THE UNIVERSITY OF MANITOBA

A STUDY OF THE GROWTH OF THE FUNCTION CONCEPT IN MATHEMATICS THROUGH SENIOR HIGH SCHOOL GRADES

BEING A THESIS SUBMITTED TO THE COMMITTEE

ON POSTGRADUATE STUDIES IN PARTIAL

FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF

EDUCATION

BY

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WINNIPEG, MANITOBA.

August, 1956.

ACKNOWLEDGEMENTS

The writer wishes to express his appreciation and thanks to Dr. H. L. Stein of the Faculty of Education, University of Manitoba, for his guidance, co-operation and assistance in the investigation, and to the teachers who assisted in the testing program.

ABSTRACT

A STUDY OF THE GROWTH OF THE FUNCTION CONCEPT IN MATHEMATICS THROUGH SENIOR HIGH SCHOOL GRADES

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Since there is, theoretically, a progression in breadth and intensity of instruction in the concept of functionality from the tenth through the twelfth grades, there should be a measureable increase in the understanding of the function concept through these grades. The purpose of this study is to determine to what extent the students of these grades have an understanding of the concept of functionality and to determine the extent of growth of functionality from grade to grade.

In this study the emphasis has been placed upon three main areas:

1. The extent to which students of the tenth, eleventh and twelfth grades are able to recognize and interpret the function as a relationship between two or more variables, to interpret a table of associated values, to change the subject of a formula and to interpret graphical representation.

2. The growth in the ability to use the function concept throughout the high school grades.

3. A comparison of the abilities of the girls and boys respectively.

The conclusions of the study are based upon the statistical information derived from the results of a test devised by the author. This test was administered as a group test in five Winnipeg and one suburban high school. Five hundred and ninety-five students were involved in the testing program and over forty-one thousand responses to seventy functional situations were examined for the study.

The major conclusions derived were:

1. Throughout the high school grades, there is a measureable increase in the understanding of the function concept by high school students.

2. A program of fuuctional mathematics must be taught rather than left to chance.

2. Courses in mathematics must be so planned and developed to present to the students a continuous program of functional situations and to provide for growth in the function concept throughout the grades.

4. There is little attempt to adapt the processes of functional thinking to problem situations.

5. Functional thinking as applied to changing the subject of a formula, to tables of associated values and to graphical representation is not developed to any great extent prior to grade twelve.

6. At the grade ten level neither sex showed any superiority in relational thinking but in successive grades the boys showed a marked superiority.

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INTRODUCTION

Purposes of the Investigation

Since there is, theoretically, a progression in breadth and intensity of instruction in the concept of functionality from the tenth through the twelfth grades, there should be a measurable increase in the understanding of the function concept through these grades. The purpose of this investigation is to determine to what extent the students of these grades have an understanding of the concept of functionality and to determine the extent of growth of functionality from grade to grade.

In the investigation the emphasis has been placed upon the following questions:

1. To what extent are the students at the end of the tenth, eleventh and twelfth grades, able to recognize a relationship between two or more variables?

2. To what extent are the students at the end of the tenth, eleventh and twelfth grades, able to interpret the function concept and express it as a relationship between the variables?

3. To what extent are the students at the end of the tenth, eleventh and twelfth grades, able to interpret a table exhibiting associated values?

4. To what extent are the students at the end of the tenth, eleventh and twelfth grades, able to interpret a graph?

5. What is the growth in the ability to use the function concepts throughout these grades?

6. How do the abilities of boys and girls respectively compare?

Justification of the Investigation

There are a number of reasons why an investigation of this nature would be of considerable value. These may be classified under the following headings:

1. The importance of functional relationship to general education.

2. The increased emphasis that has been placed upon the function concept in the teaching of mathematics at the junior and senior high school level.

3. The evaluation of the courses that are being presented at the high school level.

4. The evaluation of the progress made by the students as a result of these courses.

The Importance of the Study of Functional Relationship to General Education

To a great extent the development of newer modes of thinking and approach to problems in mathematics has made it possible for an advancement of fundamental understanding in education which in turn has been reflected in the general advancement of civilization. When man began to become aware of relationships existing between two or more quantities and through experimentation confirmed this relationship he became aware for the first time whether he realized it or not of the notion of functionality. In time, mathematics began to investigate the various possibilities of extending the relationships that had been discovered. These investigations found expression in the techniques we call modern mathematics.

In most branches of knowledge there comes a certain stage of development at which quantitative relationships between various concepts can be found and utilized. Once such a relationship is found and expressed algebraically, the road is cleared for intensive study of the topic without too much reliance upon inconvenient experimentation. Such an algebraic expression is called a functional relationship and the formula involved we call a function. In some cases this stage of development came early in the civilization of certain groups of people. Archimedes first showed how we can calculate pi and applied this knowledge to inventions based upon the wheel. His knowledge of the functional relationship between the ratio of the weight and the distances from the fulcrum gave us one of the basic principles of mechanics. The contributions of Pythagorus made it possible for the Alexandrians to devise tables of ratios of the sides of a right-angled triangle. Rapid expansion of trade and improvements of techniques in navigation necessitated a simplification of certain mathematical procedures. The introduction of logarithmic functions by Napier and subsequent tables by Briggs added to the stock of functions in mathematics and simplified many of the tedious computations occurring in business and industry.

In the Report of the Committee on the Function of Mathematics in

General Education two reasons are given for the importance of the study of functional relationship in general education. The Report states:

> In the first place, it helps the student to see how the concept of function has enabled scientists and engineers to gain an effective control (or at least an explanation) of phenomena that would otherwise seem unmanageable or mysterious. Secondly, the fact that in some cases functional relationships exist furnishes a standard in terms of which less exact relations are recognized as approximate.[1]

The concept of the function as developed in mathematics is the essence of all scientific studies. Both chemistry and physics can be truly understood only by a knowledge of the fundamental concept of mathematics. The concept of the relationship of variables is now being used to an ever increasing extent in industry, commerce and economics. The rapid advancement in the first half of this century has certainly borne out Klein's[2] belief that the world is becoming functionally minded.

To a great extent the increase in the power of mathematics as an effective instrument in general education developed with the increased recognition of the function concept. Although the term function was not used explicitly in mathematical treatises until the end of the fourteenth century, the notion of the functional relationship formed part of the earlier mathematical thinking. The development of the re-

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[1] Mathematics in General Education. A Report of the Committee on the functions of Mathematics in General Education for the Commission on Secondary School Curriculum. (New York: D. Appleton Company, Inc., 1940), p.140.

[2] H. R. Hamley, Relational and Functional Thinking in Mathematics Ninth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1934) pp. 52 - 58

lationship between a side and the area of a square, the sides and the area of a rectangle, the diameter and circumference of a circle, and the trigonometrical functions of an angle are examples of unrecognized functional relationships in this earlier mathematical thought.

More than two thousand years seem to have elapsed from the time that this fundamental mathematical concept was explicitly used until it received a name.[3] By the end of the seventeen century many functions had been studied and an attempt was made to define what was common to all of them. In 1698 John Bernoulli used the term in its modern meaning and gave a definition of it in 1718.[4] The increased attention toward the relationships between two or more variables laid the foundation for the discoveries of the fundamental relations between the trigonometrical functions and the logarithmic and exponential functions and, at a later date, the study of the complex numbers. In 1825, Derichlet suggested as a definition that "y is a function of x in a given interval when to each value attributed to x in this interval corresponds a unique and determined value of y".[5] His definition introduced the concept of correspondence between the variables x and y.

During the latter part of the century, three factors contributed to the increased consciousness of the part the function should play in

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[3] C. N. Miller, The Development of the Function Concept, School Science and Mathematics, vol. 28, May (1928), p. 506

[4] Ibid., p. 513

[5] Ibid., p. 514

school mathematics. The first of these was the increase in the number of texts published in analytical geometry, trigonometry and calculus which stressed the idea of the variable. The second factor had its roots in the development of technical education and the shifting of emphasis in mathematics from a cultural to a utilitarian aim. The third impetus was given by Felix Klein,[6] a leader of the reform movement in Germany, who indicated to teachers of mathematics the possibilities of using the function in developing a greater understanding and insight in mathematics. He was probably one of the first to realize that training in functional thinking in mathematics would contribute to the cultural and utilitarian aims of education and, for the past half century, his contribution to both a process of thought in mathematics and the realization of the contribution that mathematics can make toward the general education of the individual ranks as one of the outstanding achievements in pedagogy.

Klein's recommendations for improvement in learning and instruction in mathematics received little recognition in America until after the formation of the National Committee on Mathematical Requirements in 1916. In its Report on the Reorganization of Mathematics in Secondary Education which was published in 1923 the Committee specified:

> The primary purpose of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and of space which are necessary to a better appreciation of the progress of civilization and a better understanding of life and of the universe about us, and to develop

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[6] H. R. Hamley, op. cit. pp. 52 - 58

> those habits of thinking which will make these powers effective in the life of the individual.[7]

Georges in commenting on this section of the Report states:

> In the realization of the value of the formation of the functional thinking habit lies the true purpose or usefulness in mathematics.[8]

Commenting further about the importance of training in functional thinking he adds:

> Because of the universality of quantitative relationships, the habit of functional thinking is of utmost importance to the individual. Its acquisition should be the emphasized goal of every course dealing directly or indirectly with relations between things or processes.[9]

At a later date he again emphasized the importance of using the thought processes in mathematics in the development of functional thinking because:

> Mathematics as an integral part of the educative process is concerned in the development of functional thinking. Mathematics attains more nearly than any other subject the prototype of what is the best in functional thinking.[10]

Breslich in discussing the articulation of Junior and Senior Mathematics writes:

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[7] The Reorganization of Mathematics in Secondary Schools. A Report by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America, Inc. (Cambridge: Houghton Miffin Company, 1927), pp. 13 - 14.

[8] J. S. Georges, Functional Relations and Mathematical Training, School Science and Mathematics, vol. 21 (October, 1926), p. 690.

[9] Ibid., p. 690.

[10] J. S. Georges, Functional Thinking as an Objective of Mathematical Education, School Science and Mathematics, vol. 29 (May, 1929) pp. 508 - 509.

> One of the objectives of teaching mathematics is the power to do functional thinking. The training in functional thinking deserves a place in Secondary School mathematics.[11]

In the same article Breslich emphasized that, "At the Senior High School level, the function concept may then be made the central theme of mathematics."[12]

Hamley endorses functional thinking as the core of mathematics when he maintains:

> The concept of function may be regarded as the natural co-ordinating principle of all school mathematics.[13]

Christofferson advocates teaching of functional relationships in mathematics when he writes:

> Functional thinking is the major contribution of mathematics to the education of an individual. The use of functional thinking in mathematics is a method of emphasizing and obtaining insight and understanding.[14]

Georges makes a further appeal for increased emphasis upon teaching functional thinking when he writes:

> While functional thinking is not essential to human living, it is extremely essential to living in a world based upon scientific discoveries and inventions. If the art of functional thinking characterizes the scientific thinking of an individual, then the

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[11] E. R. Breslich, Articulation of Junior and Senior High School Mathematics, Eighth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1933), pp. 12 - 13.

[12] Ibid., p. 13

[13] Hamley, op. cit. p.13

[14] H. C. Christofferson, Teaching Functional Relationships in Elementary Algebra, School Science and Mathematics, vol.39 (May, 1939) p. 611.

> development of functional thinking should be a primary objective of our mathematical instruction.[15]

In a discussion on the solution of problems in elementary algebra, Kinney maintains that "Elementary algebra should be a model of functional thinking."[16]

These quotations from leading educators indicate the importance that has been placed upon the function concept in the development of mathematical thought and would justify an investigation into the growth of functional thinking by senior high school students from grade to grade.

It would be of interest, at this point, to consider the present status of the function as a unifying factor in mathematics. Five factors must be considered in determining the present status of the function in high school mathematics and in the evaluation of the impact a knowledge of that concept has upon the mathematical growth and insight on students of mathematics. In brief the factors that will be discussed in subsequent pages may be summarized under the following headings:

1) The Interpre tation of the Concept of Functionality.

2) Progress that has been made in devising texts and courses that use the functions as the unifying element.

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[15] J. S. Georges, Teaching Functional Thinking in Mathematics, School Science and Mathematics, vol. 46 (Nov. 1946), p.733.

[16] Jacob M. Kinney, The Function Concept in the Solution of Problems in Elementary Algebra, School Science and Mathematics, vol. 45 (Nov. 1945), p. 695.

3) The Formula as a Unifying Element.

4) The Tabular Method of Presenting the Function.

5) The Graphical Method of Presenting the Function.

1. The Interpretation of the Concept of Functionality

One of the major difficulties in making the function the unifying factor in mathematics appears to be that neither the concept nor its implications is fully understood nor appreciated by teachers of mathematics. Lennes claims that:

> The wide variations amongst teachers regarding the meaning of the function concept may be a contributing factor in the lack of emphasis placed upon the function concept as an instrument in teaching mathematics.[17]

Cronbach[18] made an extensive investigation as to the meaning attached to the word function by a group of algebra teachers. He found that there was by no means unanimity in the interpretation of functional situations by teachers actively engaged in the field of teaching mathematics. Hamley in commenting upon the confusion in the interpretation of the term function states that "Even mathematicians disagree as to what may legitimately be called a function."[19] Hedrich, in discussing functional

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[17] N. J. Lennes, The Function Concept in Elementary Algebra, Seventh Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1935), p. 55.

[18] Lee J. Cronbach, What the Word Function Means to Algebra Teachers, Mathematics Teacher, Vol.56 (May, 1943), pp. 212 - 218.

[19] Hamley, op. cit. p.13

thinking, states that "The greater part of mathematical teaching avoids functional ideas."[20]

Many leading authorities in the teaching of mathematics have attempted to interpret the notion of functions in the hope that a better understanding of this concept by teachers of mathematics would effect a more skillful adaptation in its use and greater recognition and interpretation by students of mathematics.

Georges commented as follows:

> Functional thinking embodies the following characteristics which must constitute the basis for the establishment of correct habits of thinking in general, namely: the recognition of dependence and the consciousness of the existence of functional relationship between quantities and ideas, the utilization of the quantitative or scientific method of procedure in the determination of data for the recognition and understanding of the nature of the relationship between the factors of a phenomena and finally the symbolic representation of the results of observation, mensuration and experimentation, and the statement of the discovered relationships in terms of mathematical symbols as precise, general laws.[21]

He states further that functionality is inherently present in the process of mensuration, degree of accuracy, approximations, variation in the size and shape of geometric figures, in similarity and congruence and that functionality is manifested in the study of variation, correspondence tabulation, graphs, evaluation, transformation, variance and limits.

Kinney states that "wherever we find the dependence of one thing

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[20] E. R. Hedrich, Functional Thinking, School Science and Mathematics, vol. 40 (April, 1940), p. 354.

[21] J. S. Georges, Mathematics in the Junior College, School Science and Mathematics, vol. 37 (March 1937), p. 307.

upon another, there we find the principle of functionality exemplified."[22]

A more recent interpretation was made by Bent and Kronenberg who say that:

> Function refers to a method of thinking: that is, in seeing and understanding interrelationships of time and space, continuity, limits, variables, motion, position and rhythm.[23]

Christofferson defines the term functional thinking as follows:

> Whenever two sets of quantities or variables are so related that knowing one of them determines the other or enables one to figure out the other, we say that there is a functional relationship between the quantities. Whenever I am able to analyze this relationship to the point where I am able to express it as a formula or a graph, then I have completed my functional thinking.[24]

In a further discussion he emphasizes the fact that functional thinking asks two questions that in the process must be answered. Such questions as upon what quantity or quantities does the one quantity in which we are interested depend, and, what is the precise manner in which it depends upon the other or others are the stepping stones for functional thinking.

Georges, commenting on the topic says that "functional thinking

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[22] J. M. Kinney, The Function Concept in First Year High School Mathematics, School Science and Mathematics, vol. 21 (June, 1921), p.541

[23] Rudyard K. Bent, Henry Kronenberg, Principles of Secondary Education, (New York and London: McCraw Hill Book Co., Inc., 1941), pp. 265 - 266.

[24] H. C. Christofferson, op. cit. p. 614

is concerned in recognition, rationalization or understanding, and manipulation of relationships between quantities."[25]

In a discussion on the solution of problems in algebra, Kinney emphasizes that:

> The function concept in its elementary phases coincides with the idea of interdependence of variables or quantities. It implies the dependence of one variable quantity upon one or more quantities.[26]

Lennes defines the term function for mathematical purposes in high schools as:

> Any mathematical expression, containing a variable x, that has a definite value when a number is substituted for x as a function of x. For the purpose of secondary mathematics we may say that a function is a quantity which varies in a definite way as some quantity involved in it varies.[27]

Reeve, in his recommendation for mathematical education in the Secondary Schools of the United States advocates "A more purposeful use of the idea of function, of dependence of one variable upon another."[28]

Although the emphasis of the function concept as a unifying factor in high school mathematics has been a slow process, some progress

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[25] Georges, Teaching Functional Thinking in Mathematics, op. cit., p. 734.

[26] Kinney, The Function Concept in the Solution of Problems in Elementary Algebra, op. cit. p. 693

[27] Lennes, op. cit. p. 55

[28] William D. Reeve, A Proposal for Mathematics Education in the Secondary Schools of the United States, Mathematics Teacher, vol. 36 (Jan. 1943), p. 16.

has been made. Breslich writes that "it is now generally agreed that training in functional thinking deserves a place in secondary school mathematics."[29]

Mayor, in a paper presented to the Illinois Association of the National Council of Teachers of Mathematics commented on the increased emphasis that is being placed by mathematics teachers upon the function concept and the greater understanding of the concept by students. Commenting on the improvement in College mathematics he said:

> I believe that the fine work that is being done toward improving the teaching of high school mathematics is having an effect on the teaching of college mathematics through students who come to college richer in relational thinking experiences.[30]

2. Progress in Devising Texts and Courses Using the Function Concept as the Unifying Element

Many outstanding authorities have reviewed numerous textbooks in mathematics to ascertain the degree to which writers of mathematical texts have incorporated the function concept as the central theme in the presentation of the mathematical content.

Robinson[31] reviewed the work of Breslich, Lennes and Hamley in

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[29] E. R. Breslich, The Administration of Mathematics in Secondary Schools (Chicago, Ill., University of Chicago Press, 1933), p.377

[30] J. R. Mayor, Relational Thinking as a Criterion for Success in College Mathematics, Mathematics Teacher, vol. 38 (February 1945), p. 65

[31] L. G. M. Robinson, An Investigation to Find the Understanding of the Concept of Functionality in Verbal Statements and in Formulas Among Students at the Junior High School Level. Unpublished Masters Thesis, The University of Manitoba, Winnipeg, 1936. pp. 7 - 11

their investigation of textbooks that were in common use prior to 1928. The consensus of opinion then was that the majority of the texts did not emphasize the function concept to any great degree. However Hamley observed that a change in the use of the function concept in textbooks published after 1928 was noticeable. He writes as follows:

> In the most recent works the function concept which had previously been largely incidental, began to show signs of becoming constitutional. It may be claimed by the writer of the modern textbook, with greater justice than formerly, that the function concept is the unifying principle of the whole course.[32]

Although a start seemed to have been made in revising textbooks the acceptance of the concept of functional thinking as exemplified in more recent mathematics texts has not been justified. Hedrich writes as follows:

> Many of the most popular textbooks show little or no appreciation of it, although most of them state in the preface that they do so. We don't want to see it in the preface. We do not want to see the word function at all. What we do want is to see more recognition of the idea in every topic in every subject of the mathematical curriculum.[33]

Banks, in discussing the function concept in secondary school mathematics and reviewing some of the more recent books in algebra writes:

> In an effort to get a fair picture of current practices relative to the development of the concept of mathematical function a dozen algebra texts of recent date, six first course and six second course were examined. Only three of the twelve employed the notation $y = f(x)$. Two of these introduced the idea casually in connection with graphing. Eleven out of the twelve

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[32]Hamley, <u>op. cit.</u> pp. 107 - 108

[33]Hedrich, <u>op. cit.</u> p. 354

> nowhere advanced the idea that an equation is an equality of function. Four of them contained no treatment of conditional versus identical equations. None of the twelve utilized the idea that in an identity both members represent the same function, while in the conditional equation each member represents a different function.[34]

The inadequacy both of methods of teaching and of textbooks construction in which the emphasis has been placed upon the function concept has prompted several suggested courses in which the function concept has been singled out as a unifying force. Among the first of these is the course outlined by Hamley.[35] For solution of problems in elementary algebra Kinney[36] outlined a suggested course. A program for teaching the concept of function was also suggested by Henderslop.[37] These and other programs undoubtedly have contributed in giving some direction as to how the function concept may be presented to give maximum effectiveness in understanding and insight. Because of the inadequacy of textbooks in the presentation of the function concept, groups of teachers have organized courses in functional mathematics. Possibly the most outstanding of these is the course in functional mathematics outlined by a

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[34] J. Houston Banks, The Function Concept in Secondary School Mathematics, Mathematics Teacher, vol. 47 (October 1954), p. 402.

[35] Hamley, op. cit. pp. 130 - 192

[36] Kinney, The Function Concept in the Solution of Problems in Elementary Algebra, op. cit., pp. 690 - 705

[37] W. G. Henderslop, A Suggested Program for Teaching the Function Concept in High School Algebra, Mathematics Teacher, vol. 39 (March 1946), pp. 121 - 126.

group of Florida teachers under the leadership of William A. Gager.[38]

This growing unrest among teachers of mathematics both over the inadequacy of mathematical content in texts and in the method in which the content is presented augurs well for a more functional mathematics in the future.

