

# Fourier Coefficients of Eisenstein Series and Explicit Computations of Whittaker Functionals

by

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## Abstract

This expository thesis bridges classical modular forms and the modern adelic framework, focusing on its application to the arithmetic Siegel-Weil formula. Transitioning to  $\mathrm{GL}(2, \mathbb{A})$ , we explicitly construct non-holomorphic Eisenstein series and the local Weil representation. Our primary objective is the explicit evaluation of local Whittaker functionals for unramified binary quadratic spaces. Using Yang's local density formulas, we compute the central derivatives to verify an identity, providing a complete SageMath implementation for verification.

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# 1

## Introduction

### 1.1 History

It all starts with a simple question: *Which numbers can be written as the sum of two squares?* With the accumulated works of Girard, Fermat, and Mersenne, and the foundational arithmetic observation that any number theory student can make, we know that a positive integer  $n$  can be written in the form  $n = x^2 + y^2$  if and only if each prime factor  $p \equiv 3 \pmod{4}$  of  $n$  appears to an even power.

The next natural question is to ask: *in how many different ways can we write an integer as a sum of two squares?* This question was answered by Jacobi. Denoting  $r(n)$  as the number of ways to write  $n$  as a sum of two squares, we have the exact formula

$$r(n) = 4 \sum_{d|n} \chi(d) \tag{1.1.1}$$

where  $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \rightarrow \{\pm 1\}$  is the non-trivial Dirichlet character.

What makes this formula historically and mathematically significant is that it provides the first clear connection between an analytic generating function, specifically the Jacobi theta series

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad q = \exp(2\pi i \tau) \tag{1.1.2}$$

and the normalized Eisenstein series

$$E_k^\chi(\tau) = 1 + \frac{2}{L(1-k, \chi)} \sum_{n \geq 1} \left( \sum_{d|n} \chi(d) d^{k-1} \right) q^n. \tag{1.1.3}$$

Because  $r(n)$  naturally appears as the Fourier coefficients of  $\theta^2(\tau)$ , we obtain the correspondence  $\theta^2(\tau) = E_1^X(\tau)$ . This stems from the fact that this Eisenstein series is the generator of the one-dimensional space of modular forms of weight 1 and level 4. (A detailed construction of these spaces and their dimensions is provided in Chapter 2).

This correspondence gained tremendous importance with the work of Siegel, who generalized the theta series to the weighted average of representation numbers of quadratic forms, ultimately identifying them with what are now known as Siegel Eisenstein series [15]. Briefly, let  $\Lambda$  be a positive definite quadratic lattice over  $\mathbb{Z}$  of rank  $m$  with the quadratic form  $Q : \Lambda \rightarrow \mathbb{Z}$ . The associated bilinear form is

$$(x, y) := Q(x + y) - Q(x) - Q(y), \quad \text{where } Q(x) = \frac{1}{2}(x, x). \quad (1.1.4)$$

Define the generalized representation number  $r_\Lambda(T)$  for a positive semi-definite symmetric matrix  $T$  as the number of ways to represent  $T$  by the lattice  $\Lambda$ . Siegel's theta series, which acts as a generating function for these representation numbers, is a holomorphic Siegel modular form on the Siegel upper-half plane of weight  $\frac{m}{2}$ .

To provide a representation-theoretic framework for this theory, Weil introduced the modern adelic approach. By transitioning to the adèle ring  $\mathbb{A}$  of  $\mathbb{Q}$  and formalizing necessary structural decompositions, such as the Iwasawa decomposition discussed in Chapter 3, we construct the Schrödinger model of the Weil representation on the space of Schwartz functions  $S(V(\mathbb{A})^n)$  (Chapter 4). Under this setup, the Siegel Eisenstein series associated to a standard Siegel-Weil section  $\Phi$  of the degenerate principal series representation is defined globally as

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi(\gamma g, s), \quad g \in G(\mathbb{A}), s \in \mathbb{C}. \quad (1.1.5)$$

Specifically, the section evaluates to  $\omega(g)\varphi(0)|\det a(g)|^{s-s_0}$  for a Schwartz function  $\varphi$  and a base point defined by

$$s_0 = \frac{m - (n + 1)}{2}. \quad (1.1.6)$$

This setup provides the foundation for the Siegel-Weil formula. Simply put, this formula establishes a fundamental identity: the global integral of a theta function is exactly equal to a specific value of the Siegel Eisenstein series, evaluated at  $s = s_0$ .

## 1.2 The Arithmetic Siegel-Weil Formula and Local Derivatives

There have been many modifications made to the classical Siegel-Weil formula. Simply changing the approach to writing the Eisenstein series alters its functional equation. When we lift this framework to the reductive group and compute it there, the functional equation relates  $s \leftrightarrow -s$ . Consequently, the point of symmetry shifts to  $s = 0$ , which is suitably called *the central point*.

As we will discuss, Stephen Kudla has shown that the derivatives of the Eisenstein series given by Equation 1.1.5 at this central point (the central derivatives) relate to another geometric object entirely, known as the intersection multiplicities of special cycles on Shimura varieties [13].

In our specific setting, working with quadratic forms of dimension  $m = 2$  and taking  $n = 1$ , our base evaluation point naturally aligns with the central point  $s_0 = 0$ . Our primary goal in this thesis is to explicitly compute the local factors of these central derivatives.

To prove the arithmetic Siegel-Weil formula and compute these global intersection numbers, we must look at the local factors. Analytically, the  $T$ -th Fourier coefficient of the global Eisenstein series factors into an Euler product of local Whittaker functionals:

$$E_T(g, s, \Phi) = \prod_{v \leq \infty} W_{T,v}(g_v, s, \Phi_v). \quad (1.2.1)$$

By the elementary product rule, taking the derivative with respect to  $s$  at the central point  $s = 0$  yields a sum of terms where exactly one local Whittaker functional is differentiated, while the others are evaluated at  $s = 0$ :

$$E'_T(g, 0, \Phi) = \sum_p W'_{T,p}(g_p, 0, \Phi_p) \prod_{v \neq p} W_{T,v}(g_v, 0, \Phi_v). \quad (1.2.2)$$

Consequently, having explicit formulas for the central derivatives of local  $p$ -adic Whittaker functionals is particularly useful in light of the Kudla program, which seeks to connect the global arithmetic geometry of Shimura varieties to these analytic generating functions. Using these explicit formulas developed by Yang [19], we will see how the local representation densities approach can be used to explicitly evaluate these derivatives.

## 1.3 The Expository Goal

Although the destination of this thesis is the explicit evaluation of specific local derivatives, its primary objective is expository. The literature surrounding the arithmetic Siegel-Weil formula and the Kudla program which bridges the intersection theory of arithmetic cycles on Shimura varieties with the Fourier coefficients of modular forms, heavily relies on advanced machinery from representation theory, adelic analysis, and algebraic geometry, which can present a barrier to entry.

Therefore, the core contribution of this work is to provide a detailed bridge from classical modular forms to this modern adelic framework. We systematically construct the necessary background to verify the local identity from [16, Proposition 4.4(iii)] for the specific setup we establish in Chapter 5.

## 2

# Classical Foundations

In this chapter, we give the basic foundation for the modular group and modular forms, mainly following the books Serre [17], Bump [3], and Diamond-Shurman [7]. However, the structure and presentation belong to us.

## 2.1 The Modular Group and Functions

In this section, we introduce the basic notions necessary to understand classical modular forms. Our discussion begins with the group-theoretic framework underlying their definition.

Let us consider the group

$$G = \mathrm{SL}(2, \mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left| a, b, c, d \in \mathbb{R}, \det(g) = ad - bc = 1 \right. \right\}, \quad (2.1.1)$$

and let  $K$  denote its maximal compact subgroup, given by

$$K = \mathrm{SO}(2, \mathbb{R}) = \left\{ k = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left| \theta \in \mathbb{R} \right. \right\}. \quad (2.1.2)$$

Our main object of interest is the two-dimensional coset space  $G/K = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ , which can be parametrized by the upper half-plane:

$$\mathbb{H} = \{z = x + iy \mid x, y \in \mathbb{R}, y > 0\}. \quad (2.1.3)$$

An important property of  $G$  is that it acts transitively on  $\mathbb{H}$  via Möbius trans-

formations:

$$g \cdot z = \frac{az + b}{cz + d}, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \quad (2.1.4)$$

This action preserves the upper half-plane and satisfies the identity

$$\operatorname{Im}(g \cdot z) = \frac{\operatorname{Im}(z)}{|cz + d|^2}. \quad (2.1.5)$$

Every element  $g \in G$  admits the decomposition

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (2.1.6)$$

for some  $x \in \mathbb{R}$ ,  $y > 0$ , and  $\theta \in \mathbb{R}$ .

Applying this to the point  $i \in \mathbb{H}$  shows that

$$g \cdot i = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot i = x + iy. \quad (2.1.7)$$

Hence, every point in  $\mathbb{H}$  lies in the orbit of  $i$  under the action of  $G$ .

Moreover, the isotropy subgroup of  $i$  is precisely  $K$ . This gives the identification  $\mathbb{H} \cong G/K$ .

The above factorization of  $g \in G$  is also known as the *Iwasawa decomposition* of  $G$ . It expresses each element of  $G$  as a product  $g = nak$ , where  $n$  is unipotent upper triangular,  $a$  is diagonal, and  $k \in K$ . Let us define the subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^\times, b \in \mathbb{R} \right\} \subset G, \quad (2.1.8)$$

consisting of upper triangular matrices. Then the Iwasawa decomposition implies that

$$G = B \cdot K. \quad (2.1.9)$$

Now, let us move to the discrete subgroup  $\Gamma(1) = \operatorname{SL}(2, \mathbb{Z})$  of  $G$ , and to certain subgroups of it known as *congruence subgroups*. For each positive integer  $N \geq 1$ , we define the principal congruence subgroup of level  $N$  by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}(2, \mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}. \quad (2.1.10)$$

One can observe that  $\Gamma(N)$  is a normal subgroup of  $\Gamma(1)$  of finite index. Indeed,  $\Gamma(N)$  is the kernel of the natural reduction map

$$\mathrm{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}), \quad (2.1.11)$$

given by reducing matrix entries modulo  $N$ .

Now, define the subset  $\mathcal{F}$  of  $\mathbb{H}$  by

$$\mathcal{F} = \left\{ z = x + iy \in \mathbb{H} \mid |x| \leq \frac{1}{2}, |z| \geq 1 \right\}, \quad (2.1.12)$$

which serves as a *fundamental domain* for the modular group  $\Gamma(1)$ .

Intuitively,  $\mathcal{F}$  contains exactly one representative from each orbit of the action of  $\Gamma(1)$  on  $\mathbb{H}$ , and the images of  $\mathcal{F}$  under  $\Gamma(1)$  tile the upper half-plane without overlapping interiors [17]. The boundary points correspond to equivalences induced by elements of  $\Gamma(1)$ . Therefore, another representation of  $\mathcal{F}$  is given by the quotient  $\Gamma(1) \backslash \mathbb{H}$ . This fundamental domain is depicted in Figure 2.1.

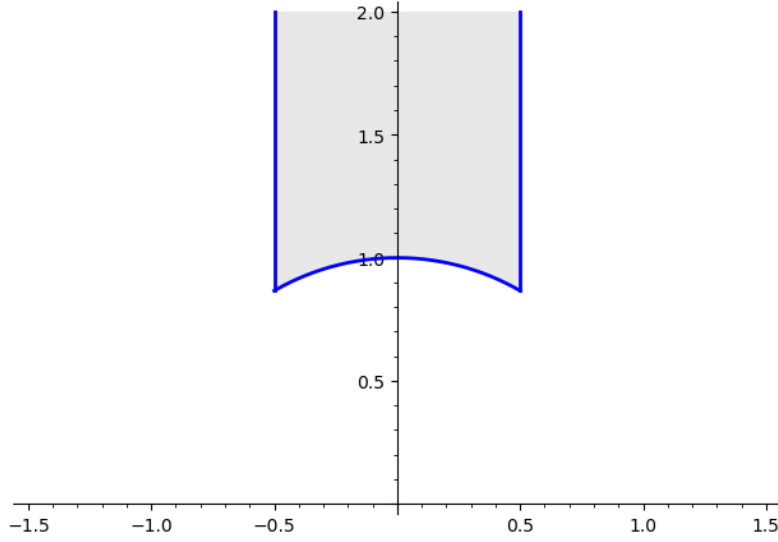


Figure 2.1: Fundamental Domain

It is important to note that  $\Gamma(1)$  acts discontinuously on  $\mathbb{H}$ , and that  $\mathcal{F}$  has finite volume with respect to the measure  $|y|^{-2} dx dy$  which is invariant under the action of  $\mathrm{SL}(2, \mathbb{R})$ . To see this invariance, let

$$\mu(A) = \int_A \frac{dx dy}{y^2}, \quad (2.1.13)$$

where  $z = x + iy$ . We want to show that for any  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ , we have

$$\mu(\gamma A) = \int_{\gamma A} \frac{dx' dy'}{(y')^2} = \int_A \frac{dx dy}{y^2} = \mu(A), \quad (2.1.14)$$

where  $x' = \Re(\gamma z)$  and  $y' = \Im(\gamma z)$ .

We can rewrite the left-hand side of (2.1.14) as

$$\int_{\gamma A} \frac{dx' dy'}{(y')^2} = \int_A \left| \frac{\partial(x', y')}{\partial(x, y)} \right| \cdot \frac{dx dy}{(y')^2}. \quad (2.1.15)$$

Using the Cauchy-Riemann equations and the fact in (2.1.5), we compute the Jacobian determinant:

$$\left| \frac{\partial(x', y')}{\partial(x, y)} \right| = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \left( \frac{\partial y'}{\partial x} \right)^2 + \left( \frac{\partial y'}{\partial y} \right)^2 = |cz + d|^{-4}. \quad (2.1.16)$$

Hence, we obtain

$$\int_A |cz + d|^{-4} \cdot \frac{dx dy}{(y')^2} = \int_A \frac{dx dy}{y^2}, \quad (2.1.17)$$

which confirms that the measure is indeed invariant under the action of  $\mathrm{SL}(2, \mathbb{R})$ .

Now, let us verify that the fundamental domain  $\mathcal{F}$  has finite volume with respect to this measure. Recalling the definition of  $\mathcal{F}$  from equation (2.1.12), we compute the volume as follows:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy dx}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} d\theta = \frac{\pi}{3}. \quad (2.1.18)$$

Now, let us move on to the compactification of the fundamental domain by adjoining certain points to it, ultimately obtaining a compact Riemann surface. This idea will become useful in the upcoming pages when we prove that the space of modular forms has finite dimension.

Before we proceed, let us briefly define the topological boundary of the upper half-plane.

Let  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$  denote the Riemann sphere. When we embed  $\mathbb{H}$  into  $\mathbb{P}^1(\mathbb{C})$ , its *topological boundary* is given by  $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ .

**Proposition 2.1.1.** *Let  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  and  $\gamma \neq \pm I$  act on the Riemann sphere  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ .*

1. If  $|\operatorname{tr}(\gamma)| < 2$ , then  $\gamma$  has two fixed points in  $\mathbb{P}^1(\mathbb{C})$ : one in  $\mathbb{H}$  and its complex conjugate. Such an element is called *elliptic*.
2. If  $|\operatorname{tr}(\gamma)| > 2$ , then  $\gamma$  has two fixed points in  $\mathbb{P}^1(\mathbb{R})$  and no other fixed points in  $\mathbb{P}^1(\mathbb{C})$ . Such an element is called *hyperbolic*.
3. If  $|\operatorname{tr}(\gamma)| = 2$ , then  $\gamma$  has a single fixed point in  $\mathbb{P}^1(\mathbb{R})$  and no other fixed points in  $\mathbb{P}^1(\mathbb{C})$ . Such an element is called *parabolic*.

*Proof.* Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}(2, \mathbb{R})$  and suppose  $\gamma$  fixes some  $z \in \mathbb{P}^1(\mathbb{C})$ . Then we have

$$\gamma(z) = \frac{az + b}{cz + d} = z \quad \Rightarrow \quad cz^2 + (d - a)z - b = 0. \quad (2.1.19)$$

This is a quadratic equation in  $z$ , whose discriminant is

$$\Delta = (d - a)^2 + 4bc. \quad (2.1.20)$$

Since  $\gamma \in \operatorname{SL}(2, \mathbb{R})$ , we have  $ad - bc = 1$ , and the trace of  $\gamma$  is  $\operatorname{tr}(\gamma) = a + d$ . Then we find the relation

$$\operatorname{tr}(\gamma)^2 - 4(ad - bc) = \operatorname{tr}(\gamma)^2 - 4 = \Delta. \quad (2.1.21)$$

Now, we analyze according to the sign of  $\Delta$ :

1. If  $|\operatorname{tr}(\gamma)| < 2$ , then  $\Delta < 0$ , so the roots are complex conjugates; one lies in  $\mathbb{H}$ , and the other in the lower half-plane.
2. If  $|\operatorname{tr}(\gamma)| > 2$ , then  $\Delta > 0$ , and the roots are real, i.e., two fixed points in  $\mathbb{P}^1(\mathbb{R})$ .
3. If  $|\operatorname{tr}(\gamma)| = 2$ , then  $\Delta = 0$ , giving a single repeated real root, i.e., a unique fixed point in  $\mathbb{P}^1(\mathbb{R})$ .

□

Now we can talk about the cusps of a congruence subgroup. Informally, these are the points where a fundamental domain for a given congruence subgroup intersects the boundary of  $\mathbb{H}$ . However, this notion admits a more general and precise formulation:

**Definition 2.1.2.** Let  $\Gamma \subset \operatorname{SL}_2(\mathbb{R})$  be a subgroup acting properly discontinuously on  $\mathbb{H}$ , such that the quotient space  $\Gamma \backslash \mathbb{H}$  has finite volume. A point  $a \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$

is called a *cuspid* if there exists a non-identity parabolic element  $\gamma \in \Gamma$  such that  $\gamma(a) = a$ .

Now, we can proceed to the notion of *compactification*. If  $\Gamma$  is a congruence subgroup, let  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$  be our topological space.

To topologize  $\mathbb{H}^*$  (that is, to formally define its open sets), we begin by assigning  $\mathbb{H}$  its usual topology. Thus, a basic open neighborhood of a point  $z \in \mathbb{H}$  is simply an open disc centered at  $z$  that is entirely contained within  $\mathbb{H}$ .

For the cusp at infinity,  $z = \infty$ , the basic open neighborhoods are defined to be

$$\mathcal{N}_M = \{z \in \mathbb{H} \mid \text{Im}(z) > M\} \cup \{\infty\} \quad (2.1.22)$$

for any real number  $M > 0$ .

For a rational cusp  $c \in \mathbb{Q}$ , we know that there exists a matrix  $\gamma \in \Gamma(1)$  such that  $\gamma(\infty) = c$ . Hence, a basic open neighborhood of  $c$  is defined by translating a neighborhood of  $\infty$  via  $\gamma$ , yielding sets of the form  $\gamma(\mathcal{N}_M)$ .

The topology defined on  $\mathbb{H}^*$  ensures that the quotient space  $X(\Gamma) = \Gamma \backslash \mathbb{H}^*$  is a compact, connected, Hausdorff space.

Next, we define the complex charts. Because the natural projection  $\pi : \mathbb{H}^* \rightarrow X(\Gamma)$  behaves differently depending on the stabilizer of the point, we must define three distinct types of charts. Recall that a complex chart is a pair  $(U, \phi)$ , where  $U \subset X(\Gamma)$  and  $V \subset \mathbb{C}$  are open subsets, and  $\phi : U \rightarrow V$  is a homeomorphism.

- If  $z \in \mathbb{H}$  has a trivial stabilizer in  $\Gamma/\{\pm I\}$ , then  $\pi$  is locally a homeomorphism at  $z$ . In this case, we simply use the inverse of the projection,  $\pi^{-1}$ , to map a small open neighborhood  $U$  of  $\pi(z)$  directly to an open disc  $V \subset \mathbb{H} \subset \mathbb{C}$ .
- If  $z_0 \in \mathbb{H}$  is fixed by a finite cyclic subgroup of  $\Gamma/\{\pm I\}$  of order  $k > 1$ , then  $\pi$  fails to be a homeomorphism around  $z_0$ . To overcome this, we apply a Cayley transform to map  $z_0$  to the origin, and then take the  $k$ -th power. The local chart  $\phi$  on  $X(\Gamma)$  is induced by the map

$$\tilde{\phi}(z) = \left( \frac{z - z_0}{z - \bar{z}_0} \right)^k. \quad (2.1.23)$$

Because this function is invariant under the stabilizer of  $z_0$ , it descends to a well-defined homeomorphism  $\phi$  on a neighborhood of  $\pi(z_0)$  in  $X(\Gamma)$ .

- For the cusp at  $\infty$ , we define the local coordinate using the exponential map,

$$\tilde{\phi}(z) = e^{2\pi iz}. \quad (2.1.24)$$

By defining the image of the cusp itself to be 0, this map descends to a homeomorphism  $\phi$  mapping a neighborhood of the cusp in  $X(\Gamma)$  to an open disc around the origin in  $\mathbb{C}$ .

Consider the two elements of  $\Gamma(1)$  defined by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.1.25)$$

Their actions on elements of  $\mathbb{H}$  are given by

$$T \cdot z = z + 1, \quad S \cdot z = -\frac{1}{z}. \quad (2.1.26)$$

**Theorem 2.1.3** (3, Proposition 1.2.3). *The group  $\Gamma(1)$  is generated by the matrices  $T$  and  $S$ .*

*Proof.* Let  $\gamma \in \Gamma(1)$ . We will show that  $\gamma$  can be written as a product of  $S$ ,  $T$ , and  $T^{-1}$ .

We know that  $\mathbb{H}$  can be covered by the union of the translates  $\gamma(\mathcal{F})$  for  $\gamma \in \Gamma(1)$ , where  $\mathcal{F}$  is the fundamental domain. Since  $\mathbb{H}$  is connected, any two such translates can be connected by a finite sequence of domains where each adjacent pair shares an edge.

Hence, we can find a sequence of matrices  $\gamma_1, \dots, \gamma_n \in \Gamma(1)$  such that  $\gamma_1(\mathcal{F}) = \mathcal{F}$  (so  $\gamma_1 = \pm I$ ),  $\gamma_n(\mathcal{F}) = \gamma'(\mathcal{F})$  (so  $\gamma_n = \pm \gamma'$ ), and for each  $k$ , the domains  $\gamma_k(\mathcal{F})$  and  $\gamma_{k+1}(\mathcal{F})$  share an edge.

Observe that the only domains adjacent to  $\mathcal{F}$  are the translates by  $S$ ,  $T$ , and  $T^{-1}$ . Moreover,  $\gamma_k(\mathcal{F})$  being adjacent to  $\gamma_{k+1}(\mathcal{F})$  implies that  $\mathcal{F}$  is adjacent to  $\gamma_k^{-1}\gamma_{k+1}(\mathcal{F})$ . Hence, the matrix  $\gamma_k^{-1}\gamma_{k+1}$  must be one of  $S$ ,  $T$ , or  $T^{-1}$  (up to sign).

Therefore,

$$\pm \gamma' = \gamma_1^{-1}\gamma_n = \prod_{k=1}^{n-1} \gamma_k^{-1}\gamma_{k+1} \quad (2.1.27)$$

shows that  $\pm \gamma'$  is generated by  $S$  and  $T$ . Since  $\Gamma(1)$  contains  $-I = S^2$ , it follows that  $\gamma'$  itself is generated by  $S$  and  $T$ .  $\square$

Finally, we observe that the matrices  $I$  and  $-I$  act trivially on the upper half-

plane  $\mathbb{H}$ , since for all  $z \in \mathbb{H}$ , we have  $(-I) \cdot z = z$ . This leads us to the following definition:

**Definition 2.1.4.** The group  $\Gamma(1)/\pm I$  is called the *modular group*. It is the image of  $\mathrm{SL}(2, \mathbb{Z})$  in  $\mathrm{PSL}(2, \mathbb{R})$ .

The modular group acts on  $\mathbb{H}$  in the same way as  $\Gamma(1)$ , and it is generated by the same matrices  $S$  and  $T$ , since  $-I$  acts trivially on  $\mathbb{H}$ .

## 2.2 The Space of Modular Forms

Let  $k$  be an even nonnegative integer, and let  $\mathbb{H}$  denote the upper half-plane.

**Definition 2.2.1.** Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  be a function that is holomorphic on  $\mathbb{H}$  and satisfies  $f(z+1) = f(z)$ . Due to its periodicity,  $f$  admits a Fourier expansion (or  $q$ -expansion) of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z} = \sum_{n=-\infty}^{\infty} a_n q^n, \quad (2.2.1)$$

where  $q = e^{2\pi i z}$ . We classify the analytic behavior of  $f$  at the cusp  $\infty$  via these Fourier coefficients:

1.  $f$  is *meromorphic at  $\infty$*  if  $a_n = 0$  for all  $n \ll 0$  (i.e.,  $f(z)$  grows at most exponentially as  $\mathrm{Im}(z) \rightarrow \infty$ ).
2.  $f$  is *holomorphic at  $\infty$*  if  $a_n = 0$  for all  $n < 0$  (i.e.,  $f(z)$  remains bounded as  $\mathrm{Im}(z) \rightarrow \infty$ ).
3.  $f$  *vanishes* (or is *cuspidal*) *at  $\infty$*  if  $a_n = 0$  for all  $n \leq 0$ , i.e.,

$$\lim_{\mathrm{Im}(z) \rightarrow \infty} f(z) = 0. \quad (2.2.2)$$

**Definition 2.2.2.** Let  $k$  be an even nonnegative integer. A *modular form of weight  $k$*  for  $\mathrm{SL}(2, \mathbb{Z})$  is a holomorphic function  $f$  on  $\mathbb{H}$  that satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \quad (2.2.3)$$

and is holomorphic at  $\infty$ .

**Definition 2.2.3.** A function  $f$  is called an *automorphic function* for  $\Gamma(1)$  if

$$f\left(\frac{az+b}{cz+d}\right) = f(z) \quad (2.2.4)$$

for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , and if  $f$  is meromorphic on  $\mathbb{H}$  and at  $\infty$ .

An automorphic function is permitted to have poles, whereas a classic modular form is strictly required to be holomorphic.

