# Quantum Mechanics with Non-Abelian Fields and Potentials

Robert Allan Corns

A thesis
presented to the University of Manitoba
in partial fulfillment of the
requirements for the degree of
DOCTOR of PHILOSOPHY
in
PHYSICS

Winnipeg, Manitoba, Canada 1988

ISBN 0-315-44094-5

# QUANTUM MECHANICS WITH NON-ABELIAN FIELDS AND POTENTIALS

BY

ROBERT ALLAN CORNS

A thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

*To my parents*

I hearby declare that I am the sole author of this thesis. I authorize the University of Manitoba to lend this thesis to other institutions or individuals for the purpose of scholarly research. I further authorize anyone to reproduce this thesis by photocopying or by other means, in total or in part, for the purposes of scholarly research.

Robert A. Corns

## ABSTRACT

The first part of this thesis explores solutions to the Schrödinger equation for systems subject to classical Yang-Mills fields. Under a weak set of assumptions on the potentials, we prove the existence of a family of operators, called *the Schrödinger evolution*, which map vectors in Hilbert space to solutions of the Schrödinger equation. By strengthening our assumptions it is possible to show that these evolution operators are integral operators. The collection of their kernels is commonly called *the propagator* in the physics literature. Through a constructive technique, an explicit formula for the propagator is found.

The second part of this dissertation derives a class of sum rules, commonly known as Levinson's theorem, for a single particle system. These rules relate the number of bound states to the energy integral of the the trace of the time delay operator. In particular we will incorporate into these rules detailed information about the spin structure of the system.

## ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to my friend and mentor Dr. Thomas Osborn. I am also deeply indebted to my friends and colleges for their valuable discussions. In particular, I wish to thank Dr. Craig Platt for his guidance in using TEX to typeset this manuscript. I would like to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada, the Province of Manitoba, and the University of Manitoba. Finally I would like to thank my parents, whose faith and inspiration have given me the strength to persevere.

# TABLE OF CONTENTS

# CHAPTER 1

## Introduction

In this treatise we address two problems arising in the theory of quantum mechanics for systems subject to non-Abelian potentials and external fields. The first is concerned with the study of a quantum system composed of $N$ particles interacting with external vector and scalar Yang-Mills fields. A special case of these fields occurs in the description of an $N$-particle atomic (or subatomic) spin system interacting with an external electromagnetic field. This latter problem manifests only a limited form of non-Abelian behaviour because the description of the electromagnetic field via the potentials $\vec{A}$ and $\phi$ is free of spin labels. All the spin interaction here is confined to the coupling of the spin magnetic moments among themselves or to the external magnetic field. For the more general Yang-Mills fields, $\vec{A}$ and $\phi$ become hermitian-matrix valued fields which gives use the most general non-Abelian structure possible. The second topic we wish to discuss is a class of sum rules that relate the number of bound states for a single particle system (without electromagnetic fields) to the energy integral of the trace of the time delay operator.

In as much as these two problems can be treated separately (although they are not wholly unrelated) we shall divide this dissertation into two parts.

## 1.1 Part 1: The Non-Abelian Time Dependent Schrödinger Equation

We wish to study a nonrelativistic $N$-body system subject to (external) classical Yang-Mills fields [Mor 83]. These fields have built into them a matrix structure which is used to describe internal degrees of freedom. For convenience we shall use "spin" as the generic name to label the matrix structure although in actuality its interpretation may be something completely different (eg. isospin) depending upon the physics being described in a given situation.

The dynamical evolution of a quantum mechanical system is determined by Schrödinger's time dependent equation of motion

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(x,t)\psi(x,t). \tag{1.1}$$

Here $x$ denotes a generic point in $\mathbb{R}^d$ that specifies the positions of all the pariticles in the system. If the $\beta^{th}$ particle has spin $s_\beta$, then $\psi(x,t)$ is a column vector of dimension

$$s = \sum_{\beta=1}^{n}(2s_\beta + 1),$$

and it is the pointwise representation of the state vector $\psi(t) \in L^2(\mathbb{R}^d;\mathbb{C}^s)$. The time parameter $t$, lies in the compact set $[0,T]$ and the Hamiltonian has the differential structure

$$H(x,t) = \frac{1}{2m}\left[\frac{\hbar}{i}\nabla I - a(x,t)\right]^2 + v(x,t). \tag{1.2}$$

Here $I$ is the unit $s \times s$ matrix and $\nabla$ is the $d$-dimensional gradient. The potential $v$ maps $\mathbb{R}^d \times [0,T] \to \mathbb{C}^{s\times s}$. For a physical situation we have the necessary pointwise condition on $v$ that for a.a. $(x,t)$, $v(x,t)$ is a hermitian matrix. The hermiticity is also useful for demonstrating the existence of solutions to (1.1) for a broad class of potentials. We will eventually make much stronger assumptions on $v$ that will allow us to relax the hermiticity requirement. Non-hermitian potentials create a source or sink for probability and are known in the physics literature as the optical potential [MS 70]. The vector potential $a$ is a $d$-dimensional vector whose components are $s \times s$ hermitian matrices.

We remark that such a Hamiltonian is sufficient to describe the $N$-body problem in atomic physics. For such systems, suppose the external electromagnetic fields $\vec{E}$ and $\vec{B}$ are generated by the vector potential $\vec{A}$ and the scalar potential $\phi$. Then the vector potential appearing in (1.2) is related to $\vec{A}$ by the formula

$$a(x,t) = \Big(q_1 \vec{A}(\vec{x}_1,t), \cdots, q_N \vec{A}(\vec{x}_N,t)\Big) I,$$

where $q_\beta$ is the charge of the $\beta^{th}$ particle and $\vec{x}_\beta$ is its position. The perturbing potential may be written in the form

$$v(x,t) = -\sum_{\beta=1}^{N} \eta_\beta \vec{S}_\beta \cdot \vec{B}(\vec{x}_\beta,t) + \sum_{\beta=1}^{N} q_\beta \phi(\vec{x}_\beta,t) I + \tilde{v}(x,t).$$

The vector $\vec{S}_\beta$ denotes the spin operator for the $\beta^{th}$ particle. For the example of a spin half particle, $\vec{S}_\beta$ is $\hbar/2$ times the vector formed from the three Pauli matrices. Substituting these expressions into (1.2), the Hamiltonian has the form

$$\begin{aligned} H(t,m) = \frac{1}{2m} \sum_{\beta=1}^{N} \left[ \frac{\hbar}{i} \vec{\nabla}_{\vec{x}_\beta} - q_\beta \vec{A}(\vec{x}_\beta,t) \right]^2 I - \sum_{\beta=1}^{N} \eta_\beta \vec{S}_\beta \cdot \vec{B}(\vec{x}_\beta,t) \\ + \sum_{\beta=1}^{N} q_\beta \phi(\vec{x}_\beta,t) I + \tilde{v}(x,t). \end{aligned} \tag{1.3}$$

We furthermore note that with a change of scale in the position variable and an adjustment of the coupling constants, we can switch from the case of all the particles having a common mass $m$, to one where the mass of the $\beta^{th}$ particle is $m_\beta$. With these changes made, the Hamiltonian is precisely in the form ascribed by Landau and Lifshitz ([LL 58], chapter XX).

Hamiltonians in the form (1.2) are sufficiently general to describe quantum particles in a classical Yang-Mills field [Mor 83] [Wo 70]. An example of their use is the study a system of quarks in an external gluon field [Ar 82].

We approach the problem of solving the Schrödinger equation from two different points of view. We first examine the solutions by solving the equation

$$i\hbar\frac{\partial}{\partial t}\psi(t) = H(t)\psi(t) \tag{1.4}$$

in the $L^2(\mathbb{R}^d;\mathbb{C}^s)$ topology. Let $C_o^\infty(\mathbb{R}^d;\mathbb{C}^s)$ denote the space of infinitely differentiable functions of compact support. We shall place sufficient conditions on $a$ and $v$ to ensure the minimal operator $H(\cdot,t)$ defined on $C_o^\infty(\mathbb{R}^d;\mathbb{C}^s)$ has a unique closed extension $H(t)$, with a domain $D\Big(H(t)\Big) = D_o$ that is time independent. Let $T_\Delta$ be the closed triangular region

$$T_\Delta = \{(t_o,t) \in [0,T]\times[0,T] : 0 \le t_o \le t \le T\}.$$

For each $t_o < T$, we will seek a solution of (1.4) that satisfies the Cauchy data problem

$$\psi(t_o,t_o) = \psi_o, \qquad \psi_o \in D_o. \tag{1.5}$$

We shall see that under a set of weak assumptions on $a$ and $v$, a unique solution exists and defines a family of bounded linear operators via the mappings $\psi_o \mapsto \psi(t,t_o)$. For each $(t_o,t)$ we denote this mapping by $U(t,t_o)$ and we call the collection of these mappings, $\{U(t,t_o)\}_{(t_o,t)\in T_\Delta}$, the *Schrödinger evolution*. We shall study equations similar to (1.4) and (1.5) and show that they also generate unique families of bounded linear operators, which we shall call an *evolution*. Properties of an evolution will be outlined in chapter 2. Here it suffices to say $U(t,t_o)$

has the domain stability property $U(t,t_o) : D_o \to D_o$; $U(t_o,t_o) = I$; and $U(t,t_o)$ is strongly continuously differentiable and satisfies the equation

$$i\hbar\frac{\partial}{\partial t}U(t,t_o)\psi = H(t)U(t,t_o)\psi, \qquad \psi \in D_o.$$

If $H(t)$ is time independent, then the Schrödinger evolution has the well known form

$$U(t,t_o) = e^{-i(t-t_o)H/\hbar}. \tag{1.6}$$

With weak restrictions placed on $a$ and $v$ we can show the existence and uniqueness of the Schrödinger evolution. These operator solutions provide us with a rigorous abstract framework in which to discuss evolutions. Under somewhat stronger assumptions on $a$ and $v$ and for times $t - t_o$ sufficiently small, it is possible to show that $U(t,t_o)$ is an integral operator. Its integral kernel is a matrix valued function, which we denote by $K(x,t;y,t_o;m)$. Another characterization of $K$ (the propagator), common through physics, is found in the Dirac bra ket statement

$$K(x,t;y,t_o;m) = \langle x|U(t,t_o)|y\rangle.$$

One should recall that the Dirac notation above assumes that every bounded operator on Hilbert space has a kernel. This is often false. Any satisfactory study of evolution must establish the existence of an integral kernel $K$. Although (with a static Hamiltonian $H$) a great deal is known about kernels associated with the analytic semigroup $e^{-tH}$, $\mathrm{Re}\, t > 0$, very little is rigorously known about the kernel of the evolution $e^{-itH}$, $t \in \mathbb{R}$. As Simon observes in his review of Schrödinger semigroups [Si 82], it is an open question whether or not for $N$-body Schrödinger operators, including the atomic Hamiltonians, $e^{-itH}$ is a weak integral operator with a jointly continuous integral kernel. We will use a constructive technique to obtain an explicit formula for the evolution's kernel and this is the principle achievement of the first part of this thesis. These kernels represent the starting point in all approximate descriptions of evolutions, such as are found in the WKB, the large mass, and the small time displacement asymptotic studies in $K$.

Let $\langle\cdot,\cdot\rangle$ denote the inner product in $\mathcal{H} = L^2(\mathbb{R}^d; \mathbb{C}^s)$ with the convention that it is linear in the right argument. We have the following definition by Simon [Si 82]:

**Definition 1.1:** A two parameter family (in $T_\Delta$, $t \neq t_o$) of functions

$$K(\cdot, t; \cdot, t_o) : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}^{s\times s}$$

that are measureable and locally integrable on $\mathbb{R}^d \times \mathbb{R}^d$, is called the *propagator* for the Schrödinger evolution if for all bounded measurable functions $\psi, \varphi$ of compact support,

$$\langle \psi, U(t, t_o)\varphi \rangle = \int \psi(x)^* K(x, t; y, t_o; m)\varphi(y)\, dy\, dx. \tag{1.7}$$

$\diamondsuit$

Kernels of the type defined by equation (1.7) are called *weak* in Simon's terminology [Si 82].

The Hamiltonian $H(t)$ may be written as a perturbation of the free Laplacian operator $H_o = -\frac{\hbar^2}{2m}\Delta$;

$$H(t) = H_o + V(t).$$

If $\{U_o(t, t_o)\}_{(t_o,t)\in T_\Delta}$ represents the evolution associated with the free Hamiltonian $H_o$, then the Schrödinger evolution satisfies the (strong) integral equation

$$U(t, t_o) = U_o(t, t_o) + \frac{1}{i\hbar}\int_{t_o}^{t} d\tau\, U_o(t, \tau)V(\tau)U(\tau, t_o). \tag{1.8}$$

We will obtain an explicit representation of $K$ by investigating the individual terms that arise from the iteration of (1.8). By iterating (1.8) an infinite number of times, the resulting series is known in the literature as the Dyson series [Dy 49,1][Dy 49,2].

To gain better control of the individual terms of the Dyson series, it will be necessary to embed our evolution problem into a larger problem in which the mass parameter $m$ shall be treated as complex. Let $\mathbb{C}_>$ denote the upper half complex plane and let $\mathbb{C}_\geq$ denote its closure with respect to $\mathbb{C}$. We shall study the evolution problem with $m \in \mathbb{C}_>$ and treat the Schrödinger equation and its solution as $\operatorname{Im} m \to 0+$ boundary value problem of this larger class of evolutions. We shall see that both the complex mass evolution and the complex mass propagator are continuous functions of $m$ and their extensions coincide on $\mathbb{C}_\geq$.

The convergence behaviour of the Dyson series is dictated by our assumptions on the potentials $a$ and $v$. Each will be assumed to be the Fourier transform of a complex matrix valued measure of compact support. For example if $\{\nu(t)\}_{t\in[0,T]}$ denotes such a family of measures over the Borel subsets of $\mathbb{R}^d$, then

$$v(x,t) = \int_{\mathbb{R}^d} e^{i\alpha\cdot x}\, d\nu(t). \tag{1.9}$$

Here, $\alpha \cdot x$ denotes the dot product

$$\alpha \cdot x = \alpha_1 x_1 + \cdots + \alpha_d x_d.$$

Similarly if $\{\gamma(t)\}_{t\in[0,T]}$ is a family of $d$-tuples whose components are complex matrix valued measures of compact support, then

$$a(x,t) = \int_{\mathbb{R}^d} e^{i\alpha\cdot x}\, d\gamma(t). \tag{1.10}$$

Sufficient $t$ differentiability properties are imposed on the measures $\nu(t)$ and $\gamma(t)$ to ensure that $v(x,t)$ and $a(x,t)$ are continuously differentiable in $t$. For each fixed $t$, the potentials will also be holomorphic functions of $x$ because of the boundedness of the support of their respective measures.

This class of potentials is very similar to that used by Ito [It 61] [It 67] and Albeverio and Høegh-Krohn [AH 76] in their studies of the Feynman path integral. In these studies, the potential has the same form as in (1.10), but $\nu(t)$ no longer need have compact support. Osborn et. al. [OF 83] [OCF 85] have studied the propagator using this same class of potentials. However in each of these references cited, $a \equiv 0$. It is the appearance of a term like $a \cdot \nabla$ in the Hamiltonian wherein lies the difficulty. Such a term leads to polynomial structures in the Fourier space for each of the terms in the Dyson series. For the $n^{th}$ term, these polynomials can be up to order $n$ and it is the compact support of the measures that provides the mechanism for controlling the polynomial growth at infinity. In another paper, Osborn et. al. [OPC 87] have addressed the same problem we are considering, but within the frame work of a spinless ($s = 1$) system. The techniques used in that paper are adaptable to spins $s > 1$ and it forms the basis for the arguments presented in part 1 of this dissertation.

In chapter 2 we discuss the properties of the Schrödinger evolution and what are sufficient conditions required of $H(t)$ in order to ensure its existence. Our assumptions on $a$ and $v$ shall be weak in order to verify the existence of the Schrödinger evolution for a broad class of Hamiltonians. Included in this class of potentials are the physically important Coulomb and Yukawa potentials. It will be necessary to allow the mass parameter to take values in $[0, \infty)$ when we consider this broad class of potentials because with a real mass we can exploit properties of a self-adjoint operator which we would not be able to utilize with a complex mass. For the last part of chapter 2 we will strengthen our assumptions on $a$ and $v$ so that we may let $m$ be complex and we can drop the hermiticity requirements on $v$. Under this setting, we investigate the $m \in \mathbb{C}_{\geq}$ continuity of the evolution $\{U(t, t_o)\}_{(t_o,t) \in T_\Delta}$.

In chapter 3, we make precise our working assumptions on the potentials to be used for the remainder of part 1. We also demonstrate that these assumptions are sufficient to verify that $a$ and $v$ will satisfy the criteria required of them in chapter 2.

Chapter 4 is concerned with the individual terms of the Dyson series and demonstrates that for sufficiently small times $t - t_o$, the series is summable and it strongly converges to a solution of the Schrödinger equation (1.4).

In chapter 5 we show that the $n^{th}$ term in the Dyson series defines an integral operator with a continuous kernel $d_n(x,t;y,t_o;m)$. Moreover, for sufficiently small times, these kernels are pointwise summable and their sum is the complex mass propagator $K(x,t;y,t_o;m)$. The complex mass propagator is shown to be continuous in the limit $\mathrm{Im}\, m \to 0+$. Using the strong continuity of the evolution $\{U(t,t_o)\}_{(t_o,t)\in T_\Delta}$ with respect to the mass parameter, we show the Schrödinger evolution (ie. $\mathrm{Im}\, m = 0$) is an integral operator whose kernel is the propagator.

It is helpful to recall in what ways the results given here for the time evolution of quantum systems in external non-Abelian fields extend those found elsewhere in the literature. The specific construction of a convergent Dyson series (in several different topologies) is the core result. These convergent results (as stated in lemma 5.5 and theorem 5.1) are new and have importance in establishing the mathematical nature of the quantum propagator and in characterizing the analytic structure of the propagator in the physical constants $\hbar$, $m$, and $q$ (the charge coupling constant). The simpler non-Abelian external field problem is already published in [OPC 87] and the revised analysis found in sections 1–5 successfully extend the idea of using the complex mass continuation technique to the non-Abelian case.

## 1.2 Part 2: Levinson's Theorem for Spin Systems

In the second part of this treatise we study Levinson's theorem for two particle scattering. Formally one may remove the center of mass motion from the problem and consider the equivalent one particle system in an external potential $v$. Again we are interested in a system possessing spin degrees of freedom. As in part 1, the term spin is a generic term and its precise interpretation depends upon the physics being described. The relevant Hilbert space for this problem will be $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^s)$. The scattering system is completely specified by the Hamiltonian pair $(H, H_o)$. The free Hamiltonian is the self-adjoint extension of the minimal operator defined by the negative Laplacian on $C_o^\infty(\mathbb{R}^3; \mathbb{C}^s)$. The interaction Hamiltonian is given as a perturbation of $H_o$;

$$H = H_o + V.$$

The operator $V$ is defined by multiplication with the matrix valued function $v : \mathbb{R}^3 \longrightarrow \mathbb{C}^{s \times s}$. For a.a. $x$, $v(x)$ will be hermitian, which is a necessary condition on $v$ if $H$ is to be self-adjoint. For many of our arguments it is sufficient that $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s \times s})$. However there is one proof where we needed to strengthen this assumption on $v$ to $v \in L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s \times s})$. The potential class $\mathcal{F}^*$ describes those hermitian matrix valued functions that are the Fourier transform of a complex matrix valued measure, similar to (1.9). This assumption was needed to provide sufficient control over the large (complex) energy behaviour of the Born series

$$R(z) - R_o(z) = \sum_{n=1}^{\infty} (-1)^n R_o(z)[V R_o(z)]^n. \tag{1.11}$$

Here $R(z) \equiv (H - z)^{-1}$ is the resolvent of $H$ and $R_o(z)$ is the resolvent of $H_o$. We shall keep these integrability assumptions on $v$ local to each claim so that in the event of improvements on controlling the large energy behaviour of the Born series, we need only modify those results directly affected, rather than having to rederive all the results.

Levinson's theorem is just one of a class of energy moment sum rules [Bo 86]. Suppose the Hamiltonian has $N_b$ bound states with eigenvalues $\lambda_j$ ($j = 1 \sim N_b$). Then for $v$ in a certain class, the sum rules are (cf. [Bo 86], equation (3.24) )

$$\int_0^\infty \lambda^N \Big\{ 2\mathrm{Im}\ \mathrm{Tr}[R(\lambda) - R_o(\lambda)] - \sum_{n=1}^{N+1} c_n \frac{1}{\lambda^{n-\frac{1}{2}}}\, d\lambda \int dx\ \mathrm{sp}\, P_n(x,x) \Big\}$$

$$= 2\pi \sum_{n=1}^{N_b} \lambda_j^N \qquad\qquad N = 0, 1, 2 \cdots. \quad (1.12)$$

Here sp corresponds to the trace in $\mathbb{C}^{s\times s}$ and the series

$$\sum_{n=1}^{N+1} c_n \frac{1}{\lambda^{n-\frac{1}{2}}} \int dx\ \mathrm{sp}\, P_n(x,x)$$

is the leading order asymptotic expansion of the function $2\mathrm{Im}\ \mathrm{Tr}[R(z) - R_o(z)]$ (cf. [OCF 85], theorem 3). Levinson's theorem corresponds to the $N = 0$ version of (1.12).

One can develop this rule further by considering time delay theory. Let $\Sigma$ be a Lebesgue measurable subset of $\mathbb{R}^3$ of finite measure. The time delay through the region $\Sigma$ is the difference in times spent by a free particle in $\Sigma$ and a particle under the influence of $v$. If we consider taking the limit $\Sigma \longrightarrow \mathbb{R}^3$, the corresponding limit of the time delay also exists and we call this the *global time delay.* The global time delay may be connected to the resolvent difference and the exploitation of this connection leads to another form of Levinson's theorem. We shall call this the *global Levinson's theorem.*

We are interested in what we shall call the *local Levinson's theorem.* It is a similar theorem to the global sum rule, but it uses the time delay associated with a finite region $\Sigma$. We further broaden our considerations by studying time delay through the region $\Sigma$ while the particle has spinor components in some fixed but otherwise arbitrary subspace $\Gamma \subset \mathbb{C}^s$. The local theory is of interest on several accounts. For the global time delay theory, there exist relationships between the time delay function, the density of states and Krein's spectral shift function [Bu 67] [JSM 72]. Initial examinations of the arguments of MacMillan and Osborn [MO 80], and Bollé et. al. [BDO 86] indicate the local time delay is related to a local representation of the density of states and what appears to be a local version of Krein's spectral shift function. The complete exploration of these aspects of the local theory we shall leave for future studies. Another use of the local theory is to study the $\Sigma \to \mathbb{R}^3$ limit. There appears in the global sum rule a term that never appears in the local version of the same rule. The origin of this term is due to the possible occurrence of a zero energy resonance [N 77] [JK 79]. Thus this limit is quite delicate and nontrivial to perform. Recent result for the local local Levinson's theorem in two dimensions have been presented by Osborn et. al. [OSBD 85].

In the above, we are only using the spatial aspect of the problem with the region $\Sigma$. We can also exploit the spinor structure in our problem. By using the arbitrariness of $\Gamma$, one may take combinations of these sum rules so that specific matrix components of $v$ and the time delay operator are singled out. This is a new contribution to the theory, allowing a detailed study of the off diagonal spin terms. Prior to this, due to a trace being performed, only information concerning the sum of the diagonal spin variables was available.

In chapter 6 we first establish the properties of the scattering system determined by the Hamiltonian pair $(H, H_o)$. A discussion of the connections between the resolvent difference and the time delay for the spatial region $\Sigma$ and spinor subspace $\Gamma$ follows. We end chapter 6 with a brief study of the Born series.

We end our study with chapter 7, which deals with the specific details of the proof of our local Levinson's theorem.

# CHAPTER 2

# Evolution Operators and the Hamiltonian

## 2.1 Evolutions in Banach Space

Before we discuss the Hamiltonian in detail we examine the properties sufficient for an operator to generate an evolution. With these properties in mind we will place restrictions on the Hamiltonian to guarantee that the Schrödinger evolution exists. A theory concerning evolutions has been worked out by Krein [Kr 71] in the more general setting of Banach spaces and we shall apply this theory to the specialised case of a Hilbert space.

The evolution problem of interest is the following: In a Banach space $E$ we consider a first order differential equation

$$\frac{d\psi}{dt} = A(t)\psi, \qquad 0 \le t \le T, \tag{2.1}$$

where $\psi : [0,T] \to E$ and $A(t)$ is a family of possibly unbounded operators on $E$. Each $A(t)$ is assumed to be closed and they all share a common domain $D\big(A(t)\big) = D(A) \subset E$ for all $t \in [0,T]$.

**Definition 2.1:** A solution of (2.1) on the segment $[t_o, T]$ for a fixed $t_o \in [0,T]$ is a function $\psi(t, t_o)$ taking values in $D(A)$ and having a strong derivative $\partial_t\psi(t,t_o)$

which satisfies (2.1) on the interval $[t_o, T]$. The problem of finding a solution $\psi(t, t_o)$ of (2.1), for each fixed $t_o \in [0, T]$, and which satisfies the initial data condition

$$\psi(t_o, t_o) = \psi_o \in D(A) \tag{2.2}$$

we shall call *the Cauchy problem on the triangle* $T_\Delta \equiv \{(t, t_o) : 0 \leq t_o \leq t \leq T\}$.
$\diamondsuit$

**Definition 2.2:** The Cauchy problem is said to be *uniformly correct* if the following statements hold:

(1) For each $t_o \in [0, T]$ and any $\psi_o \in D(A)$ there exists a unique solution $\psi(t, t_o)$ of (2.1) on the segment $[t_o, T]$ satisfying the initial data condition (2.2).

(2) The function $\psi(t, t_o)$ and its $t$ derivative $\partial_t \psi(t, t_o)$ are continuous in the triangle $T_\Delta$.

(3) The solution depends continuously on the initial data in the sense that if $\psi_{o,n} \in D(A)$ converges to zero as $n \to \infty$ then the corresponding solutions $\psi_n(t, t_o)$ converge to zero uniformly relative to $(t, t_o) \in T_\Delta$. $\diamondsuit$

When the Cauchy problem is uniformly correct we can define a linear map on $D(A)$ for each $(t, t_o) \in T_\Delta$ by the relation $\psi_o \mapsto \psi(t, t_o)$. We denote this operator by $\mathcal{U}(t, t_o)$ and we have

$$\psi(t, t_o) = \mathcal{U}(t, t_o)\psi_o. \tag{2.3}$$

From properties (1) and (3) it follows that $\mathcal{U}(t, t_o)$ is bounded and since $D(A)$ is dense we can extend $\mathcal{U}(t, t_o)$ to all of $E$. We denote the extension again by $\mathcal{U}(t, t_o)$ and we call the associated family of operators an evolution. The uniformly correct Cauchy problems leads to certain properties for the evolution $\{\mathcal{U}(t, t_o)\}_{(t_o,t) \in T_\Delta}$, summarized in the following proposition.

**Proposition 2.1:** *Suppose the Cauchy problem in the triangle* $T_\Delta$ *is uniformly correct. Then the evolution* $\{\mathcal{U}(t, t_o)\}_{(t_o,t) \in T_\Delta}$ *satisfies the following:*

(1) $\mathcal{U}(t,t_o) : D(A) \to D(A), \quad (t,t_o) \in T_\Delta$.

(2) *The operator* $\mathcal{U}(t,t_o)$ *is uniformly bounded in* $T_\Delta$.

(3) *The operator* $\mathcal{U}(t,t_o)$ *is strongly continuous in* $T_\Delta$.

(4) *The following operator identities hold in* $T_\Delta$*:*

$$\mathcal{U}(t,t_o) = \mathcal{U}(t,\tau)\mathcal{U}(\tau,t_o), \qquad 0 \le t_o \le \tau \le t \le T, \tag{2.4}$$

$$\mathcal{U}(t_o,t_o) = I, \qquad t_o \in [0,T]. \tag{2.5}$$

(5) *The restriction of the operator* $\mathcal{U}(t,t_o)$ *to the domain* $D(A)$ *is strongly differentiable in* $t \in [t_o,T]$*. Furthermore the operator* $\partial_t\mathcal{U}(t,t_o)$*, defined on* $D(A)$*, is jointly strongly continuous in* $(t,t_o) \in T_\Delta$ *and obeys the relation*

$$\partial_t\mathcal{U}(t,t_o)\psi = A(t)\mathcal{U}(t,t_o)\psi, \qquad \psi \in D(A). \tag{2.6}$$

*Proof:* See Krein [Kr 71] (pp. 193–195). ◊

The significance of proposition 2.1 is that we need only verify that the Cauchy problem (2.1) and (2.2) is uniformly correct in order to know the associated evolution $\{\mathcal{U}(t,t_o)\}_{(t_o,t)\in T_\Delta}$ exists. The next theorem states easily verified conditions on $A(t)$ that will be sufficient to guarantee the Cauchy problem is uniformly correct.

**Theorem 2.1:** *Suppose the operators* $A(t)$ $(t \in [0,T])$ *are*

(1) *densely defined and closed, with a t-invariant domain* $D(A)$*;*

(2) *strongly continuously differentiable on domain* $D(A)$*; and*

(3) *obey the resolvent estimate*

$$\|R(\lambda;A(t))\| \le \frac{1}{1+\lambda}, \qquad \lambda \ge 0. \tag{2.7}$$

*Then*

(a) *the Cauchy problem in* $T_\Delta$ *is uniformly correct;*

(b) *the restriction of the operator* $\mathcal{U}(t,t_o)$ *to the domain* $D(A)$ *is strongly continuously differentiable with respect to* $t_o \in [0,T]$ *and satisfies the equation*

$$\partial_{t_o}\mathcal{U}(t,t_o)\psi = -\mathcal{U}(t,t_o)A(t_o)\psi, \qquad (t,t_o)\in T_\Delta, \quad \psi\in D(A); \tag{2.8}$$

*and*

(c) *the operator* $\mathcal{U}(t,t_o)$ *satisfies the uniform bound*

$$\|\mathcal{U}(t,t_o)\| \le 1. \tag{2.9}$$

*Proof:* We refer the reader to Krein [Kr 71], chapter 2, section 3.1 and specifically theorem 3.11. $\diamond$

We next wish to apply this theory to the Schrödinger evolution problem. Let $E$ be the Hilbert space

$$\mathcal{H} \equiv L^2(\mathbb{R}^d;\mathbb{C}^s).$$

The Schrödinger equation with its associated Cauchy problem is

$$\begin{aligned} i\hbar\frac{d\psi}{dt} &= H(t)\psi, \\ \psi(t_o,t_o) &= \psi_o, \qquad \psi_o\in D(H). \end{aligned} \tag{2.10}$$

Here $H(t)$ is a family of self-adjoint operators on $\mathcal{H}$ with the common domain $D(H)$. We have the following result.

**Theorem 2.2:** *Let*

$$A(t) = \frac{1}{i\hbar}H(t) - cI, \tag{2.11}$$

*where c is an appropriately chosen real constant such that $A(t)$ satisfies the hypotheses of theorem 2.1 and let $\{\mathcal{U}(t,t_o)\}_{(t_o,t)\in T_\Delta}$ be its associated evolution. Then the Schrödinger equation (2.10) generates an evolution, $\{U(t,t_o)\}_{(t_o,t)\in T_\Delta}$, which satisfies the pointwise operator identity*

$$U(t,t_o) \equiv e^{c(t-t_o)}\mathcal{U}(t,t_o), \qquad (t,t_o) \in T_\Delta. \tag{2.12}$$

*The Schrödinger evolution has the analogous properties;*

(1) $U(t,t_o) : D(H) \to D(H), \quad (t,t_o) \in T_\Delta$;

(2) $U(t,t_o)$ *has the operator norm bound*

$$\|U(t,t_o)\| \le e^{c(t-t_o)}; \tag{2.13}$$

(3) $U(t,t_o)$ *is strongly continuous in* $T_\Delta$;

(4) $U(t,t_o)$ *satisfies the operator identities*

$$\begin{aligned} &U(t,t_o) = U(t,\tau)U(\tau,t_o), \qquad 0 \le t_o \le \tau \le t \le T, \\ &U(t_o,t_o) = I; \end{aligned} \tag{2.14}$$

(5) *the restriction of* $U(t,t_o)$ *to the domain* $D(H)$ *is strongly differentiable with respect to both* $t$ *and* $t_o$ *with the strong derivatives*

$$\begin{aligned} \partial_t U(t,t_o) &= \frac{1}{i\hbar}H(t)U(t,t_o), \\ \partial_{t_o} U(t,t_o) &= -\frac{1}{i\hbar}U(t,t_o)H(t_o). \end{aligned} \tag{2.15}$$

*Proof:* From the definition (2.12), theorem 2.1 and proposition 2.1 we see that properties (1) $\sim$ (4) are trivially satisfied. We show (2.15). If $\psi_o \in D(H) = D(A)$, then

$$\begin{aligned}\partial_t U(t,t_o)\psi_o &= \partial_t(e^{c(t-t_o)}\mathcal{U}(t,t_o)\psi_o)\\ &= cU(t,t_o)\psi_o + e^{c(t-t_o)}A(t)\mathcal{U}(t,t_o)\psi_o\\ &= cU(t,t_o)\psi_o + \Big(\frac{1}{i\hbar}H(t) - cI\Big)U(t,t_o)\psi_o\\ &= \frac{1}{i\hbar}H(t)U(t,t_o)\psi_o.\end{aligned}$$

The second equation in (2.15) follows similarly. ◇

### 2.2 The Hamiltonian

We are now ready to discuss what sort of assumptions are necessary to make on $H(t)$ in order that $A(t)$ defined by (2.11) have the properties (1)$\sim$(3) in theorem 2.1, needed to generate an evolution. Obviously properties (1) and (2) are satisfied if and only if the corresponding properties exist for $H(t)$.

In our studies of the Hamiltonian properties we shall discuss two different approaches in the treatment of the mass variable $m$. The first is to treat $m$ as a positive parameter. The advantage of this is it allows us to demonstrate the flexibility of Krein's evolution theory by using a broad class of potentials. The second treatment of the mass parameter allows $m$ to take up values in the upper half complex plane. The class of allowable potentials will be narrower than in the first treatment, but it is this class that we shall ultimately use to develop a pointwise representation of the propagator. The complex nature of $m$ is crucial in establishing the relation between the abstract evolution operator and the propagator.

We first make some preliminary definitions to make statements about our potentials concise and then we shall describe assumptions on the potentials sufficient to allow $A(t)$ to have the desired properties listed in theorem 2.1.

We introduce the Hilbert space $\mathcal{H}_v$ defined by

$$\mathcal{H}_v \equiv L^2(\mathbb{R}^d; (\mathbb{C}^s)^d).$$

Here $(\mathbb{C}^s)^d$ denotes the space of $d$-tuples whose components are $s$ - dimensional vectors over the complex field $\mathbb{C}$. We denote the norm on $\mathcal{H}_v$ by $\|\cdot\|_v$. We will define mappings $\mathcal{H} \to \mathcal{H}_v$ componentwise. For example we can define the momentum operator $P : \mathcal{H} \to \mathcal{H}_v$ by specifying the effect of each of its components

$$(P_j\psi)\hat{}(\alpha) = \hbar\alpha_j\hat{\psi}(\alpha), \qquad j = 1, \ldots, d.$$

Here $\hat{}$ denotes the Fourier transform mapping $\mathcal{H} \to \mathcal{H}$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$. Clearly $P_j$ has the interpretation of the generalised derivative $-i\hbar\partial/\partial x_j$. The domain of the operator $P$ is

$$D(P) = \{\psi \in \mathcal{H} : \alpha\psi \in \mathcal{H}_v\} = \{\psi \in \mathcal{H} : |\alpha|\psi \in \mathcal{H}\}.$$

**Assumption 1:** The operator $\mathbf{a}(t) : \mathcal{H} \to \mathcal{H}_v$ is a $d$-dimensional tuple whose components, $[\mathbf{a}(t)]_\mu$ are operators mapping $\mathcal{H} \to \mathcal{H}$. Each component, $[\mathbf{a}(t)]_\mu$, is defined by multiplication by the matrix valued function $[a(\cdot, t)]_\mu : \mathbb{R}^d \to \mathbb{C}^{s\times s}$ $(\mu = 1 \sim d)$. For a.a. $x$, $[a(x,t)]_\mu$ is hermitian * and hence each $[\mathbf{a}(t)]_\mu$ is a symmetric operator. If $a(\cdot, t)$ denotes the $d$-tuple formed from the $[a(\cdot, t)]_\mu$'s, then we assume $a$ satisfies the following properties:

(1) $a \in C^1(\mathbb{R}^d \times [0,T], (\mathbb{C}^{s\times s})^d)$, where $(\mathbb{C}^{s\times s})^d$ is the space of $d$-tuples whose components are $s \times s$ matrices;

---

* In the language of gauge theory, each component $[a(\cdot, t)]_\mu$ can be written pointwise as

$$[a(x,t)]_\mu = \sum_k \tilde{a}^k_\mu(x,t) F_k.$$

The coefficients $\tilde{a}^k_\mu(x,t)$ are scalar and contain all the space-time information. The matrices $F_k$ are the generators of the internal symmetry group on $\mathbb{C}^s$ and satisfy the usual commutation relations

$$[F_i, F_j] = ic_{ijk}F_k,$$

where the structure constants $c_{ijk}$ depend upon the particular group the $F_k$'s generate. For further details, cf. reference [Mor 83].

(2) $a$ and its derivatives are uniformly bounded in $x$ and $t$;

$$|a(x,t)| \le M_1,$$
$$|(\partial_x^\eta a)(x,t)| \le M_2, \qquad |\eta| = 1,$$
$$|(\partial_t a)(x,t)| \le M_3.$$

Here $\eta$ is the multi-index $(\eta_1, \ldots, \eta_d)$, with length $|\eta| = \eta_1 + \cdots + \eta_d$. We utilize the notation

$$\partial_x^\eta = (\frac{\partial}{\partial x_1})^{\eta_1} \cdots (\frac{\partial}{\partial x_d})^{\eta_d}.$$

(3) $a$ is continuously differentiable with respect to $t$ in the $L^\infty(dx; (\mathbb{C}^{s\times s})^d)$ norm. That is, there exists a measurable (with respect to $dx$) function $\dot{a}$, whose components are also hermitian $s \times s$ matrices, such that

$$\left\| \frac{1}{\delta t}[a(\cdot, t+\delta t) - a(\cdot, t)] - \dot{a}(\cdot, t) \right\|_\infty \to 0 \qquad \text{as } \delta t \to 0,$$
$$\|\dot{a}(\cdot, t) - \dot{a}(\cdot, \tau)\|_\infty \to 0 \qquad \text{as } \tau \to t.$$

(4) $\nabla \cdot a$ is continuously differentiable with respect to $t$ in the $L^\infty(dx; \mathbb{C}^{s\times s})$ norm with derivative $\nabla \cdot \dot{a}$;

$$\left\| \frac{1}{\delta t}[(\nabla \cdot a)(\cdot, t+\delta t) - (\nabla \cdot a)(\cdot, t)] - (\nabla \cdot \dot{a})(\cdot, t) \right\|_\infty \to 0 \qquad \text{as } \delta t \to 0;$$
$$\|(\nabla \cdot \dot{a})(\cdot, t) - (\nabla \cdot \dot{a})(\cdot, \tau)\|_\infty \to 0 \qquad \text{as } \tau \to t. \qquad \diamond$$

The boundedness and smoothness properties of $a$ will not be a severe restriction for physical problems such as those occuring in atomic and molecular physics. For such problems, $a$ is a multiple of the unit matrix and it is closely related to the electromagnetic vector potential $\vec{A}$. Because the spatial and temporal derivatives of $\vec{A}$ are related to the physical (external) fields $\vec{B}$ and $\vec{E}$, $\vec{A}$ can be assumed to be nicely behaved and hence $a$ will also be well behaved. It is reasonable to extend these assumptions to the case where the components of $a$ are hermitian matrix valued functions.

Let $\tilde{H}_o(m)$ be the minimal operator with domain $C_o^\infty(\mathbb{R}^d;\mathbb{C}^s)$ associated with the Laplacian in $\mathbb{R}^d$;

$$\tilde{H}_o(m) = -\frac{\hbar^2}{2m}\Delta\mathbf{I}. \tag{2.16}$$

Here $\mathbf{I}$ is the $s \times s$ unit matrix. For* $m > 0$ it is well known that $\tilde{H}_o(m)$ acting in the space $L^2(\mathbb{R}^d;\mathbb{C}^s)$ is essentially self-adjoint with a self-adjoint closure we denote by $H_o(m)$ (see reference [Ka 84], chapter V). Furthermore the spectrum of $H_o(m)$, $\sigma\Big(H_o(m)\Big)$, is the interval $[0,\infty)$. For complex $m$, by writing $\tilde{H}_o(m) = m^{-1}\tilde{H}_o(1)$ we see that $\tilde{H}_o(m)$ is closable with closure $H_o(m) = m^{-1}H_o(1)$. The domain of $H_o(m)$ for all $m \in \mathbb{C}_+$ is given by

$$D\Big(H_o(m)\Big) \equiv D_o = \{\psi \in \mathcal{H} : \alpha^2\hat{\psi}(\alpha) \in L^2(\mathbb{R}^d;\mathbb{C}^s)\}.$$

Next we consider the minimal Hamiltonian operator on $C_o^\infty(\mathbb{R}^d;\mathbb{C}^s)$ associated with the partial differential operator

$$\tilde{H}_1(t,m) = \frac{1}{2m}\Big(\frac{\hbar}{i}\nabla I - a(\cdot,t)\Big)^2. \tag{2.17}$$

Expanding out the square in (2.17) we can write $\tilde{H}_1(t,m)$ as

$$\tilde{H}_1(t,m) = \tilde{H}_o(m) + \tilde{W}(t,m). \tag{2.18}$$

The perturbing operator, $\tilde{W}(t,m)$, is given by

$$\tilde{W}(t,m) = \frac{i\hbar}{m}a(\cdot,t)\cdot\nabla + \frac{i\hbar}{2m}(\nabla\cdot a)(\cdot,t) + \frac{1}{2m}a(\cdot,t)^2, \tag{2.19}$$

* The ordering relations $<$ and $>$ have no meaning on complex numbers. Hence to say $m > 0$ implicitly implies that $m$ is real valued.

with domain $C_o^\infty(\mathbb{R}^d; \mathbb{C}^s)$. If we let $m > 0$ and we use the hermiticity of the components of $a(x,t)$, coupled with an integration by parts, we obtain

$$\langle \psi, \tilde{W}(t,m)\varphi\rangle = \langle \tilde{W}(t,m)\psi, \varphi\rangle, \qquad \psi, \varphi \in C_o^\infty.$$

Thus $\tilde{W}(t,m)$ is symmetric and hence it is closable. We denote the closure of $\tilde{W}(t,1)$ by $W(t,1)$. Since $\tilde{W}(t,m) = m^{-1}\tilde{W}(t,1)$, we see that $\tilde{W}(t,m)$ is closable for all $m \in \mathbb{C}_+$ with the closure $W(t,m) = m^{-1}W(t,1)$.

**Lemma 2.1:** *Let* $\mathbf{a}(t)$ *satisfy assumption 1. Then for all* $(t,m) \in [0,T] \times \mathbb{C}_+$ *the operator* $\tilde{H}_1(t,m)$ *has closure* $H_1(t,m)$ *satisfying*

(1) $D\Big(H_1(t,m)\Big) = D_o$;

(2) $H_1(t,m)\psi = H_o(m)\psi + W(t,m)\psi$ *for all* $\psi \in D_o$;

(3) *If* $m > 0$ *then* $H_1(t,m)$ *is self-adjoint and bounded from below by zero.*

*Proof:* We show that $\tilde{W}(t,m)$ is $\tilde{H}_o(m)$-bounded with $\tilde{H}_o(m)$-bound zero. Assumption 1 shows that both $(\nabla \cdot a)(x,t)$ and $a(x,t)^2$ give rise to bounded operators that are uniformly bounded in $t$. Thus we need only prove our assertion for $a(x,t)\cdot\nabla$. Let $\psi \in C_o^\infty$ and consider the following.

$$\begin{aligned}\|a(\cdot,t)\cdot\nabla\psi\|^2 &= \int |a(x,t)\cdot(\nabla\psi)(x)|^2\,dx \\ &\le M_1^2 \int |(\nabla\psi)(x)|^2\,dx \\ &= M_1^2 \int \alpha^2 |\hat{\psi}(\alpha)|^2\,d\alpha.\end{aligned}$$

Let $\delta > 0$ and define the set $E_\delta \equiv \{\alpha \in \mathbb{R}^d : \frac{1}{\delta}|\hat{\psi}(\alpha)| \ge \delta\alpha^2|\hat{\psi}(\alpha)|\}$. Breaking the integral above into one over $E_\delta$ and its complimentary set $\mathbb{R}^d \backslash E_\delta$ our estimate becomes

$$\begin{aligned}
\|a(\cdot,t)\cdot\nabla\psi\|^2 &\le M_1^2\left\{\frac{1}{\delta^2}\int_{E_\delta}|\hat{\psi}(\alpha)|^2\,d\alpha+\delta^2\int_{\mathbb{R}^d\setminus E_\delta}|\alpha|^4|\hat{\psi}(\alpha)|^2\,d\alpha\right\}\\
&\le M_1^2\left\{\frac{1}{\delta^2}\|\psi\|^2+\delta^2\left|\frac{2m}{\hbar^2}\right|^2\|\tilde{H}_o(m)\psi\|^2\right\}\\
&\le M_1^2\left\{\frac{1}{\delta}\|\psi\|+\delta\left|\frac{2m}{\hbar^2}\right|\|\tilde{H}_o(m)\psi\|\right\}^2 .
\end{aligned}$$

Taking the square roots of the left and right hand sides we see that $\tilde{W}(t,m)$ is $\tilde{H}_o(m)$-bounded. Furthermore $\tilde{W}(t,m)$ has $\tilde{H}_o(m)$-bound zero because $\delta$ can be made arbitrarily small. We also note that the estimate is time independent.

Let $m > 0$ for the moment. Then $\tilde{W}(t,m)$ is symmetric and an application of Kato's theorem V.4.4 [Ka 84] shows $\tilde{H}_o(m)+\tilde{W}(t,m)$ is essentially self-adjoint and its closure is given by $H_1(t,m) = H_o(m)+W(t,m)$. $H_1(t,m)$ is self-adjoint and it has the domain $D_o$. To extend the domain and closure properties to $m\in\mathbb{C}_+$ we simply note that $\tilde{H}_1(t,m)=m^{-1}\tilde{H}_1(t,1)$. Of course $H_1(t,m)$ will no longer be self-adjoint for a complex mass parameter.

Finally we show (3). Let $m>0$. To show that $H_1(t,m)\ge 0$ we must show

$$\langle\psi, H_1(t,m)\psi\rangle \ge 0 \qquad \psi\in D_o.$$

However because of the closedness of $H_1(t,m)$ we need only prove this on a core (cf. reference [Ka 84], p. 166) of $H_1(t,m)$ and then extend this to all of $D_o$. One such core is $C_o^\infty$, where the operator $H_1(t,m) = \tilde{H}_1(t,m)$. Using the definition of $\tilde{H}_1(t,m)$, the hermiticity of the components of $a(\cdot,t)$ and integrating by parts we get

$$\langle\psi,\tilde{H}_1(t,m)\psi\rangle = \frac{1}{2m}\left\|\left[\frac{\hbar}{i}\nabla I - a(\cdot,t)\right]\psi\right\|^2 \ge 0. \qquad \diamondsuit$$

From the above proof we have also shown the following result.

**Corollary 2.1:** *Under the hypotheses of lemma 2.1, $W(t,m)$ is $H_o(m)$-bounded with $H_o(m)$-bound zero. That is there exist constants $\alpha_1$ and $\beta_1$ greater than zero such that*

$$\|W(t,m)\psi\| \leq \alpha_1\|\psi\| + \beta_1\|H_o(m)\psi\|, \qquad \psi \in D_o \tag{2.20}$$

*The constant $\beta_1$ can be made arbitrarily small. Furthermore, if $\mathcal{K} \subset \mathbb{C}_+$ is any compact set and $m \in \mathcal{K}$, then the constants $\alpha_1$ and $\beta_1$ can be chosen independent of $t$ and $m$.* $\diamondsuit$

For the interaction potential $\mathbf{v}(t)$ we will consider two possible classes. The first class will allow for potentials that are relatively bounded with respect to the Laplacian. These potentials will be defined by matrix valued functions that are hermitian. For this class, we shall only consider a real mass parameter because these potentials can be unbounded and a complex $m$ leads to many difficulties in verifying the hypotheses of theorem 2.2. We wish to study these potentials because they include several important physical interactions such as the many body Coulomb and Yukawa interactions. It also demonstrates the flexibility of Krein's evolution theory. The second class of potentials we will study consists of bounded potentials defined by multiplication with complex matrix valued functions. We remove the restriction of hermiticity as the general formalism in later sections does not require this condition. It is the second class that is ultimately used in the study of the Dyson series expansion of the evolution operator.