3. The Formula as a Unifying Element

The use of the formula as a factor in developing functional thinking is recommended by leading educators. Hamley, writing about the formula states:

> Its importance now seems to be generally recognized. Most modern texts follow the lead first given by Nunn and place the formula in the forefront of their teaching.[39]

Hedrich in discussing functional thinking says that:

> Modern theory of education emphasizes the integration of related thinking. The recognition of the dependence of the required quantity upon those that are given and the formulation of the precise manner in which it depends upon the given ones, is certainly functional thinking and the most vital thing in a problem. Every problem is to be made the background for functional thinking and the formula is to be the slave.[40]

Van Zyl calls the formula the core of mathematics and states that "the teacher who fails to build his algebra courses around the formula, also

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[38] William A. Gager, Funcional Mathematics - Grades Seven Through Twelve, Mathematics Teacher, vol. 44 (May 1951), pp. 297 - 301.

[39] Hamley, op. cit. p. 119

[40] Hedrich, op. cit. p. 354

fails to give his pupils a clear idea of what algebra is."[41]

Further emphasis that has been directed toward the formula in recent years appears to indicate that something is lacking in the method by which the formula is presented in the teaching of functionality. Henderslop in reflecting upon the inability of studends to experience success with verbal problems suggests that "Algebra should be made meaningful through the recognition and representation of functional relations between variable quantities."[42] He advocates that a more effective program in functionality introduced with the formula would result in a greater understanding of verbal problems. Murray, in an article on the problems of reading in mathematics maintains that the "problems of recognizing relationship expressed by the formula, equation and graph are serious."[43]

Breslich in discussing present curricular trends in mathematics presents the various aspects of the formula that should be taught as follows:

1. A formula is a shorthand rule. Many formulas should be derived by the students.
2. A formula is an algebraic way of expressing relationships, such as dependence of one variable on one or more others.

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[41] Abraham J. Van Zyl, The Formula - The Core of Algebra, Mathematics Teacher, vol. 37 (Dec. 1944), p. 368

[42] Henderslop, op. cit. p. 121

[43] Walter I. Murray, The Problems of Reading in Mathematics, School Science and Mathematics, vol. 45 (Jan. 1945), p. 58.

3. The formula may be a general solution of a given type of Problem.
4. A formula is an equation. Both are solved by the same method.
5. A formula may be represented geometrically by a graph.[44]

The problem seems to be two-fold. First we have the inadequate development of functionality in texts from the notion of the formula, second the emphasis upon manipulative skills rather than upon functional understanding. Too frequently the formula is avoided until the students have acquired some manipulative facility with algebraic expressions. The mathematical process involved then is not one of functional thinking but rather of manipulative juggling.

A more recent development is the study of the formula through the tabular and graphical method of representing a function. Neither the table of values nor the graph are new processes in the study of algebra but the recent attempt to link the table and the graph to interpret the formula and give greater insight into relational thinking is a relatively new interpretation in high school mathematics. Georges[45] enumerates in order the following methods of studying the functions: (a) the tabular method, (b) the graph and (c) the formula.

4. The Tabular Method of Presenting the Function

The use of the tabular method is becoming increasingly recognized as a means of presenting data and as an aid in the discovery of

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[44]E. R. Breslich, Curricular Trends in High School Mathematics, Mathematics Teacher, vol. 41 (Feb. 1948), p. 66.

[45]Georges, Teaching Functional Thinking in Mathematics, op. cit., p. 738.

a function. The use of the tabular method affords opportunities for studying facts from the standpoint of functional relationship. In a review of the relationships in tabular representations of numerical facts, Breslich lists the following types of tables that may be used.

a) Tables in which the corresponding facts are not related by precise mathematical laws.
b) Tables in which precise relations exist but are not stated.
c) Tables containing facts determined from scientific studies.
d) Tables that are constructed by the pupils from precise mathematical laws.
e) Tables made for the purpose of discovering laws.
f) Tables giving a series of values of polynomials corresponding to given values of the independent variable.
g) Tables using graphic representation.[46]

The ability to formulate and interpret data is functional in character and a desirable attribute in the process of functional thinking. The tabular method of representing numerical facts is an important aid in determining the nature of the variation and dependency involved and in the graphical representation of that relationship. Georges in an article on functional relations and mathematical training writes as follows:

> Tabulation and interpretation of tabular relations is of significant importance, not only for its own direct and practical application, but also because of its direct bearing on formulation is practical, and graphical representation. A table is strictly functional in character.[47]

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[46] E. R. Breslich, Developing Functional Thinking in Secondary School Mathematics, The Third Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1928), pp. 43 - 48.

[47] Georges, Functional Relations and Mathematical Training, op. cit. p. 697

Jackson in a study of the relation between the table, verbal statements, formula, and graph outlined certain objectives that are involved in the interpretation of data. He listed the following abilities:

1. Ability to perceive relationships in data evidenced by
   - 1.1 An understanding of the meaning of symbols employed.
   - 1.2 Ability to find the value of a variable when the other variable (or variables) is known.
   - 1.3 Ability to perceive the relationship between two calculated values of dependent variables when the independent variable is increased or decreased.
   - 1.4 Ability to relate steepness of slope in line graphs to rate of change.
   - 1.5 Ability to relate like characteristics in different types of data.
2. Ability to recognize the limitations of data as evidenced by
   - 2.1 Abilities to recognize and use the kids of data appropriate for particular situations.
   - 2.2 Ability to extrapolate.
   - 2.3 Ability to Interpolate.
   - 2.4 Recognition of the domain of data.
3. Use of experimentation to discover relationship as evidenced by
   - 3.1 Recognition of necessity for accurate measurement in securing reliable data.
   - 3.2 Recognition of necessity for representative sampling before making general statements.
   - 3.3 Ability to fit data (approximately) to a curve.[48]

The tabular method makes a substantial contribution to functional thinking by emphasizing the notion of variation and the notion of dependency between numerical data. It is a potent tool for the interpretation of both the formula and the graphical representation of functions.

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[48] William N. Jackson, What Can be Done in Algebra? School Science and Mathematics, vol. 42 (February 1942), pp. 142 - 156.

5. The Graphical Method of Representing Functionality

Closely allied with the tabular method of studying functions is the graphical representation of functions. The graph was one of the earliest forms appearing in textbooks for illustrating relationship between variables. Unfortunately, the emphasis was chiefly on the graphing of equations and the techniques employed contributed little toward functional thinking. The graphical representation of the formula was largely neglected. "The presentation of graphs in our textbooks," wrote Eagle, "is not at a very high standard."[49] Commenting on the use of graphs in the teaching of functional relationship, Nyberg writes that "we must distinguish for the pupil the functional graph wherein the value of the ordinate depends on the value of the abscissa."[50] Reeve, in the Final Report of the Joint Committee as published in the Fifteen Yearbook recommends that "greater emphasis should be given to the graphic study of formulas and equations."[51] Eagle, in a review of the publication Mathematics in General Education by the Progressive

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[49] Edwin Eagle, Toward Better Graphs, Mathematics Teacher, vol. 35 (March 1942), p. 127.

[50] Jas. A. Nyberg, The Teaching of Graphs, School Science and Mathematics, vol. 21 (Jan. 1920), p. 144.

[51] William D. Reeve, The Place of Mathematics in Secondary Education, The Final Report of the Joint Commission of the Mathematical Association of Teachers of Mathematics, Fifteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1940), p. 87.

Education Association quotes from their work as follows:

> The graphical method of presenting data has recently become so prevalent that the ability to interpret the common types of graphs occurring in newspapers, magazines, and books may be considered as a necessary extension of the basic reading skills.[52]

Hartung and Erickson in an analysis of activities that should be considered in graphical representation states that:

> If graphical methods are to be made more effective, the student must learn not only the purpose of different forms of representation of data but also certain principles which influence the interpretation.[53]

Hamley writes as follows:

> The graph is one of the most useful instruments of functional thinking. The conception of the graph . . . . is essentially a mode of thinking and as such should be an integral part of the whole course.[54]

By nature, the graph is essentially functional and Murray enumerates the following abilities involved in understanding graphical representation:

a) To understand the title.
b) To understand the meaning of the reference axes.
c) To understand the scale and symbols of graphs.
d) To recognize the change in one variable corresponding to a change in the other variable.
e) To get the facts portrayed in the graph.

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[52]Eagle, op. cit. p. 127

[53]Maurice L. Hartung and Robert L. Erickson, Graphical Methods in Science and Mathematics Teaching, School Science and Mathematics, vol. 50 (Feb. 1950), p. 202.

[54]Hamley, Relational and Functional Thinking, op. cit. p. 118

f) To interpret the facts derived from the graph.
g) To restrain the interpretation within reasonable limits provided by the graphical data.[55]

Breslich attaches an additional importance to graphical representation in his review of curricular trends when he writes:

> Instruction in graphical techniques is important because they are widely used to clarify statistical facts, to make comparisons of data, and to disclose trends. The following abilities should be developed.
> a) To read, understand, and interpret graphs.
> b) To make comparisons between data represented by bar graphs.
> c) To recognize trends reflected in line graphs and to note central tendencies.
> d) To construct simple statistical graphs mainly for the purpose of attaining a better understanding of graphical techniques.[56]

Hartung and Erickson list three main types of activities in connection with the use of graphs. These they list as follows:

1) A graph may be read to determine at least approximately the value of an item of data.
2) The relationship between two items of data may be observed from a graph.
3) More general relationships or trends among several items of data may sometimes be inferred by the aid of a graph.[57]

By the above references to writings of many outstanding educationalists the position of the function at the level of high school mathematics is receiving serious consideration. Furthermore, an attention has

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[55]Murray, op. cit. p. 58

[56]Breslich, op. cit. p. 67

[57]Hartung and Erickson, op. cit. p. 200

been drawn to the chief weaknesses in the development of the function concept, and to some of the improvements in relational thinking that have been effected through the effort of outstanding teachers and writers on the subject and to indicate some of the current trends toward a better understanding of the concept of functionality. Further assessment of the degree of improvement in the understanding of the concept may be found either by standardized testing or by investigations along the line which is presented in the following pages. There appears to be a dearth both of standardized tests dealing with the functions in mathematics and of investigations into the ability to do functional thinking by students at the high school level.

CHAPTER II

THE DEVELOPMENT OF A TEST FOR EVALUATING GROWTH OF THE FUNCTION

It is the purpose of this chapter to outline the procedure that was followed in constructing the test used for this investigation, in the administration and scoring of the test.

The Construction of the Test - The original testing program began during the 1952-53 school year. It was considered desirable that a test or tests be constructed covering the following phases of competency in the function concept.

1. Recognition of functional relationship between two or more quantities and the ability to express that relationship as a formula. There were forty questions in the first test and the range of questions covered a much larger area than do the questions of sub-test II in the final draft of the test. The test was primarily experimental to determine the possible range of functional relationship that might be included in the final draft and the procedure to be followed throughout the final test. The type of questions used is illustrated by the following example:

"An express agent wishes to determine the total weight (T) of a number of articles each weighing the same amount. What information would be required in order to determine the total weight?"

The students were expected to state:

1. The weight of each article (W)
2. The number of articles (N)

The expected formula is $T = WN$

2. Recognition of how a change in one variable affects another. The attempt in this phase of the testing program was to determine in what areas and to what extent the students were able to visualize and interpret how changes in one or more variables affects another variable. In this test there were sixteen groups of five questions each covering such areas as direct, inverse and joint variation. Multiple choice type questions were used for this test and the students were instructed to circle the correct answer. The answers to the questions represented either the number of times or the amount by which one variable will change in the light of a change in another variable. An example of the type of question used is as follows:

The total cost (C) of 7 pounds of meat at "p" cents per pound may be represented by the formula $C = 7p$. What would be the change in C if:

(a) p is doubled?	2, 3, 4, 5, 6
(b) p is increased by 10 cents per pound?	50, 60, 70, 80, 90.

3. Changing the subject of a formula. There were thirty questions in this test and in each question the students were given a formula and instructed to re-arrange the formula in terms of another variable expressed in the formula. An example of the type of question used is given below:

Given Formula	Instructions	New Formula
$C = np$	Write the relationship in terms of p	$p =$

4. Finding the formula from a table of values. There were twenty-five questions in this test covering a wide range of relationship. For each set of values the students were instructed to write the formula showing the relationship between the last value in each group and the remaining values of the group. An example of the type of question that was used is as follows:

a	2	8	6	9	8
b	4	12	4	13	8
c	3	10	5	11	8

Formula c =

5. Functions as represented by graphs. This test consisted of six graphs which had been constructed and the students were instructed to study each of these graphs and write the formula which would correctly identify each graph. The graphs represented were those of direct variation, of direct variation plus a constant, inverse variation, of squares, of square roots, and of cubes.

6. Some functions arising from descriptive statements. This test consisted of a number of true-false questions which included also a few problem situations. An example of the type of question used is as follows:

If a tree 40 feet tall casts a shadow of 20 feet, then a stick 10 feet tall casts a shadow of 5 feet. ( )

In the space to the right the students were instructed to write either T or F depending whether the question was either true or false.

These tests were given to two classes of grade ten students, two classes of grade eleven students and to one class of grade twelve students. In each case the tests were administered by the same teacher who was also the mathematics teacher for these groups of students. Each of these tests was then scored with one mark assigned for each correct response. In the case of sub-test II where each question was divided into five parts one mark was assigned to each part. In the initial scoring of the test results, two score sheets were made for each of the six tests. The test papers were arranged in order of the attained scores with the highest score on top. The students' names were typed along the side of the score sheet and the question numbers along the top. For each incorrect response or omission of a question, the appropriate question number was blacked out. This score sheet served two purposes in that it gave a general pattern of the difficulty of the individual items on each test and secondly, it facilitated comparison of performances by students and formed the basis for comparative analysis of test items. A second score sheet was devised for the purpose of comparing the top and bottom twenty-seven percents of the population tested. Along the side of the score sheet was typed the name of the student who attained the highest score on the test and immediately below appeared the name of the student who had the lowest score on the test. For the next group the student obtaining the second highest score was coupled with the student obtaining the second lowest score. This grouping was repeated until the top and bottom twenty-seven percents of the population tested were grouped. Along the top of the score sheets was typed the number of the test questions. Any question answered incorrectly by the student was blocked out opposite his name. For this initial testing program no attempt was made to determine the reason for or nature of the error. Each test was regarded as an ability test and no time limit was imposed. Omissions on the test were recorded as errors. From this second score sheet it was possible to determine the number and percentages of correct responses for both the top and bottom twenty-seven percents of the population for each item in the test. Flanagan's Table[1] was used to determine the item validity coefficients from the percentages of successes in the upper and lower twenty-seven percents. An interesting feature of the initial tests was the range of item validity coefficients in each of the tests that were administered in the initial testing program. For Test I, "The Ability to Recognize Functional Relationships between two or more quantities,"[2] the item validity coefficients ranged from .11 to .74 with approximately two-thirds of the items between .50 and .59. For the second test, "Recognition of how a change in one variable affects another," the range was from -.23 to .80. On four items appearing in the test a negative item validity coefficient was obtained. Obviously, on these four items, the students in the lower twenty-seven percent of the population tested had a greater percentage of successes than that of the

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[1]Robert L. Thorndike, Personnel Selection (New York: John Wiley & Sons Inc., 1949), pp. 348 - 351.

[2]Test constructed by the Author.

students in the upper group. On two items of the test the item validity coefficients were .00. These six items were not used in the final draft of the test. The majority of the items on this test ranged in validity between .50 and .69. The items included in this test seemed to have very high discriminating powers as evidenced by the high item validity coefficients and the comparison of performance by comparative analysis. For Test III, "Changing the Subject of a Formula," the range was from .08 to .69 with the majority of the items having an item validity coefficient between .50 and .69. For Test IV, "Deriving a Formula from a Table of Values," the validity coefficients ranged between .10 and .88 with the greatest concentration between .40 and .49. Test V emphasized the ability to interpret graphs. The item validity coefficients ranged between .00 and .89. For Test VI, "Some Generalized Aspects of Functionality," the range of item validity coefficients was between -.21 and .80 with the majority of the test items having validity coefficients between .60 and .88.

By comparative analysis each question was retained or discarded depending on how the score compared. If it was found that the students who had obtained the higher scores were able to reply correctly to the question whereas the students with the lower score failed, then, for the time being, the question was retained. If there seemed to be little differentiation between the responses of students in the upper and lower levels of achievement or, if the students in the lower levels actually had a better score on a particular question, that question was eliminated.

For each question retained an item validity coefficient was determined. In addition, a difficulty index was obtained for each of these questions and those that appeared too difficult were not included in the final draft of the test. For the final test, the items were selected (a) on the basis of their discriminating power between students as evidenced by the method of comparative analysis, (b) by the validity coefficient of the individual items as determined by Flanagan's Table, and (c) by determining the difficulty index at the grade ten level for each item in the original tests. It was decided that if the item validity coefficient was greater than .15 and the difficulty index between five and seventy per cent the item was retained for the final draft. From a table of the significance level of correlation coefficients it is easily seen that for the number of degrees of freedom in our case, a coefficient of .15 is significant at between the five and ten per cent levels of confidence. We feel justified therefore in utilizing a coefficient of .15 as the lower level of acceptance for discriminating purposes of our individual items. A further selection was necessary in order to decrease the time limit for administration of the final test. In the final analysis, seventy items were selected whose item validity coefficients ranged from .25 to .77 and whose difficulty index at the grade ten level of performance ranged between eighteen to sixty-seven per cent difficulty level. It was felt that items beyond these ranges of difficulty would be too difficult for a grade ten student at one range of the scale and much too easy for a grade eleven or twelve student at the lower range of the difficulty scale. Throughout the test no single item was answered correctly by all students and no single item was answered incorrectly by all students.

In September, 1953 the final draft of the test was administered to one grade ten, one grade eleven and one grade twelve class comprising ninety-six students. All but one of these students completed the test within fifty minutes. Ninety of the students completed the test within forty-five minutes and the fastest student required twenty-four minutes to complete the test. As a result the time limit for the test was set at fifty minutes.

During the same month, the ninety-six students wrote Form A of the Foust-Schorling Test of Functional Thinking in Mathematics.[3] The scores obtained by the students on this standardized test were correlated with the scores obtained on the test used for this investigation. The correlation between these two tests was estimated as .81. In February, 1954 the function test used for this investigation was again administered to the same ninety-six students and the scores obtained on this second testing were correlated with the scores obtained on the first testing. The reliability coefficient for these two sets of scores was .86.

The original six tests were examined for curricular content by Dr. H. L. Stein, Faculty of Education, University of Manitoba and by three teachers actively engaged in the field of teaching high school

[3]Judson W. Foust, Raleigh Schorling, Foust-Schorling Test of Functional Thinking in Mathematics, Form A (New York: World Book Company, 1942).

mathematics. The final test was re-examined by Dr. H. L. Stein. This test now formed the basis for the investigation.

The Administration of the Test - At the end of May 1954 the test was administered to twenty-one classes in Winnipeg and suburban schools. Five of the six schools in which the test was given were Winnipeg schools. In the suburban school the test was given to only three grade ten classes. Table I, page 35, shows that the test was administered to seven grade ten classes, nine grade eleven classes and five grade twelve classes. In the selection of the classes for this investigation care was taken to see that the group selected was a representative cross section of the high school population in order to control, at least to some degree, the intelligence factor. For the purpose of this study it was deemed advisable to get as wide a range as possible from both the standpoint of instruction and of student personnel participating in the investigation.

From the standpoint of the school's estimate of the mathematical ability of the classes selected for this investigation two superior, three average and two below average classes were selected in grade ten, In grade eleven two superior, five average and two below average were included and for grade twelve one superior, two average and one below average class was selected. For each grade the number of students included in the superior and below average classes was approximately the same.

In each school, the testing was conducted by the mathematics teachers for the respective classes who, in each case, had some previous

TABLE I

DISTRIBUTION OF STUDENTS ACROSS SCHOOLS AND GRADES IN WHICH FINAL TEST OF FUNCTION CONCEPT WAS ADMINISTERED

School	Grade	Number of Classes	Girls	Boys	Total
I	X	3	51	43	94
II	X	2	25	38	63
V	X	2	57	--	57
II	XI	2	29	35	64
III	XI	1	25	--	25
IV	XI	2	25	30	55
V	XI	2	20	21	41
VI	XI	2	--	65	65
II	XII	1	2	17	19
III	XII	2	28	17	45
IV	XII	1	14	22	36
V	XII	1	7	24	31
TOTAL		21	283	312	595

The Table reads: In school I there were three classes of grade ten comprising fifty-one girls and forty-three boys or a total of ninety-four students, etc.

experience in standardized testing. Specific instructions[4] were issued to these teachers outlining the procedure to be followed in this particular testing program. The purpose of the instructions was to standardize as much as possible the procedure for the testing program and to equalize the conditions under which the students participated in the investigation. Each section of the test is preceded by a set of instructions which were to be read by the students and carefully followed for that particular sub-test. In no case was the teacher to interpret these instructions for the students or permitted to comment on any individual question appearing in the test. The time for the test was set at fifty minutes which did not include the time required by the teacher to read the preliminary instructions to the students. At the end of the testing period, the booklets were collected by the teacher and packaged.

