THE DEVELOPMENT OF A SEMI-AUTOMATIC POTENTIAL ANALOG APPARATUS AND ITS APPLICATION TO SOME FILTER APPROXIMATION PROBLEMS

A Thesis

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by

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ABSTRACT

This thesis presents the development of a semi-automatic potential analog apparatus which employs carbon-impregnated conducting "Teledeltos" paper and an X-Y recorder, gives a derivation--by means of the potential analogy--of functions which yield solutions to low-pass approximation problems in image parameter (and insertion loss lossless filter theory) and shows how the apparatus may be used as an aid in their practical determination. Techniques of measurement using quarter planes are investigated and the results presented. Conformal transformations (of the complex plane) are studied in order to find some which have special practical utility with the apparatus, and examples of recordings (using these planes) which yielded suitable locations for points of infinite loss are presented and compared. These--and a few other recordings--show the feasibility of using the apparatus for that purpose as well as others of passive filter (and other network) synthesis problems.

The feasibility of applications of the potential analog apparatus as an aid in solving low-pass approximation problems of lossless filter theory is studied, and the practicability of the apparatus for this purpose is evaluated and compared with templates and digital computer methods. It is found that the apparatus is more practical and sufficiently accurate in many cases. One such case, which is emphasized in this thesis, is the determination of locations of points of infinite loss so as to give prescribed attenuation characteristics.

Formulae for measurement of coefficients in partial fraction

expansions of rational functions with single and double poles are derived, and examples of actual measurements (performed manually) are presented. These illustrate the necessity for the semi-automatic analog apparatus.

PREFACE

This thesis originated with the desire to investigate applications of a potential analog apparatus originally built at the University of Manitoba by Professor E. Bridges under the supervision of Professor R. A. Johnson and with the sponsorship of the National Research Council of Canada (Grant NRCG584-1956)*. This apparatus was not suitable for most practical applications, so extensive modifications were undertaken to make it semi-automatic. Manual measurements, presented in this thesis, also showed the need for such an apparatus. After the development of this apparatus had been completed, a study of the literature on network synthesis--especially filter theory--was undertaken in order to find possible applications of the apparatus in solving approximation problems. A few of the possible applications of the potential analog apparatus are presented in this thesis. The suitability of conformal mappings for use with this apparatus are studied. Actual recordings obtained with the aid of the apparatus are presented, and its use is compared with template and computer methods.

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*E. Bridges, "A Network-Function Simulator" (Unpublished Master's Thesis, The University of Manitoba, Winnipeg, September 1958).

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CHAPTER I

THE AIMS, THEIR VALIDATION, DEFINITION OF IMPORTANT TERMS AND THE ORGANIZATION

This thesis describes the evolution of a new semi-automatic potential analog apparatus suitable for network and control system synthesis applications, gives the derivation of functions suitable for approximation of low-pass image-parameter (and, in well known ways, of low-pass insertion attenuation) functions, and presents applications of the potential analog apparatus as a design aid in determining these functions. In this thesis the presentation and/or derivation, by means of the potential analogy, of functions needed for lumped-element low-pass image parameter (and insertion loss) lossless filter synthesis and the use of the potential analog apparatus as a practical design aid in this work, is given greatest emphasis.

The functions desired are functions of a complex variable, $p=\sigma+j\omega$, usually normalized to $\omega_c=1$. In chapter V Ω is sometimes used instead of the normalized ω. The matter of independent complex variables should, however, cause no trouble.

Because of the nature of the topics, the complete method of their presentation, and their length, a historical background is not presented here, but is given in each chapter which deals with a particular topic. Further help in placing this work in proper perspective and relation to that previously done, and in showing the status of the topics up until the present time, is given by footnote references and validations and comparisons presented in each chapter of this thesis. The scope of the undertaking for each topic is also defined in the introductory section(s) of each chapter, so extra verbiage is not presented here, except that the specific aims of this thesis will be stated at the end of this chapter. These aims, when considered in relation to the historical background and justifications of this study--presented in the introductory sections of the chapters--should give the reader an initial insight into the importance and relations of the topics covered in this thesis.

I. REASONS FOR THE STUDY, AND ORGANIZATION OF THE THESIS

One of the reasons for this study was the need for a potential analog apparatus more suitable for network and control systems than any that had been previously developed. Thus, most previous analog devices were too inaccurate and tedious to work with, and yielded recordings or plots which were too small to be of practical value, or were restricted, in that only one type of conformal mapping could be used. With this need there was the desire to investigate various types of measurement techniques for potential and stream functions, and to try out the new potential analog apparatus to be described herein, in order to compare its effectiveness and practicality against point-by-point measurement techniques, as well as to investigate its over-all accuracy and modes of operation. Before the apparatus was built it was desirable to search for those features which a semi-automatic potential analog apparatus should have in order to be most useful in the contemplated fields--especially network synthesis. After the apparatus was built its errors had to be considered, its over-all accuracy assured, and further improvements suggested.

The next three chapters are concerned with the topics of the previous paragraph. Chapter II gives point-by-point measurements of phase and magnitude functions on quarter planes, and discusses and illustrates the use of various types of measurements using these quarter planes. Chapter III deals with measurement of residues of rational functions using a half plane. Equations for such measurements are derived, and point-by-point measurements carried out. Both chapters II and III demonstrate the tediousness and the rather mediocre accuracy achievable with these methods. They also give a physical appreciation of the analog. Chapter IV deals with some of the historical background of potential analog devices for network synthesis and analysis, a study of desired features of such an apparatus, with the development of a new apparatus, and with a study of its errors and over-all accuracy. Many details of the development, the calibration, and the errors, are presented in the appendices. Further improvements are suggested in chapter IV.

Another need seemed to be a thorough investigation of possible practical applications of the apparatus to solving approximation problems in filter synthesis.

The theory of the potential analogy was used in this thesis because it leads to application of the analog apparatus, and provides a physical concept for the rather complicated mathematics of approximation theory.

Chapter V is devoted to the derivation of approximating functions for image parameter filter theory by means of the potential analogy, to the investigation of possible applications of the potential analog apparatus as a design aid, to the actual implementation (with high accuracy) of the new potential analog apparatus to determination of points of infinite loss for low pass filters, to determinations and comparisons of conformal mappings to be used with the apparatus for low pass filter synthesis. Chapter VI gives a comparison of the use of the apparatus to the use of other methods yielding the same results. Although image-parameter filters were dealt with specifically, it is well-known that the approximation of image attenuation is usually used as a high quality approximation to insertion loss in the stop band(s).

II. THE DEFINITION OF THE IMPORTANT FILTER THEORY FUNCTIONS TO BE APPROXIMATED

Before the specific aims are given, a few words should be said about some important functions used in dealing with filter approximation theory.

Firstly, in dealing with image parameter filter theory one encounters the following functions of p, which are defined and described in a book by Cauer:[1]

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[1] W. Cauer, Theory of Linear Communication Networks, Vols. I andII (McGraw-Hill Book Company, Inc., Toronto, 1958), pp.121-142 and 221-291.

1. The two-terminal-pair-network propagation function, $\Gamma_1 = \alpha + j\beta$, usually abbreviated to image propagation function. α is also abbreviated to image attenuation and β to image phase.

2. The q functions, which are defined by

$$q = \sqrt{z_{11}y_{11}} = \sqrt{z_{22}y_{22}} = \coth \Gamma_1 ,$$

where z_{11} and y_{11} are, respectively, primary open- and short-circuit reactance and admittance, and z_{22} and y_{22} are the secondary open- and short-circuit reactance and admittance of a two-port lossless network. For symmetric filters one obtains

$$q = \coth \frac{\Gamma_1}{2} = \sqrt{\frac{z_a}{z_b}} ,$$

where z_b is the impedance-normalized horizontal lattice reactance, and z_a is the impedance-normalized cross-arm lattice reactance of the equivalent lattice.

3. The q' functions, defined by

$$q' = \coth\left(\frac{\Gamma_1}{2} \mp j\frac{\pi}{4}\right).$$

4. The image impedance functions

$$z_{o1} = \sqrt{\frac{z_{11}}{y_{11}}} \quad \text{and} \quad z_{o2} = \sqrt{\frac{z_{22}}{y_{22}}}$$

for unsymmetric filters, and

$$z_o = \sqrt{z_a\, z_b}$$

for symmetric filters.

Chapter V is largely concerned with the derivation of the above image parameter functions for most lossless filters by means of the potential analogy, and with their practical determination with the aid of a potential analog apparatus.

Although chapter V deals with approximation of low-pass image-parameter functions, and the application of the new potential analog apparatus to approximation of prescribed attenuation specifications, there are important relations between this theory and insertion loss theory. These are not covered in this thesis except for a few introductory statements. It should be noted, however, that the recordings of image attenuation apply to approximation of stop-band operating loss A(p), through the relation $A(p) \cong 8.686 |\alpha| - 6.02 - A_{RL}$ (in db), where A_{RL} is the return loss in the pass band.

The interested reader can find a very comprehensive, integrated and general treatment of the approximation problems of image- and insertion-loss filter theory, derived by means of the potential analogy, in a concurrent research report submitted to the National Research Council of Canada.[2] This report gives, for example, a potential analog derivation and extension of work due to Watanabe,[3] relates Watanabe's work to image-parameter theory as well as other works on insertion-loss theory, and gives additional material on solution of approximation problems of passive and active network and automatic control system synthesis.

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[2] M. Sablatash. "The Development of a Semi-Automatic Potential Analog Apparatus, an Integrated Approximation Theory for Filters Derived by Means of the Potential Analogy, and the Application of the Apparatus as an Aid in its Practical Implementation," August, 1963. A report submitted concurrently to the National Research Council of Canada under NRC Grant No. A738. Copies of the report may also be obtained from the author, presently employed by Northern Electric Co. Research and Development Laboratories, Ottawa, Canada, and from the library of the same company. Forthcoming publications will give these findings wider dissemination to scientists and engineers. This report (the material of which can only be partially included in this thesis due to limitations in scope and size) should be read for full appreciation of the subject.

[3] Hitoshi Watanabe, "Approximation Theory for Filter Networks," *IRE Transactions on Circuit Theory*, Vol. CT-9, No. 3, September, 1961.

CHAPTER II

SOME METHODS OF MEASURING PHASE AND MAGNITUDE BY USING QUARTER PLANES

I. PHASE MEASUREMENT BY USING A QUARTER COMPLEX PLANE OR ITS LOGARITHMIC TRANSFORMATION

General Theory of the Method

There is an analogy between the total current crossing the path of integration on a uniformly conducting sheet representing the complex plane or its conformal mappings and the phase difference, between the initial and final points on the path, of the rational function F(s) of a complex variable represented on the analog.[1,2] The odd symmetry of sources and sinks with respect to short-circuited boundaries shows that the total current crossing the short-circuited imaginary axis of a quarter plane, between s = 0 and s = $j\omega$ should be proportional to the angle of $\frac{F(j\omega)}{F(-j\omega)}$, when a low-resistance return path is provided to the infinity equipotential.[3,4]

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[1]S. Darlington, The potential analogue method of network synthesis, Bell Telephone System Monograph 1860 (September 1951), p. 7.

[2]E. Bridges, "A Network-Function Simulator" (unpublished Master's Thesis, The University of Manitoba, Winnipeg, September 1958), p. 6.

[3]George L. Matthei, "A General Method for Synthesis of Filter Transfer Functions as Applied to L-C and R-C Filter Examples", Technical Report No. 39, (Electronics Research Laboratory, Stanford University, California, August 31, 1952), pp. 31-40.

[4]Darlington, op. cit., pp. 16-17.

Preparation of Quarter Planes and Method of Phase Measurement

The principles of the previous section led to the preparation of the quarter plane analog sheet and its logarithmic transformation shown respectively in Figures 2.1 and 2.3. A constant current unit, previously built at the University of Manitoba, was used to supply the sources and sinks and the method of measuring total current flowing between the copper shorting bar and the infinity equipotential shown in Figures 2.2 and 2.4 was employed.[5] This current should be proportional to the phase, at $s = j\omega$, of the rational function F(s) set up on the analog sheet if the copper bar from which the current is metered is in continuous contact with the imaginary axis between $s = 0$ and $s = j\omega$. A second shorting bar (used with the logarithmic plane only), between the mapping of $s = j\omega$ and the infinity equipotential shown in Figure 2.4 merely ensured a better short-circuit for the rest of the imaginary axis than might be provided by the silver paint on the V-shaped wedges shown in Figures 2.1 to 2.4. The quarter planes were uncorrected for the effects of non-isotropic sheet conductivity.[6]

The current was metered with the milliammeter at increments of ω equal to unity. Since the scale of ω in the complex quarter plane was taken as one radian = one cm, which was the spacing of the bottom's of the cut out V's in Figure 2.1, this current was metered every time the shorting bar had been moved along the $j\omega$-axis of the quarter plane by a one-centimeter increment, and at similar points on the mapping of

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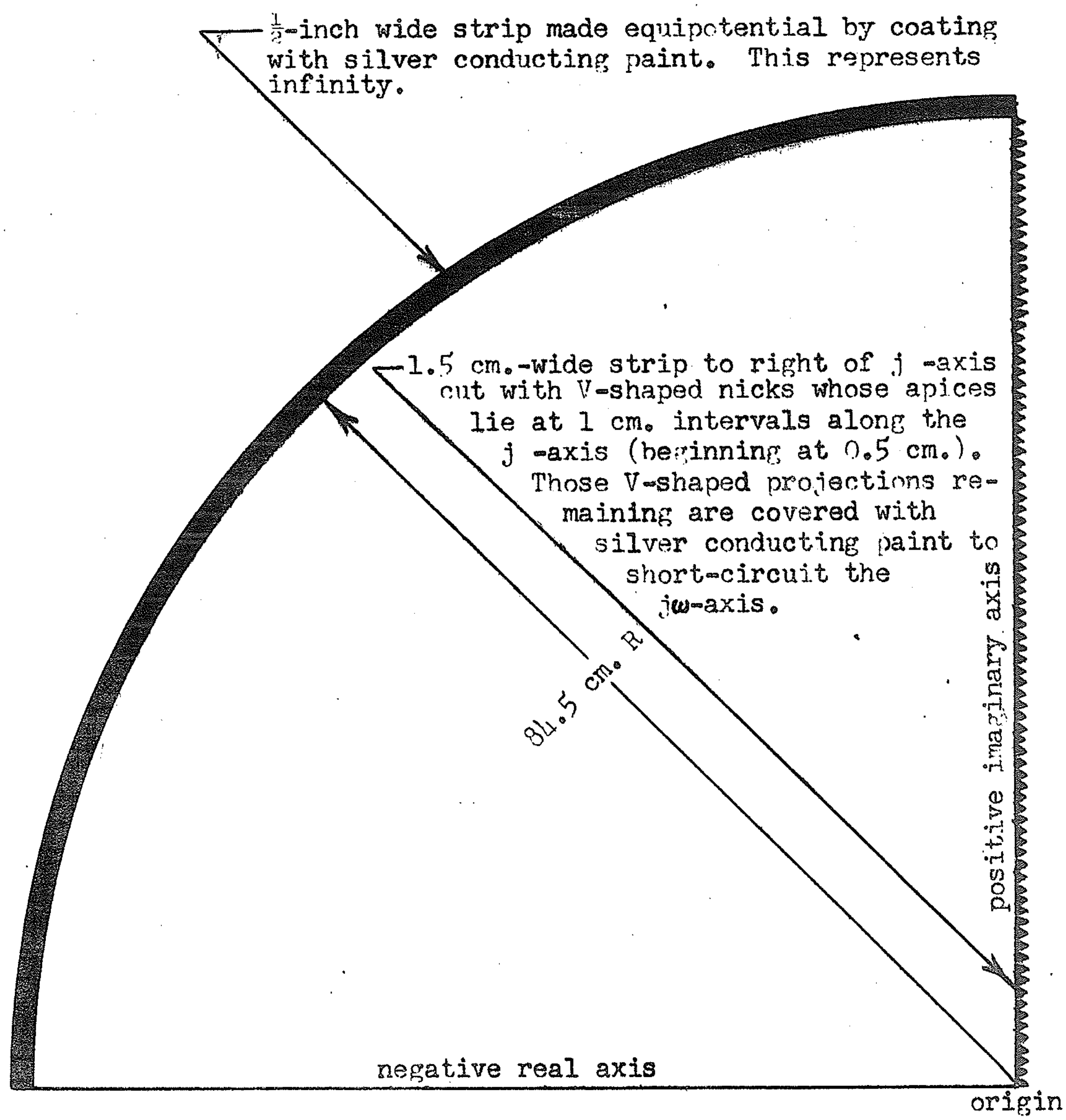
[5]Bridges, op. cit., pp. 19 and 28; figs. 2.10 - 2.14, pp. 23-27.

[6]Bridges, op. cit., pp. 68-69.



Notes:-1. Apices of nicks along jω-axis just separate the V-shaped projections.

2. Coated conducting carbon-impregnated Teledeltos paper type L-48, manufactured by Western Union Telegraph Company, used as material for above quarter plane analog sheet.

3. Method of phase measurement shown in Figure 2.2, p. .

Scale: ¼" = 4 cm.

FIGURE 2.1

QUARTER COMPLEX PLANE WITH SHORT-CIRCUITED jω-AXIS USED TO MEASURE PHASE OF RATIONAL FUNCTIONS OF A COMPLEX VARIABLE

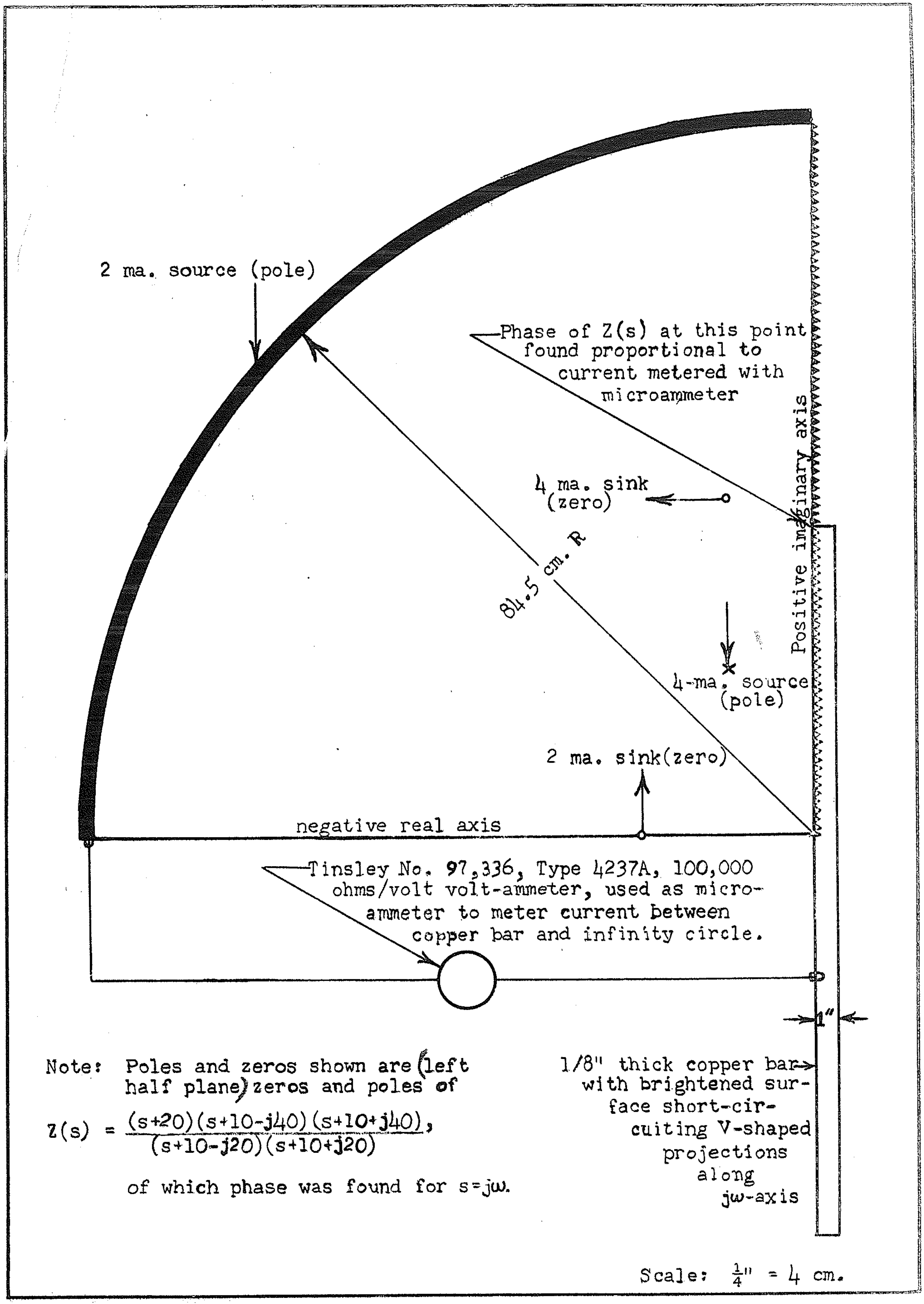
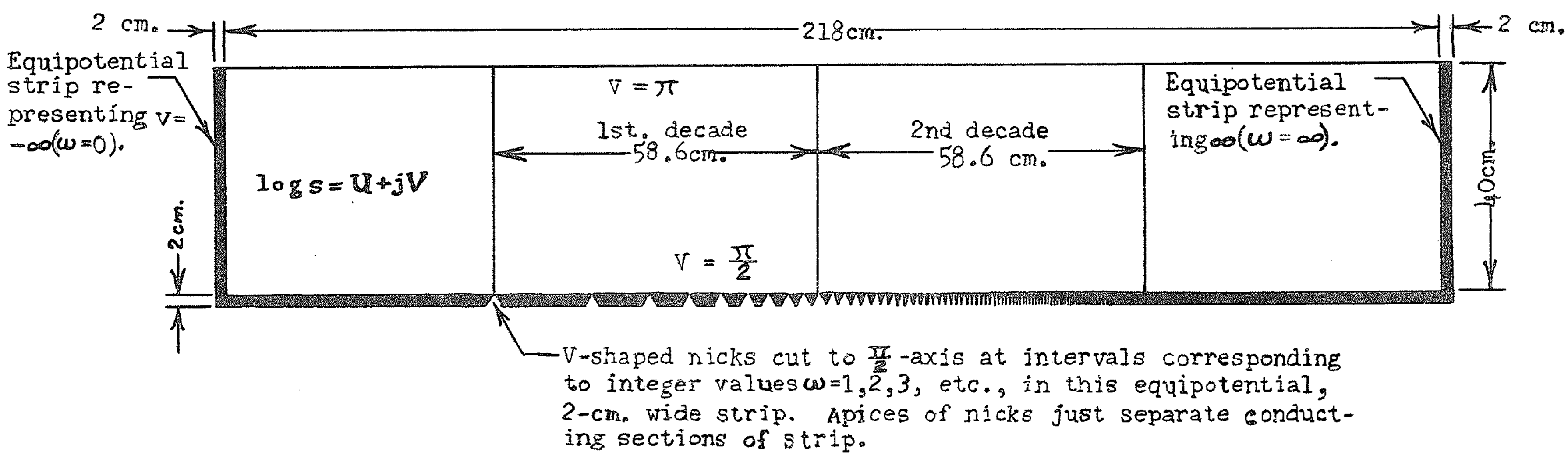


FIGURE 2.2
METHOD OF MEASURING PHASE OF RATIONAL FUNCTION OF A COMPLEX VARIABLE USING QUARTER PLANE WITH SHORT-CIRCUITED IMAGINARY AXIS



Notes:-1. Coated conducting carbon-impregnated Teledeltos paper Type L-48, manufactured by Western Union Telegraph Company, used as material for above logarithmic quarter plane analog sheet.

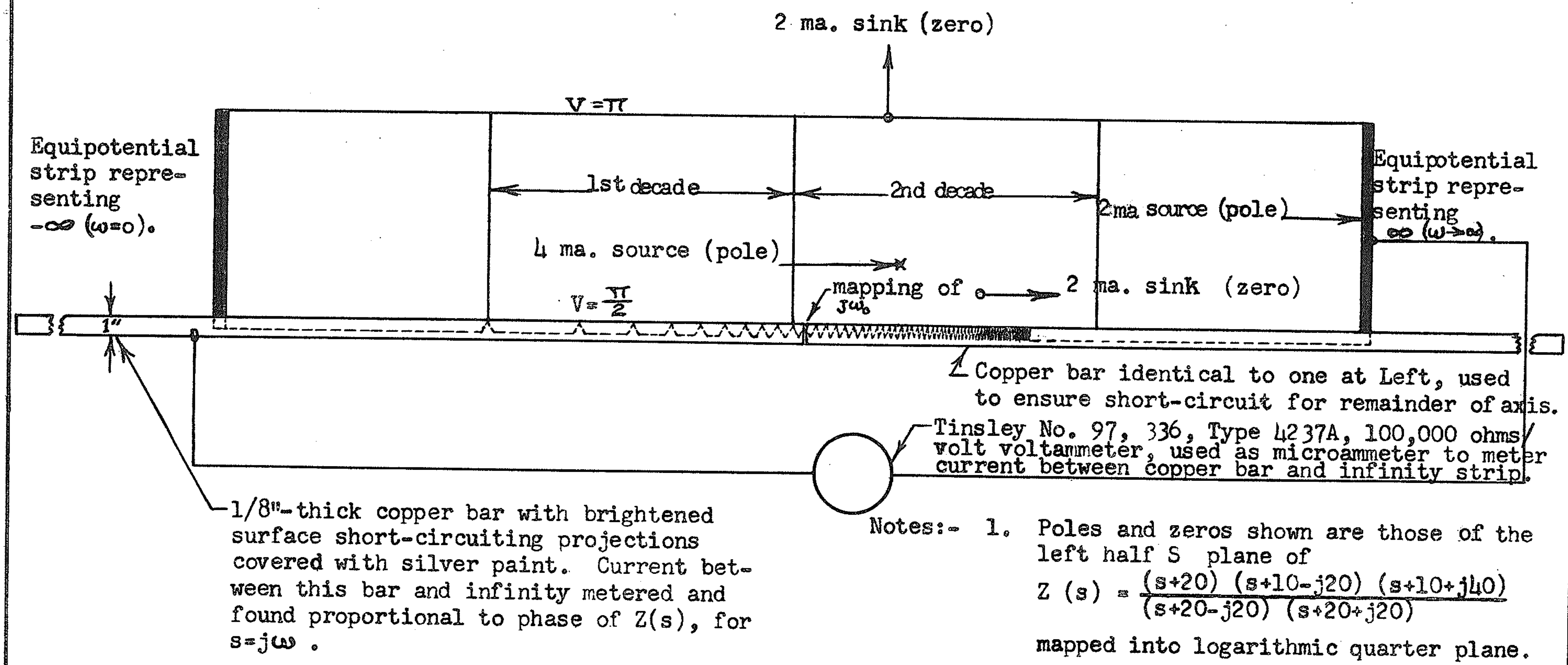
2. Equipotentials made by coating with silver conducting paint.

3. Method of phase measurement shown in Figure 2.4.

Scale: 1" = 30 cm.

FIGURE 2.3

QUARTER LOGARITHMIC PLANE WITH SHORT-CIRCUITED REAL FREQUENCY AXIS USED TO MEASURE PHASE OF RATIONAL FUNCTIONS OF A COMPLEX VARIABLE



Notes:-

1. Poles and zeros shown are those of the left half S plane of

$$Z(s) = \frac{(s+20)(s+10-j20)(s+10+j40)}{(s+20-j20)(s+20+j20)}$$

mapped into logarithmic quarter plane.

2. Phase of Z (s) was for $s=j\omega\,(v=\frac{\pi}{2})$.

3. Constant-current pole-zero supply used forsources and sinks.

4. Dimensions of analog sheet shown in Fig.2.3

Scale: 1" = 30 cm.

FIGURE 2.4
METHOD OF MEASURING PHASE OF RATIONAL FUNCTION OF A COMPLEXVARIABLE USING QUARTER LOGARITHMIC PLANE WITH SHORT-CIRCUITED AXIS

the s-plane $j\omega$-axis into the logarithmic quarter plane.

Calibration of Quarter Planes and Measurement of Phase of Test Function

The function s+20 was set up on each quarter plane, using (half-unit) current strengths of two ma., and plots obtained of degrees per milliampere vs. ω, with the milliammeter on the one milliampere range.[7] This plot for the quarter complex plane is shown in Figure 2.5, while that for its logarithmic mapping is shown in Figure 2.6. Figure 2.5 also shows the curves obtained using s+30 with 2 ma. and 6 ma. source and sink, and the mean of the three calibration curves. Figure 2.6 shows the plots for the function s+20, set up in the logarithmic transformation of the quarter complex plane with current strengths of two milliamperes. The ten-milliampere range of the milliammeter was employed. The two curves in Figure 2.6 may be compared to show the effect of changing the resistance of the milliammeter. Figures 2.5 and 2.6 thus give the calibration curves for the quarter planes.

The test function

$$Z(s) = \frac{(s+20)(s+10-j40)(s+10+j40)}{(s+10-j20)(s+10+j20)} \qquad (2.1)$$

was set up on each of the quarter planes, using unit current strengths of four milliamperes, and the readings of total current vs. ω obtained as described above.[8] Each reading was multiplied by the ordinate of the proper calibration curve at the same value of ω to obtain the phase

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[7] Since each decade of the logarithmic plane was 58.6 cm long, the finite pole and zero positions in this plane, in cm, were found from $\frac{58.6}{\ln 10} \ln s_i$, where s_i are the s-plane positions.

[8] p. 8.

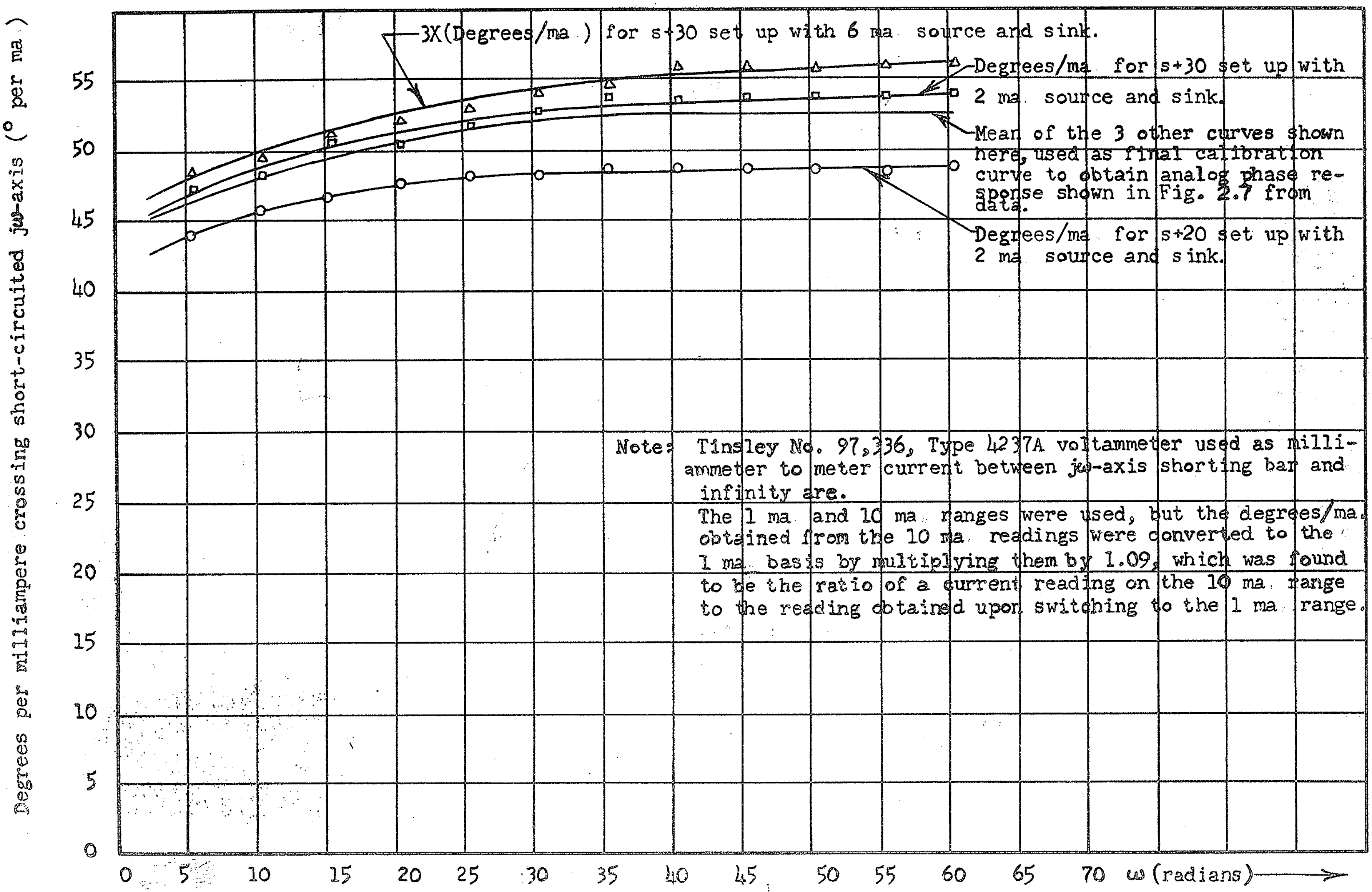

FIGURE 2.5

CALIBRATION CURVES FOR QUARTER COMPLEX PLANE USED FOR PHASE MEASUREMENT

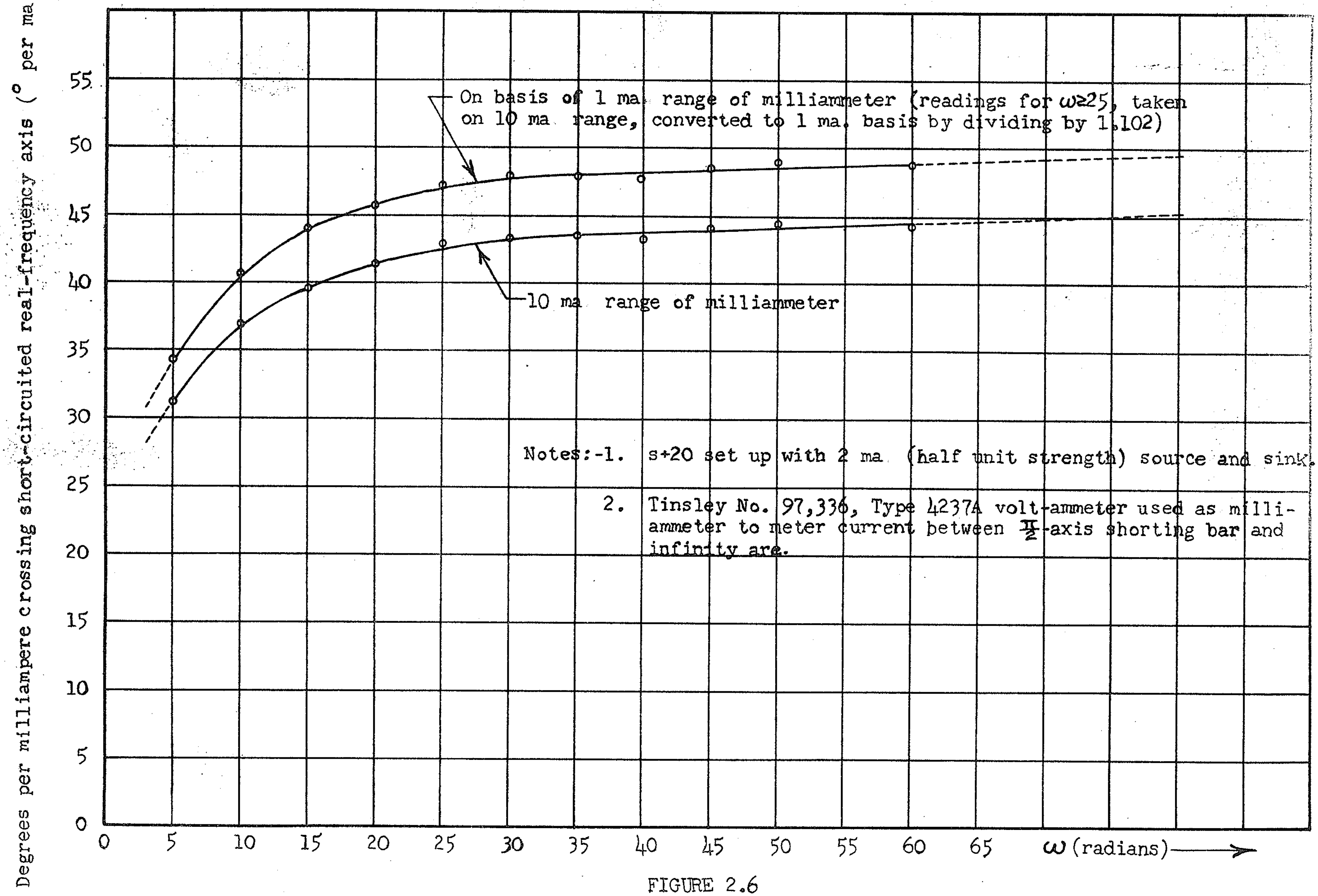

FIGURE 2.6

CALIBRATION CURVES FOR QUARTER LOGARITHMIC PLANE USED FOR PHASE MEASUREMENT of the test function. Figure 2.7 shows the actual phase of Z(s) and the measured phase obtained from the quarter plane complex analog, while Figure 2.8 shows the actual phase, and the measured phase obtained from the logarithmic quarter plane analog.

Discussion and Conclusions

The discrepancies in the three curves plotted in Figure 2.5 were due in large part to the use of an incorrect constant in converting from the readings obtained using the ten ma. range of the meter to those which would have been obtained on the one ma. range. Readings obtained in the two cases are different because of the different meter resistances. Thus, a constant multiplier must be determined from simultaneous measurements on the two scales, and used in conversion from one basis to another. Since only one such reading was taken, this multiplier may have been inaccurate. Furthermore, the technique of measurement had not been too well-developed at this time; changes in the measured current were detected when the shorting bar was held with bare hands, or when the pressure on this bar was insufficient to give good contact; furthermore, the constant current sources and sinks may have drifted a little over the hour or so that measurements were taken.

In the case of the logarithmic quarter plane, the conversion factor from 1 ma. to 10 ma. basis was more accurately determined by simultaneous measurement on both scales, and the technique of measurement was improved. The shorting bar was handled with dry gloves, pressure on it was maintained by means of weights and the constant current sources and sinks were frequently checked. Hence, the measure-

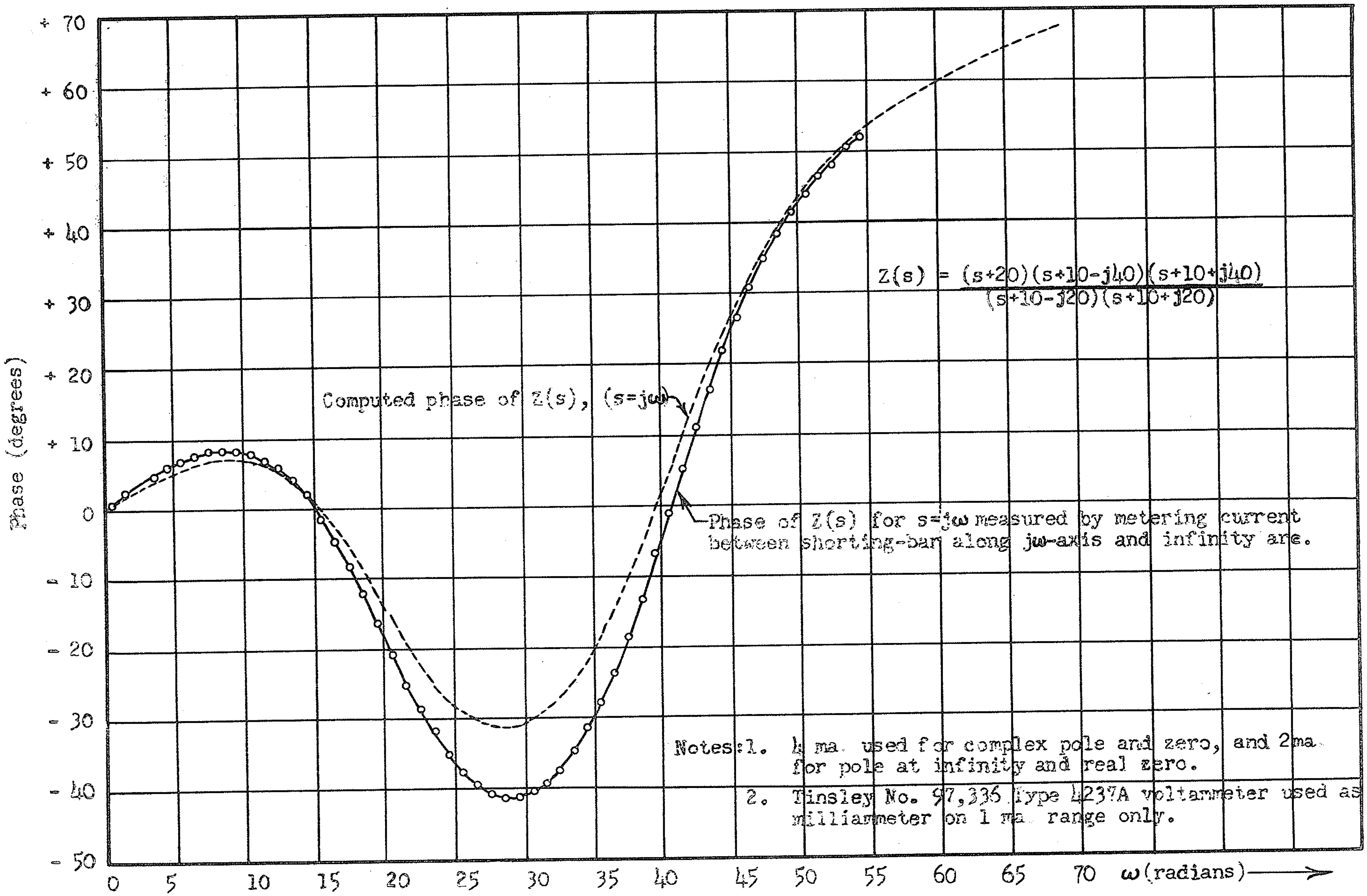

FIGURE 2.7

PHASE RESPONSE OF TEST FUNCTION FOUND BY USING QUARTER COMPLEX PLANE WITH SHORT-CIRCUITED IMAGINARY AXIS, COMPARED WITH ACTUAL PHASE

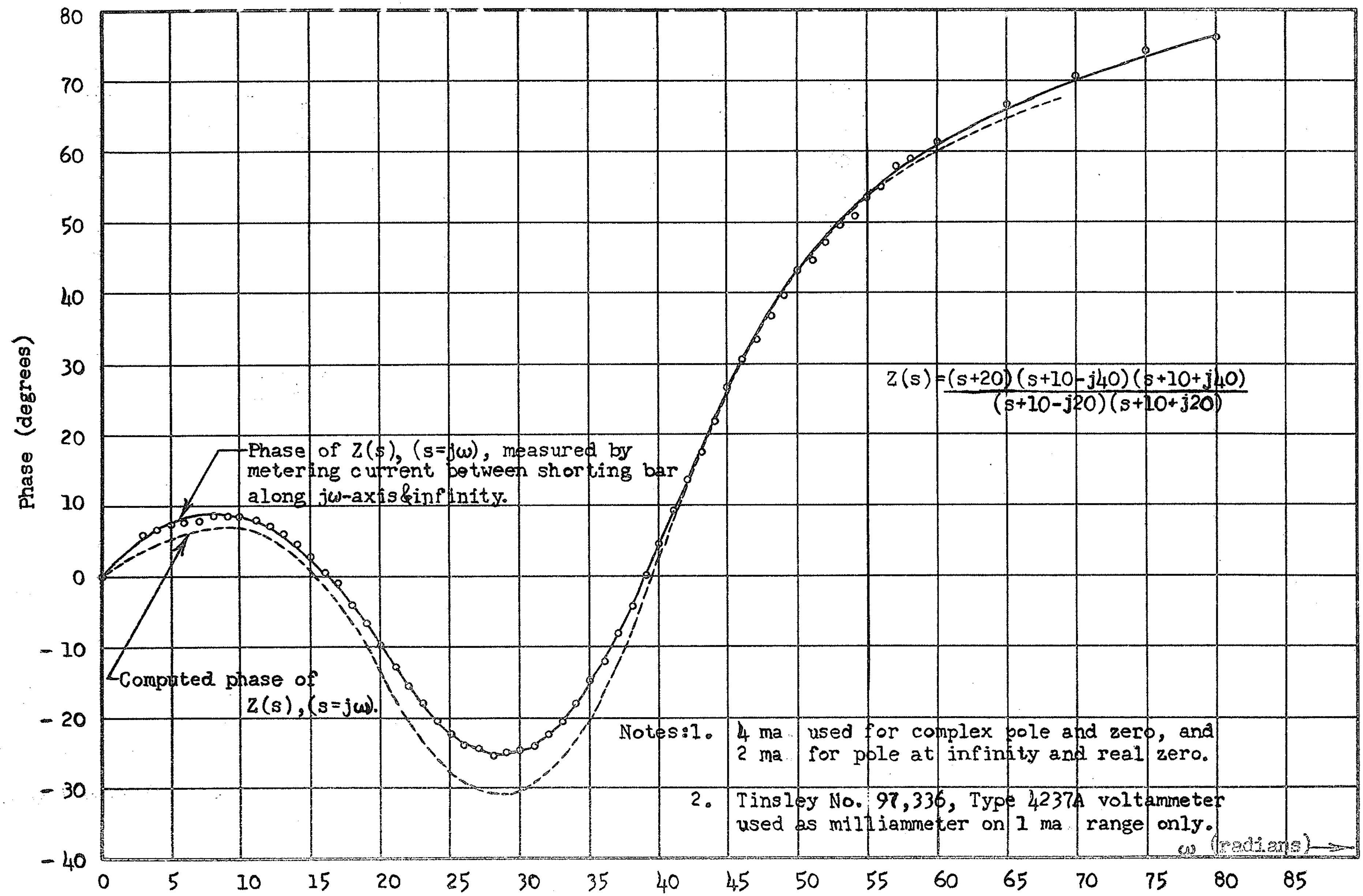

FIGURE 2.8

PHASE RESPONSE OF TEST FUNCTION, FOUND BY USING A LOGARITHMIC TRANSFORMATION OF A QUARTER COMPLEX PLANE WITH SHORT-CIRCUITED IMAGINARY AXIS, COMPARED WITH ACTUAL PHASE ments shown by Figures 2.6 and 2.8 are probably more reliable than those of Figure 2.5 and 2.7. Figure 2.5 does indicate, however, how much variation can be encountered unless great care is taken with measurement techniques.

