge$, $(P; \vee, \wedge)$. If $(P; \vee, \wedge)$ satisfies the following eight identities then it is called a weak partial lattice:

1) $a \vee a = a$

2) $a \wedge a = a$

3) $a \vee (b \vee c) = (a \vee b) \vee c$

4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

5) $a \vee b = b \vee a$

6) $a \wedge b = b \wedge a$

7) $a \vee (a \wedge b) = a$

8) $a \wedge (a \vee b) = a$

The above identities are read as follows: if the left hand side exists and if the inner segment of the right hand side exists then the entire right hand side exists and equals the left hand side.

Example. If $a \vee (b \vee c)$ and $a \vee b$ exist, then $(a \vee b) \vee c$ exists and $a \vee (b \vee c) = (a \vee b) \vee c$.

Note that in the terminology of Funayama such an algebraic structure is called a partial lattice. What we call a partial lattice, Funayama called a strong partial lattice.

It is clear that any partial lattice is a weak partial lattice. Recall that a partial order on an algebraic structure $(P;\vee,\wedge)$ was defined as follows:
for all $a,b \in P$

$a \leq_J b$ if and only if $a \vee b$ exists and $a \vee b = b$.

$a \leq_M b$ if and only if $a \wedge b$ exists and $a \wedge b = a$.

Lemma. $a \leq_J b$ if and only if $a \leq_M b$ in a weak partial lattice.
Proof. Assume $a \leq_J b$. Then $a \vee b$ exists and $a \vee b = b$.
Thus $a \vee b$ exists and so $a = a \wedge (a \vee b) = a \wedge b$.
Therefore $a \leq_M b$.

Assume $a \leq_M b$. Then $a \wedge b$ exists and $a \wedge b = a$.
Thus $b = b \vee (a \wedge b) = b \vee a$.
Therefore $a \leq_J b$.

Define $a \leq b$ if and only if $a \leq_J b$ or if $a \leq_M b$.

To clarify the idea of imbedding theorems of algebraic structures we state the following:

Definition. If $\varphi$ is a one-to-one mapping from an algebraic structure $(P; \vee, \wedge)$ into a lattice $L$, then $\varphi$ is said to be a weak embedding of $P$ into $L$ if

$x \vee y = z$ in $P$ implies that $\varphi(x) \vee \varphi(y) = \varphi(z)$ in $L$.

$x \wedge y = z$ in $P$ implies that $\varphi(x) \wedge \varphi(y) = \varphi(z)$ in $L$.

is said to be a strong embedding of $P$ into $L$ if in addition to the above

$\varphi(x) \vee \varphi(y) = \varphi(z)$ in $L$ implies that $x \vee y = z$ in $P$.

$\varphi(x) \wedge \varphi(y) = \varphi(z)$ in $L$ implies that $x \wedge y = z$ in $P$.

Example. Let $P = \{a,b,c\}$ be an algebraic structure $(P; \vee, \wedge)$ defined by $a \vee b = a$ and $a \vee c = a$. Let $L = \{\varphi(a), \varphi(b), \varphi(c), 0\}$ be a four element lattice defined by $\varphi(b) \vee \varphi(c) = \varphi(a)$. Then $\varphi$ embeds $P$ into $L$ in the weak sense but not in the strong sense.

Definition. $I$ is an ideal of a partial lattice $P$ if $I$ is a subset of $P$ satisfying the two conditions:

(i) $x \in I$, $y \leq x$, implies that $y \in I$;

(ii) if $x, y \in I$ and $x \vee y$ is defined then $x \vee y \in I$.

I is a <u>prime ideal</u> if in addition

(iii) $x \wedge y$ exists and is in I implies that $x \in I$ or $y \in I$.

<u>Definition.</u> D is a <u>dual ideal</u> of a partial lattice P if D is a subset of P satisfying the two conditions:

(i) $x \in D$, $y \geq x$, implies that $y \in D$;

(ii) if $x, y \in D$ and $x \wedge y$ exists then $x \wedge y \in D$.

D is a <u>prime dual ideal</u> if in addition

(iii) $x \vee y$ exists and is in D then $x \in D$ or $y \in D$.

We shall establish a partial order on ideals and dual ideals of a partial lattice.

If $I_1$, $I_2$ are ideals of a partial lattice then $I_1 \leq I_2$ if and only if $I_1 \subseteq I_2$.

If $D_1$, $D_2$ are dual ideals then $D_1 \leq D_2$ if and only if $D_1 \supseteq D_2$.

We shall now establish a minimal system of identities on an algebraic structure $(P; \vee, \wedge)$ such that it can be embedded strongly in a lattice.

$I\vee)$ $a \vee a = a$

$A\vee)$ $a \vee (b \vee c) = (a \vee b) \vee c$

$C\vee)$ $a \vee b = b \vee a$

$D^1\vee)$ $a \vee (a \wedge b) = a$

$D^2\vee)$ $(a \wedge b) \vee a = a$

$D^3\vee)$ $a \vee (b \wedge a) = a$

$D^4\vee)$ $(b \wedge a) \vee a = a$

$P\vee)$ $(a] \vee (b] = (c]$ implies that $a \vee b$ exists and $a \vee b = c$

$I\wedge)$ $a \wedge a = a$

$A\wedge)$ $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$C\wedge)$ $a \wedge b = b \wedge a$

$D^1\wedge)$ $a \wedge (a \vee b) = a$

$D^2\wedge)$ $(a \vee b) \wedge a = a$

$D^3\wedge)$ $a \wedge (b \vee a) = a$

$D^4\wedge)$ $(b \vee a) \wedge a = a$

$P\wedge)$ $[a) \wedge [b) = [c)$ implies that $a \wedge b$ exists and $a \wedge b = c$

The above identities are to be read as before.

Which of the above identities form a minimal system for the embedding of an algebraic structure into a lattice?

