Absolute Extreme Points of Matrix Convex Sets: Existence and Spanning

by

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Abstract

The study of matrix convex sets is a relatively young endeavour, being first proposed in 1984 by Wittstock. Matrix convex sets can be found as the matrix state spaces of operator systems, and as the solution sets of linear matrix inequalities suitably modified to allow for matrix solutions. Some concepts from classical convexity can be translated to the matrix convexity setting in more than one way, such as the notion of an extreme point of a convex set. We will see two kinds of extreme point for matrix convex sets: one called a matrix extreme point, and the other called an absolute extreme point. Important results from classical convexity theory, such as the Krein-Milman theorem, have been proven to hold in the matrix convexity setting. One such Krein-Milman-type theorem was proven by Webster and Winkler in 1999, using the notion of a matrix extreme point. In this work, we present the proof of another Krein-Milman-type theorem, first proven by Evert and Helton in 2019, that holds for a special class of compact matrix convex sets, and that uses the notion of absolute extreme points. After this, we proceed to use absolute extreme points to propose a definition of strict matrix convexity, which as yet has no agreed upon definition.

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Contents

List of Symbols

$\mathbb{M}_{m,n}$	the set of $m \times n$ matrices over $\mathbb{C}$
$\mathbb{M}_n$	$= \mathbb{M}_{n,n}$
$\mathbb{S}_n$	the set of $n \times n$ self-adjoint matrices over $\mathbb{C}$
$\mathbb{S}_n^g$	the set of g-tuples of $n \times n$ self-adjoint matrices over $\mathbb{C}$
$\mathbb{S}^g$	$= (\mathbb{S}_n^g)_{n \geqslant 1}$
$\operatorname{co} E$	the Euclidean convex hull of E
$\overline{\operatorname{co}}\, E$	the closed Euclidean convex hull of E
$\operatorname{mco} E$	the matrix convex hull of E
$\overline{\operatorname{mco}}\, E$	the closed matrix convex hull of E
$\partial^{\mathrm{top}} K$	the set of topological boundary points of K
$\partial^{\mathrm{Euc}} K$	the set of Euclidean extreme points of K
$\partial^{\mathrm{mat}} K$	the set of matrix extreme points of K
$\partial^{\mathrm{abs}} K$	the set of absolute extreme points of K
$\partial^{\mathrm{Arv}} K$	the set of Arveson boundary points of K
$X \oplus Y$	the direct sum of matrices X and Y
$X \otimes Y$	the tensor product, or Kronecker product, of matrices X and Y
L_Ω	the monic linear pencil determined by the matrix tuple Ω
Λ_Ω	the homomogeneous linear pencil determined by the matrix tuple Ω
$\mathcal{D}_\Omega$	the free spectrahedron determined by the matrix tuple Ω

1

Introduction

Matrix convex sets were first introduced by Wittstock [14] in 1984. There are two commonly cited research motivations for the study of matrix convex sets; one comes from operator theory and the other comes from convex optimization theory.

Regarding operator theory, it suffices to say that all compact matrix convex sets over a vector space can be realized as the so-called *matrix state space* of an appropriate operator system. In fact the correspondence is one-to-one, and can be expressed as a categorical duality [13, Proposition 3.5], so that the study of matrix convex sets and the study of operator systems are closely entwined. See also [6] for a paper that deals with such matters.

On the convex optimization front, a special class of convex subsets of Euclidean space called *spectrahedra* appear as the solution sets of *linear matrix inequalities.* A common task is to optimize a linear functional over a spectrahedron. The expressions involved in linear matrix inequalities also make sense with g-tuples of matrices instead of g-tuples of real numbers, and the matricial solution set of a linear matrix inequality is a special kind of matrix convex set called a *free spectrahedron.* We will learn more about free spectrahedra in Chapter 3.

This thesis does not require a familiarity with either operator theory or optimization.

A result of profound significance in classical convexity theory and functional analysis is the Krein-Milman theorem, which states that a non-empty compact convex subset of a locally convex space is the closed convex hull of its *extreme points* [1, Chapter V, Theorem 7.4]. Naturally, one wonders whether such a result holds in the matrix convexity setting. Indeed, one such result was proven by Webster and Winkler [13, Theorem 4.3]. It states that a non-empty compact matrix convex set is the closed matrix convex hull of its matrix extreme points, and it was later proven [10, Theorem 6.8] that the (non-closed) matrix convex hull is sufficient for the class of matrix convex sets we will be dealing with. There is another kind of extreme point in the matrix convexity setting, and this is called an absolute extreme point. The theorem of Webster and Winkler does not hold with absolute extreme points, for there are examples of matrix convex sets having no absolute extreme points at all! Nonetheless, Krein-Milman-type results for absolute extreme points do hold at the cost of additional restrictions on the matrix convex sets being considered.

Overall, the goal of this thesis is to present in full detail one such Krein-Milman-type result recently proven by Evert and Helton [4, Theorem 1.3]. The matrix convex sets to which their result applies are the compact free spectrahedra generated by a tuple of real symmetric matrices. Free spectrahedra are interesting objects of study in their own right. It turns out that any *compact* matrix convex set containing 0 in the interior of the first matrix level must actually be an intersection of (possibly infinitely many) free spectrahedra [3, Theorem 5.4], much like how any compact convex subset of Euclidean space is an intersection of half-spaces. This suggests that free spectrahedra aren't an especially small subclass of matrix convex sets. The proof of the Krein-Milman-type result of Evert and Helton relies heavily on a tool provided by Evert, Helton, Klep, and McCullough [5, Theorem 1.1], namely a characterization of absolute extreme points that uses the language of dilation theory.

We will frequently need to distinguish between matrix convexity notions and the corresponding classical convexity notions on which they are based. We will use the word "Euclidean" in reference to classical convexity notions, since it is suggestive of the fact that classical convexity only takes place in one Euclidean space (for example, $\mathbb{R}^g$) at a time, whereas matrix convexity requires one to consider an interacting hierarchy of such spaces (the different matrix levels). As such, we will refer to such things as "Euclidean convex combinations", "Euclidean extreme points", or "Euclidean convex hulls" in situations where it would be ambiguous to omit this extra word.

This thesis is divided into five chapters, with this introduction being Chapter 1. Chapter 2 is a general introduction to matrix convex sets. We give the definitions and basic properties of matrix convex sets, and also state the dilation-theoretic characterization of absolute extreme points which features so heavily in the proof of the main result.

In Chapter 3, free spectrahedra are properly introduced, and then the aforementioned Krein-Milman-type result of Evert and Helton is proven in full detail. A crucial technical lemma depends on the notion of a *pseudoinverse* of a matrix, and Appendix A provides the relevant definitions and properties thereof. After the Krein-Milman type result has been proven, we inspect the absolute extreme points of some specific free spectrahedra to which the main result applies.

Using what we know about absolute extreme points, in Chapter 4 we propose the notion of *strict matrix convexity*, an extension of the classical notion of strictly convex sets. We discuss the difficulty in choosing a definition which appropriately captures the essence of the classical definition. We expect that absolute extreme points will have a role to play in any good definition of strict matrix convexity. Some definitions will be ruled out entirely as untenable. We will propose another definition, given the tentative name of Property A, which is not yet known to be problematic, though analysis of some examples suggests that it may be. In anticipation of *some* definition of strict matrix convexity, whether it be ours or not, we collect various properties of the absolute extreme points and topological boundary points of free spectrahedra and several specific examples thereof. Some of this information is found across the literature, but others, particularly Theorem 4.3.3, are original contributions.

2

Matrix Convexity and Extreme Points

2.1 Convex Combinations of Matrices

We let $\mathbb{M}_{n,m}$ denote the set of complex $n \times m$ matrices, and define $\mathbb{M}_n := \mathbb{M}_{n,n}$.

In a vector space V, a convex combination of points $y_1, y_2, \ldots, y_t \in V$ is a linear combination of those points whose scalar coefficients are non-negative and sum to 1, that is,

$$x = \sum_{i=1}^{t} \lambda_i y_i, \qquad \text{where } \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{t} \lambda_i = 1.$$

Although there is no natural reason, a priori, to do so, we can select complex numbers c_i such that $\overline{c_i} c_i = |c_i|^2 = \lambda_i$. Then it follows that

$$x = \sum_{i=1}^{t} \overline{c_i} y_i c_i, \qquad \text{where } c_i \in \mathbb{C} \text{ and } \sum_{i=1}^{t} \overline{c_i} c_i = 1. \tag{1}$$

This is a needlessly convoluted presentation of the original definition in terms of non-negative scalars λ_i, but it is this latter formulation of the definition that extends nicely to situations where matrices of possibly different sizes are the objects that we wish to combine in a kind of convex combination.

Consider a simple example, where we have

$$x \in \mathbb{M}_1 \text{ and } Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \mathbb{M}_2,$$

and we want to form some kind of convex combination, whatever that may mean, of x and Y. It won't do to take a scalar $\lambda \in [0,1]$ and write $\lambda x + (1-\lambda)Y$, since λx and $(1-\lambda)Y$ do not act on the same vector space. So instead we want our extended definition of convex combination to include a means changing a matrix in $\mathbb{M}_n$ to a matrix in $\mathbb{M}_m$, where possibly $m \neq n$. Such an action can be accomplished via *conjugating by a matrix*. In the specific case of x and Y above, we could conjugate Y by, for example, the matrix $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which provides us with $C^*YC = y_{11}$. Alternatively, we could conjugate x by the matrix $D = \begin{bmatrix} 1 & 0 \end{bmatrix}$, for then

$$D^*xD = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

In either case, the conjugated matrix is readily added to the other matrix. Moreover, C could really be taken as any 2×1 complex matrix, and D any 1×2 complex matrix, although the expression for the resulting conjugated matrix would be more difficult to behold.

Thus, it seems that the following might make a good definition: a matrix X is a *matrix convex combination* of matrices $Y_1, \ldots, Y_t$ if there exist matrices $C_1, \ldots, C_t$ such that the conjugates $C_i^* Y_i C_i$ are defined and act on the same space, and for which the following properties hold:

$$X = \sum_{i=1}^{r} C_i^* Y_i C_i, \quad \text{and} \quad \sum_{i=1}^{r} C_i^* C_i = I.$$

A more careful and general statement of this definition is forthcoming, but we must first develop such language as can allow us to work not only with matrices but with tuples of matrices as well.

2.2 Matrix Tuples

In this thesis, almost all material involves tuples of self-adjoint square matrices. We reserve the letter g for the positive integer representing the number of component matrices in these tuples, which can be thought of as the "number of variables". We also reserve the letter n for the size of a square matrix or the common size of the component matrices in a matrix tuple.

We reiterate that $\mathbb{M}_{n,m}$ is the set of $n \times m$ complex matrices, and that $\mathbb{M}_n := \mathbb{M}_{n,n}$. The $n \times n$ identity matrix is I_n and $0_{n,m}$ denotes the $n \times m$ zero matrix, but in many situations there is no ambiguity if we just write I or 0. We now introduce the notation $\mathbb{S}_n$ for the set of *self-adjoint* complex $n \times n$ matrices.

We will write $\mathbb{M}_{n,m}^g$, $\mathbb{M}_n^g$, and $\mathbb{S}_n^g$ to for the set of all g-tuples $X = (X_1, X_2, \ldots, X_g)$, where each $X_i \in \mathbb{M}_{n,m}$, $X_i \in \mathbb{M}_n$, and $X_i \in \mathbb{S}_n$ respectively. Therefore $\mathbb{S}_1^g$ consists of all g-tuples of real numbers, that is, $\mathbb{S}_1^g = \mathbb{R}^g$. We define $\mathbb{S}^g$ to be the sequence $(\mathbb{S}_n^g)_{n \geqslant 1}$. One can think of $\mathbb{R}^g$ as just the first of infinitely many "levels" of the larger object $\mathbb{S}^g$. This multi-leveled space $\mathbb{S}^g$ is the setting in which all material in this thesis takes place.

Despite the fact that $\mathbb{S}^g$ is defined to be a *sequence* of sets, we will speak of *subsets* of $\mathbb{S}^g$, and write such things as $K \subseteq \mathbb{S}^g$, which really means a sequence $(K(n))_{n \geqslant 1}$ where $K(n) \subseteq \mathbb{S}_n^g$ for all $n \geqslant 1$.

We now proceed to define some of the relations and operations that exist within $\mathbb{S}^g$.

Each individual matrix level $\mathbb{S}_n^g$ comes equipped with the usual addition and scalar action over $\mathbb{R}$, performed component-wise with respect to the g variables. An operation that can combine different matrix levels is component-wise conjugation by a single matrix. That is, if $X = (X_1, \ldots, X_g) \in \mathbb{S}_n^g$ and $C \in \mathbb{M}_{n,m}$, we define

$$C^* X C = C^*(X_1, \ldots, X_g)C = (C^* X_1 C, \ldots, C^* X_g C) \in \mathbb{S}_m^g.$$

If C happens to be an isometry[1], then C^*XC is called a *compression* of X, and X is called a *dilation* of C^*XC. Note that C being an isometry forces $n \leqslant m$. Letting $k = m - n$, we say that X is a *k-dilation of C^*XC*.

Given two matrix tuples $X \in \mathbb{S}_n^g$ and $Y \in \mathbb{S}_m^g$, we say that X and Y are *unitarily equivalent* if $n = m$ and there exists a unitary $n \times n$ matrix U such that $X = U^*YU$. This equivalence relation will be denoted by $X \overset{\text{u}}{\equiv} Y$ if we do not need to be explicit about a particular choice of unitary matrix.

A handy characterization of dilations is as follows.

Proposition 2.2.1. *Let $X \in \mathbb{S}_n^g$ and $Y \in \mathbb{S}_m^g$. Then Y is a dilation of X if and only if either $Y \overset{\text{u}}{\equiv} X$, or $m > n$ and there exists $A \in \mathbb{M}_{n,m-n}^g$ and $B \in \mathbb{S}_{m-n}^g$ such that*

$$Y \overset{\text{u}}{\equiv} \begin{bmatrix} X & A \\ A^* & B \end{bmatrix}. \tag{*}$$

Proof. Suppose that Y has the form in (*). We define the isometry V to be the mapping $\mathbb{C}^n \to \mathbb{C}^m : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n, 0, \ldots, 0)$. In matrix form this looks

[1] Since we routinely identify matrices $C \in \mathbb{M}_{n,m}$ with the corresponding linear map $\mathbb{C}^m \to \mathbb{C}^n$, what we mean when we say that a matrix C is an isometry is that the linear map represented by C is an isometry (equivalently, $C^*C = I$).

like $V = \begin{bmatrix} I_n \\ 0_{m-n,n} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$. We compute

$$V^* \begin{bmatrix} X & A \\ A^* & B \end{bmatrix} V = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} X & A \\ A^* & B \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} X \\ A^* \end{bmatrix} = X.$$

Conversely, suppose Y is a dilation of X, and let $V \in \mathbb{M}_{n,m}$ be an isometry for which $X = V^*YV$. From this it already follows that $m \geqslant n$. If V happens to be unitary then we are done, so suppose not. The columns of V form an orthonormal subset of $\mathbb{C}^n$, and by extending this to an orthonormal *basis*, we obtain a unitary $U \in \mathbb{M}_n$ of the form $U = \begin{bmatrix} V & W \end{bmatrix}$ for some isometry $W \in \mathbb{M}_{n,n-m}$.

