The Symbolic Defect Sequence of Edge Ideals

by

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Abstract

Symbolic powers of homogeneous ideals are challenging to study, even for square-free monomial ideals. In particular, the containments between symbolic and ordinary powers of ideals have been widely studied for decades. The symbolic defect provides a measure of the difference between symbolic and ordinary powers of ideals, introduced by Galetto, Geramita, Shin, and Van Tuyl in 2019. In this thesis, we investigate the symbolic defect sequence of edge ideals of finite simple graphs. We classify exactly which terms of this sequence are non-zero and which terms are equal to one. Further, we present formulae for both the first and second non-zero terms. We describe a complete formula for the symbolic defect sequence of edge ideals of odd cycles, as well as a partial formula for the symbolic defect sequence of edge ideals of unicyclic graphs. We provide both lower and upper bounds on the terms that cannot be computed with this partial formula.

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Chapter 1

Introduction

The connection between commutative algebra and combinatorics via monomial ideals is a heavily studied topic. This theory was first introduced in 1990 through the work of Villarreal [51] and Fröberg [21], and has remained a popular and powerful area of research. For example, the widely studied edge ideal construction provides a one-to-one correspondence between quadratic square-free monomial ideals and finite simple graphs. This correspondence allows aspects of the ideal to be reduced to a combinatorial description, and vice versa. An often utilized example of this is that the primary decomposition of a quadratic square-free monomial ideal can be described through the minimal vertex covers of its corresponding graph. A detailed overview on classical results on this theory can be found in [30] and [53], and surveys of more recent results can be found in [19, 20, 27, 43, 50].

The combinatorics of a monomial ideal are encoded into its symbolic powers. For example, symbolic powers of square-free monomial ideals have a convenient description in terms of their primary decomposition, and so symbolic powers of edge ideals can be described in terms of the minimal vertex covers of their associated graphs. The study of symbolic powers of homogeneous ideals is deeply rooted in commutative algebra and remains an active area of research today. Symbolic powers are strongly connected to primary decompositions of ideals and play a role in many fields of mathematics, including commutative algebra, algebraic geometry and combinatorics. Classical results that make use of symbolic powers include Krull's proof of his Principal Ideal Theorem and Hartshorne's proof [29] of the Hartshorne-Lichtenbaum Vanishing Theorem on local cohomology. Another famous result is the Zariski-Nagata Theorem [44, 54], which states that if P is a prime ideal in a regular ring, then the s-th symbolic power of P consists of the functions that vanish up to order s on the variety $V(P)$. In 1985, Roberts [47] constructed a counterexample to Hilbert's Fourteenth Problem using the symbolic powers of prime ideals. In addition, the Eisenbud-Mazur Conjecture [18] considers containments for symbolic and ordinary squares of prime ideals in equicharacteristic

0, regular local rings, and can be translated to an extension of an essential step in Wiles' proof of Fermat's Last Theorem.

The interactions between symbolic powers of monomial ideals and their associated combinatorial structures give rise to powerful results in graph theory. A notable example of this is the Conforti-Cornuéjols Conjecture [9] on the max-flow min-cut and packing property of hypergraphs, a central problem in graph theory that can be reformulated in terms of symbolic powers of edge ideals. These properties have been studied from this algebraic viewpoint extensively [14, 16, 23, 24, 41]. Additionally, combinatorial optimization and linear programming problems are related to normal or normally torsion-free algebras via symbolic powers [14, 52], and Cohen-Macaulayness of symbolic powers of edge ideals may be studied through their graphs [46].

As demonstrated by the Zariski-Nagata Theorem, symbolic powers of homogeneous ideals have nicer geometric properties than those of ordinary powers. For example, symbolic powers of monomial ideals in the polynomial ring $k[x_0, \ldots, x_n]$ can be captured by the symbolic polyhedron [11], a convex polyhedron in $\mathbb{R}^{n+1}$. However, symbolic powers prove to be more complex algebraically than ordinary powers. Containments between the symbolic power and the ordinary power have attracted a great deal of interest among commutative algebraists. It is well-known that given a homogeneous ideal I, the containment $I^s \subset I^{(s)}$ holds for all $s \geq 1$, where $I^{(s)}$ denotes the s-th symbolic power of I. On the other hand, the reverse containment poses a much more complex problem. This is formalized in the Ideal Containment Problem: for a homogeneous ideal I and $s \in \mathbb{Z}^+$, find the smallest integer t such that $I^{(t)} \subset I^s$. Results of Hochster and Huneke [31], as well as Ein, Lazarfeld and Smith [17] show that if I is a homogeneous ideal in $k[x_0, \ldots, x_n]$ and $r \in \mathbb{Z}^+$, then $I^{(t)} \subset I^s$ for all $t \geq (n+1)s$, although this bound is not the best possible. The Ideal Containment Problem has been widely studied [4, 5, 25, 33], but has proven to be challenging, even in the case of square-free monomial ideals.

In 2019, Galetto, Geramita, Shin and Van Tuyl [22] introduced a different viewpoint on the containments between $I^{(s)}$ and I^s, where I is a homogeneous ideal in $R = k[x_0, \ldots, x_n]$. Since $I^s \subset I^{(s)}$, one may use the quotient $I^{(s)}/I^s$ to measure the difference between $I^{(s)}$ and I^s. Informally, if $I^{(s)}/I^s$ is "large", then the difference between $I^{(s)}$ and I^s is significant. The quotient $I^{(s)}/I^s$ is a finitely generated graded R-module and so the number of minimal generators provides one method of measuring this size, called the *s-th symbolic defect* of I and denoted $\operatorname{sdefect}(I, s)$. The sequence $\{\operatorname{sdefect}(I, s)\}_{s \in \mathbb{Z}^+}$ is called the *symbolic defect sequence* of I.

As this is a relatively new concept, there is currently little known about the symbolic defect sequence of ideals. The defining ideal of a star configuration is one of the few cases where this sequence is well-understood. Galetto, Geramita, Shin and Van Tuyl [22] investigate the second symbolic defect of the defining ideal of a star configuration, and Alba, Biermann, Galetto et. al. [2] give a formula for the third symbolic defect. Building on these results, Mantero [40] gives a complete formula for the symbolic defect sequence for star configurations. Further, Lin and Shen [37] build upon Mantero's results to describe a formula for the symbolic defect sequence of the ideals of generalized star configurations of hypersurfaces.

Many have focused on classifying the zero and non-zero terms of the symbolic defect sequence of ideals. This is a useful direction as $\operatorname{sdefect}(I, s) = 0$ if and only if $I^{(s)} = I^s$. Jafarloo and Malara [32] consider ideals of points lying on a rational normal curve defined in the projective plane or projective 3-space, and give particular cases where all terms of the symbolic defect sequence are non-zero. Kohne, Moreno, Sarmiento and Van Tuyl [6] show that the symbolic defect sequence of a principal Q-Borel ideal is the zero sequence. Harbourne, Kettinger and Zimmitti [28] classify the non-zero terms of the symbolic defect sequence for ideals of reduced fat point subschemes in $\mathbb{P}^n$. Jafarloo and Zito [33] classify all fat point ideals in $\mathbb{P}^n$ supported at three non-collinear points which have the zero sequence as their symbolic defect sequence.

Although edge ideals are an extensively studied topic in commutative algebra, there has been very little research on the symbolic defect sequence of edge ideals, and the literature on this topic is nearly exhausted in Section 2.4. The main goal of this thesis is to investigate the symbolic defect sequence of edge ideals. Chapter 2 covers the necessary preliminary material and surveys the literature on this topic. The remainder of this thesis consists of entirely original work. In Chapter 3, we consider an arbitrary edge ideal and classify exactly which terms of the symbolic defect sequence are non-zero, and which terms are equal to one. Further, we describe a simple formula for the first non-zero term in this sequence. Chapter 4 also considers an arbitrary edge ideal, and is dedicated to describing a formula for the second non-zero term.

We then begin studying the symbolic defect sequence of edge ideals of particular families of graphs. In Chapter 5, we describe a complete formula for the symbolic defect sequence of edge ideals of odd cycles. In Chapter 6, we extend some of our results from Chapter 5 to the symbolic defect sequence of edge ideals of unicyclic graphs. We provide a formula for $\operatorname{sdefect}(I, s)$ where s is small relative to the size of the cycle in the graph, and describe both lower and upper bounds for further terms in the sequence.

Chapter 2

Preliminaries

In this chapter we present the necessary definitions and preliminary results, as well as a brief survey of the literature on the symbolic defect sequence of edge ideals. Throughout the entirety of this thesis, k is an algebraically closed field. For the convenience of the reader, an index of notation can be found at the end of this thesis.

2.1 Graph theory and edge ideals

We begin by exploring the connection between graph theory and commutative algebra. The goal of this section is to establish a dictionary between these two fields and provide a brief overview of basic graph theory.

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a set of *vertices* and $E(G)$ is a set of unordered pairs from $V(G)$, called *edges*. The graph G is called *finite* if $V(G)$ is finite and *simple* if $E(G)$ contains no repeated edges and no edges of the form $\{x_i, x_i\}$. Throughout the entirety of this thesis, all graphs are assumed to be finite and simple. Graphs are in one-to-one correspondence with quadratic square-free monomial ideals, which we now define.

Let $I \subseteq k[x_0, \ldots, x_n]$ be a *monomial ideal*, meaning that I is minimally generated by a unique finite set of monomials, denoted by $\mathcal{G}(I)$. The ideal I is called *quadratic* if $\mathcal{G}(I)$ consists of monomials of degree 2, and *square-free* if $\mathcal{G}(I)$ consists of monomials of the form $x_0^{a_0} \cdots x_n^{a_n}$ with $a_0, \ldots, a_n \in \{0, 1\}$.

Definition 2.1. Let G be a finite simple graph on vertex set $\{x_0, \ldots, x_n\}$. The ideal

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E(G)) \subset k[x_0, \ldots, x_n]$$

is called the *edge ideal* of G.

Example 2.2. Consider the triangle graph G on vertex set $\{x_0, x_1, x_2\}$ with edge set $\{\{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\}\}$. The edge ideal of G is

$$I(G) = (x_0x_1, x_1x_2, x_0x_2) \subset k[x_0, x_1, x_2]$$

and $\mathcal{G}(I(G)) = \{x_0x_1, x_1x_2, x_0x_2\}$.

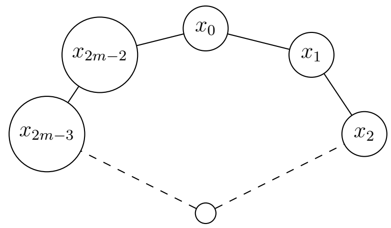
Edge ideals are quadratic square-free monomial ideals, and thus every graph corresponds to a quadratic square-free monomial ideal. Further, this construction can be reversed: given a quadratic square-free monomial ideal $I \subset k[x_0, \ldots, x_n]$, we identify I with the graph having vertex set $\{x_0, \ldots, x_n\}$ and edge set $\{\{x_i, x_j\} : x_ix_j \in \mathcal{G}(I)\}$. Note that due to this one-to-one correspondence, we often refer to vertices of a graph and variables in $k[x_0, \ldots, x_n]$ interchangeably.

Using monomial ideals to reduce algebraic problems to combinatorial descriptions was introduced in 1990. Villarreal [51] was the first to define edge ideals and used this construction to define and study Cohen-Macaulay graphs. Other early examples of studying monomial ideals through combinatorics includes the work of Fröberg [21], who studied Stanley-Reisner rings via simplicial complexes in 1990, and the work of Simis, Vasconcelos and Villarreal [48], who investigated the interactions between normality and Cohen-Macaulayness of an edge ideal and properties of its corresponding graph in 1992. Relating monomial ideals and combinatorial objects has remained a widely studied and valuable topic. The edge ideal construction can be extended to describe a one-to-one correspondence between finite simple hypergraphs and square-free monomial ideals [50, Construction 1.41], although we will not be studying this construction throughout this thesis.

We now collect some preliminary graph theory definitions. Additional preliminary graph theory notions will also be discussed in Chapter 6. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$. A *subgraph* of G is a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given $X \subseteq V(G)$, the *induced subgraph* of G on X is the graph, denoted by $G[X]$, with vertex set X and edge set $\{e \in E(G) : e \subseteq X\}$. If $F \subseteq E(G)$, then $G \setminus F$ is defined to be the graph with vertex set $V(G)$ and edge set $E(G) \setminus F$.

Definition 2.3. Given $m \geq 2$, the graph with vertex set $\{x_0, \ldots, x_{2m-2}\}$ and edge set $\{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{2m-3}, x_{2m-2}\}, \{x_{2m-2}, x_0\}\}$ is the $(2m-1)$ - *cycle*, denoted C_{2m-1} .

A $(2m-1)$ - cycle is drawn below.



The dashed lines represent the continuation of the cycle, and vertices will always be labelled clockwise in this thesis. A *chord* in a $(2m-1)$ - cycle is an edge between two vertices x_i and x_j, where $x_j \neq x_{i+1}$ and $x_i \neq x_{j+1}$, with addition on indices being done modulo $2m-1$.

Definition 2.4. A graph G is *unicyclic* if it contains a unique cycle as a subgraph.

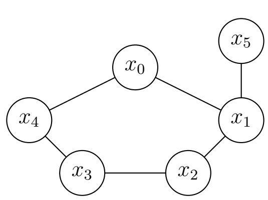
Unicyclic graphs are often defined to be connected in the literature, but for this thesis, unicyclic graphs need not be connected. Notice that in Definition 2.4, we do not require that the graph is connected. The *complete graph* on n vertices, denoted K_n, is the graph on n vertices where every pair of vertices is connected by an edge. For example, K_3 and C_3 are both the triangle graph. Given a monomial $w = x_0^{a_0} \cdots x_n^{a_n} \in k[x_0, \ldots, x_n]$, we define the *support* of w to be $\operatorname{supp}(w) = \{x_i : a_i > 0\}$.

Definition 2.5. Let G be a graph and suppose $\{x_0, \ldots, x_{2m-2}\} \subseteq V(G)$ is such that $G[\{x_0, \ldots, x_{2m-2}\}] = C_{2m-1}$. We call the monomial $x_0 \cdots x_{2m-2}$ a $(2m-1)$ - *cycle monomial.* Similarly, if $\{x_i, x_j\} \in E(G)$ we call the monomial $x_i x_j$ an *edge monomial.* In general, if H is a graph and w is a square-free monomial satisfying $G[\operatorname{supp}(w)] = H$, we call w an H *monomial.*

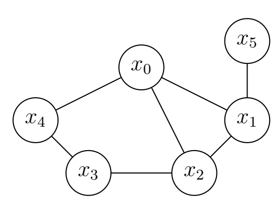
Definition 2.6. Let G and H be graphs. If $X \subseteq V(G)$ is such that $G[X] = H$, we say that G contains an *induced copy* of H on X. If $G[X] \neq H$ but there exist edges $e_0, \ldots, e_t \in E(G)$ such that $G[X] \setminus \{e_0, \ldots, e_t\} = H$, then we say that G contains a *copy* of H on X.

The terminology of Definitions 2.5 and 2.6 are not standard (with the exception of edge monomials), but will greatly simplify definitions and notation appearing in later chapters.

Example 2.7. The graph H_1 contains an induced copy of a 5 - cycle on vertices $\{x_0, x_1, x_2, x_3, x_4\}$ and the graph H_2 contains a copy of a 5 - cycle on vertices $\{x_0, x_1, x_2, x_3, x_4\}$.



H_1



H_2

Note that for the graph H_1, the monomial $x_0x_1x_2x_3x_4$ is a 5 - cycle monomial because $H_1[\{x_0, x_1, x_2, x_3, x_4\}] = C_5$. On the other hand, for the graph H_2, the monomial $x_0x_1x_2x_3x_4$ is not a 5 - cycle monomial because $H_2[\{x_0, x_1, x_2, x_3, x_4\}] \neq C_5$. Instead, this induced subgraph is a 5 - cycle with a chord.

Remark 2.8. Let G and H be graphs.

1. There is a one-to-one correspondence between induced copies of $(2m-1)$ - cycles in G and $(2m-1)$ - cycle monomials. In general, there is a one-to-one correspondence between induced copies of H in G and H monomials. We do not include the term "induced" in the definition of H monomials for sake of brevity.

2. The set of copies of H in G and the set of induced copies of H in G are disjoint. In particular, the set of induced copies of H in G is not contained in the set of copies of H in G.

The following graph theory definition allows for a combinatorial description of the symbolic powers of edge ideals (see Proposition 2.16 for a precise statement).

Definition 2.9. A *vertex cover* for a graph G is a set $X \subseteq V(G)$ such that for all edges $e \in E(G)$, $X \cap e \neq \emptyset$. A *minimal vertex cover* of G is a vertex cover that is minimal under inclusion.

Given a vertex cover X of a graph G with vertex set $\{x_0, \ldots, x_n\}$, we define the *vertex weight* of the monomial $w = x_0^{a_0} \cdots x_n^{a_n} \in k[x_0, \ldots, x_n]$ to be

$$W_X(w) = \sum_{x_i \in X} a_i.$$

We conclude this section with an important lemma regarding the vertex weights of $(2m-1)$ - cycle monomials.

Lemma 2.10. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$. Fix $m \geq 2$ and suppose that G contains a copy or an induced copy of a $(2m-1)$ - cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$. If X is a vertex cover for G, then*

$$W_X(x_0 \cdots x_{2m-2}) \geq m.$$

Proof. This is equivalent to showing that X contains at least m vertices from $\{x_0, \ldots, x_{2m-2}\}$. Let H denote the subgraph of $G[\{x_0, \ldots, x_{2m-2}\}]$ that is a $(2m-1)$ - cycle. In other words, H is obtained by removing any chords (if they exist) from this induced subgraph. Then H contains $2m-1$ edges and each vertex in H is incident to exactly two edges in H. If X

contains at most $m-1$ vertices from $\{x_0, \ldots, x_{2m-2}\}$, then the vertices in X are incident to at most $2(m-1)$ edges in H. Thus at least one edge in H does not contain any vertices from X. This same edge in G does not contain any vertices from X, contradicting the assumption that X is a vertex cover of G. Thus X contains at least m vertices from $\{x_0, \ldots, x_{2m-2}\}$ and so $W_X(x_0 \cdots x_{2m-2}) \geq m$. □

2.2 Symbolic powers of homogeneous ideals

In this section we formally define the symbolic power of an ideal and outline the relationship between symbolic powers of an edge ideal and the minimal vertex covers of its associated graph. Throughout, let R be a *Noetherian* ring, meaning that R satisfies the ascending chain condition. We begin with some preliminary definitions and facts.

The *radical* of an ideal $I \subset R$ is the ideal $\sqrt{I} = \{x \in R : \exists\, r \in \mathbb{Z}^+ \text{ such that } x^r \in I\}$. An ideal is called a *radical ideal* if it is equal to its radical. A proper ideal $I \subset R$ is called *primary* if $xy \in I$ implies that $x \in \sqrt{I}$ or $y \in \sqrt{I}$. The radical of a primary ideal is a prime ideal [1, Proposition 4.1]. A *primary decomposition* of an ideal $I \subset R$ is an expression of I as an intersection of finitely many primary ideals. A *minimal primary decomposition* of I is a primary decomposition $I = \cap_{i=0}^{t} Q_i$ where the radical of each Q_i is distinct, and no Q_i may be omitted from the intersection. By the Lasker-Noether Theorem [36, 45], every ideal in a Noetherian ring has a minimal primary decomposition, and in the case of radical ideals, this decomposition is unique. Note that square-free monomial ideals of $k[x_0, \ldots, x_n]$ are radical, and hence edge ideals are radical. When $I = I(G)$ is an edge ideal, the minimal primary decomposition of I has a convenient description in terms of the minimal vertex covers of G.

Definition 2.11. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and let $X_0, \ldots, X_t$ denote the minimal vertex covers of G. For each $0 \leq i \leq t$, define

$$(X_i) = (x_j : x_j \in X_i).$$

Proposition 2.12. *[50, Corollary 1.35] Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and let $I = I(G) \subset k[x_0, \ldots, x_n]$. If $X_0, \ldots, X_t$ denote the minimal vertex covers of G, then*

$$I = \bigcap_{i=0}^{t} (X_i)$$

is the minimal primary decomposition for I.

Example 2.13. Let $I = I(C_3) \subset k[x_0, x_1, x_2]$. The minimal vertex covers of C_3 are given by all sets of 2 vertices, and so the minimal primary decomposition of I is

$$I = (x_0, x_1) \cap (x_0, x_2) \cap (x_1, x_2).$$

Let $I \subset R$ be an ideal with minimal primary decomposition $I = \cap_{i=0}^{t} Q_i$. The set of prime ideals $\sqrt{Q_i}$ are called the *associated primes* of I, denoted $\mathrm{Ass}(I)$. The set of *minimal primes* of I, denoted $\mathrm{Min}(I)$, is the set of elements of $\mathrm{Ass}(I)$ that are minimal under inclusion. The set of *embedded primes* of I consists of the associated primes of I that are not minimal primes. Radical ideals, and thus edge ideals, do not have any embedded primes. We are now in position to define symbolic powers of ideals.

Definition 2.14. Let R be a Noetherian ring and $I \subset R$ an ideal. The *s-th symbolic power* of I, denoted $I^{(s)}$, is the ideal

$$I^{(s)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^s R_P \cap R) \subseteq R,$$

where R_P denotes the localization of R at the prime ideal P, and $- \cap R$ denotes the pre-image in R via the natural map $R \to R_P$.

See [1, Chapter 3] for a background on the process of localization. There is no consensus on the definition of symbolic powers in the literature. Some take the intersection over all associated primes of I and some take the intersection over all minimal primes of I. In general, this does result in different ideals. As noted before Definition 2.14, the set of associated primes and minimal primes coincide in the case of edge ideals and so this distinction is not relevant in this thesis. In the context of square-free monomial ideals, symbolic powers take on a simpler definition.

Theorem 2.15. *[3, Theorem 2.5] If $I \subset k[x_0, \ldots, x_n]$ is a square-free monomial ideal with minimal primary decomposition $I = \cap_{i=0}^{t} Q_i$, then for all $s \geq 1$,*

$$I^{(s)} = \bigcap_{i=0}^{t} Q_i^s.$$

From Proposition 2.12 and Theorem 2.15, we immediately obtain the following result for symbolic powers of edge ideals.

Proposition 2.16. *[3, Lemma 2.6] Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I =$*

$I(G) \subset k[x_0, \ldots, x_n]$. If $X_0, \ldots, X_t$ denote the minimal vertex covers of G, then

$$I^{(s)} = \bigcap_{i=0}^{t} (X_i)^s.$$

In particular, a monomial w lies in $I^{(s)}$ if and only if $W_{X_i}(w) \geq s$ for all $0 \leq i \leq t$.

Example 2.17. Let $I = I(C_3) \subset k[x_0, x_1, x_2]$. By Example 2.13 and Proposition 2.16,

$$\begin{aligned} I^{(2)} = (x_0, x_1)^2 \cap (x_0, x_2)^2 \cap (x_1, x_2)^2 &= (x_0^2, x_1^2, x_0x_1) \cap (x_0^2, x_2^2, x_0x_2) \cap (x_1^2, x_2^2, x_1x_2) \\ &= (x_0^2x_1^2, x_0^2x_2^2, x_1^2x_2^2, x_0x_1x_2). \end{aligned}$$

Note in particular that $I^2 \subsetneq I^{(2)}$.

We conclude this section by outlining basic properties of symbolic powers of ideals with no embedded primes in a Noetherian ring. Recall that edge ideals are in this category.

Proposition 2.18. *[25, Lemma 1.18] If I is an ideal with no embedded primes in a Noetherian ring, then*

1. *$I^{(1)} = I$;*
2. *for all $s \geq 1$, we have $I^s \subseteq I^{(s)}$;*
3. *$I^s \subseteq I^{(t)}$ if and only if $s \geq t$;*
4. *if $s \geq t$, then $I^{(s)} \subseteq I^{(t)}$;*
5. *for all $s, t \geq 1$, we have $I^{(s)} I^{(t)} \subseteq I^{(s+t)}$.*

2.3 The symbolic defect

In this section we briefly introduce the symbolic defect of an ideal. Let $I \subset R = k[x_0, \ldots, x_n]$ be a homogeneous ideal and $s \in \mathbb{Z}^+$. By Proposition 2.18, $I^s \subset I^{(s)}$, so that the R-module $I^{(s)}/I^s$ is well-defined. Further, since R is a Noetherian ring, this module is Noetherian and so $I^{(s)}/I^s$ is a finitely generated graded R-module. Thus the number of minimal generators of this module is an invariant.

Definition 2.19. Let I be a homogeneous ideal in $R = k[x_0, \ldots, x_n]$. Given $s \geq 1$, the *s-th symbolic defect* of I, denoted $\operatorname{sdefect}(I, s)$, is the number of minimal generators of $I^{(s)}/I^s$. The *symbolic defect sequence* of I is the sequence $\{\operatorname{sdefect}(I, s)\}_{s \in \mathbb{Z}^+}$.

Example 2.20. Let C_3 denote the 3 - cycle on vertices $\{x_0, x_1, x_2\}$ and $I = I(C_3) \subset k[x_0, x_1, x_2]$. As seen in Example 2.17,

$$I^{(2)}/I^2 = (x_0^2x_1^2, x_0^2x_2^2, x_1^2x_2^2, x_0x_1x_2) + I^2 = (x_0x_1x_2) + I^2.$$

Thus sdefect$(I, 2) = 1$.

Although the number of minimal generators of $I^{(s)}/I^s$ is an invariant of this module, there is not a unique minimal generating set.

Example 2.21. In Example 2.20 we showed that if $I = I(C_3) \subset k[x_0, x_1, x_2]$, then $I^{(2)}/I^2 = (x_0x_1x_2) + I^2$. However, we also have

$$(x_0x_1x_2 + x_0^2x_1^2) + I^2 = (x_0x_1x_2) + I^2 = I^{(2)}/I^2.$$

Remark 2.22. Throughout this thesis, if $w \in I^{(s)}$ is a monomial, we will often refer to the coset $w + I^s$ as a monomial in $I^{(s)}/I^s$. Although minimal generating sets for $I^{(s)}/I^s$ are not unique, in the case that I is a monomial ideal, there is a unique minimal monomial generating set for $I^{(s)}/I^s$, namely

$$\{w + I^s : w \in \mathcal{G}(I^{(s)}), w \notin I^s\}.$$

We use the following underlying method to compute or bound sdefect(I, s) throughout this thesis.

Method 2.23. *Let I be a homogeneous monomial ideal in $R = k[x_0, \ldots, x_n]$. Given $s \geq 1$, sdefect(I, s) can be computed/bounded via the following method.*

1. *Describe a monomial generating set for $I^{(s)}/I^s$.*

2. *If the monomial generating set described in (1) is not the minimal monomial generating set, then extract the minimal monomial generating set.*

3. *Count/bound the cardinality of the minimal monomial generating set.*

2.4 The symbolic defect sequence of edge ideals

We now survey the literature on the symbolic defect sequence of edge ideals. As the symbolic defect was first defined in 2019, this survey is brief yet nearly exhaustive. We begin by noting a few basic results. The following proposition is immediate by Proposition 2.18 and holds for any homogeneous ideal in $k[x_0, \ldots, x_n]$.

Proposition 2.24. *If $I \subset k[x_0, \ldots, x_n]$ is an edge ideal, then sdefect$(I, 1) = 0$.*

In 1994, work of Simis, Vasconcelos and Villarreal showed that a graph G is bipartite if and only if $I(G)^{(s)} = I(G)^s$ for all $s \geq 1$ [48, Theorem 5.9]. The following result is then immediate.

Proposition 2.25. *If G is a bipartite graph, then sdefect$(I(G), s) = 0$ for all $s \geq 1$.*

By Proposition 2.25, the symbolic defect sequence of edge ideals of bipartite graphs is the zero sequence, and is thus not particularly interesting. All graphs are assumed to be non-bipartite throughout the remainder of this thesis. Recall that a graph is non-bipartite if and only if it contains an odd cycle.

In 2019, Janssen, Kamp and Vander Woude [34] were among the first to study the symbolic defect sequence of edge ideals. They focused on edge ideals of odd cycles and used Method 2.23 to describe a partial formula for their symbolic defect sequences. To do so, they made the following definition that allows for a decomposition of $I(C_{2m-1})^{(s)}$ for $s \geq 1$.

Definition 2.26. Fix $s \geq 1$ and let $I \subset k[x_0, \ldots, x_n]$ be an edge ideal. Define

$$D(s) = \{w \in I^{(s)} : w \text{ is a monomial and } \deg(w) < 2s\}$$

and

$$L(s) = \{w \in I^{(s)} : w \text{ is a monomial and } \deg(w) \geq 2s\}.$$

Vertex covers of odd cycles, as well as numerous lemmas on the structure of monomials in $L(s)$, were used to prove the following.

Theorem 2.27. *[34, Theorem 4.4] Fix $m \geq 2$ and let $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Then $(L(s)) = I^s$ for each $s \geq 1$.*

By Theorem 2.27,

$$I(C_{2m-1})^{(s)} = (D(s)) + (L(s)) = (D(s)) + I(C_{2m-1})^s,$$

so that $D(s) + I(C_{2m-1})^s$ is a monomial generating set for $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$. Through numerous lemmas, Janssen, Kamp and Vander Woude show that for $1 \leq s \leq 2m-1$, $D(s) + I(C_{2m-1})^s$ is the minimal monomial generating set for $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$. They describe the structure of monomials in $D(s)$ for these values of s [34, Lemma 5.12] and use this to count $D(s)$, giving the following partial formula for the symbolic defect sequence of edge ideals of odd cycles.

Theorem 2.28. *[34, Corollaries 5.3 and 5.4, Theorem 5.13] Fix $m \geq 2$ and let $I = I(C_{2m-1}) \subset k[x_0, \dots, x_{2m-2}]$. Then*

$$sdefect(I, s) = \begin{cases} 0 & \text{if } 1 \leq s < m \\ \left(\!\!\binom{2m-1}{s-m}\!\!\right) & \text{if } m \leq s \leq 2m-1, \end{cases}$$

where $\left(\!\!\binom{}{}\!\!\right)$ denotes multichoose.

See [39, Notation 2.4] for the definition of multichoose. This work of Janssen, Kamp and Vander Woude has been recently generalized by Cooper and Noteboom to disjoint cycle graphs [10]. The conclusion of Theorem 2.27 holds for edge ideals of complete graphs as well.

Theorem 2.29. *[34, Theorem 6.4] Fix $n \geq 3$. If $I = I(K_n) \subset k[x_0, \dots, x_{n-1}]$, then $I^{(s)} = (D(s)) + I^s$.*

In work published soon after in 2020, Chakraborty and Mandal describe the structure of monomials in $D(s)$ for edge ideals of complete graphs [7, Lemma 4.1]. Theorem 2.29 is then used to describe a partial formula for the symbolic defect sequence of edge ideals of complete graphs.

Theorem 2.30. *[7, Theorem 4.4] Fix $n \geq 3$ and let $I = I(K_n) \subset k[x_0, \dots, x_{n-1}]$. For $2 \leq s \leq n-1$, we have*

$$sdefect(I, s) = \binom{n}{s+1}.$$

Chakraborty and Mandal also study the symbolic defect sequence of edge ideals of graphs consisting of two odd cycles joined at a single vertex. They show that the conclusion of Theorem 2.27 holds for edge ideals of such graphs for small values of s relative to the sizes of the cycles [7, Theorem 3.14]. The structure of the monomials in the minimal monomial generating set extracted from $D(s) + I^s$ are described in [7, Corollary 3.11], and the cardinality of this set is bounded to obtain the following.

Proposition 2.31. *[7, Proposition 3.17] Fix $2 \leq n < m$ and let G be a graph consisting of the two odd cycles C_{2m-1} and C_{2n-1} joined at a single vertex. Let $I = I(G)$ and $s \in \mathbb{Z}^+$ be such that $n+2 \leq s \leq m+2$. If $l_s = \lfloor s/(n+2) \rfloor$, then*

$$sdefect(I, s) \leq \begin{cases} \sum_{i=1}^{l_s} \binom{s-i(n+2)+2(n+1)+2(m+1)+1}{2(n+1)+2(m+1)+1} & \text{for } n+2 \leq s \leq m+1 \\ \sum_{i=1}^{l_s} \binom{s-i(n+2)+2(n+1)+2(m+1)+1}{2(n+1)+2(m+1)+1} + 1 & \text{for } s = m+2. \end{cases}$$

In 2020, Drabkin and Guerrieri [13] take a different approach to the symbolic defect than the work surveyed thus far. Instead of computing formulas for terms of the symbolic defect sequence, they consider the asymptotic behaviour of the sequence. They use Hilbert series to prove that the symbolic defect sequence of certain ideals is eventually a *quasi-polynomial*.

Definition 2.32. A *quasi-polynomial* in $\mathbb{Q}$ is a function $f : \mathbb{N} \to \mathbb{Q}$ such that for some $l \in \mathbb{N}$, there exist rational polynomials $f_0, \ldots, f_{l-1}$ such that for all $m \in \mathbb{N}$, $f(m) = f_r(m)$ whenever $r \equiv m \mod l$. The value l is called the *quasi-period* of f.

Given a Noetherian ring R and an ideal $I \subset R$, the *symbolic Rees algebra* of I is the blowup algebra $\mathcal{R}_s(I) = \oplus_{s \geq 0} I^{(s)} t^s \subset R[t]$. See [15] for a detailed background on the symbolic Rees algebra of edge ideals.