<u>Theorem.</u> $\sum_{\vee}^{1} = \{I\vee), A\vee), A\wedge), C\vee), C\wedge), D^1\vee), D^1\wedge), P\vee), P\wedge)\}$

$$\sum_{\wedge}^{1} = \{I\wedge), A\vee), A\wedge), C\vee), C\wedge), D^1\vee), D^1\wedge), P\vee), P\wedge)\}$$

Each of $\sum_{\vee}^{1}$ and $\sum_{\wedge}^{1}$ is a minimal system of identities which ensures that an algebraic structure $(P; \vee, \wedge)$ can be strongly embedded in a lattice $L$.

<u>Proof.</u> First, to show that $\sum$ is minimal.

Without loss of generality we will show that the system $\sum_{\vee}^{1}$ is minimal. To establish the minimality of $\sum_{\vee}^{1}$ we must exhibit nine algebraic structures each of which fails to satisfy one of the identities of $\sum_{\vee}^{1}$ but which satisfies the remaining eight identities of $\sum_{\vee}^{1}$.

1. I$\vee$) does not hold.

Consider $P = \{a\}$ where $a \vee a$ and $a \wedge a$ are not defined. In $P$ all other identities are vacuously satisfied.

2. C$\vee$) does not hold.

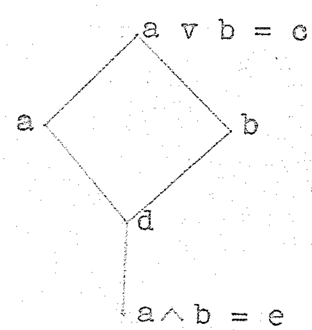
Consider $P = \{a,b,c\}$ where $a \vee b = a$, $a \wedge b = a$, $b \vee a = b$, $b \vee c = b$, $b \wedge c = c$, $c \vee b = b$, $a \vee c = a$, $a \wedge c = a$, $c \vee a = a$.

3. C$\wedge$) does not hold.

Consider $P = \{a,b,c\}$ where $a \vee b = a = b \vee a$, $b \vee c = c = c \vee b$, $a \vee c = c = c \vee a$, $a \wedge b = a$, $b \wedge c = b$, $a \wedge c = a$, $b \wedge a = b$, $c \wedge b = c$, $c \wedge a = a$.

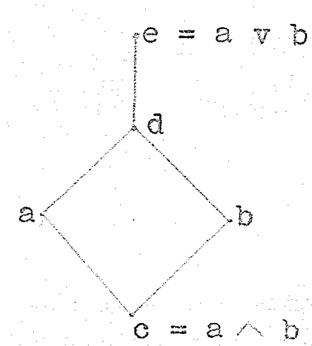
4. A$\wedge$) does not hold.

Consider $P = \{a,b,c,d,e\}$ where $a \vee c = c$, $a \wedge c = a$, $a \vee d = a$, $a \wedge d = d$, $b \vee c = c$, $b \wedge c = b$, $a \wedge b = e$, $b \vee d = b$, $e \wedge d = e$, $e \vee d = d$, $b \wedge d = d$, etc.




5. A$\vee$) does not hold.

Consider $P = \{a,b,c,d,e\}$



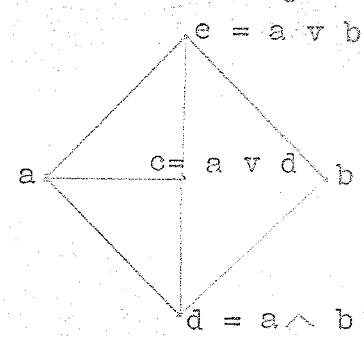
6. $D^1_\vee$) does not hold.

Consider $P = \{a, b, c, d, e\}$ where

$a \leq c$, $d \leq c$, $c \leq e$, $b \vee c = e$

$a \vee d = c$, $a \wedge b = d$, $b \wedge c = d$,

$a \vee b = e$



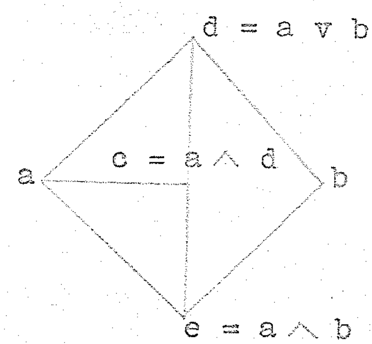

7. $D^1_\wedge$) does not hold.

Consider $P = \{a, b, c, d, e\}$ where

$c \leq a$, $c \leq d$, $e \leq c$, $b \vee c = d$,

$a \vee b = d$, $a \wedge d = e$, $b \wedge c = e$,

$a \wedge b = e$.

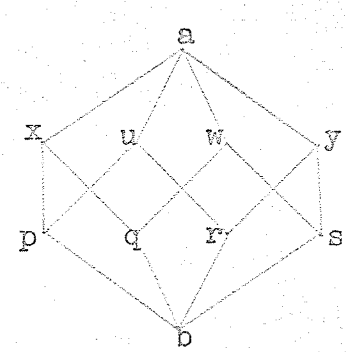



8. $P_\vee$) does not hold.

Consider $P = \{a, b, x, u, w, y, p, q, r, s\}$

where $p \vee r = u$, $q \vee s = w$,

$u \vee w = a$. Do not define $x \vee y$ to be $a$.

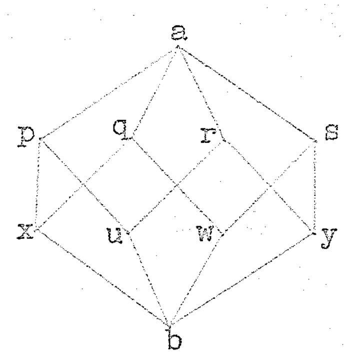



9. $P_\wedge$) does not hold.

Consider $P = \{a, b, x, u, w, y, p, q, r, s\}$

where $p \wedge r = u$, $q \wedge s = w$,

$u \wedge w = b$. Do not define $x \wedge y$ to be $b$.