For the second class we shall include the possibility of mass dependence and shall exhibit this dependence explicitly. This will play a role when considering the limit $\operatorname{Im} m \to 0$. For potentials in the first class we do not have to worry about mass continuity properties and hence the mass parameter will be considered as fixed. However for notational convenience it is simpler to write potentials in the first class in a notation matching that used for potentials in the second class.

**Assumption 2:** We define the operator $\mathbf{v}(t,m)$ by multiplication with the matrix valued function $v(\cdot,t;m)$, $t \in [0,T]$. The operator $\mathbf{v}(t,m)$ will belong to one of two possible classes.

*Class A:* For potentials in class A, we restrict $m$ to the real positive axis. Potentials in this class satisfy:

(1) For a.a. $(x,t) \in \mathbb{R}^d \times \mathbb{C}^{s\times s}$, $v(x,t;m)$ is hermitian so that the corresponding operator $\mathbf{v}(t,m)$ is symmetric;

(2) $\mathbf{v}(t,m)$ has $H_o(m)$-bound less than one. That is for all $t \in [0,T]$ the domain of $\mathbf{v}(t,m)$ satisfies $D_o \subset D(\mathbf{v}(t,m))$ and there exist finite positive constants $\beta_o < 1$ and $\alpha_o$ such that

$$\|\mathbf{v}(t,m)\psi\| \leq \alpha_o\|\psi\| + \beta_o\|H_o(m)\psi\|, \qquad \psi \in D_o. \tag{2.21}$$

Furthermore, if $\mathcal{K}_o \subset (0,\infty)$ is any compact set, we assume that $\alpha_o$ and $\beta_o$ are or can be chosen to be independent of $(t,m) \in [0,T] \times \mathcal{K}_o$;

(3) The restriction of $\mathbf{v}(t,m)$ to $D_o$ is strongly continuously differentiable. That is there exists a linear operator $\dot{\mathbf{v}}(t,m)$ with domain $D_o$ that is strongly continuous and satisfies

$$\left\|\frac{1}{\delta t}[\mathbf{v}(t+\delta t,m) - \mathbf{v}(t,m)]\psi - \dot{\mathbf{v}}(t,m)\psi\right\| \to 0 \qquad \text{as } \delta t \to 0, \quad t \in [0,T], \quad \psi \in D_o.$$

*Class B:* Functions $v(\cdot,t;m)$ defining the potentials $\mathbf{v}(t,m)$ in this class admit the following properties:

(1) Let $\mathcal{K}$ denote a compact subset of $\mathbb{C}_+$. Then the function $v(x,t;m)$, mapping $\mathbb{R}^d \times [0,T] \times \mathcal{K} \mapsto \mathbb{C}^{s\times s}$, is uniformly bounded in $(x,t,m)$; i.e.

$$\|v(\cdot,t;m)\|_\infty \leq \alpha \qquad \text{for all } (t,m) \in [0,T] \times \mathcal{K}; \tag{2.22}$$

(2) $v$ is continuously differentiable with respect to $t$ in the $L^\infty(dx; \mathbb{C}^{s\times s})$ norm. That is there exists a matrix valued function $\dot{v}(\cdot, t; m) \in L^\infty(dx; \mathbb{C}^{s\times s})$ such that for all $t \in [0, T]$

$$\left\| \frac{1}{\delta t}[v(\cdot, t+\delta t; m) - v(\cdot, t; m)] - \dot{v}(\cdot, t; m) \right\|_\infty \to 0 \qquad \text{as } \delta t \to 0;$$

$$\|\dot{v}(\cdot, t; m) - \dot{v}(\cdot, \tau; m)\|_\infty \to 0 \qquad \text{as } \tau \to t;$$

(3) the function $v(\cdot, t; m)$ is continuous with respect to the mass parameter in the $L^\infty(dx; \mathbb{C}^{s\times s})$ topology, uniformly with respect to $t$. That is

$$\sup_{t\in[0,T]} \|v(\cdot, t; m) - v(\cdot, t; m')\|_\infty \to 0 \qquad \text{as } m' \to m \in \mathcal{K}. \qquad \diamondsuit$$

**Examples:** The position vector $x$ is often a $d = 3N$-tuple describing the position of particles in space. We could write $x = (\vec{x}_1, \ldots, \vec{x}_N)$, with $\vec{x}_i$ denoting the (3 dimensional) postion vector of the $i^{th}$ particle. With this notation in mind the generalized many body Yukawa and Coulomb potentials are respectively;

$$v(x) = \sum_{j=1}^{d} \gamma_j \frac{e^{-\alpha_j |\vec{x}_j|}}{|\vec{x}_j|^\delta} I + \sum_{i<j} \gamma_{ij} \frac{e^{-\alpha_{ij}|\vec{x}_i - \vec{x}_j|}}{|\vec{x}_i - \vec{x}_j|^\delta} I, \qquad \begin{array}{c} \gamma_j, \gamma_{ij} \in \mathbb{R}, \quad \alpha_j, \alpha_{ij} > 0 \\ 0 < \delta < 3/2 \end{array}$$

and

$$v(x) = \sum_{j=1}^{d} \frac{\gamma_j}{|\vec{x}_j|^\delta} I + \sum_{i<j} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|^\delta} I \qquad \gamma_j, \gamma_{ij} \in \mathbb{R}, \quad 0 < \delta < 3/2.$$

By setting $\delta = 1$, we have the conventional Coulomb and Yukawa potentials. Kato has shown [Ka 51] that these symmetric operators have $H_o(m)$-bound zero. As these potentials are independent of $t$ and $m$, they clearly satisfy the other requirements necessary to be a member of class A.

For examples of class B potentials there are many possibilities. For instance, almost any bounded periodic potential lies in this class (provided its time dependence etc. is sufficiently smooth). Examples of non-diagonal interactions include the spin-spin interactions of the type $\vec{S}_i \cdot \vec{S}_j$, where $\vec{S}_i$ is the spin operator* associated with the $i^{th}$ particle. $\diamondsuit$

Define the family (in $t, m$) of operators

$$H(t,m) = H_1(t,m) + \mathbf{v}(t,m). \tag{2.23}$$

Since $D\Big(\mathbf{v}(t,m)\Big) \supset D\Big(H_1(t,m)\Big) = D_o$, the operators $H(t,m)$ have the common domains $D\Big(H(t,m)\Big) = D_o$.

**Lemma 2.2:** *Let* $\mathbf{a}(t)$ *and* $\mathbf{v}(t,m)$ *satisfy the assumptions 1 and 2. If* $\mathbf{v}(t,m)$ *is in class B, then for each* $(t,m) \in [0,T] \times \mathbb{C}_+$ *the operator* $H(t,m)$ *is closed. If* $\mathbf{v}(t,m)$ *is in class A (and hence* $m \in (0,\infty)$*), then* $H(t,m)$ *is self-adjoint and there exists a finite positive constant* $b_o$ *such that* $H(t,m) \geq -b_o$.

*Proof:* We first show that $\mathbf{v}(t,m)$ has an $H_1(t,m)$-bound less than 1. If $\mathbf{v}(t,m)$ is in class B this assertion is trivial. Let $\mathbf{v}(t,m)$ be in class A. From corollary 2.1 it follows that $W(t,m)$ has a $(t,m) \in [0,T] \times \mathcal{K}_o$ uniform, $H_1(t,m)$-bound zero;

$$\begin{aligned} \|W(t,m)\psi\| &\leq \alpha_1 \|\psi\| + \beta_1 \|[H_o(m) + W(t,m) - W(t,m)]\psi\| \\ &\leq \alpha_1 \|\psi\| + \beta_1 \|H_1(t,m)\psi\| + \beta_1 \|W(t,m)\psi\|; \end{aligned}$$

$$\Rightarrow \|W(t,m)\psi\| \leq \frac{\alpha_1}{1-\beta_1}\|\psi\| + \frac{\beta_1}{1-\beta_1}\|H_1(t,m)\psi\|, \qquad \psi \in D_o.$$

We note that $\alpha_1$ and $\beta_1$ are uniformly bounded in $(t,m) \in [0,T] \times \mathcal{K}_o$. As $\beta_1$ can be made arbitrarily small this proves our assertion. Next we use this bound for

* Recall for the spin half case the spin operator is $\vec{S} = \frac{\hbar}{2}(\sigma_1, \sigma_2, \sigma_3)$ where the $\sigma_i$'s are the Pauli matices. Similar representations exist for the higher dimensional spins.

$W(t,m)$ to get for the $H_1(t,m)$-bound of $\mathbf{v}(t,m)$. If $\mathbf{v}$ is in class A, bound (2.21) is valid and thus

$$\begin{aligned}
\|\mathbf{v}(t,m)\psi\| &\leq \alpha_o\|\psi\| + \beta_o\|[H_o(m) + W(t,m) - W(t,m)]\psi\| \\
&\leq \alpha_o\|\psi\| + \beta_o\|H_1(t,m)\psi\| + \beta_o\|W(t,m)\psi\| \\
&\leq \left(\alpha_o + \frac{\alpha_1\beta_o}{1-\beta_1}\right)\|\psi\| + \left(\beta_o + \frac{\beta_o\beta_1}{1-\beta_1}\right)\|H_1(t,m)\psi\|, \qquad \psi \in D_o.
\end{aligned}$$

Defining the constants $\alpha = \alpha_o + \frac{\alpha_1\beta_o}{1-\beta_1}$ and $\beta = \frac{\beta_o}{1-\beta_1}$, we have that $\mathbf{v}(t,m)$ satisfies the bound

$$\|\mathbf{v}(t,m)\psi\| \leq \alpha\|\psi\| + \beta\|H_1(t,m)\psi\|. \tag{2.24}$$

As $\beta_1$ can be arbitrarily small and $\beta_o < 1$ by assumption, we can pick $\beta_1$ so small that $\beta$ will also be less than 1. The constants $\alpha$ and $\beta$ are uniformly bounded in $(t,m) \in [0,T] \times \mathcal{K}_o$ because $\alpha_j$ and $\beta_j$ $(j = 1,2)$ have this property.

It now follows from Kato's theorem IV.1.1 [Ka 84] that $H(t,m)$ is closed and has domain $D\Big(H(t,m)\Big) = D_o$.

Let $m \in (0,\infty)$ and $\mathbf{v}(t,m)$ be in class A. Then $\mathbf{v}(t,m)$ is a symmetric operator and from lemma 2.1(3), $H_1(t,m)$ is self-adjoint. An application of Kato's theorem V.4.4 [Ka 84] shows that the Hamiltonian is self-adjoint.

Finally to show that $H(t,m)$ is bounded from below when $m \in (0,\infty)$ and the potential is in class A, we utilize another theorem of Kato, theorem IV.3.17 [Ka 84]. This states that $\xi$ is in the resolvent set of $H(t,m)$ if $\xi \in \rho\Big(H_1(t,m)\Big)$ and satisfies the estimate

$$\alpha\|R(\xi, H_1(t,m))\| + \beta\|H_1(t,m)R(\xi, H_1(t,m))\| < 1. \tag{2.25}$$

Here $\alpha$ and $\beta$ are the constants appearing in (2.24). Because $H_1(t,m) \geq 0$ we know the spectrum of $H_1(t,m)$ satisfies $\sigma\left(H_1(t,m)\right) \subset [0,\infty)$. This leads to the estimates

$$\|R(\xi, H_1(t,m))\| = \sup_{\lambda\in\sigma(H_1)} \frac{1}{|\lambda-\xi|} \leq \sup_{\lambda\in[0,\infty)} \frac{1}{|\lambda-\xi|};$$

$$\|H_1(t,m)R(\xi, H_1(t,m))\| = \sup_{\lambda\in\sigma(H_1)} \frac{\lambda}{|\lambda-\xi|} \leq \sup_{\lambda\in[0,\infty)} \frac{\lambda}{|\lambda-\xi|}.$$

If $\xi \in (-\infty, 0)$ then these estimates become

$$\|R(\xi, H_1(t,m))\| \leq \frac{1}{|\xi|};$$

$$\|H_1(t,m)R(\xi, H_1(t,m))\| \leq 1.$$

Thus we have that (2.25) is satisfied if $\frac{\alpha}{|\xi|} + \beta < 1$ or equivalently $\xi < -\frac{\alpha}{1-\beta}$. Setting $b_o = \frac{\alpha}{1-\beta}$ the spectrum of $H(t,m)$ satisfies $\sigma\Big(H(t,m)\Big) \subset [-b_o, \infty)$ and from this it follows $H(t,m) \geq -b_o$. ◊

The next property we wish to verify is the differentiability of the Hamiltonian. It is convenient to utilize the momentum operator $P : \mathcal{H} \to \mathcal{H}_v$ previously defined. In terms of the momentum operator the free Hamiltonian can be written

$$H_o(m) = \frac{1}{2m} P \cdot P$$

where $P \cdot P = \sum_{j=1}^{d} P_j^2$. Define the family of operators $\dot{H}(t,m)$ with domain $D_o$ by the equation

$$\dot{H}(t,m) = -\frac{1}{m}\dot{\mathbf{a}}(t) \cdot P + \frac{i\hbar}{2m}(\nabla \cdot \dot{a})(\cdot, t) + \frac{1}{2m}\dot{\mathbf{a}}(t) \cdot \mathbf{a}(t) + \frac{1}{2m}\mathbf{a}(t) \cdot \dot{\mathbf{a}}(t) + \dot{\mathbf{v}}(t,m). \tag{2.26}$$

By our assumptions (1) and (2) it is easy to see that $\dot{H}(t,m)$ is well defined on $D_o$ because $D_o$ is in the intersection of all the domains of the operators in the right hand side of (2.26). We claim that $\dot{H}(t,m)$ is the strongly continuous $t$-derivative of $H(t,m)$.

**Lemma 2.3:** *Let* $\mathbf{a}(t)$ *and* $\mathbf{v}(t,m)$ *satisfy the assumptions 1 and 2. Then* $H(t,m)$ *is strongly continuously differentiable on* $D_o$. *Moreover its derivative is given by the formula*

$$\frac{d}{dt}H(t,m)\psi = \dot{H}(t,m)\psi, \qquad \psi \in D_o. \tag{2.27}$$

*Proof:* Let $\delta t \neq 0$, $\psi \in D_o$ and consider the following in the limit that $\delta t \longrightarrow 0$.

$$\begin{aligned}
&\left\| \left[\frac{H(t+\delta t,m) - H(t,m)}{\delta t}\right]\psi - \dot{H}(t,m)\psi \right\| \\
&\qquad \leq \frac{1}{|m|}\left\| \frac{a(\cdot,t+\delta t) - a(\cdot,t)}{\delta t} - \dot{a}(\cdot,t)\right\|_\infty \|P\psi\|_v \\
&\qquad + \frac{\hbar}{2|m|}\left\| \frac{(\nabla\cdot a)(\cdot,t+\delta t) - (\nabla\cdot a)(\cdot,t)}{\delta t} - (\nabla\cdot\dot{a})(\cdot,t)\right\|_\infty \|\psi\| \\
&\qquad + \frac{1}{2|m|}\left\| \frac{a(\cdot,t+\delta t)^2 - a(\cdot,t)^2}{\delta t} - \dot{a}(\cdot,t)\cdot a(\cdot,t) - a(\cdot,t)\cdot\dot{a}(\cdot,t)\right\|_\infty \|\psi\| \\
&\qquad + \left\| \left[\frac{\mathbf{v}(t+\delta t,m) - \mathbf{v}(t,m)}{\delta t} - \dot{\mathbf{v}}(t,m)\right]\psi \right\|.
\end{aligned}$$

We notice that if $\mathbf{v}(t,m)$ is in class B then it is also strongly continuously differentiable like the potentials in class A. Thus we have from our assumptions 1 and 2 that the right hand side has a limit of zero as $\delta t \to 0$. This verifies (2.27). The strong continuity of $\dot{H}(t,m)$ on $D_o$ follows from the assumptions 1 and 2 with a similar argument to the one given above. $\diamondsuit$

The final step of our analysis is to show that the resolvent of $A(t,m)$ defined by equation (2.11) for a suitably chosen constant $c$, satisfies the bound (2.7). There is a simple relationship between the resolvents of $H(t,m)$ and $A(t,m)$.

$$\begin{aligned} R(\lambda, A(t,m)) &\equiv [A(t,m)-\lambda]^{-1} \\ &= \left[\frac{1}{i\hbar}H(t,m) - c - \lambda\right]^{-1} \\ &= i\hbar[H(t,m) - i\hbar(c+\lambda)]^{-1} \\ &= i\hbar R(i\hbar(c+\lambda), H(t,m)). \end{aligned} \tag{2.28}$$

We first estimate the resolvent $R\,(i\omega, H(t,m))$.

**Lemma 2.4:** *Let* $\mathbf{a}(t)$ *and* $\mathbf{v}(t,m)$ *satisfy the assumptions 1 and 2. If* $\omega > \frac{\alpha}{1-\beta}$ *then we have the estimate*

$$\|R(i\omega, H(t,m))\| \le \frac{1}{\omega - \alpha}. \tag{2.29}$$

*Here the constant* $\alpha$ *is the same as that appearing in (2.24) if* $\mathbf{v}(t,m)$ *is in class A, or (2.22) if the potential is in class B.*

*Proof:* If $\mathbf{v}(t,m)$ is in class A then $H(t,m)$ is self-adjoint and hence the spectrum is contained on the real line axis. If $d(z,\sigma(H))$ is the distance between the complex number $z$ and the spectrum of $H(t,m)$, then

$$\|R(i\omega, H(t,m))\| = \frac{1}{d(i\omega,\sigma(H))} \le \frac{1}{\omega} \le \frac{1}{\omega-\alpha}.$$

Now we suppose the potential is in class B. Because $m$ is allowed to be complex, it is convenient to write it in its polar representation; $m = |m|e^{i\phi}$, $\phi \in [0,\pi]$. The Hamiltonian and resolvent can now be rewritten

$$H(t,m) = e^{-i\phi}\{H_1(t,|m|) + e^{i\phi}\mathbf{v}(t,m)\} \equiv e^{-i\phi}\breve{H}(t,m)$$

and

$$R(i\omega, H(t,m)) = e^{i\phi} R(e^{i\phi} i\omega, \breve{H}(t,m)).$$

Recall that $H_1(t,|m|)$ is self-adjoint and that its spectrum lies in the semi-infinite interval $[0,\infty)$. Because $e^{i\phi} i\omega$ lies on the half circle $\{z : |z| = \omega, \arg z \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$. the resolvent $R(e^{i\phi} i\omega, H_1(t,|m|))$ satisfies the estimate

$$\|R(e^{i\phi} i\omega, H_1(t,|m|))\| \le \frac{1}{\omega},$$

If $\omega > \alpha$, then

$$\begin{aligned}\|e^{i\phi}\mathbf{v}(t,m)R(e^{i\phi} i\omega, H_1(t,|m|))\| &\le \|\mathbf{v}(t,m)\| \; \|R(e^{i\phi} i\omega, H_1(t,|m|))\| \\ &\le \frac{\alpha}{\omega} < 1.\end{aligned}$$

Thus the operator $I + e^{i\phi}\mathbf{v}(t,m)R(e^{i\phi} i\omega, H_1(t,|m|))$ has a bounded inverse (vis-a-vis the Neumann series) and $e^{i\phi} i\omega \in \rho(\breve{H})$. It is clear $i\omega$ is in the resolvent set of $H(t,m)$. Furthermore, from the identity

$$R(e^{i\phi} i\omega, \breve{H}(t,m)) = R(e^{i\phi} i\omega, H_1(t,|m|))\Big[I + e^{i\phi}\mathbf{v}(t,m)R(e^{i\phi} i\omega, H_1(t,|m|))\Big]^{-1},$$

we get the estimate

$$\begin{aligned}\|R(i\omega, H(t,m))\| &= \|R(e^{i\phi} i\omega, \breve{H}(t,m))\| \\ &\le \frac{1}{\omega}\frac{1}{1-\frac{\alpha}{\omega}} \\ &= \frac{1}{\omega-\alpha}.\end{aligned}$$

◇

Utilizing lemma 2.4 and equation (2.28) we can now easily estimate the resolvent of $A(t,m)$;

$$\begin{aligned}\|R(\lambda, A(t,m))\| &= \hbar \|R(i\hbar(c+\lambda), H(t,m))\| \\ &\leq \frac{\hbar}{\hbar(c+\lambda)-\alpha} \\ &= \frac{1}{c+\lambda-\frac{\alpha}{\hbar}}.\end{aligned}$$

We make the choice for the constant $c$;

$$c = 1 + \frac{\alpha}{\hbar}. \tag{2.30}$$

With this choice of $c$ it follows that $\omega = \hbar(c+\lambda)$ satisfies the estimate $\omega > \alpha$ for all $\lambda \geq 0$, so that the hypotheses of lemma 2.4 are verified. Moreover substituting this value of $c$ into the above shows the resolvent of $A(t,m)$ satisfies the estimate (2.7). We have thus shown the hypotheses of theorem 2.2 are satisfied with $c$ given by (2.30) and hence we have the existence of the Schrödinger evolution for the potentials satisfying assumptions 1 and 2.

### 2.3 The Mass Continuity of the Evolution

One final topic we wish to look at in this chapter is the continuity in the mass parameter of the complex mass Schrödinger evolution. For this topic, only potentials in class B shall be considered. This is because we need this property for our discussions about the propagator and it is only potentials in class B that we shall be considering there.

**Proposition 2.2:** *Let assumption 1 be valid and assume* $\mathbf{v}(t,m)$ *belongs to class B. Let* $U(t,t_o;m)$ *be the corresponding Schrödinger evolution operator. For each fixed* $(t,t_o) \in T_\Delta$, $U(t,t_o;m)$ *is strongly continuous with respect to* $m \in \mathbb{C}_+$.

*Proof:* As we have the relation

$$U(t, t_o; m) = e^{c(t-t_o)}\mathcal{U}(t, t_o; m)$$

where the constant $c$ is given by (2.30) and is independent of $m \in \mathcal{K}$, we see that $U(t, t_o; m)$ is strongly continuous in $m$ if and only if $\mathcal{U}(t, t_o; m)$ is strongly continuous in $m$.

Let $\psi \in D_o$ and $m, m' \in \mathcal{K}$ with $m \neq m'$, and fix $(t, t_o) \in T_\Delta$. Let $\tau \in [t_o, t]$ and consider the following in the limit that $\delta\tau \to 0$.

$$\begin{aligned}
&\left[\frac{\mathcal{U}(t, \tau + \delta\tau; m')\mathcal{U}(\tau + \delta\tau, t_o; m) - \mathcal{U}(t, \tau; m')\mathcal{U}(\tau, t_o; m)}{\delta\tau}\right]\psi \\
&\qquad = [\mathcal{U}(t, \tau + \delta\tau; m') - \mathcal{U}(t, \tau; m')]\left[\frac{\mathcal{U}(\tau + \delta\tau, t_o; m) - \mathcal{U}(\tau, t_o; m)}{\delta\tau}\right]\psi \\
&\qquad\quad + \mathcal{U}(t, \tau; m')\left[\frac{\mathcal{U}(\tau + \delta\tau, t_o; m) - \mathcal{U}(\tau, t_o; m)}{\delta\tau}\right]\psi \\
&\qquad\quad + \left[\frac{\mathcal{U}(t, \tau + \delta\tau; m') - \mathcal{U}(t, \tau; m')}{\delta\tau}\right]\mathcal{U}(\tau, t_o; m)\psi.
\end{aligned}$$

It follows from proposition 2.1 and theorem 2.1 that the second and third terms of the right hand side have the strong limits $\mathcal{U}(t, \tau; m')A(\tau, m)\mathcal{U}(\tau, t_o; m)\psi$ and $-\mathcal{U}(t, \tau; m')A(\tau, m')\mathcal{U}(\tau, t_o; m)\psi$ respectively. For the first term, we claim it goes strongly to zero. We can see this from the following;

$$\begin{aligned}
1^{\text{st}}\text{term} &= [\mathcal{U}(t, \tau + \delta\tau; m') - \mathcal{U}(t, \tau; m')]A(\tau, m)\mathcal{U}(\tau, t_o; m)\psi \\
&\quad + [\mathcal{U}(t, \tau + \delta\tau; m') - \mathcal{U}(t, \tau; m')] \\
&\qquad \times \left[\frac{\mathcal{U}(\tau + \delta\tau, t_o; m) - \mathcal{U}(\tau, t_o; m)}{\delta\tau} - A(\tau, m)\mathcal{U}(\tau, t_o; m)\right]\psi.
\end{aligned}$$

Since $\mathcal{U}(t, \tau; m')$ is strongly continuous in $\tau$ the first line of the right hand side goes to zero as $\delta\tau \to 0$. We also know that $\mathcal{U}(t_1, t_2; m')$ is bounded by 1 for all $t_1$, $t_2$, and $m'$. Thus the second line has the norm estimate

$$2\left\|\left[\frac{\mathcal{U}(\tau+\delta\tau,t_o;m)-\mathcal{U}(\tau,t_o;m)}{\delta\tau}-A(\tau,m)\mathcal{U}(\tau,t_o;m)\right]\psi\right\|,$$

which goes to zero as $\delta\tau\to 0$ by proposition 2.1. Hence we have shown the identity

$$\frac{\partial}{\partial\tau}\mathcal{U}(t,\tau;m')\mathcal{U}(\tau,t_o;m)\psi=\mathcal{U}(t,\tau;m')[A(\tau,m)-A(\tau,m')]\mathcal{U}(\tau,t_o;m)\psi. \qquad (2.31)$$

Next we show that the right hand side of (2.31) is strongly continuous in $\tau$. For the first term this is trivial because $\mathcal{U}(t,\tau;m')$ is strongly continuous and uniformly bounded by 1 and $A(\tau,m)\mathcal{U}(\tau,t_o;m)\psi=\partial_\tau\mathcal{U}(\tau,t_o;m)\psi$ is strongly continuous. The second term on the right hand side of (2.31) requires a little more work.

We know that $\mathcal{U}(t,\tau;m')A(\tau,m')$ is strongly continuous on the domain $D_o$ and that $\mathcal{U}(\tau,t_o;m)\psi\in D_o$ is also strongly continuous with respect to $\tau\in[t_o,t]$. Consider

$$\begin{aligned}
&\|\mathcal{U}(t,\tau;m')A(\tau,m')\mathcal{U}(\tau,t_o;m)\psi-\mathcal{U}(t,\tau';m')A(\tau',m')\mathcal{U}(\tau',t_o;m)\psi\|\\
&\qquad\le\|[\mathcal{U}(t,\tau;m')A(\tau,m')-\mathcal{U}(t,\tau';m')A(\tau',m')]\mathcal{U}(\tau,t_o;m)\psi\|\\
&\qquad\quad+\|\mathcal{U}(t,\tau';m')A(\tau',m')[\mathcal{U}(\tau,t_o;m)-\mathcal{U}(\tau',t_o;m)]\psi\|.
\end{aligned}$$

Clearly the first term on the right hand side tends to zero as $\tau'\to\tau$. It is again the second term that we must work on. We note the operator $A(t,m)^{-1}$ exists for all $t$ because $\lambda=0\in\rho\Big(A(t,m)\Big)$. Insert the identity operator $A(\tau,m)^{-1}A(\tau,m)$ between $\mathcal{U}(t,\tau';m')A(\tau',m')$ and $[\mathcal{U}(\tau,t_o;m)-\mathcal{U}(\tau',t_o;m)]\psi$. Now the operator $A(\tau,m')A(\tau,m)^{-1}$ is closed and defined everywhere. It follows from the closed graph theorem ([RSz 78], p. 306) that it is bounded. In particular picking $\tau=0$

shows the operator $A(0,m')A(0,m)^{-1}$ is bounded. We thus have the norm estimate

$$\begin{aligned} \|\mathcal{U}(t,\tau';m')A(\tau',m')A(\tau,m)^{-1}\| &\le \|A(\tau',m')A(\tau,m)^{-1}\| \\ &\le \|A(\tau',m')A(0,m')^{-1}\|\|A(0,m')A(0,m)^{-1}\| \\ &\quad \times \|A(0,m)A(\tau,m)^{-1}\|. \end{aligned}$$

By lemma 1.5 of Krein ([Kr 71], chapter II.§1) $A(t',m)A(t'',m)^{-1}$ is continuous with respect to $(t',t'')$ in the operator norm topology. As $(t',t'')$ belong to the compact set $[0,T]\times[0,T]$, we must have that these operators are uniformly bounded with respect to $(t',t'')$. Thus we see that $\mathcal{U}(t,\tau';m')A(\tau',m')A(\tau,m)^{-1}$ is uniformly bounded for all $\tau$ and $\tau'$. Finally we deal with

$$\begin{aligned} A(\tau,m)&[\mathcal{U}(\tau,t_o;m)-\mathcal{U}(\tau',t_o;m)]\psi \\ &= A(\tau,m)\mathcal{U}(\tau,t_o;m)\psi - A(\tau',m)\mathcal{U}(\tau',t_o;m)\psi \\ &\quad + [I - A(\tau,m)A(\tau',m)^{-1}]A(\tau',m)\mathcal{U}(\tau',t_o;m)\psi. \end{aligned}$$

The first line on the right hand side here goes to zero as $\tau' \to \tau$ because of the strong continuity of $A(\tau,m)\mathcal{U}(\tau,t_o;m)\psi$. The second line on the right hand side tends to zero because of the aforementioned continuity of $A(\tau,m)A(\tau',m)^{-1}$ in the operator topology and the fact $\|A(\tau',m)\mathcal{U}(\tau',t_o;m)\psi\|$ is uniformly bounded in $\tau'$.

Because the right hand side of (2.31) is strongly continuous we may take the strong Riemann integral ([La 69], chapter X) of (2.31) over $\tau$ from $t_o$ to $t$. The fundamental theorem of calculus is valid for the strong Riemann integral ([La 69], theorem 10.8). Using the properties of the evolution operators we obtain the relation

$$[\mathcal{U}(t,t_o;m')-\mathcal{U}(t,t_o;m)]\psi = \int_{t_o}^{t} \mathcal{U}(t,\tau;m')[A(\tau,m)-A(\tau,m')]\mathcal{U}(\tau,t_o;m)\psi \, d\tau. \tag{2.32}$$

With (2.32) we obtain the norm estimate

$$\|[\mathcal{U}(t,t_o;m') - \mathcal{U}(t,t_o;m)]\psi\| \leq \int_{t_o}^{t} \|[A(\tau,m) - A(\tau,m')]\mathcal{U}(\tau,t_o;m)\psi\| \, d\tau. \quad (2.33)$$

From the definition of $A$, equation (2.11), it easily follows that

$$\begin{aligned} A(\tau,m) - A(\tau,m') = \left(1 - \frac{m}{m'}\right) A(\tau,m) + \left(1 - \frac{m}{m'}\right) cI \\ + \frac{1}{i\hbar}[\mathbf{v}(\tau,m) - \mathbf{v}(\tau,m')] - \left(1 - \frac{m}{m'}\right) \frac{1}{i\hbar}\mathbf{v}(\tau,m). \end{aligned} \quad (2.34)$$

Substituting (2.34) into (2.33) we get the estimate

$$\begin{aligned} &\|[\mathcal{U}(t,t_o;m) - \mathcal{U}(t,t_o;m')]\psi\| \\ &\leq \left|1 - \frac{m}{m'}\right| \left\{ \int_{t_o}^{t} \|A(\tau,m)\mathcal{U}(\tau,t_o;m)\psi\| \, d\tau + \left(\frac{\alpha}{\hbar} + c\right)(t - t_o)\|\psi\| \right\} \\ &\quad + \frac{t - t_o}{\hbar} \sup_{\tau\in[0,T]} \|v(\cdot,\tau;m) - v(\cdot,\tau;m')\|_{\infty} \|\psi\|. \end{aligned}$$

The right hand side goes to zero as $m' \to m$.

Thus we have shown that $\mathcal{U}(t,t_o;m)$ restricted to the domain $D_o$ is strongly continuous with respect to the mass parameter $m$ in any compact set $\mathcal{K} \subset \mathbb{C}_+$. Because $\mathcal{U}(t,t_o;m)$ is uniformly bounded by 1 and the domain $D_o$ is dense, it is trivial to extend this continuity property to all of $\mathcal{H}$. $\diamond$

# CHAPTER 3

## Measures and the Potentials

In this chapter we give precise definitions to the class of potentials that we will be using for the remainder of part I of the thesis. The potentials under study are the Fourier images of complex matrix valued measures. We begin by discussing the relevent measure spaces.

Let the tuple $(\mathbb{R}^d, B)$ specify the measureable space consisting of the set $\mathbb{R}^d$ and the smallest $\sigma$-algebra $B$ of Borel subsets of $\mathbb{R}^d$. We let $r = d$ or 1 and we denote by $(\mathbb{C}^{s\times s})^r$, the space of complex $s \times s$ matrices grouped together as a $d$-tuple if $r = d$ or an $s \times s$ matrix if $r = 1$. A $(\mathbb{C}^{s\times s})^r$-valued measure $\gamma$ on $(\mathbb{R}^d, B)$ is a countably additive set function mapping $B \to (\mathbb{C}^{s\times s})^r$. The associated total variation measure $|\gamma|$ is a measure on $(\mathbb{R}^d, B)$ mapping $B \to [0, \infty)$ and defined by

$$|\gamma|(e) = \sup_{\pi} \sum_{e_i \in \pi} |\gamma(e_i)| \qquad e_i \in B. \tag{3.1}$$

On the right hand side of (3.1), $|\cdot|$ is the Euclidean norm for the space $(\mathbb{C}^{s\times s})^r$;

$$|\gamma(e_i)|^2 = \sum_{j=1}^{r} \sum_{\alpha,\beta=1}^{s} \left| [\gamma_{\alpha\beta}(e_i)]_j \right|^2$$

and the supremum is taken over all countable partitions $\pi$ of $e$ allowed by $B$. The measure $\gamma$ is defined to be of bounded variation if $|\gamma|(\mathbb{R}^d) < \infty$.

To the set of measures of finite total variation we adjoin, in the standard way, the operations of addition and multiplication by complex scalars. The resulting vector space we denote by $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$. We can make $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ into a normed linear space by defining the norm

$$\|\gamma\| = |\gamma|(\mathbb{R}^d), \qquad \gamma \in \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r). \tag{3.2}$$

With this norm attached it can be shown ([DS 76], pp. 160–162) that $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ is a Banach space.

It is possible and extremely useful to use a representation of $\gamma$ in terms of its total variation measure $|\gamma|$. We call this representation the *polar decomposition* of $\gamma$ and we claim there exists a Borel measureable function $\eta : \mathbb{R}^d \to (\mathbb{C}^{s\times s})^r$ such that $|\eta(\alpha)| = 1$ for all $\alpha$ and

$$\int_e d\gamma = \int_e \eta(\alpha)\, d|\gamma|, \qquad e \in B. \tag{3.3}$$

To see the existance of such an $\eta$ we note that $\gamma$ is absolutely continuous with respect to $|\gamma|$. To say a measure $\mu$ is absolutely continuous with respect to another measure $\lambda$ means that whenever we have a measurable set $e$ such that $\lambda(e) = 0$, then $\mu(e) = 0$ and we write $\mu \ll \lambda$. As $\gamma \ll |\gamma|$, we may apply the Radon-Nikodym theorem [Ru 74] which asserts the existence of the function $\eta$. The proof that $\eta$ is a function of modulus one is based on a simple modification of theorems 1.40 and 6.12 of Rudin [Ru 74].

To each measure $\gamma \in \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ we can define a function $a : \mathbb{R}^d \to (\mathbb{C}^{s\times s})^r$ via the Fourier transform of the measure,

$$a(x) = \int e^{i\alpha\cdot x}\, d\gamma. \tag{3.4}$$

The function $a$ is well defined for each $x \in \mathbb{R}^d$ because $e^{i\alpha\cdot x}$ is Borel measurable and $L^1(\mathbb{R}^d, d\gamma)$. It follows from the dominated convergence theorem that $a$ is continuous. Moreover $a$ admits the uniform bound

$$|a(x)| \leq \|\gamma\|, \qquad x \in \mathbb{R}^d. \tag{3.5}$$

We denote the Fourier image of $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ by $\mathcal{F}^r$. The Fourier transform mapping (3.4) establishes a one-to-one correspondence between $\mathcal{F}^r$ and $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ in that $a(\cdot) = 0$ if and only if $\gamma = 0$ [Ru 61]. By assigning the norm $\|a\| = \|\gamma\|$ to $\mathcal{F}^r$, $\mathcal{F}^r$ will also be a Banach space.

Prevalent in the analysis of the Dyson series analysis of the next chapter will be the use of product measures. Consider the case of two $s \times s$-matrix valued measures $\mu_1$ and $\mu_2$ over $(\mathbb{R}^d, B_1)$ and $(\mathbb{R}^d, B_2)$ respectively. The product measure $\mu_1 \times \mu_2$ is defined on the smallest $\sigma$-algebra $B_1 \times B_2$ by requiring that for every measureable rectangle $e_1 \times e_2$ of $B_1 \times B_2$,

$$(\mu_1 \times \mu_2)(e_1 \times e_2) = \mu_1(e_1)\mu_2(e_2).$$

The product on the right hand side of the above equation is the usual matrix product, so that we see that $\mu_1 \times \mu_2$ is an $s \times s$-matrix valued set function as well. We can make a Banach space $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{s\times s})$ as before with the norm

$$\|\mu_1 \times \mu_2\| = |\mu_1 \times \mu_2|(\mathbb{R}^d \times \mathbb{R}^d). \tag{3.6}$$

It is easily shown that the norm on $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{s\times s})$ satisfies the bound

$$\|\mu_1 \times \mu_2\| \leq \|\mu_1\|\|\mu_2\|. \tag{3.7}$$

The vector potential of the previous section had components whose pointwise values were required to be hermitian matrices. If $a$ defines an $r$-tuple ($r = 1, d$) of hermitian matrices and it is the Fourier image of the measure $\gamma$, then $\gamma$ will satisfy the reflection property

$$\gamma_j(e) = \gamma_j(-e)^*, \qquad e \in B, \quad j = 1, \ldots, r.$$

Here the set $-e$ is defined by $-e = \{\alpha \in \mathbb{R}^d : -\alpha \in e\}$ and the symbol $^*$ denotes the complex conjugate transpose. We denote the set of all $\gamma \in \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ that satisfy the reflection property by $\mathcal{M}^*(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ and its corresponding Fourier image by $\mathcal{F}^{r*}$. $\mathcal{F}^{r*}$ is a subspace of $\mathcal{F}^r$.

We must also discuss the convolution of two measures in order to understand terms like $a(x,t) \cdot a(x,t)$ that appeared in the Hamiltonian. Let $a$ and $a'$ be associated with the pair of measures $\gamma$ and $\gamma' \in \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$. The convolution of the two measures is a map $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r) \times \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r) \to \mathcal{M}(\mathbb{R}^d, \mathbb{C}^{s\times s})$ defined (constructively) by

$$\gamma * \gamma'(e) = \int \chi_e(\alpha + \alpha')\eta(\alpha) \cdot \eta'(\alpha')\, d|\gamma| \times |\gamma'|. \tag{3.8}$$

The functions $\eta$ and $\eta'$ are the functions associated with the polar representations of $\gamma$ and $\gamma'$ repectively. The function $\chi_e$ is the characteristic function for the set $e \in B$. The dot product between the $\eta$'s is defined as the sum over the matrix products of the components of $\eta$ and $\eta'$;

$$\eta(\alpha) \cdot \eta'(\alpha') = [\eta(\alpha)]_1 [\eta'(\alpha')]_1 + \cdots + [\eta(\alpha)]_r [\eta'(\alpha')]_r.$$

Consequently the function $\gamma * \gamma'$ takes values in $\mathbb{C}^{s\times s}$. It is easily shown that $\gamma * \gamma' \in \mathcal{M}(\mathbb{R}^d, \mathbb{C}^{s\times s})$ and satisfies the norm estimate

$$\|\gamma * \gamma'\| \le \|\gamma\| \|\gamma'\|. \tag{3.9}$$

The dot product in $\mathcal{F}^r$ is related to the convolution in $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ by the identity

$$a(x)\cdot a'(x) = \int e^{i\alpha\cdot x}\, d\gamma * \gamma'. \tag{3.10}$$

In order to be able to control the behaviour of the Dyson series we will consider a subclass of measures consisting of those measures in $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ with compact support. Let $S_k \subset \mathbb{R}^d$ be the closed ball of radius $k > 0$ and centred on the origin. We denote the subset of measures in $\mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ whose support lies in $S_k$ by $\mathcal{M}(S_k, (\mathbb{C}^{s\times s})^r)$. It is easy to show that $\mathcal{M}(S_k, (\mathbb{C}^{s\times s})^r)$ is also a Banach space. Similarly we define $\mathcal{M}^*(S_k, (\mathbb{C}^{s\times s})^r)$ to be the set of measures in $\mathcal{M}(S_k, (\mathbb{C}^{s\times s})^r)$ that satisfy the reflection property and it too is a Banach space with respect to the norm (3.2).

We next wish to discuss the idea of measures that depend upon the time and mass parameters. Let $\mathcal{K} \subset \mathbb{C}_+$ be any compact set and consider the Banach space-valued map

$$\gamma(\cdot,\cdot) : [0,T]\times\mathcal{K} \to \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r).$$

We say that $\gamma$ is jointly continuous in $(t,m) \in [0,T]\times\mathcal{K}$ if

$$\|\gamma(t',m') - \gamma(t,m)\| \to 0 \qquad \text{as } (t',m') \to (t,m) \tag{3.11}$$

for all $(t,m) \in [0,T]\times\mathcal{K}$. The measure $\gamma$ is continuously differentiable with respect to $t \in [0,T]$ if there exist a family of measures $\dot\gamma(t,m) \in \mathcal{M}(\mathbb{R}^d, (\mathbb{C}^{s\times s})^r)$ such that for each fixed $m \in \mathcal{K}$, $\dot\gamma(\cdot,m)$ is continuous in $t \in [0,T]$ and

$$\left\|\frac{\gamma(t',m) - \gamma(t,m)}{t'-t} - \dot\gamma(t,m)\right\| \to 0 \qquad \text{as } t' \to t \tag{3.12}$$

The $(t,m)$-continuity of measures $\mu(t,m) \in \mathcal{M}(\mathbb{R}^d, \mathbb{C}^{s\times s})$ implies the joint continuity of the product measures $\mu_1(t_1,m_1)\times\mu_2(t_2,m_2)\times\cdots\times\mu_n(t_n,m_n)$. We look at the case of the product of two measures.The n-fold product will follow from an induction argument.

$$\begin{aligned}\|\mu_1(t_1',m_1')\times\mu_2(t_2',m_2')-\mu_1(t_1,m_1)\times\mu_2(t_2,m_2)\|\\ \leq\|\mu(t_1',m_1')\|\|\mu_2(t_2',m_2')-\mu_2(t_2,m_2)\|\\ +\|\mu_2(t_2,m_2)\|\|\mu_1(t_1',m_1')-\mu_1(t_1,m_1)\|\end{aligned}\tag{3.13}$$

Since the joint continuity of $\mu_j(t_j,m_j)$, (j=1,2) implies the uniform boundedness of their norms, we immediately have the joint $(t_1,m_1,t_2,m_2)$-continuity of the product measure $\mu_1(t_1,m_1)\times\mu_2(t_2,m_2)$.

We are now in a position to state the hypotheses on the class of potentials under study for the remainder of part I of this thesis.

**Assumption 3:** The vector potential $a:\mathbb{R}^d\times[0,T]\to(\mathbb{C}^{s\times s})^d$ is said to be in class $\mathcal{V}_v(k)$ if $a$ is the Fourier image of a time dependent family of measures $\gamma(t)$ satisfying

(1) $\gamma(t)\in\mathcal{M}^*(S_{k/2},(\mathbb{C}^{s\times s})^d),\qquad t\in[0,T],\quad k<\infty.$

(2) $\gamma(t)$ is continuously differentiable on $[0,T]$.

**Assumption 4:** The potential $v:\mathbb{R}^d\times[0,T]\times\mathcal{K}\to\mathbb{C}^{s\times s}$ is said to be in class $\mathcal{V}(k)$ if $v$ is the Fourier image of a time and mass dependent family of measures $\nu(t,m)$ satisfying

(1) $\nu(t,m)\in\mathcal{M}(S_k,\mathbb{C}^{s\times s}),\qquad (t,m)\in[0,T]\times\mathcal{K},\quad k<\infty.$

(2) $\nu(t,m)$ is jointly continuous in $(t,m)\in[0,T]\times\mathcal{K}$ and for fixed $m\in\mathcal{K}$ it is continuously differentiable with respect to $t$ on $[0,T]$.

Because $\gamma(t)$ and $\nu(t,m)$ are continuous, their norms will also be continuous. Since the sets $[0,T]$ and $[0,T]\times\mathcal{K}$ are compact, the functions $\|\gamma(t)\|$ and $\|\nu(t,m)\|$

will attain their maximums and we have that these norms are uniformly bounded.

$$\begin{aligned} \|\gamma(t)\| &\leq \sup \|\gamma(t)\| \equiv \gamma_T \\ \|\nu(t,m)\| &\leq \sup \|\nu(t,m)\| \equiv \nu_T \end{aligned} \tag{3.14}$$

Here the first supremum is over $t \in [0,T]$ and the second is over $(t,m) \in [0,T] \times \mathcal{K}$.

Next we show as a consequence of $a \in \mathcal{V}_v(k)$ and $v \in \mathcal{V}(k)$, $a$ and $v$ will satisfy assumptions 1 and 2(B).

**Proposition 3.1:** Let $a \in \mathcal{V}_v(k)$ and $v \in \mathcal{V}(k)$. Then $a$ and $v$ satisfy the properties described in assumptions 1 and 2(B).

*Proof:* We show the proof for $v$, with the proof for $a$ following similarly. Many of the arguments given below are due to the continuity of the measures, and the compactness of their support.

From the support of $\nu(t,m)$ and an application of the dominated convergence theorem it follows that $v(x,t;m)$ is a $C^\infty$ function of $x$ with derivatives given by

$$(\partial_x^\rho v)(x,t;m) = \int i^{|\rho|} \alpha^\rho e^{i\alpha \cdot x}\, d\nu(t,m), \tag{3.15}$$

where $\rho$ is the multi-index $(\rho_1, \ldots, \rho_d)$. Further from (3.14) and the support of $\nu(t,m)$ it follows these derivatives have the $(x,t,m)$-uniform bounds

$$|(\partial_x^\rho v)(x,t;m)| \leq k^{|\rho|} \nu_T. \tag{3.16}$$

It follows from the joint continuity of $\nu(t,m)$ that $v$ is jointly continuous with respect to $(t,m) \in [0,T] \times \mathcal{K}$ using the norm $\|\cdot\|_\infty$;

$$\|v(\cdot,t';m') - v(\cdot,t;m)\|_\infty \leq \|\nu(t',m') - \nu(t,m)\| \to 0 \qquad \text{as } (t',m') \to (t,m).$$

The time derivative of the potential also has a simple form. We claim it is given by

$$(\partial_t v)(x,t;m) = \int e^{i\alpha\cdot x}\, d\dot{\nu}(t,m). \tag{3.17}$$

To see this let $\dot{v}(x,t;m)$ denote the integral on the right hand side of (3.17). From the identity

$$\frac{v(x,t';m)-v(x,t;m)}{t'-t} - \dot{v}(x,t;m) = \int e^{i\alpha\cdot x}\, d\left[\frac{\nu(t',m)-\nu(t,m)}{t'-t} - \dot{\nu}(t,m)\right],$$

we get the $x$-uniform estimate

$$\left\|\frac{v(\cdot,t';m)-v(\cdot,t;m)}{t'-t} - \dot{v}(\cdot,t;m)\right\|_\infty \leq \left\|\frac{\nu(t',m)-\nu(t,m)}{t'-t} - \dot{\nu}(t,m)\right\|.$$

The right hand side goes to zero as $t' \to t$ and our claim is proved. The $t$-continuity of $\dot{v}(\cdot,t;m)$ in the $L^\infty(dx;\mathbb{C}^{s\times s})$ topology follows from the continuity of $\dot{\nu}(t,m)$.