For the full modular group  $\Gamma(1)$ , this holomorphicity constraint is highly restrictive due to the geometry of the quotient space. Because the compactified modular curve  $\Gamma(1) \backslash \mathbb{H}^*$  is a compact Riemann surface, the maximum modulus principle implies that any globally holomorphic automorphic function on it must be constant. Consequently, any modular form for  $\Gamma(1)$  of weight 0 is necessarily a constant function.

**Lemma 2.2.4.** *If  $f$  is a modular form, then  $f$  has expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in \mathbb{C}. \quad (2.2.5)$$

*Proof.* Let (2.2.1) be the Fourier expansion of  $f(z)$ . Writing  $z = x + iy$ , we compute

$$e^{2\pi i n z} = e^{2\pi i n(x+iy)} = e^{2\pi i n x} e^{-2\pi n y}. \quad (2.2.6)$$

Thus, the Fourier coefficients  $a_n$  can be expressed as

$$a_n = \int_0^1 e^{-2\pi i n(x+iy)} f(x+iy) dx = \int_0^1 e^{-2\pi i n x} e^{2\pi n y} f(x+iy) dx, \quad (2.2.7)$$

which gives

$$a_n e^{-2\pi n y} = \int_0^1 e^{-2\pi i n x} f(x+iy) dx. \quad (2.2.8)$$

Since  $f$  remains bounded as  $y \rightarrow \infty$ , meaning that there exists a constant  $C > 0$  such that

$$|f(x+iy)| \leq C. \quad (2.2.9)$$

Then, taking norms in equation (2.2.8), we obtain

$$|a_n e^{-2\pi n y}| = \left| \int_0^1 e^{-2\pi i n x} f(x+iy) dx \right| \leq \int_0^1 |f(x+iy)| dx \leq \int_0^1 C dx = C. \quad (2.2.10)$$

Hence,

$$|a_n e^{-2\pi n y}| \leq C. \quad (2.2.11)$$

It follows from (2.2.11) that if  $n < 0$ , the exponential factor  $e^{-2\pi n y}$  grows rapidly as  $y \rightarrow \infty$ , contradicting the bound unless  $a_n = 0$ . Therefore, we must have  $a_n = 0$  for all  $n < 0$ .  $\square$

The space of modular forms of weight  $k$  for  $\Gamma(1)$  is denoted by  $\mathcal{M}_k(\Gamma(1))$ . On the previous page, we discussed that a modular form of weight  $k$  whose constant Fourier coefficient  $a_0 = 0$  is called *cuspidal*, and such forms are known as *cusp forms* of weight  $k$ . Analogously, the space of cusp forms of weight  $k$  for  $\Gamma(1)$  is denoted by  $\mathcal{S}_k(\Gamma(1))$ .

Our next goal is to show that these spaces are finite-dimensional. To do so, we begin with the following proposition:

**Proposition 2.2.5** (3, Proposition 1.3.1). *Let  $X$  be a compact Riemann surface, and let  $P_1, \dots, P_n \in X$  be distinct points with positive integers  $r_1, \dots, r_n$ . Let  $V$  be the vector space of meromorphic functions on  $X$  that are holomorphic except possibly at the points  $P_m$ , and which are holomorphic or have poles of order at most  $r_m$  at each  $P_m$ . Then  $\dim V \leq r_1 + \dots + r_n + 1$ .*

**Theorem 2.2.6** (3, Proposition 1.3.2). *The space  $\mathcal{M}_k(\Gamma(1))$  is finite-dimensional.*

*Proof.* Let  $f_0$  be a non-zero element of  $\mathcal{M}_k(\Gamma(1))$ . Because  $f_0$  is a modular form, it has a finite number of zeroes in the fundamental domain. Let  $P_1, \dots, P_m$  be the zeroes of  $f_0$  on the compact Riemann surface  $X(\Gamma(1))$ , and let  $r_1, \dots, r_m$  be the respective orders of vanishing of  $f_0$  at these points.

If  $f \in \mathcal{M}_k(\Gamma(1))$ , the quotient  $f/f_0$  is a meromorphic function on  $X(\Gamma(1))$ . Since  $f$  is holomorphic everywhere on  $\mathbb{H}$  and at the cusps, the only possible poles of  $f/f_0$  occur at the zeroes of  $f_0$ . Furthermore, the order of the pole at any  $P_j$  is bounded by  $r_j$ .

Therefore, the mapping  $f \mapsto f/f_0$  defines an embedding from  $\mathcal{M}_k(\Gamma(1))$  into the vector space  $V$  defined in Proposition 2.2.5. It follows that  $\dim \mathcal{M}_k(\Gamma(1)) \leq r_1 + \dots + r_m + 1$ .  $\square$

**Remark 2.2.7.** Let  $f \in \mathcal{M}_k(\Gamma(1))$  and  $g \in \mathcal{M}_l(\Gamma(1))$ . Then  $fg \in \mathcal{M}_{k+l}(\Gamma(1))$ .

The dimension of space of cusp forms is obviously finite since it is a subspace of the space of modular forms, and by Theorem 2.2.6, we know that  $\mathcal{M}_k(\Gamma(1))$  is finite-dimensional.

**Corollary 2.2.8.** *The space of cusp forms, denoted by  $\mathcal{S}_k(\Gamma(1))$ , is finite-dimensional with the relation*

$$\dim \mathcal{M}_k(\Gamma(1)) = \dim \mathcal{S}_k(\Gamma(1)) + 1 \quad \text{for } k \geq 4. \quad (2.2.12)$$

This follows from the fact that if there exists a modular form with a nonvanishing constant term (such as an Eisenstein series), then we can subtract a suitable multiple of it from another modular form to eliminate that constant term. This ensures the relation in Equation (2.2.12).

Now that we have constructed the space of modular forms and discussed the properties of both modular forms and cusp forms, we can illustrate these concepts with an explicit example of a modular form of weight  $k$ .

## 2.3 Classical Eisenstein Series

Let  $k \geq 4$  be an even integer. We define

$$E_k(z) = \frac{1}{2} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} (mz + n)^{-k} \quad (2.3.1)$$

to be the *Eisenstein series* of weight  $k$ .

To prove that  $E_k(z)$  defines a holomorphic function on  $\mathbb{H}$ , we must verify that this series converges absolutely and uniformly on compact subsets of  $\mathbb{H}$ . To do so, we first require the following absolute convergence lemma:

**Lemma 2.3.1.** *The series*

$$\sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} (|m| + |n|)^{-k} \quad (2.3.2)$$

*converges for every integer  $k \geq 3$ .*

*Proof.* We analyze the series as follows:

$$\sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(|m| + |n|)^k} = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{1}{|m|^k} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \sum_{m \in \mathbb{Z}} \frac{1}{(|m| + |n|)^k}. \quad (2.3.3)$$

The first sum converges for all integers  $k \geq 2$ . It remains to show that the second sum also converges for  $k \geq 3$ .

We estimate:

$$\sum_{m \in \mathbb{Z}} \frac{1}{(|m| + |n|)^k} = \frac{1}{|n|^k} + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{1}{(|m| + |n|)^k} = \frac{1}{|n|^k} + 2 \sum_{m=1}^{\infty} \frac{1}{(m + |n|)^k} \quad (2.3.4)$$

$$\leq \frac{1}{|n|^k} + 2 \int_1^{\infty} \frac{dx}{(x + |n|)^k} = \frac{1}{|n|^k} + \frac{2}{(k-1)|n|^{k-1}}. \quad (2.3.5)$$

Therefore,

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \sum_{m \in \mathbb{Z}} \frac{1}{(|m| + |n|)^k} \leq \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left( \frac{1}{|n|^k} + \frac{2}{(k-1)|n|^{k-1}} \right), \quad (2.3.6)$$

which is convergent for  $k \geq 3$ . This completes the proof.  $\square$

We now use Lemma 2.3.1 to prove that the Eisenstein series  $E_k(z)$  converges absolutely and uniformly on compact subsets of  $\mathbb{H}$ .

Let  $K \subset \mathbb{H}$  be an arbitrary compact subset. Because  $K$  is compact, it is bounded and has a strictly positive distance from the real line. Thus, there exists  $y_0 > 0$  such that  $\text{Im}(z) \geq y_0$  for all  $z \in K$ .

**Lemma 2.3.2.** *There exists a positive constant  $C_K$ , such that the lower bound*

$$|mz + n| \geq C_K(|m| + |n|) \quad (2.3.7)$$

*holds for all  $z \in K$  and all  $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ .*

*Proof.* To construct this bound, let  $S^1 = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$  denote the unit circle in  $\mathbb{R}^2$ . Consider the function  $\Psi : K \times S^1 \rightarrow \mathbb{R}$  defined by

$$\Psi(z, u, v) = |uz + v|. \quad (2.3.8)$$

The function  $\Psi$  is continuous on the product space  $K \times S^1$ . Because  $K$  and  $S^1$  are both closed and bounded,  $K \times S^1$  is compact. Furthermore,  $\Psi(z, u, v)$  is strictly positive on this domain, hence must attain a strictly positive absolute minimum value. Let this minimum be  $c_0 > 0$ , so that

$$|uz + v| \geq c_0 \quad \text{for all } z \in K \text{ and } (u, v) \in S^1. \quad (2.3.9)$$

For any pair  $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ , we can substitute  $u = \frac{m}{\sqrt{m^2 + n^2}}$  and  $v = \frac{n}{\sqrt{m^2 + n^2}}$

into this inequality to obtain

$$\left| \frac{m}{\sqrt{m^2 + n^2}}z + \frac{n}{\sqrt{m^2 + n^2}} \right| \geq c_0 \implies |mz + n| \geq c_0 \sqrt{m^2 + n^2}. \quad (2.3.10)$$

Using the standard norm inequality  $\sqrt{m^2 + n^2} \geq \frac{1}{2}(|m| + |n|)$ , we get

$$|mz + n| \geq \frac{c_0}{2}(|m| + |n|). \quad (2.3.11)$$

Setting  $C_K = \frac{c_0}{2}$  establishes the lower bound in Equation (2.3.7).  $\square$

Taking the reciprocal and raising both sides to the power of  $k$  yields

$$\frac{1}{|mz + n|^k} \leq \frac{1}{C_K^k} \frac{1}{(|m| + |n|)^k} \quad (2.3.12)$$

for all  $z \in K$ . Because  $\sum(|m| + |n|)^{-k}$  converges by Lemma 2.3.1, the Weierstrass  $M$ -test implies that the series defining  $E_k(z)$  converges absolutely and uniformly on the compact set  $K$ . Since  $K$  was an arbitrary compact subset of  $\mathbb{H}$ ,  $E_k(z)$  is holomorphic on the entire upper half-plane  $\mathbb{H}$ .

Next, let us verify that  $E_k(z)$  is indeed a modular form of weight  $k$  for  $\Gamma(1)$  by showing the transformation property.

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ , and consider the transformation  $z \mapsto \gamma z = \frac{az+b}{cz+d}$ . We compute:

$$E_k(\gamma z) = (cz + d)^k \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} (m(az + b) + n(cz + d))^{-k} \quad (2.3.13)$$

$$= (cz + d)^k \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} ((ma + nc)z + (mb + nd))^{-k} \quad (2.3.14)$$

$$= (cz + d)^k E_k(z). \quad (2.3.15)$$

The last equality follows from the fact that  $\gcd(c, d) = 1$  (since  $ad - bc = 1$ ), which ensures the set  $\{(m', n') = (ma + nc, mb + nd)\}$  runs over all of  $\mathbb{Z}^2 \setminus \{(0, 0)\}$  as  $(m, n)$  does. This confirms that  $E_k(z)$  transforms as a modular form of weight  $k$  under the action of  $\Gamma(1)$ .

The next step is to show that the Eisenstein series is analytic at infinity. To observe this, we compute its Fourier expansion. We begin with the following lemma:

**Lemma 2.3.3.** *Let  $z \in \mathbb{H}$  and let  $k \geq 2$  be an integer. Then the following identity holds:*

$$\sum_{n=-\infty}^{\infty} (n - z)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}. \quad (2.3.16)$$

*Proof.* Define

$$f(u) = (u - z)^{-k} \quad (2.3.17)$$

for  $u \in \mathbb{R}$ . Let

$$\hat{f}(v) = \int_{-\infty}^{\infty} (u - z)^{-k} e^{2\pi i u v} du \quad (2.3.18)$$

be the Fourier transform of  $f$ .

First, observe that if  $v = 0$ , then

$$\lim_{R \rightarrow \infty} \int_{-R}^R (u - z)^{-k} du = \lim_{R \rightarrow \infty} \left[ \frac{1}{(1-k)(u-z)^{k-1}} \right]_{-R}^R = 0. \quad (2.3.19)$$

We use the residue theorem to calculate this integral for  $v \neq 0$ . Let

$$g(u) = (u - z)^{-k} e^{2\pi i u v} \quad (2.3.20)$$

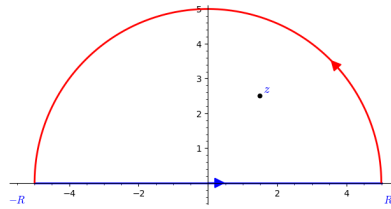
where  $v \in \mathbb{R}$ ,  $u \in \mathbb{C}$ . Observe that  $g$  has an isolated singularity at  $u = z$ .

Take a semicircular contour of radius  $R$  centered at the origin.

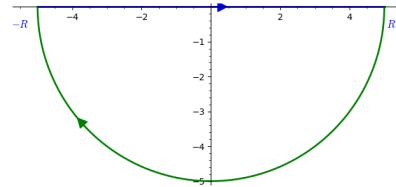
Then by the Cauchy Residue Theorem, we have:

$$\int_C g(u) = 2\pi i \cdot \text{res}((u - z)^{-k} e^{2\pi i u v})|_{u=z}. \quad (2.3.21)$$

Now, we analyze the behavior of the integrals depending on the sign of  $v$ .



(a) Upper half-plane contour



(b) Lower half-plane contour

Figure 2.2: Semicircular Contours

For the lower half-plane contour, the integral over the whole contour is 0. It is the sum of the integral over the arc and over the straight line. Using the  $M$ - $L$  estimate on the arc:

$$M = \sup_{\substack{|u|=R \\ \text{Im}u < 0}} |(u-z)^{-k} e^{2\pi i uv}| \ll R^{-k} \cdot \sup_{\substack{|u|=R \\ \text{Im}u < 0}} |e^{2\pi i uv}| \ll R^{-k} e^{-2\pi v \text{Im}u} \quad (2.3.22)$$

If  $v > 0$ ,  $M$  might not be bounded on the arc (e.g. at  $u = -iR$ ). So let  $v < 0$ .

Then

$$R^{-k} e^{-2\pi v \text{Im}u} = R^{-k} e^{-c} \leq R^{-k} \quad (2.3.23)$$

and the arc length is  $L = \pi R$ , so:

$$\lim_{R \rightarrow \infty} ML = \pi \lim_{R \rightarrow \infty} R^{1-k} = 0, \quad \text{since } k \geq 2. \quad (2.3.24)$$

Therefore, the integral over the green arc is zero, and hence the straight line integral must also be zero. Similarly, for  $v > 0$ , we do the same for the upper half-plane contour and obtain the same result.

Thus, we find:

$$\hat{f}(v) = \begin{cases} 2\pi i \cdot \text{res}((u-z)^{-k} e^{2\pi i uv})|_{u=z} & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases} \quad (2.3.25)$$

To compute the residue, we expand  $e^{2\pi i(u-z)v}$  as a Taylor series at  $u = z$ :

$$e^{2\pi i zv} \left( (u-z)^{-k} + \dots + \frac{(2\pi i v)^{k-1}}{(k-1)!} (u-z)^{-1} + \dots \right) \quad (2.3.26)$$

so the coefficient of  $(u-z)^{-1}$  gives the residue:

$$\hat{f}(v) = \begin{cases} \frac{(2\pi i)^k}{(k-1)!} v^{k-1} e^{2\pi i v z} & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases} \quad (2.3.27)$$

Finally, by the Poisson summation formula, we obtain:

$$\sum_{n=-\infty}^{\infty} (n-z)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}. \quad (2.3.28)$$

□

Now we can compute the Fourier coefficients of  $E_k(z)$ . First thing to observe is

if we let  $m = 0$ , then we see that

$$\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n^{-k} = \sum_{n=1}^{\infty} n^{-k} = \zeta(k). \quad (2.3.29)$$

Hence we can write  $E_k(z)$  as

$$E_k(z) = \zeta(k) + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (mz + n)^{-k}. \quad (2.3.30)$$

By replacing  $z$  with  $mz$  in Lemma 2.3.3, we get

$$\sum_{n=-\infty}^{\infty} (n - mz)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i m n z}. \quad (2.3.31)$$

Since  $k$  is even and  $n$  runs over all integers, the summations over  $mz + n$  and  $-mz + n$  yield the same values. Then, if we also include the summation over  $m$ , we get

$$\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} (n - mz)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i m n z}. \quad (2.3.32)$$

Let  $N = mn$ . Then we write

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i m n z} = \sum_{N=1}^{\infty} e^{2\pi i N z} \sum_{n|N} n^{k-1}. \quad (2.3.33)$$

By defining the *divisor sum* for any  $r \in \mathbb{C}$  to be

$$\sigma_r(n) = \sum_{d|n} d^r, \quad (2.3.34)$$

we see that

$$E_k(z) = \zeta(k) + \frac{(2\pi)^k (-1)^{k/2}}{(k-1)!} \sum_{N=1}^{\infty} \sigma_{k-1}(N) q^N, \quad \text{where } q = e^{2\pi i z}. \quad (2.3.35)$$

With this, we conclude that Eisenstein series are analytic at infinity. Hence  $E_k(z)$  is a modular form of weight  $k$  for  $\Gamma(1)$ .

Define

$$G_k(z) = \zeta(k)^{-1} E_k(z). \quad (2.3.36)$$

It is known that the Fourier coefficients of  $G_k(z)$  are rational numbers (see 17, Chap-

ter VII). This would imply that

$$\frac{(2\pi)^k(-1)^{k/2}}{\zeta(k)(k-1)!} \quad (2.3.37)$$

is rational.

This can be seen more clearly by recalling Euler's explicit formula relating the Riemann zeta function to the *Bernoulli numbers* (see 17, Chapter VII):

$$B_k = -\frac{\zeta(k)k!}{2^{k-1}\pi^k(-1)^{k/2}}. \quad (2.3.38)$$

It is a classical fact that  $B_k = 0$  when  $k$  is odd and greater than one, and that  $B_k \in \mathbb{Q}$  for all  $k$ . Substituting this identity into our coefficient factor yields the normalized Eisenstein series  $G_k(z)$ , whose Fourier coefficients are given by

$$G_k(z) = 1 - \frac{2k}{B_k} \sum_{N=1}^{\infty} \sigma_{k-1}(N)q^N. \quad (2.3.39)$$

## 2.4 Topological Invariants

To fully understand the structure of the space of modular forms and the of modular functions, we must view them by studying the compactified modular curve  $X(\Gamma)$ . The following theorem establishes the topological condition needed to show that a modular function (such as the  $j$ -invariant, which we will introduce later) induces an isomorphism between  $X(\Gamma)$  and the Riemann sphere.

**Theorem 2.4.1.** *Let  $M$  be a compact Riemann surface, and let  $f : M \rightarrow \mathbb{C}$  be a meromorphic function. Assume that  $f$  has only one pole, at  $m \in M$ , which is simple. Extend  $f$  to a mapping  $M \rightarrow \mathbb{P}^1(\mathbb{C})$  by setting  $f(m) = \infty$ . Then  $f$  is a biholomorphism of Riemann surfaces between  $M$  and  $\mathbb{P}^1(\mathbb{C})$ .*

To prove this theorem, we will make use of another elegant result from complex analysis:

**Theorem 2.4.2** (2, Theorem 10.8). *Suppose  $\gamma$  is a regular closed curve. If  $f$  is meromorphic inside and on  $\gamma$ , and has no zeros or poles on  $\gamma$ , and if  $\mathbf{Z}$  is the number of zeros and  $\mathbf{P}$  the number of poles of  $f$  inside  $\gamma$  (counted with multiplicity), then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} = \mathbf{Z} - \mathbf{P}. \quad (2.4.1)$$

*Proof of Theorem 2.4.1.* We begin with injectivity. Observe that  $M$  can be viewed as a union of open subsets  $\mathcal{F}$  with a boundary. Since  $f$  is meromorphic, and the boundary of  $\mathcal{F}$  is a closed curve, we can apply Theorem 2.4.2 to count the difference between the number of zeros and poles of  $f$ .

Topologically,  $M$  is obtained by gluing paired edges of the boundaries of  $\mathcal{F}$  together, so the integral in (2.4.1) vanishes. Hence, the number of poles of  $f$  equals the number of zeros. Since  $f$  has a single pole, it must have a single zero.

Suppose now that  $f(z_1) = f(z_2) = a$  for some  $z_1 \neq z_2$ . Consider the function  $f - a$ , which still has only one zero, contradicting our assumption. Thus,  $z_1 = z_2$ , and  $f$  is injective.

For surjectivity, suppose there exists  $\alpha \in \mathbb{P}^1(\mathbb{C})$  with  $\alpha \notin \text{Im}(f)$ . Consider the function  $g = \frac{1}{f-\alpha}$ , which is holomorphic on  $M$  since  $f - \alpha$  never vanishes. But  $g$  is then a holomorphic function on a compact Riemann surface, so it must be constant. This implies that  $f$  is constant, contradicting the assumption that  $f$  has a pole. Hence,  $f$  is surjective.

Finally, define  $h = f^{-1}$ . Then

$$\frac{d}{dz}[f \circ h(z)] = f'(h(z))h'(z) = 1 \quad \Rightarrow \quad h'(z) = \frac{1}{f'(h(z))}, \quad (2.4.2)$$

showing that  $h$  is meromorphic. Therefore,  $f$  is an isomorphism of Riemann surfaces.  $\square$

Let us introduce two very important modular forms:

$$G_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad G_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n. \quad (2.4.3)$$

We can immediately see that both  $G_4(z)^3$  and  $G_6(z)^2$  are modular forms of weight 12 via Remark 2.2.7, hence we can subtract them to obtain a new modular form  $G_4(z)^3 - G_6(z)^2$ .

One can easily notice that the Fourier expansion of this form has no constant term. Moreover,

$$\frac{1}{1728}(G_4^3 - G_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots \quad (2.4.4)$$

defines a cusp form of weight 12, called the *discriminant modular form* and denoted by  $\Delta(z)$ .

Define the function  $j : \mathbb{H} \rightarrow \mathbb{C}$  by

$$j(z) = \frac{G_4(z)^3}{\Delta(z)}. \quad (2.4.5)$$

The first thing to observe is that  $j$  is an automorphic function on  $\Gamma(1)$ , because both  $G_4$  and  $\Delta$  are modular forms for  $\Gamma(1)$ , and their weights cancel out (since  $G_4^3$  has weight 12 and so does  $\Delta$ ). Thus,  $j(z)$  is invariant under the action of  $\Gamma(1)$  and is meromorphic on  $\mathbb{H}$ .

Moreover, the  $j$ -function has the Fourier expansion

$$j(z) = \frac{1}{q} + 744 + 196884q + \dots \quad (2.4.6)$$

which clearly shows that  $j$  has a pole at infinity in  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} \cup \{\infty\}$ .

As Theorem 2.4.1 states,  $j$  is a bijection of the compactified space

$$\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{C}). \quad (2.4.7)$$

**Definition 2.4.3** ([7]). Let  $\Gamma$  be a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ . Topologically, the compact Riemann surface  $X(\Gamma) = \Gamma \backslash \mathbb{H}^*$  is a sphere with  $g$  handles for some  $g \in \mathbb{N}$ . This  $g$  is called the *genus* of  $X(\Gamma)$ .

From a more algebraic point of view, the genus of  $X(\Gamma)$  can be determined by computing the half-rank of the first homology group:

$$H_1(X(\Gamma), \mathbb{Z}) \cong \mathbb{Z}^{2g}. \quad (2.4.8)$$

Moreover, the second homology group satisfies

$$H_2(X(\Gamma), \mathbb{Z}) \cong \mathbb{Z}. \quad (2.4.9)$$

This topological invariant can be useful when comparing two compact Riemann surfaces via a holomorphic map, as it provides a way to relate their homological and geometric structure.

If  $f : X \rightarrow Y$  is a holomorphic mapping between compact Riemann surfaces, then the *topological degree* of  $f$  is defined to be the positive integer  $n$  such that the map

$$\mathbb{Z} \cong H_2(X, \mathbb{Z}) \rightarrow H_2(Y, \mathbb{Z}) \cong \mathbb{Z} \quad (2.4.10)$$

induced by  $f$  is multiplication by  $n$ .

Furthermore, if we define  $F_X$  and  $F_Y$  to be the fields of meromorphic functions on  $X$  and  $Y$  respectively, then  $f$  induces an injection  $F_Y \hookrightarrow F_X$ , and we have  $[F_X : F_Y] = n$ . If we focus on a point  $y \in Y$ , the fiber  $f^{-1}(y)$  generally consists of  $n$  points. However, if this number is strictly less than  $n$ , we say that  $y$  is *ramified*.

Let  $y \in Y$  be a point such that  $f^{-1}(y) = \{x_1, \dots, x_r\}$ . Let  $e_i$  be the number of points in  $f^{-1}(\eta)$  near  $x_i$  for a non-ramified point  $\eta \in Y$  close to  $y$ . Then  $e_i$  is called the *ramification index* of  $x_i$  over  $y$ , and if  $e_i > 1$ , we say that  $x_i$  is *ramified*. Clearly, we have

$$\sum_{i=1}^r e_i = n, \quad (2.4.11)$$

and if  $e_i = 1$ , then  $x_i$  is said to be unramified. We may denote this by  $e_i = e(x_i | y)$ .

The following remark describes how the genera of two compact Riemann surfaces are related via ramification data.

**Remark 2.4.4** (Hurwitz genus formula). Let  $g_X$  and  $g_Y$  be the genera of  $X$  and  $Y$ , respectively. Let  $n$  be the topological degree of  $f$ , and let  $P_1, \dots, P_u$  be the (finitely many) ramification points of  $f$  in  $X$ , with  $Q_i = f(P_i)$  their images in  $Y$ . Then the genus formula asserts that

$$2g_X - 2 = n(2g_Y - 2) + \sum_{i=1}^u (e(P_i | Q_i) - 1). \quad (2.4.12)$$

Before we see the Hurwitz formula in action, we first need a formula to compute the degree  $n$  of the covering. Let us consider the canonical mapping

$$\Gamma(N) \backslash \mathbb{H}^* \longrightarrow \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^*, \quad (2.4.13)$$

where  $\Gamma(N)$  is as defined in (2.1.10).