Now we must establish that an algebraic structure $(P; \vee, \wedge)$ satisfying $\sum^1_\vee$ is a partial lattice.

Recall the definition of $\leq$ on $(P; \vee, \wedge)$.

<u>Lemma</u>. If $a \vee b$ exists, then lub $\{a,b\} = c$ if and only if $a \vee b = c$.

<u>Proof</u>. First, assume that $c =$ lub $\{a,b\}$ .
Then $a \leq c$ and $b \leq c$. That is, $a \vee c = c$ and $b \vee c = c$.
Also by Iv) and Av) $a \leq a \vee b$ and $b \leq a \vee b$.
Thus $a \vee b$ is an upper bound of $\{a,b\}$ . Therefore $c \leq a \vee b$.
That is, $c \vee (a \vee b) = a \vee b$.

By Av) we have that $c \vee (a \vee b) = (c \vee a) \vee b = c \vee b = c$.
Thus $a \vee b = c$.

Now assume that $a \vee b = c$.

We know that $a \vee b = c$ is an upper bound of $\{a,b\}$ as above.
If there exists $d \in P$ such that $d$ is an upper bound of $\{a,b\}$ then we must show that $d \geq c$.
$a \vee (b \vee d)$ and $a \vee b$ exist so that $a \vee (b \vee d) = (a \vee b) \vee d = c \vee d$.

Assume that $d \leq c$. i.e. $c \vee d = c$.
Then $c = c \vee d = (a \vee b) \vee d = a \vee (b \vee d) = a \vee d = d$.
Therefore $c =$ lub $\{a,b\}$ .

Similarly we can prove glb $\{a,b\} = c$ if and only if $a \wedge b = c$.

To proceed with the proof of the imbedding theorem we need the following concept of ideals in a partial lattice.
Let $\sigma = \{x_\alpha\}$ be a subset of a partial lattice P. We define

$\sigma^1 = \{t \mid t \leq x_\alpha \text{ for some } x_\alpha \in \sigma\}$

$\sigma^2 = \{t \mid t \leq a \vee b \text{ where } a,b\ \sigma^1\}$

$\vdots$

$\sigma^n = \{t \mid t \leq a \vee b \text{ where } a,b\ \sigma^{n-1}\}$

The following are lemmas on the ideals of partial lattices.

Lemma. $\bigcup \sigma^n = I(\sigma)$ where $I(\sigma)$ is the ideal generated by $\sigma$.

Proof. By the definition of $\sigma^n$, if $\sigma^n \subset I(\sigma)$ then $\sigma^{n+1} \subset I(\sigma)$. $\sigma^1 \subset I(\sigma)$. Therefore $\sigma^n \subset I(\sigma)$.

It will be sufficient to prove $\bigcup \sigma^n$ is an ideal.

Consider the definition of an ideal of a partial lattice P. Condition (i) is satisfied for $\bigcup \sigma^n$.

Now for condition (ii).

Let $x,y \in \bigcup \sigma^n$ and let $x \vee y$ exist. There must exist s,t such that $x \in \sigma^s$ and $y \in \sigma^t$.

Therefore $x \vee y \in \sigma^u$ where $u = \max\{s,t\} + 1$.

Lemma. $\bigcap \sigma^n$ is an ideal of P.

Proof. Let $x \in \bigcap \sigma^n$. Let $y \leq x$. For each n, $x \in \sigma^n$. Thus $y \in \sigma^n$. Therefore $y \in \bigcap \sigma^n$.

Let $x,y \in \bigcap \sigma^n$ and let $x \vee y$ exist. Then for each n, $x,y \in \sigma^n$. And so $x \vee y \in \sigma^n$. Thus $x \vee y \in \bigcap \sigma^n$.

Lemma. The set of ideals generated by finite subsets of a partial lattice P form a lattice under $\leq$.

Proof. Let $\sigma_1, \sigma_2$ be two finite subsets of P. Then $\sigma_1 \cup \sigma_2$ is also finite. $I(\sigma_1 \cup \sigma_2)$ is the least ideal containing $I(\sigma_1)$ and $I(\sigma_2)$. Similarly for meets.

The necessity condition:

Assume that P is strongly embedded in a lattice L and that the strong embedding mapping is $\varphi$. We must show that $I(x,y) = I(z)$ implies that $z = x \vee y$. (i.e. we must show P$\vee$) ).
$I(z) = I(x,y)$ implies that $z \in \{x,y\}^n$ for some n.
$\varphi(z) \in \varphi(\{x,y\}^n) \subset \{\varphi(x), \varphi(y)\}^n \subset I(\varphi(x), \varphi(y))$.
Since L is a lattice, $\varphi(z) \leq \varphi(x) \vee \varphi(y)$ in L. But $I(z) = I(x,y)$ implies $z \geq x$ and $z \geq y$. Thus $\varphi(z) \geq \varphi(x)$ and $\varphi(z) \geq \varphi(y)$. Therefore $\varphi(z) = \varphi(x) \vee \varphi(y)$.

P is embedded strongly in L so $z = x \vee y$.
That is $I(z) = I(x,y)$ implies $z = x \vee y$.

The necessity of P$\wedge$) is proved in an analogous manner.

The sufficiency condition:

Assume that P is an algebraic structure satisfying one of $\sum_\vee^1$ or $\sum_\wedge^1$. Without loss of generality we will consider $\sum_\vee^1$.

Let L be the set of all pairs of ideals and dual ideals $(I_\alpha, D_\alpha)$ of P such that $x \in I_\sigma$ and $y \in D_\sigma$ imply $x \leq y$, where I and D may be void. We define a partial ordering $\geq$ in L as follows:

$(I_\alpha, D_\alpha) \geq (I_\beta, D_\beta)$ if and only if $I_\alpha \geq I_\beta$ and $D_\alpha \geq D_\beta$.