Because $\mathcal{M}(S_k,\mathbb{C}^{s\times s})$ is a Banach space and $\dot{\nu}(t,m)$ is the limit of measures in $\mathcal{M}(S_k,\mathbb{C}^{s\times s})$, $\dot{\nu}(t,m)$ will also have support in $S_k$. This means that the $t$-derivative of $v$ is also a $C^\infty$ function of $x$, with derivatives given by a formula like (3.15), but with $\nu$ replaced by $\dot{\nu}$. Moreover it is simple to show that the spatial derivatives of $v$ are differentiable in $t$ with the convergence for the $t$-derivative in the $L^\infty(dx;\mathbb{C}^{s\times s})$ norm. $\diamondsuit$

**Example:** As an example of our potential class, consider the case of a constant electric field $\vec{E}_o$. Define the $d$-dimensional vector $E = (q_1\vec{E}_o, q_2\vec{E}_o, \ldots, q_N\vec{E}_o)$ $(d = 3N)$ and define our $d$-dimensional vector field $a(x,t)$ by

$$\begin{aligned} a(x,t) &= \int e^{i\alpha\cdot x} EtI\, d\Upsilon(\alpha) \\ &= EtI. \end{aligned}$$

Here $I$ is the unit $s \times s$ matrix and $\Upsilon$ is a Dirac measure whose support is on the origin i.e.

$$\int \varphi(\alpha)\, d\Upsilon(\alpha) = \varphi(0).$$

Recalling the discussion leading upto the Hamiltonian in (1.3), we see that the 3-dimensional vector potential must be

$$\vec{A}(\vec{x}, t) = \vec{E}_o t.$$

From classical electromagnetism, the electric field is given by

$$\vec{E}(\vec{x}, t) = -\nabla_{\vec{x}}\phi(\vec{x}, t) - \frac{\partial}{\partial t}\vec{A}(\vec{x}, t).$$

Choosing our guage such that $\phi$ = constant and $\vec{A}$ is given by the above, our potential class describes a constant electric field. Furthermore we note that the more common choice of guage of setting $\phi(\vec{x}, t) = \vec{E}_o \cdot x$ and $\partial_t \vec{A}(x, t) = 0$ is technically more difficult to handle because $\vec{E}_o \cdot x$ is not $H_o$-bounded and consequently this choice of guage is not even a member of the class A potentials used in chapter 2.

As a final topic for this section we introduce some convenient notations for various linear combinations of measures. We first define the measure

$$\mu(t, m) = \frac{1}{2m}\gamma(t) * \gamma(t) + \nu(t, m). \tag{3.18}$$

We note that if $\gamma(t)$ has support in $\mathcal{M}(S_{k/2}, (\mathbb{C}^{s\times s})^d)$ then its convolution with itself will have support in $S_k$. As $\nu(t, m)$ also has support in $S_k$ we see that the measure $\mu(t, m)$ is a member of $\mathcal{M}(S_k, \mathbb{C}^{s\times s})$. Similar to (3.14) it is useful to define the norm bound for $\mu(t, m)$ of

$$\|\mu(t, m)\| \le \frac{1}{2|m|}\gamma_T^2 + \nu_T \equiv \mu_T. \tag{3.19}$$

This measure is a useful notation because it represents the sum of potentials

$$\frac{1}{2m}a(x,t)\cdot a(x,t)+v(x,t;m)=\int e^{i\alpha\cdot x}\,d\mu(t,m). \tag{3.20}$$

Often we will be using measures that involve the dot product between $\gamma$ and vectors in $\mathbb{R}^d$. Let $\beta$ be a fixed vector in $\mathbb{R}^d$ and define a measure $\mu_o(t,\beta)\in\mathcal{M}^*(S_{k/2},\mathbb{C}^{s\times s})$ by the formula

$$\mu_o(t,\beta)(e)=\int_e \beta\cdot\eta(t,\alpha)\,d|\gamma|(t), \qquad e\in B. \tag{3.21}$$

$\mu_o(t,\beta)$ is $\mathbb{C}^{s\times s}$-valued because of the dot product between $\beta$ and $\eta$ and the fact that the components of $\eta$ are $s\times s$ matrices. We note that $\mu_o(t,\beta)$ is a continuous measured valued function of $t$ and $\beta$.

Two more measures of similar functional form to $\mu_o(t)$ but whose $\beta$ is more complicated can be defined as follows. Let $\boldsymbol{\alpha}_n=(\alpha_1,\cdots,\alpha_n)$ be an $n$-tuple of vectors in $S_k$. For each positive index $l\le n$ we define the measures

$$\mu_l^n(t,\alpha_1,\cdots,\alpha_{l-1})(e)=\int_e\left(\frac{1}{2}\alpha+\sum_{j=1}^{l-1}\alpha_j\right)\cdot\eta(t,\alpha)\,d|\gamma|(t) \qquad e\in B$$

and

$$\hat{\mu}_l^n(t,\alpha_{l+1},\cdots,\alpha_n)(e)=\int_e\left(\frac{1}{2}\alpha+\sum_{j=l+1}^{n}\alpha_j\right)\cdot\eta(t,\alpha)\,d|\gamma|(t) \qquad e\in B.$$

$\mu_l^n$ and $\hat{\mu}_l^n$ are continuous $\mathcal{M}^*(S_{k/2},\mathbb{C}^{s\times s})$-valued functions of $t$, $(\alpha_1,\ldots,\alpha_{l-1})$ and $(\alpha_{l+1},\ldots,\alpha_n)$. If $l=1$ then the sum is absent in the expression for $\mu_l^n$ and if $l=n$

there is no sum in the expression for $\hat{\mu}_l^n$. Without these sums present it is easily seen that these measures satisfy the relations

$$\mu_1^n(t) = \mu_1^1(t) = \hat{\mu}_n^n(t) = \hat{\mu}_1^1(t). \tag{3.22}$$

Next we take linear combinations with $\mu$, $\mu_o$, $\mu_l^n$ and $\hat{\mu}_l^n$. For $l = 1, \ldots, n$ we can form the following measures in $\mathcal{M}(S_k, \mathbb{C}^{s\times s})$;

$$\lambda_l^n(t) = \mu(t,m) - \frac{\hbar}{m}[\mu_o(t,\alpha_o) + \mu_l^n(t,\alpha_1,\cdots,\alpha_{l-1})]; \tag{3.23}$$

$$\sigma_l^n(t) = \mu(t,m) - \frac{\hbar}{m}\mu_l^n(t,\alpha_1,\cdots,\alpha_{l-1}); \tag{3.24}$$

$$\hat{\lambda}_l^n(t) = \mu(t,m) - \frac{\hbar}{m}[\mu_o(t,\alpha) - \hat{\mu}_l^n(t,\alpha_{l+1},\cdots,\alpha_n)]. \tag{3.25}$$

The measure $\sigma_l^n(t)$ depends on the parameters $\alpha_1, \ldots, \alpha_{l-1}$, $\hbar$ and $m$. The measure $\lambda_l^n(t)$ also depends on these parameters, as well as it has an $\alpha_o$ dependence. Finally we note the measure $\hat{\lambda}_l^n(t)$ has a dependance upon the parameters $\alpha$, $\alpha_{l+1}, \ldots, \alpha_n$, $\hbar$ and $m$. Each of these measures are continuous with respect to these parameters in the $\mathcal{M}(S_k, \mathbb{C}^{s\times s})$ topology.

It is notationally advantageous to incorporate the variable of integration into the measure symbols. For example with the measure $\lambda_l^n(t)$ the integration variable is most often $\alpha_l$. If $h$ is any integrable function on $\mathbb{R}^d$ we now write

$$\int h\, d\lambda_l^n(t) \qquad \text{as} \qquad \int h(\alpha_l)\, d\lambda_l^n(t;\alpha_l).$$

The family of measures $\lambda_l^n(t)$ and $\hat{\lambda}_l^n(t)$ have simple norm bounds. Since it is assumed that $\alpha_j \in S_k$, $j = 1 \sim n$, we arrive at

$$\|\lambda_l^n(t)\| \le \|\mu(t)\| + \frac{\hbar}{|m|}(|\alpha_o| + nk)\|\gamma(t)\|, \tag{3.26}$$

$$\|\sigma_l^n(t)\| \le \|\mu(t)\| + \frac{\hbar}{|m|}nk\|\gamma(t)\|, \tag{3.27}$$

and

$$\|\hat{\lambda}_l^n(t)\| \leq \|\mu(t)\| + \frac{\hbar}{|m|}(|\alpha| + nk)\|\gamma(t)\|. \tag{3.28}$$

Obviously these bounds are uniform with respect to the parameters $\boldsymbol{\alpha}_n$ and the index $l$.

# CHAPTER 4

## The Dyson Series

In this chapter we examine the convergence properties of the Dyson series using a certain class of initial data functions and establish that the series constructs a solution to the Schrödinger equation. We assume $a \in \mathcal{V}_v(k)$ and $v \in \mathcal{V}(k)$ throughout this chapter.

We first define a few of the notational conventions to be used throughout. The Schwartz space of $C^\infty$, $s$-dimensional functions of rapid decrease we denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^d; \mathbb{C}^s)$. The Fourier transform convention we shall utilize is

$$\hat{h}(\alpha) = \frac{1}{(2\pi)^{d/2}} \int e^{-i\alpha\cdot x} h(x)\, dx, \qquad h \in \mathcal{S},$$

For each integer $n \geq 1$ let $\mathbf{t}_n = (t_1, \ldots, t_n)$ and for each $(t, t_o) \in T_\Delta$ define the set $\Delta_n(t, t_o) = \{\mathbf{t}_n : t_o \leq t_1 \leq \cdots \leq t_n \leq t\}$. Similarly, it is also convenient to define the set $\Delta_n(T) = \{(t, t_o, \mathbf{t}_n) : 0 \leq t_o \leq t_1 \leq \cdots \leq t \leq T\}$. We denote the $n^{\text{th}}$ order iterated time integral by

$$\int_{t_o}^{t>} d\mathbf{t}_n = \int_{t_o}^{t} dt_n \int_{t_o}^{t_n} dt_{n-1} \cdots \int_{t_o}^{t_2} dt_1.$$

If $\rho$ is any multi-index, $(\rho_1, \ldots, \rho_d)$, we define the operator $Q^\rho$ by

$$(Q^\rho \psi)(x) = x_1^{\rho_1} x_2^{\rho_2} \cdots x_d^{\rho_d} \psi(x).$$

Let $P_j$ be the partial differential operator $(\hbar/i)\partial_{x_j}$, $j = 1 \sim d$, then similarly, the partial differential operator $P^\rho$ is defined by $P_1^{\rho_1} \cdots P_d^{\rho_d}$. The domains of $Q^\rho$ and $P^\rho$ are maximally defined and both include the Schwartz space as a subset.

### 4.1 The Dyson Series

Before we actually begin the rigorous study of the Dyson series we first examine how the Dyson series arises. The purpose of this discussion is to provide motivation for studying the individual terms that appear in the series, and not to be completely rigorous in our arguments. We know that the Schrödinger evolution operator $U(t, t_o; m)$ operating on an initial data function $\psi_o \in D_o$ gives a solution to the Schrödinger equation (2.10). We wish to know what sort of integral equation $U(t, t_o; m)\psi_o$ satisfies. For the moment assume the mass parameter $m$ is positive. Let $U_o(t - t_o; m)$ denote the free Schrödinger evolution operator associated with $H_o(m)$. Because $H_o(m)$ is independent of the time, $U_o$ only depends upon the time displacement $t - t_o$ and $U_o(t - t_o; m)$ is given by the exponentiation of $H_o(m)$,

$$U_o(t - t_o; m) = \exp\{-i(t - t_o)H_o(m)/\hbar\}.$$

That aside, if we integrate the equation

$$i\hbar\partial_{t_1}[U_o(t_o - t_1; m)U(t_1, t_o; m)\psi_o] = U_o(t_o - t_1; m)\mathbf{V}(t_1, m)U(t_1, t_o; m)\psi_o$$

over $t_1$, where

$$\mathbf{V}(t_1, m) = H(t_1, m) - H_o(m),$$

we get

$$U(t, t_o; m)\psi_o = U_o(t - t_o; m)\psi_o + \frac{1}{i\hbar}\int_{t_o}^{t} dt_1\, U_o(t - t_1; m)\mathbf{V}(t_1, m)U(t_1, t_o; m)\psi_o. \tag{4.1}$$

Iterate (4.1) and in the $n^{\text{th}}$ term make the change of variables $t_j \to t_{n+1-j}$ for each $j = 1, \cdots, n$ to obtain the so called Dyson series

$$\begin{aligned} U(t, t_o; m)\psi_o &= U_o(t - t_o; m)\psi_o \\ &+ \sum_{n=1}^{\infty} \frac{1}{(i\hbar)^n} \int_{t_o}^{t>} dt_n \; U_o(t - t_n; m)\mathbf{V}(t_n, m)U_o(t_n - t_{n-1}; m) \times \cdots \\ &\times \mathbf{V}(t_1, m)U_o(t_1 - t_o; m)\psi_o. \end{aligned} \tag{4.2}$$

In (4.2) each free evolution operator will have a non-negative time difference. Consequently these operators remain bounded when the mass parameter has a positive imaginary part.

### 4.2 Dyson Iterates

In order to be able to discuss the individual terms of the Dyson series we must first understand the range stability and continuity properties of the operators $\mathbf{V}(\tau_1, m)$ and $U_o(\tau_2; m)$. We will only need the continuity properties using the $L^2(\mathbb{R}^d; \mathbb{C}^s)$ topology, but in fact we can and will show these on $L^q(\mathbb{R}^d; \mathbb{C}^s)$ $(2 \le q \le \infty)$ without too much more effort.

**Lemma 4.1:** *The operator* $\mathbf{V}(\tau, m) = H(\tau, m) - H_o(m)$ *satisfies the following:*

*(1)* $\mathbf{V}(\tau, m) : \mathcal{S} \to \mathcal{S}$;

*(2) If* $\psi \in \mathcal{S}$ *and* $\hat{\psi}$ *its Fourier transform, then* $\mathbf{V}(\tau, m)\psi$ *is given pointwise by the formula*

$$[\mathbf{V}(\tau, m)\psi](x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha_o \int d\lambda_1^1(\tau; \alpha_1)\, \hat{\psi}(\alpha_o) e^{i(\alpha_o + \alpha_1)\cdot x}. \tag{4.3}$$

*Here* $\lambda_1^1(\tau)$ *is given by the formula (3.23);*

*(3)* $Q^\rho[\mathbf{V}(\tau, m)\psi]^\wedge$ *is continuous with respect to* $\tau$ *in the* $L^p(\mathbb{R}^d; \mathbb{C}^s)$ *topology, for all* $1 \le p \le \infty$, *and* $P^\rho \mathbf{V}(\tau, m)\psi$ *is continuous with respect to* $\tau$ *in the* $L^q(\mathbb{R}^d; \mathbb{C}^s)$ *topology, for all* $2 \le q \le \infty$.

*Proof:* The restriction of $\mathbf{V}(\tau, m)$ to $\mathcal{S}$ is a partial differential operator whose coefficients are bounded $C^\infty$ functions. This implies $\mathbf{V}(\tau, m) : \mathcal{S} \to \mathcal{S}$ and (1) is proven.

To calculate $[\mathbf{V}(\tau, m)\psi](x)$, we shall consider each term of

$$\begin{aligned}[\mathbf{V}(\tau, m)\psi](x) &= \frac{i\hbar}{m} a(x,\tau) \cdot (\nabla\psi)(x) + \frac{i\hbar}{2m}(\nabla \cdot a)(x,\tau)\psi(x) \\ &\quad + \frac{1}{2m} a(x,\tau)^2 \psi(x) + v(x,\tau;m)\psi(x)\end{aligned}$$

separately.

From the inverse Fourier transform and the integral representation of $a$, the first term has the representation

$$\begin{aligned}\frac{i\hbar}{m} a(x,\tau) \cdot (\nabla\psi)(x) &= -\frac{1}{(2\pi)^{d/2}} \frac{\hbar}{m} a(x,\tau) \cdot \int d\alpha_o\, \alpha_o \hat{\psi}(\alpha_o) e^{i\alpha_o \cdot x} \\ &= -\frac{1}{(2\pi)^{d/2}} \int d\alpha_o \int d|\gamma(\tau)|\, \frac{\hbar}{m} \alpha_o \cdot \eta(\alpha_1,\tau) \hat{\psi}(\alpha_o) e^{i(\alpha_o + \alpha_1)\cdot x}.\end{aligned}$$

An application of Fubini's theorem has been used to interchange the order of the $d\alpha_o$ and the $d|\gamma(\tau)|$ integrals.

The second term can be calculated similarly if we also use the $x$-derivative (3.15), but with $v$ and $\nu$ in (3.15) replaced with $a$ and $\gamma$. We get

$$\begin{aligned}\frac{i\hbar}{2m}(\nabla \cdot a)(x,\tau)\psi(x) = -\frac{1}{(2\pi)^{d/2}} \int d\alpha_o \int d|\gamma(\tau)|\, &\frac{\hbar}{2m} \alpha_1 \cdot \eta(\alpha_1,\tau) \\ &\times \hat{\psi}(\alpha_o) e^{i(\alpha_o+\alpha_1)\cdot x}.\end{aligned}$$

The last two terms of $[\mathbf{V}(\tau, m)\psi](x)$ can be combined together using the Fourier measure in (3.20). The result is

$$[\frac{1}{2m} a(x,\tau)^2 + v(x,\tau;m)]\psi(x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha_o \int d\mu(\tau, m; \alpha_1)\, \hat{\psi}(\alpha_o) e^{i(\alpha_o+\alpha_1)\cdot x}.$$

Combining these three equations together yields (4.3).

To show (3) make the change of variables $\alpha_o \to \alpha = \alpha_o + \alpha_1$ in (4.3). With this change of variables the function $Q^\rho(\mathbf{V}(\tau,m)\psi)\hat{}$ has the pointwise value

$$\left[Q^\rho(\mathbf{V}(\tau,m)\psi)\hat{}\right](\alpha) = \alpha^\rho \int d\hat{\lambda}_1^1(\tau;\alpha_1)\,\hat{\psi}(\alpha-\alpha_1). \tag{4.4}$$

Take the difference between the left hand side of (4.4) for two different $\tau$'s. By writing out the explicit representations of the measures involved, it is easily shown this difference has the estimate

$$\begin{aligned}\Big|\left[Q^\rho(\mathbf{V}(\tau',m)\psi)\hat{}\right](\alpha) &- \left[Q^\rho(\mathbf{V}(\tau,m)\psi)\hat{}\right](\alpha)\Big| \\ &\leq \int d|\mu(\tau',m;\alpha_1) - \mu(\tau,m;\alpha_1)||\alpha|^{|\rho|}|\hat{\psi}(\alpha-\alpha_1)| \\ &\quad + \int d|\gamma(\tau';\alpha_1) - \gamma(\tau;\alpha_1)|\frac{\hbar}{|m|}|\alpha|^{|\rho|}|\hat{\psi}(\alpha-\alpha_1)|\left\{|\alpha| + \frac{k}{2}\right\}.\end{aligned}$$

The right hand side is a member of $L^1 \cap L^\infty(\mathbb{R}^d;\mathbb{C}^s)$ because $\hat{\psi} \in \mathcal{S}$ and any polynomial times a Schwartz space function is both uniformly bounded and absolutely integrable. Thus there exists a constant $C$, depending on $\psi$, such that

$$\left\|Q^\rho(\mathbf{V}(\tau',m)\psi)\hat{}-Q^\rho(\mathbf{V}(\tau,m)\psi)\hat{}\right\|_p \leq C\{\|\mu(\tau',m)-\mu(\tau,m)\|+\|\gamma(\tau',m)-\gamma(\tau,m)\|\}$$

for $p = 1$ or $\infty$. From the continuity of the measures, the right hand side tends to zero as $\tau' \to \tau$.

The $\tau$-continuity with respect to the norm $\|\cdot\|_p$ for all $1 \leq p \leq \infty$ now follows from the inequality

$$\|h\|_p \leq \|h\|_\infty^{1-1/p}\|h\|_1^{1/p}. \tag{4.5}$$

The $\tau$-continuity of $P^\rho\mathbf{V}(\tau,m)\psi$ with respect to the norm $\|\cdot\|_q$ follows from the $L^p$-continuity of $Q^\rho[\mathbf{V}(\tau,m)\psi]\hat{}$ and the Hausdorff-Young theorem of Fourier transforms ([RS 75], theorem IX.8). This theorem states that if $p$ and $q$ are conjugate indices (i.e. $p^{-1} + q^{-1} = 1$) and if $1 \le p \le 2$, then the Fourier transform is a bounded map of $L^p$ into $L^q$. ◇

Let $\psi \in \mathcal{S}$ and define the functions $\psi_n(t, t_o; \mathbf{t}_n)$ for $n = 0, 1, 2, \ldots$ recursively by the relation

$$\begin{aligned} \psi_o(t, t_o) &= U_o(t - t_o; m)\psi \\ \psi_n(t, t_o; \mathbf{t}_n) &= U_o(t - t_n; m)\mathbf{V}(t_n, m)\psi_{n-1}(t_n, t_o; \mathbf{t}_{n-1}). \end{aligned} \tag{4.6}$$

(For the case $n = 1$ we define $\psi_o(t_1, t_o; \mathbf{t}_o) \equiv \psi_o(t_1, t_o)$.) If we were to explicitly expand these recursive formulae out, it is easy to see that the $n^{\text{th}}$ function corresponds exactly to the integrand of the $n^{th}$ term in the Dyson series (4.2). A sufficient condition to ensure that that the Riemann integrals in the Dyson series exist is that the integrand be a strongly continuous function ([La 69], chapter X). The next lemma describes the continuity properties of the $\psi_n$'s.

**Lemma 4.2:** *Let $m \in \mathbb{C}_+$, $\psi \in \mathcal{S}$ and let $\hat{\psi}$ be its Fourier transform. Define the functions $\psi_n(t, t_o; \mathbf{t}_n)$ by (4.6). Then*

(1) *$\psi_n(t, t_o; \mathbf{t}_n) \in \mathcal{S}$ for $n = 0, 1, 2 \ldots$*

(2) *The $\psi_n$'s have pointwise values given by the following formulae*:*

$$\psi_o(t, t_o)(x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha_o\, \hat{\psi}(\alpha_o) e^{-\frac{i\hbar}{2m}(t-t_o)\alpha_o^2 + i\alpha_o \cdot x}\,; \tag{4.7}$$

---

* Due to the noncommutivity of the matrix structure in the measures $\lambda_j^n(t_j)$ we must write the multiple integral in (4.8) with the measures in the order shown. However each $\lambda_j^n(t_j)$ has a parametric dependence on the variables $\alpha_i$ $[i = 1 \sim (j-1)]$. The interpretation of this multiple integral is the following. Explicitly expand out the product of the sums occuring in the product measure $d\lambda_n^n(t_n; \alpha_n) \cdots d\lambda_1^n(t_1; \alpha_1)$. Then use the polarization property of the measures to factor out the matrix structure and incorporate it into the integrand. The result is a sum over scalar measures, which are various linear combinations of the product measures of $|\mu|$, $|\mu_o|$ and $|\mu_j^n|$, and an integrand that incorporates all the parametric dependence of the variables $\alpha_i$, as well as the matrix structure. Employing this latter notation this would be awkward and not particularly enlightening. Consequently we default to the notation used in (4.8).

$$\psi_n(t,t_o;\mathbf{t}_n)(x)$$

$$= \frac{1}{(2\pi)^{d/2}} \int d\alpha_o \int d\lambda_n^n(t_n;\alpha_n) \cdots \int d\lambda_1^n(t_1;\alpha_1)\, \hat{\psi}(\alpha_o) e^{i(\alpha_o+\cdots+\alpha_n)\cdot x}$$

$$\times\, e^{-\frac{i\hbar}{2m}[(t-t_n)(\alpha_n+\cdots+\alpha_o)^2+(t_n-t_{n-1})(\alpha_{n-1}+\cdots+\alpha_o)^2+\cdots+(t_1-t_o)\alpha_o^2]}$$

$$n \geq 1. \qquad (4.8)$$

(3) *$Q^\rho\hat{\psi}(t,t_o;\mathbf{t}_n)$ is jointly continuous in $(t,t_o,\mathbf{t}_n)$ with respect to the norm $\|\cdot\|_p$ for $1 \leq p \leq \infty$ and for all multi-indices $\rho$. Consequently $P^\rho\psi_n(t,t_o;\mathbf{t}_n)$ is jointly continuous in $(t,t_o,\mathbf{t}_n)$ in the $L^q(\mathbb{R}^d;\mathbb{C}^s)$ topology for all $2 \leq q \leq \infty$. In particular, $H_o(m)\psi(t,t_o;\mathbf{t}_n)$ is jointly continuous in $(t,t_o,\mathbf{t}_n)$ with respect to the norm $\|\cdot\|_q$.*

*Proof:* We recall the Fourier transform maps $\mathcal{S}$ into $\mathcal{S}$ and the free evolution operator is unitarily equivalent to multiplication by the function $\exp\{-(i\hbar/2m)\tau\alpha^2\}$. These together imply that $U_o(\tau;m): \mathcal{S} \to \mathcal{S}$. From Lemma 4.1 $\mathbf{V}(\tau_2,m)$ maps $\mathcal{S}$ into $\mathcal{S}$, so the composition $U_o(\tau_1;m)\mathbf{V}(\tau_2,m)$ also has this property. The first assertion results from induction.

Equation (4.7) follows from the aforementioned property of $U_o(\tau;m)$

$$[U_o(\tau;m)\varphi]\hat{\ }(\alpha) = e^{-\frac{i\hbar}{2m}\tau\alpha^2}\hat{\varphi}(\alpha). \qquad (4.9)$$

Equation (4.8) is the result of an inductive process on (4.7). Let $\varphi \in \mathcal{S}$. Using (4.7) and (4.4) we get the representation

$$[U_o(\tau_1;m)\mathbf{V}(\tau_2,m)\varphi](x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha\, [\mathbf{V}(\tau_2,m)\varphi]\hat{\ }(\alpha) e^{-\frac{i\hbar}{2m}\tau_1\alpha^2+i\alpha\cdot x}$$

$$= \frac{1}{(2\pi)^{d/2}} \int d\alpha \int d\hat{\lambda}_1^1(\tau_2;\alpha'')\, \hat{\varphi}(\alpha-\alpha'') e^{-\frac{i\hbar}{2m}\tau_1\alpha^2+i\alpha\cdot x}$$

$$= \frac{1}{(2\pi)^{d/2}} \int d\alpha' \int d\lambda_1^1(\tau_2;\alpha'')\, \hat{\varphi}(\alpha') e^{-\frac{i\hbar}{2m}\tau_1(\alpha'+\alpha'')^2+i(\alpha'+\alpha'')\cdot x}. \qquad (4.10)$$

We can now show that the $n = 1$ term in (4.8) is valid. In (4.10) set $\alpha' = \alpha_o$, $\alpha'' = \alpha_1$, $\tau_2 = t_1$, $\tau_1 = t - t_1$ and $\varphi = \psi_o(t_1, t_o)$. By definition the left hand side of (4.10) will equal $\psi_1(t, t_o; \mathbf{t}_1)$ and the right hand side of (4.10) yields the right hand side of (4.8) for $n = 1$. We invoke the inductive hypothesis and assume the result is true for $n - 1$. In (4.8) (with $n - 1$ in place of $n$) make the change of variables $\alpha_o \to \alpha = \alpha_o + \cdots + \alpha_{n-1}$. Then

$$\begin{aligned}
&\hat{\psi}_{n-1}(t_n, t_o; \mathbf{t}_{n-1})(\alpha) \\
&\quad = \int d\hat{\lambda}^{n-1}_{n-1}(t_{n-1}; \alpha_{n-1}) \cdots \int d\hat{\lambda}^{n-1}_1(t_1; \alpha_1)\, \hat{\psi}(\alpha - \alpha_1 - \cdots - \alpha_{n-1}) \\
&\qquad \times e^{-\frac{i\hbar}{2m}[(t_n - t_{n-1})\alpha^2 + \cdots + (t_1 - t_o)(\alpha - \alpha_1 - \cdots - \alpha_{n-1})^2]}.
\end{aligned} \tag{4.11}$$

In (4.10) set $\alpha' = \alpha$, $\alpha'' = \alpha_n$, $\tau_1 = t - t_n$, $\tau_2 = t_n$ and $\varphi = \psi_{n-1}(t_n, t_o; \mathbf{t}_{n-1})$. Then by definition, the left hand side of (4.10) is $\psi_n(t, t_o; \mathbf{t}_n)$ and (4.10) becomes the equality

$$\begin{aligned}
\psi_n(t, t_o; \mathbf{t}_n)(x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha \int d\lambda^1_1(t_n; \alpha_n)\, \hat{\psi}_{n-1}(t_n, t_o; \mathbf{t}_{n-1})(\alpha) \\
\times e^{-\frac{i\hbar}{2m}(t - t_n)(\alpha + \alpha_n)^2 + i(\alpha + \alpha_n)\cdot x}.
\end{aligned} \tag{4.12}$$

We next make the substitution of (4.11) into (4.12) and then do the change of variables $\alpha \to \alpha_o = \alpha - \alpha_1 - \cdots - \alpha_{n-1}$. It is straightforward, though tedious, to keep track of the parameters in the measures and show, after all the variable changes given above are made, that the product measure written in (4.8) is the result. This completes the proof for (2).

Finally we prove the statements in (3). The second statement about the continuity of $P^\rho \psi_n(t, t_o; \mathbf{t}_n)$ will follow from the first statement and an application of the Hausdorff-Young theorem for Fourier transforms ([RS 75], theorem IX.8). The proof of the first statement is inductive. We must first demonstrate that $Q^\rho \hat{\psi}_o(t, t_o)$ is jointly continuous in $(t, t_o)$ in the $L^p$-topology for $1 \le p \le \infty$. Because of inequality (4.5) we need only prove the continuity for $p = 1$ and $p = \infty$. $Q^\rho\hat{\psi}_o(t, t_o)$ has the pointwise representation

$$\left(Q^\rho\hat{\psi}_o(t, t_o)\right)(\alpha) = \alpha^\rho e^{-\frac{i\hbar}{2m}(t-t_o)\alpha^2}\hat{\psi}(\alpha).$$

To estimate the difference of this function for different times $(t, t_o)$ and $(t', t_o')$, we shall need the bound

$$|e^{-z} - e^{-z'}| \leq |z - z'| \qquad \mathrm{Re}\, z, \mathrm{Re}\, z' \geq 0.$$

It is then easy to see the pointwise estimate

$$\left|\left(Q^\rho\hat{\psi}_o(t', t_o')\right)(\alpha) - \left(Q^\rho\hat{\psi}_o(t, t_o)\right)(\alpha)\right| \leq \frac{\hbar}{2|m|}|\alpha|^{|\rho|+2}|\hat{\psi}(\alpha)|\left(|t' - t| + |t_o' - t_o|\right).$$

Because $\hat{\psi} \in \mathcal{S}$, the function $|\alpha|^{|\rho|+2}|\hat{\psi}(\alpha)| \in L^1 \cap L^\infty(\mathbb{R}^d)$. This implies the existence of a finite constant $C$ such that

$$\|Q^\rho\hat{\psi}_o(t', t_o') - Q^\rho\hat{\psi}_o(t, t_o)\|_p \leq C(|t' - t| + |t_o' - t_o|), \qquad p = 1, \infty,$$

which in turn implies that $Q^\rho\hat{\psi}_o(t, t_o)$ is jointly continuous in its time arguments in the $L^p(\mathbb{R}^d; \mathbb{C}^s)$ norm $(1 \leq p \leq \infty)$ (cf. equation (4.5)).

We induct this result to the $(n-1)^{th}$ term and then use this to show the $n^{th}$ term is continuous. The $n^{th}$ term's Fourier transform has the pointwise value

$$\left(Q^\rho\hat{\psi}_n(t, t_o; \mathbf{t}_n)\right)(\alpha) = \alpha^\rho e^{-\frac{i\hbar}{2m}(t-t_n)\alpha^2}[\mathbf{V}(t_n, m)\psi_{n-1}(t_n, t_o; \mathbf{t}_{n-1})]\hat{}\,(\alpha).$$

Take the difference between this function at two different time arguments $(t, t_o, \mathbf{t}_n)$ and $(t', t_o', \mathbf{t}_n')$. This difference can be estimated by adding and subtracting an intermediary term with the result

$$\begin{aligned}
&\left|[Q^\rho\hat{\psi}_n(t',t'_o;\mathbf{t}'_n)](\alpha) - [Q^\rho\hat{\psi}_n(t,t_o;\mathbf{t}_n)](\alpha)\right| \\
&\quad\le |\alpha|^{|\rho|}\left|[\mathbf{V}(t'_n,m)\psi_{n-1}(t'_n,t'_o;\mathbf{t}'_{n-1})]\hat{\ }(\alpha) - [\mathbf{V}(t_n,m)\psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})]\hat{\ }(\alpha)\right| \\
&\qquad + |\alpha|^{|\rho|}|[\mathbf{V}(t_n,m)\psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})]\hat{\ }(\alpha)|\left|e^{-\frac{i\hbar}{2m}(t'-t'_n)\alpha^2} - e^{-\frac{i\hbar}{2m}(t-t_n)\alpha^2}\right| \\
&\quad\equiv \Psi_1(\alpha) + \Psi_2(\alpha).
\end{aligned}$$

To control $\Psi_2$ we again use the bound $|e^{-z'} - e^{-z}| \le |z'-z|$. The second term has estimate

$$\Psi_2(\alpha) \le \frac{\hbar}{2|m|}|\alpha|^{|\rho|+2}\left|\Big[\mathbf{V}(t_n,m)\psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})\Big]\hat{\ }(\alpha)\right|\Big(|t'-t| + |t'_n - t_n|\Big).$$

Because $[\mathbf{V}(t_n,m)\psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})]\hat{\ } \in \mathcal{S}$, any polynomial in $\alpha$ times it will yield a function in $L^1 \cap L^\infty(\mathbb{R}^d;\mathbb{C}^s)$. It is then easliy verified that $\|\Psi_2\|_p \to 0$ in the limit $(t',t'_o,\mathbf{t}'_n) \to (t,t_o,\mathbf{t}_n)$ for all $1 \le p \le \infty$.

For the first term $\Psi_1$ we again add and subtract an intermediary term;

$$\begin{aligned}
\Psi_1(\alpha) &\le |\alpha|^{|\rho|}\left|\Big[\Big(\mathbf{V}(t'_n,m) - \mathbf{V}(t_n,m)\Big)\psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})\Big]\hat{\ }(\alpha)\right| \\
&\qquad + |\alpha|^{|\rho|}\left|\Big[\mathbf{V}(t'_n,m)\Big(\psi_{n-1}(t'_n,t'_o;\mathbf{t}'_{n-1}) - \psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})\Big)\Big]\hat{\ }(\alpha)\right| \\
&\equiv \Psi_3(\alpha) + \Psi_4(\alpha).
\end{aligned}$$

For all $1 \le p \le \infty$, $\|\Psi_3\|_p \to 0$ as $(t',t'_o,\mathbf{t}'_n) \to (t,t_o,\mathbf{t}_n)$ because lemma 4.1 showed the function $Q^\rho[\mathbf{V}(t_n,m)\varphi]\hat{\ }$ is continuous in $t_n$ with respect to these norms for all $\varphi \in \mathcal{S}$.

Finally we consider the last term $\Psi_4$. Equation (4.4) provides an explicit representation for $\Psi_4$;

$$\Psi_4(\alpha) = |\alpha|^{|\rho|} \left| \int \hat{\lambda}_1^1(t'_n;\alpha')\,[\hat{\psi}_{n-1}(t'_n,t'_o;\mathbf{t}'_{n-1})(\alpha-\alpha') - \hat{\psi}_{n-1}(t_n,t_o;\mathbf{t}_{n-1})(\alpha-\alpha')] \right|. \tag{4.13}$$

Integrating $\Psi_4$ over $\alpha$ and making the change of variables $\alpha \to \alpha_o = \alpha - \alpha'$, we get the $L^1(\mathbb{R}^d)$ estimate of $\Psi_4$ of

$$\begin{aligned}
\|\Psi_4\|_1 &\le \int d\alpha_o\, \|\lambda_1^1(t'_n)\| (|\alpha_o| + k)^{|\rho|} \left| \hat{\psi}_{n-1}(t'_n,t'_o;\mathbf{t}'_{n-1})(\alpha_o) - \hat{\psi}_{n-1}(t_n,t_o;\mathbf{t}_{n-1})(\alpha_o) \right| \\
&\le \int d\alpha_o \left\{ \|\mu(t'_n)\| + \frac{\hbar}{|m|}(|\alpha_o| + k)\|\gamma(t'_n)\| \right\} (|\alpha_o| + k)^{|\rho|} \\
&\qquad\qquad \times \left| \hat{\psi}_{n-1}(t'_n,t'_o;\mathbf{t}'_{n-1})(\alpha_o) - \hat{\psi}_{n-1}(t_n,t_o;\mathbf{t}_{n-1})(\alpha_o) \right|.
\end{aligned}$$

The second inequality follows from equation (3.26). Because $\|\mu(t'_n)\|$ and $\|\gamma(t'_n)\|$ are uniformly bounded in time and by assumption, $Q^\rho \psi_{n-1}(t_n,t_o;\mathbf{t}_{n-1})$ is jointly continuous in the norm $\|\cdot\|_1$, the above estimate shows that $\|\Psi_4\|_1 \to 0$ in the limit of $(t',t'_o,\mathbf{t}'_n) \to (t,t_o,\mathbf{t}_n)$. Similarly, starting from (4.13) it easily shown that $\|\Psi_4\|_\infty$ also tends to zero as the time arguments approach each other. This completes the proof to the lemma. $\diamondsuit$

The continuity properties of the functions $\psi_n(t,t_o;\mathbf{t}_n)$ guarantee the existence of the time ordered Riemann integrals in the spaces $L^q(\mathbb{R}^d;\mathbb{C}^s)$ for $2 \le q \le \infty$ ([La 69], chapter X). These integrals allow us to define a mapping that corresponds to the $n^{th}$ term in the Dyson series (4.2).

**Definition 4.1:** Define the $n^{th}$ *Dyson iterate operator*,

$$D_n(t,t_o;m) : \mathcal{S} \to L^q(\mathbb{R}^d;\mathbb{C}^s) \qquad 2 \le q \le \infty,$$

by the equations

$$D_n(t,t_o;m)\psi = \begin{cases} \psi_o(t,t_o), & \text{if } n=0; \\ \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_n\, \psi_n(t,t_o;\mathbf{t}_n), & \text{if } n \geq 1. \end{cases} \qquad \psi \in \mathcal{S} \tag{4.14}$$

Here the $\psi_n(t,t_o;\mathbf{t}_n)$ are defined in (4.6) and the integral is the Riemann integral for continuous functions in the $L^q(\mathbb{R}^d;\mathbb{C}^s)$ topology. $\diamondsuit$

It is important to be able to relate the pointwise value of $D_n(t,t_o;m)\psi$ to the time ordered integral of the pointwise value of $\psi_n(t,t_o;\mathbf{t}_n)$. The following lemma establishes this relation.

**Lemma 4.3** *Let* $\varphi : \Delta_n(t,t_o) \to \mathcal{S}(\mathbb{R}^d;\mathbb{C}^s)$ *be a continuous mapping in the* $L^q(\mathbb{R}^d;\mathbb{C}^s)$ *topology for* $2 \leq q \leq \infty$. *Define the vector*

$$\Phi = \int_{t_o}^{t>} d\mathbf{t}_n\, \varphi(\mathbf{t}_n) \in L^q(\mathbb{R}^d;\mathbb{C}^s). \tag{4.15}$$

*Then*

$$\Phi(x) = \int_{t_o}^{t>} d\mathbf{t}_n\, \varphi(\mathbf{t}_n)(x), \qquad x \in \mathbb{R}^d. \tag{4.16}$$

*Proof:* The $L^\infty(\mathbb{R}^d;\mathbb{C}^s)$ continuity of $\varphi(\mathbf{t}_n)$ with respect to $\mathbf{t}_n$ implies the pointwise $\mathbf{t}_n$-continuity of $\varphi(\mathbf{t}_n)(x)$ for each $x \in \mathbb{R}^d$. Thus the ordinary ($\mathbb{C}^s$-valued) Riemann integral of $\varphi(\mathbf{t}_n)(x)$ over $\mathbf{t}_n$ exists. An application of the dominated convergence theorem shows that the integral $\int_{t_o}^{t>} d\mathbf{t}_n\, \varphi(\mathbf{t}_n)(x)$ is a continuous function of $x$.

Recall what (4.15) means ([La 69], chapter X). Let $\left\{\pi_l = \{\Lambda_{\mathbf{j}}, \tau_{\mathbf{j}}\}\right\}$ be a sequence of partitions of $\Delta_n(t,t_o)$ such that $|\pi_l| \to 0$ as $l \to \infty$. Here $\mathbf{j}$ is a member of an appropiate index set $\mathcal{I}_l$; $\Lambda_{\mathbf{j}} \in \Delta_n(t, t_o)$ is a set of the form

$$\Lambda_{\mathbf{j}} = (\lambda_{j_1}, \lambda_{j_1+1}] \times \cdots \times (\eta_{j_n}, \eta_{j_n+1}],$$

where $\{\lambda_k\}, \ldots, \{\eta_k\}$ each form a partition of $[t_o, t]$; $\boldsymbol{\tau}_{\mathbf{j}} \in \Lambda_{\mathbf{j}}$ and we set $\varphi(\boldsymbol{\tau}_{\mathbf{j}}) = 0$ if $\boldsymbol{\tau}_{\mathbf{j}} \notin \Delta_n(t, t_o)$; $|\Lambda_{\mathbf{j}}|$ is the volume of $\Lambda_{\mathbf{j}}$ and $|\pi_l| = \max |\Lambda_{\mathbf{j}}|$. The integral in (4.15) means that

$$\Big\| \sum_{\mathbf{j} \in \mathcal{I}_l} \varphi(\boldsymbol{\tau}_{\mathbf{j}}) |\Lambda_{\mathbf{j}}| - \Phi \Big\|_q \longrightarrow 0, \qquad \text{as } l \longrightarrow \infty,$$

provided the limit is independent of whatever the sequence of partions $\{\pi_l\}$ is taken. Because this sum converges in norm to $\Phi$, it must converge in measure to $\Phi$ (cf. [Roy 68], §4.5). Hence we can apply Royden's proposition 4.17 [Roy 68] which shows for almost all $x$, that there exists a subsequence of partitions, $\{\pi_{l_k}\}$, such that

$$\Big| \sum_{\mathbf{j} \in \mathcal{I}_{l_k}} \varphi(\boldsymbol{\tau}_{\mathbf{j}})(x) |\Lambda_{\mathbf{j}}| - \Phi(x) \Big| \longrightarrow 0, \qquad \text{as } l_k \longrightarrow \infty.$$

But the left hand side is a Riemann sum that has the limiting value $\int_{t_o}^{t>} d\mathbf{t}_n\, \varphi(\mathbf{t}_n)(x)$. Thus (4.16) is established for almost all $x$. As the right hand side is continuous in $x$, it is used as a definition of the left hand side of (4.16) for all of $x$. ◇

In the next lemma we establish some pointwise representations of the $n^{th}$ Dyson iterate, continuity properties and its range.

**Lemma 4.4:** *Let $(t, t_o) \in T_\Delta$, $m \in \mathbb{C}_+$ and let $\rho$ be any multi-index.*

(1) *$D_n(t, t_o; m) : \mathcal{S} \to \mathcal{S}$.*

(2) *If $\psi \in \mathcal{S}$ and $n \geq 1$, then $P^\rho D_n(t, t_o; m)\psi$ has the pointwise values*

$$[P^\rho D_n(t, t_o; m)\psi](x) = \frac{1}{(i\hbar)^n} \int_{t_o}^{t>} d\mathbf{t}_n\, [P^\rho \psi_n(t, t_o; \mathbf{t}_n)](x), \tag{4.17}$$

*and similarly, its Fourier transform satisfies*

$$[P^{\rho} D_n(t, t_o; m)\psi]^{\wedge}(\alpha) = \frac{\hbar^{|\rho|}\alpha^{\rho}}{(i\hbar)^n} \int_{t_o}^{t>} d\mathbf{t}_n \, \hat{\psi}_n(t, t_o; \mathbf{t}_n)(\alpha). \tag{4.18}$$

*(3) $Q^{\rho}[D_n(t, t_o; m)\psi]^{\wedge}$ is continuous in $(t, t_o) \in T_{\Delta}$ with respect to the norm $\|\cdot\|_p$ for $1 \leq p \leq \infty$. Consequently it also follows that $P^{\rho} D_n(t, t_o; m)\psi$ is continuous in $(t, t_o) \in T_{\Delta}$ with respect to the norm $\|\cdot\|_q$ for all $2 \leq q \leq \infty$.*

*Proof:* To demonstrate (1), we have to show that for each pair of multi-indices, $\rho$ and $\rho'$, there exists a constant $C_{\rho\rho'}$ such that for all $x \in \mathbb{R}^d$,

$$\left|[Q^{\rho'} P^{\rho} D_n(t, t_o; m)\psi](x)\right| \leq C_{\rho\rho'}. \tag{4.19}$$

First we note that from lemma 4.3, setting $\varphi = \psi_n(t, t_o; \mathbf{t}_n)$, we immediately have the pointwise representation

$$[D_n(t, t_o; m)\psi](x) = \frac{1}{(i\hbar)^n} \int_{t_o}^{t>} d\mathbf{t}_n \, \psi_n(t, t_o; \mathbf{t}_n)(x).$$

We know from lemma 4.2(3) that $[P^{\rho}\psi_n(t, t_o; \mathbf{t}_n)](x)$ is continuous in $(t, t_o, \mathbf{t}_n)$, uniformly with respect to $x$. It therefore possible to interchange the derivatives $P^{\rho}$ with the integral $\int_{t_o}^{t>} d\mathbf{t}_n$ (cf. reference [Ru 76], theorem 9.42). This establishes equation (4.17). The quantity of interest has the pointwise representation

$$[Q^{\rho'} P^{\rho} D_n(t, t_o; m)\psi](x) = \frac{1}{(i\hbar)^n} \int_{t_o}^{t>} d\mathbf{t}_n \, x^{\rho'} [P^{\rho}\psi_n(t, t_o; \mathbf{t}_n)](x).$$

To show the estimate (4.19), it is enough to prove that $x^{\rho'}[P^{\rho}\psi_n(t, t_o; \mathbf{t}_n)](x)$ is uniformly bounded in $x$ and $\mathbf{t}_n$. To show this, it is sufficient to show the Fourier transform of the integrand has an $L^1$-bound independent of $\mathbf{t}_n$. The value of the

Fourier transform can be picked out of equation (4.8) (also see equation (4.11)). The result is

$$[Q^{\rho'} P^{\rho} \psi_n(t, t_o; \mathbf{t}_n)]^{\hat{}}(\alpha)$$

$$= \hbar^{|\rho|}(-i)^{\rho'} \left(\frac{\partial}{\partial \alpha}\right)^{\rho'} \alpha^{\rho} \int d\hat{\lambda}_n^n(t_n; \alpha_n) \cdots \int \hat{\lambda}_1^n(t_1; \alpha_1)\hat{\psi}(\alpha_o)$$

$$\times\, e^{-\frac{i\hbar}{2m}[(t-t_n)\alpha^2 + (t_n - t_{n-1})(\alpha-\alpha_n)^2 + \cdots + (t_1 - t_o)\alpha_o^2]},$$

where $\alpha_o$ is $\alpha_o = \alpha - \alpha_1 - \cdots - \alpha_n$.

An application of the dominated convergence theorem shows that the derivative with respect to the $\alpha_j$'s can be brought inside the integral, with due respect being paid to the parametric dependence of the $d\hat{\lambda}$'s on $\alpha$. The $d\hat{\lambda}$'s contain a polynomial structure in $\alpha$ $(\alpha_o)$ and those derivatives that strike the exponential will bring down further polynomials in the $\alpha$'s and the $t$'s. Those derivatives striking $\hat{\psi}(\alpha_o)$ will still yield a Schwartz space function. The compact support of the measures will control the polynomial growth in the $\alpha_j$'s; the compactness of the set $T_\Delta$ will control the polynomial growth in the $t_j$'s; the exponential is uniformly bounded by 1; the norm of the measures are continuous and bounded functions of the $t_j$'s; and $\hat{\psi}$ and its derivatives will control the polynomial grow of $\alpha$ $(\alpha_o)$ in the $d\alpha$ integral. All of these facts show that the Fourier transform of $Q^{\rho'} P^{\rho} \psi_n(t, t_o; \mathbf{t}_n)$ is an $L^1$ function whose norm is uniformly bounded in $\mathbf{t}_n$. This completes the proof for part (1).