Observe that for  $\tau \in \mathbb{H}^*$ , the map

$$\Gamma(N) \cdot \tau \longmapsto \mathrm{SL}(2, \mathbb{Z}) \cdot \tau \quad (2.4.14)$$

$$[\tau] \longmapsto [\tau] \quad (2.4.15)$$

simply sends the  $\Gamma(N)$ -orbit of  $\tau$  to its  $\mathrm{SL}(2, \mathbb{Z})$ -orbit. That is, it is the natural projection between the quotient spaces.

The degree  $n$  of this covering will be equal to the number of  $\Gamma(N)$ -cosets in  $\mathrm{SL}(2, \mathbb{Z})$ , which is given by the index

$$n = [\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N)]. \quad (2.4.16)$$

Since  $\Gamma(N)$  is the kernel of the mod  $N$  reduction homomorphism (see (2.1.11)), computing this index becomes manageable via a finite group computation. This is equivalent to computing the cardinality of  $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , which is given by

$$N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right). \quad (2.4.17)$$

When  $N = 2$ , we have  $I \equiv -I$ , so this formula applies directly. However, for  $N > 2$ , the matrices  $I$  and  $-I$  are no longer congruent modulo  $N$ , so we must divide the result by 2.

To see how this formula works in practice, let us consider  $\Gamma(2)$ , one of the simplest congruence subgroups. Using the degree formula, we find:

$$n = 2^3 \prod_{p|2} \left(1 - \frac{1}{p^2}\right) = 2^3 \left(1 - \frac{1}{2^2}\right) = 8 \cdot \frac{3}{4} = 6. \quad (2.4.18)$$

Since the genus of the compactified quotient  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^*$  is zero, the Hurwitz genus formula becomes:

$$-2 = 6(2g - 2) + \sum_{i=1}^u (e(P_i | Q_i) - 1). \quad (2.4.19)$$

To compute the right-hand side, we must identify the ramified points in  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^*$ , which are the cusp  $\infty$  and the elliptic points  $i$  and  $e^{2\pi i/3}$ .

We begin with the cusp  $\infty$ . Under the quotient map

$$\Gamma(2) \cdot \infty \longrightarrow \mathrm{SL}(2, \mathbb{Z}) \cdot \infty = \mathbb{Q} \cup \{\infty\}, \quad (2.4.20)$$

the preimages of  $\infty$  correspond to different  $\Gamma(2)$ -equivalence classes in  $\mathbb{Q} \cup \{\infty\}$ .

Because  $\Gamma(2)$  consists of matrices satisfying  $a \equiv d \equiv 1 \pmod{2}$  and  $b \equiv c \equiv 0 \pmod{2}$ , we find:

$$\Gamma(2) \cdot \infty = \left\{ \frac{a}{c} : a \text{ odd}, c \text{ even} \right\}. \quad (2.4.21)$$

Using 2-adic valuation, we observe that any element of this set satisfies  $v_2(a) - v_2(c) < 0$ , so the valuation is negative.

Similarly, the other equivalence classes can be described as:

$$\Gamma(2) \cdot 0 = \left\{ \frac{b}{d} : b \text{ even}, d \text{ odd} \right\} \implies v_2 > 0, \quad (2.4.22)$$

$$\Gamma(2) \cdot 1 = \left\{ \frac{a+b}{c+d} : a+b \text{ and } c+d \text{ odd} \right\} \implies v_2 = 0. \quad (2.4.23)$$

Thus, there are three distinct points in  $\Gamma(2) \backslash \mathbb{H}^*$  lying over the cusp  $\infty$ .

Since the total ramification index above  $\infty$  must add up to the degree  $n = 6$ , and there are three such points, we conclude:

$$\sum_{i=1}^3 (e(P_i | \infty) - 1) = 6 - 3 = 3. \quad (2.4.24)$$

When it comes to the elliptic points, what we are looking for are the matrix classes in  $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$  that fix the point under Möbius action. In particular, for the elliptic point  $i$ , we are interested in matrices satisfying

$$\frac{ai+b}{ci+d} \equiv i \pmod{2}. \quad (2.4.25)$$

Recall that all elements of  $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$  can be represented by the following six matrices:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}. \quad (2.4.26)$$

Among these, the only matrices that fix the point  $i$  are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.4.27)$$

Therefore, the stabilizer of  $i$  in  $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$  has order 2, and the number of distinct points lying above  $i$  is  $6/2 = 3$ .

Hence, the contribution of the ramification over  $i$  is:

$$\sum_{i=1}^3 (e(P_i | i) - 1) = 6 - 3 = 3. \quad (2.4.28)$$

Finally, let  $\omega = e^{2\pi i/3}$  so that  $\omega^3 = 1$  and  $\omega \neq 1$ . We are now interested in matrices in  $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$  that fix  $\omega$  under Möbius action modulo 2:

$$\frac{a\omega+b}{c\omega+d} \equiv \omega \pmod{2}. \quad (2.4.29)$$

Among the six coset representatives listed previously, the ones that stabilize  $\omega$  modulo 2 are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \quad (2.4.30)$$

Hence, the stabilizer of  $\omega$  in  $\mathrm{SL}(2, \mathbb{Z})/\Gamma(2)$  has order 3, and the number of distinct points lying above  $\omega$  is  $6/3 = 2$ . Therefore, the ramification contribution over  $\omega$  is

$$\sum_{i=1}^2 (e(P_i | \omega) - 1) = 6 - 2 = 4. \quad (2.4.31)$$

Returning to the Hurwitz genus formula, we have:

$$-2 = 12g - 12 + 3 + 3 + 4 = 12g - 2 \implies g = 0. \quad (2.4.32)$$

Hence, the genus of  $\Gamma(2) \backslash \mathbb{H}^*$  is 0.

# 3

## The Adelic Framework and Automorphic Forms

In this chapter, we introduce the algebraic framework required to compute the Fourier coefficients of the non-holomorphic Eisenstein series, a computation we will carry out in the final section. This machinery is necessary because, as we will see in the third section, lifting functions from the classical upper half-plane  $\mathbb{H}$  to algebraic groups allows us to study them locally. Through the local-to-global principle, we can then synthesize these local pieces to obtain global results.

### 3.1 Places

We begin by establishing the base fields for our local computations.

**Definition 3.1.1** ([5]). A *set of places* is a subset

$$S \subset \{p : \text{prime number}\} \cup \{\infty\}. \quad (3.1.1)$$

We will denote an individual place by  $v$ .

We perform our local computations over local fields  $\mathbb{Q}_v$ , which are the completions of  $\mathbb{Q}$  at the places  $v$ . The reason for this is that computing Fourier coefficients requires integrating functions over these spaces. Integration relies on limits and measure theory, which are only well-defined if the underlying field is complete with respect to the chosen metric. Since  $\mathbb{Q}$  is not complete with respect to the metric induced by the absolute value  $|\cdot|_v$  for  $v \leq \infty$  (5, Proposition 4.2.2), we must pass to its completions.

The completion at the infinite place  $v = \infty$  is the familiar completion of  $\mathbb{Q}$  with respect to the standard Euclidean absolute value:

$$|x|_\infty = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases} \quad (3.1.2)$$

This field is denoted by  $\mathbb{Q}_\infty$ , or more commonly,  $\mathbb{R}$ .

**Definition 3.1.2.** For a fixed prime  $p$ , every rational number  $r \neq 0$  can be written as

$$r = p^k \frac{m}{n}, \quad (3.1.3)$$

where  $m, n \in \mathbb{Z}$  are coprime to  $p$ . The number  $k$  is uniquely determined by  $r$ . We define the  $p$ -adic absolute value by

$$|r|_p = \left| p^k \frac{m}{n} \right|_p := p^{-k}. \quad (3.1.4)$$

We complete this definition by

$$|0|_p := 0. \quad (3.1.5)$$

Omitting the details of the formal completion process, the completion of  $\mathbb{Q}$  with respect to the  $p$ -adic metric is the field of  $p$ -adic numbers, which can be explicitly represented as the set of formal Laurent series in  $p$ :

$$\mathbb{Q}_p = \left\{ \sum_{j=-N}^{\infty} a_j p^j : N \in \mathbb{N}, 0 \leq a_j < p \right\}. \quad (3.1.6)$$

We conclude this section by stating a foundational result that will be essential for our later computations.

**Proposition 3.1.3** (5, Proposition 4.1.4). *For every  $x \in \mathbb{Q}^\times$ , we have the product formula*

$$\prod_{v \leq \infty} |x|_v = 1. \quad (3.1.7)$$

## 3.2 Strong Approximation

The product of  $\mathbb{R}$  and the restricted product over all finite places gives us an algebraic structure called the “adele ring.”

**Definition 3.2.1.** The ring of *adeles* is defined as

$$\mathbb{A} = \mathbb{R} \times \widehat{\prod_{p < \infty} \mathbb{Q}_p} \quad (3.2.1)$$

$$= \{(x_\infty, x_2, x_3, \dots) \mid x_\infty \in \mathbb{R}, x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}. \quad (3.2.2)$$

As we discussed in the previous section,  $\mathbb{Q}_p$  is the  $p$ -adic completion of  $\mathbb{Q}$ , and  $\mathbb{Z}_p$  can be defined as

$$\mathbb{Z}_p = \{x_p \in \mathbb{Q}_p : |x_p|_p \leq 1\}. \quad (3.2.3)$$

Observe that if  $x \in \mathbb{Q}$ , then  $x \in \mathbb{Z}_p$  for almost all  $p$ .

It is easily seen that the set of adeles naturally admits a ring structure.

**Theorem 3.2.2.** *The adèle ring can be written as*

$$\mathbb{A} = \mathbb{R} + \mathbb{Q}^\Delta + \widehat{\mathbb{Z}} \quad (3.2.4)$$

where  $\mathbb{Q}^\Delta$  is the diagonal embedding of the rationals into  $\mathbb{A}$  and

$$\widehat{\mathbb{Z}} = \prod_{p < \infty} \mathbb{Z}_p. \quad (3.2.5)$$

*Proof.* It is clear that  $\mathbb{A} \supset \mathbb{R} + \mathbb{Q}^\Delta + \widehat{\mathbb{Z}}$ . Hence, we only need to show that  $\mathbb{A} \subset \mathbb{R} + \mathbb{Q}^\Delta + \widehat{\mathbb{Z}}$ .

Let  $a = (a_\infty, a_2, a_3, \dots)$  be an element of  $\mathbb{A}$ . Let

$$S = \{p_i : a_{p_i} \notin \mathbb{Z}_{p_i}\}. \quad (3.2.6)$$

Our claim is that we can find  $q \in \mathbb{Q}$  such that  $a_p - q \in \mathbb{Z}_p$  for all  $p$ . Suppose for a moment that the only prime in  $S$  is 2. We know that  $a_2$  can be written as

$$a_2 = \sum_{k=-N_2}^{\infty} b_{2,k} 2^k, \quad (3.2.7)$$

where  $N_2 \in \mathbb{Z}^{>0}$  and  $b_{2,k} \in \{0, 1\}$ . By setting  $q = \sum_{k=-N_2}^{-1} b_{2,k} 2^k$ , we obtain  $a_2 - q \in \mathbb{Z}_2$ . Since the denominators of  $q$  are strictly powers of 2, we have  $|q|_p \leq 1$  for  $p \neq 2$ , ensuring  $a_p - q \in \mathbb{Z}_p$  continues to hold for all other primes  $p$ . Observe that a general

$q$  can be found for an arbitrary finite set  $S$  by summing the principal parts:

$$q = \sum_{p_i \in S} \left( \sum_{k=-N_{p_i}}^{-1} b_{p_i, k} p_i^k \right). \quad (3.2.8)$$

Having constructed  $q$  for the finite places, we turn to the infinite place, where we can always find an  $r \in \mathbb{R}$  such that  $r + q = a_\infty$ . Hence, the equality holds.  $\square$

This is a very useful theorem, known as the *strong approximation* of the ring of adèles.

**Definition 3.2.3.** The group  $\mathrm{GL}(2, \mathbb{A})$  is a restricted direct product

$$\mathrm{GL}(2, \mathbb{A}) = \mathrm{GL}(2, \mathbb{R}) \times \widehat{\prod_{p < \infty} \mathrm{GL}(2, \mathbb{Q}_p)} \quad (3.2.9)$$

$$= \{(A_\infty, A_2, A_3, \dots) \mid A_p \in \mathrm{GL}(2, \mathbb{Z}_p) \text{ for all but finitely many } p\}. \quad (3.2.10)$$

We see that strong approximation applies to  $\mathrm{GL}(2, \mathbb{A})$  as follows:

**Theorem 3.2.4.** *The group  $\mathrm{GL}(2, \mathbb{A})$  can be written as follows:*

$$\mathrm{GL}(2, \mathbb{A}) = \mathrm{GL}(2, \mathbb{R}) \cdot \mathrm{GL}(2, \mathbb{Q})^\Delta \cdot \hat{K} \quad (3.2.11)$$

where  $\mathrm{GL}(2, \mathbb{Q})^\Delta$  is the diagonal embedding of  $\mathrm{GL}(2, \mathbb{Q})$  into  $\mathrm{GL}(2, \mathbb{A})$  and

$$\hat{K} = \prod_{p < \infty} \mathrm{GL}(2, \mathbb{Z}_p). \quad (3.2.12)$$

*Proof.* We will follow a similar constructive approach to that used in the proof of Theorem 3.2.2. It is clear that the product is a subset of  $\mathrm{GL}(2, \mathbb{A})$ . Let  $A = (A_\infty, A_2, A_3, \dots) \in \mathrm{GL}(2, \mathbb{A})$ . We will construct  $Q \in \mathrm{GL}(2, \mathbb{Q})$  such that  $(QX_2, QX_3, \dots)$  where  $X_p \in \mathrm{GL}(2, \mathbb{Z}_p)$  will equal  $(A_2, A_3, \dots)$  for the finite places. We require  $QX_p = A_p$ , which implies that  $X_p = Q^{-1}A_p \in \mathrm{GL}(2, \mathbb{Z}_p)$ . If  $A_p \in \mathrm{GL}(2, \mathbb{Z}_p)$  for all  $p$ , then there is nothing further to show. Let

$$S = \{p_i : A_{p_i} \notin \mathrm{GL}(2, \mathbb{Z}_{p_i})\}, \quad (3.2.13)$$

which is, by definition, a finite set. Our claim is that we can find  $Q \in \mathrm{GL}(2, \mathbb{Q})$  such that  $Q^{-1}A_p \in \mathrm{GL}(2, \mathbb{Z}_p)$  for all  $p$ . To see how the algorithm works, suppose 2 is the only prime in  $S$ . We may assume that  $A_2$  is upper triangular by applying the

Iwasawa decomposition. By setting  $Q^{-1}$  as an upper triangular rational matrix, we have

$$\begin{pmatrix} q_1 & q_2 \\ 0 & q_4 \end{pmatrix} \cdot \begin{pmatrix} b_{2,1} & b_{2,2} \\ 0 & b_{2,3} \end{pmatrix} = \begin{pmatrix} q_1 b_{2,1} & q_1 b_{2,2} + q_2 b_{2,3} \\ 0 & q_4 b_{2,3} \end{pmatrix}. \quad (3.2.14)$$

To ensure this product is in  $\mathrm{GL}(2, \mathbb{Z}_2)$ , the matrix must have entries in  $\mathbb{Z}_2$  and its determinant must be a unit in  $\mathbb{Z}_2^\times$ . We can choose  $q_1 = 2^{-\mathrm{ord}_2(b_{2,1})}$  and  $q_4 = 2^{-\mathrm{ord}_2(b_{2,3})}$ . This forces the diagonal entries  $q_1 b_{2,1}$  and  $q_4 b_{2,3}$  to be 2-adic units. Then, we can choose a rational  $q_2$  (whose denominator is strictly a power of 2) to clear the fractional part of  $q_1 b_{2,2} + q_2 b_{2,3}$ , guaranteeing the upper-right entry lies in  $\mathbb{Z}_2$ . Because  $Q^{-1}$  relies only on powers of 2 in its denominators, multiplying by  $Q^{-1}$  preserves the condition  $Q^{-1}A_p \in \mathrm{GL}(2, \mathbb{Z}_p)$  for all  $p \neq 2$ .

Observe that a general  $Q^{-1}$  can be found by taking the product of all elementary  $Q_{p_i}^{-1}$  matrices that arise from systematically clearing the denominators of each  $A_{p_i}$  for  $p_i \in S$ .

Similar to the one-dimensional case, we can always find an  $R \in \mathrm{GL}(2, \mathbb{R})$  such that  $RQ = A_\infty$  since  $Q$  is invertible. Hence, the equality holds.  $\square$

### 3.3 Lie Algebras and Group Decompositions

In this section, our primary goal is to establish the decompositions of the local groups  $G_v = \mathrm{GL}(2, \mathbb{Q}_v)$ . These decompositions, specifically the Iwasawa and Bruhat decompositions, are the foundational building blocks for analyzing automorphic forms and defining local Whittaker functions.

**Proposition 3.3.1.** *Let  $G_v = \mathrm{GL}(2, \mathbb{Q}_v)$ ,  $P_v$  be the standard parabolic subgroup, which is the collection of upper-triangular matrices*

$$P_v = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Q}_v \right\}, \quad (3.3.1)$$

*and  $K_v$  be a maximal compact subgroup of  $G_v$ . Then  $G_v$  admits the Iwasawa decomposition*

$$G_v = P_v \cdot K_v. \quad (3.3.2)$$

If  $v = \infty$ , we will prove that  $K_\infty = \mathrm{O}(2)$ .

**Theorem 3.3.2.**  *$\mathrm{O}(2)$  is a maximal compact subgroup of  $\mathrm{GL}(2, \mathbb{R})$ .*

*Proof.* We first show that  $\mathrm{O}(2)$  is compact.

Consider the continuous map

$$m : \text{GL}(2, \mathbb{R}) \rightarrow M(2, \mathbb{R}), \quad A \mapsto AA^T. \quad (3.3.3)$$

By definition,

$$\text{O}(2) = m^{-1}(\{I\}), \quad (3.3.4)$$

which is the preimage of the closed set  $\{I\}$ . Hence  $\text{O}(2)$  is closed in  $\text{GL}(2, \mathbb{R})$ .

Next, if  $A \in \text{O}(2)$ , then  $AA^T = I$  implies that each row (and column) has a Euclidean norm of 1. Thus, the absolute value of every entry is at most 1, meaning  $\text{O}(2)$  is bounded.

Being both closed and bounded in the finite-dimensional space  $M(2, \mathbb{R})$ ,  $\text{O}(2)$  is compact.

Now, to show maximality, suppose  $K$  is a compact subgroup of  $\text{GL}(2, \mathbb{R})$  such that

$$\text{O}(2) \subset K \subset \text{GL}(2, \mathbb{R}). \quad (3.3.5)$$

Let  $A \in K$ . By the polar decomposition (see [11]), we can uniquely write

$$A = R \cdot e^X, \quad (3.3.6)$$

where  $R \in \text{O}(2)$  and  $X$  is a real symmetric matrix.

Set  $Y = e^X$ . Then  $Y$  is positive definite, so all its eigenvalues are strictly positive. Let  $PJP^{-1}$  be its Jordan canonical form. If an eigenvalue  $\lambda$  satisfies  $\lambda > 1$ , then  $Y^N = PJ^NP^{-1}$  becomes unbounded as  $N \rightarrow \infty$ , contradicting the compactness of  $K$ . If  $\lambda < 1$ , then  $Y^N$  tends to 0, which again contradicts compactness (since  $0 \notin \text{GL}(2, \mathbb{R})$ ). Therefore, all eigenvalues must be exactly 1.

Moreover, if  $J$  were not diagonal, then the superdiagonal entries of  $J^N$  would grow without bound, again contradicting compactness. Hence  $J = I$ , so  $Y = I$ .

Consequently,  $A = R \in \text{O}(2)$ , and thus  $K = \text{O}(2)$ . We conclude that  $\text{O}(2)$  is a maximal compact subgroup of  $\text{GL}(2, \mathbb{R})$ .  $\square$

Next, we proceed to analyze the decomposition  $G_\infty = P_\infty K_\infty$  in greater detail. It is clear that  $G_\infty \supset P_\infty K_\infty$ . To show equality holds, all we need to prove is  $G_\infty \subset P_\infty K_\infty$ .

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty. \quad (3.3.7)$$

Then we can factorize  $g$  into the product of a matrix  $p \in P_\infty$  and  $r \in K_\infty$  of the forms

$$\begin{pmatrix} \frac{\det(g)}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{-c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}, \quad (3.3.8)$$

respectively, which proves  $g \in P_\infty \cdot K_\infty$ .

Next, we investigate the structure of  $K_v$  in the  $p$ -adic case. To simplify the analysis, let us first consider the multiplicative group  $\mathbb{Q}_p^\times$ . Our claim is that  $\mathbb{Z}_p^\times$  is the maximal compact subgroup of  $\mathbb{Q}_p^\times$ . Since  $\mathbb{Q}_p$  is complete with respect to the  $p$ -adic metric, the closed unit ball

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \quad (3.3.9)$$

is bounded and therefore compact. Since  $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\}$  is a closed subset of the compact space  $\mathbb{Z}_p$ , it is itself compact. Hence,  $\mathbb{Z}_p^\times$  is a compact subgroup of  $\mathbb{Q}_p^\times$ .

Maximality follows immediately: the norm of powers of any element  $x \notin \mathbb{Z}_p^\times$  (meaning  $|x|_p \neq 1$ ) would be unbounded in either the positive or negative direction, contradicting compactness. Thus,  $\mathbb{Z}_p^\times$  is the maximal compact subgroup of  $\mathbb{Q}_p^\times$ .

For the general case, we will show that  $\text{GL}(n, \mathbb{Z}_p)$  is a maximal compact subgroup of  $\text{GL}(n, \mathbb{Q}_p)$ . Establishing compactness is relatively straightforward.

**Theorem 3.3.3.** *The group  $\text{GL}(n, \mathbb{Z}_p)$  is compact in  $\text{GL}(n, \mathbb{Q}_p)$ .*

*Proof.* First, observe that  $M(n, \mathbb{Z}_p)$  is compact in  $M(n, \mathbb{Q}_p)$  since it is the unit ball with respect to the sup-norm in the standard basis of  $M(n, \mathbb{Q}_p)$ . The determinant map

$$\det : M(n, \mathbb{Z}_p) \rightarrow \mathbb{Z}_p \quad (3.3.10)$$

is continuous. Therefore, the preimage  $\det^{-1}(\mathbb{Z}_p^\times) = \text{GL}(n, \mathbb{Z}_p)$  is a closed subset of a compact space. Hence,  $\text{GL}(n, \mathbb{Z}_p)$  is compact in  $M(n, \mathbb{Q}_p)$ , and consequently compact in  $\text{GL}(n, \mathbb{Q}_p)$ .  $\square$

Let us introduce some useful notions before discussing maximality.

**Definition 3.3.4.** A *lattice* in  $\mathbb{Q}_p^n$  is the  $\mathbb{Z}_p$ -span of a basis of  $\mathbb{Q}_p^n$ .

What we are genuinely interested in is the set of invertible matrices that fix a lattice.

**Definition 3.3.5.** For each lattice  $L$  in  $\mathbb{Q}_p^n$ , we define the stabilizer subgroup

$$K_L = \{g \in \mathrm{GL}(n, \mathbb{Q}_p) : g(L) = L\}. \quad (3.3.11)$$

The quintessential example of such a set is  $\mathrm{GL}(n, \mathbb{Z}_p)$ , which is the collection of matrices that fix the standard lattice  $\mathbb{Z}_p^n$ .

**Lemma 3.3.6** (4, Theorem 3.1). *Let  $H$  be a compact subgroup of  $\mathrm{GL}(n, \mathbb{Q}_p)$ . Then there exists a lattice  $M$  in  $\mathbb{Q}_p^n$  such that  $H \subset K_M$ .*

**Lemma 3.3.7** (4, Theorem 3.3). *Let  $L$  and  $L'$  be two lattices in  $\mathbb{Q}_p^n$  and suppose  $K_L \subset K_{L'}$ . Then there exists  $\lambda \in \mathbb{Q}_p^\times$  such that  $L = \lambda L'$  and  $K_L = K_{L'}$ .*

Now, we are ready to prove maximality.

**Theorem 3.3.8.** *The group  $\mathrm{GL}(n, \mathbb{Z}_p)$  is a maximal compact subgroup of  $\mathrm{GL}(n, \mathbb{Q}_p)$ .*

*Proof.* Suppose that  $\mathrm{GL}(n, \mathbb{Z}_p) \subset H$ , where  $H$  is a compact subgroup of  $\mathrm{GL}(n, \mathbb{Q}_p)$ . Then there exists a lattice  $M \subset \mathbb{Q}_p^n$  such that  $H \subset K_M$ . Since  $K_{\mathbb{Z}_p^n} = \mathrm{GL}(n, \mathbb{Z}_p)$ , we have the chain of inclusions

$$K_{\mathbb{Z}_p^n} \subset H \subset K_M. \quad (3.3.12)$$

By Lemma 3.3.7, the inclusion  $K_{\mathbb{Z}_p^n} \subset K_M$  forces  $K_M = K_{\mathbb{Z}_p^n}$ , and hence  $H = \mathrm{GL}(n, \mathbb{Z}_p)$ .  $\square$

It follows from the above that  $\mathrm{GL}(2, \mathbb{Z}_p)$  is a maximal compact subgroup of  $\mathrm{GL}(2, \mathbb{Q}_p)$ . Therefore, we set  $K_p = \mathrm{GL}(2, \mathbb{Z}_p)$  for finite places.

Similar to the infinite case, it is clear that  $P_p \cdot K_p \subset G_p$ . To show the reverse inclusion, let  $g_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$ . If  $c = 0$ , then  $g_p \in P_p$  and we are done. If  $c \neq 0$ , we have two cases based on the  $p$ -adic valuations: either  $|c/d|_v \leq 1$  (meaning  $c/d \in \mathbb{Z}_p$ ), or  $|d/c|_v < 1$  (meaning  $d/c \in \mathbb{Z}_p$ ).

If  $|c|_p \leq |d|_p$ , we can factorize  $g_p$  as follows:

$$g_p = \begin{pmatrix} a - \frac{bc}{d} & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}, \quad (3.3.13)$$

where the first matrix is in  $P_p$  and the second is in  $K_p$  since  $c/d \in \mathbb{Z}_p$ .