Comparison of the phase responses in Figures 2.7 and 2.8 with that phase response obtained by graphical integration of phase slope measurements shows that the results obtainable with the methods presented here are superior[9]. Bridges, for example, obtained a maximum error of about 15^{o} in the phase of the same function in the range $\omega = 0$ to $\omega = 55$ and an error of about 5^{o} at $\omega = 30$.

For practical phase measurement, however, the method presented here is probably not too useful, because it is rather time-consuming. It was also found that electronic integration of phase slope found by measurement of potential between two moving probes was adaptable to the semi-automatic system described in Chapter IV, while the method presented here was not. However, the method or an extension of it may be employed in some future device.

The results that have been presented also provide a reasonable experimental verification of this as a method of phase measurement, according to the ideas set forth in the general theory.

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[9]Bridges, op. cit., pp. 52 - 53.

II. A MORE PRACTICAL METHOD OF MEASURING PHASE BY USING QUARTER PLANES[10, 11]

A much more practical way of measuring phase than the one described in the last section can be employed. A quarter plane with the edge of an unbroken equipotential (that is, without the V's of the planes described in the last section), about one half to one inch wide, along the real frequency axis is employed. The voltage gradient across the strip of width $\Delta\alpha$ (physically five millimeters or less wide) with one edge along the real frequency axis is then measured at equal increments along the real frequency axis and then summed in the well-known fashion to yield phase, since this summation is proportional to the total current flowing across the real frequency axis.

It will easily be seen that this method is particularly suitable for use with the semi-automatic potential analogue system described in Chapter IV, because it allows any smaller value of α (which does not impose the other limitations discussed in Appendices referred to in Chapter IV) than the spacing between voltage probes, to be used. Only one probe is needed for the phase measurement by this method. The

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[10] R. Staffin, _Network Synthesis Procedures with a Potential Analog Computer_, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-391-54, PIB-324, June 6,1954, p. 10.

[11] P.M. Liebman, _Simultaneous Gain-Phase Approximation with a Potential Analog Computer_, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-617-57, PIB-545, August 22, 1957, pp. 16-17 and 18-20.

accuracy of phase measurement should be high because the phase of $\frac{F(s)}{F(-s)}$ (twice the phase of F(s)) is measured, and because $\Delta\alpha$ can be made small.

By recording only the differences in potential between the probe and the equipotential; the slope of the phase with respect to the real frequency axis scale is obtained. For example, in the ordinary quarter s-plane one obtains $\frac{\partial \phi}{\partial \omega}$, while in the logarithmic quarter plane one obtains $\frac{\partial \phi}{\partial \log \omega}$. Using the s-plane analogue one thus obtains delay.

III. MAGNITUDE MEASUREMENT BY USING THE LOGARITHMIC TRANSFORMATION OF A QUARTER PLANE WITH OPEN-CIRCUITED REAL FREQUENCY AXIS

General Theory of the Method

There is a well-known analogy between the potential on a uniformly conducting sheet representing the complex plane or its conformal mapping and the logarithm of the magnitude of a rational function F(s) of a complex variable represented on the analog[12, 13]. The even symmetry of sources and sinks with respect to open-circuited boundaries shows that the potential on the open-circuited real-frequency axis of the quarter plane or any of its conformal mappings at $s = j\omega$ or, respectively, its transformed position, is proportional to the logarithm of the magnitude

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[12]Darlington, op. cit., pp. 3-10.

[13]Bridges, op. cit., pp. 4-5.

of $|F(j\omega)|^2 = F(j\omega)\,F(-j\omega)$.[14, 15]

Preparation of Logarithmic Quarter Plane and Method of Magnitude Measurement

The principles of the previous section led to the preparation of the logarithmic quarter plane shown in Figure 2.9. The same unit was used to provide constant-current sources and sinks as for the phase measurement. In Figure 2.9, the method of voltage measurement is also indicated.

Calibration of Logarithmic Quarter Plane and Measurement of Magnitude of Test Function

The function $F(s) = s$ was set up on the plane using (half-unit for sinks or sources on the real axis or at infinity) current strengths of two ma, and the plot, shown in Figure 2.10, was obtained of potential in volts vs. ω, with ω plotted to a logarithmic scale. The slope of the straight line obtained was the coefficient K_1 in the equation

$$V = K_1 \log_{10}\left|\frac{F(s)}{F(s_o)}\right| \tag{2.2}$$

which relates the potential at s measured on the analog with respect to a point s_o, to $\left|\frac{F(s)}{F(s_o)}\right|$.[16] From Figure 2.10, $K_1 = 5.33$.

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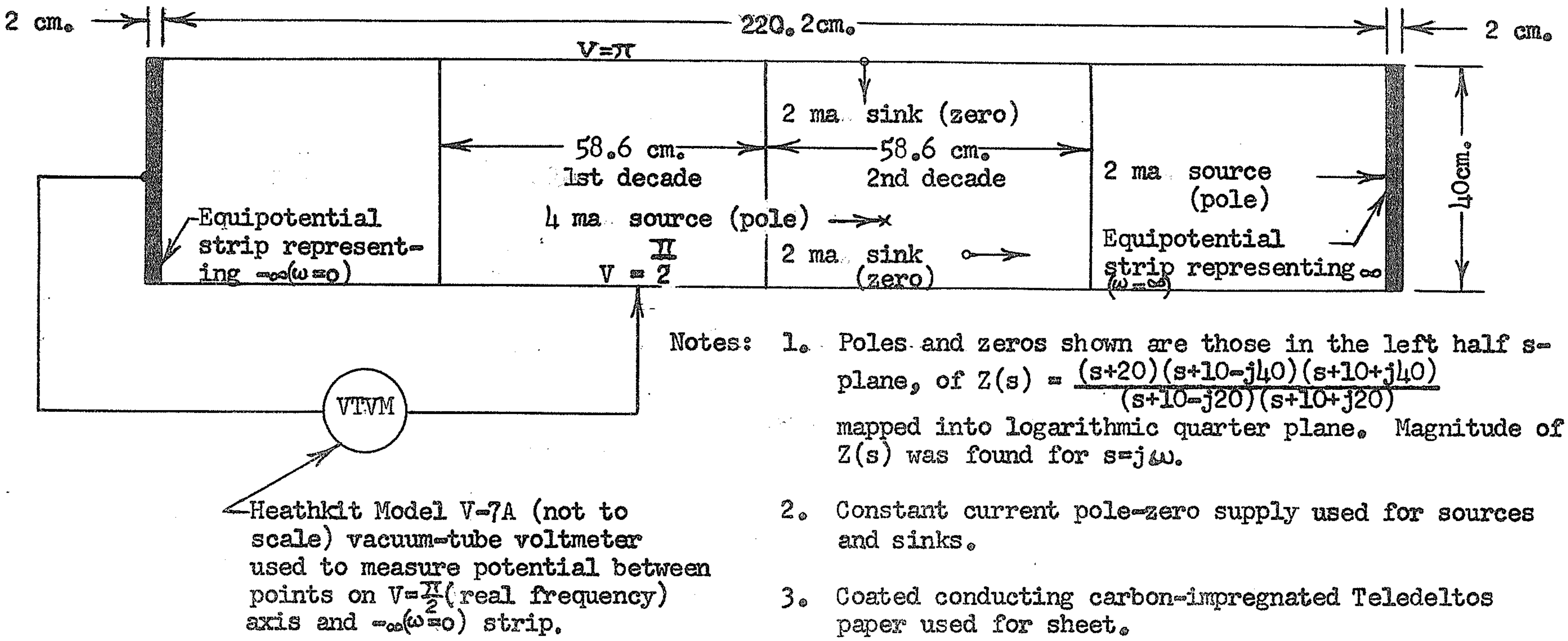
[14] Matthei, op. cit.

[15] Darlington, op. cit.

[16] Bridges, op. cit., p. 5.



Notes:

1. Poles and zeros shown are those in the left half s-plane, of $Z(s) = \dfrac{(s+20)(s+10-j40)(s+10+j40)}{(s+10-j20)(s+10+j20)}$ mapped into logarithmic quarter plane. Magnitude of Z(s) was found for $s=j\omega$.

2. Constant current pole-zero supply used for sources and sinks.

3. Coated conducting carbon-impregnated Teledeltos paper used for sheet.

Scale 1" = 30 cm.

FIGURE 2.9

QUARTER LOGARITHMIC PLANE WITH OPEN-CIRCUITED REAL FREQUENCY AXIS AND METHOD USED FOR FINDING MAGNITUDE OF RATIONAL FUNCTION OF A COMPLEX VARIABLE

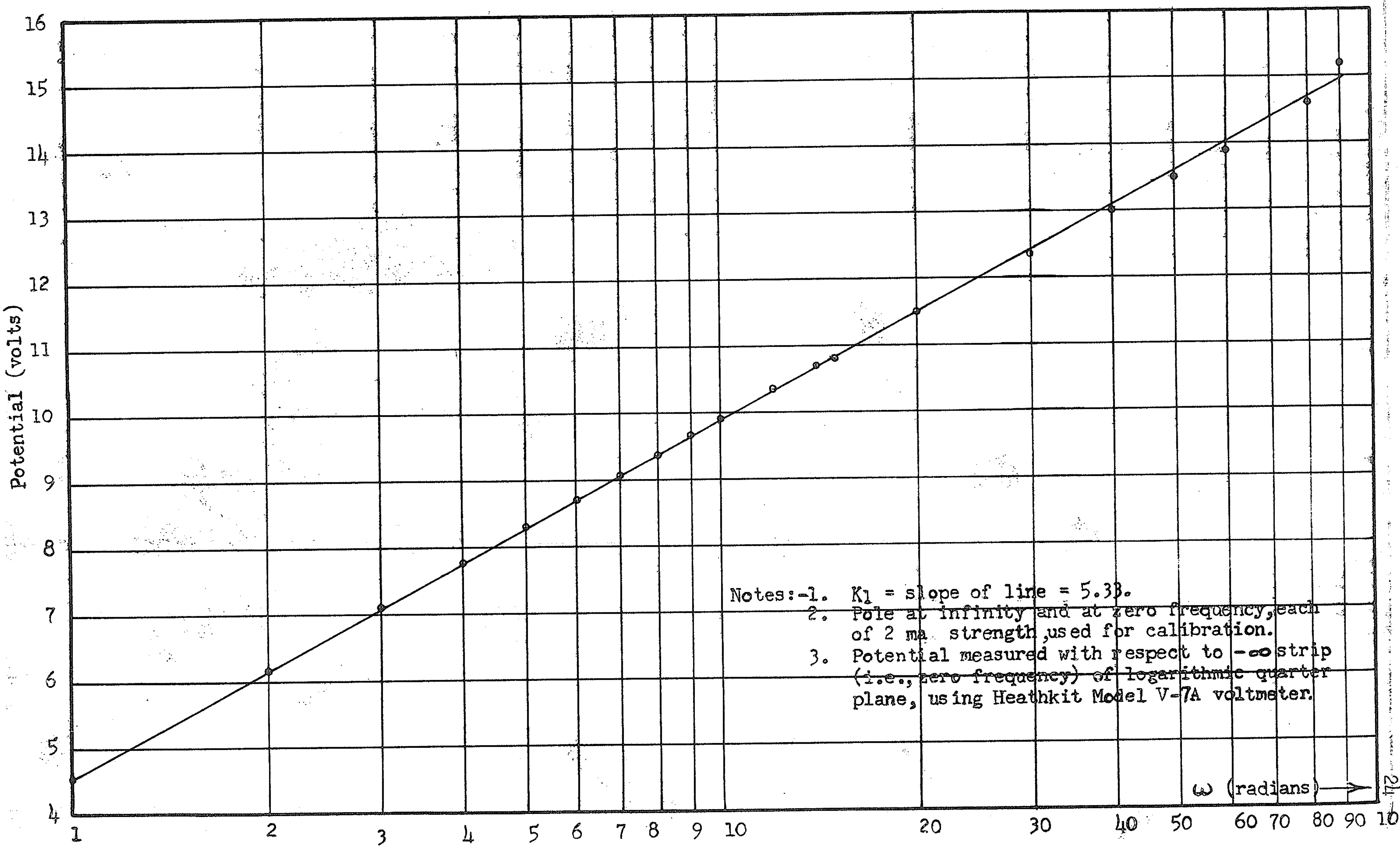


FIGURE 2.10

CALIBRATION CURVE FOR DETERMINING COEFFICIENT K_1 FOR QUARTER LOGARITHMIC PLANE WITH OPEN-CIRCUITED IMAGINARY AXIS, TO BE USED FOR MAGNITUDE MEASUREMENTS

The test function (2.1) was set up on the plane, using unit current strengths of four milliamperes (on the real axis and at infinity these strengths must be half-unit strength) and readings of potential vs. ω were obtained. From (2.1), $Z(o) = 68$. Using equation (2.2), a plot of $|Z(j\omega)|$ vs. ω was obtained; it is displayed in Figure 2.11, together with the computed curve.

Discussion and Conclusions

Excellent agreement between the measured analog plot of $|Z(j\omega)|$ and the actual curve was obtained, as Figure 2.11 shows. The measured points also lie extremely close to the straight line of Figure 2.10. A comparison with magnitude measurements using a half logarithmic plane shows that the results found here were even slightly better than those found using the half logarithmic plane.[17]

Because the potential measured on quarter planes is twice that measured on half planes, it is preferable to use quarter planes for measuring magnitude functions or those with quadrantal symmetry.[18] The use of a quarter plane for such functions is practical. This is particularly true since quarter planes can be used with the semi-automatic system described in Chapter IV.

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[17] Bridges, *op. cit.*, pp. 54-55.

[18] The same result is obtained if the function F(s) F(-s) is set up in the half plane, but this requires twice the amount of conducting paper.

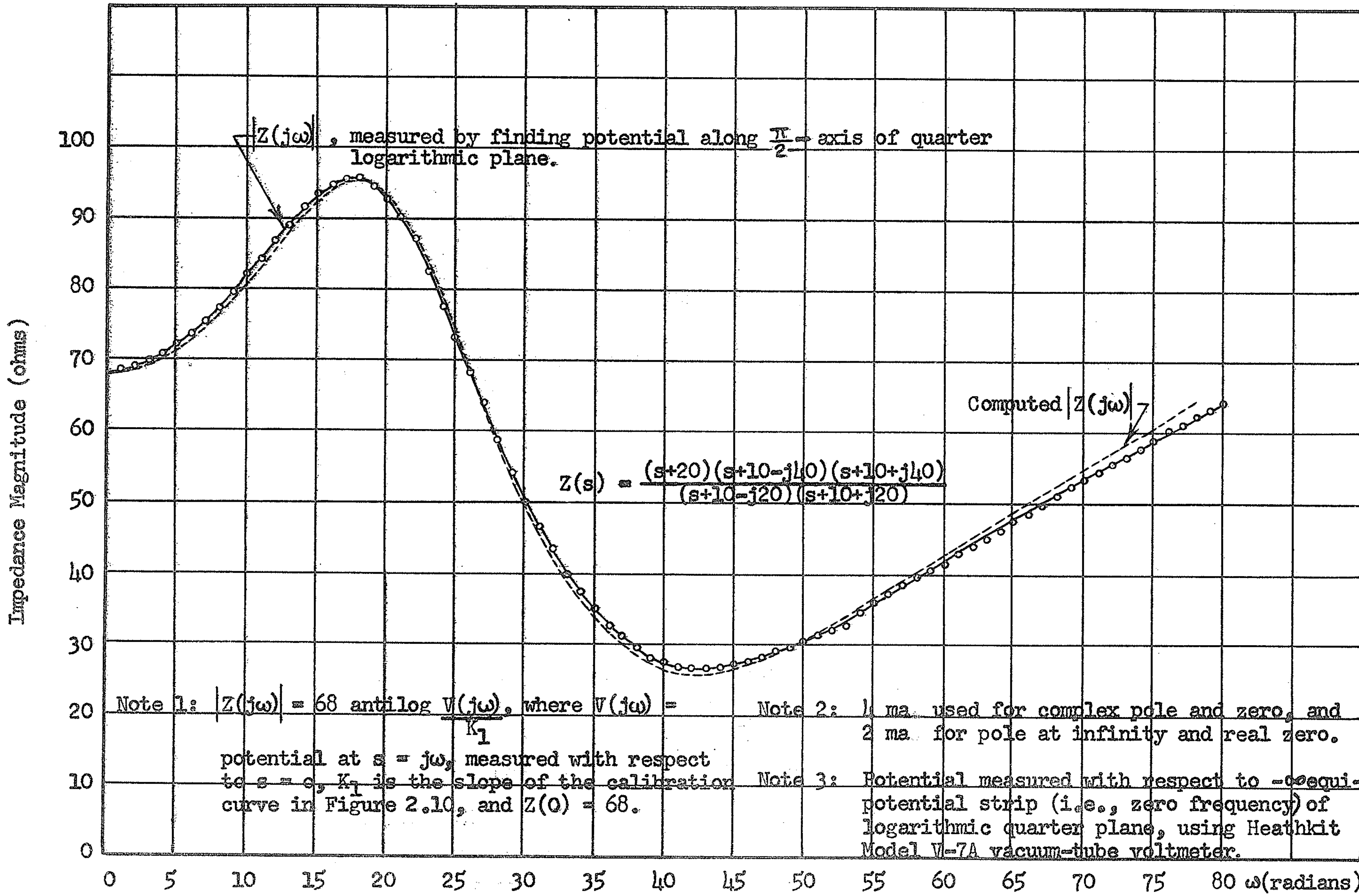

FIGURE 2.11

MAGNITUDE RESPONSE OF TEST FUNCTION, FOUND BY USING A LOGARITHMIC QUARTER PLANE WITH OPEN-CIRCUITED IMAGINARY AXIS, COMPARED WITH ACTUAL MAGNITUDE

CHAPTER III

DETERMINATION OF PARTIAL FRACTION EXPANSIONS OF RATIONAL FUNCTIONS OF A COMPLEX VARIABLE FROM MEASUREMENTS IN A HALF PLANE[1]

I. INTRODUCTION

A rational function of a complex variable with at most a simple pole at infinity may be expanded as follows:

$$Z(s) = k_1 + k s + \frac{k_2}{s-s_2} + \frac{k_4}{s-s_4} + \ldots. \qquad (3.1)$$

If there are double poles, an expansion of the form

$$Z(s) = k_1 + k s + \frac{k_{21}}{s-s_2} + \frac{k_{22}}{(s-s_2)^2} + \frac{k_{41}}{s-s_4} + \frac{k_{42}}{(s-s_4)^2} + \ldots \qquad (3.2)$$

can be effected.

The usefulness of such expansions in mathematics, circuit theory, network synthesis and control system analysis and design is well-known, and need not be elaborated upon here. In general, these expansions permit the determination of the transient response of a physical system.

Expressions have been derived for the determination of the

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[1]An attempt to derive expressions which would enable determination of coefficients in partial fraction expansions by using open-or-short circuited quarter planes led to the conclusion that such planes could not be used for this purpose.

coefficients in these expansions in terms of quantities that can be measured on a circular potential analog[2]. In the following, expressions are found which enable these coefficients to be found from measurements on a half-plane analog. One example is given of the determination, from analog measurements, of the residues at single poles in expansions of the form given in equation (3.1) and another example in which coefficients in the terms of the expansion due to a double pole -- as illustrated by (3.2) -- were found from analog measurements in a half plane. The measured values are compared with the actual, calculated values.

The discussion of accuracy and of errors introduced by the point-by-point measurements (obtained manually with a voltmeter or -- when much smaller potential differences were to be measured -- with the aid of a simple circuit[3] employing a microammeter for a null indicator), by the use of coated Teledeltos paper, and by other sources of error, and the time consumed in making the measurements, indicated that the semi-automatic system using uncoated Teledeltos paper, described in the following chapter, should prove to be quite useful in

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[2]A.R. Boothroyd, E.C. Cherry, and R. Makar, "An Electrolytic Tank for the Measurement of Steady-State Response, Transient Response, and Allied Properties of Networks", Proceedings of the I.E.E. (British), 96: 169-170 and 176-177, Part 1, 1949.

[3]Fig. A.2, Appendix A, p. 168.

facilitating and greatly improving the accuracy of such measurements.

III. DERIVATION OF EXPRESSIONS ENABLING RESIDUES AT FIRST-ORDER POLES TO BE FOUND WITH THE AID OF MEASUREMENTS ON A SEMI-CIRCULAR HALF PLANE

The residue at a first-order pole of $Z(s)$ is given by

$$k_p = (s-s_p)\, Z(s)\Big|_{s=s_p}. \tag{3.3}$$

When the semicircular representation of the complex plane is used, the removal of a pole at s_p also removes its conjugate, which is reflected across the real axis, and results in a remainder function

$$Z_2(s) = (s-s_p)\,(s-s_p^*)\, Z(s), \tag{3.4}$$

in which $s_p^* = \alpha - j\beta$ is the conjugate of $s_p = \alpha + j\beta$. For $s=s_p$,

$$(s-s_p)\,(s-s_p^*)\, Z(s)\Big|_{s=s_p} = k_p\,(s-s_p^*)\Big|_{s=s_p} + k_p^*\,(s-s_p)\Big|_{s=s_p}$$

$$= k_p\,(s_p-s_p^*) = k_p\,(2j\beta). \tag{3.5}$$

Hence, (3.5) and (3.4) yield

$$k_p = \frac{(s-s_p)\,(s-s_p^*)\,Z(s)}{2j\beta}\Big|_{s=s_p} = \frac{Z_2(s)}{2j\beta}\Big|_{s=s_p}. \tag{3.6}$$

The magnitude of k_p may be found, therefore, by measuring the potential at $s=s_p$ with respect to the origin when the pole at $s=s_p$ of $Z(s)$ has been moved to the infinity equipotential (so that $Z_2(s)$ remains set up on the plane) and then using

$$V(s_p) = K_1 \ln \left| \frac{2\beta\, k_p}{(\alpha^2+\beta^2)\quad k_1} \right|, \tag{3.7}$$

in which K_1 is the familiar constant found during calibration of the sheet for magnitude measurements and, from equation (3.4),

$$Z_2(o) = (\alpha^2 + \beta^2) \quad k_1 \ . \tag{3.8}$$

The angle, Θ_p, of the residue k_p is found according to the results of the following argument.

Since

$$Z_2(s) = (s-s_p)\ (s-s_p^*)\ Z(s) = (s-s_p^*)\ Z_1(s), \tag{3.9}$$

$$Z_2(s_p) = 2j\omega_p\ Z_1(s_p), \tag{3.9a}$$

and

$$\text{Arg } Z_2(s_p) = \tfrac{1}{2}\pi + \text{Arg } Z_1(s_p) = \tfrac{1}{2}\pi + \Theta_p. \tag{3.10}$$

Hence,

$$\Theta_p = \text{Arg } Z_2(s_p) - \tfrac{1}{2}\pi. \tag{3.11}$$

Therefore, Θ_p is found by measuring Arg $Z_2(s_p)$. This was done by removing the pole at s_p and integrating to s_p the voltage gradient at right angles to the perpendicular dropped from s_p to the real axis. If there are zeros or poles to the right of this line, an elementary analysis shows that 2π must be respectively added or subtracted for each multiplicity of each of them. For real zeros or poles the last statement applies with the 2π replaced by π.

In the manner presented above the residues at all simple poles may be measured using the semicircular representation of the complex plane or its conformal mapping (with, of course, appropriate transformations of lines used for measurement in the complex plane).

III. DERIVATION OF EXPRESSIONS ENABLING COEFFICIENTS IN THE TERMS OF PARTIAL FRACTION EXPANSIONS DUE TO DOUBLE POLES TO BE FOUND WITH THE AID OF MEASUREMENTS ON A SEMI-CIRCULAR HALF PLANE

The problem is, specifically, to find the expansion

$$Z(s) = \frac{k_{p1}}{s-s_p} + \frac{k_{p2}}{(s-s_p)^2} + \frac{k^*_{p1}}{(s-s_p^*)} + \frac{k^*_{p2}}{(s-s_p^*)^2} + R, \qquad (3.12)$$

in which R is a remainder function. If the double pole at s_p in the half plane is moved to infinity, the function

$$Z_4(s) = (s-s_p)^2 (s-s_p^*)^2 Z(s) = k_{p1}(s-s_p)(s-s_p^*)^2 + k_{p2}(s-s_p^*)^2$$
$$+ k^*_{p1} (s-s_p)^2 (s-s_p^*) + k^*_{p2} (s-s_p)^2 + (s-s_p)^2 (s-s_p^*)^2 R \qquad (3.13)$$

is represented on the complex half plane analog sheet, if the reflections across the real axis are considered. Evaluation of (3.13) at $s=s_p=\alpha+j\beta$ results in

$$Z_4(s)_{s=s_p} = 4\beta^2 \left| k_{p2} \right|. \qquad (3.14)$$

The magnitude of k_{p2} may, therefore, be obtained by measuring the voltage $V_4(s)$ at $s=s_p$ with respect to some convenient point. If this point is chosen as $s=0$, $\left|k_{p2}\right|$ may be found from

$$V_4(s\ ) = K_1 \ln \left| \frac{4\beta^2 k_{p2}}{(\alpha^2+\beta^2)^2 \quad k_1} \right|. \qquad (3.15)$$

The angle of k_{p2} may be found according to the results derived in the following short analysis.

The function

$$Z_3(s) = (s-s_p)^2\, Z(s) \tag{3.16}$$

is defined first. Then the relation

$$Z_4(s_p) = (s_p-s_p^*)^2\, Z_3(s_p) = -4\beta^2\, Z_3(s_p) \tag{3.17}$$

holds. Hence,

$$\operatorname{Arg} Z_4(s_p) = \pi + \operatorname{Arg} Z_3(s_p) = \pi + \operatorname{Arg} k_{p2}. \tag{3.18}$$

Therefore,

$$\operatorname{Arg} k_{p2} = \operatorname{Arg} Z_4(s_p) - \pi, \tag{3.19}$$

from which the angle of k_{p2} may be found from analog measurements as in the case of the residue at a simple pole described in the last section. As mentioned there, a correction of $\pm n\pi$ may have to be applied to the measured angle if there are zeros or poles to the right of the perpendicular dropped from s_p to the real axis, along which is performed the integration to determine the phase of $Z_4(s)$.

In order to find an expression for k_{p1} in terms of quantities which can be measured on a half plane analog, the following analysis is applied. First,

$$\frac{d}{ds}\,[Z_4(s)] = (s-s_p^*)^2\, \frac{dZ_3(s)}{ds} + 2(s-s_p^*)Z_3(s). \tag{3.20}$$

One defines

$$Z_4(s) = Re^{j\phi}. \tag{3.21}$$

Then, with the aid of the Cauchy-Riemann conditions, and the use of equation (3.20), one obtains

$$\left(\frac{R\partial\phi}{\partial\omega}\cos\phi + \frac{\partial R}{\partial\omega}\sin\phi\right) + j\left(\frac{R\partial\phi}{\partial\omega}\sin\phi - \frac{\partial R}{\partial\omega}\cos\phi\right)$$

$$= (s-s_p^*)^2\,\frac{dZ_3(s)}{ds} + 2(s-s_p^*)\,Z_3(s). \qquad (3.22)$$

Hence,

$$k_{p1} = \frac{dZ_3(s)}{ds}\Big|_{s=s_p} - \frac{1}{4\beta^2}\left[\left(\frac{R\partial\phi}{\partial\omega}\cos\phi + \frac{\partial R}{\partial\omega}\sin\phi\right)\right.$$

$$\left. + j\left(\frac{R\partial\phi}{\partial\omega}\sin\phi - \frac{\partial R}{\partial\omega}\cos\phi - 4\beta Z_3(s\ \right)\right]_{s=s_p}. \qquad (3.23)$$

Since it has been shown how to obtain

$$k_{p2} = Z_3(s_p),$$

and the R, $\frac{\partial\phi}{\partial\omega}$, ϕ and $\frac{\partial R}{\partial\omega}$ can be obtained from analog measurements, k_{p1} may be determined using (3.23). The amount of calculation required to find k_{p1} from this equation, using the five quantities measured on the analog, is relatively small.

Similar derivations would enable the coefficients in the expansions of rational functions with multiple poles of order greater than two to be found from potential analog measurements. However, the expressions become quite complicated, and the accuracy in finding the residues and other coefficients decreases. Furthermore, one does not often encounter, in practice, multiple poles of order greater than two.

IV. EXPERIMENTAL DETERMINATION OF RESIDUES AT FIRST-ORDER POLES AND OF COEFFICIENTS OF TERMS OF A PARTIAL-FRACTION EXPANSION DUE TO DOUBLE POLES[4]

Preparation and Calibration of the Potential Analogue Sheet

A semi-elliptical coated conducting sheet of Teledeltos paper, obtained by correction of a semi-circular sheet for the effects of non-isotropy, along real and imaginary axes, was used as the analog sheet.

Since the complex potential function

$$\phi(s) = V(s) + j\Psi(s) \tag{3.24}$$

and a rational function F(s) of the complex variable, s, are related by

$$\phi(s) = V(s) + j\Psi(s) = K_o + K_1 \ln F(s), \tag{3.25}$$

$$V(s) = K_o + 2.303\, K_1 \log_{10} |F(s)|, \tag{3.25a}$$

and

$$\Psi(s) = K_1 \operatorname{Arg} F(s). \tag{3.25b}$$

Hence, it was sufficient to perform a calibration of magnitude only, to obtain K_1.

The Functions for which Residues and Coefficients were Determined

The calibrated sheet was used to determine, according to the theory presented in sections II and III of this chapter, the residues

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[4]Details about the preparation and calibration of the plane, and the measurement techniques, are presented in Appendix A, pp. 165-174.

at the finite poles of the test function (2.1), and the residues and coefficients of terms in the partial fraction expansion of

$$Z'(s) = \frac{(s+20)(s+10-j40)(s+10+j40)(s+30-j30)(s+30+j30)}{(s+10-j20)^2\ (s+10+j20)^2}. \quad (3.26)$$

Thus, for $Z'(s)$ it was proposed to find k_{21} and k_{22} (and hence, k_{21}^* and k_{22}^*) in the partial fraction expansion

$$Z'(s) = k_1' + s + \frac{k_{21}}{s+10-j20} + \frac{k_{22}}{(s+10-j20)^2}$$

$$+ \frac{k_{21}^*}{s+10+j20} + \frac{k_{22}^*}{(s+10+j20)^2}. \quad (3.27)$$

Details of the measurements are presented in an Appendix.[5]

<u>Comparison of Measured and Calculated Residue of Function with First-Order Poles</u>

The test function (2.1) was expanded in partial fractions to find the calculated residues. This calculation yielded

$$Z(s) = 68 + s + \frac{671\underline{/-26°33'}}{s+10-j20} + \frac{671\underline{/26°33'}}{s+10+j20}.$$

Comparison with the value of $k_2 = 732\underline{/-25°}$ determined from the analog measurements is reasonably good, considering the rather crude method employed for measuring phase, and the fact that there may have been considerable "infinity error" due to the finite position of the equipotential periphery of the analog. A more complete discussion of errors and a discussion of methods which could be used to improve the accuracy in determining residues by using potential analog measurements is

[5]Appendix A, pp. 166-174.

given in the last two sections of this chapter.

Comparison of Measured and Calculated Coefficients of Terms of a Partial Fraction Expansion Due to Double Poles

The partial fraction expansion of $Z'(s)$ given by equation (3.27) was calculated as

$$Z'(s) = 245 + s + \frac{1640\angle{-40.33^\circ}}{s+10-j20} + \frac{1640\angle{40.33^\circ}}{s+10+j20}$$

$$+ \frac{20200\angle{-74.95^\circ}}{(s+10-j20)^2} + \frac{20200\angle{74.95^\circ}}{(s+10+j20)^2}.$$

In addition, for

$$Z_4(s) = Re^{j\phi} = (s+20)(s+10-j40)(s+10+j40)(s+30-j30)(s+30+j30),$$

the values

$$\frac{\partial \phi}{\partial \omega} = \frac{1}{\sec^2\phi} \; \frac{\partial(\tan\phi)}{\partial\omega} = 0.0668 \text{ radians per radian,}$$

$$\frac{\partial R}{\partial \omega} = 126000 \text{ ohms per radian,}$$

$$R = 32.3X10^6 \text{ ohms,}$$

and

$$\phi = 1.7505 \text{ radians}$$

for

$$s = -10+j20$$

were calculated, for the sake of comparison with the measured values of these quantities. The calculated quantities were, incidentally, substituted into equation (3.23), and a check obtained for the calculated value of k_{21}. Table I presents the measured and calculated quantities for comparison.

From Table I, and from the description and graphs of the previous sections, it is clear that measurement of the coefficients in partial fraction expansions of rational functions with double poles was not too accurate as well as time-consuming, when done by this manual procedure.

TABLE I

COMPARISON BETWEEN MEASURED AND CALCULATED QUANTITIES OBTAINED IN DETERMINATION OF COEFFICIENTS OF PARTIAL FRACTION EXPANSION OF RATIONAL FUNCTION WITH DOUBLE COMPLEX POLES

Quantity	Measured Value	Calculated Value
k_{22}	$23365 \angle -72.5^{\circ}$	$20200 \angle -74.95^{\circ}$
k_{21}	$2050 \angle -40.75^{\circ}$	$1640 \angle -40.33^{\circ}$
$R(-10+j20)$	37.5×10^{6} ohms	32.3×10^{6} ohms
$\phi(-10+j20)$	1.877 radians	1.7505 radians
$\frac{\partial \phi}{\partial \omega}$, for $s = -10+j20$	0.0744 radians per radian	0.0668 radians per radian
$\frac{\partial R}{\partial \omega}$, for $s = -10+j20$	0.145×10^{6} ohms per radian	0.126×10^{6} ohms per radian

V. DISCUSSION OF ACCURACY AND SOURCES OF ERROR

The fact that there were a large number of quantities entering into the expression (3.23) for finding k_{21}, made k_{21} particularly subject to error. Accurate determination of $\frac{\partial R}{\partial \omega}$ using the above manual methods of measurement was extremely difficult. The method of finding phase slopes (and phase) was time-consuming; it also was inaccurate because the data was only sampled by this means of measurement.

Considerable variations are to be observed among the curves of phase vs. ω shown in Figure A.3 and among the curves of $Z_4(-10+j\omega)$ vs. ω shown in Figure A.5. The conditions under which the data for each curve was taken are also shown in the Figures. It has also been previously noted that one of the current sources was feeding the upper limit of 8 ma., and, although the milliammeter metering this current was closely watched, there were some variations in this current.

The cause of the variations among the curves of Figure A.3 and among the curves of Figure A.5, and the cause of errors in all other measurements were:

I. Most important of all the sources of error and probably the cause of most of the variations among the curves was the fact that Teledeltos paper with a non-conducting coating was used for the analog. The reasons for this deduction are:

A. A change in pressure on a voltage-measuring probe, or a change in the material beneath the Teledeltos paper often resulted in considerable change in a reading.

B. Although no proof is presented here, it seems plausible that leakage of charge over the high-resistance path between lead and conducting paper, as well as the effect of the potential of the Teledeltos paper would cause some of the rather erratic readings and movements of indicating instrument needles that were observed.

C. When coated Teledeltos paper was used later in obtaining recordings on a brush oscillograph and -- finally -- on an X-Y recorder, the surface irregularities of the coated Teledeltos paper resulted in extremely irregular recordings until a great many recordings had been made, and the coating worn away. Even after many "runs" some irregularities persisted. When the uncoated Teledeltos paper was used with the semi-automatic system, no irregularities were ever obtained in the recordings.

II. The variation in a current source operating at maximum current. This variation may have been great enough to cause noticeable, important fluctuations in measurements, although it is barely noticeable on the milliammeter used for metering it. By using two sources, each feeding half the original current, such variations may be made negligibly small.

III. The effect of a finite "infinity ring".

IV. The use of sampled data for plotting curves or for integration.

V. The time involved in making measurements; over such extended periods the possibility of variations increases.

VI. Noise due to power cables, etc.

VII. Non-isotropy of the Teledeltos paper.

In summary, the main sources of error were probably due to the use of coated Teledeltos paper, the use of sampled (rather than continuous) data, the time involved in making measurements, stray noise, the finite "infinity", and instability of current sources and sinks operating at too high a current level.

VI. IMPROVING SPEED AND ACCURACY OF MEASUREMENTS

A potential analog system is described in the following chapter which, when employing uncoated Teledeltos paper, yielded such excellent results in measurement of magnitude and phase, that there is no doubt the determination of the residues and coefficients according to the theory in the preceding material of this chapter would be speedily and accurately accomplished by its use. Uncoated Teledeltos paper proved to be far superior to the coated paper, as explained in the previous section. Plots were obtained quickly and semi-automatically on an X-Y recorder from continuous data by using the system, thus minimizing effects of possible instability of current sources and sinks, noise and other possible fluctuations. It is to be noted that current sources and sinks were always, in later work, kept within optimum operating ranges by using two or more sources or sinks to feed one probe if a high single current was required. Finally, by using conformal mappings to eliminate or minimize the effect of a finite "infinity", the resulting plots obtained by using this system proved to be highly accurate, conveniently calibrated and speedily executed.

As this chapter and the one preceding it illustrate, making the potential analog semi-automatic was essential; this potential analog system finally enables the accurate, practical, speedy application of the potential analogy to solving problems such as finding partial fraction expansions of rational functions. If a more highly isotropic conducting paper or other material were found, the accuracy obtained by using this system would be more than sufficient for even exacting requirements.

In conclusion, the measurement of residues and coefficients in partial fraction expansions by using a potential analog is inaccurate, tedious and slow if coated Teledeltos paper and the manual methods of this chapter are used; more accurate, less tedious and still quite slow if uncoated Teledeltos paper and the same manual methods were employed, and probably quite accurate and speedy if the semi-automatic potential analogue system described in the next chapter were employed. Only the last method would make such measurements practicable.[6]

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[6]M. L. Morgan, "Algebraic Function Calculations Using Potential Analog Pairs," Proceedings of the IRE, 49: 276-282, January, 1961. The ESIAC (trade name) potential analog computer described in this article is extremely practical for functions readily and accurately treated on the logarithmic plane. It works on a rather novel principle, different from the one described in this thesis.

CHAPTER IV

A PRACTICAL SEMI-AUTOMATIC GAIN-PHASE POTENTIAL ANALOG

I. INTRODUCTION

The existing pole-zero machine at the University of Manitoba was not completely suitable for practical, speedy solution of problems.[1] In this chapter are presented the functions and desired practical performance of a gain-phase potential analog machine envisioned after a few months' study of the existing and other gain-phase potential analog machines and their possible applications to electrical problems, particularly in the fields of electric circuit theory and synthesis, and automatic control systems. The extensive modifications of the original pole-zero machine are briefly mentioned, and reference is made to a detailed treatment in an Appendix of the limitations of this original machine and of the modifications which, consequently, had to be made to it. A description of the general construction, calibration and operation of the new potential analog system is presented.[2] Details of the dry

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[1]E. Bridges, "A Network-Function Simulator", (unpublished Master's Thesis, The University of Manitoba, Winnipeg, September 1958), pp. 2-40.

[2]The new potential analog machine connected with a commercial X-Y recorder is referred to as "the potential analog system", or simply as "the system".

electrolytic tank, of the construction of new electrical circuits and mechanical apparatus which were added to the portion of the original pole-zero machine still left intact, and of the calibration and operation of the system, are relegated to the Appendices. An outline of the sources of error in the system and the means by which they may be minimized is presented; this outline is again enlarged upon in an Appendix.

The overall accuracy of the potential analog system is discussed. By means of discussion and examples, some justification is provided that the system meets the requirements originally envisioned for a practical gain-phase potential analog machine, that the system is flexible, convenient and accurate, and that problems can be solved quickly by its use -- in short, that the system works well. Further proof is provided by the examples of the ensuing Chapter. Finally, some possible modifications and additions are suggested which might improve the new potential analog system.

II. THE REQUIREMENTS ON A PRACTICAL GAIN-PHASE POTENTIAL ANALOG MACHINE EMPLOYING AN ISOTROPIC RESISTIVE MATERIAL FOR REPRESENTATION OF THE COMPLEX PLANE OR ITS CONFORMAL MAPPINGS[3]

For a gain-phase potential analog machine which could be used with facility to solve problems in electric circuits and synthesis, in automatic control systems and in many applications in which electric field problems arise, the following characteristics were deemed desirable:

1. The machine should use a resistive material such as Teledeltos paper, rather than an electrolytic solution.

This conclusion was reached after a study of literature relating to the building and use of both wet-electrolyte potential analog machines and those employing isotropic (or nearly so) resistive paper. A study and comparison of many potential analog machines can be made by referring to the extensive literature given in the bibliographies of an excellent historical survey by Higgins dealing with applications of the potential analogy and in a summarizing paper by Cherry[4,5]. Moore also gives some

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[3]Only two-dimensional field analogs are considered, although wet-electrolyte tanks have been built and used for investigation of three-dimensional problems.

[4]Thomas J. Higgins, "Electroanalogic Methods", Applied Mechanics Reviews, Vol. 9, No. 1, January, 1956.

[5]E. C. Cherry, "Application of the Electrolytic Tank Techniques to Network Synthesis", Proceedings of Symposia on Modern Network Synthesis, 1: 140-160, April, 1952.

historical review and refers to several earlier bibliographies[6]. He presents a comparison among a number of the earlier models, and summarizes some of the difficulties encountered with wet-electrolyte tanks. He also describes a wet-electrolyte logarithmic plane which he built and used for network analysis. Machines using Teledeltos carbon-impregnated paper for the conducting medium have been built at Stanford University, The Massachusetts Institute of Technology, The Polytechnic Institute of Brooklyn and the University of Manitoba.[7,8,9,10,11]

2. For best contact with probes which pick up voltages from the resistive paper, the paper should be uncoated.[12]

The coat on the paper forms an insulating surface which may cause spurious signals to be picked off the paper. It was found later that

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[6]A. D. Moore, "The Potential Analogy in Network Synthesis" (unpublished Master's thesis, Queen's University, Kingston, Ontario, September, 1949), pp. 14-43.

[7]Stanford University, Electrical Eng. Tech. Memorandum PZM, No. 12, c. 1950. Probably due to Dr. J. M. Pettit.

[8]R. E. Scott, An Analog Device for Solving the Approximation Problem of Network Synthesis, Massachusetts Institute of Technology, Research Laboratory of Electronics, Technical Report 137, June, 1950.

[9]Stanley Lehr, Solution of the Approximation Problem of Network Synthesis With an Analog Computer, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-327-53, PIB-263, June 18, 1953.

[10]Paul M. Liebman, Simultaneous Gain-Phase Approximation with a Potential Analog Computer, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-617-57, PIB-545, August 22, 1957.

[11]Bridges, loc.cit.

[12]The most easily obtainable resistive paper (Teledeltos paper)

this coating produced recordings which were bumpy or jagged when moving probes were used. Use of the uncoated paper resulted in smooth curves on recordings.

3. Such a machine would be most useful and effective if two closely-spaced moving probes which could be carried at constant speed in continuous contact with the paper's surface were used (rather than the many fixed probes or needles employed in some of the previous designs) for the detection of potentials along lines on the paper.