We have already indicated how (4.17) comes about. Identity (4.18) is a consequence of using Fubini's theorem to interchange the time ordered integral with the integral appearing in the definition of the Fourier transform.

As before, the second assertion in (3) follows from the first and hence we shall only provide the proof for the first one. Let $(t', t_o') \in T_\Delta$ differ from $(t, t_o) \in T_\Delta$.

The continuity of $Q^\rho[D_n(t,t_o;m)\psi]\hat{}$ can be shown with the addition and subtraction of an intermediary term which gives the estimate

$$\begin{aligned}
&\|Q^\rho[D_n(t',t_o';m)\psi]\hat{}- Q^\rho[D_n(t,t_o;m)\psi]\hat{}\|_p \\
&\qquad \le \frac{1}{\hbar^n}\left\| \left\{ \int_{t_o'}^{t'>} d\mathbf{t}_n - \int_{t_o}^{t>} d\mathbf{t}_n \right\} Q^\rho \hat{\psi}_n(t',t_o';\mathbf{t}_n) \right\|_p \\
&\qquad + \frac{1}{\hbar^n}\int_{t_o}^{t>} d\mathbf{t}_n \left\| Q^\rho[\hat{\psi}_n(t',t_o';\mathbf{t}_n) - \hat{\psi}_n(t,t_o;\mathbf{t}_n)] \right\|_p .
\end{aligned}$$

The second term tends to zero in the limit $(t',t_o') \to (t,t_o)$ by lemma 4.2(3). For the first term the integrand is uniformly bounded in $\mathbf{t}_n$. Thus the first term is bounded by a constant times $|t'-t|^n + |t_o'-t_o|^n$. ◊

### 4.3 A Solution of the Schrödinger Equation

We are now ready to consider the summability of the Dyson series and show that it sums to a solution of the Schrödinger-Cauchy problem (2.10). We note an immediate consequence of Lemma 4.4 is that the $n^{th}$ Dyson iterate, $D_n(t,t_o;m)\psi$, is a member of the set $D_o$. Thus $H_o(m)D_n(t,t_o;m)\psi$ makes sense, as does the full Hamiltonian acting on $D_n(t,t_o;m)\psi$.

We begin by examining the sum of the vectors $P^\rho D_n(t,t_o;m)\psi$, but with a slightly more restricted class of test vectors $\psi$. In particular the values of $|\rho| = 0, 1$ and 2 will be of importance for establishing the Dyson series is summable and that it generates a solution to the Schrödinger equation.

**Lemma 4.5:** *Assume $\hat{\psi} \in C_o^\infty(\mathbb{R}^d, \mathbb{C}^s)$ and let the support of $\hat{\psi}$ lie in a ball $B_{bk}$ which is centered on the origin and has radius $bk$. Here, $k$ is the radius of the ball which contains the support of the measures $\gamma(t)$ and $\nu(t,m)$ and $0 < b < \infty$ is a scaling constant. Let $\rho$ be any multi-index, $m \in \mathbb{C}_+$ and $(t,t_o) \in T_\Delta$. Then there* *exists a constant $C$, depending only on $\rho$, $\psi$ and the index $q$ of the Banach space $L^q(\mathbb{R}^d;\mathbb{C}^s)$ $(2 \le q \le \infty)$, such that*

$$\left\| P^\rho D_n(t,t_o;m)\psi \right\|_q \le C\frac{1}{n!}\left(\frac{t-t_o}{\hbar}\right)^n (b+n)^{|\rho|}[\mu_T + \frac{\hbar}{|m|}(b+n)k\,\gamma_T]^n. \qquad (4.20)$$

*Furthermore if*

$$t - t_o < \frac{|m|}{ek\gamma_T}, \qquad (e = 2.718\cdots) \qquad (4.21)$$

*then the sequence $\sum_{n=0}^N P^\rho D_n(t,t_o;m)\psi$ converges with respect to the norm $\|\cdot\|_q$, for all $2 \le q \le \infty$, as $N \to \infty$.*

*Proof:* In the following, let $q$ and $p$, $1 \le p \le 2$, be conjugate indices for the Banach spaces $L^q(\mathbb{R}^d;\mathbb{C}^s)$ and $L^p(\mathbb{R}^d;\mathbb{C}^s)$ (i.e. $p^{-1}+q^{-1}=1$). For $n=0$, there is no integral involved with the definition of $D_o(t,t_o;m)$ and hence it must be treated separately. An application of the Hausdorff-Young theorem ([RS 75], theorem IX.8) for Fourier transforms yields

$$\begin{aligned}\left\| P^\rho D_o(t,t_o;m)\psi \right\|_q &\le (2\pi)^{d(\frac{1}{2}-\frac{1}{q})}\hbar^{|\rho|}\left\| e^{-\frac{i\hbar}{2m}(t-t_o)|Q|^2} Q^\rho\hat{\psi}\right\|_p \\ &\le (2\pi)^{d(\frac{1}{2}-\frac{1}{q})}\hbar^{|\rho|}(bk)^{|\rho|}\|\hat{\psi}\|_p.\end{aligned}$$

For $n \ge 1$ we will again apply the Hausdorff-Young theorem.

$$\begin{aligned}\left\| P^\rho D_n(t,t_o;m)\psi \right\|_q &\le \hbar^{-n}\int_{t_o}^{t>} d\mathrm{t}_n \left\| P^\rho \psi_n(t,t_o;\mathrm{t}_n)\right\|_q \\ &\le (2\pi)^{d(\frac{1}{2}-\frac{1}{q})}\hbar^{|\rho|-n}\int_{t_o}^{t>} d\mathrm{t}_n \left\| Q^\rho \hat{\psi}_n(t,t_o;\mathrm{t}_n)\right\|_p \qquad (4.22)\end{aligned}$$

The integrand in the second inequality has the explicit formula (cf. equation (4.11))

$$\left\|Q^\rho \hat{\psi}_n(t,t_o;\mathbf{t}_n)\right\|_p^p = \int d\alpha \left| \int d\hat{\lambda}_n^n(t_n;\alpha_n)\cdots d\hat{\lambda}_1^n(t_1;\alpha_1)\,\alpha^\rho \hat{\psi}(\alpha_o) \right.$$

$$\left. \times\, e^{-\frac{i\hbar}{2m}[(t-t_n)\alpha^2+\cdots+(t_1-t_o)\alpha_o^2]} \right|^p ,$$

where once again $\alpha_o = \alpha - \alpha_1 - \cdots - \alpha_n$. In the outer most integral, make the change of variables $\alpha \to \alpha_o$. Working these variable changes into the parametric dependence of the measures we find that $\hat{\lambda}_j^n(t_j) \to \lambda_j^n(t_j)$. Next we bring the absolute value through the integrals to majorize the $d\alpha$'s integrand. The compactness of the supports of the $\lambda$'s is used to bound the polynomial growth of the $\alpha_l$'s in the resulting integral. Finally we use the norm estimates (3.26), (3.14) and (3.19), plus the compactness of the support of $\hat{\psi}$ to get the bound

$$\left\|Q^\rho \hat{\psi}_n(t,t_o;\mathbf{t}_n)\right\|_p \le \|\hat{\psi}\|_p [k(b+n)]^{|\rho|}\{\mu_T + \frac{\hbar}{|m|}(b+n)k\gamma_T\}^n.$$

Substituting this estimate into (4.22) leads to (4.20), with $C$ given by

$$C = (2\pi)^{d(\frac{1}{2}-\frac{1}{q})}(\hbar k)^{|\rho|}\|\hat{\psi}\|_p.$$

Equation (4.20) can be rewritten for $n \ge 1$ as follows,

$$\left\|P^\rho D_n(t,t_o;m)\psi\right\|_q \le C\left(1+\frac{b}{n}\right)^{|\rho|}\frac{n^{n+|\rho|}}{n!}\left[\frac{(t-t_o)k\gamma_T}{|m|}\right]^n \{1+[b+\frac{|m|}{\hbar k\gamma_T}\mu_T]\frac{1}{n}\}^n$$

$$\equiv B_n. \qquad (4.23)$$

Consider the series $\sum_{n=1}^N B_n$. An application of the ratio test shows

$$\frac{B_{n+1}}{B_n} \to (t-t_o)\frac{ek\gamma_T}{|m|} \qquad \text{as } n\to\infty, \qquad (e = 2.718\cdots).$$

Consequently the series converges if $t - t_o$ satisfies inequality (4.21). To show that the series $\sum_{n=0}^{N} P^\rho D_n(t,t_o;m)\psi$ converges in the (complete) space $L^q(\mathbb{R}^d;\mathbb{C}^s)$, it is enough to show it is a Cauchy sequence;

$$\begin{aligned}\Big\|\sum_{N_1}^{N_2} P^\rho D_n(t,t_o;m)\psi\Big\|_q &\le \sum_{N_1}^{N_2}\Big\|P^\rho D_n(t,t_o;m)\psi\Big\|_q \\ &\le \sum_{N_1}^{N_2} B_n \to 0 \qquad \text{as } N_1, N_2 \to \infty. \qquad \diamondsuit\end{aligned}$$

This establishes the convergence of the term-by-term (spatial) differentiation of the Dyson series. We will see shortly that the order of the partial differential operator and the summation may be interchanged. This will be necessary in showing that the Dyson series gives a solution to the Schrödinger equation. The other aspect of the Schrödinger equation is the time derivative, and we next establish the differentiabilty of the $n^{th}$ Dyson iterate with respect to $t$ and the summability of that series.

Henceforth we shall only work in the $L^2$ topology as opposed to the more general $L^q$ Banach spaces for $2 \le q \le \infty$. To work in $L^q(\mathbb{R}^d;\mathbb{C}^s)$ would require knowledge of the closure properties of $H_o(m)$ and $\mathbf{V}(t,m)$ in $L^q$. We have only described these operators fully in $L^2(\mathbb{R}^d;\mathbb{C}^s)$. From a physical standpoint, $L^2$ will suffice because it is the only $L^q$ space that is also a Hilbert space.

**Lemma 4.6:** *Let $\psi \in \mathcal{S}$ and $m \in \mathbb{C}_+$.*

(1) *The mapping $\psi_n(\cdot,t_o;\mathbf{t}_n) : [t_o,T] \to \mathcal{S}$ is strongly continuously differentiable and satisfies the formula*

$$i\hbar\frac{\partial}{\partial t}\psi_n(t,t_o;\mathbf{t}_n) = H_o(m)\psi_n(t,t_o;\mathbf{t}_n) \qquad n = 0,1,2,\ldots \,. \tag{4.24}$$

*Moreover, the limiting process in taking the derivative is uniform with respect to $t$, $t_o$ and $\mathbf{t}_n$.*

(2) *The $n^{th}$ Dyson iterate $D_n(\cdot, t_o; m)\psi : [t_o, T] \to \mathcal{S}$, is strongly continuously differentiable. Its derivative we denote by $\dot{D}_n(t, t_o; m)\psi$, and it satisfies the recurrence relation*

$$i\hbar\dot{D}_n(t, t_o; m)\psi = H_o(m)D_n(t, t_o; m)\psi + \mathbf{V}(t, m)D_{n-1}(t, t_o; m)\psi, \tag{4.25}$$

*where $D_{n-1} = 0$ if $n = 0$. Here also, the limiting process in taking the derivative is uniform with respect to $(t, t_o) \in T_\Delta$.*

*Proof:* We note that if $t_n = t$, the derivative appearing in (4.24) is the right sided derivative. If $t_n = t$, let $\delta > 0$, otherwise pick $\delta$ such that $0 < |\delta| \leq t - t_n$. Again we treat the $n = 0$ case separately from the $n \leq 1$ case.

Recall that $\psi_o(t, t_o)$ is defined in (4.6). Let $\{\varphi_o(\delta)\}$ be a family of vectors in $L^2(\mathbb{R}^d; \mathbb{C}^s)$ defined by

$$\begin{aligned}\varphi_o(\delta) &= i\hbar\frac{1}{\delta}[\psi_o(t+\delta, t_o) - \psi_o(t, t_o)] - H_o(m)\psi_o(t, t_o)\\ &= i\hbar\frac{1}{\delta}[U_o(t+\delta-t_o; m) - U_o(t-t_o; m)]\psi - H_o(m)U_o(t-t_o; m)\psi.\end{aligned}$$

To prove our claim, it is enough to prove that $\varphi_o(\delta)$ tends to zero uniformly with respect to $(t, t_o, t_n)$. From the Plancherel theorem for Fourier transforms ([Ru 73], §7.9) we have the equality

$$\|\varphi_o(\delta)\| = \|\hat{\varphi}_o(\delta)\|.$$

Thus it is enough to prove $\|\hat{\varphi}_o(\delta)\| \to 0$ as $\delta \to 0$. Pointwise, $\hat{\varphi}_o(\delta)$ is given by the formula

$$\hat{\varphi}_o(\delta)(\alpha) = \left\{ i\hbar\frac{1}{\delta}[e^{-\frac{i\hbar}{2m}(t+\delta-t_o)\alpha^2} - e^{-\frac{i\hbar}{2m}(t-t_o)\alpha^2}] - \frac{\hbar^2}{2m}\alpha^2 e^{-\frac{i\hbar}{2m}(t-t_o)\alpha^2} \right\} \hat{\psi}(\alpha).$$

To estimate this pointwise formula we need the following bound. Let $\mathrm{Re}\, z$, $\mathrm{Re}\, z' \geq 0$. Then

$$\left| \frac{e^{-z'} - e^{-z}}{z' - z} + e^{-z} \right| \leq \frac{3}{2} |z' - z|. \tag{4.26}$$

Setting $z = (i\hbar/2m)(t - t_o)\alpha^2$ and $z' = (i\hbar/2m)(t + \delta - t_o)\alpha^2$ leads us to the inequality

$$|\hat{\varphi}_o(\delta)(\alpha)| \leq \frac{3}{2} \frac{\hbar^3}{(2|m|)^2} |\alpha|^4 \left| \hat{\psi}(\alpha) \right| |\delta|.$$

Consequently we see that

$$\|\hat{\varphi}_o(\delta)\| \leq \frac{3\hbar^3}{8|m|} \left\| |Q|^4 \hat{\psi} \right\| |\delta| \to 0 \qquad \text{as } \delta \to 0$$

where $|Q|^4 \equiv \left( \sum_{j=1}^d Q_j^2 \right)^2$. Since the right hand side of this last inequality is independent of $t$ and $t_o$, this shows the convergence is uniform in $t$ and $t_o$.

For the case $n \geq 1$, define the family of vectors $\{\varphi_n(\delta)\}$ by

$$\varphi_n(\delta) = i\hbar \frac{1}{\delta} [\psi_n(t + \delta, t_o; \mathbf{t}_n) - \psi_n(t, t_o; \mathbf{t}_n)] - H_o(m) \psi_n(t, t_o; \mathbf{t}_n).$$

As before, it is enough to prove that $\|\hat{\varphi}_n(\delta)\| \to 0$ as $\delta \to 0$, uniformly in $(t, t_o, \mathbf{t}_n)$. From (4.6) and (4.9), the pointwise representation of $\hat{\varphi}_n(\delta)$ is

$$\begin{aligned} \hat{\varphi}_n(\delta)(\alpha) = \Big\{ \frac{i\hbar}{\delta} [e^{-\frac{i\hbar}{2m}(t+\delta-t_n)\alpha^2} - e^{-\frac{i\hbar}{2m}(t-t_n)\alpha^2}] - \frac{\hbar^2}{2m} \alpha^2 e^{-\frac{i\hbar}{2m}(t-t_n)\alpha^2} \Big\} \\ \times [\mathbf{V}(t_n, m) \psi_{n-1}(t_n, t_o; \mathbf{t}_{n-1})]\hat{\ }(\alpha). \end{aligned}$$

Exactly as before we use bound (4.26) to obtain the desired estimate. If we also note that

$$\mathbf{V}(t_n, m) \psi_{n-1}(t_n, t_o; \mathbf{t}_{n-1}) = \psi_n(t_n, t_o; \mathbf{t}_n),$$

this estimate can be written as

$$\|\varphi_n(\delta)\| \leq \frac{3\hbar^3}{8|m|^2}\left\||Q|^4\hat{\psi}_n(t_n,t_o;\mathbf{t}_n)\right\||\delta|.$$

Now $|Q|^4 = \left(\sum_{j=1}^d Q_j^2\right)^2$, so by lemma 4.2 we see that $\left\||Q|^4\hat{\psi}_n(t_n,t_o;\mathbf{t}_n)\right\|$ is jointly continuous in $(t_n,t_o,\mathbf{t}_n)$ on the compact set $\Delta_n(T)$, and hence it will attain its maximum. Thus there exists a constant $C$, independent of $(t,t_o;\mathbf{t}_n)$, such that

$$\|\varphi_n(\delta)\| \leq C|\delta| \to 0 \qquad \text{as } \delta \to 0.$$

This finishes the proof of (1)

In order to ensure that all the vectors in the following exist, it is necessary to consider the left and right $t$-derivatives of $D_n(t,t_o;m)\psi$ separately. As both proofs are somewhat similar, only the right derivative proof shall be shown.

Let $\delta > 0$ and define the family of vectors

$$\begin{aligned}\Phi(\delta) &= \frac{1}{\delta}[D_n(t+\delta,t_o;m)\psi - D_n(t,t_o;m)]\psi \\ &= \frac{1}{(i\hbar)^n}\frac{1}{\delta}\left\{\int_{t_o}^{t+\delta>} d\mathbf{t}_n\,\psi_n(t+\delta,t_o;\mathbf{t}_n) - \int_{t_o}^{t>} d\mathbf{t}_n\,\psi_n(t,t_o;\mathbf{t}_n)\right\}.\end{aligned}$$

Next we add and subtract a cross term into the definition of $\Phi(\delta)$. It is at this point that the left and right derivative arguments differ. The right derivative must use the following cross term, whereas the left must utilize the other possible cross term. The use of this particular cross term ensures that the time arguments lie in a range upon which the vectors with those time labels have meaning. (The same can be said for the cross term used in the left sided derivative.)

$$\Phi(\delta) = \frac{1}{(i\hbar)^n}\frac{1}{\delta}\left[\int_{t_o}^{t+\delta>} d\mathbf{t}_n - \int_{t_o}^{t>} d\mathbf{t}_n\right]\psi_n(t+\delta, t_o; \mathbf{t}_n)$$
$$+\frac{1}{(i\hbar)^n}\frac{1}{\delta}\int_{t_o}^{t>} d\mathbf{t}_n\left[\psi_n(t+\delta, t_o; \mathbf{t}_n) - \psi_n(t, t_o; \mathbf{t}_n)\right]$$
$$\equiv \Phi_1(\delta) + \Phi_2(\delta). \tag{4.27}$$

We first consider $\Phi_1(\delta)$. Due to the nature of the iterated integrals, it is necessary to distinguish between the $n = 1$ and $n \geq 2$ cases.

$$\Phi_1(\delta) = \begin{cases} \frac{1}{i\hbar}\frac{1}{\delta}\int_t^{t+\delta} dt_1\, \psi_1(t+\delta, t_o; t_1), & n = 1; \\ \frac{1}{(i\hbar)^n}\frac{1}{\delta}\int_t^{t+\delta} dt_n \int_{t_o}^{t_n>} d\mathbf{t}_{n-1}\, \psi_n(t+\delta, t_o; \mathbf{t}_n), & n \geq 2. \end{cases}$$

We wish to add and subtract a term to $\Phi_1(\delta)$, but it is necessary to modify the notation for $\psi_n$ a little. We expand the argument $\mathbf{t}_n$ to $t_n, \mathbf{t}_{n-1}$ because we will wish to replace $t_n$ by $t$ and have this explicitly exhibited. For $n = 1$ the train of argument is

$$\Phi_1(\delta) = \Phi_1(\delta) - \frac{1}{i\hbar}\psi_1(t, t_o; t) + \frac{1}{i\hbar}\psi_1(t, t_o; t),$$
$$\equiv \Phi_3(\delta) + \frac{1}{i\hbar}\psi_1(t, t_o; t),$$
$$= \Phi_3(\delta) + \frac{1}{i\hbar}\mathbf{V}(t, m)\psi_o(t, t_o),$$
$$= \Phi_3(\delta) + \frac{1}{i\hbar}\mathbf{V}(t, m)D_o(t, t_o; m)\psi.$$

For $n \geq 2$ it proceeds as

$$\begin{aligned}
\Phi_1(\delta) &= \Phi_1(\delta) - \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\psi_n(t, t_o; t, \mathbf{t}_{n-1}) \\
&\quad + \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\psi_n(t, t_o; t, \mathbf{t}_{n-1}), \\
&\equiv \Phi_3(\delta) + \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\psi_n(t, t_o; t, \mathbf{t}_{n-1}), \\
&= \Phi_3(\delta) + \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\mathbf{V}(t,m)\psi_{n-1}(t, t_o; \mathbf{t}_{n-1}), \\
&= \Phi_3(\delta) + \frac{1}{i\hbar}\mathbf{V}(t,m)\frac{1}{(i\hbar)^{n-1}}\int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\psi_{n-1}(t, t_o; \mathbf{t}_{n-1}), \\
&= \Phi_3(\delta) + \frac{1}{i\hbar}\mathbf{V}(t,m)D_{n-1}(t, t_o; m)\psi.
\end{aligned}$$

In these equalities we have used that if $A$ is a closed operator and both $\int \varphi(\tau)\,d\tau$ and $\int A\varphi(\tau)\,d\tau$ exist as strong Riemann integrals, then ([Hi 72], theorem 10.2.3)

$$A\int \varphi(\tau)\,d\tau = \int A\varphi(\tau)\,d\tau. \tag{4.28}$$

If we can show that $\|\Phi_3(\delta)\| \to 0$ as $\delta \to 0$, uniformly in $(t, t_o)$, then we can conclude that $\|\Phi_1(\delta) - \frac{1}{i\hbar}\mathbf{V}(t,m)D_{n-1}(t, t_o; m)\psi\| \to 0$ as $\delta \to 0$, uniformly in $(t, t_o)$. The vector $\Phi_3(\delta)$ is given by the formula

$$\Phi_3(\delta) = \frac{1}{i\hbar}\frac{1}{\delta}\int_t^{t+\delta} dt_1\,[\psi_1(t+\delta, t_o; t_1) - \psi_1(t, t_o; t)], \qquad n = 1;$$

$$\Phi_3(\delta) = \frac{1}{(i\hbar)^n}\frac{1}{\delta}\int_t^{t+\delta} dt_n \left[\int_{t_o}^{t_n>} d\mathbf{t}_{n-1}\,\psi_n(t+\delta, t_o; t_n, \mathbf{t}_{n-1}) - \int_{t_o}^{t>} d\mathbf{t}_{n-1}\,\psi_n(t, t_o; t, \mathbf{t}_{n-1})\right], \quad n \geq 2.$$

Again, because of differences in the iterated integrals, we shall have to handle the $n = 1$, $n = 2$, and $n \geq 3$ cases separately.

For $n = 1$ it is a simple matter to estimate the $L^2(\mathbb{R}^d; \mathbb{C}^s)$-norm of $\Phi_3(\delta)$;

$$\|\Phi_3(\delta)\| \leq \frac{1}{\hbar\delta} \int_t^{t+\delta} dt_1 \, \|\psi_1(t+\delta, t_o; t_1) - \psi_1(t, t_o; t)\|.$$

The integrand on the right hand side is uniformly continuous and hence given $\epsilon > 0$, there exists $\delta_o$, independent of $(t, t_o, t_1)$ such that $\|\psi_1(t+\delta, t_o; t_1) - \psi_1(t, t_o; t)\| \leq \epsilon$ for all $0 < \delta \leq \delta_o$. Thus for all $\delta \leq \delta_o$,

$$\|\Phi_3(\delta)\| \leq \frac{\epsilon}{\hbar}.$$

This shows the uniform continuity of $\Phi_3(\delta)$ for $n = 1$.

For $n = 2$, we split apart the inner integral into one over $t_o$ to $t$ and the other over $t$ to $t_2$;

$$\Phi_3(\delta) = \frac{1}{(i\hbar)^2} \frac{1}{\delta} \int_t^{t+\delta} dt_2 \left\{ \int_t^{t_2} dt_1 \, \psi_2(t+\delta, t_o; \mathrm{t}_2) \right.$$
$$\left. + \int_{t_o}^{t} dt_1 \, [\psi_2(t+\delta, t_o; t_2, t_1) - \psi_2(t, t_o; t, t_1)] \right\}.$$

To estimate the $L^2$-norm of $\Phi_3$, we bring $\|\cdot\|$ through the integrals. Given $\epsilon > 0$, there exists $\delta_o$ such that for all $0 < \delta \leq \delta_o$, $\|\psi_2(t+\delta, t_o; t_2, t_1) - \psi_2(t, t_o; t, t_1)\| \leq \epsilon$. Furthermore, since $\|\psi_2(t+\delta, t_o; \mathrm{t}_2)\|$ is jointly continuous in its arguments on a compact set, it will attain its maximum. Hence there exists a constant $C$ such that $\|\psi_2(t+\delta, t_o; \mathrm{t}_2)\| \leq C$ for all $(t, t_o, \mathrm{t}_2, \delta)$. If $\delta \leq \delta_o$, then it is easily shown that

$$\|\Phi_3(\delta)\| \leq \frac{C}{2\hbar^2}\delta + \frac{T}{\hbar^2}\epsilon,$$

and consequently $\|\Phi_3(\delta)\| \to 0$ as $\delta \to 0$, uniformly in $(t, t_o)$.

For the case $n \geq 3$, we split up the inner integrals in the definition of $\Phi_3(\delta)$ as shown below;

$$\Phi_3(\delta) = \frac{1}{(i\hbar)^n}\frac{1}{\delta}\int_t^{t+\delta} dt_n \int_t^{t_n} dt_{n-1} \int_{t_o}^{t_{n-1}>} d\mathbf{t}_{n-2}\, \psi_n(t+\delta, t_o; \mathbf{t}_n)$$
$$+ \frac{1}{(i\hbar)^n}\frac{1}{\delta}\int_t^{t+\delta} dt_n \int_{t_o}^{t>} d\mathbf{t}_{n-1}[\psi_n(t+\delta, t_o; t_n, \mathbf{t}_{n-1}) - \psi_2(t, t_o; t, \mathbf{t}_{n-1})]$$

Following a similar argument to the $n = 1$ and 2 cases we again arrive at the desired continuity property.

We consider $\Phi_2(\delta)$ next. To the definition of $\Phi_2(\delta)$ we add and subtract a term;

$$\Phi_2(\delta) = \Phi_2(\delta) - \frac{1}{i\hbar}H_o(m)D_n(t, t_o; m)\psi + \frac{1}{i\hbar}H_o(m)D_n(t, t_o; m)\psi$$
$$\equiv \Phi_4(\delta) + \frac{1}{i\hbar}H_o(m)D_n(t, t_o; m)\psi.$$

We shall show that $\Phi_2(\delta)$ converges to $\frac{1}{i\hbar}H_o(m)D_n(t, t_o; m)\psi$ as $\delta \to 0$, uniformly with respect to $(t, t_o)$. This is equivalent to showing $\Phi_4(\delta)$ converges to zero uniformly. Writing out the integral representation of $D_n(t, t_o; m)\psi$ and using the result leading to (4.28) to interchange the operator $H_o(m)$ with the strong Riemann integral $\int_{t_o}^{t>} d\mathbf{t}_n$ , $\Phi_4(\delta)$ can be written

$$\Phi_4(\delta) = \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} d\mathbf{t}_n \left\{\frac{1}{\delta}[\psi_n(t+\delta, t_o; \mathbf{t}_n) - \psi_n(t, t_o; \mathbf{t}_n)] - \frac{1}{i\hbar}H_o(m)\psi_n(t, t_o; \mathbf{t}_n)\right\}.$$

Take the norm of $\Phi_4(\delta)$ and estimate it by bringing the norm through the integral in the last equation above. From part (1) of this lemma, given $\epsilon > 0$ there exists a $\delta_o$, independent of $(t, t_o; \mathbf{t}_n)$, such that for all $\delta < \delta_o$, the norm of the above integrand is bounded by $\epsilon$. Consequently we obtain the bound

$$\|\Phi_4(\delta)\| \leq \frac{\epsilon T^n}{n!\hbar^n}.$$

This implies the desired result for $\Phi_4(\delta)$.

Combining all these results for the $\Phi_j(\delta)$'s, we have proven the existence of the strong right derivative of $D_n(t, t_o; m)\psi$ and that it is given by

$$\frac{\partial}{\partial t_+} D_n(t, t_o; m)\psi = \frac{1}{i\hbar} H_o(m) D_n(t, t_o; m)\psi + \frac{1}{i\hbar} \mathbf{V}(t, m) D_{n-1}(t, t_o; m)\psi. \quad (4.29)$$

We have also shown the limiting process of taking the right derivative is uniform with respect to $(t, t_o)$. Because the right hand side of (4.29) is strongly continuous, it follows that $D_n(t, t_o; m)\psi$ is strongly continuously right differentiable.

For the left sided derivative, a similar argument applies except we add and subtract the cross term

$$\frac{1}{(i\hbar)^n} \frac{1}{\delta} \int_{t_o}^{t+\delta>} d\mathbf{t}_n\, \psi_n(t, t_o; \mathbf{t}_n), \qquad \delta < 0,$$

as we have previously indicated. We will end up with an equation corresponding to (4.29), but with the left handed derivative instead of the right. These two equations together imply the existence of the derivative and the equation (4.25). The continuity of the left and right derivatives, and the uniform convergence of the limiting process of taking either of those two derivatives implies the like properties in the total derivative. $\diamondsuit$

With the differentiabilty properties of each term in the Dyson series established, we next examine the possibilty of exchanging the the infinite sum forming the Dyson series with these differential operators. Recall in lemma 4.5 we saw that if $\hat{\psi} \in C_o^\infty(\mathbb{R}^d, \mathbb{C}^s)$ and $t - t_o$ satisfied estimate (4.21), then the sequence

$\sum_{n=0}^{N} D_n(t,t_o;m)\psi$ is Cauchy with respect to $\|\cdot\|$. Hence as $N \to \infty$ this sum converges to a unique element of $L^2(\mathbb{R}^d, \mathbb{C}^s)$, which we denote by

$$\psi(t,t_o;m) = \mathrm{s}-\lim_{N\to\infty} \sum_{n=0}^{N} D_n(t,t_o;m)\psi. \tag{4.30}$$

**Proposition 4.1:** *Assume that $\hat{\psi} \in C_o^\infty(\mathbb{R}^d, \mathbb{C}^s)$ and that the support of $\hat{\psi}$ is a subset of the closed ball $B_{bk}$ for some $0 < b < \infty$. Let $m \in \mathbb{C}_+$, $(t,t_o) \in T_\Delta$ and let $t-t_o$ satisfy (4.21). Then $\psi(t,t_o;m) \in D_o$ and $\psi(t,t_o;m)$ is strongly continuously differentiable with respect to $t$. Moreover $\psi(t,t_o;m)$ is a solution of the Cauchy-Schrödinger problem:*

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(t,t_o;m) &= H(t,m)\psi(t,t_o;m) \\ \psi(t_o,t_o;m) &= \psi. \end{aligned} \tag{4.31}$$

*Proof:* Because $H_o(m) = (2m)^{-1}\sum_{j=1}^{d} P_j^2$, it follows from lemma 4.5 that the series

$$H_o(m)\sum_{n=0}^{N} D_n(t,t_o;m)\psi = \sum_{n=0}^{N} H_o(m) D_n(t,t_o;m)\psi$$

is strongly convergent as $N \to \infty$. Now $H_o(m)$ is a closed operator and both the sequences $\sum_{n=0}^{N} D_n(t,t_o;m)\psi$ and $H_o(m)\sum_{n=0}^{N} D_n(t,t_o;m)\psi$ converge. This leads us to conclude that $\psi(t,t_o;m) \in D_o$ and

$$H_o(m)\psi(t,t_o;m) = \mathrm{s}-\lim_{N\to\infty}\sum_{n=0}^{N} H_o(m)D_n(t,t_o;m)\psi.$$

The initial condition aspect of (4.31) is easily verified. We make use of equation (4.20) to show

$$\begin{aligned}
\|\psi(t,t_o;m)-\psi\| &\le \|D_o(t,t_o;m)\psi-\psi\| + \sum_{n=1}^{\infty}\|D_n(t,t_o;m)\psi\| \\
&\le \|[U_o(t-t_o;m)-I]\psi\| \\
&\quad + \sum_{n=1}^{\infty} C\frac{1}{n!}\left(\frac{t-t_o}{\hbar}\right)^n (b+n)^{|\rho|}[\mu_T + \frac{\hbar}{|m|}(b+n)k\,\gamma_T]^n .
\end{aligned}$$

In the second inequality, the first term goes to zero as $t \longrightarrow t_o$ because $U_o(t-t_o;m)$ is strongly continuous and $U_o(0;m) = I$. The second term goes to zero in the limit $t \longrightarrow t_o$ because it is of the form $(t-t_o)h(t-t_o)$, where $h(\tau)$ is defined by a convergent Taylor series.

It remains to verify the differential equation in (4.31). The key to showing this result is the recurrence relation (4.25). We can easily differentiate any finite partial sum in the Dyson series with the result

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\sum_{n=0}^{N} D_n(t,t_o;m)\psi &= H_o(m)D_o(t,t_o;m)\psi \\
&\quad + \sum_{n=1}^{N}\Big\{H_o(m)D_n(t,t_o;m)\psi + \mathbf{V}(t,m)D_{n-1}(t,t_o;m)\psi\Big\} \\
&= H(t,m)\sum_{n=0}^{N-1} D_n(t,t_o;m)\psi + H_o(m)D_N(t,t_o;m)\psi. \qquad (4.32)
\end{aligned}$$

We note that the sum $\sum_{n=0}^{N-1} H(t,m)D_n(t,t_o;m)\psi$ will converge because $\mathbf{V}(t,m)$ is $H_o(m)$-bounded and

$$\begin{aligned}
&\left\| \sum_{n=M}^{M'} H(t,m)D_n(t,t_o;m)\psi \right\| \\
&\qquad \leq \sum_{n=M}^{M'} \|H(t,m)D_n(t,t_o;m)\psi\| \\
&\qquad \leq \sum_{n=M}^{M'} \|H_o(m)D_n(t,t_o;m)\psi\| + \sum_{n=M}^{M'} \|\mathbf{V}(t,m)D_n(t,t_o;m)\psi\| \\
&\qquad \leq (1+\beta) \sum_{n=M}^{M'} \|H_o(m)D_n(t,t_o;m)\psi\| + \alpha \sum_{n=M}^{M'} \|D_n(t,t_o;m)\psi\| \\
&\qquad \to 0 \qquad \text{as } M, M' \to \infty,
\end{aligned} \tag{4.33}$$

which shows it is strongly Cauchy.

Because the sums $H(t,m)\sum_{n=0}^{N} D_n(t,t_o;m)\psi$ and $\sum_{n=0}^{N} H(t,m)D_n(t,t_o;m)\psi$ are both strongly convergent and because $H(t,m)$ is a closed operator, we have that $\psi(t,t_o;m)$ is in the domain of $H(t,m)$ and that

$$\mathrm{s}-\lim_{N\to\infty} H(t,m) \sum_{n=0}^{N} D_n(t,t_o;m)\psi = H(t,m)\psi(t,t_o;m).$$

Furthermore, as $\sum_{n=0}^{N} H_o(m)D_n(t,t_o;m)\psi$ is strongly convergent we necessarily have that

$$\mathrm{s}-\lim_{N\to\infty} H_o(m)D_N(t,t_o;m)\psi = 0.$$

These results can be combined, yielding

$$\mathrm{s}-\lim_{N\to\infty} i\hbar \frac{\partial}{\partial t} \sum_{n=0}^{N} D_n(t,t_o;m)\psi = H(t,m)\psi(t,t_o;m).$$

It remains to be shown that

$$\text{s}-\lim_{N\to\infty} i\hbar\frac{\partial}{\partial t}\sum_{n=0}^{N} D_n(t,t_o;m)\psi = i\hbar\frac{\partial}{\partial t}\psi(t,t_o;m).$$

The standard theorems of analysis concerning the differentiation of uniformly convergent sequences of functions remain valid on the $L^2(\mathbb{R}^d;\mathbb{C}^s)$ topology (cf. references [Kr 71], p. 4 and [La 83], theorem 5.9.1).

Let $0<\delta<|m|(ek\gamma_T)^{-1}$ and $t_o\le t\le t_o+\delta$ (and therefore $t-t_o$ satisfies (4.21)). Define $\varphi_N(t)=\sum_{n=0}^{N} D_n(t,t_o;m)\psi$. Then each $\varphi_N(t)$ is differentiable on $[t_o,t_o+\delta]$ and $\|\varphi_N(t)-\psi(t,t_o;m)\|\to 0$ as $N\to\infty$. If we can show that $\partial_t\varphi_N(t)$ converges strongly and uniformly with respect to $t$ on $[t_o,t_o+\delta]$, then

$$i\hbar\frac{\partial}{\partial t}\psi(t,t_o;m) = \text{s}-\lim_{N\to\infty} i\hbar\frac{\partial}{\partial t}\varphi_N(t),$$

and our proof is complete. For details to this claim, we refer to reference [Ru 76], theorem 7.17. In the notation using $\varphi_N(t)$, we rewrite (4.32) as

$$i\hbar\frac{\partial}{\partial t}\varphi_N(t) = \sum_{n=0}^{N} H(t,m)D_n(t,t_o;m)\psi + H_o(m)D_N(t,t_o;m)\psi.$$

It is enough to prove that each of the two terms on the right hand side are separately uniformly Cauchy. Because $\mathbf{V}(t,m)$ is $H_o(m)$-bounded, it is enough to prove that $\sum_{n=0}^{N}\|H_o(m)D_n(t,t_o;m)\psi\|$ and $\sum_{n=0}^{N}\|D_n(t,t_o;m)\psi\|$ are uniformly Cauchy (cf. (4.33)). More generally still, it is enough to prove that $\sum_{n=0}^{N}\|P^\rho D_n(t,t_o;m)\psi\|$ is uniformly Cauchy for all multi-indices $\rho$. By using inequality (4.20) these partial sums can be magorized termwise and the magorizing sequence will converge uniformly with respect to $t\in[t_o,t_o+\delta]$ and $t_o\in[0,T]$. Hence the sequence $\sum_{n=0}^{N}\|P^\rho D_n(t,t_o;m)\psi\|$ is uniformly Cauchy. Equation (4.20) also shows that $H_o(m)D_N(t,t_o;m)\psi$ converges to zero uniformly in $t$ and $t_o$ as $N\to\infty$.

Thus we have verified the Dyson series satisfies the Schrödinger equation in the strong sense for $t$ in the interval $[t_o, t_o + \frac{|m|}{ek\gamma_T})$ and $t_o$ in the interval $[0, T)$. The arguments for the special case $t_o = t = T$ are somewhat simpler, and will not be presented here. $\diamondsuit$

# CHAPTER 5

## Dyson Kernels and the Propagator

Proposition 4.1 showed for a certain class of initial test functions and a time interval $t - t_o$ sufficiently small, the Dyson series converged to a solution of the Schrödinger equation. For $m \in \mathbb{C}_+$ we shall demonstrate that each of the Dyson operators, $D_n(t, t_o; m)$, is an integral operator. Furthermore their integral kernels are summable and the resulting function we show to be the propagator for the complex mass problem. Again, throughout chapter 5 we have the universal assumption that $a \in \mathcal{V}_v(k)$ and $v \in \mathcal{V}(k)$.

### 5.1 Product Measures and Their Combinatorics

To begin, we first develop some convenient notations for product measures that appear in the ensuing discussions. Let $n$ be the order of the Dyson iterate under consideration, and let $r$ be an integral index between 0 and $n$. If $r > 0$, define an $r$-tuple with non-negative integer arguments by $\mathbf{j}_r = (j_1, \ldots, j_r)$, and if $r = 0$, let $\mathbf{j}_0$ simply be a label whose meaning will be made clear below. Define the ordered index set $J_{n,r}$ by

$$J_{n,r} = \begin{cases} \emptyset, & \text{if } r = 0; \\ \{\mathbf{j}_r : 1 \le j_1 < j_2 < \cdots < j_r \le n\}, & \text{if } 1 \le r \le n. \end{cases}$$

For $r > 0$, $J_{n,r}$ can be thought of as the set of all ways of picking $r$ distinct numbers out of $1, \ldots, n$. If $r = 0$, then $J_{n,r}$ is the empty set and $\mathbf{j}_0$ is a label that reflects the choice of picking no numbers out of $n$. There are $\binom{n}{r}$ elements in $J_{n,r}$. To each $\mathbf{j}_r$ we associate an $n$-fold product measure $\Lambda_n(\mathbf{j}_r, \mathbf{t}_n)$, defined by

$$\Lambda_n(\mathbf{j}_r, \mathbf{t}_n) = |\sigma_n^n(t_n)| \times \cdots \times |\gamma(t_{j_r})| \times \cdots \times |\gamma(t_{j_1})| \times \cdots \times |\sigma_1^n(t_1)|. \tag{5.1}$$

Here we recall the definition of $\sigma_l^n(t_l)$ given in equation (3.24) and write it in its polar decomposition form

$$d\sigma_l^n(t_l) = \varsigma_l^n(\alpha, t_l)\, d|\sigma_l^n(t_l)|.$$

The right hand side of (5.1) is to be understood in the following sense. If $r = 0$, the measure only involves the product of the $|\sigma_l^n(t_l)|$'s. When $r > 0$, replace the measure $|\sigma_{j_l}^n(t_{j_l})|$ in the measure $\Lambda_n(\mathbf{j}_0, \mathbf{t}_n)$, by the measure $|\gamma(t_{j_l})|$, for $j_l = j_1, \ldots, j_r$. For example if $j_r = n-1$, then $|\gamma(t_{n-1})|$ replaces $|\sigma_{n-1}^n(t_{n-1})|$. From equations (3.13) and (3.27) it is evident that $\Lambda_n(\mathbf{j}_r, \mathbf{t}_n)$ has the uniform bound

$$\|\Lambda_n(\mathbf{j}_r, \mathbf{t}_n)\| \leq \left(\mu_T + \frac{\hbar}{|m|} nk\gamma_T\right)^{n-r} \gamma_T^r, \qquad \mathbf{j}_r \in J_{n,r}, \quad \mathbf{t}_n \in \Delta_n(t, t_o). \tag{5.2}$$

To establish the existence of the Dyson kernels, we shall work in the Fourier transform space first and pull back these results to the co-ordinate space representation. To be able to pull back these results, we first need a couple of results to provide the necessary mathematical tools. The proofs of these elementary results are straightforward and shall only be sketched.

**Lemma 5.1:** *Let* $\operatorname{Im} z \geq 0$ *(*$z \neq 0$*),* $\beta, \xi \in \mathbb{R}^d$*. Assume that* $\psi \in \mathcal{S}$ *and its Fourier transform is given by* $\hat{\psi}$*. Then*

$$\int d\alpha\, e^{-\frac{i}{2z}(\alpha+\xi-z\beta)^2} \hat{\psi}(\alpha) = (-iz)^{d/2} \int dy\, e^{\frac{iz}{2}y^2 + iy\cdot(\xi - z\beta)} \psi(y); \tag{5.3}$$

*and*

$$(-1)^{|\rho|}\int d\alpha\,(i\alpha)^{\rho}e^{-\frac{i}{2z}(\alpha+\xi-z\beta)^2}\hat{\psi}(\alpha)=(-iz)^{d/2}\int dy\,\psi(y)\partial_y^{\rho}e^{\frac{iz}{2}y^2+iy\cdot(\xi-z\beta)}\psi(y). \tag{5.4}$$

*Proof:* Introduce into the integrand of the left hand side of (5.3) the characteristic function $\chi_R(\alpha)$ for the hypercube $|\alpha_j|\leq R$, $(1\leq j\leq d)$. One can show with an application of the dominated convergence theorem that the resulting integral converges to the left hand side of (5.3) as $R\to\infty$. On the other hand, writing out the explicit representation of the Fourier transform of $\psi$ and applying Fubini's theorem, we get

$$\int dy\,\psi(y)\int d\alpha\,\chi_R(\alpha)e^{-\frac{i}{2z}(\alpha+\xi-z\beta)^2-i\alpha\cdot y}.$$

The $R\to\infty$ limit for this integral is considered. If $\mathrm{Im}\,z>0$ it easily shown that the inner $d\alpha$ integal above is uniformly bounded in $y$ and $R$. If $\mathrm{Im}\,z=0$ $(z\neq 0)$, we have to be more careful with the estimate, as the exponential is then pure oscillatory with no decay. An application of the lemma below shows this Fresnel type integral can also be uniformly bounded in $y$ and $R$. In either case we can apply the dominated convergence theorem to bring the $R\to\infty$ limit through the $dy$ integral, with the result

$$\int dy\,\psi(y)\prod_{j=1}^{d}e^{-\frac{i}{2z}(\xi_j-z\beta_j)^2}\int_{-\infty}^{\infty}d\alpha_j\,e^{-\frac{i}{2z}\alpha_j^2-i[\frac{1}{z}(\xi_j-z\beta_j)+y_j]\alpha_j}.$$

The inner integral can be split up into ones over the intervals $(-\infty,0]$ and $[0,\infty)$. If $\mathrm{Im}\,z>0$ then these integrals have a standard result and we refer to reference [GR 80], 3.322.2. For $\mathrm{Im}\,z=0$ with $z\neq 0$, the inner intergral is a Fresnel type integral which can be evaluated by contour integration techniques. Both cases for $z$ lead to the right hand side of (5.3). The branch of the square root being taken is $\arg z\in(-\pi,\pi)$ (i.e. the cut is along the negative real axis).

The second equation (5.4) is shown by starting with the right hand side, integrating by parts until all the derivatives strike $\psi$ and then use (5.3), but with $\psi$ replaced by $\partial_y^\rho \psi(y)$. Using the identity $(\partial^\rho \psi)\hat{}(\alpha) = i^{|\rho|} \alpha^\rho \hat{\psi}(\alpha)$, we see that equation (5.4) results. $\diamondsuit$

**Lemma 5.2:** *For all $a, b \in \mathbb{R} \backslash \{0\}$ we have the $a$-uniform estimate*

$$\left| \int_0^a e^{-ibt^2}\, dt \right| \leq C(|b|)$$

*where*

$$C(|b|) = \left( \frac{\pi}{4|b|} \right)^{1/2} + \frac{\pi}{4} \max\{1, |b|^{-1}\}.$$

*Proof:* Since the integrand is an even function of $t$, without loss of generality we may assume that $a > 0$. If $b < 0$ then by taking the complex conjugate of the above integral we could estimate the equivalent function

$$\left| \int_0^a e^{-i|b|t^2}\, dt \right|.$$

Thus without loss of generality, we may take $b > 0$. The proof is now identical to Truman's lemma 1 [Tru 77]. $\diamondsuit$

For the next lemma we introduce the following notation. If $r$ is an integer between 0 and $n$, then the symbol $[r/2]$ means the greatest integer less than or equal to $r/2$. Let $l$ be an integer between 0 and $[r/2]$. The summation sign $\sum'_{r,l}$ will denote the sum over the division of $r$ objects into certain sets. Label these objects by the indices $j = 1 \sim r$. For a given $r$ and $l$, pick $r - 2l$ of these objects. Of the remaining $2l$ objects, we pick $l$ pairs. The result is to partition these objects like

$\{j_1, \ldots, j_{r-2l}; (j_{r-2l+1}, j_{r-2l+2}), \ldots, (j_{r-1}, j_r)\}$. The sum is taken over all distinct choices of this type. The number of terms in this sum is

$$\Lambda_{r,l} = \frac{r!}{2^l (r-2l)! l!}. \tag{5.5}$$

**Lemma 5.3:** *Let $\{\eta_j\}_{j=1}^r$ be a set of $r$ constant $d$-dimensional vectors. Let $\nabla_y$ denote the $d$-dimensional gradient with respect to the variable $y \in \mathbb{R}^d$. Then the following formula is valid;*

$$(\eta_r \cdot \nabla_y) \cdots (\eta_1 \cdot \nabla_y) e^{ay^2}$$
$$= (2a)^r e^{ay^2} \sum_{l=0}^{[r/2]} \frac{1}{(2a)^l} {\sum_{r,l}}' (\eta_{i_1} \cdot y) \cdots (\eta_{i_{r-2l}} \cdot y)(\eta_{i_{r-2l+1}} \cdot \eta_{i_{r-2l+2}}) \cdots (\eta_{i_{r-1}} \cdot \eta_{i_r}),$$
$$a \in \mathbb{C}. \tag{5.6}$$

*Proof:* The result is trivially seen for the case $r = 1$. The general case will follow from a tedious but straight forward induction argument. The one dimensional case ($d = 1$) of formula (5.6) can be found in reference [GR 80], 0.432.2 . $\diamond$

### 5.2 Dyson Kernels

With these tools in hand we are ready to show that the $n^{th}$ Dyson iterate is an integral operator.