On the other hand, if  $|c|_p > |d|_p$ , we can factorize  $g_p$  as:

$$g_p = \begin{pmatrix} c & a \\ 0 & -\frac{\det(g_p)}{c} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{d}{c} \end{pmatrix}. \quad (3.3.14)$$

The first matrix is upper triangular, hence in  $P_p$ . Because  $|d/c|_p < 1$ , the second matrix has entries in  $\mathbb{Z}_p$  and its determinant is  $-1 \in \mathbb{Z}_p^\times$ , ensuring it belongs to  $K_p$ . This proves  $G_p \subset P_p \cdot K_p$ .

Next, we shift our focus to the Bruhat decomposition.

**Proposition 3.3.9** (8, 4.1). *The Bruhat decomposition of  $G_v = \mathrm{GL}(2, \mathbb{Q}_v)$  is*

$$G_v = P_v \sqcup P_v w N_v, \quad \text{where } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.3.15)$$

with  $P_v$  the parabolic subgroup of  $G_v$  and  $N_v$  its unipotent radical, which is the collection of matrices of the form

$$N_v = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_v \right\}. \quad (3.3.16)$$

We call  $P_v$  the *little cell*, which consists of matrices with  $c = 0$ , and we call  $P_v w N_v$  the *big cell*, consisting of matrices of the form

$$P_v w N_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Q}_v) \mid c \neq 0 \right\}. \quad (3.3.17)$$

Our next goal is to explicitly express the coset space  $P_v \backslash G_v$  using the Bruhat decomposition.

We claim that

$$P_v \backslash G_v = P_v \backslash P_v \cup P_v \backslash (P_v w N_v) \cong \{1\} \cup w \cdot ((w^{-1} P_v w \cap N_v) \backslash N_v) \cong \{1\} \cup w N_v, \quad (3.3.18)$$

which asserts that  $w N_v \cong P_v \backslash (P_v w N_v)$ . To formally establish this, we must construct an isomorphism between these two sets.

Observe that we have a natural sequence of maps

$$w N_v \xrightarrow{j} P_v w N_v \xrightarrow{p} P_v \backslash (P_v w N_v) \quad (3.3.19)$$

where the first map,  $j$ , is the canonical inclusion (which is clearly injective), and the second map,  $p$ , is the canonical projection (which is surjective). We want to show that the composition  $p \circ j$  is a bijection.

Because  $P_v w N_v$  is exactly the orbit of  $w N_v$  under the left action of  $P_v$ , every coset in  $P_v \backslash (P_v w N_v)$  contains a representative from  $w N_v$ . Thus,  $p \circ j$  is surjective.

To show injectivity, suppose  $n, n' \in N_v$  such that  $p(j(wn)) = p(j(wn'))$ . This means they lie in the same left  $P_v$ -coset:

$$\begin{aligned} P_v wn' = P_v wn &\iff wn'(wn)^{-1} \in P_v \\ &\iff wn'n^{-1}w^{-1} \in P_v \\ &\iff n'n^{-1} \in w^{-1}P_v w. \end{aligned}$$

Let  $n'n^{-1} = h$ . Since  $N_v$  is a subgroup,  $h \in N_v$ . This implies that

$$h \in w^{-1}P_v w \cap N_v. \quad (3.3.20)$$

However, the intersection  $w^{-1}P_v w \cap N_v$  consists only of the identity matrix, because  $w^{-1}P_v w$  contains only lower-triangular matrices, while  $N_v$  contains strictly upper-triangular unipotent matrices. Hence,  $h = I$ , which forces  $n' = n$ . This shows injectivity. Therefore,  $p \circ j$  is a bijection, and the claim holds.

### 3.4 From Domains to Groups

We now shift our focus from domains to groups. This perspective allows us to utilize powerful representation-theoretic tools. Let  $f$  be a function on  $\mathbb{H}$ . By the standard isomorphism  $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R}) \cong \mathbb{H}$ , we can transfer this function to the group  $\mathrm{SL}(2, \mathbb{R})$  via the map  $g\mathrm{SO}(2, \mathbb{R}) \mapsto g \cdot i$  and define a function  $F$  as

$$F(g) = f(g \cdot i), \quad g \in \mathrm{SL}(2, \mathbb{R}). \quad (3.4.1)$$

The function  $F$  is right  $\mathrm{SO}(2, \mathbb{R})$ -invariant, since  $\mathrm{SO}(2, \mathbb{R})$  is the isotropy subgroup of  $i$ . Conversely, any right  $\mathrm{SO}(2, \mathbb{R})$ -invariant function on  $\mathrm{SL}(2, \mathbb{R})$  naturally descends to a function on  $\mathbb{H}$ .

In the first chapter, we defined holomorphic modular forms of weight  $k$  via Equation (2.2.3), which can also be rewritten as

$$f\left(\frac{az+b}{cz+d}\right) = j(\gamma, z)^k f(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \quad (3.4.2)$$

where  $j(\gamma, z) = cz + d$  is the *factor of automorphy*. It satisfies the essential cocycle condition

$$j(\alpha\beta, z) = j(\alpha, \beta z)j(\beta, z). \quad (3.4.3)$$

Given a modular form  $f : \mathbb{H} \rightarrow \mathbb{C}$  of weight  $k$ , we define a new complex-valued function  $\phi_f$  on  $\mathrm{SL}(2, \mathbb{R})$  by

$$\phi_f(g) = j(g, i)^{-k} f(g \cdot i), \quad (3.4.4)$$

where  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ . This lifted function satisfies the following four fundamental properties (see 9, Chapter 2):

- (i)  $\phi_f(\gamma g) = \phi_f(g)$  for all  $\gamma \in \Gamma(1)$ .

This left-invariance property follows directly from the cocycle condition:

$$\begin{aligned} \phi_f(\gamma g) &= j(\gamma g, i)^{-k} f(\gamma g \cdot i) \\ &= j(\gamma, gi)^{-k} j(g, i)^{-k} j(\gamma, gi)^k f(g \cdot i) \\ &= j(g, i)^{-k} f(g \cdot i) = \phi_f(g). \end{aligned}$$

- (ii)  $\phi_f(gr(\theta)) = e^{-ik\theta} \phi_f(g)$  for all  $r(\theta) \in K$ .

Here,  $K$  denotes the maximal compact subgroup  $\mathrm{SO}(2, \mathbb{R})$  of the Lie group  $\mathrm{SL}(2, \mathbb{R})$ , and  $r(\theta)$  is the rotation matrix

$$r(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (3.4.5)$$

which fixes the point  $i \in \mathbb{H}$ . When  $r(\theta)$  acts on  $g$  from the right, we evaluate the factor of automorphy at  $i$ :

$$\begin{aligned} \phi_f(gr(\theta)) &= j(gr(\theta), i)^{-k} f(gr(\theta) \cdot i) \\ &= j(g, r(\theta)i)^{-k} j(r(\theta), i)^{-k} f(g \cdot i) \\ &= e^{-ik\theta} j(g, i)^{-k} f(g \cdot i) = e^{-ik\theta} \phi_f(g). \end{aligned}$$

- (iii)  $\phi_f(g)$  is bounded if  $f \in \mathcal{S}_k(\Gamma)$ . In particular, its  $L^2$ -norm is finite:

$$\int_{\Gamma(1) \backslash G} |\phi_f(g)|^2 dg < \infty, \quad (3.4.6)$$

where  $G = \mathrm{SL}(2, \mathbb{R})$ .

To justify this property, we invoke the following classical lemma.

**Lemma 3.4.1** (9, Lemma 1.5). *Suppose  $f \in \mathcal{M}_k(\Gamma)$ . Then  $f \in \mathcal{S}_k(\Gamma)$  if and only if  $y^{k/2}|f(x+iy)| < M$  for some constant  $M$  independent of  $x$ .*

*Proof.* Define the function

$$h(z) = \text{Im}(z)^{k/2}|f(z)|, \quad (3.4.7)$$

which is clearly  $\Gamma$ -invariant. Thus, it suffices to show that  $h(z) < M$  if and only if  $f$  vanishes at every cusp  $s$  of  $\Gamma$ .

Let  $\sigma \in \Gamma(1)$  be an element that maps  $\infty$  to  $s$ . Then,

$$h(\sigma z) = \text{Im}(\sigma z)^{k/2}|f(\sigma z)| = \text{Im}(z)^{k/2}|j(\sigma, z)|^{-k}|f(\sigma z)|. \quad (3.4.8)$$

This shows that if  $f \in \mathcal{S}_k(\Gamma)$ , then  $h(w) \rightarrow 0$  as  $w \rightarrow s$  in the topology of the extended upper half-plane  $\mathbb{H}^*$ . Hence,  $h$  extends to a continuous function on the compact space  $\Gamma \backslash \mathbb{H}^*$ , and thus must be bounded by some  $M$ .

Conversely, if  $h(z) < M$  is bounded globally, then  $f$  must vanish at all cusps to prevent the imaginary part of the expansion from blowing up.  $\square$

Because  $h(g \cdot i) = |\phi_f(g)|$ , it immediately follows that  $\phi_f(g)$  is bounded over the fundamental domain, guaranteeing a finite  $L^2$ -norm.

(iv)  $\phi_f(g)$  is cuspidal; i.e., for all  $g \in G$  and  $\sigma \in \Gamma(1)$ ,

$$\int_0^1 \phi_f \left( \sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g \right) dx = 0, \quad (3.4.9)$$

where  $h \in \mathbb{Z}$ .

To prove this property, we parameterize  $g$  using the Iwasawa decomposition of  $\text{SL}(2, \mathbb{R})$ . Observe that since

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (3.4.10)$$

we have the straightforward action  $g \cdot i = x + iy = z$ . Hence, the integral

becomes

$$\int_0^1 \phi_f \left( \sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g \right) dx = \int_0^1 j \left( \sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g, i \right)^{-k} f \left( \sigma \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g \cdot i \right) dx \quad (3.4.11)$$

$$= \int_0^1 j \left( \sigma, \begin{pmatrix} 1 & xh \\ 0 & 1 \end{pmatrix} g \cdot i \right)^{-k} f(\sigma(z + xh)) dx \quad (3.4.12)$$

$$= \int_0^1 (f|_k \sigma)(z + xh) dx. \quad (3.4.13)$$

This final expression is exactly the zeroth Fourier coefficient of  $f$  at the cusp  $s = \sigma(\infty)$ . Since all such constant terms vanish by definition, and hence the integral is zero.

To fully appreciate the significance of the  $L^2$ -norm described in property (iii), we must briefly introduce the natural inner product on the space of classical cusp forms, known as the *Petersson inner product*. For two modular forms  $f_1, f_2 \in S_k(\Gamma)$ , it is defined as

$$\langle f_1, f_2 \rangle := \iint_{\mathcal{F}} f_1(z) \overline{f_2(z)} y^k \frac{dx dy}{y^2}, \quad (3.4.14)$$

where  $\mathcal{F}$  is the fundamental domain of  $\Gamma \backslash \mathbb{H}$ . This integral equips the space  $S_k(\Gamma)$  with a geometric structure, allowing us to define orthogonality: two cusp forms are called orthogonal if their Petersson inner product vanishes, meaning  $\langle f_1, f_2 \rangle = 0$ .

Now, let us compute the general  $L^2$ -inner product of two lifted functions  $\phi_{f_1}$  and  $\phi_{f_2}$  over the group quotient  $\Gamma(1) \backslash G$ . Because the lifted functions  $\phi_f$  are left- $\Gamma(1)$ -invariant, their pointwise product  $\phi_{f_1}(g) \overline{\phi_{f_2}(g)}$  is a well-defined function on the quotient  $\Gamma(1) \backslash G$ . Furthermore, since  $G = \mathrm{SL}(2, \mathbb{R})$  is unimodular, its Haar measure  $dg$  is bi-invariant. The left-invariance of  $dg$  ensures it induces a well-defined quotient measure on  $\Gamma(1) \backslash G$ .

Using the  $(x, y, \theta)$  parameterization stemming from the Iwasawa decomposition of  $g \in G$ , the normalized Haar measure on  $G = \mathrm{SL}(2, \mathbb{R})$  is given by  $dg = \frac{1}{2\pi} \frac{dx dy}{y^2} d\theta$ . We can evaluate the integral over the quotient by integrating over a fundamental domain  $\mathcal{F}_G$  for  $\Gamma(1)$  in  $G$ . Since  $\Gamma(1)$  acts solely on the upper half-plane component, we can identify  $\mathcal{F}_G$  with  $\mathcal{F} \times [0, 2\pi)$ , where  $\mathcal{F}$  is the standard fundamental domain for  $\Gamma(1) \backslash \mathbb{H}$ .

Expanding the integral yields:

$$\int_{\Gamma(1)\backslash G} \phi_{f_1}(g) \overline{\phi_{f_2}(g)} dg = \int_{\mathcal{F}_G} \phi_{f_1}(g) \overline{\phi_{f_2}(g)} dg \quad (3.4.15)$$

$$= \frac{1}{2\pi} \iint_{\mathcal{F}} \int_0^{2\pi} (f_1(z) y^{k/2} e^{-ik\theta}) \overline{(f_2(z) y^{k/2} e^{-ik\theta})} \frac{dx dy}{y^2} d\theta \quad (3.4.16)$$

$$= \frac{1}{2\pi} \iint_{\mathcal{F}} \int_0^{2\pi} f_1(z) \overline{f_2(z)} y^k \frac{dx dy}{y^2} d\theta \quad (3.4.17)$$

$$= \iint_{\mathcal{F}} f_1(z) \overline{f_2(z)} y^k \frac{dx dy}{y^2} \quad (3.4.18)$$

$$= \langle f_1, f_2 \rangle. \quad (3.4.19)$$

This evaluation is highly significant: it demonstrates that the lift  $f \mapsto \phi_f$  is an isometric embedding from the space of cusp forms  $S_k(\Gamma)$  to the space of square-integrable functions  $L^2(\Gamma(1)\backslash G)$ . This provides the crucial theoretical justification for studying modular forms via representation theory, as it guarantees that the Petersson inner product is preserved. Consequently, orthogonal modular forms on the upper-half plane map directly to orthogonal functions on the group.

Finally, we extend these functions from  $\mathrm{SL}(2, \mathbb{R})$  to  $\mathrm{GL}(2, \mathbb{R})^+$ , the group of real matrices with positive determinant, which also preserves the upper half-plane. Observe that the center

$$Z_{\mathbb{R}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}^\times \right\} \quad (3.4.20)$$

trivially stabilizes  $i$ , and we already know  $\mathrm{SO}(2, \mathbb{R})$  is the isotropy subgroup within  $\mathrm{SL}(2, \mathbb{R})$ . Hence, we deduce the extended isomorphism

$$\mathrm{GL}(2, \mathbb{R})^+ / Z_{\mathbb{R}}^+ \mathrm{SO}(2, \mathbb{R}) \cong \mathbb{H}. \quad (3.4.21)$$

Thus, we can naturally extend our lift to

$$F(g) = f(g \cdot i), \quad g \in \mathrm{GL}(2, \mathbb{R})^+, \quad (3.4.22)$$

which is left  $\mathrm{GL}(2, \mathbb{Z})^+$ -invariant, right  $\mathrm{SO}(2, \mathbb{R})$ -invariant, and  $Z_{\mathbb{R}}$ -invariant. To ensure the lift continues to satisfy the cocycle condition for matrices with arbitrary positive determinants, we introduce a normalizing adjustment to the factor of auto-

morphism:

$$j(g, z) = (cz + d)^k (\det g)^{-k/2}. \quad (3.4.23)$$

With this modification, the analogous representation-theoretic results for  $\mathrm{GL}(2, \mathbb{R})^+$  follow directly.

### 3.5 Non-holomorphic Eisenstein Series

We define the Eisenstein series on  $\mathbb{H}$  as the sum

$$E_k(s, z) = \sum_{\substack{c, d \in \mathbb{Z} \\ \gcd(c, d) = 1}} \frac{y^s}{(cz + d)^k |cz + d|^{2s}}, \quad \text{for } z \in \mathbb{H} \text{ and } \Re(s) > 1. \quad (3.5.1)$$

By invoking Equation (2.1.5), we can rewrite the Eisenstein series over the cosets  $\Gamma_\infty \backslash \Gamma$  as

$$E_k(s, z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{(\mathrm{Im}(\gamma z))^s}{j(\gamma, z)^k}, \quad (3.5.2)$$

where  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ ,  $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d$ , and  $\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma \right\}$ .

Observe that we can extend  $E_k(s, z)$  to both the upper and lower half-planes by substituting  $\mathrm{Im}(\gamma z)$  with  $|\mathrm{Im}(\gamma z)|$ , which yields

$$E_k(s, z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{|\mathrm{Im}(\gamma z)|^s}{j(\gamma, z)^k}, \quad \text{for } z \in \mathbb{H} \cup \overline{\mathbb{H}}, \quad (3.5.3)$$

where  $\Gamma = \mathrm{GL}(2, \mathbb{Z})$  and  $\Gamma_\infty$  is defined as above.

Then, using (3.4.22), we can lift  $E_k(s, z)$  from  $\mathbb{H} \cup \overline{\mathbb{H}}$  to  $\mathrm{GL}(2, \mathbb{R})$  by setting

$$E_k^{\mathrm{group}}(g) = E_k(g \cdot i) |\det g|^{\frac{k}{2}} j(g, i)^{-k}, \quad \text{for } g \in \mathrm{GL}(2, \mathbb{R}). \quad (3.5.4)$$

To study the most general picture, let us define the following function via the Iwasawa decomposition  $G_v = P_v K_v$  for  $v \leq \infty$ . For  $g_v = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbf{k}$ , we define

$$\varphi_v \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbf{k} \right) = \begin{cases} \left| \frac{a}{d} \right|_v^s, & \text{for } v < \infty, \\ \left| \frac{a}{d} \right|_\infty^s e^{ik\theta}, & \text{for } v = \infty, \end{cases} \quad (3.5.5)$$

where

$$\mathbf{k} \in K_v = \begin{cases} \mathrm{GL}(2, \mathbb{Z}_v), & \text{for } v < \infty, \\ \mathrm{SO}(2), & \text{for } v = \infty, \text{ with } k = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \end{cases} \quad (3.5.6)$$

Clearly,  $a, d \in \mathbb{Q}_v^\times$  and  $b \in \mathbb{Q}_v$  in all cases.

For simplicity, we will take  $k = 0$  to compute the Fourier coefficients of the Eisenstein series. By writing  $E_s(z) = E_0(s, z)$  and invoking Equation (2.1.5), we can rewrite the Eisenstein series over the cosets  $\Gamma_\infty \backslash \Gamma$  as

$$E_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}(\gamma z))^s, \quad (3.5.7)$$

where  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$  and  $\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma \right\}$ .

If we define a global function  $\varphi$  on the adèle group  $G_\mathbb{A} = \mathrm{GL}(2, \mathbb{A})$  at the outset by evaluating the product over all places of the local functions above, then for  $g = (g_v) \in G_\mathbb{A}$ , we define:

$$\varphi(g) = \prod_v \varphi_v(g_v). \quad (3.5.8)$$

This infinite product is well-defined because for any adèle  $g \in G_\mathbb{A}$ , its components  $g_v$  belong to the maximal compact subgroup  $K_v = \mathrm{GL}(2, \mathbb{Z}_v)$  for almost all finite places  $v$ . By our local normalizations,  $\varphi_v(k_v) = 1$  for all  $k_v \in K_v$ . Thus, the product evaluates to 1 for almost all places, reducing it to a well-defined finite product.

Note that  $\varphi$  is ultimately a function on the adèle group  $G_\mathbb{A} = \mathrm{GL}(2, \mathbb{A})$ . By the product formula, we observe that  $\varphi$  is left  $P_\mathbb{Q}$ -invariant. Since  $A, D \in \mathbb{Q}^\times$ , their global adelic norm is 1, so:

$$\varphi \left( \begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} k \right) = \prod_{v \leq \infty} \left| \frac{Aa}{Dd} \right|_v^s = \left| \frac{A}{D} \right|_\mathbb{A}^s \left| \frac{a}{d} \right|_\mathbb{A}^s = \left| \frac{a}{d} \right|_\mathbb{A}^s = \varphi \left( \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} k \right). \quad (3.5.9)$$

We will show that the following proposition holds:

**Proposition 3.5.1.** *Let  $\varphi$  be the function on  $\mathrm{GL}(2, \mathbb{A})$  defined above. Then we have*

$$E_s(z) = \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot (g_\infty, 1, 1, \dots)), \quad (3.5.10)$$

where  $z = x + iy \in \mathbb{H}$ , and  $g_\infty \in \mathrm{GL}(2, \mathbb{R})$  is chosen such that  $g_\infty \cdot i = z$  (for example,  $g_\infty = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ ). Here,  $(g_\infty, 1, 1, \dots)$  denotes the natural embedding of the real matrix into the adèle group  $G_{\mathbb{A}}$ .

Moreover, we can extend this series globally to the group  $G_{\mathbb{A}} = \mathrm{GL}(2, \mathbb{A})$  as

$$E_s^{\mathrm{adele}}(g) := \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma \cdot g). \quad (3.5.11)$$

(We note that this series converges absolutely in the right half-plane  $\mathrm{Re}(s) > 1$ , a standard property of Eisenstein series, which justifies this definition).

This expression defines a function that is both left  $G_{\mathbb{Q}}$ -invariant and  $Z_{\mathbb{A}}$ -invariant, where

$$Z_{\mathbb{A}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{A}^\times \right\}. \quad (3.5.12)$$

There are three parts that need to be proved to establish Proposition 3.5.1: Equation (3.5.10) gives  $\sum_{\gamma \in P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}} |\mathrm{Im}(\gamma g_\infty \cdot i)|^s$ , there is a bijection between the cosets  $P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}$  and  $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$ , where  $G_{\mathbb{Z}} = \mathrm{GL}(2, \mathbb{Z})$ , and the summation over  $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$  is well-defined. The last case is obvious by Equation 3.5.9.

1. To establish the bijection between the cosets  $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$  and  $P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}$ , we use the geometric fact that each line in  $\mathbb{Q}^2$  can be uniquely represented by a *primitive*  $\mathbb{Z}$ -line in  $\mathbb{Z}^2$ . Specifically, if  $(x, y) \in \mathbb{Z}^2$  is a primitive vector with  $\mathrm{gcd}(x, y) = 1$ , then the set

$$L = \{(kx, ky) \mid k \in \mathbb{Z}\} \quad (3.5.13)$$

defines the associated primitive  $\mathbb{Z}$ -line. Using this, we will show that the canonical map

$$\begin{aligned} P_{\mathbb{Z}} \backslash G_{\mathbb{Z}} &\rightarrow P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}, \\ P_{\mathbb{Z}} g &\mapsto P_{\mathbb{Q}} g \end{aligned} \quad (3.5.14)$$

is surjective.

First, observe that  $G_{\mathbb{Q}}$  acts transitively on the set of lines in  $\mathbb{Q}^2$ . Moreover,  $P_{\mathbb{Q}}$  is exactly the stabilizer of the base line  $\{(0 \ *)\}$ . Therefore, there is a natural bijection between the cosets  $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$  and the set of lines in  $\mathbb{Q}^2$ .

Let  $l$  be a line in  $\mathbb{Q}^2$ . Then  $l$  is the  $\mathbb{Q}$ -span of some vector  $(x \ y) \in \mathbb{Z}^2$  with  $\mathrm{gcd}(x, y) = 1$ . The intersection of this line with  $\mathbb{Z}^2$  gives the  $\mathbb{Z}$ -span of  $(x \ y)$ , which is a free rank-one  $\mathbb{Z}$ -module generated by the primitive vector  $(x \ y)$ .

Note that  $\mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{GL}(2, \mathbb{Z})$  is already transitive on primitive  $\mathbb{Z}$ -lines. Indeed, there exist integers  $b, d \in \mathbb{Z}$  such that

$$bx + dy = \gcd(x, y) = 1, \quad (3.5.15)$$

and therefore

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} y & b \\ -x & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \quad (3.5.16)$$

The same transitivity naturally holds for  $\mathrm{GL}(2, \mathbb{Z})$ .

Since

$$P_{\mathbb{Z}} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{Z}^{\times}, b \in \mathbb{Z} \right\} \quad (3.5.17)$$

stabilizes the  $\mathbb{Z}$ -line spanned by  $\begin{pmatrix} 0 & 1 \end{pmatrix}$ , we conclude that the map (3.5.14) is surjective.

Injectivity follows immediately from the intersection property  $P_{\mathbb{Z}} = G_{\mathbb{Z}} \cap P_{\mathbb{Q}}$ . Hence, the bijection holds.

2. Observe that there is a natural injective map

$$j_v : \mathrm{GL}(2, \mathbb{Q}) \rightarrow \mathrm{GL}(2, \mathbb{Q}_v), \quad (3.5.18)$$

and thus we can define the natural diagonal embedding  $j$ , which identifies  $\mathrm{GL}(2, \mathbb{Q})$  with its diagonal image  $\mathrm{GL}(2, \mathbb{Q})^{\Delta}$ :

$$j = \prod_v j_v : \mathrm{GL}(2, \mathbb{Q}) \xrightarrow{\sim} \mathrm{GL}(2, \mathbb{Q})^{\Delta} \subset \prod_v \mathrm{GL}(2, \mathbb{Q}_v). \quad (3.5.19)$$

We can therefore evaluate  $\varphi(\gamma \cdot g_{\infty})$  component-wise as

$$\varphi(\gamma \cdot g_{\infty}) = \varphi_{\infty}(j_{\infty}(\gamma) \cdot g_{\infty}) \cdot \prod_{v < \infty} \varphi_v(j_v(\gamma)). \quad (3.5.20)$$

Let us start with the infinite part. We want to show that

$$\varphi_{\infty}(g_{\infty}) = |\mathrm{Im}(g_{\infty} \cdot i)|^s. \quad (3.5.21)$$

We know that  $\varphi_{\infty}$  is determined by the ratio of the diagonal entries of  $p_{\infty} \in P_{\infty}$  arising from the Iwasawa decomposition  $G_{\infty} = P_{\infty} \cdot K_{\infty}$ . If  $\det(g_{\infty}) = 1$ , we

obtain

$$\varphi_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varphi_\infty \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & * \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} = \left| \frac{1}{c^2+d^2} \right|^s, \quad (3.5.22)$$

which is exactly  $|\mathrm{Im}(g_\infty \cdot i)|^s$ .