Under the partial ordering $\geq$ L is a complete lattice. Let $\{(I_\beta, D_\beta)\}$ be any subset of L. Then $(\vee I_\beta, \vee D_\beta) = \text{lub}\{(I_\beta, D_\beta)\}$ where I is the least ideal containing every $I_\beta$ and $\vee D_\beta$ is the largest dual ideal contained in every $D_\beta$. In a similar manner $(\wedge I_\beta, \wedge D_\beta) = \text{glb}\{(I_\beta, D_\beta)\}$.

Now we must show that P is strongly embedded in L. Let I(x) ( D(x) ) be the principal (dual) ideal generated by x. Let $\varphi$ be a mapping from P into L such that $\varphi(x) = (I(x), D(x))$. $\varphi$ is a one-to-one mapping of P into L. If $z = x \vee y$ then $I(z) = I(x) \vee I(y)$. $D(z) = D(x) \vee D(y)$. Then $\varphi(z) = (I(z), D(z)) = (I(x) \vee I(y), D(x) \vee D(y)) = \varphi(x) \vee \varphi(y)$. Now $\varphi(z) = \varphi(x) \vee \varphi(y)$ implies that $I(z) = I(x) \vee I(y)$. Using the identity Pv) $z = x \vee y$. Similarly using the identity P$\wedge$) $\varphi(z) = \varphi(x) \wedge \varphi(y)$ implies $z = x \wedge y$. This completes the proof of the theorem in Chapter One.

CHAPTER TWO

The purpose of this chapter is to give an algorithm for the extension of a partial lattice to a lattice.

Consider an algebraic structure $P(x_1,\ldots,x_n ; S)$ consisting of elements $\{x_1,\ldots,x_n\}$ and a partial (or incomplete) Cayley table S where S satisfies the following identities for all a, b, c $\in$ P :

1) $a * a = a$

2) $a * b = b * a$

3) $a * (b * c) = (a * b) * c$

4) $a * (a *' b) = a$

5) $(a] \vee (b] = (c]$ implies $a \vee b$ exists and $a \vee b = c$.

6) $[a) \wedge [b) = [c)$ implies $a \wedge b$ exists and $a \wedge b = c$,

where $*$ is either $\vee$ or $\wedge$ and $*'$ is the alternate operation to $*$.

We define the <u>completion</u> of S, $\overline{S}$, to be the Cayley table forced under the identities 1) through 6).

Then $P(x_1,\ldots,x_n ; \overline{S})$ is a <u>partial lattice.</u>

A Cayley table, S , is called <u>irreducible</u> if x is a relation in S then $\overline{S - \{x\}} \neq \overline{S}$ .

Using the results of Chapter One, a free extension of a partial lattice $P(x_1,\ldots,x_n ; S)$ is a lattice, defined by the generating elements $x_1,\ldots,x_n$ and a system of defining relations $\overline{S}$. Denote the free extension by FL(P).

Conversely, if L is a finite lattice, then it is a free extension of some finite partial lattice. This partial lattice can be found using Evans algorithm (1).

Evans Algorithm for the Construction of FL(P)

If we are given an algebraic structure $P(x_1, \ldots, x_n ; S)$, then we complete S to obtain $\overline{S}$. If all the entries of the table $\overline{S}$ are filled, then P is a lattice and FL(P) $\equiv$ P.

But suppose some entry, say (i,j) is empty. Then we add a new element $x_{n+1}$ to P and add to $\overline{S}$ either the relation $x_i \vee x_j = x_{n+1}$ or $x_i \wedge x_j = x_{n+1}$ depending which one is undefined. Now we must substitute $x_{n+1}$ into the identities 1) through 6) and thereby fill the entries that we are forced to define (i.e. we must complete the new Cayley table). The result of this operation is a partial lattice $P_1(x_1, \ldots, x_n, x_{n+1} ; S_1)$.

If all the entries of the table $S_1$ are filled, then the process is complete. That is FL(P) $\equiv P_1$. Otherwise, we must fill another empty entry in $\overline{S}$ and so construct a partial lattice $P_2$ in a similar manner.

In this filling proceedure we get either a finite or a countable sequence of partial lattices, each of which can be embedded in the following one.

A partial lattice $P_1$ is said to be weakly embedded in a partial lattice $P_2$ by an embedding mapping if $x, y, z \in P_1$ and $x \underset{P_1}{*} y = z$ implies $\varphi(x) \underset{P_2}{*} \varphi(y) = \varphi(z)$.

A partial lattice $P_1$ is said to be strongly embedded in a partial lattice $P_2$ by an embedding mapping $\varphi$ if $P_1$ is weakly embedded in $P_2$ and for $x,y,z \in P_1$ if $\varphi(x) \underset{P_2}{*} \varphi(y) = \varphi(z)$ implies that $x \underset{P_1}{*} y = z$.

If the strong embedding of a partial lattice $P_1$ into a partial lattice $P_2$ is denoted by $P_1 \Rightarrow P_2$ then we obtain a sequence of partial lattices $P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \ldots \Rightarrow P_k \Rightarrow \ldots$ . This sequence of partial lattices will be called the sequence of extensions of the partial lattice P.

Theorem 1. (Funayama (3) ). A partial lattice can be strongly embedded in a lattice L if and only if identities 1) through 6) hold for the partial lattice.

Corollary. If a partial lattice $P_k$ belongs to a certain sequence of extensions of a partial lattice P, then $FL(P_k) \cong FL(P)$.

Proof: By theorem 1 $FL(P_1) \cong FL(P)$.

The remainder of the theorem is proved by induction on k.

Let us assume the element $x_l$ and the relation $x_{s_l} \vee x_{t_l} = x_l$ have been added at each step l of the extension.

Assume the theorem is true for $k = m - 1$. Now for $k = m$.

Use the identity map $\varphi: P_{m-1} \to P_m$ . This map extends to a map $\bar{\varphi}$ defined by $\bar{\varphi}(x_i) = x_i$

$$\bar{\varphi}(x_{s_m} \vee x_{t_m}) = x_m .$$

Then $\bar{\varphi}$ is the required isomorphism.