**Lemma 5.4:** *Let $m \in \mathbb{C}_+$, $(t, t_o) \in T_\Delta$ and $\psi \in \mathcal{S}$.*

(1) *For a.a. $x \in \mathbb{R}^d$, each operator $D_n(t, t_o; m)$ ($n \geq 0$) has a Fourier integral representation*

$$[D_n(t, t_o; m)\psi](x) = \int d\alpha_o \; \hat{d}_n(x, t; \alpha_o, t_o; m) \hat{\psi}(\alpha_o). \tag{5.7}$$

*The functions* $\hat{d}_n(x,t;\alpha_o,t_o;m)$ *are defined by*

$$\hat{d}_o(x,t;\alpha_o,t_o;m) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{i\hbar}{2m}(t-t_o)\alpha_o^2 + i\alpha_o\cdot x} I;$$

$$\begin{aligned} &\hat{d}_n(x,t;\alpha_o,t_o;m) \\ &\quad = \frac{1}{(2\pi)^{d/2}}\frac{1}{(i\hbar)^n}\int_{t_o}^{t>} dt_n \int d\lambda_n^n(t_n;\alpha_n)\cdots\int d\lambda_1^n(t_1;\alpha_1)\, e^{ix\cdot(\alpha_o+\cdots\alpha_n)} \\ &\qquad\qquad \times e^{-\frac{i\hbar}{2m}\left[\sum_{i,j=1}^n (t-t_{i\vee j})\alpha_i\cdot\alpha_j + 2\sum_{i=1}^n (t-t_i)\alpha_i\cdot\alpha_o + (t-t_o)\alpha_o^2\right]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 1, \end{aligned} \tag{5.8}$$

*where* $i \vee j = \max(i,j)$ *and* $I$ *in* $\hat{d}_o$ *is the* $s \times s$ *unit matrix.*

(2) *The* $n^{th}$ *Dyson operator is an integral operator with kernel* $d_n(x,t;y,t_o;m)$. *That is,* $D_n(t,t_o;m)$ *satisfies*

$$[D_n(t,t_o;m)\psi](x) = \int dy\, d_n(x,t;y,t_o;m)\psi(y), \qquad a.a.\ x. \tag{5.9}$$

*The functions* $d_n(x,t;y,t_o;m)$ *are defined by the formulae*

$$d_o(x,t;y,t_o;m) = \left[\frac{m}{2\pi i\hbar(t-t_o)}\right]^{d/2} e^{\frac{im}{2\hbar(t-t_o)}(x-y)^2} I;$$

$$\begin{aligned} d_n(x,t;y,t_o;m) &= \left[\frac{m}{2\pi i\hbar(t-t_o)}\right]^{d/2} \frac{1}{(i\hbar)^n}\int_{t_o}^{t>} dt_n \\ &\quad \times \int \left[d\sigma_n^n(t_n;\alpha_n) - \frac{i\hbar}{m}\eta(\alpha_n,t_n)\cdot\nabla_y\, d|\gamma|(t_n;\alpha_n)\right] \times \cdots \\ &\quad \times \int \left[d\sigma_1^n(t_1;\alpha_1) - \frac{i\hbar}{m}\eta(\alpha_1,t_1)\cdot\nabla_y\, d|\gamma|(t_1;\alpha_1)\right] \\ &\quad \times e^{ix\cdot(\alpha_1+\cdots+\alpha_n) + \frac{im}{2\hbar(t-t_o)}(y-\chi_n)^2 - \frac{i\hbar}{2m}\sum_{j,l=1}^n (t-t_{j\vee l})\alpha_j\cdot\alpha_l}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad n \geq 1. \end{aligned} \tag{5.10}$$

*Here, $\chi_n \in \mathbb{C}^d$ is given by the formula*

$$\chi_n = \begin{cases} x, & \text{if } n = 0; \\ x - \frac{\hbar}{m}\sum_{j=1}^{n}(t-t_j)\alpha_j, & \text{if } n \geq 1. \end{cases} \tag{5.11}$$

*The branch of the square root is $-\pi < \arg z < \pi$. The $y$ derivatives $\nabla_y$ will only act on the exponential in the formula for $d_n$, and not on the test function $\psi$ in equation (5.9).*

*Proof:* For the $n = 0$ case, we have that $D_o(t, t_o; m)\psi = U_o(t-t_o; m)\psi$, whence equations (5.7) and (5.8) follow immediately (cf. equation (4.9)).

For $n \geq 1$, equations (4.17) (with $|\rho| = 0$) and (4.8) and an application of Fubini's theorem imply (5.7) and (5.9) up to an apparent difference in the exponential. The identity

$$(t - t_n)(\alpha_o + \cdots + \alpha_n)^2 + \cdots + (t_1 - t_o)\alpha_o^2 = \sum_{i,j=0}^{n}(t - t_{i\vee j})\alpha_i \cdot \alpha_j, \tag{5.12}$$

shows that the argument to both exponentials are indeed the same (once one expands out the $i = 0$ and $j = 0$ parts of the right hand side of (5.12)). This identity is trivial to show for $n = 0$ and the general case can be proven by an inductive argument. We note that identity (5.12) shows that

$$\sum_{i,j=0}^{n}(t - t_{i\vee j})\alpha_i \cdot \alpha_j \geq 0. \tag{5.13}$$

For $n = 0$, part (1) implies the mixed integral representation

$$[D_o(t, t_o; m)\psi](x) = \frac{1}{(2\pi)^{d/2}} \int d\alpha_o\, e^{-\frac{i\hbar}{2m}(t-t_o)\left(\alpha_o - \frac{m}{\hbar(t-t_o)}x\right)^2 + i\frac{m}{2\hbar(t-t_o)}x^2}\, \hat{\psi}(\alpha_o)$$

But this is in the form of the integral in lemma 5.1(1), with $z = m[\hbar(t - t_o)]^{-1}$, $\xi = 0$ and $\beta = x$. Applying lemma 5.1 leads us immediately to (5.9) and (5.10).

To show part (2) for $n \geq 1$, start by using the results in part (1). In (5.8), explicitly separate out the parametric dependence on $\alpha_o$ in the measures $\lambda_l^n(t_l)$, writing them as

$$d\lambda_l^n(t_l;\alpha_l) = \Big[d\sigma_l^n(t_l;\alpha_l) - \frac{\hbar}{m}\Big(\eta(\alpha_l,t_l)\cdot\alpha_o\Big)\, d|\gamma|(t_l;\alpha_l)\Big].$$

Next we expand out the product of these sums as the sum of the products. Taking care to preserve the order of the matrix products, the resulting measure is

$$\begin{aligned}
&\Big[d\sigma_n^n(t_n;\alpha_n) - \frac{\hbar}{m}\Big(\eta(\alpha_n,t_n)\cdot\alpha_o\Big)\, d|\gamma|(t_n;\alpha_n)\Big] \times \cdots \\
&\qquad\qquad \times \Big[d\sigma_1^n(t_1;\alpha_1) - \frac{\hbar}{m}\Big(\eta(\alpha_1,t_1)\cdot\alpha_o\Big)\, d|\gamma|(t_1;\alpha_1)\Big] \\
&= \sum_{r=0}^{n}\sum_{J_{n,r}} \Big(-\frac{\hbar}{m}\Big)^r d\Lambda_n(\mathbf{j}_r,\mathbf{t}_n)\varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{j_r},t_{j_r})\cdot\alpha_o\Big)\times\cdots \\
&\qquad \times\Big(\eta(\alpha_{j_1},t_{j_1})\cdot\alpha_o\Big)\cdots\varsigma_1^n(\alpha_1,t_1).
\end{aligned}$$

Substituting this into (5.7) and (5.8), and then using Fubini's theorem to interchange the $d\alpha_o$ integral with the time ordered and complex measure integrals, we obtain for an expression of the $n^{th}$ Dyson iterate;

$$\begin{aligned}
&[D_n(t,t_o;m)\psi](x) \\
&\quad = \sum_{r=0}^{n}\sum_{J_{n,r}} \Big(-\frac{\hbar}{m}\Big)^r \frac{1}{(i\hbar)^n}\frac{1}{(2\pi)^{d/2}}\int_{t_o}^{t>} d\mathbf{t}_n \int d\Lambda_n(\mathbf{j}_r,\mathbf{t}_n) \\
&\quad \times e^{-\frac{i\hbar}{2m}\sum_{i,j=1}^{n}(t-t_{i\vee j})\alpha_i\cdot\alpha_j + ix\cdot(\alpha_1+\cdots+\alpha_n)} \\
&\quad \times \int d\alpha_o\, \varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{j_r},t_{j_r})\cdot\alpha_o\Big)\times\cdots \\
&\quad \times \Big(\eta(\alpha_{j_1},t_{j_1})\cdot\alpha_o\Big)\cdots\varsigma_1^n(\alpha_1,t_1)\hat{\psi}(\alpha_o)e^{-\frac{i\hbar}{2m}(t-t_o)\alpha_o^2 + i\chi_n\cdot\alpha_o}.
\end{aligned} \tag{5.14}$$

Let

$$\varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{j_r},t_{j_r})\cdot\alpha_o\Big)\cdots\Big(\eta(\alpha_{j_1},t_{j_1})\cdot\alpha_o\Big)\cdots\varsigma_1^n(\alpha_n,t_n)\equiv\sum_{|\rho|=r}C_\rho\alpha_o^\rho,$$

where $\rho$ is the multi-index $(\rho_1,\ldots,\rho_d)$ and the matrices $C_\rho$ have a dependence on the $\alpha_l$'s and $t_l$'s not explicitly displayed. Then the $d\alpha_o$ integral can be written

$$\begin{aligned}
&\int d\alpha_o\,\varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{j_r},t_{j_r})\cdot\alpha_o\Big)\cdots\Big(\eta(\alpha_{j_1},t_{j_1})\cdot\alpha_o\Big)\times\cdots\\
&\qquad\qquad\times\varsigma_1^n(\alpha_1,t_1)\hat\psi(\alpha_o)e^{-\frac{i\hbar}{2m}(t-t_o)\alpha_o^2+i\chi_n\cdot\alpha_o}\\
&=\sum_{|\rho|=r}C_\rho\,e^{\frac{im}{2\hbar(t-t_o)}\chi_n^2}\int d\alpha_o\,\alpha_o^\rho\hat\psi(\alpha_o)\,e^{-\frac{i\hbar}{2m}(t-t_o)\left(\alpha_o-\frac{m}{\hbar(t-t_o)}\chi_n\right)^2}\\
&=\sum_{|\rho|=r}C_\rho\,e^{\frac{im}{2\hbar(t-t_o)}\chi_n^2}\int d\alpha_o\,\alpha_o^\rho\,\hat\psi(\alpha_o)e^{-\frac{i\hbar}{2m}(t-t_o)\left(\alpha_o+\sum_{j=1}^n\frac{t-t_j}{t-t_o}\alpha_j-\frac{m}{\hbar(t-t_o)}x\right)^2}.
\end{aligned}\tag{5.15}$$

In the second equality we have merely expanded out the definition of $\chi_n$. We can now apply lemma 5.1 if we make the associations $z=\frac{m}{\hbar(t-t_o)}$, $\xi=\sum_{j=1}^n\frac{t-t_j}{t-t_o}\alpha_j$ and $\beta=x$. Then the right hand side of (5.15) becomes

$$\begin{aligned}
&\sum_{|\rho|=r}C_\rho\frac{e^{\frac{im}{2\hbar(t-t_o)}\chi_n^2}}{(-i)^{|\rho|}}\left(-\frac{im}{\hbar(t-t_o)}\right)^{d/2}\int dy\,\psi(y)\,\partial_y^\rho e^{\frac{im}{2\hbar(t-t_o)}y^2+iy\cdot\left(\sum_{j=1}^n\frac{t-t_j}{t-t_o}\alpha_j-\frac{m}{\hbar(t-t_o)}x\right)}\\
&=\left(\frac{m}{i\hbar(t-t_o)}\right)^{d/2}\int dy\sum_{|\rho|=r}i^{|\rho|}C_\rho\psi(y)\,\partial_y^\rho e^{\frac{im}{2\hbar(t-t_o)}(y-\chi_n)^2}\\
&=\left(\frac{m}{i\hbar(t-t_o)}\right)^{d/2}\int dy\,\varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{j_r},t_{j_r})\cdot i\nabla_{y_1}\Big)\times\cdots\\
&\qquad\times\Big(\eta(\alpha_{j_1},t_{j_1})\cdot i\nabla_{y_1}\Big)\cdots\varsigma_1^n(\alpha_1,t_1)\,e^{\frac{im}{2\hbar(t-t_o)}y_1^2}\Big|_{y_1=y-\chi_n}\psi(y).
\end{aligned}\tag{5.16}$$

After substituting the expressions (5.16) and (5.15) into (5.14), our equation for $D_n(t,t_o;m)\psi$ becomes

$$[D_n(t,t_o;m)\psi](x) = \sum_{r=0}^{n}\sum_{J_{n,r}}\left(-\frac{\hbar}{m}\right)^r \frac{1}{(i\hbar)^n}\frac{1}{(2\pi)^{d/2}}\int_{t_o}^{t>} d\mathbf{t}_n$$
$$\times \int d\Lambda_n(\mathbf{j}_r,\mathbf{t}_n)\, e^{-\frac{i\hbar}{2m}\sum_{i,j=1}^{n}(t-t_{i\vee j})\alpha_i\cdot\alpha_j + ix\cdot(\alpha_1+\cdots+\alpha_n)}$$
$$\times \left(\frac{m}{i\hbar(t-t_o)}\right)^{d/2}\int dy\, \varsigma_n^n(\alpha_n,t_n)\cdots\left(\eta(\alpha_{j_r},t_{j_r})\cdot i\nabla_{y_1}\right)\times\cdots$$
$$\times\left(\eta(\alpha_{j_1},t_{j_1})\cdot i\nabla_{y_1}\right)\cdots\varsigma_1^n(\alpha_1,t_1)\, e^{\frac{im}{2\hbar(t-t_o)}y_1^2}\Big|_{y_1=y-\chi_n}\psi(y). \tag{5.17}$$

Notice that the combinatorics that resulted in the sums $\sum_{r=0}^{n}$ and $\sum_{J_{n,r}}$ in (5.14) are exactly the same in (5.17), but with $\alpha_o$ replaced by $i\nabla_y$. The result we are after now follows, provided we can justify the interchange of the integrals $\int_{t_o}^{t>} d\mathbf{t}_n$ and $\int d\Lambda_n(\mathbf{j}_r,\mathbf{t}_n)$ with $\int dy$ and then show the derivatives $\nabla_y$ can be brought outside of the time and complex measure integrals.

Grouping together the exponentials in (5.17), their combined argument is

$$i\Big[x\cdot\sum_{j=1}^{n}\alpha_j - (x-y)\cdot\sum_{j=1}^{n}\frac{t-t_j}{t-t_o}\alpha_j\Big] + \frac{im}{2\hbar(t-t_o)}(x-y)^2$$
$$-\frac{i\hbar}{2m}\sum_{j,l=1}^{n}\left[(t-t_{j\vee l}) - \frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l.$$

Consider the third term in this expression. If $t_{j_o} < t$ and $t_{j_o+1} = \cdots = t_n = t$, then

$$\sum_{j,l=1}^{n}\left[(t-t_{j\vee l}) - \frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l = \sum_{j,l=1}^{j_o}\left[(t-t_{j\vee l}) - \frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l.$$

Without loss of generality we assume that $t_n < t$ (and hence all $t_j$'s are less than $t$ because $t_j \le t_n$). It is a simple algebraic exercise to show that

$$\sum_{j,l=1}^{n}\left[(t-t_{j\vee l})-\frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l.=\sum_{j=1}^{n}\left[\frac{1}{t-t_j}-\frac{1}{t-t_{j-1}}\right]\Big(\sum_{l=j}^{n}(t-t_l)\alpha_l\Big)^2,\qquad n\ge 1. \tag{5.18}$$

Because of the ordering relation $t_j \ge t_{j-1}$, it is easy to see that the right hand side of (5.18) is non-negative. This allows us to bound the exponentials in (5.17) for all $m \in \mathbb{C}_+$;

$$\left|e^{i\left[x\cdot\sum_{j=1}^{n}\alpha_j-(x-y)\cdot\sum_{j=1}^{n}\frac{t-t_j}{t-t_o}\alpha_j\right]+\frac{im}{2\hbar(t-t_o)}(x-y)^2-\frac{i\hbar}{2m}\sum_{j,l=1}^{n}\left[(t-t_{j\vee l})-\frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l}\right|$$
$$\le e^{-\frac{\operatorname{Im} m}{2\hbar(t-t_o)}(x-y)^2}. \tag{5.19}$$

Next we use lemma 5.3 to explicitly expand out the effect of the derivatives $\eta\cdot\nabla_y$ acting on the exponential. The result is another useful expression for the $n^{th}$ Dyson iterate. Its kernel will be given by

$$d_n(x,t;y,t_o;m)$$
$$=\sum_{r=0}^{n}\sum_{J_{n,r}}\sum_{l=0}^{[r/2]}\sum_{r,l}{}'\left(\frac{im}{\hbar(t-t_o)}\right)^{r-l}\left(-\frac{\hbar}{m}\right)^{r}\frac{1}{(i\hbar)^n}\left(\frac{m}{2\pi i\hbar(t-t_o)}\right)^{d/2}$$
$$\times\int_{t_o}^{t>}d\mathbf{t}_n\int d\Lambda_n(\mathbf{j}_r,\mathbf{t}_n)\,\varsigma_n^n(\alpha_n,t_n)\cdots\Big(\eta(\alpha_{q_h},t_{q_h})\cdot(y-\chi_n)\Big)\times\cdots$$
$$\times\Big\{\eta(\alpha_{q_i},t_{q_i})\Big\}_1\cdots\Big(\eta(\alpha_{q_j},t_{q_j})\cdot(y-\chi_n)\Big)\cdots\Big\{\eta(\alpha_{q_k},t_{q_k})\Big\}_1\times\cdots$$
$$\times\Big\{\eta(\alpha_{q_p},t_{q_p})\Big\}_l\cdots\Big\{\eta(\alpha_{q_u},t_{q_u})\Big\}_l\cdots\varsigma_1^n(\alpha_1,t_1)$$
$$\times\, e^{i\left[x\cdot\sum_{j=1}^{n}\alpha_j-(x-y)\cdot\sum_{j=1}^{n}\frac{t-t_j}{t-t_o}\alpha_j\right]+\frac{im}{2\hbar(t-t_o)}(x-y)^2-\frac{i\hbar}{2m}\sum_{j,l=1}^{n}\left[(t-t_{j\vee l})-\frac{(t-t_j)(t-t_l)}{t-t_o}\right]\alpha_j\cdot\alpha_l}. \tag{5.20}$$

A brief explanation of this sum is in order. We would like to group the $\eta$'s together as in lemma 5.3, but their matrix structure, and that of the $\varsigma$'s add an additional complication not present in the Abelian case. We must preserve the order of their multiplication, while at the same time we must take dot products between $\eta$'s that may be separated by $\varsigma$'s or other $\eta$'s. We use the curly parenthesis with a subscript to indicate between which pairs of $\eta$'s dot products are being taken. There are $r$ $\eta$'s appearing in the summand and each $\eta$ appears in a dot product with $y - \chi_n$ or another $\eta$. The labels $q_i$ reflect the possible ways of making these pairings, as per the summation convention in lemma 5.3. Our expression for the $n^{th}$ Dyson iterate at this point is similar to (5.20), but with $\int dy$ as the inner most integral and the factor $\psi(y)$ appearing after $\varsigma_1^n(\alpha_1, t_1)$. With the derivatives explicitly evaluated, we can now proceed to estimate the integrand in our expression for $D_n(t, t_o; m)\psi$. From equation (5.19), the exponential function is uniformly bounded by 1. The compactness of the support of $\Lambda_n(\mathbf{j}_r, \mathbf{t}_n)$ controls the polynomial growth of the $\alpha_j$'s, and the fact $\psi \in \mathcal{S}$ is used in controlling the polynomial growth in $y$. We recall $|\eta(\alpha_j, t_j)| = 1$ for all $\alpha_j \in \mathbb{R}^d$ and $t_j \in [t_o, t]$ and we also recall the norm estimate (5.2). These combined with the fact that all the sums appearing in (5.20) are of finite order, show that the integral is absolutely integrable and hence we can apply Fubini's theorem. Similarly, it is easily shown, via the dominated convergence theorem, that the derivatives $\nabla_y$ can be interchanged with the various integrals involved, to arrive at (5.9) and (5.10). $\diamondsuit$

With the existence of the Dyson kernels established, the next step is to study their properties. But first we make note of the bound

$$|y - \chi_n| \leq |x - y| + \frac{n\hbar k}{|m|}(t - t_o) \equiv Z_n, \tag{5.21}$$

valid for all $\alpha_j \in S_k$ and all $x, y \in \mathbb{R}^d$. As a majorizing function to the $n^{th}$ Dyson kernel we introduce the function

$$g(x;t,t_o,m) = \left(\frac{|m|}{2\pi\hbar(t-t_o)}\right)^{d/2} e^{\frac{1}{2\hbar}[-c_o|x|^2+c_1|x|+c_2]}, \tag{5.22}$$

where the constants $c_i$ are

$$c_o = \frac{\mathrm{Im}\, m}{t-t_o} \qquad c_1 = \frac{|m|}{k(t-t_o)} \qquad c_2 = \frac{|m|}{2k^2(t-t_o)} + \frac{|m|\mu_T}{k\gamma_T}. \tag{5.23}$$

To properly describe the convergence properties of the sum over the Dyson kernels, we will use the parameters

$$\begin{aligned} \theta &= \frac{2ek\gamma_T}{|m|}(t-t_o); \\ \theta_- &= \frac{2ek\gamma_T}{m_-}(t-t_o). \end{aligned} \tag{5.24}$$

Here $m_-$ represents the smallest value $|m|$ will attain on a compact set of $\mathbb{C}_+$.

**Lemma 5.5:** *Let $d_n(x,t;y,t_o;m)$ be defined as in lemma 5.4 and let $g$ and $\theta$ be defined as above. Then*

(1) *For all $x, y \in \mathbb{R}^d$, $d_n(x,t;y,t_o;m)$ satisfies the pointwise estimate*

$$|d_n(x,t;y,t_o;m)| \leq s^{1/2}\theta^n g(x-y;t,t_o,m) \qquad n \geq 0 \tag{5.25}$$

(2) *For all $(x,y,m) \in \boldsymbol{\mathcal{L}}$, where $\boldsymbol{\mathcal{L}}$ is a compact subset of $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{C}_+$, the Dyson kernels $d_n(\cdot,t;\cdot,t_o;\cdot)$ are jointly continuous.*

(3) *If $m \in \mathbb{C}_>$, then $d_n(\cdot,t;y,t_o;m)$ and $d_n(x,t;\cdot,t_o;m)$ are both members of $L^1 \cap L^\infty(\mathbb{R}^d, \mathbb{C}^{s\times s})$. Furthermore their $L^p$ norms are uniformly bounded with respect*

*to $x$ and $y$ and satisfy the estimates*

$$\|d_n(x,t;\cdot,t_o;m)\|_p \le s^{1/2}\theta^n \|g(\cdot;t,t_o,m)\|_p$$
$$\|d_n(\cdot,t;y,t_o;m)\|_p \le s^{1/2}\theta^n \|g(\cdot;t,t_o,m)\|_p \tag{5.26}$$

(4) *If $0<\theta<1$, then for each $m \in \mathbb{C}_+$ and $x,y \in \mathbb{R}^d$ the sum over $n = 0 \sim \infty$ of the Dyson kernels $d_n(x,t;y,t_o;m)$ is absolutely convergent. The pointwise value of this series is defined to be*

$$K(x,t;y,t_o;m) = \sum_{n=0}^{\infty} d_n(x,t;y,t_o;m). \tag{5.27}$$

*and the function $K(x,t;y,t_o;m)$ has the pointwise estimate*

$$|K(x,t;y,t_o;m)| \le s^{1/2}\frac{1}{1-\theta}\, g(x-y;t,t_o,m). \tag{5.28}$$

(5) *If $(x,y,m) \in \boldsymbol{\mathcal{L}}$, $m_-$ is the smallest value of $|m|$ in $\boldsymbol{\mathcal{L}}$ and $\theta_-$, defined in (5.24), is less than 1, then $K(\cdot,t;\cdot,t_o;\cdot)$ is jointly continuous on $\boldsymbol{\mathcal{L}}$.*

(6) *If $m \in \mathbb{C}_>$ then for each $x \in \mathbb{R}^d$ the series over $n$ of $d_n(x,t;\cdot,t_o;m)$ converges in the $L^p(\mathbb{R}^d;\mathbb{C}^{s\times s})$ topology to $K(x,t;\cdot,t_o;m)$ for $1 \le p \le \infty$. The corresponding statement holds for each $y \in \mathbb{R}^d$, $d_n(\cdot,t;y,t_o;m)$ and $K(\cdot,t;y,t_o;m)$. The $L^p$ norms of these functions satisfy*

$$\|K(x,t;\cdot,t_o;m)\|_p \le s^{1/2}\frac{1}{1-\theta}\|g(\cdot;t,t_o,m)\|_p$$
$$\|K(\cdot,t;y,t_o;m)\|_p \le s^{1/2}\frac{1}{1-\theta}\|g(\cdot;t,t_o,m)\|_p \tag{5.29}$$

*Proof:* For $n = 0$ the first result is trivial. It is only for the $n = 0$ case that the $s^{1/2}$ factor appears and this is due to the unit matrix in the equation for $d_o(x,t;y,t_o;m)$. As $s^{1/2} \ge 1$, there is no harm done in adjoining this factor to all of the estimates. However, we note that if we estimate $d_n(x,t;y,t_o;m)$ acting on any spinor (a $\mathbb{C}^s$-valued vector), then this factor can be made to disappear because the unit matrix structure in the $0^{th}$ order term disappears into the spinor structure.

For $n \geq 1$, we utilize the representation (5.20) for the kernel $d_n(x,t;y,t_o;m)$. To estimate the summand in (5.20), we recall equations (5.2), (5.19) and (5.21). From these it is easily shown that

$$
\begin{aligned}
|d_n(x,t;y,t_o;m)| \leq s^{1/2} &\left(\frac{|m|}{2\pi\hbar(t-t_o)}\right)^{d/2} \left(\frac{t-t_o}{\hbar}\right)^n \frac{1}{n!} e^{-\frac{\operatorname{Im} m}{2\hbar(t-t_o)}|x-y|^2} \\
&\times \sum_{r=0}^{n} \binom{n}{r} \gamma_T^r \{\mu_T + \frac{\hbar k}{|m|}\gamma_T n\}^{n-r} \\
&\times \sum_{l=0}^{[r/2]} \frac{r!}{2^l (r-2l)!l!} Z_n^{r-2l} \frac{1}{(t-t_o)^{r-l}} \left(\frac{\hbar}{|m|}\right)^l
\end{aligned} \tag{5.30}
$$

To arrive at (5.30), we have used $\sum_{J_{n,r}} = \binom{n}{r}$ and $\sum'_{r,l} = \frac{r!}{2^l(r-2l)!l!}$.

We examine the sum $\sum_{l=0}^{[r/2]}$ on the right hand side of (5.30);

$$
\begin{aligned}
\sum_{l=0}^{[r/2]} \frac{r!}{(r-2l)!l!} \left(\frac{Z_n}{t-t_o}\right)^{r-l} \left(\frac{\hbar}{2|m|Z_n}\right)^l &\leq \sum_{l=0}^{[r/2]} \frac{r!n^l}{l!(r-l)!} \left(\frac{Z_n}{t-t_o}\right)^{r-l} \left(\frac{\hbar}{2|m|Z_n}\right)^l \\
&\leq \sum_{l=0}^{r} \binom{r}{l} \left(\frac{Z_n}{t-t_o}\right)^{r-l} \left(\frac{n\hbar}{2|m|Z_n}\right)^l \\
&= \left[\frac{Z_n}{t-t_o} + \frac{n\hbar}{2|m|Z_n}\right]^r .
\end{aligned}
$$

Using this to further our estimate in (5.30) we get

$$
\begin{aligned}
|d_n(x,t;y,t_o;m)| &\le s^{1/2}\left(\frac{|m|}{2\pi\hbar(t-t_o)}\right)^{d/2}\left(\frac{t-t_o}{\hbar}\right)^n \frac{1}{n!}e^{-\frac{\operatorname{Im} m}{2\hbar(t-t_o)}|x-y|^2} \\
&\quad\times\sum_{r=0}^{n}\binom{n}{r}\{\mu_T+\frac{\hbar k}{|m|}\gamma_T n\}^{n-r}\left[\frac{Z_n\gamma_T}{t-t_o}+\frac{n\hbar\gamma_T}{2|m|Z_n}\right]^r. \\
&\le s^{1/2}\left(\frac{|m|}{2\pi\hbar(t-t_o)}\right)^{d/2}\left(\frac{t-t_o}{\hbar}\right)^n \frac{1}{n!}e^{-\frac{\operatorname{Im} m}{2\hbar(t-t_o)}|x-y|^2} \\
&\quad\times\left\{\mu_T+\frac{\hbar k}{|m|}\gamma_T n+\frac{Z_n\gamma_T}{t-t_o}+\frac{n\hbar\gamma_T}{2|m|Z_n}\right\}^n \\
&\le s^{1/2}\left(\frac{|m|}{2\pi\hbar(t-t_o)}\right)^{d/2}\frac{n^n}{n!}\left(\frac{2k(t-to)\gamma_T}{|m|}\right)^n \\
&\quad\times\left\{1+\left[\frac{c_1}{2\hbar}|x-y|+\frac{c_2}{2\hbar}\right]\frac{1}{n}\right\}^n e^{-\frac{1}{2\hbar}c_o|x-y|^2} \\
&\le s^{1/2}\theta^n g(x-y;t,t_o,m).
\end{aligned}
$$

In the last inequality we have used that $\frac{n^n}{n!}\le e$ and $(1+\frac{\xi}{n})^n\le e^{\xi}$ for all $\xi\ge 0$. This establishes (5.25).

The joint continuity of the Dyson kernels with respect to $(x,y,m)\in\boldsymbol{\mathcal{L}}$ follows easily from the formula (5.20). If we explicitly expand out the measures to their lowest possible representation in terms of $\gamma(\tau)$ and $\nu(\tau,m)$, we readily find that the resulting integrand is a jointly continuous function of $(x,y,m)$. (Recall the measure $\nu(\tau,m)$ is continuous in $m$ in the norm defined on $\mathcal{M}(S_k,\mathbb{C}^{s\times s})$). Furthermore the integrand is a uniformly bounded function of all its arguments. This will suffice to allow us to apply the dominated convergence theorem to conclude the kernels are jointly continuous.

If $\operatorname{Im} m>0$, the function $g(\cdot;t,t_o,m)\in L^1\cap L^\infty(\mathbb{R}^d)$. The $L^p(dx;\mathbb{R}^d;\mathbb{C}^{s\times s})$ and $L^p(dy;\mathbb{R}^d;\mathbb{C}^{s\times s})$ norms of the kernels follows immediately from this and equation (5.26).

To demonstrate the pointwise convergence of the sum of the Dyson kernels, we again employ equation (5.26). It is enough to show that the series forms a Cauchy sequence in the space $\mathbb{C}^{s\times s}$:

$$\begin{aligned}\left|\sum_{n=N_1}^{N_2} d_n(x,t;y,t_o;m)\right| &\leq \sum_{n=N_1}^{N_2} |d_n(x,t;y,t_o;m)| \\ &\leq s^{1/2} g(x-y;t,t_o,m) \sum_{n=N_1}^{N_2} \theta^n \\ &\rightarrow 0 \qquad \text{as } N_1, N_2 \rightarrow \infty.\end{aligned}$$

This line of argument also shows the estimate of $K(x,t;y,t_o;m)$ to be

$$|K(x,t;y,t_o;m)| \leq s^{1/2} g(x-y;t,t_o,m) \sum_{n=0}^{\infty} \theta^n = \frac{s^{1/2}}{1-\theta} g(x-y;t,t_o,m).$$

Next we prove the joint continuity of $K(x,t;y,t_o;m)$ on $\boldsymbol{\mathcal{L}}$. Because each $d_n(x,t;y,t_o;m)$ is jointly continuous on $\boldsymbol{\mathcal{L}}$, from a standard theorem in analysis (cf. [Ru 76], theorem 7.12), it is enough to show the partial sums of $d_n(x,t;y,t_o;m)$ converge uniformly to $K(x,t;y,t_o;m)$. Let $m_+$ and $m_-$ denote the upper and lower bounds of $|m|$ and let $|x_+ - y_+|$ denote the upper bound of $|x-y|$, for $(x,y,m) \in \boldsymbol{\mathcal{L}}$. Then

$$\begin{aligned}&\left|K(x,t;y,t_o;m) - \sum_{n=0}^{N} d_n(x,t;y,t_o;m)\right| \\ &\qquad \leq s^{1/2} \left(\frac{m_+}{2\pi\hbar(t-t_o)}\right)^{d/2} e^{c_1'|x_+-y_+|+c_2'} \sum_{n=N+1}^{\infty} \theta_-^n \\ &\qquad \rightarrow 0 \qquad \text{as } N \rightarrow \infty.\end{aligned}$$

Here, the constants $c_j'$ are the same as in (5.23), but with $m$ replaced by $m_+$. Notice also that the sum converges because it is assumed $\theta_- < 1$. The right hand side converges to zero independently of $(x, y, m)$ and hence the partial sums of $d_n(x, t; y, t_o; m)$ converge $K(x, t; y, t_o; m)$ uniformly with respect to $(x, y, m) \in \boldsymbol{\mathcal{L}}$.

Finally, if $m \in \mathbb{C}_>$, then by the triangle inequality,

$$\begin{aligned} \|K(\cdot, t; y, t_o; m)\|_p &\leq \sum_{n=0}^{\infty} \|d_n(\cdot, t; y, t_o; m)\|_p \\ &\leq s^{1/2} \|g(\cdot; t, t_o, m)\|_p \sum_{n=0}^{\infty} \theta^n \\ &= \frac{s^{1/2}}{1-\theta} \|g(\cdot; t, t_o, m)\|_p. \end{aligned}$$

The corresponding argument holds for $\|K(x, t; \cdot, t_o; m)\|_p$. ◇

### 5.3 The Propagator

If $\operatorname{Im} m > 0$, the function $K(x, t; y, t_o; m)$ can be used as a kernel to define a bounded integral operator acting on $\mathcal{H}$. We can also use the kernel $d_n(x, t; y, t_o; m)$ to extend the domain of the $n^{th}$ Dyson operator from $\mathcal{S}$ to all of $\mathcal{H}$.

**Definition 5.1:** Let $m \in \mathbb{C}_>$ and $0 < \theta < 1$. For each $\psi \in \mathcal{H}$, define (pointwise) the functions

$$[\bar{D}_n(t, t_o; m)\psi](x) = \int d_n(x, t; y, t_o; m)\psi(y)\, dy \tag{5.31}$$

and

$$[K(t, t_o; m)\psi](x) = \int K(x, t; y, t_o; m)\psi(y)\, dy. \tag{5.32}$$

◇

The integrals in (5.31) and (5.32) are well defined for each $x$ because both $d_n(x, t; \cdot, t_o; m)$ and $K(x, t; \cdot, t_o; m)$ are $L^2(\mathbb{R}^d; \mathbb{C}^{s\times s})$ functions, so that the integrands will be $L^1(\mathbb{R}^d; \mathbb{C}^s)$ functions. The next lemma shows that these functions define bounded operators which we denote by $\bar{D}_n(t, t_o; m)$ and $K(t, t_o; m)$ respectively. We shall also prove some of their important properties.

**Lemma 5.6:** *Let* $m \in \mathbb{C}_>$ *and* $0 < \theta < 1$.

(1) $\bar{D}_n(t, t_o; m)$ *and* $K(t, t_o; m)$, *as defined by definition 5.1, are bounded linear mappings of* $\mathcal{H}$ *into* $\mathcal{H}$. *They satisfy the operator norm estimates*

$$\|\bar{D}_n(t, t_o; m)\| \leq \theta^n \|g(\cdot; t, t_o; m)\|_1 \tag{5.33}$$

*and*

$$\|K(t, t_o; m)\| \leq \frac{1}{1-\theta} \|g(\cdot; t, t_o; m)\|_1 . \tag{5.34}$$

(2) *The partial sums* $\sum_{n=0}^{N} \bar{D}_n(t, t_o; m)$ *converge in the operator norm topology to* $K(t, t_o; m)$.

(3) *If* $\{U(t, t_o; m)\}_{(t_o,t) \in T_\Delta}$ *is the complex mass Schrödinger evolution, then for all* $(t_o, t) \in T_\Delta$

$$K(t, t_o; m) = U(t, t_o; m). \tag{5.35}$$

*Proof:* As we have previously commented on, any unit matrices appearing in $d_n(x, t; y, t_o; m)$ and $K(x, t; y, t_o; m)$ will be absorbed into the spinor structure of $\psi$, so that the factor $s^{1/2}$ will not be present.

From lemma 5.5 we obviously have the convolution bound

$$\begin{aligned} \left|[\bar{D}_n(t, t_o; m)\psi](x)\right| &\leq \theta^n \int g(x - y; t, t_o, m) |\psi(y)|\, dy \\ &\equiv \theta^n [g(\cdot; t, t_o, m) * |\psi|](x). \end{aligned}$$

Applying the Hausdorff-Young estimate for convolutions ([HS 78], theorem 12.2) we have immediately,

$$\|\bar{D}_n(t,t_o;m)\psi\| \le \theta^n \|g(\cdot;t,t_o,m)\|_1 \Big\| |\psi| \Big\|_{L^2(\mathbb{R}^d)} = \theta^n \|g(\cdot;t,t_o,m)\|_1 \|\psi\|.$$

This demonstrates (1) for $\bar{D}_n(t,t_o;m)$. Similar arguments will apply for $K(t,t_o;m)$.

From (5.33), it is obvious that the partial sum $\sum_{n=0}^{N} \bar{D}_n(t,t_o;m)$ forms a Cauchy sequence in $\mathcal{B}$ and hence it converges to some bounded operator. Consider the difference between this sum and $K(t,t_o;m)$ acting on the function $\psi$;

$$\begin{aligned}
\Big\|\Big(\sum_{n=0}^{N} \bar{D}_n(t,t_o;m) - K(t,t_o;m)\Big)\psi\Big\|^2 &= \int \Big| \int \sum_{n=N+1}^{\infty} d_n(x,t;y,t_o;m)\psi(y)\,dy \Big|^2 \,dx \\
&\le \Big(\sum_{n=N+1}^{\infty} \theta^n\Big)^2 \|g(\cdot;t,t_o,m) * |\psi|\,\|^2 \\
&\le \left(\frac{\theta^{N+1}}{1-\theta}\right)^2 \|g(\cdot;t,t_o,m)\|_1^2 \|\psi\|^2.
\end{aligned}$$

This implies

$$\Big\|\sum_{n=0}^{N} \bar{D}_n(t,t_o;m) - K(t,t_o;m)\Big\| \le \frac{\theta^{N+1}}{1-\theta}\|g(\cdot;t,t_o,m)\|_1 \to 0 \qquad \text{as } N \to \infty.$$

To show $K(t,t_o;m)$ is the complex mass Schrödinger evolution, we first consider a smaller class of test functions than all of $L^2(\mathbb{R}^d;\mathbb{C}^s)$. Assume $\hat{\psi} \in C_o^\infty(\mathbb{R}^d,\mathbb{C}^s)$. From (4.30) and proposition 4.1, the strong limit $\psi(t,t_o;m) = \sum_{n=0}^{\infty} D_n(t,t_o;m)\psi$ is a solution of the complex mass valued Schrödinger equation. Note too that

$\bar{D}_n(t,t_o;m)$ is the extension of $D_n(t,t_o;m)$. The abstract evolution theory of chapter 2 showed the solution to this equation to be unique and hence

$$U(t,t_o;m)\psi = \sum_{n=0}^{\infty} D_n(t,t_o;m)\psi = \sum_{n=0}^{\infty} \bar{D}_n(t,t_o;m)\psi = K(t,t_o;m)\psi.$$

This demonstrates the two bounded operators $U(t,t_o;m)$ and $K(t,t_o;m)$ coincide on the dense set $C_o^\infty(\mathbb{R}^d,\mathbb{C}^s)$. This is possible only if the two operators are equal. $\diamondsuit$

It remains to be shown that lemma 5.6 remains valid in some sense in the limit $\operatorname{Im} m \to 0$. Let $L_o^p(\mathbb{R}^d,\mathbb{C}^s)$ denote the space of compactly supported $L^p(\mathbb{R}^d,\mathbb{C}^s)$ functions.

**Theorem 5.1:** *Let $0<\theta<1$, $m\in\mathbb{C}_+$ and $\psi\in L_o^2(\mathbb{R}^d,\mathbb{C}^s)$. Then for almost all $x\in\mathbb{R}^d$,*

$$[U(t,t_o;m)\psi](x) = \int K(x,t;y,t_o;m)\psi(y)\,dy, \tag{5.36}$$

*where $\{U(t,t_o;m)\}_{(t,t_o)\in T_\Delta}$ is the Schrödinger evolution and $K(x,t;y,t_o;m)$ is as in lemma 5.5.*

*Proof:* As $\psi\in L_o^2(\mathbb{R}^d,\mathbb{C}^s)$, it is also in $L_o^1(\mathbb{R}^d,\mathbb{C}^s)$. Fix $x\in\mathbb{R}^d$ and let the mass $m'$ be in a closed neighbourhood of $m$ whose diameter is less than $\varepsilon$. Then $(x,y,m')$ will be in a compact subset of $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{C}_+$ and for $\varepsilon$ sufficiently small, $\theta' = 2|m'|^{-1}ek\gamma_T(t-t_o) < 1$. Hence $K(\cdot,t;\cdot,t_o;\cdot)$ is a continuous function of $(x,y,m')$. and consequently we can consider the $m'\to m$ ($\varepsilon\to 0$) limit of

$$[U(t,t_o;m')\psi](x) = [K(t,t_o;m')\psi](x) = \int K(x,t;y,t_o;m')\psi(y)\,dy.$$

As $K(x,t;y,t_o;m)$ is jointly continuous on a compact set, its modulo will attain its maximum and hence there exists a constant $C$, depending on $t,t_o$ and $\psi$, but independent of $m'$ such that

$$|K(x,t;y,t_o;m')\psi(y)| \leq C|\psi(y)| \in L^1(\mathbb{R}^d, dy).$$

This shows we can apply the dominated convergence theorem to conclude

$$\lim_{m'\to m} [U(t,t_o;m')\psi](x) = \int K(x,t;y,t_o;m)\psi(y)\,dy.$$

On the other hand, we know from proposition 2.2 that $U(t,t_o;m')$ is strongly continuous in $m'$ on $\mathbb{C}_+$. Set $m' = m + in^{-1}$. Then there exists a subsequence (reference [HS 78], lemma 3.9) $\{n_j\}$, such that for almost all $x$,

$$\lim_{n_j\to\infty} [U(t,t_o;m+\frac{i}{n_j})\psi](x) = [U(t,t_o;m)\psi](x).$$

These two limits combined together yield proof of identity (5.36). $\diamond$

There are two comments that come to mind immediately on examination of theorem 5.1. The first is that the class of functions used in (5.36) can be extended to all of $L^2(\mathbb{R}^d;\mathbb{C}^s)$. For an $L^2(\mathbb{R}^d;\mathbb{C}^s)$ function, the integral representation becomes

$$[U(t,t_o;m)\psi](x) = \ell.i.m. \int K(x,t;y,t_o;m)\psi(y)\,dy.$$

The second observation is that the theory leading up to theorem 5.1 appears to be limited to small times only, in that the parameter $\theta < 1$ (recall $\theta$ is proportional to $t - t_o$). In the case of a complex mass parameter ($\mathrm{Im}\, m > 0$), it is possible to define an extended kernel, also denoted $K(x,t;y,t_o;m)$, such that our theory remains valid for times allowed in the operator valued evolution problem, i.e. all $(t_o,t) \in T_\Delta$.

**Corollary 5.1:** *Assume $m \in \mathbb{C}_>$ and that $(t,t_o) \in T_\Delta$.*

(1) *The complex mass evolution operator* $U(t,t_o;m)$ *is an integral operator with a jointly continuous kernel* $K(\cdot,t;\cdot,t_o;m)$. *For a fixed* $x$ *or* $y$, $K(\cdot,t;y,t_o;m)$ *and* $K(x,t;\cdot,t_o;m)$ *are both in* $L^1\cap L^\infty(\mathbb{R}^d;\mathbb{C}^{s\times s})$.

(2) *The kernel* $K$ *is Gaussian bounded. That is there exist finite positive constants* $C_o$ *and* $C_1$, *depending on* $m$, $t$, *and* $t_o$ *such that*

$$|K(x,t;y,t_o;m)|\le C_o e^{-C_1(x-y)^2}\,. \tag{5.37}$$

(3) $K(x,t;y,t_o;m)$ *obeys the composition rule*

$$K(x,t;y,t_o;m)=\int K(x,t;x',\tau;m)K(x',\tau;y,t_o;m)\,dx',\qquad 0\le t_o\le\tau\le t\le T. \tag{5.38}$$

*Proof:* We show these results by induction. First we demonstrate the corollary for $\theta<2$. Suppose that $t-\tau$ and $\tau-t_o$ are such that their corresponding $\theta$'s are less than 1. Then the representation (5.36) is valid and similar equations exist for $U(t,\tau;m)$ and $U(\tau,t_o;m)$. Now $U(t,t_o;m)=U(t,\tau;m)U(\tau,t_o;m)$ (cf. (2.14)) and from this we see

$$\begin{aligned}[U(t,t_o;m)\psi](x)&=\int K(x,t;x',\tau;m)[U(\tau,t_o;m)\psi](x')\,dx'\\&=\int K(x,t;x',\tau;m)\left\{\int K(x',\tau;y,t_o;m)\psi(y)\,dy\right\}\,dx'.\end{aligned}$$

Equation (5.28) serves to show that the integrand is absolutely integrable over $dy\,dx'$. Applying the Fubini theorem allows the interchange of the integrals and this shows $U(t,t_o;m)$ is an integral operator for times $t-t_o$ such that $\theta<2$. We also see the kernel satisfies the relation (5.38). Moreover this integral relation is easily shown to be independent of the choice of $\tau$.

For $\theta < 1$ and $K$ defined in lemma 5.5, we can further estimate $K(x,t;y,t_o;m)$ in inequality (5.28). It is easy to show there exist constants $C'$ and $C''$ such that

$$|K(x,t;y,t_o;m)| \le C' e^{-C''(x-y)^2}, \qquad \theta < 1.$$

Using this and the free heat kernel compostition identity

$$\frac{1}{[4\pi(\beta_1+\beta_2)]^{d/2}} e^{-\frac{(x-y)^2}{4(\beta_1+\beta_2)}} = \frac{1}{[(4\pi\beta_1)(4\pi\beta_2)]^{d/2}} \int e^{-\frac{(x-x')^2}{4\beta_1}} e^{-\frac{(x'-y)^2}{4\beta_2}} dx',$$

(5.37) is easily shown for $\theta < 2$.

The continuity of the function $K$ is easily seen because of the joint continuity of the kernels appearing in the integrand in (5.38) and an application of the dominated convergence theorem.

Finally the $L^p(\mathbb{R}^d; \mathbb{C}^{s\times s})$ nature of the kernel is easily seen for the Guassian bound (5.37). This completes the proof of our statements for times $t - t_o$ such that $\theta < 2$.