Then by computation we have

$$U^*YU = \begin{bmatrix} V^* \\ W^* \end{bmatrix} Y \begin{bmatrix} V & W \end{bmatrix} = \begin{bmatrix} V^*YV & V^*YW \\ W^*YV & W^*YW \end{bmatrix} = \begin{bmatrix} X & V^*YW \\ (V^*YW)^* & W^*YW \end{bmatrix},$$

which is of the desired form. □

The *direct sum* of two matrices $X \in \mathbb{S}_n$ and $Y \in \mathbb{S}_m$ is defined as

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathbb{S}_{n+m}.$$

Then we define the direct sum of matrix *tuples* $X \in \mathbb{S}_n^g$ and $Y \in \mathbb{S}_m^g$ as

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} := (X_1 \oplus Y_1,\ X_2 \oplus Y_2,\ \ldots,\ X_g \oplus Y_g)\,.$$

By the previous proposition, the direct sum of X and Y is an example of a dilation of X (and of Y).

Evidently, $X \oplus Y$ admits a non-trivial subspace of $\mathbb{C}^{n+m}$ which is reducing for each component matrix $X_i \oplus Y_i$. A matrix tuple $Z \in \mathbb{S}_n^g$ is *irreducible* if there is *no* non-trivial subspace of $\mathbb{C}^n$ which is simultaneously a reducing subspace for each component matrix Z_i. Since the matrices are self-adjoint, there is no distinction between a reducing subspace and an invariant subspace.

An inductive argument can be used to show that any self-adjoint matrix tuple can be expressed, up to unitary equivalence, as a direct sum of irreducible self-adjoint matrix tuples.

Proposition 2.2.2. *Let $X \in \mathbb{S}_n^g$. Then there exist irreducible tuples $Y_1, \ldots, Y_t \in \mathbb{S}^g$ for some $t \geqslant 1$ such that $X \overset{\mathrm{u}}{\equiv} Y_1 \oplus \cdots \oplus Y_t$.*

Proof. We use induction on n. In case $X \in \mathbb{S}_1^g$ then X is necessarily irreducible and $X \overset{\mathrm{u}}{\equiv} X$ so we are done. Now suppose $n \geqslant 2$ and that the statement is true for all elements of $\mathbb{S}_k^g$ $(1 \leqslant k \leqslant n-1)$. If X is irreducible, then we are done already. If X is reducible, then $X \overset{\mathrm{u}}{\equiv} Y \oplus Z$ for some $Y \in \mathbb{S}_{m_1}^g$ and $Z \in \mathbb{S}_{m_2}^g$, and necessarily $m_1, m_2 < n$. By assumption, there exist irreducible matrix tuples $Y_1, \ldots, Y_s \in \mathbb{S}^g$ and $Z_1, \ldots, Z_t \in \mathbb{S}^g$ such that $Y \overset{\mathrm{u}}{\equiv} Y_1 \oplus \cdots \oplus Y_s$ and $Z \overset{\mathrm{u}}{\equiv} Z_1 \oplus \cdots \oplus Z_t$. Therefore $X \overset{\mathrm{u}}{\equiv} Y_1 \oplus \cdots \oplus Y_s \oplus Z_1 \oplus \cdots \oplus Z_t$, as required. $\square$

Irreducibility will become an important notion once we are ready to talk about extreme points of matrix convex sets.

2.3 Matrix Convex Combinations and Sets

Now that we have some operations under which $\mathbb{S}^g$ is closed, we want to consider certain subsets of $\mathbb{S}^g$ which are also closed under these operations. We will actually come at this from a different direction, by giving the common definition of a matrix convex set and then showing that the defining property is equivalent to closure under direct sums and compressions.

Definition 2.3.1 (Matrix convex combination)**.** A matrix tuple $X \in \mathbb{S}_n^g$ is said to be a *matrix convex combination* of the matrix tuples $Y_1, \ldots, Y_t$ (with each $Y_i \in \mathbb{S}_{n_i}^g$) if there exist $C_i \in \mathbb{M}_{n,n_i}$ $(i = 1, \ldots, t)$ satisfying

$$X = \sum_{i=1}^{N} C_i^* Y_i C_i, \quad \text{and } \sum_{i=1}^{t} C_i^* C_i = I_n.$$

The conjugating matrices C_i in such an expression will be called the *coefficients*.

Example 2.3.2. For a fixed n, a Euclidean convex combination of points in $\mathbb{S}_n^g$ is also a matrix convex combination of those same points. For, let $Y_1, \ldots, Y_t \in \mathbb{S}_n^g$, and consider scalars $\lambda_i \geqslant 0$ such that $\sum_{i=1}^t \lambda_i = 1$ (thus $\sum_{i=1}^t \lambda_i Y_i$ is the Euclidean convex combination we are interested in). Define $C_i := \sqrt{\lambda_i} I_n$. Then $C_i \in \mathbb{M}_n^g$, and we have

$$\sum_{i=1}^{t} C_i^* C_i = \sum_{i=1}^{t} (\sqrt{\lambda_i} I_n)^* (\sqrt{\lambda_i} I_n) = \left(\sum_{i=1}^{t} \lambda_i \right) I_n = I_n,$$

and

$$\sum_{i=1}^{t} C_i^* Y_i C_i = \sum_{i=1}^{t} (\sqrt{\lambda_i} I_n)^* Y_i (\sqrt{\lambda_i} I_n) = \sum_{i=1}^{t} \lambda_i Y_i.$$

Example 2.3.3. Direct sums are matrix convex combinations. For suppose $Z = X \oplus Y$, where $X \in \mathbb{S}_n^g$ and $Y \in \mathbb{S}_m^g$. Consider the linear map $P : \mathbb{C}^{n+m} \to \mathbb{C}^n$ given by $P(v_1, \ldots, v_{n+m}) := (v_1, \ldots, v_n)$, and $Q : \mathbb{C}^{n+m} \to \mathbb{C}^m$ given by $Q(v_1, \ldots, v_{n+m}) := (v_{n+1}, \ldots, v_{m+n})$. Then P^*P is the orthogonal projection of $\mathbb{C}^{n+m}$ onto the first n coordinates, and Q the orthogonal projection onto the final m coordinates. Hence $P^*P + Q^*Q = I_{m+n}$, and we compute

$$(P^*XP + Q^*YQ)(u \oplus v) = P^*Xu + Q^*Yv = ((Xu) \oplus 0) + (0 \oplus (Yv))$$

$$= (Xu) \oplus (Yv) = (X \oplus Y)(u \oplus v) = Z(u \oplus v).$$

Thus $Z = X \oplus Y$ is a matrix convex combination of X and Y.

Example 2.3.4. Compressions (in particular, unitary conjugates) are matrix convex combinations. This is easy to see: suppose $X \in \mathbb{S}_n^g$ and let $V : \mathbb{C}^m \to \mathbb{C}^n$ be an isometry. Then $V^*V = I_m$, and so V^*XV satisfies the definition of a matrix convex combination (in a somewhat degenerate way, as we have only one term and one coefficient). Thus a compression of X is a matrix convex combination of X.

Although the definition of a matrix convex combination is very clearly inspired by the classical notion of a convex combination, it is not always the easiest to work with. As a loose converse to the previous two examples of matrix convex combinations, we can always express a matrix convex combination of matrix tuples $Y_1, \ldots, Y_t$ as a "compression of a direct sum" of the very same matrix tuples.

Example 2.3.5. Consider a matrix convex combination of $Y_i \in \mathbb{S}_{n_i}^g$ $(i = 1, \ldots, t)$, say

$$X := \sum_{i=1}^{t} C_i^* Y_i C_i, \quad I_n = \sum_{i=1}^{t} C_i^* C_i, \quad C_i \in \mathbb{M}_{n_i,n}.$$

We can assemble all the coefficients into one isometry, and use it to compress the direct sum of all the tuples Y_i. This can be done in a relatively straightforward way:

$$Y := Y_1 \oplus \cdots \oplus Y_t, \text{ and } C := \begin{bmatrix} C_1 \\ \vdots \\ C_t \end{bmatrix}.$$

We have that C is an isometry, since

$$C^*C = \sum_{i=1}^{t} C_i^* C_i = I_n.$$

Since C has codomain $\mathbb{C}^{n_1+\cdots+n_t}$, which is the domain of Y, we may compress Y by

C, and we get

$$C^*YC = \begin{bmatrix} C_1^* & \cdots & C_t^* \end{bmatrix} \begin{bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_t \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_t \end{bmatrix} = \sum_{i=1}^{t} C_i^* Y_i C_i = X.$$

Often it is more convenient to express matrix convex combinations in this way, rather than dealing with the more complicated form given in Definition 2.3.1.

Just as Euclidean convex sets are defined as those subsets of a vector space that are closed under the formation of Euclidean convex combinations of their points, so shall we define matrix convex sets.

Definition 2.3.6 (Matrix convex set)**.** A subset $K \subseteq \mathbb{S}^g$ is called *matrix convex* if it is closed under the formation of matrix convex combinations. That is, whenever $Y_i \in K$ $(i = 1, \ldots, t)$ and $X \in \mathbb{S}^g$ is a matrix convex combination of the points Y_i, then it follows that $X \in K$. We denote the n-th matrix level of K by $K(n)$.

Each matrix level $K(n)$ of a matrix convex set K is Euclidean convex. This is not deep: as we already saw, Euclidean convex combinations of points can be expressed as matrix convex combinations of the same points, so closure under the latter entails closure under the former.

As an immediate consequence of Examples 2.3.3, 2.3.4, and 2.3.5, we have the following equivalent condition for a set to be matrix convex.

Corollary 2.3.7. *A subset $K \subseteq \mathbb{S}^g$ is matrix convex if and only if it is closed under compression and the formation of direct sums.*

For a simple example of matrix convex sets, consider the *matrix intervals*.

Example 2.3.8 (Matrix intervals)**.** In this example we have $g = 1$, that is, instead of matrix tuples we just have matrices. Let $a, b \in \mathbb{R}$ with $a < b$. Define

$$[aI, bI] := \{X \in \mathbb{S} : aI \leqslant X \leqslant bI\}.$$

Note that the first matrix level is the closed interval $[a, b] \subseteq \mathbb{R}$, which justifies the name *matrix interval*.

We will show that $[aI, bI]$ is a matrix convex set. If $X, Y \in [aI, bI]$, with $X \in \mathbb{S}_n$ and $Y \in \mathbb{S}_m$, then $aI_n \leqslant X$ and $aI_m \leqslant Y$. It follows that $aI_{n+m} = aI_n \oplus aI_m \leqslant X \oplus Y$. Similarly, $X \oplus Y \leqslant bI_{n+m}$, whence $X \oplus Y \in [aI, bI]$. If $X \in [aI, bI]$ and V is an isometry, then $aI = V^*(aI)V \leqslant V^*XV$. Similarly $V^*XV \leqslant bI$, whence $V^*XV \in [aI, bI]$. Closure under direct sums and compressions shows that $[aI, bI]$ is matrix convex.

Example 2.3.9 (OH-ball)**.** Consider the *OH-ball*, defined to be the set $B_g^{\mathrm{OH}} := \{X \in \mathbb{S}^g : \sum_{i=1}^g X_i^2 \leqslant I\}$. The name comes from [7, Section 14], where "OH" refers to "operator" and "Hilbert", although [5, Section 7.2] calls B_2^{OH} the "wild disk".

The OH-ball has the closed unit ball $\mathbb{B}_g \subseteq \mathbb{R}^g$ as its first matrix level. To see that B_g^{OH} is matrix convex, consider $X \in B_g^{\mathrm{OH}}(n)$ and $Y \in B_g^{\mathrm{OH}}(m)$. We have

$$\sum_{i=1}^g (X_i \oplus Y_i)^2 = \left(\sum_{i=1}^g X_i^2\right) \oplus \left(\sum_{i=1}^g Y_i^2\right) \leqslant I_n \oplus I_m = I_{n+m},$$

which shows that $X \oplus Y \in B_g^{\mathrm{OH}}$. If $X \in B_g^{\mathrm{OH}}(n)$ and $V \in \mathbb{M}_{n,m}$ is an isometry, then using the fact that $VV^* \leqslant I$ and $V^*V = I$, we compute

$$\sum_{i=1}^g (V^*X_iV)^2 = \sum_{i=1}^g (X_iV)^*(VV^*)(X_iV) \leqslant \sum_{i=1}^g (X_iV)^*(X_iV)$$

$$= \sum_{i=1}^g V^*X_i^2V = V^*\left(\sum_{i=1}^g X_i^2\right)V \leqslant V^*IV = I.$$

Hence $V^*XV \in B_g^{\mathrm{OH}}$. This proves that B_g^{OH} is matrix convex.

Although it will be a while before they are needed, we at least wish to introduce the topological notions of matrix convex sets. The topology of a single matrix level $\mathbb{S}_n^g$ is determined by the norm

$$\|X\| := \left(\sum_{i=1}^{g} \|X_i\|^2\right)^{\frac{1}{2}}, \quad \forall X \in \mathbb{S}_n^g,$$

where $\|X_i\|$ is the operator norm. A matrix convex set $K \subseteq \mathbb{S}^g$ is defined to be *bounded* if there is a real number $C > 0$ such that $\sum_{j=1}^{g} X_j^2 \leqslant CI$ for all $X \in K$, and this is equivalent to there being a real number $D > 0$ such that $\|X\| \leqslant D$ for all $X \in K$. It might seem unjustified to demand that all matrix levels in a bounded matrix convex set must share a common bound, but in fact by [10, Proposition 1.5], boundedness in the strong sense so defined is equivalent to the boundedness of each matrix level $K(n)$, or merely the boundedness of $K(1)$.

We do not define a topological structure on the entire space $\mathbb{S}^g$ collectively, but rather, we say $K \subseteq \mathbb{S}^g$ is *closed* (respectively *open*, *compact*) if $K(n)$ is closed (respectively open, compact) in $\mathbb{S}_n^g$ for all $n \geqslant 1$. This is **not** the same as giving $\mathbb{S}^g$ the disjoint union topology of all the matrix levels; for example, compactness in the disjoint union topology is not the same as compactness of each matrix level. The topological boundary[2] of K is defined to be the sequence $\partial^{\mathrm{top}} K := (\partial^{\mathrm{top}} K(n))_{n \geqslant 1}$. Likewise, the topological interior of K is $\operatorname{Int} K := (\operatorname{Int} K(n))_{n \geqslant 1}$.

2.4 Matrix Convex Hulls

We have said that matrix convexity "extends" or "adapts" convexity from the setting of $\mathbb{R}^g$ to the setting of $\mathbb{S}^g$, which is clear enough from the fact that the definitions

[2]The word "boundary" is overloaded in this field of study, so it is wise to be unambiguous by including the word "topological".

are visibly similar to one another. This does not mean, however, that a Euclidean convex subset of $\mathbb{R}^g$ is an example of matrix convex set.