On the other hand,

$$\varphi_v(j_v(\gamma)) = \varphi_v(\gamma) = 1, \quad (3.5.23)$$

since  $\gamma \in \mathrm{GL}(2, \mathbb{Q})$  is mapped to  $\mathrm{GL}(2, \mathbb{Z}_v)$  at all finite places  $v$ . Hence, the product over the finite places is equal to 1.

Therefore, we obtain

$$\sum_{\gamma \in P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}} \varphi(\gamma \cdot g_\infty) = \sum_{\gamma \in P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}} \varphi_\infty(\gamma \cdot g_\infty) \cdot 1 = \sum_{\gamma \in P_{\mathbb{Z}} \backslash G_{\mathbb{Z}}} |\mathrm{Im}(\gamma g_\infty \cdot i)|^s, \quad (3.5.24)$$

as claimed.

With this, we demonstrated that Proposition 3.5.1 holds.

To evaluate the global Eisenstein series  $E_s(g_\infty \cdot i)$  explicitly, we must sum over the coset space  $P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$ . By invoking the Bruhat decomposition,  $G_{\mathbb{Q}} = P_{\mathbb{Q}} \sqcup P_{\mathbb{Q}} w N_{\mathbb{Q}}$ , the sum splits over two orbits:

$$E_s(g_\infty \cdot i) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma \cdot g_\infty) = \varphi(g_\infty) + \sum_{\gamma \in w N_{\mathbb{Q}}} \varphi(\gamma \cdot g_\infty). \quad (3.5.25)$$

Since  $g_\infty \cdot i = x + iy$ , we can write the contribution from the little cell explicitly as

$$\varphi((g_\infty, 1, 1, \dots)) = \varphi_\infty(g_\infty) \cdot \prod_{v < \infty} \varphi_v(1) = |y|_\infty^s. \quad (3.5.26)$$

For the contribution from the big cell, we have

$$\sum_{\gamma \in w N_{\mathbb{Q}}} \varphi(\gamma \cdot g_\infty) = \sum_{\gamma \in w N_{\mathbb{Q}}} \left( \varphi_\infty(\gamma \cdot g_\infty) \cdot \prod_{v < \infty} \varphi_v(\gamma) \right). \quad (3.5.27)$$

Let us begin with the archimedean factor. Since  $\varphi_v$  is right  $K_v$ -invariant for all

places, we evaluate the action of  $\gamma = w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  with  $t \in \mathbb{Q}$  to obtain

$$\varphi_\infty \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \varphi_\infty \left( \begin{pmatrix} 0 & -1 \\ y & x+t \end{pmatrix} \right). \quad (3.5.28)$$

Multiplying on the right by a suitable matrix in  $O(2)$  gives

$$\begin{pmatrix} 0 & -1 \\ y & x+t \end{pmatrix} \begin{pmatrix} \frac{x+t}{\sqrt{(x+t)^2+y^2}} & \frac{y}{\sqrt{(x+t)^2+y^2}} \\ \frac{-y}{\sqrt{(x+t)^2+y^2}} & \frac{x+t}{\sqrt{(x+t)^2+y^2}} \end{pmatrix} = \begin{pmatrix} \frac{y}{\sqrt{(x+t)^2+y^2}} & * \\ 0 & \sqrt{(x+t)^2+y^2} \end{pmatrix}. \quad (3.5.29)$$

Hence,

$$\varphi_\infty \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \left| \frac{y}{(x+t)^2+y^2} \right|_\infty^s. \quad (3.5.30)$$

The non-archimedean factor is computed similarly, accounting for valuation sizes.

We have

$$\varphi_v \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \varphi_v \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right). \quad (3.5.31)$$

If  $|t|_v \leq 1$ , then the product already lies in  $GL(2, \mathbb{Z}_v)$ , so

$$\varphi_v \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = 1, \quad \text{for } |t|_v \leq 1. \quad (3.5.32)$$

On the other hand, if  $|t|_v > 1$ , we must multiply

$$\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad (3.5.33)$$

on the right by a suitable element of  $K_v$  to obtain an upper-triangular form. Indeed,

$$\begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & * \\ 0 & t \end{pmatrix}, \quad (3.5.34)$$

because when  $|t|_v > 1$ , we have  $-t^{-1} \in \mathbb{Z}_v$ , ensuring  $\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \in K_v$ . Therefore,

reading the diagonal entries yields

$$\varphi_v \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = |t|_v^{-2s}, \quad \text{for } |t|_v > 1. \quad (3.5.35)$$

Combining both cases, the product across all finite places is

$$\prod_{v < \infty} \varphi_v(\gamma) = \prod_{v < \infty} \begin{cases} 1, & \text{if } |t|_v \leq 1, \\ |t|_v^{-2s}, & \text{if } |t|_v > 1. \end{cases} \quad (3.5.36)$$

Thus, combining the archimedean and non-archimedean parts gives the full evaluation:

$$\varphi \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \left| \frac{y}{(x+t)^2 + y^2} \right|_\infty^s \cdot \prod_{v < \infty} \begin{cases} 1, & \text{if } |t|_v \leq 1, \\ |t|_v^{-2s}, & \text{if } |t|_v > 1, \end{cases} \quad (3.5.37)$$

as claimed.

Now, let  $t = d/c$  with  $\gcd(c, d) = 1$ . Then  $|t|_v > 1 \iff |d/c|_v > 1 \iff p \mid c$ . Hence, the product can be elegantly rewritten as

$$\prod_{v < \infty} \begin{cases} 1, & \text{if } |d/c|_v \leq 1, \\ |d/c|_v^{-2s}, & \text{if } |d/c|_v > 1, \end{cases} = \prod_{v < \infty} |c|_v^{2s} = \frac{1}{|c|_\infty^{2s}}. \quad (3.5.38)$$

Using (3.5.37), the global expression collapses to

$$\left| \frac{y}{(x + \frac{d}{c})^2 + y^2} \right|_\infty^s \cdot \frac{1}{|c|_\infty^{2s}} = \left| \frac{y}{(cx + d)^2 + (cy)^2} \right|_\infty^s = \frac{y^s}{|cz + d|^{2s}}, \quad (3.5.39)$$

which is exactly the classical summand.

The next step is to use these explicit local forms to compute the constant coefficient of the Fourier expansion of  $E_s(z)$ .

The constant coefficient of a modular form  $f(x + iy)$  can be classically computed via

$$Cf(iy) = \int_{\mathbb{R}/\mathbb{Z}} f(x + iy) dx, \quad (3.5.40)$$

which can be expressed over the quotient of the unipotent group as

$$Cf(iy) = \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} f(n \cdot iy) dn \quad (3.5.41)$$

since we have the isomorphism  $N_{\mathbb{R}} \simeq (\mathbb{R}, +)$  for the unipotent group  $N$ . Note that, using the injection  $\mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{A}/\mathbb{Q}$ , we can rewrite  $N_{\mathbb{Z}} \backslash N_{\mathbb{R}}$  globally as

$$N_{\mathbb{Z}} \backslash \left( N_{\mathbb{R}} \cdot \left( N_{\mathbb{A}} \cap \prod_{v < \infty} K_v \right) \right) = N_{\mathbb{Q}} \backslash N_{\mathbb{A}}, \quad (3.5.42)$$

since

$$N_{\mathbb{Q}} \cdot \left( N_{\mathbb{R}} \cdot \left( N_{\mathbb{A}} \cap \prod_{v < \infty} K_v \right) \right) = N_{\mathbb{A}}.$$

To make this precise, we must establish the Haar measure  $dn$  on  $N_{\mathbb{A}}$ . Because  $N$  is isomorphic to additive group, we construct this globally by first defining local additive Haar measures  $dn_v$  on each  $\mathbb{Q}_v$ . For the archimedean place,  $dn_{\infty}$  is the standard Lebesgue measure on  $\mathbb{R}$ . For each finite place  $p$ , we normalize the Haar measure  $dn_p$  such that the compact open subgroup  $\mathbb{Z}_p$  has a volume of 1. The global Haar measure on  $N_{\mathbb{A}}$  is then the product measure:

$$dn = \prod_v dn_v. \quad (3.5.43)$$

Under this standard normalization, the volume of the quotient space  $N_{\mathbb{Q}} \backslash N_{\mathbb{A}}$  is 1. This follows from the fact that a fundamental domain for  $\mathbb{Q} \backslash \mathbb{A}$  can be chosen as  $[0, 1) \times \prod_p \mathbb{Z}_p$ . Taking the product measure over this domain yields  $1 \times \prod_p 1 = 1$ . With the measure established, we can now rewrite the classical constant term globally.

Suppose  $F$  is right-invariant under the maximal compact subgroup  $K = \prod_p K_p$ . Because the volume of the finite unipotent subgroup  $N_{\text{fin}} \cap K = \prod_p \mathbb{Z}_p$  is 1, for  $g_{\infty} \in G_{\infty}$ ,  $n_{\infty} \in N_{\infty}$ , and  $n_{\text{fin}} \in N_{\text{fin}}$ , we have

$$CF(g_{\infty}) = \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} F(n_{\infty} \cdot g_{\infty}) dn_{\infty} \quad (3.5.44)$$

$$= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} \int_{N_{\mathbb{A}} \cap \prod_{v < \infty} K_v} F(n_{\infty} \cdot g_{\infty} \cdot n_{\text{fin}}) dn_{\text{fin}} dn_{\infty} \quad (3.5.45)$$

$$= \int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} \int_{N_{\mathbb{A}} \cap \prod_{v < \infty} K_v} F(n_{\infty} \cdot n_{\text{fin}} \cdot g_{\infty}) dn_{\text{fin}} dn_{\infty} \quad (3.5.46)$$

$$= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} F(n \cdot g_{\infty}) dn. \quad (3.5.47)$$

Applying this formulation to the global adelic Eisenstein series  $E_s^{\text{adele}}(g)$  defined

previously, the constant coefficient is evaluated via the integral

$$CE_s^{\text{adele}}(g) = \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s^{\text{adele}}(ng) \, dn, \quad (3.5.48)$$

and by inserting the adelic summation formulation from Equation (3.5.10), we obtain

$$\begin{aligned} \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s^{\text{adele}}(ng) \, dn &= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma ng) \, dn \\ &= \sum_{w \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}} / N_{\mathbb{Q}}} \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in P_{\mathbb{Q}} \backslash P_{\mathbb{Q}} w N_{\mathbb{Q}}} \varphi(\gamma ng) \, dn. \end{aligned}$$

This decomposes into a single contribution from the small cell and a sum over the large cell:

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng) \, dn + \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in N_{\mathbb{Q}}} \varphi(w\gamma ng) \, dn = \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng) \, dn + \int_{N_{\mathbb{A}}} \varphi(wng) \, dn. \quad (3.5.49)$$

Since  $\varphi$  is left  $N_{\mathbb{A}}$ -invariant, the small cell contribution trivially evaluates to

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng) \, dn = \varphi(g) \cdot \text{vol}(N_{\mathbb{Q}} \backslash N_{\mathbb{A}}) = \varphi(g), \quad (3.5.50)$$

which, for  $g = (g_{\infty}, 1, 1 \dots)$ , yields exactly  $\varphi(g) = y^s$ .

The large cell contribution factors over the places as

$$\int_{N_{\mathbb{A}}} \varphi(wng) \, dn = \prod_{v \leq \infty} \int_{N_v} \varphi_v(wng_v) \, dn_v. \quad (3.5.51)$$

We now use Equation (3.5.36) to explicitly compute the large cell contribution in the non-archimedean cases (where  $v = p$ ). Since  $\mathbb{Q}_p$  is locally compact, we normalize the Haar measure on  $\mathbb{Q}_p$  such that the total measure of  $\mathbb{Z}_p$  is 1. Because the units  $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \backslash p\mathbb{Z}_p$  have measure  $1 - \frac{1}{p}$ , it follows that the scaled shell  $p^{-l}\mathbb{Z}_p^{\times}$  has measure  $p^l \left(1 - \frac{1}{p}\right)$ .

Thus, the integral splits as

$$\begin{aligned} \int_{|t|_p \leq 1} 1 \, dt + \int_{|t|_p > 1} |t|_p^{-2s} \, dt &= 1 + \sum_{l=1}^{\infty} |p^{-l}|_p^{-2s} \int_{p^{-l}\mathbb{Z}_p^{\times}} 1 \, dt \\ &= 1 + \sum_{l=1}^{\infty} (p^l)^{-2s} \cdot p^l \left(1 - \frac{1}{p}\right) \end{aligned}$$

$$\begin{aligned}
&= 1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} (p^l)^{1-2s} \\
&= \frac{1 - p^{-2s}}{1 - p^{1-2s}} = \frac{\zeta_p(2s-1)}{\zeta_p(2s)},
\end{aligned}$$

where  $\zeta_p(s) = (1 - p^{-s})^{-1}$  denotes the  $p$ th Euler factor of the Riemann zeta function.

Recall the standard Euler product for the Riemann zeta function over all finite primes, given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p < \infty} (1 - p^{-s})^{-1} = \prod_{p < \infty} \zeta_p(s). \quad (3.5.52)$$

Therefore, the combined contribution from the large cell across all finite places is exactly

$$\prod_{p < \infty} \int_{N_p} \varphi_p(wng_p) dn_p = \prod_{p < \infty} \frac{\zeta_p(2s-1)}{\zeta_p(2s)} = \frac{\zeta(2s-1)}{\zeta(2s)}. \quad (3.5.53)$$

The archimedean case follows smoothly using Equation (3.5.30):

$$\begin{aligned}
\int_{\mathbb{R}} \left| \frac{y}{(x+t)^2 + y^2} \right|_{\infty}^s dt &= \int_{\mathbb{R}} \frac{y^s}{((x+t)^2 + y^2)^s} dt \\
&= y^s \int_{\mathbb{R}} \frac{1}{(t^2 + y^2)^s} dt,
\end{aligned}$$

where we used the translation invariance of the integral. By substituting  $t \mapsto yt$  and then employing the trigonometric change of variables  $t = \tan \theta$ , we obtain

$$y^{1-s} \int_{\mathbb{R}} \frac{1}{(1+t^2)^s} dt = y^{1-s} 2 \int_0^{\frac{\pi}{2}} (\cos \theta)^{2s-2} d\theta. \quad (3.5.54)$$

Recall the Euler beta function

$$B(s_1, s_2) = \int_0^1 u^{s_1-1} (1-u)^{s_2-1} du = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)}, \quad (3.5.55)$$

which admits the trigonometric representation

$$B(s_1, s_2) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2s_1-1} (\cos \theta)^{2s_2-1} d\theta. \quad (3.5.56)$$

Setting  $2s_1 - 1 = 0 \iff s_1 = \frac{1}{2}$  and  $2s_2 - 1 = 2s - 2 \iff s_2 = s - \frac{1}{2}$ , we

evaluate the integral as

$$\begin{aligned} y^{1-s} 2 \int_0^{\frac{\pi}{2}} (\cos \theta)^{2s-2} d\theta &= y^{1-s} \frac{\Gamma(s - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)} \\ &= y^{1-s} \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)}, \end{aligned}$$

since  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .

Rewriting this fraction in terms of the completed archimedean zeta factors, we find

$$y^{1-s} \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} = y^{1-s} \frac{\pi^{-(s-\frac{1}{2})} \Gamma(s - \frac{1}{2})}{\pi^{-s} \Gamma(s)} = y^{1-s} \frac{\zeta_{\infty}(2s-1)}{\zeta_{\infty}(2s)}, \quad (3.5.57)$$

where  $\zeta_{\infty}(s) = \pi^{-s/2} \Gamma(s/2)$  is the special archimedean Euler factor.

Let  $\xi(s) = \zeta_{\infty}(s) \cdot \zeta(s)$  denote the fully completed zeta function. As is standard in the literature, this completion is special so that  $\xi(s)$  admits the symmetric functional equation  $\xi(s) = \xi(1-s)$  (see 3, Chapter 1). Combining Equations (3.5.50), (3.5.53), and (3.5.57), the constant coefficient of the Eisenstein series  $E_s(z)$  is

$$CE_s(z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}. \quad (3.5.58)$$

**Definition 3.5.2.** Let  $\psi$  be the standard non-trivial additive character of  $N_{\mathbb{Q}} \backslash N_{\mathbb{A}} \cong \mathbb{Q} \backslash \mathbb{A}$ , defined on local components  $t_v \in \mathbb{Q}_v$  by:

$$\psi_v(t_v) = \begin{cases} e^{2\pi i t_{\infty}}, & \text{if } v = \infty, \\ e^{-2\pi i \{t_p\}_p}, & \text{if } v = p, \end{cases} \quad (3.5.59)$$

where  $\{t_p\}_p \in \mathbb{Q}$  denotes the fractional part of the  $p$ -adic number  $t_p$ . For any  $m \in \mathbb{Q}$ , we denote by  $\psi_m$  the character obtained by scaling  $\psi$ , i.e.,  $\psi_m(n) = \psi(mn)$ , which decomposes locally as  $\psi_{m,v}(t_v) = \psi_v(mt_v)$ .

For the global adelic non-holomorphic Eisenstein series  $E_s^{\text{adele}}(g)$ , the  $m$ -th Fourier coefficient is defined by the adelic integral

$$F_m(E_s^{\text{adele}}(g)) = \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s^{\text{adele}}(ng) \overline{\psi_m(n)} dn. \quad (3.5.60)$$

This adelic integral recovers the classical  $m$ -th Fourier coefficient of  $E_s(z)$ .

By invoking Equation (3.5.10) and the Bruhat cell decomposition once more, we

expand the integral as

$$\begin{aligned} \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} E_s^{\text{adele}}(ng) \overline{\psi_m(n)} dn &= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \left( \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(\gamma ng) \right) \overline{\psi_m(n)} dn \\ &= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng) \overline{\psi_m(n)} dn + \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in N_{\mathbb{Q}}} \varphi(w\gamma ng) \overline{\psi_m(n)} dn. \end{aligned}$$

The contribution of the little cell simplifies to

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(ng) \overline{\psi(n)} dn = \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \varphi(g) \overline{\psi(n)} dn = \varphi(g) \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \overline{\psi(n)} dn, \quad (3.5.61)$$

since  $\varphi$  is right  $N_{\mathbb{A}}$ -invariant. Because  $\psi$  is a non-trivial character, its integral over the fundamental domain vanishes:

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \overline{\psi(n)} dn = 0. \quad (3.5.62)$$

Hence, the contribution of the little cell to the non-constant coefficients is strictly zero.

Let us turn our attention to the big cell. Observe that we can apply a change of variables by setting  $\tilde{n} = \gamma n$ , which means  $n = \gamma^{-1} \tilde{n}$ . Because  $\psi_m$  is defined as an additive character on the quotient space  $N_{\mathbb{Q}} \backslash N_{\mathbb{A}}$ , it is trivial on  $N_{\mathbb{Q}}$  by definition. Since  $\gamma \in N_{\mathbb{Q}}$ , we immediately have  $\psi_m(\gamma) = 1$ . Thus, the character can be rewritten simply as

$$\overline{\psi_m(n)} = \overline{\psi_m(\gamma^{-1} \tilde{n})} = \psi_m(\gamma) \overline{\psi_m(\tilde{n})} = \overline{\psi_m(\tilde{n})}. \quad (3.5.63)$$

By standard unfolding of the integral, we can combine the sum over the discrete subgroup  $N_{\mathbb{Q}}$  with the integral over the fundamental domain for  $N_{\mathbb{Q}} \backslash N_{\mathbb{A}}$  to obtain a single integral over the entire adelic group  $N_{\mathbb{A}}$ . Since the integrand is a product of local functions, this global integral factors into a product of local integrals:

$$\begin{aligned} \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \sum_{\gamma \in N_{\mathbb{Q}}} \varphi(w\gamma ng) \overline{\psi_m(n)} dn &= \int_{N_{\mathbb{A}}} \varphi(w\tilde{n}g) \overline{\psi_m(\tilde{n})} d\tilde{n} \\ &= \prod_{v \leq \infty} \int_{N_{\mathbb{Q}_v}} \varphi_v(w\tilde{n}g_v) \overline{\psi_{m,v}(\tilde{n})} d\tilde{n}. \end{aligned}$$

Let  $p < \infty$  be a finite prime. Recall that to recover the classical Fourier coefficients, we evaluate the adelic Eisenstein series at the specific element  $g = (g_{\infty}, 1, 1, \dots) \in G_{\mathbb{A}}$ . Thus, the local component  $g_p$  is exactly the identity matrix 1, and the local

integral reduces to

$$\int_{\mathbb{Q}_p} \varphi_p \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) \overline{\psi_{m,p}(t)} dt. \quad (3.5.64)$$

Invoking the evaluations from (3.5.36), we introduce the function

$$\max(1, |t|_p) = \begin{cases} 1, & \text{if } t \in \mathbb{Z}_p, \\ |t|_p, & \text{otherwise.} \end{cases} \quad (3.5.65)$$

For a fixed  $m \in \mathbb{Q}^\times$ , let  $\psi_{m,p}$  be the character of  $N_{\mathbb{Q}_p} \cong \mathbb{Q}_p$  given by  $\psi_{m,p}(t) = \psi_p(mt) = e^{-2\pi i \{mt\}_p}$ .

Thus, each non-archimedean factor can be computed by splitting the domain:

$$\int_{\mathbb{Q}_p} \max(1, |t|_p)^{-2s} e^{-2\pi i \{mt\}_p} dt = \int_{\mathbb{Z}_p} e^{-2\pi i \{mt\}_p} dt + \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |t|_p^{-2s} e^{-2\pi i \{mt\}_p} dt.$$

Here, we introduce the *p-adic Gaussian* indicator:

$$\gamma_p(m) = \begin{cases} 1, & \text{if } m \in \mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases} \quad (3.5.66)$$

By the orthogonality of characters, the integral over  $\mathbb{Z}_p$  is precisely  $\gamma_p(m)$ .

The second integral over the fractional shells requires an expansion:

$$\begin{aligned} \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |t|_p^{-2s} e^{-2\pi i \{mt\}_p} dt &= \sum_{l=1}^{\infty} \int_{p^{-l}\mathbb{Z}_p^\times} |t|_p^{-2s} e^{-2\pi i \{mt\}_p} dt \\ &= \sum_{l=1}^{\infty} p^{-2sl} \int_{p^{-l}\mathbb{Z}_p^\times} e^{-2\pi i \{mt\}_p} dt. \end{aligned}$$

Assume  $m \in \mathbb{Z}_p$ . Then its valuation is  $|m|_p = p^{-k}$  for some integer  $k \geq 0$ , meaning we can write  $m = p^k u$  for a unit  $u \in \mathbb{Z}_p^\times$ . Because multiplication by a unit is measure-preserving,  $u$  can be absorbed into the Haar measure without changing the integral. We thus have:

$$\sum_{l=1}^{\infty} p^{-2sl} \int_{p^{-l}\mathbb{Z}_p^\times} e^{-2\pi i \{p^k t\}_p} dt = \sum_{l=1}^{\infty} p^{l-2sl} \int_{\mathbb{Z}_p^\times} e^{-2\pi i \{p^{k-l} t\}_p} dt \quad (3.5.67)$$

$$= \sum_{l=1}^{\infty} p^{l-2sl} \left( \int_{\mathbb{Z}_p} e^{-2\pi i \{p^{k-l} t\}_p} dt - p^{-1} \int_{\mathbb{Z}_p} e^{-2\pi i \{p^{k-l+1} t\}_p} dt \right) \quad (3.5.68)$$

$$= \sum_{l=1}^{\infty} p^{l-2sl} (\gamma_p(p^{k-l}) - p^{-1} \gamma_p(p^{k-l+1})) . \quad (3.5.69)$$

There are two cases. If  $l \leq k$ , both  $\gamma_p(p^{k-l}) = 1$  and  $\gamma_p(p^{k-l+1}) = 1$ , yielding the sum:

$$\sum_{l=1}^k p^{l-2sl} (1 - p^{-1}) = \frac{p^{1-2s} - p^{(k+1)(1-2s)}}{1 - p^{1-2s}} (1 - p^{-1}). \quad (3.5.70)$$

If  $l = k + 1$ , then  $\gamma_p(p^{k-l}) = 0$  but  $\gamma_p(p^{k-l+1}) = 1$ , generating the term:

$$- \frac{p^{(k+1)(1-2s)}}{p}. \quad (3.5.71)$$

By merging these two terms and simplifying algebraically, we obtain:

$$(1 - p^{-2s}) \frac{1 - p^{(1-2s)} |m|_p^{2s-1}}{1 - p^{1-2s}} - 1. \quad (3.5.72)$$

Conversely, if  $m \notin \mathbb{Z}_p$ , its valuation is  $|m|_p = p^{-k}$  for some  $k < 0$ . In this case, the sum strictly vanishes:

$$\sum_{l=1}^{\infty} p^{l-2sl} \int_{\mathbb{Z}_p^\times} e^{-2\pi i \{p^{k-l}t\}_p} dt = 0, \quad (3.5.73)$$

because  $p^{k-l} \notin \mathbb{Z}_p$  for all  $l \geq 1$ . Hence, by summing  $\gamma_p(m)$  and (3.5.72), the complete local integral evaluates to

$$\int_{\mathbb{Q}_p} \max(1, |t|_p)^{-2s} e^{-2\pi i \{mt\}_p} dt = \gamma_p(m) (1 - p^{-2s}) \frac{1 - p^{(1-2s)} |m|_p^{2s-1}}{1 - p^{1-2s}}. \quad (3.5.74)$$

Taking the product over all finite places isolates the Riemann zeta function factor:

$$\prod_{p < \infty} \left( \gamma_p(m) (1 - p^{-2s}) \frac{1 - p^{(1-2s)} |m|_p^{2s-1}}{1 - p^{1-2s}} \right) = \zeta(2s)^{-1} \prod_{p < \infty} \left( \gamma_p(m) \frac{1 - p^{(1-2s)} |m|_p^{2s-1}}{1 - p^{1-2s}} \right). \quad (3.5.75)$$

Observe that by the definition of  $\gamma_v(m)$ , we see that the product of this function over all finite places is

$$\prod_{v < \infty} \gamma_v(m) = \begin{cases} 1, & \text{if } m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.5.76)$$

Thus, the entire expression is non-zero only when  $m \in \mathbb{Z}$ . Let  $m = p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$ . Recall the classical divisor sum function, defined by  $\sigma_\nu(m) := \sum_{d|m} d^\nu$ . For a prime power  $p^\beta$ ,  $\sigma_{1-2s}(p^\beta)$  is evaluated as

$$\sigma_{1-2s}(p^\beta) = 1 + p^{1-2s} + p^{2(1-2s)} + \dots + p^{\beta(1-2s)} = \frac{1 - p^{(1-2s)(\beta+1)}}{1 - p^{1-2s}}. \quad (3.5.77)$$

Because  $\sigma_{1-2s}(m)$  is multiplicative and  $|m|_p = p^{-\beta}$ , the remaining product evaluates exactly to this divisor sum:

$$\prod_{p < \infty} \left( \gamma_p(m) \frac{1 - p^{(1-2s)} |m|_p^{2s-1}}{1 - p^{1-2s}} \right) = \sigma_{1-2s}(m). \quad (3.5.78)$$

Thus, the contribution of the finite places is:

$$\prod_{p < \infty} \int_{N_{\mathbb{Q}_p}} \varphi_p(w\tilde{n}) \overline{\psi_p(\tilde{n})} d\tilde{n} = \frac{\sigma_{1-2s}(m)}{\zeta(2s)}. \quad (3.5.79)$$

Finally, we address the archimedean factor where  $v = \infty$ :

$$\int_{\mathbb{R}} \varphi_\infty \left( w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \overline{\psi_\infty(t)} dt. \quad (3.5.80)$$

Using (3.5.30) and the standard continuous additive character  $\psi_\infty(t) = e^{2\pi i m t}$ , this integral takes the explicit form:

$$\int_{\mathbb{R}} \left| \frac{y}{(x+t)^2 + y^2} \right|^s e^{-2\pi i m t} dt. \quad (3.5.81)$$

To compute this, we apply the change of variables  $u = t + x$ . The integral transforms into

$$e^{2\pi i m x} y^s \int_{\mathbb{R}} \frac{e^{-2\pi i m u}}{(u^2 + y^2)^s} du. \quad (3.5.82)$$

This Fourier transform is a well-known classical identity that yields the modified Bessel function of the second kind,  $K_\nu(z)$ . Specifically, we utilize the relation (see 10, 3.771.2)

$$\int_{\mathbb{R}} \frac{e^{-2\pi i m u}}{(u^2 + y^2)^s} du = \frac{2\pi^s |m|^{s-1/2}}{\Gamma(s) y^{s-1/2}} K_{s-1/2}(2\pi |m| y). \quad (3.5.83)$$

Substituting this evaluation back, the archimedean contribution simplifies to

$$e^{2\pi imx} \frac{2\pi^s |m|^{s-1/2} y^{1/2}}{\Gamma(s)} K_{s-1/2}(2\pi|m|y). \quad (3.5.84)$$

Combining the archimedean and non-archimedean factors, we have the following result:

**Proposition 3.5.3.** *The  $m$ th Fourier coefficient of the Eisenstein series  $E_s(z)$  is given by*

$$F_m(E_s(z)) = \frac{2\pi^s |m|^{s-1/2} \sigma_{1-2s}(m)}{\Gamma(s) \zeta(2s)} y^{1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi imx}. \quad (3.5.85)$$

This explicit formula is useful because it clearly separates the arithmetic and analytic properties of the Eisenstein series. The non-archimedean places contribute the arithmetic data: the divisor sum  $\sigma_{1-2s}(m)$  and the Riemann zeta function in the denominator. In contrast, the archimedean place provides the analytic geometry, yielding the modified Bessel function  $K_{s-1/2}(2\pi|m|y)$  that forces the coefficients to decay exponentially as  $y \rightarrow \infty$ . This factorization highlights the main advantage of the adelic approach.