As the extended kernel $K$ has the same properties as the kernel defined in lemma 5.5, we see this argument may be repeated as often as we desire and so we can extend our results to the full time domain of $[0,T]$. $\diamondsuit$

For future studies, we note it is possible to factor out the free evolution kernel out of each of the Dyson kernels. The free evolution kernel is defined by

$$K_o(x,t;y,t_o;m) = \left[\frac{m}{2\pi i\hbar(t-t_o)}\right]^{d/2} e^{\frac{im}{2\hbar(t-t_o)}(x-y)^2} I. \tag{5.39}$$

By examining equation (5.20), we see there exists a function, $\tilde{d}_n(x,t;y,t_o;m^{-1})$, such that

$$d_n(x,t;y,t_o;m) = K_o(x,t;y,t_o;m)\tilde{d}_n(x,t;y,t_o;m^{-1}). \tag{5.40}$$

Moreover these functions are absolutely summable, and we denote that sum by $F(x,t;y,t_o;m^{-1})$. It is easily seen that $K(x,t;y,t_o;m)$ admits the factorization

$$K(x,t;y,t_o;m) = K_o(x,t;y,t_o;m)F(x,t;y,t_o;m^{-1}). \tag{5.41}$$

These identities are important because they allow a study of the analytic structure of the propagator in the mass variable. On inspection of (5.39), we see there is an essential singularity in $m$ as $m \to 0$. By making the the factorizations (5.40) and (5.41), it should be possible to show that the functions $\tilde{d}_n$ and $F$ are smooth in the mass variable. A recent study of this problem for the Abelian case has been done by Papiez et. al. [POM t.a.]. They have found that the function $F$ admits an asymptotic expansion in $m^{-1}$ fo the form

$$F(x,t;y,t_o;m^{-1}) \sim e^{\frac{1}{i\hbar}J(x,t;y,t_o)}\{1 + m^{-1}T_1(x,t;y,t_o) + m^{-2}T_2(x,t;y,t_o) + \cdots\} \tag{5.42}$$

The exponentiated factor $J(x,t;y,t_o)$ carries all the gauge dependence for the problem and the coefficients $T_j(x,t;y,t_o)$ are explicitly gauge invariant. For the case of atomic physics, (ie. both the vector potential $a$ and the scalar potential $\phi$ are scalar functions times the unit $s \times s$ matrix) we should expect an expansion similar to (5.42) because of the Abelian nature of the fields $a$ and $\phi$. There still would be differences between $J$ and the coefficients $T_j$ we would obtain and those found by Papiez et. al. [POM t.a.] because of the non-Abelian nature of $v$. However, we would expect that our $J$ should carry all the gauge dependence and the $T_j$'s would be manifestly gauge invariant.

There is yet another closely related set of results available. If one considers the Bloch equation (also called the heat equation) instead of the Schrödinger equation a similar series expansion of the corresponding heat propagator exists. The Bloch equation is

$$-\frac{\partial}{\partial\beta}\psi(\beta) = H\psi(\beta), \qquad \beta > 0$$

and the solution satisfies an appropiate Cauchy data problem;

$$\psi(0) = \psi_o, \qquad \psi_o \in D(H).$$

The connection between the Schrödinger and Bloch equations is the following: We first analytically extend the Bloch equation onto the right half complex $\beta$ plane ($\mathrm{Re}\,\beta > 0$). Then using the continuity properties of this equation and its solution with respect to the complex $\beta$ parameter, we extend it to the imaginary axis boundary, whereby it becomes the Schrödinger equation. Provided the potentials are now explicitly time independent, this connection is made clear from the variable change

$$\frac{i}{\hbar}t \leftrightarrow \beta.$$

The connections between these two equations for the Abelian case having only scalar interactions has been studied in detail by Osborn and Fujiwara [OF 83]. For the special case of the Bloch equation subject to external electromagnetic fields, the heat propagator has been studied under the context of the WKB approximation and the Wigner-Kirkwood expansion [TLR 83] [BR 84] [BR 85] [BR 86] [Z 86]. The work of Zuk [Z 86] is of particular interest to us because he examines a system subject to non-Abelian potentials in the context of a Wigner-Kirkwood expansion.

The WKB approximation is an expansion of the heat propagator in the limit $\hbar \to 0$ of the form

$$K(x,y) = K_o(x,y)e^{S(x,y)}$$

where

$$S(x,y) = \sum_{n=-1}^{\infty} \hbar^n S_n(x,y).$$

This may be rearranged [BR 84] to yield

$$K(x,y) = K_o(x,y) e^{\frac{1}{\hbar} S_{-1}(x,y) + S_o(x,y)} \{1 + \hbar S_1(x,y) + O(\hbar^2)\}. \tag{5.43}$$

The Wigner-Kirkwood expansion is a large mass expansion of the heat propagator, similar to the expansion given by (5.41) and (5.42). The connections between these two expansions have been explored by Osborn and Molzahn [OM 86]. We further note that both of these expansions are non-perturbative — in the sense that one always keeps some terms involving all powers of the potential $v$.

# CHAPTER 6

## Transit Time Operators in Spinor Space

This chapter deals with the problems of describing the transit times (and their associated operators) of a scattering state $\psi \in L^2(\mathbb{R}^3; \mathbb{C}^s)$ through a finite region $\Sigma \subset \mathbb{R}^3$. The difference between these times for the free system and the system subject to an interaction is the time delay for that region.

We shall also study the Born series expansion of the resolvent difference. Specifically we are interested in the analytic structure of the trace of the resolvent difference.

### 6.1 The Hamiltonian

Our ambient space and Hamiltonians have changed somewhat from part one. We shall reduce the dimensionality of the underlying coordinate space from $\mathbb{R}^d$ to $\mathbb{R}^3$, so that the working Hilbert space is now $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^s)$. We shall also extensively use the Hilbert space of Schmidt class operators, $\mathcal{B}_2(\mathcal{H})$, and the Banach space of trace class operators, $\mathcal{B}_1(\mathcal{H})$.

The Hamiltonian pair, $(H, H_o)$, we will study is of the form

$$H = H_o + V$$

where the free Hamiltonian is the extension of the negative Laplacian;

$$H_o = -\Delta I.$$

We have set $\hbar = 2m = 1$, as they are not of interest here. The quantity $I$ above denotes the unit $s \times s$ matrix.

The potential $V$ is defined by multiplication with the hermitian valued matrix $v(\cdot)$. Our assumptions on $v$ for many of the proofs can be quite weak. However there are a few proofs in which much stronger restrictions on $v$ were imposed to verify the validity of the statements made in the lemma, proposition or theorem. It is in our best interest to keep the assumptions made on the potential in any given claim local to that claim. The two main classes under consideration will be $L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ and $L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$. Recall $v \in \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$ means that $v$ is the Fourier transform of a complex matrix valued measure and that $v(x)$ is a hermitian matrix for a.a. $x$. Also note that $L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s}) \subset L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$.

An important difference between the Hamiltonians in part 1 and the Hamiltonians here is the absence of an explicit time dependence. This results in a markedly simpler representation of the evolution family;

$$U(t, t_o) = e^{-i(t-t_o)H}; \qquad U_o(t, t_o) = e^{-i(t-t_o)H_o}.$$

If $H$ and $H_o$ were bounded, then the exponential operators could be interpreted as their appropiate Taylor series expansion, which would converge in the operator norm topology. However, $H$ and $H_o$ are not bounded. One can make sense of the exponential operator $e^{-itH}$ as the strong limit

$$e^{-itH} = \mathrm{s} - \lim_{n\to\infty} \left( I + \frac{it}{n} H \right)^{-n}$$

and a similar statement holds for $e^{-itH_o}$. For a complete dicussion of the evolution operator defined in this manner, we refer to Kato [Ka 84], chapter IX. Since $H$ and $H_o$ are also self-adjoint, one can also make sense of these exponential operators by using the spectral theorem. For a general discussion of the spectral representation of self-adjoint operators, we refer to Riesz and Sz.Nagy [RSz 78], chapters IX and X. Without loss of generality we may set $t_o = 0$ because the total and free evolution operators depend upon the single parameter $t - t_o$.

We shall also make the small shifts in notation by denoting the total and free evolution families by $U_t$ and $U_t^o$ and the total and free resolvents by $R(z)$ and $R_o(z)$.

The following factorization scheme will be used extensively. For each $x \in \mathbb{R}^3$, $v(x)$ is a hermitian matrix. The absolute value of $v(x)$, denoted $[v(x)]$, is a positive matrix that is the square root of the matrix $v(x)^* v(x)$. There exists a unitary matrix, denoted $W(x)$, such that

$$\begin{aligned} v(x) &= W(x)[v(x)] = W(x)^*[v(x)^*] \\ [v(x)] &= W(x)^* v(x). \end{aligned} \tag{6.1}$$

Defining the two matrix valued functions

$$\begin{aligned} u(x) &= [v(x)]^{1/2} \\ w(x) &= W(x)u(x), \end{aligned} \tag{6.2}$$

$v(x)$ has the representation $v(x) = w(x)u(x)$. With $|\cdot|$ denoting the Euclidean norm, we note that these functions satisfy the estimates

$$|u(x)| \leq |v(x)|^{1/2}; \qquad |w(x)| \leq |v(x)|^{1/2}.$$

Let $\Pi$ denote the cut plane $\mathbb{C}\backslash[0,\infty)$. If $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$, we can define an important integral operator $A(z)$ $(z \in \Pi)$ via its kernel

$$A(x,y;z) = u(x)R_o(x,y;z)w(y). \tag{6.3}$$

Here $R_o(x, y; z)$ is the kernel of the free resolvent and has the well known formula

$$R_o(x, y; z) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}I,$$

where $I$ is the unit $s \times s$ matrix. The branch of the square root taken satisfies $0 < \arg z < 2\pi$.

Control over the integrability properties of $A(x, y; z)$ is given by a class of inequalties known as Sobolev inequalities (cf. reference [Si 71], p. 9). In brief they are the following: consider a function $f \in L^p(\mathbb{R}^n)$ and a function $h \in L^r(\mathbb{R}^n)$ and suppose $\lambda < n$ is such that

$$\frac{1}{p} + \frac{1}{r} + \frac{\lambda}{n} = 2.$$

Then there exists a constant $C$, depending on $p$, $r$, $\lambda$, and $n$ such that

$$\int \frac{|f(x)||h(y)|}{|x-y|^\lambda} \, d^n x \, d^n y < C\|f\|_p\|h\|_r. \tag{6.4}$$

The Sobolev inequality (with $n = 3$) will suffice to show that $A(x, y; z)$ is an $L^2(dx\,dy; \mathbb{C}^{s\times s})$ kernel for all pertinent values of $z$. These include the boundaries $z = \lambda \pm i0$, $\lambda \geq 0$. Moreover, because $e^{i\sqrt{z}|x-y|}$ is a continuous function of $z$ and it is uniformly bounded by 1, we have that $A(\cdot, \cdot; z)$ is continuous in $z$ with respect to the $L^2(dx\,dy; \mathbb{C}^{s\times s})$ norm, for all $z \in \Pi \cup [0, \infty)$. From the equality

$$\|A(z)\|_{\mathcal{B}_2} = \|A(\cdot, \cdot; z)\|_{L^2(dx\,dy; \mathbb{C}^{s\times s})}$$

we see $A(z)$ is a Schmidt class operator for all $z \in \Pi \equiv \rho(H_o)$ and it has a continuous extention onto the boundary of $\Pi$ with respect to the $\mathcal{B}_2$ topology. There will be two extensions of $A(z)$ onto the positive real axis; one from below the axis and one from above. Let $\Pi_c$ denote the closure of $\Pi$ that maintains the distinction between these extensions of $A(z)$.

**Proposition 6.1:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian. Then the operator $A(z)$ is $\mathcal{B}_2$ holomorphic in $\Pi$ and has a $\mathcal{B}_2$ continuous extension in $\Pi_c$. If $z \to \lambda \pm i0$, then this extension is uniform in $\lambda$, for $\lambda$ in a compact subset of $\mathbb{R}\backslash\{0\}$. Furthermore, $\|A(z)\|$, $\|A(z)^2\|_{\mathcal{B}_2}$ and $\|A(z)^*A(z)\|_{\mathcal{B}_2}$ all tend to zero as $|z| \to \infty$.*

*Proof:* Theorem I.22 in reference [Si 71] shows $A(z)$ is $\mathcal{B}_2$ holomorphic on $\Pi$. The uniformity of the continuity with respect to $\lambda$ on a compact set $\mathcal{K}$ follows easily from the joint $(\lambda, \eta)$ continuity of $A(\lambda + i\eta)$ on the compact set $\mathcal{K} \times [0, 1]$.

Finally we prove the last statement. We first note the inequality

$$\|A(z)\|^4 \leq \|A(z)^*A(z)\|^2_{\mathcal{B}_2}.$$

Thus we need only concern ourselves with $A(z)^2$ and $A(z)^*A(z)$. The proofs for these two operators are virtually identical, so we only demonstrate it for $A(z)^2$. If sp denotes the trace in $\mathbb{C}^{s\times s}$ (i.e. the sum over the diagonal elements), then

$$\|A(z)^2\|_{\mathcal{B}_2} = \frac{1}{(4\pi)^4} \int dx\, dy\, dx'\, dy' \, \frac{\mathrm{sp}\, w(y)^* v(x')[v(x)]v(y')w(y)}{|x-x'|\,|x'-y|\,|y-y'|\,|y'-x|}$$
$$\times\, e^{-\mathrm{Im}\sqrt{z}\{|x-x'|+|x'-y|+|y-y'|+|y'-x|\}+i\mathrm{Re}\sqrt{z}\{|y-y'|+|y'-x|-|x-x'|-|x'-y|\}}.$$

It is easily shown by using the Sobolev inequality (6.4) that

$$\frac{\mathrm{sp}\, w(y)^* v(x')[v(x)]v(y')w(y)}{|x-x'|\,|x'-y|\,|y-y'|\,|y'-x|} \in L^1(dx\, dy\, dx'\, dy').$$

We also note the exponential in the above is uniformly bounded by 1. Thus we can apply the dominated convergence theorem if $\mathrm{Im}\sqrt{z} \to \infty$ or the Riemann-Lebesgue lemma ([Ru 73], theorem 7.5) if $\mathrm{Re}\sqrt{z} \to \infty$ to conclude $\|A(z)^2\|_{\mathcal{B}_2} \to 0$ as $|z| \to \infty$.

$\diamondsuit$

The implication of $\|A(z)\| \to 0$ as $|z| \to \infty$ is that $[1 + A(z)]^{-1}$ will exist via the Neumann series for all $z$ sufficiently large. This will be important in our study of the Born series. It also has consequences in terms of what can be said about the singular spectrum of $H$. We summarize the important properties of the Hamiltonians in the following.

**Theorem 6.1:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian and define the set*

$$\mathcal{E} = \{0\} \cup \{\lambda \in \mathbb{R}\backslash\{0\} : \ 1 + A(\lambda + i0) \text{ or } 1 + A(\lambda - i0) \text{ is not injective}\}.$$

*Then*

(1) *The Hamiltonian is self-adjoint with domain $D(H) = D(H_o)$.*

(2) *For every $z \in \rho(H_o) \cap \rho(H)$, $[1 + A(z)]^{-1} \in \mathcal{B}$ and one has the resolvent equation*

$$R(z) - R_o(z) = -R_o(z)w[1 + A(z)]^{-1}uR_o(z). \tag{6.5}$$

*Furthermore the mapping $z \mapsto [1 + A(z)]^{-1}$ is $\mathcal{B}$ holomorphic in $\Pi$.*

(3) *$\mathcal{E}$ is a closed and bounded set of Lebesgue measure zero.*

(4) *The Møller wave operators*

$$\Omega^{\pm} = \mathrm{s} - \lim_{t\to\mp\infty} U_{-t}U_t^o$$
$$\tilde{\Omega}^{\pm} E_{\mathbb{R}\backslash\mathcal{E}} = \mathrm{s} - \lim_{t\to\mp\infty} U_{-t}^o U_t E_{\mathbb{R}\backslash\mathcal{E}}$$

*exist. The scattering system defined by the Hamiltonian pair $(H, H_o)$ is asymptotically complete in the sense that*

$$\text{Range}\,\Omega^+ = \text{Range}\,\Omega^- = \mathcal{H}_{ac}(H) = E_{\mathbb{R}\backslash\mathcal{E}}\mathcal{H}$$
$$\mathcal{H}_s(H) \subseteq E_{\mathcal{E}}\mathcal{H}.$$

*Proof:* The self-adjointness of $H$ and its domain equality are a consequence of the Rellich-Kato theorem (cf. propositions 8.5 and 8.7 of reference [AJS 77]).

Equation (6.5) is shown in reference [Si 71], theorem II.34. The first part of statement (2) is a consequence of the Fredholm alternative: Suppose $\varphi$ is a vector in $\mathcal{H}$ such that $A(z)\varphi = -\varphi$. We show $A(z)\varphi$ and hence $\varphi$ is in the domain of $w$. Pointwise $wA(z)\varphi$ is given by

$$[wA(z)\varphi](x) = \int dy\, w(x)u(x)\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}w(y)\varphi(y).$$

It is enough to prove this function is square integrable in $x$. We have

$$\begin{aligned}\int dx\, |[wA(z)\varphi](x)|^2 &= \int dx \left| \int dy\, w(x)u(x)\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}w(y)\varphi(y)\right|^2 \\ &\le \frac{1}{(4\pi)^2}\int dx\, |v(x)| \left[\left(\int dy \frac{|v(y)|}{|x-y|^2}\right)^{\frac{1}{2}} \left(\int dy'\, |\varphi(y')|^2\right)^{\frac{1}{2}}\right]^2 \\ &= \frac{1}{(4\pi)^2}\|\varphi\|^2 \int dx\, dy\, \frac{|v(x)||v(y)|}{|x-y|^2} \\ &< \infty.\end{aligned}$$

Thus $\varphi \in D(w)$ and we may define $\psi = w\varphi$. Left multiplying $A(z)\varphi = -\varphi$ by $R(z)w$ we obtain $R(z)VR_o(z)\psi = -R(z)\psi$. The left hand side here can be rewritten using the second resolvent equation $R(z) - R_o(z) = -R(z)VR_o(z)$. As a result, we see $R_o(z)\psi = 0$. This in turn implies $\varphi = -uR_o(z)\psi = 0$ and thus by the Fredholm alternative, $[1 + A(z)]^{-1}$ exists and is bounded. The $\boldsymbol{B}$ holomorphy of $[1 + A(z)]^{-1}$ follows from the $\boldsymbol{B}_2$ holomorphy of $A(z)$.

Statement (3) follows with observation that $\|A(\lambda \pm i0)\| \to 0$ as $|\lambda| \to \infty$ and an application of the Fredholm alternative.

Statement (4) is the result of an application of the Kato-Levine theory (cf. [AJS 77], proposition 9.16) $\diamondsuit$

## 6.2 The Trace Theorem and Time Delay

We shall employ the *spectral representation of $\mathcal{H}$ relative to $H_o$* extensively. For a discussion of spectral representations we refer the reader to [AJS 77], chapter 5.7. Let $S^{(2)}$ denote the unit sphere in $\mathbb{R}^3$ and let $\mathcal{H}_o \equiv L^2(S^{(2)})$. Define the Hilbert space $\mathcal{G} = L^2\Big([0,\infty), \mathcal{H}_o\Big)$ and again denote the Fourier transform of $\psi$ by $\hat{\psi}$. Consider writing $\hat{\psi}$ as a function using spherical co-ordinates and use $\lambda$ to denote the modulo square of the wave vector. We can define a unitary transformation $\mathcal{U}$ mapping $\mathcal{H}$ onto $\mathcal{G}$ by

$$(\mathcal{U}\psi)_\lambda(\omega) \equiv \psi_\lambda(\omega) = \frac{\lambda^{1/4}}{2^{1/2}}\hat{\psi}(\lambda^{1/2}\omega), \qquad \omega \in S^{(2)}. \tag{6.6}$$

It is trivial to verify that $H_o$ is unitarily equivalent to multiplication by $\lambda$ under this mapping. The operator $\mathcal{U}$ defines the spectral representation of $\mathcal{H}$ relative to $H_o$. With it, we can associate to each vector $\psi$ in $\mathcal{H}$ a family of vectors $\{\psi_\lambda\}$, where for a.a. $\lambda$, $\psi_\lambda \in L^2(S^{(2)})$. The isometric nature of $\mathcal{U}$ can be seen by the equation

$$\langle \psi, \varphi \rangle = \int_\Lambda \langle \psi_\lambda, \varphi_\lambda \rangle_o \, d\lambda \tag{6.7}$$

where $\Lambda = [0,\infty)$ is the spectrum of $H_o$.

The next theorem, developed by Jauch et. al. [JSM 72], allows one to find a representation for any trace class operator in the spectral representation of $\mathcal{H}$ with respect to $H_o$.

**Theorem 6.2:** *Let $U_t^o$ be a unitary group with an absolutely continuous self-adjoint generator $H_o$, whose spectrum is $\Lambda = [0,\infty)$. Let $\mathcal{D}_o$ denote the dense set $\{\psi \in \mathcal{H} : \operatorname{ess\,sup} \|\psi_\lambda\|_o < \infty\}$, where the* ess sup *is taken over $\Lambda$. Then:*

(1) *For each $T \in \mathcal{B}_1$ the sesquilinear form*

$$B_T(\psi, \varphi) = \int_{-\infty}^{\infty} \langle \psi, U_t^{o*} T U_t^o \varphi \rangle \, dt \tag{6.8}$$

*is finite for all* $\psi, \varphi \in \mathcal{D}_o$. *Furthermore there exist a family of trace class operators* $\{t(\lambda)\}$ *such that*

$$B_T(\psi, \varphi) = \int_\Lambda \langle \psi_\lambda, t(\lambda)\varphi_\lambda \rangle_o \, d\lambda. \tag{6.9}$$

*Suppose* $T$ *has the canonical representation*

$$T = \sum_k \alpha_k \langle \phi_k, \cdot \rangle \tilde{\phi}_k \tag{6.10}$$

*where* $\{\phi_k\}$ *and* $\{\tilde{\phi}_k\}$ *are sets of eigenvectors of* $T^*T$ *and* $TT^*$ *respectively and* $\alpha_k^2$, $(\alpha_k^2 \geq \alpha_{k+1}^2 \geq 0)$, *is the common eigenvalue of* $\phi_k$ *and* $\tilde{\phi}_k$. *If* $\phi_k$ $(\tilde{\phi}_k)$ *is unitarily equivalent to* $\{\phi_{k,\lambda}\}$ $(\{\tilde{\phi}_{k,\lambda}\})$, *then* $t(\lambda)$ *is defined by*

$$t(\lambda) = 2\pi \sum_k \alpha_k \langle \phi_{k,\lambda}, \cdot \rangle_o \tilde{\phi}_{k,\lambda}. \tag{6.11}$$

(2) *The following relations are valid:*

$$\frac{1}{2\pi} \int_\Lambda \operatorname{tr} t(\lambda) \, d\lambda = \operatorname{Tr} T; \tag{6.12}$$

$$\frac{1}{2\pi} \int_\Lambda \|t(\lambda)\|_{\mathcal{B}_1} \, d\lambda \leq \|T\|_{\mathcal{B}_1}, \tag{6.13}$$

*where* Tr *and* tr *are the trace in* $\mathcal{H}$ *and* $\mathcal{H}_o$ *respectively and* $\|\cdot\|_{\mathcal{B}_1}$ *is the appropriate trace norm.*

(3) *Let* $\mathcal{D}_\Theta$ *be a dense set in* $\mathcal{H}$ *defined by*

$$\mathcal{D}_\Theta = \left\{ \psi \in \mathcal{H} : \operatorname*{ess\,sup}_{\lambda \in \Lambda} \Theta(\lambda) \|\psi_\lambda\|_o < \infty \right\}, \tag{6.14}$$

*where* $\Theta$ *is a non-negative function on* $\Lambda$. *Suppose* $B(\psi, \varphi)$ *is a sesquilinear form that is finite for all* $\psi, \varphi \in \mathcal{D}_\Theta$ *and has a diagonal representation,* $\{b(\lambda)\}$, *of operators* *acting on $\mathcal{H}_o$ which are bounded a.e. and satisfy*

$$B(\psi,\varphi) = \int_\Lambda \langle \psi_\lambda, b(\lambda)\varphi_\lambda \rangle_o \, d\lambda. \tag{6.15}$$

*Then if $\{b(\lambda)'\}$ is a second diagonal representation of $B$, we have for a.a. $\lambda \in \operatorname{supp}\Theta$, $b(\lambda) = b(\lambda)'$.*

*Proof:* The proofs of parts (1) and (2) are simple extensions of those ideas presented by Jauch et. al. [JSM 72]. The proof of part (3) is given by MacMillan and Osborn [MO 80], although there is one subtle point they don't address but which warrants a discussion. Their argument proceeds as follows.

If $B$ is represented by both $b(\lambda)$ and $b(\lambda)'$, then for all $\psi, \varphi \in \mathcal{D}_\Theta$ we must have

$$\int_\Lambda \langle \psi_\lambda, [b(\lambda) - b(\lambda)']\varphi_\lambda \rangle_o \, d\lambda = 0.$$

Let $\beta(\lambda)$ be the phase argument of $\langle \psi_\lambda, [b(\lambda) - b(\lambda)']\varphi_\lambda \rangle_o$. If $\psi$ is in $\mathcal{D}_\Theta$, then so is $\tilde{\psi} \simeq \{e^{i\beta(\lambda)}\psi_\lambda\}$. Repeating the above reasoning with $\psi$ replaced by $\tilde{\psi}$ leads to the equation

$$\int_\Lambda \left| \langle \psi_\lambda, [b(\lambda) - b(\lambda)']\varphi_\lambda \rangle_o \right| \, d\lambda = 0.$$

Thus it is clear that

$$\langle \psi_\lambda, [b(\lambda) - b(\lambda)']\varphi_\lambda \rangle_o = 0 \qquad \text{a.a. } \lambda. \tag{6.16}$$

MacMillan and Osborn [MO 80] argue that the functions $\{\psi_\lambda : \psi \in \mathcal{D}_\Theta\}$ is a dense set in $\mathcal{H}_o$ and hence because the bounded operators $b(\lambda)$ and $b(\lambda)'$ coincide on a dense set, they must be equal a.e.

Herein lies the subtlety of the proof. Let $\Delta$ denote the set of measure zero for which (6.16) does not remain valid. In general, $\Delta$ will depend upon the choice of $\psi$ and $\varphi$. If we were to allow these Hilbert space vectors to vary freely, then it is possible that the operators $b(\lambda)$ and $b(\lambda)'$ could be nonequal on a set of measure strictly greater than zero. We can circumvent this difficulty. Let $\{\varepsilon_n\}$ be a total orthonormal set in $\mathcal{H}_o$ and let $\nu(\lambda)$ be a nonzero, positive, measurable function of $\lambda$ such that $\nu \in L^2(\Lambda; d\lambda)$ and $\operatorname{ess\,sup} \Theta(\lambda)\nu(\lambda) < \infty$. Define $e_{n,\lambda} = \nu(\lambda)\varepsilon_n$. Then for each fixed $n$, $\{e_{n,\lambda}\}$ is isomorphic to a vector in $\mathcal{H}$, which we denote by $e_n$. Moreover $e_n \in \mathcal{D}_\Theta$. Thus by (6.16) we have

$$\langle e_{n,\lambda}, [b(\lambda) - b(\lambda)']e_{m,\lambda}\rangle_o = 0$$

and $\nu \neq 0$ implies

$$\langle \varepsilon_n, [b(\lambda) - b(\lambda)']\varepsilon_m\rangle_o = 0. \tag{6.17}$$

Let $\Delta_{n,m}$ denote the set of measure zero where (6.17) is not valid. Take the union over these sets, $\bigcup_{m,n} \Delta_{n,m}$, and denote this set by $\Delta$. We note $\Delta$ will still depend upon the choice of the basis $\{\varepsilon_n\}$. As $\Delta$ is the countable union of sets of measure zero, it too is a set of measure zero. Now (6.17) holds for all $\lambda \notin \Delta$, independent of $m$ and $n$. Using this, the boundedness of the $b$'s, and the totality of $\{\varepsilon_n\}$ we can now conclude that for all $\lambda \notin \Delta$,

$$b(\lambda) = b(\lambda)'. \qquad \diamondsuit$$

**Corollary 6.1:** *If $A$ is any positive trace class operator on $\mathcal{H}$ and $\{a(\lambda)\}$ is its induced representation on $\mathcal{H}_o$, then for a.a. $\lambda$, $a(\lambda)$ is a positive operator.*

*Proof:* The proof is identical to that presented in [MO 80]. $\diamondsuit$

In the next lemma we introduce an explicit manifestation of the spin structure. The chief difference between the ideas presented here and those in [MO 80] is in the treatment of the projection operator used. Let $\Sigma$ be any measurable subset of $\mathbb{R}^3$ with finite Lebesgue measure and let $\Gamma$ denote any subspace of $\mathbb{C}^s$. We will use the projection operator $P(\Sigma,\Gamma) = P(\Sigma)P(\Gamma) = P(\Gamma)P(\Sigma)$. $P(\Sigma)$ is defined by multiplication with the characteristic function $\chi_\Sigma(x)I$ (here $I$ is the unit matrix on $\mathbb{C}^{s\times s}$), while $P(\Gamma)$ is the projection operator into the space of spinors that lie in $\Gamma$. In [MO 80], they only have the projection operator $P(\Sigma)$ because they are considering the spinless ($s = 1$) case.

**Lemma 6.1:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian and let the projection operator $P(\Sigma,\Gamma)$ be defined as above. Then the operators $R_o(z)^* P(\Sigma,\Gamma) R_o(z)$ and $R_o(z)^* \Omega^{\pm *} P(\Sigma,\Gamma)\Omega^{\pm} R_o(z)$ are trace class for all $z \in \rho(H_o) \cap \rho(H)$.*

*Proof:* The proof is almost identical to the corresponding lemma 1 presented in reference [MO 80]. By writing $P(\Sigma,\Gamma) = P(\Gamma)P(\Sigma)$ we have exactly as in [MO 80] that $P(\Sigma)R_o(z)$ and $P(\Sigma)\Omega^{\pm}R_o(z)$ are Schmidt class operators. Since the product of a Schmidt class operator with a bounded operator is again Schmidt class we see both $P(\Sigma,\Gamma)R_o(z)$ and $P(\Sigma,\Gamma)\Omega^{\pm}R_o(z)$ must be Schmidt class. Next we recall the adjoint of a Schmidt class operator is Schmidt class and the product of two Schmidt class operators is trace class. This suffices to complete our proof. $\diamondsuit$

We need another lemma along similar lines to the one just given. If $\{E^o_\lambda\}$ denotes the spectral family $H_o$, we replace the resolvent in lemma 6.1 by the spectral projector $E^o_\lambda$.

**Lemma 6.2:** *Take $(H, H_o)$ and $P(\Sigma,\Gamma)$ as in lemma 6.1. Then for finite $\lambda$, the operators $P(\Sigma,\Gamma)E^o_\lambda$ and $P(\Sigma,\Gamma)\Omega^{\pm}E^o_\lambda$ are Schmidt class.*

*Proof:* The proof is almost identical to the one given in lemma 2 of [MO 80]. The obvious modifications are made for using $P(\Sigma,\Gamma)$ instead of $P(\Sigma)$ as was done in the proof of the previous lemma. $\diamondsuit$

We are now in a position to discuss the theory of the time delay for the region $\Sigma$. For an arbitrary state $\phi \in \mathcal{H}$, the associated free and interaction evolution states are $\varphi_t = U_t^o\phi$ and $\psi_t^{\pm} = U_t\Omega^{\pm}\phi$. These vectors are asymptotically related to each other for large time;

$$\psi_t^{\pm} \sim \varphi_t \qquad \text{as } t \to \mp\infty.$$

The transit time of $\varphi_t$ and $\psi_t^{\pm}$ through the spatial region $\Sigma$ while having spin components in the spinor subspace $\Gamma$, will be given respectively by

$$\int_{-\infty}^{\infty} \|P(\Sigma,\Gamma)\varphi_t\|^2\, dt \qquad \text{and} \qquad \int_{-\infty}^{\infty} \|P(\Sigma,\Gamma)\psi_t^{\pm}\|^2\, dt. \tag{6.18}$$

We define sesquilinear forms $Q^o$ and $Q^{\pm}$ on $\mathcal{H}$, with diagonal elements $Q^{\alpha}(\phi,\phi)$ ($\alpha = 0, \pm$) given by (6.18). Clearly we see that $Q^{\alpha}(\phi,\phi) \geq 0$. From the polarization identity for inner products and the intertwining relation $U_t\Omega^{\pm} = \Omega^{\pm}U_t^o$, the off diagonal values of $Q^o$ and $Q^{\pm}$ are

$$\begin{aligned} Q^o(\psi,\varphi) &= \int_{-\infty}^{\infty} \langle \psi, U_t^{o*}P(\Sigma,\Gamma)U_t^o\varphi\rangle\, dt; \\ Q^{\pm}(\psi,\varphi) &= \int_{-\infty}^{\infty} \langle \psi, U_t^{o*}\Omega^{\pm*}P(\Sigma,\Gamma)\Omega^{\pm}U_t^o\varphi\rangle\, dt. \end{aligned} \tag{6.19}$$

The time delay for a particle moving through the spatial region $\Sigma$ while having spinor components lying in $\Gamma$ is defined by the integral

$$\mathcal{Q}^{\pm}(\psi,\psi) = \int_{-\infty}^{\infty} \{\|P(\Sigma,\Gamma)U_t\Omega^{\pm}\psi\|^2 - \|P(\Sigma,\Gamma)U_t^o\psi\|^2\}\, dt. \tag{6.20}$$

The sesquilinear form $\mathcal{Q}^{\pm}$ will have the non-diagonal components

$$\begin{aligned} \mathcal{Q}^{\pm}(\psi,\varphi) &= Q^{\pm}(\psi,\varphi) - Q^o(\psi,\varphi) \\ &= \int_{-\infty}^{\infty} \Big\langle \psi, U_t^{o*}\Big[\Omega^{\pm*}P(\Sigma,\Gamma)\Omega^{\pm} - P(\Sigma,\Gamma)\Big]U_t^o\varphi\Big\rangle\, dt. \end{aligned} \tag{6.21}$$

There exists an important distinction between the use of the spinor subspace $\Gamma$ and the spin channel formalism in scattering theory*. In the spin channel formalism, we would set $\Gamma = \mathbb{C}^s$ (i.e. $P(\Gamma) = I$) and consider scattering from spin channel $\alpha$ to spin channel $\beta$. Associated with these channels are the channel indexed Møller operators

$$\Omega_\gamma^\pm = \Omega^\pm P_\gamma,$$

where $P_\gamma$ projects onto the spin state $\gamma$ ($\gamma = \alpha, \beta$). Then the time delay associated with the scattering process $\alpha \to \beta$ through the spatial region $\Sigma$ is defined by the sesquilinear form

$$\begin{aligned} \mathcal{Q}_{\alpha\beta}(\psi,\varphi) &= \int_{-\infty}^{\infty} \left\langle \psi, U_t^{o*}\left[\Omega_\alpha^{+*} P(\Sigma)\Omega_\beta^+ - P_\alpha P_\beta P(\Sigma)\right] U_t^o \varphi \right\rangle \\ &= \int_{-\infty}^{\infty} \left\langle \psi, U_t^{o*}\left[P_\alpha \Omega^{+*} P(\Sigma)\Omega^+ P_\beta - P_\alpha P_\beta P(\Sigma)\right] U_t^o \varphi \right\rangle \end{aligned} \tag{6.22}$$

Comparing (6.22) with (6.21), we see that $P(\Gamma)$ appears between the Møller operators, whereas the projection operators $P_\alpha$ and $P_\beta$ are exterior to the Møller operators.

In the following proposition we show that the sesquilinear forms $Q^\alpha$ generate a family of operators, $T^\alpha(\lambda; \Sigma, \Gamma)$, similar to those found in theorem 6.2. We also establish certain properties these $T^\alpha$'s satisfy and develop a formula for their trace.

**Proposition 6.2:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$. Then the sesquilinear forms defined by (6.19) induce in $\mathcal{H}_o$ a family of positive trace class operators $\{T^\alpha(\lambda; \Sigma, \Gamma)\}$ which satisfy*

$$Q^\alpha(\psi,\varphi) = \int_\Lambda \langle \psi_\lambda, T^\alpha(\lambda;\Sigma,\Gamma)\varphi_\lambda\rangle_o \, d\lambda. \tag{6.23}$$

---

* The spin channel formalism is structurally the same as the channel formalism of multiparticle systems. For a description of the multichannel formalism in regards to time delay, we refer to Bollé and Osborn [BO 76]. In particular cf. equations (4.8) $\sim$ (4.12) for comparisons.

*The operators $T^\alpha(\lambda;\Sigma,\Gamma)$ have traces (in $\mathcal{H}_o$) given for a.a. $\lambda$ by the following limits;*

$$\begin{aligned}\lim_{\mu\to 0+} \operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im} R_{ac}(\lambda+i\mu)P(\Sigma,\Gamma) &= \frac{1}{2}\operatorname{tr} T^{\pm}(\lambda;\Sigma,\Gamma) \qquad \text{a.a. } \lambda;\\ \lim_{\mu\to 0+} \operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im} R_{o}(\lambda+i\mu)P(\Sigma,\Gamma) &= \frac{1}{2}\operatorname{tr} T^{o}(\lambda;\Sigma,\Gamma) \qquad \text{a.a. } \lambda.\end{aligned} \tag{6.24}$$

*Here* $\operatorname{Im} R_{ac}(\lambda+i\mu) \equiv \frac{1}{2i}[R_{ac}(\lambda+i\mu) - R_{ac}(\lambda+i\mu)^*]$ *and* $R_{ac}(\lambda+i\mu)$ *is the the absolutely continuous part of the resolvent of* $H$. *A similar definition applies to* $\operatorname{Im} R_o(\lambda+i\mu)$.

*Proof:* We would like to be able to apply theorem 6.2 to $Q^\alpha$, but we cannot because $P(\Sigma,\Gamma)$ and $\Omega^{\pm *}P(\Sigma,\Gamma)\Omega^{\pm}$ are not trace class operators. We can circumvent this by considering closely related forms. For each $z\in\rho(H_o)\cap\rho(H)$, consider the operators

$$\Upsilon^o(z) = R_o(z)^* P(\Sigma,\Gamma)R_o(z)$$

and

$$\Upsilon^{\pm}(z) = R_o(z)^*\Omega^{\pm *}P(\Sigma,\Gamma)\Omega^{\pm}R_o(z).$$

By lemma 6.1, $\Upsilon^\alpha(z)$ is trace class and by theorem 6.2 the sesquilinear forms $Q_{\Upsilon^\alpha(z)}$ generate families of trace class operators acting in $\mathcal{H}_o$;

$$Q_{\Upsilon^\alpha(z)} \to \frac{1}{|\lambda-z|^2}T^\alpha(\lambda;\Sigma,\Gamma,z). \tag{6.25}$$

The factor $|\lambda-z|^{-2}$ is always finite for finite $z$ and $\lambda$, so we were free to introduce it into the family of trace class operators as indicated and adjust the definition of $T^\alpha$ accordingly.

We claim the operator family $T^\alpha(\lambda;\Sigma,\Gamma,z)$ is independent of $z$. Of course the consequence of this is all of the $z$ dependence for the operator family generated by $Q_{\Upsilon^\alpha(z)}$ is contained in the $|\lambda-z|^{-2}$ factor. Let $\psi,\varphi\in\mathcal{D}_\Theta$, where we set, for some fixed $z \in \rho(H_o) \cap \rho(H)$, $\Theta(\lambda) = |\lambda - z|^2$. It is easily verified that $\mathcal{D}_\Theta$ is dense in $\mathcal{H}$. We note that $\psi \in \mathcal{D}_\Theta$ implies $\|\psi_\lambda\|_o \leq C|\lambda - z|^{-2}$ and consequently $\mathcal{D}_\Theta \subset D_o$. The relationship between $Q^\alpha$ and $Q_{\Upsilon^\alpha(z)}$ is

$$\begin{aligned} Q^\alpha(\psi, \varphi) &= \int_{-\infty}^{\infty} \langle \psi, U_t^{o*}(H_o - z)^* \Upsilon^\alpha (H_o - z) U_t^o \varphi \rangle \, dt \\ &= \int_{-\infty}^{\infty} \langle (H_o - z)\psi, U_t^{o*} \Upsilon^\alpha U_t^o (H_o - z)\varphi \rangle \, dt \\ &= Q_{\Upsilon^\alpha(z)}\Big((H_o - z)\psi, (H_o - z)\varphi\Big). \end{aligned}$$

This identity and the representation of $Q_{\Upsilon^\alpha(z)}$ given by theorem 6.2, lead to

$$\begin{aligned} Q^\alpha(\psi, \varphi) &= \int_\Lambda \langle (\lambda - z)\psi_\lambda, \frac{1}{|\lambda - z|^2} T^\alpha(\lambda; \Sigma, \Gamma, z)(\lambda - z)\varphi_\lambda \rangle_o \, d\lambda \\ &= \int_\Lambda \langle \psi_\lambda, T^\alpha(\lambda; \Sigma, \Gamma, z)\varphi_\lambda \rangle_o \, d\lambda. \end{aligned}$$

But the left hand side is independent of $z$, hence we also have

$$Q^\alpha(\psi, \varphi) = \int_\Lambda \langle \psi_\lambda, T^\alpha(\lambda; \Sigma, \Gamma, z_o)\varphi_\lambda \rangle_o \, d\lambda,$$

for some fixed $z_o \in \rho(H) \cap \rho(H_o)$.

To apply theorem 6.2(3) we are required to show that $T^\alpha(\lambda; \Sigma, \Gamma, z)$ is bounded almost everywhere. This follows easily from theorem 6.2(2);

$$\begin{aligned} \frac{1}{2\pi} \int_\Lambda \frac{1}{|\lambda - z|^2} \|T^\alpha(\lambda; \Sigma, \Gamma, z)\|_{\mathcal{B}_1} \, d\lambda &\leq \|\Upsilon^\alpha(z)\|_{\mathcal{B}_1} < \infty \\ \Rightarrow \qquad \|T^\alpha(\lambda; \Sigma, \Gamma, z)\|_{\mathcal{B}_1} &< \infty \qquad \text{a.a. } \lambda. \end{aligned}$$

We are now free to apply the uniqueness property of theorem 6.2(3) to obtain the equality

$$\begin{aligned} T^{\alpha}(\lambda;\Sigma,\Gamma,z) &= T^{\alpha}(\lambda;\Sigma,\Gamma,z_o) \\ &\equiv T^{\alpha}(\lambda;\Sigma,\Gamma), \qquad \text{a.a. } \lambda. \end{aligned}$$

This demonstrates equation (6.23).

Because $\Upsilon^{\alpha}(z)$ is positive, corollary 6.1 shows $T^{\alpha}(\lambda;\Sigma,\Gamma)$ is positive for a.a. $\lambda$ as well. Consequently theorem 6.2(2) implies

$$\begin{aligned} \|\Upsilon^{\alpha}(z)\|_{\mathcal{B}_1} = \operatorname{Tr}\Upsilon^{\alpha}(z) &= \frac{1}{2\pi}\int_{\Lambda}\frac{\operatorname{tr}T^{\alpha}(\lambda;\Sigma,\Gamma)}{|\lambda-z|^2}\,d\lambda \\ &= \frac{1}{2\pi}\int_{\Lambda}\frac{\|T^{\alpha}(\lambda;\Sigma,\Gamma)\|_{\mathcal{B}_1}}{|\lambda-z|^2}\,d\lambda. \end{aligned} \tag{6.26}$$

Moreover (6.26) implies that $\|T^{\alpha}(\lambda;\Sigma,\Gamma)\|_{\mathcal{B}_1} \in L^1_{loc}(\Lambda,d\lambda)$. This is easy to see because $|\lambda-z|^2$ is locally bounded and $|\lambda-z|^{-2}\|T^{\alpha}(\lambda;\Sigma,\Gamma)\|_{\mathcal{B}_1}$ is integrable on $\Lambda$.

Next we demonstrate the almost everywhere equality

$$\operatorname{tr}T^{+}(\lambda;\Sigma,\Gamma) = \operatorname{tr}T^{-}(\lambda;\Sigma,\Gamma).$$

Let $\Delta$ be an arbitrary measurable subset of $\Lambda$ and consider the trace class operators

$$\Upsilon^{\alpha}_{\Delta}(z) = \Upsilon^{\alpha}(z)E^{o}_{\Delta}.$$

Since $E^{o}_{\Delta}$ is unitarily equivalent to $\{\chi_{\Delta}(\lambda)I\}$, it is easy to verify that $\Upsilon^{\alpha}_{\Delta}(z)$ induces the family of operators

$$\Upsilon^{\alpha}_{\Delta}(z) \to \frac{1}{|\lambda-z|^2}\chi_{\Delta}(\lambda)T^{\alpha}(\lambda;\Sigma,\Gamma).$$

Recall that $\Omega^{\pm}\Omega^{\pm *}$ is the projection operator onto the absolutely continuous part of $\mathcal{H}$. To say $R_{ac}(z)$ is the absolutely continuous part of $R(z)$ means

$$R_{ac}(z) = R(z)\Omega^{\pm}\Omega^{\pm *} = R(z)E_{\mathbb{R}\setminus\mathcal{E}}.$$

The operator $R_{ac}(z)$ will satisfy the Hilbert identity

$$\begin{aligned} 2i\mathrm{Im}\, R_{ac}(z) &\equiv R_{ac}(z) - R_{ac}(z)^* \\ &= (z - z^*)R_{ac}(z)R_{ac}(z)^*. \end{aligned}$$

We also bring our attention to the formula below, which results from an application of the intertwining relation;

$$\Omega^{\pm}R_o(z)R_o(z)^*\Omega^{\pm *} = R(z)\Omega^{\pm}\Omega^{\pm *}R(z)^* = R_{ac}(z)R_{ac}(z)^*.$$

Let $z = \lambda + i\mu$ and let $\{E_\lambda\}$ be the family of spectral projections associated with $H$. Then using the cyclic properties of the trace, the intertwining relation and the above identities, we have

$$\begin{aligned} &\mathrm{Tr}\, P(\Sigma,\Gamma)E_\Delta \mathrm{Im}\, R_{ac}(\lambda + i\mu)P(\Sigma,\Gamma) \\ &\qquad = \mu\,\mathrm{Tr}\, P(\Sigma,\Gamma)E_\Delta R_{ac}(\lambda + i\mu)R_{ac}(\lambda + i\mu)^*P(\Sigma,\Gamma) \\ &\qquad = \mu\,\mathrm{Tr}\, P(\Sigma,\Gamma)E_\Delta \Omega^{\pm}R_o(\lambda + i\mu)R_o(\lambda + i\mu)^*\Omega^{\pm *}P(\Sigma,\Gamma) \\ &\qquad = \mu\,\mathrm{Tr}\, P(\Sigma,\Gamma)\Omega^{\pm}R_o(\lambda + i\mu)E^o_\Delta R_o(\lambda + i\mu)^*\Omega^{\pm *}P(\Sigma,\Gamma) \\ &\qquad = \mu\,\mathrm{Tr}\, R_o(\lambda + i\mu)^*\Omega^{\pm *}P(\Sigma,\Gamma)\Omega^{\pm}R_o(\lambda + i\mu)E^o_\Delta \\ &\qquad = \mu\,\mathrm{Tr}\, \Upsilon^{\pm}_\Delta(\lambda + i\mu) \\ &\qquad = \frac{1}{2\pi}\int_\Delta \frac{\mu}{(\lambda - \lambda')^2 + \mu^2}\,\mathrm{tr}\, T^{\pm}(\lambda';\Sigma,\Gamma)\, d\lambda'. \end{aligned} \tag{6.27}$$

The left hand side of (6.27) is independent of the $\pm$ sign. This implies

$$\int_\Delta \frac{\mu}{(\lambda-\lambda')^2+\mu^2}[\operatorname{tr} T^+(\lambda';\Sigma,\Gamma) - \operatorname{tr} T^-(\lambda';\Sigma,\Gamma)]\,d\lambda' = 0.$$

As $\Delta$ is an arbitrary set, this is only possible if the integrand is zero for a.a. $\lambda'$. Since $\frac{\mu}{(\lambda-\lambda')^2+\mu^2} \neq 0$, this means

$$\operatorname{tr} T^+(\lambda';\Sigma,\Gamma) = \operatorname{tr} T^-(\lambda';\Sigma,\Gamma) \qquad \text{a.a. } \lambda'.$$

Suppose now $E_\Delta = I$. A repetition of the argument leading to (6.27) yields the expression

$$\operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im} R_{ac}(\lambda+i\mu)P(\Sigma,\Gamma) = \frac{1}{2\pi}\int_\Lambda \frac{\mu}{(\lambda-\lambda')^2+\mu^2}\operatorname{tr} T^\pm(\lambda';\Sigma,\Gamma)\,d\lambda' \quad (6.28)$$

and similarily,

$$\operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im} R_o(\lambda+i\mu)P(\Sigma,\Gamma) = \frac{1}{2\pi}\int_\Lambda \frac{\mu}{(\lambda-\lambda')^2+\mu^2}\operatorname{tr} T^o(\lambda';\Sigma,\Gamma)\,d\lambda'. \quad (6.29)$$

Theorem 13 of §1.16 of Titchmarsh [Ti 48] provides the necessary result to evaluate the limit $\mu \to 0$ taken in equations (6.28) and (6.29). Taking this limit, we get (6.24). $\diamond$

Define the function

$$q(\lambda;\Sigma,\Gamma) = \operatorname{tr} T^\pm(\lambda;\Sigma,\Gamma) - \operatorname{tr} T^o(\lambda;\Sigma,\Gamma). \quad (6.30)$$

It can be interpreted as the average time delay through the region $\Sigma$ for particles having spin lying in $\Gamma$ and energy equal to $\lambda$. From the equation

$$\frac{1}{2\pi}\int_0^\infty \frac{\operatorname{tr} T^\alpha(\lambda;\Sigma,\Gamma)}{1+\lambda^2}\,d\lambda = \|\Upsilon^\alpha(i)\|_{\mathcal{B}_1} < \infty$$

and the fact that $\operatorname{tr} T^\alpha(\lambda;\Sigma,\Gamma)$ is positive for a.a. $\lambda$, we see that

$$\int_0^\infty \frac{|q(\lambda;\Sigma,\Gamma)|}{1+\lambda^2}\, d\lambda < \infty. \tag{6.31}$$

Combining formulae (6.12) and (6.25) yields the result

$$\frac{1}{2\pi}\int_0^\infty \frac{q(\lambda;\Sigma,\Gamma)}{|\lambda - z|^2}\, d\lambda = \operatorname{Tr} R_o(z)^*[\Omega^{+*}P(\Sigma,\Gamma)\Omega^+ - P(\Sigma,\Gamma)]R_o(z), \tag{6.32}$$

while (6.28) and (6.29) lead to

$$\frac{1}{2\pi}\int_0^\infty q(\lambda;\Sigma,\Gamma)\operatorname{Im}\frac{1}{\lambda - z}\, d\lambda = \operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im}[R(z)E_{ac} - R_o(z)]P(\Sigma,\Gamma). \tag{6.33}$$

We shall also require certain properties of $q$ with respect to the sets $\Gamma$ and $\Sigma$. Suppose that $\Gamma_1$ and $\Gamma_2$ are two orthogonal subspaces of $\mathbb{C}^s$. Then the projection operator onto $\Gamma_1 \cup \Gamma_2$ is

$$P(\Gamma_1 \cup \Gamma_2) = P(\Gamma_1) + P(\Gamma_2) \tag{6.34}$$

and $q$ satisfies the formula

$$q(\lambda;\Sigma,\Gamma_1\cup\Gamma_2) = q(\lambda;\Sigma,\Gamma_1) + q(\lambda;\Sigma,\Gamma_2).$$

The proof of this is easily shown by using definition (6.30) and the formulae in (6.24). The cyclic property of the trace is utilized to combine the two projection operators in (6.24) into one projection. This resulting formula for $q(\lambda;\Sigma,\Gamma_1\cup\Gamma_2)$ can be split into the sum of two terms by using (6.34) and the linearity of the trace. The sum of these two formulae gives exactly $q(\lambda;\Sigma,\Gamma_1) + q(\lambda;\Sigma,\Gamma_2)$.