Conclusion: If the sequence $P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \dots$ of a partial lattice P is finite and ends at $P_k$ then $FL(P) \cong P_k$. If, however, the sequence is infinite then $FL(P) \cong \bigcup_{i=1}^{\infty} P_i$.

Definition. A partial lattice Q will be called a free extension of a finite partial lattice P if Q belongs to some sequence of extensions of a partial lattice P.

We shall now establish the concept of basis for a partial lattice.

Definition. A partial lattice $P_o$ will be called a basis of a partial lattice P if P is a finitely free extension of $P_o$, and $P_o$ is not a finitely free extension of any partial lattice different from itself.

Theorem 2. If $P_1(x_1,\dots,x_{m-1} ; S_1) \Rightarrow P_2(x_1,\dots,x_m ; S_2)$ then for any partial lattice $P_2$ to be a finitely free extension of a partial lattice $P_1$ it is necessary and sufficient that any irreducible system of defining relations of the partial lattice $P_2$ which includes the system of defining relations of the partial lattice $P_1$, contains a relation of the form $x_i * x_j = x_m$ where $x_i \neq x_m$ , $x_j \neq x_m$, and $S_2$ does not contain any other relation with $x_m$.

Proof. (Necessity) Assume that $P_1 \Rightarrow P_2$ and suppose the new relation added is $x_i * x_j = x_m$ (1)

Suppose $T_1$ is an irreducible system of defining relations of

$P_1$. Let $T_2$ be an irreducible system of defining relations of $P_2$ which contains $T_1$. We are required to prove that $T_2$ contains exactly one relation of the form $x_{i_1} * x_{j_1} = x_m$. First, $T_2$ must contain at least one relation involving $x_m$ because $T_2 - \{\text{all relations involving } x_m\} \neq S_2$. Let a relation containing $x_m$ be denoted by $\Delta$. If (1) is a relation of $T_2$ then we are finished as $T_2$ is irreducible, i.e. the relations of $T_2$ are the relations of $T_1$ and (1).

If (1) is not a relation of $T_2$ then we know that (1) is a consequence of $T_2$ because (1) is a relation in $S_2$. If we are now able to show that (1) is a consequence of $T_1$ and then we will be finished because $T_1 \_ T_2$, is a relation of $T_2$, and $T_1$ + is essentially a basis of the consequences of $T_2$.

Assume the statement is false. That is, in the extension from $P_1$ to $P_2$ in which we added the new element $x_m$ and the relation $\Delta$, that $x_i * x_j$ was not defined. Note that $x_i * x_j$ could not have been defined as anything different because there is no collapsing in the finitely free extension of P. The extension is $P_1 \Rightarrow P_1 + \Delta$. Let us perform another extension which fills (i,j) square.

$$P_1 \Rightarrow P_1 + \Delta \Rightarrow P_1 + \Delta + x_i * x_j = x_{m+1} .$$

If we are able to show that $\Delta$ must have the form $x_{i_1} * x_{j_1} = x_m$, then $x_i * x_j = x_{m+1}$ implies that $x_{i_1} * x_{j_1} = x_{m+1}$ (because $\Delta$ is a consequence of $T_1$ + (1) ), which is a contradiction.

Now we must show that $\triangle$ has the form $x_{i_1} * x_{j_1} = x_m$ . Assume the contrary, i.e. $\triangle$ is not in the form

$$x_{i_1} * x_{j_1} = x_m \text{ where } i_1, j_1 < m.$$

Then the tabular system $S_1$ consists of relations of the form

$$\begin{cases} x_i * x_j = x_k & i,j,k < m \\ x_i * x_m = x_s & i < m \end{cases}$$

If we are able to prove that $S_1$ is closed under the identities 1) - 6) then $x_i * x_j = x_m$ where $i, j < m$ will not be a consequence of $S_1$.

It is obvious that $S_1$ is closed under 1) and 2).

Now we must show $S_1$ is closed under 4). If $a \wedge (a \vee x_m) = a$ then $x_m \wedge (x_m \vee a) = x_m$ i.e. if the $(i,j)$ position is filled then 4) forces the $(j,i)$ position to be filled by the opposite element. Therefore $S_1$ is closed under 4).

We next show $S_1$ is closed under 3). Let $* \equiv \vee$. Let us assume $a \vee (x_m \vee c) = (a \vee x_m) \vee c$ where $x_m \vee c = d$ and $a \vee x_m = e$. Also assume that $a \vee d$ exists and $a \vee d$ is a relation of $S_1$. Therefore $e \vee c$ exists and $e \vee c = a \vee d$. Now we assume that $e \vee c$ is not a relation of $S_1$. i.e. $e \vee c = x_m$ where $e \neq x_m$ and $c \neq x_m$. Therefore $a \vee d = e \vee c = x_m$. But $a \vee d$ is a relation of $S_1$. Thus, either $a = x_m$ or $d = x_m$. The statement is obvious if $a = x_m$.

If $d = x_m$ then $a \vee d = a \vee x_m = e$. Thus $e = x_m$ . This is a contradiction. In a similar manner we can show that $S_1$ is closed under $* \equiv \wedge$.

Finally, we must show that $S_1$ is closed under 5) (Note that 6) will be the dual case.):

$(a] \vee (b] = (x_m]$ implies $a \vee b$ exists and $a \vee b = x_m$ where $a \neq x_m$ and $b \neq x_m$.

To prove the above we must describe $(a] \vee (b]$ .

Define $K_0 = \{(a] \cup (b]\}$

$\vdots$

$K_{i+1} = \{x \mid x \leq c \vee d \text{ where } c,d \in K_i \text{ and } c \vee d \text{ exists}\}$

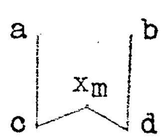
Then $(a] \vee (b] = \bigcup_{0}^{\infty} K_i$ .

The proof of the above statement is by induction on i.

(i) $i = 0$. $x_m \in K_0$ . i.e. $x_m \in (a]$ or $x_m \in (b]$ .