For, since direct sums are a special kind of matrix convex combination, a matrix convex set K must contain matrix tuples of arbitrarily large size. In particular, a Euclidean convex subset of $\mathbb{R}^g = \mathbb{S}^g_1$, or of any $\mathbb{S}^g_n$, cannot be matrix convex by itself. Moreover, since compressions are also a special kind of matrix convex combination, and since we can compress a matrix tuple to any lower matrix level, we see that if $K(n)$ is non-empty then $K(1)$, $K(2)$, …, $K(n-1)$ are non-empty as well. Combining both of these observations, it must be that a matrix convex set contains at least one element of every size $n \geqslant 1$.

Therefore matrix convex sets do not “generalize” Euclidean convex sets. Rather, they “extend” them, and not just by having a similar definition, but also in the following sense. In the classical setting, if we have a subset E of a vector space that fails to be convex, then we can form its convex hull. We denote this by $\operatorname{co}^{\mathrm{Euc}} E$. This can be defined as the intersection of all convex sets that contain E, or equivalently as the set of all points which are convex combinations of the points in E. Either way, one obtains the convex hull, which is a convex set that is in some sense the closest approximation of E by a convex set. We will return to this line thinking after properly introducing the definition of matrix convex hull.

Just like in classical convexity, we can extend subsets of $\mathbb{S}^g$ to matrix convex sets in a minimal way. Consider any subset $E \subseteq \mathbb{S}^g$. Note that $\mathbb{S}^g$ is itself a matrix convex set which contains E. It is trivial to see that the intersection of any family of matrix convex sets containing E is yet another matrix convex set that contains E. Hence there is a smallest matrix convex set containing E, which we call the *matrix convex hull* of E, and denote it by $\operatorname{mco} E$.

When topology is a concern, we can also consider only the **closed** matrix convex sets containing E, whose intersection will be the smallest closed matrix convex set containing E. This is called the *closed matrix convex hull* of E and is denoted by $\overline{\mathrm{mco}}\, E$. It is a simple exercise to see that $\overline{\mathrm{mco}}\, E$ is the topological closure of $\mathrm{mco}\, E$, although we won't need this particular fact.

The Euclidean convex hull $\mathrm{co}^{\mathrm{Euc}}\, E$ of a set of points E is exactly equal to the set of all Euclidean convex combinations of points in E. The analogous fact remains true in the matrix convexity setting.

Proposition 2.4.1. *Let $E \subseteq \mathbb{S}^g$. Then $\mathrm{mco}\, E$ is precisely the set of all matrix convex combinations of elements of E. It is also the set of all compressions of direct sums of elements of E:*

$$\mathrm{mco}\, E = \{V^*(X_1 \oplus X_2 \oplus \cdots \oplus X_t)V : \ t \geqslant 1,\ X_i \in E,\ V \ \textit{an isometry}\}\,.$$

Proof. First note that by Example 2.3.5, matrix convex combinations in E are exactly the same thing as compressions of direct sums in E. So the first statement and second statement are equivalent. Hence, we will only prove the second statement directly.

Let K denote the collection of all compressions of direct sums of elements of E. We want to prove that $\mathrm{mco}\, E = K$. Since direct sums and compressions are matrix convex combinations, then certainly E will be contained in every matrix convex set containing E. It will henceforth suffice to prove that K is **itself** a matrix convex set containing E (and clearly it does contain E).

Let $X, Y \in K$. Then by definition $X = V^*(S_1 \oplus \cdots \oplus S_r)V$ and $Y = W^*(T_1 \oplus \cdots \oplus T_s)W$, where V and W are isometries and $S_i, T_j \in E$. But now $V \oplus W$ is an isometry, and

$$X \oplus Y = (V \oplus W)^*(S_1 \oplus \cdots \oplus S_r \oplus T_1 \oplus \cdots \oplus T_s)(V \oplus W) \in K.$$

So K is closed under direct sums.

Let $X \in K$ be as above, and now let W be an isometry such that W^*XW is defined. Then

$$W^*XW = (VW)^*(S_1 \oplus \cdots \oplus S_r)(VW) \in K,$$

since VW is also an isometry. Hence K is closed under compression. By Corollary 2.3.7, K is therefore matrix convex. $\square$

One piece of additional structure that appears in passing from Euclidean to matrix convexity is the matrix convex hull of a single point. The Euclidean convex hull of a single point x consists only of x itself. For the matrix convex hull of a point, however, the previous proposition gives the following.

Corollary 2.4.2. *Given a single point $X \in \mathbb{S}^g$, then*

$$\operatorname{mco}\{X\} = \left\{ V^*(\underbrace{X \oplus X \oplus \cdots \oplus X}_{t\ times})V : \ t \geqslant 1 \text{ and } V \text{ an isometry} \right\}.$$

In particular, considering just the point $0 \in \mathbb{S}^g_1$, we have that $\operatorname{mco}\{0\}$ consists of the zero matrices of every possible size.

Although the matrix convex hull of one point can contain many other points, it must be compact [2, Proposition 2.5], and so is small in the topological sense at least. From this it follows that $\operatorname{mco} E$ is compact for any finite subset $E \subseteq \mathbb{S}^g$, by considering the identity

$$\operatorname{mco} E = \operatorname{mco}\left(\bigoplus_{X \in E} X \right),$$

which requires E to be finite for the direct sum to be defined.

But we can now form the matrix convex hull $\operatorname{mco}(A)$, which we can think of as the best approximation of the Euclidean convex set A by a matrix convex set. One does wonder whether "extending" a Euclidean convex set in this way produces any new points at level 1. That is, if $A \subseteq \mathbb{R}^g$ is convex, do we have $\operatorname{co}^{\mathrm{Euc}} A = (\operatorname{mco} A)(1)$?

Corollary 2.4.3. *Let* $x_1, \ldots, x_t \in \mathbb{S}_1^g = \mathbb{R}^g$. *Then*

$$\mathrm{co}^{\mathrm{Euc}}\{x_1, \ldots, x_t\} = \left\{V^*(x_1 \oplus \cdots \oplus x_t)V : \ V \text{ an isometry } \mathbb{C}^1 \to \mathbb{C}^t\right\}.$$

Proof. Let $x = \sum_{i=1}^{t} \lambda_i x_i$ be a Euclidean convex combination. We have implicitly assumed that no point x_i is repeated, since any like terms can be combined (this is not so simple for matrix convex combinations). In Example 2.3.2 we saw how x can be expressed as a matrix convex combination of the same points x_i. Then by Proposition 2.4.1, $x = V^*(x_1 \oplus \cdots \oplus x_t)V$ for some isometry V.

Conversely, consider any $V^*(x_1 \oplus \cdots \oplus x_t)V$ where V is an isometry $\mathbb{C}^1 \to \mathbb{C}^t$. Then V has the form

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_t \end{bmatrix}, \quad v_i \in \mathbb{C}$$

and $V^*V = I_1$ means that $\sum_i |v_i|^2 = 1$. We also have that

$$V^*(x_1 \oplus \cdots \oplus x_t)V = \sum_i |v_i|^2 x_i,$$

which is a Euclidean convex combination. Hence we have containment in both directions. □

An immediate corollary is:

Corollary 2.4.4. *Let* $E \subseteq \mathbb{R}^g$. *Then*

$$\mathrm{co}^{\mathrm{Euc}} E = (\mathrm{mco}\, E)(1).$$

In case $E \subseteq \mathbb{R}^g$ is already Euclidean convex, the corollary says that $E = (\mathrm{mco}\, E)(1)$. So in a sense $\mathrm{mco}\, E$ is merely E extended to all the higher matrix levels in a natural and minimal way.

We note that for any given Euclidean convex subset $E \subseteq \mathbb{R}^g$, there is *not*, in general, just one matrix convex set K for which $E = \operatorname{mco} K(1)$. In Section 3.5, we will see examples of two matrix convex sets that each have the unit ball as the first matrix level. Refer to [12] for a paper that delves deeper into the question of distinct matrix convex sets which have the same first matrix level. In particular, Theorem 4.1 of that paper states that simplices are the only compact convex subsets of $\mathbb{R}^g$ which appear as the first matrix level of a *unique* matrix convex set.

2.5 Extreme Points of Matrix Convex Sets

Now that we have introduced the general concept of matrix convexity and matrix convex hulls, we turn to a notion of particular relevance to all the remaining material: that of extreme points. A common statement of the definition of an extreme point in the Euclidean setting is as follows.

Definition 2.5.1 (Euclidean extreme point)**.** Let V be a vector space, let $E \subseteq V$, and $x \in E$. Then x is called a *Euclidean extreme point* of E if whenever $y_1, y_2 \in E$, and $0 < t < 1$, and $x = ty_1 + (1-t)y_2$, then necessarily $y_1 = y_2$. We will use $\partial^{\mathrm{Euc}} E$ to denote the set of Euclidean extreme points of E.

The Euclidean convexity setting often allows one to only consider convex combinations of just two points, as in the definition above, rather than an arbitrary finite number of points. There is no reason to assume that the same convenience is available when working with matrix convexity, and so we seek a definition of extreme points which is given in terms of convex combinations of any possible number of points.

In order to translate this notion to the matrix convexity setting, we must first understand that the parameter t in Definition 2.5.1 really stands for a pair of convex coefficients $\lambda_1 = t$ and $\lambda_2 = 1 - t$. To say that $0 < t < 1$ is to say that neither λ_1

nor λ_2 are 0. This seems easily extended to the matrix convexity setting where the "coefficients" are matrices, although we will see two different ways of making such an extension.

Consider the Euclidean convex combination

$$x = \sum_{i=1}^{t} \lambda_i y_i, \quad y_i \in E.$$

Here are two ways in which this convex combination might be considered "degenerate". First, there could be a coefficient λ_j equal to 0, in which case the j-th term in the combination is superfluous and could be discarded to obtain a simpler convex combination that still equals x. Second, some point y_j could equal x itself, and provided that the coefficient λ_j is not 1 (in which case every other term has coefficient 0), we can discard the j-th term by appropriately scaling the remaining coefficients to ensure that their sum is 1. That is, if $y_j = x$ and $\lambda_j \neq 1$ then we can write x as a convex combination of the remaining y_i $(i \neq j)$ as follows:

$$x = \sum_{i \neq j} \frac{\lambda_i}{1 - \lambda_j} y_i.$$

It turns out that Euclidean extreme points are precisely the points which can only be represented as a convex combination provided that the combination is "degenerate" in the two ways just described.

Proposition 2.5.2. *Let V be a vector space and $E \subseteq V$ a convex set. A point $x \in E$ is a Euclidean extreme point of E if and only if whenever*

$$x = \sum_{i=1}^{t} \lambda_i y_i, \quad y_1, \ldots, y_t \in E, \quad \lambda_1, \ldots, \lambda_t \geqslant 0, \quad 1 = \sum_{i=1}^{t} \lambda_i \tag{*}$$

then it follows that for each i, $\lambda_i = 0$ or $y_i = x$.

Proof. Assume x satisfies the given condition. If we have $y_1, y_2 \in E$, and $0 < t < 1$, and that $x = ty_1 + (1-t)y_2$, then letting $\lambda_1 = t$ and $\lambda_2 = 1-t$, we have $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 y_1 + \lambda_2 y_2$. By hypothesis, each $\lambda_i = 0$ or $y_i = x$. We also assumed $0 < t < 1$, from which it follows that $\lambda_1, \lambda_2 \neq 0$. We conclude that $y_1 = x = y_2$, which shows that x is a Euclidean extreme point of E.

Conversely, suppose x is a Euclidean extreme point of E and that x is represented as a convex combination as in (*). Without loss of generality we may assume that any coefficients equal to 0 have had their terms discarded (and hence the conclusion we seek is that $y_i = x$ for all i).

If there is only one term in the sum, then the sum is $x = y_1$, which is exactly what we want. If there are exactly two terms, we apply the hypothesis to deduce that $y_1 = y_2$. It follows that $x = \lambda_1 y_1 + \lambda_2 y_1 = y_1$, whence $y_1 = x = y_2$, as required.

Henceforth we assume that there are at least 3 terms. We can transform the given convex combination into another which has only two points, namely

$$x = \lambda_1 y_1 + (1-\lambda_1)\sum_{i=2}^{t}\left(\frac{\lambda_i}{1-\lambda_1} y_i\right) = \lambda_1 y_1 + (1-\lambda_1)z.$$

Note that z is itself a convex combination of points of E, which is assumed convex, so that $z \in E$. Now we apply the hypothesis which yields $y_1 = z$. From this it follows that $x = z$ as well. But this expresses x as convex combination of the remaining y_i $(i = 2, \ldots, t)$. Repeating this process if needed, we eventually conclude that $y_2, \ldots, y_t = x$ as well. $\square$

In the proof, we used a procedure for transforming a convex combination into another with exactly two points, and this required division by $1 - \lambda_i$ where λ_i is a convex coefficient. In the matrix convex setting it is not clear whether a similar procedure exists for reducing the number of points in a convex combination, which is why we state the definition of extreme points using matrix convex combinations with any possible number of terms.

Now we define the first of two notions of extreme point for matrix convex sets. Both notions can be seen to extend the characterization of Euclidean extreme points that we just saw.

Definition 2.5.3 (Absolute extreme point)**.** Let $K \subseteq \mathbb{S}^g$ be a matrix convex set. A point $X \in K$ is called an *absolute extreme point* of K if, whenever X is expressed as a matrix convex combination of points $Y_1, \dots, Y_t \in K$ with all coefficients non-zero, then for each i either $Y_i \overset{\mathrm{u}}{\equiv} X$ or else there is $Z \in K$ such that $Y_i \overset{\mathrm{u}}{\equiv} X \oplus Z$. The set of absolute extreme points of K will be denoted $\partial^{\mathrm{abs}} K$.

This is nearly a direct extension of Euclidean extreme points to the matrix convexity setting, except that the conclusion is no longer just that $Y_i = X$, but also possibly that Y_i has X as a direct summand. We can think of this as a natural consequence of our points belonging to possibly different matrix levels; for any Y_i that are the same size as X, the conclusion is still that Y_i equals X (up to unitary equivalence).

There is another closely related notion of extreme point in the matrix convexity setting which deserves to be mentioned here. It is perhaps the more natural extension of the Euclidean extreme point definition because its conclusion more closely matches that of the Euclidean definition.

Definition 2.5.4 (Matrix extreme point)**.** Let $K \subseteq \mathbb{S}^g$ be a matrix convex set. A point $X \in K$ is called a *matrix extreme point* of K if whenever X is a matrix convex combination of points $Y_1, \dots, Y_t \in K$ with all coefficients surjective, then $Y_i \overset{\mathrm{u}}{\equiv} X$ for all i. The set of matrix extreme points of K will be denoted $\partial^{\mathrm{mat}} K$.

The crucial difference between this definition and the previous is that we now demand that the matrix convex coefficients are surjective instead of merely non-zero. Surjective coefficients guarantee that the points Y_i come from matrix levels of K no larger than that of X, which excludes the possibility of having $Y_i \overset{\mathrm{u}}{\equiv} X \oplus Z$. Thus the *only* real difference between the two definitions is the restriction on the coefficients.

It is clear from these definitions that all absolute extreme points are also matrix extreme points. In light of Proposition 2.5.2, it is also true that all matrix extreme points are Euclidean extreme points of the matrix level to which they belong. For a matrix convex set K we define $\partial^{\mathrm{Euc}} K := (\partial^{\mathrm{Euc}} K(n))_{n\geqslant 1}$.