# 4

## The Weil Representation

Beginning with this chapter, we will slowly shift our focus to local Whittaker functionals. To do so, we first need to introduce the most important representation-theoretic tool, the Schrödinger model of the local Weil representation. We will introduce the basics of local quadratic spaces and Schwartz-Bruhat functions, adapting the notation in Kudla [13] and Kudla-Yang [13]. However, the details of these subjects will be limited to only what we need to cover our upcoming computations.

### 4.1 Local Quadratic Spaces

**Definition 4.1.1** ([6]). Let  $V$  be an  $R$ -module. A quadratic form on  $V$  is a function  $Q : V \rightarrow R$  which satisfies:

- (i) For all  $r \in R$  and  $x \in V$ ,

$$Q(r \cdot x) = r^2 \cdot Q(x). \quad (4.1.1)$$

- (ii) The function  $B : V \times V \rightarrow R$  defined by

$$B(x, y) = Q(x + y) - Q(x) - Q(y) \quad (4.1.2)$$

is a bilinear form on  $V$ .

The pair  $(V, Q)$  is called a quadratic space over  $R$ .

We are mainly interested in quadratic spaces over local fields, specifically over finite places. For a fixed place  $v$ , let  $(V, Q)$  be a quadratic space over  $\mathbb{Q}_v$ , i.e.,  $V$  is a vector space equipped with a quadratic form

$$Q : V \rightarrow \mathbb{Q}_v. \quad (4.1.3)$$

Given a quadratic form, we obtain the associated bilinear form

$$(x, y) := \frac{1}{2}(Q(x + y) - Q(x) - Q(y)) \quad (4.1.4)$$

in the usual way.

Let  $\chi_v : \mathbb{Q}_v^\times \rightarrow \{\pm 1\}$  be the quadratic character defined by

$$\chi_v(a) := (a, (-1)^{\frac{m(m-1)}{2}} \det V)_v, \quad (4.1.5)$$

where  $(\cdot, \cdot)_v$  denotes the quadratic Hilbert symbol at  $v$ , defined by

$$(a, b)_v = \begin{cases} 1, & \text{if the equation } ax^2 + by^2 = z^2 \text{ has a nontrivial solution in } \mathbb{Q}_v, \\ -1, & \text{otherwise.} \end{cases} \quad (4.1.6)$$

Here,  $\det V$  is defined as follows; consider the matrix

$$[Q(\mathbf{x})]_{i,j} = \frac{1}{2}(x_i, x_j), \quad (4.1.7)$$

where  $\mathbf{x} = (x_1, \dots, x_m)$  is a basis for  $V$  and  $Q(\mathbf{x}) \in M_m(\mathbb{Q}_v)$  denotes the corresponding symmetric matrix. Thus,

$$\det(V) := \det Q(\mathbf{x}) \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2, \quad (4.1.8)$$

which is independent of the choice of basis.

## 4.2 Schwartz-Bruhat Functions

**Definition 4.2.1.** Let  $V$  be a finite-dimensional vector space over  $\mathbb{Q}_v$  (where  $v$  can be finite or infinite).

- (i) For  $v = \infty$  (so  $V \cong \mathbb{R}^m$ ), a function  $\phi : V \rightarrow \mathbb{C}$  is called a *Schwartz function* if  $\phi$  is infinitely differentiable and  $\phi$  along with all its partial derivatives are rapidly decreasing, meaning that  $\phi$  decays at infinity faster than the inverse of any polynomial.
- (ii) For  $v < \infty$ , a function  $\phi : V \rightarrow \mathbb{C}$  is called a *Schwartz-Bruhat function* if it is locally constant and compactly supported.

The vector space  $S(V)$  consisting of these functions is called the *Schwartz space*.

A classical example of  $\phi \in S(\mathbb{R}^m)$  is the Gaussian

$$\phi(x) = e^{-\pi(x_1^2 + x_2^2 + \cdots + x_m^2)}. \quad (4.2.1)$$

For the  $p$ -adic case, we may consider

$$\mathbf{1}_L(x) = \begin{cases} 1, & x \in L, \\ 0, & \text{otherwise,} \end{cases} \quad (4.2.2)$$

where  $L \subset V$  is any open  $\mathbb{Z}_p$ -lattice. We call  $\mathbf{1}_L$  the *indicator function* (or *characteristic function*) of  $L$ .

**Lemma 4.2.2.** *Let  $L \subset V$  be an open lattice and  $a \in V$ . Let  $\mathbf{1}_{a+L}(x)$  denote the characteristic function of the coset  $a + L$ .*

- (i)  $\mathbf{1}_{a+L}$  is an element of  $S(V)$ .
- (ii)  $S(V)$  is spanned by characteristic functions of the form in part (i).

*Proof.* (i) We need to check that  $\mathbf{1}_{a+L}$  is both compactly supported and locally constant.

Since  $L$  is a lattice in  $V$ , it is compact. Translating it by  $a$  preserves compactness, hence  $a + L$  is compact. The support of  $\mathbf{1}_{a+L}$  is exactly  $a + L$ , so it is compactly supported.

To show local constancy, we must show that for every  $x \in V$ , there exists an open neighborhood  $U_x$  of  $x$  such that  $\mathbf{1}_{a+L}$  is constant on  $U_x$ . In the  $p$ -adic topology, open lattices form a neighborhood basis of the identity, so we can choose  $U_x = x + L$  as an open neighborhood.

Case 1:  $x \in a + L$ . Then  $x = a + l$  for some  $l \in L$ . Let  $y \in U_x = x + L$ . Then  $y = x + l' = a + l + l' \in a + L$ , hence  $\mathbf{1}_{a+L}(y) = 1$ .

Case 2:  $x \notin a + L$ . Then  $x + L$  and  $a + L$  are distinct cosets. Since cosets of  $L$  are either identical or disjoint, we have

$$(x + L) \cap (a + L) = \emptyset, \quad (4.2.3)$$

hence  $\mathbf{1}_{a+L}(y) = 0$  for all  $y \in x + L$ .

In both cases,  $\mathbf{1}_{a+L}$  is constant on  $U_x$ , and therefore locally constant.

We have shown that  $\mathbf{1}_{a+L} \in S(V)$ .

(ii) Let  $\phi \in S(V)$ . By definition,  $\phi$  is locally constant and compactly supported.

Because  $\phi$  has compact support, its local constancy is uniform. Thus, there exists a single open lattice  $L \subset V$  such that  $\phi(x+y) = \phi(x)$  for all  $x \in V$  and all  $y \in L$ . This means that  $\phi$  is constant on each coset  $a + L$ .

Since  $\phi$  has compact support, it is nonzero on only finitely many cosets  $a_1 + L, \dots, a_n + L$ . Let  $c_i = \phi(a_i)$  for  $i = 1, \dots, n$ . Then for all  $x \in V$ , we can write

$$\phi(x) = \sum_{i=1}^n c_i \mathbf{1}_{a_i+L}(x), \quad (4.2.4)$$

which is a finite linear combination of characteristic functions of the form in part (i).

Hence  $S(V)$  is spanned by functions  $\mathbf{1}_{a+L}$ .

□

## 4.3 The Local Weil Representation

Let  $v$  be a place of  $\mathbb{Q}$  and  $(V, Q)$  a quadratic space over  $\mathbb{Q}_v$  as defined above. Let us denote  $\dim(V)$  by  $m$ . To define the Weil representation, we fix a non-trivial additive character  $\psi_v$  of  $\mathbb{Q}_v$ . Let  $dy$  denote the self-dual Haar measure on  $V$  with respect to  $\psi_v$ , and let  $\gamma_{V,v}$  denote the local Weil index associated to  $(V, Q)$  and  $\psi_v$ , which is fundamentally an eighth root of unity that arises as a normalizing factor for the Fourier transform on  $V$  (see 12, I.3).

The local Weil representation (in the Schrödinger model) is a representation  $\omega_{V,v} : \mathrm{SL}(2, \mathbb{Q}_v) \rightarrow \mathrm{Aut}(S(V))$ . Since  $\mathrm{SL}(2, \mathbb{Q}_v)$  is generated by the diagonal, unipotent, and Weyl elements, the representation is uniquely determined by its action on these generators. The explicit formulas are given as follows (see 12, II.4):

1. Let  $a \in \mathbb{Q}_v^\times$ . Then, for  $\phi \in S(V)$ ,

$$\left[ \omega_{V,v} \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \phi \right] (x) = |a|_v^{\frac{m}{2}} \chi_v(a) \phi(ax), \quad (4.3.1)$$

where  $|\cdot|_v$  is the  $v$ -adic absolute value.

2. Let  $b \in \mathbb{Q}_v$ . Then, for  $\phi \in S(V)$ ,

$$\left[ \omega_{V,v} \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \phi \right] (x) = \psi_v(bQ(x))\phi(x). \quad (4.3.2)$$

3. Let  $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then, for  $\phi \in S(V)$ ,

$$[\omega_{V,v}(w)\phi](x) = \gamma_{V,v}^{-1} \int_V \psi_v((x, y))\phi(y) dy, \quad (4.3.3)$$

where  $dy$  is the self-dual Haar measure on the vector space  $V$ .

# 5

## Explicit Computations with Local Whittaker Functionals

In this chapter, we will examine the building blocks of the non-holomorphic Eisenstein series that we discussed in detail in the previous chapter.

### 5.1 Induced Representations

**Definition 5.1.1.** For  $v \leq \infty$ , let  $\chi_v : \mathbb{Q}_v^\times \rightarrow \mathbb{C}^\times$  be a multiplicative character. For  $s \in \mathbb{C}$ , we define the *degenerate principal series representation*  $I_v(s, \chi_v)$  to be the space of smooth functions

$$\Phi(\cdot, s) : \mathrm{SL}(2, \mathbb{Q}_v) \rightarrow \mathbb{C} \quad (5.1.1)$$

such that

$$\Phi \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g, s \right) = |a|_v^{s+1} \chi_v(a) \Phi(g, s) \quad (5.1.2)$$

for all  $a \in \mathbb{Q}_v^\times$ ,  $b \in \mathbb{Q}_v$ , and  $g \in \mathrm{SL}(2, \mathbb{Q}_v)$ .

Let the global induced representation  $I(s, \chi)$  be defined as the restricted tensor product of the local induced representations

$$I(s, \chi) = \bigotimes'_v I_v(s, \chi_v). \quad (5.1.3)$$

Here,  $\bigotimes'$  denotes that for all but finitely many places  $v$ , the local components belong to the family of spherical (unramified) vectors (see Definition 5.1.2).

**Definition 5.1.2.** Let  $p$  be a prime integer. Let  $\chi_p$  be an unramified character (i.e.  $\chi_p(a) = 1$  for all  $a \in \mathbb{Z}_p^\times$ ). A standard section  $\Phi_p^0(s) \in I_p(s, \chi_p)$  is called *spherical* (or unramified) if it is invariant under the right action of the maximal compact subgroup  $K_p$ . That is,  $\Phi_p^0(gk, s) = \Phi_p^0(g, s)$  and  $\Phi_p^0(1, s) = 1$  for all  $g \in \mathrm{SL}(2, \mathbb{Q}_p)$  and  $k \in K_p$ .

**Definition 5.1.3.** A *section*  $\Phi(s)$  is a family of vectors  $\{\Phi(\cdot, s) \mid s \in \mathbb{C}\}$ . We say a section is holomorphic or meromorphic if for a fixed  $g \in \mathrm{GL}(2, \mathbb{A})$ , the mapping

$$s \mapsto \Phi(g, s) \quad (5.1.4)$$

is holomorphic or meromorphic.

A section is factorizable if it can be written as a product over all places  $v \leq \infty$

$$\Phi(g, s) = \prod_v \Phi_v(g_v, s), \quad g = (g_v) \in G(2, \mathbb{A}), \quad (5.1.5)$$

such that for almost all finite places  $v$ , the local section  $\Phi_v(\cdot, s) \in I_v(s, \chi_v)$  is spherical section.

**Definition 5.1.4.** A section  $\Phi(s) \in I(s, \chi)$  is called *standard* if its restriction to the maximal compact subgroup  $K \subset G_{\mathbb{A}}$  is independent of  $s$ .

Let  $(V, Q)$  be a quadratic space of dimension  $m$  over  $\mathbb{Q}_v$ , and let  $\chi_v$  be the quadratic character associated with  $V$ . Define

$$s_0 := \frac{m}{2} - 1. \quad (5.1.6)$$

Given the Schwartz-Bruhat space  $S(V)$ , we can construct an element in  $I_v(s_0, \chi_v)$  via the Weil representation  $\omega_v$ . Specifically, the standard section attached to a function  $\phi \in S(V)$  is given at  $s = s_0$  by

$$\Phi(g, s_0) = (\omega_V(g)\phi)(0). \quad (5.1.7)$$

To see why this naturally defines an element of  $I_v(s_0, \chi_v)$ , we verify its transformation behavior under the parabolic subgroup. Let  $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P_v$ . From the explicit formulas for the Weil representation, we get

$$\Phi(pg, s_0) = (\omega_V(p)\omega_V(g)\phi)(0) = \chi_v(a)|a|_v^{m/2}(\omega_V(g)\phi)(0). \quad (5.1.8)$$

Because we defined our base point as  $s_0 = \frac{m}{2} - 1$ , the exponent is exactly  $s_0 + 1$ . This yields  $\Phi(pg, s_0) = \chi_v(a)|a|_v^{s_0+1}\Phi(g, s_0)$ , which is precisely the required transformation law for the principal series representation.

**Lemma 5.1.5.** *Fix any  $s_0 \in \mathbb{C}$  and vector  $\Phi_0 \in I_v(s_0, \chi_v)$ . Then there exists a unique standard section  $\Phi_v(\cdot, s)$  such that  $\Phi_v(s_0) = \Phi_0$ .*

*Proof.* We will construct a standard section  $\Phi_v$  that satisfies  $\Phi_v(s_0) = \Phi_0$ . Let  $f : K_v \rightarrow \mathbb{C}$  be a smooth function satisfying

$$f(pk) = |a|_p^{s+1}\chi_v(a)f(k) \quad (5.1.9)$$

for all  $k \in K_v$  and  $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P_v \cap K_v$ . By the Iwasawa decomposition, any  $g \in \mathrm{SL}(2, \mathbb{Q}_v)$  can be written as  $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} k_v$ . We define  $\tilde{\Phi}$  as

$$\tilde{\Phi}(g, s) = |a|_v^{s+1}\chi_v(a)f(k_v). \quad (5.1.10)$$

Observe that the restriction  $\tilde{\Phi}|_{K_v}$  is determined solely by the function  $f$  and is independent of  $s$ , hence  $\tilde{\Phi}$  is a standard section.

Now, let  $f_0(k_v) = \Phi_0(k_v)$  for all  $k_v \in K_v$ , and construct the section

$$\Phi_v(g, s) = |a|_v^{s+1}\chi_v(a)f_0(k_v). \quad (5.1.11)$$

By the above reasoning,  $\Phi_v$  is a standard section. Moreover, evaluating at  $s = s_0$ , we obtain

$$\Phi_v(g, s_0) = |a|_v^{s_0+1}\chi_v(a)f_0(k_v) = |a|_v^{s_0+1}\chi_v(a)\Phi_0(k_v) = \Phi_0(g). \quad (5.1.12)$$

Hence, a section satisfying  $\Phi_v(s_0) = \Phi_0$  exists.

To show uniqueness, suppose  $\Phi_{v,1}(g, s)$  and  $\Phi_{v,2}(g, s)$  are two standard sections such that  $\Phi_{v,1}(s_0) = \Phi_{v,2}(s_0) = \Phi_0$ . Set  $\Psi(g, s) = \Phi_{v,1}(g, s) - \Phi_{v,2}(g, s)$ , which is again a standard section. By assumption,  $\Psi(g, s_0) = 0$  for all  $g$ . Since  $\Psi$  is a standard section, its restriction to  $K_v$  is independent of  $s$ , which implies that  $\Psi(k_v, s) = 0$  for all  $s$ . Because standard sections are completely determined by their values on  $K_v$ ,  $\Psi$  must be identically zero. Therefore,  $\Phi_{v,1}(g, s) = \Phi_{v,2}(g, s)$ , proving uniqueness.  $\square$

**Definition 5.1.6.** For a global standard section  $\Phi(s) \in I(s, \chi)$ , we define the Eisen-

stein series as

$$E(g, s, \Phi) := \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \Phi(\gamma g, s). \quad (5.1.13)$$

**Remark 5.1.7.** The Eisenstein series given by Equation 3.5 can be recovered as a special case of this global adelic framework. Specifically, the previous setting corresponds to inducing from the *trivial character*,  $\chi = 1$ .

## 5.2 Whittaker Functional

In this section, we transition to the local Whittaker functional, which is the essential ingredient for understanding the Fourier expansion of our Eisenstein series. By computing these functionals explicitly over given quadratic spaces, we can determine the exact local factors of the global coefficients.

**Definition 5.2.1.** Let  $\Phi_v(\cdot, s)$  be a standard section. The local Whittaker functional is defined as

$$W_{m,v}(g, s, \Phi_v) := \int_{\mathbb{Q}_v} \Phi_v(w n(b_v) g_v, s) \psi_v(-m b_v) db_v, \quad (5.2.1)$$

where  $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is the Weyl element,  $n(b_v) = \begin{pmatrix} 1 & b_v \\ 0 & 1 \end{pmatrix}$  for  $m, b_v \in \mathbb{Q}_v$ , and  $\psi_v$  is the local additive character of  $\mathbb{Q}_v$  defined in Definition 3.5.2.

**Theorem 5.2.2.** For any factorizable holomorphic section  $\Phi(\cdot, s) = \prod_v \Phi_v(\cdot, s) \in I(s, \chi)$ , the  $m$ -th Fourier coefficient of the global Eisenstein series decomposes into a product of local Whittaker functionals over all places  $v \leq \infty$  as

$$E_m(g, s, \Phi) = \begin{cases} \prod_{v \leq \infty} W_{m,v}(g, s, \Phi_v), & m \neq 0, \\ \Phi(g, s) + \prod_{v \leq \infty} (M_v(s) \Phi_v)(g_v, s), & m = 0. \end{cases} \quad (5.2.2)$$

where  $g = (g_v) \in G(2, \mathbb{A})$  and  $(M_v(s) \Phi_v)(g_v, s)$  is the local intertwining operator

$$\prod_{v \leq \infty} \int_{\mathbb{Q}_v} \Phi_v \left( w \begin{pmatrix} 1 & b_v \\ 0 & 1 \end{pmatrix} g_v, s \right) db_v. \quad (5.2.3)$$

*Proof.* The  $m$ -th Fourier coefficient of the Eisenstein series along  $N$  is given by the integral

$$E_m(g, s, \Phi) = \int_{\mathbb{Q} \backslash \mathbb{A}} E \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g, s, \Phi \right) \psi(-mb) db \quad (5.2.4)$$

where  $\psi = \prod_{v \leq \infty} \psi_v$  is a non-trivial additive character  $\mathbb{A}$  trivial on  $\mathbb{Q}$ . Expanding the global Eisenstein series yields

$$E_m(g, s, \Phi) = \int_{\mathbb{Q} \backslash \mathbb{A}} \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \Phi \left( \gamma \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g, s \right) \psi(-mb) db. \quad (5.2.5)$$

Via applying the Bruhat decomposition we get a summation over the identity coset and the Weyl element.

- Let  $m \neq 0$ . Then the identity coset contribution vanishes because of the integral of a non-trivial character  $\psi(-mb)$  over  $\mathbb{Q} \backslash \mathbb{A}$  is zero. Unfolding the remaining sum over the Weyl element orbit gives us the integral

$$E_m(g, s, \Phi) = \int_{\mathbb{A}} \Phi \left( w \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g, s \right) \psi(-mb) db. \quad (5.2.6)$$

Since both the global section  $\Phi = \prod_v \Phi_v$  and the measure  $db = \prod_v db_v$  are restricted tensor products, the integral factorizes into local integrals

$$E_m(g, s, \Phi) = \prod_{v \leq \infty} \int_{\mathbb{Q}_v} \Phi_v \left( w \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g_v, s \right) \psi_v(-mb) db_v \quad (5.2.7)$$

where each local integral matches exactly the definition of the local Whittaker functional  $W_{m,v}(g_v, s, \Phi_v)$ .

- Let  $m = 0$ . The identity coset contribution is

$$\int_{\mathbb{Q} \backslash \mathbb{A}} \Phi \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g, s \right) db = \int_{\mathbb{Q} \backslash \mathbb{A}} \Phi(g, s) db = \Phi(g, s) \int_{\mathbb{Q} \backslash \mathbb{A}} db = \Phi(g, s) \quad (5.2.8)$$

since the volume of  $\mathbb{Q} \backslash \mathbb{A}$  under normalized measure is 1.