$x_m \leq a \leq a \vee b = x_m$. So $a = x_m$. This is a contradiction.

(ii) $i = 1$. $x_m \in K_1$. Then $x_m \leq c \vee d$ where $c \leq a$ and $d \leq b$.




If $c \neq x_m$, $d \neq x_m$ then $x_m \leq c \vee d \leq a \vee b = x_m$. Therefore $c \vee d = x_m$. This is a contradiction. If $c = x_m$ then (ii) reduces to (i).

Assume true for $k = i$.

(iii) $k = i + 1$. $x_m \in K_{i+1}$. Then $x_m \leq c \vee d$ where $c,d \in K_i$ and $c \vee d$ exists.

$$
\begin{array}{lll}
c \in K_i & d \in K_i & \\
c \leq c_1 \vee c_2 & d \leq d_1 \vee d_2 & \text{where } c_1,c_2,d_1,d_2 \in K_{i-1} \\
c_1 \leq c_{1_1} \vee c_{2_1} & d_1 \leq d_{1_1} \vee d_{2_1} & \text{where } c_{1_1},c_{2_1},d_{1_1},d_{2_1} \in K_{i-2} \\
\vdots & \vdots & \\
c_{1_{i-1}} \leq c_{1_i} \vee c_{2_i} & d_{1_{i-1}} \leq d_{1_i} \vee d_{2_i} & \text{where } c_{1_i},c_{2_i},d_{1_i},d_{2_i} \in K_o
\end{array}
$$

If $c \neq x_m$, $d \neq x_m$ then $x_m \leq c \vee d \leq a \vee b = x_m$ implies that $x_m = c \vee d$. This is a contradiction. If $c = x_m$ (iii) reduces to $K_{i-1}$.

Thus $S_1$ is closed under the identities 1) through 6). Hence $x_i * x_j = x_m$ where $i,j < m$ is not a consequence of $S_1$. This is a contradiction. Thus $\Delta$ must be in the form $x_{j_1} * x_{j_1} = x_m$ where $i_1, j_1 < m$. The sufficiency is evident.

Theorem 3. There exists an algorithm to find a basis of a finite partial lattice $P(x_1,\ldots,x_n : S)$ in a finite number of steps. The algorithm is:

Remove from P an element $x_i$ and remove from S all relations in which $x_i$ occur. The result $P^{(i)}(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n ; S^{(i)})$ is a partial lattice.

Use Theorem 2 to determine whether P is a finite free extension of $P^{(i)}$.

If P is not the finite free extension of $P^{(i)}$ for $i = 1,\ldots,n$ then P is its own basis.

If P is a finite free extension of $P^{(i)}$ for some i then apply the above proceedure to the partial lattice $P^{(i)}$.

Each time this process is completed the number of elements of the resulting partial lattice is decreased by one. The number of elements in P is finite. Therefore in a finite number of steps we will get a partial lattice. $P_0$ which is a basis for P.

We shall now establish that a basis is unique.

Definition. An element $x_1$ of a partial lattice $P(x_1, \ldots, x_n ; S)$ will be called k-removable if P is a finite free extension of some partial lattice with (n - k) elements not containing $x_1$.

An element $x_1$ will be called removable if it is removable for some k; in that case k will be called the order of removability of the element $x_1$.

The following is believed to be a true statement but as yet no proof exists:

Theorem 4. A removable element of a finite partial lattice cannot be a member of its basis. Using this conjecture we could prove the following interesting results:

Theorem 5. Any partial lattice has a unique basis.

Proof. Let P have bases R and S. If $x \in R$ and $x \notin S$ then x must be a removable element of P. Using theorem 4, $x \notin R$.

Corollary 1. Let $P_1$ and $P_2$ be two partial lattices. A necessary and sufficient condition for the lattices $FL(P_1)$ and $FL(P_2)$ to be isomorphic is that the bases of the partial lattices $P_1$ and $P_2$ must be isomorphic.

Proof. The sufficiency follows from the uniqueness of a free extension and the necessity from the uniqueness of the basis in a partial lattice.

Corollary 2. The group of automorphisms of the lattice FL(P) is isomorphic to the group of automorphisms of the basis of the partial lattice P. This group will be isomorphic to a certain subgroup of the symmetric group $S_n$, where n is the number of elements in the basis of P.

## CHAPTER THREE

The purpose of Chapter Three is to consider the free product of lattices with a finite number of generators.

A. G. Gruško (6) proved that if a free group S with a finite number of generators can be mapped homomorphically onto a group G, which can be decomposed into a free product of its subgroups $A_1, A_2, \ldots, A_k$ then it is possible to choose in S a system of free factors such that under the given homomorphism, each generator can be mapped into one of the free factors $A_1$, $A_2, \ldots, A_k$. A. I. Žukov (10) proved a similar theorem for non associative algebras.

In lattice theory, the analogous theorem is false. P. M. Whitman (9) proved that under the homomorphism mapping from the free lattice with three generators, FL(3), onto the free lattice with two generators, FL(2), there does not exist a system of generators of FL(3) such that each generator can be mapped into one of the free factors of the lattice FL(2) = FL(1) * FL(1). Therefore, it is not possible to map each generator of a lattice into one of the free factors.

We shall prove a theorem for lattices which is analogous to one of the basic corollaries of Gruško's theorem one, namely:

Theorem. If $L = L_1 * L_2 * \ldots * L_k$ is an arbitrary decomposition into a free product of a lattice L with a finite number of generators, then the minimum number of generators of the lattice L is equal to the sum of the number of generators of each of the free factors $L_1, \ldots, L_k$.

We will consider the case when $k = 2$ i.e. $L = L_1 * L_2$.

Definition. An element $u \in L$ will be called intrinsic with respect to $L_i$ if there exist $x_1, x_2 \in L_i$ such that $x_1 \leq u \leq x_2$ where $i = 1$ or $2$. We shall say an element $u \in L$ is intrinsic if $u$ is intrinsic with respect to $L_1$ or to $L_2$. We note, by the definition of free product of lattices, that elements of the lattices $L_1$ and $L_2$ are intrinsic and that no element can be intrinsic with respect to both lattices.