Scoring and Recording Test Results - In scoring the test, one mark was assigned for each correct response. The responses of the students tested were recorded on three score sheets. One score sheet was used for each grade. These score sheets formed the basis for eighteen additional score sheets, one for each grade and one for each of the six sections of the test. On these score sheets the total correct responses for boys, girls and total of boys and girls were recorded for each question of the six sub-tests. Each score sheet also contained a

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[4]See the Appendix for the complete instructions.

distribution summary of the total number of correct responses for that particular sub-test and at the end of each sub-test a combined distribution summary for the three grades. These score sheets were later used in comparing the performance by the students in each sub-test on a grade-sex basis, on a grade to grade basis, and, in estimating the possible growth in the function concept from grade to grade.

CHAPTER III

AN ANALYSIS, BY MEANS OF THE TEST, OF THE STUDENTS' ABILITIES TO USE THE FUNCTION CONCEPTS IN THE ENTIRE TEST

It is the purpose of this chapter to present the statistical information that has been obtained as a result of the testing program, and to examine this information in the light of the investigation that is being conducted. In analyzing the statistics we assumed the null hypothesis that:

1. There is no difference between the respective abilities of girls and boys.

2. There is no difference between grades in the students' abilities to think in terms of functionality.

Tables have been constructed showing the distribution of scores on the entire test by grade and sex and various combinations of these tables have been used in an attempt to determine the possible improvement by grade and sex in the ability to do functional thinking.

Table II, page 39, shows a distribution of the scores on the final test for the entire group of students tested for this investigation. The scores were grouped in intervals of five and for each interval the scores for the girls, boys and total have been recorded for each grade. The last column is a summary of the scores for the three grades for the girls, boys and total for each interval.

The actual range of the scores for the grades was from a total score of eleven to sixty-two for grade ten, thirteen to sixty-eight for grade eleven, and from twenty-six to sixty-eight for grade twelve. Nine per cent of the students tested at the grade eleven level and sixteen per cent of the students at the grade twelve level received scores greater than the highest score attained by a grade ten student. At the lower end of the scale there were approximately twelve per cent of grade ten and eight per cent of the grade eleven students who received scores lower than the lowest grade twelve score. There were students at the grade ten level who attained a score equivalent to, or better than that attained by eighty-four per cent of the grade twelve students. The top score in grade eleven was the same as the top score for grade twelve.

TABLE II

DISTRIBUTION OF SCORES ON TEST
BY GRADE AND SEX

Score	Grade X			Grade XI			Grade XII			Grades X-XII		
	G	B	T	G	B	T	G	B	T	G	B	T
66-70	-	-	-	1	6	7	-	10	10	1	16	17
61-65	1	-	1	5	20	25	2	17	19	8	37	45
56-60	10	1	11	13	24	37	9	20	29	32	45	77
51-55	9	10	19	9	24	33	7	15	22	25	49	74
46-50	24	16	40	11	22	33	9	9	18	44	47	91
41-45	17	12	29	10	17	27	10	2	12	37	31	68
36-40	19	12	31	13	14	27	11	4	15	43	30	73
31-35	24	11	35	13	14	27	-	2	2	37	27	64
26-30	11	11	22	7	7	14	3	1	4	21	19	40
21-25	12	4	16	14	2	16	-	-	-	26	6	32
16-20	5	4	9	3	-	3	-	-	-	8	4	12
11-15	1	-	1	-	1	1	-	-	-	1	1	2
TOTAL	133	81	214	99	151	250	51	80	131	283	312	595

The table reads as follows: At the grade ten level there were no students that received a score between 66 and 70, in grade eleven there was one girl and six boys, a total of seven students, that received a total score in this interval, etc.

For comparative purposes a further analysis of the results was necessary. Table III is a summary of the grade-sex and grade means for the entire test used in this investigation.

TABLE III

SUMMARY OF THE MEANS FOR THE GIRLS AND BOYS FOR EACH OF THE HIGH SCHOOL GRADES AND FOR THE ENTIRE PROGRAM

Grade	Sex	Number of Students	Mean	$S.E._m$
X	Girls	133	39.09	.94
	Boys	81	39.05	1.19
	Girls & Boys	214	39.09	.73
XI	Girls	99	41.16	1.28
	Boys	151	48.79	.93
	Girls & Boys	250	45.74	.80
XII	Girls	51	46.72	1.21
	Boys	80	55.87	1.09
	Girls & Boys	131	52.31	.86
X-XII	Girls	283	41.18	.61
	Boys	312	48.08	.69
	Girls & Boys	595	44.78	.55

This table should be read as follows: In grade ten there were 133 girls who participated in the investigation whose adjusted mean score was 39.09 ± .94 standard error of the mean, etc.

An analysis of Table III, page 40, reveals an improvement in the ability to do functional thinking by both girls and boys from grade to grade. The table indicates that the mean score for the girls and boys is practically the same at the grade ten level. For the girls, the improvement in the mean score from grade ten to grade eleven is 2.07 score points and from grade eleven to grade twelve the increase is 5.56 score points. On the other hand the improvement in the mean score for the boys is much more pronounced since the improvement in the mean score from grade ten to grade eleven is 9.74 score points and from grade eleven to grade twelve 7.08 score points.

The mean scores for each group of students in Table III is only a sample mean and is only one of the possible values that might arise through a random sampling of the high school population. Table IV, page 42, indicates the probable range within which all of these possible means will lie. In each case the range is for a confidence level of .01.[1]

At the grade ten level there is considerable overlapping in the distribution of scores for the girls and boys. At this grade level there appears to be no significant difference in the performance of girls and boys in their ability to do functional thinking. A further examination of Table IV indicates that the range in mean scores is

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[1]Henry E. Garrett, Statistics in Psychology and Education, (New York: Longmans, Green and Company, 1947), p. 190

TABLE IV

THE PROBABLE RANGE WITHIN WHICH ALL POSSIBLE SAMPLE MEANS WILL LIE AT THE ONE PERCENT LEVEL OF CONFIDENCE

	Grade X	Grade XI	Grade XII	Grades X - XII
Girls	36.69 - 41.49	37.89 - 44.43	43.63 - 49.81	39.62 - 42.74
Boys	36.01 - 42.09	46.41 - 51.17	53.09 - 58.65	46.32 - 49.84
Total	37.13 - 40.95	43.70 - 47.78	50.31 - 54.51	43.38 - 46.18

This table should be read as follows: In grade X the range for the possible sample means for the girls at the one percent level of confidence lies between 36.69 and 41.49, etc.

practically the same for the girls and boys. It should be noted that there is a possible overlapping of the mean scores for the girls from grade to grade. This is particularly noticeable for grades ten and eleven. On the other hand the possible mean scores for the boys do not overlap. The improvement from grade to grade has been greater for the boys as compared to the improvement in the mean scores for the girls.

A further comparison of the performance between the girls and boys within the grade is presented in Table V, page 43. This table presents the following additional information: The differences between the mean scores of the girls and boys for each grade, the standard deviation of the scores for the girls and boys in each grade, the differences between the standard errors of the means, the critical ratio and whether or not the differences between the mean scores are significant at the .01 confidence level.

TABLE V

THE GRADE-SEX COMPARISON OF THE MEAN SCORES WITHIN THE GRADE

Grade	Sex	No. Cases	Mean Score	D.	S.D.	$S.E._m$	$S.E._{diff.}$	C.R.	Signif. level
X	Girls	133	39.09		10.9	.94			
	Boys	81	39.05	.04	10.7	1.19	1.51	.03	N.S.
XI	Girls	99	41.16		12.8	1.28			
	Boys	151	48.79	7.63	11.5	.93	1.58	4.19	.01
XII	Girls	51	46.72		8.65	1.21			
	Boys	80	55.87	9.15	9.77	1.09	1.62	5.65	.01
X-XII	Girls	283	41.18		10.2	.61			
	Boys	312	48.08	6.90	12.2	.69	.92	7.50	.01

This table should read as follows: In grade ten the number of cases for girls and boys is 133 and 81 respectively, the mean scores are 39.09 and 39.05, the difference between the respective mean scores is .04, the standard deviation of the scores for the girls and boys is 10.9 and 10.7 respectively, the standard error of the difference between the means is 1.15, the critical ratio is .02, and the difference between the means is not significant at the .01 confidence level. The statistical constants used have their usual meaning and have been determined in the usual manner. D represents the difference between the mean scores, S.D. the standard deviation of the distribution, $S.E._m$ refers to the standard error of the means, $S.E._{diff}$ means the standard error of the difference between two means, C.R. the critical ratio and finally the last colum refers to the confidence level at which a difference between the means does exist. A complete list of the statistical formulas used in this investigation appears in the Appendix.

In a normal distribution approximately 68.26 per cent of the scores lie between the mean and plus and minus one standard deviation of the scores.[2] In grade ten this percentage of the scores would lie between 28.19 - 49.99 and between 28.35 - 49.75 for the girls and boys respectively. In grade eleven the range is between 28.36 - 53.96 and 37.29 - 60.29 respectively, and in grade twelve between 38.07 - 55.37 and 46.10 - 65.64 for the girls and boys respectively. For the three high school grades the range for the girls is 30.98 - 51.38 as compared to a range between 35.88 - 60.28 for the boys. A significant factor as revealed by this information is that whereas the performance for the boys and girls is the same at the end of grade X, the girls at the end of grade XII are not performing quite at the level of the boys at the end of grade XI.

Garrett[3] states that for a distribution as large as the one used in this investigation a critical ratio of 1.65 is required at the .10 confidence level. The critical ratio for the girls and boys at the grade ten level was .02. This critical ratio falls far below the required ratio of 1.65 at the .10 confidence level, and consequently the difference between the means of the girls and boys at the grade ten level is not significant. There is no difference in the mean performance of the girls and boys at the end of grade ten in their ability to do functional thinking. On the other hand, we have a critical ratio of 4.19 and 5.65 in grades

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[2] *Ibid.*, p.115

[3] *Ibid.*, p. 190

eleven and twelve respectively. Critical ratios of 2.59 and 2.62 are required at the .01 level. Since both 4.19 and 5.65 are far above the required ratios the corresponding differences of 7.63 and 9.15 are significant at the .01 level and there is a marked difference in the ability to do functional thinking between the boys and girls in grades eleven and twelve.

In table VI a further comparative study was made of the mean

TABLE VI

COMPARATIVE MEAN SCORES FOR THE GIRLS FROM GRADE TO GRADE

Grade	Number of Cases	Mean Score	D	S.D.	$S.E._m$	$S.E._{diff}$	C.R.	Level of Significance
X	133	39.09		10.9	.94			
XI	99	41.16	2.07	12.8	1.28	1.59	1.30	N.S.
XI	99	41.16		12.8	1.28			
XII	51	46.72	5.56	8.65	1.21	1.76	3.16	.01
X	133	39.09		10.9	.94			
XII	51	46.72	7.63	8.65	1.21	1.53	4.98	.01

This Table reads as Table V.

score for the girls from grade to grade. The statistical information compiled in this table shows that the difference between the mean scores for the girls in grades ten and eleven is 2.07 and the standard deviation of the differences between the two means is 1.59. With a critical ratio of 1.30 we cannot say that the difference between the mean scores for the girls in grades ten and eleven is high enough to make the results statistically significant. A critical ratio of 1.96 is required to make the difference between the means significant at the five per cent level of confidence.[4] We must accept the null hypothesis and state that no true difference exists in the ability of the girls at the grade ten and eleven level to do functional thinking. A comparison of the difference between the mean scores for the girls in grades eleven and twelve shows that there is a difference of 5.56 score points in their respective means. This gives rise to a critical ratio of 3.16. If a critical ratio is 2.58 or more, we reject the null hypothesis since only once in a hundred trials would a larger difference arise from sampling errors, when a true difference is zero.[5] We may therefore be confident that there are at least ninety-nine chances out of a hundred that the difference was greater than zero and that the grade twelve girls are really superior to the grade elven girls in mean attainment. Similarly, a comparison of the mean performances of the girls in grades ten and twelve shows a critical ratio of 4.98. We may reject the null hypothesis at .01 level of confidence and be confident that there are practically one hundred chances out of a hundred that the difference was greater than zero.

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[4] Ibid., p. 115

[5] Ibid., p. 203

Table VII compares the mean scores attained by the boys at various grade levels. The table shows that there is a mean difference of 9.74 score points between the two means for the boys in grades ten and eleven. This gives rise to a critical ratio of 6.20 and we may

TABLE VII

COMPARATIVE MEAN SCORES FOR THE BOYS FROM GRADE TO GRADE

Grade	Number of Cases	Mean Score	D	S.D.	S.E.$_m$	S.E.$_{diff}$	C.R.	Level of Significance
X	81	39.05		10.7	1.19			
XI	151	48.79	9.74	11.5	.93	1.57	6.20	.01
XI	151	48.79		11.5	.93			
XII	80	55.87	7.08	9.77	1.09	1.43	4.95	.01
X	81	39.05		10.7	1.19			
XII	80	55.87	16.82	9.77	1.09	1.61	10.45	.01

This Table reads as Table VI.

reject the null hypothesis and be confident that a true difference does exist in the mean performance between the boys at the grades ten and eleven levels in their ability to do functional thinking. We may say the boys in grade eleven are superior to the boys in grade ten in their mean attainment. Comparison of the mean performances of the boys in grades eleven and twelve shows that there is a difference of 7.08 in their means. With a critical ratio of 4.95 we may again reject the null hypothesis and be confident that there are practically one hundred chances out of a hundred that the difference was greater than zero. We may say that the boys in grade twelve are really superior to the boys in grade eleven in their ability to do functional thinking.

Table VIII reveals that for the entire group of students tested there is a constant improvement from grade to grade. There was only a

TABLE VIII

COMPARATIVE MEAN SCORES FOR GIRLS AND BOYS FROM GRADE TO GRADE

Grade	Number of Cases	Mean Score	D	S.D.	$S.E._m$	$S.E._{diff}$	C.R.	Level of Significance
X	214	39.09						
XI	250	45.74	6.65	12.7	.80	1.13	5.88	.01
XI	250	45.74		12.7	.73			
XII	131	52.31	6.57	9.9	.86	1.17	5.61	.01

This Table reads as Table VII.

difference of .08 between the differences of the means scores from grade to grade and with critical ratios of 5.88 and 5.61 respectively we may reject the null hypothesis and be confident that a true difference does exist in the mean performance between students from grade to grade.

CHAPTER IV

THE EXTENT TO WHICH THE STUDENTS ARE ABLE TO USE THE FUNCTION CONCEPT IN EACH OF THE SIX SECTIONS OF THE TEST.

In the previous chapter the test as a whole was considered in order to determine to what extent the entire group of questions were mastered by the students at the various grade levels. Comparisons were made between the mean performances of the girls and boys both within and between grades. It is the purpose of this chapter to present the performances of the students on each of the six phases which comprised the entire testing program. Each phase attempts to test a particular ability inherent in the program. The following abilities were tested:

1. The ability to recognize functional relationships between two or more quantities. In sub-test I some generalized aspects of functionality were tested.

2. Recognition of functional relationship between two or more quantities and the ability to express that relationship as a formula.

3. Recognition of how a change in one variable affects another.

4. The ability to derive a formula from a table of values.

5. The ability to change the subject of a formula.

6. Functional relationship as expressed by graphs.

In order to estimate the quality of the performance of the high school students in each of the six sub-tests, three sets of tables were constructed. The first three sets, comprising Tables IX, X and XI, compares the mean performances of the girls and boys for each of the three high school grades. In each table the following information is presented:

(a) The number of questions in each set.

(b) The number of girls and boys participating in the test.

(c) The mean scores for each sex.

(d) The differences between the mean scores.

(e) The standard deviation of the mean scores for each sex on

TABLE IX

COMPARISON OF THE MEAN PERFORMANCES FOR THE GIRLS AND BOYS OF GRADE X IN EACH OF THE SUB-TESTS

Sub-Test	Sex	Number Cases	No. Questions	Mean Score	D.	S.D.	$S.E._m$	$S.E._{diff}$	C.R.	Sig. Level
I	Girls	133	10	7.48		1.25	.11			
	Boys	81	10	7.58	.10	1.36	.15	.18	.55	N.S.
II	Girls	133	11	5.55		2.95	.26			
	Boys	81	11	5.02	.53	3.01	.33	.42	1.26	N.S.
III	Girls	133	24	14.77		5.04	.44			
	Boys	81	24	14.59	.18	4.62	.51	.67	.27	N.S.
IV	Girls	133	6	1.79		1.71	.15			
	Boys	81	6	1.87	.06	1.94	.22	.26	.23	N.S.
V	Girls	133	5	1.76		1.26	.11			
	Boys	81	5	1.48	.28	1.14	.13	.17	1.64	N.S.
VI	Girls	133	14	7.64		2.93	.25			
	Boys	81	14	8.52	.88	3.14	.35	.53	1.66	N.S.

The Table reads as Table VIII.

each of the sub-tests.

(f) The standard deviation of the means.

(g) The Standard deviation of the differences of the means.

(h) The critical ratio

(i) Whether or not the respective scores are significant at the .01 level of confidence.

Table IX, page 50, reveals that there is no statistical significant difference between the mean performances of the girls and boys at the grade X level. This confirms information that was obtained in the previous chapter but, in addition, reveals the fact that there is no one area of the entire test in which either the girls or boys have a marked difference in their mean performance. For no sub-test can we say that at the .01 level of confidence the mean performance of one or the other sex is better. The greatest variation appears to be in sub-tests V and VI where critical ratios of 1.64 and 1.66 respectively indicates that at the .10 level of confidence there may be a difference in the mean performance of the girls and boys in these areas.

Table X, page 52, presents an entirely different situation. In five out of the six sub-tests the mean performance of the boys outranks that of the girls. With critical ratios of 3.21 or better, we may be very confident that the boys are really superior to the girls in mean attainment. The only area for which this claim cannot be made is for sub-test V. Apparently there is no significant difference in the mean performance of the boys and girls to derive a formula from a table of values. With one exception, it would seem probable that there is a real gain by the boys of the population tested to do functional thinking at the grade eleven level.

TABLE X

COMPARISON OF THE MEAN PERFORMANCES FOR THE GIRLS AND BOYS OF GRADE XI IN EACH OF THE SUB-TESTS

Sub-Test	Sex	No. Cases	No. of Questions	Mean Score	D	S.D.	$S.E._m$	$S.E._{diff.}$	C.R.	Sig. level.
I	Girls	99	10	7.44		1.57	.16			
	Boys	151	10	8.14	.70	1.36	.11	.19	3.68	.01
II	Girls	99	11	6.29		2.89	.29			
	Boys	151	11	7.46	1.17	2.41	.19	.34	3.44	.01
III	Girls	99	24	14.95		5.35	.53			
	Boys	151	24	17.82	2.87	4.67	.38	.65	4.41	.01
IV	Girls	99	6	2.26		1.85	.18			
	Boys	151	6	3.03	.77	1.97	.16	.24	3.21	.01
V	Girls	99	5	2.57		1.63	.16			
	Boys	151	5	2.92	.35	1.46	.12	.20	1.75	N.S.
VI	Girls	99	14	7.69		4.18	.42			
	Boys	151	14	0.38	1.69	3.79	.31	.52	3.25	.01

This Table reads as Table IX.

A comparison of the scores for the girls and boys at the grade twelve level is shown in Table XI, page 53. An examination of this table and a comparison with the mean scores of the previous table shows that the girls and boys, with one exception, have improved their respective mean scores. In the first four sub-tests the girls have made the greater gain as in each case the difference in the mean score between the girls and boys has decreased to a marked extent. In the first two sub-tests, where there was a significant difference in the respective mean performances between the girls and boys at the grade eleven level, there is not a significant difference in their mean performances at the grade twelve level. The table shows that the results are not statistically significant at the .01 level of confidence. In Sub-test V there has been an increase in the mean scores for both sexes but the difference between

TABLE XI

COMPARISON OF THE MEAN PERFORMANCES FOR THE GIRLS AND BOYS OF GRADE XII IN EACH OF THE SUB-TESTS

Sub-Test	Sex	No. Cases	No. of Questions	Mean Score	D	S.D.	$S.E._m$	$S.E._{diff.}$	C.R.	Sig. Level
I	Girls	51	10	8.17		1.00	.14			
	Boys	80	10	8.36	.19	1.01	.11	.17	1.18	N.S.
II	Girls	51	11	7.67		1.90	.27			
	Boys	80	11	8.31	.64	1.99	.21	.34	1.88	N.S.
III	Girls	51	24	18.73		3.82	.53			
	Boys	80	24	20.63	1.90	3.15	.35	.63	3.01	.01
IV	Girls	51	6	2.63		2.07	.29			
	Boys	80	6	4.27	1.64	1.84	.20	.35	4.57	.01
V	Girls	51	5	3.49		1.97	.28			
	Boys	80	5	4.12	.63	1.39	.16	.31	2.03	N.S.
VI	Girls	51	14	6.02		4.77	.66			
	Boys	80	14	9.98	3.96	3.86	.43	.78	5.08	.01

This table should be read the same as Table X their means has not been sufficiently great for a confidence level of .01. In three sub-tests the boys have maintained their superiority of performance. In particular, these areas are as follows:

(a) The recognition of how a change in one variable affects another.

(b) The ability to change the subject of a formula, and

(c) Functional relationship as expressed by graphical representation.

The second group of tables comprises Tables XII, XIII and XIV. The previous set of tables compared the mean performances of the boys and girls at each of the three grade levels. The second group of tables compares the mean performance for each sex and for the total population tested from grade to grade.