While we are comparing the different notions of extreme point, let us also recall the standard fact from classical convexity that Euclidean extreme points of a convex subset E of a vector space must lie on the topological boundary of E: if a point $x \in E$ is not on the topological boundary, then it is an interior point, and any open U such that $x \in U \subseteq E$ will contain a non-degenerate line segment with x strictly between the two endpoints (which belong to E). We thus obtain the following set inclusions.

Proposition 2.5.5. *Let $K \subseteq \mathbb{S}^g$ be matrix convex. Then*

$$\partial^{\mathrm{abs}} K \subseteq \partial^{\mathrm{mat}} K \subseteq \partial^{\mathrm{Euc}} K \subseteq \partial^{\mathrm{top}} K.$$

We reiterate that our interest is primarily in the more restrictive of the two non-classical types of extreme point: the absolute extreme points.

2.6 Dilation-Theoretic Characterization of Extreme Points

It is the purpose of this section to state and discuss, but not prove, the dilation-theoretic characterization of absolute extreme points given by Evert, Helton, Klep, and McCullough [5, Theorem 1.1]. This characterization is a crucial tool used in the proof of Evert and Helton's Krein-Milman-type theorem [4, Theorem 1.3], which we present in the next chapter. This kind of dilation-theoretic characterization is not entirely unprecedented, for even Euclidean extreme points have such a characterization.

Proposition 2.6.1 (See [5], Proposition 1.2)**.** *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set and let $X \in K(n)$. Then $X \in \partial^{\mathrm{Euc}} K(n)$ if and only if for all $A \in \mathbb{M}_n^g$,*

$$\begin{bmatrix} X & A \\ A^* & X \end{bmatrix} \in K(2n) \qquad \Longrightarrow \qquad A = 0.$$

Recall that a matrix tuple $X \in \mathbb{S}^g$ is *irreducible* if there are no non-trivial subspaces which are simultaneously invariant for each component matrix X_i. Also note that since the matrix tuples are self-adjoint, there is no distinction to be made between "invariant" and "reducing". Equivalently, $X \in \mathbb{S}^g$ is irreducible if it is not unitarily equivalent to a direct sum of at least two other elements of $\mathbb{S}^g$. With relative ease, we can prove that absolute extreme points must be irreducible. The same is true of matrix extreme points, but we don't bother proving it.

Proposition 2.6.2. *Suppose $K \subseteq \mathbb{S}^g$ is a matrix convex set. If $X \in \partial^{\mathrm{abs}} K$, then X is irreducible.*

Proof. If $X \in K(n)$ is reducible then it has the form $X \overset{\mathrm{u}}{\equiv} Y \oplus Z$ for some $Y \in K(m_1)$ and $Z \in K(m_2)$. Evidently $1 \leqslant m_1, m_2 < n$. But as we saw in Example 2.3.3, the direct sum of Y and Z is a matrix convex combination of Y and Z whose coefficients are non-zero, say $X = C^*YC + D^*ZD$. If X is an absolute extreme point of K, then by definition we have $Y \overset{\mathrm{u}}{\equiv} X$ or $Y \overset{\mathrm{u}}{\equiv} X \oplus W$ for some $W \in K$. It follows that $m_1 = n$ or $m_1 > n$, a contradiction in either case. Hence $X \notin \partial^{\mathrm{abs}} K$. $\square$

For a given point X in a matrix convex set K, there always exist dilations of X in K. For, any direct sum $X \oplus Z$, where $Z \in K$, is a dilation of X in K. We consider these dilations to be trivial, in the sense that they must always exist in abundance.

Definition 2.6.3 (Arveson boundary point)**.** If $K \subseteq \mathbb{S}^g$ is a matrix convex set, then a point $X \in K(n)$ is called an *Arveson boundary point* of K if whenever we have a

1-dilation

$$\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in K$$

for some $\beta \in \mathbb{M}_{n,1}^g$ and some $\gamma \in \mathbb{S}_1^g$, then it can only be that $\beta = 0$. The collection of Arveson boundary points of K is denoted by $\partial^{\mathrm{Arv}} K$.

The use of lowercase Greek letters for the blocks of a 1-dilation is intentional, and meant to emphasize the smallness of a 1-dilation compared to k-dilations for $k \geqslant 2$.

Although the definition only considers 1-dilations of X, it is equivalent to the statement that *all* dilations of X are trivial [5, Lemma 3.5].

What follows is the dilation-theoretic characterization of absolute extreme points, which turn out to be exactly the irreducible Arveson boundary points.

Theorem 2.6.4 (See [5], Theorem 1.1, Theorem 3.10)**.** *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set and $X \in K(n)$. Then $X \in \partial^{\mathrm{abs}} K$ if and only if X is irreducible and $X \in \partial^{\mathrm{Arv}} K$.*

There is a very similar characterization of matrix extreme points. It involves the notion of an *n-block diagonalizable* matrix tuple, which means a matrix tuple that is unitarily equivalent to a direct sum of matrix tuples, each with size at most n. We will not be needing this concept later on, though we bring it up for the sake of completeness.

Theorem 2.6.5 (See [5] Theorem 1.1, Theorem 4.1)**.** *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set and $X \in K(n)$. Then $X \in \partial^{\mathrm{mat}} K$ if and only if X is irreducible and whenever we have an n-block diagonalizable dilation*

$$\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in K$$

for some $Y \in \mathbb{M}_{n,m}^g$ and some $Z \in \mathbb{S}_m^g$, then it can only be that $Y = 0$.

Note that this matrix extreme point characterization does not only involve 1-dilations, but dilations up to any finite size. It is a fortunate consequence of the definition of absolute extreme points that they can be characterized by a 1-dilation criterion.

We can use the characterization of absolute extreme points to prove that the Arveson boundary of a matrix convex set consists of direct sums of absolute extreme points, up to unitary equivalence. This is the task to which we now turn, before we are ready to proceed into the study of Krein-Milman-type matrix convexity results.

Lemma 2.6.6. *Let $K \subseteq \mathbb{S}^g$ be matrix convex. Then $\partial^{\mathrm{Arv}} K$ is closed with respect to unitary equivalence.*

Proof. Suppose $X \in \partial^{\mathrm{Arv}} K(n)$ and that $U \in \mathbb{M}_n$ is unitary. Consider a 1-dilation of U^*XU in K, say

$$Z := \begin{bmatrix} U^*XU & \beta \\ \beta^* & \gamma \end{bmatrix} \in K.$$

Let $V = U^* \oplus 1 = \begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix}$, which is also unitary. We have

$$V^*ZV = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U^*XU & \beta \\ \beta^* & \gamma \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} X & U\beta \\ (U\beta)^* & \gamma \end{bmatrix} \in K.$$

Since X is an Arveson boundary point of K, it follows that $U\beta = 0$. Since U is unitary, $\beta = 0$. Hence $U^*XU \in \partial^{\mathrm{Arv}} K$. □

Lemma 2.6.7. *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set and $X, Y \in K$. Then $X \oplus Y \in \partial^{\mathrm{Arv}} K$ if and only if $X \in \partial^{\mathrm{Arv}} K$ and $Y \in \partial^{\mathrm{Arv}} K$.*

Proof. Suppose X and Y both are Arveson boundary points, and that we have a

1-dilation of $X \oplus Y$ in K, namely

$$Z := \begin{bmatrix} X & 0 & \alpha \\ 0 & Y & \beta \\ \alpha^* & \beta^* & \gamma \end{bmatrix} \in K.$$

Since Z is a dilation of $X \in \partial^{\mathrm{Arv}} K$, we have $\alpha = 0$. Since K is closed under compressions, we have

$$\begin{bmatrix} Y & \beta \\ \beta^* & \gamma \end{bmatrix} \in K,$$

which is a dilation of $Y \in \partial^{\mathrm{Arv}} K$. Hence $\beta = 0$ too, which shows that $X \oplus Y \in \partial^{\mathrm{Arv}} K$.

Conversely, assume $X \oplus Y \in \partial^{\mathrm{Arv}} K$, and that we have a 1-dilation of Y in K, namely

$$Z := \begin{bmatrix} Y & \beta \\ \beta^* & \gamma \end{bmatrix} \in K.$$

Then also

$$\begin{bmatrix} X & 0 & 0 \\ 0 & Y & \beta \\ 0 & \beta^* & \gamma \end{bmatrix} = X \oplus Z \in K,$$

which is a dilation of $X \oplus Y \in \partial^{\mathrm{Arv}} K$. Hence $\beta = 0$, and $Y \in \partial^{\mathrm{Arv}} K$. To see that $X \in \partial^{\mathrm{Arv}} K$, note that $Y \oplus X \overset{\mathrm{u}}{\equiv} X \oplus Y \in \partial^{\mathrm{Arv}} K$, so that Lemma 2.6.6 gives $Y \oplus X \in \partial^{\mathrm{Arv}} K$. Applying the same argument as before to $Y \oplus X$ gives that $X \in \partial^{\mathrm{Arv}} K$. □

The property of being closed under unitary equivalence and direct sums makes $\partial^{\mathrm{Arv}} K$ an example of a *free set*. The additional property of being closed under direct summands makes $\partial^{\mathrm{Arv}} K$ a *fully free set*.

Proposition 2.6.8. *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set and $X \in K$. Then $X \in$*

$\partial^{\text{Arv}} K$ *if and only if there exist* $Y_1, \ldots, Y_t \in \partial^{\text{abs}} K$ *such that* $X \overset{u}{\equiv} Y_1 \oplus \cdots \oplus Y_t$.

Proof. Suppose $X \in \partial^{\text{Arv}} K$. By Proposition 2.2.2, we have $X \overset{u}{\equiv} \bigoplus_{i=1}^{t} Y_i$ for some irreducible matrix tuples $Y_1, \ldots, Y_t \in \mathbb{S}^g$. For each i, necessarily $Y_i \in K$ by virtue of being a compression of X. Since $X \in \partial^{\text{Arv}} K$, Lemma 2.6.7 implies that $Y_i \in \partial^{\text{Arv}} K$. Since Y_i is irreducible and in the Arveson boundary, it follows by Theorem 2.6.4 that $Y_i \in \partial^{\text{abs}} K$.

Conversely, suppose that $X \overset{u}{\equiv} \bigoplus_{i=1}^{t} Y_i$ for $Y_1, \ldots, Y_t \in \partial^{\text{abs}} K$. For each i, Theorem 2.6.4 says that $Y_i \in \partial^{\text{Arv}} K$. Then Lemma 2.6.7 implies that $\bigoplus_{i=1}^{t} Y_i \in \partial^{\text{Arv}} K$. Since this direct sum is unitarily equivalent to X, Lemma 2.6.6 gives $X \in \partial^{\text{Arv}} K$. $\square$

3

Krein-Milman-type Results in Matrix Convexity

The Krein-Milman theorem of classical convexity states that a compact convex subset K of a locally convex vector space V is equal to the closed convex hull of its extreme points [1, Chapter V, Theorem 7.4], that is,

$$K = \overline{\operatorname{co}}\, \partial^{\mathrm{Euc}} K.$$

Moreover, for a subset $E \subseteq K$, we have $K = \overline{\operatorname{co}}\, E$ if and only if $\partial^{\mathrm{Euc}} K \subseteq \overline{E}$ (the topological closure of E), which says that the extreme points of K are the *minimal* subset of K whose closed convex hull gives the entire set K.

In particular, the Krein-Milman theorem also guarantees the *existence* of extreme points, even in infinite-dimensional vector spaces.

As an example of a Krein-Milman-type theorem that holds in the matrix convexity setting, the following result is well-known [13, Theorem 4.3, Theorem 4.6].

Theorem 3.0.1 (Webster-Winkler 1999)**.** *Let $K \subseteq \mathbb{S}^g$ be a nonempty compact matrix convex set. Then $\partial^{\mathrm{mat}} K$ is nonempty, and $\overline{\operatorname{mco}}\, \partial^{\mathrm{mat}} K = K$. Moreover, given a subset $E \subseteq K$ that is closed topologically, closed under compressions, and satisfies*

$\overline{\text{mco}}(E) = K$, *it follows that* $\partial^{\text{mat}} K \subseteq E$.

The Krein-Milman-type result of Webster and Winkler actually applies to a much larger class of matrix convex sets than what we are concerned with in this thesis. Our matrix convex sets always reside in the space $\mathbb{S}^g$ for an integer $g \geqslant 1$, but it is possible to alter one's perspective on $\mathbb{S}^g$ in such a way that many other spaces become natural settings in which to study matrix convexity. Nonetheless, the spaces $\mathbb{S}^g$ are the only ones we consider in this thesis. For matrix convex subsets of $\mathbb{S}^g$, Kriel [10, Theorem 6.8] gave a much more elementary proof of the Webster-Winkler result, with the stronger conclusion that the (non-closed) matrix convex hull of a compact matrix convex set is sufficient to span the entire set.

The absolute extreme points generally form a proper subset of the matrix extreme points, so we should anticipate that fewer matrix convex sets are spanned by their absolute extreme points than are spanned by their matrix extreme points. What does hold in full generality is the minimality statement: that spanning sets, subject to a few modest constraints, contain all the absolute extreme points. Since this is relatively straightforward to prove, we handle it immediately.

Proposition 3.0.2. *Suppose* $K \subseteq \mathbb{S}^g$ *is a matrix convex set, and that* $E \subseteq K$ *has the following properties:*

1. E *is closed under unitary equivalence;*
2. *Each* $X \in E$ *is irreducible;*
3. $\operatorname{mco} E = K$.

Then $\partial^{\text{abs}} K \subseteq E$.

Proof. If $\partial^{\text{abs}} K = \varnothing$ we are already done, so assume contrariwise. Consider an absolute extreme point $X \in \partial^{\text{abs}} K(n)$. Since $X \in K = \operatorname{mco} E$, there exist $Y_1, \ldots, Y_s \in E$

for some $s \geqslant 1$, and non-zero maps $C_i : \mathbb{C}^n \to \mathbb{C}^{n_i}$ such that

$$X = \sum_{i=1}^{s} C_i^* Y_i C_i, \quad \sum_{i=1}^{s} C_i^* C_i = I_n.$$

Now X is an absolute extreme point, so by definition each Y_i is unitarily equivalent to either X or a direct sum of X with another matrix tuple. But each Y_i is irreducible, so only the former can happen. Thus $X \overset{\mathrm{u}}{\equiv} Y_i \in E$. Since E is closed under unitary equivalence, it follows that $X \in E$. $\square$

Of course, if we hadn't assumed that E is closed under unitary equivalence, we merely need to change the conclusion to say that E contains all absolute extreme points up to unitary equivalence.

What happens if we don't assume that every element of the spanning set E is irreducible? Let $K = \operatorname{mco} E$ for some $E \subseteq \mathbb{S}^g$. By Proposition 2.2.2, each $X \in E$ is unitarily equivalent to a direct sum of irreducible elements of K. Let F be the set of all irreducible direct summands of elements of E. Then $E \subseteq \operatorname{mco} F$ since $\operatorname{mco} F$ contains all direct sums of elements of F, and also $F \subseteq \operatorname{mco} E$ since $\operatorname{mco} E$ contains all compressions, and therefore direct summands, of elements of E. Hence $\operatorname{mco} E = \operatorname{mco} F$, and F consists of irreducible matrix tuples. By Proposition 3.0.2, $\partial^{\mathrm{abs}} K \subseteq F$, up to unitary equivalence. Thus we obtain the following corollary to the last result.