Two closely-spaced probes are necessary to enable the difference of potential (or its integral, which is a measure of total current flow), approximating current flow at a point, to be determined on half planes. Although a quarter plane with an equipotential strip along the real frequency axis requires the use of only one voltage wiper (probe) for measurement of phase or its derivative, and although this measurement in a quarter plane is more accurate, magnitude functions cannot be measured on the same quarter plane. Since it is convenient to be able to measure phase or its derivative, and magnitude, on one plane -- which is especially true if a number of measurements of phase or its derivative and of magnitude must be made --, two probes should be provided, for one can still effect measurements on quarter planes by using only one of the probes. Furthermore, with two probes there is the possibility of simultaneous display of phase, or its derivative, and magnitude. Thus, the voltage

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is coated with a non-conducting surface. However, the uncoated paper can be obtained from the Sunshine Scientific Instrument Co., 1810 Grant Ave., Philadelphia, Pa.

on each probe would be summed to obtain twice the average voltage on the two probes, and the difference of the two voltages would be integrated to obtain phase (as in this potential analog system). A dual-beam scope, or dual-input X-Y recorder could then be used to obtain simultaneous gain-phase displays.[13] The system described in this thesis is flexible in that it allows either half or quarter planes to be used, and would (with the addition of an amplifier) allow for simultaneous gain-phase display if desired. Scott shows, however, that the use of the average voltage on the two probes is not accurate if the probes are spaced rather far apart compared to the distance of a pole or zero from the probes, and analyzes the error.[14] Thus, the machine described in this thesis permits more accurate measurement of magnitude by using one probe travelling along the real frequency axis.

Constant speed of probe travel is required so that a recorder with its own linear time base may be used, and so that an integration of the potential between the two moving probes with respect to time may be performed to yield, for example, the phase of a rational function.

Probes in continuous contact with the paper would provide continuous information, as compared with the sampling obtainable with fixed-probe systems.

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[13]Liebman, op. cit., pp. 18-21. Liebman compares the use of two quarter planes with the use of one half plane for simultaneous gain-phase display, and concludes that a half plane employing a double voltage probe is most practical.

[14]Scott, op. cit., pp. 43-44.

The use of only two moving probes, rather than a large number of fixed probes, eliminates the need for a great deal of equipment, some of which is difficult to construct.[15,16] Using such extra equipment may also result in loss of information.

4. In order to permit expansion or contraction of desired regions of the complex plane, to allow raising potentials or current flow to greater magnitudes (thus resulting in greater accuracy of measurement), or to allow choice of the mapping on which a problem may be most effectively solved, the system should permit the facile use of conformal mappings.

5. For accurate representations of network functions, it should be possible to employ large sheets of resistive paper -- possibly three to four feet to a side.

6. DC constant-current sources or sinks should be used if an X-Y recorder is to be employed. Since highly stable dc amplifiers can be built or bought commercially, the additional circuitry is very simply constructed. On the other hand, if ac were used the ac would have to be converted to dc -- with possible loss of accuracy -- in order for an X-Y recorder to be used.

7. Because the speed of movement of the probes would be relatively slow, the voltages picked off the conducting paper would have to be

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[15] Ibid.

[16] Bridges, loc. cit.

increased by direct current amplifiers.[17]

8. In order to measure the current flow function or its derivative, means must be provided for integrating, with respect to time, the difference between the voltages picked up by the probes as they move at constant velocity across the resistive paper while in continuous contact with it. If quarter planes were used, the potential between one probe and the equipotential along the real frequency axis would be integrated.

9. It must be possible to sufficiently amplify the voltages picked off the paper, their difference, or the integral of their difference, so that a recording device can be actuated and calibrated with a convenient scale.

10. The recordings of potential, potential difference or its integral should be large and easily calibrated in convenient units, such as, respectively, db, milliseconds or degrees.

11. Recordings should be made quickly and easily. Calibration should not take long. There should be means for checking and adjusting calibrations periodically while a problem is being solved.

12. The recording scheme should allow a curve to be drawn by hand on a sheet of the recording paper, and then approximated during successive trials by adjusting the complex potential field over the conducting resistive paper.

13. The overall system must have sufficient accuracy for a wide range of practical problems; it must be a working, practical apparatus,

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[17]Extremely stable dc amplifiers with very high (over 50,000) open-loop gains are commercially available.

suitable for day-to-day use.

Although the above has been written under the inevitable influence of hindsight, it is a summary of the functions that a practical machine, having wide and flexible possibilities of application to the fields that have been previously mentioned, should perform -- and, to a certain extent, how it should perform these functions.

III. GENERAL CONSTRUCTION AND OPERATION OF THE NEW POTENTIAL ANALOG SYSTEM

Modifications to an Earlier Pole-Zero Machine

The pole-zero machine built by Bridges at the University of Manitoba had certain limitations which restricted its use for practical problem-solving in the areas of circuit theory, network synthesis, control systems and field theory. These limitations were compared to the required functions and mode of operation listed in part II of this Chapter. This comparison brought forth the limitations and the consequent items for improvement or replacement which are listed and discussed in an Appendix.[18] A consideration of the points presented there, and the many frustrating experiments performed with the earlier apparatus as work proceeded on improving it, finally resulted in a large number of modifications being made to the earlier machine. The details of these modifications and short general descriptions of the new apparatus providing the similar function, with which the old apparatus was replaced, are presented

[18] Appendix B-1, pp. 175-178.

in an Appendix.[19] These new modifications and additions, together with the original constant-current supply and distribution unit, resulted in the desired potential analog machine, which had to a high degree the desired functions, accuracy and other characteristics originally envisioned as desirable.

General Physical Description of the New Potential Analog Machine

A photograph of the equipment layout for the new potential analog system is shown in Figure 4.1. A description of the apparatus shown there is presented next. References are made to Appendices for details of various parts of the system.

At the right is a work table to which steel blocks are tightly clamped. The steel blocks support rigidly between them two parallel steel bars three-quarters of an inch in diameter, whose centers are two and one-quarter inches apart. At one end of the bars, bolted to a small wooden platform -- which, in turn, is bolted with U-bolts to the steel bars -- is a small electric motor which, through a speed-reducing gear system, drives the large (three and one-half inches in diameter) sheave at the same end. At the other end a small idler sheave is mounted. Over the two sheaves runs a fine-stranded copper cable covered with nylon cloth. Tension in the cable is maintained by a fairly stiff spring in the upper loop. This pulley cable is attached in its lower loop to a carriage which is supported on the parallel steel bars by four brass blocks deeply notched on their under-side so as to travel upon, and be

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[19] Appendix B-1, pp. 175-178.

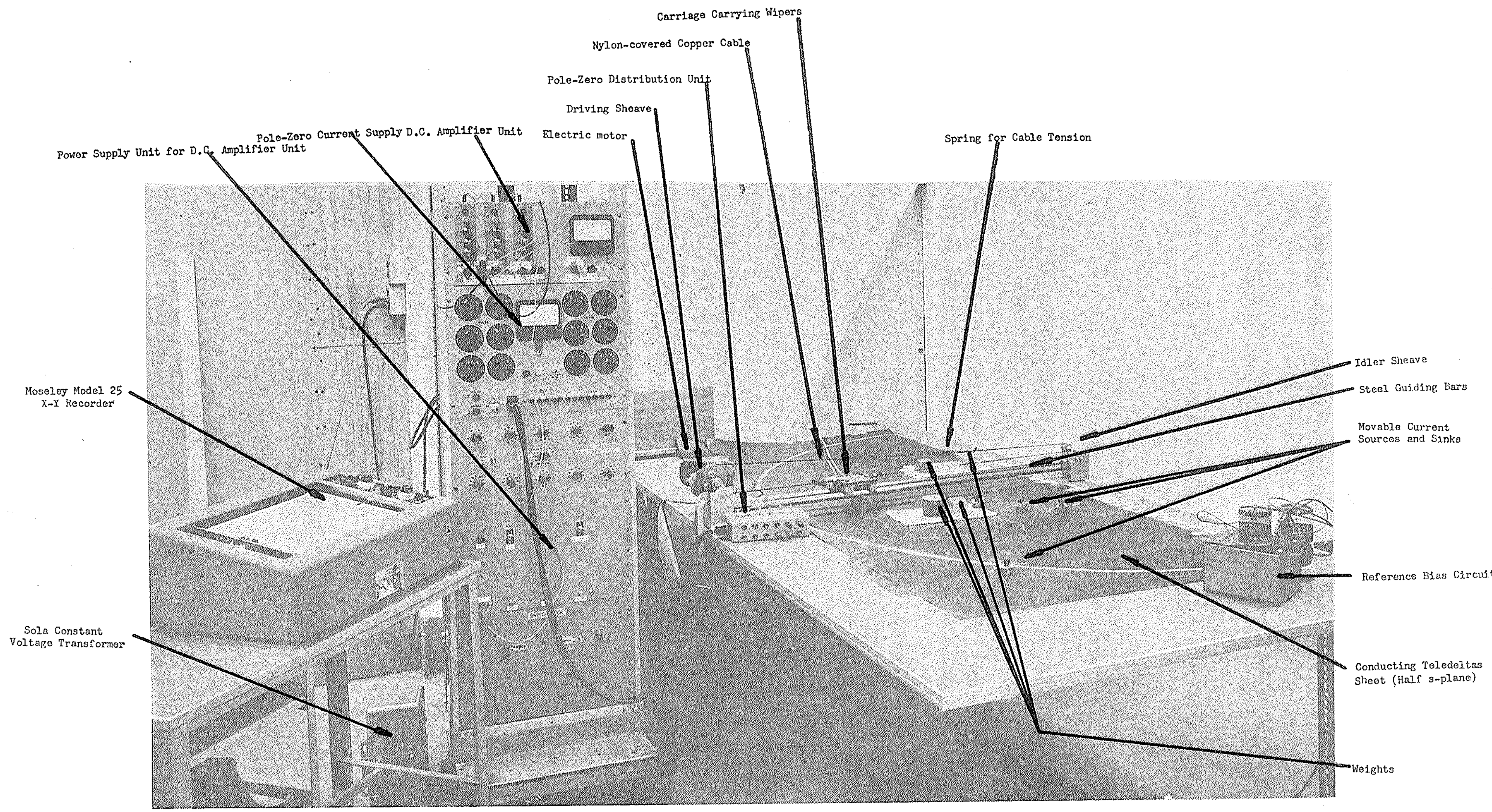

FIGURE 4.1

THE NEW SEMI-AUTOMATIC POTENTIAL ANALOG SYSTEM guided by, these two parallel bars. This carriage carries (below and between the bars) two curved 360° potentiometer wipers which are spring-loaded to maintain continuous contact with the Teledeltos paper. Two wires, one from each wiper, lead to a supporting hook upon the wall, and thence to the amplifier inputs.[20]

The Teledeltos paper is laid smoothly over a large sheet of bakelite or similar hard, smooth-surfaced material, and taped to this surface and the table at its edges. Weights are placed on insulating sheets on top of the Teledeltos paper to keep it motionless while the wipers are travelling over it.

For the moving-probe system the coated Teledeltos type L-48 carbon-impregnated, conducting paper formerly used was found unsuitable. The insulating coating on this paper caused irregularities in the recordings. Furthermore, removal of this coat with acetone affected adversely its isotropic and linear properties. Thus, uncoated paper was used for the dry electrolytic tank of this system; its use resulted in smooth recordings.[21,22] It was also found that one sheet could be used for at

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[20]Details of the moving probes, their mounting, and of the carriage and carriage pulley system -- i.e., of the apparatus which has been described in this paragraph -- may be found in the descriptions and Figures of Appendix B-2, pp. 179-184.

[21]The black uncoated paper was purchased from the Sunshine Instrument Co., 1810 Grant Ave., Philadelphia, Pennsylvania.

[22]The shape of the tank depended upon the conformal mapping used. Correction for non-isotropy (done only in the s-plane) resulted in an elliptical tank.

least one hundred runs without becoming excessively worn by the wipers.

On the table, at the far right of Figure 4.1, rest a 50,000 - ohm potentiometer and two dry cells for adjusting the reference level of the floating ground of the dc amplifier unit with respect to the potential of a point on the Teledeltos paper.

A current distribution unit, shown in the Figure, is used to plug in the various movable probes which are used as point sources and sinks. Four of these sources and sinks are visible in Figure 4.1.[23]

On the tall portable rack in the Figure are mounted the electronic units. The top deck in the rack holds the three Heath Co. dc amplifiers which, with the aid of associated passive circuits and switches, are used for amplification, inversion or integration of signals from the moving probes.[24] At the right of this deck is a meter for zeroing the amplifiers. The amplifier output is fed through a twin-tee resistance-capacitance circuit rejecting sixty cycles per second, and through a shielded cable, to the input of an X-Y recorder. The Heath Co. power supply for the amplifiers is in the second deck from the bottom of the rack.

The second deck from the top in the rack contains the original constant-current pole-zero supply, which operated very satisfactorily,

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[23]Bridges, op. cit., p. 19; Fig. 2.9, p. 22; and Fig. 2.14, p. 27.

[24]Detailed functional, electrical and physical descriptions of the dc amplifier unit, together with a block diagram and a circuit diagram, are given in Appendix B-3, pp. 184-193.

and hence was left unchanged from that employed in the original pole-zero machine.[25]

The third deck from the top and the bottom deck are no longer used; they are remnants of the older machine which have been left in place for the sake of appearance.

On the floor, to the left of the rack, is a constant-voltage transformer through which the mains voltage is supplied to the electronic units.

A Moseley X-Y recorder stands on the table at the left of the Figure.[26] The Y-input to this recorder is fed from the output of the amplifier unit in the top deck of the rack.

General Description of Potential Analog and Recording System Operation

General description of system operation. Figure 4.2 shows a functional block diagram of the potential analog system. The pole-zero constant-current unit feeds the point sources on the uniformly conducting resistance paper. A potential field is thus set up over this paper. A small electric motor turns the driving sheave, which moves (at constant speed) the carriage carrying the potentiometer wipers in constant contact with the paper. The signals from the two wipers are fed into the dc amplifier unit.[27] A rotary ceramic wafer switch is used to select the

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[25]Bridges, op. cit., pp. 19 and 28; Figures 2.10 - 2.13, pp. 23-26.

[26]Model 2S Moseley Autograph.

[27]Detailed functional, electrical and physical descriptions of the dc amplifier unit, including block and circuit diagrams, are given in Appendix B-3, pp.184-193.

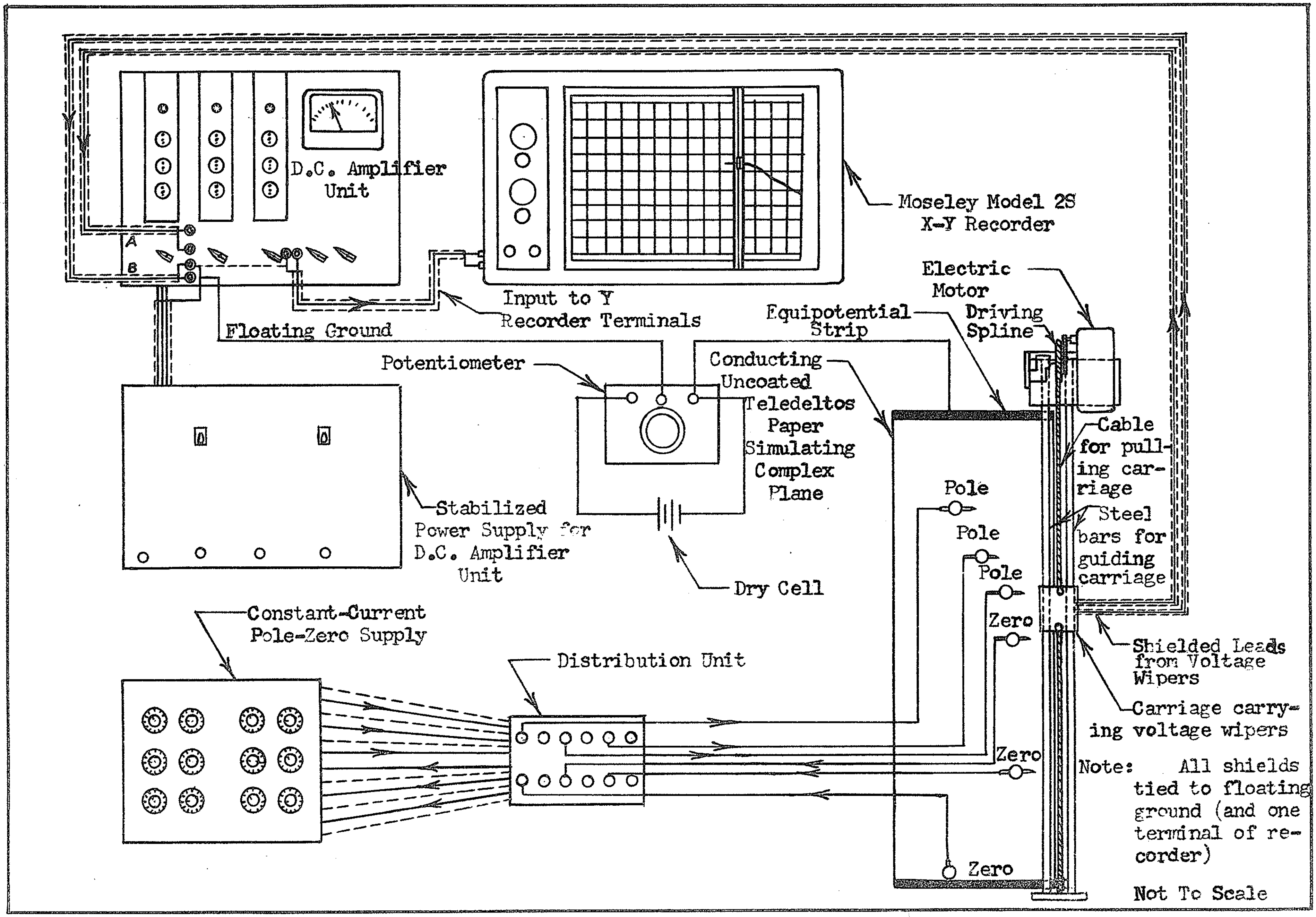

FIGURE 4.2

FUNCTIONAL BLOCK DIAGRAM OF NEW POTENTIAL ANALOG SYSTEM desired potential — voltage from one wiper, difference between voltages at the wipers, or integrated difference between wiper voltages — to be recorded. If only the voltage from one probe (analogous, for example, to the logarithm of the magnitude of a rational function of a complex variable) is desired, the inverter and integrator are not used when the proper function is selected with the rotary switch.[28] The signal from the one moving probe is multiplied by a constant (which is selected by a rotary switch) by the adder amplifier. This signal then passes through a resistance-capacitance twin-tee network rejecting line frequency noise, via a shielded cable, to the input of the X-Y recorder. If the difference between the voltages on the probes (corresponding in a half plane, for example, to phase slope of a rational function of a complex variable) is desired, one of the probe voltages is passed through an inverter, and then to the adder amplifier input, while the other goes directly to the adder. This is effected by a rotary switch. The output of the adder then feeds the Y-input of the X-Y recorder, just as in the previous case. If the integral, with respect to time, of the difference in potential between the two probes (corresponding in a half plane, for example, to phase of a rational function of a complex variable) is desired, the output of the adder in the last case (that is, the difference in potential between the probes, multiplied by a real constant) is fed to the

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[28] If it is desired to measure phase derivative (with respect to the mapping variable) on a quarter plane, with an equipotential along the real frequency axis, only one probe and only the adder amplifier are used, and the equipotential is placed at the potential of the floating ground.

input of the integrator.[29] This selection is effected by means of a rotary switch. The output of the integrator is then fed, via the twin-tee network, to the input of the X-Y recorder.

The reference level of a recording of magnitude may be changed by adjusting the reference level biasing unit to fix the potential of the conducting sheet at a small voltage above or below the floating ground potential.

This completes the general description of the potential analog system's operation. A few words will next be written about the recording system, which was essential to making the potential analog system practical.

The recording system. The most convenient, speedy, and flexible way to use the new potential analog machine was to employ a Moseley X-Y recorder to plot the output of the dc amplifier unit on sixteen and one-half by ten inch sheets of squared paper.[30]

The X-Y recorder's pen moves in the X-direction at constant velocity by means of the recorder's own time base, while the probes are driven across the conducting sheet at constant velocity by the small electric motor. The X-Y recorder's pen moves in the Y-direction in proportion to the voltage output of the dc amplifier unit. Large, accurate

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[29]If a quarter plane with equipotential real frequency axis is used for phase measurement, the inverter is not used, and the voltage on one probe (close to the edge of the equipotential) is fed to the adder, with the equipotential at floating ground potential. The adder output is then integrated to obtain phase.

[30]Model 2S Moseley autograph.

plots of magnitude, delay and phase were obtained. These were easily calibrated to a suitable scale according to the procedures described in great detail in the Appendices.[31,32,33] The simplicity and accuracy of the calibrating procedures, and the convenience of the vertical scales established during this procedure are the oustanding features of this potential analog system; these are features contributing greatly to the speed and convenience of its use, and to its practicability. The use of the X-Y recorder enabled the new potential analog machine to be used as a practical tool for studies in network synthesis, control systems and electric field theory. In this work the potential analog machine and X-Y recorder are referred to as the "potential analog system".

IV. SOURCES OF ERROR, THEIR RELATIVE IMPORTANCE AND THEIR MINIMIZATION

Errors Due to the Physical Analog

The sources of error due to the limitations in setting up an exact physical analog were:

1. Non-linearity and non-isotropy of sheet conductivity.

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[31]Calibration and operation to obtain plots of functions analogous to potential (usually referred to as "logarithm of magnitude of a rational function" in electric network synthesis), Appendix B-4, section IV, pp. 200-202.

[32]Calibration and operation to obtain plots of functions analogous to the derivative of the function which is the conjugate harmonic to the potential (referred to as "delay" in network synthesis), Appendix B-4, section V, pp. 203-205.

[33]Calibration and operation to obtain plots of functions analogous to the conjugate harmonic of the potential function (referred to as "phase", in network synthesis), Appendix B-4, section VI, pp. 205-207.

2. The approximate physical representation of the infinite complex plane, or a conformal mapping of it, by a finite sheet of conducting paper with appropriate boundary conditions.

3. Fluctuations in the strengths of source and sink currents.

4. The change in sheet conductivity due to current density.

5. Finite size of probes used for current sources and sinks, finite size of voltage-measuring wipers, and effect of holes left in the paper by the current probes.

6. Finite width and imperfect conductivity of silver paint strips used to represent equipotentials.

Detailed discussion of these errors is presented in the Appendix.[34] That discussion shows:

1. There appears to be some conflict among those who have studied the non-linear and non-istropic properties of the Teledeltos paper.

Bridges and the author of this thesis have found the sheets to be non-isotropic in the direction of rolling and the perpendicular direction. The average resistance perpendicular to the direction of rolling is ten to twelve percent greater.[35] Bridges derives formulae for correction of this non-isotropy for circular, semi-circular or quarter-circle s-plane sheets. Unfortunately, this compensation is not generally desirable when conformal mappings of the complex plane are used, because of the inconveniently curved coordinate systems introduced by compensation. On the other hand,

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[34]Appendix B-5, pp. 208-218.

[35]Bridges, op. cit., pp. 66-67.

the advantages of mappings which eliminate error due to representation of an infinite complex plane by a finite one, as well as other advantages of mappings, generally outweigh those resulting from facility in compensating for non-isotropy.

Bridges and the author also found variations of the order of eight to fifteen percent in conductivities of 11-inch squares of Teledeltos paper in the same direction, and a conductivity change of ten percent in a thirty-three inch sheet.[36]

Scott and Lehr's measurements, on the other hand, indicate much greater uniformity in sheet conductivity.[37,38] Scott found less than one-tenth percent change of resistivity in fifty-five inches, based on measurements of thirty-five samples taken in succession from a single roll. The results of measurements of random variations in conductivity by Scott would indicate a maximum error of voltage of 3.7 percent between the voltage wipers of the machine described in this thesis. Lehr's measurements indicated variations less than 0.4 percent in non-linearity.

The results obtained when employing the ordinary s-plane showed that significant increase in accuracy was attained by using sheets corrected for non-isotropy.

However, results obtained using conformal mappings uncorrected for

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[36]Bridges, op. cit., pp. 49-53.

[37]Scott, op. cit., pp. 33-35.

[38]Lehr, op. cit., p. 8.

non-isotropy or non-linearity have often been accurate to within one percent.[39] Graduate students using the new potential analog system were also able to obtain better than two percent accuracy in magnitude and phase measurements of fourth-order Butterworth and Chebyshev functions in a half s-plane corrected for non-isotropy; these results indicate that the effect of non-linearity and "finite infinity" must have fortuitously compensated or that their effects were fairly small.

The evidence seems to cast doubt on the measurements by Bridges and the author as valid indications of accuracy to be expected from the analog when used in solving problems. In this matter, Scott's and Lehr's measurements seem to be much more likely criteria. The author's experience with uncorrected elliptic and logarithmic planes shows that conformal mappings may decrease the effects of the non-linearity and non-isotropy. Finally, it is proposed that a more satisfactory way be found for measuring and correlating the effects of sheet non-linearity and non-isotropy with the experimental results of magnitude and other measurements <u>or</u> that sufficient data for the latter be obtained so that the limits of the new potential analog system will be thoroughly known.

2. The second source of error may be made negligible or completely eliminated (or it can be calculated) -- with other advantages, such as a more suitable scale, often resulting -- by means of conformal mappings. Although one is generally no longer conveniently able to

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[39] Figure 2.11, p. 26; Figures 4.3 and 4.4, pp. 67 and 68; various Figures in Chapter V.

correct for non-isotropy in mappings of the s-plane, the advantages of eliminating "infinity" error and the introduction of particularily useful scales usually far outweigh the desirability of correcting for non-isotropy. If the number (counting multiplicities) of poles and zeros in the interior of the plane are equal, boundary conditions are probably not important.

3. The error due to the finite size of probes used for current sources and sinks is negligible except very near such points, where it is not important.

4. The other sources of error are negligibly small, or can be made small by simple precautions, as explained in Appendix B-5.

5. All the above errors were small enough to yield results for phase and magnitude that were in error by two percent or less at most points, as long as infinity error was made negligible by conformal mapping.

In summary, the only sources of errors of any practical importance (if the precautions explained in the Appendices have been observed) which are due to the limitations of the physical analog, and which cannot be eliminated or minimized, are the non-linearity and non-isotropy of the conducting paper, if conformal mapping has been used (as is usually desirable) to correct for the second source of error in the above list.

Errors Due to the Methods of Measurement and the Recording System

These errors are those occurring in the rest of the system. They are due to:

1. Finite spacing and positioning of the voltage-measuring wipers (probes) transported by the carriage. The former error

occurs, of course, only when phase slope or phase is being measured.

2. Non-linear amplification, drift and other errors in the dc amplifier unit.

3. The X-Y recorder.

Detailed discussion of these errors is presented in the Appendix.[40] The gist of that discussion is:

1. Error in phase derivative or phase due to finite spacing of voltage-measuring wipers cannot be eliminated, but can generally be made quite small by using a short spacing between them or by achieving the same result by the use of quarter planes. Greater error is introduced when singularities or zeros are very close to the line of probe travel. Nearly perfect results have been obtained by graduate students at the University of Manitoba for phase measurements of Butterworth, Chebyshev and other functions in half planes corrected for non-isotropy, when a probe spacing of about seven millimeters was used (and the function $\frac{F(s)}{F(-s)}$ set up).

2. The dc amplifier unit introduces no detectable error due to non-linear amplification. The integrator will integrate accurately over periods of time as long as five minutes. Phase or delay measurements are affected by a small amount of drift. This drift makes certain adjustments during phase measurements somewhat sensitive, and may introduce error in

[40] Appendix B-5, pp. 218-224.

this measurement. The possibility that such an error has occurred may be checked immediately after a recording has been taken, and subsequent recordings may be taken, if necessary, which indicate that this error has not occurred. In this way, and by using adder gains less than or equal to unity, the error due to drift can be made negligibly small.

3. The X-Y recorder introduced negligibly small errors. These were of such an order of smallness that they could not be detected on the recordings. The recorder was admirably suited for use in the new potential analog system.

V. ACCURACY OF THE OVERALL POTENTIAL ANALOG SYSTEM

Just as one interested in the use of apparatus for solving problems need not concern himself with the non-linearities and accuracy of every single circuit of that apparatus, as long as the overall system works well in its applications, and produces accurate results, one need not be too concerned with data on the accuracy of various parts of the potential analog system, as long as one is aware of them, and takes steps to minimize them while using the system. The real proof of the value of the system lies in the fact that it works accurately and flexibly on practical problems, when the parts of it are used in an intelligent manner, based upon a knowledge of what causes the errors and how they are minimized.

An example of a magnitude measurement of the function

$$Z(s) = \frac{(s+20)(s+10-j40)(s+10+j40)}{(s+10-j20)(s+10+j20)}$$

(this is the function given by equation(2.1),made in an uncorrected half s-plane using the potential analog system, is shown by the recording of Figure 4.3, and an example of phase measurement is shown in Figure 4.4 by the recording of the phase of the same function. The actual magnitude and phase of Z(s) are also plotted on Figures 4.3 and 4.4, respectively, for the sake of comparison. Comparison with the results obtained by Bridges, using point-by-point manual measurements, shows that the results obtained by using the semi-automatic system are superior.[41] Furthermore, they were obtained quickly, were easily calibrated with convenient scales, and were automatically traced on a large sheet of paper.

Further proof of the flexibility, speed, convenience, and accuracy of the system is provided by the examples of the following chapter.

VI. SUGGESTED IMPROVEMENTS AND MODIFICATIONS

Although the new potential analog machine is fully operational, and is a practical, convenient tool, it was built from the cheapest possible components. Higher quality components and certain refinements would improve performance even further.

1. The carriage and its guides could be made more streamlined.
2. The voltage wipers could be spaced more closely, or a variable spacing device provided. Their mountings could also be made more elegant.
3. A reversing motor should be provided to enable automatic returning of the carriage. Microswitches actuated by the

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[41] Bridges, op. cit., pp. 49 and 52.

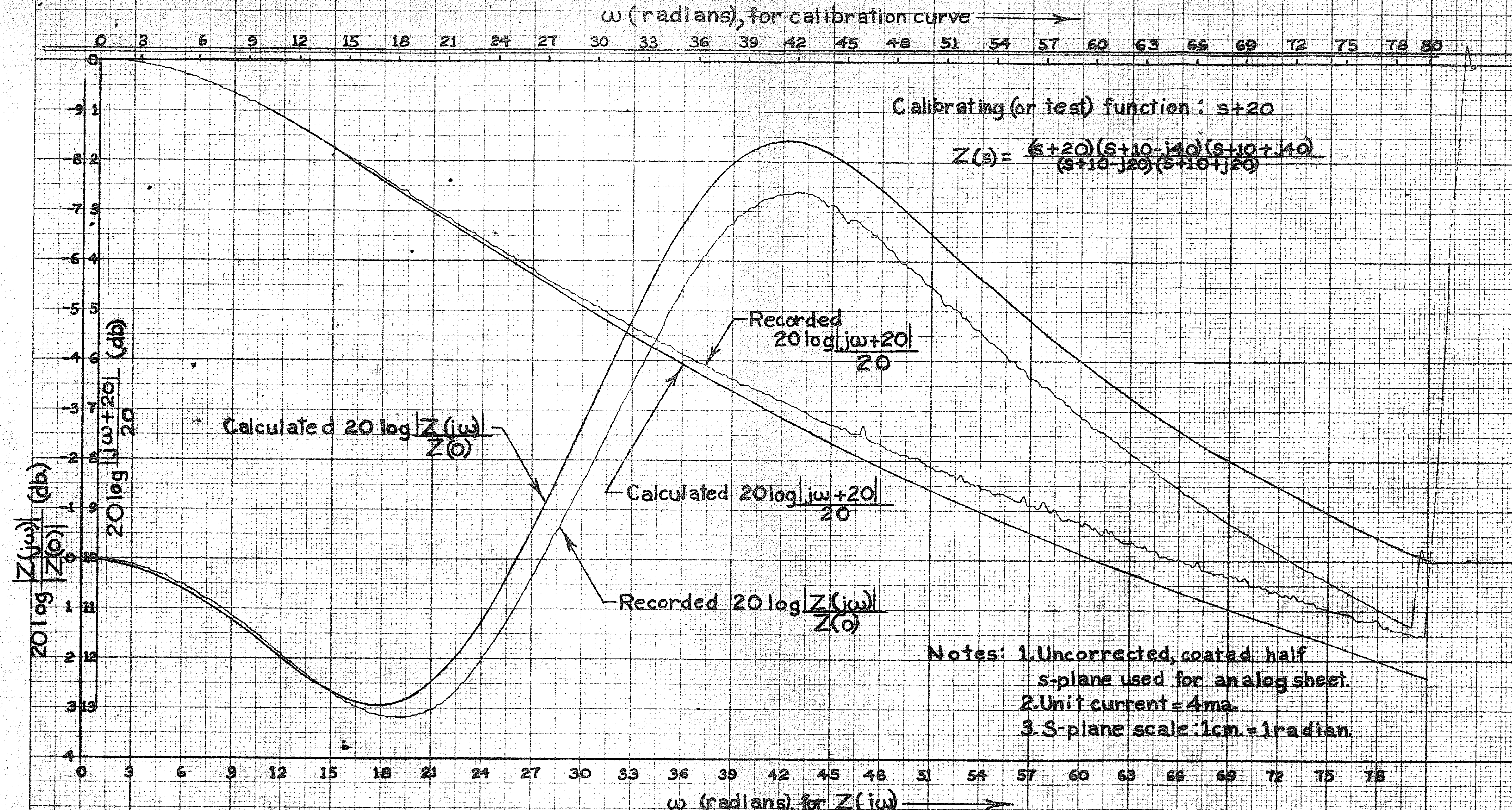

FIGURE 4.3

LOGARITHMIC IMPEDANCE MAGNITUDE RECORDING FOUND BY USE OF SEMI-AUTOMATIC POTENTIAL ANALOG

M. L.
Sept. 30 1959

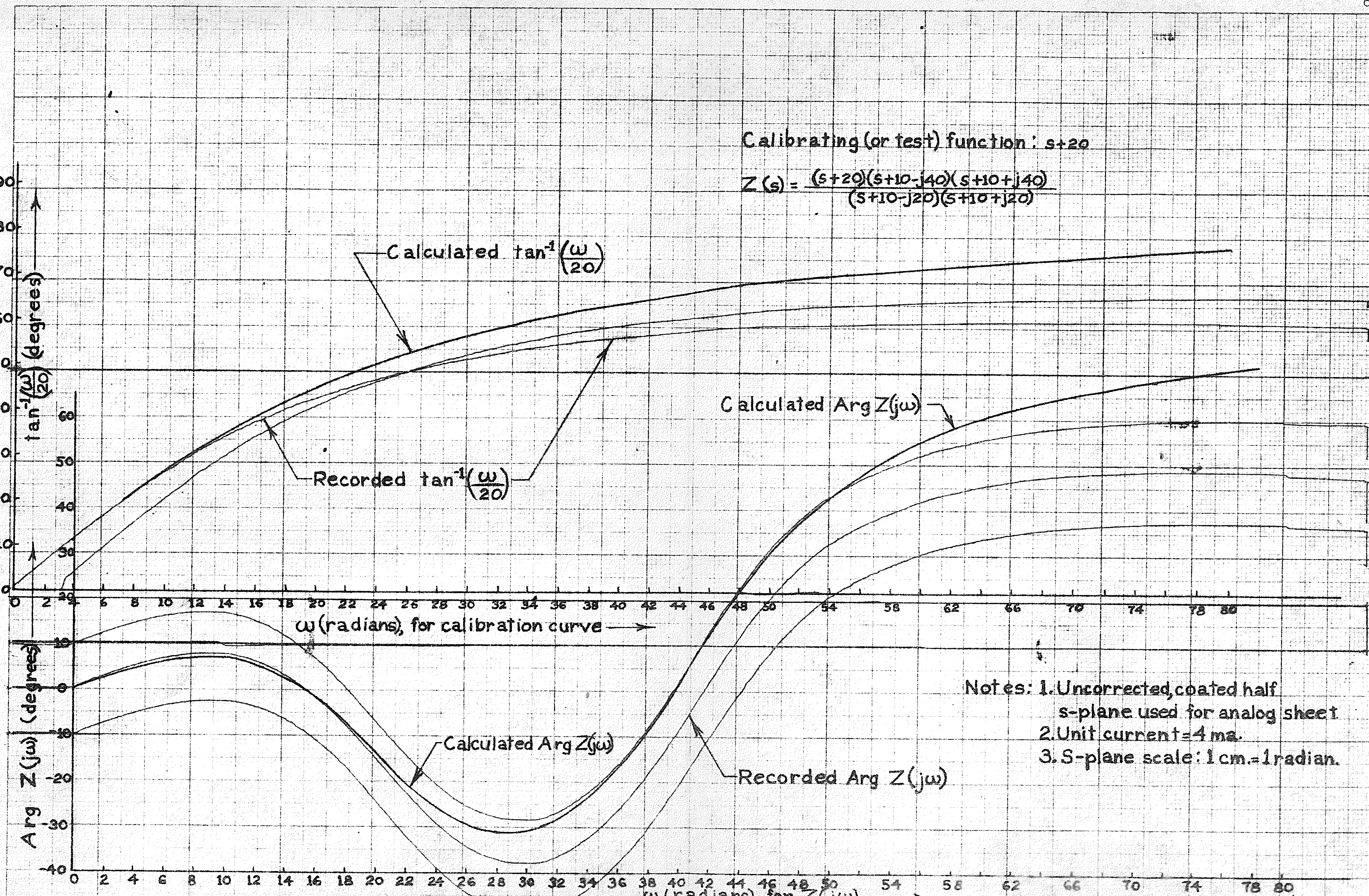

FIGURE 4.4

IMPEDANCE PHASE RECORDING FOUND WITH SEMI-AUTOMATIC POTENTIAL ANALOG

Sept 30, 1959

carriage at each end of its run could be used to reverse and stop the motor.

4. The carriage motor and X-Y recorder should start simultaneously. This would enable using exactly the same origin for all plots.

5. Consideration should be given to calibrating the X-Y recorder's Y-gain control accurately in db, degrees and suitable delay units.

6. A more rigid gear system should be built to link the motor shaft to the driving pulley sheave.

7. A still more stable current source and sink supply unit could be built. Constant-current supplies which are accurate to within $\pm$ 0.1% are commercially available.

8. Chopper-stabilized direct current amplifiers (and their required power supply) should replace the Heath Co. amplifiers and their power supply. This would eliminate all problems with drift and permit making plots of phase more easily.

9. Consideration should be given to obtaining simultaneous plots of magnitude and phase by using half-planes. A dual Y-input X-Y recorder would be required, or two single Y-input X-Y recorders could be used.

   An extra amplifier and simple changes in circuitry and switching would be required to find the sum of the two wiper inputs (proportional, to a good degree of approximation, to the logarithm of the magnitude halfway between the wipers). Although this method of magnitude measurement is not accurate when poles or zeros lie close to the axis of measurement, it suffices in many cases.[42] For more accurate measurement of magnitude, provision should be left for using only one wiper travelling exactly along the desired axis.

10. Finally, consideration should be given to construction of a real-part potential analog machine or to a combined gain-phase and real-part potential analog such as mentioned by Scott.[43] Such an analog would enable one to obtain plots of gain, phase, real part, imaginary part and impulse response.

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[42] Scott, _op. cit._, pp. 43-44.

[43] R.E. Scott, "Potential Analogs in Network Synthesis", _IRE Convention Record_, Part 2, Circuit Theory, 2-8, 1955.

CHAPTER V

THEORY AND APPLICATIONS OF THE POTENTIAL ANALOG APPARATUS IN THE SOLUTION OF APPROXIMATION PROBLEMS OF IMAGE PARAMETER FILTER DESIGN

I. INTRODUCTION

This chapter deals with the theory and applications of the potential analog apparatus in the solution of approximation problems occurring in image parameter lossless filter design. The approximation of specifications for the image parameters only is treated; that is, as is usual in this method of design, it is assumed that allowances have been made for the effects of reflection and interaction factors.[1,2,3] It is also assumed, as is usual, that the effects of dissipation have been considered in advance and are calculated after the lossless design has been completed.[4,5,6,7] After these allowances have been made, the

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[1] W. Cauer, Theory of Linear Communication Networks, Vol. I (McGraw-Hill Book Company, Inc., Toronto, 1958), pp. 130-142 and 248-253.

[2] J. H. Mole, Filter Design Data for Communication Engineers (E. & F. N. Spon Ltd., London, 1952), pp. 57-86.

[3] Vitold Belevitch, "Elements in the Design of Conventional Filters," Electrical Communication, March, 1949, pp. 84-98.

[4] Mole, op. cit., pp. 140-160.

problem remains to approximate the specifications on the image parameters.

One may wonder why the image parameter problem should be considered at all in view of the growing use of insertion loss theory. A justification of this is given in the following section.

The solution of the approximation problems for the low pass filter is sufficient for the treatment, through reactance and bilinear transformations, of most high and low pass, band stop, symmetrical band pass and some other band pass filter problems.[8,9,10] Hence, the approximation problems for low pass filters are presented, suitable potential analogies derived, and some examples given of such filters whose image attenuation requirements were approximated by using the potential analog apparatus. Frequency-asymmetrical band pass filter approximation problems are not treated in this thesis, although the author has derived all the pertinent functions by means of the potential analogy in his report, and shown that the potential analog apparatus may be used to solve the approximation problems in a practical way.[11] The theory for the multiple-band

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[5]Cauer, op. cit., pp. 333-340.

[6]J. Bohse, "Die Verlustdämpfung im Durchlaßbereich von Wellenfiltern bei unterschiedlicher Spulengüte," Frequenz, Bd. 12, Nr. 12, 1958, pp. 380-383.

[7]Lars-Olof Nordmark, "Template Methods for Studying the Influence of Losses on the Image Parameters of Reactance Fourpoles," The Royal Institute of Technology, Stockholm 70, Sweden, Department of Telegraphy-Telephony, Report No. 28, August 16, 1962. The potential analogy is used here to develop the templates.

[8]Cauer, op. cit., pp. 296-332.

[9]Ernst A. Guillemin, Communication Networks, Vol. II. (John Wiley & Sons, Inc., Toronto, 1935), pp. 388-394.

problem is also presented in the report.

Whether or not a solution to the approximation problems can be effected with the potential analog apparatus depends upon whether or not a conformal transformation which will result in sufficient accuracy can be found. With this in mind several mappings were applied to some low pass filter problems so that recordings could be compared. The results showed that the mappings defined by

$$\frac{p}{\omega_o} = \frac{\zeta^2 - 1}{2\zeta}$$

and

$$\frac{-jp}{\omega_o} = \mathrm{sn}(z,k)$$

were suitable for use on most conceivable low pass filter problems. After suitable bilinear transformations have first been applied, these mappings may also be used when solving asymmetrical band pass filter problems. Many conformal transformations exist, of course, which have not yet been applied to the problems presented in this chapter.[12] Those who use the results of this work may discover more suitable transformations for at least some of them.

Examples are not given of all the analogies derived in this

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[10]H. Piloty, "Beiträge zur Berechnung von Wellenfiltern," Elektrische Nachrichtentechnik, Bd. 15, H. 2, 1958, pp. 45-54.

[11]R. V. Brandt, "Ein einheitliches System der Dimensionierung von Bandpässen nach Zobel und Laurent," Frequenz, Bd.7, Nr.6, 1955, p.167.

[12]H. Kober, Dictionary of Conformal Representations (London: Admiralty Computing Service, British Admiralty, 1945).

chapter, but those that are presented should be sufficient to point the way towards the application of the other analogies. The "tricks of the trade," which are sometimes applied by those who have spent years in the field, are not considered.[13,14] In certain cases such "tricks" enable a designer to dispense with an element, or shape a network characteristic, while still meeting specifications.

A truly satisfactory experimental solution to the problem of approximating prescribed phase has not been found, though a trial- and-error solution based on Bode's and Dietzold's paper is possible.[15,16] In general, the approximating procedures presented below are based on trial and error. Hence, rapid convergence to a solution is possible (for reasonably complex functions) only when sources and/or sinks lie along fixed lines such as the real frequency axis. This still allows one to solve the simpler general filtering problems (i.e., including phase specifications) and the (most important) problems of finding

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[13] Vilhelm Peterson, Matching of Image Parameter Filters and Associated Problems, Transmission Dept., Telefonaktiebolaget LM Ericsson, Stockholm; circa 1959.

[14] Mole, loc. cit.

[15] H. W. Bode and R. L. Dietzold, "Ideal Wave Filters," Bell System Technical Journal; Vol. 14 (April, 1935), pp. 215-252.

[16] Guillemin, op. cit., pp. 412-422.

filters whose image attenuation and image impedance meet prescribed specifications. Thus, the practical aspects of approximating the most general specifications by using the potential analog apparatus are far from complete.

On the other hand, included in this chapter are nearly all analogies for low pass, lossless image parameter functions to be found in works of Zobel, Cauer, Bode and others, as well as those functions obtained, after suitable transformation, for frequency asymmetrical band pass filters.[17,18,19,20,21,22,23] Analogies are given for all of Cauer's q and "admissible" q functions, and his q' functions, as well as the image attenuation and phase functions corresponding to these.[24]

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[17]O. J. Zobel, "Extensions to the Theory and Design of Electric Wave Filters", Bell System Technical Journal, Vol. 10, 1931 : 2, pp. 284-341.

[18]Cauer, op. cit., pp. 221-340.

[19]H. W. Bode, A General Theory of Electric Wave Filters, Bell Telephone System Monograph B-843 1934 .

[20]Guillemin, op. cit., pp. 299-459.

[21]Brandt, op. cit., pp. 167-180.

[22]J. Oswald, "Filtres en Échelle Élémentaires", Cables et Transmission, 7e Année, No. 4, 1953, pp. 325-358.