Theorem 2.33. *[13, Theorem 2.4] Let R be a Noetherian local or graded-local ring, and I an ideal (or homogeneous ideal) of R such that $\mathcal{R}_s(I)$ is a Noetherian ring. Let $d_1, \ldots, d_t$ be the degrees of the generators of $\mathcal{R}_s(I)$ as an R-algebra. Then sdefect(I, s) is eventually a quasi-polynomial in s with quasi-period $d = lcm(d_1, \ldots, d_t)$.*

Edge ideals fall in the category of ideals considered in Theorem 2.33 [38, Proposition 1], and so the symbolic defect sequence of edge ideals is eventually a quasi-polynomial. Drabkin and Guerrieri use Theorem 2.33 and symbolic Rees algebras to study the symbolic defect sequence of cover ideals, which are another class of ideals that are in one-to-one correspondence with finite simple graphs (see [50] for a detailed background on cover ideals). Their main results include a classification of exactly which graphs have cover ideals with the second symbolic defect equal to one [13, Theorem 4.9] and a complete recursive formula for the symbolic defect sequence of the cover ideals of graphs satisfying certain conditions [13, Theorem 5.5].

Most recently in 2022, Mandal and Pradhan [39] describe a complete formula for the symbolic defect sequence of edge ideals of unicyclic graphs. Note that Chapter 5 of this thesis describes a complete formula for the symbolic defect sequence of edge ideals of odd cycles, and Chapter 6 describes a partial formula for the symbolic defect sequence of edge ideals of unicyclic graphs, as well as lower and upper bounds for the terms that cannot be computed with this partial formula. These formulae and bounds are generally different from those of Mandal and Pradhan, and were proven independently. Despite this, there are some similarities which will be noted as they arise. Mandal and Pradhan use the following decomposition of symbolic powers of edge ideals of unicyclic graphs in terms of their ordinary powers.

Lemma 2.34. *[26, Theorem 3.4] Fix $m \geq 2$ and let G be a unicyclic graph with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G)$. Fix $s \geq 1$. If $s = qm + r$ with $q \in \mathbb{Z}$ and $0 \leq r < m$, then*

$$I^{(s)} = \sum_{d=0}^{q} (x_0 \cdots x_{2m-2})^d I^{s-dm}.$$

If G and s are as in Lemma 2.34, then

$$I^{(s)}/I^s = \left(\sum_{d=0}^{q} (x_0 \cdots x_{2m-2})^d I^{s-dm} \right) /I^s = \left(I^s + \sum_{d=1}^{q} (x_0 \cdots x_{2m-2})^d I^{s-dm} \right) /I^s.$$

It follows that the cosets corresponding to generators of $\sum_{d=1}^{q} (x_0 \cdots x_{2m-2})^d I^{s-dm}$ form a monomial generating set for $I^{(s)}/I^s$. In Chapters 5 and 6, we will extract and algebraically describe the minimal monomial generating set for $I^{(s)}/I^s$ to provide formulae and bounds for the symbolic defect sequence. Mandal and Pradhan instead describe the minimal monomial generating set combinatorially and recursively. They make use of *minimum edge covers* of graphs to do so.

Definition 2.35. An *edge cover* of a graph G is a set $L \subset E(G)$ such that for every vertex $x \in V(G)$, there exists an edge $e \in L$ such that $x \in e$. An edge cover is a *minimum edge cover* if it has minimum cardinality among all edge covers.

Definition 2.36. Fix $m \geq 2$ and let G be a unicyclic graph with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G)$. Given $s > m$, let $\mathfrak{G}_{s-m}$ denote the set of elements of $\mathcal{G}(I^{s-m})$ that are not multiples of some element of $\mathcal{G}(I^{s-2m}(x_0 \cdots x_{2m-2}))$.

Mandal and Pradhan count $|\mathfrak{G}_{s-m}|$ in terms of the quantity of edge sets containing a minimum edge cover of G, resulting in a complex and lengthy formula [39, Lemma 4.7]. Through numerous technical lemmas, they use this notation to present the following recursive formula for the symbolic defect sequence of edge ideals of unicyclic graphs.

Theorem 2.37. *[39, Theorem 4.8] Fix $m \geq 2$ and let G be a unicyclic graph with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G)$. Fix $s \geq 1$ and write $s = qm + r$, with $q \in \mathbb{Z}$ and $0 \leq r < m$. Then*

- *sdefect*$(I, s) = \left(\!\!\binom{|E(G)|}{s-m}\!\!\right)$ *for* $m \leq s \leq 2m - 1$, *and*
- *sdefect*$(I, s) = |\mathfrak{G}_{s-m}| +$ *sdefect*$(I, s - m)$ *for* $s > 2m - 1$.

Mandal and Pradhan expand this recursive formula [39, Corollary 4.9] and illustrate that this formula is quite difficult to simplify, although it is possible to arrive at the quasi-polynomial whose existence is guaranteed by Theorem 2.33 [39, Example 4.11].

Chapter 3

Non-zero terms of the symbolic defect sequence

In this chapter, we investigate the non-zero terms of the symbolic defect sequence of edge ideals. In particular, with Proposition 3.3, we show that the sequence is zero up to a certain term, at which point all following terms are non-zero. With Theorem 3.7, we provide a simple formula for the first non-zero term of the sequence, and with Proposition 3.10, we classify exactly which terms of the sequence may be equal to one. We begin with a lemma that is crucial for the proofs of many results throughout this thesis.

Lemma 3.1. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$, $I = I(G) \subset k[x_0, \ldots, x_n]$ and $m \geq 2$. Let w be a monomial such that $w + I^m \in I^{(m)}/I^m$ is non-zero. If $H = G[supp(w)]$, then H contains an induced copy of an odd cycle of length at most $2m-1$.*

Proof. Suppose first that H does not contain an induced copy of an odd cycle. Note that the existence of a copy of an odd cycle in H implies the existence of an induced copy of an odd cycle in H. Indeed, if there is a copy of an odd cycle in H on vertex set Y, then by definition, $H[Y]$ is an odd cycle with at least one chord. The existence of a chord in $H[Y]$ implies the existence of a smaller odd cycle in $H[Y]$, so that the smallest odd cycle in $H[Y]$ is an induced copy of an odd cycle in H.

Thus H not containing an induced copy of an odd cycle implies that H also does not contain a copy of an odd cycle. Then H does not contain an odd cycle as a subgraph, and so H is a bipartite graph. By Proposition 2.25, $I(H)^{(m)} = I(H)^m$. Let $X \subseteq V(H)$ be a minimal vertex cover of H. Then $X' = X \cup (V(G) \setminus V(H))$ is a vertex cover for G. Since $w \in I^{(m)}$, by Proposition 2.16,

$$m \leq W_{X'}(w) = W_{X' \cap \operatorname{supp}(w)}(w) = W_{X' \cap V(H)}(w) = W_X(w).$$

Thus $w \in I(H)^{(m)} = I(H)^m \subseteq I^m$. This contradicts the assumption that $w + I^m$ is non-zero in $I^{(m)}/I^m$, and so H contains an induced copy of an odd cycle.

If H contains an induced copy of an odd cycle of length greater than or equal to $2m+1$, then we claim $w \in I^m$. Indeed, suppose that H contains an induced copy of an odd cycle of length $2s+1$ on vertices $\{x_0, \ldots, x_{2s}\}$, with $s \geq m$. Then $\{x_0, \ldots, x_{2s}\} \subseteq V(H) = \mathrm{supp}(w)$ and so the monomial $x_0 \cdots x_{2s}$ divides w. However,

$$x_0 \cdots x_{2s} = (x_0 x_1)(x_2 x_3) \cdots (x_{2s-2} x_{2s-1}) x_{2s} \in I^s,$$

so that $w \in I^s \subseteq I^m$. This contradicts the assumption that $w + I^m$ is non-zero in $I^{(m)}/I^m$, and so H cannot contain an induced copy of an odd cycle of length greater than or equal to $2m+1$. We conclude that H contains an induced copy of an odd cycle of length at most $2m-1$. $\square$

The following corollary is a rewording of Lemma 3.1 using terminology introduced in Section 2.1. This version of Lemma 3.1 will be more useful within the proofs of some results in this thesis.

Corollary 3.2. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$, $I = I(G) \subset k[x_0, \ldots, x_n]$ and $m \geq 2$. If w is a monomial such that $w + I^m \in I^{(m)}/I^m$ is non-zero, then there exists $2 \leq s \leq m$ such that w is divisible by a $(2s-1)$ - cycle monomial.*

Proof. By Lemma 3.1, there exists $2 \leq s \leq m$ such that $H = G[\mathrm{supp}(w)]$ contains an induced copy of a $(2s-1)$ - cycle. Without loss of generality, say this cycle lies on vertices $\{x_0, \ldots, x_{2s-2}\}$. Since $\{x_0, \ldots, x_{2s-2}\} \subseteq V(H) = \mathrm{supp}(w)$, the $(2s-1)$ - cycle monomial $x_0 \cdots x_{2s-2}$ divides w. $\square$

3.1 Classification of non-zero terms

We are now ready to begin studying terms of the symbolic defect sequence of edge ideals. The following proposition classifies exactly which terms of these sequences are non-zero. Equivalent results have been shown in [46, Lemma 3.10], [35, Corollary 4.5] and [12, Theorem 4.13]. We include an original proof that more closely matches the techniques used throughout the rest of this chapter. The converse direction of this proof is very similar to that given in [46].

Proposition 3.3. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$, $I = I(G) \subset k[x_0, \ldots, x_n]$ and $s \geq 2$. Then sdefect$(I, s) > 0$ if and only if G contains an induced copy of an odd cycle of length at most $2s-1$.*

Proof. Assume first that $\operatorname{sdefect}(I, s) > 0$, meaning there exists a monomial w such that $w + I^s \in I^{(s)}/I^s$ is non-zero. By Lemma 3.1, $G[\operatorname{supp}(w)]$ contains an induced copy of an odd cycle of length at most $2s - 1$, and hence G contains an induced copy of an odd cycle of length at most $2s - 1$.

Conversely, assume G contains an induced copy of an odd cycle of length at most $2s - 1$ and let $2 \leq t \leq s$ be such that G contains an induced copy of a $(2t - 1)$ - cycle. Without loss of generality, suppose this cycle lies on vertices $\{x_0, \dots, x_{2t-2}\}$ and define the monomial

$$w = x_0^{s-t+1} x_1^{s-t+1} x_2 \cdots x_{2t-2}.$$

Let X be a vertex cover of G. By Lemma 2.10, $W_X(x_0 \cdots x_{2t-2}) \geq t$. Further, since $\{x_0, x_1\} \in E(G)$, X contains at least one vertex from $\{x_0, x_1\}$. Thus $W_X(x_0^{s-t} x_1^{s-t}) \geq s - t$. Then

$$\begin{aligned} W_X(w) &= W_X((x_0^{s-t} x_1^{s-t})(x_0 \cdots x_{2t-2})) \\ &= W_X(x_0^{s-t} x_1^{s-t}) + W_X(x_0 \cdots x_{2t-2}) \\ &\geq (s - t) + t \\ &= s, \end{aligned}$$

so that $w \in I^{(s)}$ by Proposition 2.16. Further,

$$\deg(w) = \deg(x_0^{s-t} x_1^{s-t}) + \deg(x_0 \cdots x_{2t-2}) = 2(s - t) + (2t - 1) = 2s - 1 < 2s,$$

so that $w \notin I^s$. Thus $w + I^s$ is non-zero in $I^{(s)}/I^s$ and so we conclude that $\operatorname{sdefect}(I, s) > 0$. □

Remark 3.4. Given a graph G, if the smallest induced copy of an odd cycle in G has length $2m - 1$, then by Proposition 3.3,

$$\operatorname{sdefect}(I(G), t) = 0 \text{ for all } \ 1 \leq t < m$$

and

$$\operatorname{sdefect}(I(G), s) > 0 \text{ for all } s \geq m.$$

In other words, $\operatorname{sdefect}(I(G), m)$ is the first non-zero term of the symbolic defect sequence of $I(G)$, and all following terms are non-zero as well.

Example 3.5. Let G be any graph that contains a triangle. By Proposition 3.3,

$$\operatorname{sdefect}(I(G), 1) = 0$$

and

$$\operatorname{sdefect}(I(G), s) > 0 \text{ for all } s \geq 2.$$

Example 3.6. Let $m \geq 2$ and $G = C_{2m-1}$. By Proposition 3.3,

$$\operatorname{sdefect}(I(G), t) = 0 \text{ for all } t < m$$

and

$$\operatorname{sdefect}(I(G), s) > 0 \text{ for all } s \geq m.$$

3.2 A formula for the first non-zero term

Next, we give a simple formula for the first non-zero term of the symbolic defect sequence of any edge ideal.

Theorem 3.7. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, m)$ is the first non-zero term in the symbolic defect sequence of I, then sdefect(I, m) is equal to the number of induced copies of $(2m-1)$ - cycles in G.*

Proof. Note first that by Proposition 2.24, $m \geq 2$. Let

$$B := \{w : w \text{ is a } (2m-1) \text{ - cycle monomial}\}.$$

We will show that $B + I^m$ is the minimal monomial generating set for $I^{(m)}/I^m$, so that sdefect$(I, m) = |B|$. By Remark 2.8, there is a one-to-one correspondence between induced copies of $(2m-1)$ - cycles in G and $(2m-1)$ - cycle monomials, so that $|B|$ is equal to the number of induced copies of $(2m-1)$ - cycles in G.

First, note that $B \neq \emptyset$ by Proposition 3.3. Now, suppose $w \in B$ and let X be a vertex cover of G. By Lemma 2.10, $W_X(w) \geq m$ and so by Proposition 2.16, $w \in I^{(m)}$. Further, $\deg(w) = 2m - 1 < 2m$, and so $w \notin I^m$. Thus $w + I^m$ is a non-zero monomial in $I^{(m)}/I^m$. We conclude that $B + I^m$ consists of non-zero monomials in $I^{(m)}/I^m$.

Next, we want to show that every non-zero monomial in $I^{(m)}/I^m$ is divisible by a monomial in $B + I^m$. Let $u + I^m$ be a non-zero monomial in $I^{(m)}/I^m$. By Corollary 3.2, there exists $2 \leq s \leq m$ such that u is divisible by a $(2s-1)$ - cycle monomial. Since sdefect(I, m) is the first non-zero term in the symbolic defect sequence of I, G does not contain any induced copies of odd cycles of length less than $2m-1$ by Proposition 3.3. Thus there does not exist any $(2s-1)$ - cycle monomial for $s < m$. It follows that u must be divisible by a $(2m-1)$ - cycle monomial, which lies in B by definition.

We have shown that $B + I^m$ is a monomial generating set for $I^{(m)}/I^m$. To see that $B + I^m$ is the minimal monomial generating set, we need only note that for all $w \in B$, $\deg(w) = 2m-1$. Thus no monomial in $B + I^m$ can properly divide any other monomial in $B + I^m$, and so $B + I^m$ is the minimal monomial generating set for $I^{(m)}/I^m$. $\square$

Example 3.8. Let $n \geq 3$ and $I = I(K_n)$. Since K_n contains a triangle, sdefect$(I, 2)$ is the first non-zero term in the symbolic defect sequence of I by Proposition 3.3. By Theorem 3.7, sdefect$(I, 2) = \binom{n}{3}$, the number of triangles in K_n.

Example 3.9. Let $m \geq 2$ and $I = I(C_{2m-1})$. By Proposition 3.3, sdefect(I, m) is the first non-zero term in the symbolic defect sequence of I, and by Theorem 3.7, sdefect$(I, m) = 1$.

3.3 Classification of terms that are equal to one

The final result of this chapter classifies exactly which terms of the symbolic defect sequence of an edge ideal are equal to one. These terms are of particular interest because when sdefect$(I, m) = 1$, $I^{(m)}$ is "almost" equal to I^m.

Proposition 3.10. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Given $m \geq 2$, sdefect$(I, m) = 1$ if and only if G contains a unique induced copy of a $(2m-1)$ - cycle and does not contain an induced copy of any odd cycles of length less than $2m-1$.*

Proof. Let $m \geq 2$ and suppose that sdefect$(I, m) = 1$. By Proposition 3.3, there exists some $2 \leq s \leq m$ such that G contains an induced copy of a $(2s-1)$ - cycle. Fix s_0 to be the smallest such s, and suppose $s_0 < m$. Without loss of generality, suppose this $(2s_0 - 1)$ - cycle lies on vertices $\{x_0, \ldots, x_{2s_0-2}\}$. Let

$$w_1 = x_0^{m-s_0+1} x_1^{m-s_0+1} x_2 \cdots x_{2s_0-2}$$

and

$$w_2 = \begin{cases} x_0 x_1^{m-1} x_2^{m-1} & \text{if } s_0 = 2 \\ x_0 x_1^{m-s_0+1} x_2^{m-s_0+1} x_3 \cdots x_{2s_0-2} & \text{if } s_0 > 2. \end{cases}$$

Note that $s_0 < m$ implies that $m - s_0 + 1 > 1$, so that $w_1 \neq w_2$. We will show that both $w_1 + I^m$ and $w_2 + I^m$ are minimal monomial generators of $I^{(m)}/I^m$, implying sdefect$(I, m) \geq 2$

and contradicting our assumption that $\operatorname{sdefect}(I, m) = 1$. By Remark 2.22, it will suffice to show that $w_1, w_2 \in \mathcal{G}(I^{(m)}) \setminus I^m$.

Let X be a minimal vertex cover of G. Since $x_0 \cdots x_{2s_0-2}$ is a $(2s_0 - 1)$ - cycle monomial, by Lemma 2.10, $W_X(x_0 \cdots x_{2s_0-2}) \geq s_0$. Further, since $\{x_0, x_1\}, \{x_1, x_2\} \in E(G)$, X contains at least one vertex from each of $\{x_0, x_1\}$ and $\{x_1, x_2\}$. Thus $W_X(x_0^{m-s_0} x_1^{m-s_0}) \geq m - s_0$ and $W_X(x_1^{m-s_0} x_2^{m-s_0}) \geq m - s_0$. Then

$$\begin{aligned} W_X(w_1) &= W_X((x_0 \cdots x_{2s_0-2})(x_0^{m-s_0} x_1^{m-s_0})) \\ &= W_X(x_0 \cdots x_{2s_0-2}) + W_X(x_0^{m-s_0} x_1^{m-s_0}) \\ &\geq s_0 + (m - s_0) \\ &= m, \end{aligned}$$

so that by Proposition 2.16, $w_1 \in I^{(m)}$. Similarly, note that whether $s_0 = 2$ or $s_0 > 2$, $w_2 = (x_0 \cdots x_{2s_0-2})(x_1^{m-s_0} x_2^{m-s_0})$. Thus we may perform similar calculations to conclude that $w_2 \in I^{(m)}$ as well.

Next we show that no proper divisors of w_1 or w_2 lie in $I^{(m)}$, so that $w_1, w_2 \in \mathcal{G}(I^{(m)})$. We first consider w_1. Let

$$X_1 = \{x_1, x_3, x_5, \ldots, x_{2s_0-5}, x_{2s_0-3}, x_{2s_0-2}\} \cup (V(G) \setminus \{x_0, \ldots, x_{2s_0-2}\})$$

and

$$X_2 = \{x_0, x_2, x_4, \ldots, x_{2s_0-2}\} \cup (V(G) \setminus \{x_0, \ldots, x_{2s_0-2}\}).$$

Both X_1 and X_2 are vertex covers for G, and we see

$$W_{X_1}(x_0 \cdots x_{2s_0-2}) = W_{X_2}(x_0 \cdots x_{2s_0-2}) = s_0$$

and

$$W_{X_1}(x_0^{m-s_0} x_1^{m-s_0}) = W_{X_2}(x_0^{m-s_0} x_1^{m-s_0}) = m - s_0.$$

Thus for $k = 1$ and $k = 2$, we have

$$W_{X_k}(w_1) = W_{X_k}(x_0 \cdots x_{2s_0-2}) + W_{X_k}(x_0^{m-s_0} x_1^{m-s_0}) = s_0 + (m - s_0) = m.$$

Then for $0 \leq i \leq 2s_0 - 2$, i odd, we have

$$W_{X_1}\left(\frac{w_1}{x_i}\right) = W_{X_1}(w_1) - W_{X_1}(x_i) = m - 1 < m.$$

By Proposition 2.16, $w_1/x_i \notin I^{(m)}$. Similarly, for $0 \le j \le 2s_0 - 2$, j even, we have

$$W_{X_2}\left(\frac{w_1}{x_j}\right) = W_{X_2}(w_1) - W_{X_2}(x_j) = m - 1 < m,$$

so that $w_1/x_j \notin I^{(m)}$. Thus no proper divisor of w_1 lies in $I^{(m)}$, so that $w_1 \in \mathcal{G}(I^{(m)})$.

Now, if $s_0 > 2$, we apply an identical argument with $x_0^{m-s_0}x_1^{m-s_0}$ replaced by $x_1^{m-s_0}x_2^{m-s_0}$ to show that $w_2 \in \mathcal{G}(I^{(m)})$. If $s_0 = 2$, recall that $w_2 = x_0x_1^{m-1}x_2^{m-1}$. This case can be handled similarly, using slightly different vertex covers. Define

$$X_1' = \{x_0, x_1\} \cup (V(G) \setminus \{x_0, x_1, x_2\})$$

and

$$X_2' = \{x_0, x_2\} \cup (V(G) \setminus \{x_0, x_1, x_2\}).$$

Both are vertex covers for G, and we see that

$$W_{X_1'}(x_0x_1x_2) = W_{X_2'}(x_0x_1x_2) = 2$$

and

$$W_{X_1'}(x_1^{m-2}x_2^{m-2}) = W_{X_2'}(x_1^{m-2}x_2^{m-2}) = m - 2.$$

Thus for $k = 1$ and $k = 2$, we have

$$W_{X_k'}(w_2) = W_{X_k'}(x_0x_1x_2) + W_{X_k'}(x_1^{m-2}x_2^{m-2}) = 2 + (m-2) = m.$$

Then for $i = 0$ and $i = 2$, we have

$$W_{X_2'}\left(\frac{w_2}{x_i}\right) = W_{X_2'}(w_2) - W_{X_2'}(x_i) = m - 1 < m,$$

so that $w_2/x_i \notin I^{(m)}$ for $i = 0, 2$. Similarly,

$$W_{X_1'}\left(\frac{w_2}{x_1}\right) = W_{X_1'}(w_2) - W_{X_1'}(x_1) = m - 1 < m,$$

so that $w_2/x_1 \notin I^{(m)}$. Thus no proper divisor of w_2 lies in $I^{(m)}$, and so $w_2 \in \mathcal{G}(I^{(m)})$. Finally, note that

$$\deg(w_1) = \deg(x_0 \cdots x_{2s_0-2}) + \deg(x_0^{m-s_0}x_1^{m-s_0}) = (2s_0 - 1) + 2(m - s_0) = 2m - 1 < 2m.$$

Similarly, $\deg(w_2) = 2m-1$, regardless of whether $s_0 = 2$ or $s_0 > 2$. Thus neither w_1 nor w_2 lie in I^m. It follows that $w_1 + I^m$ and $w_2 + I^m$ are both minimal monomial generators of $I^{(m)}/I^m$, as desired. We conclude that $s_0 = m$, so that G contains an induced copy of a $(2m-1)$ - cycle and does not contain an induced copy of an odd cycle of length less than $2m-1$. By Proposition 3.3, sdefect(I, m) is the first non-zero term of the symbolic defect sequence of I, and by Theorem 3.7, G contains a unique induced copy of a $(2m-1)$ - cycle.

Conversely, suppose that G contains a unique induced copy of a $(2m-1)$ - cycle and does not contain an induced copy of any odd cycles of length less than $2m-1$. By Proposition 3.3, sdefect(I, m) is the first non-zero term of the symbolic defect sequence of I, and by Theorem 3.7, sdefect(I, m) is equal to the number of induced copies of $(2m-1)$ - cycles in G, which is one. □

Remark 3.11. Given an edge ideal I, Proposition 3.10 shows that there exists at most one value of m such that sdefect$(I, m) = 1$. In particular, if there exists m such that sdefect$(I, m) = 1$, then sdefect(I, m) is the first non-zero term of the symbolic defect sequence of I, and for all $s > m$, sdefect$(I, s) > 1$.

Example 3.12. Let G be any graph with a unique odd cycle, say of length $2m-1$ with $m \geq 2$. Then by Proposition 3.3 and Proposition 3.10, for $1 \leq s \leq m$,

$$\text{sdefect}(I(G), s) = \begin{cases} 0 & \text{if } s < m \\ 1 & \text{if } s = m \end{cases}$$

and for $s > m$, sdefect$(I(G), s) > 1$.

Chapter 4

The second non-zero term of the symbolic defect sequence

We now seek a formula for the second non-zero term in the symbolic defect sequence of edge ideals. We consider two cases: first, the edge ideals of graphs containing a triangle, and afterwards, the edge ideals of triangle-free graphs. We remind the reader of the index of notation at the end of this thesis.

Throughout, $\operatorname{sdefect}(I, m)$ will always denote the first non-zero term in the symbolic defect sequence of an edge ideal I. Recall then that by Remark 3.4, $\operatorname{sdefect}(I, m+1)$ is the second non-zero term. Also, note that if $I = I(G) \subset k[x_0, \ldots, x_n]$, then by Theorem 3.7, there exists at least one $(2m-1)$ - cycle in G. Thus it is implicitly assumed that $|V(G)| = n + 1 \geq 2m - 1$.

4.1 Edge ideals of graphs containing a triangle

In this section, we consider only edge ideals of graphs that contain triangles. By Proposition 3.3, this is equivalent to edge ideals for which the second term of the symbolic defect sequence, $\operatorname{sdefect}(I, 2)$, is greater than zero. With Lemma 4.2, we describe a monomial generating set for $I^{(3)}/I^3$, and in Lemma 4.5, we extract the minimal monomial generating set. By counting this minimal monomial generating set, we give a formula for $\operatorname{sdefect}(I, 3)$ in Theorem 4.16. Some parameters within this formula are generally not easy to compute, and so Corollaries 4.20 and 4.21 demonstrate special cases where this formula is simplified.

4.1.1 The minimal monomial generating set

We first introduce the notation needed to describe a monomial generating set for $I^{(3)}/I^3$.

Definition 4.1. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then define

$$C_E(I) = \{ce : c \text{ is a triangle monomial and } e \text{ is an edge monomial}\},$$

$$K_4(I) = \{w : w \text{ is a } K_4 \text{ monomial}\},\ C_5(I) = \{c : c \text{ is a 5 - cycle monomial}\},$$

and

$$A_3(I) = C_E(I) \cup K_4(I) \cup C_5(I).$$

Note that if a graph G contains a triangle as a subgraph, this subgraph is an induced copy of a triangle. In general, if $n \geq 1$ is such that G contains K_n as a subgraph, it is an induced copy of K_n. Throughout this section, we refer to induced copies of triangles and K_4's in G simply as triangles and K_4's in G.

Lemma 4.2. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then $A_3(I) + I^3$ is a generating set for $I^{(3)}/I^3$.*

Proof. By Proposition 3.3, there exists a triangle in G, and so there exists a triangle monomial arising from G. The existence of edge monomials is clear, and so $C_E(I) \neq \emptyset$. Therefore $A_3(I) \neq \emptyset$.

We next show that $A_3(I) \subset I^{(3)}$. Let $ce \in C_E(I)$, where c is a triangle monomial and e is an edge monomial, and let X be a vertex cover of G. By Proposition 2.10, $W_X(c) \geq 2$, and so $c \in I^{(2)}$. Then $ce \in I^{(2)}I \subseteq I^{(3)}$ by Proposition 2.18, so that $C_E(I) \subset I^{(3)}$. Similarly, let $w \in K_4(I)$. Since any vertex cover of G contains at least 3 vertices from any K_4 in G, $W_X(w) \geq 3$ and so $w \in I^{(3)}$. Thus $K_4(I) \subset I^{(3)}$. Lastly, let $w \in C_5(I)$. By Lemma 2.10, $W_X(w) \geq 3$, so that $w \in I^{(3)}$ and so $C_5(I) \subset I^{(3)}$. We conclude that $A_3(I) = C_E(I) \cup K_4(I) \cup C_5(I) \subset I^{(3)}$. Further, monomials in $A_3(I)$ have degrees 4 and 5, so that $A_3(I) \cap I^3 = \emptyset$. Thus all monomials in $A_3(I) + I^3$ are non-zero in $I^{(3)}/I^3$.

Now, let w be a monomial such that $w + I^3 \in I^{(3)}/I^3$ is non-zero. We will show that w is divisible by a monomial in $A_3(I)$. First, note that by Corollary 3.2, w is divisible by a $(2s-1)$ - cycle monomial for some $2 \leq s \leq 3$. If w is divisible by a 5 - cycle monomial, then we are done, since $C_5(I) \subset A_3(I)$ consists of all 5 - cycle monomials. So assume that w is not divisible by a 5 - cycle monomial, so that w must be divisible by a triangle monomial. Without loss of generality, assume that $c = x_0x_1x_2$ is a triangle monomial dividing w.

Suppose first that supp(w/c) is not an independent set of vertices in G, meaning that there exists at least one edge in G among the vertices in supp(w/c). Then there exists an edge monomial e dividing w/c, and so the monomial $ce \in C_E(I) \subseteq A_3(I)$ divides w.

Now, suppose that $\mathrm{supp}(w/c)$ is an independent set of vertices. We will show that w is divisible by a monomial in $C_E(I) \cup K_4(I) \subset A_3(I)$ by contradiction. In particular, assuming that w is not divisible by a monomial in $C_E(I) \cup K_4(I)$, we will produce a vertex cover X for G that satisfies $W_X(c) = 2$ and $W_X(w/c) = 0$, implying that $W_X(w) = W_X(c) + W_X(w/c) = 2 < 3$. This will give the desired contradiction by Proposition 2.16, since $w \in I^{(3)}$.

To this end, assume that w is not divisible by a monomial in $C_E(I) \cup K_4(I)$. Recall that we are also assuming that w is not divisible by a monomial in $C_5(I)$, so that together, we are assuming that w is not divisible by a monomial in $A_3(I)$. Producing a vertex cover X with $W_X(w/c) = 0$ is equivalent to producing a vertex cover X with $X \cap \mathrm{supp}(w/c) = \emptyset$. To produce such a vertex cover, we need to ensure vertices that have neighbours in $\mathrm{supp}(w/c)$ all lie in X. Thus we define

$$U = \mathrm{supp}(w) \cap \left\{ x_i : x_i \text{ is adjacent to some vertex in supp} \left(\frac{w}{c}\right) \right\}.$$

Notice that since $\mathrm{supp}(w/c)$ is assumed to be an independent set, $U \cap \mathrm{supp}(w/c) = \emptyset$. Thus

$$U \subseteq \mathrm{supp}(w) \setminus \mathrm{supp}(w/c) \subseteq \mathrm{supp}(c) = \{x_0, x_1, x_2\}.$$

In particular, $|U| \leq 3$.

Claim 4.1. Define U as above. If w is not divisible by a monomial in $A_3(I)$, then $|U| \leq 2$.

Proof. Suppose that $|U| > 2$, so that $U = \{x_0, x_1, x_2\}$. We will produce distinct $x_0', x_1', x_2' \in \mathrm{supp}(w/c)$ such that x_i is adjacent to x_i' for $i = 0, 1, 2$. Since x_0', x_1', x_2' are all distinct, the monomial $x_0'x_1'x_2'$ divides w/c and so $cx_0'x_1'x_2'$ divides w. However,

$$cx_0'x_1'x_2' = (x_0x_1x_2)x_0'x_1'x_2' = (x_0x_0')(x_1x_1')(x_2x_2') \in I^3,$$

contradicting the assumption that $w + I^3$ is non-zero in $I^{(3)}/I^3$. We may then conclude that $|U| \leq 2$.

We begin by choosing x_0' to be any vertex in $\mathrm{supp}(w/c)$ that is adjacent to x_0. Such an x_0' exists by definition of U. Next, we want to choose $x_1' \in \mathrm{supp}(w/c)$ adjacent to x_1 and distinct from x_0'. To do so, we will show that x_0' is not adjacent to x_1. Then we will be able to choose x_1' to be any vertex in $\mathrm{supp}(w/c)$ adjacent to x_1. Again, such an x_1' exists by definition of U.

Indeed, suppose that x_1 is adjacent to x_0'. Note that since $U \cap \mathrm{supp}(w/c) = \emptyset$, x_0' is distinct from all of x_0, x_1 and x_2, so that there is a triangle in G on vertices $\{x_0, x_1, x_0'\}$. If x_2 is also adjacent to x_0', then there is a K_4 in G on the vertices $\{x_0, x_1, x_2, x_0'\}$. Then the monomial $cx'_0 = x_0x_1x_2x'_0 \in K_4(I)$ divides w, contradicting our assumption that w is not divisible by a monomial in $A_3(I)$. Thus x_2 is not adjacent to x'_0. Since $x_2 \in U$, there exists $x'_2 \in \text{supp}(w/c)$ adjacent to x_2 and distinct from x'_0. Because $x'_0, x'_2 \in \text{supp}(w/c)$ are distinct, the monomial $cx'_0x'_2$ divides w. Then since $x_0x_1x'_0$ is a triangle monomial and $x_2x'_2$ is an edge monomial,

$$cx'_0x'_2 = (x_0x_1x_2)x'_0x'_2 = (x_0x_1x'_0)(x_2x'_2) \in C_E(I).$$

Thus w is divisible by a monomial in $C_E(I)$, contradicting our assumption that w is not divisible by a monomial in $A_3(I)$. We conclude that x_1 is not adjacent to x'_0, and so we may choose $x'_1 \in \text{supp}(w/c)$ adjacent to x_1 and distinct from x'_0.