Corollary 3.0.3. *If $E \subseteq \mathbb{S}^g$ and $K = \operatorname{mco} E$, then for each $X \in \partial^{\mathrm{abs}} K$, we have $X \oplus Z \in E$ for some $Z \in K$, up to unitary equivalence.*

As mentioned before, Proposition 3.0.2 is a minimality condition on $\partial^{\mathrm{abs}} K$. Since the absolute extreme points of K must be found within any irreducible spanning set of K, it is natural to wonder when the absolute extreme points themselves comprise such a spanning set. That is, under what conditions do we have $K = \operatorname{mco} \partial^{\mathrm{abs}} K$? Recalling that matrix convex combinations of elements of a set E are precisely the "compressions of direct sums" of elements of E, and that direct sums of absolute extreme points are precisely the Arveson boundary points, we can equivalently phrase our question as: when is it true that every point of K dilates to an Arveson boundary point of K?

It is our goal to present a class of matrix convex sets for which this is true. This Krein-Milman-type result was first seen as Theorem 1.3 of [4]. However, it is not proven for all compact matrix convex sets. We must first restrict our attention to compact *free spectrahedra*, and even then it is only a further subclass to which our main result applies. As such, we must now become acquainted with such free spectrahedra theory as is needed to state and prove the main result.

3.1 Linear Pencils, Matrix Inequalities, and Free Spectrahedra

The definition and study of free spectrahedra requires the notion of the *tensor product* of matrices, also known as the *Kronecker product*, and the basic algebraic properties thereof. We will state these for reference, but the details can be found in many textbooks, for example [9].

Definition 3.1.1 (Tensor product of matrices)**.** Let $A \in \mathbb{M}_{m,n}$ and $B \in \mathbb{M}_{p,q}$. Denote the i, j-th entry of A by $a_{i,j}$. Then $A \otimes B$ is defined to the be the matrix in $\mathbb{M}_{mp,nq}$ given here in block form as

$$A \otimes B := \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{bmatrix}.$$

We will make frequent use of the fact that tensor products respect block matrix notation, up to unitary equivalence in some cases. For example, with (2×2)-block matrices we have the identities

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes E = \begin{bmatrix} A \otimes E & B \otimes E \\ C \otimes E & D \otimes E \end{bmatrix},$$

and

$$E \otimes \begin{bmatrix} A & B \\ C & D \end{bmatrix} \overset{\mathrm{u}}{\equiv} \begin{bmatrix} E \otimes A & E \otimes B \\ E \otimes C & E \otimes D \end{bmatrix}.$$

Definition 3.1.2 (Linear pencils, free spectrahedron)**.** Fix $\Omega \in \mathbb{S}_d^g$. We denote by Λ_Ω the *homogeneous linear pencil* of Ω, which is defined as

$$\Lambda_\Omega(X) := \sum_{i=1}^{g} \Omega_i \otimes X_i, \quad \forall\ m, n \geqslant 1,\ \forall\ X \in \mathbb{M}_{n,m}^g.$$

Usually we only need to evaluate Λ_Ω at elements of $\mathbb{S}^g$.

We denote by L_Ω the *monic linear pencil* of Ω, which is defined as

$$L_\Omega(X) := I - \Lambda_\Omega(X) = I_{dn} - \sum_{i=1}^{g} \Omega_i \otimes X_i, \quad \forall\ n \geqslant 1,\ \forall\ X \in \mathbb{S}_n^g.$$

The *free spectrahedron defined by* Ω is

$$\mathcal{D}_\Omega := \{X \in \mathbb{S}^g :\ L_\Omega(X) \geqslant 0\} = \{X \in \mathbb{S}^g : \Lambda_\Omega(X) \leqslant I\}.$$

In other words, the free spectrahedron defined by Ω is the solution set of the *linear matrix inequality* $L_\Omega(X) \geqslant 0$, although such terminology will not be used here and is more prevelant in an optimization setting.

We always have $0 \in \mathcal{D}_\Omega$, since $L_\Omega(0) = I - \sum_{i=1}^{g} \Omega_i \otimes 0 = I \geqslant 0$, this holding for the 0 element of any matrix level.

The next proposition collects various algebraic identities of the linear pencils L_Ω

and Λ_Ω. They are proven simply by expanding the definition of L_Ω or Λ_Ω, and then applying the basic algebraic properties of the tensor product.

Proposition 3.1.3. *Let $\Omega \in \mathbb{S}^g$. If $X, Y \in \mathbb{S}^g_n$, then*

(1) $L_\Omega(X+Y) = L_\Omega(X) - \Lambda_\Omega(Y)$;

(2) $L_\Omega(X)^* = L_\Omega(X)$.

If $X, Y \in \mathbb{M}^g_{m,n}$, and $c \in \mathbb{R}$, then

(3) $\Lambda_\Omega(X+Y) = \Lambda_\Omega(X) + \Lambda_\Omega(Y)$;

(4) $\Lambda_\Omega(cX) = c\Lambda_\Omega(X)$;

(5) $\Lambda_\Omega(X)^* = \Lambda_\Omega(X^*)$.

The next set of pencil properties deserve some more in-depth justification.

Proposition 3.1.4. *For $\Omega \in \mathbb{S}^g_d$, the following identities hold.*

(1) If $X \in \mathbb{S}^g_n$, $Y \in \mathbb{M}^g_{n,m}$, and $Z \in \mathbb{S}^g_m$, then

$$L_\Omega\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right) \overset{\mathrm{u}}{\equiv} \begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(Y) \\ -\Lambda_\Omega(Y)^* & L_\Omega(Z) \end{bmatrix}.$$

(2) If $X_{11} \in \mathbb{S}^g_{n_1}$, $X_{22} \in \mathbb{S}^g_{n_2}$, $X_{33} \in \mathbb{S}^g_{n_3}$, $X_{12} \in \mathbb{M}^g_{n_1,n_2}$, $X_{13} \in \mathbb{M}^g_{n_1,n_3}$, and $X_{23} \in \mathbb{M}^g_{n_2,n_3}$, then

$$L_\Omega\left(\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix}\right) \overset{\mathrm{u}}{\equiv} \begin{bmatrix} L_\Omega(X_{11}) & -\Lambda_\Omega(X_{12}) & -\Lambda_\Omega(X_{13}) \\ -\Lambda_\Omega(X_{12})^* & L_\Omega(X_{22}) & -\Lambda_\Omega(X_{23}) \\ -\Lambda_\Omega(X_{13})^* & -\Lambda_\Omega(X_{23})^* & L_\Omega(X_{33}) \end{bmatrix}.$$

(3) If $X^1, \dots, X^t \in \mathbb{S}^g$, then

$$L_\Omega(X^1 \oplus \cdots \oplus X^t) \overset{\mathrm{u}}{\equiv} L_\Omega(X^1) \oplus \cdots \oplus L_\Omega(X^t).$$

(4) If $X \in \mathbb{S}_n^g$ and $V \in \mathbb{M}_{n,m}$ is an isometry, then

$$L_\Omega(V^*XV) \overset{\mathrm{u}}{\equiv} (I_d \otimes V)^* L_\Omega(X)(I_d \otimes V).$$

Proof. For (1), we compute

$$L_\Omega\left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}\right) = I - \sum_{i=1}^{g} \Omega_i \otimes \begin{bmatrix} X_i & Y_i \\ Y_i^* & Z_i \end{bmatrix} \overset{\mathrm{u}}{\equiv} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \sum_{i=1}^{g} \begin{bmatrix} \Omega_i \otimes X_i & \Omega_i \otimes Y_i \\ \Omega_i \otimes Y_i^* & \Omega_i \otimes Z_i \end{bmatrix}$$

$$= \begin{bmatrix} I - \sum \Omega_i \otimes X_i & -\sum \Omega_i \otimes Y_i \\ -(\sum \Omega_i \otimes Y_i)^* & I - \sum \Omega_i \otimes Z_i \end{bmatrix} = \begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(Y) \\ -\Lambda_\Omega(Y)^* & L_\Omega(Z) \end{bmatrix}.$$

Similarly (2) follows from (1).

Property (3) follows from (1) by setting $Y = 0$ to obtain the result for a direct sum of two matrix tuples. Then the result extends inductively to any number of direct summands.

Now we prove property (4). We have

$$L_\Omega(V^*XV) = I_{dn} - \sum_{i=1}^{g} \Omega_i \otimes (V^*X_iV) = I_{dn} - \sum_{i=1}^{g} (I_d^*\Omega_i I_d) \otimes (V^*X_iV)$$

$$= I_{dn} - \sum_{i=1}^{g} (I_d \otimes V)^*(\Omega_i \otimes X_i)(I_d \otimes V) = (I_d \otimes V)^* \left(I_{dn} - \sum_{i=1}^{g} \Omega_i \otimes X_i \right) (I_d \otimes V)$$

$$= (I_d \otimes V)^* L_\Omega(X)(I_d \otimes V).$$

We have used the fact that V is an isometry which gives $(I_d \otimes V)^*(I_d \otimes V) = (I_d^* I_d) \otimes V^*V = I_d \otimes I_n = I_{dn}$. $\square$

The last two properties are essential ingredients in proving that free spectrahedra are matrix convex.

Proposition 3.1.5. *A free spectrahedron is a closed matrix convex set.*

Proof. Let $\mathcal{D}_\Omega$ be a free spectrahedron. We need to show is that $\mathcal{D}_\Omega$ is closed under compressions and direct sums. This follows from Proposition 3.1.4. Indeed, if $X_1, \ldots, X_t \in \mathcal{D}_\Omega$ then we observe that

$$L_\Omega(X_1 \oplus \cdots \oplus X_t) \overset{\mathrm{u}}{\equiv} L_\Omega(X_1) \oplus \cdots \oplus L_\Omega(X_t),$$

and since each $L_\Omega(X_i) \succcurlyeq 0$, it follows that $L_\Omega(X_1 \oplus \cdots \oplus X_t) \succcurlyeq 0$. Hence $X_1 \oplus \cdots \oplus X_t \in \mathcal{D}_\Omega$.

Now let $X \in \mathcal{D}_\Omega$ and let V be an isometry of appropriate size to compress X. Then

$$L_\Omega(V^*XV) \overset{\mathrm{u}}{\equiv} (I \otimes V)^* L_\Omega(X)(I \otimes V) \succcurlyeq 0$$

since $L_\Omega(X) \succcurlyeq 0$, whence $V^*XV \in \mathcal{D}_\Omega$.

We have shown that $\mathcal{D}_\Omega$ is a matrix convex set, and it remains to show that $\mathcal{D}_\Omega$ is closed, meaning that every matrix level $\mathcal{D}_\Omega(n)$ is closed. So suppose we have a sequence $(X^{(j)})_{j \in \mathbb{N}} \subseteq \mathcal{D}_\Omega(n)$ that converges to $X \in \mathbb{S}_n^g$ (we put the sequence indices as a superscript to visually separate them from the g-tuple indices). This means that $L_\Omega(X^{(j)}) \succcurlyeq 0$ for all $j \geqslant 0$, and we want to show that $L_\Omega(X) \succcurlyeq 0$. We note that all the algebraic operations involved in building the expression $I - \sum_{i=1}^g \Omega_i \otimes X_i^{(j)}$ are continuous so we can conclude that L_Ω, as a function on $\mathbb{S}_n^g$, is norm-continuous. Hence

$$L_\Omega(X) = L_\Omega(\lim_j X^{(j)}) = \lim_j L_\Omega(X^{(j)}).$$

Thus $L_\Omega(X)$ is the limit of a sequence of positive semidefinite matrices, and must therefore be positive semidefinite itself. $\square$

Consider what happens if we have a defining tuple in the first matrix level, say $y \in \mathbb{S}_1^g = \mathbb{R}^g$. Then $\Lambda_y(X) = y_1 X_1 + \cdots + y_g X_g$, a mere linear combination of the components of X. In case we restrict the X to the first matrix level as well, we have

$\Lambda_y(x) = y_1x_1 + \cdots + y_gx_g$ for all $x \in \mathbb{R}^g$. Thus Λ_y restricted to $\mathbb{R}^g$ is nothing more than a continuous linear functional $\mathbb{R}^g \to \mathbb{R}$. The condition $\Lambda_y(x) \leqslant 1$ defines a *closed half-space* in $\mathbb{R}^g$, particularly one with 0 in its interior. Thus, if $y \in \mathbb{R}^g$ then the free spectrahedron $\mathcal{D}_y$ has the property that $\mathcal{D}_y(1)$ is a closed half-space in $\mathbb{R}^g$.

A single free spectrahedron can encode multiple inequalities at once. The next result shows why.

Proposition 3.1.6. *If* $\Omega^1, \ldots, \Omega^t \in \mathbb{S}^g$, *then*

$$\mathcal{D}_{\Omega^1 \oplus \cdots \oplus \Omega^t} = \mathcal{D}_{\Omega^t} \cap \cdots \cap \mathcal{D}_{\Omega^t}.$$

Proof. Let $X \in \mathbb{S}^g$. Applying Proposition 3.1.4, we have

$$\begin{aligned} X \in \mathcal{D}_{\Omega^1 \oplus \cdots \oplus \Omega^t} &\iff L_{\Omega^1 \oplus \cdots \oplus \Omega^t}(X) \succcurlyeq 0 \\ &\iff L_{\Omega^1}(X) \oplus \cdots \oplus L_{\Omega^t}(X) \succcurlyeq 0 \\ &\iff L_{\Omega^i}(X) \succcurlyeq 0 \text{ for all } i = 1, \ldots, t \\ &\iff X \in \mathcal{D}_{\Omega^i} \text{ for all } i = 1, \ldots, t \\ &\iff X \in \mathcal{D}_{\Omega^1} \cap \cdots \cap \mathcal{D}_{\Omega^t}. \end{aligned}$$

$\square$

In particular, if we have vectors $y_1, \ldots, y_t \in \mathbb{R}^g$ and let $\Omega := y_1 \oplus \cdots \oplus y_t$, then $\mathcal{D}_\Omega = \mathcal{D}_{y_1} \cap \cdots \cap \mathcal{D}_{y_t}$. In this case $\mathcal{D}_\Omega(1)$ is an intersection of finitely many closed half-spaces. For example, a g-dimensional polytope is the first matrix level of some free spectrahedron. In this way, free spectrahedra extend, at least roughly, the notion of polytopes to the matrix convexity setting: first by letting the defining inequalities be determined by matrix tuples instead of mere vectors in $\mathbb{R}^g$, and also by allowing the solution set to consist of matrix tuples instead of mere vectors in $\mathbb{R}^g$.