[23]J. E. Colin, "Généralisation des Filtres en Échelle du Genre Zobel", Cables et Transmission, 12e Année, No. 3, 1958, pp. 185-205.

[24]Cauer, loc. cit.

(The image impedance is, of course, a q function).

It shall be noted that the use of the potential analogy to derive image parameter theory can be generalized and used as an aid in understanding a most general insertion loss filter theory presented in a recent paper by Watanabe, and to the use of the pole-zero apparatus as an aid in solving approximation problems according to this insertion loss filter theory.[25,26] Applications to insertion loss filter theory are not presented here although the author has submitted a concurrent report to the National Research Council of Canada which gives an exhaustive treatment of these topics.[27]

In summary, this chapter covers the following points:

1. Reasons for treating the image parameter theory and the corresponding potential analogies.
2. The approximation problems of image parameter theory (especially for the low pass case).
3. The potential analogies for the low pass (reactance) image attenuation and phase. These include the image attenuation

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[25] S. Darlington, "Synthesis of Reactance 4-Poles which Produce Prescribed Insertion Loss Characteristics," Journal of Mathematics and Physics, Vol. 18 (Sept. 1939), pp.312-320.

[26] Hitoshi Watanabe, "Approximation Theory for Filter Networks," IRE Transactions on Circuit Theory, Vol. CT-9, No. 3, September, 1961, pp.341-356.

[27] This thesis is only a part of the report entitled "The Development of a Semi-Automatic Potential Analog Apparatus, an Integrated Approximation Theory for Filters Derived by Means of the Potential Analogy, and the Application of the Apparatus as an Aid in their Practical Implementation", August, 1963, NRC Grant No. A738.

and phase corresponding to Cauer's q, admissible q, and q' functions.

4. Implementation of the pole-zero apparatus for approximation of the low pass image attenuation and phase of 3.
5. Approximation of constant image impedance for low-pass bands by means of the potential analogy.
6. The potential analogies for q, admissible q, and q' functions for reactance low pass filters.
7. Some examples of simple ladder symmetric and antimetric filters synthesized with the aid of the potential analog system, using various conformal representations.
8. A comparison of the suitability of the various mappings used in this chapter for application to solution of filter problems with the aid of the potential analog apparatus.

II. REASONS FOR TREATING THE IMAGE PARAMETER FILTER THEORY AND DESIGN, AND THE CORRESPONDING POTENTIAL ANALOGIES

Although the modern insertion loss method of synthesis is now used widely, particularly where digital computing facilities are available, there are reasons for the treatment of filter design according to the computationally simpler image parameter method. These are:

1. The unavailability of digital computation facilities to all filter designers.
2. The lack of, or waiting period for, digital computer time,

coupled with the fact that many practical requirements may be satisfied with an image-parameter design at little or no extra cost in components above that necessary for the insertion-loss design meeting the requirements.[28,29,30] The relationships between these and similar factors may well result in an image-parameter design being employed.

3. No study has been made to show that certain filter problems are not better solved by using image parameter methods rather than insertion loss methods. For example, no comparative study has been made to show that Bode's and Dietzold's method of parameter determination to obtain a linear-phase filter yields a more or less economical design than the insertion loss methods of Bennett, Beletskiy and others.[31,32,33] As another example, the works of Herzog,

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[28] Johannes B. Fischer, "Über elektrische Wellenfilter mit vorgegeben Betriebseigenschaften," Archiv der Elektrischen Übertragung, Band 14, Heft 7, Juli, 1960, pp. 283-298.

[29] Viktor Fetzer, "Vergleich von Filtern nach der Wellenparametertheorie mit den Filtern der Betriebsparametertheorie und die neutzeitlichen Methoden der Filterberechnung," Archiv der Elektrischen Übertragung, Band 10, Heft 6, Juni, 1956, pp. 225-240.

[30] Henry Simon, "Comparative Investigation of Image and Insertion Parameter Filters," K. Tekn. Hogsk. Handl. (Sweden), #156, 1960.

[31] H. W. Bode and R. L. Dietzold, loc. cit.

[32] Byron J. Bennett, "Linear Phase Electric Filters," Technical Report No. 43 (Electronics Research Laboratory, Stanford University, February 14, 1952), pp. 73-164.

[33] A. F. Beletskiy, "Synthesis of Filters with Linear Phase Characteristics," Telecommunications, No. 4, April, 1961, pp. 39-48.

Idjoudjian, and Poschenrieder lead one to suspect that many crystal filters may best be designed on the image-parameter basis.[34,35,36,37]

4. A knowledge of image-parameter theory and design has often contributed to the solution of outstanding problems in insertion-loss theory.[38,39,40,41]

5. The potential analog approach has also led to fruitful results. One outstandingly practical example is the work due to Bennett.[42]

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[34] Werner Herzog, Siebschaltungen mit Schwingkristallen (Braunschweiz, Germany: Friedr. Vieweg und Sohn, 1962).

[35] D. Idjoudjian, Les Filtres a Cristaux Piézoéléctriques (Paris: Gauthier-Villars, 1953).

[36] W. Poschenrieder, "Steile Quarzfilter großer Bandbreite in Abzweigschaltung," Nachrichtentechnik Zeitschrift (N.T.Z.), Heft 12, 9 Jg. (Dezember, 1956), pp. 561-565.

[37] Leo Storch, D. B. Pike, and D.I. Kosowsky, "Crystal Filter Design Viewed from the Perspective of the Filter Design Literature," by Storch, and the two following replies by Pike and Kosowsky: IRE Transactions on Circuit Theory, Vol. CT-7, No. 1, March, 1960, pp. 67-69.

[38] Darlington, op. cit., pp. 312-320.

[39] Cauer, op. cit., pp. 548-560.

[40] A.R. Boothroyd, "Design of Electric Wave Filters with the Aid of the Electrolytic Tank," Proc. I.E.E. (British), pt. 4, Vols. 90-100, 1951-53. Also Monograph #8, Radio Section.

[41] R. Rubini, "Graphical Procedures for Solving the Approximation Problem of Electrical Filters," Alta Frequenza, Vol. 30, 1961, pp. 198-215.

[42] Byron J. Bennett, "Synthesis of Electrical Filters with Arbitrary Phase Characteristics," IRE National Convention Record, Pt. 5, Circuit Theory, 1953, pp. 19-26.

6. The point of view adopted here is that an integrated knowledge of network synthesis is sound and desirable. It is hoped that new insights may be provided by the application of the potential analogy to the solution of the approximation problems of image parameters. A larger, more comprehensive work due to the author shows how this integration leads to an understanding of an advanced general insertion-loss filter theory, and how the potential analog apparatus may be used as an aid in designing filters according to this insertion-loss theory.[43]

III. THE APPROXIMATION PROBLEMS OF IMAGE-PARAMETER FILTER THEORY

In this section the basis for the introduction of the various approximation problems arising in image-parameter theory are referred to and briefly summarized, and the various approximation problems are stated. Work done by Tuttle is drawn upon in this and following sections, although the treatment here is much more concise.[44] It is assumed here that the reader is familiar at least with the image-parameter theory as explained by Bode, Cauer and Guillemin in their well known works.[45,46,47]

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[43] M. Sablatash, *op. cit.*

Two related approximation problems arise in connection with the symmetric low pass filter when only attenuation specifications are designed for. Cauer states these problems as

$$z_o = \sqrt{z_b z_a} \doteq 1 \qquad \text{for } 0 < \Omega < 1 \tag{5.1}$$

and

$$q = \sqrt{\frac{z_a}{z_b}} \doteq 1 \qquad \text{for } \Omega > 1, \tag{5.2}$$

where Ω is the normalized radian frequency, z_b is the normalized series (horizontal) lattice reactance, z_a is the normalized cross-arm lattice reactance, z_o is the normalized image impedance and q is the q function, related to the two-terminal-pair-network propagation function, Γ_1, by

$$\coth\left(\frac{\Gamma_1}{2}\right) = q \ . \tag{5.3}$$

These relations are discussed in detail in Cauer's work.[48]

For unsymmetrical filters the relations between attenuation functions

$$q_z = \coth \Gamma_1 = \sqrt{z_{11}y_{11}} = \sqrt{z_{22}y_{22}} \tag{5.4}$$

and left- and right-side image impedances

$$z_{01} = \sqrt{\frac{z_{11}}{y_{11}}} \quad \text{and} \quad z_{02} = \sqrt{\frac{z_{22}}{y_{22}}} \tag{5.5}$$

(where the notations for image impedances are changed from w_1 and w_2 in Cauer's work) are assumed to be known.[49,50] Cauer has shown, for

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[44] D.F. Tuttle, An Introduction to Network Synthesis, Unpublished text for Stanford University course 237, 1951.

[45] Bode, loc. cit. [46] Cauer, op. cit., Vol. I.

[47] Guillemin, op. cit., pp. 279-460.

[48] Cauer, op. cit., pp. 121-142, and 224-248.

[49] Ibid., pp. 124-126 and 264-283.

example, that the pass and stop conditions for an asymmetric normalized low pass filter are[51]

$$z_{01} \doteq 1, \qquad z_{02} \doteq 1, \text{ for } 0 < \Omega < 1, \tag{5.6}$$

and

$$q_z \doteq 1 \text{ for } \Omega > 1. \tag{5.7}$$

For any reactance filter,

$$\coth \Gamma_1 = \coth(\alpha + j\beta) = \frac{1 + j \tanh\alpha \tan\beta}{\tanh\alpha + j \tan\beta}, \tag{5.8}$$

and one finds that

$$\coth \Gamma_1 = \frac{1}{j \tan\beta} \quad \text{when } \alpha = 0, \tag{5.9}$$

$$\coth \Gamma_1 = \frac{1}{\tanh\alpha} \quad \text{when } \beta = 0, \tag{5.10a}$$

and

$$\coth \Gamma_1 = \tanh\alpha \quad \text{when } \beta = \pm n\frac{\pi}{2} \quad . \tag{5.10b}$$

The pass bands are defined as the bands of frequencies in which $\alpha = 0$ and the stop bands as the bands of frequencies in which $\alpha > 0$. Using equations (5.1) and (5.2) with equations (5.9) - (5.10b) one finds that when z_{11} and y_{11} (z_{22} and y_{22}) have the same sign, z_{01} and z_{02} are purely imaginary, $\beta = n\frac{\pi}{2}$, where n is an integer, and $\alpha > 0$. Conversely, when $\alpha > 0$, $\beta = n\frac{\pi}{2}$ and z_{01} and z_{02} are purely imaginary. When z_{11} and y_{11}(z_{22} and y_{22}) have opposite signs, z_{01} and z_{02} are purely real, $\alpha = 0$ and β varies with frequency. Conversely, when $\alpha = 0$, β varies while z_{01} and z_{02} are purely real. Similar arguments hold with regard to (5.1), (5.2) and (5.3) for the symmetric filters. With these basic facts in mind the approximation problems and the potential analogies can be considered.

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[50] Bode, *op. cit.*, pp.13-17 and 42-46. [51] Cauer, *op. cit.*, p.266.

Three approximation problems present themselves for filter problems. These are:

1. Approximation of prescribed stop-band attenuation specifications. For ideal stopping, $\alpha \rightarrow \infty$ and $q_z \doteq 1$ in the stop bands. For ideal symmetric filters the condition $q \doteq 1$ should hold.

2. Approximation of prescribed phase, β, in the pass bands. Coupled with this there may also be a requirement on stop-band attenuation.[52] For ideal filtering the pass-band phase, β, is linear, pass-band attenuation, α, zero, and stop-band attenuation, α, infinite.[53] In this work, potential analogies for Γ_1 (or $\frac{\Gamma_1}{2}$) are found, but only approximation of α in the low pass case is considered in detail.

3. Approximation of resistive terminations by the image impedances, in the pass band. That is, $z_{01} \doteq 1$ and $z_{02} \doteq 1$ in the pass-band range. The approximations of unity by q, q_z and q' functions are treated below as separate but (particularly for q and q_z functions) complementary problems. The potential analogies necessary for the solution of the

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[52] Bode and Dietzold, loc. cit.

[53] Strictly speaking this is to hold for operating conditions, although these approximations for the image parameters result in reasonably good approximations to operating conditions as long as the terminations are approximated by the image impedances.

first two problems for the normalized low-pass reactance filter will be considered next.

IV. THE POTENTIAL ANALOGIES FOR Γ_1 AND $\frac{\Gamma_1}{2}$ IN THE LOW PASS CASE

Boothroyd and Tuttle have treated this problem (using the potential analogy) although from somewhat different points of view.[54,55] The work here will be somewhat more complete, in that all possible low-pass propagation attenuation functions will be derived using the potential analogy. All the propagation functions treated by Cauer by means of q functions, "admissible q functions" for asymmetric filters, and q' functions for antimetric filters, are included.[56] In this section the points of infinite attenuation are at first restricted to the real frequency axis while proving the analogy for simple charge distributions. (Also, the most efficient approximation of α results from such a restriction). The charges are then allowed to take the most general distribution and the resulting complex potential found by superposition of the simpler solutions. It is found that the complex potential is analogous to Γ_1 (or $\frac{\Gamma_1}{2}$ for symmetric and antimetric filters) so that

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[54] Boothroyd, op. cit., pp. 65-72.

[55] Tuttle, op. cit., pp. 410-429.

[56] Cauer, op. cit., pp.221-291.

α and β may be approximated by using the new potential analog system. Band pass attenuation specifications of many types may be satisfied by using appropriate frequency transformations, as noted in the introduction.

Approximation of Γ_1 Using Attenuation Poles at Infinity Only

A part of what is found below has been treated by Tuttle.[57] By means of the mapping

$$p = \sigma + j\omega = \sinh w = \sinh (u + jv), \qquad (5.11)$$

whose properties are described by Tuttle in great detail, a conducting plate along the pass band in the p-plane, imposing the constraint $\alpha = 0$ in this region, is mapped into the whole w-plane v-axis.[58] By placing positive charge Q at infinity (and an equal negative charge on the plate) on each one of the two sheets (of each of which there is an infinite number) of each pair joined through the branch cut along the pass band, and mapping each sheet, with its positive charges, into a strip in the right half of the w-plane, the potential in the latter becomes

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[57]Tuttle, op. cit., pp. 417-425.

[58]D. F. Tuttle, Network Synthesis, Vol. I (John Wiley & Sons, Inc., New York, 1958), pp. 861-890.

$$W = -Qw, \quad u > 0. \tag{5.12}$$

With equation (5.11), the potential in the p-plane is found to be

$$W = -Q \sinh^{-1} p \tag{5.13}$$

$$= -Q \ln \left\{ p + \sqrt{p^2 + 1} \right\}. \tag{5.14}$$

Hence,

$$\coth \frac{\Gamma_1}{-Q} = \frac{\sqrt{1 + p^2}}{p}, \tag{5.15}$$

where $\Gamma_1 = \alpha + j\beta = W$.

With

$$Q = -N, \tag{5.16}$$

where N is an integer, the functions $\coth \Gamma_1$ become "admissible q-functions" with one-points only at infinity.[59] With Q = -1 one has the simplest admissible q-function, with a double one-point only at infinity, i.e.,

$$q_z = \coth \Gamma_1 = \frac{\sqrt{1 + p^2}}{p}. \tag{5.17}$$

By composition according to the equation

$$q_{2z} = \frac{q_{1z}^2 + 1}{2\, q_{1z}}, \tag{5.18}$$

where q_{1z} is the admissible q function (5.17) or any other such function derived from it by repeated use of equation (5.18), all admissible q functions

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[59]Cauer, op. cit., pp. 271-274.

for low-pass filters, with one-points only at infinity, can be derived. In the potential analogy this clearly corresponds to addition of integral charges at infinity. In the pass band the following relations are easily found to hold:

$$\beta = N \sin^{-1}\omega, \; \alpha = 0, \tag{5.19}$$

In the upper stop band, $\omega \geqslant 1$,

$$\Gamma_1 = \alpha + j\beta = N \cosh^{-1}\omega + j N \frac{\pi}{2}. \tag{5.20}$$

In the lower stop band, $\omega \leqslant -1$,

$$\Gamma_1 = \alpha + j\beta = N \cosh^{-1}\omega - j N \frac{\pi}{2}. \tag{5.21}$$

W has thus been found analogous to Γ_1. One finds Γ_1 using the potential analog system by painting a highly-conducting strip of silver paint along the pass band in the p-plane or one of its conformal mappings. Unit current, corresponding to N=1, may be assigned and then injected at the point at infinity and withdrawn from the pass band strip. Calibration is performed by adjusting the scale of the sheet on the X-Y recorder according to equation (5.21), usually with a db scale. Multiples of the unit current may be used in order to meet specifications on α, which is found by recording the voltage. β may be found by recording current flow perpendicular to the conducting pass band strip. A recording of α found for N=2 by using the potential analog system, and the elliptic sine mapping of Figure 5.1, is shown in Figure 5.2, together with the computed curve.

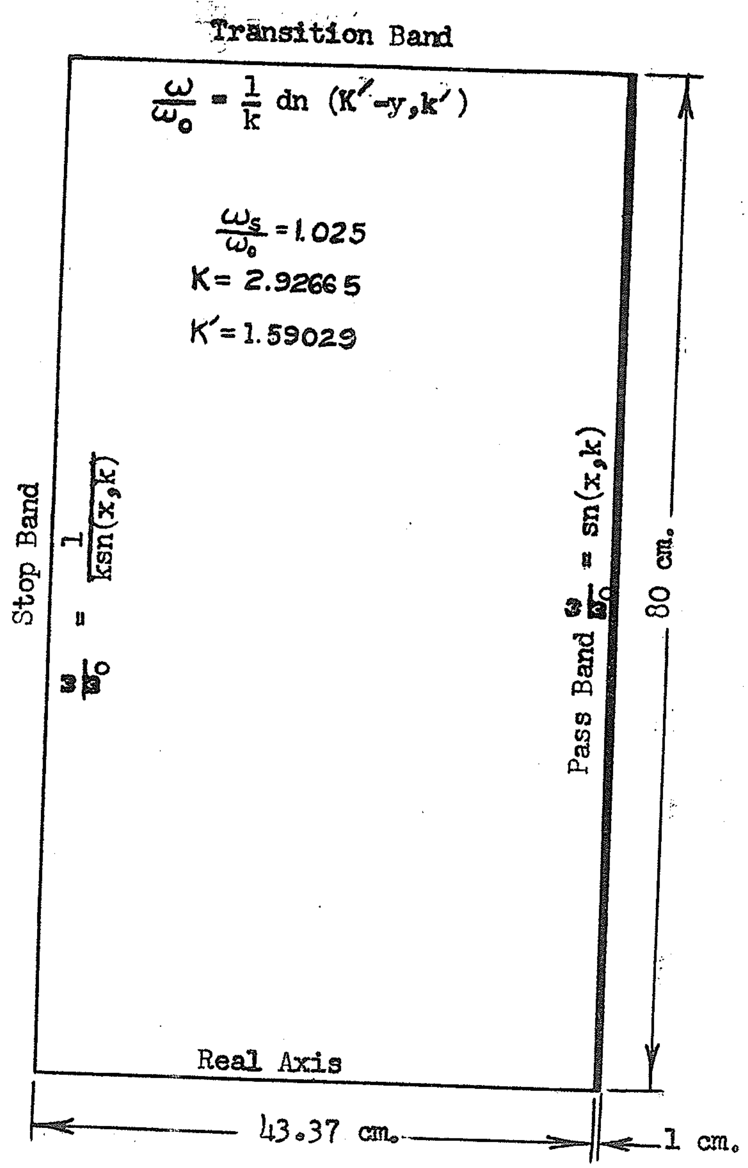
A potential analogy suitable for representation of $\frac{\Gamma_1}{2}$, related

Mapping: $-j\frac{p}{\omega_o} = sn(z,k)$; $p = \sigma + j\omega$, $z = x + jy$

$$k = \frac{\omega_o}{\omega_s} \quad K = \int_0^1 \frac{dp}{\sqrt{(1-p^2)(1-k^2p^2)}}$$

$$k' = \sqrt{1-k^2} \quad K' = \int_0^1 \frac{dp}{\sqrt{(1-p^2)(1-k'^2p^2)}}$$

$$\frac{\text{Pass Band}}{\text{Transition Band}} = \frac{K}{K'}$$



Uncoated Carbon-Impregnated Teledeltos Paper Used For Quarter Plane

FIGURE 5.1

ELLIPTIC SINE QUARTER PLANE MAPPING WITH FORMULAE, AND CALCULATIONS AND DIMENSIONS USED FOR $\frac{\omega_s}{\omega_o} = 1.025$

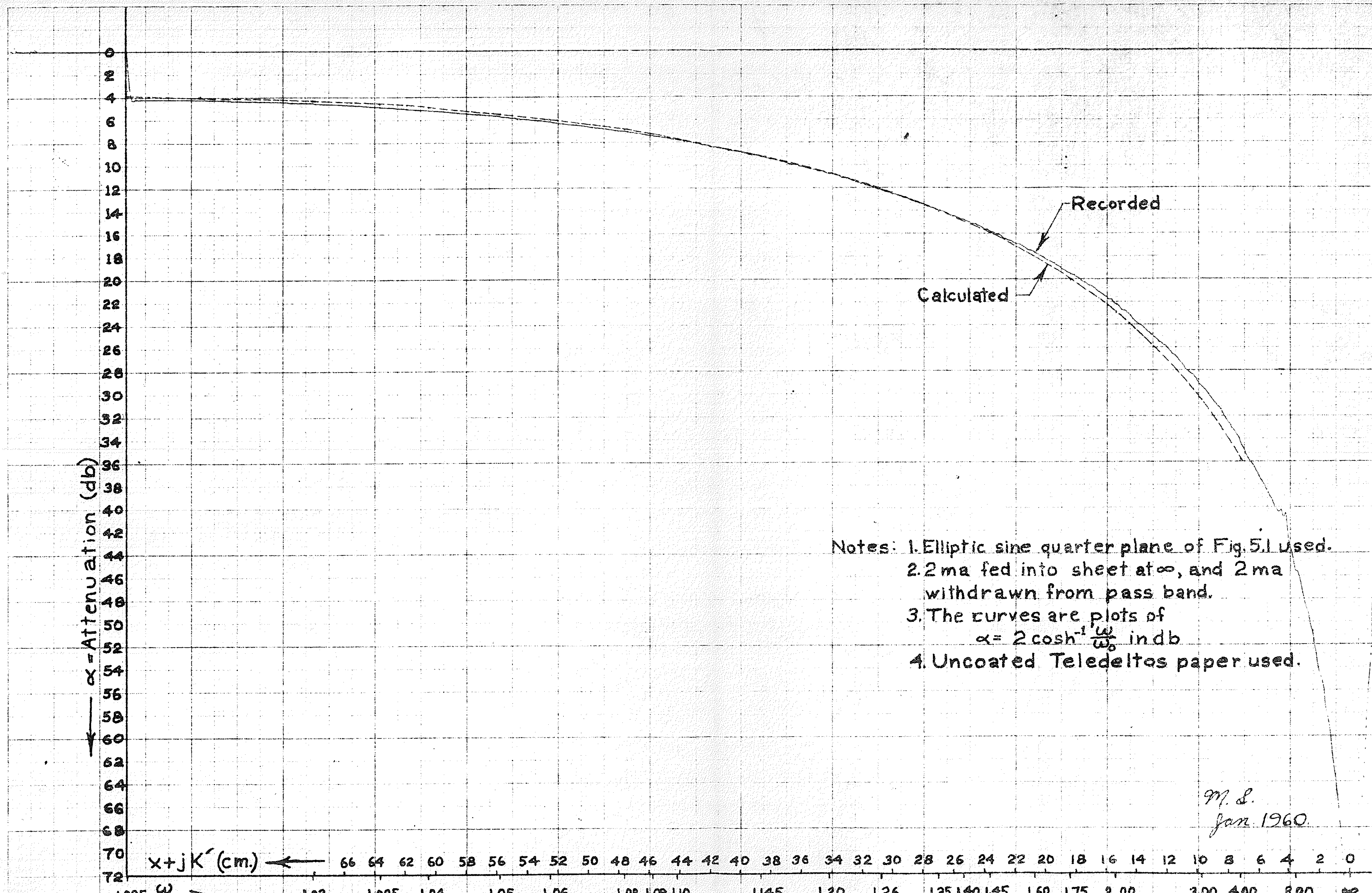


FIGURE 5.2 IMAGE PROPAGATION ATTENUATION OF CONSTANT-K FULL SECTION

to Cauer's q functions for a symmetric low-pass filter by

$$\coth \frac{\Gamma_1}{2} = q = \sqrt{\frac{z_a}{z_b}}, \qquad (5.22)$$

is found by considering that when all attenuation poles are at infinity the simplest such functions are given by

$$\coth \frac{\Gamma_1}{2} = \frac{\sqrt{1 + p^2}}{p} \qquad (5.23)$$

or

$$\coth \frac{\Gamma_1}{2} = \frac{p}{\sqrt{1 + p^2}}. \qquad (5.24)$$

The first of these has been treated.[60] The second yields the same $\frac{\alpha}{2}$ as the first, but the phase $\frac{\beta}{2}$ at $p = 0$ is $-\frac{\pi}{2}$, while it is 0 at $p = j1$. The same analogy may be used, therefore, as for equation 5.23, except that the reference for phase, $\beta/2$, is fixed at the value $-\frac{\pi}{2}$ at $p = 0$ on the recordings. This holds for all q-functions with a zero at zero.

For the functions $\frac{\Gamma_1}{2} \pm j\frac{\pi}{4}$, related to Cauer's q' functions for antimetric filters by

$$q' = \tanh\left(\frac{\Gamma_1}{2} \pm j\frac{\pi}{4}\right) = \coth\left(\frac{\Gamma_1}{2} \mp j\frac{\pi}{4}\right), \qquad (5.25)$$

a potential analog representation for the case when all one-points of

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[60]With simple replacement of Γ_1 by $\frac{\Gamma_1}{2}$ in the previous case.

q' are at infinity is obtained by using the analog developed above, but by placing odd multiples of half unit charges at infinity, and adding $\frac{\pi}{4}$ to $\frac{\beta}{2}$ at $p = 0$.[61]

Approximation of Γ_1 Using Attenuation Poles at Finite Real Frequencies and Extension of the Results to Arbitrary Complex Poles

Tuttles's solution for the potential due to two conjugate positive charges, each of value Q, above and below the pass band, and a conducting plate with charge of opposite sign along the pass band, is summarized in the following. The potential analog representation of W (analogous to Γ_1) in the p-plane can be mapped to the w-plane ($p = \sinh w$), resulting in the potential problem depicted in Figure 5.3. By application of the method of images, the conductor may be removed if the positive charges in the left half-plane are replaced by negative charges. Then the new potential problem of Figure 5.4 is solved and the right half-plane solution retained and transformed into the p-plane. The complex potential in Figure 5.4 is

$$W = Q\sum_{m=-\infty}^{+\infty} \ln \left[1 - \frac{w}{-a + j\frac{\pi}{2}(1 + 2m)}\right] - Q\sum_{m=-\infty}^{+\infty} \ln \left[1 - \frac{w}{a + j\frac{\pi}{2}(1 + 2m)}\right]. \tag{5.26}$$

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[61] Cauer, op. cit., pp. 284-288.

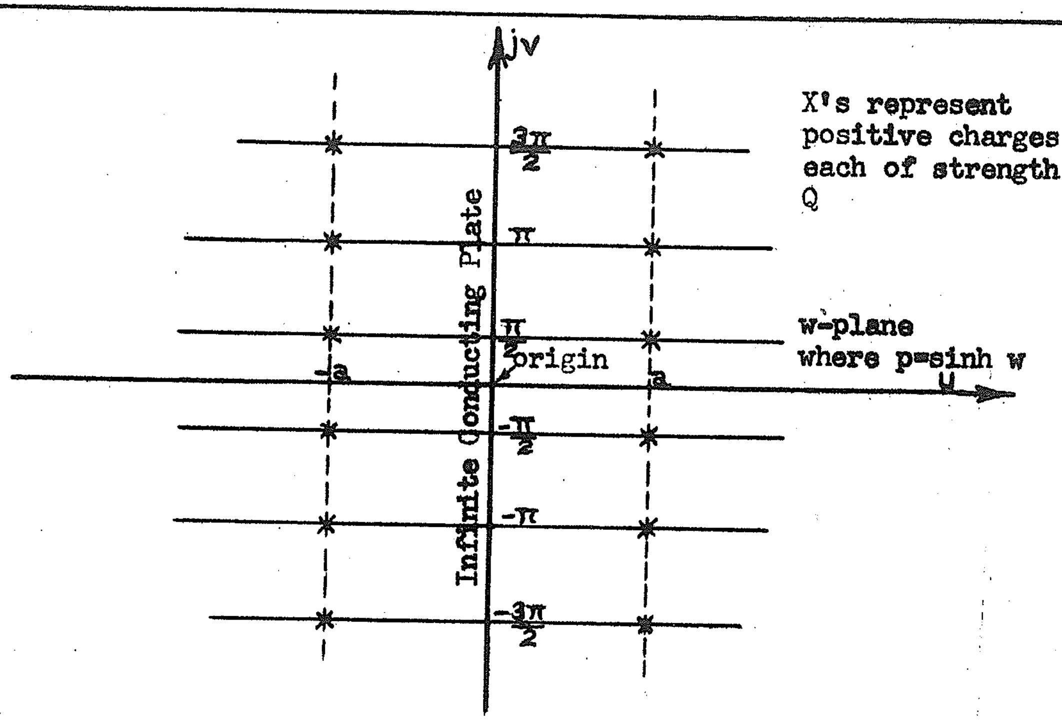

FIGURE 5.3

MAPPING TO W-PLANE OF POTENTIAL PROBLEM FOR TWO COMPLEX CONJUGATE CHARGES ON IMAGINARY P-AXIS AND CONDUCTOR ALONG PASS BAND

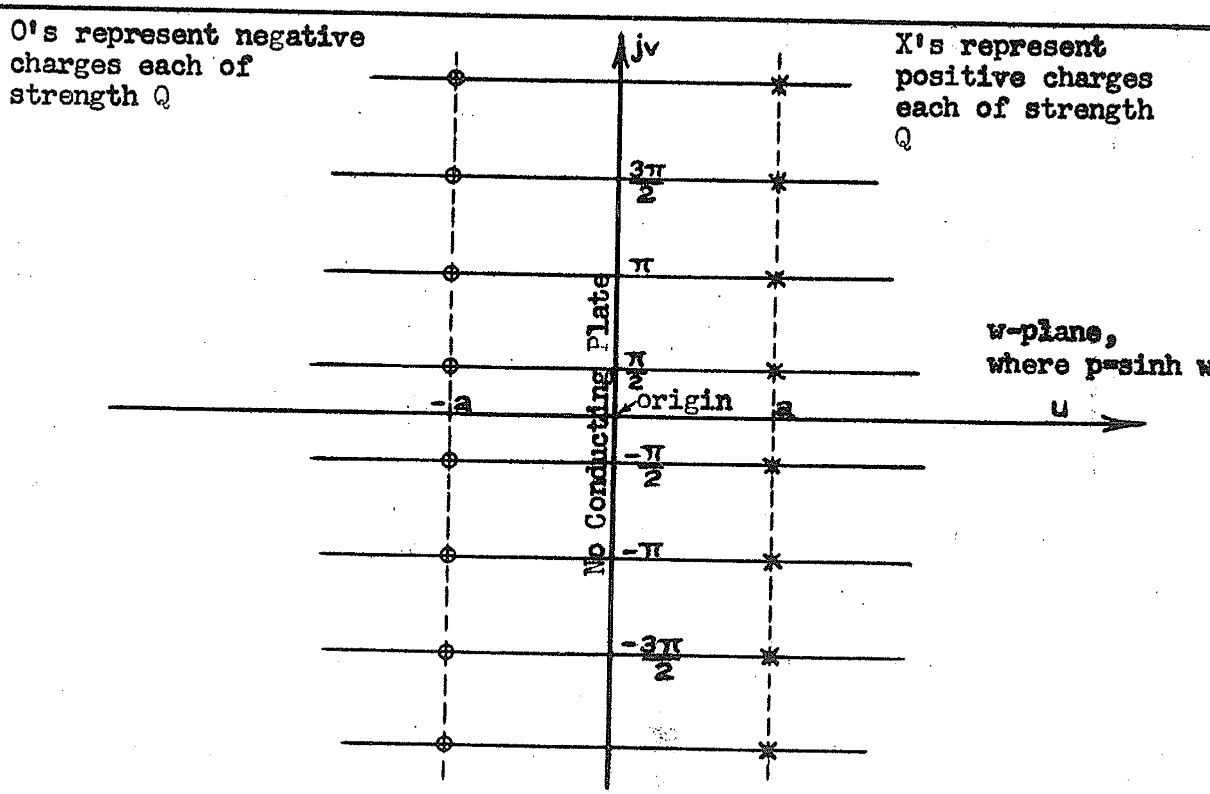

FIGURE 5.4

POTENTIAL PROBLEM EQUIVALENT TO THAT OF FIGURE 5.3 IN RIGHT HALF PLANE

Justification of convergence is not attempted; the final solution is proof enough from an "engineering" standpoint. By association of conjugate charges one obtains

$$W = Q\sum_{m=o}^{+\infty} \ln \left\{ \frac{1 + \left[\dfrac{w + a}{\frac{\pi}{2}(1 + 2m)}\right]^2}{1 + \left[\dfrac{a}{\frac{\pi}{2}(1 + 2m)}\right]^2} \right\}$$

$$-Q\sum_{m=o}^{+\infty} \ln \left\{ \frac{1 + \left[\dfrac{w - a}{\frac{\pi}{2}(1 + 2m)}\right]^2}{1 + \left[\dfrac{a}{\frac{\pi}{2}(1 + 2m)}\right]^2} \right\}. \qquad (5.27)$$

Hence,

$$e^{W} = \exp\left\{ Q \ln \frac{\cosh (w + a)}{\cosh a} - Q \ln \frac{\cosh (w - a)}{\cosh (-a)} \right\}$$

$$= \left\{ \frac{\cosh (w + a)}{\cosh (w - a)} \right\}^{Q}, \qquad (5.28)$$

and

$$e^{\frac{W}{Q}} = \frac{\cosh (w + a)}{\cosh (w - a)}, \; u > 0. \qquad (5.29)$$

For $w = 0^+ + jv$,

$$\frac{W}{Q} = 0 + j\,2\tan^{-1}\left[(\tanh a)\,(\tan v)\right], \tag{5.30}$$

which shows that flux increases with increasing v.

For $w = u + \frac{j\pi}{2}$, i.e., a mapping of the real frequency axis above the pass band,

$$\frac{W}{Q} = \ln \frac{\sinh(a + u)}{\sinh(a - u)} + j\pi, \quad 0 \leq u < a \tag{5.31}$$

$$= \ln \frac{\sinh(u + a)}{\sinh(u - a)}, \quad a < u < \infty. \tag{5.32}$$

In the p-plane, with $\Gamma_1 = W$ in anticipation of the analogy,

$$e^{\frac{W}{Q}} = e^{\frac{\Gamma_1}{Q}} = \frac{\sqrt{1 + p^2} + m\,p}{\sqrt{1 + p^2} - m\,p}, \tag{5.33}$$

where

$$m = \tanh a, \tag{5.34}$$

so that

$$\sinh a = \frac{1}{\sqrt{1 - m^2}} \tag{5.35}$$

and

$$\cosh a = \frac{1}{\sqrt{1 - m^2}}. \tag{5.36}$$

With $p = j\,u$ and $\omega \leq 1$,

$$\Gamma_1 = 0 + j\beta = 0 + j\,2\,Q\,\tan^{-1}\frac{m\,\omega}{\sqrt{1-\omega^2}}\,. \tag{5.37}$$

For $\omega > 1$,

$$\omega = \cosh\,u\,. \tag{5.38}$$

Then the frequency of infinite loss is

$$\omega_\infty = \cosh\,a. \tag{5.39}$$

Hence,

$$\omega_\infty = \frac{1}{\sqrt{1-m^2}}\,. \tag{5.40}$$

A sketch of α and β for real positive ω is shown in Figure 5.5.

The following is a collection of formulae expressing the complex potential in various ways.

$$\cosh\frac{\Gamma_1}{Q} = \frac{1+(1+m^2)\,p^2}{1+(1-m^2)\,p^2}\,. \tag{5.41}$$

$$\sinh\frac{\Gamma_1}{Q} = \frac{2\,mp\,\sqrt{1+p^2}}{1+(1-m^2)\,p^2}\,. \tag{5.42}$$

$$\tanh\frac{\Gamma_1}{Q} = \frac{2\,mp\,\sqrt{1+p^2}}{1+(1+m^2)\,p^2}\,. \tag{5.43}$$

$$\coth\left(\frac{\Gamma_1/Q}{2}\right) = \frac{\sqrt{1+p^2}}{m\,p}\,. \tag{5.44}$$

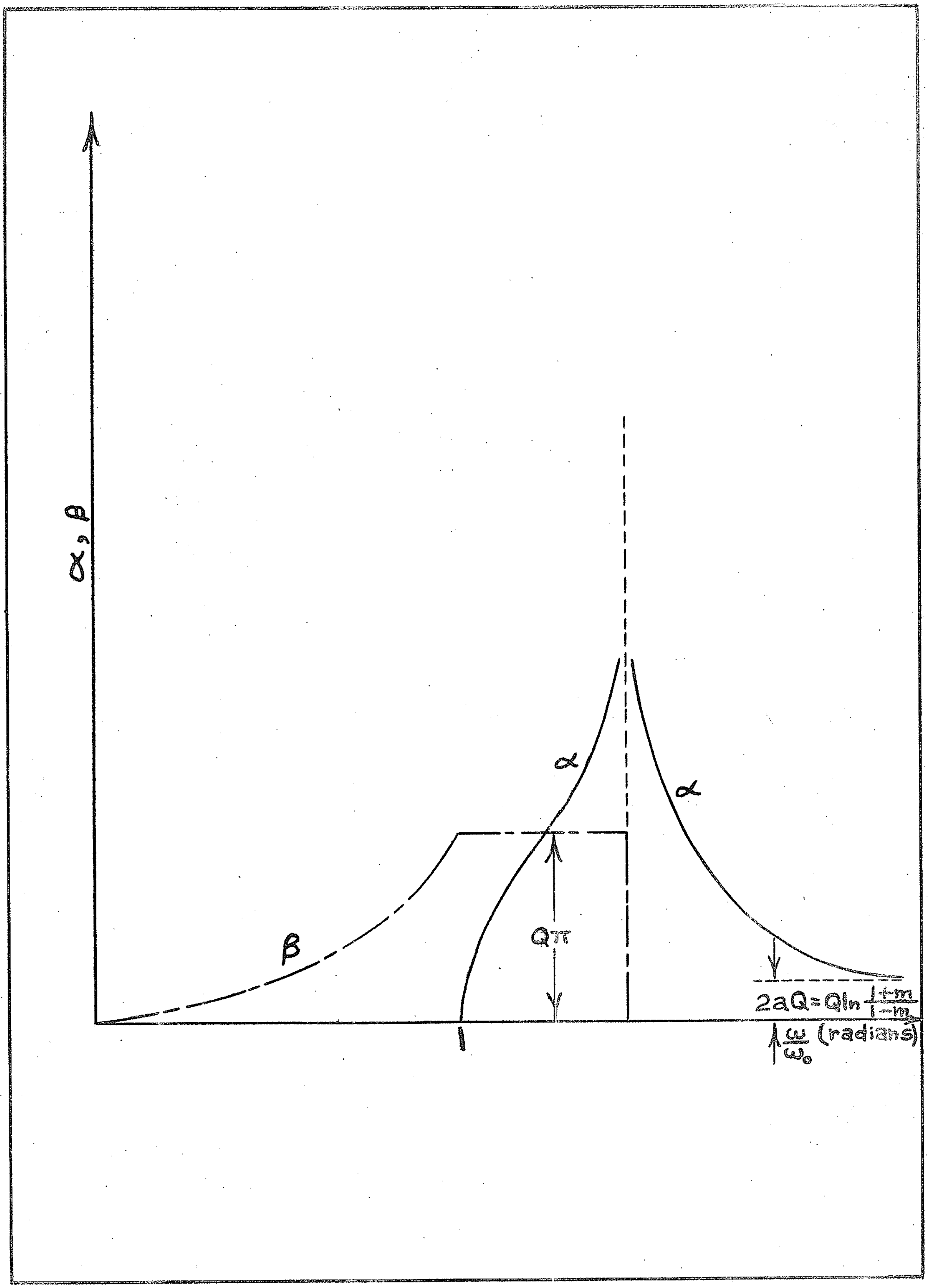

FIGURE 5.5

POTENTIAL AND STREAM FUNCTION ALONG POSITIVE REAL FREQUENCY AXIS WHEN THERE IS A CONDUCTOR ALONG THE PASSBAND AND TWO COMPLEX CONJUGATE POSITIVE CHARGES ON THE REAL FREQUENCY AXIS BELOW AND ABOVE THE PASSBAND

For $m = 1$,

$$\coth\left(\frac{\Gamma_1/Q}{2}\right) = \frac{\sqrt{1+p^2}}{p}, \qquad (5.45)$$

so that with $Q = \frac{1}{2}$, one obtains the admissible q function of equation (5.17). With $Q = \frac{1}{2}$, equation (5.44) becomes an admissible q function with simple one-points at conjugate values of p on the imaginary axis. With $Q = 1$,

$$\coth \frac{\Gamma_1}{2} = \frac{\sqrt{1+p^2}}{mp}, \qquad (5.46)$$

and W is analogous to Γ_1 with double attenuation poles (double one-points of coth Γ_1) along the real frequency axis. On the other hand W may be made analogous to $\frac{\Gamma_1}{2}$, in which case $Q = \frac{1}{2}$, and the q function has simple finite one-points on the real frequency axis. By subtracting $\frac{\pi}{2}$ from the phase of $\frac{\Gamma_1}{2}$ in $\coth \frac{\Gamma_1}{2} = \frac{\sqrt{1+p^2}}{mp}$, the analogy for $\frac{\Gamma_1}{2}$ in

$$\coth \frac{\Gamma_1}{2} = \frac{m\,p}{\sqrt{1+p^2}} \qquad (5.47)$$

is obtained.

Now that the potential analogies for Γ_1 corresponding to admissible q functions, $\frac{\Gamma_1}{2}$ corresponding to ordinary q functions, and $\frac{\Gamma_1}{2} \div j\frac{\pi}{4}$ corresponding to

$$q' = \sqrt{\frac{\Omega+1}{\Omega-1}} \qquad \text{or} \qquad \sqrt{\frac{\Omega-1}{\Omega+1}}, \ (p = j\Omega)$$

for poles at infinity in all three cases, and for finite poles in the first two cases, has been found, the analogy when there are any number of finite real frequency poles is easily found by superposition. This corresponds to composition of q-functions, admissible q-functions or q' functions according to the rules given by Cauer.[62] A more involved method of obtaining the same result is to solve the potential problem as above.

Although points of infinite loss in the complex plane (i.e., off the imaginary axis) are not generally used when the attenuation is of prime concern, the potential analogy is easily extended to this case. The admissible attenuation functions have one-points of even multiplicity off the imaginary axis. Thus, corresponding to one-points at real p, one has

$$q_z = \coth \Gamma_1 = \frac{m + m^{-1}}{2} \; \frac{p^2 + [1/(1 + m^2)]}{p\sqrt{p^2 + 1}} \qquad (5.48)$$

for the generating function, while for conjugate complex (not imaginary) one-points one has

$$q_z = \coth \Gamma_1 = \frac{(1 + m_o^2)\, p^4 + (1 + 2a^2 m_o^2)\, p^2 + a^4 m_o^2}{2m_o p(p^2 + a^2)\sqrt{p^2 + 1}} \qquad (5.49)$$

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[62] Cauer, op. cit., pp. 237-243, pp. 271-274 and pp. 285-286

as the generating function.[63] Hence, one places charges of Q=1 at real p and conjugate complex p (not imaginary), in the analogy for Γ_1.

Since the generating functions for ordinary q functions with real and conjugate complex (not imaginary) one-points are

$$q = \frac{m\ p}{\sqrt{p^2 + 1}} \tag{5.50}$$

and

$$q = \frac{m_o(p^2 + a^2)}{p\sqrt{p^2 + 1}}, \tag{5.50a}$$

respectively, charges of $Q=\frac{1}{2}$ are placed at these points in the analogy for $\frac{\Gamma_1}{2}$.

Since the q' functions are formed by composition of

$$q' = \sqrt{\frac{\Omega + 1}{\Omega - 1}} \quad \text{or} \quad \sqrt{\frac{\Omega - 1}{\Omega + 1}}$$

with

$$q = \frac{\sqrt{p^2 + 1}}{m\ p},$$

corresponding to simple one-points at imaginary or real p, and

$$q = \frac{m_o\ (p^2 + a^2)}{p\sqrt{p^2 + 1}}$$

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[63] Cauer, op. cit., pp. 242-243 and p. 271. The symbols m_o and a are constants explained here.

corresponding to conjugate complex (not imaginary) simple one-points, charges of $Q = \frac{1}{2}$ are placed at all points, including infinity, in the analogy for $\frac{\Gamma_1}{2} \pm j\frac{\pi}{4}$.