It is obvious that the free product of the lattices $L_1$ and $L_2$ coincides with the free extension FL(P) of the partial lattice P where P is the cardinal sum $L_1 + L_2$ of the lattices $L_1$ and $L_2$. Recall that Chapter Two characterized the free extension of the partial lattice P.

Definition. The minimum number of generators of a partial lattice $P_\alpha$ will be called its rank, denoted by rank $P_\alpha$.

Consider the sequence of partial lattices in the extension of the partial lattice P:

$$P = P_0 \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \ldots \Rightarrow P_{n-1} \Rightarrow P_n \quad \ldots$$

Suppose that the relation added to the irreducible set of defining relations in the extension $P_{n-1} \Rightarrow P_n$ is of the form $x_i \vee x_j = x_n$ where $i, j < n$.

Lemma 1. In the partial lattice $P_n$, the element $x_n$ cannot be decomposed into an intersection of two elements distinct from $x_n$. i.e. if in $P_n$ there is a relation of the form $x_s \wedge x_t = x_n$ then either $x_s = x_n$ or $x_t = x_n$.

Proof. This follows immediately from theorem two of Chapter Two.

Lemma 2. If the relation $x_n \wedge x_s = x_t$ where $x_s \neq x_n$, $x_t \neq x_n$ occurs in $P_n$ then there exists a set of elements $w_1, w_2, \ldots, w_q$ in $P_n$ such that:

1. $w_i > x_n \quad i = 1, \ldots, q$
2. $\bigwedge_{i=1}^{q} (w_i \wedge x_s) = x_t$

Proof. Assume there does not exist $y \in P_n$ such that $y > x_n$ where $x_n$ is added in the extension $P_{n-1} \Rightarrow P_n$ by the relation $x_i \vee x_j = x_n$. Then if the relation $x_n \wedge x_s = x_t$ where $x_s \neq x_n$, $x_t \neq x_n$, $x_s \neq x_t$ occurs in $P_n$ then $x_s \not> x_n$. We also know that $x_n \not> x_s$. Therefore $x_n$ must be incomparable with $x_s$. In the extension $P_{n-1} \Rightarrow P_n$ how could the relation $x_n \wedge x_s = x_t$ have been forced? It is evident that it was not by 1), 2), 4), or 6). The only possibility is 3), i.e. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ where the right hand side is defined and $b \wedge c$ is defined.

Consider the case when $b \wedge c = x_n$. By Lemma 1 this can occur only if either $b = x_n$ or if $c = x_n$. Assume $b = x_n$.

Then $a = x_s$ and $(a \wedge b) \wedge c = x_t$. So $x_n \wedge c = x_n$, i.e. $c > x_n$, which is a contradiction.

When $a = x_n$, $b \wedge c = x_s$ is defined, $x_n \wedge b$ is defined, and $(x_n \wedge b) \wedge c = x_t$ is defined then we must determine the way in which $x_n \wedge b$ was defined in $P_n$. By theorem two of Chapter Two, there exist $y > x_n$ in $P_n$. The set of all such elements in $P_n$ is $\{y_1, \ldots, y_m\}$. Now we consider the set of all covers of $x_n$; call them $\{w_1, w_2, \ldots, w_q\}$. We note that $\{w_1, \ldots, w_q\} \subset \{y_1, \ldots, y_m\}$. Also $u_i = w_i \wedge x_s \geq x_t$. $x_t = x_n \wedge x_s = x_n \wedge (x_s \wedge u_i) = (x_n \wedge x_s) \wedge u_i = x_t \wedge u_i$. And

$$\bigwedge_{i=1}^{q} (w_i \wedge x_s) = x_t.$$

<u>Lemma 3</u>. If $u \in P_{n-1}$ and in $P_n$ $u = \varphi(a_1, \ldots, a_m, x_n)$ where $a_i \neq x_n$ then $u = \Psi(a_1, \ldots, a_m, w_1, w_2, \ldots)$ where $w_1, w_2, \ldots$ is the set $W$ of all elements which cover $x_n$ in $P_n$.

<u>Proof</u>. The proof is by induction on $l(\varphi)$.

If $l(\varphi) = 2$ then either $u = a \wedge x_n$ or $u = a \vee x_n$. When $u = a \wedge x_n$ we know by lemma 2 that there exist $w_1, w_2, \ldots \in P_n$ such that $w_i > x_n$ and $\bigwedge_i (w_i \wedge a) = u$. Let $\Psi$ have the form $u = \bigwedge_i (w_i \wedge a)$.

When $u = a \vee x_n$ we know that there exists a smallest $w$ such that $u \geq w \geq x_n$ and such that $w$ covers $x_n$. Let $\Psi$ have the form $u = w \vee a$.

Assume true for $l(\varphi) = n - 1$.

When $l(\varphi) = n$, $\varphi$ has two possibilities:

either $u = \varphi_1 \vee \varphi_2$

or $u = \varphi_1 \wedge \varphi_2$ where $l(\varphi_1)$, $l(\varphi_2) < n$.

This case has been covered if neither $\varphi_1 = x_n$ nor $\varphi_2 = x_n$.

Without loss of generality we may consider $\varphi_2 = x_n$. Then $\varphi$ either has the form $u = \varphi_1 \wedge x_n$ or $u = \varphi_1 \vee x_n$ where $l(\varphi_1) < n$.

When $u = \varphi_1 \vee x_n$ there exists a smallest $w$ such that $u \geqslant w > x_n$ such that $w$ covers $x_n$. Let $\psi$ have the form $u = \varphi_1 \vee w$.

When $u = \varphi_1 \wedge x_n$ then let $\psi$ be of the form $u = \bigwedge_i (w_i \wedge \varphi_1)$ where $w_i$ are covers of $x_n$.