A classical result states that a closed convex subset of a locally convex topological vector space is the intersection of all the closed half-spaces that contain it [1, Chapter IV, Corollary 3.11]. A result of Effros and Winkler implies that a compact matrix convex set that has 0 in the first matrix level is an intersection of (possibly infinitely-many) free spectrahedra [3, Theorem 5.4]. It is not immediately clear that Effros and Winkler's result implies this, but the difficult details can be found in Appendix X.4 of [10]. This result provides further support for the idea that the free spectrahedra of matrix convexity play a similar role to the closed half-spaces or polytopes of classical convexity.

Example 3.1.7 (Matrix intervals)**.** We saw the matrix intervals previously, in Example 2.3.8, and proved directly that they are matrix convex sets. Here we consider the matrix intervals $[aI, bI]$ for which $a < 0 < b$. We will prove that $[aI, bI]$ is a free spectrahedra. By definition,

$$[aI, bI] = \{X \in \mathbb{S}^g : \ aI \leqslant X \leqslant bI\} = \{X \in \mathbb{S} : aI \leqslant X\} \cap \{X \in \mathbb{S}^g : X \leqslant bI\}.$$

Defining $\Omega = b^{-1}$ and $\Psi = a^{-1}$, we have that

$$L_\Omega(X) \geqslant 0 \iff I - b^{-1}X \geqslant 0 \iff X \leqslant bI,$$

and

$$L_\Psi(X) \geqslant 0 \iff I - a^{-1}X \geqslant 0 \iff aI \leqslant X.$$

Hence $[aI, bI] = \mathcal{D}_\Omega \cap \mathcal{D}_\Psi = \mathcal{D}_{\Omega \oplus \Psi}$.

Example 3.1.8 (OH-ball)**.** Consider once again the OH-ball, defined as $B_g^{\mathrm{OH}} = \{X \in \mathbb{S}^g : \sum_{i=1}^g X_i^2 \leqslant I\}$. This was seen previously in Example 2.3.9. It happens to be a free spectrahedron with defining tuple

$$\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \right) \in \mathbb{S}_{g+1}^{g},$$

so that we could equivalently define

$$B_g^{\mathrm{OH}} = \left\{ X \in \mathbb{S}^g : L_\Omega(X) = \begin{bmatrix} I & -X_1 & -X_2 & \cdots & -X_g \\ -X_1 & I & 0 & \cdots & 0 \\ -X_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_g & 0 & 0 & \cdots & I \end{bmatrix} \succeq 0 \right\}.$$

To see this, let $Y = \begin{bmatrix} -X_1 & -X_2 & \cdots & -X_g \end{bmatrix}$. Then $L_\Omega(X) = \begin{bmatrix} I & Y \\ Y^* & I \end{bmatrix} \succeq 0$ if and only if $\begin{bmatrix} I & Y^* \\ Y & I \end{bmatrix} \succeq 0$. By Proposition A.0.4, $L_\Omega(X) \succeq 0$ if and only if

$$0 \preceq I - YY^* = I - \begin{bmatrix} -X_1 & -X_2 & \cdots & -X_g \end{bmatrix} \begin{bmatrix} -X_1 \\ -X_2 \\ \vdots \\ -X_g \end{bmatrix} = I - \sum_{i=1}^{g} X_i^2.$$

Therefore the OH-ball is a free spectrahedron.

There do exist matrix convex sets that are not free spectrahedra. For example the interior of a free spectrahedron is matrix convex but fails to be closed and so fails to be a free spectrahedron. More interestingly, the set

$$\left\{ X \in \mathbb{S}^g : I - \sum_{i=1}^{g} x_i X_i \succeq 0 \text{ for all } x \in \mathbb{R}^g \text{ with } \|x\| = 1 \right\},$$

is matrix convex but not a free spectrahedron. This is a consequence of [5], Corollary 6.3 (or Corollary 8.3 in the corrected version of the paper), though it takes some effort to see why. Also, note that it is an intersection of infinitely many free spectrahedra, one for each unit-norm $x \in \mathbb{R}^g$.

Before proceeding, it is worth mentioning a recent Krein-Milman-type result, namely Theorem 1.2 of [5], that is different from the one we aim to prove in this thesis. This result uses the notion of the *free polar dual* of a matrix convex set, which is beyond the scope of this thesis. For the purpose of stating the result, it suffices to say that K° denotes the free polar dual of a matrix convex set K.

Theorem 3.1.9 (See [5], Theorem 1.2)**.** *Suppose $K \subseteq \mathbb{S}^g$ is a closed matrix convex set with $0 \in K(1)$. If K° is a free spectrahedron, then $K = \operatorname{mco}\{\Omega\}$ for some $\Omega \in \partial^{\text{Arv}} K$, or equivalently $K = \operatorname{mco}\{\Omega^1, \ldots, \Omega^t\}$ for some $\Omega^1, \ldots, \Omega^t \in \partial^{\text{abs}} K$. In particular, $K = \operatorname{mco} \partial^{\text{abs}} K$.*

Conversely, for any $\Omega \in \mathbb{S}^g$ we have $\mathcal{D}_\Omega = (\operatorname{mco}\{\Omega\})^\circ$.

In the theorem, $E = \{\Omega^1, \ldots, \Omega^t\}$ is a set of irreducible matrix tuples satisfying $K = \operatorname{mco} E$. By Proposition 3.0.2, it follows that $\partial^{\text{abs}} K \subseteq E$ up to unitary equivalence. From this it follows that such a matrix convex set K has just finitely many absolute extreme points, up to unitary equivalence.

3.2 Dilations in Free Spectrahedra

Heretofore, we dealt strictly with matrices whose entries were allowed to be complex numbers. Presently, however, we need to consider matrices whose entries must only be real numbers. We shall prove the main result under that assumption first (Theorem 3.3.3), and then proceed to partially extend the result to the complex case with relative ease (Theorem 3.4.4).

Accordingly, we will let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$ in situations where the choice is irrelevant. We use $(\mathbb{S}^{\mathbb{K}})^g$ to denote all g-tuples of self-adjoint matrices over $\mathbb{K}$. We use $\mathcal{D}_\Omega^{\mathbb{K}}$ for the set of $X \in (\mathbb{S}^{\mathbb{K}})^g$ such that $L_\Omega(X) \geqslant 0$. Other notation will be adapted in like manner.

The dilation-theoretic characterization of absolute extreme points that we recorded in Theorem 2.6.4 was originally proven over $\mathbb{C}$ in [5, Theorem 3.1], but can also be proven over $\mathbb{R}$ using the same arguments, the details of which can be found in [4, Section 5.2].

3.2.1 The Dilation Subspace

We aim to prove that any element of a compact free spectrahedron $\mathcal{D}_\Omega^{\mathbb{K}}$ generated by a real tuple can be dilated to an Arveson boundary point. To this end, we introduce a way to measure how "far off" a point X is from being an Arveson boundary point, and then show how dilations can produce points that are "closer" to being Arveson boundary points.

Loosely speaking, to a point X there corresponds a certain integer $l \geqslant 0$. It will be seen that $l = 0$ if and only if X is an Arveson boundary point. It will also be shown that if $l > 0$ then X can be dilated to another point X' of $\mathcal{D}_\Omega^{\mathbb{K}}$ whose corresponding l is strictly smaller than that of X. Thus by stringing together enough dilations, one obtains a dilation of X whose particular $l = 0$, and is consequently an Arveson boundary point of $\mathcal{D}_\Omega^{\mathbb{K}}$.

Following the paper [4], we begin with the notion of the *dilation subspace* at a point in a free spectrahedron. The integer l in the preceding discussion will correspond to the dimension of the dilation subspace.

Definition 3.2.1 (Dilation subspace)**.** Given $\Omega \in (\mathbb{S}_d^{\mathbb{K}})^g$ and $X \in (\mathbb{S}_n^{\mathbb{K}})^g$, the *dilation subspace of* $\mathcal{D}_\Omega^{\mathbb{K}}$ *at* X is defined as

$$\mathcal{K}_\Omega^{\mathbb{K}}(X) := \left\{ \beta \in (\mathbb{M}_{n,1}^{\mathbb{K}})^g : \ \exists c > 0, \ \exists \gamma \in \mathbb{R}^g, \ \begin{bmatrix} X & c\beta \\ c\beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}} \right\}.$$

Loosely speaking, the dilation subspace at X consists of all "off-diagonal blocks", up to positive scaling, of 1-dilations of X in $\mathcal{D}_\Omega^{\mathbb{K}}$.

Proposition 3.2.2. *For all* $X \in \mathbb{S}^g$*, we have* $X \in \partial^{\mathrm{Arv}} \mathcal{D}_\Omega^{\mathbb{K}}$ *if and only if* $\mathcal{K}_\Omega^{\mathbb{K}}(X) = \{0\}$.

Proof. First, since $X, 0 \in \mathcal{D}_\Omega^{\mathbb{K}}$, we have $\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} = X \oplus 0 \in \mathcal{D}_\Omega^{\mathbb{K}}$ too. Therefore $0 \in \mathcal{K}_\Omega^{\mathbb{K}}(X)$.

If $X \in \partial^{\mathrm{Arv}} \mathcal{D}_\Omega^{\mathbb{K}}$ and $\beta \in \mathcal{K}_\Omega^{\mathbb{K}}(X)$, then there exists $c > 0$ and $\gamma \in \mathbb{R}^g$ such that $\begin{bmatrix} X & c\beta \\ c\beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$. By the definition of the Arveson boundary, we have $c\beta = 0$, which implies $\beta = 0$. Hence $\mathcal{K}_\Omega^{\mathbb{K}}(X) = \{0\}$.

Conversely, suppose that the dilation subspace is trivial. If $\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix}$ is any 1-dilation of X which belongs to $\mathcal{D}_\Omega^{\mathbb{K}}$, then taking $c = 1$ we have that β is in the dilation subspace. Hence $\beta = 0$, which implies that $X \in \partial^{\mathrm{Arv}} K$. $\square$

We will see that the dilation subspace of a point $X \in \mathcal{D}_\Omega^{\mathbb{K}}(n)$ is, in fact, a linear subspace of $(\mathbb{M}_{n,1}^{\mathbb{K}})^g$, but not until after we have characterized the dilation subspace in a way that makes this fact obvious. In the meantime its vector space structure is of no consequence.

A Suite of Kernel Containment Results

Consider a generic self-adjoint 1-dilation of $X \in (\mathbb{S}_n^{\mathbb{K}})^g$, which takes the form $W = \begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix}$ for some $\beta \in (\mathbb{M}_{n,1}^{\mathbb{K}})^g$ and $\gamma \in \mathbb{R}^g$. Then, as we saw in Proposition 3.1.4,

$$W \in \mathcal{D}_\Omega^{\mathbb{K}} \quad \Longleftrightarrow \quad L_\Omega\left(\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix}\right) \succeq 0 \quad \Longleftrightarrow \quad \begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(\beta) \\ -\Lambda_\Omega(\beta)^* & L_\Omega(\gamma) \end{bmatrix} \succeq 0.$$

It turns out the containment $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(\beta)^*$ is closely related to the positivity of the rightmost matrix, a fact which we can use to obtain a characterization of the dilation subspace of $\mathcal{D}_\Omega^{\mathbb{K}}$ at X.

The following is a special case of what is sometimes called the row-inclusion property [9, Observation 7.1.10]. It will be utilized in several arguments to come.

Lemma 3.2.3. *Let $X \in \mathbb{M}_n^{\mathbb{K}}$, $Z \in \mathbb{M}_m^{\mathbb{K}}$, $Y \in \mathbb{M}_{n,m}^{\mathbb{K}}$. Then*

$$\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \succeq 0 \quad \Longrightarrow \quad \operatorname{Ker} X \subseteq \operatorname{Ker} Y^* \quad \textit{and} \quad \operatorname{Ker} Z \subseteq \operatorname{Ker} Y.$$

Proof. The proof in the case of $\mathbb{K} = \mathbb{R}$ is a simplified version of the proof for $\mathbb{K} = \mathbb{C}$. As such, we will assume that $\mathbb{K} = \mathbb{C}$ in the proof.

Let $W = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$ and let $h \in \operatorname{Ker} X$. For any $k \in \mathbb{C}^m$, $r \in \mathbb{R}$, and $\theta \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leqslant \left\langle W(h \oplus re^{i\theta}k),\ h \oplus re^{i\theta}k \right\rangle \\ &= \langle Xh, h\rangle + 2r\operatorname{Re}(e^{i\theta}\langle Y^*h, k\rangle) + r^2\langle Zk, k\rangle \\ &= 2r\operatorname{Re}(e^{i\theta}\langle Y^*h, k\rangle) + r^2\langle Zk, k\rangle. \end{aligned}$$

Whenever $r > 0$, we therefore have

$$0 \leqslant 2\,\mathrm{Re}(e^{i\theta}\langle Y^*h, k\rangle) + r\langle Zk, k\rangle.$$

Letting $r \to 0$ we then have that $0 \leqslant \mathrm{Re}(e^{i\theta}\langle Y^*h, k\rangle)$. The only way this can hold for all θ is if $\langle Y^*h, k\rangle = 0$. Since this holds for all k, it must be that $Y^*h = 0$, and so $h \in \operatorname{Ker} Y^*$. Therefore $\operatorname{Ker} X \subseteq \operatorname{Ker} Y^*$.

Applying what we just proved to

$$\begin{bmatrix} Z & Y^* \\ Y & X \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \geqslant 0$$

gives that $\operatorname{Ker} Z \subseteq \operatorname{Ker} Y$. □

Lemma 3.2.4 (A Positivity and Kernel Containment Property)**.** *Let $X, Y \in \mathbb{M}_n^{\mathbb{K}}$. If $X \geqslant 0$, Y is self-adjoint, and* $\operatorname{Ker} X \subseteq \operatorname{Ker} Y$*, then there is a real number $d > 0$ such that $X \pm cY \geqslant 0$ for all $c \in \mathbb{R}$ with $0 < c \leqslant d$.*

Proof. Decompose the underlying vector space $\mathbb{K}^n = (\operatorname{Ker} X)^\perp \oplus \operatorname{Ker} X$. With respect to this decomposition, and because X is self-adjoint, X takes the form $X = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, where A is invertible and $A \geqslant 0$. Hence $A \geqslant \delta I$ for some $\delta > 0$. Now, since $\operatorname{Ker} X \subseteq \operatorname{Ker} Y$, and since Y is self-adjoint, it follows that $Y = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition $\mathbb{K}^n = (\operatorname{Ker} X)^\perp \oplus \operatorname{Ker} X$. Let $d > 0$ be chosen so that $\pm dB \leqslant \delta I$ (equivalently, $d\,\|B\| \leqslant \delta$), and fix $0 < c \leqslant d$. Since $\pm cB \leqslant \delta I \leqslant A$, we have $A \pm cB \geqslant 0$. Finally we compute

$$X \pm cY = \begin{bmatrix} A \pm cB & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0.$$

$\square$

The next two results are stated and proved in more generality than is required simply because they are no harder to prove. This is also the first time we appeal to the Appendix. In particular, Corollary A.0.5 from the Appendix states that the 2×2 block matrix $\begin{bmatrix} X & Y \\ Y^* & I \end{bmatrix} \geqslant 0$ if and only if $X - YY^* \geqslant 0$. From this point onward, several of our arguments will depend on the results found in the Appendix.