Next, we want to choose $x'_2 \in \text{supp}(w/c)$ adjacent to x_2 and distinct from both x'_0 and x'_1. We will again show that x_2 cannot be adjacent to x'_0 or x'_1. Since $x'_0 \neq x'_1$, the monomial $cx'_0x'_1$ divides w. If x_2 is adjacent to x'_0, then $x_0x_2x'_0$ is a triangle monomial. Since $x_1x'_1$ is an edge monomial,

$$cx'_0x'_1 = (x_0x_1x_2)x'_0x'_1 = (x_0x_2x'_0)(x_1x'_1) \in C_E(I)$$

divides w, a contradiction. Similarly, if x_2 is adjacent to x'_1, then $x_1x_2x'_1$ is a triangle monomial. Since $x_0x'_0$ is an edge monomial,

$$cx'_0x'_1 = (x_0x_1x_2)x'_0x'_1 = (x_1x_2x'_1)(x_0x'_0) \in C_E(I)$$

divides w. Thus x_2 is not adjacent to x'_0 or x'_1, meaning that we may choose $x'_2 \in \text{supp}(w/c)$ adjacent to x_2 and distinct from x'_0 and x'_1. This completes the proof of Claim 4.1.

$\square$

We are assuming that w is not divisible by a monomial in $A_3(I)$, and so by Claim 4.1, $|U| \leq 2$. Let $U' \subset \{x_0, x_1, x_2\}$ be any subset such that $U \subseteq U'$ and $|U'| = 2$. We now claim that

$$X = U' \cup (V(G) \setminus \text{supp}(w))$$

is our desired vertex cover. First, we show that X is a vertex cover for G. Let $e = \{x_i, x_j\} \in E(G)$. If at least one of x_i or x_j is not in $\text{supp}(w)$, then $e \cap X \neq \emptyset$. If both x_i and x_j lie in $\text{supp}(c)$, then $e \cap U' \neq \emptyset$ and so $e \cap X \neq \emptyset$. Since $\text{supp}(w/c)$ is assumed to be an independent set of vertices, it is not possible for both x_i and x_j to be in $\text{supp}(w) \setminus \text{supp}(c) \subseteq \text{supp}(w/c)$. The only remaining possibility is that $x_i \in \text{supp}(c)$ and $x_j \in \text{supp}(w) \setminus \text{supp}(c) \subseteq \text{supp}(w/c)$. Then $x_i \in U \subseteq U'$, so that $e \cap X \neq \emptyset$. Thus X is a vertex cover for G.

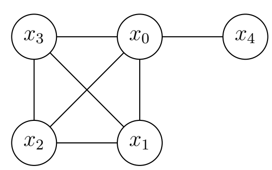
It is clear that $W_X(c) = |U'| = 2$, and so all that remains to show is that $W_X(w/c) = 0$. Suppose there exists $x_j \in X \cap \operatorname{supp}(w/c)$. This implies $x_j \in U' \subseteq \{x_0, x_1, x_2\}$. Since both vertices in $\{x_0, x_1, x_2\} \setminus \{x_j\}$ are adjacent to x_j, it follows by definition of U that $\{x_0, x_1, x_2\} \setminus \{x_j\} \subseteq U \subseteq U'$. Hence $U' = \{x_0, x_1, x_2\}$, a contradiction since $|U'| \leq 2$. Thus $X \cap \operatorname{supp}(w/c) = \emptyset$, and so $W_X(w/c) = 0$. Then

$$W_X(w) = W_X(c) + W_X\left(\frac{w}{c}\right) = 2 + 0 = 2 < 3$$

and we have reached our desired contradiction. We conclude that w is divisible by a monomial in $A_3(I)$, and so $A_3(I) + I^3$ is a monomial generating set for $I^{(3)}/I^3$.

□

Example 4.3. Let G be the following graph on vertex set $\{x_0, x_1, x_2, x_3, x_4\}$ and $I = I(G) \subset k[x_0, x_1, x_2, x_3, x_4]$.



By Proposition 3.3, sdefect$(I, 2) > 0$. Let

$$w_1 := (x_1x_2x_3)(x_0x_4) = x_0x_1x_2x_3x_4 \in C_E(I) \subseteq A_3(I)$$

and

$$w_2 := x_0x_1x_2x_3 \in K_4(I) \subseteq A_3(I).$$

The monomial w_2 strictly divides w_1, and so $A_3(I) + I^3$ is not the minimal monomial generating set for $I^{(3)}/I^3$.

We introduce more notation to extract the minimal monomial generating set for $I^{(3)}/I^3$ from $A_3(I) + I^3$.

Definition 4.4. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then define

$$C_E^{\min}(I) = \{w \in C_E(I) : w \text{ is not divisible by a } K_4 \text{ monomial}\}$$

and

$$B_3(I) = C_E^{\min}(I) \cup K_4(I) \cup C_5(I).$$

Lemma 4.5. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then $B_3(I) + I^3$ is the minimal monomial generating set for $I^{(3)}/I^3$.*

Proof. By Proposition 3.3, there exists a triangle in G, and so there exists a triangle monomial arising from G. Suppose $x_0x_1x_2$ is a triangle monomial. Then x_0x_1 is an edge monomial and so $(x_0x_1x_2)(x_0x_1) = x_0^2x_1^2x_2 \in C_E(I)$. Since $|\text{supp}(x_0^2x_1^2x_2)| = 3$, this monomial is not divisible by a K_4 monomial and so $x_0^2x_1^2x_2 \in C_E^{\min}(I) \subseteq B_3(I)$. Thus $B_3(I) \neq \emptyset$.

Since $B_3(I) \subseteq A_3(I)$, $B_3(I) + I^3$ consists of non-zero monomials in $I^{(3)}/I^3$ by Lemma 4.2. Next, let w be a monomial such that $w + I^3 \in I^{(3)}/I^3$ is non-zero. We will show that w is divisible by a monomial in $B_3(I)$. By Lemma 4.2, w is divisible by a monomial $u \in A_3(I)$. If $u \in K_4(I)$ or $C_5(I)$, then $u \in B_3(I)$ and we are done. If $u \in C_E(I)$ and u is not divisible by a K_4 monomial, then $u \in C_E^{\min}(I) \subseteq B_3(I)$ and we are done. If $u \in C_E(I)$ and u is divisible by a K_4 monomial, this implies that u, and thus w, is divisible by a monomial in $K_4(I) \subseteq B_3(I)$. Thus $B_3(I) + I^3$ is a monomial generating set for $I^{(3)}/I^3$.

All that remains to show is that this monomial generating set is minimal. All monomials in $C_E^{\min}(I)$ and $C_5(I)$ have degree 5, and all monomials in $K_4(I)$ have degree 4. By definition, no monomials in $C_E^{\min}(I)$ are divisible by monomials in $K_4(I)$. Also by definition, 5 - cycle monomials cannot be divisible by K_4 monomials, and so no monomials in $C_5(I)$ are divisible by monomials in $K_4(I)$. It follows that no monomial in $B_3(I)$ properly divides another, and so $B_3(I) + I^3$ is the minimal monomial generating set for $I^{(3)}/I^3$. $\square$

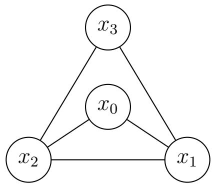
4.1.2 A formula for the third term

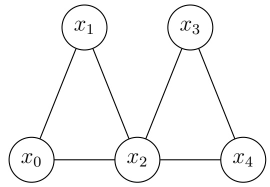
To describe a formula for sdefect$(I, 3)$, we need only count the cardinality of the minimal monomial generating set $B_3(I) + I^3$ for $I^{(3)}/I^3$ found in Lemma 4.5. By definition, 5 - cycle monomials cannot be divisible by triangle monomials, and so $C_E^{\min}(I) \cap C_5(I) = \emptyset$. It easily follows that $B_3(I) = C_E^{\min}(I) \sqcup K_4(I) \sqcup C_5(I)$, where $\sqcup$ denotes disjoint union. Thus

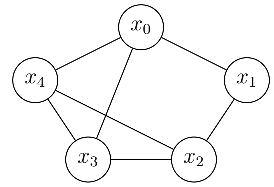
$$|B_3(I)| = |C_E^{\min}(I)| + |K_4(I)| + |C_5(I)|.$$

By Remark 2.8, there is a one-to-one correspondence between 5 - cycle monomials and induced copies of 5 - cycles in G, and so $|C_5(I)|$ is equal to the number of induced copies of 5 - cycles in G. Similarly, $|K_4(I)|$ is equal to the number of K_4's in G. On the other hand, computing $|C_E^{\min}(I)|$ is not straightforward as monomials in $C_E^{\min}(I)$ may have multiple representations as the product of a triangle monomial and an edge monomial.

Example 4.6. Consider the following graphs.







H_1 H_2 H_3

For each $i = 1, 2, 3$, $\text{sdefect}(I(H_i), 2) = 2$ by Theorem 3.7. To compute $\text{sdefect}(I(H_i), 3)$, we need only count $|B_3(I(H_i))| = |C_E^{\min}(I(H_i))| + |K_4(I(H_i))| + |C_5(I(H_i))|$. However, note that

$$x_0x_1^2x_2x_3 = (x_0x_1x_2)(x_1x_3) = (x_1x_2x_3)(x_0x_1) \in C_E^{\min}(I(H_1)),$$
$$x_0x_1x_2x_3x_4 = (x_0x_1x_2)(x_3x_4) = (x_2x_3x_4)(x_0x_1) \in C_E^{\min}(I(H_2))$$

and

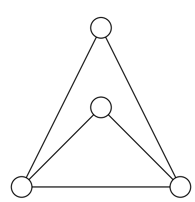
$$x_0x_1x_2x_3x_4 = (x_0x_3x_4)(x_1x_2) = (x_2x_3x_4)(x_0x_1) \in C_E^{\min}(I(H_3))$$

all have two different representations as the product of a triangle monomial and an edge monomial.

We will soon see that the graphs given in Example 4.6 are the only types of subgraphs that may give rise to monomials in $C_E^{\min}(I)$ that do not have a unique representation as the product of a triangle monomial and an edge monomial. We now introduce the necessary notation to work towards computing $|C_E^{\min}(I)|$.

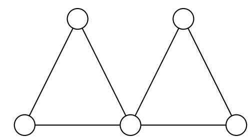
Definition 4.7.

1. Let H_e denote the graph on 4 vertices that consists of two triangles sharing an edge, shown below.



H_e

2. Let H_v denote the graph on 5 vertices that consists of two triangles sharing a vertex, shown below.



H_v

Definition 4.8. Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If $\operatorname{sdefect}(I, 2) > 0$, then define

1. $C_E^{H_e}(I) = \{w \in C_E^{\min}(I) : G[\operatorname{supp}(w)] = H_e\}$;
2. $C_E^{H_v}(I) = \{w \in C_E^{\min}(I) : G[\operatorname{supp}(w)] = H_v\}$;
3. $C_E^{C_5}(I) = \{w \in C_E^{\min}(I) : G[\operatorname{supp}(w)]$ is a copy of a 5 - cycle$\}$; and
4. $C'_E(I) = C_E^{\min}(I) \setminus (C_E^{H_e}(I) \cup C_E^{H_v}(I) \cup C_E^{C_5}(I))$.

Note that

$$C_E^{\min}(I) = C_E^{H_e}(I) \sqcup C_E^{H_v}(I) \sqcup C_E^{C_5}(I) \sqcup C'_E(I),$$

and so

$$|C_E^{\min}(I)| = |C_E^{H_e}(I)| + |C_E^{H_v}(I)| + |C_E^{C_5}(I)| + |C'_E(I)|. \tag{4.1}$$

We will compute each term on the right-hand side of Equation 4.1 through a series of lemmas. First, we introduce more necessary notation.

Definition 4.9. Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. Assume $\operatorname{sdefect}(I, 2) > 0$.

1. Let $h_e(G)$ denote the number of induced copies of H_e in G, $h_v(G)$ denote the number of induced copies of H_v in G, and $k_4(G)$ denote the number of K_4's in G.
2. Let $c_5^{i.c.}(G)$ denote the number of induced copies of 5 - cycles in G, and $c_5^{c}(G)$ denote the number of copies of 5 - cycles in G that do not contain a K_4.
3. Let $c_5(G) = c_5^{i.c.}(G) + c_5^{c}(G)$, the total number of 5 - cycles existing as subgraphs in G that do not contain a K_4.
4. Let $C_3(I) = \{c : c$ is a triangle monomial$\}$.

5. Let

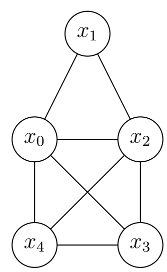
$$\sigma(I) = \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{H_e}(I) \cup C_E^{H_v}(I) \cup C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\}|.$$

6. Let

$$\sigma_{C_5}^{K_4}(I) = \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\}|.$$

Notice that if a monomial ce lies in $C_E(I) \setminus C_E^{\min}(I)$, then ce is divisible by a K_4 monomial. This motivates the notation for $\sigma_{C_5}^{K_4}(I)$.

Example 4.10. Consider the following graph G on vertex set $\{x_0, x_1, x_2, x_3, x_4\}$ and let $I = I(G) \subset k[x_0, x_1, x_2, x_3, x_4]$.



By Theorem 3.7, $\operatorname{sdefect}(I, 2) = 5$. We see

$$C_3(I) = \{x_0x_1x_2, x_0x_2x_3, x_0x_2x_4, x_0x_3x_4, x_2x_3x_4\}.$$

For each $c \in C_3(I)$, let

$$\begin{aligned} E_c &= \{e \in \mathcal{G}(I) : ce \in C_E^{H_e}(I) \cup C_E^{H_v}(I) \cup C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\} \\ &= \{e \in \mathcal{G}(I) : G[\operatorname{supp}(ce)] = H_e \text{ or } H_v \text{ or is a copy of a 5 - cycle, or contains a } K_4\}. \end{aligned}$$

Consider $c = x_0x_1x_2$. We see that $G[\operatorname{supp}(cx_0x_3)] = G[\operatorname{supp}(cx_2x_3)]$ and $G[\operatorname{supp}(cx_0x_4)] = G[\operatorname{supp}(cx_2x_4)]$ are all equal to H_e, and that $G[\operatorname{supp}(cx_3x_4)]$ is a copy of a 5 - cycle (and also contains a K_4). For all other edge monomials e, $G[\operatorname{supp}(ce)]$ is not equal to H_e, H_v, a copy of a 5 - cycle, nor contains a K_4. Thus $|E_{x_0x_1x_2}| = |\{x_0x_3, x_2x_3, x_0x_4, x_2x_4, x_3x_4\}| = 5$. Similarly, $|E_{x_0x_2x_3}| = |\{x_0x_1, x_1x_2, x_0x_4, x_2x_4, x_3x_4\}| = 5$, $|E_{x_0x_2x_4}| = |\{x_0x_1, x_1x_2, x_0x_3, x_2x_3, x_3x_4\}| = 5$, $|E_{x_0x_3x_4}| = |\{x_0x_2, x_2x_3, x_2x_4, x_1x_2\}| = 4$, and $|E_{x_2x_3x_4}| = |\{x_0x_1, x_0x_2, x_0x_3, x_0x_4\}| = 4$. Thus

$$\sigma(I) = \sum_{c \in C_3(I)} |E_c| = 23.$$

Similarly, let

$$\begin{aligned} E'_c &= \{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\} \\ &= \{e \in \mathcal{G}(I) : G[\operatorname{supp}(ce)] \text{ is a copy of a 5 -cycle or contains a } K_4\}. \end{aligned}$$

Then

$$|E'_{x_0x_1x_2}| = |\{x_3x_4\}| = 1,\ |E'_{x_0x_2x_3}| = |\{x_0x_4, x_2x_4, x_3x_4\}| = 3,$$

$$|E'_{x_0x_2x_4}| = |\{x_0x_3, x_2x_3, x_3x_4\}| = 3,\ |E'_{x_0x_3x_4}| = |\{x_0x_2, x_2x_3, x_2x_4, x_1x_2\}| = 4,$$

and

$$|E'_{x_2x_3x_4}| = |\{x_0x_2, x_0x_3, x_0x_4, x_0x_1\}| = 4,$$

so that $\sigma_{C_5}^{K_4}(I) = 15$.

Lemma 4.11. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I, 2) > 0$, then*

$$|C_E^{H_v}(I)| = h_v(G).$$

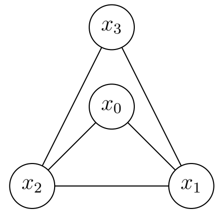
Proof. Monomials in $C_E^{H_v}(I)$ have degree 5 because they lie in $C_E^{\min}(I)$. Monomials in $C_E^{H_v}(I)$ also have support size 5, and so all monomials in $C_E^{H_v}(I)$ are square-free. It follows by definition that $C_E^{H_v}(I)$ is the set of H_v monomials, and so by Remark 2.8, $|C_E^{H_v}(I)| = h_v(G)$. □

Lemma 4.12. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I, 2) > 0$, then*

$$|C_E^{H_e}(I)| = 2h_e(G).$$

Proof. We will establish a two-to-one correspondence between monomials in $C_E^{H_e}(I)$ and induced copies of H_e in G , implying by Remark 2.8 that $|C_E^{H_e}(I)| = 2h_e(G)$. Given a monomial $w \in C_E^{H_e}(I)$, there clearly exists a unique induced copy of H_e in G on the vertex set $\operatorname{supp}(w)$, namely $G[\operatorname{supp}(w)]$. Next, we will show that given an induced copy of H_e in G on vertex set X, there exists exactly two monomials in $C_E^{H_e}(I)$ with support being X.

Indeed, suppose there is an induced copy of H_e in G on the vertex set $X = \{x_0, x_1, x_2, x_3\}$, with vertices labelled as below.



Suppose that $w \in C_E^{H_e}(I)$ is such that $\operatorname{supp}(w) = X$. Since $C_E^{H_e}(I) \subseteq C_E^{\min}(I) \subseteq C_E(I)$, we may write $w = ce$, where c is a triangle monomial and e is an edge monomial. There is a one-to-one correspondence between triangle monomials dividing w and triangles in $G[\operatorname{supp}(w)] = G[X]$, so that $c = x_0x_1x_2$ or $c = x_1x_2x_3$.

Suppose first that $c = x_0x_1x_2$. Then since $\operatorname{supp}(w) = X$, the variable x_3 divides w. Since $x_3 \nmid c$, we must have $x_3 \mid e$. The only edges in $G[X]$ incident to x_3 are $\{x_1, x_3\}$ and $\{x_2, x_3\}$, and so this forces $e = x_1x_3$ or $e = x_2x_3$. Thus

$$w = (x_0x_1x_2)(x_1x_3) = x_0x_1^2x_2x_3 := w_1$$

or

$$w = (x_0x_1x_2)(x_2x_3) = x_0x_1x_2^2x_3 := w_2.$$

Similarly, suppose instead that $c = x_1x_2x_3$. Then since $\operatorname{supp}(w) = X$, the variable x_0 divides w. Since $x_0 \nmid c$, we must have $x_0 \mid e$. The only edges in $G[X]$ incident to x_0 are $\{x_0, x_1\}$ and $\{x_0, x_2\}$, and so this forces $e = x_0x_1$ or x_0x_2. Thus

$$w = (x_1x_2x_3)(x_0x_1) = x_0x_1^2x_2x_3 = w_1$$

or

$$w = (x_1x_2x_3)(x_0x_2) = x_0x_1x_2^2x_3 = w_2.$$

In either case, $w = w_1$ or w_2. We conclude that w_1 and w_2 are the only two monomials in $C_E^{H_e}(I)$ with support being X. $\square$

Lemma 4.13. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then*

$$|C_E^{C_5}(I)| = c_5^c(G).$$

Proof. Recall from Definition 4.9 that $c_5^c(G)$ is the number of copies of 5 - cycles in G that do not contain a K_4. We will establish a one-to-one correspondence between monomials in $C_E^{C_5}(I)$ and copies of 5 - cycles in G that do not contain a K_4. Let $w \in C_E^{C_5}(I)$, and recall that $C_E^{\min}(I)$ is the set of monomials in $C_E(I)$ that are not divisible by a K_4 monomial. Since $C_E^{C_5}(I) \subseteq C_E^{\min}(I)$, there is a unique copy of a 5 - cycle in G that does not contain a K_4 on vertex set $\operatorname{supp}(w)$, namely $G[\operatorname{supp}(w)]$. Next, we will show that given a copy of a 5 - cycle in G that does not contain a K_4 on vertex set X, there exists a unique monomial in $C_E^{C_5}(I)$ with support being X.

Indeed, suppose there is a copy of a 5 - cycle that does not contain a K_4 in G on vertex set $X = \{x_0, x_1, x_2, x_3, x_4\}$. Recall that $G[X]$ being a copy of a 5 - cycle means it is a 5 - cycle with at least one chord. Any chord in a 5 - cycle creates a triangle, and so we may assume without loss of generality that $\{x_0, x_2\}$ is a chord in $G[X]$, creating a triangle on vertices $\{x_0, x_1, x_2\}$. Let $w = x_0x_1x_2x_3x_4$ and note $\text{supp}(w) = X$. Further, since $G[X] = G[\text{supp}(w)]$ does not contain a K_4, w is not divisible by a K_4 monomial. Then

$$w = (x_0x_1x_2)(x_3x_4) \in C_E^{C_5}(I),$$

and so there exists a monomial in $C_E^{C_5}(I)$ with support being X. To see that this monomial is unique, note that all monomials in $C_E^{C_5}(I)$ have degree 5 and support size 5. Thus all monomials in $C_E^{C_5}(I)$ are square-free. It easily follows that w is the unique monomial in $C_E^{C_5}(I)$ with support being X. $\square$

Lemma 4.14. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, 2) > 0$, then*

$$|C'_E(I)| = \textit{sdefect}(I, 2)|E(G)| - \sigma(I).$$

Proof. Throughout, we drop the I from all notation with the exception of $C_3(I)$ and sdefect$(I, 2)$. Note first that

$$\begin{aligned} C'_E &= \{w \in C_E^{\min} : w \notin C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5}\} \\ &= \bigcup_{c \in C_3(I)} \{ce : e \in \mathcal{G}, ce \notin C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}. \end{aligned}$$

If all monomials in C'_E have a unique representation as the product of a triangle monomial and an edge monomial, then the above union is disjoint. In this case,

$$\begin{aligned} |C'_E| &= \sum_{c \in C_3(I)} |\{ce : e \in \mathcal{G}, ce \notin C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\ &= \sum_{c \in C_3(I)} |\{e \in \mathcal{G} : ce \notin C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\ &= \sum_{c \in C_3(I)} |\mathcal{G} \setminus \{e \in \mathcal{G} : ce \in C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\ &= \sum_{c \in C_3(I)} |\mathcal{G}| - |\{e \in \mathcal{G} : ce \in C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\ &= \text{sdefect}(I, 2)|E(G)| - \sigma \end{aligned}$$

where the final equality holds because $|\mathcal{G}| = |E(G)|$ and $|C_3(I)| = \text{sdefect}(I, 2)$ by Theorem 3.7. Thus it will suffice to show that all monomials in C'_E have a unique representation as the product of a triangle monomial and an edge monomial.

To this end, suppose that $w = ce, w' = c'e' \in C'_E$ are equal. Write $c = x_0x_1x_2$, $c' = x'_0x'_1x'_2$, $e = x_ix_j$ and $e' = x'_ix'_j$. We want to show that $c = c'$ and $e = e'$. We consider three cases, depending on the intersection of $\mathrm{supp}(c)$ and $\mathrm{supp}(e)$.

Case 1: Suppose that $\mathrm{supp}(e) \subset \mathrm{supp}(c)$. This happens if and only if $|\mathrm{supp}(w)| = |\mathrm{supp}(c)| = 3$. Since $w = w'$, this means $|\mathrm{supp}(w')| = 3$, and so $\mathrm{supp}(e') \subset \mathrm{supp}(c')$ as well. Now, note that $\mathrm{supp}(e) \subset \mathrm{supp}(c)$ implies that x_i and x_j are the only variables whose squares divide w. Similarly, x'_i and x'_j are the only variables whose squares divide w'. We conclude that $e = x_ix_j = x'_ix'_j = e'$. This implies $c = c'$ as well, and we are done.

Case 2: Suppose that $|\mathrm{supp}(c) \cap \mathrm{supp}(e)| = 1$. This happens if and only if $|\mathrm{supp}(w)| = |\mathrm{supp}(c)| + 1 = 4$. Since $w = w'$, this means $|\mathrm{supp}(w')| = 4$ and so $|\mathrm{supp}(c') \cap \mathrm{supp}(e')| = 1$ as well. Assume that $x_i \in \mathrm{supp}(c)$, $x_j \notin \mathrm{supp}(c)$, $x'_i \in \mathrm{supp}(c')$ and $x'_j \notin \mathrm{supp}(c')$. Note that this implies that x_i is the only variable whose square divides w, and x'_i is the only variable whose square divides w'. Thus $x_i = x'_i$.

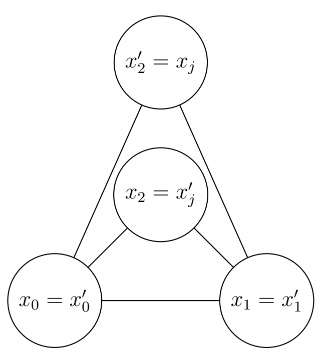
If $x_j = x'_j$ as well, then $e = e'$, implying $c = c'$, and we are done. Thus we may assume $x_j \neq x'_j$. Then, since

$$w = w' \implies cx_ix_j = c'x'_ix'_j \implies cx_j = c'x'_j \implies c = \frac{c'x'_j}{x_j} = \left(\frac{c'}{x_j}\right) x'_j,$$

it follows that x_j divides c' and x'_j divides c. In other words, $x_j \in \{x'_0, x'_1, x'_2\}$ and $x'_j \in \{x_0, x_1, x_2\}$. Without loss of generality, suppose that $x'_j = x_2$ and $x_j = x'_2$. Then

$$c = \frac{c'x'_j}{x_j} \implies x_0x_1x_2 = \frac{x'_0x'_1x'_2x_2}{x'_2} \implies x_0x_1 = x'_0x'_1.$$

Without loss of generality, assume $x_0 = x'_0$ and $x_1 = x'_1$. Then $G[\mathrm{supp}(w)]$ contains the following graph as a subgraph.



Since $x_i = x'_i$ is assumed to lie in $\mathrm{supp}(c) \cap \mathrm{supp}(c') = \{x_0, x_1\}$, all the vertices in $\mathrm{supp}(w)$ are shown in the graph above. If $G[\mathrm{supp}(w)]$ is equal to the graph above, then $G[\mathrm{supp}(w)] = H_e$, a contradiction since $w \notin C_E^{H_e}$. Similarly, if $G[\text{supp}(w)]$ is not the above graph, this forces $G[\text{supp}(w)] = K_4$, contradicting that $w \in C_E^{\min}$.

Case 3: Suppose that $\text{supp}(e) \cap \text{supp}(c) = \emptyset$. This happens if and only if $|\text{supp}(w)| = |\text{supp}(c)| + 2 = 5$. Since $w = w'$, this means $|\text{supp}(w')| = 5$ and so $\text{supp}(e') \cap \text{supp}(c') = \emptyset$ as well. Note that

$$w = w' \implies cx_ix_j = c'x_i'x_j' \implies c = \frac{c'x_i'x_j'}{x_ix_j}.$$

If x_ix_j divides $x_i'x_j'$, then $e = e'$, implying $c = c'$, and we are done. Thus we may assume that at least one of x_i or x_j does not divide $x_i'x_j'$. We consider the case that exactly one of x_i or x_j divide $x_i'x_j'$, and the case that neither x_i nor x_j divide $x_i'x_j'$ separately. We show that both scenarios lead to a contradiction.

Subcase 3a: Suppose that only x_j divides $x_i'x_j'$. Without loss of generality, assume $x_j = x_j'$ and $x_i \neq x_i'$. Since $c = (c'x_i'x_j')/(x_ix_j) = (c'x_i')/x_i = (c'/x_i)x_i'$, this implies that x_i divides c' and x_i' divides c. In other words, $x_i \in \{x_0', x_1', x_2'\}$ and $x_i' \in \{x_0, x_1, x_2\}$. Without loss of generality, suppose that $x_i = x_2'$ and $x_i' = x_2$. Then

$$c = \frac{c'x_i'}{x_i} \implies x_0x_1x_2 = \frac{x_0'x_1'x_2'x_2}{x_2'} \implies x_0x_1 = x_0'x_1'.$$

We may then assume $x_0 = x_0'$ and $x_1 = x_1'$. Then there is a copy of a 5 - cycle in $G[\text{supp}(w)]$ on vertices

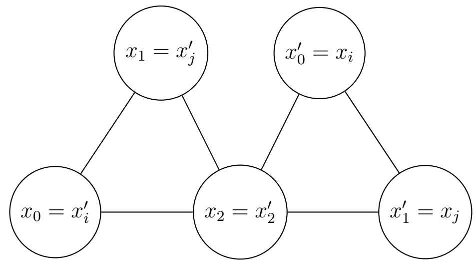
$$\{x_0' = x_0, x_1, x_2 = x_i', x_j' = x_j, x_i = x_2'\}.$$

Since the above set of vertices is exactly $\text{supp}(w)$, it follows that $G[\text{supp}(w)]$ is equal to a copy of a 5 - cycle, a contradiction since $w \notin C_E^{C_5}$.

Subcase 3b: Suppose that x_i and x_j both do not divide $x_i'x_j'$. Since $c = (c'x_i'x_j')/(x_ix_j) = (c'/(x_ix_j))x_i'x_j'$, this implies that x_i and x_j both divide c' and x_i' and x_j' both divide c. In other words, $x_i, x_j \in \{x_0', x_1', x_2'\}$ and $x_i', x_j' \in \{x_0, x_1, x_2\}$. We may assume without loss of generality that $x_i = x_0'$, $x_j = x_1'$, $x_i' = x_0$ and $x_j' = x_1$. Then

$$c = \frac{c'x_i'x_j'}{x_ix_j} \implies x_0x_1x_2 = \frac{x_0'x_1'x_2'x_0x_1}{x_0'x_1'} \implies x_2 = x_2'.$$

Then $G[\text{supp}(w)]$ contains the following graph as a subgraph.



If $G[\mathrm{supp}(w)]$ is equal to this graph, then $G[\mathrm{supp}(w)] = H_v$, a contradiction since $w \notin C_E^{H_v}$. Otherwise, the addition of any edges to the subgraph above turns it into a copy of a 5 - cycle, contradicting the assumption that $w \notin C_E^{C_5}$.

Thus in all possible cases, we arrive at the conclusion that $c = c'$ and $e = e'$. We conclude that all monomials in C'_E have a unique representation as the product of a triangle monomial and an edge monomial. $\square$

Our final lemma before describing a formula for sdefect$(I, 3)$ provides a description of $\sigma(I)$ in terms of other parameters of G.

Lemma 4.15. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I, 2) > 0$, then*

$$\sigma(I) = 4h_e(G) + 2h_v(G) + \sigma_{C_5}^{K_4}(I).$$

Proof. Throughout, we drop the I from all notation with the exception of $C_3(I)$. Since $C_E^{H_e}$, $C_E^{H_v}$ and $C_E^{C_5} \cup (C_E \setminus C_E^{\min})$ are all disjoint, we see that

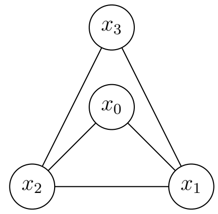
$$\begin{aligned}
\sigma &= \sum_{c \in C_3(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_e} \cup C_E^{H_v} \cup C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\
&= \sum_{c \in C_3(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_e}\}| + \sum_{c \in C_3(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_v}\}| + \\
&\quad \sum_{c \in C_3(I)} |\{e \in \mathcal{G} : ce \in C_E^{C_5} \cup (C_E \setminus C_E^{\min})\}| \\
&= \sigma_{C_5}^{K_4} + \sum_{c \in C_3(I)} |\{ce : e \in \mathcal{G} \text{ and } ce \in C_E^{H_e}\}| + \sum_{c \in C_3(I)} |\{ce : e \in \mathcal{G} \text{ and } ce \in C_E^{H_v}\}| \\
&= \sigma_{C_5}^{K_4} + \sum_{c \in C_3(I)} |\{ce \in C_E^{H_e}\}| + \sum_{c \in C_3(I)} |\{ce \in C_E^{H_v}\}|.
\end{aligned}$$

We first claim that

$$\sum_{c \in C_3(I)} |\{ce \in C_E^{H_e}\}| = 2|C_E^{H_e}|.$$

It will suffice to show that each monomial in $C_E^{H_e}$ has exactly two representations as the product of a triangle monomial and an edge monomial. Let $w \in C_E^{H_e}$, so that $G[\mathrm{supp}(w)] =$

H_e, and label the vertices of $G[\text{supp}(w)]$ as below.