Table XII, page 55, reveals that in Sub-test I the girls of grade ten had a higher mean score than the girls of grade eleven. In Sub-test VI, the girls of grade eleven had a higher mean score than the girls of grade twelve. In the latter case, the decrease is sufficiently alarming to make it statistically significant, at least, at the .05 level of confidence. In comparing the mean scores of the girls in grades ten and eleven, the ability to derive a formula from a table of values shows the only real gain made by the grade eleven girls at the .01 confidence level. The greatest gain has been made by the grade twelve girls as in four of the six sub-tests the differences of the mean scores are significant at the .01 level of confidence. There appears to be no significant change in grade twelve student's ability to change the subject of a formula and in the test on graphical representation of a function there is actually a decrease in the mean performance.

TABLE XII

THE PERFORMANCE OF THE GIRLS ON EACH OF THE SUB-TESTS FROM GRADE TO GRADE

Sub-Test	Grade	Number Cases	Mean Score	D	S.D.	S.E.$_m$	S.E.$_{diff.}$	C.R.	Significance Level
I	X	133	7.48		1.25	.11			
	XI	99	7.44	-.04	1.57	.16	.19	-.21	N.S.
	XII	51	8.17	.73	1.00	.14	.21	3.48	.01
II	X	133	5.55		2.95	.26			
	XI	99	6.29	.74	2.89	.29	.39	1.87	N.S.
	XII	51	7.67	1.38	1.90	.27	.39	3.54	.01
III	X	133	14.77		5.04	.44			
	XI	99	14.95	.18	5.35	.53	.68	.27	N.S.
	XII	51	18.73	3.78	3.82	.53	.75	5.04	.01
IV	X	133	1.79		1.71	.15			
	XI	99	2.26	.47	1.85	.18	.23	2.04	N.S.
	XII	51	2.63	.37	2.07	.29	.33	1.12	N.S.
V	X	133	1.76		1.26	.11			
	XI	99	2.57	.81	1.63	.16	.19	4.26	.01
	XII	51	3.49	.92	1.97	.28	.32	2.87	.01
VI	X	133	7.64		2.93	.25			
	XI	99	7.69	.05	4.18	.42	.48	.10	N.S.
	XII	51	6.02	-1.67	4.77	.66	.78	-2.14	-.03

The table should be read as follows: For sub-test I, 133 grade ten girls attained a mean score of 7.48 with a standard deviation of 1.25, a standard deviation of the mean of .11. For grade eleven, 99 students attained a mean score of 744, a decrease of .04 in the mean, a standard deviation of 1.57, standard error of the mean of .16, standard error of the difference of .19, critical ratio of .21 and the scores are not significant at the .01 level of confidence, etc.

Table XIII compares the mean scores of the boys on each of the sub-tests from grade to grade. In four of the six sub-tests we may say that at the .01 level of confidence there has been a real growth from grade ten to grade eleven and from grade eleven to grade twelve in the ability of the boys to do functional thinking. From the results obtained

TABLE XIII

THE PERFORMANCE OF THE BOYS ON EACH OF THE SUB-TESTS FROM GRADE TO GRADE

Sub-Test	Grade	Number Cases	Mean Score	D	S.D.	$S.E._m$	$S.E._{diff.}$	C.R.	Significance Level
I	X	81	7.58		1.36	.15			
	XI	151	8.14	.56	1.36	.11	.27	2.07	.05
	XII	80	8.36	.22	1.01	.11	.15	1.46	N.S.
II	X	81	5.02		3.01	.33			
	XI	151	7.46	2.44	2.41	.19	.38	6.42	.01
	XII	80	8.31	.85	1.99	.21	.28	3.03	.01
III	X	81	14.59		4.62	.51			
	XI	151	17.82	3.23	4.67	.38	.63	5.12	.01
	XII	80	20.63	2.81	3.15	.35	.51	5.51	.01
IV	X	81	1.87		1.94	.22			
	XI	151	3.03	1.16	1.97	.16	.27	4.29	.01
	XII	80	4.27	1.24	1.84	.20	.25	4.96	.01
V	X	81	1.48		1.14	.13			
	XI	151	2.92	1.44	1.46	.12	.17	8.47	.01
	XII	80	4.12	1.20	1.39	.16	.20	6.00	.01
VI	X	81	8.52		3.14	.35			
	XI	151	9.38	.86	3.79	.31	.46	1.87	N.S.
	XII	80	9.98	.60	3.86	.43	.53	1.13	N.S.

This Table reads as Table XII.

in sub-test I, it would appear that from the viewpoint of a generalized aspect of functionality, the boys attain a fairly high standard of performance after which the increase in the mean score is less pronounced as compared to the other sections of the test. The results also show that the boys attain a more pronounced increase in their ability to apply the function concepts expressed in Sub-tests II, III, IV and V. In no case was there a decrease in the mean score throughout the three grades tested. With one exception, the increase in the mean scores was greater for grade eleven than what it was for grade twelve. Although there was an increase in the mean scores from grade to grade in the ability to recognize functional relationship as expressed by graphs, the increase was not significant at the .01 confidence level.

Table XIV, page 58, indicates a measureable increase in the ability to do functional thinking from grade to grade in each area of the entire test except in the ability to interpret graphical representation of functional relationships. In this particular sub-test, we actually had a decrease in the mean score at the grade eleven level. Generally, the increase in the mean scores for the entire population tested is fairly regular as in no instance is there a difference greater than 1.5 points. In sub-tests I and IV, the increase is constant. For the first five of the six sub-tests the increase in mean performance of the students in the three high-school grades is significant at the .01 confidence level.

The third groups of tables comprises Tables XV, XVI and XVII.

TABLE XIV

COMPARISON OF THE MEAN PERFORMANCES FOR THE POPULATION TESTED FROM GRADE TO GRADE ON EACH SUB-TEST

Sub-Test	Grade	No. of Cases	Mean Score	D	S.D.	S.E.$_m$	S.E.$_{diff.}$	C.R.	Significance Level
I	X	214	7.52		1.29	.08			
	XI	250	7.90	.38	1.49	.09	.12	3.16	.01
	XII	131	8.29	.39	1.01	.08	.12	3.25	.01
II	X	214	5.45		2.99	.20			
	XI	250	7.08	1.63	2.58	.16	.26	6.27	.01
	XII	131	8.06	.98	1.98	.17	.23	4.26	.01
III	X	214	14.71		4.90	.33			
	XI	250	16.73	2.02	5.16	.33	.46	4.39	.01
	XII	131	19.89	3.16	3.55	.31	.45	7.02	.01
IV	X	214	1.82		1.79	.12			
	XI	250	2.75	.90	1.97	.12	.17	5.29	.01
	XII	131	3.63	.88	2.09	.18	.22	4.00	.01
V	X	214	1.67		1.22	.08			
	XI	250	2.78	1.11	1.57	.15	.17	6.53	.01
	XII	131	3.87	1.09	1.67	.14	.20	5.45	.01
VI	X	214	7.97		3.04	.63			
	XI	250	7.84	-.13	3.63	.23	.67	-.19	N.S.
	XII	131	8.44	.60	4.65	.41	.47	1.27	N.S.

The Table reads: like Table XIII

These tables show the mastery coefficients for the girls, boys and total population respectively. For each test the total possible, the actual number of correct responses and the resulting mastery coefficient has been determined for each grade and for the entire high school population tested, together with the ranking of the test for each grade and for the high school.

Table XV, page 60, shows that with two exceptions there is an increase in the mastery coefficients from grade to grade for each of the six sub-tests. The exceptions are an almost negligible decrease in sub-test I in the mastery coefficient for grade eleven girls in general proficiency in recognition of functional relationship and, a rather pronounced decrease for grade twelve girls in the inability to interpret graphical representation of functional relationship as revealed by sub-test VI. The table shows that the girls' abilities to ascertain a generalized aspect of functionality ranked first for each grade. The ability to recognize how a change in one variable affects another variable ranked second for each grade. Sub-test II ranked third for grades eleven and twelve and for the entire group of girls tested but ranked fourth for grade ten. This sub-test consisted of two separate abilities. For each situation presented the girls were asked to determine the relationship involved and secondly to express that relationship as a formula. A further analysis will be found in subsequent tables to determine which of these two factors presented the greater difficulty. The ability to change the subject of a formula proved the most difficult for grades ten and eleven girls. The improvement in position for grade twelve girls in this phase of the testing program is probably due not so much to a better understanding of the functional relationship involved, but rather a greater facility in algebraic manipulation. The master coefficient for grade twelve girls was determined as 43.79 per cent, which is still relatively low. The second most difficult area of functionality for grades ten and

TABLE XV

THE NUMBER AND PERCENTAGE OF CORRECT REPONSES FOR THE GIRLS TOGETHER WITH MASTERY COEFFICIENTS AND RANKING OF SUB-TESTS

	Grade X	Grade XI	Grade XII	Grades X-XII	Ranking X	Ranking XI	Ranking XII	Ranking X-XII
Sub-test I								
No. of questions	1330	990	512	2830				
No. correct responses	995	736	418	2149				
Mastery coefficient	74.81	74.34	81.96	75.94	1	1	1	1
Sub-test II								
No. of questions	1463	1089	561	3113				
No. correct responses	748	626	392	1766				
Mastery coefficient	51.12	57.48	69.87	56.73	4	3	3	3
Sub-test III								
No. of questions	3192	2376	1224	6792				
No. Correct responses	1963	1478	956	4397				
Mastery coefficient	61.49	62.20	78.10	64.73	2	2	2	2
Sub-test IV								
No. of questions	798	594	306	1698				
No. correct responses	240	224	134	598				
Mastery coefficient	30.07	37.71	43.79	35.25	6	6	5	6
Sub-test V								
No of questions	665	495	255	1415				
No. correct responses	237	254	178	669				
Mastery coefficient	35.64	51.31	69.80	35.35	5	5	4	5
Sub-test VI								
No. of questions	1862	1386	714	3962				
No. correct responses	1016	766	307	2089				
Mastery coefficient	54.56	55.26	42.99	52.72	3	4	6	4

This table should be read as follows: For sub-test I there were 995 out of a possible of 1330 correct responses and a mastery coefficient of 74.81 per cent, for the grade X girls, etc. Sub-test I ranks first in achievement in grades ten, eleven and twelve and for high school girls.

eleven girls is the ability to derive a formula from a table of values. The better relative standing for the grade twelve girls in this area is possibly due to increased use of tables by the senior students. Functional relationship as expressed by graphs ranked higher for the grade ten girls than for the girls of grades eleven and twelve. Two factors may have some bearing on this ranking. In the first place, proximity to instruction may have been one factor, and the nature of the graphical representation used may be another. By the nature of the curricular content and improper emphasis on functional relationship the girls of grade twelve have lost their ability to interpret functional relationship by means of graphs. The table also shows that for five of the six sub-tests at least two grades had the same ranking as that for the entire high school population of girls tested.

TABLE XVI, page 62, shows the mastery coefficient for the boys for each of the sub-tests and for each grade together with the corresponding ranking for the sub-tests. There are much greater variations in the actual ranking of the sub-tests from grade to grade than what was the case for the girls. From grade to grade there was no decrease in the mastery coefficients in the sub-tests. In four of the six sub-tests two grades had the same rankings as for the entire group of boys tested. In grade eleven, the ranking for the boys remains the same as the ranking for the grade elven girls. In grade ten, there was an interchange in the rankings of sub-tests four and five, but otherwise the rankings remain the same. The greatest change is in grade twelve

TABLE XVI

THE NUMBER AND PERCENTAGE OF CORRECT RESPONSES FOR THE BOYS TOGETHER WITH MASTERY COEFFICIENTS AND RANKING OF SUB-TESTS

	Grade X	Grade XI	Grade XII	Grades X-XII	Ranking X	Ranking XI	Ranking XII	Ranking H.S.
Sub-test I								
No. of questions	810	1510	800	3120				
No. correct responses	614	1229	668	2511				
Mastery coefficient	75.80	81.39	83.50	80.48	1	1	2	1
Sub-test II								
No. of questions	891	1661	880	3432				
No. correct responses	407	1123	664	2194				
Mastery coefficient	45.67	67.61	75.45	63.92	4	3	4	4
Sub-test III								
No. of questions	1944	3624	1920	7488				
No. correct responses	1183	2693	1649	5525				
Mastery coefficient	60.85	74.31	85.88	73.78	2	2	1	2
Sub-test IV								
No. of questions	486	906	480	1872				
No. correct responses	151	454	341	946				
Mastery coefficient	31.07	50.11	71.04	50.53	5	6	6	6
Sub-test V								
No. of questions	405	755	400	1560				
No. correct responses	119	440	329	888				
Mastery coefficient	29.38	58.28	82.25	56.92	6	5	3	5
Sub-test VI								
No. of questions	1134	2114	1120	4368				
No. correct responses	690	1417	798	2905				
Mastery coefficient	60.84	67.03	71.25	66.51	3	4	5	3

This Table should be read as Table XV.

where in no sub-test is the ranking the same for the girls and boys. In grades ten and eleven, the rankings for sub-tests I, II, III and VI remain the same. The only changes are the interchanges in ranking for sub-tests IV and V at the grade ten level. With four exceptions, all of them at the grade ten level, the mastery coefficients for the boys are greater than those for the girls.

Table XVII, page 64, is a summary of the percentages of correct responses for each of the sub-tests for the entire population tested. Sub-test I ranks as the test which showed the greatest understanding for the entire group of students. The second easiest test is sub-test II. The ability to recognize functional relationship between two or more quantities and the interpretation of how a change in one variable affects another seems to some extent to be generally established at the high school level. The expression of the relationships as algebraic formulas, the interchange of the subjects of the formula in order to obtain new relationships is not developed to the same degree. The interpretation of a table of values is not developed to any appreciable extent in grades ten and eleven but there is a decided improvement for grade XII students in the understanding of this concept. The table shows that the students of grades ten and eleven are more capable of interpreting graphical representation of functional relationship. For the entire high school population the mastery coefficient rankings were as follows:

(a) The ability to recognize functional relationships between two or more quantities.

TABLE XVII

NUMBER AND PERCENTAGE OF CORRECT RESPONSES FOR THE POPULATION TESTED TOGETHER WITH THE MASTERY COEFFICIENT AND RANKING FOR THE SUB-TESTS

	Grade X	Grade XI	Grade XII	Grades X-XII	Ranking X	XI	XII	H.S.
Sub-test I								
No. of questions	2140	2500	1310	5950				
No. of correct responses	1609	1965	1086	4660				
Mastery coefficient	75.18	78.60	82.90	78.32	1	1	1	1
Sub-test II								
No. of questions	2354	2750	1441	6545				
No. of correct responses	1155	1749	1056	3960				
Mastery coefficient	49.05	63.60	73.28	60.50	4	3	4	3
Sub-test III								
No. of questions	5136	6000	3144	14280				
No. of correct responses	3146	4171	2605	9922				
Mastery coefficient	61.25	69.52	82.54	69.48	2	2	2	2
Sub-test IV								
No. of questions	1284	1500	786	3570				
No. of correct responses	391	678	475	1544				
Mastery coefficient	30.45	45.20	60.43	43.25	6	6	5	6
Sub-test V								
No. of questions	1070	1250	655	2975				
No. of correct responses	356	694	507	1557				
Mastery coefficient	33.27	55.52	77.40	52.33	5	5	3	5
Sub-test VI								
No. of questions	2996	3500	1834	8330				
No. of correct responses	1706	2183	1105	4994				
Mastery coefficient	56.94	62.37	60.25	59.95	3	4	6	4

This Table should be read as Table XVI.

(b) Recognition of how a change in one variable affects another variable.

(c) The ability to express the relationship between two or more variables as a formula.

(d) Functional relationship as interpreted through graphical representation.

(e) The ability to change the subject of a formula.

(f) The ability to derive a formula from a table of values.

CHAPTER V

A DETAILED ANALYSIS OF THE FUNCTIONAL SITUATIONSPRESENTED IN THE TEST AND THE RESPONSES TO THE SITUATION BY STUDENTS OF THE SENIOR HIGH SCHOOL

This chapter deals primarily with the degree of success by the students in the individual functional situations presented by the test. The chapter also attempts to clarify, wherever possible, the types of errors made by the students and to give a comparison of the errors for the various groups and grade levels.

For Sub-test I, which dealt with the recognition of generalized aspects of functional relationship and was answered on a true-false basis, little can be deduced as to the quality of the error, but, for subsequent sub-tests, a more detailed analysis of the students' responses is possible.

The tables of the chapter show the mastery or error coefficients for each question and, where possible, the types of error made on each of the seventy questions of the entire testing program. The following reference list will be used for question responses:

R - represents a correct response to the question.

X - represents an incorrect response to the question.

A - represents an incorrect response in the recognition of the functional relationship involved in the question.

B - represents an error in setting up the required formula. For this particular situation the students have been successful in determining the relationships involved but have not been able to combine these relationships into a representative formula. Symbols A and B were used for Sub-test II where two factors were involved for a correct solution to the particular situation.

Sub-test I.[1] The Ability to Recognize Functional Relationship between Two or More Variables.

Table XVIII, page 68, shows the percentages of passes and failures for each question in Sub-test I. The performance of the girls, boys, and that of the entire population for each grade and for the high school is presented. The table also shows the performance level of each question on this sub-test from the easiest to the most difficult.

From the table, five inferences may be made. First, in simple direct variation, students at all grade levels have little difficulty in ascertaining the relationship between the variables. Questions I, II, IV, VIII and IX showed mastery coefficients in excess of eighty-two per cent for each sex and grade. Secondly, unfamiliarity with particular situations tends toward a pronounced decrease in the ability to visualize possible relationship. This was particularly true in questions such as "An increase in the number of pages in a yearbook decreases the cost of publishing the book", or in a situation such as "The rate of depreciation on a machine depends upon the length of the expected life of the machine." This information seems to indicate that the range and scope of the functional experiences at the senior high school level is somewhat limited. Thirdly, depth in functional thinking seems to be lacking in the population tested. The range of mastery coefficients for the following

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1 See Appendix

TABLE XVIII

THE NUMBER AND PERCENTAGE OF CORRECT RESPONSES AND THE ACHIEVEMENT RATING FOR EACH QUESTION IN SUB-TEST I

Grade	X			XI			XII			H.S.			Rank
Sex	G	B	T	G	B	T	G	B	T	G	B	T	
Students	133	81	214	99	151	250	51	80	131	283	312	595	
Question 1													
R	114	68	182	88	145	233	49	78	127	251	291	542	
X	19	13	32	11	6	17	2	2	4	32	21	53	
Mastery Coeff.	85.71	83.95	85.05	88.89	96.03	92.40	96.08	96.95	97.99	87.99	93.59	91.09	2
Question 2													
R	118	72	190	87	144	231	48	75	123	221	291	542	
X	15	9	24	12	7	19	3	5	8	30	21	53	
Mastery Coeff.	88.72	88.89	88.79	87.88	95.36	92.40	94.12	93.75	93.13	89.40	93.27	91.43	1
Question 3													
R	85	53	138	61	105	166	34	51	85	180	209	389	
X	48	28	76	38	46	84	17	29	46	103	103	206	
Mastery Coeff.	63.16	65.43	64.49	61.62	69.53	66.40	66.67	63.75	64.89	63.60	66.99	65.38	9
Question 4													
R	113	74	187	88	128	216	45	72	117	246	274	520	
X	20	7	27	11	23	34	6	8	14	37	38	75	
Mastery Coeff.	84.96	91.36	87.38	88.89	84.77	86.40	88.24	90.00	89.31	86.92	87.82	87.39	3
Question 5													
R	119	60	179	87	132	219	42	67	109	248	259	507	
X	14	21	35	12	19	31	9	13	22	35	53	88	
Mastery Coeff.	89.47	74.32	83.64	87.88	87.41	87.60	82.35	83.75	83.21	87.63	83.01	85.21	6
Question 6													
R	93	55	148	73	113	186	45	59	104	211	227	338	
X	40	26	66	26	38	64	6	21	27	72	85	157	
Mastery Coeff.	69.92	67.90	69.16	73.74	75.49	74.40	88.24	73.75	79.39	74.56	72.75	73.61	8
Question 7													
R	104	69	173	71	134	205	43	72	115	228	275	501	
X	29	12	41	28	17	45	8	8	16	55	37	94	
Mastery Coeff.	78.19	85.18	80.84	71.72	88.74	82.00	84.31	90.00	87.78	80.56	87.50	84.20	7
Question 8													
R	114	67	181	82	130	212	47	76	123	243	273	516	
X	19	14	33	17	21	38	4	4	8	40	39	79	
Mastery Coeff.	85.71	82.71	84.58	82.83	86.09	84.80	92.16	95.00	92.37	85.86	87.52	86.72	5
Question 9													
R	111	69	180	83	136	219	45	76	121	239	281	520	
X	22	12	34	16	15	31	6	4	10	44	31	75	
Mastery Coeff.	83.46	85.18	84.11	83.89	90.07	87.60	88.24	95.00	92.37	84.45	90.07	87.39	3
Question 10													
R	24	27	51	16	62	78	20	42	62	60	131	291	
X	109	54	163	83	89	172	31	38	69	223	181	304	
Mastery Coeff.	18.05	33.33	23.83	16.16	41.06	31.20	39.21	52.50	47.33	21.20	41.98	48.91	10

The Table reads as follows: In Grade X there were 133 girls who wrote the test. For question 1, there were 114 correct and 19 incorrect responses. The mastery coefficient was 85.71. The question ranked as the second easiest question of the test.

question, "The perimeter of a square depends on the area of the square," was from 16.16 per cent for the grade eleven girls to 52.50 per cent for the grade twelve boys. It was the only question for which the mastery coefficient dropped below fifty per cent in this group of questions. Fourthly, inverse variation presented greater difficulty than direct variation. Finally, in seven of the ten questions comprising this group there is an improvement in the mastery coefficient, from grade to grade, for the total population tested. The three exceptions are questions three, four and five. In question three, the grade ten students attained a mastery coefficient of 64.49, the grade eleven students 66.40 and the grade twelve students 64.89. In question four, there was a slight decrease in the mastery coefficient for grade eleven as the mastery coefficients for the three high school grades were 87.58, 86.40 and 89.31 respectively. The third exception was question five where the mastery coefficient for grade twelve was lower than that for grade eleven. These coefficients were 83.64, 87.60 and 83.21 respectively.