Lemma 3.2.5. *If $X \in \mathcal{D}_\Omega^\mathbb{K}(n)$, $Y \in (\mathbb{M}_{n,m}^\mathbb{K})^g$, and $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(Y)^*$, then $\begin{bmatrix} X & cY \\ cY^* & 0 \end{bmatrix} \in \mathcal{D}_\Omega^\mathbb{K}$ for some $c > 0$.*

Proof. Note that we also have $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(Y)\Lambda_\Omega(Y)^*$, so we apply the previous result to $L_\Omega(X)$ and $\Lambda_\Omega(Y)\Lambda_\Omega(Y)^*$ to get a $d > 0$ such that $L_\Omega(X) - d\Lambda_\Omega(Y)\Lambda_\Omega(Y)^* \geqslant 0$. Letting $c = \sqrt{d}$, this last inequality can be written as $L_\Omega(X) - (-\Lambda_\Omega(cY))(-\Lambda_\Omega(cY))^* \geqslant 0$. By Corollary A.0.5, it follows that

$$\begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(cY) \\ -\Lambda_\Omega(cY)^* & I \end{bmatrix} \geqslant 0.$$

By Proposition 3.1.4, we conclude that $\begin{bmatrix} X & cY \\ cY^* & 0 \end{bmatrix} \in \mathcal{D}_\Omega^\mathbb{K}$, as desired. $\square$

Corollary 3.2.6. *If $X \in \mathcal{D}_\Omega^\mathbb{K}$ and we have a dilation $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in \mathcal{D}_\Omega^\mathbb{K}$, then $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(Y)^*$ and $\begin{bmatrix} X & cY \\ cY^* & 0 \end{bmatrix} \in \mathcal{D}_\Omega^\mathbb{K}$ for some $c > 0$.*

Proof. We have

$$\begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(Y) \\ -\Lambda_\Omega(Y)^* & L_\Omega(Z) \end{bmatrix} \geqslant 0.$$

As we saw in Lemma 3.2.3, this implies $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(Y)^*$. Applying the previous result yields a scalar $c > 0$ such that $\begin{bmatrix} X & cY \\ cY^* & 0 \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$. ☐

At long last, we are ready to give the "kernel containment" characterization of the dilation subspace. We have already done all the work; it is just a matter of collecting all of it together.

Proposition 3.2.7. *Let $\Omega \in (\mathbb{S}_d^{\mathbb{K}})^g$, $X \in (\mathbb{S}_n^{\mathbb{K}})^g$, and $\beta \in (\mathbb{M}_{n,1}^{\mathbb{K}})^g$. The following are equivalent:*

1. *$\beta \in \mathcal{K}_\Omega^{\mathbb{K}}(X)$, that is, $\exists c > 0, \exists \gamma \in \mathbb{R}^g$, $\begin{bmatrix} X & c\beta \\ c\beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$;*
2. *$\exists c > 0$, $\begin{bmatrix} X & c\beta \\ c\beta^* & 0 \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$;*
3. *$\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(\beta)^*$.*

Proof. Corollary 3.2.6 and Lemma 3.2.5 show that (1) implies (3) and that (3) implies (2). It is immediate that (2) implies (1). ☐

This new criterion for membership in the dilation subspace makes it an easy matter to prove that the dilation subspace of $X \in \mathcal{D}_\Omega^{\mathbb{K}}(n)$ does form a linear subspace of $(\mathbb{M}_{n,1}^{\mathbb{K}})^g$.

Corollary 3.2.8. *If $\Omega \in (\mathbb{S}_d^{\mathbb{K}})^g$ and $X \in \mathcal{D}_\Omega^{\mathbb{K}}(n)$, then $\mathcal{K}_\Omega^{\mathbb{K}}(X)$ is a linear subspace of $(\mathbb{M}_{n,1}^{\mathbb{K}})^g$.*

Proof. Suppose we have $\beta_1, \beta_2 \in \mathcal{K}_\Omega^{\mathbb{K}}(X)$, and scalars $a_1, a_2 \in \mathbb{K}$. Then taking a vector $x \in \operatorname{Ker} L_\Omega(X)$ we have $x \in \operatorname{Ker} \Lambda_\Omega(\beta_1)^*$ and $x \in \operatorname{Ker} \Lambda_\Omega(\beta_2)^*$, and so

$$\Lambda_\Omega(a_1\beta_1 + a_2\beta_2)^* x = a_1 \Lambda_\Omega(\beta_1)^* x + a_2 \Lambda_\Omega(\beta_2)^* x = 0.$$

Hence $\operatorname{Ker} L_\Omega(X) \subseteq \operatorname{Ker} \Lambda_\Omega(a_1\beta_1 + a_2\beta_2)^*$, whence $a_1\beta_1 + a_2\beta_2 \in \mathcal{K}_\Omega^{\mathbb{K}}(X)$. ☐

The vector space structure of the dilation subspace matters, since the proof of our main result will argue using the dimension of the dilation subspace.

3.2.2 Maximal 1-dilations

Consider any matrix convex set $K \subseteq (\mathbb{S}^{\mathbb{K}})^g$, and suppose $W = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in K$. Since X and Z are compressions of W, we have $X \in K$ and $Z \in K$. It then follows that $X \oplus Z \in K$. Now, by taking Euclidean convex combinations of $X \oplus Z$ and W, we see that

$$\begin{bmatrix} X & cY \\ cY^* & Z \end{bmatrix} = (1-c)\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} + c\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in K, \quad \forall 0 \leqslant c \leqslant 1.$$

So we can imagine continuously shrinking the off-diagonal components of a 2×2 block matrix tuple until they disappear entirely, and the matrix tuples so obtained all remain in K. We naturally wonder how *big* the scaling factor can be while still ensuring that the resulting matrix tuple remains in K. The notion of a maximal 1-dilation touches on this.

Definition 3.2.9 (Maximal 1-dilation)**.** Let $K \subseteq (\mathbb{S}^{\mathbb{K}})^g$ be a matrix convex set and $X \in K(n)$. Consider a 1-dilation of X:

$$Y = \begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in K(n+1).$$

We say that Y is a *maximal* 1*-dilation of* X if the following two conditions hold:

(a) $1 = \max\left\{ c \in \mathbb{R} : \exists \eta \in \mathbb{R}^g \text{ such that } \begin{bmatrix} X & c\beta \\ c\beta^* & \eta \end{bmatrix} \in K \right\}.$

(b) $\|\gamma\| = \max\left\{ \|\eta\| : \eta \in \mathbb{R}^g \text{ and } \begin{bmatrix} X & \beta \\ \beta^* & \eta \end{bmatrix} \in K \right\}.$

If $X \in \partial^{\mathrm{Arv}} K$, then X has no maximal 1-dilation. The only 1-dilations would have $\beta = 0$, which would allow us to choose any $c > 0$ with any $\eta \in \mathbb{R}^g$ whatsoever.

We will now identify a class of matrix convex sets in which maximal 1-dilations of non-Arveson-boundary points can always be found. The next observation is actually the first time that we use the assumption of compactness of a matrix convex set. So recall now that a matrix convex set $K \subseteq (\mathbb{S}^{\mathbb{K}})^g$ is *closed* if every matrix level $K(n)$ is closed. As well, recall that K is *bounded* if there is a constant $C > 0$ such that $C - \sum_{i=1}^g X_i^2 \geqslant 0$ for all $X \in K$. Finally, we say that K is *compact* if $K(n)$ is compact for every $n \geqslant 1$.

Proposition 3.2.10. *If $\mathcal{D}_\Omega^{\mathbb{K}}$ is a compact free spectrahedron and $X \in \mathcal{D}_\Omega^{\mathbb{K}} \setminus \partial^{\mathrm{Arv}} \mathcal{D}_\Omega^{\mathbb{K}}$, then there exists a maximal 1-dilation of X in $\mathcal{D}_\Omega^{\mathbb{K}}$.*

Proof. Since X is not an Arveson boundary point, there exists some 1-dilation of X in $\mathcal{D}_\Omega^{\mathbb{K}}$, say

$$\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}},$$

with $\beta \neq 0$. We claim that that $\mathcal{A} = \left\{ c \in \mathbb{R} : \exists \gamma' \in \mathbb{R}^g \text{ s.t. } \begin{bmatrix} X & c\beta \\ c\beta^* & \gamma' \end{bmatrix} \in K \right\}$ is bounded, and to prove it we assume to the contrary that $\mathcal{A}$ is unbounded.

Among the g components of β, there is some $\beta_{i_0} \neq 0$, and so $\beta_{i_0}^* \neq 0$ as well. Fix some unit-norm $x \in \mathbb{K}^n$ such that $\beta_{i_0}^* x \neq 0$. Then $x^* \beta_{i_0} \beta_{i_0}^* x \neq 0$ as well.

Let $d > 0$ and choose $c > 0$ in $\mathcal{A}$ which is large enough to satisfy

$$\sum_{i=1}^{g} (x^* X_i^2 x + c^2 x \beta_i \beta_i^* x) > d.$$

Since c belongs to $\mathcal{A}$, there exists $\eta \in \mathbb{R}^g$ such that $Y := \begin{bmatrix} X & c\beta \\ c\beta^* & \eta \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$. We claim that $\sum_{i=1}^g Y_i^2 \nleq dI$, for which it suffices to exhibit a unit vector $y \in \mathbb{K}^{n+1}$ such that $y^* \left(\sum_{i=1}^g Y_i^2\right) y > y^*(dI)y = d$. This is accomplished by $y := \begin{bmatrix} x \\ 0 \end{bmatrix}$. We have, first of all,

$$Y_i y = \begin{bmatrix} X_i & c\beta_i \\ c\beta_i^* & \eta_i \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} X_i x \\ c\beta_i^* x \end{bmatrix},$$

and then

$$\begin{aligned} y^* \left(\sum_{i=1}^g Y_i^2 \right) y &= \sum_{i=1}^g (Y_i y)^* (Y_i y) \\ &= \sum_{i=1}^g \begin{bmatrix} x^* X_i & c x^* \beta_i \end{bmatrix} \begin{bmatrix} X_i x \\ c\beta_i^* x \end{bmatrix} \\ &= \sum_{i=1}^g (x^* X_i^2 x + c^2 x^* \beta_i \beta_i^* x) \\ &> d. \end{aligned}$$

We have therefore shown that no matter which $d > 0$ we start with, there exists $Y \in \mathcal{D}_\Omega^{\mathbb{K}}$ such that $\sum_{i=1}^g Y_i^2 \nleq dI$. Since this implies that $\mathcal{D}_\Omega^{\mathbb{K}}$ is unbounded, we have contradicted the compactness of $\mathcal{D}_\Omega^{\mathbb{K}}$. This means that the set $\mathcal{A}$ must be bounded above, and since $\mathcal{D}_\Omega^{\mathbb{K}}$ is closed, the supremum of $\mathcal{A}$ belongs to $\mathcal{A}$. Let $d = \max \mathcal{A}$, let $\beta' = d\beta$, and take γ' such that $Z := \begin{bmatrix} X & \beta' \\ \beta'^* & \gamma' \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}}$. Then this Z satisfies the first condition for being a maximal 1-dilation, but it may not satisfy the second condition.

Let $\mathcal{B} = \left\{ \eta \in \mathbb{R}^g : \begin{bmatrix} X & \beta' \\ \beta'^* & \eta \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{K}} \right\}$. It is quite easy to see that $\mathcal{B}$ is bounded in norm: if $\eta \in \mathcal{B}$ then η is a compression of an element of $\mathcal{D}_\Omega^{\mathbb{K}}$, and therefore $\eta \in \mathcal{D}_\Omega^{\mathbb{K}}$.

Since $\mathcal{D}_\Omega^\mathbb{K}$ is bounded, then so must be $\mathcal{B} \subseteq \mathcal{D}_\Omega^\mathbb{K}(1)$.

We now claim that $\mathcal{B}$ is closed. For, suppose we have a sequence $(\eta_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ converging in norm to $\eta \in \mathbb{R}^g$. Then by continuity,

$$L_\Omega\left(\begin{bmatrix} X & \beta' \\ \beta'^* & \eta_n \end{bmatrix}\right) \to L_\Omega\left(\begin{bmatrix} X & \beta' \\ \beta'^* & \eta \end{bmatrix}\right), \text{ as } n \to \infty.$$

Since each term $L_\Omega\left(\begin{bmatrix} X & \beta' \\ \beta'^* & \eta_n \end{bmatrix}\right) \succcurlyeq 0$, it follows that $L_\Omega\left(\begin{bmatrix} X & \beta' \\ \beta'^* & \eta \end{bmatrix}\right) \succcurlyeq 0$. That is, $\eta \in \mathcal{B}$, whence $\mathcal{B}$ is closed.

This proves that $\mathcal{B}$ is compact, and so we can choose some $\eta' \in \mathcal{B}$ of maximum norm. Having done so, the dilation $\begin{bmatrix} X & \beta' \\ \beta'^* & \eta' \end{bmatrix}$ is a maximal 1-dilation of X. $\square$

3.3 The Krein-Milman-type Theorem Over $\mathbb{R}$

The imminent Theorem 3.3.2 relies heavily on the next lemma, which itself depends heavily on both Proposition A.0.4 and Proposition A.0.6 from Appendix A. The appendix is meant to be a brief introduction to the *Moore-Penrose pseudoinverse* $X^\dagger$ of a matrix X, giving only the material that is directly necessary for our purposes. The following lemma encapsulates all the dependencies on the appendix so that the proof of the main result flows more smoothly.

Lemma 3.3.1. *Let $P \in \mathbb{S}_n^\mathbb{R}$, $Q \in \mathbb{M}_{n,m}^\mathbb{R}$, $R \in \mathbb{S}_m^\mathbb{R}$, and $S \in \mathbb{S}_m^\mathbb{R}$, and $c > 0$. If*

$$\begin{bmatrix} P & Q & cQ \\ Q^* & R & S \\ cQ^* & S & I \end{bmatrix} \succcurlyeq 0,$$

then there exists $b > 0$ such that $1 \pm bc \geqslant 0$ and

$$\begin{bmatrix} P & \sqrt{1 \pm bc}Q \\ \sqrt{1 \pm bc}Q^* & R \pm bS \end{bmatrix} \geqslant 0.$$

Proof. By Proposition A.0.4 applied to $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geqslant 0$, we have that $R - Q^*P^\dagger Q \geqslant 0$. Since $S = S^*$, we may therefore apply Proposition A.0.6 to get $\mathrm{Ker}(R - Q^*P^\dagger Q) \subseteq \mathrm{Ker}(S - cQ^*P^\dagger Q)$. Then note that $(S - cQ^*P^\dagger Q)^* = S - cQ^*P^\dagger Q$ since S and $P^\dagger$ are self-adjoint (since P is self-adjoint, Proposition A.0.3 says $P^\dagger$ is self-adjoint). Applying Lemma 3.2.4 to this containment of kernels gives a scalar $b > 0$, which we may choose small enough so that $1 \pm bc \geqslant 0$, such that $(R - Q^*P^\dagger Q) \pm b(S - cQ^*P^\dagger Q) \geqslant 0$. This can be rearranged to yield $(R \pm bS) - (\sqrt{1 \pm bc}Q)^*P^\dagger(\sqrt{1 \pm bc}Q) \geqslant 0$.