Since $w \in C_E^{\min} \subset C_E$, we may write $w = ce$, where c is a triangle monomial and e is an edge monomial. Since there is a one-to-one correspondence between triangle monomials dividing w and triangles in $G[\text{supp}(w)]$, we must have $c = x_0x_1x_2$ or $c = x_1x_2x_3$. Then since $\text{supp}(w) = \{x_0, x_1, x_2, x_3\}$, we must have either

$$w = (x_0x_1x_2)(x_2x_3) = (x_1x_2x_3)(x_0x_2)$$

or

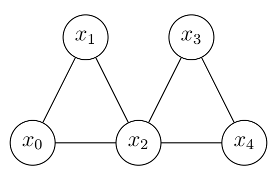
$$w = (x_0x_1x_2)(x_1x_3) = (x_1x_2x_3)(x_0x_1).$$

In either case, w has exactly two different representations as the product of a triangle monomial and an edge monomial.

Next, we prove in a similar fashion that

$$\sum_{c \in C_3(I)} |\{ce \in C_E^{H_v}\}| = 2|C_E^{H_v}|.$$

It will suffice to show that each monomial in $C_E^{H_v}$ has exactly two representations as the product of a triangle monomial and an edge monomial. Let $w \in C_E^{H_v}$, so that $G[\text{supp}(w)] = H_v$, and label the vertices of $G[\text{supp}(w)]$ as below.



Since $w \in C_E^{\min} \subset C_E$, we may write $w = ce$, where c is a triangle monomial and e is an edge monomial. Since there is a one-to-one correspondence between triangle monomials dividing w and triangles in $G[\text{supp}(w)]$, we must have $c = x_0x_1x_2$ or $c = x_2x_3x_4$. Then since $\text{supp}(w) = \{x_0, x_1, x_2, x_3, x_4\}$, it must be the case that

$$w = (x_0x_1x_2)(x_3x_4) = (x_2x_3x_4)(x_0x_1).$$

Thus w has exactly two different representations as the product of a triangle monomial and an edge monomial. Then by Lemmas 4.11 and 4.12,

$$\sigma = 2|C_E^{H_e}| + 2|C_E^{H_v}| + \sigma_{C_5}^{K_4} = 4h_e(G) + 2h_v(G) + \sigma_{C_5}^{K_4}.$$

□

With Lemmas 4.11 - 4.15, Equation 4.1 can be written as

$$|C_E^{\min}(I)| = c_5^c(G) + \text{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I). \tag{4.2}$$

A formula for sdefect$(I,3)$ now easily follows.

Theorem 4.16. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I,2) > 0$, then*

$$\textit{sdefect}(I,3) = c_5(G) + k_4(G) + \textit{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I).$$

Proof. Recall that by Remark 2.8, $|C_5(I)| = c_5^{i.c.}(G)$ and $|K_4(I)| = k_4(G)$. By Lemma 4.5 and Equation 4.2,

$$\begin{aligned}
\text{sdefect}(I,3) &= |B_3(I)| \\
&= |C_5(I)| + |K_4(I)| + |C_E^{\min}(I)| \\
&= c_5^{i.c.}(G) + k_4(G) + c_5^c(G) + \text{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I) \\
&= c_5(G) + k_4(G) + \text{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I).
\end{aligned}$$

□

4.1.3 Growth between the second and third term

In this section, we use Lemma 4.5 to bound the growth between the second and third terms of the symbolic defect sequence of an edge ideal.

Corollary 4.17. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I,2) > 0$, then*

$$3\textit{sdefect}(I,2) \le \textit{sdefect}(I,3) \le \textit{sdefect}(I,2)|E(G)| + c_5^{i.c.}(G) + k_4(G).$$

Proof. We first prove the claimed lower bound on sdefect$(I,3)$. Note that if c is a triangle monomial and e is one of the three edge monomials satisfying $e \mid c$, then $G[\text{supp}(ce)] =$

$G[\mathrm{supp}(c)]$ is a triangle, so that $G[\mathrm{supp}(ce)]$ cannot be equal to H_e, H_v, a copy of a 5 - cycle, nor can it contain a K_4. Thus $ce \in C_E^{\min}(I)$ and $ce \notin C_E^{H_e}(I) \cup C_E^{H_v}(I) \cup C_E^{C_5}(I)$, and so

$$\{ce \in C_E(I) : e \mid c\} \subseteq C'_E(I).$$

Since $C'_E(I) \subseteq C_E^{\min}(I) \subseteq B_3(I)$ by definition, it follows that

$$|\{ce \in C_E(I) : e \mid c\}| \leq |C'_E(I)| \leq |B_3(I)| = \mathrm{sdefect}(I, 3).$$

We will show that $|\{ce \in C_E(I) : e \mid c\}| = 3\mathrm{sdefect}(I, 2)$, completing the proof for the lower bound.

We first claim that monomials in $\{ce \in C_E(I) : e \mid c\}$ have a unique representation as the product of a triangle monomial and an edge monomial. Indeed, suppose that $ce = c'e' \in \{ce \in C_E(I) : e \mid c\}$. Write $e = x_i x_j$ and $e' = x'_i x'_j$. Then x_i and x_j are the only variables whose squares divide ce, and x'_i and x'_j are the only variables whose squares divide $c'e'$. It follows that $e = e'$, immediately implying that $c = c'$ as well. Thus

$$\begin{aligned}
|\{ce \in C_E(I) : e \mid c\}| &= \left| \bigsqcup_{c \in C_3(I)} \{ce : e \in \mathcal{G}(I), e \mid c\} \right| \\
&= \sum_{c \in C_3(I)} |\{ce : e \in \mathcal{G}(I), e \mid c\}| \\
&= \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : e \mid c\}| \\
&= \sum_{c \in C_3(I)} 3 \\
&= 3\mathrm{sdefect}(I, 2),
\end{aligned}$$

where the final equality holds because $|C_3(I)| = \mathrm{sdefect}(I, 2)$ by Theorem 3.7.

We now prove the claimed upper bound on $\mathrm{sdefect}(I, 3)$. We have

$$\mathrm{sdefect}(I, 3) = |B_3(I)| = |C_E^{\min}(I)| + |K_4(I)| + |C_5(I)| = |C_E^{\min}(I)| + k_4(G) + c_5^{i.c.}(G),$$

and since $C_E^{\min}(I) \subseteq C_E(I)$,

$$\begin{aligned}
|C_E^{\min}(I)| &\leq |C_E(I)| \\
&= |\{ce : c \text{ is a triangle monomial and } e \text{ is an edge monomial}\}| \\
&= \left| \bigcup_{c \in C_3(I)} \{ce : e \in \mathcal{G}(I)\} \right| \\
&\leq \sum_{c \in C_3(I)} |\{ce : e \in \mathcal{G}(I)\}| \\
&= \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I)\}| \\
&= \operatorname{sdefect}(I, 2)|E(G)|.
\end{aligned}$$

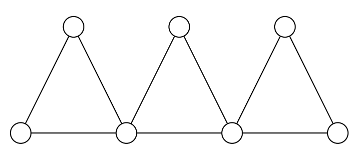
Thus

$$\operatorname{sdefect}(I, 3) \leq \operatorname{sdefect}(I, 2)|E(G)| + k_4(G) + c_5^{i.c.}(G).$$

□

Remark 4.18. Given a graph G containing a triangle, Corollary 4.17 shows that the third term of the symbolic defect sequence of $I(G)$ is always strictly larger than the second term. Also, although the difference between the third and second term is bounded above for any particular edge ideal, in general, this difference may be arbitrarily large.

Example 4.19. For $j \geq 1$, let T_j denote the graph that consists of a chain of j triangles, each sharing exactly one vertex with the triangle before it. For example, T_3 is drawn below.



By Theorem 3.7, for each $j \geq 1$, $\operatorname{sdefect}(I(T_j), 2) = j$. By Corollary 4.17, as $j \to \infty$,

$$\operatorname{sdefect}(I(T_j), 3) - \operatorname{sdefect}(I(T_j), 2) \to \infty.$$

4.1.4 Special cases

Using Theorem 4.16 to compute $\operatorname{sdefect}(I, 3)$ may not always be convenient, as $\sigma_{C_5}^{K_4}(I)$ is not easily calculated in general. We next illustrate families of graphs where Theorem 4.16 gives a simpler formula.

Corollary 4.20. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If sdefect$(I, 2) = 1$, then*

$$\textit{sdefect}(I,3) = |E(G)| + c_5^{i.c.}(G).$$

Proof. By Theorem 4.16,

$$\text{sdefect}(I,3) = c_5(G) + k_4(G) + |E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I).$$

Note that both H_e and H_v contain two triangles. By Proposition 3.10, G contains a unique triangle, and so G cannot contain any induced copies of H_e or H_v. Thus $h_e(G) = h_v(G) = 0$. Similarly, K_4 contains four triangles, so that G cannot contain a K_4 as well and so $k_4(G) = 0$.

Now, since G contains a unique triangle, there exists a unique triangle monomial. Write $C_3(I) = \{c\}$ and note that since G does not contain any K_4's, $C_E(I) = C_E^{\min}(I)$. Then

$$\begin{aligned}
\sigma_{C_5}^{K_4}(I) &= \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\}| \\
&= |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I) \cup (C_E(I) \setminus C_E^{\min}(I))\}| \\
&= |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I)\}| \\
&= |\{ce : e \in \mathcal{G}(I) \text{ and } ce \in C_E^{C_5}(I)\}| \\
&= |C_E^{C_5}(I)| \\
&= c_5^c(G),
\end{aligned}$$

where the final equality follows from Lemma 4.13. Thus

$$\begin{aligned}
\text{sdefect}(I,3) &= c_5(G) + |E(G)| - \sigma_{C_5}^{K_4}(I) \\
&= (c_5^{i.c.}(G) + c_5^c(G)) + |E(G)| - c_5^c(G) \\
&= |E(G)| + c_5^{i.c.}(G).
\end{aligned}$$

□

Corollary 4.21. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. Assume that sdefect$(I, 2) > 0$. If G does not contain any K_4's, and all copies of 5 - cycles in G contain exactly one chord, then*

$$\textit{sdefect}(I,3) = \textit{sdefect}(I,2)|E(G)| + c_5^{i.c.(G)} - 2h_e(G) - h_v(G).$$

Proof. By Theorem 4.16,

$$\operatorname{sdefect}(I,3) = c_5(G) + k_4(G) + \operatorname{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I).$$

By assumption, $k_4(G) = 0$, implying also that $C_E(I) = C_E^{\min}(I)$. We see

$$\begin{aligned}
\sigma_{C_5}^{K_4}(I) &= \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I) \cup (C_E(I)) \setminus C_E^{\min}(I)\}| \\
&= \sum_{c \in C_3(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_5}(I)\}| \\
&= \sum_{c \in C_3(I)} |\{ce \in C_E^{C_5}(I)\}|.
\end{aligned}$$

We now claim that $\sigma_{C_5}^{K_4}(I) = |C_E^{C_5}(I)|$. Indeed, we see

$$C_E^{C_5}(I) = \bigcup_{c \in C_3(I)} \{ce \in C_E^{C_5}(I)\},$$

and so it will suffice to show that the above union is disjoint. Suppose that $ce = c'e' \in C_E^{C_5}(I)$, where c, c' are triangle monomials and e, e' are edge monomials. By assumption, $G[\operatorname{supp}(ce)] = G[\operatorname{supp}(c'e')]$ is a 5 - cycle with exactly one chord. This unique chord creates a unique triangle in $G[\operatorname{supp}(ce)] = G[\operatorname{supp}(c'e')]$. It follows that $c = c'$ must be the corresponding triangle monomial. Thus

$$\sigma_{C_5}^{K_4}(I) = |C_E^{C_5}(I)| = c_5^c(G)$$

where the last equality holds by Lemma 4.13. Together, this gives

$$\begin{aligned}
\operatorname{sdefect}(I,3) &= c_5(G) + k_4(G) + \operatorname{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I) \\
&= c_5(G) + \operatorname{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - \sigma_{C_5}^{K_4}(I) \\
&= (c_5^{i.c.}(G) + c_5^c(G)) + \operatorname{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G) - c_5^c(G) \\
&= c_5^{i.c.}(G) + \operatorname{sdefect}(I,2)|E(G)| - 2h_e(G) - h_v(G).
\end{aligned}$$

□

Example 4.22. Every odd cycle of length at least 5 in a non-perfect graph (see [49] for the definition of and a detailed background on perfect graphs) has at most one chord [42]. Thus non-perfect graphs that contain at least one triangle and do not contain a K_4 fall in the class of graphs considered in Corollary 4.21. Note that non-perfect graphs that contain at least one triangle and do not contain a K_4 are exactly the non-perfect graphs with chromatic number equal to 3.

Note that in Corollary 4.20, we achieve the upper bound on $\operatorname{sdefect}(I, 3)$ given in Corollary 4.17.

Example 4.23. Consider the triangle C_3 and let $I = I(C_3)$. By Proposition 3.10, $\operatorname{sdefect}(I, 2) = 1$, and by Corollary 4.20, $\operatorname{sdefect}(I, 3) = |E(C_3)| = 3$. Thus

$$|E(C_3)|\operatorname{sdefect}(I, 2) = 3\operatorname{sdefect}(I, 2) = 3 = \operatorname{sdefect}(I, 3),$$

so that the lower and upper bounds in Corollary 4.17 are equal and attained.

4.2 Edge ideals of triangle-free graphs

In this section, we consider only edge ideals of graphs that do not contain triangles. By Proposition 3.3, this is equivalent to edge ideals for which the second term of the symbolic defect sequence, $\operatorname{sdefect}(I, 2)$, is zero. This case mirrors the case of edge ideals of graphs containing triangles from Section 4.1, with analogous results. With Lemma 4.25, we will describe the minimal monomial generating set for $I^{(m+1)}/I^{m+1}$, where $\operatorname{sdefect}(I, m+1)$ is the second non-zero term of the symbolic defect sequence of I. By counting this minimal monomial generating set, we give a formula for the second non-zero term in Theorem 4.36. Some parameters within this formula are generally not easy to compute, and so Corollaries 4.40 and 4.41 demonstrate special cases where this formula is simplified.

4.2.1 The minimal monomial generating set

We begin with the notation needed to describe the minimal monomial generating set for $I^{(m+1)}/I^{m+1}$.

Definition 4.24. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that $\operatorname{sdefect}(I, m) > 0$, then define

$$C_{2m+1}(I) = \{c : c \text{ is a } (2m+1) \text{ - cycle monomial}\},$$

$$C_E(I) = \{ce : c \text{ is a } (2m-1) \text{ - cycle monomial and } e \text{ is an edge monomial}\},$$

and

$$B_{m+1}(I) = C_{2m+1}(I) \cup C_E(I).$$

By definition, $(2m+1)$ - cycle monomials cannot be divisible by any $(2m-1)$ - cycle monomial. Thus $C_{2m+1}(I) \cap C_E(I) = \emptyset$ and so $B_{m+1}(I) = C_{2m+1}(I) \sqcup C_E(I)$.

Lemma 4.25. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then $B_{m+1}(I) + I^{m+1}$ is the minimal monomial generating set for $I^{(m+1)}/I^{m+1}$.*

Proof. This proof uses a similar argument to that of Lemma 4.2. By Theorem 3.7, there exists an induced copy of a $(2m-1)$ - cycle in G, and so there exists a $(2m-1)$ - cycle monomial arising from G. The existence of edge monomials is clear, and so $C_E(I) \neq \emptyset$. Therefore $B_{m+1}(I) \neq \emptyset$.

We next show that $B_{m+1}(I) \subset I^{(m+1)}$. Let $w \in C_{2m+1}(I)$ and X be a vertex cover of G. By Lemma 2.10, $W_X(w) \geq m+1$. By Proposition 2.16, $w \in I^{(m+1)}$, and so $C_{2m+1}(I) \subset I^{(m+1)}$. Similarly, let $w = ce \in C_E(I)$, with c a $(2m-1)$ - cycle monomial and e an edge monomial. By Proposition 2.10, $W_X(c) \geq m$, and so $c \in I^{(m)}$. Then since $e \in I$, $w = ce \in I^{(m)}I \subseteq I^{(m+1)}$ by Proposition 2.18. Thus $C_E(I) \subset I^{(m+1)}$, and so $B_{m+1}(I) = C_{2m+1}(I) \sqcup C_E(I) \subset I^{(m+1)}$. Further, every monomial in $B_{m+1}(I)$ has degree $2m+1 < 2(m+1)$ and so $B_{m+1}(I) \cap I^{m+1} = \emptyset$. Thus every monomial in $B_{m+1}(I) + I^{m+1}$ is non-zero in $I^{(m+1)}/I^{m+1}$.

Now, let w be a monomial such that $w + I^{m+1} \in I^{(m+1)}/I^{m+1}$ is non-zero. We will show that w is divisible by a monomial in $B_{m+1}(I)$. First, note that by Corollary 3.2, w is divisible by a $(2s-1)$ - cycle monomial for some $2 \leq s \leq m+1$. Since sdefect(I, m) is the first non-zero term of the symbolic defect sequence of I, by Proposition 3.3, there does not exist an induced copy of an odd cycle in G of length less than $2m-1$. Thus there does not exist a $(2s-1)$ - cycle monomial for $2 \leq s < m$, and so w is divisible by a $(2m-1)$ - cycle monomial or a $(2m+1)$ - cycle monomial.

If w is divisible by a $(2m+1)$ - cycle monomial, then we are done, since $C_{2m+1}(I) \subseteq B_{m+1}(I)$ consists of all $(2m+1)$ - cycle monomials. So assume that w is not divisible by a $(2m+1)$ - cycle monomial, so that w must be divisible by a $(2m-1)$ - cycle monomial. Without loss of generality, assume that $c = x_0 \cdots x_{2m-2}$ is a $(2m-1)$ - cycle monomial dividing w.

Suppose first that supp(w/c) is not an independent set of vertices in G, meaning that there exists at least one edge in G among the vertices in supp(w/c). Then there exists an edge monomial e dividing w/c. The monomial $ce \in C_E(I) \subseteq B_{m+1}(I)$ then divides w, and we are done.

Now, suppose that supp(w/c) is an independent set of vertices. We will prove by contradiction that w is divisible by a monomial in $C_E(I)$. In particular, assuming that w is not divisible by a monomial in $C_E(I)$, we will produce a vertex cover X for G that satisfies $W_X(c) = m$ and $W_X(w/c) = 0$, implying that $W_X(w) = W_X(c) + W_X(w/c) = m < m+1$. This gives the desired contradiction by Proposition 2.16, since $w \in I^{(m+1)}$.

To this end, assume that w is not divisible by a monomial in $C_E(I)$. Recall that we are also assuming that w is not divisible by a monomial in $C_{2m+1}(I)$, so that together, we are assuming that w is not divisible by a monomial in $B_{m+1}(I)$. Producing a vertex cover X with $W_X(w/c) = 0$ is equivalent to producing a vertex cover X with $X \cap \operatorname{supp}(w/c) = \emptyset$. To produce such a vertex cover, we need to ensure vertices that have neighbours in $\operatorname{supp}(w/c)$ all lie in X. Thus we define

$$U = \operatorname{supp}(w) \cap \left\{ x_i : x_i \text{ is adjacent to some vertex in } \operatorname{supp}\left(\frac{w}{c}\right) \right\}.$$

Notice that since $\operatorname{supp}(w/c)$ is assumed to be an independent set, $U \cap \operatorname{supp}(w/c) = \emptyset$. Thus $U \subseteq \operatorname{supp}(w) \setminus \operatorname{supp}(w/c) \subseteq \operatorname{supp}(c)$.

Case 1: Suppose first that $U = \emptyset$. Let

$$X = \{x_0, x_2, x_4, \dots, x_{2m-2}\} \cup (V(G) \setminus \operatorname{supp}(w)).$$

We claim that this is our desired vertex cover for G. We first show that X is indeed a vertex cover. Let $e = \{x_i, x_j\} \in E(G)$. If at least one of x_i or x_j is not in $\operatorname{supp}(w)$, then $e \cap X \neq \emptyset$. If both x_i and x_j lie in $\operatorname{supp}(c)$, then $e \cap \{x_0, x_2, x_4, \dots, x_{2m-2}\} \neq \emptyset$, and so $e \cap X \neq \emptyset$. Since it is assumed that $\operatorname{supp}(w/c)$ is an independent set of vertices, it is not possible for x_i and x_j to both be in $\operatorname{supp}(w) \setminus \operatorname{supp}(c) \subseteq \operatorname{supp}(w/c)$. Lastly, since it is assumed that $U = \emptyset$, it is not possible for $x_i \in \operatorname{supp}(c)$ and $x_j \in \operatorname{supp}(w) \setminus \operatorname{supp}(c) \subseteq \operatorname{supp}(w/c)$, for this would imply $x_i \in U$. Thus X is a vertex cover for G.

It is clear that $W_X(c) = m$. All that remains to show is that $W_X(w/c) = 0$. Note that $U = \emptyset$ implies $\operatorname{supp}(c) \cap \operatorname{supp}(w/c) = \emptyset$. Indeed, if $x_i \in \operatorname{supp}(c) \cap \operatorname{supp}(w/c)$, then $x_{i-1}, x_{i+1} \in \operatorname{supp}(c)$ both lie in U, where addition on indices is done modulo $2m-1$. Now since $\{x_0, x_2, x_4, \dots, x_{2m-2}\} \cap \operatorname{supp}(w/c) \subseteq \operatorname{supp}(c) \cap \operatorname{supp}(w/c)$, this implies that $\{x_0, x_2, x_4, \dots, x_{2m-2}\} \cap \operatorname{supp}(w/c) = \emptyset$. Thus $X \cap \operatorname{supp}(w/c) = \emptyset$, and so $W_X(w/c) = 0$. Hence

$$W_X(w) = W_X(c) + W_X\left(\frac{w}{c}\right) = m + 0 = m < m+1$$

and we have reached our desired contradiction.

Case 2: Assume that $U \neq \emptyset$ and recall that $U \subset \operatorname{supp}(c)$. Shift the labelling of the vertices in $\operatorname{supp}(c)$ if necessary so that $x_0 \in U$. Write

$$U = \{x_0 = x_{i_0}, x_{i_1}, \ldots, x_{i_t}\},$$

with $0 = i_0 < i_1 < \cdots < i_t \leq 2m-2$ and $0 \leq t \leq 2m-2$.

Claim 4.2. Define U as above. If w is not divisible by a monomial in $B_{m+1}(I)$, then one of the following two cases hold.

1. All indices $i_0, \ldots, i_t$ are even, or
2. There exists some $0 \leq r < t$ such that the indices $i_0, \ldots, i_r$ are all even and $i_{r+1}, \ldots, i_t$ are all odd.

Proof. We use an argument similar to that used to prove Claim 4.1 in the proof of Lemma 4.5. Throughout, all addition on the indices of variables is done modulo $2m-1$. Suppose neither of the above two cases hold. This means that there exists p and q with $1 \leq p < q \leq t$ such that the index i_p is odd and the index i_q is even. We will find distinct vertices x'_0, $x'_{i_p}, x'_{i_q} \in \operatorname{supp}(w/c)$ such that x_0 is adjacent to x'_0, x_{i_p} is adjacent to x'_{i_p} and x_{i_q} is adjacent to x'_{i_q}. Since x'_0, x'_{i_p}, x'_{i_q} are all distinct, the monomial $x'_0 x'_{i_p} x'_{i_q}$ divides w/c and so $c x'_0 x'_{i_p} x'_{i_q}$ divides w. However,

$$\begin{aligned} c x'_0 x'_{i_p} x'_{i_q} &= (x_0 \cdots x_{2m-2}) x'_0 x'_{i_p} x'_{i_q} \\ &= (x_0 x'_0)(x_1 x_2)(x_3 x_4) \cdots (x_{i_p-2} x_{i_p-1})(x_{i_p} x'_{i_p})(x_{i_p+1} x_{i_p+2})(x_{i_p+3} x_{i_p+4}) \cdots (x_{i_q-2} x_{i_q-1}) \\ &\quad (x_{i_q} x'_{i_q})(x_{i_q+1} x_{i_q+2})(x_{i_q+3} x_{i_q+4}) \cdots (x_{2m-3} x_{2m-2}) \end{aligned}$$

is the product of $m+1$ edge monomials. Thus $w \in I^{m+1}$, contradicting our initial assumption that $w + I^{m+1}$ is non-zero in $I^{(m+1)}/I^{m+1}$. We may then conclude that (1) or (2) holds.

We begin by choosing x'_0 to be any vertex in $\operatorname{supp}(w/c)$ that is adjacent to x_0. Such an x'_0 exists by definition of U. Next, we want to choose $x'_{i_p} \in \operatorname{supp}(w/c)$ adjacent to x_{i_p} and distinct from x'_0. To do so, we will show that x'_0 is not adjacent to x_{i_p}, and so we will be able to choose x'_{i_p} to be any vertex in $\operatorname{supp}(w/c)$ adjacent to x_{i_p}. Again, such an x'_{i_p} exists by definition of U.

Suppose first that $x'_0 \in \operatorname{supp}(c)$. This implies that $x'_0 = x_1$ or x_{2m-2}. Since i_p is odd, x_{i_p} cannot be adjacent to x_1. So suppose that $x'_0 = x_{2m-2}$ and x_{i_p} is adjacent to x_{2m-2}. This forces $i_p = 2m-3$. Then since $p < q \leq t$ implies that $i_p < i_q \leq 2m-2$, this forces $i_q = 2m-2$. Thus $x_{i_q} = x_{2m-2} = x'_0 \in U \cap \operatorname{supp}(w/c)$. When defining U, we noted that $U \cap \operatorname{supp}(w/c) = \emptyset$, and so this is a contradiction. Thus if $x'_0 \in \operatorname{supp}(c)$, then x'_0 is not adjacent to x_{i_p}.

Now, suppose that $x_0' \notin \operatorname{supp}(c)$ and x_{i_p} is adjacent to x_0'. Then there exists a copy or an induced copy of on odd cycle in G on vertices $\{x_0, x_1, \ldots, x_{i_p}, x_0'\}$. Since $1 \leq i_p \leq 2m-3$, this cycle has length at most $2m-1$. Recall that by Proposition 3.3, G does not contain any induced copies of odd cycles of length less than $2m-1$, and so this must be an induced copy of a $(2m-1)$ - cycle. This forces $i_p = 2m-3$. In turn, since $i_p < i_q \leq 2m-2$, this forces $i_q = 2m-2$. We claim now that $x_{i_q} = x_{2m-2}$ cannot be adjacent to x_0'. Indeed, since $U \cap \operatorname{supp}(w/c) = \emptyset$, x_0' is not equal to x_0 or x_{2m-2}. Then if x_0' is adjacent to x_{2m-2}, there is a triangle in G on vertices $\{x_0, x_0', x_{2m-2}\}$, contradicting the assumption that G is triangle-free. Thus there exists $x_{i_q}' = x_{2m-2}' \in \operatorname{supp}(w/c)$ adjacent to x_{2m-2} and distinct from x_0'. Because $x_0', x_{2m-2}' \in \operatorname{supp}(w/c)$ are distinct, $cx_0'x_{2m-2}'$ divides w. Then since $x_0 \cdots x_{i_p} x_0' = x_0 \cdots x_{2m-3} x_0'$ is a $(2m-1)$ - cycle monomial and $x_{2m-2}x_{2m-2}'$ is an edge monomial,

$$cx_0'x_{2m-2}' = (x_0 x_1 \cdots x_{2m-3} x_0')(x_{2m-2}x_{2m-2}') \in C_E(I).$$

Thus w is divisible by a monomial in $C_E(I)$, contradicting our assumption that w is not divisible by a monomial in $B_{m+1}(I)$. We conclude that x_{i_p} is not adjacent to x_0', and so we may choose some vertex $x_{i_p}' \in \operatorname{supp}(w/c)$ adjacent to x_{i_p} and distinct from x_0'.

Next, we want to choose $x_{i_q}' \in \operatorname{supp}(w/c)$ adjacent to x_{i_q} and distinct from both x_0' and x_{i_p}'. We will again show that x_{i_q} cannot be adjacent to x_0' or x_{i_p}'. These arguments are nearly identical to those used to show that x_{i_p} is not adjacent to x_0'.

Suppose first that $x_0' \in \operatorname{supp}(c)$. This implies that $x_0' = x_1$ or x_{2m-2}. Since i_q is even and $0 < i_q$, x_{i_q} cannot be adjacent to x_{2m-2}. So suppose $x_0' = x_1$ and x_{i_q} is adjacent to x_0'. This forces $i_q = 2$. Then since $1 \leq i_p < i_q = 2$, this forces $i_p = 1$. Thus $x_{i_p} = x_1 = x_0' \in U \cap \operatorname{supp}(w/c) = \emptyset$, a contradiction. Thus if $x_0' \in \operatorname{supp}(c)$, then x_0' is not adjacent to x_{i_q}.

Now, suppose $x_0' \notin \operatorname{supp}(c)$ and x_{i_q} is adjacent to x_0'. Then there exists a copy or an induced copy of an odd cycle in G on vertices $\{x_{i_q}, x_{i_q+1}, \ldots, x_{2m-2}, x_0, x_0'\}$. Since $2 \leq i_q \leq 2m-2$, this cycle has length at most $2m-1$. Recall that G does not contain any induced copies of odd cycles of length less than $2m-1$. Thus this must be an induced copy of a $(2m-1)$ - cycle. This forces $i_q = 2$. In turn, since $1 \leq i_p < i_q$, this forces $i_p = 1$. Recall that $x_{i_p}' = x_1' \in \operatorname{supp}(w/c)$ was chosen to be distinct from $x_0' \in \operatorname{supp}(w/c)$. It follows that $cx_0'x_1' \mid w$. Then since $x_{i_q}x_{i_q+1} \cdots x_{2m-2}x_0x_0' = x_2x_3 \cdots x_{2m-2}x_0x_0'$ is a $(2m-1)$ - cycle monomial and x_1x_1' is an edge monomial,

$$cx_0'x_1' = (x_2x_3 \cdots x_{2m-2}x_0x_0')(x_1x_1') \in C_E(I).$$

Thus w is divisible by a monomial in $C_E(I)$, contradicting our assumption that w is not divisible by a monomial in $B_{m+1}(I)$. We conclude that x_{i_q} is not adjacent to x_0'.

Finally, we show that x_{i_q} is not adjacent to x_{i_p}'. Suppose first that $x_{i_p}' \in \operatorname{supp}(c)$. This implies that $x_{i_p}' = x_{i_p-1}$ or x_{i_p+1}. Since i_p+1 and i_q are both even and $2 \leq i_p+1 \leq i_q \leq 2m-2$, it is not possible for x_{i_p+1} to be adjacent to x_{i_q}. So suppose that $x_{i_p}' = x_{i_p-1}$. Then since $i_p < i_q$, x_{i_q} is adjacent to x_{i_p-1} if and only if $i_p - 1 = 0$ and $i_q = 2m-2$. But then $x_{i_p}' = x_0 \in U \cap \operatorname{supp}(w/c) = \emptyset$, a contradiction. Thus, if $x_{i_p}' \in \operatorname{supp}(c)$, we conclude that x_{i_q} is not adjacent to x_{i_p}'.

Now, suppose that $x_{i_p}' \notin \operatorname{supp}(c)$ and x_{i_q} is adjacent to x_{i_p}'. Then there exists a copy or an induced copy of an odd cycle in G on vertices $\{x_{i_p}, x_{i_p+1}, \ldots, x_{i_q}, x_{i_p}'\}$. Since $1 \leq i_p < i_q \leq 2m-2$, this cycle has length at most $2m-1$. Recall that G does not contain any induced copies of odd cycles of length less than $2m-1$. Thus this must be an induced copy of a $(2m-1)$ - cycle. This forces $i_p = 1$ and $i_q = 2m-2$. Recall that $x_{i_p}' = x_1' \in \operatorname{supp}(w/c)$ was chosen to be distinct from $x_0' \in \operatorname{supp}(w/c)$. It follows that $cx_0'x_1' \mid w$. Then since $x_{i_p}x_{i_p+1}\cdots x_{i_q}x_{i_p}' = x_1 \cdots x_{2m-2}x_1'$ is a $(2m-1)$ - cycle monomial and x_0x_0' is an edge monomial,

$$cx_0'x_1' = (x_1 \cdots x_{2m-2}x_1')(x_0x_0') \in C_E(I).$$

Thus w is divisible by a monomial in $C_E(I)$, contradicting our assumption that w is not divisible by a monomial in $B_{m+1}(I)$. Thus we conclude that x_{i_q} is not adjacent to x_{i_p}'.

We have shown that x_{i_q} is not adjacent to x_0' or x_{i_p}'. Thus we may choose a vertex $x_{i_q}' \in \operatorname{supp}(w/c)$ adjacent to x_{i_q} and distinct from both x_0' and x_{i_p}'. We conclude that there exists distinct $x_0', x_{i_p}', x_{i_q}' \in \operatorname{supp}(w/c)$ such that x_0 is adjacent to x_0', x_{i_p} is adjacent to x_{i_p}' and x_{i_q} is adjacent to x_{i_q}'. As noted at the beginning of the proof of this claim, this implies that $w \in I^{m+1}$, a contradiction. We conclude that (1) or (2) holds, completing the proof of Claim 4.2.