Similarly, if $\Sigma_1$ and $\Sigma_2$ are two disjoint measurable sets of finite Lebesgue measure, then $P(\Sigma_1 \cup \Sigma_2) = P(\Sigma_1) + P(\Sigma_2)$ and the time delay function satisfies the formula $q(\lambda; \Sigma_1 \cup \Sigma_2, \Gamma) = q(\lambda; \Sigma_1, \Gamma) + q(\lambda; \Sigma_2, \Gamma)$.

Consider the spin channel scattering formalism once again. There exists a natural connection between the $S$-matrix and the global time delay operator. For scattering from spin $\alpha$ to spin $\beta$, the scattering operator is $S_{\beta\alpha} = \Omega_\beta^{-*}\Omega_\alpha^{+}$. The scattering operator $S_{\beta\alpha}$ commutes with the free Hamiltonian and consequently it admits a spectral decomposition with respect to $H_o$. We denote this decomposition by $s_{\beta\alpha}(\lambda)$, and it is commonly called the $S$-matrix. If we take the limit $\Sigma \longrightarrow \mathbb{R}^3$, then the $\alpha \longrightarrow \beta$ global time delay function, $\tilde{q}_{\alpha\beta}$, satisifies the formula (cf. [BO 76], equation* (4.12))

$$\tilde{q}_{\alpha\beta} = -i \sum_{\gamma=1}^{s} \operatorname{tr} s_{\gamma\alpha}(\lambda)^* \frac{d}{d\lambda} s_{\gamma\beta}(\lambda).$$

Whether or not our time delay function has an analogous relation with the $S$-matrix we leave as an open question for future studies.

### 6.3 The Born Series

The Born series is a high energy expansion of the resolvent that is obtained by iterating the second resolvent equation

$$R(z) = R_o(z) - R_o(z)VR(z).$$

In our particular problem, we will have the Born series pre- and post- multiplied by the operator $P(\Sigma, \Gamma)$ and hence we shall include this feature in our study here. The advantage of this is it will allow us to verify convergence properties of the series

* Bollé and Osborn use the operator analog to this equation.

under the $\mathcal{B}_1$ topology for all $z \in \Pi_c$. We first identify a region where the series converges. For each $\theta \in (0,1)$ let $\Lambda_\theta$ be the infinum of the set

$$\{\Lambda \in \mathbb{R}^+ : \|A(z)\| \leq \theta < 1, \forall z \in \Pi_c \text{ with } |z| > \Lambda\}.$$

The Born dominant region of $\Pi_c$ will contain the set $\{z \in \Pi_c : |z| > \Lambda_\theta\}$ as a subset for each $0 < \theta < 1$. Obviously in this region, $[1 + A(z)]^{-1} \in \mathcal{B}$; it is given by the Neumann series

$$[1 + A(z)]^{-1} = 1 + \sum_{n=1}^{\infty} \Big(-A(z)\Big)^n$$

with the sum converging in the $\mathcal{B}$ norm; and it has the $z$ uniform norm estimate

$$\|[1 + A(z)]^{-1}\| \leq \frac{1}{1-\theta}.$$

Notice that for each $0 < \theta < 1$, $\mathcal{E}$ is contained in the closed interval $[-\Lambda_\theta, \Lambda_\theta]$. Our next lemma concerns itself with the convergence properties of the Born series.

**Lemma 6.3:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian and fix $0 < \theta < 1$. Then for all $|z| > \Lambda_\theta$ the Born series expansion*

$$P(\Sigma,\Gamma)\{R(z) - R_o(z)\}P(\Sigma,\Gamma) = \sum_{n=1}^{\infty}(-1)^n P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma) \quad (6.35)$$

*converges in the $\mathcal{B}_1$ norm. Moreover the convergence is uniform in $z$.*

*Proof:* We first show that the resolvent difference is trace class for all $z \in \Pi_c$. Recall the resolvent equation (6.5). Then

$$P(\Sigma,\Gamma)[R(z) - R_o(z)]P(\Sigma,\Gamma) = -P(\Sigma,\Gamma)R_o(z)w[1 + A(z)]^{-1}uR_o(z)P(\Sigma,\Gamma).$$

Since $[1+A(z)]^{-1} \in \mathcal{B}$ for all $|z| > \Lambda_\theta$, it is enough to prove that $P(\Sigma,\Gamma)R_o(z)w$ and $uR_o(z)P(\Sigma,\Gamma)$ are Schmidt class. Suppose the orthonormal vectors $\{\zeta_j\}_{j=1}^{s_o}$ span the spinor subspace $\Gamma$. Then the projection operator $P(\Gamma)$ is unitarily equivalent to multiplication with the matrix $M_\Gamma$;

$$P(\Gamma) \simeq \sum_{j=1}^{s_o} (\zeta_j, \cdot)\zeta_j = \sum_{j=1}^{s_o} \zeta_j \otimes \zeta_j^* \equiv M_\Gamma. \tag{6.36}$$

Here the symbol $(\cdot,\cdot)$ is the inner product on $\mathbb{C}^s$ and $\zeta_j \otimes \zeta_j^*$ is the tensor product between the column vector $\zeta_j$ and the row vector $\zeta_j^*$. With this notation set, then the $\mathcal{B}_2$ norm of $uR_o(z)P(\Sigma,\Gamma)$ has the estimate

$$\begin{aligned} \|uR_o(z)P(\Sigma,\Gamma)\|_{\mathcal{B}_2}^2 &= \int dx\,dy \left| \frac{u(x)M_\Gamma \chi_\Sigma(y) e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} \right|^2 \\ &\leq C_\Gamma \int dx\,dy\, \frac{|v(x)|\chi_\Sigma(y)}{|x-y|^2} \end{aligned}$$

where the constant $C_\Gamma$ is independent of $z$. The right hand side is finite via the Sobolev inequality (6.4) and it is independent of $z \in \Pi_c$. We note that it is the projection operators on each side of the resolvent difference that permits us to consider all of $\Pi_c$. For the operator $P(\Sigma,\Gamma)R_o(z)w$ we have a similar estimate that is also independent of $z \in \Pi_c$. This establishes our claim.

Next we study the convergence properties of the Born series in the $\mathcal{B}_1$ topology. Recall $A(z) \equiv uR_o(z)w$. Then we can use the identity

$$[1 + A(z)]^{-1} = 1 + \sum_{n=1}^{N} \Big(-A(z)\Big)^n + [1 + A(z)]^{-1}\Big(-A(z)\Big)^{N+1}$$

to write

$$
\begin{aligned}
&P(\Sigma,\Gamma)[R(z)-R_o(z)]P(\Sigma,\Gamma)\\
&\quad = -P(\Sigma,\Gamma)R_o(z)wuR_o(z)P(\Sigma,\Gamma)\\
&\qquad + \sum_{n=1}^{N} P(\Sigma,\Gamma)(-1)^{n+1}R_o(z)wA(z)^n uR_o(z)P(\Sigma,\Gamma)\\
&\qquad + (-1)^{N+2}P(\Sigma,\Gamma)R_o(z)w[1+A(z)]^{-1}A(z)^{N+1}uR_o(z)P(\Sigma,\Gamma). \qquad (6.37)
\end{aligned}
$$

Consider the $\mathcal{B}_1$ norm of the third term in (6.37).

$$
\begin{aligned}
&\|P(\Sigma,\Gamma)R_o(z)w[1+A(z)]^{-1}A(z)^{N+1}uR_o(z)P(\Sigma,\Gamma)\|_{\mathcal{B}_1}\\
&\qquad \le \|[1+A(z)]^{-1}\|\|A(z)\|^{N+1}\|P(\Sigma,\Gamma)R_o(z)w\|_{\mathcal{B}_2}\|uR_o(z)P(\Sigma,\Gamma)\|_{\mathcal{B}_2}\\
&\qquad \le C_\Gamma \frac{\theta^{N+1}}{1-\theta}\int dx\,dy\,\frac{\chi_\Sigma(x)|v(y)|}{|x-y|^2}\\
&\qquad \to 0 \qquad \text{as } N\to\infty.
\end{aligned}
$$

Moreover, we note the convergence is uniform with respect to $z$. To show the second term in right hand side of (6.37) converges in the trace norm, it is enough to prove it forms a Cauchy sequence, because $\mathcal{B}_1$ is a Banach space. We will prove even more in that we claim the sequence is uniformly Cauchy with respect to $z$, and hence the series will converge uniformly with respect to $z$. As with the third term, we have the $z$-uniform majorant

$$
\begin{aligned}
&\left\|\sum_{n=N_1}^{N_2}(-1)^n P(\Sigma,\Gamma)R_o(z)wA(z)^n uR_o(z)P(\Sigma,\Gamma)\right\|_{\mathcal{B}_1}\\
&\qquad\qquad \le C_\Gamma\int dx\,dy\,\frac{\chi_\Sigma(x)|v(y)|}{|x-y|^2}\sum_{n=N_1}^{N_2}\theta^n.
\end{aligned}
$$

Notice the right hand side converges to zero as $N_1, N_2 \to \infty$, uniformly with respect to $z$.

Thus we have shown that $P(\Sigma,\Gamma)[R(z)-R_o(z)]P(\Sigma,\Gamma)$ has a series representation that converges in the trace norm, uniformly with respect to $z \in \Pi_c$. To relate this series to the Born series, we note the termwise identity

$$R_o(z) w A(z)^n u R_o(z) = R_o(z)[V R_o(z)]^{n+1}. \tag{6.38}$$

◇

As an immediate consequence of this theorem we have

$$\operatorname{Tr} P(\Sigma,\Gamma)[R(z)-R_o(z)]P(\Sigma,\Gamma) = \sum_{n=1}^{\infty}(-1)^n \operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma) \tag{6.39}$$

and that this series converges uniformly with respect to $z$.

We would like to be able to perform various operations, such as integration, termwise to the series (6.39). The $z$-uniform convergence of this series does much to prove the analyticity, continuity and integrability of the series, provided each of the individual terms have these properties.

**Lemma 6.4:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian. Then for each $n \geq 1$, the function $\operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma)$ is holomorphic in $\Pi$ and continuous in $\Pi_c$.*

*Proof:* Utilizing identity (6.38), and the useful relation

$$\operatorname{Tr} ST = \langle S^*, T\rangle_{\mathcal{B}_2} \qquad S,T \in \mathcal{B}_2,$$

we have

$$\begin{aligned}&\operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[V R_o(z)]^{n+1} P(\Sigma,\Gamma)\\ &\qquad = \Big\langle \Big(P(\Sigma,\Gamma)R_o(z)w\Big)^*, A(z)^n u R_o(z) P(\Sigma,\Gamma)\Big\rangle_{\mathcal{B}_2}.\end{aligned}$$

It is enough to prove that each argument of the inner product is separately $\mathcal{B}_2$ holomorphic in $\Pi$ and $\mathcal{B}_2$ continuous in $\Pi_c$. As $A(z)$ has these properties and it is uniformly bounded in $z$, it is enough to prove these statements for $P(\Sigma,\Gamma)R_o(z)w$ and $uR_o(z)P(\Sigma,\Gamma)$. We consider $uR_o(z)P(\Sigma,\Gamma)$ and note that similar arguments can be applied to the other operator.

The operator $uR_o(z)P(\Sigma,\Gamma)$ is an integral operator with the matrix valued kernel

$$u(x)\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}\chi_\Sigma(y)M_\Gamma.$$

Now $|x-y|^{-1}\chi_\Sigma(y)u(x)M_\Gamma \in L^2(dx\,dy;\mathbb{C}^{s\times s})$ via the Sobolev inequality (6.4). Since the exponential is a continuous function of $z \in \Pi_c$ and it is uniformly bounded by 1, an application of the dominated convergence theorem shows that the kernel of $uR_o(z)P(\Sigma,\Gamma)$ is continuous in the $L^2(dx\,dy;\mathbb{C}^{s\times s})$ norm for all $z \in \Pi_c$. We also have $\chi_\Sigma(y)u(x)M_\Gamma \in L^2(dx\,dy;\mathbb{C}^{s\times s})$, which allows us to define a Schmidt class operator whose kernel is

$$\frac{ie^{i\sqrt{z}|x-y|}}{8\pi\sqrt{z}}\chi_\Sigma(y)u(x)M_\Gamma. \tag{6.40}$$

It is easily shown that the kernel of $uR_o(z)P(\Sigma,\Gamma)$ is differentiable with respect to $z \in \Pi$ in the $L^2(dx\,dy;\mathbb{C}^{s\times s})$ topology via an application of the dominated convergence theorem and that its derivative yields the $L^2(dx\,dy;\mathbb{C}^{s\times s})$ function defined in (6.40). Because of the isometry between $L^2(dx\,dy;\mathbb{C}^{s\times s})$ and $\mathcal{B}_2$, we see that $uR_o(z)P(\Sigma,\Gamma)$ is $\mathcal{B}_2$ differentiable with respect $z \in \Pi$. $\diamondsuit$

**Corollary 6.2:** *Assume the hypothesis of lemmas 6.3 and 6.4. Then the function* $\operatorname{Tr} P(\Sigma,\Gamma)[R(z) - R_o(z)]P(\Sigma,\Gamma)$ *is holomorphic in* $\Pi$ *and continuous in* $\Pi_c$. *If* $C$ *is any curve that lies in a compact subset of* $\Pi_c \cap \{z : |z| > \Lambda_\theta\}$, *then we may integrate the series (6.39) termwise over the contour* $C$. $\diamondsuit$

As a final topic to discuss about the Born series, we write an explicit representation for each of the terms $\mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)$. In the next lemma the potential will be in the class $L^1 \cap \mathcal{F}^*(\mathbb{R}^3;\mathbb{C}^{s\times s})$ and so it is the Fourier transform of a bounded complex matrix measure. For $v \in \mathcal{F}^*(\mathbb{R}^3;\mathbb{C}^{s\times s})$, we have the representation

$$\begin{aligned} v(x) &= \int d\mu(\alpha)\, e^{i\alpha\cdot x} \\ &= \int d|\mu|(\alpha)\, \eta(\alpha) e^{i\alpha\cdot x} \end{aligned}$$

Since $v$ is also in $L^1(\mathbb{R}^3;\mathbb{C}^{s\times s})$, its classical Fourier transform exists and we note the relation

$$d\mu(\alpha) = \frac{1}{(2\pi)^{3/2}} \hat{v}(\alpha)\, d\alpha.$$

Before proceeding with the next lemma, we first define some notation. We express the following two multiple integrals with the shorthands

$$\int_0^{1>} d^n\xi = \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{n-1}} d\xi_n$$

and

$$\begin{aligned} \int d^n\mu &= \int d\mu(\alpha_n) \int d\mu(\alpha_{n-1}) \cdots \int d\mu(\alpha_1) \\ &= \int d^n|\mu|\, \eta(\alpha_n) \cdots \eta(\alpha_1). \end{aligned}$$

**Lemma 6.5:** *Let $v \in L^1 \cap \mathcal{F}^*(\mathbb{R}^3;\mathbb{C}^{s\times s})$. Then for all $z \in \Pi$,*

$$\begin{aligned} &\mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma) \\ &\quad = \frac{\Gamma(n-\frac{1}{2})}{2^{3/2}} \int_0^{1>} d^n\xi \int d^n|\mu|\, \frac{\hat{\chi}_\Sigma^*(\alpha_1 + \cdots + \alpha_n)}{(a_n - z)^{n-\frac{1}{2}}}\, \mathrm{sp}\, M_\Gamma \eta(\alpha_n) \cdots \eta(\alpha_1) M_\Gamma \end{aligned} \qquad (6.41)$$

*where the function* $a_n$ *is*

$$a_n = \sum_{l,m=1}^{n} \theta(\xi_l, \xi_m)\alpha_l \cdot \alpha_m \geq 0 \tag{6.42}$$

*and*

$$\theta(\xi_l, \xi_m) = \min\{\xi_l(1-\xi_m), \xi_m(1-\xi_l)\}. \tag{6.43}$$

*Proof:* The idea of the proof is to exploit the relationship between the resolvent kernel and the kernel of the semigroup $e^{-\beta H}$ ($\mathrm{Re}\,\beta > 0$) to obtain a formula for the kernel of $P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)$. For potentials in $\mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$, the semigroup kernel has the known representation [OCF 85]

$$K(x,y;\beta) = K_o(x,y;\beta)F(x,y;\beta). \tag{6.44}$$

Here $K_o(x,y;\beta)$ is the free semigroup kernel and it is given by the formula

$$K_o(x,y;\beta) = \frac{e^{-\frac{|x-y|^2}{4\beta}}}{(4\pi\beta)^{3/2}} I.$$

The matrix valued function $F(x,y;\beta)$ is the series

$$F(x,y;\beta) = \sum_{n=0}^{\infty} B_n(x,y;\beta)$$

where

$$\begin{aligned} B_o(x,y;\beta) &= I \\ B_n(x,y;\beta) &= (-\beta)^n \int_0^{1>} d^n\xi \int d^n\mu \, e^{-\beta a_n + i b_n}. \end{aligned} \tag{6.45}$$

Here the function $a_n$ is given by (6.42) and the function $b_n$ is defined by the formula

$$b_n = \sum_{l=1}^{n} [(1-\xi_l)x + \xi_l y] \cdot \alpha_l \tag{6.46}$$

We note we have obtained a similar expansion for the propagator in chapter 5. Specifically we refer to equation (5.41). We should expect this because the propagator represents the $\mathrm{Re}\,\beta \to 0$ limit of the semigroup problem.

Introduce into the potential the coupling constant $\gamma$ so that $v \to \gamma v$ and $\mu \to \gamma\mu$. Then our expansion for the semigroup kernel remains valid provided we rewrite $B_n(x, y; \beta)$ as

$$B_n(x, y; \beta, \gamma) = (-\gamma\beta)^n \int_0^{1>} d^n\xi \int d^n\mu \, e^{-\beta a_n + i b_n}.$$

For the moment, let $z$ satisfy $\mathrm{Re}\, z < -|\gamma| \|\mu\|$ and $|z| > \Lambda_\theta$. Then the resolvent kernel is given as the Laplace transform of the semigroup kernel;

$$R(x, y; z) = \int_0^\infty d\beta \, e^{\beta z} K(x, y; \beta).$$

For proof of this result we refer to reference [OCF 85], proposition 4. We note that in proposition 4, the integral is along a contour $L_\delta^\pm$. The integral along this contour was used to consider values of $z$ in a much larger domain than is being considered here. For $\mathrm{Re}\, z < -\|\mu\|$ the integral along $[0, \infty)$ is equivalent to the integral along $L_\delta^\pm$. Indeed, in the proof of proposition 4, Osborn et. al. used the integral we are considering here; showed it was equivalent to the one along $L_\delta^\pm$; and then analytically extended the domain of allowed values of $z$.

We claim we can interchange the $\sum_{n=0}^\infty$ appearing in the definition of $F$ and the $d\beta$ integral. Because the partial series $K_o(x, y; \beta) \sum_{n=0}^N B_n(x, y; \beta)$ admits the

$N$ independent estimate $e^{|\gamma|\beta\|\mu\|}$ and

$$\int_0^\infty d\beta\, e^{(\mathrm{Re}\, z - |\gamma|\|\mu\|)\beta} < \infty,$$

we can apply the dominated convergence theorem to interchange the sum and integral. Thus we have

$$R(x,y;z,\gamma) = R_o(x,y;z) + \sum_{n=1}^{\infty}(-\gamma)^n \int_0^\infty d\beta\, e^{\beta z}\beta^n \times \int_0^{1>} d^n\xi \int d^n\mu\, K_o(x,y;\beta)e^{-\beta a_n + ib_n}. \quad (6.47)$$

To get (6.47) we have used lemma 3 of reference [OCF 85] which states

$$R_o(x,y;z) = \int_0^\infty d\beta\, e^{\beta z} K_o(x,y;\beta) \qquad \mathrm{Re}\, z < 0.$$

Thus the operator $P(\Sigma,\Gamma)[R(z,\gamma) - R_o(z)]P(\Sigma,\Gamma)$ has the kernel

$$\chi_\Sigma(x)\sum_{n=1}^{\infty}(-\gamma)^n \int_0^\infty d\beta\, e^{\beta z}\beta^n \int_0^{1>} d^n\xi \int d^n|\mu| M_\Gamma \eta(\alpha_n)\cdots\eta(\alpha_1)M_\Gamma \times K_o(x,y;\beta)e^{-\beta a_n + ib_n}\chi_\Sigma(y). \quad (6.48)$$

Because $z$ is in the region of Born dominance and we can consider the Born series

$$P(\Sigma,\Gamma)[R(z,\gamma) - R_o(z)]P(\Sigma,\Gamma) = \sum_{n=1}^{\infty}(-\gamma)^n P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma).$$

In lemma 6.3 we established this series converges in the $\boldsymbol{B}_1$ topology. Hence it also converges in the $\boldsymbol{B}_2$ topology which in turn implies the sum of the kernels of these operators converge in the $L^2(dx\, dy; \mathbb{C}^{s\times s})$ topology. Consequently this kernel sum converges in measure (cf. [Roy 68], chapter 4§5) and hence there is a convergent subsequence such that

$$\sum_{n=1}^{N_j}(-\gamma)^n\Big[P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\Big](x,y)$$
$$\to \sum_{n=1}^{\infty}(-\gamma)^n\Big[P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\Big](x,y) \quad (6.49)$$

for a.a. $(x,y)$. On comparing the terms of equal order in $\gamma$ between (6.48) and (6.49) we obtain

$$(-1)^n\Big[P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\Big](x,y) = \int_0^\infty d\beta\, e^{\beta z} K_o(x,y;\beta)B_n(x,y;\beta). \quad (6.50)$$

Next we take the trace of $P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)$. Let $T = RS$ be a trace class operator, given by the product of the two Schmidt class operators $R$ and $S$. Suppose that $R$ and $S$ have kernels $K_R$ and $K_S$ and $T$ has kernel $K_T$. Then

$$\begin{aligned}\mathrm{Tr}\, T = \langle R^*, S\rangle_{B_2} &= \int dx\, dy\ \mathrm{sp}\, K_R(x,y)K_S(y,x)\\ &= \int dx\ \mathrm{sp}\, K_T(x,x). \end{aligned} \quad (6.51)$$

For proof of this result we refer to [Sch 70], theorem 2.4. Applying this to equation (6.50) we have

$$\begin{aligned}&\mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\\ &= \int dx \int_0^\infty d\beta \int_0^{1>} d^n\xi \int d^n|\mu|\ \mathrm{sp}\, M_\Gamma \eta(\alpha_n)\cdots\eta(\alpha_1)M_\Gamma\, \frac{\chi_\Sigma(x)}{(4\pi)^{3/2}}\\ &\quad\times \beta^{n-\frac{3}{2}} e^{\beta(z-a_n)} e^{ix\cdot(\alpha_1+\cdots+\alpha_n)}.\end{aligned}$$

The integrand is absolutely integrable and hence we may interchange the orders of integration via Fubini's theorem. We note the $\beta$ integral is in the form of a gamma function and the $x$ integral is a Fourier integral. After explicitly evaluating the $\beta$ and $x$ integrals, the resulting formula is equation (6.41), but with the restriction $\mathrm{Re}\, z < -\|\mu\|$ $(\gamma = 1)$ and $|z| > \Lambda_\theta$.

Now lemma 6.4 showed that $\mathrm{Tr}\, P(\Sigma, \Gamma) R_o(z) [V R_o(z)]^n P(\Sigma, \Gamma)$ is holomorphic in the region $\Pi$. Furthermore the right hand side of (6.41) is easily shown to be holomorphic in $\Pi$ as well. As these two holomorphic functions coincide on $\mathrm{Re}\, z < -\|\mu\|$, they must coincide everywhere on $\Pi$. $\diamondsuit$

# CHAPTER 7

## Sum Rules for Spin Systems

For this chapter, we assume the same working hypotheses as stated in chapter 6. Again we shall keep our assumptions on the integrability of $v$ specific to each of our claims. We first state the main result of this chapter, with the detailed proof following in the ensuing lemmas and propositions.

### 7.1 The Sum Rule

Let $\Gamma$ be an arbitrary, but fixed subspace of $\mathbb{C}^s$ and suppose the set of orthonormal spinors $\{\zeta_j\}_{j=1}^{s_o}$ span $\Gamma$. We will extensively use a partial trace (in $\mathbb{C}^s$) of $v(x)$ on the subspace $\Gamma$. Denote this trace by

$$v_\Gamma(x) = \sum_{j=1}^{s_o} (\zeta_j, v(x)\zeta_j). \tag{7.1}$$

**Theorem 7.1:** *Let $v \in L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$. Let $\Sigma$ be a Lebesgue measurable set of finite measure and let $\Gamma$ be a subset of $\mathbb{C}^s$. If $q(\lambda;\Sigma,\Gamma)$ denotes the time delay for the region $\Sigma$, the spinor subspace $\Gamma$ and the energy $\lambda$, then*

$$\begin{aligned} \lim_{b\to\infty} \left\{ \int_0^b d\lambda\, q(\lambda;\Sigma,\Gamma) + \frac{1}{2\pi}\sqrt{b} \int_\Sigma dx\, v_\Gamma(x) + M(b;\Sigma,\Gamma) \right\} \\ = -2\pi \operatorname{Tr} P(\Sigma,\Gamma) E_s P(\Sigma,\Gamma). \end{aligned} \tag{7.2}$$

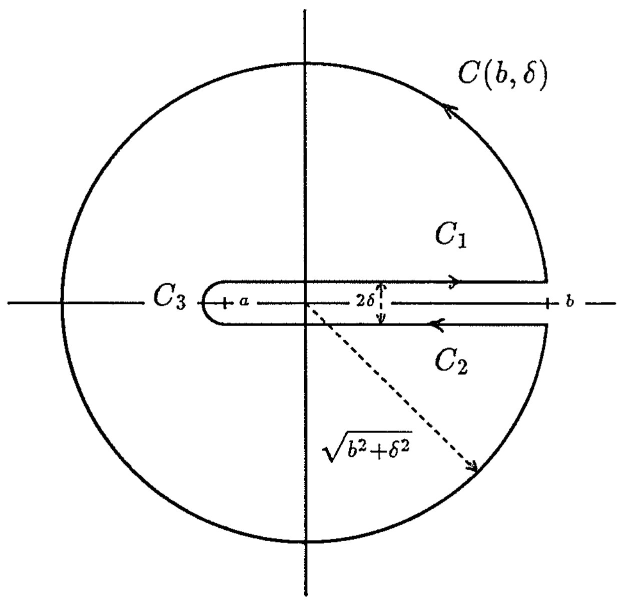
*where $E_s$ is the projection operator associated with the singular spectrum of $H$ and*

$$M(b;\Sigma,\Gamma) = \frac{1}{8\pi}\int d\alpha\,[b - \frac{\alpha^2}{4}]\frac{\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)}{|\alpha|}\ln\left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 - \frac{1}{4\pi}\sqrt{b}\int_\Sigma dx\, v_\Gamma(x). \tag{7.3}$$

*Proof:* Let $\theta \in (0,1)$ be fixed and consider the contour integral of the function

$$\mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma)$$

about the contour $C$ shown below.




$$C = C_1 + C_2 + C_3 + C(b,\delta)$$

**Figure 7.1**

Here the interval $(a,b) \supset [-\Lambda_\theta, \Lambda_\theta]$. From lemma 6.4 and corollary 6.2 $\mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma)$ is holomorphic inside and on $C$.

Thus by Cauchy's theorem,

$$\oint_C dz \ \mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma) = 0$$

Formula (7.2) is the result of calculating the contributions of $C_j$ ($j = 1,2,3$) and $C(b,\delta)$ in the limit $b \longrightarrow \infty$ and $\delta \longrightarrow 0$. We shall calculate these limits in the following sections. ◇

A couple of remarks are in order here. The first comment concerns the term $M(b,\Sigma,\Gamma)$ given in (7.3). The second term on the right hand side of this equation is expected to cancel out the leading order large $b$ behaviour of the first term of the right hand side. To get a feeling for the large $b$ behaviour of $M(b;\Sigma,\Gamma)$ suppose $\hat{v}$ has compact support. For $b$ sufficiently large, a ball centered on the origin and of radius $b$ will contain the support of $\hat{v}$. Then for all $\alpha \in \mathrm{supp}\,\hat{v}$, the ln function appearing in the definition of $M(b;\Sigma,\Gamma)$ has the asymptotic expansion

$$\ln\left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 \sim 2\frac{|\alpha|}{\sqrt{b}} + O\left(\frac{|\alpha|^3}{b^{3/2}}\right).$$

Using this expansion, $M(b;\Sigma,\Gamma)$ behaves as

$$M(b;\Sigma,\Gamma) = \left\{\frac{\sqrt{b}}{4\pi}\int_\Sigma dx\, v_\Gamma(x) + O(b^{-1/2})\right\} - \frac{\sqrt{b}}{4\pi}\int_\Sigma dx\, v_\Gamma(x) = O(b^{-1/2})$$

in the limit $b \longrightarrow \infty$. For the more general $v$, the ordering behaviour of $M(b;\Sigma,\Gamma)$ is much more difficult to determine. However, it is implicit that if the integral over the contour $C(b,\delta)$ tends to zero in the limits $\delta \longrightarrow 0$ and $b \longrightarrow \infty$, then for any singular structure in $b$ occuring in $M(b;\Sigma,\Gamma)$, there must be a corresponding structure in the energy integral of $q(\lambda;\Sigma,\Gamma)$ that exactly cancels with it.

Next we have a remark concerning the spin subspace $\Gamma$. By restricting the dimension of $\Gamma$ to 1, but otherwise leaving it free, we can obtain via the polarization identity* on $\mathbb{C}^s$, a sum rule which picks out the $ij^{th}$ component of $v$. If $\{\zeta_j\}_{j=1}^s$ is an orthonormal basis of $\mathbb{C}^s$, define

$$\begin{aligned} v_{jl}(x) &= \Big(\zeta_j, v(x)\zeta_l\Big) \\ &= \frac{1}{4}\Bigg\{\Big(\zeta_j+\zeta_l, v(x)[\zeta_j+\zeta_l]\Big) - \Big(\zeta_j-\zeta_l, v(x)[\zeta_j-\zeta_l]\Big) \\ &\qquad - i\Big(\zeta_j+i\zeta_l, v(x)[\zeta_j+i\zeta_l]\Big) + i\Big(\zeta_j-i\zeta_l, v(x)[\zeta_j-i\zeta_l]\Big)\Bigg\}. \end{aligned}$$

This is the matrix representation of $v(x)$ with respect to the basis $\{\zeta_j\}_{j=1}^s$. Let $\Gamma_1$ be the subspace spanned by $\zeta_j+\zeta_l$; $\Gamma_2$ be the subspace spanned by $\zeta_j-\zeta_l$; $\Gamma_3$ be the subspace spanned by $\zeta_j+i\zeta_l$; and $\Gamma_4$ be the subspace spanned by $\zeta_j-i\zeta_l$. Then for each $\Gamma_k$, we have a sum rule corresponding to (7.2). We may take appropriate linear combinations of these four sum rules which would allow us to invoke the polarization identity to recover the components $v_{jl}(x)$ and $\hat{v}_{jl}(\alpha)$. We combine the linear combinations of the four time delay functions into a single function;

$$\boldsymbol{q}_{jl}(\lambda;\Sigma) \equiv \frac{1}{2}\Big\{q(\lambda;\Sigma,\Gamma_1) - q(\lambda;\Sigma,\Gamma_2) - iq(\lambda;\Sigma,\Gamma_3) + iq(\lambda;\Sigma,\Gamma_4)\Big\}.$$

Clearly we may form a matrix from the $\boldsymbol{q}_{jl}(\lambda;\Sigma)$'s. In fact we claim this matrix will be hermitian. To see this, interchange the indices $j$ and $l$ above. The vectors $\zeta_l+\zeta_j$ and $\zeta_l-\zeta_j = -(\zeta_j-\zeta_l)$ will still span $\Gamma_1$ and $\Gamma_2$ respectively. However, the

---

* The polarization identity we refer to applies to any sesquilinear functional on a Hilbert space. If $\langle\cdot,\cdot\rangle$ denotes the inner product of a Hilbert space $\mathcal{H}$, then

$$\begin{aligned} \langle f, Ag\rangle = \frac{1}{4}\{&\langle f+g, A(f+g)\rangle - \langle f-g, A(f-g)\rangle \\ &- i\langle f+ig, A(f+ig)\rangle + i\langle f-ig, A(f-ig)\rangle\}. \end{aligned}$$

where $A$ is an operator on $\mathcal{H}$ with domain $D(A)$ and $f, g \in D(A)$.

vector $\zeta_l + i\zeta_j = i(\zeta_j - i\zeta_l)$ spans $\Gamma_4$ and the vector $\zeta_l - i\zeta_j = -i(\zeta_j + i\zeta_l)$ spans $\Gamma_3$. Then $\boldsymbol{q}_{lj}(\lambda;\Sigma)$ will be given by

$$\boldsymbol{q}_{lj}(\lambda;\Sigma) = \frac{1}{2}\Big\{q(\lambda;\Sigma,\Gamma_1) - q(\lambda;\Sigma,\Gamma_2) - iq(\lambda;\Sigma,\Gamma_4) + iq(\lambda;\Sigma,\Gamma_3)\Big\}.$$

As the $q(\lambda;\Sigma,\Gamma_i)$'s are real valued ($i = 1 \sim 4$), we see that $\boldsymbol{q}_{lj}(\lambda;\Sigma)^* = \boldsymbol{q}_{jl}(\lambda;\Sigma)$. This establishes our claim. We caution again, that the function $\boldsymbol{q}_{jl}(\lambda;\Sigma)$ is not necessarily the same as the time delay function one obtains from the $j$ and $l$ spin channels in the channel formalism of scattering theory discussed briefly in chapter 6. In the context presented here, $\boldsymbol{q}_{jl}(\lambda;\Sigma)$ is a useful construct for obtaining sum rules that explicitly involve off diagonal components of $v(x)$. Following the procedure outlined, we have constructed the sum rule

$$\begin{aligned}\lim_{b\to\infty}&\left\{\int_0^b d\lambda\, \boldsymbol{q}_{jl}(\lambda;\Sigma) + \frac{1}{2\pi}\sqrt{b}\int_\Sigma dx\, v_{jl}(x) + M_{jl}(b;\Sigma,\Gamma)\right\}\\
&= -\pi\Big\{\operatorname{Tr} P(\Sigma,\Gamma_1)E_sP(\Sigma,\Gamma_1) - \operatorname{Tr} P(\Sigma,\Gamma_2)E_sP(\Sigma,\Gamma_2)\\
&\qquad - i\operatorname{Tr} P(\Sigma,\Gamma_3)E_sP(\Sigma,\Gamma_3) + i\operatorname{Tr} P(\Sigma,\Gamma_4)E_sP(\Sigma,\Gamma_4)\Big\},\end{aligned} \tag{7.4}$$

where

$$M_{jl}(b;\Sigma,\Gamma) = \frac{1}{8\pi}\int d\alpha\left[b - \frac{\alpha^2}{4}\right]\frac{\hat{\chi}^*_\Sigma(\alpha)\hat{v}_{jl}(\alpha)}{|\alpha|}\ln\left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 - \frac{1}{4\pi}\sqrt{b}\int_\Sigma dx\, v_{jl}(x). \tag{7.5}$$

**7.2 The $C_1$, $C_2$, and $C_3$ Contributions**

The $C_3$ contribution is easily found. Without loss of generality we may pick $a$ so large that $[1+A(z)]^{-1}$ exists via its Neumann series representation. Let $0 < \theta < 1$ be a fixed parameter and let $\Lambda_\theta$ be the set in $\Pi_c$ as described at the start of section 6.3. Then for all $z \in \Lambda_\theta$,

$$
\begin{aligned}
&|\operatorname{Tr} P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma)| \\
&\qquad \equiv |\operatorname{Tr} P(\Sigma,\Gamma)R_o(z)wA(z)[1 + A(z)]^{-1}uR_o(z)P(\Sigma,\Gamma)| \\
&\qquad \leq c_\Gamma \frac{\theta}{1-\theta} \int dx\, dy\, \frac{\chi_\Sigma(x)|v(y)|}{|x-y|^2} < \infty
\end{aligned}
$$

where $c_\Gamma$ is a constant independent of $z$. Thus there exists a constant $c$ such that

$$
\left| \int_{C_3} dz \, \operatorname{Tr} P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma) \right| \leq c\delta
$$

and the right hand side tends to zero in the limit $\delta \to 0$.

The $C_1 + C_2$ contribution requires more work. We note that if $f$ is a holomorphic function satisfying $f(z^*) = f(z)^*$, then

$$
\begin{aligned}
\int_{C_1+C_2} dz\, f(z) &= \int_a^b d\lambda\, [f(\lambda + i\delta) - f(\lambda - i\delta)] \\
&= 2i \int_a^b d\lambda \operatorname{Im} f(\lambda + i\delta).
\end{aligned}
$$

In particular, $\operatorname{Tr} P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma)$ is such a function.

**Proposition 7.1:** *Suppose $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ is hermitian. For each fixed $\theta \in (0,1)$ and every finite interval $(a,b) \supset [-\Lambda_\theta, \Lambda_\theta]$,*

$$\lim_{\delta\to 0+} \int_a^b d\lambda \; \mathrm{Tr}\, P(\Sigma,\Gamma)\mathrm{Im}\,[R(\lambda+i\delta) - R_o(\lambda+i\delta)]P(\Sigma,\Gamma)$$
$$= \frac{1}{2}\int_0^b d\lambda\, q(\lambda;\Sigma,\Gamma) + \pi\, \mathrm{Tr}\, P(\Sigma,\Gamma)E_s P(\Sigma,\Gamma). \qquad (7.6)$$

*Here $E_s$ is the spectral projector associated with singular spectrum of $H$.*

*Proof:* Let $\delta > 0$ and consider the two functions

$$\begin{aligned} I_{ac}(\delta) &= \int_a^b d\lambda \; \mathrm{Tr}\, P(\Sigma,\Gamma)\mathrm{Im}\,[R(\lambda+i\delta)E_{ac} - R_o(\lambda+i\delta)]P(\Sigma,\Gamma) \\ I_s(\delta) &= \int_a^b d\lambda \; \mathrm{Tr}\, P(\Sigma,\Gamma)\mathrm{Im}\, R(\lambda+i\delta)E_s P(\Sigma,\Gamma) \end{aligned} \qquad (7.7)$$

As $E_{ac} + E_s = I$, we clearly have that $I_{ac}(\delta) + I_s(\delta)$ is the left hand side of (7.6) before the $\delta \to 0+$ limit is taken. Before proceeding further, we first verify that the traces of the above operators exist. As the imaginary part of the resolvent can be expressed as the difference of the resolvent at two different points, we can use the Hilbert indentity $R(z) - R(z') = (z - z')R(z)R(z')$ to get

$$P(\Sigma,\Gamma)\mathrm{Im}\, R(\lambda+i\delta)P(\Sigma,\Gamma) = \delta P(\Sigma,\Gamma)R(\lambda+i\delta)R(\lambda-i\delta)P(\Sigma,\Gamma).$$

We claim $R(z)P(\Sigma,\Gamma) \in \mathcal{B}_2$. Using the second resolvent equation we have

$$\begin{aligned} R(z)P(\Sigma,\Gamma) &= R_o(z)P(\Sigma,\Gamma) - R(z)VR_o(z)P(\Sigma,\Gamma) \\ &= R_o(z)P(\Sigma,\Gamma) - [P(\Sigma,\Gamma)R_o(z^*)VR(z^*)]^*. \end{aligned}$$

Now in the proof of lemma 6.1 we showed that $R_o(z)P(\Sigma,\Gamma)$ and $P(\Sigma,\Gamma)R_o(z^*)$ are both Schmidt class operators. Furthermore, $V$ is $H$-bounded and consequently $VR(z^*)$ is a bounded operator. Thus $P(\Sigma,\Gamma)R_o(z^*)VR(z^*)$ and its adjoint are also Schmidt class. This proves that $P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)P(\Sigma,\Gamma)\in\mathcal{B}_1$. Similarly $P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)E_{ac}P(\Sigma,\Gamma)$ is trace class and from the identity

$$\begin{aligned}&P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)E_sP(\Sigma,\Gamma)\\ &\qquad=P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)P(\Sigma,\Gamma)-P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)E_{ac}P(\Sigma,\Gamma),\end{aligned}$$

we must have $P(\Sigma,\Gamma)\mathrm{Im}\,R(\lambda+i\delta)E_sP(\Sigma,\Gamma)\in\mathcal{B}_1$.

Consider the $\delta\to0+$ limit of $I_{ac}(\delta)$ first. From equation (6.33) we get the representation

$$I_{ac}(\delta)=\frac{1}{2\pi}\int_a^b d\lambda\int_0^\infty d\mu\,\frac{\delta}{(\mu-\lambda)^2+\delta^2}q(\mu;\Sigma,\Gamma). \tag{7.8}$$

Note that

$$\int_0^\infty d\mu\,\frac{\delta}{(\mu-\lambda)^2+\delta^2}q(\mu;\Sigma,\Gamma)$$

is a continuous function of $\lambda\in[a,b]$ via an application of the dominated convergence theorem, so that $I_{ac}(\delta)$ exists for each $\delta>0$. Next we wish to interchange integral orders. To apply Fubini's theorem, it is enough to prove that the the integrand in (7.8) is absolutely integrable;

$$\begin{aligned}\int_a^b d\lambda\int_0^\infty d\mu\,\frac{\delta}{(\mu-\lambda)^2+\delta^2}|q(\mu;\Sigma,\Gamma)|&=\int_a^b d\lambda\int_0^\infty d\mu\,\frac{(1+\mu^2)\delta}{(\mu-\lambda)^2+\delta^2}\frac{|q(\mu;\Sigma,\Gamma)|}{1+\mu^2}\\ &\le c_\delta(b-a)\int_0^\infty d\mu\,\frac{|q(\mu;\Sigma,\Gamma)|}{1+\mu^2}\\ &<\infty.\end{aligned}$$

Hence we are justified in interchanging orders of integration. The $d\lambda$ integral is elementary, and after evaluating it we have

$$I_{ac}(\delta)=\frac{1}{2\pi}\int_o^\infty d\mu\,q(\mu;\Sigma,\Gamma)\left[\arctan\frac{b-\mu}{\delta}+\arctan\frac{\mu-a}{\delta}\right] \tag{7.9}$$

Next we wish to bring the limit $\delta \to 0+$ through the $d\mu$ integral, via an application of the dominated convergence theorem. Consider splitting the $d\mu$ integral into a part where $\mu > 2b$ and a part where $\mu \leq 2b$. Let us examine the $\mu > 2b$ integral first. Recall the arctan identity (cf. [AS 72], 4.4.34)

$$\arctan z_1 \pm \arctan z_2 = \arctan \frac{z_1 \pm z_2}{1 \mp z_1 z_2}$$

and the simple estimate $|\arctan \xi| \leq |\xi|$ ($\xi \in \mathbb{R}$). Then for all $0 \leq \delta \leq 1$,

$$\begin{aligned} \left| \arctan \frac{b-\mu}{\delta} + \arctan \frac{\mu - a}{\delta} \right| &\leq \frac{\delta(b-a)}{(\mu-a)(\mu-b)+\delta^2} \\ &\leq \frac{(b-a)}{(\mu-a)(\mu-b)} \\ &\leq \frac{c_{a,b}}{1+\mu^2} \end{aligned}$$

Thus for the $\mu > 2b$ integral, the integrand is majorized by the $\delta$ independent $L^1(d\mu)$ function $c_{a,b}(1+\mu^2)^{-1}|q(\mu;\Sigma,\Gamma)|$.

For the $\mu \leq 2b$ integral, we note that

$$\frac{|q(\mu;\Sigma,\Gamma)|}{1+\mu^2} \in L^1(d\mu) \Rightarrow q(\mu;\Sigma,\Gamma) \in L^1_{loc}(d\mu)$$

and the arctan's are uniformly bounded by $\pi$. Thus the integrand is bounded by an $L^1(d\mu;[0,2b])$ function that is independent of $\delta$.

The above shows we may apply the dominated convergence theorem to bring the $\delta \to 0+$ limit through the $d\mu$ integral. Evaluating this limit on the arctan's is a simple exercise, with the result

$$\lim_{\delta \to 0+} I_{ac}(\delta) = \frac{1}{2} \int_0^b d\mu\, q(\mu;\Sigma,\Gamma).$$

This gives us the first term on the right hand side of (7.6).

The evaluation of $I_s(\delta)$ should yield us the second term in (7.6). We quote the following useful result.

Suppose we have a sequence of bounded operators, $\{S_n\}_{n=1}^{\infty}$, that converge strongly to $S$ and suppose that $R$ and $T$ are Schmidt class operators. Then

$$\lim_{n\to\infty} \operatorname{Tr} RS_nT = \operatorname{Tr} RST. \tag{7.10}$$

This is a result of lemma 8.23 of reference [AJS 77].

The operator $P(\Sigma,\Gamma)E_s \in \boldsymbol{B}_2$ because

$$P(\Sigma,\Gamma)E_s = P(\Sigma,\Gamma)R(z)(H - z)E_s,$$

$(H-z)E_s \in \boldsymbol{B}$, and we have shown $P(\Sigma,\Gamma)R(z)$ to be Schmidt class. This of course implies that $P(\Sigma,\Gamma)E_sP(\Sigma,\Gamma) \in \boldsymbol{B}_1$.