The performances of the boys in sub-test I was better than that of the girls. For the girls, there were only two questions in which there was an increase in the mastery coefficient from grade to grade. The boys showed an increase in the mastery coefficients from grade to grade in six out of the ten questions. The results also show that in five questions the mastery coefficients for the girls greater at the grade ten level than what they were at the grade eleven level. In each case direct variation was involved. For questions four and nine, both of which involve variation with a constant factor, the mastery coefficient was greatest for grade eleven students. For question five, the mastery coefficient for the girls was greatest at the grade ten level. On the other hand, the mastery coefficient in questions three, five and six were greater for the boys at the grade eleven level than at the grade twelve level. For questions four the mastery coefficient was greater at the grade ten level than at the grade eleven level.

Sub-test II.[2] Recognition of Functional Relationship and the Ability to Express that Relationship as a Formula.

Table XIX, page 71, shows that the mastery coefficients for the questions in Sub-test II are considerably lower than the mastery coefficients in Sub-test I. In Sub-test II, there are two factors involved. The first is the recognition of the relationships for each particular situation and, secondly, the ability to write the existing relationship as a formula. The response to any question of this group was considered incorrect if any one of the two parts of the question was either incorrect or incomplete. For Sub-test I possible existing relationships were indicated by the nature of the wording of the question. Sub-test II required more depth in functional thinking as no clues were given to possible relationship between the variables. The lower mastery coefficients were then in all probability due to two factors, namely, a greater depth in relational thinking and the regrouping of the existing

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[2] See Appendix

TABLE XIX

THE NUMBER OF CORRECT AND INCORRECT RESPONSES, THE MASTERY COEFFICIENT AND ACHIEVEMENT RATING FOR EACH QUESTION IN SUB-TEST II

Grade	X			XI			XII			X - XII			Rank
Sex	G	B	T	G	B	T	G	B	T	G	B	T	
Students	133	81	214	99	151	250	51	80	131	283	312	595	
Question 11													
R	63	18	81	62	104	166	41	57	98	166	179	345	
X	70	63	133	37	47	84	10	23	33	117	133	250	
Mastery Coeff.	47.29	22.22	37.84	62.62	68.87	66.40	80.39	71.25	74.81	58.66	57.38	57.98	9
Question 12													
R	72	35	107	62	106	168	29	63	92	163	204	367	
X	61	46	107	37	45	82	22	17	39	120	108	228	
Mastery Coeff.	54.13	43.21	50.00	62.62	70.19	67.20	56.86	78.75	70.23	57.60	65.39	61.51	7
Question 13													
R	88	53	141	56	124	180	28	66	94	172	248	420	
X	45	28	73	43	27	70	23	14	37	111	64	175	
Mastery Coeff.	66.16	65.43	65.88	56.56	82.12	72.00	54.90	82.50	72.06	60.78	77.89	70.59	2
Question 14													
R	72	47	119	65	116	181	44	71	115	181	234	415	
X	61	34	95	34	35	69	7	9	16	102	78	180	
Mastery Coeff.	54.13	58.02	55.61	65.65	76.82	72.40	86.27	88.75	87.79	63.96	75.00	69.75	3
Question 15													
R	94	53	147	79	117	196	47	74	121	220	244	464	
X	39	28	67	20	34	54	4	6	10	63	68	131	
Mastery Coeff.	70.67	65.43	68.69	79.79	77.48	78.40	92.16	92.50	92.37	77.74	78.21	77.98	1
Question 16													
R	82	40	122	64	113	177	40	59	99	186	212	398	
X	51	41	92	35	38	73	11	21	32	97	100	197	
Mastery Coeff.	61.66	49.38	57.01	64.64	74.83	70.80	78.55	73.75	75.57	65.73	67.95	66.89	5
Question 17													
R	33	18	51	39	65	104	26	40	66	218	123	221	
X	100	63	163	60	86	146	25	40	65	185	189	374	
Mastery Coeff.	24.81	22.22	23.83	39.39	43.05	41.60	50.98	50.00	50.38	34.63	39.42	37.13	10
Question 18													
R	82	46	128	57	125	182	39	65	104	178	236	414	
X	51	35	86	42	26	68	12	15	27	105	76	181	
Mastery Coeff.	61.66	56.79	59.81	57.57	83.17	72.80	76.47	81.25	79.39	62.90	75.64	69.58	4
Question 19													
R	72	39	111	68	103	171	42	62	104	182	204	386	
X	61	42	103	31	48	79	9	18	27	101	108	209	
Mastery Coeff.	54.13	48.15	51.87	68.68	68.21	68.40	82.35	77.50	79.39	64.32	65.39	64.88	6
Question 20													
R	80	41	121	46	86	132	36	67	103	162	194	256	
X	53	40	93	53	65	118	15	13	28	121	118	239	
Mastery Coeff.	60.15	50.61	56.54	46.46	56.95	52.80	70.59	83.75	78.63	57.25	62.18	59.83	8
Question 21													
R	10	17	27	28	64	92	20	40	60	64	121	185	
X	117	64	181	71	87	158	31	40	71	219	191	410	
Mastery Coeff.	7.52	20.98	12.61	28.28	42.38	36.80	39.21	50.00	45.80	22.62	38.79	31.10	11

The Table reads the same as Table XVIII.

variables so that they may be written as an algebraic formula.

For the purpose of this investigation, Table XIX reveals a number of pertinent facts. In the first place, the mastery coefficients for the entire high school population tested show considerable range. In only five of the eleven questions of this test do we find mastery coefficients of sixty-six per cent or better. The range is from 31.10 to 77.98. Secondly, with one exception, the total mastery coefficients increase from grade to grade for each functional situation presented in this sub-test. The exception is question twenty in which the students were asked to determine upon what factor the circumference of a circle would depend and to write the existing relationship. Thirdly, for grade ten, only one question had a mastery coefficient greater than sixty-six per cent, for grade eleven eight questions had a mastery coefficient of sixty-six per cent or better, whereas for grade twelve there were nine questions that showed this percentage or better. The range in mastery coefficients for grade ten is from 12.61 to 68.69, a range from 36.80 to 78.40 for grade eleven and from 45.80 to 92.37 for grade twelve.

Since two factors are involved in determining the mastery coefficients for the situations in Sub-test II, an attempt was made to determine which of these two factors presented the greater difficulty. In order to determine the error coefficients a count was made of the errors that may be attributed to each of the factors. The errors were classified into two main groups as follows: Errors A represented errors in the question that were attributed to the inability of the student to determine the relationships or variables involved. The error coefficients were determined on the total possible responses. Errors B represents the errors made in arranging the variables into an algebraic formula. In each case the students had completed correctly the first part of the question. In each case the total possible responses were decreased by the number of errors made in determining the relationship involved and the percentage of errors for Errors B was determined on this new total.

A review of Tables XX, XXI and XXII, page 74, reveals the following information pertinent to this investigation. In the first instance, for the population tested, the ability to determine the functional relationships involved between two or more factors appears to be easier than the ability to set up the corresponding relationship as an algebraic formula. In each case, the total error coefficient is lower for errors A than for the corresponding Errors B. In only three cases is an increase in the total error coefficient recorded as the performances of the students are compared from grade to grade. In Tables XX, XXI and XXII the total error coefficients for errors A for the girls in each of the high school grades are respectively 22.14, 11.20 and 15.86. Where there is a decrease in the error coefficient from grade ten to grade eleven we find an increase in the error coefficient from grade eleven to grade twelve. For the corresponding grades the total error coefficients for the boys were 23.22, 9.69 and 8.41. For both girls and boys the greatest improvement in the ability to recognize relationships between factors takes place between grades ten and eleven. A second feature of the tables is the decrease in range of the error coefficients in grades ten and eleven and the increase in the range for both sexes in grade twelve. An examination of the error coefficients for the boys and

TABLE XX

PERCENTAGES OF ERRORS FOR GRADE X STUDENTS IN DETERMINING THE FUNCTIONAL RELATIONSHIP INVOLVED (A) AND THE INABILITY TO SET UP THE CORRESPONDING FORMULA (B)

Question	Errors A						Errors B					
	No. of errors			Percentage of errors			No. of errors			Percentage of errors		
	G	B	T	G	B	T	G	B	T	G	B	T
11	20	20	40	15.04	24.69	18.68	50	43	93	44.25	70.49	53.45
12	53	43	96	39.85	53.08	44.86	8	3	11	10.00	7.89	9.32
13	42	27	69	31.57	33.33	32.24	3	1	4	3.30	1.85	2.76
14	18	18	36	13.53	22.22	16.82	43	16	59	37.39	25.39	33.15
15	9	6	15	6.76	7.41	7.01	30	22	52	24.19	29.33	26.13
16	14	7	21	10.52	8.64	9.81	37	34	71	31.09	45.95	31.61
17	16	3	19	12.03	3.74	8.88	84	60	144	71.79	76.92	73.89
18	26	21	47	19.55	25.92	21.96	25	14	39	23.36	23.33	23.35
19	53	39	92	39.85	48.15	42.99	8	3	11	10.00	7.14	9.01
20	18	10	28	13.53	12.31	13.07	35	30	65	30.43	42.25	34.95
21	55	13	68	41.36	16.05	31.77	62	51	113	79.48	75.00	77.39
Total	324	207	531	22.14	23.22	22.56	385	277	662	33.80	40.49	36.31

The Table reads as follows: For question 11 in determining the relationships involved the girls, boys and combined girls and boys had 20, 20 and 40 errors respectively with corresponding percentages of errors as 15.04, 24.69 and 18.68. In writing the corresponding formula, the girls, boys and total population of Grade X made 50, 43 and 93 errors respectively with the corresponding percentages of errors as 44.25, 70.49 and 53.45 respectively, etc.

TABLE XXI

PERCENTAGES OF ERRORS FOR GRADE XI STUDENTS IN DETERMINING THE FUNCTIONAL RELATIONSHIP INVOLVED (A) AND THE INABILITY TO SET UP THE CORRESPONDING FORMULA (B)

Question	Errors A						Errors B					
	No. of errors			Percentage of errors			No. of errors			Percentage of errors		
	G	B	T	G	B	T	G	B	T	G	B	T
11	6	5	11	6.06	3.31	4.40	31	42	73	32.98	28.76	30.54
12	10	26	36	10.10	17.22	14.40	27	19	46	30.34	15.20	21.49
13	29	16	45	29.19	10.59	18.00	14	11	25	20.00	8.15	12.19
14	1	5	6	1.01	3.31	2.40	33	30	63	33.67	20.54	25.82
15	3	15	18	3.03	9.93	7.20	17	19	36	17.71	13.97	15.52
16	1	9	10	1.01	5.96	3.60	34	29	63	34.69	20.42	26.25
17	10	16	26	10.10	10.59	10.40	50	70	120	56.18	51.85	53.57
18	5	10	15	5.05	6.62	6.00	37	16	53	39.36	11.35	22.55
19	18	37	55	18.18	24.50	22.00	13	11	24	16.05	9.65	12.31
20	14	16	30	14.14	10.59	12.00	39	49	88	45.86	36.29	40.00
21	25	6	31	25.25	3.98	12.40	46	81	127	62.16	55.86	57.99
Total	122	161	283	11.20	9.69	10.29	341	377	718	35.26	25.13	28.08

The Table reads as Table XX.

TABLE XXII

PERCENTAGES OF ERRORS FOR GRADE XII STUDENTS IN DETERMINING THE FUNCTIONAL RELATIONSHIP INVOLVED (A) AND THE INABILITY TO SET UP THE CORRESPONDING FORMULA (B)

Question	Errors A						Errors B					
	No. of errors			Percentage of errors			No. of errors			Percentage of errors		
	G	B	T	G	B	T	G	B	T	G	B	T
11	5	7	12	9.80	8.75	9.15	5	16	21	10.87	21.92	17.64
12	20	11	31	39.21	13.75	23.66	2	6	8	6.45	8.69	8.00
13	18	10	28	35.29	12.50	21.37	5	4	9	15.15	7.14	8.73
14	--	--	--	-----	-----	-----	7	9	16	13.71	11.45	12.21
15	4	2	6	7.84	2.50	4.57	-	4	4	-----	5.13	3.20
16	2	2	4	3.92	2.50	3.05	9	19	28	18.37	24.35	22.05
17	13	22	35	25.49	27.50	26.71	12	18	30	31.58	31.04	31.25
18	4	--	4	7.84	-----	3.05	8	15	23	17.02	18.75	18.11
19	9	10	19	17.65	12.50	14.50	-	8	8	-----	11.43	7.14
20	4	3	7	7.84	3.75	5.34	11	10	21	23.40	12.98	16.94
21	10	7	17	19.60	8.75	12.97	21	33	54	51.22	39.76	47.37
Total	89	74	163	15.86	8.41	11.31	80	142	222	16.95	17.62	17.37

The Table reads as Table XXI.

girls in grades ten and eleven shows that question 17 is the only question for which the error coefficient for grade eleven exceeds that for grade ten. The boys of grade ten show ten percent better understanding of the relationship involved. In determining the formula that relates the variables, the girls and boys of grade ten had a lower error coefficient in three questions. These were question 12, in which the students were asked to write the perimeter of a triangle, question 13 in which the students were asked to write the volume of a cube, and in question 19 in which they were to determine the formula connecting the amount of money spent during the month, their monthly allowance, the amount of money earned during the month and the amount they had on deposit at the end of the month. For question 20, which asked the students to determine the formula connecting the circumference and diameter of a circle, the error coefficient for the grade eleven girls was greater than that of the grade ten girls.

Comparing the performance of grades eleven and twelve students in their ability to recognize the relationship involved, an examination of Tables XXI and XXII reveals that in seven of the eleven questions in the sub-test the girls of grade eleven had a lower error coefficient in determining the relationship involved. In comparing the performance of the boys of grades eleven and twelve, the latter had a lower error coefficient in seven of the eleven questions. It appears that fairly rapid improvement is made by the students in grade eleven in determining the relationships involved between two or more variables. A review of

Tables XX and XXI shows a marked decrease in the error coefficients in errors A from grades ten to eleven. A comparison of the performance of grades eleven and twelve students indicates that there is a slight decrease in their ability to recognize existing relationship for the girls and a slight increase in the ability of the boys. Whereas, there was a total error coefficient of 10.29 for the grade eleven students, there is a total error coefficient of 11.31 for grade twelve students. The difference in performance is not significant. What is important is the fact that an increased ability to recognize functional relationship is not maintained from grade to grade.

With respect to the ability for the students to set up the corresponding algebraic formula, Tables XX, XXI and XXII show that the total error coefficients are greater for Errors B than for Errors A. The tables show that the error coefficients for the grade eleven students is approximately one-half those of the grade ten students. With the exception of the total error coefficients between the grades ten and eleven girls, there is a fairly constant increase in the ability to set up the corresponding algebraic relationship. Two factors are involved. The first of these is an increased ability to recognize the functional relationship and expressing that relationship as an algebraic equivalent and, secondly, the increased facility with the language of algebra. The second is possibly the lesser of the two because by the end of the first year of high school grade ten students have a sufficient working knowledge of the language of algebra to satisfactorily complete the formulas required in this part of the test. In some cases the error coefficients for grade ten students are actually lower than those of grades eleven or twelve students. In all probability the fact that that particular concept has recently been taught in grade ten may be the determining factor.

The individual items on this sub-test were carefully examined in order to classify the types of errors that appeared in the sub-test. In order to discuss adequately the types of errors made in the questions the following reference list will be used.

O - meaning an omission of the question

U - meaning a response that is unrelated to the particular situation

P - meaning a partial response is given. In question eleven two areas are required in order to determine the difference between the areas of the two circles. In some cases the students have only determined the area of one circle.

Q - meaning the quality of the response. In this case the students did not pursue the identification far enough to assure the successful completion of the relationship. In question 11 some of the students merely indicated that the difference in areas of two circles depended on the size of the two circles. They did not pursue the relationship any farther to indicate that the size of the circles depended on the size of their respective radii. If the studend followed this partial answer with a correct formula then credit was given for the entire question.

M - meaning an error in interpretation of the language of algebra.

X - meaning an incorrect response

A - an error attributed to the interpretation of the relationship involved.

B - an error in setting up the algebraic formula.

For each question in this sub-test a table is presented which shows the type of error that has been made and the per cent of incorrect responses that may be attributed to a particular type of error. In the answer to the functional situations presented in this sub-test the most common correct response has been stated. Other responses may be possible and the students were given credit for these alternate responses.

Sub-Test II - Question 11. The difference (D) in area between two circles would depend on:

(a) The radius of the larger circle . . . . R

(b) The radius of the smaller circle. . . . r

The required formula $D = \pi \left(R^2 - r^2\right)$

One feature of the majority of the questions in this sub-test is the fairly high percentage of students that have omitted the question. It is rather surprising that grade eleven students should have difficulty in recognizing the relationships involved. Table XXIII, page 79, shows eleven students omitted to state the required relationships. In grade ten many unrelated variables were stated. Some of these were as follows:

TABLE XXIII

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 11

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	42.50	100.00	58.33	O	20.43	23.29	33.33
Q	27.50		41.67	P	47.31	24.66	
U	30.00			X	32.26	52.05	66.67
Total errors	40	11	12		93	73	12

The table should read as follows: In grade ten, 42.50 per cent of the incorrect responses were due to omissions, 27.50 per cent were due to inadequate definition of the relationship involved and 30 per cent of the responses had no connection with the particular functional situation, etc.

The relationship between the areas of the two circles depends upon the distances between the two circles, the length and width of the circles, the circumference of each area, the distance, rate and time between the two circles. Thirty per cent of the errors at the grade ten level were of this nature. There was a complete absence of responses of this nature in grades eleven and twelve. It will be noted that even at the grades eleven and twelve level there are still a number of students who are not too sure of the functional relationships involved in problem situations dealing with the area of a circle. In errors B there was a considerable range in the formulas presented by the students in illustrating the functional relationship and the formulation of that relationship as an algebraic formula. Some of the students believed that the required formula should be expressed as the difference between the two radii, the differences between the diameters, differences between their circumferences, the ratio between the circumference and diameter, product of their diameters and circumferences, product of radii and diameters, the ratio of their areas, the product of their areas. These were not single errors nor are they typical for a particular grade. Usually the frequency of the error was much greater in grade ten but we cannot say that these errors were confined to grade ten. In most cases they were found through the entire high school population tested.

Sub-test II - Question 12 The Perimeter (P) of any triangle would depend on:

(a) The length of one side (o)

(b) The length of the second side (s)

(c) The length of the third side (t)

Formula: $P = o + s + t$

One factor in this question was the vast number of unrelated variables that were stated and the high incidence of omissions at the grades eleven and twelve level. A second factor was the number of students who stated that in order to determine the perimeter of a triangle we must know the length of the base and the perpendicular height of the triangle. Partly, the statement is correct but, beyond the comprehension of a grade ten student. The errors B portion almost invariably showed the resulting formula as the area of a triangle instead of the perimeter.

TABLE XXIV

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 12

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	14.56	47.22	54.84	O	18.17	4.35	50.00
Q	22.33		16.13	R	81.83	95.65	50.00
U	63.11	52.78	29.03				
Errors	103	36	31		11	46	8

This Table should read in the same manner as Table XXIII However, for errors A, the students have been given the benefit of the doubt and the error has been classed as Q. For errors B there wasn't any doubt and consequently the error has been classed as a reading error.

In determining the relationship between variables the following were frequently mentioned: the length and height of the sides, the distance and the height, length of the radius, the diameter and circumference. For the corresponding formula the area of a triangle and the perimeter of a rectangle were frequently stated as the required relationship.

<u>Sub-test II - Question 13</u> The volume (V) of a regular cube will depend upon:

(a) The length of a side . . . s

Formula: $V = s^3$

The results of Table XXV seem to indicate that the concept of the volume of a cube is better understood by grade ten students than it is for students of grades eleven and twelve. Actually, at the grade ten level there is the possibility of recency in instruction and that the concept, since it is little used by grade eleven and twelve students, has been forgotten. For determining the relationship some of the responses were that the independent variable would be the circumference of a circle, the width and the length, radius and height, diameter and height, circumference and height, the density. In stating the formula the following were frequently given as the required

TABLE XXV

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION No. 13

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	57.97	66.67	100.00	O	100.00	35.00	77.78
R	14.49	26.67				47.50	22.22
U	27.54	6.66				17.50	
Errors	69	45	28		4	40	6

This Table should read in the same manner as Table XXIV.

formula: The area of a circle with radius s, surface area of a cube, product of height and weight, the area of one side of the cube.