Implementation of the New Potential Analog System for Approximation of Γ_1, $\frac{\Gamma_1}{2}$ and $\frac{\Gamma_1}{2} \mp j\frac{\pi}{4}$ Corresponding Respectively to Cauer's q_z, q and q' Functions

One begins with a suitable quarter plane (rarely the p-plane representation) which is a suitable conformal mapping of the p-plane, and paints a highly conducting strip of silver paint (usually about one-half to one cm. wide) along the mapping of the normalized low-pass band, where $\alpha = 0$. The elliptic sine or elliptic tangent mapping has been found useful for the low-pass problem, especially when an equal-ripple stop band approximation is to be found. These mappings have the advantage that the scale is suitably expanded and that infinity is mapped to a finite point.[64] Unit current is determined by injecting a current, usually at infinity (or any other point, if desired), and withdrawing it from the pass band (or injecting it at the pass band and withdrawing it at infinity, if the plot is to be inverted), and then

[64]Boothroyd, A.R., "Design of Electric Wave Filters with the Aid of the Electrolytic Tank," Proc. I.E.E.(British), pt. 4, Vols. 90-100, 1951-53. Also Monograph #8, Radio Section.

adjusting the scale on the recording sheet to read the correct number of db above the pass band value $\alpha = 0$, at a point in the stop band. The details about how one sets the reference level on the recording, adjusts the scale, and takes a recording (if desired) of the calibrating function, are given in Appendix B-4, pp. 200-202. Once the system is calibrated for attenuation measurements the unit current is taken to be a multiple, usually one, of the current injected at infinity, and the attenuation specification in the stop band is met by using the appropriate number and positions of current probes. (An equal current is, of course, always withdrawn from the pass band). This is quickly done by making a sequence of recordings for different trial probe positions. In order to arrive at reactance ladder networks without mutual coupling the poles of attenuation must be along the real frequency axis. This also results in the most efficient approximation to attenuation specifications, although it is not suitable for approximation of linear phase (constant delay). The values of current that were appropriate for the various possible positions of attenuation poles for the three different functions representable on the analog have been discussed. One must consider only half these currents, of course, if the probes are along edges of quarter planes, at positions corresponding to finite points on the real or imaginary axis of the p-plane, and one-quarter of the current each at zero and infinity, and recall that the calibration is done by using current injected at infinity in the quarter plane. The details are so obvious that these points need no further comment. An example is probably of more help.

Figure 5.6 shows an attenuation curve recording found by using the potential analog system, the attenuation specifications that it meets and a ladder network having the calculated attenuation. The elliptic sine mapping of Fig. 5.1, page 88 was used to obtain a quarter plane. Calibration was performed by placing one current probe at infinity and one of opposite type in the pass band, and adjusting these currents to 2 ma. The reference level was set by moving the A wiper onto the pass band and using the potentiometer. Then the A wiper was moved to a point in the stop band, and the gain on the X-Y recorder adjusted to obtain the suitable db scale. Currents of 2 ma were then moved about along the stop band until the curve shown in Figure 5.6 was obtained. The finite poles transformed to the p-plane are at $\omega_{m1} = 1.07011$ and $\omega_{m2} = 1.19859$. This attenuation, α, corresponds to a q function given (according to composition of such functions from those with only a pair of simple conjugate one-points each) by

$$q = \coth \frac{\Gamma_1}{2} = \frac{p^3 + 0.906700905p}{(p^2 + 0.440731282)\sqrt{p^2 + 1}} .$$

The details of calibration for phase are found in Appendix B-4, pp. 205-207. Again a simple function is used. The phase is recorded along the pass band strip and the special considerations for q functions with zeros at zero, and q' functions, discussed in the previous section, are applied. For linear phase in the pass band one must use attenuation poles off the imaginary axis. It is to be remembered that linear phase

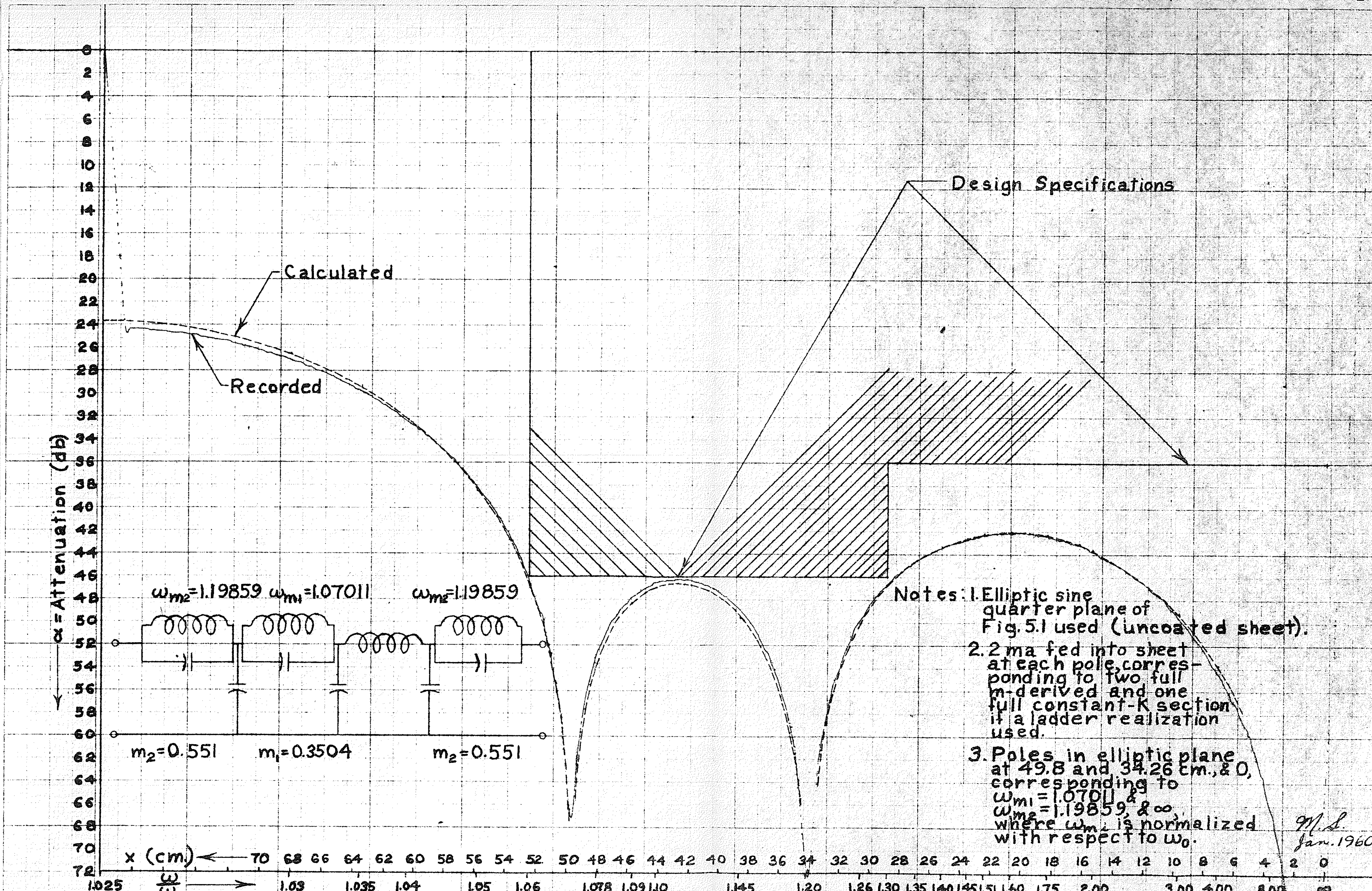

FIGURE 5.6 α MEETING SPECIFICATIONS AND NETWORK SCHEMATIC

in the p-plane does not, in general, map into linear phase in the conformal mappings.

V. APPROXIMATION OF CONSTANT IMAGE IMPEDANCE FOR A LOW-PASS BAND BY MEANS OF THE POTENTIAL ANALOGY

This section is concerned with the problem of finding normalized image impedances for a symmetric or asymmetric low-pass reactance filter, such that this impedance approximates unity in the pass band. Since a Chebyshev approximation to unity is most efficient, this type of approximation is particularly important, although other types (probably rarely used because of poorer matching, extra cost of components and inconvenient resulting filter structure) are possible. The problems considered, then, are the approximation problem

$$z_0 = \sqrt{z_b\, z_a} \doteq 1 \quad \text{for } 0 < \omega < 1 \tag{5.51}$$

for symmetric low-pass filters and the approximation problems

$$z_{o1} = \sqrt{\frac{z_{11}}{y_{11}}} \doteq 1 \text{ and } z_{o2} = \sqrt{\frac{z_{22}}{y_{22}}} \doteq 1 \quad \text{for } 0 < \omega < 1 \tag{5.52}$$

for asymmetric low-pass filters.[65] It is easy to see, from the properties of q-functions, that in the pass band z_o has the same zeros

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[65] Cauer, op. cit., pp. 225 and 265-267.

as z_a has poles or vice versa, at the pass band edge only one of z_a and z_b has a zero or pole, while in the stop band z_a and z_b have the same poles and the same zeros. Exactly the same statements may be made about (5.52) if z_{11} and $\frac{1}{y_{11}}$ replace z_a and z_b in the last statement.[66] Hence, z_0, z_{01} and z_{02} are purely resistive in the pass band and purely reactive in the stop band, with a branch point of order $\frac{1}{2}$ at the theoretical pass-band limit. This shows that it is sufficient to treat only the problem of approximating z_0.[67]

Tuttle has obtained the first two (simplest) classes of z_0 functions (or their inverses) with all one-points on the imaginary axis only, by using the potential analogy, and has indicated how one may use this analogy to derive z_0 functions of higher order with all one-points along $p = j\omega$.[68] It will be easy to see how the latter case, as well as the practically much less important case when all one-points are not at real frequencies, can be determined experimentally using the potential analog system, after Tuttle's solutions for the two simplest cases are summarized.

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[66] H.W. Bode, "A General Theory of Electric Wave Filters," Bell Telephone System Monograph B-843, November, 1934, pp. 28-46.

[67] It is to be recalled that z_{01}, z_{02} and q_z for unsymmetrical filters are specially related. However, this relationship is not needed here. The z_0, z_{01} and z_{02} are all q functions, as defined by Cauer, op. cit., p. 232.

[68] D.F. Tuttle, An Introduction to Network Synthesis, Unpublished text for Stanford University course 237, c.1951.

Approximation of z_o by Use of the Potential Analogy

Tuttle shows (as can be seen easily from a consideration of the properties of q functions) that for the complex potential $W=V + j\Psi$, which is analoguous to $\ln z_o$,

1. At cutoff ($p = \pm j\,1$) Ψ has a discontinuity of $\pm \frac{\pi}{2}$. Therefore, there can be only half unit charges at the pass band edges.
2. In the stop band Ψ may have discontinuities of $\pm \pi$. Therefore, there can be only unit charges at points in the stop band.
3. In the pass band there can be no discontinuities in Ψ. Therefore, there can be no charges in the pass band.
4. Between the above discontinuities Ψ is constant (except in the pass band, of course).

For the simplest z_o function the analogy is obtained by placing charges Q, of the same sign, at $p = \pm j\,1$. Then the complex potential is

$$W = -Q \ln (p^2 + 1). \tag{5.53}$$

Because of the first property above $Q = \pm \frac{1}{2}$, so

$$W = \pm \tfrac{1}{2} \ln (p^2 + 1), \tag{5.54}$$

and

$$e^W = \left(\sqrt{p^2 + 1}\,\right)^{\pm 1} . \tag{5.55}$$

Since a constant μ_o only changes the level, the simplest image impedances are

$$z_o = \frac{1}{\mu_o} \sqrt{p^2 + 1} \tag{5.56}$$

or
$$z_o = \frac{\mu_o}{\sqrt{p^2 + 1}} \, . \tag{5.57}$$

Because of the alternation of poles and zeros of z_o, the charge following the branch point charge must be of opposite sign to the latter. By mapping the p-plane potential problems, when there is only one charge in addition to that at the branch point, to the w plane, where p= sinh w, one obtains the w-plane potential problems shown in Figure 5.7(a) or 5.7(b), since the w-plane unit charges occur at the mapping of the p-plane branch points, at which there are half-unit charges. The exponential potential for the first case (Figure 5.7(a)) is

$$e^W = (\cosh w) \frac{\cosh a}{\cosh (w + a)} \quad \frac{\cosh (-a)}{\cosh (w - a)} \tag{5.58}$$

With $p = \sinh w$, $\cosh w = \sqrt{1 + p^2}$ and

$$m \triangleq \tanh a, \tag{5.59}$$

and the introduction of the real constant μ_o, (5.58) becomes

$$e^W = \frac{\sqrt{p^2 + 1}}{\mu_o \left[1 + (1 - m^2)\, p^2\right]} \, . \tag{5.60}$$

Solution of the potential problem of Figure 5.7(b) results in the inverse of (5.60). Hence, the analogy for the z_o function of second-order complexity has been proven.[69]

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[69] Cauer, op. cit., p. 232.

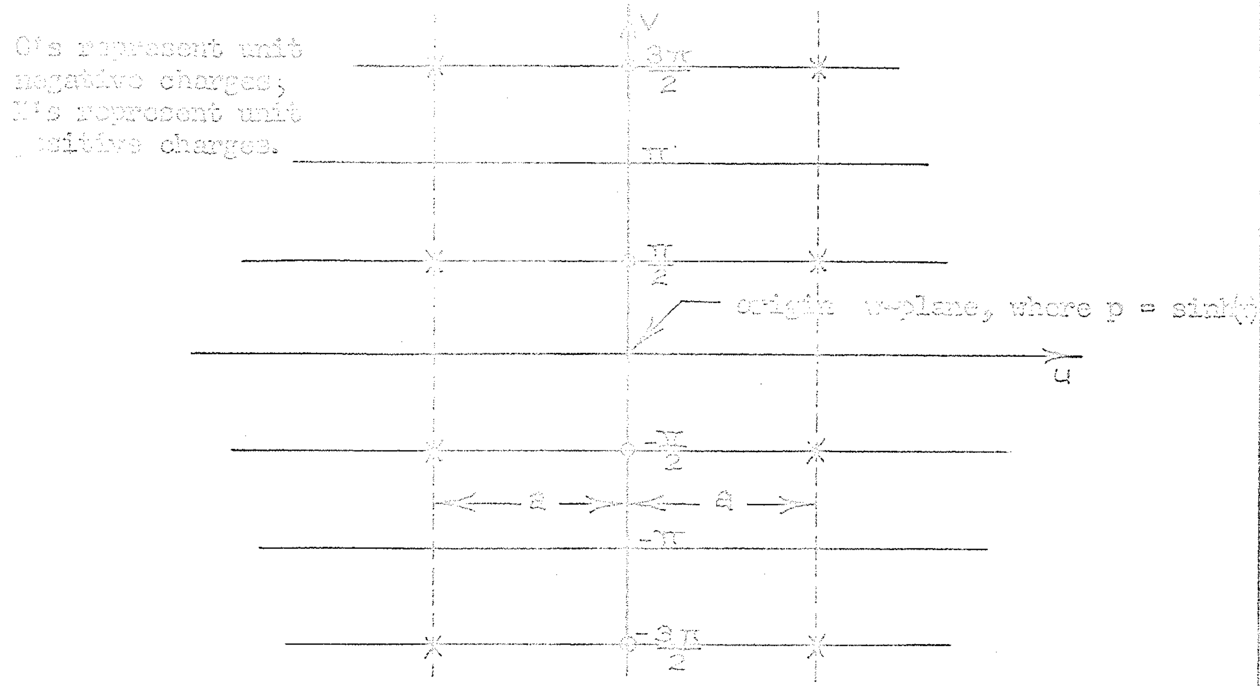

FIGURE 5.7(a)

MAPPING OF POTENTIAL PROBLEM, FOR 1/2 - UNIT NEGATIVE CHARGES AT $p = \pm j$ AND A PAIR OF UNIT POSITIVE CHARGES IN STOP BAND, INTO W-PLANE

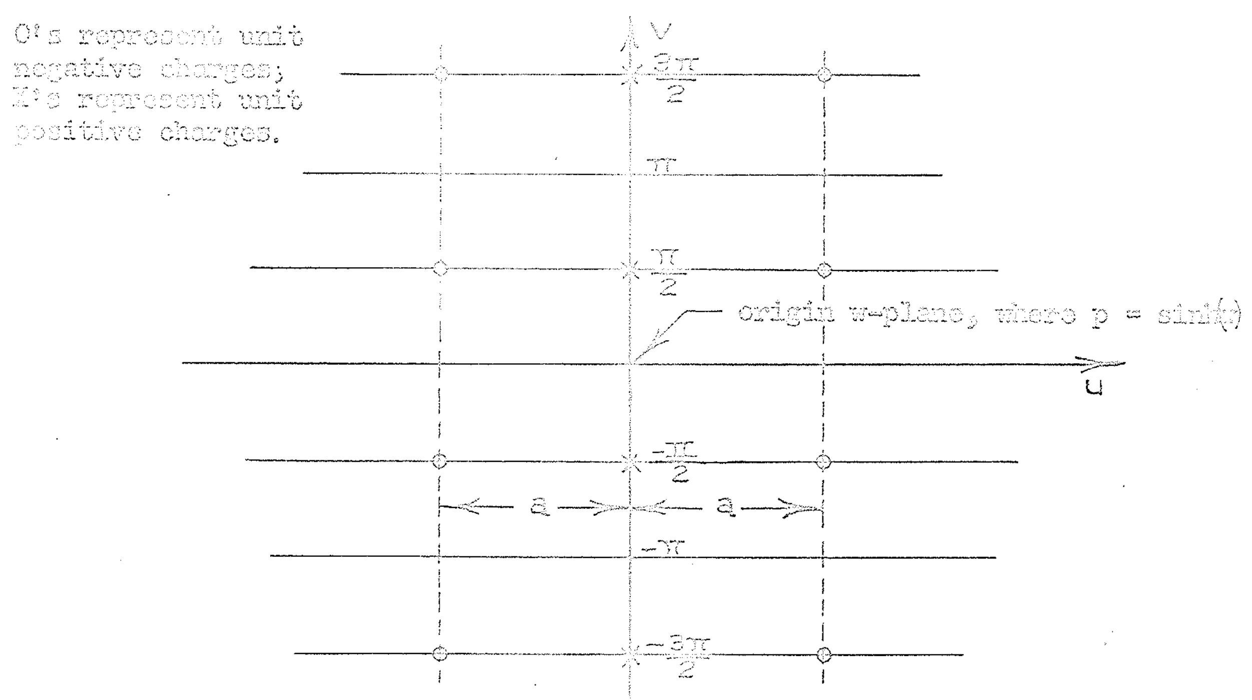

FIGURE 5.7(b)

MAPPING OF POTENTIAL PROBLEM, FOR 1/2 - UNIT POSITIVE CHARGES AT $p = \pm j$ AND A PAIR OF UNIT NEGATIVE CHARGES IN STOP BAND, INTO W-PLANE

The extension to z_o functions of any order of complexity is easily effected by placing additional charges, all of alternating sign, along the imaginary axis of the p-plane. For the most efficient approximation, it is desirable to position these charges so as to obtain Chebyshev approximation to unity in the pass band. Then all one-points of z_o will be in the pass band. To facilitate obtaining such an approximation it is most desirable to use a quarter elliptic sine conformal mapping, with pass band, transition band and stop band each mapping to three different sides of the sheet.[70] In this plane it is evident from physical considerations of symmetry (and the fact that all charges are of unit magnitude) that the desired charge positions are equally spaced along the mapping of the stop band. A unit charge of either sign must be present at both the mapping of the branch point and at infinity.

With these ideas it is easy to see how the potential analog apparatus may be implemented to obtain desired approximations to a constant unity image impedance in the pass band, even in the (rare) case where one-points of z_o are not imaginary.

It is interesting to note that the deviation, s, of an image impedance from unity in the pass band, is related to the attenuation

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[70] Such a plane is also useful for the least complicated functions z_o.

function A_1 in the stop band, attained with the q function obtained by replacing p by $\frac{1}{p}$ in a z_o function of the same degree, by

$$s = \coth \frac{A_1}{2} - 1, \qquad (5.61)$$

where the relation holds at frequencies inverse with respect to the cutoff frequency. Equation (5.61) also holds for admissible q functions, with $\frac{A_1}{2}$ replaced by A_1. These ideas follow easily from the fact that z_o and q functions are obtained from each other by replacing p by $\frac{1}{p}$. Equation (5.61) thus provides a way of measuring s by means of a measurement of $\frac{A_1}{2}$ or A_1 according to the analogies and methods developed for $\frac{\Gamma_1}{2}$ and Γ_1. This may be useful if s_{max} is very small, and difficult to measure accurately.

The above has shown the analogy between z_o and the exponential complex potential, and provided a method of obtaining the most general possible function, z_o, for the low-pass reactance filter by means of rapid measurements with the new pole-zero apparatus. Since q functions are related to z_o functions by the transformation $p \rightarrow \frac{1}{p}$, it is evidently possible to represent q functions and q_z functions by the potential analog method. Furthermore, q' functions for antimetric low-pass filters can be represented on the analog. The following section treats the representation of q, admissible q, and q' functions by the potential analog method.

VI. THE POTENTIAL ANALOGIES FOR q, ADMISSIBLE q AND q′ FUNCTIONS FOR REACTANCE LOW-PASS FILTERS

From the fact that q functions are obtained from z_o functions by replacing p by $\frac{1}{p}$ in the latter, it is easy to see that the potential analogy for both q and admissible q functions is obtained by placing one-half unit charges at the branch points in the p plane and alternating unit charges along the pass band, including zero frequency. The admissible q functions have a pole at zero frequency, while ordinary q functions may have either a pole or zero there. Also, the charges in the pass band represent coincident poles and zeros of z_b and $\frac{1}{z_a}$ (or z_a and $\frac{1}{z_b}$) in the case of ordinary q functions, while they represent coincident poles and zeros of z_{11} and y_{11} for the admissible q functions.

For measurement of q and admissible q functions it is usually most convenient to use a conformal representation such as the quarter elliptic sine mapping. It is clear that the real potential is proportional to $\ln|q|$, unless an antilogarithmic amplifier is used.

q' functions are represented in the analogy by placing half-unit charges of opposite sign at the two branch points and unit charges of opposite and alternating sign on the $j\omega$-axis, symmetrically located about p=0. For measurement using the potential analog system it is probably most convenient to use a half plane such as an elliptic sine mapping.

VII. SOME EXAMPLES OF SIMPLE LADDER REACTANCE FILTERS SYNTHESIZED WITH THE AID OF THE POTENTIAL ANALOG SYSTEM

The example given by Storer was done by using the potential analog system.[71] Figure 5.8 shows the image loss, α, recorded by using the potential analog system, the calculated image loss and the lossless filter schematic. The ordinary p half plane described on Figure 5.8 was used for obtaining the recording. One notes that the scale near the stop band edge was not sufficiently expanded to permit accurate recording in this region.

Figure 5.9 shows the same problem done by using the ζ-plane, where

$$\frac{p}{\omega_0} = \frac{\zeta^2 - 1}{2\zeta} \tag{5.62}$$

(The attenuation for the full constant-k section and one of the others is also recorded). The transition region is greatly expanded now and the point $p = \infty$ maps into a point.

Figure 5.10, recorded for the same problem, shows that the w-plane, where

$$w = \frac{-1}{p/\omega_0 + j} \tag{5.63}$$

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[71] J. E. Storer , Passive Network Synthesis (Toronto: McGraw-Hill Book Co., Inc., 1957), pp. 105-107.

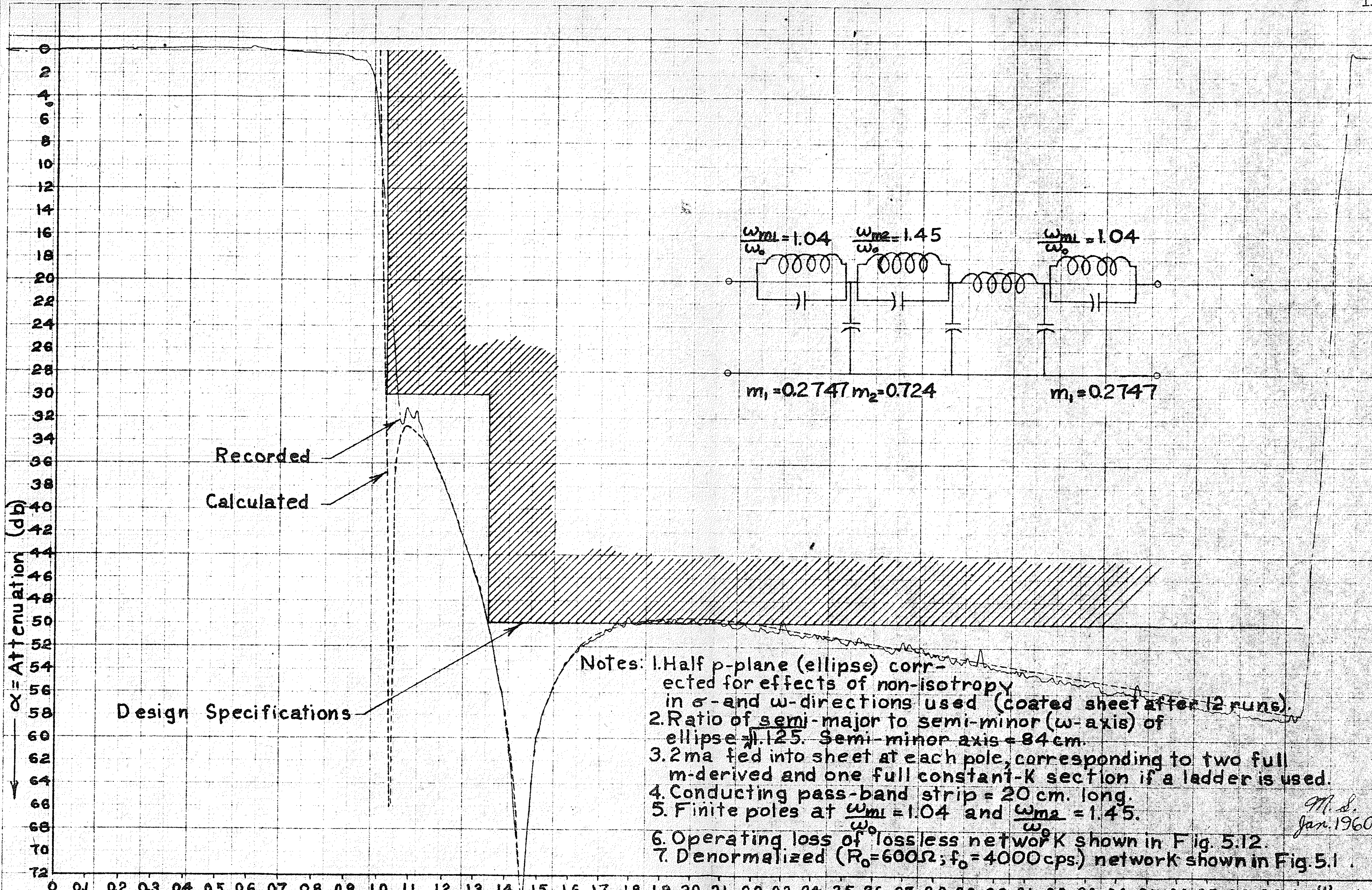

FIGURE 5.8 RECORDED α USING P HALF PLANE, CALCULATED α, SPECIFICATIONS & NETWORK SCHEMATIC

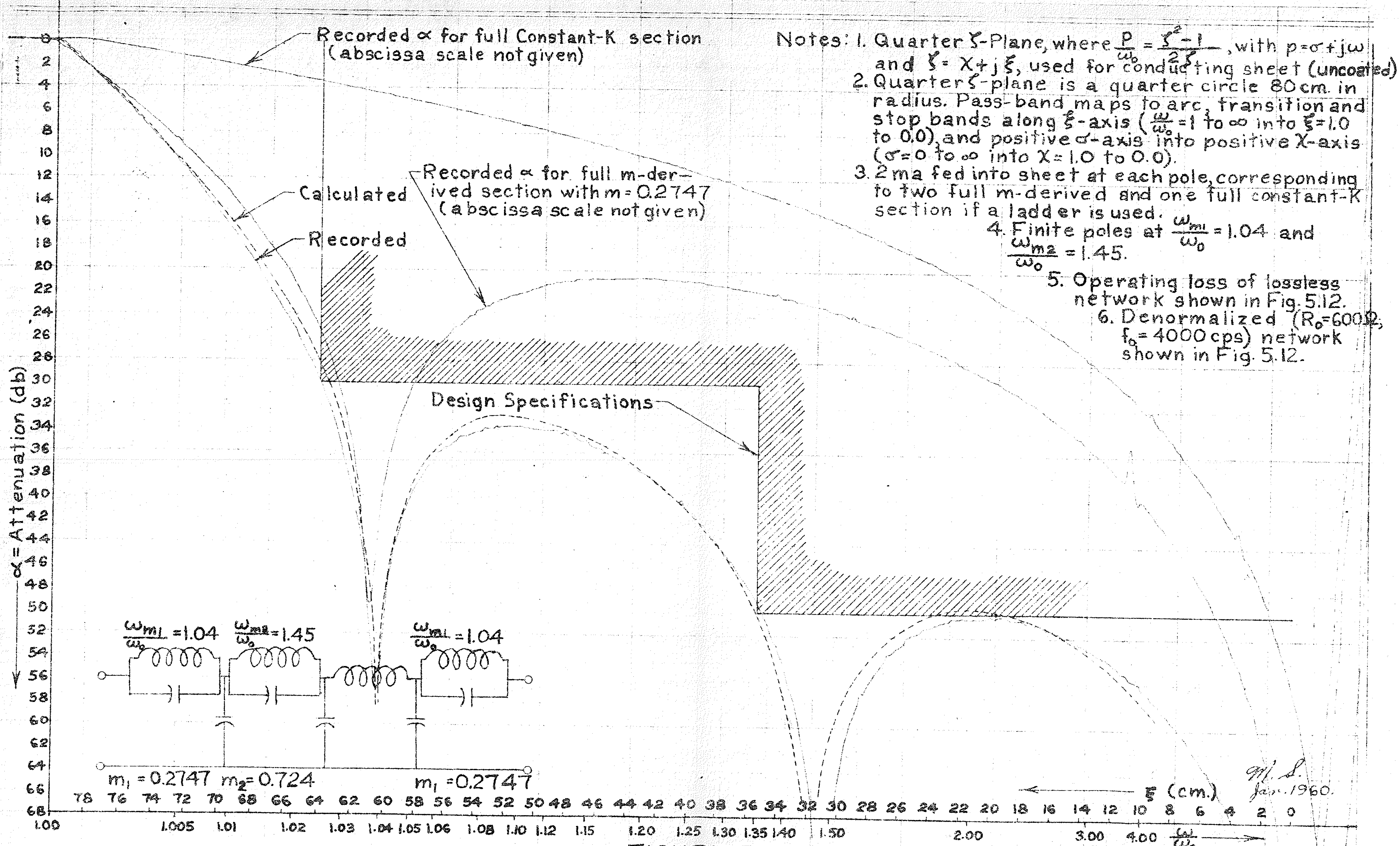

FIGURE 5.9
RECORDED α USING ζ QUARTER PLANE, CALCULATED α, SPECIFICATIONS & NETWORK SCHEMATIC

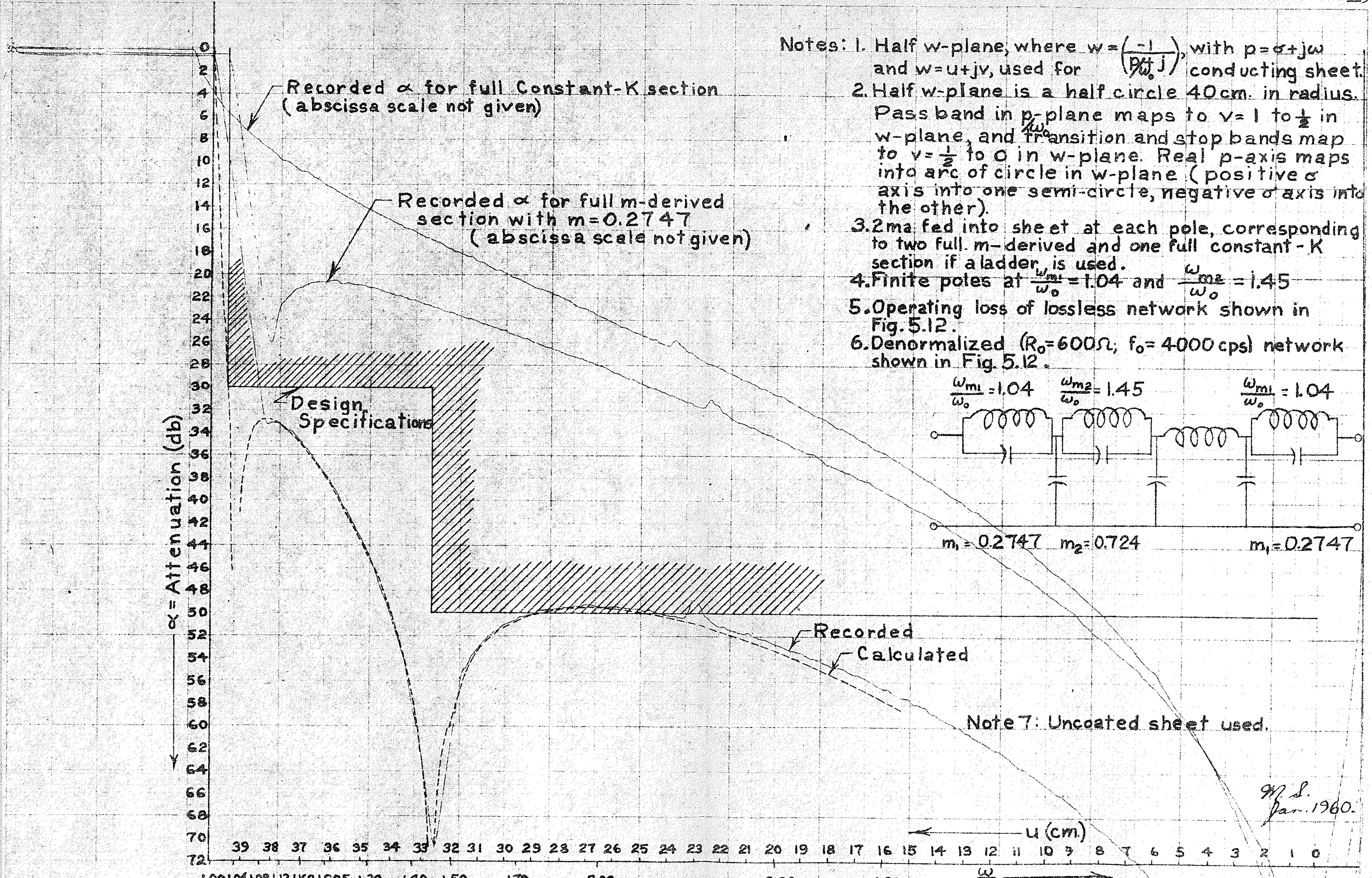

FIGURE 5.10 RECORDED α USING W HALF PLANE, CALCULATED α, SPECIFICATIONS & NETWORK SCHEMATIC

is not suitable for this problem because of the contracted mapping in the vicinity of the stop-band edge.

Figure 5.11 shows Storer's filter problem done on an elliptic sine quarter plane described in the Figure. Evidently one pole occurs on a side of this plane along which the potential was not recorded. Otherwise, this plane is perfectly suitable.

Figure 5.12 shows the final mapping and recording for this problem. The elliptic sine quarter plane of Figure 5.1, page 88, was completely suitable, although the transition band is not recorded as in Figure 5.9. The network resulting from denormalization to an ideal cutoff frequency of 4000 cps and terminations of 600 ohms, which the image impedance matches in a Chebyshev fashion in the pass band, is also shown in Figure 5.12. Chebyshev matching of the image impedance results if the limit of this matching is at

$$f_c = f_{\infty 1} \sqrt{2 - \left(\frac{f_{\infty 1}}{f_o}\right)^2}$$

and the relative deviation, θ, of z_o from unity is found from

$$\theta^2 = \frac{\left(\frac{f_{\infty 1}}{f_o}\right)^2}{2\sqrt{\left(\frac{f_{\infty 1}}{f_o}\right)^2 - 1}} ,$$

as is easily shown from equation (5.60), or found in various books.[72]

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[72] Cauer, op. cit., p. 361. Design procedures and formulae for element values may be found in Appendix 2, pp. 357-384.

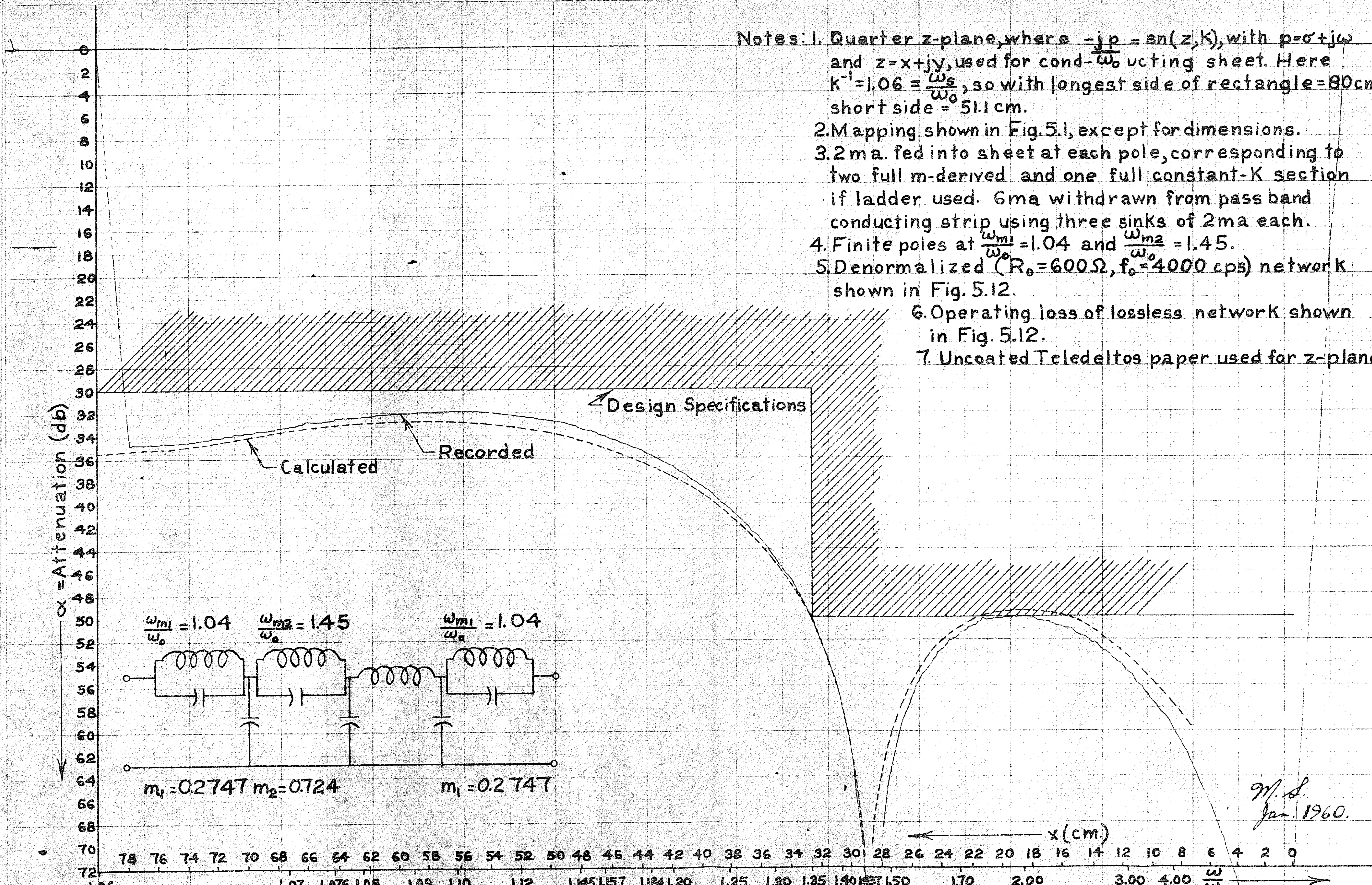

FIGURE 5.11 RECORDED α USING SN QUARTER PLANE, CALCULATED α, SPECIFICATIONS & SCHEMATIC

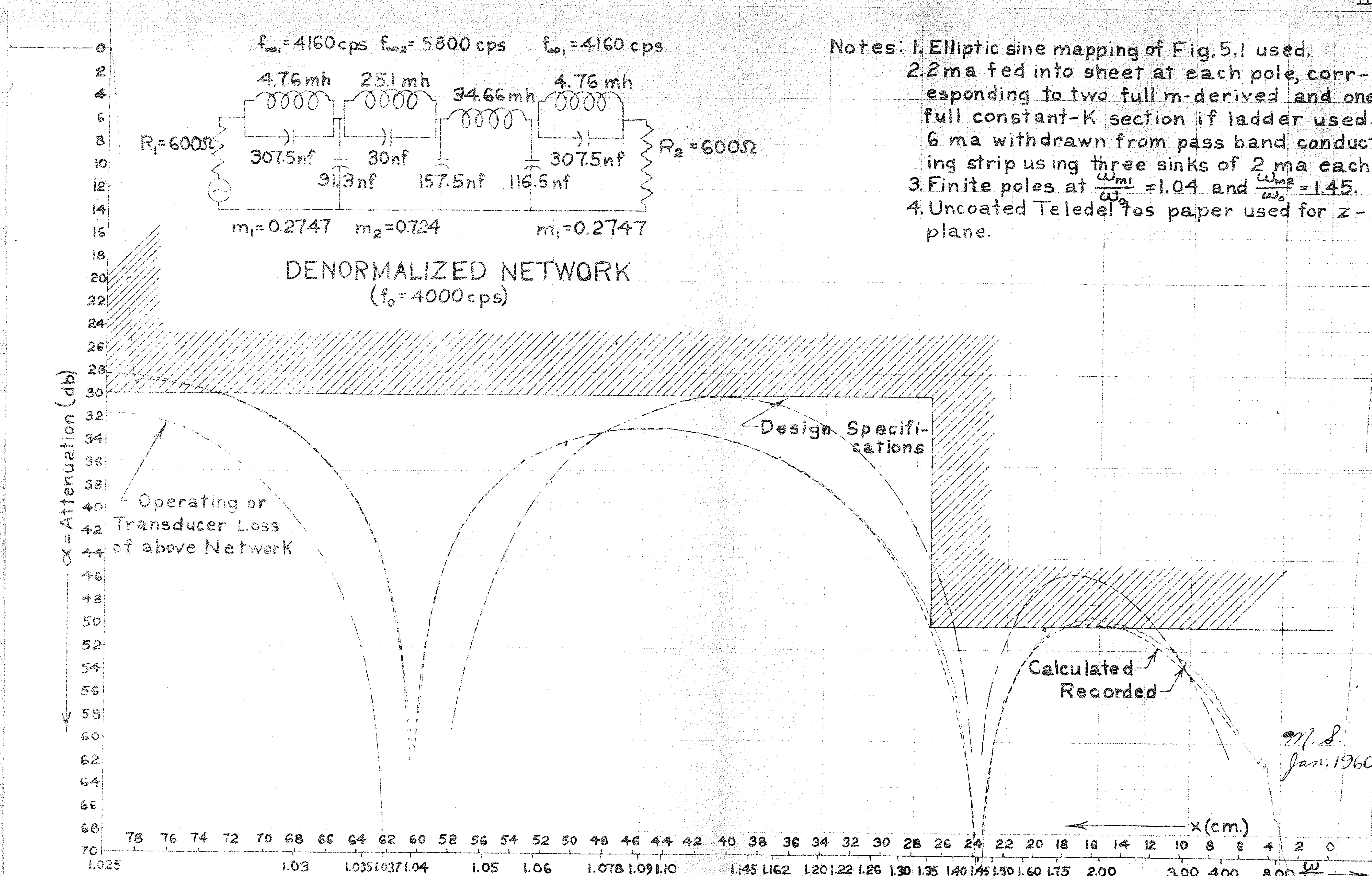

FIGURE 5.12
RECORDED α USING SN QUARTER PLANE, CALCULATED α, SPECIFICATIONS, NETWORK & OPERATING LOSS

The operating loss of the lossless network is also plotted in Figure 5.12.[73]

Figures 5.13 and 5.14 show recorded and calculated image attenuation for a more complicated example. The same ζ plane as in Figure 5.9 was used for Figure 5.13, while the same elliptic sine plane as in Figure 5.11 was used for Figure 5.14. Figure 5.15 (for the same example) also shows the network denormalized to an ideal cut-off frequency of 10 kc and Chebyshev-matched terminations of 135 ohms. The operating loss of the lossless network is also plotted.

Figure 5.16 shows recorded and calculated image attenuation, the denormalized network, and its operating loss, for an antimetric low-pass filter of class 7/2 with experimentally determined Chebyshev behavior of stop band image attenuation.[74] The terminations were again determined to match the image impedance in a Chebyshev manner. The image impedance at the generator end of the network is of class β^* and the right side image impedance is of class β.[75] The network elements were obtained with the aid of the Appendix in Cauer's book (Footnote 72). It is interesting, incidentally, to note the effect of the reflection losses in increasing the "strength" of the finite pole

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[73]For a definition of operating loss one may consult Cauer, ibid., pp. 130-131.