□

We are assuming that w is not divisible by a monomial in $C_{2m+1}(I) \sqcup C_E(I) = B_{m+1}(I)$ and so we may assume that one of (1) or (2) from Claim 4.2 holds. Relabel the vertices if necessary so that (1) holds, i.e.

$$U \subseteq \{x_0, x_2, x_4, \ldots, x_{2m-2}\},$$

and $x_0 \in U$. Note that this is indeed possible: if all vertices in U have even indices, there is no need to relabel. Otherwise, if i_r is the least odd index so that $x_{i_r} \in U$, then for each $0 \leq i \leq 2m-2$, relabel x_i to be x_{i-i_r}, with subtraction on indices done modulo $2m-1$. We now claim that

$$X = \{x_0, x_2, x_4, \ldots, x_{2m-2}\} \cup (V(G) \setminus \mathrm{supp}(w))$$

is our desired vertex cover. First, we show that X is a vertex cover for G. Let $e = \{x_i, x_j\} \in E(G)$. If at least one of x_i or x_j is not in $\mathrm{supp}(w)$, then $e \cap X \neq \emptyset$. If both x_i and x_j lie in $\mathrm{supp}(c)$, then $e \cap \{x_0, x_2, x_4, \ldots, x_{2m-2}\} \neq \emptyset$, and so $e \cap X \neq \emptyset$. Since $\mathrm{supp}(w/c)$ is assumed to be an independent set of vertices, it is not possible for both x_i and x_j to be in $\mathrm{supp}(w) \setminus \mathrm{supp}(c) \subseteq \mathrm{supp}(w/c)$. The only remaining possibility is that $x_i \in \mathrm{supp}(c)$ and $x_j \in \mathrm{supp}(w) \setminus \mathrm{supp}(c) \subseteq \mathrm{supp}(w/c)$. Then $x_i \in U \subseteq \{x_0, x_2, x_4, \ldots, x_{2m-2}\}$, so that $e \cap X \neq \emptyset$. Thus X is a vertex cover for G.

It is clear that $W_X(c) = m$, and so all that remains to show is that $W_X(w/c) = 0$. Suppose there exists $x_j \in X \cap \mathrm{supp}(w/c)$. This implies $x_j \in \{x_0, x_2, x_4, \ldots, x_{2m-2}\}$. Then $x_{j-1}, x_{j+1} \in U$, where addition on indices is done modulo $2m-1$. Since $U \subseteq \{x_0, x_2, x_4, \ldots, x_{2m-2}\}$, this implies that all of x_{j-1}, x_j, and x_{j+1} lie in $\{x_0, x_2, x_4, \ldots, x_{2m-2}\}$, which is a clear contradiction. Thus $X \cap \mathrm{supp}(w/c) = \emptyset$, and so $W_X(w/c) = 0$. Then

$$W_X(w) = W_X(c) + W_X\left(\frac{w}{c}\right) = m < m+1$$

and we have reached our desired contradiction. We conclude that w must be divisible by a monomial in $B_{m+1}(I)$, and so $B_{m+1}(I) + I^{m+1}$ is a monomial generating set for $I^{(m+1)}/I^{m+1}$.

Finally, to see that $B_{m+1}(I)+I^{m+1}$ is the minimal monomial generating set for $I^{(m+1)}/I^{m+1}$, we need only note that all monomials in $B_{m+1}(I)$ have degree $2m+1$. Thus no monomial in $B_{m+1}(I)$ properly divides another.

□

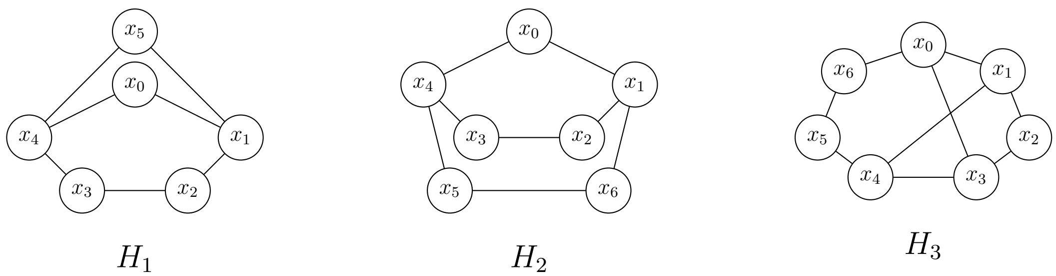
4.2.2 A formula for the second non-zero term

To describe a formula for $\mathrm{sdefect}(I, m+1)$, we need only count the cardinality of the minimal monomial generating set $B_{m+1}(I) + I^{m+1}$ for $I^{(m+1)}/I^{m+1}$ described in Lemma 4.25. Since $B_{m+1}(I) = C_{2m+1}(I) \sqcup C_E(I)$,

$$|B_{m+1}(I)| = |C_{2m+1}(I)| + |C_E(I)|.$$

As noted in Remark 2.8, there is a one-to-one correspondence between $(2m+1)$ - cycle monomials and induced copies of $(2m+1)$ - cycles in G, and so $|C_{2m+1}(I)|$ is equal to the number of induced copies of $(2m+1)$ - cycles in G. On the other hand, computing $|C_E(I)|$ is not straightforward, as monomials in $C_E(I)$ may have multiple representations as the product of a $(2m-1)$ - cycle monomial and an edge monomial.

Example 4.26. Consider the following graphs.



For each $i = 1, 2, 3$, $\text{sdefect}(I(H_i), 2) = 0$ by Proposition 3.3, and $\text{sdefect}(I(H_i), 3) = 2$ by Theorem 3.7. To compute $\text{sdefect}(I(H_i), 4)$, we need only count $|B_4(I(H_i))| = |C_7(I(H_i))| + |C_E(I(H_i))|$. However, note that

$$x_0x_1x_2x_3x_4^2x_5 = (x_0x_1x_2x_3x_4)(x_4x_5) = (x_1x_2x_3x_4x_5)(x_0x_4) \in C_E(I(H_1)),$$

$$x_0x_1x_2x_3x_4x_5x_6 = (x_0x_1x_2x_3x_4)(x_5x_6) = (x_0x_1x_6x_5x_4)(x_2x_3) \in C_E(I(H_2))$$

and

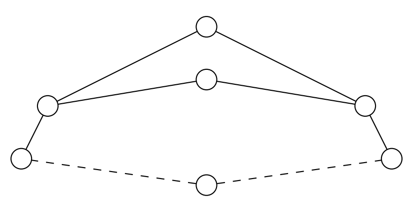
$$x_0x_1x_2x_3x_4x_5x_6 = (x_0x_3x_4x_5x_6)(x_1x_2) = (x_0x_1x_4x_5x_6)(x_2x_3) \in C_E(I(H_3))$$

all have two different representations as the product of a 5 - cycle monomial and an edge monomial.

We will soon see that the graphs given in Example 4.26 are the only types of subgraphs that may give rise to monomials in $C_E(I)$ that do not have a unique representation as the product of a $(2m-1)$ - cycle monomial and an edge monomial. We now introduce the necessary notation to work towards computing $|C_E(I)|$, which parallels the notation of Section 4.2.

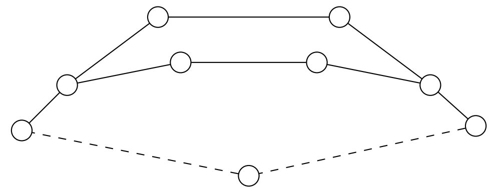
Definition 4.27. Let $m \geq 3$.

1. Let H_2^m denote the graph on $2m$ vertices that consists of two $(2m-1)$ - cycles sharing all but two edges, shown below.



H_2^m

2. Let H_3^m denote the graph on $2m+1$ vertices that consists of two $(2m-1)$ - cycles sharing all but three edges, shown below.



H_3^m

Note that in Example 4.26, H_1 is the graph H_2^3 and H_2 is the graph H_3^3.

Definition 4.28. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that $\operatorname{sdefect}(I, m) > 0$, then define

1. $C_E^{H_2^m}(I) = \{w \in C_E(I) : G[\operatorname{supp}(w)] = H_2^m\}$;
2. $C_E^{H_3^m}(I) = \{w \in C_E(I) : G[\operatorname{supp}(w)] = H_3^m\}$;
3. $C_E^{C_{2m+1}}(I) = \{w \in C_E(I) : G[\operatorname{supp}(w)]$ is a copy of a $(2m+1)$ - cycle$\}$; and
4. $C'_E(I) = C_E(I) \setminus (C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I))$.

Note that

$$C_E(I) = C_E^{H_2^m}(I) \sqcup C_E^{H_3^m}(I) \sqcup C_E^{C_{2m+1}}(I) \sqcup C'_E(I)$$

and so

$$|C_E(I)| = |C_E^{H_2^m}(I)| + |C_E^{H_3^m}(I)| + |C_E^{C_{2m+1}}(I)| + |C'_E(I)|. \tag{4.3}$$

We will compute each term on the right-hand side of Equation 4.3 through a series of lemmas. First, we introduce more necessary notation.

Definition 4.29. Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Let $m \geq 3$ be the least integer such that $\operatorname{sdefect}(I, m) > 0$.

1. Let $h_2^m(G)$ denote the number of induced copies of H_2^m in G and $h_3^m(G)$ denote the number of induced copies of H_3^m in G.
2. Let $c_{2m+1}^{i.c.}(G)$ denote the number of induced copies of $(2m+1)$ - cycles in G, and $c_{2m+1}^{c}(G)$ denote the number of copies of $(2m+1)$ - cycles in G.

3. Let $c_{2m+1}(G) = c^{i.c.}_{2m+1}(G) + c^{c}_{2m+1}(G)$, the total number of $(2m+1)$ - cycles existing as subgraphs in G.

4. Let $C_{2m-1}(I) = \{c : c \text{ is a } (2m-1) \text{ - cycle monomial}\}$.

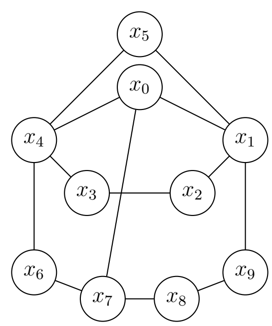
5. Let

$$\sigma(I) = \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}|.$$

6. Let

$$\sigma_{C_{2m+1}}(I) = \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_{2m+1}}(I)\}|.$$

Example 4.30. Consider the following graph G on vertex set $\{x_0, \ldots, x_9\}$ and let $I = I(G) \subset k[x_0, \ldots, x_9]$.



By Proposition 3.3, sdefect$(I, 3)$ is the first non-zero term of the symbolic defect sequence of I, so that $m = 3$ in this case. We see that $C_5(I) = \{x_0x_1x_2x_3x_4, x_1x_2x_3x_4x_5, x_0x_1x_9x_8x_7\}$. For each $c \in C_5(I)$, let

$$\begin{aligned} E_c &= \{e \in \mathcal{G}(I) : ce \in C_E^{H_2^3}(I) \cup C_E^{H_3^3}(I) \cup C_E^{C_7}(I)\} \\ &= \{e \in \mathcal{G}(I) : G[\text{supp}(ce)] = H_2^3 \text{ or } H_3^3, \text{ or is a copy of a 7 - cycle}\}. \end{aligned}$$

Consider $c = x_0x_1x_2x_3x_4$. We see that $G[\text{supp}(cx_1x_5)] = G[\text{supp}(cx_4x_5)] = H_2^3$, and for all other edge monomials e, $G[\text{supp}(ce)]$ is not equal to H_2^3, H_3^3 or a copy of a 7 - cycle. Thus $|E_{x_0x_1x_2x_3x_4}| = |\{x_1x_5, x_4x_5\}| = 2$. Similarly, $|E_{x_1x_2x_3x_4x_5}| = |\{x_0x_1, x_0x_4\}| = 2$ and $|E_{x_0x_1x_9x_8x_7}| = |\{x_4x_6\}| = 1$. Thus

$$\sigma(I) = \sum_{c \in C_5(I)} |E_c| = 5.$$

Similarly, one may compute

$$\begin{aligned}
\sigma_{C_7}(I) &= \sum_{c\in C_5(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_7}(I)\}| \\
&= \sum_{c\in C_5(I)} |\{e \in \mathcal{G}(I) : G[\text{supp}(ce)] \text{ is a copy of a 7 - cycle}| \\
&= |\{e \in \mathcal{G}(I) : G[\text{supp}(x_0x_1x_9x_8x_7)e] \text{ is a copy of a 7 - cycle}\}| \\
&= |\{x_4x_6\}| \\
&= 1.
\end{aligned}$$

Lemma 4.31. *Let G be a graph on vertex set $\{x_0,\ldots,x_n\}$ and $I = I(G) \subset k[x_0,\ldots,x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I,m) > 0$, then*

$$|C_E^{H_3^m}(I)| = h_3^m(G).$$

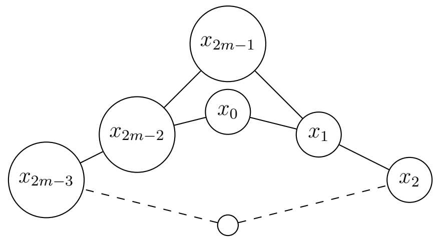
Proof. Monomials in $C_E^{H_3^m}(I)$ have degree $2m+1$ because they lie in $C_E(I)$. Monomials in $C_E^{H_3^m}(I)$ also have support size $2m+1$, and so all monomials in $C_E^{H_3^m}(I)$ are square-free. It follows by definition that $C_E^{H_3^m}(I)$ is the set of H_3^m monomials, and so $|C_E^{H_3^m}(I)| = h_3^m(G)$. □

Lemma 4.32. *Let G be a graph on vertex set $\{x_0,\ldots,x_n\}$ and $I = I(G) \subset k[x_0,\ldots,x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I,m) > 0$, then*

$$|C_E^{H_2^m}(I)| = 2h_2^m(G).$$

Proof. This proof uses a similar argument to that of Lemma 4.12. We will establish a two-to-one correspondence between monomials in $C_E^{H_2^m}(I)$ and induced copies of H_2^m in G, implying that $|C_E^{H_2^m}(I)| = 2h_2^m(G)$. Given a monomial $w \in C_E^{H_2^m}(I)$, there clearly exists a unique induced copy of H_2^m in G on the vertex set supp(w), namely $G[\text{supp}(w)]$. Next, we will show that given an induced copy of H_2^m in G on vertex set X, there exists exactly two monomials in $C_E^{H_2^m}(I)$ with support being X.

Indeed, suppose there is an induced copy of H_2^m in G on vertex set $X = \{x_0,\ldots,x_{2m-1}\}$, with vertices labelled as below.



Suppose that $w \in C_E^{H_2^m}(I)$ is such that $\mathrm{supp}(w) = X$. Since $C_E^{H_2^m}(I) \subset C_E(I)$, we may write $w = ce$, where c is a $(2m-1)$ - cycle monomial and e is an edge monomial. By Remark 2.8, there is a one-to-one correspondence between $(2m-1)$ - cycle monomials dividing w and induced copies of $(2m-1)$ - cycles in $G[\mathrm{supp}(w)] = G[X]$, and so it follows that $c = x_0x_1 \cdots x_{2m-2}$ or $x_1x_2 \cdots x_{2m-1}$.

Suppose first that $c = x_0x_1 \cdots x_{2m-2}$. Then since $\mathrm{supp}(w) = X$, the variable x_{2m-1} divides w. Since $x_{2m-1} \nmid c$, we must have $x_{2m-1} \mid e$. The only edges in $G[X]$ incident to x_{2m-1} are $\{x_{2m-2}, x_{2m-1}\}$ and $\{x_1, x_{2m-1}\}$, and so this forces $e = x_{2m-2}x_{2m-1}$ or $e = x_1x_{2m-1}$. Thus

$$w = (x_0x_1 \cdots x_{2m-2})(x_{2m-2}x_{2m-1}) = x_0x_1 \cdots x_{2m-3}x_{2m-2}^2x_{2m-1} := w_1$$

or

$$w = (x_0x_1 \cdots x_{2m-2})(x_1x_{2m-1}) = x_0x_1^2x_2 \cdots x_{2m-1} := w_2.$$

Similarly, suppose instead that $c = x_1x_2 \cdots x_{2m-1}$. Then since $\mathrm{supp}(w) = X$, the variable x_0 divides w. Since $x_0 \nmid c$, we must have $x_0 \mid e$. The only edges in $G[X]$ incident to x_0 are $\{x_0, x_1\}$ and $\{x_0, x_{2m-2}\}$, and so this forces $e = x_0x_1$ or $e = x_0x_{2m-2}$. Thus

$$w = (x_1x_2 \cdots x_{2m-1})(x_0x_1) = x_0x_1^2x_2 \cdots x_{2m-1} = w_2$$

or

$$w = (x_1x_2 \cdots x_{2m-1})(x_0x_{2m-2}) = x_0x_1 \cdots x_{2m-3}x_{2m-2}^2x_{2m-1} = w_1.$$

In either case, $w = w_1$ or w_2. We conclude that w_1 and w_2 are the only two monomials in $C_E^{H_2^m}(I)$ with support being X.

$\square$

Lemma 4.33. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then*

$$|C_E^{C_{2m+1}}(I)| = c_{2m+1}^c(G).$$

Proof. This proof uses a similar argument to that of Lemma 4.13. Recall from Definition 4.29 that $c_{2m+1}^c(G)$ is the number of copies of $(2m+1)$ - cycles in G. We will establish a one-to-one correspondence between monomials in $C_E^{C_{2m+1}}(I)$ and copies of $(2m+1)$ - cycles in G. Let $w \in C_E^{C_{2m+1}}(I)$. It is clear that there is a unique copy of a $(2m+1)$ - cycle in G on vertex set $\text{supp}(w)$, namely $G[\text{supp}(w)]$. Next, we show that given a copy of a $(2m+1)$ - cycle in G on vertex set X, there exists a unique monomial in $C_E^{C_{2m+1}}(I)$ with support being X.

Indeed, suppose there is a copy of a $(2m+1)$ - cycle in G on vertex set $X = \{x_0, \dots, x_{2m}\}$. Recall that this means that $G[X]$ is a $(2m+1)$ - cycle with at least one chord. Suppose that $\{x_i, x_j\}$ is a chord in $G[X]$, with $0 \leq i < j \leq 2m$. By Proposition 3.3, there does not exist an induced copy of an odd cycle in G with length less than $2m-1$. This forces $j = i+3$ or $i = j+3 \mod 2m+1$, for otherwise, this chord would create an odd cycle of length less than $2m-1$ in G. Assume first that $j = i+3$. It follows that $G[X \setminus \{x_{i+1}, x_{i+2}\}]$ is an induced copy of a $(2m-1)$ - cycle in G. Let $w = x_0 \cdots x_{2m}$ and note $\text{supp}(w) = X$. Then $w/(x_{i+1}x_{i+2})$ is a $(2m-1)$ - cycle monomial and $x_{i+1}x_{i+2}$ is an edge monomial. Thus

$$w = \left(\frac{w}{x_{i+1}x_{i+2}}\right)(x_{i+1}x_{i+2}) \in C_E^{C_{2m+1}}(I),$$

and so there exists a monomial in $C_E^{C_{2m+1}}(I)$ with support being X. If we are instead in the case that $i = j+3 \mod 2m+1$, then replace all instances of x_{i+1}, x_{i+2} above with x_{j+1}, x_{j+2}, where addition on the indices is done modulo $2m+1$, to arrive at the same conclusion.

To see that this monomial is unique, note that all monomials in $C_E^{C_{2m+1}}(I)$ have degree $2m+1$ and support size $2m+1$. Thus all monomials in $C_E^{C_{2m+1}}(I)$ are square-free. It easily follows that w is the unique monomial in $C_E^{C_{2m+1}}(I)$ with support being X.

□

Lemma 4.34. *Let G be a graph on vertex set $\{x_0, \dots, x_n\}$ and $I = I(G) \subset k[x_0, \dots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then*

$$|C'_E(I)| = \textit{sdefect}(I, m)|E(G)| - \sigma(I).$$

Proof. This proof uses a similar argument to that of Lemma 4.14. Notice that

$$\begin{aligned} C'_E(I) &= \{w \in C_E(I) : w \notin C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\} \\ &= \bigcup_{c \in C_{2m-1}(I)} \{ce : e \in \mathcal{G}(I), ce \notin C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}. \end{aligned}$$

If all monomials in $C'_E(I)$ have a unique representation as the product of a $(2m-1)$ - cycle monomial and an edge monomial, then the above union is disjoint. In this case,

$$\begin{aligned}
|C'_E(I)| &= \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}(I), ce \notin C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : ce \notin C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\mathcal{G}(I) \setminus \{e \in \mathcal{G}(I) : ce \in C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} \left(|\mathcal{G}(I)| - |\{e \in \mathcal{G}(I) : ce \in C_E^{H_2^m}(I) \cup C_E^{H_3^m}(I) \cup C_E^{C_{2m+1}}(I)\}| \right) \\
&= \text{sdefect}(I, m)|E(G)| - \sigma,
\end{aligned}$$

where the final equality holds because $|\mathcal{G}(I)| = |E(G)|$ and $|C_{2m-1}(I)| = \text{sdefect}(I, m)$ by Theorem 3.7. Thus it will suffice to show that all monomials in $C'_E(I)$ have a unique representation as the product of a $(2m-1)$ - cycle monomial and an edge monomial.

To this end, suppose that $w = ce, w' = c'e' \in C'_E(I)$ are equal. Write $c = x_0 \cdots x_{2m-2}$, $c' = x'_0 \cdots x'_{2m-2}$, $e = x_i x_j$ and $e' = x'_i x'_j$. We want to show that $c = c'$ and $e = e'$. We consider three cases, depending on the intersection of $\text{supp}(c)$ and $\text{supp}(e)$.

Case 1: Suppose that $\text{supp}(e) \subset \text{supp}(c)$. This happens if and only if $|\text{supp}(w)| = |\text{supp}(c)| = 2m-1$. Since $w = w'$, this means $|\text{supp}(w')| = 2m-1$, and so $\text{supp}(e') \subset \text{supp}(c')$ as well. Now, note that $\text{supp}(e) \subset \text{supp}(c)$ implies that x_i and x_j are the only variables whose squares divides w. Similarly, x'_i and x'_j are the only variables whose squares divides w'. We conclude that $e = x_i x_j = x'_i x'_j = e'$. This implies $c = c'$ as well.

Case 2: Suppose that $|\text{supp}(e) \cap \text{supp}(c)| = 1$. This happens if and only if $|\text{supp}(w)| = |\text{supp}(c)|+1 = 2m$. Since $w = w'$, this means $|\text{supp}(w')| = 2m$ and so $|\text{supp}(e') \cap \text{supp}(c')| = 1$ as well. Assume that $x_i \in \text{supp}(c)$, $x_j \notin \text{supp}(c)$, $x'_i \in \text{supp}(c')$ and $x'_j \notin \text{supp}(c')$. Note that this implies that x_i is the only variable whose square divides w and x'_i is the only variable whose square divides w'. Thus $x_i = x'_i$.

If $x_j = x'_j$ as well, then $e = e'$, implying $c = c'$, and we are done. Thus we may assume that $x_j \neq x'_j$. Then since

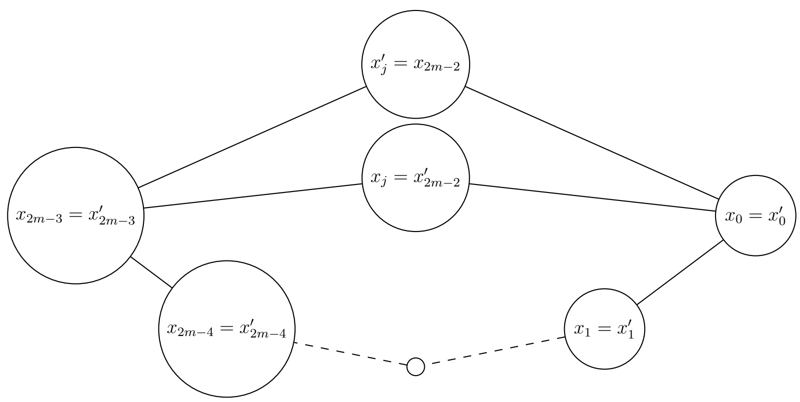
$$w = w' \implies c x_i x_j = c' x'_i x'_j \implies c x_j = c' x'_j \implies c = \frac{c' x'_j}{x_j} = \left(\frac{c'}{x_j} \right) x'_j,$$

it follows that x_j divides c' and x'_j divides c. In other words, $x_j \in \{x'_0, \dots, x'_{2m-2}\}$ and $x'_j \in \{x_0, \dots, x_{2m-2}\}$. Without loss of generality, suppose that $x'_j = x_{2m-2}$ and $x_j = x'_{2m-2}$.

Then

$$c = \frac{c' x_j'}{x_j} \implies x_0 \cdots x_{2m-2} = \frac{x_0' \cdots x_{2m-2}' x_{2m-2}}{x_{2m-2}'} \implies x_0 \cdots x_{2m-3} = x_0' \cdots x_{2m-3}'.$$

It follows that $\{x_0, \ldots, x_{2m-3}\} = \{x_0', \ldots, x_{2m-3}'\}$. Note that x_0 and x_{2m-3} are the only vertices in $\{x_0, \ldots, x_{2m-3}\}$ that have exactly one neighbour in $\{x_0, \ldots, x_{2m-3}\}$. Similarly, x_0' and x_{2m-3}' are the only vertices in $\{x_0', \ldots, x_{2m-3}'\}$ that have exactly one neighbour in $\{x_0', \ldots, x_{2m-3}'\}$. Thus $\{x_0, x_{2m-3}\} = \{x_0', x_{2m-3}'\}$, and so we may assume without loss of generality that $x_0 = x_0'$ and $x_{2m-3} = x_{2m-3}'$. It follows that $x_k = x_k'$ for all $0 \leq k \leq 2m-3$. We now claim that $G[\mathrm{supp}(w)]$ is the following graph.



Indeed, note first that since $x_i = x_i'$ is assumed to lie in $\mathrm{supp}(c) \cap \mathrm{supp}(c') = \{x_0, \ldots, x_{2m-3}\}$, all the vertices in $\mathrm{supp}(w)$ are shown in the graph above. By Proposition 3.3, there are no induced copies of odd cycles of length less than $2m-1$ in G. The addition of any edges to the graph above creates an odd cycle of length less than $2m-1$, and so the graph above is indeed $G[\mathrm{supp}(w)]$. Thus $G[\mathrm{supp}(w)] = H_2^m$, a contradiction since $w \notin C_E^{H_2^m}(I)$.

Case 3: Suppose that $\mathrm{supp}(e) \cap \mathrm{supp}(c) = \emptyset$. This happens if and only if $|\mathrm{supp}(w)| = |\mathrm{supp}(c)| + 2 = 2m+1$. Since $w = w'$, this means $|\mathrm{supp}(w')| = 2m+1$ and so $\mathrm{supp}(e') \cap \mathrm{supp}(c') = \emptyset$ as well. Note that

$$w = w' \implies c x_i x_j = c' x_i' x_j' \implies c = \frac{c' x_i' x_j'}{x_i x_j}.$$

If $x_i x_j$ divides $x_i' x_j'$, then $e = e'$ and so $c = c'$ as well. Thus we may assume that at least one of x_i or x_j does not divide $x_i' x_j'$. We consider the case that exactly one of x_i or x_j divide $x_i' x_j'$, and the case that neither x_i nor x_j divide $x_i' x_j'$ separately. We show that both scenarios lead to a contradiction.

Subcase 3a: Suppose that only x_j divides $x_i' x_j'$. Without loss of generality, assume

$x_j = x'_j$ and $x_i \neq x'_i$. Since $c = (c'x'_ix'_j)/(x_ix_j) = (c'x'_i)/x_i = (c'/x_i)x'_i$, this implies that x_i divides c' and x'_i divides c. In other words, $x_i \in \{x'_0, \ldots, x'_{2m-2}\}$ and $x'_i \in \{x_0, \ldots, x_{2m-2}\}$. Without loss of generality, suppose that $x_i = x'_{2m-2}$ and $x'_i = x_{2m-2}$. Then

$$c = \frac{c'x'_i}{x_i} \implies x_0 \cdots x_{2m-2} = \frac{x'_0 \cdots x'_{2m-2}x_{2m-2}}{x'_{2m-2}} \implies x_0 \cdots x_{2m-3} = x'_0 \cdots x'_{2m-3}.$$

By the same reasoning used in Case 2, we may assume without loss of generality that $x_k = x'_k$ for all $0 \leq k \leq 2m-3$. Then there is a copy of a $(2m+1)$ - cycle in $G[\text{supp}(w)]$ on vertices

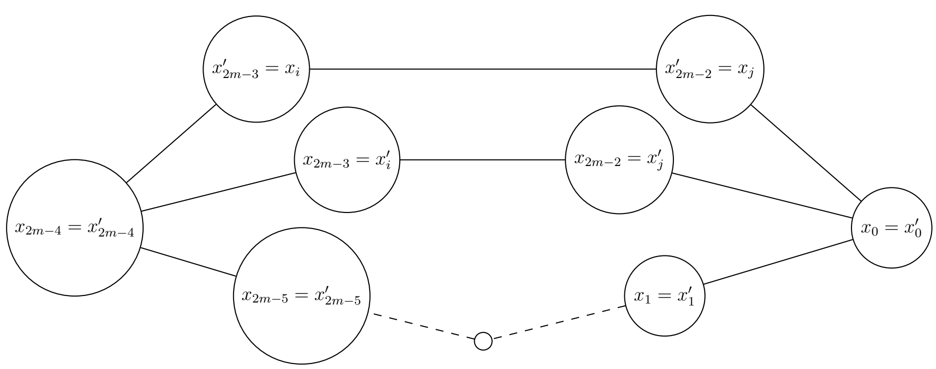
$$\{x'_0 = x_0, x_1, x_2, \ldots, x_{2m-3}, x_{2m-2} = x'_i, x'_j = x_j, x_i = x'_{2m-2}\}.$$

Since the above set of vertices is exactly $\text{supp}(w)$, it follows that $G[\text{supp}(w)]$ is equal to a copy of a $(2m+1)$ - cycle, a contradiction since $w \notin C_E^{C_{2m+1}}(I)$.

Subcase 3b: Suppose that x_i and x_j both do not divide $x'_ix'_j$. Since $c = (c'x'_ix'_j)/(x_ix_j) = (c'/(x_ix_j))x'_ix'_j$, this implies that x_i and x_j both divide c' and x'_i and x'_j both divide c. In other words, $x_i, x_j \in \{x'_0, \ldots, x'_{2m-2}\}$ and $x'_i, x'_j \in \{x_0, \ldots, x_{2m-2}\}$. Since $G[\text{supp}(c')]$ and $G[\text{supp}(c)]$ are both induced copies of $(2m-1)$ - cycles, and thus do not contain any chords, this implies that $\{x_i, x_j\}$ is an edge on the cycle $G[\text{supp}(c')]$ and $\{x'_i, x'_j\}$ is an edge on the cycle $G[\text{supp}(c)]$. We may assume without loss of generality that $x_i = x'_{2m-3}$, $x_j = x'_{2m-2}$ $x'_i = x_{2m-3}$ and $x'_j = x_{2m-2}$. Then

$$c = \frac{c'x'_ix'_j}{x_ix_j} \implies x_0 \cdots x_{2m-2} = \frac{x'_0 \cdots x'_{2m-2}x_{2m-3}x_{2m-2}}{x'_{2m-3}x'_{2m-2}} \implies x_0 \cdots x_{2m-4} = x'_0 \cdots x'_{2m-4}.$$

By the same reasoning used in Case 2, we may assume without loss of generality that $x_k = x'_k$ for all $0 \leq k \leq 2m-4$. Then $G[\text{supp}(w)]$ contains the following graph as a subgraph.



If $G[\text{supp}(w)]$ is equal to this graph, then $G[\text{supp}(w)] = H_3^m$, a contradiction since $w \notin C_E^{H_3^m}(I)$. Otherwise, the only additional edges that may exist in $G[\text{supp}(w)]$ are $\{x'_i, x_j\}$ or

$\{x_i, x'_j\}$, since any other additional edges create an odd cycle in G of length less than $2m-1$. Suppose that $\{x'_i, x_j\}$ is an edge in $G[\text{supp}(w)]$. Then there is a copy of a $(2m+1)$ - cycle in $G[\text{supp}(w)]$ on vertices

$$\{x'_i, x_j, x_i, x'_{2m-4}, x'_{2m-5}, \ldots, x'_1, x'_0, x'_j\},$$

a contradiction since $w \notin C_E^{C_{2m+1}}(I)$. Similarly, if $\{x_i, x'_j\}$ is an edge in $G[\text{supp}(w)]$, then there is a copy of a $(2m+1)$ - cycle in $G[\text{supp}(w)]$ on vertices

$$\{x'_j, x_i, x_j, x_0, x_1, \ldots, x_{2m-5}, x_{2m-4}, x'_i\},$$

again contradicting the assumption that $w \notin C_E^{C_{2m+1}}(I)$.

Thus in all possible cases, we arrive at the conclusion that $c = c'$ and $e = e'$. We conclude that if $ce = w = w' = c'e'$, then $c = c'$ and $e = e'$. Thus all monomials in $C'_E(I)$ have a unique representation as the product of a $(2m-1)$ - cycle monomial and an edge monomial, as desired.

$\square$

Our final lemma before describing a formula for $\text{sdefect}(I, m+1)$ provides a description of $\sigma(I)$ in terms of other parameters of G.