Sub-test II - question 14. During a motor trip I average 15 miles to the gallon. The number of gallons (G) I used during the trip would depend on:

(a) The total number of miles travelled . . . (M)

The required formula is $G = \frac{M}{15}$

From the standpoint of errors A, this question ranks as the second easiest question of this sub-test. The most common incorrect response to the first part of the question was that the number of gallons used depends on the cost of the gasoline, the speed, and the number of persons in the car.

TABLE XXVI

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 14

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	36.11	100.00	--	O	18.64	15.87	18.75
X	63.89	--	--	X	81.36	84.13	81.25
Errors	36	6	0		59	63	16

This Table should read as Table XXV.

For errors B the chief error was the improper statement of relationship between the number of gallons used, the average distance per gallon and the total distance travelled. In each grade approximately 80 per cent of the students indicated the required formula as G = 15 D.

Sub-test II - Question 15. The number of dozens (D) oranges I can buy for $1.00 will depend on:

(a) The cost per dozen (C).

The required formula is $D = \frac{\$1.00}{C}$

TABLE XXVII

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 15

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
0	100.00	100.00	100.00	o	74.19	44.44	100.00
				X	25.81	55.56	--
Errors	15	18	6		55	36	4

This Table should read as Table XXVI

Of the 595 students who participated in the investigation 6.55 per cent of the students omitted to state the relationships involved in this question. This question ranked as the easiest of the group presented in this sub-test. The majority of the type X errors that occurred in this question were due to the fact that the students stated that the number of dozens of oranges that could be bought for $1.00 was equal to the product of the price per dozen and one dollar or, the cost per dozen divided by the total cost of the oranges.

Sub-test II - Question 16. A tinsmith wishes to determine the area (A) of a circular piece of tin. What must be known in order to determine the area of the piece of tin?

(a) Radius of the piece of tin? . . . . . (r)

The required formula is $A = \pi r^2$

TABLE XXVIII

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 16

Grade / Type	Errors A X	Errors A XI	Errors A XII	Type	Errors B X	Errors B XI	Errors B XII
O	66.67	60.00	100.00	O	14.08	22.22	--
U	33.36	40.00	--	U	85.92	77.78	100.00
Errors	21	10	4		71	63	28

This Table should read as Table XXVII.

The chief error in Errors A was the fact the students maintained the area of the circular piece of tin depended on the length and width of the tin. In writing the area of the tin as an algebraic formula, the students wrote the circumference of the circle, the product of pi and the square of the diameter, the product of the circumference and the diameter, the formula for twice the area of a circle, the product of pi and the radius, the product of the circumference and the diameter, and the ratio of the circumference to the diameter.

It is rather surprising that one-third of the high school population are unable to recognize and interpret the relationship between the radius and the area of a circle. Even at the grade twelve level twenty-five per cent of the students were unable to cope with the functional situation presented.

Sub-test II - Question 17. At the beginning of the year James deposited $200.00 in a bank account which he has just opened. No further deposits nor withdrawals were made. The first interest payment was made at the end of the first six months. What further information would be required in order to determine the total amount of money (A) in the account after the interest payment has been made?

(a) The interest rate per year. . . . (R)

The required formula is $A = 200 \neq 200\ RT$

For this question the only group of students who received a mastery coefficient greater than fifty per cent was the grade twelve group. It must be assumed that the eighty students who omitted to state the functional relationship involved in this situation could not identify the variables involved. Actually the number of errors that have been classified under errors B in Table XXIX, page 87, does not present an entirely true picture as to the functional understanding by the pupils of the situation presented. The errors classed under type O and U are relatively few as compared to the errors classed under type P. In the latter case a part of the required formula was presented. In each case the students wrote the formula for the simple interest for the half year instead of the accumulated amount.

TABLE XXIX

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 17

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	100.00	100.00	100.00	O	17.36	27.78	6.67
				P	81.26	69.17	93.33
				U	.38	3.05	--
Errors	19	26	35		144	120	30

This Table reads as Table XXVIII.

Sub-test II - Question 18. A car leaves a town at 9.00 a.m. and by travelling continuously arrives at a second town at noon. The distance (D) travelled would depend upon:

(a) The average speed per hour . . . . (R)

The required formula is D = 3 R

Again we find a fairly high rate of omission particularly in the first part of the question. It has been noted that wherever the students have some acquaintance with the relationships that are involved they are less inclined to guess at the answer. Rather than guess they are more apt to omit the question entirely. This has been noticed particularly with the easier questions in this sub-test.

TABLE XXX

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 18

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
O	76.09	100.00	100.00	O	25.64	20.75	--
X	23.91	--	--	X	74.36	79.25	100.00
Errors	47	15	4		39	53	23

This Table should read as Table XXIX.

The chief error that was classed under errors A was that the required variable was the time or the distance between the towns instead of the average rate per hour. From the nature of the responses by the 181 students whose responses were incorrect it was obvious they know little about the relationship between the distance, rate, and time involved. In writing the algebraic formula the chief error was that the students expressed the distance between the towns as a quotient of the distance and rate, the rate and time, the distance and time, and, the product of the distance and time. These types of responses for this question were common throughout the entire high school grades.

Sub-test II - Question 19. If you received your allowance at the beginning of the month and you earned a certain sum of money during the month, what would you need to know in order to determine what amount of money you spent (S) during the month?

(a) Amount of the Allowance (A)

(b) Amount Earned (E)

(c) Amount left at the end of the month (L)

The required formula is: $S = A + E - L$

TABLE XXXI

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 19

	Errors A				Errors B		
Grade	X	XI	XII		X	XI	XII
Type				Type			
0	47.82	49.09	57.89	0	100.00	91.67	--
X	52.18	50.91	42.11	X	--	8.33	100.00
Errors	92	55	19		11	24	8

This Table should read as Table XXX.

In most cases the students did not indicate that they must know the amount of money that was left at the end of the month. In the cases of errors B the formula was written in terms of the amount of money that remained at the end of the month instead of in terms of the amount of money that was spent during the month.

<u>Sub-test II - Question 20.</u> The circumference (C) of a circle will depend upon:

(a) The diameter (d)

(b) The constant (pi)

The required formula is: C = pi X d.

TABLE XXXII

NATURE AND PER CENT OF ERROR FROM GRADE TO GRADE IN QUESTION 20

Grade	X	Errors A XI	XII		X	Errors B XI	XII
Type							
0	100.00	100.00	100.00	0	10.76	17.05	23.81
				R	53.85	44.32	38.09
				X	35.39	38.63	39.10
Errors	28	30	7		65	88	21

This Table should read in the same manner as Table XXXI.

As in the case of question 16 which dealt with the area of a circle, it is rather surprising at the number of errors in questions dealing with the circle. A mastery coefficient of 59.83 is rather low for this particular concept. It is not a new concept for high school students but is apparently a concept that is easily forgotten. Students of grades ten and twelve do use the concept to some extent. By the nature of the curricular content there is very little occasion to use the circle in grade eleven algebra. Of the three grades the grade eleven students had the lowest mastery coefficients for this question.

It is difficult to determine whether or not type R error is actually a reading error but once again the student has been given the benefit of the doubt. In each case the student determined the area of the circle instead of the circumference. In writing the corresponding formula some of the more common errors were as follows: circumference of a semi-circle, double the circumference of a circle, product of the radius and the area of the circle, product of the diameter and the area of the circle, product of the radius and the diameter, double the diameter, ratio of the area and radius, of the area and diameter and of the radius and diameter.

Sub-test II - Question 21. The ratio (R) of completed passes thrown by a quarterback will depend upon:

(a) The number of passes thrown . . . . (T)

(b) The number of completed passes . . (C)

The required formula is: $R = C/T$

This question proved to be the most difficult question in the group of questions in this sub-test both from the standpoint of determining the relationships involved and also in expressing the relationship as an algebraic formula. Approximately eighteen per cent of the students omitted the question entirely and approximately sixty per cent of the remaining students were unable to express the relationships of the variables as an algebraic formula. Lack of familiarity with the situation presented seemed to have been the chief reason for the low mastery coefficient. In setting up the formula the chief error was

TABLE XXXIII

NATURE AND PER CENT OF ERROR FROM GRADE
TO GRADE IN QUESTION 21

Grade	Errors A X	XI	XII		Errors B X	XI	XII
Type							
O	100.00	100.00	100.00	O	19.47	25.98	9.26
				P	40.71	44.09	42.51
				X	39.82	29.93	48.15
Errors	68	31	17		113	127	54

This Table should read as Table XXXII.

the fact that the formula in most cases was inverted. Another common error was stating the ratio in terms of the incompleted passes and finally the misinterpretation of the term ratio. In seven cases the term was interpreted as the difference between the total number of passes thrown and the number completed.

In summary, it should be noted that in most of the situations presented in sub-test II there was a growth in the ability to think functionally by the students as they progressed from grade to grade. There were however exceptions, and in each case mathematical curricular content for the grade was primarily responsible. The indications are that a program of functionality must stress continuous use of functional situations in mathematics and constant growth in their use.

Sub-test III.[3] Recognition of How a Change in One Variable Affects Another.

Table XVII, page 64, revealed that Sub-test III ranked second in performance with a mastery coefficient of 69.48 for the entire population tested. Each group was made up of four or five questions designed to test the understanding of the function involved. To the right of each question five statements were given. One value represented either the amount or the number of times by which the function will increase or decrease. The student was required to underline the statement that would correctly define the relationship. The first set consisted of a group of five questions based on a direct variation. The group read as follows:

22-26. The total cost (C) of a number of articles will depend upon the number (n) of articles purchased and the price (p) per article. This is represented by the formula $C = np$. What change takes place in C if:

22.	"n" is doubled and "p" remains the same?	no change, 1/2, 1/4, 2, 4.
23.	"p" is doubled and "n" remains the same?	no change, 1/2, 1/4, 2, 4.
24.	Both "n" and "p" are doubled?	no change, 1/2, 1/4, 2, 4.
25.	"n" is doubled and "p" is trebled?	no change, 2/3, 4, 6, 8.
26.	"n" is doubled and "p" is halved?	no change, 2, 3, 4, 6.

Table XXXIV, page 94, shows that the students have a good grasp of how in direct variation a change in one variable will affect a change in another. The table shows that the mastery coefficients increase from grade to grade. The table also shows that the increase for grade twelve

[3] See Appendix

is more than double that recorded for grade eleven. The table also shows that for the entire population tested the mastery coefficient for this group of questions is 85.61 which ranks first in performance of the five groups in sub-test III.

Table XXXIV

THE MASTERY COEFFICIENTS FOR QUESTIONS 22 - 26

Grade	X			XI			XII			X-XII	Rank
	G	B	T	G	B	T	G	B	T		
Ques.											
22	81.96	85.19	83.17	86.86	89.40	88.40	96.08	98.75	97.71		
23	81.95	87.65	84.58	83.83	88.09	86.40	90.19	96.25	93.89		
24	85.71	83.95	85.05	81.81	87.41	85.20	92.16	98.75	96.18		
25	78.19	81.48	79.91	71.71	87.41	81.20	78.43	86.25	83.21		
26	75.18	76.54	75.70	74.74	90.73	84.40	94.12	95.00	94.66		
Total	80.75	82.96	81.62	79.80	88.61	85.12	90.19	95.00	93.13	85.61	1

The table reads: For question 22 the mastery coefficient for the girls of grade ten is 81.96 per cent, the mastery coefficient for grade ten boys is 85.19 and for the total grade ten students the mastery coefficient is 83.17 per cent, etc.

The most common incorrect response for questions 22 and 23 was the selection of item 2 in each case. In other words, the students maintained that if we doubled the number of articles purchased or doubled the cost per article it would half the total cost. This response accounted for 61 per cent of the errors in question 22 and 59 per cent in question 23.

In 58.1 per cent of the incorrect responses for question 24 the students stated that if "n" and "p" are each doubled then the total cost would be doubled, and in 28.4 per cent of the incorrect responses the students stated that there would be no change in the total cost.

In forty per cent of the incorrect responses in question 25 the students stated that as "n" is doubled and "p" is trebled then the total cost would be two-thirds of its original cost. In another forty per cent of the incorrect responses the students stated that the total cost would be eight times the original cost.

Forty-five per cent of the incorrect responses in question 26 was attributed to the selection of item two in the answers to the right of the initial statement and twenty-four per cent of the incorrect responses was due to the selection of item three by the students.

The second group of questions read as follows:

27-30 The area (A) of a square with side (X) is represented by the formula $A = X^2$. What changes take place in A if "X" is

27. Doubled?	1/2, 1/4, 2, 4, 8.
28. Trebled?	2, 3, 4, 6, 9.
29. Halved?	1/2, 1/4, 2, 4, 8.
30. How much greater will "X" be if "A" is made four times as great?	1/2, 1/4, 2, 4, 8.

From Table XXXV, page 96, three important facts may be deduced. The first of these is that direct variation where one of the variables is squared is a much more difficult concept than variation where the variables are of the first degree. The mastery coefficients for the questions of this group are considerably lower than the mastery coefficients for the previous group of questions. This is particularly true at the grades ten and eleven levels where there are differences as large as thirty percentage points. A second feature of the table is that the mastery coefficients show increases from grade to grade for both sexes with the more pronounced increase at the grade twelve

TABLE XXXV

THE MASTERY COEFFICIENTS FOR QUESTIONS 27 - 30

Grade	X			XI			XII			X-XII	Rank
	G	B	T	G	B	T	G	B	T		
Ques.											
27	65.41	64.19	64.95	70.70	82.12	77.60	84.31	95.00	90.84		
28	57.89	64.19	60.28	64.64	78.14	72.80	84.31	91.25	87.79		
29	54.89	55.56	55.14	59.59	70.86	66.40	78.43	87.50	83.97		
30	39.85	41.98	40.65	54.54	63.05	59.60	62.74	78.75	72.52		
Total	54.51	56.48	55.26	62.37	73.51	69.10	77.45	88.12	83.97	75.79	2

This Table reads: as Table XXXIV.

level. Grade Twelve students attain a fairly high level of performance and the greater understanding of the functional relationships involved in this group of situations seems to be acquired in grade twelve. The third observation is that in question 30 there is a change in the composition of the relationship as the formula shows a direct variation to a square root. The mastery coefficients are considerably lower for this question than for the other questions of the group. However, there is an increase in the mastery coefficients for both sexes from grade to grade.

The third group of questions reads as follows:

31-35. The time (T) to make a certain journey depends upon the distance (D) travelled and the rate (R) per hour. This is represented by the formula T = D / R. What change takes place in T if:

31. D is doubled and R remains the same? no change, 1/2, 2, 3, 4.

32. R is trebled and D is doubled? no change, 2/3, 3/2, 2, 3.

33. Both D and R are doubled? no change, 1/2, 1/4, 2, 4.

34. D is halved and R is doubled? no change, 1/2, 1/4, 2, 4.

35. D is doubled and R is halved? no change, 1/2, 1/4, 2, 4.

TABLE XXXVI

THE MASTERY COEFFICIENTS FOR QUESTIONS 31 - 35

Grade		X			XI			XII		X-XII	Rank
Ques.	G	B	T	G	B	T	G	B	T		
31	81.95	80.24	81.77	77.77	92.06	86.40	96.08	97.50	96.95		
32	48.87	46.91	48.13	48.48	63.05	57.20	60.78	68.75	65.65		
33	66.91	69.14	67.76	73.73	88.09	86.40	86.23	92.50	90.07		
34	36.84	34.57	35.98	36.36	56.29	48.40	58.82	66.25	63.36		
35	57.14	56.79	57.01	66.66	78.14	73.60	82.35	90.00	87.02		
Total	58.49	57.53	58.13	60.61	75.49	69.60	76.86	83.00	80.60	67.89	3

This Table reads as Table XXXV.

In this group of questions the time depends upon the total distance and the average rate per hour. The understanding of the concepts involved is by no means as advanced as the understanding of the direct variation as presented in the first group of this sub-test. The only question that approximates the mastery coefficients of Table XXXIV, page 94, is question 31 of this group of questions. In this particular question the value of only one independent variable was changed and as such the situation was a direct variation between two variables. Table XXXVI, page 97, also indicates that there is little improvement in the achievement of the grade eleven girls as compared with the performance of the grade ten girls. The table also reveals that in three of the five questions in this group the grade ten girls had a higher mastery coefficient than the grade eleven girls. On the other hand there is an average increase of sixteen percentage points by the grade twelve girls in the same group of questions. The table also shows that for the boys and the total population the mastery coefficients do increase from grade to grade but in only two questions does the total mastery coefficient reach a mark of ninety per cent or better.

The fourth group of questions in this sub-test reads as follows:

36-40. If y = 10/x, what changes take place in "y" if "x" is

36	Doubled?	1/2, 1/3, 1/4, 2, 4.
37	Trebled?	1/2, 1/3, 1/4, 2, 3.
38	Halved?	1/2, 1/4, 2, 4, 6.
39	Whatever value is given to "x" the product of that value and the corresponding value of "y" must always equal	1, 2, 5, 10, 15.

40. If "x" is given a number of values as 1, 1/2, 1/3, 1/4 etc., then each value of "y" will differ from its previous values by 1/2, 1/3, 2, 5, 10.

For this group of questions the product of the two variables is constant, hence the one quantity is said to vary inversely as the other. Question 39 in particular attempts to focus the attention of the student to the fact that the product of the variables is a constant. The mastery coefficient as recorded in Table XXXVII indicates that the concept was not understood by the majority of the students. The total mastery coefficients for this question ranged from 31.31

TABLE XXXVII

THE MASTERY COEFFICIENTS FOR QUESTIONS 36 - 40

Grade		X			XI			XII		X-XII	Rank
Ques.	G	B	T	G	B	T	G	B	T		
36	76.69	81.48	78.50	70.70	88.09	81.20	94.12	96.25	95.43		
37	70.68	72.84	71.49	67.67	83.44	73.20	82.35	92.50	87.79		
38	66.91	71.61	68.69	61.61	85.43	76.00	88.23	90.00	89.31		
39	33.83	27.16	31.31	33.33	50.33	43.60	54.90	80.00	70.23		
40	17.29	12.34	15.42	27.27	31.13	27.60	49.02	56.25	53.43		
Total	53.08	53.08	53.09	52.12	67.68	61.52	73.72	83.00	79.39	62.42	5

This Table reads as Table XXXVI.

for grade ten to 70.23 for grade twelve. The most difficult situation in this series of questions was question 40 in which total mastery coefficients ranged from 15.42 for grade ten to 53.43 for grade twelve. The question itself borders the calculus and it is suspected that whatever success the students have had was on a basis of mathematical manipulation rather than on the basis of understanding of the concept involved. On the basis of the responses on the first three questions of this group it would appear that the students had a fair knowledge of the relationships involved but the results of the last two questions of the group indicates that the depth of the understanding is not too great. On the entire group of questions there is an increase from grade to grade in the understanding of the relationship involved. The total mastery coefficients increase from grade to grade with the greater increase between grades eleven and twelve.

The last group of questions in this sub-test reads as follows:

41-45. If the principal depends upon the interest, rate and time we might express this relationship by the formula $P = I/RT$ where P represents the principal, I the interest per year, R the rate per year and T the time in years. What is the effect on P if

41.	I is doubled and R and T remain the same?	no change, 2, 3, 4, 8.
42.	R is halved and I and T remain the same?	no change, 1/2, 2, 3, 4.
43.	R is halved, T. is doubled and I remains the same?	no change, 1/2, 1/4, 2, 3.
44.	I, R and T are each doubled?	no change, 1/2, 1/4, 2, 3.
45.	T is doubled, R is halved, T is trebled?	no change, 1/2, 1/4, 4/3, 2.

The formula employed in this group is similar to that employed in the 31-35 group but the situation is different and an extra variable has been introduced. Between the two situations there does not appear to be an appreciable difference in the total mastery coefficients. In

TABLE XXXVIII

THE MASTERY COEFFICIENTS FOR QUESTIONS 41 - 45

Grade / Ques.	X G	X B	X T	XI G	XI B	XI T	XII G	XII B	XII T	X-XII	Rank
41	81.20	79.01	80.37	79.79	90.73	86.40	90.19	95.00	93.13		
42	45.11	35.80	41.59	42.42	54.30	47.60	68.63	73.75	71.76		
43	65.41	70.37	67.29	66.66	80.13	74.80	78.43	90.00	85.49		
44	36.09	19.75	29.91	30.30	33.78	32.40	54.90	58.75	57.25		
45	68.42	61.73	65.88	61.61	71.52	67.60	68.63	86.25	79.39		
TOTAL	59.25	53.33	57.01	56.16	66.09	62.16	72.16	80.75	77.41	63.66	4

The Table reads as Table XXXVII.

general, two interesting features are revealed. The first of these is that if one factor is changed and the remaining factor or factors remain constant, the performances in questions 31-35 group and 41-45 group are the same. It is interesting to note that the total mastery coefficients for grade eleven is the same in each case as the comparison of Table XXXVI, page 97, and Table XXXVIII will reveal. The second feature is that with the exception of questions 41 and 43, the relationship is not understood by the students of grades ten and eleven but concept is fairly well established by the students of grade twelve. The one area in which the grade twelve students experienced a great deal of difficulty was when each of the independent variables was doubled. This did not present the same difficulty in question 33 where the same situation was presented. The difference in performances may be partially due to the introduction of the additional variable and consequently greater difficulty in comprehending the relationship involved or the introduction of the extra variable may have resulted in more frequent manipulative errors. In either case it does show a lack of understanding of the principle involved.

Sub-test IV The Ability to Derive a Formula from a Table of Values.