[74]The classification is due to Cauer, ibid., p. 283.

[75]Ibid., p. 282. The image impedances could have been studied by using an analog of section V of this chapter to obtain a suitable recording.

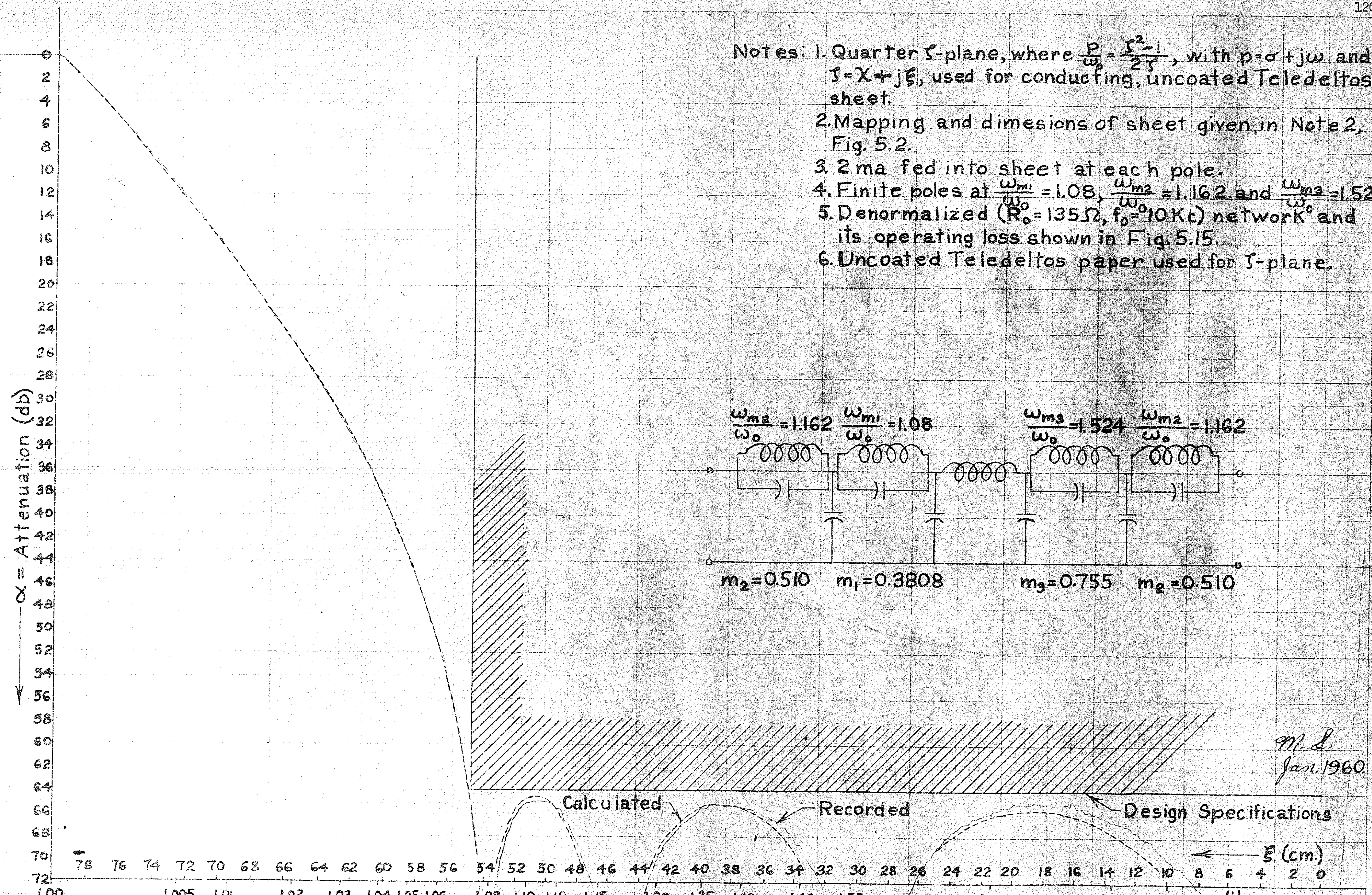


FIGURE 5.13 RECORDED α USING ζ QUARTER PLANE, CALCULATED α, SPECIFICATIONS & SCHEMATIC

Some additional values of transition band attenuation. The measured values were found by connecting a metal probe to the "A" input jack and reading attenuation at the left scale of this sheet when it lay in the X-Y recorder.

$\frac{\omega}{\omega_o}$	Measured (db)	Calculated (db)
1.005	11.6	12.2
1.01	17.2	17.4
1.02	23.6	25.25
1.04	35.7	38.0
1.05	42.0	44.3

Notes:
1. Quarter elliptic sine plane described in Notes 1&2, Fig. 5.11, also used here.
2. 2ma fed into sheet at each pole, & 8ma withdrawn from pass band conducting strip using four sinks of 2 ma each, distributed along the strip.
3. Uncoated Teledeltos paper used for z-plane.
4. Finite poles at $\frac{\omega_{m1}}{\omega_o} = 1.08$, $\frac{\omega_{m2}}{\omega_o} = 1.162$ & $\frac{\omega_{m3}}{\omega_o} = 1.524$.
5. Denormalized ($R_o = 135\Omega$, $f_o = 10Kc$) network and its operating loss shown in Fig. 5.15.

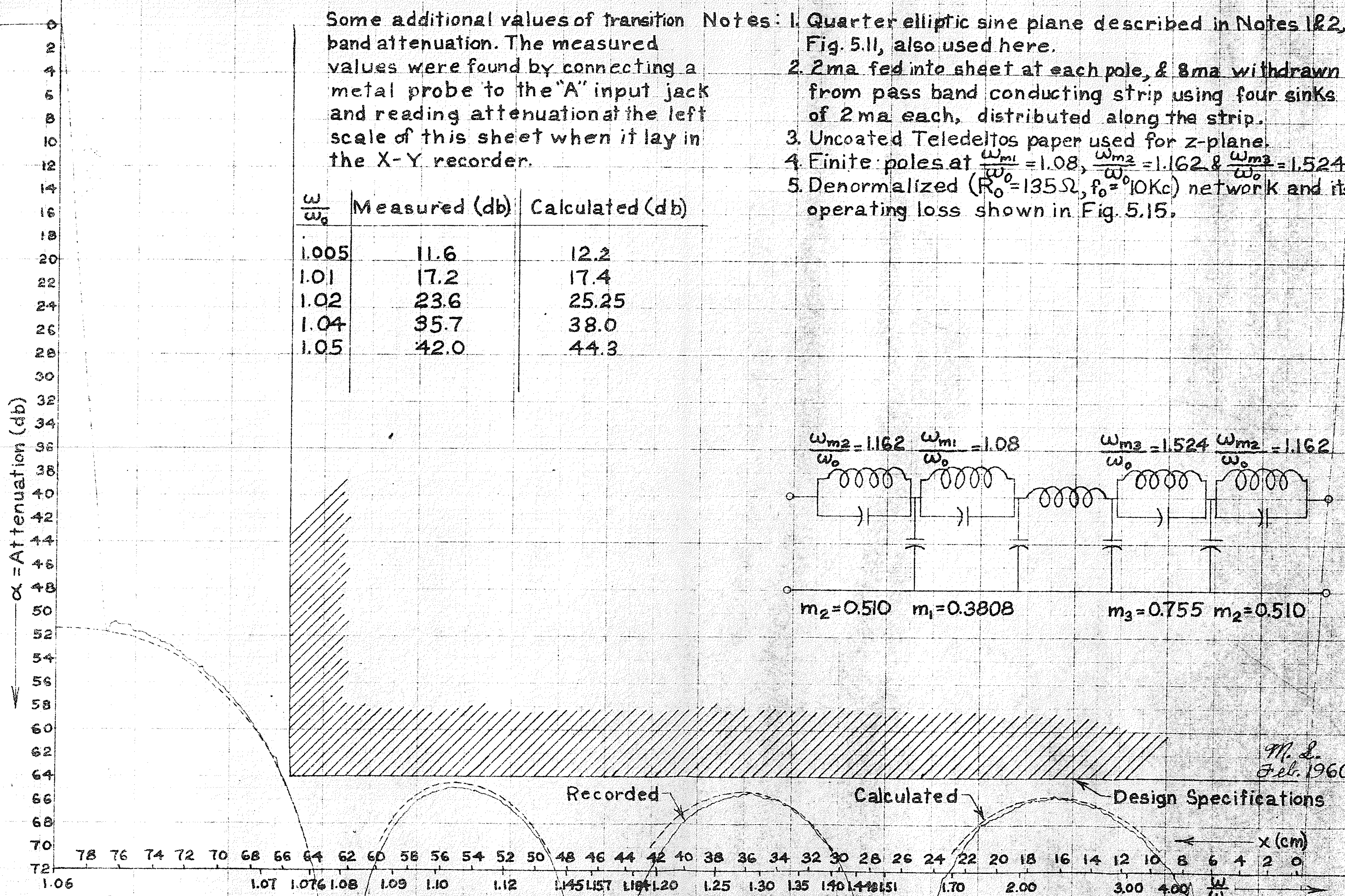
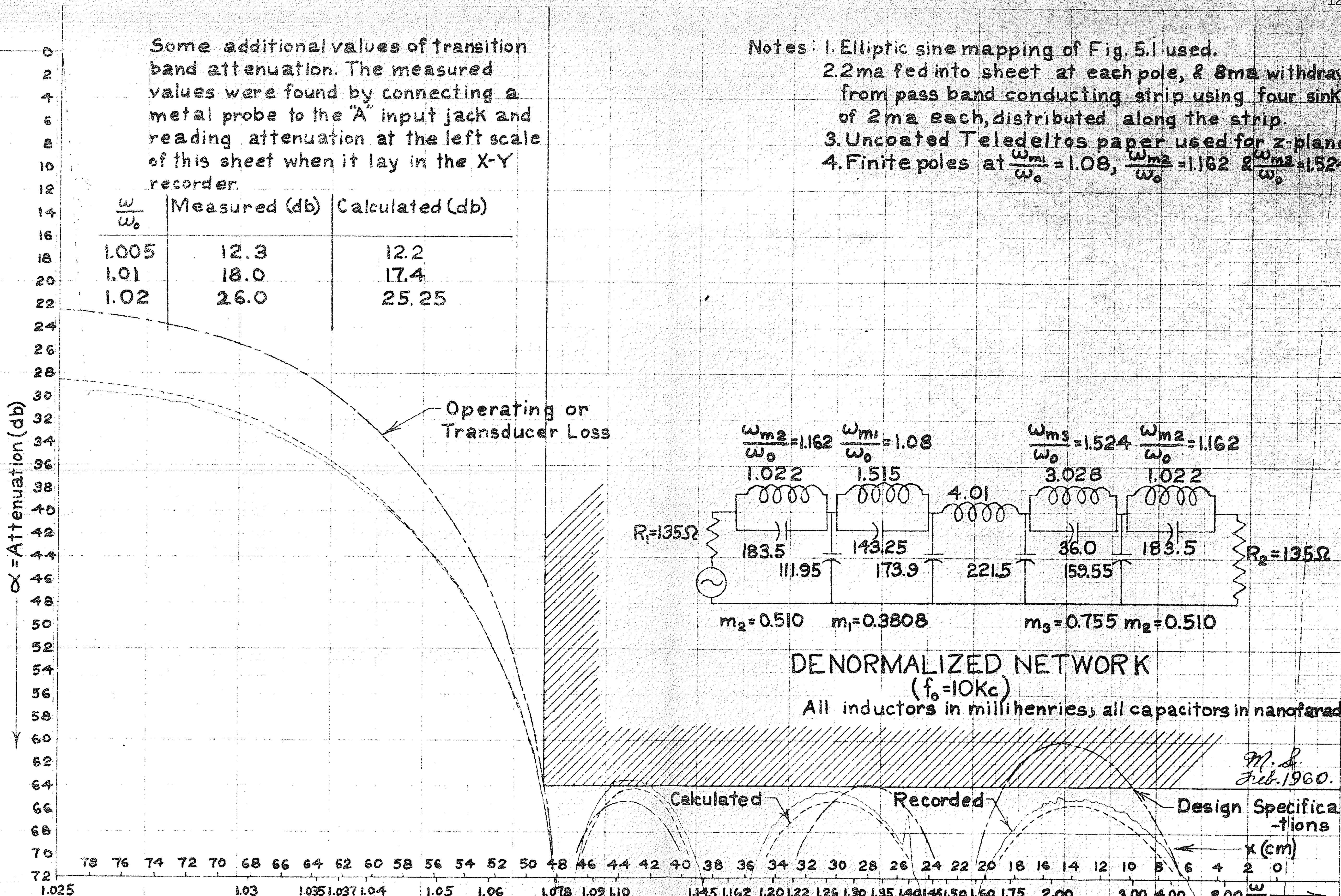


FIGURE 5.14 RECORDED α USING SN QUARTER PLANE, CALCULATED α, SPECIFICATIONS & SCHEMATIC



Some additional values of transition band attenuation. The measured values were found by connecting a metal probe to the "A" input jack and reading attenuation at the left scale of this sheet when it lay in the X-Y recorder.

$\frac{\omega}{\omega_0}$	Measured (db)	Calculated (db)
1.005	12.3	12.2
1.01	18.0	17.4
1.02	26.0	25.25

Notes: 1. Elliptic sine mapping of Fig. 5.1 used.
2. 2 ma fed into sheet at each pole, & 8 ma withdrawn from pass band conducting strip using four sinks of 2 ma each, distributed along the strip.
3. Uncoated Teledeltos paper used for z-plane.
4. Finite poles at $\frac{\omega_{m1}}{\omega_0} = 1.08$, $\frac{\omega_{m2}}{\omega_0} = 1.162$ & $\frac{\omega_{m3}}{\omega_0} = 1.524$.

FIGURE 5.15 RECORDED α USING SN QUARTER PLANE, CALCULATED α, SPECIFICATIONS, NETWORK & OPERATING LOSS

Some additional values of transition band attenuation. The measured values were found by connecting a metal probe to the "A" input jack and reading attenuation at the left scale of this sheet when it lay in the X-Y recorder.

$\frac{\omega}{\omega_0}$	Measured (db)	Calculated (db)
1.005	14.9	15.35
1.01	22.4	22.7
1.02	35.9	36.3

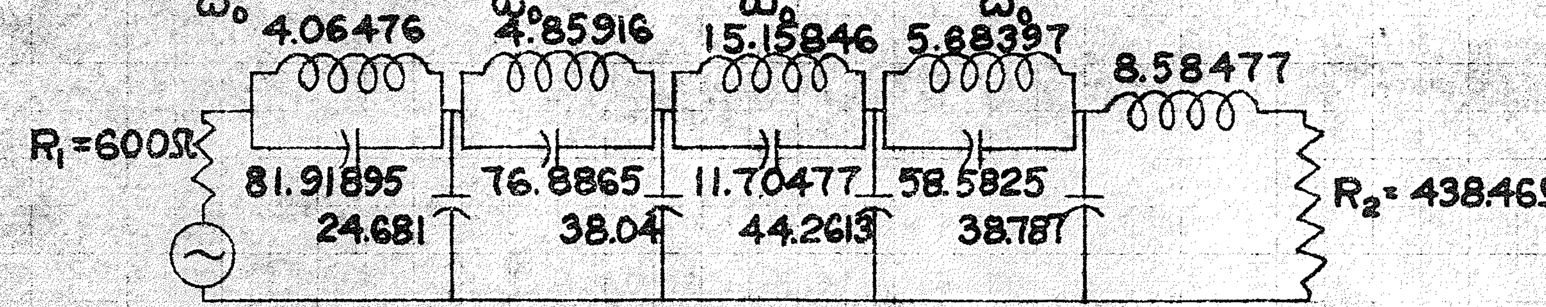
Notes: 1. Elliptic sine mapping of Fig. 5.1 used.

2. 2 ma fed into sheet at each finite pole, and 1 ma at the infinite frequency pole. 7 ma withdrawn from pass band conducting strip using 3 sinks of 2 ma a 1 sink of 1 ma, all distributed along the strip.

3. Uncoated Teledeltos paper used for z-plane.

4. Finite poles at $\frac{\omega_{m1}}{\omega_0} = 1.02957$, $\frac{\omega_{m2}}{\omega_0} = 1.090235$ & $\frac{\omega_{m3}}{\omega_0} = 1.493555$.

$\frac{\omega_{m2}}{\omega_0} = 1.090235$ $\frac{\omega_{m1}}{\omega_0} = 1.02957$ $\frac{\omega_{m3}}{\omega_0} = 1.4936$ $\frac{\omega_{m2}}{\omega_0} = 1.090235$



$m_2 = 0.39835$ $m_1 = 0.23795$ $m_3 = 0.74277$ $m_2 = 0.39835$

DENORMALIZED NETWORK
($f_0 = 8$ Kc)

All inductors in millihenries; all capacitors in nanofarads

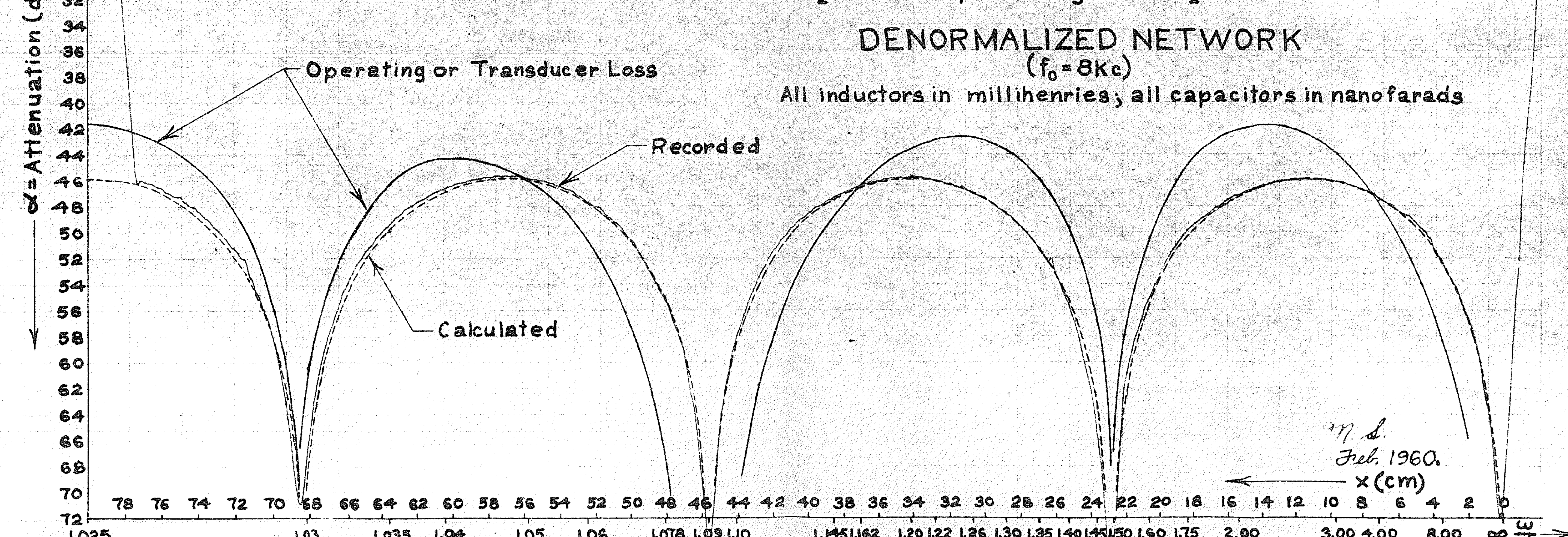


FIGURE 5.16 RECORDED α, CALCULATED α, NETWORK & OPERATING LOSS OF ANTIMETRIC LOW PASS FILTER

used in the matching sections.

In the above examples it has been assumed that the reader is sufficiently familiar with the design methods to obviate the necessity for more detailed explanations.

The calculated image attenuations were all found with the aid of a Bendix G-15 Digital Computer by using the pole positions found from the analog. The operating loss of each filter was found by using an IBM 1620 Digital Computer.

From the recordings it is evident that the ζ-plane and the elliptic planes are very suitable for the solution of the approximation problems for most low-pass filter specifications occurring in practice. The other two planes tried are obviously not suitable.

CHAPTER VI

SUMMARY, CONTRIBUTIONS AND FUTURE INVESTIGATIONS

Successful point-by-point measurements of phase were performed by using total current flow across the boundary of a short-circuited quarter plane as the analog to phase. Another method by which this could be accomplished using such quarter planes was considered. Successful point-by-point measurements of a magnitude function were also carried out on an open-circuited quarter p-plane and its logarithmic transformation. New equations for measurement, in a half plane, of residues of rational functions with simple and double poles were derived, and reasonably successful point-by-point measurements of such residues were carried out on Teledeltos sheets. All these point-by-point measurements, though successful to at least a reasonable degree, showed that the inaccuracy, tediousness and time involved in making them necessitated the construction of a semi-automatic potential analog apparatus in order to expedite the solution of approximation and other problems in network and control system design. The above point-by-point measurements also gave a good physical appreciation of the analog--a salutary step towards later applications.

Following suggestions due to Bridges, a new semi-automatic potential analog apparatus employing an X-Y recorder was developed.[1]

This new apparatus was designed around the original constant-current unit built by Bridges, and incorporated the most desirable features which, according to the studied reasons presented in chapter IV, such an apparatus should have in order to be most useful for network and control system analysis and synthesis especially network synthesis.[2] (This apparatus is also suitable for solution of other problems involving, basically, Laplace's equation in two dimensions). The main features of the new apparatus which contributed to its flexibility, accuracy and speedy, practical operation were:

1. The use of uncoated Teledeltos paper.
2. The voltage pick-off system, which used two wipers travelling at constant speed in continuous contact with the paper.
3. Facile use of fairly large conformal mappings.
4. Reasonably stable amplifiers for the voltages picked off the Teledeltos sheet.
5. The use of an X-Y recorder yielding large recordings.

The use of the voltage wipers, and the particular combination of equipment for this potential analog apparatus seems to be new.

The over-all accuracy of the apparatus in measuring magnitude (attenuation) was found to be high, often (Chapter V) within one per

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[1] E. Bridges, "A Network-Function Simulator" (Unpublished Masters Thesis, The University of Manitoba, Winnipeg, September, 1958), pp. 62-63.

[2] *Ibid*., pp. 19-27.

cent. Similar accuracy has been achieved, in certain cases, for phase measurement, but this may not be possible if singularities are close to the wipers. The use of still more stable (probably chopper-stabilized) dc amplifiers would improve and facilitate phase and delay measurements.

Once the potential analog apparatus had been developed and its practicability shown, a comprehensive study of literature on image parameter and insertion loss filter theory--including applications of the potential analogy and potential analog devices in these areas-- was under-taken, the necessary derivations of equations in terms of the potential analogy carried out, and some of the applications of the new potential analog apparatus as a design aid were pointed out.

Although time, circumstances and space prevented the use of the apparatus on examples of all these applications those that were done definitely proved its accuracy and practicability in at least those problems involving only attenuation (gain, magnitude) measurements. With these examples on hand it was easy enough to show that the apparatus could be used in a practical way on other--often related--problems. Extensive applications to measurement of phase and delay have not been performed and, although some excellent results of phase measurements have been made by graduate students, only one curve of phase (taken on an uncorrected half p-plane) is given in this thesis, and no curves of delay. It is suggested that future work might be done in applying the analog apparatus to problems suggested by this thesis which involve obtaining recordings of phase and/or delay; this would establish the extent of the apparatus' value in such problems.

In Chapter V the reasons for treating image parameter theory and design, especially according to the potential analog method, were first presented. Then the approximation problems of lossless image parameter filter theory were presented.

The potential analogies were then worked out for all the low-pass image parameter functions, with a view to applying the potential analog apparatus as an aid in determining and studying them. The low-pass image parameter functions included are all those normally found in the image parameter filter literature; for example, the material by Cauer, Bode and others.[3,4]

It was found that the potential analog apparatus was especially suitable for solution of the low-pass approximation problems occurring when Γ_1 had attenuation poles only along the real frequency axis and/or the real axis. The case of poles occurring only at real frequencies is the most common in practice--that is, when attenuation only is of prime concern. Several examples of low-pass symmetric and antimetric filters which were synthesized with the aid of the potential analog apparatus to meet specified stop band attenuation characteristics were presented. Actual tracings of transition and stop-band attenuation

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[3]W. Cauer, Theory of Linear Communication Networks, Vol. I (McGraw-Hill Book Company, Inc., Toronto, 1958).

[4]H. W. Bode, A General Theory of Electric Wave Filters, Bell Telephone System Monograph B - 843 (1934).

obtained by implementation of the potential analog apparatus, using various conformal mappings of the quarter p-plane, were presented, and compared with calculated curves. The agreement was often within 1% when a suitable conformal mapping was employed.

The simultaneous approximation of phase and attenuation in the low pass case is not as simply effected with the potential analog apparatus because the pole distribution cannot be calculated and cannot be located except by trial and error -- a virtually impossible situation except in simpler cases involving a fairly low-order network.

By using the same methods as for Γ_1 one could also study the image reflection function in the pass band.

By employing open-circuited sheets the potential analog apparatus may be used to study q, q' and image impedance (z_o) functions. No examples are given since the application is evident from the discussions.

Throughout this thesis (and the author's complete report) conformal transformations serve as a powerful theoretical and practical tool. They are necessary if desired accuracy is to be attained with the potential analog apparatus. It was found that the ordinary p plane and the mapping $w = \frac{-\omega_o}{p + j\omega_o}$ yielded impractical scales for filter problems. However, the mappings

$$\frac{p}{\omega_o} = \frac{\zeta^2 - 1}{2\zeta}$$

and

$$\frac{-jp}{\omega_o} = \mathrm{sn}\,(z, k)$$

were admirably suited for practical solution of the low pass problems, the asymmetrical band-pass problems after transformation to the special low-pass type problem, and the double pass-band problem. For multiple pass-band problems the hyper-elliptic mappings should be well suited. The workers in this field should investigate other mappings, of which many are known and catalogued.

Because of the symmetries involved, one may use quarter sheets in studying all Γ_1 functions, all q functions and all z_0 functions, but for studying q' functions one must use half planes.

After application of bilinear transformations and/or reactance transformations, high-pass, band-stop and symmetrical and other band-pass problems are transformed to equivalent low-pass problems, and so are treated as in the low-pass case above.[5] In order to obtain minimum inductance band-pass frequency-symmetrical filters the transformations due to Laurent may be used on antimetric low-pass filters.[6] This transformation has been given in a more recent article by Saal and Ulbrich.[7]

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[5]Cauer, <u>op. cit.</u>, pp. 296-332.

[6]T. Laurent, "Le Filtre Zig-Zag", <u>Ericsson Tech.</u>, Vol. 4, January, 1936, pp. 3-28.

[7]R. Saal and E. Ulbrich, "On the Design of Filters by Synthesis", <u>IRE Transactions on Circuit Theory</u>, Vol. CT-5, No. 4, December 1958, pp. 300-301.

The idea that the potential analog apparatus can be used as a practical tool, of sufficient accuracy for many filter problems, and that this apparatus might serve to replace the template methods of Rumpelt in both low-pass and asymmetrical band-pass cases seems not to have been generally accepted previously, or even greatly considered.[8] This chapter shows, with illustrative examples of low-pass symmetric and antimetric filters, that such a replacement is feasible and would probably result in greater speed of application than Rumpelt's template methods. Furthermore, the potential analog apparatus may be used to find the contributions due to real-axis and complex poles.

Recently, there has been considerable activity in techniques using digital computers to arrive at optimum pole positions.[9,10] These methods have been limited to low-pass and band-pass filters and to poles lying along the real frequency axis. Furthermore, one cannot easily build "judgment" into these digital computer programs; that is, one cannot change pole positions to allow for greater tolerances at desired places.[11] One can, on the other hand, easily use the analog apparatus to obtain desired results in such cases. Finally, the analog apparatus can be

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[8] E. Rumpelt, "Schablonenverfahren fur den Entwurf elektrischer Wellenfilter auf der Grundlage der Wellenparameter," _Telegraphen-Fernsprech- Funk- und Fernseh-Technik_, 31 Jahrgang, Heft 8, August, 1942, pp. 203-210.

[9] T. Fujisawa, "Optimization of Low-Pass Attenuation Characteristics by a Digital Computer," Paper presented at the Midwest Symposium on Circuit Theory, May, 1963.

used more cheaply for the determination of pole positions than can a digital computer, besides being immediately applicable to arbitrary distributions of these poles--and for the many other purposes not discussed in this thesis.

Finally, it is usual in theses to deal with possible future investigations. In order to point out where future investigations might lie, it is necessary to refer the reader to the author's complete work.[12] Limitations on size and scope of this thesis have not permitted the inclusion of that material here.

In his report the author has obtained all the functions for image-parameter asymmetrical band-pass filters, using the potential analogy, and shown that the mappings used successfully in this thesis, plus another, can be employed with the potential analog apparatus

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[10] B. R. Smith and G. C. Temes, "An Iterative Procedure for the Automatic Synthesis of General Parameter Lowpass and Bandpass Filters," Northern Electric Research and Development Laboratories Technical Memorandum TM8134-63-2, July 2, 1963.

[11] Private discussion with J. D. Macdonald, Northern Electric Research and Development Laboratories, Aug. 6, 1963.

[12] M. Sablatash, "The Development of a Semi-Automatic Potential Analog Apparatus, an Integrated Approximation Theory for Filters Derived by Means of the Potential Analogy, and the Application of the Apparatus as an Aid in its Practical Implementation," August, 1963. A concurrent report to the National Research Council of Canada, under NRC Grant No. A738. Copies of the report may also be obtained from the author who is presently employed by Northern Electric Co. Research and Development Laboratories, Ottawa, Canada, and from the library of his employer. Although other publications are forthcoming, this report will give the reader the most thorough treatment.

(after a previous transformation) to solve the image-parameter approximation problems. Linear-phase image-parameter filter approximation problems are treated and possible practical uses of the potential analog apparatus are presented. The theory is then generalized to multi-band filters, and a general image propagation function is given. This function is a sum of hyperelliptic integrals and, in potential analogy terms, is a complex potential. The residues of the integrand of this integral are just the positive charge multiplicities. A similar integral form is given to describe the complex potential when there are no point charges, but only conducting plates with positive or negative charge along the imaginary axis.

The sixth chapter of the author's report first presents a historical development of the insertion loss theory for filters, with relationships between the works. A general insertion loss theory for filters with Chebyshev pass-band behaviour is then presented, using the potential analogy. The theory parallels, to a great extent, that derived by Watanabe in a strictly mathematical way.[13] However, the potential analogy gives easy visualization of the mathematics of approximation. The ideal transmission function without poles is just the complex potential function for the case given in the previous paragraph. That for the ideal transmission function with poles is the

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[13] Hitoshi Watanabe, "Approximation Theory for Filter Networks," IRE Transactions on Circuit Theory, Vol. CT-9, No. 3, September, 1961, pp. 341-356.

complex potential problem for the corresponding image parameter problem, except that positive <u>and</u> negative charges are permitted in the insertion loss theory. Watanabe's integrals around branch cuts correspond to encirclement of charge and cutting of flux. The potential analog apparatus may be used to approximate the stopband operating (transducer) loss through use of the formula

$$A(p) \cong 8.686|\alpha| - 6.02 - A_{RL} \text{ (in db)},$$

where $|\alpha|$ is the real part of the ideal transmission function. This is set up on the analog apparatus in exactly the same way (except that now negative charges are also permitted) as shown in this thesis.

In the above way <u>all</u> possible insertion-loss functions with Chebyshev pass band behaviour are included. The author's report shows how all the previous known such functions are special cases and relates them to the image parameter theory usually used for the derivation. Limiting cases of the theory are given by compressing plates to point charges; this yields all previous maximally flat cases. Most other insertion-loss filters, including linear-phase filters are treated and related to the general theory in order to obtain an integrated result.

The approximation theory is also described for RC filters, one-port functions, constant-phase-difference networks, distributed networks and (very briefly) amplifiers and automatic control systems. This theory is related and integrated with the whole.

In all the above the possible applications of the potential analog apparatus are constantly noted and described.

An exhaustive study of the literature has been done on all topics in the report, and related to the material given there.

Now that the reader has been shown how to get knowledge at frontiers of the field, suggestions for future investigation (given also in the author's report) can be presented. These are:

1. Conformal mappings for the various problems, other than those used in this thesis might be found and tried.

2. One might present more thorough physical pictures of the Riemann surfaces defined by the complex potential functions used in the report in the thorough manner, perhaps, that is presented by Tuttle in his treatment of one-port approximation problems.[14] There really is hardly room for such explanations even in the author's complete work.

3. To be very thorough one should experimentally apply the potential analog apparatus to all the filter types considered in the report. All sorts of selectivities should be tried, and the limits of accuracy for the apparatus more exactly established. Thus, one should find out whether all filters that can be built with present-day components can also be designed with the aid of the analog apparatus. In particular,

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[14] D. F. Tuttle, Jr., Network Synthesis Vol. I (New York: John Wiley & Sons, Inc., 1958), pp. 832 - 1050.

the designs for Chebyshev pass band double pass band filters should be tried, especially those for which the analog apparatus must be used to experimentally solve the transcendental equation containing an elliptic integral of the third kind. The use of double pass band filters should result in a considerable saving in components from that necessary when paralleling band-pass filters to obtain band separation or mixing.

4. One should try to apply the general insertion loss theory of this chapter to the design of crystal filters. The special forms of characteristic functions of Nth degree which are suitable for realization of filters with $\frac{N-2}{2}$ coils and those of odd degree, as well as others for which both positive and negative charges are employed in the complex potential problem analogous to the ideal transmission function, might prove to be useful for crystal filter design.

5. As mentioned in Watanabe's paper, it may also be possible to design comb-filters, and filters with negative as well as positive resistances, besides L and C.[15]

6. The use of hyper-elliptic mappings for filters with a greater number of bands should be investigated. This involves finding suitable ways of calculating such mappings, and

[15] Watanabe, op. cit., p. 348.

their experimental use in finding multi-band filters. Thus, experimental quantization would have to be used to find the zeros of the characteristic functions. One should be able to use a bilinear transformation to effectively reduce the number of bands that have to be considered, and to obtain formulae for design from the general theory of this chapter.

7. An antilogarithmic amplifier might be added to the potential analog system in order to obtain direct plots of magnitudes of characteristic functions and voltage ratios, although this is merely a luxury, not a necessity.

8. The properties of the Abelian integrals should be thoroughly investigated, and calculating procedures found for those of higher order.

9. Explicit design formulae for the numerous filter types evolving from the general theory should be worked out in detail.

10. There seems to be a conflict between Matthei's contention that positive and negative point charges give rise to non-Chebyshev pass band behaviour, and the statement by Watanabe to the contrary, as is mentioned in section II of Chapter VI. of the author's report.[16,17] This conflict should be resolved.

11. An attempt should be made to solve the problem of designing filters with given pass band (especially flat) delay and Chebyshev pass and stop band attenuation characteristics. The potential analogy ideas of the author's complete work should be useful in this connection.

12. One might attempt to treat narrow-band approximation problems with the aid of Baum's theory. For this purpose a prototype "low pass" problem involving complex coefficients is to be considered

13. Application of the analog apparatus, according to principles developed by the author, and their extensions, to the design of practical networks, amplifiers and control systems of all types: those thoroughly considered in his work and those only briefly discussed.

14. Investigation of the economic feasibility of using the potential analog apparatus to obtain initial approximations to be improved upon by iterative methods employing digital computers.

Applications and numerous other topics for further investigation are presented in the author's report. The above seem to be the most important.

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[16] George L. Matthei, _A General Method for Synthesis of Filter Transfer Functions as Applied to L-C and R-C Filter Examples,_ Technical Report No. 39 (Stanford Electronics Research Laboratory, Aug. 31, 1951, pp. 61-67

[17] Sablatash, _loc. cit._

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APPENDICES

APPENDIX A

EXPERIMENTAL DETERMINATION OF RESIDUES AT FIRST-ORDER POLES AND OF COEFFICIENTS OF TERMS OF A PARTIAL FRACTION EXPANSION DUE TO DOUBLE POLES

Preparation and Calibration of the Potential Analog Sheet

For the corrected semi-elliptical conducting sheet the ratio of real to imaginary axis scale was $\sqrt{\frac{\sigma_{XX}}{\sigma_{YY}}} = \sqrt{1.12}$ (where σ_{XX} was the conductivity in the direction of rolling, which was made the X-axis, and σ_{YY} was the conductivity perpendicular to this direction)[1]. The dimensions of the semi-elliptical analog sheet, measured from the inside edges of the silver paint around the periphery, were 177 cm. along the major axis and 83.6 cm along the minor semi-axis. The corrected finite pole and zero positions of the calibrating and test functions were marked on the sheet with non-conducting pencil.[2]

As shown in chapter III, section IV, it was sufficient to calibrate for magnitude only.[3] To this end, a pole was placed at s = 0 and a zero on the infinity equipotential, each with half-unit current

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[1]E. Bridges, "A Network-Function Simulator" (unpublished Master's Thesis, The University of Manitoba, Winnipeg, September 1958) pp. 41 and 65-69. An example is given on pp. 65-67 of typical measurements showing the non-isotropic properties of the Teledeltos paper. On pp. 67-69, the derivation is presented of the correction to be applied to the semi-circular analog sheet.

[2]Bridges, ibid., p. 41.

[3]p. 34.

strengths of two ma. The plot of voltage vs. ω shown in Figure A.1 was obtained, and K_1 determined from the slope of the straight line.[4] This gave $K_1 = 1.174$.

Measurement of Magnitude and Phase of the Residue at a Single Complex Pole

The test function given by equation (2.1) was set up on the sheet with its finite poles and zeros at the corrected positions. A unit current of 4 ma. was used. The finite pole was moved to the infinity equipotential and the potential at its original position, with respect to the potential at s = 0, was found to be -0.175 volts. Since $K_1 = 1.174$, $\alpha = -10$, $\beta = 20$ and $Z(o) = 68$, equation (3.7) yielded

$$k_2 \quad = 850 \ln \frac{-1V(s_2)}{K_1} = 732.$$

To measure the angle, θ_2, of k_2 on the analog, the circuit in Figure A.2 was used to measure voltage gradient at right angles to the perpendicular drawn from s_2 to the real axis. From the readings,

$$\sum_i \frac{\Delta V_i}{\Delta \alpha_i} \Delta\omega_i = \sum_i V_i = 1.3324,$$

since $\Delta\omega_i = \Delta\alpha_i = 1$ cm, so that

$$\operatorname{Arg} Z_2(s_2) = \frac{57.25}{K_1} \sum_i V_i = 65^\circ.$$[5]

Equation (3.11) then yielded

$$\operatorname{Arg} k_2 = 65^\circ - 90^\circ = -25^\circ.$$

[4]The scale on the analog sheet along the real frequency axis was 1 cm. = 1 radian.

[5]Bridges, op. cit., pp. 6 and 98.

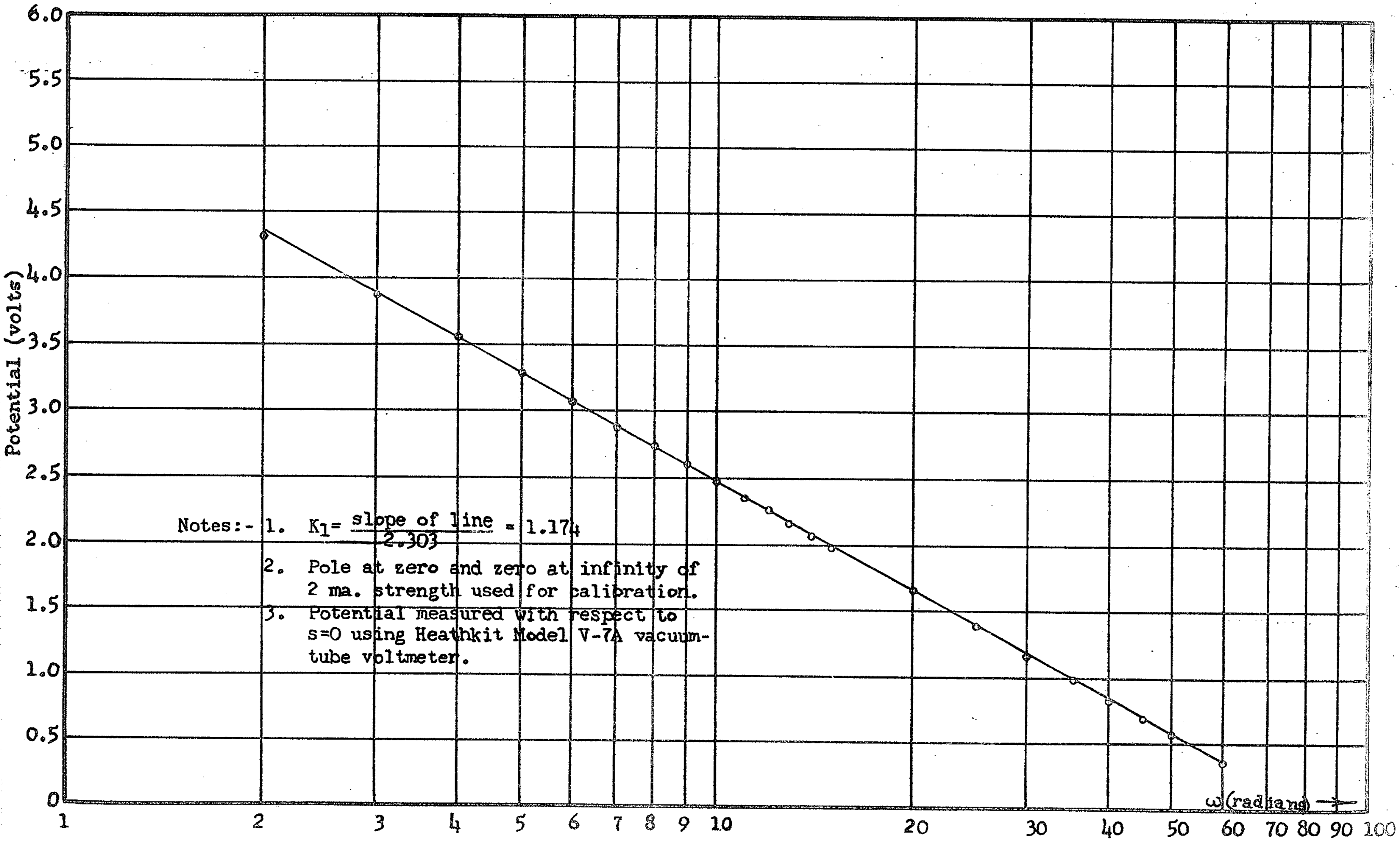
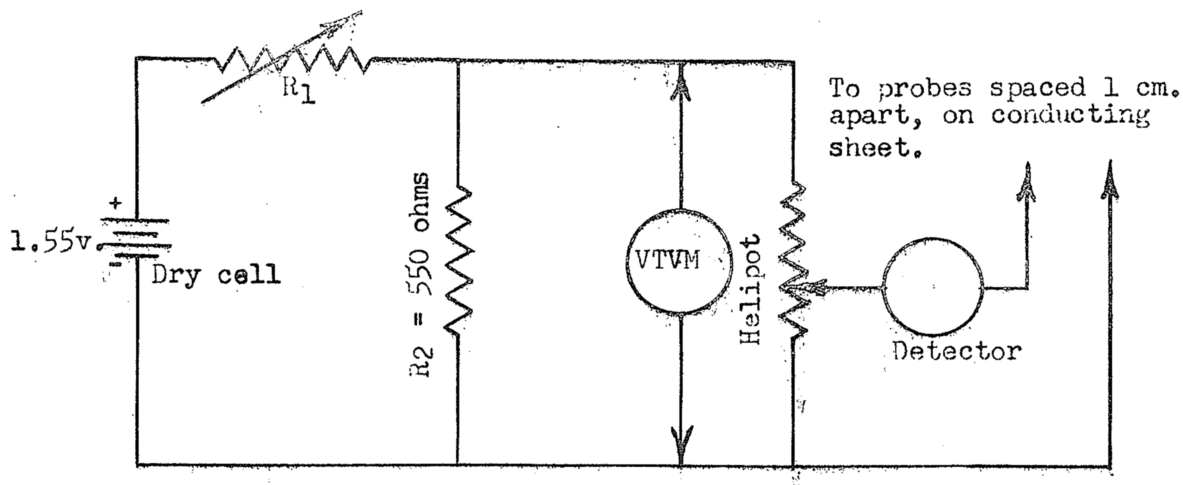


FIGURE A.1

CALIBRATION CURVE FOR DETERMINING COEFFICIENT K FOR CORRECTED HALF COMPLEX PLANE USED FOR MEASUREMENT OF RESIDUES AND OTHER COEFFICIENTS IN PARTIAL-FRACTION EXPANSIONS OF RATIONAL FUNCTIONS OF A COMPLEX VARIABLE



Note 1: VTVM was General Radio Type No. 1800-A, on 0.5v range.

Note 2: Detector was either General Electric Type Do-71 microammeter, or Honeywell Detector.