Lemma 4.35. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then*

$$\sigma(I) = 4h_2^m(G) + 2h_3^m(G) + \sigma_{C_{2m+1}}(I).$$

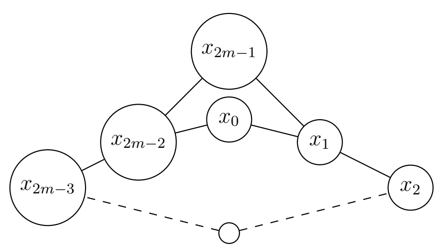
Proof. This proof uses a similar argument to that of Lemma 4.15. Throughout, we drop the I from all notation with the exception of $C_{2m-1}(I)$. Since $C_E^{H_2^m}$, $C_E^{H_3^m}$ and $C_E^{C_{2m+1}}$ are all disjoint, we see that

$$\begin{aligned}
\sigma &= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_2^m} \cup C_E^{H_3^m} \cup C_E^{C_{2m+1}}\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_2^m}\}| + \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G} : ce \in C_E^{H_3^m}\}| + \\
&\quad \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G} : ce \in C_E^{C_{2m+1}}\}| \\
&= \sigma_{C_{2m+1}} + \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}, ce \in C_E^{H_2^m}\}| + \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}, ce \in C_E^{H_3^m}\}| \\
&= \sigma_{C_{2m+1}} + \sum_{c \in C_{2m-1}(I)} |\{ce \in C_E^{H_2^m}\}| + \sum_{c \in C_{2m-1}(I)} |\{ce \in C_E^{H_3^m}\}|.
\end{aligned}$$

We first claim that

$$\sum_{c \in C_{2m-1}(I)} |\{ce \in C_E^{H_2^m}\}| = 2|C_E^{H_2^m}|.$$

It will suffice to show that each monomial in $C_E^{H_2^m}$ has exactly two representations as the product of a $(2m-1)$ - cycle monomial and an edge monomial. Let $w \in C_E^{H_2^m}$, so that $G[\text{supp}(w)] = H_2^m$, and label the vertices of $G[\text{supp}(w)]$ as below.



Since $w \in C_E$, we may write $w = ce$, where c is a $(2m-1)$ - cycle monomial and e is an edge monomial. Since there is a one-to-one correspondence between $(2m-1)$ - cycle monomials dividing w and induced copies of $(2m-1)$ - cycle monomials in $G[\text{supp}(w)]$, we must have $c = x_0 \cdots x_{2m-2}$ or $c = x_1 \cdots x_{2m-1}$. Then since $\text{supp}(w) = \{x_0, \ldots, x_{2m-1}\}$, we must have either

$$w = (x_0 \cdots x_{2m-2})(x_1 x_{2m-1}) = (x_1 \cdots x_{2m-1})(x_0 x_1)$$

or

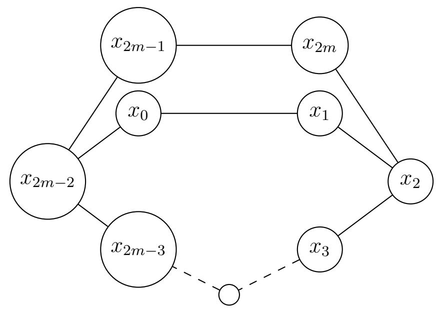
$$w = (x_0 \cdots x_{2m-2})(x_{2m-2} x_{2m-1}) = (x_1 \cdots x_{2m-1})(x_0 x_{2m-2}).$$

In either case, w has exactly two different representations as the product of a $(2m-1)$ - cycle monomial and an edge monomial.

Next, we prove in a similar fashion that

$$\sum_{c \in C_{2m-1}(I)} |\{ce \in C_E^{H_3^m}\}| = 2|C_E^{H_3^m}|.$$

It will suffice to show that each monomial in $C_E^{H_3^m}$ has exactly two representations as the product of a $(2m-1)$ - cycle monomial and an edge monomial. Let $w \in C_E^{H_3^m}$, so that $G[\text{supp}(w)] = H_3^m$, and label the vertices of $G[\text{supp}(w)]$ as below.



Since $w \in C_E$, we may write $w = ce$, where c is a $(2m-1)$ - cycle monomial and e is an edge monomial. Then we must have $c = x_0 \cdots x_{2m-2}$ or $c = x_2 x_3 \cdots x_{2m}$. Then since $\text{supp}(w) = \{x_0, \ldots, x_{2m}\}$, we must have

$$w = (x_0 \cdots x_{2m-2})(x_{2m-1}x_{2m}) = (x_2 x_3 \cdots x_{2m})(x_0 x_1).$$

Thus w has exactly two different representations as the product of a $(2m-1)$ - cycle monomial and an edge monomial. Then by Lemmas 4.31 and 4.32,

$$\sigma(I) = 2|C_E^{H_2^m}| + 2|C_E^{H_3^m}| + \sigma_{C_{2m+1}} = 4h_2^m(G) + 2h_3^m(G) + \sigma_{C_{2m+1}}.$$

$\square$

With Lemmas 4.31 - 4.35, Equation 4.3 can be written as

$$|C_E(I)| = c_{2m+1}^c(G) + \text{sdefect}(I, m)|E(G)| - 2h_2^m(G) - h_3^m(G) - \sigma_{C_{2m+1}}(I). \tag{4.4}$$

A formula for sdefect$(I, m+1)$ now easily follows.

Theorem 4.36. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then*

$$\textit{sdefect}(I, m+1) = c_{2m+1}(G) + \textit{sdefect}(I, m)|E(G)| - 2h_2^m(G) - h_3^m(G) - \sigma_{C_{2m+1}}(I).$$

Proof. Recall that by Remark 2.8, $|C_{2m+1}(I)| = c^{i.c.}_{2m+1}(G)$. By Lemma 4.25 and Equation 4.4,

$$\begin{aligned}
\text{sdefect}(I, m+1) &= |B_{m+1}(I)| \\
&= |C_{2m+1}(I)| + |C_E(I)| \\
&= c^{i.c.}_{2m+1}(G) + c^{c}_{2m+1}(G) + \text{sdefect}(I,m)|E(G)| - 2h^m_2(G) - h^m_3(G) \\
&\quad - \sigma_{C_{2m+1}}(I) \\
&= c_{2m+1}(G) + \text{sdefect}(I,m)|E(G)| - 2h^m_2(G) - h^m_3(G) - \sigma_{C_{2m+1}}(I).
\end{aligned}$$

□

4.2.3 Growth between the first two non-zero terms

In this section, we use Lemma 4.25 to bound the growth between the first two non-zero terms of the symbolic defect sequence of an edge ideal.

Corollary 4.37. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m \geq 3$ is the least integer such that sdefect$(I, m) > 0$, then*

$$(2m-1)\textit{sdefect}(I,m) \leq \textit{sdefect}(I, m+1)$$

and

$$\textit{sdefect}(I, m+1) \leq \textit{sdefect}(I,m)|E(G)| + c^{i.c.}_{2m+1}(G).$$

Proof. This proof uses a similar argument to that of Corollary 4.17. We first prove the claimed lower bound on sdefect$(I, m+1)$. Note that if c is a $(2m-1)$ - cycle monomial and e is one of the $2m-1$ edge monomials where $e \mid c$, then $G[\text{supp}(ce)] = G[\text{supp}(c)]$ is an induced copy of a $(2m-1)$ - cycle, so that $G[\text{supp}(ce)]$ cannot be equal to H^m_2, H^m_3, or a copy of a $(2m+1)$ - cycle. Thus $ce \notin C^{H^m_2}_E(I) \cup C^{H^m_3}_E(I) \cup C^{C_{2m+1}}_E(I)$, and so

$$\{ce \in C_E(I) : e \mid c\} \subseteq C'_E(I).$$

Since $C'_E(I) \subseteq C_E(I) \subseteq B_{m+1}(I)$ by definition, it follows that

$$|\{ce \in C_E(I) : e \mid c\}| \leq |C'_E(I)| \leq |C_E(I)| \leq |B_{m+1}(I)| = \text{sdefect}(I, m+1).$$

We will show that $|\{ce \in C_E(I) : e \mid c\}| = (2m-1)\text{sdefect}(I,m)$, completing the proof for the lower bound.

We first claim that monomials in $\{ce \in C_E(I) : e \mid c\}$ have a unique representation as the product of a $(2m-1)$ - cycle monomial and an edge monomial. Indeed, suppose that $ce = c'e' \in \{ce \in C_E(I) : e \mid c\}$. Write $e = x_i x_j$ and $e' = x'_i x'_j$. Then x_i and x_j are the only variables whose squares divide ce, and x'_i and x'_j are the only variables whose squares divide $c'e'$. It follows that $e = e'$, immediately implying that $c = c'$ as well. Thus

$$\begin{aligned}
|\{ce \in C_E(I) : e \mid c\}| &= \left| \bigsqcup_{c \in C_{2m-1}(I)} \{ce : e \in \mathcal{G}(I), e \mid c\} \right| \\
&= \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}(I), e \mid c\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : e \mid c\}| \\
&= \sum_{c \in C_{2m-1}(I)} (2m-1) \\
&= (2m-1)\mathrm{sdefect}(I, m),
\end{aligned}$$

where the final equality holds because $|C_{2m-1}(I)| = \mathrm{sdefect}(I, m)$ by Theorem 3.7.

We now prove the claimed upper bound on $\mathrm{sdefect}(I, m+1)$. We have

$$\mathrm{sdefect}(I, m+1) = |B_{m+1}(I)| = |C_E(I)| + |C_{2m+1}(I)| = |C_E(I)| + c^{i.c.}_{2m+1}(G),$$

and

$$\begin{aligned}
|C_E(I)| &= |\{ce : c \text{ is a } (2m-1) \text{ - cycle monomial and } e \text{ is an edge monomial}\}| \\
&= \left| \bigcup_{c \in C_{2m-1}(I)} \{ce : e \in \mathcal{G}(I)\} \right| \\
&\leq \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |E(G)| \\
&= |E(G)|\mathrm{sdefect}(I, m).
\end{aligned}$$

Thus

$$\text{sdefect}(I, m+1) \leq \text{sdefect}(I, m)|E(G)| + c^{i.c.}_{2m+1}(G).$$

$\square$

Remark 4.38. Corollaries 4.17 and 4.37 show that the second non-zero term of the symbolic defect sequence of an edge ideal I is always strictly larger that the first non-zero term. In particular, the second non-zero term is always at least three times larger than the first non-zero term. Also, although the difference between the second and first non-zero term is bounded above for any particular edge ideal, in general, this difference may be arbitrarily large.

Example 4.39. For $m \geq 3$, consider the $(2m-1)$ - cycle C_{2m-1}. By Proposition 3.3 and Theorem 3.7, $\text{sdefect}(I(C_{2m-1}), m) = 1$ is the first non-zero term in the symbolic defect sequence of $I(C_{2m-1})$. By Corollary 4.37, as $m \to \infty$,

$$\text{sdefect}(I(C_{2m-1}), m+1) - \text{sdefect}(I(C_{2m-1}), m) \to \infty.$$

4.2.4 Special cases

Using Theorem 4.36 to compute the second non-zero term in the symbolic defect sequence of an edge ideal may not always be convenient, as $\sigma_{C_{2m+1}}(I)$ is not easily calculated in general. We next illustrate families of graphs where Theorem 4.36 gives a simpler formula.

Corollary 4.40. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If sdefect$(I, m) = 1$ is the first non-zero term of the symbolic defect sequence of I and $m \geq 3$, then*

$$\textit{sdefect}(I, m+1) = |E(G)| + c^{i.c.}_{2m+1}(G).$$

Proof. This proof uses a similar argument to that of Corollary 4.20. By Theorem 4.36,

$$\text{sdefect}(I, m+1) = c_{2m+1}(G) + |E(G)| - 2h^m_2(G) - h^m_3(G) - \sigma_{C_{2m+1}}(I).$$

Notice that the graphs H^m_2 and H^m_3 both contain two induced copies of $(2m-1)$ - cycles. By Proposition 3.10, G contains a unique induced copy of a $(2m-1)$ - cycle, and so G cannot contain any induced copies of H^m_2 or H^m_3. Thus $h^m_2(G) = h^m_3(G) = 0$.

Now, since G contains a unique induced copy of a $(2m-1)$ - cycle, there exists a unique

$(2m-1)$ - cycle monomial arising from G. Write $C_{2m-1}(I) = \{c\}$. We see

$$\begin{aligned}
\sigma_{C_{2m+1}}(I) &= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_{2m+1}}(I)\}| \\
&= |\{e \in \mathcal{G}(I) : ce \in C_E^{C_{2m+1}}(I)\}| \\
&= |\{ce : e \in \mathcal{G}(I) \text{ and } ce \in C_E^{C_{2m+1}}(I)\}| \\
&= |C_E^{C_{2m+1}}(I)| \\
&= c^c_{2m+1}(G),
\end{aligned}$$

where the final equality follows from Lemma 4.33. Thus

$$\begin{aligned}
\text{sdefect}(I, m+1) &= c_{2m+1}(G) + |E(G)| - 2h_2^m(G) - h_3^m(G) - \sigma_{C_{2m+1}}(I) \\
&= c_{2m+1}(G) + |E(G)| - \sigma_{C_{2m+1}}(I) \\
&= (c^{i.c.}_{2m+1}(G) + c^c_{2m+1}(G)) + |E(G)| - c^c_{2m+1}(G) \\
&= |E(G)| + c^{i.c.}_{2m+1}(G).
\end{aligned}$$

$\square$

Corollary 4.41. *Let G be a graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Assume that sdefect$(I, m)$ is the first non-zero term in the symbolic defect sequence of I and $m \geq 3$. If all copies of $(2m+1)$ - cycles in G contain exactly one chord, then*

$$\textit{sdefect}(I, m+1) = \textit{sdefect}(I, m)|E(G)| + c^{i.c.}_{2m+1}(G) - 2h_2^m(G) - h_3^m(G).$$

Proof. This proof uses an argument similar to that of Corollary 4.21. Throughout, we drop the G from all notation. By Theorem 4.36,

$$\text{sdefect}(I, m+1) = c_{2m+1} + \text{sdefect}(I, m)|E| - 2h_2^m - h_3^m - \sigma_{C_{2m+1}}(I).$$

We see

$$\begin{aligned}
\sigma_{C_{2m+1}}(I) &= \sum_{c \in C_{2m-1}(I)} |\{e \in \mathcal{G}(I) : ce \in C_E^{C_{2m+1}}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{ce : e \in \mathcal{G}(I) \text{ and } ce \in C_E^{C_{2m+1}}(I)\}| \\
&= \sum_{c \in C_{2m-1}(I)} |\{ce \in C_E^{C_{2m+1}}(I)\}|.
\end{aligned}$$

We now claim that $\sigma_{C_{2m+1}}(I) = |C_E^{C_{2m+1}}(I)|$. Indeed, we see

$$C_E^{C_{2m+1}}(I) = \bigcup_{c \in C_{2m-1}} \{ce \in C_E^{C_{2m+1}}(I)\},$$

and so it will then suffice to show that the above union is disjoint. Suppose that $ce = c'e' \in C_E^{C_{2m+1}}(I)$, where c, c' are $(2m-1)$ - cycle monomials and e, e' are edge monomials. By assumption, $G[\text{supp}(ce)] = G[\text{supp}(c'e')]$ is a $(2m+1)$ - cycle with exactly one chord. By Proposition 3.3, G does not contain any odd cycles of length less than $2m-1$, and so this unique chord creates a unique induced copy of a $(2m-1)$ - cycle in $G[\text{supp}(ce)] = G[\text{supp}(c'e')]$. It follows that $c = c'$ must be the $(2m-1)$ - cycle monomial corresponding to this $(2m-1)$ - cycle. Thus

$$\sigma_{C_{2m+1}}(I) = |C_E^{C_{2m+1}}(I)| = c_{2m+1}^{c},$$

where the last equality holds by Lemma 4.33. Together, this gives

$$\begin{aligned}
\text{sdefect}(I, m+1) &= c_{2m+1} + \text{sdefect}(I, m)|E| - 2h_2^m - h_3^m - \sigma_{C_{2m+1}}(I) \\
&= (c_{2m+1}^{i.c.} + c_{2m+1}^{c}) + \text{sdefect}(I, m)|E| - 2h_2^m - h_3^m - c_{2m+1}^{c} \\
&= \text{sdefect}(I, m)|E| + c_{2m+1}^{i.c.} - 2h_2^m - h_3^m.
\end{aligned}$$

□

Example 4.42. As in Example 4.22, non-perfect graphs that are triangle-free fall in the class of graphs considered in Corollary 4.41,

Note that in Corollary 4.40, we achieve the upper bound on $\text{sdefect}(I, m+1)$ given in Corollary 4.37.

Example 4.43. Consider the $(2m-1)$ - cycle C_{2m-1} with $m \geq 3$, and let $I = I(C_{2m-1})$. By Proposition 3.3 and Theorem 3.7, $\text{sdefect}(I, m) = 1$ is the first non-zero term of the symbolic defect sequence of I, and by Corollary 4.40, $\text{sdefect}(I, m+1) = |E(C_{2m-1})| = 2m-1$. Thus

$$|E(C_{2m-1})|\text{sdefect}(I, m) = (2m-1)\text{sdefect}(I, m) = 2m-1 = \text{sdefect}(I, m+1),$$

so that the lower and upper bounds from Corollary 4.37 are equal and attained.

Chapter 5

The symbolic defect sequence of edge ideals of odd cycles

In this chapter, we consider the symbolic defect sequence of the edge ideal of the odd cycle C_{2m-1}, where $m \geq 2$. By Proposition 3.3, Theorem 3.7, and Corollaries 4.20 and 4.40,

$$\operatorname{sdefect}(I(C_{2m-1}), s) = \begin{cases} 0 & \text{if } s < m \\ 1 & \text{if } s = m \\ 2m-1 & \text{if } s = m+1. \end{cases}$$

We now seek a formula for sdefect$(I(C_{2m-1}), s)$ for $s > m+1$. With Lemma 5.6, we describe a monomial generating set for $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$, and in Lemma 5.10, we extract the minimal monomial generating set. We count this set to present a formula for sdefect$(I(C_{2m-1}), s)$ in Theorems 5.11 (for the case $m+1 < s < 2m$), 5.21 (for the case $s \geq 2m$ and $m = 2$) and 5.26 (for the case $s \geq 2m$ and $m \geq 3$). Throughout, unless otherwise stated, we will always assume that C_{2m-1} represents the $(2m-1)$ - cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$, and that $s > m+1$.

5.1 The minimal monomial generating set

We begin with an important preliminary lemma that was proven independently by Mandal and Pradhan [39, Lemma 4.1].

Definition 5.1. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. For each $0 \leq i \leq 2m-2$, let ϵ_i denote the edge monomial $x_i x_{i+1}$, where addition on indices is done modulo $2m-1$.

Lemma 5.2. *Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. If $a_0, \ldots, a_{2m-2}, b_0, \ldots, b_{2m-2}$ $\in \mathbb{Z}_{\geq 0}$ are such that*

$$\epsilon_0^{a_0} \cdots \epsilon_{2m-2}^{a_{2m-2}} = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}},$$

then $a_i = b_i$ for all $0 \leq i \leq 2m-2$.

Proof. We see that

$$\begin{aligned} \epsilon_0^{a_0} \cdots \epsilon_{2m-2}^{a_{2m-2}} &= \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \\ \implies (x_0x_1)^{a_0}(x_1x_2)^{a_1} \cdots (x_{2m-2}x_0)^{a_{2m-2}} &= (x_0x_1)^{b_0}(x_1x_2)^{b_1} \cdots (x_{2m-2}x_0)^{b_{2m-2}} \\ \implies x_0^{a_0+a_{2m-2}} x_1^{a_0+a_1} \cdots x_{2m-2}^{a_{2m-3}+a_{2m-2}} &= x_0^{b_0+b_{2m-2}} x_1^{b_0+b_1} \cdots x_{2m-2}^{b_{2m-3}+b_{2m-2}}. \end{aligned}$$

Equating exponents gives the system of $(2m-1)$ linear equations

$$\begin{aligned} a_0 + a_{2m-2} &= b_0 + b_{2m-2} \\ a_0 + a_1 &= b_0 + b_1 \\ &\vdots \\ a_{2m-3} + a_{2m-2} &= b_{2m-3} + b_{2m-2}, \end{aligned}$$

which has the solution $a_i = b_i$ for all $0 \leq i \leq 2m-2$. □

Remark 5.3. Lemma 5.2 greatly facilitates counting $|\mathcal{G}(I(C_{2m-1})^s)|$ for any $s \geq 0$ (by convention, $I(C_{2m-1})^0 = k[x_0, \ldots, x_{2m-2}]$). Visualize the edge monomials $\epsilon_0, \ldots, \epsilon_{2m-2}$ as distinct boxes. The monomial $\epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I(C_{2m-1})^s)$ then represents the scenario that the box ϵ_i contains b_i objects for each $0 \leq i \leq 2m-2$, where all objects are identical. Thus $|\mathcal{G}(I(C_{2m-1})^s)|$ is the number of ways to distribute s identical objects into $2m-1$ distinct boxes, where boxes may receive multiple objects. It follows that

$$|\mathcal{G}(I(C_{2m-1})^s)| = \left(\!\!\binom{2m-1}{s}\!\!\right),$$

where $(\!(\;)\!)$ represents multichoose.

This visualization of edge monomials as distinct boxes and exponents as identical objects in Remark 5.3 will prove very useful throughout the process of obtaining a formula for sdefect$(I(C_{2m-1}), s)$. The conclusion of Remark 5.3 is also noted by Mandal and Pradan [39, Remark 4.3].

Definition 5.4. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Let c denote the $(2m-1)$ - cycle monomial $x_0 \cdots x_{2m-2}$.

Definition 5.5. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Given $s > m+1$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Define

$$A_s(I) = \{c^d u : 1 \leq d \leq q, u \in \mathcal{G}(I^{s-dm})\}.$$

Lemma 5.6. *Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. If $s > m+1$, then $A_s(I) + I^s$ is a generating set for $I^{(s)}/I^s$.*

Proof. Write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. By Proposition 2.34,

$$I^{(s)} = \sum_{d=0}^{q} (c)^d I^{s-dm} = \left(\sum_{d=1}^{q} (c)^d I^{s-dm} \right) + I^s.$$

Then $A_s(I)$ is a generating set for $\sum_{d=1}^{q} (c)^d I^{s-dm}$, so that $A_s(I) + I^s$ is a generating set for $I^{(s)}/I^s$. □

Example 5.7. Let $I = I(C_5) \subset k[x_0, x_1, x_2, x_3, x_4]$, so that $m = 3$ in this case, and $s = 7 = 2(3) + 1$. By definition,

$$A_7(I) = \{(x_0x_1x_2x_3x_4)^d u : 1 \leq d \leq 2, u \in \mathcal{G}(I^{7-3d})\}.$$

Then

$$w_1 := (x_0x_1x_2x_3x_4)\epsilon_0\epsilon_1\epsilon_2\epsilon_4 = x_0^3x_1^3x_2^3x_3^2x_4^2 \in A_7(I)$$

and

$$w_2 := (x_0x_1x_2x_3x_4)^2\epsilon_0 = x_0^3x_1^3x_2^2x_3^2x_4^2 \in A_7(I).$$

The monomial w_2 strictly divides w_1, and so $A_7(I) + I^7$ is not the minimal monomial generating set for $I^{(7)}/I^7$.

We introduce more notation to extract the minimal monomial generating set for $I^{(s)}/I^s$ from $A_s(I) + I^s$.

Definition 5.8. Let $m \geq 2$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $A_s = A_s(I)$. Given $s > m+1$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Define

$$B_s(I) = \left\{ c^d u \in A_s : \text{either } c \nmid u \text{ or } \forall\ 0 < \delta \leq q - d \text{ such that } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\}.$$

Example 5.9. Let $I = I(C_5) \subset k[x_0, x_1, x_2, x_3, x_4]$, so that $m = 3$ in this case, and $s = 7 =$

$2(3)+1$. Recall that we showed in Example 5.7 that

$$w_1 = (x_0x_1x_2x_3x_4)\epsilon_0\epsilon_1\epsilon_2\epsilon_4 = x_0^3x_1^3x_2^3x_3^2x_4^2 \in A_7(I)$$

is not a minimal generator of $I^{(7)}/I^7$ because

$$w_2 = (x_0x_1x_2x_4x_4)^2\epsilon_0 = x_0^3x_1^3x_2^2x_3^2x_4^2 \in A_7(I)$$

strictly divides w_1. We see $w_2 \in B_7(I)$ because $c \nmid \epsilon_0$. On the other hand, by taking $\delta = 1$, we see $w_1 \notin B_7(I)$ because c divides $\epsilon_0\epsilon_1\epsilon_2\epsilon_4 = x_0^2x_1^2x_2^2x_3x_4$ and

$$\frac{\epsilon_0\epsilon_1\epsilon_2\epsilon_4}{c} = \frac{x_0^2x_1^2x_2^2x_3x_4}{x_0x_1x_2x_3x_4} = x_0x_1x_2 = \epsilon_0x_2 \in I = I^{7-3(1+1)}.$$

Proposition 5.10. *Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. If $s > m+1$, then $B_s(I) + I^s$ is the minimal monomial generating set for $I^{(s)}/I^s$.*

Proof. Let $B_s := B_s(I)$, $A_s := A_s(I)$, and write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. We see first that $c\epsilon_0^{s-m} \in B_s$ since $c = x_0 \cdots x_{2m-2}$ does not divide ϵ_0^{s-m}, and so $B_s \neq \emptyset$. Now, since $B_s + I^s \subseteq A_s + I^s$, by Lemma 5.6 we need only show that

1. no monomial in $(A_s \setminus B_s) + I^s$ is a minimal generator of $I^{(s)}/I^s$, and

2. all monomials in $B_s + I^s$ are minimal generators of $I^{(s)}/I^s$.

We first show (1). This is equivalent to showing that all monomials in $A_s \setminus B_s$ are properly divisible by another monomial in A_s. Let $w = c^d u \in A_s \setminus B_s$. Since $w \notin B_s$, there exists $0 < \delta \leq q - d$ such that $c^\delta \mid u$ and $u/(c^\delta) \in I^{s-m(d+\delta)}$. Note here that $0 < d + \delta \leq q$. Let $v \in \mathcal{G}(I^{s-m(d+\delta)})$ be such that v divides $u/(c^\delta)$. This implies that $c^\delta v$ divides u, so that $c^d(c^\delta v) = c^{d+\delta}v \in A_s$ divides $c^d u = w$. Further,

$$\begin{aligned}
\deg(w) &= \deg(c^d u) \\
&= d(2m-1) + 2(s - dm) \\
&= 2s - d \\
&> 2s - (d+\delta) \\
&= (d+\delta)(2m-1) + 2(s - m(d+\delta)) \\
&= \deg(c^{d+\delta}v),
\end{aligned}$$

so that $c^{d+\delta}v$ properly divides w.

We now show (2). This is equivalent to showing that all monomials in B_s are not properly divisible by any other monomial in A_s. Let $w = c^d u \in B_s$ and suppose that w is properly divisible by some $w' = c^{d'} u' \in A_s$. Then

$$2s - d' = \deg(w') < \deg(w) = 2s - d,$$

so that $d < d'$. Let $\delta = d' - d$, so that $0 < \delta \leq q - d$ and $d + \delta = d'$. By definition of A_s, $u' \in \mathcal{G}(I^{s-d'm}) = \mathcal{G}(I^{s-m(d+\delta)})$. Further, since $c^{d'}u'$ divides w, it follows that u' divides $w/(c^{d'}) = w/(c^{d+\delta})$. Thus $w/(c^{d+\delta}) \in I^{s-m(d+\delta)}$. But

$$\frac{w}{c^{d+\delta}} = \frac{c^d u}{c^{d+\delta}} = \frac{u}{c^\delta},$$

so that $u/(c^\delta) \in I^{s-m(d+\delta)}$. This is a clear contradiction since $w \in B_s$. Thus w is not divisible by any other monomial in A_s, and we conclude that $B_s + I^s$ is the minimal monomial generating set for $I^{(s)}/I^s$. □

5.2 A formula for the symbolic defect sequence

Using the minimal monomial generating set from Proposition 5.10, we can easily describe a simple formula for sdefect$(I(C_{2m-1}), s)$ when $m + 1 < s < 2m$. The following theorem aligns with that found in [34, Theorem 5.13], although our proof is entirely different, and is also a special case of [39, Theorem 4.8].

Theorem 5.11. *Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. If $m + 1 < s < 2m$, then*

$$\textit{sdefect}(I, s) = \left(\!\!\binom{2m-1}{s-m}\!\!\right).$$

Proof. Throughout, "s.t." denotes the term "such that". Write $s = qm + r$, where $q \in \mathbb{Z}^+$ and $0 \leq r < m$ and, let $A_s := A_s(I)$. Since $m + 1 < s < 2m$, we have $q = 1$ and $r = s - m$.

Then

$$\begin{aligned}
B_s(I) &= \left\{ c^d u \in A_s : \text{either } c \nmid u \text{ or } \forall\ 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\} \\
&= \left\{ c^d u : 1 \leq d \leq q, u \in \mathcal{G}(I^{s-dm}) \text{ and either } c \nmid u \text{ or } \forall\ 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \right. \\
&\qquad \left. \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\} \\
&= \{ cu : u \in \mathcal{G}(I^{s-m}) \text{ and } c \nmid u \},
\end{aligned}$$

where the last equality follows because $\{d : 1 \leq d \leq q\} = \{1\}$, and there does not exist δ satisfying $0 < \delta \leq q - d = 1 - 1 = 0$. Then by Proposition 5.10,

$$\operatorname{sdefect}(I, s) = |B_s(I)| = \left| \{ cu : u \in \mathcal{G}(I^{s-m}) \text{ and } c \nmid u \} \right|.$$

Since $s < 2m$, it follows that $s - m \leq m - 1$. Then for any $u \in \mathcal{G}(I^{s-m})$,

$$\deg(u) = 2(s-m) \leq 2(m-1) < 2m - 1 = \deg(c),$$

and so $c \nmid u$ for any $u \in \mathcal{G}(I^{s-m})$. It follows that

$$\operatorname{sdefect}(I, s) = |\{ cu : u \in \mathcal{G}(I^{s-m}) \}| = |\mathcal{G}(I^{s-m})| = \left(\!\!\binom{2m-1}{s-m}\!\!\right),$$

where the final equality follows from Remark 5.3. $\square$

Obtaining a formula for $\operatorname{sdefect}(I(C_{2m-1}), s)$ when $s \geq 2m$ is a lengthy task, and will be done through a series of lemmas.

Definition 5.12. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Given $s \geq 2m$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. For $1 \leq d \leq q - 1$, define

$$U_{s,d}(I) = \left\{ u \in \mathcal{G}(I^{s-dm}) : \exists\ 0 < \delta \leq q - d \text{ such that } c^\delta \mid u \text{ and } \frac{u}{c^\delta} \in I^{s-m(d+\delta)} \right\}.$$

Lemma 5.13. *Let $m \geq 2$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. If $s = qm + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then*

$$|B_s(I)| = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1} \left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |U_{s,d}(I)| \right).$$

Proof. Let $B := B_s(I)$ and $U_d := U_{s,d}(I)$. Throughout, "s.t." denotes the term "such that". By definition,

$$\begin{aligned}
B &= \left\{ c^d u \in A_s(I) : \text{either } c \nmid u, \text{ or } \forall\, 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\} \\
&= \Big\{ c^d u : 1 \leq d \leq q, u \in \mathcal{G}(I^{s-dm}) \text{ and either } c \nmid u, \text{ or } \forall\, 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \\
&\qquad \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \Big\} \\
&= \bigcup_{d=1}^{q} \left\{ c^d u : u \in \mathcal{G}(I^{s-dm}) \text{ and either } c \nmid u, \text{ or } \forall\, 0 < \delta \leq q-d \text{ s.t } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\}.
\end{aligned}$$

We claim that the above union is disjoint. Indeed, suppose that $c^d u, c^{d'} u' \in B$ are equal. Then

$$\begin{aligned}
\deg(c^d u) = \deg(c^{d'} u') &\implies d(2m-1) + 2(s-dm) = d'(2m-1) + 2(s-d'm) \\
&\implies 2dm - d + 2s - 2dm = 2d'm - d' + 2s - 2d'm \\
&\implies d = d'.
\end{aligned}$$

Thus

$$|B| = \sum_{d=1}^{q} \left| \left\{ u \in \mathcal{G}(I^{s-dm}) : \text{either } c \nmid u, \text{ or } \forall\, 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\} \right|.$$

For $1 \leq d \leq q-1$, note that

$$\mathcal{G}(I^{s-dm}) \setminus U_d = \left\{ u \in \mathcal{G}(I^{s-dm}) : \text{ either } c \nmid u, \text{ or } \forall\, 0 < \delta \leq q-d \text{ s.t. } c^\delta \mid u, \frac{u}{c^\delta} \notin I^{s-m(d+\delta)} \right\}$$

and

$$|\mathcal{G}(I^{s-dm}) \setminus U_d| = |\mathcal{G}(I^{s-dm})| - |U_d| = \left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |U_d|,$$

where the final equality follows from Remark 5.3.