In this sub-test the following directions and questions were presented to the students.

Directions: Study each of the following groups of numbers. The values for the last mentioned letter for each group depends for its value upon the other values in the group. Write the formula showing how the last letter in each group depends upon the remaining letters of the group.

46.

F	1	2	3	4	6	
S	1	4	9	16	36	S =

47.

X	1	2	3	4	5	
Y	3	5	7	9	11	Y =

48.

X	2	4	5	6	8	
Y	3	4	4	3	5	
A	3	8	10	9	20	A =

49.

P	100	1000	2000	3000	
R	.03	.04	.05	.05	
A	103	1040	2100	3150	A =

50.	A	1	2	3	4	6	
	T	1	8	27	64	216	T =

51.	A	2	2	3	4	5	
	W	2	5	4	6	5	
	P	8	14	14	20	20	P =

The situations selected were those that high school students undoubtedly have had some experience with in earlier grades. They included the relationship between the sides and area of a square and of a cube, the relationship between the base, height and area of a triangle, the sides and perimeter of a rectangle, the principal, rate of interest, time and accumulated amount of a single sum of money, and the direct variation between two variables when a constant factor is employed.

In the initial testing program many additional tables of values were examined. Some of these were not included in the present test as the results showed a perfect score for the entire population tested in the pilot test. Included among these were two questions that involved a direct variation such as $x = y$ and $x = 3y$. Others that were not included in the final draft because they appeared to be too easy were such relationships as $c = a/b$, $K = abc$, $M = a + b$, and $D = a + b + c$. On the other hand for relationships that are involved in such formulas as $y = 36/x$, $c^2 =-a^2 + b^2$, $T = a + b - c$ and $T = R - nd$, the validity coefficients were so low that they could not be included in the final draft.

The results of Table XXXIX, page 104, shows that with the exception of questions forty-six the mastery coefficients on this part of the testing program was very low. Three factors are indicated. Firstly, with the exception of direct variation in which the students of the pilot test had a perfect score, the understanding of the functional relationship as expressed by a table of values is not developed to any appreciable extent by the high school students.

TABLE XXXIX

THE MASTERY COEFFICIENTS FOR QUESTIONS 46 - 51

Grade / Ques.	X G	X B	X T	XI G	XI B	XI T	XII G	XII B	XII T	X-XII	Rank
46	66.16	56.79	62.61	70.70	81.85	77.20	64.71	90.00	80.13		1
47	24.81	23.45	24.31	37.37	44.37	41.60	56.86	70.00	64.89		3
48	18.79	28.39	22.89	19.19	35.09	28.80	33.33	65.00	52.67		4
49	13.53	13.58	13.55	13.13	27.81	22.00	19.61	56.25	41.98		6
50	34.27	43.21	37.38	54.55	68.19	62.80	54.89	86.25	74.05		2
51	23.31	20.99	22.43	31.31	43.71	38.80	33.33	58.75	48.85		5
Total	30.07	31.07	30.45	37.54	50.11	45.20	43.79	71.04	60.43	43.25	

The Table reads: For question 46 the grade ten girls attained a mastery coefficient of 66.16 per cent, the boys 56.79 per cent and the total girls and boys for grade ten the mastery coefficient was 62.61, etc.

Secondly, in most cases an improvement in the mastery coefficients is noted from grade to grade. The greatest improvement is in the performance of the grade twelve students. This is to be expected because students at that level are confronted with a greater variety of tabular information. Thirdly, the table reveals that at the grades eleven and twelve levels the performances of the boys on this sub-test was considerably better than that of the girls.

Sub-test V. The Ability to Change the Subject of a Formula.

In this group of questions the students were given five formulas each of which represented a different type of relationship. These were represented as follows: a direct variation, a direct variation plus a constant, inverse variations, direct variation with the independent variable squared, a direct variation with more than two variables.

Directions: Sometimes it becomes necessary to change a formula in terms of another quantity. Rewrite each of the following in terms of the quantity stated:

52. If P = 12s then s =

53. If y = mx ≠ b then x =

54. If A = 40/B then B =

55. If A = k^2 then k =

56. If V = ABC, then A =

The mastery coefficients of this group of questions are given in Table XL, page 106. Direct variation seems to have been the easiest situation for the students to understand. The boys of grades eleven and twelve again showed their superiority in the interpretation of the change in the relationship. Direct variation which included a constant factor as in question 53 presented the greatest difficulty in each of the three grades. The grade twelve students appear to have a much greater understanding of the concept involved but even here the functional relationships have not been adequately established. In the light of the fact that grade twelve students spend much of their time studying the straight line, their performance on this question was not very encouraging.

TABLE XL

THE MASTERY COEFFICIENTS FOR QUESTIONS 52 - 56

Grade	X			XI			XII			X-XII	Rank
	G	B	T	G	B	T	G	B	T		
Ques.											
52	69.17	65.43	67.76	79.79	88.08	84.80	78.43	92.50	87.02		1
53	12.03	7.41	10.28	24.24	27.81	26.40	52.94	66.25	61.07		5
54	45.11	38.27	42.52	54.54	68.87	63.20	70.59	86.25	80.15		3
55	7.52	9.88	8.41	38.38	34.44	40.00	78.43	82.50	80.92		2
56	44.36	25.92	37.39	59.59	65.56	63.20	69.02	83.75	77.86		4
Total	35.64	29.38	33.27	51.31	58.29	55.52	69.81	82.25	77.40	52.33	

The table reads: For question 52 the grade ten girls attained a mastery coefficient of 69.17 per cent, the boys 65.43 per cent and the total girls and boys the mastery coefficient for grade ten was 67.76 etc.

Inverse variation presented a great deal of difficulty to the students of grades ten and eleven. The better perforances by the grade twelve students is probably due to the fact that a more detailed study of ratio, proportion and variation is a part of the grade twelve mathematics program.

Square root was another area in which grades ten and eleven students showed little success. This is a bit surprising as grade eleven students have much of their program in algebra based on surds and indices. Unfortunately, much of the work appears to be on a manipulative basis and the results of the investigation shows that manipulation devoid of functional understanding is of little value.

A comparison of the mastery coefficients of Table XL, page 106, with that of Table XXXIV, page 94 and with Table XXXV, page 96, indicates that the interpretation of dependence is more difficult in formulas than in verbal statements. The nature of many of the responses in this group of questions strongly supports the suspicion that the attempt was of the nature of manipulative juggling instead of thinking about the dependence or relationship between the variables of the algebraic formula.

Sub-test VI. Functional Relationship as Represented by Graphs.

In the initial program seven types of graphs were considered. These were as follows:

1. The graph of direct proportion which may be represented by the formula $y = ax$ where "a" represents a constant.

2. The graph of direct proportion plus a constant in the form $y = mx + b$.

3. The graph of inverse proportion in the form $y = a/x$

4. The graph of squares in the form $y = ax^2$ where "a" represents unity.

5. The graph of squares in the form $y = ax^2$ where "A" is a positive number other than unity.

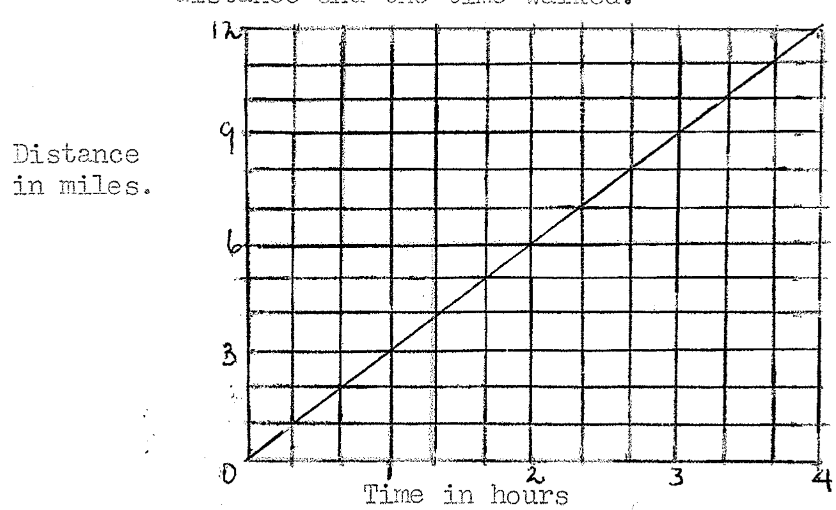
6. The graph of square roots in the form $y = a\sqrt{x}$

7. The graph of cubes in the form $y = ax^3$ where "a" represents unity.

The students were asked to consider the seven types of graphs that had been constructed and below each write the formula that would show the relationship between the variables. Type two proved to be too easy for students of the three high school grades. The validity coefficients for types 3, 6 and 7 did not warrant the inclusion of that type of graph in the final draft. In each case the validity coefficient was less than .10.

In the final draft three graphs were included in the test. Each graph was constructed in sub-test VI and for each graph a number of questions were asked to determine the depth of understanding in reading graphical representation of a functional situation.

57 - 61. Below is a graph which James prepared after a hiking trip with some of his friends. He attempted to show the relationship between the distance and the time walked.



57. What was the average rate per hour they walked? ____

58. How many hours did they walk? ____

59. How far did they walk in 2 hours 20 minutes? ____

60. Write the formula which would represent the relationship between the distance and the time. ____

The mastery coefficients for each of these questions may be found in table XLI. A surprising feature of these mastery coefficients

TABLE XLI

THE MASTERY COEFFICIENTS FOR QUESTIONS 57 - 61

Grade	X			XI			XII			X-XII
Ques.	G	B	T	G	B	T	G	B	T	
57	91.73	92.59	92.06	82.82	92.05	88.40	66.67	92.50	82.44	
58	91.73	85.18	89.25	80.81	89.34	86.00	66.67	86.25	78.62	
59	70.67	76.25	72.43	60.61	76.82	70.40	56.90	82.50	72.52	
60	70.67	76.25	72.43	64.65	86.09	77.60	58.82	87.50	76.33	
61	48.87	55.56	51.40	43.43	70.86	60.00	43.14	72.50	61.07	
Total	74.73	76.79	74.86	66.46	83.05	76.48	58.43	84.25	74.19	75.63

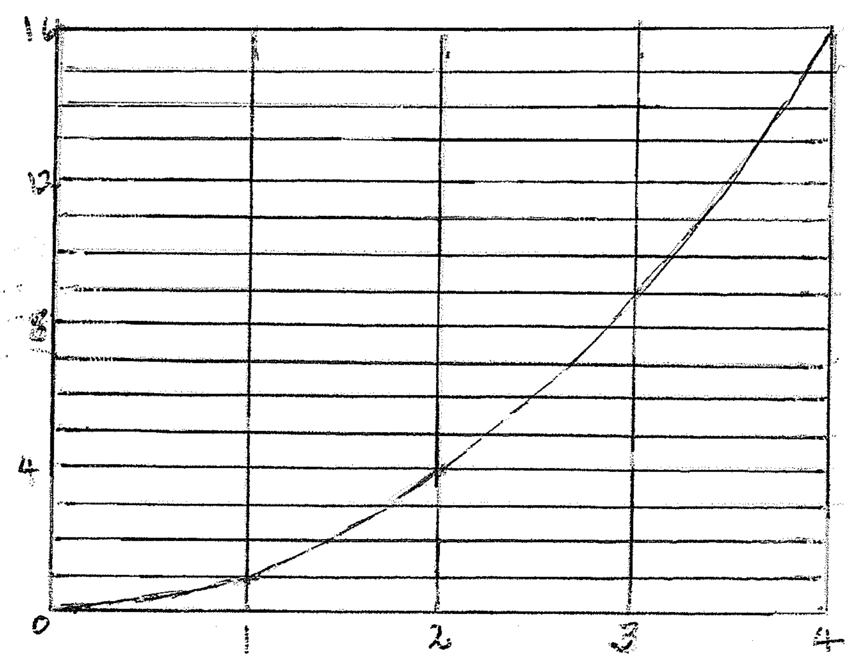
This Table reads as Table XL.

is that the coefficients for grades ten and eleven girls are higher than the mastery coefficients for the grade twelve girls. For each question the mastery coefficients for the girls decreased from grade to grade. For the boys there is in most cases a slight increase in the mastery coefficient from grade to grade. The most pronounced trend toward an increase was in the ability to write the algebraic relationship that existed between the variables.

The ability to interpret graphical representation does not appear to be developed to any great extent. The principles involved in questions 57 and 58 were the same as those involved in questions 59 and 60. The correlation between these two sets of questions should be very high yet there is a considerable variance in their mastery coefficients.

The second group of questions was based on a graph of squares in which the coefficient of the squared variable is unity.

62-66. A boy has four squares each of different size. He plotted the relationship between the length of the sides of the squares and the areas of the squares.



62. What do the numbers at the bottom of the graph represent? ________

63. What do the numbers along the side of the graph represent? ________

64. Read from the graph the area of a square 3 inches in length. ________

65. What is the increase in the differences between the areas of the squares as the sides of the squares increase by one unit? ________

66. Write the formula that would show the relationship between these two groups of values. ________

TABLE XLII

THE MASTERY COEFFICIENTS FOR QUESTIONS 62 - 66

Grade / Ques.	X G	X B	X T	XI G	XI B	XI T	XII G	XII B	XII T	X-XII
62	64.66	74.07	68.22	69.69	77.48	74.40	50.98	80.13	70.99	
63	59.39	72.84	64.48	69.69	76.82	74.00	49.02	81.25	68.70	
64	57.14	59.26	47.94	63.63	71.52	68.40	50.98	80.00	68.70	
65	9.77	11.11	10.28	10.10	21.85	17.20	1.96	30.00	19.08	
66	24.06	30.86	26.63	31.31	44.37	39.20	21.57	50.00	38.91	
TOTAL	43.01	49.39	45.51	48.89	58.41	54.64	34.90	65.00	53.29	51.06

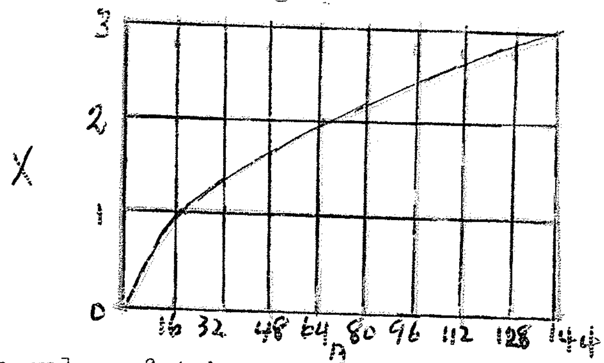
The Table reads as Table XLI.

The interpretation of this graph was much more difficult for the students than the interpretation of the graph of direct proportion. As in the previous graph, the girls of grade ten showed a better understanding of the principles involved than the girls of grade twelve. The boys show a fairly regular increase in their ability to interpret the graphical representation. Approximately one-third of the students could not determine the elements of the graph. Very few of the students were aware of the fact that as the size of the square increased the difference in area between successive squares form an arithmetic progression with the difference between the terms of the progression as 2. There appears to be little, if any, attention directed toward the determining and construction of graphs by the use of gradients. The use of gradients in graphical representation would not only aid in the construction of graphical representations but would also aid students in the interpretation of the relationship existing between variables. Finally, the interpretation of this graph and the representation of the algebraic formula was very unsatisfactory.

The final graph was similar to the preceding graph in that it was also a graph of squares but it varied in that the coefficient of the squared variable was not unity. With the exception of writing the formula connecting the variables, the questions that were asked concerning the graph attempted to investigate further understandings of graphical representation.

With mastery coefficients as low as those in Table XLIII, page 114, the interpretation of graphical representation is not developed to any great extent amongst our high school students. In question 67, the students were asked to determine graphically how a change in one

67-70. Study the graph below and answer the questions that are based on this graph.



67. As the value of A increased the value of X ____________

68. As X varies from 0 to 3, A varies from________to__________

69. Complete the table of corresponding values of A and X

X	0	2	1
A			

70. Write the formula that would illustrate the relationship between X and A ____________________

variable affects another variable. The mastery coefficient ranged from 54.88 for the girls in grade twelve to 77.50 for grade twelve boys. In three of the four questions in this group the grade ten girls did better than the grade twelve girls. In two of the four questions the grade ten boys had a higher mastery coefficient than the boys of grade twelve. In question 68 the students were asked to determine the range in one variable as the other variable increases between certain limits. The girls of grade twelve showed the lowest ability to interpret the situation and the boys of grade ten had the highest mastery coefficient for the

TABLE XLIII

THE MASTERY COEFFICIENTS FOR QUESTIONS 67 - 70

Grade	X			XI			XII			X-XII
	G	B	T	G	B	T	G	B	T	
Ques.										
67	66.92	76.25	70.09	67.67	72.85	70.80	54.88	77.50	68.70	
68	64.66	80.24	70.56	65.65	74.17	70.80	41.18	76.25	62.59	
69	41.35	60.49	48.59	56.56	72.18	66.00	33.33	71.25	56.48	
70	2.25	3.70	2.80	7.07	11.92	10.00	5.88	26.25	18.32	
Total	43.79	43.95	48.01	49.24	57.78	54.40	33.82	62.81	51.52	51.49

This Table reads as Table XLII

entire group of students. Throughout the grades approximately fifty per cent of the students were able to read the corresponding table of values from the graph but most of the students were unable to set up the formula that would illustrate the relationship between the two variables.

CHAPTER VI

SUMMARY AND CONCLUSIONS

This chapter presents, first, a summary of the most important information arising out of the investigation; second, certain conclusions that might be suggested as a result of the investigation; and third, suggested lines along which further investigations might be made in order to determine additional information relative to the improvement of relational thinking by students of mathematics.

Summary of the Findings of this Investigation

1. There is a difference in the mean scores from grade to grade in the high school population that was tested. The respective means for grades ten, eleven and twelve were 39.09, 45.74 and 52.51. This gives increase to mean differences of 6.65 and 6.57 from grades ten to eleven and from grades eleven to twelve respectively.

2. Although there was a constant rise in the mean scores from grade to grade on the entire test, there is little evidence of constant rise in the mean scores on the individual sections of the test.

3. In grade ten, the mean scores for the girls and boys were 39.09 and 39.05 respectively. With a difference of only .04 in the mean scores between the girls and boys we may say that their ability to do functional thinking at this grade level is the same.

4. A comparison of the mean scores of grades ten and eleven girls shows that the difference of their mean scores is not significant at the

.01 level of confidence. The difference between the mean scores of grades eleven and twelve girls in the samples tested is significant at the .01 level of confidence.

5. For the boys the difference of the mean scores is significant at each level with the greater difference at the grade eleven level.

6. For the entire population tested for this investigation, the differences of the mean scores from grade to grade are significant at the .01 level of confidence.

7. By the end of grade eleven there is a mean difference of approximately one year achievement between the boys and girls. The mean score for the grade eleven boys is slightly higher than the mean score for the grade twelve girls.

8. There is a wide overlapping of the score distribution for the three grades. The possible score for the test is 70. The individual scores range from 11 to 68. Twenty-five per cent of the grade ten students received a total score greater than the mean score for the grade eleven students and approximately twelve per cent of the grade ten students received scores higher than the mean score for grade twelve students. Thirty-six per cent of the grade eleven students received scores larger than the mean score for grade twelve students. On the lower range of the scale, thirty-four per cent of grade eleven students received scores lower than the mean scores for grade ten, and thirteen per cent of the grade twelve students had a score lower than

the mean score for grade ten. Twenty-three per cent of grade twelve students obtained scores below the mean score for grade eleven.

9. From grade to grade there is a possible overlapping of the mean scores for the girls, but the mean scores for the boys do not overlap.

10. For the entire testing program there is no one area of the six sub-tests in which either sex shows any superiority at the grade ten level. At the grade eleven level the only area in which there is no significant difference at the .01 level of confidence between the means of the girls and boys is in the ability to change the subject of a formula. At the grade twelve level there are three areas in which there are no significant differences between the means of the girls and boys. These areas are respectively: the ability to recognize functional relationships in some generalized aspects of functionality, recognition of functional relationship between two or more variables, the ability to express that relationship as a formula, and the ability to change the subject of a formula.

11. For the girls there was only one area of the test in which there was a significant difference between the means of grade ten and eleven students. This was in the area of changing the subject of a formula. This was probably due to greater facility in manipulative ability than to understanding of functional relationships involved. Between grades eleven and twelve there are two areas where there is no significant difference between the means at the .01 confidence level.

In the first of these, the ability to derive a formula from a table of values, there is some improvement in the mean scores from the end of grade eleven to the end of grade twelve. In the area of functional relationship as expressed by graphs the mean of the grade eleven girls was better at the .03 level of confidence than the mean of the grade twelve girls.

12. For the boys there is no significant difference between the means of successive grades in the generalized aspects of functionality and in the ability to express functional relationship in graphical representation. In all other areas there is a significant difference between the means at the .01 level of confidence from grade to grade.

13. For the population tested, the only area that does not show a significant difference between the means of successive grades is in the interpretation of functional relationship by graphical representation.

14. In the ranking of the six sub-tests there were only two areas which had the same ranking for each grade and for the entire population. These were the area of generalized functionality and the recognition of how a change in one variable affects another. The ranking for the grade eleven students on the six sub-tests was the same as the ranking for the entire population tested.

15. For the girls in grades ten and eleven there were only two areas in which these girls received a mastery coefficient of 60 per cent or better. For each grade a more general aspect of functionality and the recognition of how a change in one variable affects another were the

only areas in which a mastery coefficient of 60 per cent were obtained. For the grade twelve girls there were two areas in which they did not attain a mastery coefficient of 60 per cent. These areas were respectively the ability to derive a formula from a table of values and functional relationship as expressed by graphs.