FIGURE A.2

APPARATUS FOR METERING VOLTAGE GRADIENT[6]

Measurement of Coefficients of Terms of a Partial Fraction Expansion Due to Double Poles

The rational function

$$Z'(s) = \frac{(s+20)(s+10-j40)(s+10+j40)(s+30-j30)(s+30+j30)}{(s+10-j20)^2\ (s+10+j20)^2} \qquad (A.1)$$

was set up on the sheet with the poles and zeros at the corrected positions marked on it. A unit current of 4 ma. was used for the finite complex zeros, 8 ma. for the finite, double complex pole and 2 ma. each for the real zero and pole at infinity. The problem was to find k_{22} and

[6]Bridges, op. cit., Fig. 4.3, p. 50.

k_{21} as given by the partial fraction expansion (3.27).

First, measurements were made to determine k_{22}. To this end, the pole at s_2 = -10+j20 (corrected position -10.6+j20) was removed and placed on the infinity arc; this resulted in the analog representation of $Z_4(s)$, according to equation (3.13). The potential, with respect to the origin, at the original finite position of this pole was then measured using a vacuum-tube voltmeter. This potential was V_4 = -0.583 volts. Equation (3.15), with $\alpha = -10$, $\beta = 20$, $Z(o) = 245$ and $K_1 = 1.174$ then yielded

$$|k_{22}| = 23,300.$$

Another measurement gave V_4 = -0.576 volts, so that

$$|k_{22}| = 23,430.$$

To find the phase of k_{22}, the angle of $Z_4(-10+j20)$ was found in exactly the same manner as described for finding the phase of the residue at a single complex pole. Five sets of readings were taken because of some difficulties in measurement due to the poor contact caused by the thin non-conducting coating of the Teledeltos paper (which probably caused the detector variations when pressure was applied to the voltage-measuring probes), and possible current source fluctuations (probably due to the use of one current source at its maximum rating of 8 ma.). Plots of $Z_4(-10+j\omega)$ obtained from each of the five sets of measurements are given in Figure A.3, in which the conditions under which data for each curve was obtained are also shown. The mean angle $\phi = \text{Arg } Z_4(-10+j\omega)$ is plotted versus ω in Figure A.4. From this Figure the ordinate at $\omega = 20$ was read and, with the aid of equation

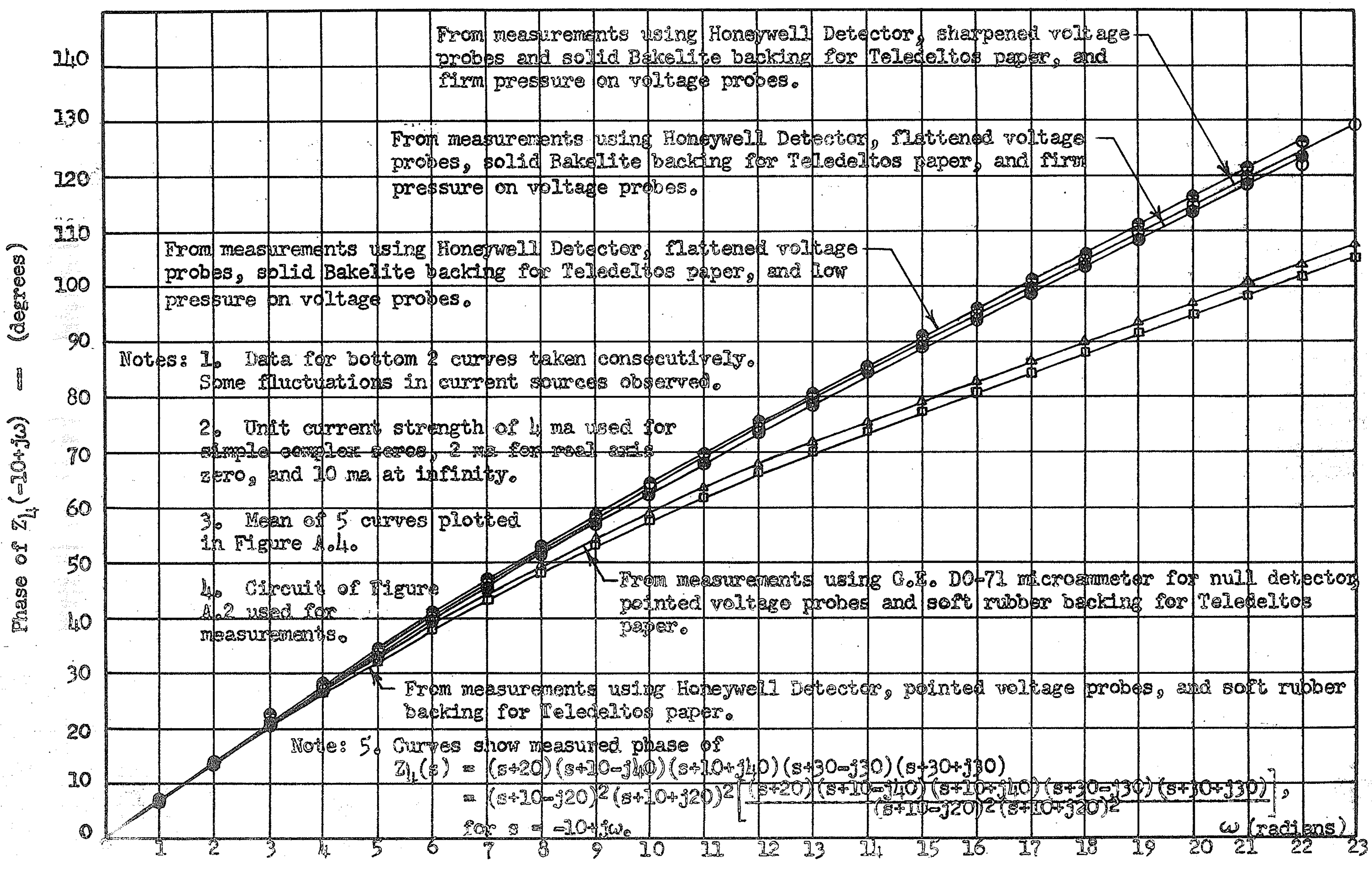


FIGURE A.3

PHASE OF $Z_4(-10+j\omega)$ MEASURED ON ANALOG UNDER VARIOUS PHYSICAL CONDITIONS

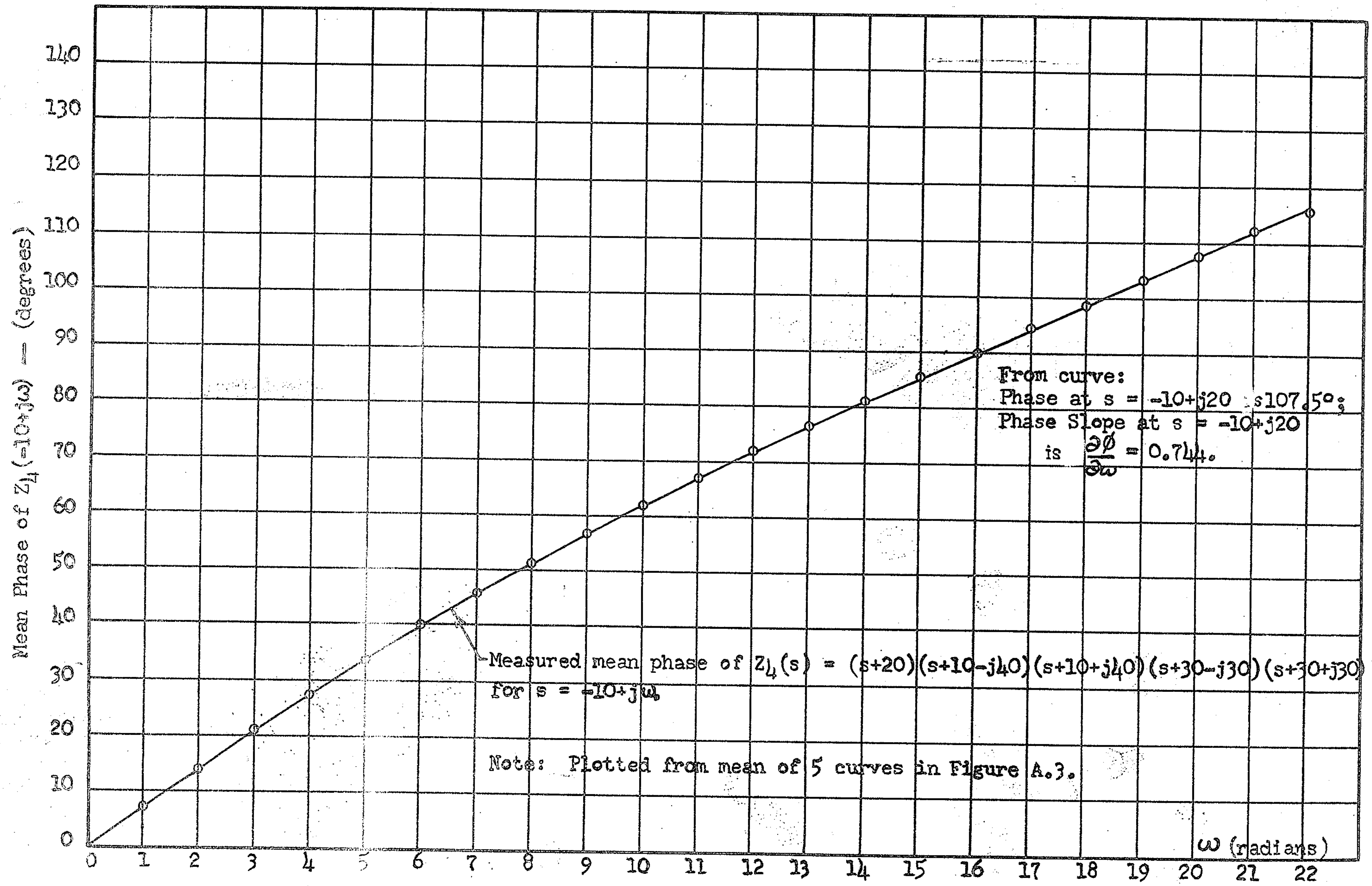


FIGURE A.4

MEAN PHASE OF $Z_4(-10+j\omega)$ MEASURED ON ANALOG AND USED IN DETERMINING THE COEFFICIENTS OF THE PARTIAL FRACTION EXPANSION OF A RATIONAL FUNCTION OF A COMPLEX VARIABLE WITH DOUBLE COMPLEX POLES

(3.19),

$$\text{Arg } k_2 = \text{Arg } Z_4(-10+j20)-\pi = -72.5^\circ$$

was found. Thus,

$$k_{22} = 23365 \angle -72.5^\circ,$$

where the mean of the two measured magnitudes was used.

In order to determine k_{21} from equation (3.23), the quantities $\frac{\partial \phi}{\partial \omega}$, R and $\frac{\partial R}{\partial \omega}$ had to be obtained. The first, $\frac{\partial \phi}{\partial \omega}$, was easily obtained as the slope of the curve in Figure A.4 at $\omega = 20$. Hence,

$$\frac{\partial \phi}{\partial \omega} = 0.0744 \text{ radians/radian.}$$

To obtain the voltage gradient $\frac{\partial R}{\partial \omega}$, readings of potential with respect to the origin were taken along the line $s = -10+j\omega$, and values of $|Z_4(-10+j\omega)|$ were calculated at each point of measurement, using

$$V_4 = K_1 \ln \left| \frac{Z_4(-10+j\omega)}{Z(o)} \right| ,$$

with $K_1 = 1.174$. The circuit of Figure A.2 was used for making the measurements. Difficulty was again encountered in obtaining the measurements. As discussed in the second last section of this chapter, the probable cause of the trouble was the use of coated Teledeltos paper (in later work the uncoated variety yielded much better results) and the use of a current source (pole) operating at too high a current level.

Three sets of data were used for the curves of $|Z_4(-10+j\omega)|$ vs. ω shown in Figure A.5, in which are also shown the conditions of measurement for the data from which each curve was obtained. The lowest curve in this Figure was not used for finding $\frac{\partial R}{\partial \omega}$ because of the lack of

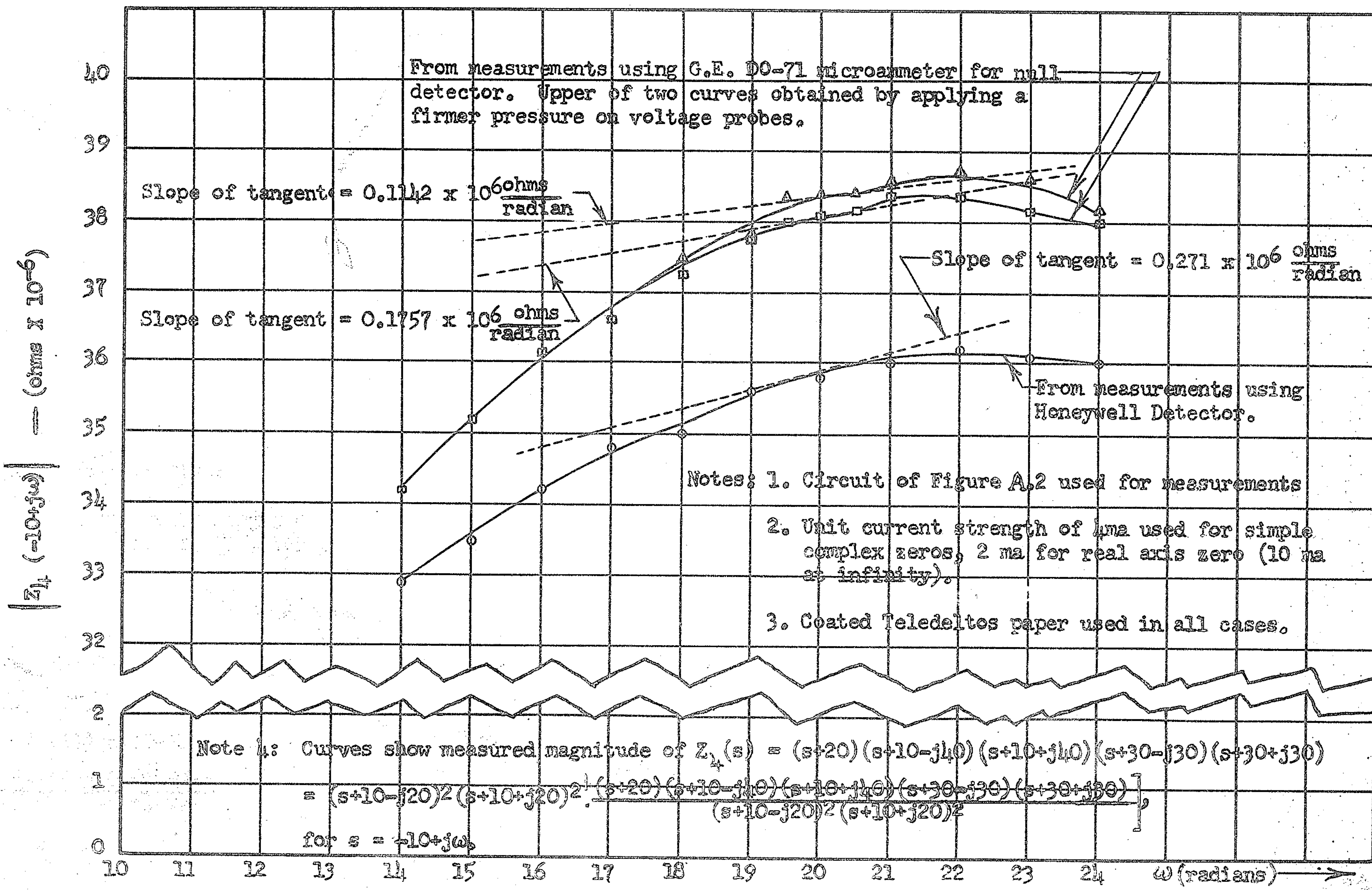


FIGURE A.5

MAGNITUDE OF Z_4 $(-10+j\omega)$ MEASURED ON ANALOG sufficient data in the vicinity of $\omega = 20$, the uncertainty of the data and the high value of $\frac{\partial R}{\partial \omega}$ compared to that found from the other two curves. The measured value of $\frac{\partial R}{\partial \omega}$ at $\omega = 20$, taken as the mean of the values from the two top curves of Figure A.5, was then,

$$\frac{\partial R}{\partial \omega} = 0.145 \text{ X } 10^6 \text{ ohms/radian.}$$

With a small weight given to the lowest curve in Figure A.5, the measured value of

$$R = Z_4(-10+j20) = 37.5 \text{ X } 10^6 \text{ ohms.}$$

Equation (4.23), with

$$R = Z_4(-10+j20) = 37.5 \text{ X } 10^6 \text{ ohms,}$$

$$\phi = 1.877 \text{ radians,}$$

$$\frac{\partial \phi}{\partial \omega} = 0.0744 \text{ radians per radian,}$$

$$\frac{\partial R}{\partial \omega} = 0.145 \text{ X } 10^6 \text{ ohms/radian,}$$

$$\beta = 20,$$

and

$$Z_3(-10+j20) = k_{22} = 23365\underline{/-72.5^\circ} = 7020-j22280,$$

yields

$$k_{21} = 2050\underline{/-40.75^\circ},$$

from measurements on the potential analog.

APPENDIX B-1

THE MODIFICATIONS TO THE EARLIER POLE-ZERO MACHINE[1]

A comparison of the original pole-zero machine with a machine meeting the requirements described in section II, chapter IV, brought forth the following limitations to, difficulties encountered with, and items for improvement of, this original machine:

1. The oscilloscope on the early pole-zero machine was too small to provide practical, accurate recordings, and its controls interacted rather badly. Hence, it was decided -- after some experimenting -- to use some other means for recording. (A brush oscillograph gave traces with a maximum ordinate of one and one-half inches). An X-Y recorder proved ideal for the final version of the system.[2]

2. The integrator was not functioning, so that recordings of integrated current flow across lines on the sheet could not be obtained.[3] Thus, for example, the phase of rational functions of a complex variable could not be obtained by automatic recording, but only by slow and tedious measurements with a voltmeter. Hence, an integrator with a long time constant had to be built.

3. The rotary motor-driven break-before-make switch used to pick

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[1]E. Bridges, "A Network-Function Simulator" (unpublished Master's Thesis, The University of Manitoba, Winnipeg, Sept. 1958), pp. 2-40.

[2]Model 2S Moseley Autograph.

[3]Bridges, op. cit., p. 32.

up the voltages on the fixed voltage probes resting on the Teledeltos paper introduced a great deal of "chatter" as the rotating wiper passed from one switch contact to the next.[4] So much "chatter" was introduced by the switch that phase or delay measurements could not be made at all, and magnitude measurements could be made only on the inadequate brush oscillograms by using the envelope of the recorded waveform. The waveform showed every make and break of switch contact by jumping to the envelope of the curve, and then back to zero again. All attempts at filtering for obtaining smooth curves by using this rotary switch were unsuccessful in producing an undistorted, smooth curve on the brush oscillogram. The X-Y recorder, furthermore, could not even follow the rapid voltage variations introduced by the rotary switch. These difficulties with the rotary switch resulted in consideration being given to replacing this part of the apparatus with either a commutator-type mechanism or moving probes, continuously in contact with the Teldeltos paper, for picking off the voltages.[5,6,7,8,9]

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[4]Bridges, op. cit., p. 32.

[5]E. Bridges, who suggested that moving voltage probes such as described in Germain's article (footnote 9) might be used.

[6]R. E. Scott, An Analog Device for Solving the Approximation Problem of Network Synthesis, Massachusetts Institute of Technology, Research Laboratory of Electronics, Technical Report 137, June, 1950, pp. 19-20. A commutator was used to pick off voltages on consecutive, fixed, closely-spaced voltage probes.

[7]Stanley Lehr, Solution of the Approximation Problem of Network Synthesis With an Analog Computer, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-327-53, PIB-263, June 18, 1953, pp. 6-7. A rolling voltage probe attached to a moving carriage was used in the potential analog machine described here.

4. The chassis with fixed probes on it, the rotary switch and all the associated wiring were cumbersome.[10]

5. The fixed-probe system did not permit all possible information from the sheet to be obtained, because of the finite spacing of the probes in each row.

6. There was a possibility that an X-Y recorder would work with the moving-probe system. This would enable large, easily-calibrated recordings with the most convenient scales to be made semi-automatically.

The above considerations, and the many frustrating experiments performed with the earlier apparatus as work on improving it proceeded, finally resulted in the following modifications being made to the earlier machine:

1. The oscilloscope unit was removed. An X-Y recorder was finally employed for the recordings.[11]

2. The probe chassis, transposition frame, rotary switch and its motor, manual selector switch and all associated connecting

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[8]Paul M. Liebman, Simultaneous Gain-Phase Approximation with a Potential Analog Computer, Polytechnic Institute of Brooklyn, Microwave Research Institute, Research Report R-617-57, PIB-545, August 22, 1957, p. 23. A double-roller voltage probe attached to a moving carriage was used in the potential analog machine. This was a modification of the one described by Lehr (previous footnote).

[9]P. Germain, "Electric Analog Methods for the Study of Problems related to the Laplace Equation", (in French), La Revue H.F., 2,9, 1954. A moving probe was used.

[10]Bridges, op. cit., pp. 14-19.

[11]Model 2S Moseley Autograph.

wires were removed, and replaced by apparatus employing two closely-spaced probes, driven at constant speed, and in continuous contact with the paper, to pick the voltages off the Teledeltos paper.

3. The dc amplifier unit and its regulated power supply were removed, and replaced by a dc amplifier unit consisting of three Heath Co. Model ES-201 dc amplifiers and their associated Heath Co. power supply. Circuits were built, suitable multiple-pole wafer switches used, and the amplifiers associated with these passive components in such a manner as to give an adder, inverter and integrator which could be used to obtain potential between two probes or the time integral of the latter, with suitable amplification.

4. The moving voltage probes (wipers) were connected to the new amplifier unit by two shielded cables which were rather loosely suspended from a point on the wall above the moving carriage. The output of the amplifier unit was fed to the Y-input of the X-Y recorder, via an R-C twin-tee filter rejecting line frequency.

APPENDIX B-2

THE MOVING PROBES, THEIR MOUNTING, THE CARRIAGE AND THE CARRIAGE PULLEY SYSTEM

The moving-probe system was one of the important improvements over the existing potential analog machine at the University of Manitoba, and one of the main reasons for the success of the new potential analog machine. This idea of using potentiometer wipers on a carriage as voltage probes eliminated the need for the cumbersome and hard-to-construct probe chassis, for switches or commutators, for the many wires leading from the probes to switch sections or commutator bars, and for filters to exclude switch or commutator "hash". It also increased accuracy because it resulted in maximum information transmission from the sheet to the amplifiers, and it also made possible large, smooth recordings on an X-Y recorder fed from the outputs of the amplifiers.

In Figure 4.1, page 52, one may see the carriage in position on its steel guiding bars.[12] These steel bars are 3/4" in diameter and 4' 1/4" long, their centers 2 1/4" apart, and 1 11/16" above the table surface. The ends of the steel bars are inserted in holes drilled in end blocks which are 3/4" thick, 2 7/8" high and 5" long (as they rest on the table), and the steel bars are kept in position by 1/4" set screws tapped through vertical holes in the end blocks. In Figure 4.1 one can see that two plates (flat sides facing the viewer) are fastened to the ends of the steel end

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[12]The steel bars are greased with vaseline to keep them smooth and shiny, and to aid easy movement of the carriage.

blocks, so as to be at right angles to them. These plates are of steel and are 1/4" thick, 3 3/8" high and 2 1/2" wide. They are fastened to the end blocks by means of countersunk screws. To the plates are fastened mountings which hold the shaft for the driving sheave at one end of the steel bars and for the idler sheave at the other end. The mountings are of white metal, but there are bearing surfaces for the shafts.[13] A small (about 1" diameter) aluminum sheave with concavely rounded edge was used for the idler.

To the driving sheave, which is 3 1/2" in diameter, is bolted securely a concentric (on the same shaft as the driving sheave) aluminum gear tooth with 159 teeth.[14] This gear tooth is 2 1/2" in diameter. It is driven by a gear made out of aluminum which has 24 teeth and is 3/8" in diameter. The latter gear (together with an unused larger one -- that can be seen in Figure 4.1 -- with which it was integrally cast) was fastened to the electric motor shaft. This gear system resulted in a speed reduction of 6.625, and a carriage speed of about 3 ft. in 20 seconds.

The small electric motor, which can be seen in Figure 4.1 at the left end of the steel bars, was manufactured by the Emerson Electric Co., St. Louis, Missouri, and had the following data on its nameplate:

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[13]More sturdy and durable mountings probably should be used.

[14]A harder metal should have been used for the gears. It would be best to have a new gear assembly manufactured from steel.

Frame F 32 GF	HP 2 oz. ft. Cy. 60
Style 301	RPM 1700/69
Volts 115	Amps. 0.4

This motor is bolted to a plywood platform (part of which is visible in Figure 4.1, protruding past the right of the steel side plate at the left end of the steel bars) which has dimensions of 3/4" X 4 1/4" X 8 7/8". The dimension of 8 7/8" is perpendicular to the steel bars. The plywood platform is fastened to the steel bars by means of four U-bolts 3/16" in diameter (two of these may be seen in Figure 4.1 protruding through the top of the plywood). A switch for turning the electric motor on and off is mounted on the end of the plywood board. This may be seen in Figure 4.1, to the right and below the motor. The switch is turned off by a spring steel arm mounted on the carriage, when the latter gets near the end of its travel.

Over the V-notched driving sheave runs a cloth-covered, stranded copper cable, as can be seen in Figure 4.1. It is kept taught by a fairly stiff spring in its upper loop. In its lower loop it is attached to the carriage. Provision for tightening the cable was provided by looping it back onto itself after it was passed around the hook on the carriage. A Marr connector was passed over this "doubled-up" cable, leaving a small loop to attach to the hook on the carriage. The Marr connector has a small screw in its (cylindrical) wall. When the cable must be tightened this small screw may be loosened, more of the cable end pulled through the Marr connector and the screw tightened again.

The underside of the carriage, and the details of the wipers (voltage probes) are shown in Figure B-2.1. The four V-grooved brass blocks were machined from 1" cubes of brass. They are tapped and bolted to a piece of Permali 5/16" X 3 1/4" X 4", where the 4-inch dimension is parallel to the steel bars. The projecting plywood centrepiece, which holds the mountings for the two probes, is fastened to the Permali by means of two right-angled pieces of aluminum.

Each wiper and its holder, as well as the short shaft through the two wiper holders, were taken from a type RV2 potentiometer manufactured by Technology Instrument Corp., Acton, Massachusetts. The wiper holders are mounted -- insulated from each other -- rigidly on the shaft, and this assembly is prevented from rotating by means of a nail driven through the plywood upright just above the shaft, and bent around the top of one of the holders, as can be seen in Figure B-2.1. The two wipers themselves, as can be seen in Figure B-2.1, are rounded, and are soldered near their tips to keep them from splitting. They are welded to flat, springy copper strips. A line between the points of contact of the wipers when they are on the Teledeltos paper is at right angles to the direction of travel, parallel to the steel bars, and the distance between them is about 7 millimeters. The carriage may be moved in both directions along the steel bars without catching the wipers on the Teledeltos paper, because of the curvature at their ends, when they have the proper pressure (not critical) against the paper.

At one leading corner of the carriage (as shown at the left of Figure B-2.1) a right-angled piece of aluminum is bolted to the Permali. A thin piece of blue spring steel is bolted to this piece of aluminum.

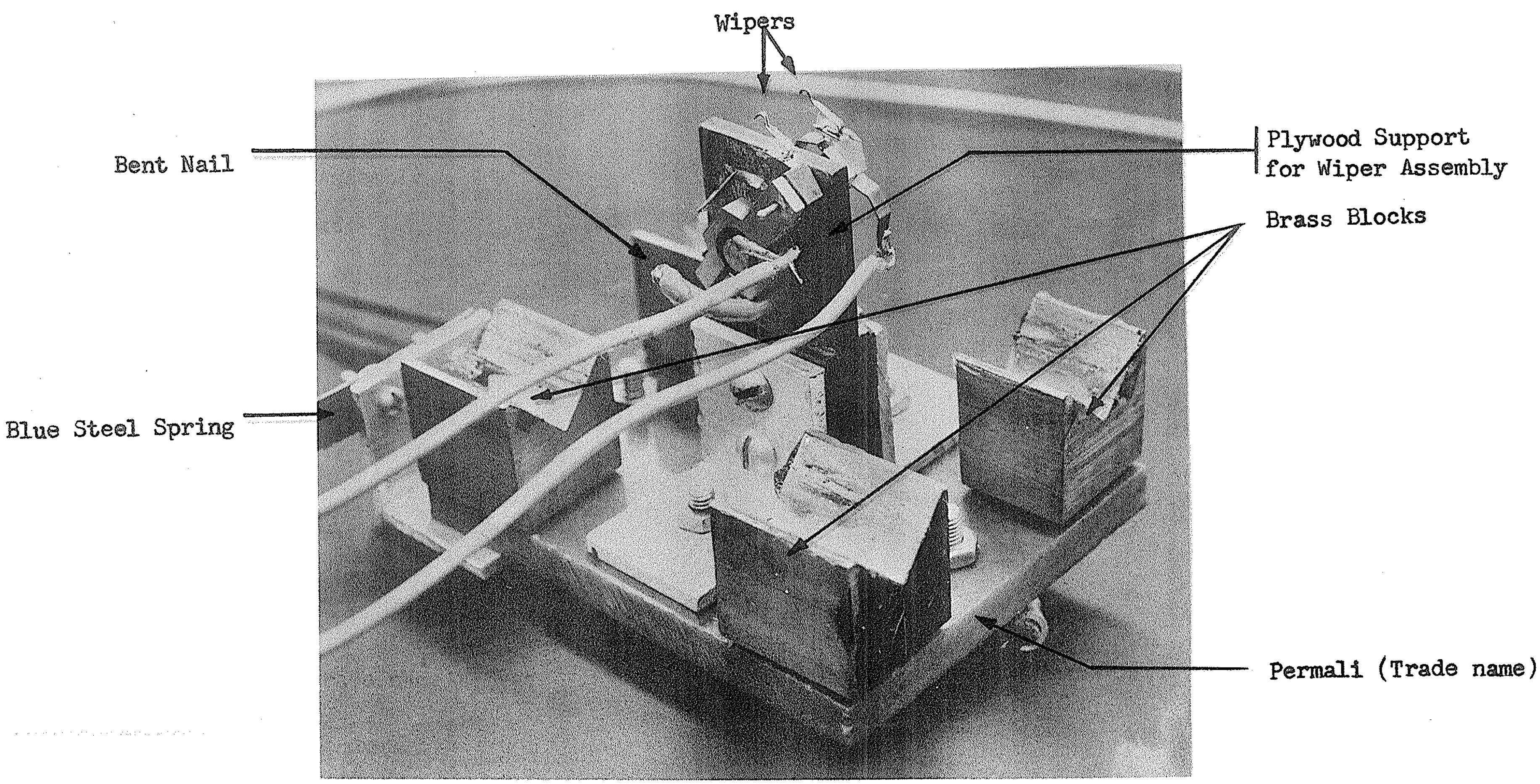

FIGURE B-2.1

DETAILS OF THE UNDER-SIDE OF THE CARRIAGE

AND WIPER ASSEMBLY

This serves to turn off the switch for the electric motor without causing undue sudden stress on the gears and other parts of the carriage and pulley system.

With a little time and money the moving-carriage system which has been described above may be improved. Some way may be found to space the wipers more closely. The carriage may be streamlined. A reversing motor might be used for returning the carriage. A much more finely machined system could be designed.

APPENDIX B-3

DETAILS OF THE DC AMPLIFIER UNIT

The block circuit diagram of the amplifier unit is given by Figure B-3.1, the detailed circuit diagram (in which the Heath Co. amplifier units, however, are indicated by appropriate blocks) of the dc amplifier unit and reference biasing circuit by Figure B-3.2, and photographs of the front, back and bottom (with details of wiring exposed) of the unit are shown in Figures B-3.3(a)-(c), respectively. A functional physical and electrical description of the dc amplifier unit is presented next.

Each signal from the two wipers is carried by a shielded audio cable to a double banana plug under the word "INPUTS" shown in Figure B-3.3(a).[15] The upper of the two cables is referred to as the A input

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[15]The two audio cables are the light-coloured leads coming over the top of the right side of the amplifier unit in Figure B-3.3(a).

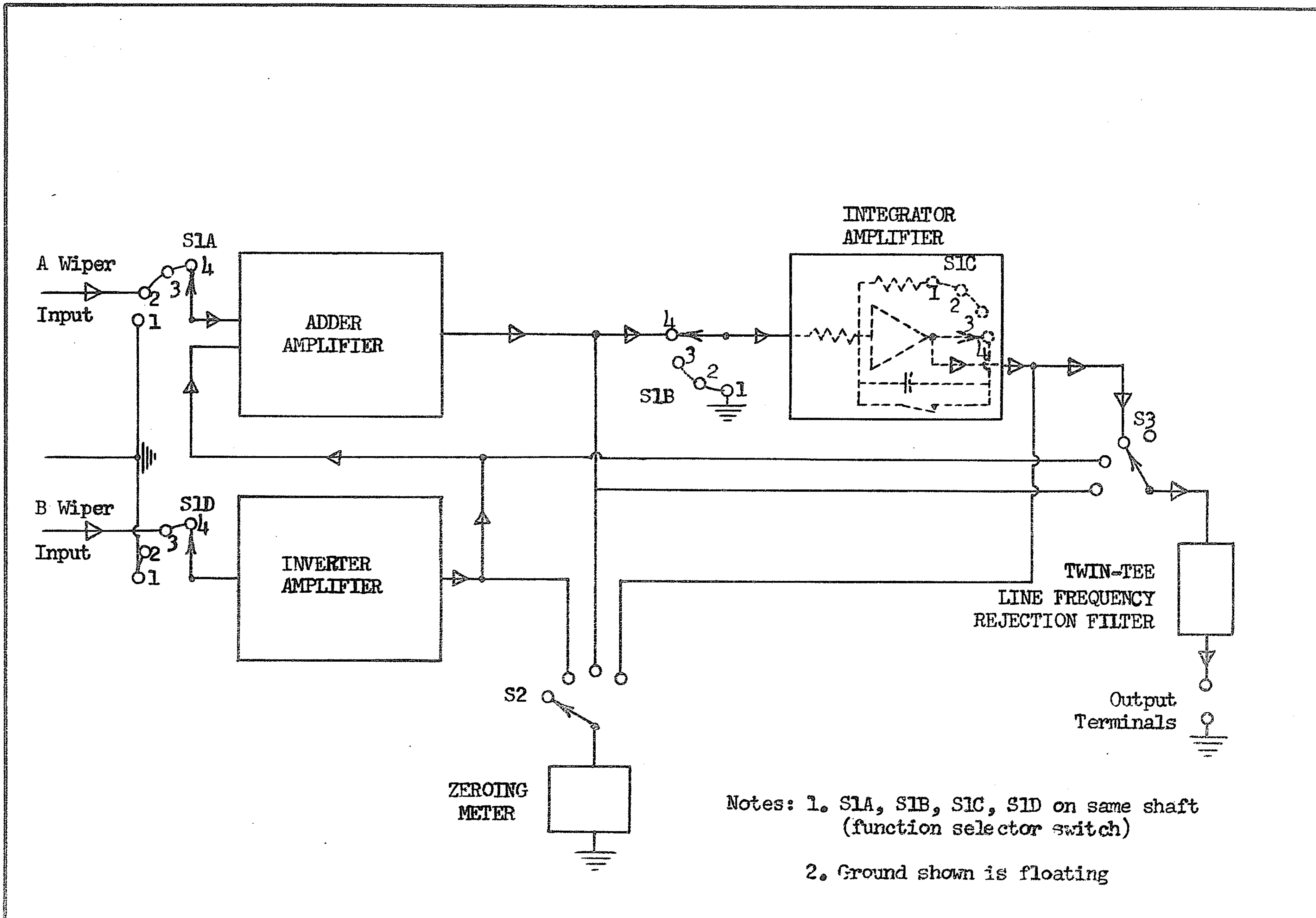

FIGURE B-3.1

BLOCK DIAGRAM OF D.C. AMPLIFIER UNIT

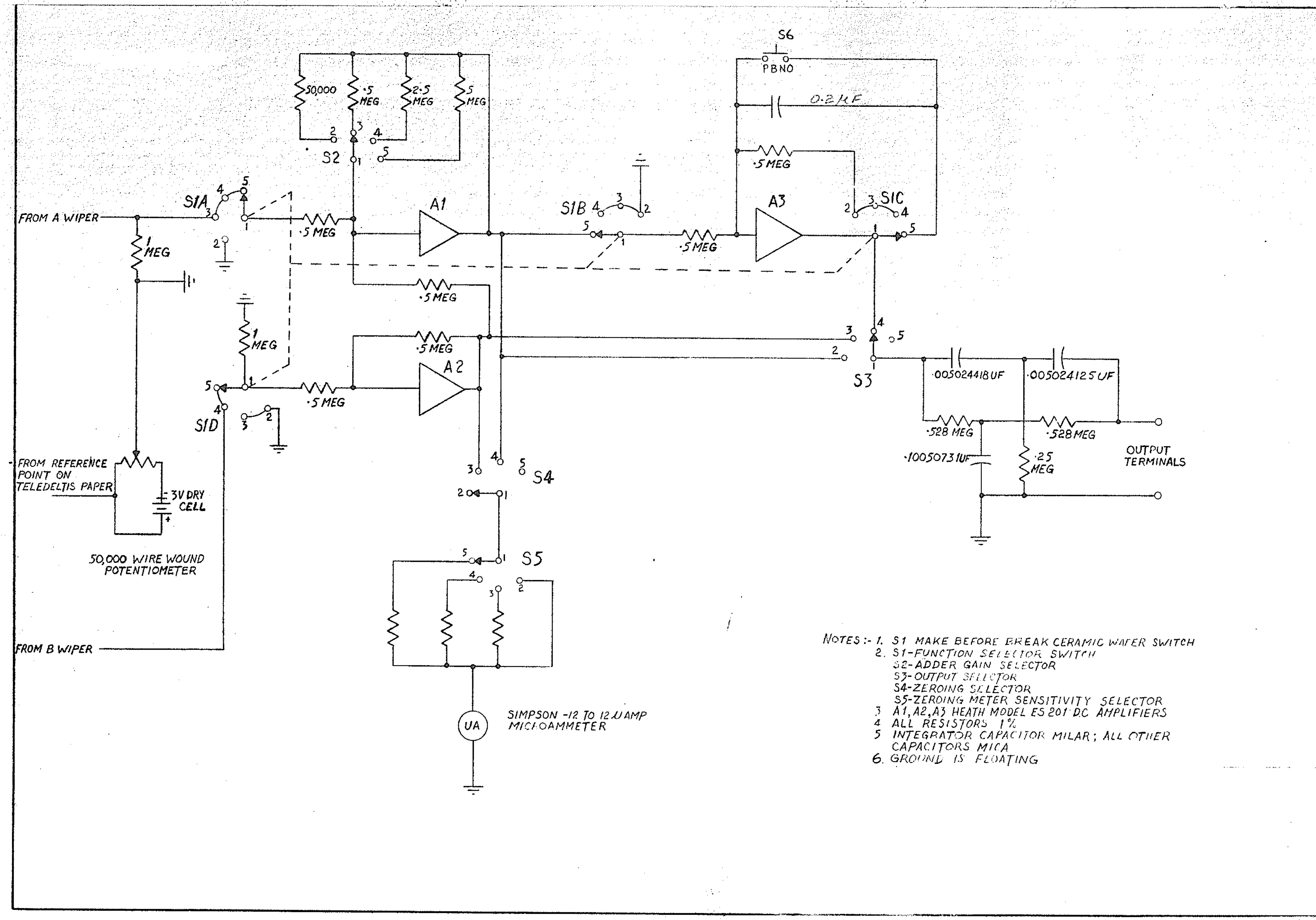

FIGURE B-3.2

SCHEMATIC OF DC AMPLIFIER UNIT AND REFERENCE BIASING CIRCUIT

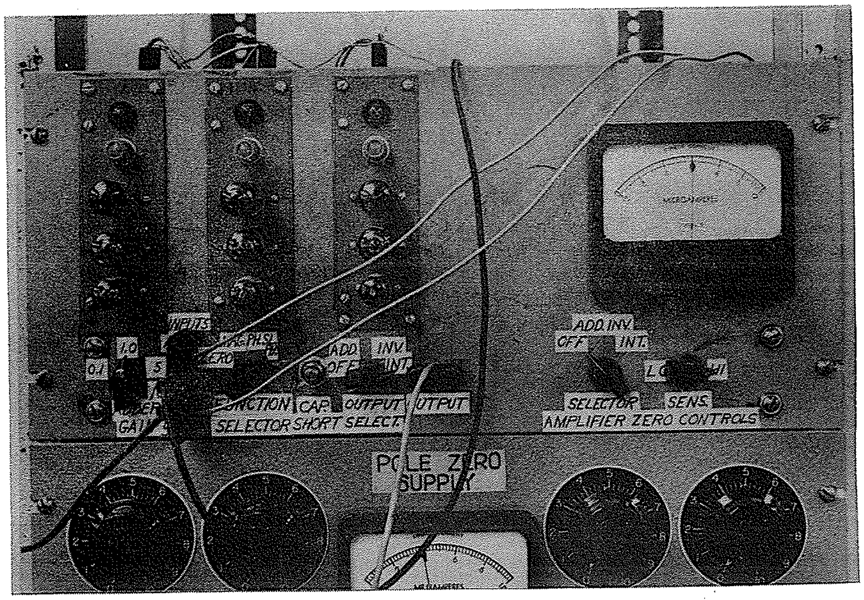

FIGURE B-3.3 (a)

FRONT OF DC AMPLIFIER UNIT

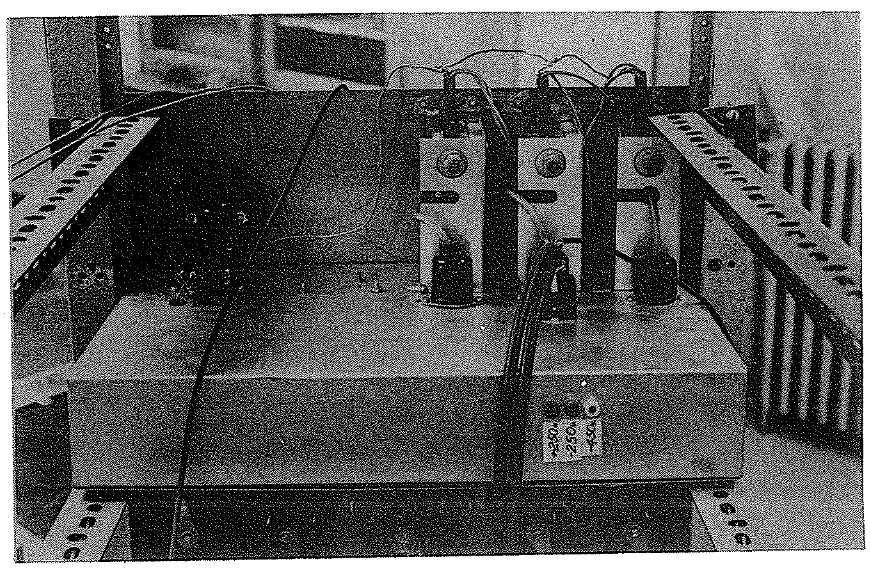

FIGURE B-3.3 (b)

BACK OF DC AMPLIFIER UNIT

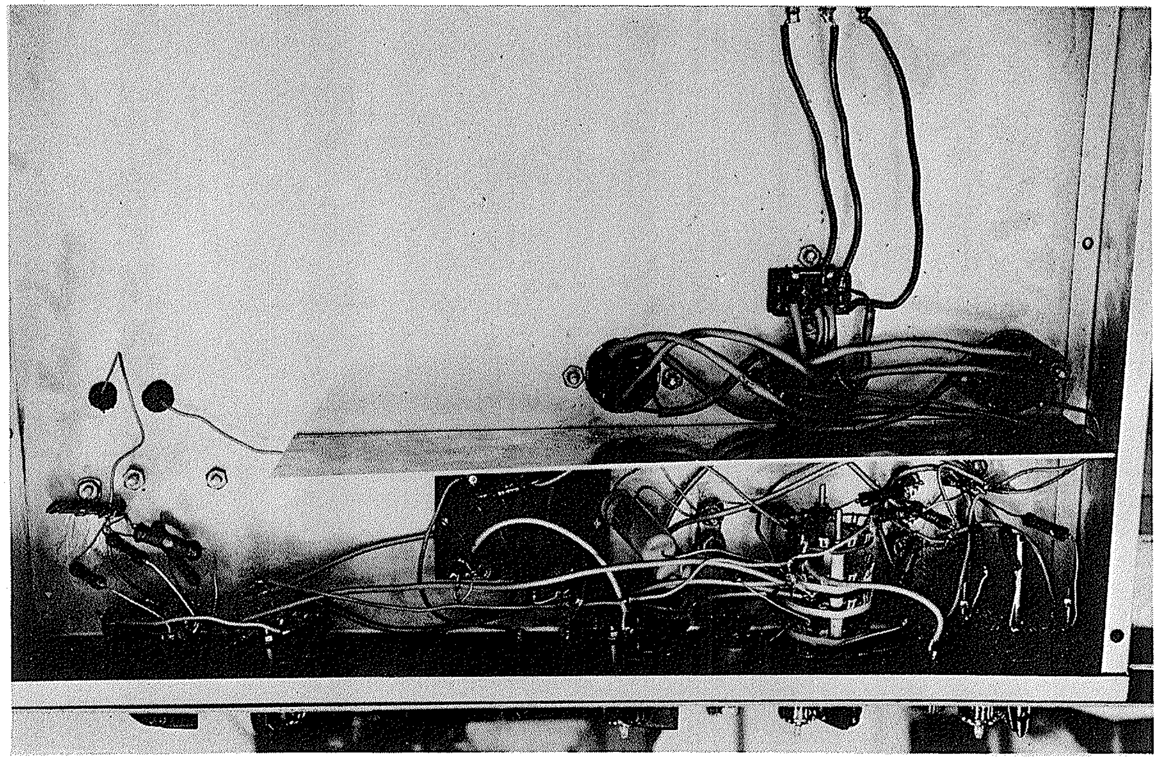

FIGURE B-3.3 (c)

UNDERSIDE OF DC AMPLIFIER UNIT, WITH BOTTOM PLATE REMOVED and the lower one as the B input. The outside shielding of each audio cable is tied to the floating ground through one of the plugs of its double banana plug.