For $d = q$, there does not exist δ satisfying $0 < \delta \leq q - d = q - q = 0$. Thus the q-th term in this sum is equal to

$$|\left\{ u \in \mathcal{G}(I^{s-qm}) : c \nmid u \right\}|.$$

Note that $s - qm = r \leq m - 1$. Thus for all $u \in \mathcal{G}(I^{s-qm})$,

$$\deg(u) = 2(s - qm) \leq 2(m-1) < 2m - 1 = \deg(c),$$

implying $c \nmid u$. Thus

$$\left\{u \in \mathcal{G}(I^{s-qm}) : c \nmid u\right\} = \mathcal{G}(I^{s-qm}) = \mathcal{G}(I^r).$$

Then when $d = q$, by Remark 5.3, $|\mathcal{G}(I^{s-qm})| = |\mathcal{G}(I^r)| = \left(\!\binom{2m-1}{r}\!\right)$, and so

$$|B| = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1}\left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |U_d|\right).$$

$\square$

By Proposition 5.10 and Lemma 5.13, if $m \geq 2$, $s \geq 2m$ and $s = qm + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then

$$\operatorname{sdefect}(I(C_{2m-1}), s) = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1}\left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |U_{s,d}(I)|\right). \tag{5.1}$$

Our next goal is to compute $|U_{s,d}(I)|$ for $1 \leq d \leq q-1$. To do so, we introduce a new definition that will simplify our description of $U_{s,d}(I)$.

Definition 5.14. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Given $s \geq 2m$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. For $1 \leq d \leq q-1$, define

$$\tilde{U}_{s,d}(I) = \left\{u \in \mathcal{G}(I^{s-dm}) : c \mid u \text{ and } \frac{u}{c} \in I^{s-m(d+1)}\right\}.$$

A lemma of Mandal and Pradhan gives rise to the containment $U_{s,1}(I) \subseteq \tilde{U}_{s,1}(I)$ [39, Lemma 4.5]. The following lemma gives equality of these sets for all values of d and was proven independently.

Lemma 5.15. *Let $m \geq 2$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. If $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then for all $1 \leq d \leq q-1$,*

$$U_{s,d}(I) = \tilde{U}_{s,d}(I).$$

Proof. Fix $1 \leq d \leq q-1$ and let $U_d := U_{s,d}(I)$, $\tilde{U}_d := \tilde{U}_{s,d}(I)$. It is clear by definition that $\tilde{U}_d \subseteq U_d$ (by taking $\delta = 1$), and so we need only show that $U_d \subseteq \tilde{U}_d$. Let $u \in U_d$, so that

$u \in \mathcal{G}(I^{s-dm})$ and there exists $0 < \delta \le q - d$ such that $c^\delta = (x_0 \cdots x_{2m-2})^\delta$ divides u and $u/(c^\delta) \in I^{s-m(d+\delta)}$. If $\delta = 1$, then $u \in \tilde{U}_d$ and we are done. Thus we may assume that $\delta > 1$. The assumption that $c^\delta \mid u$ immediately implies that $c \mid u$, so we need only show that $u/c \in I^{s-m(d+1)}$. To do so, we show that $u/(c^\delta) \in I^{s-m(d+\delta)}$ implies that $u/(c^{\delta-1}) \in I^{s-m(d+\delta-1)}$. We may then repeat this argument $\delta - 1$ times to conclude that $u/c \in I^{s-m(d+1)}$.

Write

$$\frac{u}{c^\delta} = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} x_0^{a_0} \cdots x_{2m-2}^{a_{2m-2}},$$

where each $b_0, \ldots, b_{2m-2}, a_0, \ldots, a_{2m-2} \in \mathbb{Z}_{\ge 0}$ and $b_0 + \cdots + b_{2m-2} = s - m(d+\delta)$. Recall here that ϵ_i is the edge monomial $x_i x_{i+1}$, where addition on indices is done modulo $2m-1$. Note that

$$\begin{aligned}
\deg\left(\frac{u}{c^\delta}\right) &= \deg(u) - \deg(c^\delta) \\
&= 2(s - dm) - \delta(2m-1) \\
&= 2(s - m(d+\delta)) + \delta \\
&> 2(s - m(d+\delta)),
\end{aligned}$$

so that $\dfrac{u}{c^\delta} \neq \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}}$. Thus there exists some $0 \le i \le 2m-2$ such that $a_i > 0$. Then

$$\begin{aligned}
\frac{u}{c^{\delta-1}} &= \left(\frac{u}{c^\delta}\right) c \\
&= \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} x_0^{a_0} \cdots x_{2m-2}^{a_{2m-2}} c \\
&= \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \left(\frac{x_0^{a_0} \cdots x_{2m-2}^{a_{2m-2}}}{x_i}\right) c x_i.
\end{aligned}$$

Note that $cx_i \in I^m$ since

$$cx_i = \begin{cases} \epsilon_0 \epsilon_2 \epsilon_4 \cdots \epsilon_{2m-2} & \text{if } i = 0 \\ \epsilon_0 \epsilon_2 \epsilon_4 \cdots \epsilon_{i-1} \epsilon_i \epsilon_{i+2} \epsilon_{i+4} \cdots \epsilon_{2m-3} & \text{if } i \text{ is odd} \\ \epsilon_1 \epsilon_3 \epsilon_5 \cdots \epsilon_{i-1} \epsilon_i \epsilon_{i+2} \epsilon_{i+4} \cdots \epsilon_{2m-2} & \text{if } i > 0 \text{ is even.} \end{cases}$$

Thus $\epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} c x_i \in I^{s-m(d+\delta)+m} = I^{s-m(d+\delta-1)}$ and divides $u/(c^{\delta-1})$. Thus $u/(c^{\delta-1}) \in I^{s-m(d+\delta-1)}$, as desired. $\square$

By Lemma 5.15, Equation 5.1 can be written as

$$\operatorname{sdefect}(I(C_{2m-1}), s) = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1}\left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |\tilde{U}_{s,d}(I)|\right). \tag{5.2}$$

Thus our next goal is to compute $|\tilde{U}_{s,d}(I)|$. To do so, we introduce one more definition that provides an alternate description of $\tilde{U}_{s,d}(I)$. Although this new description appears to be more complex, it greatly facilitates computing $|\tilde{U}_{s,d}(I)|$.

Definition 5.16. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Given $s \geq 2m$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. For $1 \leq d \leq q-1$ and $0 \leq i \leq 2m-2$, define

$E^i_{s,d}(I) = \{\epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm}) : b_j > 0$ for all $j < i$ with different parity than i, and for all $j \geq i$ with the same parity as i$\}$.

Example 5.17. Consider the 7 - cycle C_7 on vertices $\{x_0, \ldots, x_6\}$, so that $m = 4$ in this case, and let $I = I(C_7) \subset k[x_0, \ldots, x_6]$. Set $s = 9 = 2(4) + 1$. Then

$$\begin{aligned}
E^0_{9,1}(I) &= \{\epsilon_0^{b_0} \cdots \epsilon_6^{b_6} \in \mathcal{G}(I^{9-1(4)}) : b_0, b_2, b_4, b_6 > 0\},\\
E^1_{9,1}(I) &= \{\epsilon_0^{b_0} \cdots \epsilon_6^{b_6} \in \mathcal{G}(I^{9-1(4)}) : b_0, b_1, b_3, b_5 > 0\},\\
E^2_{9,1}(I) &= \{\epsilon_0^{b_0} \cdots \epsilon_6^{b_6} \in \mathcal{G}(I^{9-1(4)}) : b_1, b_2, b_4, b_6 > 0\},
\end{aligned}$$

and

$$E^3_{9,1}(I) = \{\epsilon_0^{b_0} \cdots \epsilon_6^{b_6} \in \mathcal{G}(I^{9-1(4)}) : b_0, b_2, b_3, b_5 > 0\}.$$

Remark 5.18. Let $m \geq 2$ and $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$. Given $s \geq 2m$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Fix some $1 \leq d \leq q-1$. By Lemma 5.2, a monomial $u \in \mathcal{G}(I^{s-dm})$ is in $E^i_{s,d}(I)$ if and only if u is divisible by one of

- $\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2} = cx_0$ if $i = 0$,
- $\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{i-1}\epsilon_i\epsilon_{i+2}\epsilon_{i+4} \cdots \epsilon_{2m-3} = cx_i$ if i is odd, or
- $\epsilon_1\epsilon_3\epsilon_5 \cdots \epsilon_{i-1}\epsilon_i\epsilon_{i+2}\epsilon_{i+4} \cdots \epsilon_{2m-2} = cx_i$ if $i > 0$ is even,

and $u/(cx_i) \in I^{s-m(d+1)}$.

A lemma of Mandal and Pradhan gives rise to the equality $\tilde{U}_{s,1}(I) = \cup_{i=0}^{2m-2} E^i_{s,1}(I)$, where $I = I(C_{2m-1})$ [39, Lemma 4.4]. The following lemma gives this equality for all values of d and was proven independently.

Lemma 5.19. *Let $m \geq 2$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. If $s = qm + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then for each $1 \leq d \leq q-1$,*

$$\tilde{U}_{s,d}(I) = \bigcup_{i=0}^{2m-2} E^i_{s,d}(I).$$

Proof. Fix $1 \leq d \leq q-1$ and let $\tilde{U} := \tilde{U}_{s,d}(I)$ and $E^i := E^i_{s,d}(I)$. We first show that $\tilde{U} \subseteq \cup_{i=0}^{2m-2} E^i$. Let $u \in \tilde{U}$. By definition of $\tilde{U}$, $u \in \mathcal{G}(I^{s-dm})$, $c = x_0 \cdots x_{2m-2}$ divides u and $u/c \in I^{s-m(d+1)}$. We see

$$\begin{aligned}
\deg\left(\frac{u}{c}\right) &= \deg(u) - \deg(c) \\
&= 2(s-dm) - (2m-1) \\
&= 2(s - m(d+1)) + 1.
\end{aligned}$$

Thus we may write

$$\frac{u}{c} = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} x_j$$

for some $0 \leq j \leq 2m-2$, where $b_0 + \cdots + b_{2m-2} = s - m(d+1)$. It follows that

$$u = (\epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}})(cx_j).$$

By Remark 5.18, $u \in E^j$ and so $u \in \bigcup_{i=0}^{2m-2} E^i$. Thus $\tilde{U} \subseteq \bigcup_{i=0}^{2m-2} E^i$.

Now, let $u \in \cup_{i=0}^{2m-2} E^i$. Then there exists $0 \leq j \leq 2m-2$ such that $u \in E^j$. By Remark 5.18, u is divisible by cx_j and $u/(cx_j) \in I^{s-m(d+1)}$. Since c divides cx_j, this means that u is divisible by c. Further, since u/c is divisible by $u/(cx_j)$, it follows that $u/c \in I^{s-m(d+1)}$. Thus $u \in \tilde{U}$ and so $\cup_{i=0}^{2m-2} E^i \subseteq \tilde{U}$.

$\square$

Note that

$$\bigcup_{i=0}^{2m-2} E^i_{s,d}(I) = E^0_{s,d}(I) \sqcup \left(\bigsqcup_{i=1}^{2m-2} \left(E^i_{s,d}(I) \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}(I) \right) \right) \right),$$

and so

$$|\tilde{U}_{s,d}(I)| = \left| \bigcup_{i=0}^{2m-2} E^i_{s,d}(I) \right| = |E^0_{s,d}(I)| + \sum_{i=1}^{2m-2} \left| E^i_{s,d}(I) \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}(I) \right) \right|.$$

Thus if $I = I(C_{2m-1})$ and $E^i_{s,d} = E^i_{s,d}(I)$, then Equation 5.2 can be written as

$$\operatorname{sdefect}(I, s) = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1}\left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |E^0_{s,d}| - \sum_{i=1}^{2m-2}\left|E^i_{s,d} \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}\right)\right|\right). \tag{5.3}$$

Our final goal before describing a formula for $\operatorname{sdefect}(I(C_{2m-1}), s)$ is to calculate $|E^0_{s,d}(I)|$ and $|E^i_{s,d}(I) \setminus (\cup_{j=0}^{i-1} E^j_{s,d}(I))|$ for each $1 \le d \le q-1$ and $1 \le i \le 2m-2$. With the tools of Remark 5.3, this is a basic counting problem.

Lemma 5.20. *Let $m \ge 2$, $I = I(C_{2m-1}) \subset k[x_0, \dots, x_{2m-2}]$ and $s \ge 2m$. If $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \le r < m$, then for all $1 \le d \le q-1$ we have*

1. $|E^0_{s,d}(I)| = \left(\!\binom{2m-1}{s-m(d+1)}\!\right)$;
2. $|E^1_{s,d}(I) \setminus E^0_{s,d}(I)| = \left(\!\binom{2m-1}{s-m(d+1)}\!\right) - \left(\!\binom{2m-1}{s-m(d+2)+1}\!\right)$; *and*
3. $|E^2_{s,d}(I) \setminus (E^0_{s,d}(I) \cup E^1_{s,d}(I))| = \left(\!\binom{2m-2}{s-m(d+1)}\!\right)$.

Proof. Fix $1 \le d \le q-1$ and let $E^i := E^i_{s,d}(I)$ for $i = 0, 1, 2$. Throughout, we use the same visualization as in Remark 5.3 of edge monomials as boxes and exponents as identical objects.

1. By definition, $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm})$ lies in E^0 if and only if $b_0, b_2, b_4, \dots, b_{2m-2}$ are all positive. Thus counting $|E^0|$ is equivalent to counting the number of ways to distribute $s - dm$ identical objects into the $2m-1$ boxes $\epsilon_0, \dots, \epsilon_{2m-2}$, where boxes may receive multiple objects, and the boxes corresponding to $\epsilon_0, \epsilon_2, \epsilon_4, \dots, \epsilon_{2m-2}$ are all non-empty. To do so, we first distribute an object into each of the m boxes we require to be non-empty, meaning there are $(s-dm) - m = s - m(d+1)$ objects left to distribute among $2m-1$ boxes. Thus
$$|E^0| = \left(\!\!\binom{2m-1}{s-m(d+1)}\!\!\right).$$
2. By definition, $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm})$ lies in $E^1 \setminus E^0$ if and only if
   (a) $b_0, b_1, b_3, b_5, \dots, b_{2m-3}$ are all positive (since $u \in E^1$); and
   (b) there exists an even index j such that $b_j = 0$ (since $u \notin E^0$).

Together, this implies that $u \in E^1 \setminus E^0$ if and only if the following two conditions are both satisfied:

Condition 1: $b_0, b_1, b_3, b_5, \ldots, b_{2m-3}$ are all positive;
Condition 2: at least one of $b_2, b_4, b_6, \ldots, b_{2m-2}$ is zero.

We first count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying Condition 1. This is equivalent to counting the number of ways to distribute $s-dm$ identical objects among the $2m-1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where the boxes may receive multiple objects and the boxes $\epsilon_0, \epsilon_1, \epsilon_3, \epsilon_5, \ldots, \epsilon_{2m-3}$ are all non-empty. To do so, we first distribute an object into each of the m boxes we require to be non-empty, meaning there are $s-dm-m = s-m(d+1)$ objects left to distribute among $2m-1$ boxes. Thus there are $\left(\!\binom{2m-1}{s-m(d+1)}\!\right)$ monomials in $\mathcal{G}(I^{s-dm})$ satisfying Condition 1.

Next, we count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying Condition 1 but not satisfying Condition 2. This is equivalent to counting the number of ways to distribute $s-dm$ identical objects among the $2m-1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where the boxes may receive multiple objects and the boxes $\epsilon_0, \epsilon_1, \epsilon_3, \epsilon_5, \ldots, \epsilon_{2m-3}$ and $\epsilon_2, \epsilon_4, \epsilon_6, \ldots, \epsilon_{2m-2}$ are all non-empty. To do so, we first distribute an object into each of these $m+(m-1) = 2m-1$ boxes, meaning there are $s-dm-(2m-1) = s-m(d+2)+1$ objects left to distribute among $2m-1$ boxes. Thus there are $\left(\!\binom{2m-1}{s-m(d+2)+1}\!\right)$ monomials in $\mathcal{G}(I^{s-dm})$ satisfying Condition 1 and not satisfying Condition 2.

Notice that if $d = q-1$, it is possible for $s-m(d+2)+1$ to be strictly negative. In this case, there does not exist a monomial in $\mathcal{G}(I^{s-dm})$ satisfying Condition 1 and not satisfying Condition 2. This formula still holds in this case, since if $s-m(d+2)+1 < 0$ then $\left(\!\binom{2m-1}{s-m(d+2)+1}\!\right) = 0$ by convention.

Together, this means there are

$$\left(\!\binom{2m-1}{s-m(d+1)}\!\right) - \left(\!\binom{2m-1}{s-m(d+2)+1}\!\right)$$

monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2.

3. By definition, $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm})$ is in $E^2 \setminus (E^0 \cup E^1)$ if and only if

   (a) $b_1, b_2, b_4, b_6, \ldots, b_{2m-2}$ are all positive (since $u \in E^2$);

   (b) at least one of $b_0, b_1, b_3, b_5, \ldots, b_{2m-3}$ is zero (since $u \notin E^1$); and

   (c) there exists an even index j such that $b_j = 0$ (since $u \notin E^0$).

The first and third item together imply that $b_0 = 0$, which immediately implies that the second item is satisfied as well. Thus $u \in E^2 \setminus (E^0 \cup E^1)$ if and only if $b_1, b_2, b_4, b_6, \ldots, b_{2m-2}$ are all positive and $b_0 = 0$. Therefore counting $|E^2 \setminus (E^0 \cup E^1)|$ is equivalent to counting the number of ways to distribute $s - dm$ identical objects among the $2m-1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where boxes may receive multiple objects, the boxes corresponding to $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_6, \ldots, \epsilon_{2m-2}$ are all non-empty, and the box ϵ_0 is empty. To do so, we first distribute an object into each of the m boxes we require to be non-empty, meaning there are $s - dm - m = s - m(d+1)$ objects left to distribute. Since ϵ_0 must be empty, there are only $2m-2$ boxes available to distribute the remaining objects. Thus

$$|E^2 \setminus (E^0 \cup E^1)| = \left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right).$$

□

Theorem 5.21. *Given $s \geq 2$, write $s = 2q + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < 2$. If $I = I(C_3) \subset k[x_0, x_1, x_2]$, then*

$$\textit{sdefect}(I, s) = \begin{cases} 3q - 2 & \textit{if } s \textit{ is even} \\ 3q & \textit{if } s \textit{ is odd.} \end{cases}$$

Proof. By Theorem 3.7 and Corollary 4.20, the statement holds for $s = 2$ and $s = 3$. Suppose that $s \geq 4$ and let $E^i_{s,d} := E^i_{s,d}(I)$. By Equation 5.3 and Lemma 5.20,

$$\begin{aligned}
\text{sdefect}(I, s) &= \left(\!\!\binom{3}{r}\!\!\right) + \sum_{d=1}^{q-1} \left(\left(\!\!\binom{3}{s-2d}\!\!\right) - |E^0_{s,d}| - |E^1_{s,d} \setminus E^0_{s,d}| - |E^2_{s,d} \setminus (E^0_{s,d} \cup E^1_{s,d})| \right) \\
&= \left(\!\!\binom{3}{r}\!\!\right) + \sum_{d=1}^{q-1} \Bigg(\left(\!\!\binom{3}{s-2d}\!\!\right) - \left(\!\!\binom{3}{s-2(d+1)}\!\!\right) - \left(\!\!\binom{3}{s-2(d+1)}\!\!\right) \\
&\quad + \left(\!\!\binom{3}{s-2(d+2)+1}\!\!\right) - \left(\!\!\binom{2}{s-2(d+1)}\!\!\right) \Bigg) \\
&= \left(\!\!\binom{3}{r}\!\!\right) + \sum_{d=1}^{q-1} \Bigg(\left(\!\!\binom{3}{s-2d}\!\!\right) - 2\left(\!\!\binom{3}{s-2(d+1)}\!\!\right) + \left(\!\!\binom{3}{s-2(d+2)+1}\!\!\right) \\
&\quad - \left(\!\!\binom{2}{s-2(d+1)}\!\!\right) \Bigg).
\end{aligned}$$

For a fixed $1 \leq d \leq q-1$, we have

$$\left(\!\!\binom{3}{s-2d}\!\!\right) - 2\left(\!\!\binom{3}{s-2(d+1)}\!\!\right) + \left(\!\!\binom{3}{s-2(d+2)+1}\!\!\right) - \left(\!\!\binom{2}{s-2(d+1)}\!\!\right) = 3,$$

so that

$$\operatorname{sdefect}(I,s) = \left(\!\!\binom{3}{r}\!\!\right) + \sum_{d=1}^{q-1} 3 = \left(\!\!\binom{3}{r}\!\!\right) + 3(q-1).$$

If s is even, i.e. $r=0$, then

$$\left(\!\!\binom{3}{0}\!\!\right) + 3(q-1) = 1 + 3(q-1) = 3q-2,$$

and if s is odd, i.e. $r=1$, then

$$\left(\!\!\binom{3}{1}\!\!\right) + 3(q-1) = 3 + 3(q-1) = 3q.$$

□

Remark 5.22. Given $s \geq 2$, write $s = 2q + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < 2$, and let $I = I(C_3) \subset k[x_0, x_1, x_2]$. Notice that $q = (s-r)/2$. Thus by Theorem 5.21, $\operatorname{sdefect}(I,s)$ is a quasi-polynomial in s with quasi-period 2. This aligns with the work of Drabkin and Guerrieri in [13, Theorem 2.4].

Computing $\operatorname{sdefect}(I(C_{2m-1}), s)$ for $m \geq 3$ and $s \geq 2m$ requires just a few more lemmas. The following lemma will greatly facilitate counting $|E^i_{s,d}(I) \setminus (\cup_{j=0}^{i-1} E^j_{s,d}(I))|$ for $3 \leq i \leq 2m-2$.

Lemma 5.23. *Let $m \geq 3$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. If $s = qm + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then for all $1 \leq d \leq q-1$ and $3 \leq i \leq 2m-2$ we have*

$$E^i_{s,d}(I) \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}(I) \right) = \begin{cases} E^i_{s,d}(I) \setminus (E^0_{s,d}(I) \cup E^{i-2}_{s,d}(I)) & \text{if } i \text{ is odd} \\ E^i_{s,d}(I) \setminus (E^1_{s,d}(I) \cup E^{i-2}_{s,d}(I)) & \text{if } i \text{ is even.} \end{cases}$$

Proof. Fix $1 \leq d \leq q-1$ and let $E^i := E^i_{s,d}(I)$. Note first that regardless of the parity of i, both $E^0 \cup E^{i-2}$ and $E^1 \cup E^{i-2}$ are subsets of $\cup_{j=0}^{i-1} E^j$. Thus

$$E^i \setminus \left(\bigcup_{j=0}^{i-1} E^j \right) \subseteq E^i \setminus (E^0 \cup E^{i-2}) \text{ and } E^i \setminus \left(\bigcup_{j=0}^{i-1} E^j \right) \subseteq E^i \setminus (E^1 \cup E^{i-2}),$$

so that we need only show the reverse inclusion.

Suppose first that i is odd, and let $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in E^i \setminus (E^0 \cup E^{i-2})$. By definition, this means that $u \in \mathcal{G}(I^{s-dm})$ and

1. $b_0, b_2, b_4, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-3}$ are all positive (since $u \in E^i$);
2. there exists an even j such that $b_j = 0$ (since $u \notin E^0$); and
3. at least one of $b_0, b_2, b_4, \ldots, b_{i-5}, b_{i-3}, b_{i-2}, b_i, b_{i+2}, \ldots, b_{2m-3}$ is zero (since $u \notin E^{i-2}$).

The first and third item together imply that $b_{i-2} = 0$. For all odd j satisfying $0 \leq j \leq i-1$, the index j has the same parity as $i-2$ and $i-2 \geq j$ (since $i-1 \geq j$ and $i-1$ is even, j is odd). Thus by definition, $u \notin E^j$ for any odd $0 \leq j \leq i-1$. Similarly, the first and second item together imply that there exists an even index l satisfying $i < l \leq 2m-2$ with $b_l = 0$. For all even $0 \leq j \leq i-1$, the index j has the same parity as l, and $l > j$. Thus by definition, $u \notin E^j$ for any even $0 \leq j \leq i-1$. Together, this means $u \notin E^j$ for all $0 \leq j \leq i-1$, and so $u \in E^i \setminus (\cup_{j=0}^{i-1} E^j)$.

The case where i is even is handled similarly. We include the details for completeness. Let $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in E^i \setminus (E^1 \cup E^{i-2})$. By definition, this means that $u \in \mathcal{G}(I^{s-dm})$ and

1. $b_1, b_3, b_5, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-2}$ are all positive (since $u \in E^i$);
2. at least one of $b_0, b_1, b_3, b_5, \ldots, b_{2m-3}$ is zero (since $u \notin E^1$); and
3. at least one of $b_1, b_3, b_5, \ldots, b_{i-5}, b_{i-3}, b_{i-2}, b_i, b_{i+2}, \ldots, b_{2m-2}$ is zero (since $u \notin E^{i-2}$).

The first and third item together imply that $b_{i-2} = 0$. For all even j satisfying $0 \leq j \leq i-1$, the index j has the same parity as $i-2$ and $i-2 \geq j$ (since $i-1 \geq j$ and $i-1$ is odd, j is even). Thus by definition, $u \notin E^j$ for any even $0 \leq j \leq i-1$. Similarly, the first and second item together imply that either $b_0 = 0$, or there exists an odd index l satisfying $i < l \leq 2m-2$ with $b_l = 0$. If $b_0 = 0$, then for all odd j satisfying $0 \leq j \leq i-1$, j has a different parity than 0 and $0 < j$. Thus by definition, $u \notin E^j$ for any odd $0 \leq j \leq i-1$. Similarly, if there exists an odd index l satisfying $i < l \leq 2m-2$ with $b_l = 0$, then for all odd j satisfying $0 \leq j \leq i-1$, j has the same parity as l and $l > j$. Thus by definition, $u \notin E^j$ for any odd $0 \leq j \leq i-1$. Together, this means $u \notin E^j$ for all $0 \leq j \leq i-1$, and so $u \in E^i \setminus (\cup_{j=0}^{i-1} E^j)$.

$\square$

Lemma 5.24. *Let $m \geq 3$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. Write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. For each $1 \leq d \leq q-1$ and $3 \leq i \leq 2m-2$ with i being odd,*

$$\left| E^i_{s,d}(I) \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}(I) \right) \right| = \left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right) - \left(\!\!\binom{2m-2}{s-m(d+2)+(i+1)/2}\!\!\right).$$

Proof. Fix $1 \le d \le q-1$ and $3 \le i \le 2m-2$, with i being odd. Let $E^i := E^i_{s,d}(I)$. By Lemma 5.23,

$$E^i \setminus \left(\bigcup_{j=0}^{i-1} E^j \right) = E^i \setminus (E^0 \cup E^{i-2}).$$

By definition, $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm})$ lies in $E^i \setminus (E^0 \cup E^{i-2})$ if and only if

1. $b_0, b_2, b_4, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-3}$ are all positive (since $u \in E^i$);

2. there exists an even j such that $b_j = 0$ (since $u \notin E^0$); and

3. at least one of $b_0, b_2, b_4, \ldots, b_{i-5}, b_{i-3}, b_{i-2}, b_i, b_{i+2}, \ldots, b_{2m-3}$ is zero (since $u \notin E^{i-2}$).

The first and third item together imply that $b_{i-2} = 0$. The first and second item together imply that there exists an even index $j > i$ such that $b_j = 0$. Thus $u \in E^i \setminus (E^0 \cup E^{i-2})$ if and only if each of the following conditions are satisfied:

Condition 1: $b_0, b_2, b_4, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-3}$ are all positive;
Condition 2: $b_{i-2} = 0$;
Condition 3: there exists an even index $j > i$ such that $b_j = 0$.

We first count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2. This is equivalent to counting the number of ways to distribute $s - dm$ identical objects among the $2m-1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$ where the boxes may receive multiple objects, the boxes $\epsilon_0, \epsilon_2, \epsilon_4, \ldots, \epsilon_{i-3}, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+2}, \epsilon_{i+4}, \ldots, \epsilon_{2m-3}$ are all non-empty, and the box ϵ_{i-2} is empty. To do so, we first distribute an object into each of the m boxes we require to be non-empty, meaning there are $s-dm-m = s-m(d+1)$ objects left to distribute. Since ϵ_{i-2} must be empty, there are only $2m-2$ boxes available to distribute these remaining objects into. Thus there are $\left(\!\binom{2m-2}{s-m(d+1)}\!\right)$ monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2.

Next, we count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2, but not satisfying Condition 3. This is equivalent to counting the number of ways to distribute $s - dm$ identical objects among the $2m-1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where the boxes may receive identical objects, the boxes $\epsilon_0, \epsilon_2, \epsilon_4, \ldots, \epsilon_{i-3}, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+2}, \epsilon_{i+4}, \ldots, \epsilon_{2m-3}$ and all boxes lying in $\{\epsilon_j : j > i, j \text{ even}\}$ are non-empty, and the box ϵ_{i-2} is empty. To do so, we first distribute an object into each of the boxes we require to be non-empty. We see

$$\{\epsilon_j : j > i, j \text{ even}\} = \{\epsilon_{i+1}, \epsilon_{i+3}, \ldots, \epsilon_{2m-2}\} = \{\epsilon_{2\left(\frac{i+1}{2}\right)}, \epsilon_{2\left(\frac{i+1}{2}+1\right)}, \ldots, \epsilon_{2(m-1)}\},$$

so that

$$|\{\epsilon_j : j > i, j \text{ even}\}| = (m-1) - \left(\frac{i+1}{2}\right) + 1 = m - \left(\frac{i+1}{2}\right).$$

Thus there are $s - dm - m - (m - (i+1)/2) = s - m(d+2) + (i+1)/2$ objects left to distribute. Since ϵ_{i-2} must be empty, there are only $2m-2$ boxes available to distribute these remaining objects into. Thus there are $\left(\!\binom{2m-2}{s-m(d+2)+(i+1)/2}\!\right)$ monomials satisfying both Conditions 1 and 2, but not satisfying Condition 3.

Notice that if $d = q-1$, it is possible for $s-m(d+2)+(i+1)/2$ to be strictly less than zero. In this case, there does not exist a monomial in $\mathcal{G}(I^{s-dm})$ satisfying Conditions 1 and 2, but not satisfying Condition 3. The formula still holds in this case, since if $s-m(d+2)+(i+1)/2 < 0$, then $\left(\!\binom{2m-2}{s-m(d+2)+(i+1)/2}\!\right) = 0$.

Together, this means there are

$$\left(\!\binom{2m-2}{s-m(d+1)}\!\right) - \left(\!\binom{2m-2}{s-m(d+2)+(i+1)/2}\!\right)$$

monomials in $\mathcal{G}(I^{s-dm})$ satisfying all of Conditions 1, 2 and 3. $\square$

The following lemma has a very similar proof to that of Lemma 5.24.

Lemma 5.25. *Let $m \geq 3$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. Write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. For each $1 \leq d \leq q-1$ and $3 \leq i \leq 2m-2$ with i being even,*

$$\left| E^i_{s,d}(I) \setminus \left(\bigcup_{j=0}^{i-1} E^j_{s,d}(I) \right) \right| = \left(\!\binom{2m-2}{s-m(d+1)}\!\right) - \left(\!\binom{2m-2}{s-m(d+2)+i/2}\!\right).$$

Proof. Fix $1 \leq d \leq q-1$ and $3 \leq i \leq 2m-2$ with i being even. Let $E^i := E^i_{s,d}(I)$. By Lemma 5.23,

$$E^i \setminus \left(\bigcup_{j=0}^{i-1} E^j \right) = E^i \setminus (E^1 \cup E^{i-2}).$$

By definition, $u = \epsilon_0^{b_0} \cdots \epsilon_{2m-2}^{b_{2m-2}} \in \mathcal{G}(I^{s-dm})$ lies in $E^i \setminus (E^1 \cup E^{i-2})$ if and only if

1. $b_1, b_3, b_5, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-2}$ are all positive (since $u \in E^i$);

2. at least one of $b_0, b_1, b_3, b_5, \ldots, b_{2m-3}$ is zero (since $u \notin E^1$); and

3. at least one of $b_1, b_3, b_5, \ldots, b_{i-5}, b_{i-3}, b_{i-2}, b_i, b_{i+2}, \ldots, b_{2m-2}$ is zero (since $u \notin E^{i-2}$).

The first and third item together imply that $b_{i-2} = 0$. The first and second item together imply that either $b_0 = 0$, or there exists an odd index $j > i$ such that $b_j = 0$. Thus

$u \in E^i \setminus (E^1 \cup E^{i-2})$ if and only if each of the following conditions are satisfied:

Condition 1: $b_1, b_3, b_5, \ldots, b_{i-3}, b_{i-1}, b_i, b_{i+2}, b_{i+4}, \ldots, b_{2m-2}$ are all positive;
Condition 2: $b_{i-2} = 0$;
Condition 3: either $b_0 = 0$ or there exists an odd index $j > i$ satisfying $b_j = 0$.

We first count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2. This is equivalent to counting the number of ways to distribute $s - dm$ identical objects among the $2m - 1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where boxes may receive multiple objects, the boxes $\epsilon_1, \epsilon_3, \epsilon_5, \ldots, \epsilon_{i-3}, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+2}, \epsilon_{i+4}, \ldots, \epsilon_{2m-2}$ are all non-empty, and the box ϵ_{i-2} is empty. To do so, we first distribute an object into each of the m boxes we require to be non-empty, meaning there are $s-dm-m = s-m(d+1)$ objects left to distribute. Since ϵ_{i-2} must be empty, there are only $2m - 2$ boxes available to distribute these remaining objects into. Thus there are $\left(\!\binom{2m-2}{s-m(d+1)}\!\right)$ monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2.