16. For the grade ten boys there were three areas in which they did not attain a mastery coefficient of 60 per cent. This was in the sub-test that dealt with the recognition of functional relationships and the expression of the relationship as a formula; the ability to derive a formula from a table of values; and the ability to change the subject of a formula. In grade eleven there were two areas, namely the ability to derive a formula from a table of values and the ability to change the subject of a formula. For grade twelve boys the mastery coefficients were in excess of 71 per cent.

17. With two exceptions the mastery coefficients for each grade-sex group in each sub-test increases from grade to grade. The two exceptions, both arising in sub-test VI, are the sharp decline in the mastery coefficient for the grade twelve girls, and the decrease in the total mastery coefficient for grade twelve. The latter is due to the pronounced decrease in the mastery coefficients for the grade twelve girls.

18. In the ability to determine the required variables inherent in a verbal statement and to express these variables in an algebraic formula, the mastery coefficients for grade ten were quite low. For

the eleven questions in the group there were only two at the grade ten level for which the mastery coefficients were in excess of sixty per cent, eight in grade eleven and nine in grade twelve. Of the two abilities involved in Sub-test II the ability to arrange the variables into an algebraic formula was more difficult than the selection of the variables involved in the functional situation.

19. Direct variation between two variables was the concept best understood by the students of each grade-sex group. This was in turn followed by direct variation where the independent variable is a square, inverse variation, and finally direct variation to a square root.

20. With the possible exception of direct variation, functional relationship as expressed by a table of values is not developed to any appreciable understanding at even the grade twelve level. If direct variation where one of the variables is a square was deleted from the questions that were given in the test, the remaining questions would in most cases show mastery coefficients below fifty per cent.

21. Prior to grade twelve, the ability to change the subject of a formula is not developed to any great extent. In the majority of cases the mastery coefficients in grades ten and eleven were below fifty per cent. With one exception, the total mastery coefficients for grade twelve were better than seventy-seven per cent.

22. For graphical representation of functional relationship the graph of direct proportion was the easiest type of graph to interpret. In reading the information from the graph, the students at each level

had mastery coefficients in excess of sixty per cent. In the interpretation of the corresponding algebraic formula from the graph, the mastery coefficients for the girls at each grade level were below fifty per cent. For the boys the mastery coefficients were 55.55, 70.86 and 72.50 for grades ten, eleven and twelve respectively. The results of the final two graphs indicate that the students can read with reasonable accuracy the quantities or variables represented but have little understanding of such concepts as gradient, the table of values, or the formula relating the variables.

Conclusions in the Findings of this Investigation

As a result of the investigation the following are the conclusions:

1. Throughout the high school grades, there is a measureable increase in the understanding of the function concept by high school students. The increase in the mean from grade to grade was regular and significant at the .01 level of confidence.

2. With one exception, increases in the means of the individual sub-tests were recorded. The increases, however, were not regular for each of the phases of functionality tested. In three sub-tests the increase in the mean from grade to grade was regular. The differences of the means on the remaining sub-tests did not exceed .75. The indications are that there is a fairly regular increase from grade to grade in the ability of high school students to do relational thinking in the various aspects of functionality.

3. The students of the high school have an understanding of a fair range of functional concepts. Of the concepts presented the grade ten students answered fifty-five per cent of the questions correctly, the grade eleven students answered sixty-five per cent correctly, and the grade twelve students answered seventy-five per cent correctly.

4. In grade ten neither sex showed any superiority in relational thinking. In grade eleven the boys far outstripped the girls in the ability to think functionally. In grade twelve the girls began to regain some of the ground they lost in grade eleven.

5. Functionality must be taught rather than left to chance. Many of the situations presented in the test the students had encountered throughout their junior and senior high school grades. When students were asked to think about these situations in terms of functionality it soon became evident that relational thinking is not regarded as the unifying element in high school mathematics. There was too much attempt to arrive at a solution through mechanical manipulation.

6. To a great extent, depth in functional thinking seems to be lacking at the high school level. In the initial testing program one test contained two questions which were subsequently discarded because the difficulty index was .00. The questions tested whether or not there was a relationship between the length of the side of a square and the perimeter of the same square, and between the length

of a side of a square and its area. In the final test a question was included which tested the student's ability to determine whether or not there is a relationship between the perimeter and the area of a square. The question was answered correctly by twenty-three per cent of the grade ten students, thirty-one per cent of the grade eleven students, and forty-seven per cent of the grade twelve students. Students are not challenged sufficiently to think in terms of functional situations, and there appears to be little attempt to adapt the processes of functional thinking to problem situations.

7. Functional thinking as applied to tables of values, and to changing the subject of a formula is not developed to any great extent prior to grade twelve. Mastery coefficients of 30.45 and 33.27 respectively for grade ten students, and 45.20 and 55.52 for grade eleven students, indicate low ability in these two areas of functional thinking.

8. In the process of functional thinking the students develop at an early stage of high school training the ability to select the component parts of a functional situation but do not develop to the same extent in their ability to re-arrange these parts into an algebraic formula. For each grade the ability to select the variables of a functional situation was approximately twice as great as was the ability to arrange these variables into an algebraic formula. It was evident from many of the responses given to functional situations that the students depended a great deal on rote learning rather than on functional understanding.

9. From the above eight observations it is clear that courses in mathematics must be so planned and developed to present to the students a continuous program of conscious functional situations and to provide for growth in the function concept throughout the high school grades.

AREAS IN WHICH FURTHER INVESTIGATIONS SHOULD BE DEVELOPED

Much has been written about the development and uses of the function concept but with respect to this concept there are very few investigations that have been made at either the elementary, junior high, senior high school or university level. In order to create a better understanding and greater use of the function concept in the teaching of mathematics many additional investigations should be made. A few investigations that might be suggested as the result of this investigation are as follows:

1. Investigation into the development of various phases of functionality and the growth of the functional concepts throughout the elementary, junior, and senior high school grades.

2. Investigation into the method of presenting functional situations and concepts to students at the elementary, junior and senior high school grades.

3. Investigations into the development of courses of studies in mathematics at the elementary, junior, and senior high school level

based on the function concept.

4. Investigation into the use of the function concept in the solution of problems.

5. The function concept as applied to geometry.

6. The relationship between a student's intelligence and his ability to interpret functional situations in mathematics.

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APPENDIX I

INSTRUCTIONS FOR GIVING THE TEST

1. The maximum time allotted for the entire test is 50 minutes. This does not include the time required to distribute the booklets and for the teachers' instructions to the students.

2. Each student is to be issued one booklet and is asked to place it face downward on the desk until such time that he is instructed to complete the information at the beginning of the test.

3. As soon as the students have been given the signal to begin the test, do not make any comments about the test and do not answer any questions by the students.

4. Each student should have either a pen or pencil - preferably a pencil.

5. Read the following explanations and instructions to the students:

"Each of you have before you a test which attempts to measure your ability to express how two or more values or quantities are related and how changes in any one of these will affect changes in others. If, for example, you went to a store and bought five pounds of meat then the total cost of the meat would depend upon the cost per pound. Or, if the cost per pound of meat was doubled then the total cost of the meat would be doubled.

The test is divided into six sub-tests. At the beginning of each read the directions carefully and then complete the sub-tests. In some cases, as a guide, sample questions precede the sub-test. When you have completed the first sub-test then immediately continue with the second sub-test. You must do all six sub-tests within a fifty minute time limit.

You must observe the following rules:

a. Write your name, the name of your school, your grade, date, and whether male or female in the proper space provided at the beginning of the test.

b. You may use either pen or pencil - preferably pencil.

c. Do not begin the test until you have been instructed to do so.

d. If an error has been made black out the incorrect response and mark the answer you believe to be correct.

e. You are to read the directions preceding each sub-test and then complete the sub-test. Work steadily until each sub-test has been completed.

f. Do not guess at the answers.

g. Now turn your booklet and complete the information at the top of the first page. When you have completed the information then turn the booklet face downward.

h. We are now ready for our test. Turn the booklet and...begin. The fifty minute time is taken from this point."

APPENDIX II

STUDENT'S NAME ______________________ GRADE __________ SEX ________

SCHOOL ______________________________ DATE __________ SCORE ________

FUNCTION TEST IN MATHEMATICS

SUB-TEST I

DIRECTIONS: In the parenthesis after each of the following statements, write "T" if the statement is true, and "F" if the statement is false.

1. If the diameter of a wheel is increased, the number of revolutions the wheel would make in travelling one mile is decreased. ( )
2. An increase in the number of base hits by a batter decreases his batting average. ( )
3. The rate of depreciation of a machine depends upon the length of the expected life of the machine. ( )
4. The profit on an article is equal to the difference between the selling price and the cost price of the article. ( )
5. An increase in the number of pages in a yearbook decreases the cost of publishing the book. ( )
6. If A = 36/B, a decrease in the value of B decreases the value of A. ( )
7. If $100 is placed in a bank at the beginning of a year and is allowed to accumulate interest for one year then the simple interest at 4 per cent is greater than the interest at 4 per cent compounded semi-annually. ( )
8. If a tree 40 feet tall casts a shadow of 20 feet, then a stick 10 feet tall casts a shadow of 5 feet. ( )

9. If $20 + x = + y$ then an increase in the value of x will increase the value of y. ( )

10. The perimeter of a square depends upon the area of the square. ( )

SUB-TEST II

DIRECTIONS: In each of the following situations one quantity depends upon one or more other quantities for its value. Fill in the blank spaces stating the quantity or quantities that are required for the solution of each problem. Let each quantity be represented by a letter and use the letter or letters you have chosen to write a formula which would represent the relationship of the quantities. (If an area is involved, do not just state area but show how that area is obtained.)

Sample: An express agent wishes to determine the total weight (T) of a number of articles each weighing the same amount. What information would be required in order to determine the total weight?

1. The weight of one article . . . . (W)
2. The number of articles . . . . (N)
3.

Formula: $T = WN$

11. The difference (D) in area between two circles would depend upon:

1.

2.

3.

Formula: $D =$

12. The perimeter (P) of any triangle would depend upon:

1.

2.

3.

Formula: $P =$

13. The volume (V) of a regular cube will depend upon:

1.

2.

3.

Formula: V =

14. During a motor trip I averaged 15 miles to the gallon. The number of gallons (G) I used during the trip would depend upon:

1.

2.

3.

Formula: G =

15. The number of dozens (D) of oranges I can buy for $1.00 will depend upon:

1.

2.

3.

Formula: D =

16. A tinsmith wishes to determine the area (A) of a circular piece of tin. What must he know in order to determine the area of the piece of tin?

1.

2.

3.

Formula: A =

17. At the beginning of the year James deposited $200 in a bank account which he has just opened. No further deposits or withdrawals were made. The first interest payment was made at the end of the first six months. What further information would be required in

order to determine the total amount of money (A) in the account after the interest payment has been made?

1.

2.

3.

Formula: A =

18. A car leaves a town at 9.00 a.m. and by travelling continuously arrives at a second town at noon. The distance (D) travelled would depend upon:

1.

2.

3.

Formula: D =

19. If you received your allowance at the beginning of the month and you earned a certain sum of money during the month, what would you need to know in order to determine what amount of money you spent (S) during the month?

1.

2.

3.

Formula: S =

20. The circumference (C) of a circle will depend upon:

1.

2.

3.

Formula: C =

21. The ratio (R) of completed passes thrown by a quarterback will depend upon:

1.

2.

3.

Formula: R =

SUB-TEST III

DIRECTIONS: In the following group of questions circle the answer to the right of each question which best answers the question. The value to the right represents either the amount or the number of times by which the function will increase or decrease.

22 - 26. The total cost (C) of a number of articles will depend upon the number (n) of articles purchased and the price (p) per article. This is represented by the formula $C = np$. What changes take place in C if:

22. n is doubled and p remains the same?	no change, 1/2, 1/4, 2, 4.
23. p is doubled and n remains the same?	no change, 1/2, 1/4, 2, 4.
24. Both n and p are doubled?	no change, 1/2, 1/4, 2, 4.
25. n is doubled and p is trebled?	no change, 2/3, 4, 6, 8.
26. n is doubled and p is halved?	no change, 2, 3, 4, 6.

27 - 30. The area (A) of a square with side (X) is represented by the formula $A = X^2$. What change takes place in A if X is:

27. Doubled?	1/2, 1/4, 2, 4, 8.
28. Trebled?	2, 3, 4, 6, 9.
29. Halved?	1/2, 1/4, 2, 4, 8.
30. By how much greater will X be if A is made four times as great?	1/2, 1/4, 2, 4, 8.

31 - 35. The time (T) to make a certain journey depends upon the distance (D) travelled and the average rate (R) per hour. This is represented by the formula $T = D/R$. What changes take place in T if:

31. D is doubled and R remains the same?	no change, 1/2, 2, 3, 4.
32. R is trebled and D is doubled?	no change, 2/3, 3/2, 2, 3.
33. Both D and R are doubled?	no change, 1/2, 1/4, 2, 4.

34. D is halved and R is doubled? no change, 1/2, 1/4, 2, 4.

35. D is doubled and R is halved? no change, 1/2, 1/4, 2, 4.

36 - 40. If $y = 10/x$, what changes take place in y if x is:

36. Doubled? 1/2, 1/3, 1/4, 2, 4.

37. Trebled? 1/2, 1/3, 1/4, 2, 3.

38. Halved? 1/2, 1/4, 2, 4, 6.

39. Whatever value is given to x the product of that value and the corresponding value of y must always equal:. 1, 2, 5, 10, 15.

40. If x is given a number of values as 1, 1/2, 1/3, 1/4 etc., then each value of y will differ from its previous value by: 1/2, 1/3, 2, 5, 10.

41 - 45. If the principal depends upon the interest, rate, and time we might express the relationship by the formula $P = I/RT$ where P represents the principal, I the interest per year, R the yearly rate of interest and T the time in years. What is the effect on P if:

41. I is doubled and R and T remain the same? no change, 2, 3, 4, 8.

42. R is halved and I and T remain the same? no change, 1/2, 2, 3, 4.

43. R is halved, T is doubled and I remains the same? no change, 1/2, 1/4, 2, 3.

44. I, R and T are each doubled? no change, 1/2, 1/4, 2, 3.

45. T is doubled, R is halved, T is trebled? no change, 1/2, 1/4, 4/3, 2.

SUB-TEST IV

DIRECTIONS: Study each of the following groups of numbers. The values for the last mentioned letter for each group depends for its value upon the other values in the group. Write the formula showing how the last letter in each group depends upon the remaining letters of the group.

46.

F	1	2	3	4	6
S	1	4	9	16	36

S =

47.

X	1	2	3	4	5
Y	3	5	7	9	11

Y =

48.

X	2	4	5	6	8	
Y	3	4	4	3	5	
A	3	8	10	9	20	A =

49.

P	100	1000	2000	3000	
R	.03	.04	.05	.05	
A	103	1040	2100	3150	A =

50.

A	1	2	3	4	6	
T	1	8	27	64	216	T =

51.

A	2	2	3	4	5	
W	2	5	4	6	5	
P	8	14	14	20	20	P =

SUB-TEST V

DIRECTIONS: Sometimes it becomes necessary to change a formula in terms of some other quantity. Re-write each of the following in terms of the quantity stated.

Sample: If A = b - c, then b = A ≠ c

52. If P = 12S then S = ____________

53. If y = mx ≠ b then x = ____________

54. If A = 40/B then B = ____________

55. If A = K^2 then K = ____________

56. If V = ABC then A = ____________

SUB-TEST VI

DIRECTIONS: Study carefully each of the following graphs and then answer the questions pertaining to each graph. Write your answer in the space to the right of the question.

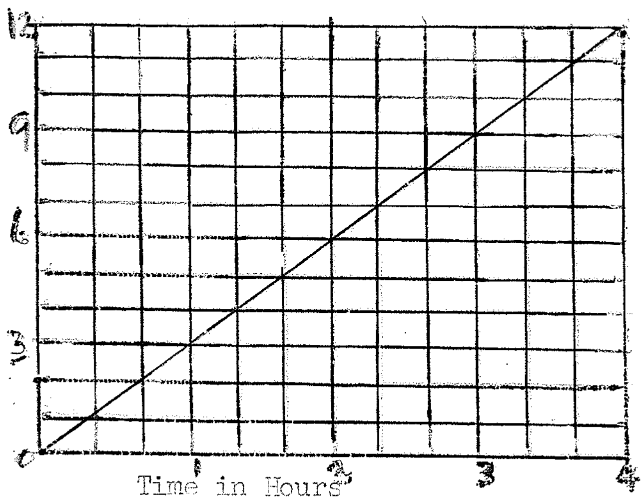
57 - 61. Below is a graph which James prepared after a hiking trip with some of his friends. He attempted to show the relationship

between the distance and the time walked.

Distance

in

Mil s



57. What was the average rate per hour they walked? __________

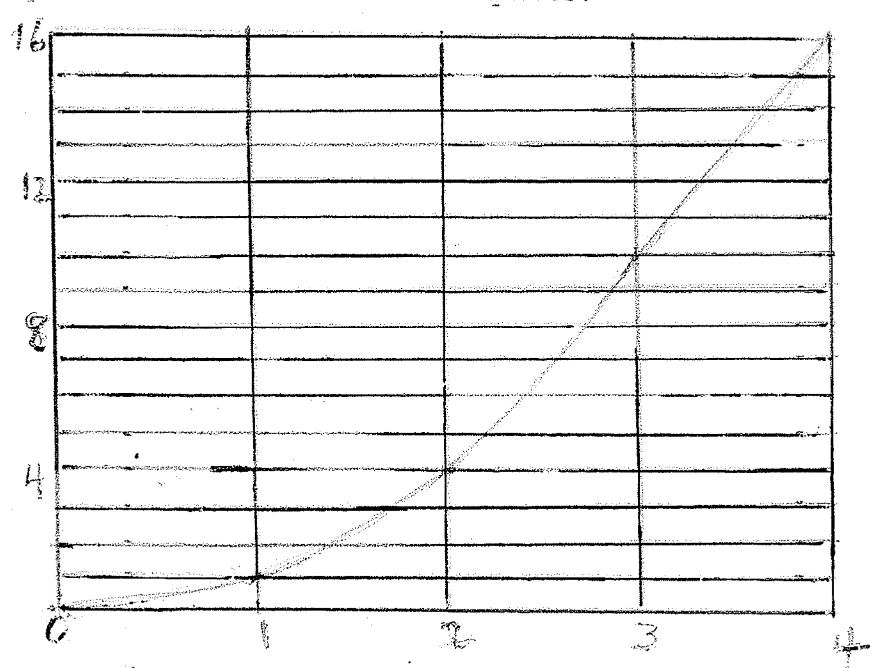
58. How many hours did they walk? __________

59. How far did they walk in 2 hours 20 minutes? __________

60. How long did it take them to walk 10 miles? __________

61. Write the formula that would represent the relationship between the distance and the time. __________

62 - 66. A boy has four squares each of different size. He plotted the relationship between the length of the sides of the squares and the areas of the squares.



62. What do the numbers at the bottom of the graph represent? ________

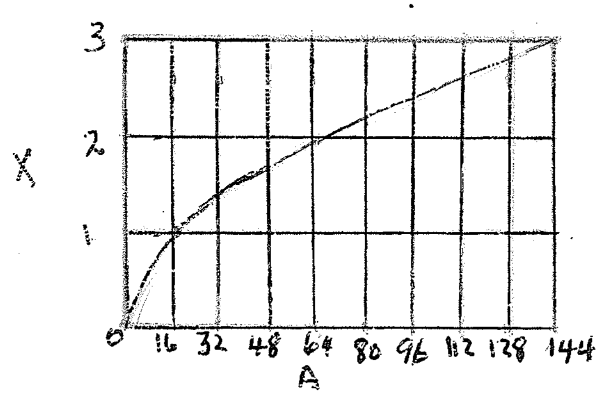
63. What do the numbers along the side of the graph represent? ________

64. Read from the graph the area of a square 3 inches in length. ______

65. What is the increase in the differences between the areas of the squares as the sides of the square increase by one unit? ______

66. Write the formula that would show the relationship between these two groups of values. ____________

67 - 70. Study the graph below and answer the questions that are based on this graph.



67. As the value of A increases the value of X ____________________

68. As X varies from 0 to 3, A varies from __________ to ___________

69. Complete the table of corresponding values of A and X.

X	0	2	1
A			

70. Write the formula that would illustrate the relationship between X and A. ________________________

APPENDIX III

THE STATISTICAL FORMULAS USED IN THE INVESTIGATION

1. $M = \frac{\sum fx}{N}$

2. $SD = \sqrt{\frac{fx^2}{N}}$

3. $SE_{mean} = \frac{SD}{\sqrt{N}}$

4. $SE_{diff} = \sqrt{\sigma^2_{M_1} + \sigma^2_{M_2}}$

5. $CR = t = D/\sigma_D$

6. $$r = \frac{\sum f_x d_x f_y d_y - \frac{\sum f_x d_x \cdot \sum f_y d_y}{N}}{\sqrt{\left[\sum f_x d_x^2 - \frac{(\sum f_x d_x)^2}{N}\right]\left[\sum f_y d_y^2 - \frac{(\sum f_y d_y)^2}{N}\right]}}$$

7. $\text{Reliability} = 1 - \frac{\sum x^2}{N\sigma^2}$