<u>Lemma 4</u>. If $T = \{u_1, u_2, \ldots, u_r, x_n\}$ is a system of generators of the partial lattice $P_n$ then $T_1 = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots\}$ where $w_i$ W is also a system of generators of $P_n$. The proof follows immediately from lemma 3.

<u>Lemma 5</u>. If an element $u$ appears in an irreducible system of generators of the lattice L, then it is an intrinsic element.

<u>Proof</u>. Every irreducible system of generators of a lattice L appears in some partial lattice $P_\alpha$ of the extension

$$L_1 + L_2 = P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_{\alpha-1} \Rightarrow P_\alpha \Rightarrow \ldots$$

Hence to prove this lemma we can induct on $\alpha$. The induction will be of the form if $T = \{u_1, \ldots, u_r, x_n\}$ is an irreducible system of generators of $P_n$ then $x_n$ is intrinsic.

$\alpha = 0$. Here $P_0 = L_1 + L_2$. Every element of a cardinal sum is intrinsic by definition.

Assume the statement is true when $\alpha = n-1$, i.e. every element of an irreducible system of generators of $P_{n-1}$ is intrinsic. Consider $\alpha = n$. Assume, without loss of generality, that $x_n$ and the relation $x_s \vee x_t = x_n$ has been added to the partial lattice $P_{n-1}$. Let $T = \{u_1, \ldots, u_r, x_n\}$ be an irreducible system of generators of $P_n$, i.e. $T \subseteq P_n$. First, $\{u_1, \ldots, u_r\}$ does not generate $P_{n-1}$. Secondly, $\{u_1, \ldots, u_r\} \cup W$ generates $P_{n-1}$ where W is the set of covers of $x_n$ in $P_n$. Therefore there exists T", an irreducible set of generators of $P_{n-1}$, and there exists some $w_{i_0} \in T"$ such that $w_{i_0}$ covers $x_n$. i.e. $w_{i_0}$ is intrinsic.

Now we assume there does not exist an element $u \in T"$ such that $x_n > u$, i.e. for all elements $y \in T"$ either $x_n < y$ or $x_n$ is incomparable with y. Thus neither $x_s$ nor $x_t \in T"$ and so $x_s = \varphi(y_1, \ldots, y_r)$ and $x_t = \psi(y_1, \ldots, y_r)$. Neither $\varphi$ nor $\psi$ has the form $y_i \vee y_j$ for any i or j. Therefore we can say that $\varphi$ has the form $x_s = y_1 \wedge y_2$ and $x_t = y_3 \wedge y_4$. In this case T" is reducible. This is a contradiction. Thus there exists an intrinsic element $u \in T"$ such that $x_n > u$. Therefore $x_n$ is also intrinsic.

<u>Lemma 6.</u> There does not exist a set of four intrinsic elements

$u', u'', v', v''$ in any partial lattice $P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \ldots \Rightarrow P_\alpha \Rightarrow \ldots$ satisfying the following three conditions:

1) $u' \neq u''$, $v' \neq v''$
2) $u'$ covers $v''$ , $u''$ covers $v'$
3) $u' > v'$ , $u'' > v''$

Proof. $u'$ covers $v''$ therefore $u'' > v''$. $u' > v''$ and $u'' > v''$. Suppose that $u'' < u'$. This is impossible as $u'$ covers $v''$. Suppose that $u' < u''$. This is impossible as $u''$ covers $v'$. Therefore $u'$ must be incomparable to $u''$. Since $u'$ and $u''$ are intrinsic elements define $u' \wedge u'' = u$. Now $u \geqslant v''$. This is impossible since $u'$ covers $v''$. Therefore $u' \wedge u'' = v''$. In a similar manner we can show that $u' \wedge u'' = v'$. Therefore $v' = v''$. This is a contradiction.

Lemma 7. Rank L = rank P.

Proof. It will be sufficient to prove that rank $P_n$ = rank $P_{n-1}$. Every system of generators of the partial lattice $P_{n-1}$ is also a system of generators of the partial lattice $P_n$. Therefore, we have rank $P_n \leqslant$ rank $P_{n-1}$. We must now show that rank $P_{n-1} \leqslant$ rank $P_n$. It will be sufficient to prove that if $T = \{u_1, \ldots, u_r, x_n\}$ is an irreducible system of generators of $P_n$ then there exists a system of generators of $P_{n-1}$ which does not contain more than r+1 elements.

Let T be a system of generators of $P_n$ and let $x_n = x_s \vee x_t$. Then $T' = \{u_1, \ldots, u_r, x_s, x_t\}$ is also a system of generators of

$P_{n-1}$. The theorem is proved if T' is reducible. Suppose that T' is irreducible. Then every element of T' is intrinsic, i.e. $x_s$ and $x_t$ are intrinsic. Using lemma 4 we know that $T_1 = \{u_1, \ldots, u_r, w_1, w_2, \ldots\}$ is a system of generating elements of $P_{n-1}$ where $w_i \in W$ is the set of covers of $x_n$ in $P_n$. Let $T_2 \subseteq T_1$ be an irreducible system of generators of $P_n$. Using the same argument as in lemma 5 we know that at least one element of W must occur in $T_2$. Suppose there are two elements, $w_1$, $w_2 \in W$ which belong to $T_2$. Then we have four intrinsic elements of $P_{n-1}$ such that $w_1$ covers $x_s$, $w_2$ covers $x_t$, $x_1 > x_t$, $w_2 > x_s$, $x_s \neq x_t$, $w_1 \neq w_2$. By lemma 6 this is impossible. Therefore $w_1 = w_2$, i.e. $T_2$ contains only one element from W. $T_2 \subseteq P_{n-1}$. Therefore rank $P_{n-1} \leq$ rank $P_n$.

Lemma 7 states that the minimum number of generators of the lattice L is equal to the minimum number of generators of the partial lattice $P = L_1 + L_2$, and this proves the theorem of Chapter Three.

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