Next, we count the number of monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2, but not satisfying Condition 3. This is equivalent to counting the number of ways to distribute $s - dm$ identical objects among the $2m - 1$ distinct boxes $\epsilon_0, \ldots, \epsilon_{2m-2}$, where the boxes may receive identical objects, the boxes $\epsilon_1, \epsilon_3, \epsilon_5, \ldots, \epsilon_{i-3}, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+2}, \epsilon_{i+4}, \ldots, \epsilon_{2m-2}$ and all boxes lying in $\{\epsilon_0\} \sqcup \{\epsilon_j : j > i, j \text{ odd}\}$ are non-empty, and the box ϵ_{i-2} is empty. To do so, we first distribute an object into each of the boxes we require to be non-empty. We see

$$\{\epsilon_j : j > i, j \text{ odd}\} = \{\epsilon_{i+1}, \epsilon_{i+3}, \ldots, \epsilon_{2m-3}\} = \{\epsilon_{2\left(\frac{i}{2}+1\right)-1}, \epsilon_{2\left(\frac{i}{2}+2\right)-1}, \ldots, \epsilon_{2(m-1)-1}\},$$

so that

$$|\{\epsilon_j : j > i, j \text{ odd}\}| = (m-1) - \left(\frac{i}{2} + 1\right) + 1 = m - \frac{i}{2} - 1.$$

Thus there are $s-dm-m-1-(m-i/2-1) = s-m(d+2)+i/2$ objects left to distribute. Since ϵ_{i-2} must be empty, there are only $2m-2$ boxes available to distribute these remaining objects into. Thus there are $\left(\!\binom{2m-2}{s-m(d+2)+i/2}\!\right)$ monomials in $\mathcal{G}(I^{s-dm})$ satisfying both Conditions 1 and 2, but not satisfying Condition 3.

Notice that if $d = q-1$, it is possible for $s-m(d+2)+i/2$ to be strictly less than zero. In this case, there does not exist a monomial in $\mathcal{G}(I^{s-dm})$ satisfying Conditions 1 and 2, but not satisfying Condition 3. This formula still holds in this case, since if $s - m(d+2) + i/2 < 0$, then $\left(\!\binom{2m-2}{s-m(d+2)+i/2}\!\right) = 0$.

Together, this means there are

$$\left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right) - \left(\!\!\binom{2m-2}{s-m(d+2)+i/2}\!\!\right)$$

monomials in $\mathcal{G}(I^{s-dm})$ satisfying Conditions 1, 2 and 3. $\square$

Mandal and Pradhan give formulas for the symbolic defect sequence of edge ideals of unicyclic graphs in both recursive form [39, Theorem 4.8] and non-recursive form [39, Corollary 4.9]. The following formula for the symbolic defect sequence of edge ideals of odd cycles is less computationally demanding than that of Mandal and Pradhan, and was derived independently.

Theorem 5.26. *Fix $m \geq 3$, $I = I(C_{2m-1}) \subset k[x_0, \ldots, x_{2m-2}]$ and $s \geq 2m$. Write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Then*

$$\begin{aligned}\textit{sdefect}(I, s) = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1} \Bigg(& \left(\!\!\binom{2m-1}{s-dm}\!\!\right) - 2\left(\!\!\binom{2m-1}{s-m(d+1)}\!\!\right) + \\ & \left(\!\!\binom{2m-1}{s-m(d+2)+1}\!\!\right) - (2m-3)\left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right) + \\ & \sum_{i=3}^{2m-2} \left(\!\!\binom{2m-2}{s-m(d+2)+\lceil i/2 \rceil}\!\!\right) \Bigg).\end{aligned}$$

Proof. For each $1 \leq d \leq q-1$ and $0 \leq i \leq 2m-2$, let $E_d^i := E_{s,d}^i(I)$. By Equation 5.3,

$$\operatorname{sdefect}(I, s) = \left(\!\!\binom{2m-1}{r}\!\!\right) + \sum_{d=1}^{q-1} \left(\left(\!\!\binom{2m-1}{s-dm}\!\!\right) - |E^0| - \sum_{i=1}^{2m-2} \left| E_d^i \setminus \left(\bigcup_{j=0}^{i-1} E_d^j \right) \right| \right).$$

By Lemma 5.20,

$$\begin{aligned}|E_d^0| &= \left(\!\!\binom{2m-1}{s-m(d+1)}\!\!\right), \\ |E_d^1 \setminus E_d^0| &= \left(\!\!\binom{2m-1}{s-m(d+1)}\!\!\right) - \left(\!\!\binom{2m-1}{s-m(d+2)+1}\!\!\right), \\ |E_d^2 \setminus (E_d^1 \cup E_d^0)| &= \left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right).\end{aligned}$$

Now, note that when i is odd, $(i+1)/2 = \lceil i/2 \rceil$, and when i is even, $i/2 = \lceil i/2 \rceil$. Thus by

Lemmas 5.24 and 5.25,

$$\begin{aligned}\sum_{i=3}^{2m-2} \left| E_d^i \setminus \left(\bigcup_{j=0}^{i-1} E_d^j \right) \right| &= \sum_{i=3}^{2m-2} \left(\left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right) - \left(\!\!\binom{2m-2}{s-m(d+2)+\lceil i/2 \rceil}\!\!\right) \right) \\ &= (2m-4)\left(\!\!\binom{2m-2}{s-m(d+1)}\!\!\right) - \sum_{i=3}^{2m-2} \left(\!\!\binom{2m-2}{s-m(d+2)+\lceil i/2 \rceil}\!\!\right).\end{aligned}$$

The formula now easily follows.

$\square$

Remark 5.27. The formula for sdefect$(I(C_{2m-1}), s)$ given in Theorem 5.26 simplifies to be a quasi-polynomial in s with quasi-period m. This aligns with the work of Drabkin and Guerrieri in [13, Theorem 2.4].

Chapter 6

The symbolic defect sequence of edge ideals of unicyclic graphs

In this chapter, we consider the symbolic defect sequence of edge ideals of unicyclic graphs (see Definition 2.4). If G is a unicyclic graph containing a $(2m-1)$ - cycle, by Proposition 3.3, Theorem 3.7 and Corollaries 4.20 and 4.40,

$$\operatorname{sdefect}(I(G), s) = \begin{cases} 0 & \text{if } s < m \\ 1 & \text{if } s = m \\ |E(G)| & \text{if } s = m+1. \end{cases}$$

We now seek a formula for $\operatorname{sdefect}(I(G), s)$ for $s > m+1$. With Lemma 6.10, we describe a monomial generating set for $I(G)^{(s)}/I(G)^s$, and in Proposition 6.14, we extract the minimal monomial generating set. We describe a formula for $\operatorname{sdefect}(I(G), s)$ only in the case that $m+1 < s < 2m$ in Theorem 6.15. When $s \geq 2m$, we instead provide upper and lower bounds for $\operatorname{sdefect}(I(G), s)$ in Proposition 6.16 and Corollary 6.20.

Recall that we are only considering non-bipartite graphs throughout this thesis. Unless otherwise stated, it is always assumed that G is a unicyclic graph containing a $(2m-1)$ - cycle for some $m \geq 2$, $I = I(G)$ and that $s > m+1$. Note then that it is implicitly assumed that $|V(G)| \geq 2m-1$, as in Chapter 4.

6.1 The minimal monomial generating set

We begin with some basic graph theory results.

Definition 6.1. A *leaf* in a graph is a vertex that is adjacent to exactly one other vertex.

Lemma 6.2. *If a graph G is unicyclic and there exists an edge in G that does not lie on its cycle, then G contains a leaf.*

Proof. Let C denote the unique cycle in G. Let P be a longest path in G that does not contain any edges of C. Note that since G contains at least one edge that does not lie on C, this path contains at least two vertices. Since G is unicyclic, there exists at most one vertex on P that lies on C (otherwise, there is an additional cycle in G). Suppose that x_j is an endpoint of P that does not lie on C, and the vertex adjacent to x_j on P is labelled x_i. Since P is the longest path that does not contain any edges of C and x_j does not lie on C, all neighbours of x_j lie on P. Since G is unicyclic, x_j cannot be adjacent to any vertices on P other than x_i, and so x_j is a leaf. □

Definition 6.3. A graph is *acyclic* if it does not contain any cycles.

Lemma 6.4. *[8, Theorem 4.3] Acyclic graphs with at least one edge contain a leaf.*

Definition 6.5. Let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $\{x_i, x_j\} \in E(G)$, let $\epsilon_{i,j}$ denote the edge monomial $x_i x_j$.

The following lemma was proven independently by Mandal and Pradhan [39, Lemma 4.2].

Lemma 6.6. *Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $\{a_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \cup \{b_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \subset \mathbb{Z}_{\geq 0}$ are such that*

$$\prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{a_{i,j}} = \prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{b_{i,j}},$$

then $a_{i,j} = b_{i,j}$ for all $\epsilon_{i,j} \in \mathcal{G}(I)$.

Proof. Let $\epsilon^a := \prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{a_{i,j}}$ and $\epsilon^b := \prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{b_{i,j}}$. If $\epsilon^a = \epsilon^b$, then in particular, $\deg(\epsilon^a) = \deg(\epsilon^b)$. We proceed via induction on $\deg(\epsilon^a)$. Note that since ϵ^a is a product of edge monomials, $\deg(\epsilon^a)$ is even.

If $\deg(\epsilon^a) = 2$, then $\epsilon^a = \epsilon^b$ is a single edge monomial, and the result is clear. Let $l \geq 2$ be even and assume that if $\{a_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \cup \{b_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \subset \mathbb{Z}_{\geq 0}$ are such that $\epsilon^a = \epsilon^b$ and $\deg(\epsilon^a) = l$, then $a_{i,j} = b_{i,j}$ for all $\epsilon_{i,j} \in \mathcal{G}(I)$. Let $\{\alpha_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \cup \{\beta_{i,j} : \epsilon_{i,j} \in \mathcal{G}(I)\} \subset \mathbb{Z}_{\geq 0}$ be such that

$$\epsilon^\alpha := \prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{\alpha_{i,j}} = \prod_{\epsilon_{i,j} \in \mathcal{G}(I)} \epsilon_{i,j}^{\beta_{i,j}} =: \epsilon^\beta,$$

and $\deg(\epsilon^\alpha) = l+2$. We want to show that $\alpha_{i,j} = \beta_{i,j}$ for all $\epsilon_{i,j} \in \mathcal{G}(I)$.

If $\operatorname{supp}(\epsilon^\alpha) \subseteq \{x_0, \ldots, x_{2m-2}\}$, then $\epsilon^\alpha = \epsilon^\beta$ is a product of edge monomials in $\mathcal{G}(I(C_{2m-1}))$ and we are done by Lemma 5.2. Otherwise, suppose $\operatorname{supp}(\epsilon^\alpha) \nsubseteq \{x_0, \ldots, x_{2m-2}\}$ and consider $G[\operatorname{supp}(\epsilon^\alpha)]$. Since G is unicyclic, this induced subgraph is either acyclic or unicyclic. Since ϵ^α is the product of edge monomials, this graph contains at least one edge. Further, since $\operatorname{supp}(\epsilon^\alpha) \nsubseteq \{x_0, \ldots, x_{2m-2}\}$, there exists an edge in $G[\operatorname{supp}(\epsilon^\alpha)]$ that is not an edge on the cycle in G. Thus by Lemmas 6.2 and 6.4, this subgraph contains at least one leaf. Let x_p denote some leaf in $G[\operatorname{supp}(\epsilon^\alpha)]$ and suppose $\epsilon_{p,q}$ is the edge monomial corresponding to the unique edge in $G[\operatorname{supp}(\epsilon^\alpha)]$ incident to x_p. Since x_p is a vertex in $G[\operatorname{supp}(\epsilon^\alpha)]$, x_p divides ϵ^α, forcing $\alpha_{p,q} > 0$. Since $G[\operatorname{supp}(\epsilon^\alpha)] = G[\operatorname{supp}(\epsilon^\beta)]$, x_p divides ϵ^β as well, so that $\beta_{p,q} > 0$. Thus

$$\left(\prod_{\epsilon_{i,j} \in \mathcal{G}(I)\setminus\{\epsilon_{p,q}\}} \epsilon_{i,j}^{\alpha_{i,j}}\right) \epsilon_{p,q}^{\alpha_{p,q}-1} = \frac{\epsilon^\alpha}{\epsilon_{p,q}} = \frac{\epsilon^\beta}{\epsilon_{p,q}} = \left(\prod_{\epsilon_{i,j} \in \mathcal{G}(I)\setminus\{\epsilon_{p,q}\}} \epsilon_{i,j}^{\beta_{i,j}}\right) \epsilon_{p,q}^{\beta_{p,q}-1}.$$

Since $\epsilon^\alpha/\epsilon_{p,q} = \epsilon^\beta/\epsilon_{p,q}$ and $\deg(\epsilon^\alpha/\epsilon_{p,q}) = l$, we apply the induction hypothesis to conclude that $\alpha_{i,j} = \beta_{i,j}$ for all $\epsilon_{i,j} \in \mathcal{G}(I) \setminus \{\epsilon_{p,q}\}$ and $\alpha_{p,q} - 1 = \beta_{p,q} - 1$, clearly implying that $\alpha_{p,q} = \beta_{p,q}$ as well.

□

Remark 6.7. Note that Lemma 6.6 mirrors Lemma 5.2, and so the observations of Remark 5.3 hold in the case of unicyclic graphs as well. By using this same visualization of edge monomials as boxes and exponents as identical objects, we conclude that $|\mathcal{G}(I(G)^s)| = \left(\!\binom{|E(G)|}{s}\!\right)$ for any unicyclic graph G and $s \geq 0$.

Definition 6.8. Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Let c_G denote the $(2m-1)$ - cycle monomial $x_0 \cdots x_{2m-2}$.

The generating set $A_s(I(C_{2m-1})) + I(C_{2m-1})$ for $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$ of Definition 5.8 extends naturally to the unicyclic case.

Definition 6.9. Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Given $s > m+1$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Define

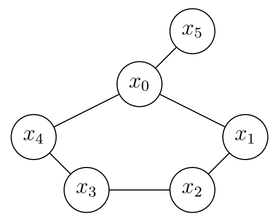
$$A_s(I) = \{c_G^d u : 1 \leq d \leq q, u \in \mathcal{G}(I^{s-dm})\}.$$

Lemma 6.10. *Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $s > m+1$, then $A_s(I) + I^s$ is a generating set for $I^{(s)}/I^s$.*

Proof. As in the proof of Lemma 5.6, this follows directly from Proposition 2.34.

□

Example 6.11. Let G denote the following graph and let $I = I(G) \subset k[x_0, x_1, x_2, x_3, x_4, x_5]$.



The graph G contains a 5 - cycle and so $m = 3$ in this case. Let $s = 7 = 2(3) + 1$. By definition,

$$A_7(I) = \{(x_0x_1x_2x_3x_4)^d u : 1 \leq d \leq 2, u \in \mathcal{G}(I^{7-3d})\}.$$

Then

$$w_1 := (x_0x_1x_2x_3x_4)\epsilon_{1,2}\epsilon_{3,4}\epsilon_{0,4}\epsilon_{0,5} = x_0^3x_1^2x_2^2x_3^2x_4^3x_5 \in A_7(I)$$

and

$$w_2 := (x_0x_1x_2x_3x_4)^2\epsilon_{0,5} = x_0^3x_1^2x_2^2x_3^2x_4^2x_5 \in A_7(I).$$

The monomial w_2 strictly divides w_1, and so $A_7(I) + I^7$ is not the minimal monomial generating set for $I^{(7)}/I^7$.

We introduce more notation to extract the minimal monomial generating set for $I^{(s)}/\, I^s$ from $A_s(I) + I^s$. Again, the minimal monomial generating set $B_s(I(C_{2m-1})) + I(C_{2m-1})^s$ for $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$ of Definition 5.8 extends naturally to the unicyclic case.

Definition 6.12. Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Given $s > m+1$, write $s = qm + r$, with $q \in \mathbb{Z}^+$ and $0 \leq r < m$. Define

$$B_s(I) = \left\{ c_G^d u \in A_s(I) : c_G \nmid u \text{ or } \forall\ 0 < \delta \leq q - d \text{ such that } c_G^\delta \mid u, \frac{u}{c_G^\delta} \notin I^{s-m(d+\delta)} \right\}.$$

Example 6.13. Define G as in Example 6.11 and let $I = I(G) \subset k[x_0, \ldots, x_5]$. Recall that G contains a 5 - cycle, so that $m = 3$ in this case. Let $s = 7 = 2(3) + 1$. Recall that we showed in Example 6.11 that

$$w_1 = (x_0x_1x_2x_3x_4)\epsilon_{1,2}\epsilon_{3,4}\epsilon_{0,4}\epsilon_{0,5} = x_0^3x_1^2x_2^2x_3^2x_4^3x_5 \in A_7(I)$$

is not a minimal generator of $I^{(7)}/I^7$ because

$$w_2 = (x_0x_1x_2x_3x_4)^2\epsilon_{0,5} = x_0^3x_1^2x_2^2x_3^2x_4^2x_5 \in A_7(I)$$

strictly divides w_1. We see $w_2 \in B_7(I)$ because $c_G \nmid \epsilon_{0,5}$. On the other hand, by taking $\delta = 1$, we see $w_1 \notin B_7(I)$ because c_G divides $\epsilon_{1,2}\epsilon_{3,4}\epsilon_{0,4}\epsilon_{0,5} = x_0^2x_1x_2x_3x_4^2x_5$ and

$$\frac{\epsilon_{1,2}\epsilon_{3,4}\epsilon_{0,4}\epsilon_{0,5}}{c_G} = \frac{x_0^2x_1x_2x_3x_4^2x_5}{x_0x_1x_2x_3x_4} = x_0x_4x_5 = \epsilon_{0,5}x_4 \in I = I^{7-3(1+1)}.$$

Proposition 6.14. *Fix $m \geq 2$ and let G be a unicyclic graph on vertex set $\{x_0, \ldots, x_n\}$ with its unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $s > m+1$, then $B_s + I^s$ is the minimal monomial generating set for $I^{(s)}/I^s$.*

Proof. This proof is identical to that of Proposition 5.10 with Definition 5.8 replaced by Definition 6.12 and Lemma 5.6 replaced by Lemma 6.10. □

6.2 A partial formula for the symbolic defect sequence

Using the minimal monomial generating set from Proposition 6.14, we can easily describe a simple formula for sdefect$(I(G), s)$ when G is a unicyclic graph containing a $(2m-1)$ - cycle and $m+1 < s < 2m$. The following theorem was proven independently by Mandal and Pradhan [39, Theorem 4.8].

Theorem 6.15. *Fix $m \geq 2$ and let G be a unicyclic graph on vertices $\{x_0, \ldots, x_n\}$ with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. If $m+1 < s < 2m$, then*

$$\text{sdefect}(I, s) = \left(\!\!\binom{|E(G)|}{s-m}\!\!\right).$$

Proof. This proof is identical to that of Theorem 5.11 with Definition 5.8 replaced by Definition 6.12, Proposition 5.10 replaced by Proposition 6.14 and Remark 5.3 replaced by Remark 6.7. □

Obtaining a formula for sdefect$(I(G), s)$ when $s \geq 2m$ is not an easy task. The techniques used in Chapter 5 to compute sdefect$(I(C_{2m-1}), s)$ do not easily generalize to the unicyclic case. In particular, if $\tilde{U}_{s,d}(I(C_{2m-1}))$ of Definition 5.14 and $E^i_{s,d}(I(C_{2m-1}))$ of Definition 5.16 are naturally extended to the unicyclic case, then it is simple to find an example of a unicyclic graph G for which $\tilde{U}_{s,d}(I(G)) \neq \cup_{i=0}^{2m-2} E^i_{s,d}(I(G))$, meaning that Lemma 5.19 does not generalize to the unicyclic case.

6.3 Bounds on the symbolic defect sequence

In this section, we describe both lower and upper bounds on sdefect$(I(G), s)$, where G is a unicyclic graph containing a $(2m-1)$ - cycle and $s \geq 2m$. Note that Mandal and Pradhans work provides a formula for these values of s [39, Theorem 4.8]. The bounds presented in this section are less computationally demanding than this formula.

Proposition 6.16. *Fix $m \geq 2$ and let G be a unicyclic graph on vertices $\{x_0, \ldots, x_n\}$ with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Fix $s \geq 2m$. If $s = qm + r$ with $q \in \mathbb{Z}^+$ and $0 \leq r < m$, then*

$$\textit{sdefect}(I, s) < \sum_{d=1}^{q} \left(\!\!\binom{|E(G)|}{s-dm}\!\!\right).$$

Proof. For each $0 \leq i \leq 2m-2$, let ϵ_i denote the edge monomial $x_i x_{i+1}$, where addition on indices is done modulo $2m-1$. Let $u = (\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2})\epsilon_0^{s-2m} \in \mathcal{G}(I^{m+(s-2m)}) = \mathcal{G}(I^{s-m})$, so that $c_G u \in A_s(I)$. Note that $\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2} = c_G x_0$, so that $c_G \mid u$ and

$$\frac{u}{c_G} = \frac{(\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2})\epsilon_0^{s-2m}}{c_G} = \frac{c_G x_0 \epsilon_0^{s-2m}}{c_G} = x_0\epsilon_0^{s-2m} \in I^{s-2m} = I^{s-m(1+1)}.$$

Thus by taking $\delta = 1$, we see $c_G u \notin B_s(I)$, and so $B_s(I)$ is a proper subset of $A_s(I)$. Then

$$\begin{aligned}
\text{sdefect}(I, s) = |B_s(I)| &< |A_s(I)| \\
&= |\{c_G^d u : 1 \leq d \leq q, u \in \mathcal{G}(I^{s-dm})\}| \\
&= \left| \bigcup_{d=1}^{q} \{c_G^d u : u \in \mathcal{G}(I^{s-dm})\} \right| \\
&\leq \sum_{d=1}^{q} |\{c_G^d u : u \in \mathcal{G}(I^{s-dm})\}| \\
&= \sum_{d=1}^{q} |\{u \in \mathcal{G}(I^{s-dm})\}| \\
&= \sum_{d=1}^{q} \left(\!\!\binom{|E(G)|}{s-dm}\!\!\right),
\end{aligned}$$

where the final equality follows from Remark 6.7. ∎

Example 6.17. Fix $m \geq 2$ and let G be a unicyclic graph on vertices $\{x_0, \ldots, x_n\}$ with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. For each $0 \leq i \leq 2m-2$, let ϵ_i denote the edge monomial $x_i x_{i+1}$, where addition on indices is done modulo

$2m-1$. Given $s \geq 2m$, write $s = q_s m + r_s$, where $q_s \in \mathbb{Z}^+$ and $0 \leq r_s < m$. Note that $q_s \geq 2$. Then for each $1 \leq d \leq q_s - 1$, the monomial

$$u_d := (\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2})\epsilon_0^{s-m(d+1)} \in \mathcal{G}(I^{m+s-m(d+1)}) = \mathcal{G}(I^{s-dm}),$$

and so $c_G^d u_d \in A_s(I)$. Thus $c_G^d u_d$ is counted at least once in the sum $\sum_{d=1}^{q_s} \left(\!\binom{|E(G)|}{s-dm}\!\right)$. However, note that $\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2} = c_G x_0$, so that $c_G \mid u_d$ and

$$\frac{u_d}{c_G} = \frac{(\epsilon_0\epsilon_2\epsilon_4 \cdots \epsilon_{2m-2})\epsilon_0^{s-m(d+1)}}{c_G} = \frac{(c_G x_0)\epsilon_0^{s-m(d+1)}}{c_G} = x_0\epsilon_0^{s-m(d+1)} \in I^{s-m(d+1)}.$$

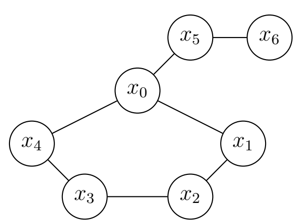
Thus by taking $\delta = 1$, we see $c_G^d u_d \notin B_s(I)$ and is therefore not counted in $\mathrm{sdefect}(I, s)$. By comparing degrees, one can easily deduce that $c_G^d u_d = c_G^{d'} u_{d'}$ if and only if $d = d'$, so that each of these monomials are distinct. Thus

$$\sum_{d=1}^{q_s} \left(\!\binom{|E(G)|}{s-dm}\!\right) - \mathrm{sdefect}(I, s) \geq q_s - 1.$$

As $s \to \infty$, $q_s \to \infty$ as well, and so

$$\sum_{d=1}^{q_s} \left(\!\binom{|E(G)|}{s-dm}\!\right) - \mathrm{sdefect}(I, s) \to \infty \text{ as } s \to \infty.$$

Example 6.18. For $l \geq 2$, let G_l denote the graph on vertices $\{x_0, \ldots, x_{3+l}\}$ that consists of a 5 - cycle with a path of length l, denoted P_l, beginning at one of the vertices on the 5 - cycle. For example, G_3 is drawn below.



Since this graph contains a 5 - cycle, $m = 3$ in this case. For each $0 \leq i \leq 4$, let ϵ_i denote the edge monomial $x_i x_{i+1}$, where addition on indices is done modulo 5. Note that for any $l \geq 2$, $c_{G_l} = x_0x_1x_2x_3x_4$, and so we let $c_G := c_{G_l}$. Let $s = 7 = 2(3)+1$. For any edge monomial $e \in \mathcal{G}(I(G_l))$, the monomial

$$u_e := (\epsilon_0\epsilon_2\epsilon_4)e \in \mathcal{G}(I(G_l)^4) = \mathcal{G}(I(G_l)^{7-3(1)}),$$

and so $c_G u_e \in A_7(I(G_l))$. Thus the monomial $c_G u_e$ is counted at least once in the sum

$\sum_{d=1}^{2} \left(\!\!\binom{|E(G_l)|}{7-3d}\!\!\right)$. However, note that $\epsilon_0\epsilon_2\epsilon_4 = c_G x_0$, so that $c_G \mid u_e$ and

$$\frac{u_e}{c_G} = \frac{(\epsilon_0\epsilon_2\epsilon_4)e}{c_G} = \frac{c_G x_0 e}{c_G} = x_0 e \in I(G_l) = I(G_l)^{7-3(1+1)}.$$

Therefore by taking $\delta = 1$, we see $c_G u_e \notin B_7(I(G_l))$, and is therefore not counted in sdefect$(I(G_l), 7)$. We conclude that for each $e \in \mathcal{G}(I(G_l))$, the monomial $c_G u_e$ is counted at least once in the sum $\sum_{d=1}^{2} \left(\!\!\binom{|E(G_l)|}{7-3d}\!\!\right)$, but is not counted in sdefect$(I(G_l), s)$. By Lemma 6.6, each of the monomials $c_G u_e$ are distinct, and so

$$\sum_{d=1}^{2} \left(\!\!\binom{|E(G_l)|}{7-3d}\!\!\right) - \text{sdefect}(I(G_l), 7) \geq |E(G_l)|.$$

As $l \to \infty, |E(G_l)| \to \infty$ as well. Thus

$$\sum_{d=1}^{2} \left(\!\!\binom{|E(G_l)|}{7-3d}\!\!\right) - \text{sdefect}(I(G_l), 7) \to \infty \text{ as } l \to \infty.$$

To describe a lower bound on sdefect$(I(G), s)$ when $s \geq 2m$, we first make a general observation that holds for all finite simple graphs.

Proposition 6.19. *Let G be a graph on vertices $\{x_0, \ldots, x_n\}$. If H is an induced subgraph of G, then sdefect$(I(H), s) \leq$ sdefect$(I(G), s)$ for each $s \geq 1$.*

Proof. If $s = 1$, then sdefect$(I(H), 1) = 0 =$ sdefect$(I(G), 1)$. Assume that $s \geq 2$ and $w + I(H)^s$ is a minimal monomial generator of $I(H)^{(s)}/I(H)^s$. We will show that $w + I(G)^s$ is also a minimal monomial generator of $I(G)^{(s)}/I(G)^s$, so that sdefect$(I(H), s) \leq$ sdefect$(I(G), s)$.

Since $w \in I(H)^{(s)}$ and thus supp$(w) \subseteq V(H)$, it is clear to see that $w \notin I(H)^s$ implies that $w \notin I(G)^s$. Let X be a vertex cover of G. Then $X \cap V(H)$ is a vertex cover of H. By Proposition 2.16,

$$W_X(w) = W_{X \cap V(H)}(w) \geq s,$$

and so $w \in I(G)^{(s)}$. Finally, if u is a monomial such that $u + I(G)^s \in I(G)^{(s)}/I(G)^s$ is non-zero and properly divides $w + I(G)^s$, then u properly divides w. This implies supp$(u) \subseteq$ supp$(w) \subseteq V(H)$. Then for any vertex cover Y of H, $Y \cup (V(G) \setminus V(H))$ is a vertex cover for G, so that

$$W_Y(u) = W_{Y \cup (V(G) \setminus V(H))}(u) \geq s.$$

Thus $u \in I(H)^{(s)}$. Clearly $u \notin I(G)^s$ implies $u \notin I(H)^s$, and so $u + I(H)^s$ is non-zero in $I(H)^{(s)}/I(H)^s$ and properly divides $w + I(H)^s$. This contradicts the assumption that

$w + I(H)^s$ is a minimal monomial generator of $I(H)^{(s)}/I(H)^s$. $\square$

Corollary 6.20. *Fix $m \geq 2$ and let G be a unicyclic graph on vertices $\{x_0, \ldots, x_n\}$ with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Assume that G has an edge that does not lie on its cycle. If $s \geq 2m$, then*

$$\textit{sdefect}(I(C_{2m-1}), s) < \textit{sdefect}(I, s).$$

Proof. By Proposition 6.19, $\text{sdefect}(I(C_{2m-1}), s) \leq \text{sdefect}(I, s)$. Let e denote an edge monomial corresponding to an edge in G that does not lie on its cycle. Then $ce^{s-m} \in B_s(I)$, so that $ce^{s-m} + I^s$ is a minimal monomial generator of $I^{(s)}/I^s$ that does not lie in $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$, and so is certainly not a minimal monomial generator of $I(C_{2m-1})^{(s)}/I(C_{2m-1})^s$. Thus $\text{sdefect}(I(C_{2m-1}), s) < \text{sdefect}(I, s)$. $\square$

Notice that by Proposition 6.19, if G is any finite simple graph (not necessarily unicyclic) containing an induced copy of a $(2m-1)$ - cycle, then $\text{sdefect}(I(C_{2m-1}), s) \leq \text{sdefect}(I(G), s)$.

Example 6.21. Fix $m \geq 2$ and let G be a unicyclic graph on vertices $\{x_0, \ldots, x_n\}$ with unique cycle on vertices $\{x_0, \ldots, x_{2m-2}\}$ and $I = I(G) \subset k[x_0, \ldots, x_n]$. Assume that there exists an edge in G that does not lie on its cycle, and denote its corresponding edge monomial by e. Given $s \geq 2m$, write $s = q_s m + r_s$, where $q_s \in \mathbb{Z}^+$ and $0 \leq r_s < m$. Note that $q_s \geq 2$. Then for each $1 \leq d \leq q_s - 1$, the monomial $c_G^d e^{s-dm}$ lies in $B_s(I)$, and is thus counted in $\text{sdefect}(I, s)$. Since e is not an edge on the cycle and $d \leq q_s - 1$, implying that $s - dm > 0$, the monomial $c_G^d e^{s-dm} \notin B_s(I(C_{2m-1}))$, and is therefore not counted in $\text{sdefect}(I(C_{2m-1}), s)$. By comparing degrees, one can easily deduce that $c_G^d e^{s-dm} = c_G^{d'} e^{s-d'm}$ if and only if $d = d'$. Thus each of the monomials $c_G^d e^{s-dm}$ are distinct, and so

$$\text{sdefect}(I, s) - \text{sdefect}(I(C_{2m-1})) \geq q_s - 1.$$

As $s \to \infty$, $q_s \to \infty$ as well and so

$$\text{sdefect}(I, s) - \text{sdefect}(I(C_{2m-1})) \to \infty \text{ as } s \to \infty.$$

Example 6.22. For $l \geq 2$, let G_l denote the graph from Example 6.18. As before, let $c_G := c_{G_l}$ for $l \geq 2$ and $s = 7 = 2(3)+1$. For any edge monomial e in $\mathcal{G}(I(G_l))$ corresponding to an edge that lies on the path in G_l, the monomial $c_G^2 e \in B_7(I(G_l))$, and is therefore counted in $\text{sdefect}(I(G_l), 7)$. On the other hand, since e is not an edge on the cycle, $c_G^2 e \notin B_7(I(C_5))$ and is therefore not counted in $\text{sdefect}(I(C_5), 7)$. Each of these monomials are clearly distinct, and so

$$\text{sdefect}(I(G_l), 7) - \text{sdefect}(I(C_5), 7) \geq |E(P_l)|.$$

As $l \to \infty$, $|E(P_l)| \to \infty$ as well and so

$$\text{sdefect}(I(G_l), 7) - \text{sdefect}(I(C_5), 7) \to \infty \text{ as } l \to \infty.$$

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