A black cable may be seen in Figure B-3.3(a) leading from the floating ground terminal of the A banana plug (which is electrically connected to the ground terminal of the B double banana plug), to which the shields of the audio cables are tied, over the top of the middle of the dc amplifier unit. The other end of this cable leads to one end or to the middle tap of a potentiometer (usually 50,000 ohms or thereabouts, so as not to drain the battery) across which is placed a constant dc battery of a few volts. One of the other potentiometer taps (depending on the desired polarity) is then connected to a convenient point (usually a strip of silver paint, though any other point which will not interfere with the travel of the carriage is suitable) on the sheet of conducting paper. By this means the input signal from one wiper may be adjusted to any desired dc level above or below the floating ground. The potentiometer can be used, therefore, to set the Y-position of the X-Y recorder pen to a desired initial value, if only one wiper signal is used--as in recording only the logarithm of the magnitude of an immittance function.

Another black cable, shown in Figure B-3.3(a) leading to the lower left from the floating ground on the input double banana plugs, goes to the shielding on the dc amplifier power cables, as may be seen in Figure B-3.3(b).

Three cheap Heath Co. dc amplifiers were used in the dc amplifier unit.[16] These are clearly seen from the front in Figure B-3.3(a) and from the back in Figure B-3.3(b). From left to right in Figure B-3.3(a), the amplifiers were used with simple and well-known passive circuitry -- as shown in the circuit schematic of Figure B-3.2 -- to accomplish addition (and multiplication by constants 0.1, 1.0, 5.0 or 10.0, as seen marked on the front of the unit in Figure B-3.3(a), above the adder gain control and below the adder amplifier), inversion (that is, multiplication by -1) and integration, respectively.

The function selector switch knob shown in Figure B-3.3(a) is used to turn a ceramic wafer switch which makes various connections (as may be seen from the circuit schematic of Figure B-3.2) between the circuit components. When the function selector is turned to zero, the inputs of all the amplifiers are connected to the floating ground. Each of the amplifiers may then be zeroed by selecting the proper amplifier; the selection is done with the amplifier selector switch shown under the left side of the Simpson microammeter in Figure B-3.3(a). When this last switch is turned to "OFF", one side of the microammeter is connected to ground, while the other side is left open. When one of the amplifiers is selected with this switch (by turning it to "ADD", "INV.", or "INT.", as desired), the microammeter is connected, through a resistance (which is selected by the SENS. switch under the microammeter in Figure B-3.3(a)), to the amplifier output. Except for a minor interaction between the adder and inverter amplifiers (due to the 500 K resistor connected between the inverter output and adder input), each amplifier is zeroed separately, by using its

[16]Heath Model ES-201 dc Amplifiers.

zero control. This zero control is located just above the three tubes on each amplifier, as can be seen in Figure B-3.3(a).

When the function selector is turned to "MAG" (magnitude), the A input is connected, through a 500 K series input resistor, to the input of the adder amplifier. The inputs to the other two amplifiers remain grounded through one-megohm resistors.

When the function selector is turned to "PH.SL." (phase slope), the A input cable remains connected to the adder amplifier in the same way as when the selector was turned to "MAG", and the B input is also connected to the inverter amplifier in this way. The output of the inverter amplifier is connected, through a 500 K resistor, to the input of the adder amplifier. Hence, the adder amplifier subtracts the A signal from the B signal. The integrator amplifier remains grounded through a one-megohm resistor.

When the function selector switch is set to "PH." (phase), the input connections to the adder and inverter remain unchanged from those when it was set to "PH.SL." However, the output of the adder is connected to the input of the integrator through a 500 K series resistor, and the output of the integrator amplifier is connected to one side of a capacitor (which is normally shorted, but can be opened by pressing the "CAP. SHORT" button shown in Figure B-3.3(a)), whose other side is permanently connected to the input of the integrator amplifier. Thus, the integrator is used for integration of the difference between the A and B input signals.

The output of any one of the three amplifiers may be selected by means of the output selector switch which can be seen in Figure B-3.3(a) under the integrator amplifier. This switch is used to connect the output signal of the desired amplifier to the input of a twin-tee RC filter rejecting line frequency, whose other side is permanently connected to a double banana jack shown just above "OUTPUT" in Figure B-3.3(a). A shielded cable, with its shield tied to the floating ground connection of the output banana jack, feeds the output signal to the Y input of the X-Y recorder.

Shielded cable was employed in almost all wiring, as is evident in Figure B-3.3(a) - (c), in order to eliminate problems of noise and amplifier oscillation. All of the wiring, except the obviously very short wiring, and the wiring that did not affect the operation of the unit (for example, the connections to the zeroing meter) was done with shielded cable. Figure B-3.3(c) also shows the use of a shield between the power cables and the rest of the associated circuit wiring. It was found that the amplifier tended to oscillate if the power cables were not properly shielded from the rest of the amplifier wiring. It was also necessary to use separate cables for the filament and dc leads from the power supply. The amplifier unit power supply is the specified one available from the Heath Co. Controls for adjusting the power supply voltages are on the front panel of the second rack from the bottom in Figure 4.1, page 52. The voltages may be measured at the jacks shown at the back of the amplifier unit in Figure B-3.3(b). Switches for turning the power supply off and on are on the front panel of the power supply unit.

All the power from the mains to the amplifier unit power supply, as well as to the pole-zero power supply, is fed through the Sola Co. constant voltage regulating transformer shown in Figure 4.1, page 52, on the floor to the left of the tall rack. The use of this transformer was found to effect an improvement in zero stability of the dc amplifiers and the pole-zero constant-current sources and sinks.

APPENDIX B-4

DETAILED CALIBRATION AND OPERATION OF THE POTENTIAL ANALOG MACHINE USED WITH AN X-Y RECORDER

I. Some General Points on Maintenance and Operation

1. To keep the steel bars from rusting and the carriage running smoothly, the steel bars should be greased with vaseline.

2. The electric motor and gear system should be oiled occasionally. Care should be taken, however, not to get oil on the pulley cable or driving sheave.

3. A small lead weight of about three to five cubic inches should be placed on the carriage to keep it firmly on the steel bars, thus minimizing variable contact pressure of the wipers with the conducting sheet. A large weight should not be placed on the carriage, since such a weight might cause inconstancy in its velocity, particularly during the first few seconds after the electric motor has been turned on.

4. Care should be taken to ensure that voltages of the power supply for the dc amplifier unit are at their correct values. It is not sufficient to measure them with a vacuum-tube voltmeter. A good way to

measure them is to find the exact voltages of each of four 45-volt batteries on the 50-volt scale of a reliable, high-quality meter. One of the best batteries is then used as a standard to measure the voltage of the other three in series, by means of a Tinsley Potentiometer. The three batteries are then connected in series with the power supply with such a polarity as to reduce the voltage of the batteries in series with the power supply from that of the power supply voltage. The Tinsley Potentiometer is then set to such a value that, with the accurately measured battery used as a standard (measured with the voltmeter while it is in the circuit), the value of the voltage across the potentiometer is equal to the nominal voltage of the power supply minus the voltage of the three batteries in series. The power supply is then adjusted to obtain a null on the potentiometer.

5. A glossy-finished sheet of bakelite, glass, or other similar material, about 0.1 inch thick, and larger in all other dimensions than any conducting sheets which are to be used, is laid underneath the steel bars and carriage so all conducting sheets can be laid on top of the glossy surface. The glossily-surfaced sheet is fastened firmly to the table by using masking tape at its edges, weights or other means.

6. A warm-up period of at least one hour should be allowed the potential analog machine and X-Y recorder.

II. <u>Preparation of the Conducting Analog Sheet</u>

1. A conformal mapping is selected which emphasizes the regions of the complex plane which are of interest, and also (if possible) eliminates the "infinity error" by mapping infinity into a point, or makes

it negligibly small in some other fashion. The sheet may, for certain mappings only, be corrected for the effects of non-isotropy. Naturally, if the carriage is to be used, only those mappings and corrections are permitted which leave the axes of interest as straight lines. Otherwise, magnitude or phase slope measurements can be made by using one probe or two closely spaced (0.4 - 0.7 cm) metal voltage probes, respectively, and reading the values of magnitude or delay at various points along the desired curved lines of measurement on the conducting sheet, from the vertical scale along the left edge of the graph paper on the X-Y recorder, which has been calibrated according to procedures given in the following two sections. These values may then be plotted manually on the recording paper.

2. The largest possible sheet of Teledeltos paper is selected. Three or four inches should be allowed between the point at which the wipers first begin their movement at the beginning of a run and the point at which they first contact the conducting sheet, and the last point of interest on the sheet should be recorded before the spring arm on the carriage contacts the switch to turn off the electric motor.

3. The desired sheet is cut out of the uncoated Teledeltos paper, and strips of silver paint are painted on the sheet wherever equipotentials are desired. If these strips are at edges of the sheet they should be about one-half inch wide to ensure a good equipotential. If they are at other positions, they may have to be made thinner, but more heavily painted if, for example, they are to represent equipotential lines. On the other hand, other equipotentials are needed if an equipotential object is

represented.

4. Some values of the simple function which will be used to calibrate the system are worked out and recorded for a few points along the desired line of measurement. The points are chosen so as to fall within the region of the plane which is of greatest usefulness. They will be used when calibrating the recording sheets.

5. The points of the last step are marked with non-conducting red pencil, beside the desired line(s) of measurement, with the value of the frequency (or other independent complex variable) at each point. The last point of interest along the line of wiper travel on the conducting sheet is also marked. These points will be used in fixing the X-scale on the recordings. Expected or known positions of all finite poles and zeros should also be marked on the conducting sheet. Care should be taken not to place the non-conducting markings (except for the first and last points) along the line of travel of the probes, so as to prevent loss of contact at these markings.

6. The prepared conducting sheet is placed under the carriage and steel bars, with the initial point on the line of measurement about three or four inches from the starting point of the probe movement.

If the magnitude is to be measured, the conducting sheet is moved into such a position that the wiper (connected to the A input) which will pick the voltage off the conducting sheet will travel along the entire line of measurement on this sheet. This is accomplished by placing some small weights on the sheet and moving it about until the carriage can be moved manually (by stretching the spring in the top loop of the driving cable, and pulling the carriage with it as the cable slips over its sheaves) so that the desired wiper travels between the initial and final points on the line of measurement.

If phase slope or phase is to be measured on a half plane, the line of measurement is midway between the two wipers.

If phase slope or phase is to be measured on a quarter plane, the line of (one) probe travel is as close to the equipotential strip -- whose edge is the real frequency axis -- as possible without introducing undesirable noise due to wiper contact pressure fluctuation, or signal levels which would result in amplifier drift problems and insufficient Y gain on the X-Y recorder.

Care must be taken that the last point of interest marked on the sheet is contacted by the wiper(s) before the spring arm on the carriage touches the switch shutting off the electric motor.

The sheet is then smoothly fastened to the glossily-surfaced sheet with masking tape, and a few small lead weights placed on small squares of insulating paper set on the conducting sheet, to prevent the conducting sheet from rising before the wipers when they are in motion.

7. A piece of smooth, nonconducting paper with at least one straight edge is taped down to the conducting sheet with its straight edge intersecting, facing, and perpendicular to, the line of measurement, at the final point marked on this line. The tape should be placed on either side of, and close to the line of measurement, so as to keep the edge of the nonconducting sheet down firmly, yet so as not to interfere with the wipers.

8. The carriage is moved to its initial position (where the wipers are not in contact with the conducting paper).

III. Readying the X-Y Recorder for Calibration, Zeroing the Amplifiers, and Setting up the Simple Calibrating Function[17]

1. A sheet of squared recording paper (11" X 16 1/2" size) is placed roughly in position on the X-Y recorder, but the vacuum pump of the recorder is not turned on. The stylus of the recorder is made to travel automatically in the X-direction, and the paper moved until the point of the stylus exactly follows a horizontal line on the squared paper. The initial X-position of the stylus is then adjusted so that the stylus point is just above the leftmost vertical line on the graph paper, or a little to the left of it. When the recording paper has been adjusted, the vacuum pump of the recorder is turned on, and the recording paper smoothed so that it is firmly held on the recorder.

2. The X-Y recorder is set to operate the X-axis sweep from its own time base.

3. The poles and zeros of the simple calibrating function are plugged into the distribution unit and placed at the marked positions on the conducting sheet. The switches on the distribution unit for these poles and zeros are turned on.

4. The currents for the poles and zeros are adjusted so that the greatest current from any one pole or zero for the functions that would

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[17]It is assumed that the output jack on the dc amplifier unit has been connected to the Y-input of the X-Y recorder, with the recorder ground tied to the floating ground.

be measured using the calibration would not exceed six to eight milliamperes when the current strength per pole or zero used for the calibration was chosen as the unit strength. (It must be remembered, however, that more than one source or sink may be used to feed a probe requiring high current). Similar remarks apply to the least permissible current strength (1.5 milliamperes per source or sink).

5. The function selector switch is set to "ZERO", the amplifier selector switch set to "OFF" and the amplifier sensitivity set to "LO".[18] After this, the adder and inverter amplifiers are zeroed if magnitude or phase slope are to be obtained, and all three amplifiers zeroed if integration is desired -- except that the inverter amplifier is treated according to footnote 19, below, if quarter planes are used. The reason for zeroing both adder and inverter amplifiers when only magnitude is desired, is that there is a small interaction between adder and inverter amplifiers.[19] The desired amplifier for zeroing is selected by using the amplifier zero selector switch.[20] As the null point is approached, the amplifier zero sensitivity control is turned towards "HI." After zeroing, the amplifier zero selector should be set to "OFF" and the amplifier sensitivity control set to "LO".

6. The pole and zero current strengths are checked, and readjusted

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[18]Figure B-3.3(a), p. 187.

[19]If magnitude on any plane, or phase (or its derivative) on a quarter plane is measured, the inverter amplifier is not needed, so its output may be shorted to the input B banana jack during such measurements. Then it may be ignored during zeroing. A still better scheme would be to open the connection from inverter output to adder input.

[20]Figure B-3.3(a), p. 187.

if they are not exactly at the correct values. Normally, the currents remain quite stable over long periods of time, and require only occasional small adjustments.

IV. Calibration and Operation to Obtain Plots of Magnitude

1. The steps outlined in sections I - III of this Appendix (B-4) are performed.

2. The carriage is moved manually so that the wiper connected to the A input jack just touches the conducting sheet at the first point of the desired line of measurement, the function selector set to "MAG", and the output selector set to "ADD". The potentiometer of the reference potential biasing circuit is then turned so as to move the stylus of the X-Y recorder to a desired position for the Y zero line. (Consideration should be given, at this point, to the expected vertical range of the functions which are to be plotted on the calibrated recording sheets). Then the carriage is moved manually so that the A wiper is at one of the marked positions between the initial and final positions along the line of measurement, at which the calibrating function has been calculated. The value of the simple function at this point being known, the vertical movement of the stylus from the Y zero line can be set to a value that results in a convenient vertical scale -- such as one db per inch, one neper per two inches, or one volt per inch, depending on the field from which the problem was taken -- by using the vertical gain controls of the X-Y recorder, and the proper adder gain. (To avoid amplifier drift, it is best to use the lowest possible adder gain). This calibration should

be checked and, perhaps, slightly adjusted after moving the A wiper to other marked points along the line of travel of the carriage, where the simple calibrating function has been calculated, and checking that the proper value of the simple function is obtained at these points, using the scale set by the calibration at the preceding points. Naturally, one does not carry the calibration to points on the sheet at which infinity error or other inaccuracy may occur. If some discrepancies occur in the values of the simple calibrating function, it might be wise to set the calibration at their mean. Normally, however, there are no observable discrepancies in the readings at different points, from the calculated values.

3. For practical use complete calibrating curves need not be run when magnitude is to be measured. The above simple calibration is usually sufficient for setting the vertical scale; the values at the other points usually check exactly. However, a complete curve may be run, if desired, by starting the electric motor after the carriage has been placed at its initial position (three or four inches from the conducting paper), and starting the X-Y recorder's horizontal sweep before the wiper contacts the conducting paper. The probe will be lifted off the conducting paper by the piece of non-conducting paper on its path near the end of its travel, giving a jump in the recording. Since the point at which the probe first touched the conducting paper may also be identified on the recording, the horizontal scale of the recording can be found from the knowledge of the conformal mapping used for obtaining the conducting sheet. The actual magnitude of the simple calibrating function may be manually plotted on the recording sheet, and compared with the recording. The ordinates of

other magnitude curves traced by the X-Y recorder may then be multiplied by the ratio of the actual to measured calibrating curves at the corresponding values of the abscissae, to obtain the measured magnitudes.

4. A new recording sheet is placed on the X-Y recorder, properly positioned except for the location of the Y zero line.

5. The desired poles and zeros of the function whose magnitude is to be measured are set up on the conducting sheet, using the proper current strengths based on the unit strength used for calibration.

6. The Y zero line is reset by using the reference potential bias circuit, as before. This does not affect the vertical scale set during calibration. This scale calibration is extremely stable, and re-calibration need normally not be performed over periods of six hours or more if magnitudes requiring only one scale are measured. The calibration should periodically be checked, however. The vertical scale is marked along the left edge of the recording paper.

7. The carriage is moved to its initial position (with the wipers three or four inches from the conducting paper), the electric motor switch turned on, and the X sweep of the X-Y recorder started just before the wiper touches the conducting paper. A recording of the magnitude function set up on the conducting sheet is thus obtained. The horizontal scale is fixed from the knowledge of the intial and final points on the recording (corresponding to the initial and final points of the line of measurement on the conducting paper) and the conformal mapping used. If the magnitude is to approximate a desired curve, this curve is plotted on the recording paper and the poles and zeros are moved about until an approximating magnitude curve has been obtained.

V. <u>Calibration and Operation to Obtain Plots of Phase Slope (Delay in the s-Plane)</u>

1. The steps outlined in sections I - III of this Appendix (B-4) are carried out.

2. If phase slope (with respect to the X-scale of the conformal mapping) in a half plane or full plane is desired, the carriage is moved manually so that both of its probes just touch the conducting paper at the first points on each side of the desired line of measurement, the function selector set to "PH.SL.", and the output selector set to "ADD". If a quarter plane is used, the potential between one probe and ground is desired, so this probe voltage is fed to the adder amplifier (A input), the inverter output is shorted to the B input jack, the function selector set to "MAG" and the output selector set to "ADD". A better scheme than shorting the inverter output to the B input jack is to open the connection between the inverter output and adder input. The adder and recorder gains are adjusted so that a reasonable vertical deflection of the stylus is obtained at the first point and the other marked points along the desired line of measurement. In the case of a half-plane measurement, if the deflection is in the wrong direction to that indicated by calculating the simple function at the various marked points, the A and B banana plugs at the amplifier inputs are interchanged. The adder gain and recorder Y-gain controls are adjusted to give the desired scale to the phase slope -- for example, the side of one large square on the recording paper equal to one millisecond, when the s-plane is used. (To avoid amplifier drift one should use the low est possible adder gain). Caution should be exercised

in choosing the Y-scale for phase slope so that the expected phase slope from functions to be measured later will yield recordings not too large for the X-Y recorder, yet with a convenient Y-scale for these functions. For practical use, the potential analog system is suitably calibrated at this point.

3. If a more elaborate calibration is desired, a complete recording of the phase slope of the simple calibrating function may be obtained. The carriage is pulled manually to its initial position, where the wipers are three or four inches from the first point on the line of measurement, the electric motor started and the X-sweep of the X-Y recorder begun just before the wipers contact the conducting paper. The recording stylus will jump when the wipers first touch the conducting sheet and when they are lifted off this sheet. Since the X-scale on the recording between these two points is proportional to the scale of the conformal mapping used for the conducting sheet, this X-scale may be marked on the recording and the actual phase slope of the simple calibrating function plotted on the recording sheet to the vertical scale found in the last step. Ordinates of other phase slope curves traced by the X-Y recorder may then be multiplied by the ratio of actual to measured calibrating function ordinates to find the phase slope.

4. Steps 4 and 5 in section IV of this Appendix are executed, except that the words "phase slope" are to be substituted for the word "magnitude" in step 5.

5. Step 6 of section IV of this Appendix is executed, except that the reference potential bias circuit is not used, and the desired

Y zero position is set with the Y zero control on the recorder, and the word "magnitudes" is replaced by "phase slopes."

6. Step 7 of section IV of this Appendix is executed, except that the word "wiper" is replaced by the word "wipers" and the word "magnitude" by the word "phase slope".

VI. Calibration and Operation to Obtain Plots of Phase

1. The steps outlined in sections I - III of this Appendix (B-4) are executed.

2. The carriage is moved manually to its initial position (where its wipers are not in contact with the conducting paper), the function selector set to "PH.", and the output selector set to "INT." If a quarter plane is used, however, the inverter is treated according to footnote 19, page 199.

3. The Y zero position of the stylus is set so that expected plots of phase will use the largest possible portion of the recording. The output selector is set to "INT" and the "CAP. SHORT" button is depressed, shorting the input and output of the integrator amplifier, and then released, leaving the capacitor open between amplifier input and output. If there is any vertical motion of the recorder stylus after the button is released, the integrator amplifier is not quite balanced. In that case, the integrator amplifier's balancing control is turned until there is no vertical motion of the stylus. If there is some difficulty in obtaining this condition, the adder and inverter amplifiers should be zeroed again, and further attempts made to achieve this condition.[21] As soon as the stylus

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[21]The inverter amplifier is not operational, of course, if quarter planes are used.

shows no vertical motion, the function selector is set to "PH.", the electric motor started and the X-sweep of the X-Y recorder begun just before the wipers contact the conducting paper, and a plot of the phase of the calibrating function is thus obtained. The recording will jump very slightly when the wipers first touch the conducting sheet and when they are lifted off the sheet. If there is an excessive vertical movement of the stylus after the wipers have come to rest, out of contact with the conducting sheet, the amplifiers are zeroed again (for some imbalance of the amplifiers is indicated by the movement of the stylus), the integrator amplifier checked and adjusted so that, as before, no vertical motion of the stylus occurs (with the output selector set to "INT."), and the recording is taken again. A rough value of the phase at various abscissae on the recordings is calculated. If this indicates the Y-scale for phase to be obviously unsuitable, the Y-gain on the X-Y recorder is adjusted to obtain a better scale. The adder gain should be kept as low as possible to eliminate integration of drifts in adder and inverter amplifiers. The balancing procedures are repeated and recordings taken with new values of Y-gain until a curve with a reasonable vertical scale has been attained. If desired, more recordings of the phase of the simple function are taken, and the vertical scale adjusted so that a convenient scale such as two degrees per division has been exactly attained. However, this usually requires too much adjustment, and is not practical unless many plots of phase are desired. Generally, it is more practical to use a scale based on a recording when the vertical scale -- whether or not it is simply expressible in terms of basic divisions of the recording paper --

is suitable for the phase of the simple calibrating function and the expected phase of functions to be recorded later. The known phase of the calibrating function is plotted on the recording paper, using the same X-scale and origin as the recording, to such a scale that the actual calibrating function matches the recorded phase along portions of the latter curve where errors due to the limitations of the conformal mapping (such as finite infinity equipotential of the ordinary complex plane) are known to be small. The vertical scale for phase recordings is thus determined. Although it is not often necessary in practice (because of the close match between the two curves over a wide range of the X-scale), the recorded phase of other curves may be multiplied by the ratio of the actual to recorded phase of the simple calibrating function at the corresponding abscissae, to obtain the correct phase.

4. Steps 4 and 5 in section IV of this Appendix are executed, except that the word "phase" is to be substituted for the word "magnitude" in step 5.

5. Step 6 of section IV of this Appendix is executed, except that the reference potential bias circuit is not used, a desired Y zero line is set with the Y zero control on the recorder, and the word "magnitudes" in replaced by "phases".

6. Recordings of phase are obtained, just as recordings of the phase of the calibrating function have already been obtained. If the phase is to approximate a desired curve, this curve is plotted on the recording paper and the poles and zeros are moved about until an approximating phase curve has been obtained.

APPENDIX B-5

DETAILED DISCUSSION OF SOURCES OF ERROR AND OF THEIR MINIMIZATION

Errors Due to the Physical Analog

1. Non-linearity and non-isotropy of sheet conductivity. The non-linearity and non-isotropy of Teledeltos paper have been measured, and are discussed in some detail in the literature.

Karplus has tabulated the properties of various conducting materials and given precautions to be used with each material.[22] Bridges has measured an average non-isotropy in sheet conductivity in the direction of rolling and the perpendicular direction, and found the average resistance perpendicular to the direction of rolling to be about 12% greater.[23] Bridges derives formulae for correction of this non-isotropy for circular, semi-circular or quarter-circle s-plane sheets by changing the shape to that of an ellipse, semi-ellipse or quarter-ellipse, respectively, and applying a real shift to the positions of poles and zeros. In the course of making measurements of non-isotropy, he found variations of the order of 8 to 15% in sheet conductivities of 11-inch squares in one direction. Measurements of linearity of sheet conductivity also showed a conductivity change of 10% in a 33-inch long sheet.[24] No correction was proposed for

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[22] W. J. Karplus, Analog Simulation (Toronto: McGraw - Hill Book Company Inc., 1958).

[23] Bridges, op. cit., pp. 66 - 67.

[24] Ibid., pp. 70 - 72.

the last effect because the change in linearity over the effectively useful area would be less than 5%, and because such a correction would be rather involved.

Scott measured gradual changes such as those due to a change in spacing of rollers while the paper was being made, and random changes in resistivity such as those caused by impurities and inhomogeneities in the material.[25] In 35 55-inch long samples taken in succession from a single roll he found less than 1/10 percent change of resistivity in 55 inches. Using a sheet 7.55 inches wide fed with a uniform current distribution, which should have given rise to straight line equipotentials, he found a maximum deviation in equipotential lines of 0.037 inch. If the logarithmic plane is used for magnitude measurements by setting up F(s) F(-s) (or a quarter plane is used), the errors can arise only from a shift of equipotential lines along the frequency axis. According to this reasoning, Scott arrives at a constant error of 0.85 percent in the frequency at which a given potential occurs. With a spacing of 7 mm. used for the probes of the potential analog machine of this thesis, the maximum error in measurement of voltage between probes would be 3.7 percent, according to the method of calculation used by Scott. Because of smoothing, the error in phase would probably be less.

Lehr states that his measurements showed that the paper had a resistance of "1800 ohms per square" with variations of less than 0.4 percent.[26]

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[25] Scott, _op. cit._, pp. 33 - 35.

[26] Lehr, _loc.cit._

Although measurements performed at the University of Manitoba by Bridges and the author of this thesis indicated random variations of as much as 15 percent in the resistivity of the Teledeltos paper, as well as the non-isotropy, the experimental results for phase and magnitude measurements obtained using planes for which infinity error was negligible or non-existent and non-isotropy was not corrected for, showed such high accuracy (often within 1%) that doubt is cast on the latter resistivity measurements as indicative of performance to be expected when conformal mapping has been used. On the other hand, the results obtained when employing the ordinary s-plane showed that significant increase in accuracy was attained by using sheets corrected for non-isotropy, as described by Bridges.

Thus, the effects of non-isotropy and non-linearity may be more important in certain mappings than in others. Practical experience shows that much higher accuracy is attainable than Bridge's and the author's measurements of non-linearity and non-isotropy would indicate. Scott's measurements of sheet properties seem to be in greater agreement with the excellent accuracy in magnitude and phase curves obtained with the analog.

2. The approximate physical representation of the infinite complex plane or a conformal mapping of it by a finite sheet of conducting paper with appropriate boundary conditions.

Hansen and Lundstrom have derived formulae and given curves enabling determination of errors in measurements of gain and phase in finite tanks with circular equipotential boundaries, representing the ordinary s-plane.[27] Using Hansen and Lundstrom's results, Moore gives the maximum error in potential along the real frequency axis of such a circular tank as about $\frac{a^2}{2}$, where a is the usable fraction of tank radius, and the portion of the tank which must be used to yield an allowable error of one percent as one-seventh of the tank radius.[28] Bridges calculates the error in measuring, at $j\frac{R}{2}$, the magnitude function $|F(s)| = \frac{1}{\left|s+\frac{R}{2}\right|}$ set up in a circular plane of radius R, as 2.86 percent.[29] Scott has analyzed, using a procedure similar to that of Hansen and Lundstrom, the error due to finite size of a circular sheet and shown how to apply his results to conformal mappings of the s-plane.[30,31] He has also verified his results by actual measurements. These results show that if a conformal mapping represents an s-plane of radius seven times that of the closest finite zero or pole, the error in voltage measurement in the plane is nowhere greater than one percent. Thus, a logarithmic mapping with errors from this source of less than one percent can easily be used for the potential analog machine described in

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[27] W. W. Hansen and O. C. Lundstrom, "Experimental Determination of Impedance Functions by the Use of an Electrolytic Tank", Proceedings of the IRE, 33:529-533, August, 1945.

[28] A. D. Moore, "The Potential Analogy in Network Analysis," (unpublished Master's Thesis, Queen's University, Kingston, Ontario, September 1949), p. 18.

[29] Bridges, op. cit. pp. 88 - 89.

[30] Hansen and Lundstrom, loc. cit.

[31] Scott, op. cit., pp. 35 - 38.

this thesis. If an elliptic function is used in the conformal mappings, the error due to finite boundaries disappears, since infinity is mapped into a point. On the other hand, as discussed in the following paragraph, there are functions which are insensitive to boundary conditions.

Intuitively, it seems that the lines of force due to an equal number of finite positive and negative charges, when there are none at infinity, should mostly cancel each other at the boundaries of a finite tank. Hence, the boundary conditions on a finite sheet should not greatly affect the measurements of all-pass functions or others with no poles or zeros at infinity. Liebman provides a reasonable experimental verification of this idea, and uses it in some examples to obtain very good results with the representation of the complex plane by a rectangular sheet.[32]

Farr and Keen have investigated the use of conformal transformation to improve accuracy of analogs, and have thus been able to solve problems in electrostatics which were previously intractable.[33]

The often impractical scales that have to be used with the ordinary complex s-plane, and the errors due to its finite size, usually make it extremely desirable to use a conformal mapping. However, due to the introduction of non-linear scales by the mapping, non-isotropy can no longer be corrected by simple changes in the shape of the sheet. Furthermore, such changes of shape would generally distort the straight-line.

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[32] Liebman, op. cit., pp. 56 - 73.

[33] H. K. Farr and W. A. Keen, Jr., "Improving Field Analogs through Conformal Mapping", Communications and Electronics, A.I.E.E., Pt. I, 395 - 399, July, 1955.

mappings of the axes along which measurements are required, making it impossible to use the carriage. Thus, the errors due to non-isotropy must be accepted. As discussed above, however, these errors are probably quite tolerable. Results using logarithmic and elliptic mappings have shown errors of less than one per cent in both magnitude and phase measurements. If diverse examples indicate such accuracy in future, they will provide the most important proof of the accuracy, and indicate limitations of the potential analog system.

3. Fluctuations in the strengths of source and sink currents. These were found negligibly small over the time that a recording was taken, as long as currents for individual probes did not exceed about six milliamperes or go below about one and one-half milliamperes.

A large milliameter scale is essential for accuracy in metering currents. Furthermore, adjustment on only one scale range is probably wise, unless the shunts for the milliameter are known to be accurate.

Over periods of as long as an hour, occasional checking and adjustment of currents is advisable. Adjustment of currents should be carefully done so as to secure equal distribution of the variable resistances among the sources and sinks. This is indicated by the scales on the potentiometers on the front of the panel. With proper adjustment the currents tend to stabilize each other and fluctuations become negligibly small.

Little trouble was experienced with current fluctuations when the above precautions were taken.

4. Change in sheet conductivity with change in current density. Scott has found the allowable current density in the Teledeltos paper to be 9 ma for no heating and 40 ma for burning.[34] (Hence, the diameters of the current probes cannot be too small or the paper will burn.). Scott used current probes 0.025 inch in diameter and a current strength of 0.835 ma. He has found that this is about the maximum that can be used without running into nonlinear effects in the vicinity of the probe, although values greater than this can be used without introducing errors of significance except in the neighbourhood of the probes.[35]

Harries has found that current flow through Teledeltos paper should be kept below about one milliampere through areas of a few square inches having minimum path lengths of the order of a centimeter, in order to keep nonlinear resistance effects small.[36]

Bridges has made measurements to determine the effect of current density on sheet conductivity.[37] He has found that the conductivity one-quarter inch from a probe carrying four ma is only one percent greater than in a region of low current density.

These investigations indicate that the effect of current density due to probes carrying six or less milliamperes can be considered negligible

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[34] Scott, op. cit., pp. 15 - 16.

[35] As Scott states, the current is not changed by a change of surface resistance adjacent to the probe.

[36] J. H. O. Harries, "The Rubber Membrane and Resistance Paper Analogies", Proceedings of the IRE, 44: 245, February, 1956.

[37] Bridges, op. cit., pp. 73 - 76.

at distances greater than about one-quarter inch from the probes. Furthermore, high accuracy near these probes is seldom required. The results of previous investigators were borne out by examples (given in the following chapter) in which the attenuation very close to the line of measurement was recorded. The accuracy remained high except very close to the poles, where it was not needed.

Conformal mappings can be used to expand the scale in the vicinity of poles or zeros where high accuracy is required. Elliptic mappings, for example, greatly expand the scale near the edges of filter pass bands, near which both poles and zeros are generally situated. Thus, the importance of using conformal mappings is again to be noted.

5. Finite sizes of probes used for current sources and sinks, finite size of voltage-measuring wipers, and effect of holes left in the plane by the current probes.

Scott has analyzed these errors thoroughly.[38] The errors due to the current probes are:

(i) The effect of conformal mapping (Scott treats only the logarithmic mapping although similar analysis applies to others).

(ii) Their finite size.

(iii) The holes they sometimes burn in the paper.

The voltage wipers also introduce an error because of their finite size.

Scott shows that a circle in the s-plane maps into a circle in the

[38] Scott, op. cit., pp. 38 - 42.

logarithmic plane whose radius in the latter is the radius, r, in the s-plane, divided by its distance, R, from the origin of the s-plane, provided $\frac{R}{r} \geqslant 10$. Conversely, a circle of constant size in the logarithmic plane maps into a circle of the s-plane which decreases in radius in proportion to the distance from the origin. Hence, the percentage error due to the probe size is constant everywhere in the logarithmic plane. For other conformal mappings a similar analysis could easily be performed. Since the probes have diameters of less than one-sixteenth of an inch where they contact the paper, the errors introduced by these probes can generally be considered neglibly small at short distances from these probes. Clearly, the error decreases if the conformal mapping has expanded a region of the s-plane, for the probe diameters are of constant size.

Due to their finite sizes, the current and voltage probes distort the field in their immediate vicinity. The same is true of holes left in the paper. Scott's analysis, valid for any conformal transformation which maps small circles into circles and makes the effects of finite boundaries negligible, shows that at a radius of ten times the probe radius the error in the field due to the finite current-probe size or of holes in the paper is of the order of one-tenth percent and the error due to the effect of finite voltage probe size is of the order of one percent. Thus, these errors are all negligible unless voltages are measured very close to current probes or holes. There is also a small error due to the finite size of the voltage-measuring wiper. Since each of these wipers probably has a contact area with the paper of less than half that of a current probe, this error must be negligibly small.

6. Finite width and imperfect conductivity of silver paint strips used to simulate equipotentials.

In certain problems equipotential conducting strips, lines or (other) shapes must be used for simulation. If an "equipotential strip" is used for an equipotential as a boundary of a sheet -- as, for example, the equipotential around the curved periphery of semi-circular sheets -- the strip may be made as much as an inch wide, if desired. The inside edge of the strip should smoothly and accurately follow the prescribed arc or line of the boundary. This may be quite important in cases where the boundary is, theoretically, supposed to be a line, although it does not greatly matter, for example, when zero and infinity are represented by the strips at each end of a logarithmic plane. The former case (i.e., a boundary is to be a line equipotential) occurs, for example, when problems in image parameter theory are dealt with, for the image propagation attenuation function is zero in the pass band of filters. Care must be taken in such cases not to cover the part of the sheet which represents the desired mapping, with the conducting paint. However, since the edge at which the strip of silver conducting paint meets the sheet has imperfections, the strips of paint should overlap the plane by about a millimeter. In both cases discussed so far, the strip should be wide enough to ensure an equipotential if current is to be fed into or withdrawn at these equipotentials. Several probes should be distributed along them and the required current divided equally among them.[39]

If line equipotentials inside the sheet boundaries are to be represented, the silver paint conducting strips must be as thin as possible, yet heavily painted, and current probes, if any, must be distributed as in the previous cases.

Other equipotential shapes should be accurately painted on the conducting sheets. A number of current probes (if any are needed), distributed over the shape, should be used.

Errors Due to the Methods of Measurement and the Recording System

1. Finite spacing and positioning of the voltage-measuring wipers (probes) carried by the carriage.

When measuring magnitude, a wiper is positioned so as to travel exactly along a line. The wiper travel has extremely small deviations from a straight line, and the paper can be set up so the wiper travels (as well as can be observed by eye) exactly along the required line. According to analysis such as Scott's, the errors due to positioning of wipers are negligibly small.[40]

The finite spacing (7 mm.) of the wipers affects phase and phase slope measurements because the spacing of the wipers is not the infinitesimally small distance required for ideal measurement. An exact analysis of the error for simple cases can easily be performed by using field

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[39]Scott, ibid., p. 43.

[40]Scott, ibid., pp. 43 - 44. The analysis here applies to error introduced by measuring magnitude as the mean of voltages measured with probes at equal distances on each side of the axis, but a similar analysis can be applied to phase slope measurement.

theory.[41] It is clear, however, such errors can be rather high if sources or sinks lie near the wipers' line of travel.

Measurements (by graduate students at the University of Manitoba) of phase of Butterworth and Chebyshev pass band functions of fourth order in a half s-plane corrected for non-isotropy have yielded nearly perfect results. The function $\frac{F(s)}{F(-s)}$ was set up for making these measurements. The phase recording shown on page 68 also shows good accuracy. Therefore, high or sufficiently high accuracy can be obtained in many cases.

One method by means of which the accuracy in phase and delay measurement can be improved is by setting up the function $\frac{F(s)}{F(-s)}$ in half planes or using a quarter plane with a short-circuited real frequency axis. In the latter case, one probe travels parallel to, and close to, the edge of the conducting strip, while the other probe travels on the conducting strip. Thus, the spacing between probes is no longer a problem when measurements are made in such a quarter plane.

Another method of improving accuracy would be to decrease the spacing between the wipers. As mentioned in the concluding sentence of the last paragraph, this is not necessary if the quarter plane is used. It is to be noted, however, that if this spacing were very small, higher adder gain may have to be used, causing a variable drift in the input to the integrator amplifier. The integrator would also be difficult to adjust so as to eliminate vertical motion of the recorder stylus, with

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[41] Scott, *ibid.*, pp. 43 - 44. The analysis here applies to error introduced by measuring magnitude as the mean of voltages measured with probes at equal distances on each side of the axis, but a similar analysis can be applied to phase slope measurement.

the amplifier's feedback capacitor open, just before taking the recording. The adder drift would also cause an error in delay measurements. If the probe spacing were very small, there would also be fluctuations in the recording of phase slope, because the slight variations in pressure of the probes against the conducting paper during carriage travel would cause variable voltage differences between them. These differences are smoothed out during integration, but they may still cause an error in phase. In the case of phase slope, however, there would probably be fluctuations which the recorder could not even follow. Another effect is the distortion in the field caused by the finite area of contact of the two probes with the paper. This effect would be greater if the probes were very close together.

The probe spacing of seven millimeters resulted in high accuracy in many problems solved in a half plane. In others, the accuracy would be high enough for practical purposes. Higher accuracy may be obtained by using a quarter plane with an equipotential strip along the real frequency axis, and running one wiper as close to the edge of the strip as the limitations discussed above will permit.

2. Non-linear amplification, drift and other errors in the dc amplifier unit.

Ormrod has analyzed the low frequency response of the Heath amplifiers used in the dc electronic unit.[42] He has discussed the special considerations

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[42] J. H. Ormrod, "Operational Amplifiers", (unpublished Bachelor's Thesis, The University of Manitoba, Winnipeg, 1959), pp. 3 - 7.

in the design of these dc amplifiers for computer use.[43] He has analyzed the errors which the amplifiers have when they are used for multiplication by a constant, addition and integration.[44] Finally, he has proposed a system similar to the one used in the dc electronic unit.[45] Ormrod experimentally verified the accuracy of his adder circuit and his integrator circuit.

Using Ormrod's work it can be shown -- neglecting, for the moment, the effects of drift -- that the adder and inverter unit are accurate to less than one-half of one percent. By integrating a constant voltage for a period of five minutes and taking simultaneous readings of output voltage and time during this period, he found that the measured slope of the plot of output voltage versus time was within three percent of the calculated value, and that all but the last two final readings did lie on a straight line. The three percent difference between calculated and experimental results was within the experimental error since all components were one percent and the full scale error of the meter was three percent. The final output voltage of the integrator was 23.5 volts.

Particularly when the adder gain is greater than unity, phase or phase slope measurements can be noticeably affected by the small amounts

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[43]Ibid., pp. 8 - 15.

[44]Ibid., pp. 16 - 21.

[45]Ibid., pp. 22 - 24.

of drift. This drift shows up a little when the adder and inverter amplifiers are being zeroed with the adder gain greater than unity. Greater difficulty with drift is experienced, however, when adjusting the integrator amplifier balance so that no vertical motion of the X-Y recorder stylus results when the capacitor across the integrator is open, prior to making a recording of phase. The amplifier drift can make this adjustment quite difficult if the adder gain is over unity.[46]

The adder amplifier does not introduce significant error when magnitude only is being measured. In this case the input signal level to the adder amplifier is high enough to make the effects of drift negligibly small. The adder gain in this case may also be made greater than unity if desired. However, the higher signal level permits use of the X-Y recorder Y gain control, and this is always preferable to increasing the adder gain.

To minimize the error during phase measurements, the Y drift of the recorder stylus, before the recording is taken, must be stopped by using the integrator amplifier zero control, and the Y drift of the stylus at the end of the recording (with the voltage wipers not contacting the conducting paper) should be observed. If the latter drift is excessive, the recording should be taken again.

To minimize the error from drift in all cases, the adder amplifier gain control should be kept as low as possible and the Y gain control of the X-Y recorder used.

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[46]Fortunately, it was found that the X-Y recorder Y gain was sufficiently great, so that the adder gain could be left at unity during phase measurements.

The amplifier drift has been kept as low as possible by using the proper Heath power supply for the Heath ES-201 amplifiers, by accurately adjusting the voltages of this power supply (as described in Section 1.4 of Appendix B-4), and by feeding the power supply with the voltage from a constant voltage regulating transformer to eliminate line voltage fluctuations.

The problem of amplifier drift could be entirely eliminated by using more expensive chopper-stabilized dc amplifiers.

3. The X-Y recorder.

The accuracy of the X-Y recorder is exactly described in the literature supplied with it by the manufacturer. Briefly, it can be said on the basis of considerable experience, that the X-Y recorder was admirably suited for use with the potential analog system. With the speed of travel used for the carriage, no difficulty was encountered in the speed of response of the X-Y recorder. In the immediate vicinity of singularities near the line of probe travel, the response may not have been perfectly accurate, but an exact knowledge near such singularities was never desired. The magnitude response may not have been perfectly accurate at the start of the recording, when the voltage wiper first touched the conducting paper, if the first point was not at ground potential. This occurred, for example, whenever the response along the "stop band side" of an elliptic function mapping was taken. There was almost no detectable oscillation of the X-Y recorder's stylus if the proper settings of recorder sensitivity were used.

The recordings (after the initial jump in many attenuation responses) were smooth, except for occasional slight roughnesses due to the probe's passing over the surface of the conducting sheet. The roughness was not present in recordings of phase; this showed that it was not due to the X-Y recorder.

The stylus could easily be positioned to within the width of the fine inked line.

The calibrations and operating procedures -- and the pertinent errors -- are described and discussed in Appendix B-4. Scales established by using simple calibrating functions often remained stable over periods as long as three or four hours. Periodic checks ensured accurate calibration at all times.

In conclusion, the X-Y recorder introduced no error of any practical importance, and was remarkably appropriate for use with the potential analog machine.