Consider the maps

$$\lambda \mapsto \operatorname{Im} R(\lambda + i\delta) \tag{7.11}$$

$$\lambda \mapsto \operatorname{Tr} P(\Sigma,\Gamma)E_s \operatorname{Im} R(\lambda + i\delta)E_sP(\Sigma,\Gamma). \tag{7.12}$$

The mapping (7.11) is $\lambda$ continuous in the $\boldsymbol{B}$ norm for each $\delta > 0$, and therefore $\operatorname{Im} R(\lambda + i\delta)$ has a strong Riemann integral over $\lambda$. Using the result (7.10), the mapping (7.12) is continuous and hence its Riemann integral over $\lambda$ exists. Using the identity $R(z)E_s = E_sR(z)E_s$; the definition of the two integrals

$$\int_a^b d\lambda \ \operatorname{Tr} P(\Sigma,\Gamma)E_s \operatorname{Im} R(\lambda + i\delta)E_sP(\Sigma,\Gamma)$$

and

$$\int_a^b d\lambda \operatorname{Im} R(\lambda + i\delta);$$

the linearity of the trace; and (7.10), we have

$$I_s(\delta) = \operatorname{Tr} P(\Sigma,\Gamma)E_s\left[\int_a^b d\lambda \operatorname{Im} R(\lambda+i\delta)\right]E_sP(\Sigma,\Gamma). \tag{7.13}$$

Now neither $a$ nor $b$ are singular points of $H$. Thus we can use the standard result (cf. [AJS 77] p. 360)

$$\mathrm{s}-\lim_{\delta\to 0+}\int_a^b d\lambda \operatorname{Im} R(\lambda+i\delta) = \pi E_{[a,b]}.$$

This in conjunction with (7.10) applied to (7.13) yields

$$\begin{aligned}\lim_{\delta\to 0+} I_s(\delta) &= \pi \operatorname{Tr} P(\Sigma,\Gamma)E_sE_{[a,b]}E_sP(\Sigma,\Gamma)\\ &= \pi \operatorname{Tr} P(\Sigma,\Gamma)E_sP(\Sigma,\Gamma)\end{aligned}$$

which is the second term in the right hand side of (7.6). ◇

Next we discuss the contribution of $\operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im} R_o(\lambda+i\delta)VR_o(\lambda+i\delta)P(\Sigma,\Gamma)$. We first derive a representation of this function in the limit $\delta\to 0+$.

**Lemma 7.1:** *Let $v\in L^1\cap L^2(\mathbb{R}^3;\mathbb{C}^{s\times s})$ be hermitian. Then for all $\lambda\geq 0$,*

$$\begin{aligned}&\operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im}\left[R_o(\lambda+i0)VR_o(\lambda+i0)\right]P(\Sigma,\Gamma)\\ &\qquad = \frac{1}{16\pi}\int d\alpha\,\frac{\hat{\chi}_\Sigma^*(\alpha)\hat{v}_\Gamma(\alpha)}{|\alpha|}\ln\left(\frac{2\sqrt{\lambda}+|\alpha|}{2\sqrt{\lambda}-|\alpha|}\right)^2\end{aligned} \tag{7.14}$$

*where $\sqrt{\lambda}=\sqrt{\lambda+i0}$.*

*Proof:* Let $\delta>0$. From lemma 6.4, the function

$$J(\lambda,\delta) = \operatorname{Tr} P(\Sigma,\Gamma)\operatorname{Im}\left[R_o(\lambda+i\delta)VR_o(\lambda+i\delta)\right]P(\Sigma,\Gamma) \tag{7.15}$$

is continuous in $\delta$. Moreover if $\lambda < 0$, then

$$J(\lambda, 0) \equiv J_\lambda = 0.$$

Expanding out $\operatorname{Im}[R_o(\lambda + i\delta)VR_o(\lambda + i\delta)]$ explicitly, our expression for $J_\lambda$ is

$$\begin{aligned} J_\lambda = \frac{1}{2i} \operatorname{Tr}\Big\{ & P(\Sigma,\Gamma)R_o(\lambda + i0)V[R_o(\lambda + i0) - R_o(\lambda - i0)]P(\Sigma,\Gamma) \\ & + P(\Sigma,\Gamma)[R_o(\lambda + i0) - R_o(\lambda - i0)]VR_o(\lambda - i0)]P(\Sigma,\Gamma)\Big\}. \end{aligned}$$

Recalling the relationship between the trace of an operator and its associated kernel (cf. equation (6.50)), the expression for $J_\lambda$ can be written as

$$\begin{aligned} J_\lambda &= \frac{\sqrt{\lambda}}{(4\pi)^2} \int dx\, dy \left[ \frac{e^{2i\sqrt{\lambda}|x-y|} - e^{-2i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}|x-y|} \right] \frac{\chi_\Sigma(y) v_\Gamma(x)}{|x-y|} \\ &= \frac{\sqrt{\lambda}}{(4\pi)^2} \int dx\, dy \left\{ \int_{-1}^{1} d\xi\, e^{2i\sqrt{\lambda}|x-y|\xi} \right\} \frac{\chi_\Sigma(y) v_\Gamma(x)}{|x-y|} \\ &= \frac{2\sqrt{\lambda}}{8} \frac{1}{(2\pi)^3} \int dx\, dy \int d\Omega\, e^{2i\sqrt{\lambda}\hat{n}\cdot(x-y)} \frac{\chi_\Sigma(y) v_\Gamma(x)}{|x-y|}. \end{aligned}$$

Here, $\hat{n}$ is a unit vector in a Cartesian coordinate system whose third axis coincides with the line $x - y$ and the integral over $d\Omega$ is the integral over the unit sphere of this coordinate system. Define the vector

$$\alpha_o = 2\sqrt{\lambda}\hat{n}.$$

We note that $|x - y|^{-1}\chi_\Sigma(y)v_\Gamma(x) \in L^1(dx\, dy)$, which follows from an application of the Sobolev inequality (6.4). Then it is possible to interchange integral orders via the Fubini theorem and our expression for $J_\lambda$ becomes

$$J_\lambda = \frac{|\alpha_o|}{8}\frac{1}{(2\pi)^3}\int d\Omega \int dx\, dy\, e^{i\alpha_o\cdot(x-y)}\frac{\chi_\Sigma(y)v_\Gamma(x)}{|x-y|}$$
$$= \frac{|\alpha_o|}{8}\frac{1}{(2\pi)^{3/2}}\int d\Omega \int dx'\, e^{i\alpha_o\cdot x'}\frac{1}{|x'|}(\chi_\Sigma * \Pi v_\Gamma)(x'). \tag{7.16}$$

In the second equality of (7.16), we have made a change of variables $x \longrightarrow x' = x - y$ and we are using the definition of the convolution

$$(\psi * \varphi)(x) = \frac{1}{(2\pi)^{3/2}}\int dy\, \psi(x-y)\varphi(y).$$

We are also using the parity mapping

$$(\Pi\psi)(x) = \psi(-x).$$

Let $\epsilon > 0$ and define

$$J_\lambda^\epsilon = \frac{|\alpha_o|}{8}\frac{1}{(2\pi)^{3/2}}\int d\Omega \int dx\, \frac{e^{-\epsilon|x|}}{|x|}(\chi_\Sigma * \Pi v_\Gamma)(x)e^{i\alpha_o\cdot x}.$$

As $|x|^{-1}(\chi_\Sigma * \Pi v_\Gamma)(x) \in L^1(d\Omega\, dx)$ and the exponentials are uniformly bounded by 1, the dominated convergence theorem shows

$$\lim_{\epsilon\to 0+} J_\lambda^\epsilon = J_\lambda.$$

Define

$$\hat{f}_\epsilon(x) = \frac{e^{-\epsilon|x|}}{|x|}$$
$$\hat{g} = \chi_\Sigma * \Pi v_\Gamma. \tag{7.17}$$

It is easily verified that $\hat{f}_\epsilon \in L^1 \cap L^2(\mathbb{R}^3)$. We also have $\hat{g} \in L^1 \cap L^2(\mathbb{R}^3)$ via the Hausdorff-Young inequality;

$$\|\psi * \varphi\|_p \leq c\|\psi\|_1 \|\varphi\|_p \tag{7.18}$$

Then

$$J_\lambda^\epsilon = \frac{|\alpha_o|}{8} \int d\Omega \left[F^{-1}(\hat{f}_\epsilon \hat{g})\right](\alpha_o). \tag{7.19}$$

Here the symbol $F^{-1}$ denotes the inverse Fourier transform.

Recall if $\psi$ and $\varphi$ are both $L^1$ functions, then

$$(\psi * \varphi)\hat{} = \hat{\psi}\hat{\varphi}$$

(cf. [Ru 73], theorem 7.2). We would like to be able to use this result for our problem, but it will turn out $f_\epsilon$ and $g$ are not so nicely behaved as to apply this result directly. Nevertheless, as we shall prove in the lemma to follow,

$$F^{-1}(\hat{f}_\epsilon \hat{g}) = f_\epsilon * g.$$

Let us calculate $f_\epsilon$ and $g$:

The inverse Fourier transform of $\hat{f}_\epsilon$ is elementary to do and it is given by

$$f_\epsilon(\alpha) = \frac{2}{(2\pi)^{1/2}} \frac{1}{\epsilon^2 + \alpha^2}. \tag{7.20}$$

For the inverse Fourier transform of $\hat{g}$, we can use the result just quoted above to get

$$\begin{aligned} g(\alpha) &= [F^{-1}(\chi_\Sigma * \Pi v_\Gamma)](\alpha) \\ &= \hat{\chi}_\Sigma(-\alpha)(\Pi v_\Gamma)\hat{}(-\alpha) \\ &= \hat{\chi}_\Sigma^*(\alpha)\hat{v}_\Gamma(\alpha). \end{aligned} \tag{7.21}$$

Notice that as $\hat{\chi}_\Sigma$ and $\hat{v}_\Gamma$ are both in $L^2 \cap L^\infty(\mathbb{R}^3)$, we have $g \in L^1 \cap L^\infty(\mathbb{R}^3)$.

Thus our expression (7.19) for $J^\epsilon_\lambda$ becomes

$$J^\epsilon_\lambda = \frac{|\alpha_o|}{8} \int d\Omega \, \frac{1}{(2\pi)^{3/2}} \int d\alpha \, \frac{2}{(2\pi)^{1/2}} \frac{1}{\epsilon^2 + (\alpha_o - \alpha)^2} \hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha).$$

As the integrand is absolutely integrable, we may intechange integral orders by using the Fubini theorem. The $d\Omega$ integral may then be explicitly performed with the result

$$J^\epsilon_\lambda = \frac{1}{16\pi} \int d\alpha \, \frac{\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)}{|\alpha|} \ln \left[ \frac{\epsilon^2 + (|\alpha_o| + |\alpha|)^2}{\epsilon^2 + (|\alpha_o| - |\alpha|)^2} \right].$$

We are interested in the $\epsilon \to 0+$ limit of this function and we would like to be able to bring this limit through the integral. The integrand converges pointwise to the function

$$\frac{\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)}{|\alpha|} \ln \left( \frac{|\alpha_o| + |\alpha|}{|\alpha_o| - |\alpha|} \right)^2. \tag{7.22}$$

Furthermore we notice that

$$0 \leq \ln \left[ \frac{\epsilon^2 + (|\alpha_o| + |\alpha|)^2}{\epsilon^2 + (|\alpha_o| - |\alpha|)^2} \right] \leq \ln \left( \frac{|\alpha_o| + |\alpha|}{|\alpha_o| - |\alpha|} \right)^2.$$

Thus to apply the dominated convergence theorem, we need only study the absolute integrability of the function in (7.22). It suffices to study its integrability at $\infty$; in a neighbourhood of $\alpha_o$; and in a neighbourhood of the origin.

If $\alpha_o = 0$, the function is identically zero which is obviously $L^1(d\alpha)$. Let $|\alpha_o| > 0$. For $|\alpha| \to \infty$ we have

$$\ln \left( \frac{|\alpha_o| + |\alpha|}{|\alpha_o| - |\alpha|} \right)^2 = O\left( \frac{|\alpha_o|}{|\alpha|} \right),$$

$|\alpha|^{-1} < 1$ and $\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha) \in L^1(d\alpha)$. Thus (7.22) is absolutely integrable at $\infty$.

We are free to choose a neighbourhood of $\alpha_o \neq 0$, so as to exclude the origin. Then $|\alpha|^{-1}\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)$ is bounded on this neighbourhood and it is easily verified that $\ln\left(\frac{|\alpha_o|+|\alpha|}{|\alpha_o|-|\alpha|}\right)$ is an $L^1(d\alpha)$ function in this neighbourhood of $\alpha_o$. Thus (7.22) is absolutely integrable about $\alpha_o$.

Finally if $\alpha_o \neq 0$, there is a neighbourhood of the origin such that for all $\alpha$ in this neighbourhood,

$$\ln\left(\frac{|\alpha_o|+|\alpha|}{|\alpha_o|-|\alpha|}\right)^2 < 1.$$

Since $\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha) \in L^\infty(d\alpha)$ and $|\alpha|^{-1}$ is integrable at the origin, we see that (7.22) is absolutely integrable at the origin.

Thus we may take the $\epsilon \to 0+$ limit through the $d\alpha$ integral. Recalling $J^\epsilon_\lambda \to J_\lambda$ and $|\alpha_o| = 2\sqrt{\lambda}$, we arrive at (7.17) $\diamond$

**Lemma 7.2:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian and let $\epsilon > 0$. Then*

$$F^{-1}[\hat{f}_\epsilon(\chi_\Sigma * \Pi v_\Gamma)] = f_\epsilon * (\hat{\chi}^*_\Sigma \hat{v}_\Gamma)$$

*where $\hat{f}_\epsilon$ is defined in (7.17).*

*Proof:* It is enough to prove that

$$(f_\epsilon * g)\hat{} = \hat{f}_\epsilon \hat{g}$$

where $\hat{g}$ is defined in (7.17). From the formula (7.20) we see $f_\epsilon \in L^{\frac{3}{2}+} \cap L^\infty(\mathbb{R}^3)$, and we have previously argued that $g \in L^1 \cap L^\infty(\mathbb{R}^3)$. Thus it follows from the Hausdorff-Young inequality (cf. (7.18)) that

$$f_\epsilon * g \in L^{\frac{3}{2}+} \cap L^\infty(\mathbb{R}^3).$$

In particular it is in $L^2(\mathbb{R}^3)$, so that its Fourier transform exists in the $\ell.i.m.$ sense. That is, if $\chi_R$ denotes the characteristic function for a ball of radius $R$, centered on the origin, then

$$(f_\epsilon * g)\hat{}= \mathrm{s}-\lim_{R\to\infty}[\chi_R(f_\epsilon * g)]\hat{}. \tag{7.23}$$

By using Fubini's theorem and making a change of integration variables we can get for the pointwise representation of $[\chi_R(f_\epsilon * g)]\hat{}$,

$$[\chi_R(f_\epsilon * g)]\hat{}(x) = \frac{1}{(2\pi)^{3/2}} \int d\alpha\, e^{-i\alpha\cdot x} \hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)(f_\epsilon \tau_{-\alpha}\chi_R)\hat{}(x).$$

Here $\tau_\alpha$ denotes the translation operator

$$(\tau_\alpha\psi)(\alpha') = \psi(\alpha' - \alpha).$$

Then

$$\begin{aligned}
&\left\|\hat{f}_\epsilon\hat{g} - [\chi_R(f_\epsilon * g)]\hat{}\right\|^2 \\
&= \frac{1}{(2\pi)^3}\int dx \left|\int d\alpha\, e^{-i\alpha\cdot x}\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)\left[\hat{f}_\epsilon(x) - (f_\epsilon\tau_{-\alpha}\chi_R)\hat{}(x)\right]\right|^2 \\
&\le \frac{\|v_\Gamma\|^2}{(2\pi)^3}\int dx\, d\alpha\, |\hat{\chi}_\Sigma(\alpha)|^2 \left|\hat{f}_\epsilon(x) - (f_\epsilon\tau_{-\alpha}\chi_R)\hat{}(x)\right|^2 \\
&= \frac{\|v_\Gamma\|^2}{(2\pi)^3}\int d\alpha\, |\hat{\chi}_\Sigma(\alpha)|^2 \|\hat{f}_\epsilon - (f_\epsilon\tau_{-\alpha}\chi_R)\hat{}\|^2 \\
&= \frac{\|v_\Gamma\|^2}{(2\pi)^3}\int d\alpha\, |\hat{\chi}_\Sigma(\alpha)|^2 \|f_\epsilon - f_\epsilon\tau_{-\alpha}\chi_R\|^2 \\
&= \frac{\|v_\Gamma\|^2}{(2\pi)^3}\int d\alpha\, d\alpha' |\hat{\chi}_\Sigma(\alpha)|^2 |f_\epsilon(\alpha')|^2 \left|1 - \chi_R(\alpha' + \alpha)\right|^2.
\end{aligned}$$

The integrand tends to zero as $R \to \infty$ and it is bounded by $4|\hat{\chi}_\Sigma(\alpha)|^2|f_\epsilon(\alpha')|^2$, which is integrable. Thus we can apply the dominated convergence theorem to conclude the right hand side tends to zero as $R \to \infty$. This yields the result

$$\mathrm{s}-\lim_{R\to\infty}[\chi_R(f_\epsilon * g)]\hat{} = \hat{f}_\epsilon(\chi_\Sigma * \Pi v_\Gamma).$$

On comparing this with (7.23) we are finished. $\diamondsuit$

Next we explore the integrabilty of $J(\lambda,\delta)$ and $J(\lambda,0) \equiv J_\lambda$.

**Lemma 7.3:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$. Then $J(\lambda,\delta)$, as defined by equation (7.15), is Riemann integrable on finite intervals and*

$$\lim_{\delta\to 0+}\int_a^b d\lambda\, J(\lambda,\delta) = \int_0^b d\lambda\, J_\lambda.$$

*Proof:* From lemma 6.4, the function $\mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)VR_o(z)P(\Sigma,\Gamma)$ is continuous with respect to $z \in \Pi_c$. Thus $J(\lambda,\delta)$, which is comprised of linear combinations of these functions, is jointly continuous in $\lambda$ and $\delta$. Hence its Riemann integral with respect to $\lambda$ over any finite interval exists. Furthermore, recall that $J(\lambda,0) \equiv J_\lambda = 0$ for all $\lambda < 0$. Finally we note that

$$|\,\mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)VR_o(z)P(\Sigma,\Gamma)| \le \frac{1}{(4\pi)^2}\int dx\, dy\, \frac{\chi_\Sigma(x)|v_\Gamma(y)|}{|x-y|^2}$$

for all $z \in \Pi_c$. This implies that $J(\lambda,\delta)$ is uniformly bounded in $\lambda$ and $\delta$, which will enable us to apply the dominated convergence theorem to arrive at our conclusion. $\diamondsuit$

**Proposition 7.2:** *Let $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ be hermitian. Then*

$$\lim_{\delta\to 0+}\int_a^b d\lambda\ \mathrm{Tr}\, P(\Sigma,\Gamma)\mathrm{Im}\,[R_o(\lambda+i\delta)VR_o(\lambda+i\delta)]P(\Sigma,\Gamma)$$

$$= \frac{\sqrt{b}}{4\pi} \int_\Sigma dx\, v_\Gamma(x) + \frac{1}{2} M(b; \Sigma, \Gamma), \tag{7.24}$$

*where $M(b;\Sigma,\Gamma)$ is defined in (7.3).*

*Proof:* From lemma 7.1, equation (7.15) and lemma 7.3, the left hand side of (7.24) is

$$\frac{1}{16\pi} \int_0^b d\lambda \int d\alpha\, \frac{\hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha)}{|\alpha|} \ln \left( \frac{2\sqrt{\lambda} + |\alpha|}{2\sqrt{\lambda} - |\alpha|} \right)^2 . \tag{7.25}$$

We wish to interchange integral orders to perform the $d\lambda$ integral. It is enough to prove the the integrand is absolutely integrable and then apply Fubini's theorem. Take the absolute value of the integrand and integrate this over $d\lambda$ and $d\alpha$. We are free to interchange the order of integration here because the absolute value of the integrand is positive. Furthermore we note that the ln function is positive, and hence equal to its own absolute value. Define

$$h(b, |\alpha|) = \int_0^b d\lambda \ln \left( \frac{2\sqrt{\lambda} + |\alpha|}{2\sqrt{\lambda} - |\alpha|} \right)^2 .$$

This integrable is exact and consists of linear combinations of one dimensional integrals of the type ($\xi^2 \equiv \lambda$)

$$\int d\xi\, \xi \ln(c_1\xi + c_2) = \frac{1}{2} \left[ \xi^2 - \left( \frac{c_2}{c_1} \right)^2 \right] \ln(c_1\xi + c_2) + \frac{c_2}{2c_1}\xi - \frac{1}{4}\xi^2.$$

Using this identity and after some elementary algebra we get

$$h(b, |\alpha|) = 2\sqrt{b}|\alpha| + \left[ b - \frac{\alpha^2}{4} \right] \ln \left( \frac{2\sqrt{b} + |\alpha|}{2\sqrt{b} - |\alpha|} \right)^2 . \tag{7.26}$$

Notice that $h(b,|\alpha|) \geq 0$, because the integrand was positive. In particular, for $|\alpha| \geq 2\sqrt{b}$ this implies

$$2\sqrt{b}|\alpha| \geq \left[\frac{\alpha^2}{4} - b\right] \ln \left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 . \tag{7.27}$$

Then

$$\begin{aligned}
\int_0^b d\lambda \int d\alpha \, \frac{|\hat{\chi}_\Sigma(\alpha)||\hat{v}_\Gamma(\alpha)|}{|\alpha|} \ln \left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 \\
&= \int d\alpha \, \frac{|\hat{\chi}_\Sigma(\alpha)||\hat{v}_\Gamma(\alpha)|}{|\alpha|} h(b,|\alpha|) \\
&= 2\sqrt{b} \int d\alpha \, |\hat{\chi}_\Sigma(\alpha)||\hat{v}_\Gamma(\alpha)| \\
&\quad + \int d\alpha \, \frac{|\hat{\chi}_\Sigma(\alpha)||\hat{v}_\Gamma(\alpha)|}{|\alpha|} \left[b - \frac{\alpha^2}{4}\right] \ln \left(\frac{2\sqrt{b}+|\alpha|}{2\sqrt{b}-|\alpha|}\right)^2 .
\end{aligned}$$

In the second equality on the right hand side, the first term is finite because $\chi_\Sigma$ and $v_\Gamma$ are both $L^2(\mathbb{R}^3)$ functions. For the second term, split the $d\alpha$ integral into a part where $|\alpha| < 2\sqrt{b}$ and a part where $|\alpha| \geq 2\sqrt{b}$. For the $|\alpha| < 2\sqrt{b}$ integral, $|\alpha|^{-1}$ is in $L^1_{loc}(d\alpha)$, while the rest of the integrand is bounded. Hence this integral is finite. For the integral outside the ball of radius $2\sqrt{b}$, we use estimate (7.27) to estimate this integral by

$$2\sqrt{b} \int_{|\alpha| \geq 2\sqrt{b}} d\alpha \, |\hat{\chi}_\Sigma(\alpha)||\hat{v}_\Gamma(\alpha)| < \infty.$$

Hence we have shown we can interchange integral orders in (7.25). But we have already evaluated the $d\lambda$ integral with the result (7.26). Substituting this formula into (7.25) and using the Plancherel theorem to write

$$\int d\alpha \, \hat{\chi}^*_\Sigma(\alpha)\hat{v}_\Gamma(\alpha) = \int dx \, \chi_\Sigma(x) v_\Gamma(x)$$

we get (7.24). ◊

This concludes our study of the contributions over $C_1$, $C_2$ and $C_3$. On examining these terms we see they comprise our sum rule, provided the contribution from $C(b,\delta)$ tends to zero in the limit $\delta \to 0+$ and $b \to \infty$. This is the topic of our next section.

### 7.3 The $C(b,\delta)$ Contribution

We now wish to study the limit

$$\lim_{b\to\infty}\lim_{\delta\to 0+}\int_{C(b,\delta)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma).$$

Because the integrand is continuous and uniformly bounded with respect to $z \in \Pi_c$, it is trivial to take the $\delta \to 0+$ limit. From corollary 6.2 we have the result

$$\int_{C(b,0)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z)VR_o(z)]P(\Sigma,\Gamma)$$
$$= \sum_{n=2}^{\infty}\int_{C(b,0)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma).$$

Thus we are motivated to study the individual terms of this series first.

**Lemma 7.4:** *Let $v \in L^1 \cap \mathcal{F}^*(\mathbb{R}^3;\mathbb{C}^{s\times s})$. Then for all $n \geq 2$,*

$$\lim_{b\to\infty}\int_{C(b,0)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma) = 0.$$

*Proof:* Let $\gamma > 0$ be fixed but otherwise arbitrary. We shall make our choice for $\gamma$ more definite in the proposition to follow. Consider breaking the curve $C(b,0)$

into two pieces. The first piece is

$$\tilde{C}_1(b,\gamma) = \{z \in C(b,0) : 0 \leq \arg z \leq \theta_\gamma \text{ or } 2\pi - \theta_\gamma \leq \arg z \leq 2\pi\}$$

where $\sin\theta_\gamma = \gamma/b$. The second piece is

$$\tilde{C}_2(b,\gamma) = C(b,0)\backslash\tilde{C}_1(b,\gamma).$$

We examine contribution along $\tilde{C}_1(b,\gamma)$ first. This curve has an arc length of

$$\ell\Big(\tilde{C}_1(b,\gamma)\Big) = 2b\theta_\gamma = 2b\arcsin\frac{\gamma}{b}.$$

Now for $0 \leq \xi \leq \sqrt{3}/2$, $\arcsin\xi \leq 2\xi$. If $\gamma$ is fixed, then for $b$ sufficiently large, $\gamma/b < \sqrt{3}/2$ and the arc length of the curve satisfies the esimate

$$\ell\Big(\tilde{C}_1(b,\gamma)\Big) \leq 4\gamma.$$

Next we estimate $\operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)$. Let $0 < \theta < 1$ be fixed but otherwise arbitrary. Then for all $b > \Lambda_\theta$ and all $z$ such that $|z| = b$, we know that $\|A(z)\| \leq \theta < 1$ and

$$|\operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)| \leq c_\Gamma \theta^{n-1}$$

where

$$c_\Gamma = \frac{|M_\Gamma|^2}{(4\pi)^2}\int dx\, dy\, \frac{\chi_\Sigma(x)|v(y)|}{|x-y|^2}.$$

With these inequalities, the contribution over $\tilde{C}_1(b,\gamma)$ has the estimate

$$\left|\int_{\tilde{C}_1(b,\gamma)} dz\ \operatorname{Tr} P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\right| \leq 4c_\Gamma\gamma\theta^{n-1}, \qquad n \geq 2. \qquad (7.28)$$

Next we consider the contribution from $\tilde{C}_2(b,\gamma)$. Utilizing expression (6.41) in lemma 6.5, we have

$$\int_{\tilde{C}_2(b,\gamma)} dz \; \mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)$$

$$= \frac{\Gamma(n-\frac{1}{2})}{2^{3/2}} \int_{\tilde{C}_2(b,\gamma)} dz \int_0^{1>} d^n\xi \int d^n|\mu| \, \frac{\hat{\chi}^*_\Sigma(\alpha_1+\cdots+\alpha_n)}{(a_n-z)^{n-\frac{1}{2}}}$$
$$\times \,\mathrm{sp}\, M_\Gamma \eta(\alpha_n)\cdots\eta(\alpha_1)M_\Gamma$$

where we recall

$$a_n = \sum_{l,m=1}^{n} \theta(\xi_l,\xi_m)\alpha_l\cdot\alpha_m \geq 0$$

and

$$\theta(\xi_l,\xi_m) = \min\{\xi_l(1-\xi_m),\xi_m(1-\xi_l)\}.$$

Now for $z \in \tilde{C}_2(b,\gamma)$ we have the estimate

$$\frac{1}{|a_n-z|} \leq \begin{cases} \gamma^{\frac{1}{2}-n}, & \text{if } |\mathrm{Im}\, z| \geq \gamma; \\ b^{\frac{1}{2}-n}, & \text{if } |\mathrm{Im}\, z| \leq \gamma. \end{cases}$$

With this, it is easily shown that the integrand is an $L^1(dz\, d^n\xi\, d^n|\mu|)$ function and consequently we may interchange the order of integrals. The $dz$ integral may now be evaluated explicitly, with the result

$$\int_{\tilde{C}_2(b,\gamma)} dz \, \frac{1}{(a_n-z)^{n-\frac{1}{2}}} = \frac{1}{n-\frac{3}{2}} \left\{ \frac{1}{(a_n-\tilde{b}-i\gamma)^{n-\frac{3}{2}}} - \frac{1}{(a_n-\tilde{b}+i\gamma)^{n-\frac{3}{2}}} \right\}.$$

Here the constant $\tilde{b} = \sqrt{b^2 - \gamma^2}$. Substituting in this result, the contribution over $\tilde{C}_2(b,\gamma)$ is

$$\int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma)$$

$$= \frac{\Gamma(n-\frac{1}{2})}{(n-\frac{3}{2})2^{3/2}} \int_0^{1>} d^n\xi \int d^n|\mu|\, \hat{\chi}^*_\Sigma(\alpha_1+\cdots+\alpha_n)\,\mathrm{sp}\, M_\Gamma \eta(\alpha_n)\cdots\eta(\alpha_1)M_\Gamma$$

$$\times \left\{ \frac{1}{(a_n - \tilde{b} - i\gamma)^{n-\frac{3}{2}}} - \frac{1}{(a_n - \tilde{b} + i\gamma)^{n-\frac{3}{2}}} \right\}.$$

This integral has the estimate

$$\left| \int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma) \right|$$

$$\leq \frac{|M_\Gamma|^2 m(\Sigma)\Gamma(n-\frac{1}{2})}{(n-\frac{3}{2})2^{1/2}} \int_0^{1>} d^n\xi \int d^n|\mu| \frac{1}{[(a_n-\tilde{b})^2+\gamma^2]^{\frac{n}{2}-\frac{3}{4}}}. \tag{7.29}$$

Here, $m(\Sigma)$ is the Lebesgue measure of $\Sigma$. For all $n \geq 2$ the integrand on the right hand side of (7.29) tends to zero as $b \to \infty$ and it is uniformly bounded by

$$\gamma^{\frac{3}{2}-n} \in L^1(d^n\xi\, d^n|\mu|).$$

Therefore we can apply the dominated convergence theorem to conclude

$$\lim_{b\to\infty} \left| \int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma) \right| = 0 \qquad (n \geq 2). \tag{7.30}$$

For purposes in the next proposition, we further esimate the right hand side of (7.29) to obtain the $b$ independent bound

$$\left| \int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z)[V R_o(z)]^n P(\Sigma,\Gamma) \right| \leq \frac{\gamma^{3/2}|M_\Gamma|^2 m(\Sigma)\Gamma(n-\frac{1}{2})}{(n-\frac{3}{2})n!2^{1/2}} \left(\frac{\|\mu\|}{\gamma}\right)^n. \tag{7.31}$$

Statement (7.30) means given $\epsilon > 0$, there exists $b_o$ such that for all $b > b_o$

$$\left| \int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z) [V R_o(z)]^n P(\Sigma,\Gamma) \right| \leq \frac{\epsilon}{2}.$$

In the estimate (7.28), we are free to pick $\theta$ as small as we please. In particular, we may pick it such that $4c_{\Gamma}\gamma\theta^{n-1} < \epsilon/2$. Implicit in (7.28) is that there exists $b_1$ such that (7.28) is valid for all $b > b_1$.

Thus we have shown given $\epsilon > 0$ and for each $n \geq 2$, there exist $b'$ such that for all $b > b'$

$$\begin{aligned} &\left| \int_{C(b,0)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z) [V R_o(z)]^n P(\Sigma,\Gamma) \right| \\ &\qquad \leq \left| \int_{\tilde{C}_1(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z) [V R_o(z)]^n P(\Sigma,\Gamma) \right| \\ &\qquad\quad + \left| \int_{\tilde{C}_2(b,\gamma)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z) [V R_o(z)]^n P(\Sigma,\Gamma) \right| \\ &\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\diamondsuit$

**Proposition 7.3:** *Let $v \in L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$. Then*

$$\lim_{b\to\infty} \lim_{\delta\to 0+} \int_{C(b,\delta)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z) V R_o(z)] P(\Sigma,\Gamma) = 0.$$

*Proof:* We have already discussed the $\delta \to 0+$ limit and we know

$$\begin{aligned} I(b) &\equiv \int_{C(b,0)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma)[R(z) - R_o(z) + R_o(z) V R_o(z)] P(\Sigma,\Gamma) \\ &= \sum_{n=2}^{\infty} (-1)^n \int_{C(b,0)} dz \ \mathrm{Tr}\, P(\Sigma,\Gamma) R_o(z) [V R_o(z)]^n P(\Sigma,\Gamma). \end{aligned}$$

Dividing the curve $C(b,0)$ up as in the proof of lemma 7.4, $I(b)$ can be estimated by

$$\begin{aligned}|I(b)| \leq& \sum_{n=2}^{\infty}\left|\int_{\tilde{C}_1(b,\gamma)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\right| \\ &+\sum_{n=2}^{\infty}\left|\int_{\tilde{C}_2(b,\gamma)} dz\ \mathrm{Tr}\, P(\Sigma,\Gamma)R_o(z)[VR_o(z)]^n P(\Sigma,\Gamma)\right| \\ \equiv& I_1(b)+I_2(b).\end{aligned}$$

From inequality (7.28) we have

$$I_1(b) \leq 4\gamma c_\Gamma \sum_{n=2}^{\infty}\theta^{n-1} = 4\gamma c_\Gamma \frac{\theta}{1-\theta}.$$

The right hand side can be made as small as desired, with the corresponding $\Lambda_\theta$'s increasing in size. This demonstrates that

$$\lim_{b\to\infty} I_1(b) = 0.$$

We need to be more careful with $I_2(b)$. In the proof of lemma 7.4 we demonstrated that each term in the series for $I_2(b)$ tended to zero as $b \to \infty$. From inequality (7.31), the series for $I_2(b)$ is majorized by the $b$ independent series

$$c\sum_{n=2}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{(n-\frac{3}{2})n!}\left(\frac{\|\mu\|}{\gamma}\right)^n \leq \sqrt{\pi}c\sum_{n=2}^{\infty}\frac{1}{(n-1)n}\left(\frac{\|\mu\|}{\gamma}\right)^n.$$

Here we have grouped all the constants on the right hand side of (7.31) into $c$. The series on the right hand side above converges if $\|\mu\| \leq \gamma$. Since $\gamma$ was an arbitrary parameter, we are free to pick it equal to $\|\mu\|$. Thus the series for $I_2(b)$ converges uniformly with respect to $b$ and consequently

$$\lim_{b\to\infty} I_2(b) = 0. \qquad \diamondsuit$$

In conclusion, we have shown the existence of a new class of sum rules given by formulas (7.2) and (7.4). The first term on the left hand side of (7.2) (before the $b \to \infty$ limit is taken) is the integral over energy of the time delay function $q(\lambda; \Sigma, \Gamma)$. This integral does not converge as the upper energy limit tends to $\infty$. From equations (6.24) and (6.30), we see the relationship between $q(\lambda; \Sigma, \Gamma)$ and the resolvent difference $P(\Sigma, \Gamma)[R(z) - R_o(z)]P(\Sigma, \Gamma)$ is

$$q(\lambda; \Sigma, \Gamma) = 2 \lim_{\mu \to 0+} \Big\{ \mathrm{Tr}\, P(\Sigma, \Gamma) \mathrm{Im}\, R_{ac}(\lambda + i\mu) P(\Sigma, \Gamma) \\ - \mathrm{Tr}\, P(\Sigma, \Gamma) \mathrm{Im}\, R_o(\lambda + i\mu) P(\Sigma, \Gamma) \Big\}.$$

By using the Born series expansion of the resolvent difference, it is possible to cancel exactly the large energy divergence of the integral of $q(\lambda; \Sigma, \Gamma)$. The first term of the Born series suffices to control this singularity and this is the origin of the second and third terms on the left hand side of (7.2). The right hand side of (7.2) is related to the number of bound states that have support in the spatial region $\Sigma$ and spinor subspace $\Gamma$. Its originates from the pole contributions of the exact resolvent to the contour integral performed in chapter 7.

The sum rules given in (7.2) and (7.4) are structurally different from those rules obtained via the spin channel formalism. The difference in structure occurs between the placement of the spin projection operators relative to the Møller operators (cf. equations (6.21) and (6.22)). The spin channel formalism places these projection operators to the exterior of the product of the two Møller operators. This has the advantage of allowing one to decompose the $S$-matrix into a matrix over the spin channels and the global time delay may be related directly to the $S$-matrix (cf. reference [BO 76], equation (4.12)). Our formalism places the projection operator between the Møller operators. While we have a particularly simple physical interpretation of the time delay we have studied, its connection to the $S$-matrix is more opaque than the spin channel case and we leave this as a problem to be addressed to in future studies. We note that our class of rules can also incorporate a spin channel formalism. This is because we have not yet exploited any degree of freedom in the choice of the asymptotic in and out states. By picking particular polarizations of these states, we obtain the spin channel formalism.

Finally some remarks about the restriction of $v$ to $L^1 \cap \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$. A possible method for showing the contribution along $C(b,\delta)$ tends to 0 is given by Buslaev [Bu 67]. In his technique, he uses an elliptical co-ordinate system to remove the $|x-y|$ singularity in the kernel of $R_o(z)$. By making smoothness assumptions on the potential he was able to integrate by parts one of the integrals that appear in the trace (cf. equation (6.50)). Each integration by parts would pull down a factor of $(\sqrt{z})^{-1}$ from the exponential appearing in the free resolvent kernel $R_o(x,y;z)$. After a sufficient number of integrations by parts, there exist enough decay in $z$ that the contribution around the contour $C(b,\delta)$ will tend to zero as $b \to \infty$. This technique has the advantage of using only smoothness properties of the potential so that the assumption $v \in L^1 \cap L^2(\mathbb{R}^3; \mathbb{C}^{s\times s})$ should only require minimal strengthening.

On the other hand, if one is interested in higher moment sum rules like those in equation (1.12), then one must use a large energy ($z$) asymptotic expansion of $\operatorname{Tr} P(\Sigma,\Gamma)[R(z) - R_o(z)]P(\Sigma,\Gamma)$. A closely related expansion has been worked out by Osborn et. al. ([OCF 85], theorem 3). We remark that this asymptotic series is in the variable $z$, whereas the Born series is a series in the coupling constant that can be associated with $v$. It is because these two series coincide to lowest order that we were able to utilize the Born series in our analysis. For the higher moment sum rules we must subtract away the leading order terms of the large $z$ asymptotic expansion of $\operatorname{Tr} P(\Sigma,\Gamma)[R_(z) - R_o(z)]P(\Sigma,\Gamma)$ in order to cancel the energy growth at $\infty$. The higher the power in the moment, the more terms in the asymptotic expansion that must be used. To be able to utilize this expansion, we must subsume the hypotheses of Osborn et. al. [OCF 85] that $v \in \mathcal{F}^*(\mathbb{R}^3; \mathbb{C}^{s\times s})$. In theorem 3 of reference [OCF 85], a further smoothness in the potential was assumed so that the asymptotic expansion of the difference between the total and free resolvent kernels could be carried out to higher orders. We would also still require $v \in L^1(\mathbb{R}^3; \mathbb{C}^{s\times s})$ in order to gain control over the contribution of that part of $C(b, \delta)$ near the positive real axis. This should enable us to utilize arguments similar to those presented in section 7.3. In light of these details, the class of potentials studied for our sum rule would be consistent with the class used to study the higher moment sum rules.

# REFERENCES


[AS 72] M. Abramowitz and I. A. Stegun, **Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables** (Dover, New York, 1972)

[AH 76] S. Albeverio and R. J. Høegh-Krohn, **Mathematical Theory of Feynman Path Integrals** (Springer-Verlag, Berlin, 1976)

[AJS 77] W. O. Amrein, J. M. Jauch, and K. B. Sinha, **Scattering in Quantum Mechanics** (Benjamin, Reading MA., 1977)

[Ar 82] H. Arodź, *Colored, Spinning Classical Paritcle in an External Non-Abelian Gauge Field*, Phys. Lett. **116B**(4), pp. 251–254 (1982)

[Bo 86] D. Bollé, *Sum Rules in Scattering Theory and its Application to Statistical Mechanics*, in **Mathematics and Physics: Lectures on Recent Results** Volume 2 (World Scientific, Singapore, 1986)

[BDO 86] D. Bollé, C. Danneels, and T. A. Osborn, *Local and Global Spectral Shift Functions in* $\mathbb{R}^2$, University of Manitoba Preprint (1986)

[BO 76] B. Bollé and T. A. Osborn, *Concepts of Multiparticle Time Delay*, Phys. Rev. D **13**(2), pp. 299–311 (1976)

[BR 84] D. Bollé and D. Roekaerts, *Equation for the WKB Approximation and the Wigner-Kirkwood Expansions for Systems with Magnetic Field*, Phys. Rev. A **30**, pp. 2024–2038 (1984)

[BR 85] D. Bollé and D. Roekaerts, *Expansion for the Thermal Kernel of Systems with Nonuniform Magnetic Fields*, Phys. Rev. A **31**, pp. 1667–1680 (1985)

[BR 86] D. Bollé and D. Roekaerts, Erratum: *Expansion for the Thermal Kernel of Systems with Nonuniform Magnetic Fields*, Phys. Rev. A **33**, p. 1427 (1986)

[Bu 67] V. S. Buslaev, *Trace Formulas and Some Asymptotic Estimates of the Resolvent Kernel of the Three-Dimensional Schroedinger Equation*, in Spectral Theory and Wave Processes, Volume 1 (Consultants Bureau, London, 1967) pp. 69–85

[DS 76] N. Dunford and J. T. Schwartz, **Linear Operators** Part 1 (Interscience, New York, 1976) pp. 160–162

[Dy 49,1] F. J. Dyson, *The Radiation Theories of Tomonaga, Schwinger, and Feynman*, Phys. Rev. **75**, pp. 486–502 (1949)

[Dy 49,2] F. J. Dyson, *The S Matrix in Quantum Electrodynamics*, Phys. Rev. **75**, pp. 1736–1754 (1949)

[GR 80] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series and Products** Corrected and Enlarged Edition (Academic Press, New York, 1980)

[HS 78] P. R. Halmos and V. S. Sunder, **Bounded Integral Operators on $L^2$ Spaces** (Springer-Verlag, Berlin, 1978)

[Hi 72] E. Hille, **Methods in Classical and Functional Analysis** (Addison-Wesley, Reading Massachusetts, 1972)

[It 61] K. Ito, *Wiener Integral and Feynman Integral*, in Proc. Fourth Berkeley Symp. Math. Stat. and Prob., Volume 2 (Univ. California Press, Berkeley, 1961) pp. 227–238

[It 67] K. Ito, *Generalized Uniform Complex Measures in Hilbertian Metric Space with Their Application to Feynman Integral*, in Proc. Fifth Berkeley Symp. Math. Stat. and Prob., Volume 2 Part 1 (Univ. California Press, Berkeley, 1967) pp. 145–161

[JK 79] Arne Jensen and Tosio Kato, *Spectral Properties of Schrödinger Operators and Time-Decay of the Wave Functions*, Duke Math. J. **46**(3), pp. 583–611 (1979)

[JSM 72] J. M. Jauch, K. B. Sinha, and B. N. Misra, *Time-Delay in Scattering Processes*, Helvetica Physica Acta **45**, pp. 398–426 (1972)

[Ka 84] T. Kato, **Perturbation Theory for Linear Operators** Second Edition (Springer-Verlag, Berlin, 1984)

[Ka 51] T. Kato *Fundamental Properties of Hamiltonian Operators of the Schrödinger Type*, Trans. Am. Math. Soc. **70**, pp. 195–211 (1951)

[Kr 71] S. G. Kreĭn, **Linear Differential Equations in Banach Space** (American Mathematical Society, Providence, 1971)

[LL 58] L. D. Landau and E. M. Lifshitz, **Quantum Mechanics: Nonrelativistic Theory** (Pergamon Press, London, 1958)

[La 69] S. Lang, **Analysis I** Second Printing (Addison-Wesley, Reading Massachusetts, 1969)

[La 83] S. Lang, **Real Analysis** Second Edition (Addison-Wesley, Reading Massachusetts, 1983)

[MS 70] P. Marmier and E. Sheldon, **Physics of Nuclei and Particles** Volume II (Academic Press, New York, 1970), §14.1 pp.1087–1127

[MO 80] L. W. MacMillan and T. A. Osborn, *Transit Time Measures in Scattering Theory*, Ann. Phys. **126**(1), pp. 1–19 (1980)

[Mor 83] K. Moriyasu, **An Elementary Primer for Gauge Theory** (World Scientific, Singapore, 1983)

[N 77] Roger G. Newton, *Noncentral Potentials: The Generalized Levinson Theorem and the Structure of the Spectrum*, J. Math. Phys. **18**(7), pp. 1348–1357 (1977)

[OCF 85] T. A. Osborn, R. A. Corns, and Y. Fujiwara, *Schrödinger Semigroups for Vector Fields*, J. Math. Phys. **26**(3), pp. 453–464 (1985)

[OF 83] T. A. Osborn and Y. Fujiwara, *Time Evolution Kernels: Uniform Asymptotic Expansions*, J. Math. Phys. **24**(5), pp. 1093–1103 (1983)

[OM 86] T. A. Osborn and F. H. Molzahn, *Structural Connections Between Two Semiclassical Approximations: The WKB and the Wigner-Kirkwood Approximations*, Phys. Rev. A **34**, pp. 1696–1707 (1986)

[OPC 87] T. A. Osborn, L. Papiez, and R. Corns, *Constructive Representations of Propagators for Quantum Systems with Electromagnetic Fields*, J. Math. Phys. **28**(1), pp. 103–123 (1987)

[OSBD 85] T. A. Osborn, K. B. Sinha, D. Bollé, and C. Danneels, *Spectral Sum Rules for Time Delay in* $\mathbb{R}^2$, J. Math. Phys. **26**(11), pp. 2796–2802 (1985)

[POM t.a.] L. Papiez, T. A. Osborn, and F. H. Molzahn, *Quantum Systems with External Electromagnetic Fields: The Large Mass Asymptotics*, to appear J. Math. Phys.

[RS 75] M. Reed and B. Simon, **Fourier Analysis, Self-Adjointness** (Academic Press, New York, 1975)

[RSz 78] F. Riesz and B. Sz.-Nagy, **Functional Analysis** Seventh Printing, (Ungar, New York, 1978)

[Roy 68] H. L. Royden, **Real Analysis** Second Edition (Macmillan, New York, 1968)

[Ru 74] W. Rudin, **Real and Complex Analysis** Second Edition (McGraw-Hill, New York, 1974)

[Ru 61] W. Rudin, **Fourier Analysis on Groups** (Interscience, New York, 1961)

[Ru 76] W. Rudin, **Principles of Mathematical Analysis** Third Edition (McGraw-Hill, New York, 1976)

[Ru 73] W. Rudin, **Functional Analysis** (McGraw-Hill, New York, 1973)

[Sch 70] R. Schatten, **Norm Ideals of Completely Continuous Operators** Second Printing (Sringer-Verlag, Berlin, 1970)

[Si 82] B.Simon, *Schrödinger Semigroups*, Bull. Am. Math. Soc. **7**, pp. 447–526 (1982)

[Si 71] B. Simon, **Quantum Mechanics for Hamiltonians Defined as Quadratic Forms** (Princeton University Press, Princeton, 1971)

[TLR 83] E. Tirapegui, F. Langouche, and D. Roekaerts *Wigner-Kirkwood Expansion of the N-Body Green's Function: The Case with Magnetic Field*, Phys. Rev. A **27**, pp. 2649–2655 (1983)

[Ti 48] E. C. Titchmarsh, **Introduction to the Theory of Fourier Integrals** Second Edition (Oxford University Press, Glasgow, 1948)

[Tru 77] A. Truman, *Classical Action in Nonrelativistic Quantum Mechanics*, J. Math. Phys. **18**, pp. 1499–1509 (1977)

[Wo 70] S. K. Wong, *Field and Particle Equations for Classical Yang-Mills Field and Particles with Isotopic Spin*, Il Nuovo Cimento **65A**(4), pp. 689–694 (1970)

[Z 86] Josef A. Zuk, *Wigner-Kirkwood Expansion for Matrix Potentials*, Phys. Rev. A **33**(6), pp. 4342–4354 (1986)