The Weyl element coset contribution is

$$\int_{\mathbb{A}} \Phi \left( w \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g, s \right) db. \quad (5.2.9)$$

Again, since  $\Phi$  and  $db$  are factorizable, we obtain

$$\prod_{v \leq \infty} \int_{\mathbb{Q}_v} \Phi_v \left( w \begin{pmatrix} 1 & b_v \\ 0 & 1 \end{pmatrix} g_v, s \right) db_v. \quad (5.2.10)$$

□

If  $g$  is of the form  $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$  and  $\Phi_v$  is spherical, then we can directly use Equation 5.1.2 to evaluate this integral explicitly. If this is not the case, we must employ an alternative approach. To demonstrate this, let us work on a concrete example with the following setup:

Fix a prime  $p \neq 2$ . Let  $k_p/\mathbb{Q}_p$  be an unramified quadratic extension (i.e., the prime  $p$  remains a uniformizer in  $k_p$ ), and let  $N : k_p \rightarrow \mathbb{Q}_p$  be the norm map, i.e.,

$$N_{k_p/\mathbb{Q}_p}(x) = x \cdot \bar{x}, \quad (5.2.11)$$

for  $x \in k_p$ , where  $\bar{x}$  denotes the Galois conjugate of  $x$ . We consider two 2-dimensional quadratic spaces:

$$V_1 = (k_p, N), \quad V_2 = (k_p, pN). \quad (5.2.12)$$

**Lemma 5.2.3.** *Let  $V_1$  and  $V_2$  be the quadratic spaces as set up above. Then their associated quadratic characters are equal, meaning  $\chi_{V_1} = \chi_{V_2}$ .*

*Proof.* Let  $k_p = \mathbb{Q}_p(\sqrt{u})$  where  $u \in \mathbb{Z}_p^\times$  is a non-square unit, and choose the basis  $\{1, \sqrt{u}\}$ . Then the associated bilinear form is

$$(x, y) := \frac{1}{2}(N(x + y) - N(x) - N(y)). \quad (5.2.13)$$

Computing the entries:

$$(1, 1) = \frac{1}{2}(N(1 + 1) - N(1) - N(1)) = \frac{1}{2}(4 - 1 - 1) = 1, \quad (5.2.14)$$

$$(1, \sqrt{u}) = \frac{1}{2}(N(1 + \sqrt{u}) - N(1) - N(\sqrt{u})) = \frac{1}{2}(1 - u - 1 - (-u)) = 0, \quad (5.2.15)$$

$$(\sqrt{u}, \sqrt{u}) = \frac{1}{2}(N(2\sqrt{u}) - N(\sqrt{u}) - N(\sqrt{u})) = \frac{1}{2}(-4u - (-u) - (-u)) = -u. \quad (5.2.16)$$

Hence the matrix of the bilinear form in this basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & -u \end{pmatrix}, \quad (5.2.17)$$

so that the determinant is

$$\det(V_1) := \det \begin{pmatrix} 1 & 0 \\ 0 & -u \end{pmatrix} = -u. \quad (5.2.18)$$

Therefore, the local quadratic character is given by the Hilbert symbol:

$$\chi_{V_1,p}(a) = (a, -\det(V_1))_p = (a, u)_p = \begin{cases} 1, & \text{if } ax^2 + uy^2 = z^2 \text{ has a nontrivial solution in } \mathbb{Q}_p, \\ -1, & \text{otherwise.} \end{cases} \quad (5.2.19)$$

Equivalently, we can write

$$\chi_{V_1,p}(a) = 1 \quad \text{if and only if } a \text{ is a norm from } k_p. \quad (5.2.20)$$

For the scaled space  $V_2$ , the bilinear form is multiplied by  $p$ , so its determinant is  $\det(V_2) = p^2(-u) = -up^2$ . We observe that

$$\chi_{V_2,p}(a) = (a, -\det(V_2))_p = (a, up^2)_p. \quad (5.2.21)$$

Because  $p^2$  is trivially a square in  $\mathbb{Q}_p^\times$ , the Hilbert symbol simplifies to  $(a, up^2)_p = (a, u)_p(a, p^2)_p = (a, u)_p$ . Equivalently,

$$\chi_{V_2,p}(a) = 1 \quad \text{if and only if } a \text{ is a norm from } k_p. \quad (5.2.22)$$

This shows that  $\chi_{V_1,p} = \chi_{V_2,p}$ . □

Now let us consider the ring of integers  $\mathcal{O}_{k_p}$ . This is naturally a  $\mathbb{Z}_p$ -lattice in  $k_p$ . Let  $\varphi_0$  be the characteristic function of  $\mathcal{O}_{k_p}$ . Since the underlying vector spaces for  $V_1$  and  $V_2$  are identically  $k_p$ , we can view  $\varphi_0 \in S(V_1)$  and  $\varphi_0 \in S(V_2)$  as a Schwartz-Bruhat function on either space. Then we obtain two standard sections

$$\Phi_1(s), \Phi_2(s) \in I_p(s, \chi_p), \quad \chi_p = \chi_{V_1,p} = \chi_{V_2,p}, \quad (5.2.23)$$

associated to the Weil representation, such that at  $s_0 = 0$ , we have

$$\Phi_i(g, 0) = (\omega_{V_i}(g)\varphi_0)(0). \quad (5.2.24)$$

**Lemma 5.2.4.**  $\mathcal{O}_{k_p}$  is self-dual in  $V_1$ .

*Proof.* Let  $\mathcal{O}_{k_p}^*$  be the dual of  $\mathcal{O}_{k_p}$ , defined as

$$\mathcal{O}_{k_p}^* = \{x \in k_p \mid (x, y) \in \mathbb{Z}_p \text{ for all } y \in \mathcal{O}_{k_p}\}. \quad (5.2.25)$$

Let  $k_p = \mathbb{Q}_p(\sqrt{u})$ , and choose the standard basis  $\{1, \sqrt{u}\}$  for  $\mathcal{O}_{k_p}$ . First, we show that  $\mathcal{O}_{k_p}^* \subset \mathcal{O}_{k_p}$ .

Let  $x \in \mathcal{O}_{k_p}^*$  with  $x = m + n\sqrt{u}$  and  $y \in \mathcal{O}_{k_p}$  with  $y = \alpha + \beta\sqrt{u}$ . Using the bilinear form computed previously, we have

$$|(x, y)|_p = \left| \frac{1}{2} (N(x + y) - N(x) - N(y)) \right|_p = |m\alpha - n\beta u|_p \leq 1. \quad (5.2.26)$$

By evaluating this at our basis elements, setting  $y = 1$  ( $\alpha = 1, \beta = 0$ ) gives  $|m|_p \leq 1$ . Similarly, setting  $y = \sqrt{u}$  ( $\alpha = 0, \beta = 1$ ) yields  $|-nu|_p \leq 1$ . Since  $u$  is a unit in  $\mathbb{Z}_p^\times$ , we have  $|u|_p = 1$ , which forces  $|n|_p \leq 1$ . Therefore,  $m, n \in \mathbb{Z}_p$ , which ensures that  $x \in \mathcal{O}_{k_p}$ .

The reverse inclusion  $\mathcal{O}_{k_p} \subset \mathcal{O}_{k_p}^*$  is clear since the bilinear form evaluated on any two elements of  $\mathcal{O}_{k_p}$  inherently yields an element in  $\mathbb{Z}_p$ .  $\square$

**Lemma 5.2.5** (14, Lemma 4.1). *Assume that  $p$  is a finite prime, and  $L$  is a unimodular  $\mathbb{Z}_p$ -lattice in  $V$ , i.e.,  $L^* = L$ , where*

$$L^* = \{x \in V \mid (x, y) \in \mathbb{Z}_p \text{ for all } y \in L\}. \quad (5.2.27)$$

*Then the standard section associated to the indicator function of  $L$  is the spherical section  $\Phi_p^0(s) \in I_p(s, \chi_p)$ , i.e.,*

$$\Phi_p^0(gk, s) = \Phi_p^0(g, s), \quad \Phi_p^0(1, s) = 1, \quad (5.2.28)$$

*for any  $k \in K_p = \text{SL}(2, \mathbb{Z}_p)$ .*

Observe that Lemma 5.2.4 and Lemma 5.2.5 together ensure that  $\Phi_1$  is spherical.

## 5.3 Local Identity

Let us now compute the explicit values of these Whittaker functionals. While the general identity connecting Whittaker functionals on non-split spaces is established by Sankaran [16, Proposition 4.4(iii)], our specific contribution lies in making this explicit for our chosen pair of quadratic spaces  $V_1$  and  $V_2$ . Specifically, Sankaran establishes an identity relating the derivative of a Whittaker functional on a space  $V_v$  (which does not represent  $m$ ) to the value of a Whittaker functional on the complementary space  $V'_v$  (which represents  $m$ ).

In our setting, the pair  $V_1$  and  $V_2$  plays the role of  $V_v$  and  $V'_v$ . Which space fails to represent  $m$  is determined precisely by the parity of  $\text{ord}_p(m)$ . The following

proposition explicitly generalizes Sankaran's identity to our chosen sections  $\Phi_1$  and  $\Phi_2$  across both parity cases.

**Proposition 5.3.1.** *For odd  $\text{ord}_p(m)$ , we have*

$$\left. \frac{d}{ds} [W_{m,p}(e, s, \Phi_1) - W_{p^{-2}m,p}(e, s, \Phi_1)] \right|_{s=0} = W_{m,p}(e, 0, \Phi_2) \log p. \quad (5.3.1)$$

*On the other hand, for even  $\text{ord}_p(m)$ , we have*

$$\left. \frac{d}{ds} [W_{m,p}(e, s, \Phi_2) - W_{p^{-2}m,p}(e, s, \Phi_2)] \right|_{s=0} = W_{m,p}(e, 0, \Phi_1) \log p. \quad (5.3.2)$$

For the remainder of this section, we focus on proving the case where  $\text{ord}_p(m)$  is odd. We start with the left-hand side, beginning with the evaluation of

$$W_{m,p}(e, s, \Phi_1) = \int_{\mathbb{Q}_p} \Phi_1(wn(b), s) \psi_p(-mb) db. \quad (5.3.3)$$

We evaluate this by splitting the domain of integration into two main cases:

- If  $b \in \mathbb{Z}_p$ , then  $wn(b) \in K_p$ . Since  $\Phi_1$  is spherical, it is right- $K_p$ -invariant, and thus  $\Phi_1(wn(b), s) = \Phi_1(e, s) = 1$ . Hence, this part of the integral yields

$$\int_{\mathbb{Z}_p} \psi_p(-mb) db = \int_{\mathbb{Z}_p} e^{2\pi i \{mb\}} = \begin{cases} 1, & m \in \mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases} \quad (5.3.4)$$

- If  $b \notin \mathbb{Z}_p$ , we first observe the matrix multiplication

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}. \quad (5.3.5)$$

To apply the transformation rule for the induced section, we utilize the right- $K_p$ -invariance of  $\Phi_1$  by multiplying on the right by a suitable element of  $K_p = \text{SL}(2, \mathbb{Z}_p)$  to obtain an upper triangular matrix:

$$\begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & -1 \\ 0 & b \end{pmatrix}. \quad (5.3.6)$$

Because  $-b^{-1} \in \mathbb{Z}_p$ , the rightmost matrix is indeed in  $K_p$ . Thus, we have

$$\Phi_1 \left( \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}, s \right) = \Phi_1 \left( \begin{pmatrix} b^{-1} & -1 \\ 0 & b \end{pmatrix}, s \right) = |b^{-1}|_p^{s+1} \chi_p(b^{-1}). \quad (5.3.7)$$

The integral over  $\mathbb{Q}_p \setminus \mathbb{Z}_p$  then becomes

$$\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |b^{-1}|_p^{s+1} \chi_p(b^{-1}) \psi_p(-mb) db. \quad (5.3.8)$$

We decompose the domain by writing  $b = p^{-k}v$  where  $v \in \mathbb{Z}_p^\times$  and  $k \geq 1$ , and  $dv$  denotes the normalized additive Haar measure on  $\mathbb{Q}_p$  restricted to  $\mathbb{Z}_p^\times$ . Then  $|b^{-1}|_p = p^{-k}$ . Noting that the Haar measure scales as  $db = p^k dv$ , the integral becomes the sum

$$\sum_{k=1}^{\infty} \int_{\mathbb{Z}_p^\times} (p^{-k})^{s+1} \chi_p(p^k v^{-1}) e^{2\pi i \{-mp^{-k}v\}} p^k dv = \sum_{k=1}^{\infty} p^{-ks} \int_{\mathbb{Z}_p^\times} \chi_p(b^{-1}) e^{2\pi i \{-mb\}} dv. \quad (5.3.9)$$

Since  $\chi_p(b^{-1}) = (b^{-1}, u)_p$  and  $u$  is a non-square unit modulo  $p$ , the Hilbert symbol simplifies to

$$(b^{-1}, u)_p = \left( \frac{u}{p} \right)^k = (-1)^k. \quad (5.3.10)$$

Let  $m = p^l w$ . We can rewrite the sum as

$$\sum_{k=1}^{\infty} p^{-ks} \int_{\mathbb{Z}_p^\times} (-1)^k e^{-2\pi i \{p^{l-k}vw\}} dv. \quad (5.3.11)$$

Observe that, as the same oscillatory behavior managed in Chapter 3 and demonstrated in the proof of 14, Theorem 4.3, the integral evaluates to

$$\int_{\mathbb{Z}_p^\times} e^{-2\pi i \{p^{l-k}vw\}} dv = \begin{cases} 1 - p^{-1}, & \text{if } k \leq l, \\ -p^{-1}, & \text{if } k = l + 1, \\ 0, & \text{if } k > l + 1. \end{cases} \quad (5.3.12)$$

Hence, the final sum is

$$\sum_{k=1}^{\infty} p^{-ks} (-1)^k \int_{\mathbb{Z}_p^\times} e^{-2\pi i \{p^{l-k}vw\}} dv = (1 - p^{-1}) \sum_{k=1}^l (-1)^k p^{-ks} + (-1)^{l+1} p^{-s(l+1)} (-p^{-1}), \quad (5.3.13)$$

which simplifies to

$$(1 - p^{-1}) \sum_{k=1}^l (-1)^k p^{-ks} + (-1)^{l+2} p^{-s(l+1)-1}. \quad (5.3.14)$$

By merging the two cases, we obtain

$$W_{m,p}(e, s, \Phi_1) = 1 + (1 - p^{-1}) \sum_{k=1}^l (-1)^k p^{-ks} + (-1)^{l+2} p^{-s(l+1)-1}. \quad (5.3.15)$$

Replacing  $m$  with  $p^{-2}m$  shifts the valuation  $l \mapsto l - 2$ . Thus,

$$W_{p^{-2}m,p}(e, s, \Phi_1) = 1 + (1 - p^{-1}) \sum_{k=1}^{l-2} (-1)^k p^{-ks} + (-1)^l p^{-s(l-1)-1}. \quad (5.3.16)$$

Subtracting Equation 5.3.16 from Equation 5.3.15 gives the polynomial

$$W_{m,p}(e, s, \Phi_1) - W_{p^{-2}m,p}(e, s, \Phi_1) \quad (5.3.17)$$

$$= (1 - p^{-1}) [(-1)^{l-1} p^{-(l-1)s} + (-1)^l p^{-ls}] + (-1)^{l+2} p^{-s(l+1)-1} - (-1)^l p^{-s(l-1)-1}. \quad (5.3.18)$$

By analyzing the parity of  $\text{ord}_p(m) = l$  and computing the derivative  $\frac{d}{ds}$  at  $s = 0$ , we obtain

$$\frac{d}{ds} [W_{m,p}(e, s, \Phi_1) - W_{p^{-2}m,p}(e, s, \Phi_1)] \Big|_{s=0} = \begin{cases} (1 + p^{-1}) \log p, & \text{if } \text{ord}_p(m) \text{ is odd,} \\ -(1 + p^{-1}) \log p, & \text{if } \text{ord}_p(m) \text{ is even.} \end{cases} \quad (5.3.19)$$

For the right-hand side of the proposition, we compute the functional for  $V_2$ . Evaluating the section at  $s = 0$  yields  $\Phi_2(wN(b), 0) = (\omega_{V_2}(wN(b))\varphi_0)(0)$ . Expanding this definition gives

$$W_{m,p}(e, 0, \Phi_2) = \int_{\mathbb{Q}_p} \Phi_2(wN(b), 0) \psi_p(-mb) db = \int_{\mathbb{Q}_p} \left( \int_{V_2} \psi_p(bpN(y)) \varphi_0(y) dy \right) \psi_p(-mb) db. \quad (5.3.20)$$

By applying Fubini's theorem to swap the integrals, we get

$$\int_{V_2} \varphi_0(y) \left( \int_{\mathbb{Q}_p} \psi_p(b(pN(y) - m)) db \right) dy. \quad (5.3.21)$$

The inner integral over  $\mathbb{Q}_p$  vanishes unless  $pN(y) - m = 0$ . Therefore, the non-zero

contribution to the integral over  $V_2$  comes strictly from the set where  $pN(y) = m$ . To evaluate this, we analyze the  $p$ -adic valuation of this condition. Since  $k_p/\mathbb{Q}_p$  is an unramified quadratic extension, the valuation  $\text{ord}_p(N(y))$  is always even for any  $y \in k_p^\times$ . This means  $\text{ord}_p(pN(y))$  must be odd.

The inner integral can be written as

$$\int_{\mathbb{Z}_p} \psi_p(b(pN(y) - m)) db + \sum_{k=1}^{\infty} \int_{p^{-k}\mathbb{Z}_p^\times} \psi_p(b(pN(y) - m)) db. \quad (5.3.22)$$

Substituting this back into the double integral and distributing the integration over  $V_2$ , we obtain

$$\int_{V_2} \varphi_0(y) \left( \int_{\mathbb{Z}_p} \psi_p(b(pN(y) - m)) db \right) dy + \sum_{k=1}^{\infty} \int_{V_2} \varphi_0(y) \left( \int_{p^{-k}\mathbb{Z}_p^\times} \psi_p(b(pN(y) - m)) db \right) dy. \quad (5.3.23)$$

Since  $\varphi_0$  is the characteristic function of  $\mathcal{O}_{k_p}$ , we should take  $y \in \mathcal{O}_{k_p}$  for a non-zero evaluation.

Assuming  $m \in \mathbb{Z}_p$ , the term  $pN(y) - m$  is an element of  $\mathbb{Z}_p$ . Thus, the inner integral over  $\mathbb{Z}_p$  evaluates to 1, and the entire first term reduces to the integral  $\int_{\mathcal{O}_{k_p}} dy$  where  $dy$  is a measure on the quadratic space  $V_2$ . By 18, 2.1.3(a), the measure is the standard, normalized additive Haar measure scaled by  $|\det V_2|_p^{\frac{1}{2}}$ . Hence, the integral evaluates to

$$\text{vol}(\mathcal{O}_{k_p}) |\det V_2|_p^{\frac{1}{2}} = |\det V_2|_p^{\frac{1}{2}} = |-up^2|_p^{\frac{1}{2}} = p^{-1}. \quad (5.3.24)$$

For the second term, let  $b = p^{-k}v$  where  $v \in \mathbb{Z}_p^\times$ , so the measure scales as  $db = p^k dv$ . Then, we get

$$\sum_{k=1}^{\infty} p^k \int_{\mathcal{O}_{k_p}} \int_{\mathbb{Z}_p^\times} \psi_p(p^{-k}v(pN(y) - m)) dv dy. \quad (5.3.25)$$

We swap the order of integration after rewriting the additive character as an exponential

$$\sum_{k=1}^{\infty} p^k \int_{\mathbb{Z}_p^\times} e^{-2\pi i \{p^{-k}vm\}} \left( \int_{\mathcal{O}_{k_p}} e^{2\pi i \{p^{-k+1}vN(y)\}} dy \right) dv. \quad (5.3.26)$$

**Lemma 5.3.2.** *For the given setup above, we evaluate*

$$\int_{\mathcal{O}_{k_p}} e^{2\pi i \{p^{-k+1}vN(y)\}} dy = (-1)^{k-1} p^{-k}. \quad (5.3.27)$$

*Proof.* When  $k = 1$ , the exponent is scaled by  $p^0 = 1$ . Since  $y \in \mathcal{O}_{k_p}$ , its norm

$N(y) \in \mathbb{Z}_p$ . Thus,  $vN(y) \in \mathbb{Z}_p$ , and its fractional part vanishes, so the integral is the volume of the lattice, which is  $p^{-1}$  under the self-dual Haar measure.

For  $k \geq 2$ , we have

$$\int_{\mathcal{O}_{k_p}} e^{2\pi i \{p^{-k+1}vN(y)\}} dy = \sum_{x \in \mathcal{O}_{k_p}/p^{k-1}\mathcal{O}_{k_p}} \left( \int_{x+p^{k-1}\mathcal{O}_{k_p}} e^{2\pi i \{p^{-k+1}vN(y)\}} dy \right) \quad (5.3.28)$$

Focus on the integral over a single coset  $x + p^{k-1}\mathcal{O}_{k_p}$ . We perform a change of variables by letting  $y = x + z$ , where  $z \in p^{k-1}\mathcal{O}_{k_p}$ . Since the Haar measure is translation-invariant,  $dy = dz$ . Writing  $z = p^{k-1}w$  for  $w \in \mathcal{O}_{k_p}$ , we expand the norm

$$N(x + z) = N(x) + \text{tr}_{k_p/\mathbb{Q}_p}(x\bar{z}) + N(z) = N(x) + p^{k-1} \text{tr}_{k_p/\mathbb{Q}_p}(x\bar{w}) + p^{2(k-1)}N(w). \quad (5.3.29)$$

Multiplying by  $p^{-k+1}v$ , we obtain

$$p^{-k+1}vN(x + z) = p^{-k+1}vN(x) + v \text{tr}_{k_p/\mathbb{Q}_p}(x\bar{w}) + p^{k-1}vN(w). \quad (5.3.30)$$

Because  $k_p/\mathbb{Q}_p$  is unramified, the trace of any integer is in  $\mathbb{Z}_p$ . Thus,  $v \text{tr}_{k_p/\mathbb{Q}_p}(x\bar{w}) \in \mathbb{Z}_p$ . Since  $k \geq 2$ , the term  $p^{k-1}vN(w)$  is also in  $\mathbb{Z}_p$ . The fractional part omits these integer terms, yielding

$$\{p^{-k+1}vN(x + z)\} = \{p^{-k+1}vN(x)\}. \quad (5.3.31)$$

Because the integrand is constant on the coset, the integral over the coset evaluates to the constant value multiplied by the volume of the subgroup  $p^{k-1}\mathcal{O}_{k_p}$ . The index of  $p^{k-1}\mathcal{O}_{k_p}$  in the 2-dimensional  $\mathbb{Z}_p$ -module  $\mathcal{O}_{k_p}$  is  $p^{2(k-1)}$ . Thus, the coset volume is  $p^{-1}p^{-2(k-1)} = p^{-2k+1}$ .

The total integral then reduces to a finite sum over the quotient ring

$$p^{-2k+1} \sum_{x \in \mathcal{O}_{k_p}/p^{k-1}\mathcal{O}_{k_p}} e^{2\pi i \{p^{-k+1}vN(x)\}}. \quad (5.3.32)$$

Since  $k_p/\mathbb{Q}_p$  is an unramified quadratic extension, we can fix a basis  $\mathcal{O}_{k_p} = \mathbb{Z}_p \oplus \mathbb{Z}_p \sqrt{u}$ , where  $u \in \mathbb{Z}_p^\times$  is a quadratic non-residue modulo  $p$ . Writing  $x = x_1 + x_2 \sqrt{u}$ , the norm is  $N(x) = x_1^2 - ux_2^2$ . The 2-dimensional sum splits into a product of two 1-dimensional

Gauss sums

$$\left( \sum_{x_1 \bmod p^{k-1}} e^{2\pi i \{p^{-k+1} v x_1^2\}} \right) \left( \sum_{x_2 \bmod p^{k-1}} e^{2\pi i \{p^{-k+1} (-uv) x_2^2\}} \right). \quad (5.3.33)$$

Let  $c = k - 1$  and let

$$G(a, c) = \sum_{m \bmod p^c} e^{2\pi i \{p^{-c} a m^2\}} \quad (5.3.34)$$

Using the property  $G(a, c) = \left(\frac{a}{p}\right)^c G(1, c)$  (see 1, Chapter 8.5), the product is

$$G(v, c)G(-uv, c) = \left(\frac{v}{p}\right)^c \left(\frac{-uv}{p}\right)^c G(1, c)^2. \quad (5.3.35)$$

By the multiplicativity of the Legendre symbol,  $\left(\frac{v}{p}\right) \left(\frac{-uv}{p}\right) = \left(\frac{-uv^2}{p}\right) = \left(\frac{-u}{p}\right) = -\left(\frac{-1}{p}\right)$ . Applying 1, Exercise 8.16, we obtain

$$G(v, c)G(-uv, c) = (-1)^c \left(\frac{-1}{p}\right)^c \left(\frac{-1}{p}\right)^c p^c = (-1)^c p^c = (-p)^{k-1}. \quad (5.3.36)$$

Finally, multiplying this finite sum by the coset volume factor  $p^{-2k+1}$ , the integral evaluates to

$$p^{-2k+1}(-p)^{k-1} = p^{-2k+1}(-1)^{k-1}p^{k-1} = (-1)^{k-1}p^{-k}. \quad (5.3.37)$$

This completes the proof. □

Substituting this into Equation 5.3.26 yields

$$\sum_{k=1}^{\infty} p^k \int_{\mathbb{Z}_p^\times} e^{-2\pi i p^{-k} v m} ((-1)^{k-1} p^{-k}) dv = \sum_{k=1}^{\infty} (-1)^{k-1} \int_{\mathbb{Z}_p^\times} e^{-2\pi i p^{-k} v m} dv. \quad (5.3.38)$$

Let  $l = \text{ord}_p(m)$ , so we can write  $m = p^l w$  for some unit  $w \in \mathbb{Z}_p^\times$ . The integral over  $\mathbb{Z}_p^\times$  evaluates to

$$\int_{\mathbb{Z}_p^\times} e^{-2\pi i p^{l-k} v w} dv = \begin{cases} 1 - p^{-1}, & \text{if } k \leq l, \\ -p^{-1}, & \text{if } k = l + 1, \\ 0, & \text{if } k > l + 1. \end{cases} \quad (5.3.39)$$

Because the integral vanishes for  $k > l + 1$ , we get

$$(1 - p^{-1}) \sum_{k=1}^l (-1)^{k-1} + (-1)^{(l+1)-1} (-p^{-1}) = \begin{cases} 1, & \text{if } l \text{ is odd,} \\ -\frac{1}{p}, & \text{if } l \text{ is even.} \end{cases} \quad (5.3.40)$$

Recall that we evaluated the first term to be  $1/p$ , hence

$$W_{m,p}(e, 0, \Phi_2) = \begin{cases} 1 + \frac{1}{p}, & \text{if } l \text{ is odd,} \\ 0, & \text{if } l \text{ is even.} \end{cases} \quad (5.3.41)$$

Substituting this back into the right-hand side of the identity for odd  $\text{ord}_p(m)$ , we get

$$W_{m,p}(e, 0, \Phi_2) \log p = (1 + p^{-1}) \log p. \quad (5.3.42)$$

This matches the derivative we computed above, verifying Equation 5.3.1.

To prove Equation 5.3.2, we will instead use the explicit formulas developed by Kudla and Yang in Appendix A.

# 6

## Conclusion

### 6.1 Reasoning Behind the Parity Difference

To conclude, we offer some insight into why the parity of  $\text{ord}_p(m)$  dictates the vanishing of the local Whittaker functionals. Recall that if

$$\varphi \in S(V) \mapsto \Phi \in I(s_0, \chi) \quad (6.1.1)$$

under the Weil representation, the Whittaker functional has the integral representation

$$W_{m,p}(e, s, \Phi) = \int_V \varphi(y) \left( \int_{\mathbb{Q}_p} \psi_p(b(m - Q(y))) db \right) dy. \quad (6.1.2)$$

If the quadratic space  $V$  does not represent  $m$  (meaning the equation  $Q(x) = m$  has no solutions  $x \in V$ ), the inner integral over  $b$  evaluates to 0, and consequently  $W_{m,p} = 0$ .

For our unramified space  $V_1 = \langle 1, -u \rangle$ , any element  $x \in V_1$  can be factored as  $x = p^l v$  for some unit  $v \in \mathcal{O}_{k_p}$ . Therefore, the norm is

$$N(x) = p^{2l} N(v). \quad (6.1.3)$$

Because  $N(v)$  is a unit, we have  $\text{ord}_p(N(x)) \in 2\mathbb{Z}$ . Thus,  $V_1$  cannot represent any  $m$  where  $\text{ord}_p(m)$  is odd.

Conversely, for the scaled space  $V_2$  with the form  $pN(x)$ , the valuation is  $\text{ord}_p(pN(x)) \in 1 + 2\mathbb{Z}$ . Hence,  $V_2$  cannot represent any  $m$  with an even valuation. This parity difference governs the swapping of the local functionals.

Since our global section  $\Phi = \otimes_v \Phi_v$  is factorizable, the  $m$ -th Fourier coefficient

of the Eisenstein series decomposes into local Whittaker functionals as given by Theorem 5.2.2

$$E_m(g, s, \Phi) = \prod_{v \leq \infty} W_{m,v}(g, s, \Phi_v). \quad (6.1.4)$$

We evaluate these local functionals, and their derivatives, specifically at  $s = 0$  because it is the center of symmetry for the functional equation of the global Eisenstein series.

Observe that if  $l = \text{ord}_p(m)$  is even, then  $m$  is not represented by  $V_2$ . By 16, Proposition 2.2, this non-representation guarantees that the central value of the functional  $W_{m,p}(e, 0, \Phi_2)$  must vanish. Conversely, if  $l$  is odd, the functional for  $\Phi_1$  vanishes at  $s = 0$ . Because of the aforementioned symmetry, this vanishing is expected. In these cases, it is necessary to compute the central derivative  $\frac{d}{ds} \big|_{s=0}$  to capture the non-trivial arithmetic information hidden at the central point.

While this parity explicitly determines when the central value vanishes, a complete geometric explanation as to why the derivative of the vanishing section exactly yields the value of the alternate section lies deep within the arithmetic intersection theory of the Kudla program, which remains beyond the scope of this thesis.

## 6.2 Summary

The primary objective of this thesis was to systematically bridge the gap between classical modular forms and the modern adelic framework of automorphic representations, using the arithmetic Siegel-Weil formula as our primary motivating application.

We began by establishing the classical theory of the holomorphic modular forms and Eisenstein series. To make these analytic tools suitable for modern arithmetic geometry, we transitioned to the adelic setting on  $\text{GL}(2, \mathbb{A})$ . By utilizing  $p$ -adic completions and group decompositions, we explicitly constructed the non-holomorphic Eisenstein series and the local Weil representation, which translates the algebraic geometry of quadratic spaces into the language of automorphic forms.

With this structural foundation, the thesis culminated in the explicit computation of local Whittaker functionals. For the specific unramified binary quadratic spaces  $V_1$  and  $V_2$ , we showed that the representation of an element  $m \in \mathbb{Q}_p$  is dictated entirely by the parity of its  $p$ -adic valuation. This parity governs the vanishing of the local sections  $\Phi_1$  and  $\Phi_2$ .

Using the explicit formulas for local representation densities developed by Yang,

we computed these derivatives and verified the local identity stated in Proposition 5.3.1. Finally, we provided a complete SageMath implementation in Appendix to algorithmically verify these theoretical formulas.

# Appendix A

## Kudla-Yang Formulas

Another way to show these identities hold is to utilize 14, Theorem 4.3. Let us introduce the necessary notation first.

Let  $L \cong \mathbb{Z}_p^n$  be an integral lattice with Gram matrix  $S = \text{diag}(2\epsilon_1 p^{l_1}, \dots, 2\epsilon_n p^{l_n})$ , where  $\epsilon_i \in \mathbb{Z}_p^\times$  and  $l_i \geq 0$  are integers. The dual lattice is then given by  $L^* = \bigoplus_{i=1}^n p^{-l_i} \mathbb{Z}_p$ .

For an element  $\mu = (\mu_1, \dots, \mu_n) \in V \cong \mathbb{Q}_p^n$ , we define the following data:

$$\begin{aligned} H_\mu &= \{i \mid 1 \leq i \leq n, \mu_i \in \mathbb{Z}_p\}, \\ K_0(\mu) &= \begin{cases} \infty, & \text{if } \mu \in L, \\ \min\{l_i + \text{ord}_p \mu_i \mid i \notin H_\mu\}, & \text{if } \mu \notin L, \end{cases} \\ L_\mu(k) &= \{i \in H_\mu \mid l_i - k < 0 \text{ is odd}\}, \\ l_\mu(k) &= |L_\mu(k)|, \\ d_\mu(k) &= k + \frac{1}{2} \sum_{i \in H_\mu} \min(l_i - k, 0), \\ \epsilon_\mu(k) &= \left(\frac{-1}{p}\right)^{[l_\mu(k)/2]} \prod_{i \in L_\mu(k)} \left(\frac{\epsilon_i}{p}\right), \end{aligned}$$

where the modified Legendre symbol is defined as

$$\left(\frac{\alpha}{p}\right) = \begin{cases} (\alpha, p)_p, & \text{if } \alpha \in \mathbb{Z}_p^\times, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A.0.1})$$

Finally, we define the indicator function

$$f_1(\alpha p^a) = \begin{cases} -\frac{1}{p}, & \text{if } l_\mu(a+1) \text{ is even,} \\ \left(\frac{\alpha}{p}\right) \frac{1}{\sqrt{p}}, & \text{if } l_\mu(a+1) \text{ is odd.} \end{cases} \quad (\text{A.0.2})$$

**Theorem A.0.1** (14, Theorem 4.3). *Let the notation be as above and fix  $\mu \in L^*$ .*

(i) *If  $m \notin Q(\mu) + \mathbb{Z}_p$ , then  $W_p(s + s_0, m, \mu) = 0$ .*

(ii) *If  $m \in Q(\mu) + \mathbb{Z}_p$ , set*

$$t_\mu = t_\mu(m) = m - \sum_{i \notin H_\mu} \epsilon_i p^{l_i} \mu_i^2, \quad a = a_\mu(m) = \text{ord}_p t_\mu, \quad (\text{A.0.3})$$

*and let  $X = p^{-s}$ .*

*If  $0 \leq a < K_0$ , then*

$$W_p(s + s_0, m, \mu) = 1 + \left(1 - \frac{1}{p}\right) \sum_{\substack{0 < k \leq a \\ l_\mu(k) \text{ even}}} \epsilon_\mu(k) p^{d_\mu(k)} X^k + \epsilon_\mu(a+1) f_1(t_\mu) p^{d_\mu(a+1)} X^{a+1}. \quad (\text{A.0.4})$$

*If  $a \geq K_0$ , then*

$$W_p(s + s_0, m, \mu) = 1 + \left(1 - \frac{1}{p}\right) \sum_{\substack{0 < k \leq K_0 \\ l_\mu(k) \text{ even}}} \epsilon_\mu(k) p^{d_\mu(k)} X^k. \quad (\text{A.0.5})$$

The Whittaker functionals in Theorem A.0.1 differ from our standard notation by a scaling factor. To write them in terms of our framework, we scale them as

$$\frac{W_{m,p}(e, s, \Phi_i)}{\gamma(V_i) |\det(S)|_p^{\frac{1}{2}}} = W_p(s_0 + s, m, \mu), \quad (\text{A.0.6})$$

where  $\gamma(V_i)$  is the Weil index of the quadratic space and  $\det(S)$  is the determinant of the corresponding Gram matrix. Note that the Weil indices for both quadratic spaces are 1.

All we have to do is determine what each of these functions evaluates to for our specific Whittaker functionals,  $W_{m,p}(e, 0, \Phi_1)$  and  $W_{m,p}(e, s, \Phi_2)$ .

Let us test this first on  $W_{m,p}(e, 0, \Phi_1)$ . Observe that since  $L = L^*$  for the unramified space  $V_1$ , we have  $\mu = \mathbf{0}$ . Furthermore, the valuation of the matrix entries is

zero, so  $l_1 = l_2 = 0$ . Evaluating the associated data for our specific case yields:

$$\begin{aligned}
H_\mu &= \{1, 2\}, \\
K_0(\mu) &= \infty, \\
L_\mu(k) &= \begin{cases} \{1, 2\}, & \text{if } k \text{ is odd,} \\ \emptyset, & \text{otherwise,} \end{cases} \\
l_\mu(k) &= \begin{cases} 2, & \text{if } k \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases} \\
d_\mu(k) &= 0.
\end{aligned}$$

For the sign character  $\epsilon_\mu(k)$ , recall that the determinant of the Gram matrix is  $4\epsilon_1\epsilon_2 = -u$ . When  $k$  is odd, the product of the Legendre symbols simplifies as

$$\left(\frac{\epsilon_1}{p}\right)\left(\frac{\epsilon_2}{p}\right) = \left(\frac{-u}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{u}{p}\right) = \left(\frac{-1}{p}\right)(-1),$$

where we used the fact that  $u$  is a non-square unit, forcing  $(u/p) = -1$ . This evaluation yields:

$$\epsilon_\mu(k) = \begin{cases} -1, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

Since  $l_\mu(k) \in \{0, 2\}$  is always even regardless of the parity of  $k$ , the indicator function reduces to the constant  $f_1 = -\frac{1}{p}$ . Furthermore, since  $\det(S) = -u$  for  $V_1$ , and  $u$  is a unit, its  $p$ -adic valuation is 0. This means  $|\det(S)|_p = 1$ , making our scaling factor exactly 1. Hence,  $W_{m,p}(e, 0, \Phi_1) = W_p(0, m, \mathbf{0})$ . Evaluating the sum gives

$$W_p(0, m, \mathbf{0}) = 1 + \left(1 - \frac{1}{p}\right) \sum_{k=1}^a (-1)^k + \frac{(-1)^{a+2}}{p} \quad (\text{A.0.7})$$

$$= \begin{cases} 0, & \text{if } a \text{ is odd,} \\ 1 + \frac{1}{p}, & \text{if } a \text{ is even.} \end{cases} \quad (\text{A.0.8})$$

The evaluation of these functions is slightly different for  $V_2$ . We keep  $\mu = 0$  because the standard section  $\Phi_2$  is still associated to the characteristic function of

the lattice  $L = \mathcal{O}_{k_p}$ . However, because the bilinear form on  $V_2$  is scaled by  $p$ , the valuation of the Gram matrix entries increases by one, meaning  $l_1 = l_2 = 1$ . Evaluating the associated data for  $V_2$  yields:

$$\begin{aligned} H_\mu &= \{1, 2\}, \\ K_0(\mu) &= \infty, \\ L_\mu(k) &= \begin{cases} \emptyset, & \text{if } k \text{ is odd,} \\ \{1, 2\}, & \text{if } k \text{ is even,} \end{cases} \\ l_\mu(k) &= \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 2, & \text{if } k \text{ is even,} \end{cases} \\ d_\mu(k) &= 1. \end{aligned}$$

For the sign character  $\epsilon_\mu(k)$ , recall that the determinant of the scaled Gram matrix is  $4\epsilon_1\epsilon_2p^2 = -up^2$ . When  $k$  is even,  $L_\mu(k) = \{1, 2\}$ , and the product of the Legendre symbols simplifies as

$$\left(\frac{\epsilon_1}{p}\right)\left(\frac{\epsilon_2}{p}\right) = \left(\frac{-u}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{u}{p}\right) = \left(\frac{-1}{p}\right)(-1),$$

where we once again used the fact that  $u$  is a non-square unit. Multiplying this by  $\left(\frac{-1}{p}\right)^{\lfloor l_\mu(k)/2 \rfloor} = \left(\frac{-1}{p}\right)^1$  yields:

$$\epsilon_\mu(k) = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ -1, & \text{if } k \text{ is even.} \end{cases}$$

Since  $l_\mu(k) \in \{0, 2\}$  is always even, the indicator function again reduces to  $f_1 = -\frac{1}{p}$ . For  $V_2$ , we have  $\det(S) = -up^2$ , which has a  $p$ -adic valuation of 2. Thus,  $|\det(S)|_p^{1/2} = (p^{-2})^{1/2} = p^{-1}$ , and our scaled relation is  $W_{m,p}(e, s, \Phi_2) = p^{-1}W_p(s, m, \mathbf{0})$ .

For  $m$ , we obtain the polynomial

$$W_p(s, m, \mathbf{0}) = 1 + \left(1 - \frac{1}{p}\right) \sum_{k=1}^a (-1)^{k+1} p^{-sk+1} + (-1)^{a+3} p^{-s(a+1)}, \quad (\text{A.0.9})$$

and for the shifted argument  $p^{-2}m$ , we have

$$W_p(s, p^{-2}m, \mathbf{0}) = 1 + \left(1 - \frac{1}{p}\right) \sum_{k=1}^{a-2} (-1)^{k+1} p^{-sk+1} + (-1)^{a+1} p^{-s(a-1)}. \quad (\text{A.0.10})$$

The difference between these evaluations is

$$W_p(s, m, \mathbf{0}) - W_p(s, p^{-2}m, \mathbf{0}) = \left(1 - \frac{1}{p}\right) [(-1)^a p^{-s(a-1)+1} + (-1)^{a+1} p^{-sa+1}] \quad (\text{A.0.11})$$

$$+ (-1)^{a+3} p^{-s(a+1)} - (-1)^{a+1} p^{-s(a-1)}. \quad (\text{A.0.12})$$

By analyzing the parity of  $\text{ord}_p(m) = a$  and computing the derivative  $\frac{d}{ds}$  at  $s = 0$ , we obtain

$$\frac{d}{ds} [W_p(s, m, \mathbf{0}) - W_p(s, p^{-2}m, \mathbf{0})] \Big|_{s=0} = \begin{cases} -(1+p) \log p, & \text{if } a \text{ is odd,} \\ (1+p) \log p, & \text{if } a \text{ is even.} \end{cases} \quad (\text{A.0.13})$$

Multiplying by our scaling factor  $p^{-1}$ , we get the final derivative

$$\frac{d}{ds} [W_{m,p}(e, s, \Phi_2) - W_{p^{-2}m,p}(e, s, \Phi_2)] \Big|_{s=0} = \begin{cases} -(1+p^{-1}) \log p, & \text{if } a \text{ is odd,} \\ (1+p^{-1}) \log p, & \text{if } a \text{ is even.} \end{cases} \quad (\text{A.0.14})$$

This confirms Equation 5.3.2.

# Appendix B

## Source Code

To compute these Whittaker functionals and ultimately verify the identity by utilizing 14, Theorem 4.3, we implement them in SageMath. Each function that paves the pathway is defined and functions as follows:

- Since the code was initially written for the identity check, we begin by determining the Gram matrices for each quadratic space. We then feed the code an odd prime  $p$ , a vector  $\mu$  belonging to the dual lattice (as defined in the previous section), and the Gram matrices `gram1` and `gram2` for the vector spaces  $V_1$  and  $V_2$ , respectively.
- The function `exponents_and_units(grm1, grm2)` extracts the  $p$ -adic order and the unit appearing in the factorization of each diagonal element of the Gram matrices. It returns the exponent lists `exp1_list` and `exp2_list` for the respective Gram matrices, alongside a single unit list `unit_list`. The reason we obtain only one list for the units is that  $V_2$  is a scaled version of  $V_1$ , thereby yielding the identical unit list.
- The functions `H_mu_func(vector)`, `L_mu_func(k, expo)`, `d_mu_func(k, expo)`, `K_0_func(vector_dual, expo)`, `epsilon_mu_func(k, expo)`, and `f1_func(mm, expo)` compute the functions  $H_\mu$ ,  $L_\mu(k)$ ,  $d_\mu(k)$ ,  $K_0(\mu)$ ,  $\epsilon_\mu(k)$ , and  $f_1$ , respectively.
- The function `whittaker_func(m_int, expo)` computes the Whittaker functional  $W_{m,p}(e, s, \Phi)$  by utilizing 14, Theorem 4.3.
- Finally, the functions `local_identity_check(mmm, expo1, expo2)` and `tables_are_turned(mmm, expo1, expo2)` verify Proposition 5.3.1 for odd and even  $\text{ord}_p(m)$ , respectively.

```

1  p = 5
2  mu = [0, 0]
3  gram1 = [1, -3]
4  gram2 = [p, p*(-3)]
5
6  def exponents_and_units(grm1, grm2):
7      exp1_list = []
8      exp2_list = []
9      unit_list = []
10     for i in grm1:
11         li = i.valuation(p)
12         exp1_list.append(li)
13         unt = Zp(p)(i).unit_part()
14         unit_list.append(unt/2)
15     for i in grm2:
16         lj = i.valuation(p)
17         exp2_list.append(lj)
18     return exp1_list, exp2_list, unit_list
19
20 exponents1, exponents2, units = exponents_and_units(gram1,
21     gram2)
22
23 Zp_ring = Zp(p)
24 Qp_field = Qp(p)
25
26 def H_mu_func(vector):
27     index_set = []
28     for i in range(len(vector)):
29         if vector[i].valuation(p) >= 0:
30             index_set.append(i+1)
31     return index_set
32
33 H_mu = H_mu_func(mu)
34
35 def L_mu_func(k, expo):
36     L_mu_list = []
37     for i in H_mu:
38         if (expo[i-1] - k) < 0 and (expo[i-1] - k) % 2 == 1:

```

```

38         L_mu_list.append(i)
39     return L_mu_list
40
41 def d_mu_func(k, expo):
42     sum_part = 0
43     for i in H_mu:
44         sum_part += min(expo[i-1]-k,0)
45     return k + (sum_part)/2
46
47 def K_0_func(vector_dual, expo):
48     summ = 0
49     for i in range(len(vector_dual)):
50         if vector_dual[i] in Zp_ring:
51             summ += 1
52     if summ == len(vector_dual):
53         return oo
54     else:
55         choose_min = []
56         for i in range(len(vector_dual)):
57             if vector_dual[i] not in Zp_ring:
58                 summm = expo[i] + vector_dual[i].valuation(p)
59                 choose_min.append(summm)
60         minn = min(choose_min)
61         return minn
62
63 def epsilon_mu_func(k, expo):
64     product = 1
65     for i in L_mu_func(k, expo):
66         product *= kronecker(units[i-1], p)
67     final_prod = product * kronecker(-1, p)**floor(len(
        L_mu_func(k, expo))/2)
68     return final_prod
69
70 def f1_func(mm, expo):
71     if len(L_mu_func(mm.valuation(p) + 1, expo)) % 2 == 0:
72         return -1/p
73     else:
74         return kronecker(Zp(p)(mm).unit_part(), p) * 1/sqrt(p)

```

```

75
76 def whittaker_func(m_int, expo):
77     var('q, s')
78     X = q^(-s)
79     if m_int.valuation(p) < 0:
80         return 0
81     else:
82         some_sum = 0
83         for i in range(1, len(H_mu)+1):
84             if i not in H_mu:
85                 some_sum += units[i-1]*p^(expo[i-1])*mu[i
                        -1]**2
86         t_mu = m_int - some_sum
87         a = t_mu.valuation(p)
88         if a < K_0_func(mu, expo):
89             poly1 = 0
90             for k in range(1, a+1):
91                 if len(L_mu_func(k, expo)) % 2 == 0:
92                     poly1 += epsilon_mu_func(k, expo)*(q**
                        d_mu_func(k, expo))*X**k
93             Whittaker_poly = 1 + (1- 1/q)*poly1 +
                        epsilon_mu_func(a+1, expo)*f1_func(t_mu, expo)
                        *(q**d_mu_func(a+1, expo))*X**(a+1)
94         elif a >= K_0_func(mu, expo):
95             poly2 = 0
96             for k in range (1, K_0_func(mu, expo) + 1):
97                 if len(L_mu_func(k, expo)) % 2 == 0:
98                     poly2 += epsilon_mu_func(k, expo)*(q**
                        d_mu_func(k, expo))*X**k
99             Whittaker_poly = 1 + (1 - 1/q)*poly2
100         return Whittaker_poly
101
102 def local_identity_check(mmm, expo1, expo2):
103     sum_expo1 = 0
104     for i in expo1:
105         sum_expo1 += i
106     LHSWhit = whittaker_func(mmm, expo1) * p^(-sum_expo1/2)
107     LHSWhit_scaled = whittaker_func(mmm*p^(-2), expo1) * p^(-

```

```

        sum_expo1/2)
108     sum_expo2 = 0
109     for i in expo2:
110         sum_expo2 += i
111     RHS = whittaker_func(mmm, expo2).subs(s=0) * p^(-sum_expo2
        /2)
112     LHS = diff(LHSWhit - LHSWhit_scaled, s).subs(s=0)
113     check = (LHS/log(q) - RHS).subs(q=p)
114     return check
115
116 def tables_are_turned(mmm, expo1, expo2):
117     sum_expo2 = 0
118     for i in expo2:
119         sum_expo2 += i
120     LHSWhit = whittaker_func(mmm, expo2) * p^(-sum_expo2/2)
121     LHSWhit_scaled = whittaker_func(mmm*p^(-2), expo2) * p^(-
        sum_expo2/2)
122     sum_expo1 = 0
123     for i in expo1:
124         sum_expo1 += i
125     RHS = whittaker_func(mmm, expo1).subs(s=0) * p^(-sum_expo1
        /2)
126     LHS = diff(LHSWhit - LHSWhit_scaled, s).subs(s=0)
127     check = (LHS/log(q) - RHS).subs(q=p)
128     return check

```

# Appendix C

## Examples

### C.1 Identity Check

Now that we have the necessary codes to implement Kudla-Yang formulas, we can showcase how they work.

The implementation begins by defining the base prime  $p$ , the lattice shift  $\mu$ , and the Gram matrices for the quadratic spaces  $V_1$  and  $V_2$ . Because Proposition 5.3.1 evaluates standard sections associated to the characteristic function of the ring of integers  $\mathcal{O}_{k_p}$ , we work on the base lattice and fix the shift parameter as  $\mu = [0, 0]$ .

When altering the parameters to test different quadratic spaces, it is crucial to preserve the condition that  $k_p/\mathbb{Q}_p$  is an unramified quadratic extension.

For a basis  $\{1, \sqrt{u}\}$ , the Gram matrix of  $V_1$  takes the form  $\text{diag}(1, -u)$ , where  $u \in \mathbb{Z}_p^\times$ . To ensure the extension does not split,  $u$  must be chosen as a quadratic non-residue modulo  $p$ . This means we require the Legendre symbol to be

$$\left(\frac{u}{p}\right) = -1. \tag{C.1.1}$$

For example, by default we use  $p = 5$  and  $u = 3$ , since 3 is a non-residue modulo 5. Thus, our unscaled Gram matrix is  $[1, -3]$ , and the scaled matrix for  $V_2$  is  $[5, -15]$ . If the reader wishes to test other primes,  $u$  must be adjusted accordingly:

- For  $p = 3$ , the only non-residue is 2. Use `gram1 = [1, -2]`.
- For  $p = 7$ , non-residues are 3, 5, 6. Use `gram1 = [1, -3]`.
- For  $p = 11$ , non-residues are 2, 6, 7, 8, 10. Use `gram1 = [1, -2]`.

The function `local_identity_check` evaluates the derivative on  $V_1$  and the constant term on  $V_2$  (testing the case where  $\text{ord}_p(m)$  is odd). The function `tables_are_turned` evaluates the derivative on  $V_2$  and the constant term on  $V_1$  (testing the case where  $\text{ord}_p(m)$  is even). In all cases, the functions subtract the right-hand side from the left-hand side, returning 0 when the identity holds.

#### Example Cases

```

1 # Case 1: The Odd Valuation (m = 5)
2 # Tests the standard identity where ord_p(m) = 1.
3 sage: local_identity_check(5, exponents1, exponents2)
4 0
5
6 # Case 2: The Even Valuation (m = 25)
7 # Proves that the roles of V_1 and V_2 swap when ord_p(m) =
8   2.
9 sage: tables_are_turned(25, exponents1, exponents2)
10 0
11
12 # Case 3: Unit Twists (m = 10, m = 15)
13 # Proves the identity relies purely on valuation, holding
14   for m = 2*5 and m = 3*5.
15 sage: local_identity_check(10, exponents1, exponents2)
16 0
17
18 # Case 4: The Trivial Case (m = 1/5)
19 # Forces the algorithm into the negative valuation,
20   verifying it correctly vanishes.
21 sage: local_identity_check(1/5, exponents1, exponents2)
22 0

```

## C.2 Whittaker Functionals

We can also generate Whittaker functionals in higher dimensions by increasing the size of the Gram matrices. We still retain both Gram matrices and the vector  $\mu$ , as the functions leading to the Whittaker functional are defined based on them.

## </> Example Cases

```
1      # Case 1: Dimension 2
2      sage: gram1 = [1, -3]
3      sage: gram2 = [p, p*(-3)]
4      sage: mu = [0, 0]
5      sage: print(whittaker_func(9*p^6, exponents1))
6      (1/q^s - 1/q^(2*s) + 1/q^(3*s) - 1/q^(4*s) + 1/q^(5*
7      s) - 1/q^(6*s))*(1/q - 1) + 1/7/q^(7*s) + 1
8      sage: print(whittaker_func(9*p^7, exponents2))
9      -(q/q^s - q/q^(2*s) + q/q^(3*s) - q/q^(4*s) + q/q
10     ^ (5*s) - q/q^(6*s) + q/q^(7*s))*(1/q - 1) + 1/7*q
11     /q^(8*s) + 1
12
13     # Case 2: Dimension 4
14     sage: gram1 = [2, 6, 28, p]
15     sage: gram2 = [1, 1, 1, 1]
16     sage: mu = [0, 0, 0, 0]
17     sage: print(whittaker_func(9*p^15, exponents1))
18     -(1/q^s - 1/(q*q^(2*s)) + 1/(q^2*q^(3*s)) - 1/(q^3*q
19     ^ (4*s)) + 1/(q^4*q^(5*s)) - 1/(q^5*q^(6*s)) + 1/(
20     q^6*q^(7*s)) - 1/(q^7*q^(8*s)) + 1/(q^8*q^(9*s))
21     - 1/(q^9*q^(10*s)) + 1/(q^10*q^(11*s)) - 1/(q^11*
22     q^(12*s)) + 1/(q^12*q^(13*s)) - 1/(q^13*q^(14*s))
23     + 1/(q^14*q^(15*s)))*(1/q - 1) + 1/7/(q^15*q
24     ^ (16*s)) + 1
25
26     # Case 3: Dimension 5
27     sage: gram1 = [p^6, 1, -1, 5, 17]
28     sage: gram2 = [1, 1, 1, 1, 1, 1]
29     sage: mu = [1/p, 0, 0, 0, 0]
30     sage: print(whittaker_func(9*p^6, exponents1))
31     (1/q - 1)*(1/(q*q^s) - 1/(q^2*q^(2*s)) + 1/(q^3*q
32     ^ (3*s)) - 1/(q^4*q^(4*s))) + 1/3/(q^5*q^(5*s)) +
33     1
```

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