We now want to apply Lemma A.0.4 to the matrix

$$\begin{bmatrix} P & \sqrt{1 \pm bc}Q \\ \sqrt{1 \pm bc}Q^* & R \pm bS \end{bmatrix},$$

and so far we have that $P \geqslant 0$ and that $(R \pm bS) - (\sqrt{1 \pm bc}Q)^*P^\dagger(\sqrt{1 \pm bc}Q) \geqslant 0$. The last condition we need is that $(I - PP^\dagger)(\sqrt{1 \pm bc}Q) = 0$, but in fact since $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geqslant 0$, we have by Lemma A.0.4 that $(I - PP^\dagger)Q = 0$, which still holds when we scale Q by $\sqrt{1 \pm bc}$.

Applying Lemma A.0.4, we have

$$\begin{bmatrix} P & \sqrt{1 \pm bc}Q \\ \sqrt{1 \pm bc}Q^* & R \pm bS \end{bmatrix} \geqslant 0.$$

$\square$

Theorem 3.3.2 (See [4], Theorem 2.4)**.** *Let $\Omega \in (\mathbb{S}_d^{\mathbb{R}})^g$ and suppose that $\mathcal{D}_\Omega^{\mathbb{R}}$ is compact. Let $X \in \mathcal{D}_\Omega^{\mathbb{R}}$ and assume that X is **not** an Arveson boundary point of $\mathcal{D}_\Omega^{\mathbb{R}}$. Then there exists a maximal 1-dilation $Y \in \mathcal{D}_\Omega^{\mathbb{R}}$ of X, and any such Y satisfies*

$$\dim \mathcal{K}_\Omega^{\mathbb{R}}(Y) < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X).$$

Proof. We saw already in Proposition 3.2.10 that maximal dilations exist in compact free spectrahedra, such as $\mathcal{D}_\Omega^{\mathbb{R}}$. So consider a maximal 1-dilation of X:

$$Y := \begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{R}}(n+1).$$

Let $P : (\mathbb{M}_{(n+1),1}^{\mathbb{R}})^g \to (\mathbb{M}_{n,1}^{\mathbb{R}})^g$ be the linear map defined as

$$P\left(\begin{bmatrix} \eta \\ \sigma \end{bmatrix}\right) = \eta, \quad \forall \eta \in (\mathbb{M}_{n,1}^{\mathbb{R}})^g,\ \sigma \in \mathbb{R}^g.$$

Our present and intermediate goal is to show that $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$.

<u>Step 1. $P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] \subseteq \mathcal{K}_\Omega^{\mathbb{R}}(X)$ and hence $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] \leqslant \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$</u>.

For, suppose $\eta \in P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)]$. Then there exists $\sigma \in \mathbb{R}^g$ such that $\begin{bmatrix} \eta \\ \sigma \end{bmatrix} \in \mathcal{K}_\Omega^{\mathbb{R}}(Y)$.
By definition of the dilation subspace $\mathcal{K}_\Omega^{\mathbb{R}}(Y)$, there is $c > 0$ such that

$$\begin{bmatrix} X & \beta & c\eta \\ \beta^* & \gamma & c\sigma \\ c\eta^* & c\sigma & 0 \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{R}}.$$

Since $\mathcal{D}_\Omega^{\mathbb{R}}$ is a matrix convex set,

$$\begin{bmatrix} X & c\eta \\ c\eta^* & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X & \beta & c\eta \\ \beta^* & \gamma & c\sigma \\ c\eta^* & c\sigma & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{R}}.$$

Hence, we have that $\eta \in \mathcal{K}_\Omega^{\mathbb{R}}(X)$. This proves that $P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] \subseteq \mathcal{K}_\Omega^{\mathbb{R}}(X)$, and hence $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] \leqslant \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$.

<u>Step 2. $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$</u>. Assume towards a contradiction that the dimensions are equal. This, combined with the containment of $P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)]$ in $\mathcal{K}_\Omega^{\mathbb{R}}(X)$, gives that $P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] = \mathcal{K}_\Omega^{\mathbb{R}}(X)$. Since $\beta \in \mathcal{K}_\Omega^{\mathbb{R}}(X)$, we therefore have $\beta \in P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)]$. So there is $\sigma \in \mathbb{R}^g$ and $c > 0$ such that

$$L_\Omega\left(\begin{bmatrix} X & \beta & c\beta \\ \beta^* & \gamma & \sigma \\ c\beta^* & \sigma & 0 \end{bmatrix}\right) \geqslant 0.$$

Note that σ, consisting of real scalars, satisfies $\sigma^* = \sigma$. By Proposition 3.1.4, this last inequality is equivalent to

$$\begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(\beta) & -c\Lambda_\Omega(\beta) \\ -\Lambda_\Omega(\beta)^* & L_\Omega(\gamma) & -\Lambda_\Omega(\sigma) \\ -c\Lambda_\Omega(\beta)^* & -\Lambda_\Omega(\sigma) & I \end{bmatrix} \geqslant 0.$$

Let us temporarily simplify the notation by writing

$$\begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(\beta) & -c\Lambda_\Omega(\beta) \\ -\Lambda_\Omega(\beta)^* & L_\Omega(\gamma) & -\Lambda_\Omega(\sigma) \\ -c\Lambda_\Omega(\beta)^* & -\Lambda_\Omega(\sigma) & I \end{bmatrix} = \begin{bmatrix} P & Q & cQ \\ Q^* & R & S \\ cQ^* & S & I \end{bmatrix} \geqslant 0. \tag{1}$$

Applying Lemma 3.3.1, we obtain a $b > 0$ such that $1 \pm bc \geqslant 0$ and

$$0 \leqslant \begin{bmatrix} P & \sqrt{1 \pm bc}Q \\ \sqrt{1 \pm bc}Q^* & R \pm bS \end{bmatrix} = \begin{bmatrix} L_\Omega(X) & -\sqrt{1 \pm bc}\Lambda_\Omega(\beta) \\ -\sqrt{1 \pm bc}\Lambda_\Omega(\beta)^* & L_\Omega(\gamma) \mp b\Lambda_\Omega(\sigma) \end{bmatrix}$$

$$= \begin{bmatrix} L_\Omega(X) & -\Lambda_\Omega(\sqrt{1 \pm bc}\beta) \\ -\Lambda_\Omega(\sqrt{1 \pm bc}\beta^*) & L_\Omega(\gamma \pm b\sigma) \end{bmatrix} \overset{\mathrm{u}}{\equiv} L_\Omega\left(\begin{bmatrix} X & \sqrt{1 \pm bc}\beta \\ \sqrt{1 \pm bc}\beta^* & \gamma \pm b\sigma \end{bmatrix}\right).$$

By maximality of the original dilation $\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix}$, we must have $\sqrt{1 \pm bc} \leqslant 1$. Hence $bc = 0$, which is a contradiction since both b and c are strictly positive. Therefore, $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$.

<u>Step 3. $\dim \mathcal{K}_\Omega^{\mathbb{R}}(Y) < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$.</u> Assume to the contrary that $\dim \mathcal{K}_\Omega^{\mathbb{R}}(Y) \geqslant \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$. Since $\dim P[\mathcal{K}_\Omega^{\mathbb{R}}(Y)] < \dim \mathcal{K}_\Omega^{\mathbb{R}}(X)$, there are $\eta \in \mathbb{M}_{n\times 1}(\mathbb{R})^g$, and $\sigma_1, \sigma_2 \in \mathbb{R}^g$ such that $\begin{bmatrix} \eta \\ \sigma_1 \end{bmatrix}, \begin{bmatrix} \eta \\ \sigma_2 \end{bmatrix} \in \mathcal{K}_\Omega^{\mathbb{R}}(Y)$ but $\sigma_1 \neq \sigma_2$. This implies that $\begin{bmatrix} 0 \\ \sigma_1 - \sigma_2 \end{bmatrix} \in \mathcal{K}_\Omega^{\mathbb{R}}(Y)$. Now define $\sigma := \sigma_1 - \sigma_2$. Then $\sigma \neq 0$ and $\begin{bmatrix} 0 \\ \sigma \end{bmatrix} \in \mathcal{K}_\Omega^{\mathbb{R}}(Y)$.

By Proposition 3.2.7, there is $c > 0$ such that

$$L_\Omega\left(\begin{bmatrix} X & \beta & 0 \\ \beta^* & \gamma & c\sigma \\ 0 & c\sigma & 0 \end{bmatrix}\right) \geqslant 0.$$

Similar to the previous step, we let $P = L_\Omega(X)$, $Q = -\Lambda_\Omega(\beta)$, $R = L_\Omega(\gamma)$, and

$S = -\Lambda_\Omega(\sigma)$, so that the above inequality implies

$$\begin{bmatrix} P & Q & 0 \\ Q^* & R & cS \\ 0 & cS & I \end{bmatrix} \geqslant 0.$$

We apply Lemma 3.3.1 to obtain $b > 0$ such that $\begin{bmatrix} P & Q \\ Q^* & R \pm bcS \end{bmatrix} \geqslant 0$, which implies that $L_\Omega\left(\begin{bmatrix} X & \beta \\ \beta^* & \gamma \pm bc\sigma \end{bmatrix}\right) \geqslant 0$. By maximality of the original dilation, we have

$$\|\gamma \pm bc\sigma\| \leqslant \|\gamma\|.$$

By basic convexity of the ball of radius $\|\gamma\|$, these inequalities cannot both hold unless $bc\sigma = 0$. But b, c, and σ are all non-zero, which is a contradiction. Hence $\dim \mathcal{K}_\Omega^\mathbb{R}(Y) < \dim \mathcal{K}_\Omega^\mathbb{R}(X)$. □

At this point we've all but proven the main result over the real numbers, except for a basic counting argument.

Theorem 3.3.3. *Suppose the real free spectrahedron $\mathcal{D}_\Omega^\mathbb{R}$ is compact. Then $\mathcal{D}_\Omega^\mathbb{R} = \operatorname{mco} \partial^{\text{abs}} \mathcal{D}_\Omega^\mathbb{R}$.*

Proof. We first prove that every non-Arveson-boundary point of K dilates to an Arveson boundary point. Assume to the contrary that this is not true, so that there is $X \in \mathcal{D}_\Omega^\mathbb{R} \setminus \partial^{\text{Arv}} \mathcal{D}_\Omega^\mathbb{R}$ which has no Arveson boundary point as a dilation.

Let $Y_0 = X$. By Theorem 3.3.2, there is a dilation Y_1 of X such that $\dim \mathcal{K}_\Omega^\mathbb{R}(X) > \dim \mathcal{K}_\Omega^\mathbb{R}(Y_1)$. Now assuming we have constructed a sequence $Y_0, Y_1, Y_2, \ldots, Y_n$ of dilations of X such that $\dim \mathcal{K}_\Omega^\mathbb{R}(Y_i) > \dim \mathcal{K}_\Omega^\mathbb{R}(Y_{i+1})$ for all i, we apply Theorem 3.3.2 to obtain a dilation Y_{n+1} of Y_n such that $\dim \mathcal{K}_\Omega^\mathbb{R}(Y_n) > \dim \mathcal{K}_\Omega^\mathbb{R}(Y_{n+1})$. Since

Y_n is a dilation of X and Y_{n+1} is a dilation of Y_n, then indeed Y_{n+1} is a dilation of X. By induction, we obtain a strictly decreasing sequence of non-negative integers, $(\dim \mathcal{K}_\Omega^{\mathbb{R}}(Y_i))_{i\geqslant 1}$. This is a contradiction. Hence every non-Arveson-boundary point dilates to an Arveson boundary point.

Let $X \in \mathcal{D}_\Omega^{\mathbb{R}}$. Then there is $Y \in \partial^{\mathrm{Arv}} \mathcal{D}_\Omega^{\mathbb{R}}$ and an isometry V such that $V^*YV = X$. By Proposition 2.6.8, $Y \overset{\mathrm{u}}{\equiv} Y_1 \oplus \cdots \oplus Y_t$ for some $Y_i \in \partial^{\mathrm{abs}} \mathcal{D}_\Omega^{\mathbb{K}}$. Hence $X \overset{\mathrm{u}}{\equiv} V^*(Y_1 \oplus \cdots \oplus Y_t)V \in \operatorname{mco} \partial^{\mathrm{abs}} \mathcal{D}_\Omega^{\mathbb{R}}$. $\square$

3.4 The Krein-Milman-type Theorem Over $\mathbb{C}$

We now turn to the task of proving the main result for compact, complex free spectrahedra which are defined by a tuple of real symmetric matrices, leveraging what we already proved in Theorem 3.3.3.

First, we show that a complex free spectrahedron has a real defining tuple if and only if it is closed under complex conjugation (Lemma 3.4.1). From this it follows that any element X of a complex free spectrahedron $\mathcal{D}_\Omega^{\mathbb{C}}$, with real defining tuple, can be dilated to a tuple of real symmetric matrices in $\mathcal{D}_\Omega^{\mathbb{C}}$ (Lemma 3.4.2). Third, we show that the Arveson boundary points of $\mathcal{D}_\Omega^{\mathbb{R}}$ are precisely the Arveson boundary points of $\mathcal{D}_\Omega^{\mathbb{C}}$ whose component matrices have real entries (Lemma 3.4.3). Then we put these all together to prove the main result, Theorem 3.4.4.

We denote the complex conjugate of a complex number $x \in \mathbb{C}$ by $\overline{x}$. Given a matrix $X \in \mathbb{M}_{n,m}^{\mathbb{C}}$, $\overline{X}$ denotes the entrywise complex conjugate of X. A standard fact about matrix tensor products is that $\overline{X \otimes Y} = \overline{X} \otimes \overline{Y}$. Given a matrix *tuple* $X \in (\mathbb{M}_{n,m}^{\mathbb{C}})^g$, we define $\overline{X} := \left(\overline{X_1},\ \overline{X_2}, \ldots,\ \overline{X_g}\right)$.

Lemma 3.4.1. *Let $\Omega \in (\mathbb{S}^{\mathbb{C}})^g$. Then $\mathcal{D}_\Omega^{\mathbb{C}}$ is closed under complex conjugation if and only if there exists $\Psi \in (\mathbb{S}^{\mathbb{R}})^g$ such that $\mathcal{D}_\Omega^{\mathbb{C}} = \mathcal{D}_\Psi^{\mathbb{C}}$.*

Proof. Suppose $\mathcal{D}_\Omega^{\mathbb{C}}$ is closed under complex conjugation. Then $X \in \mathcal{D}_\Omega^{\mathbb{C}}$ if and only if both $L_\Omega(X) \succcurlyeq 0$ and $L_\Omega(\overline{X}) \succcurlyeq 0$. But

$$L_\Omega(\overline{X}) = I - \sum_{i=1}^{g} \Omega_i \otimes \overline{X_i} = \overline{I - \sum_{i=1}^{g} \overline{\Omega_i} \otimes X_i} = \overline{L_{\overline{\Omega}}(X)}.$$

Thus $L_\Omega(\overline{X}) \succcurlyeq 0$ if and only if $L_{\overline{\Omega}}(X) \succcurlyeq 0$. Thus, recalling the linear pencil properties of Proposition 3.1.4,

$$X \in \mathcal{D}_\Omega^{\mathbb{C}} \iff L_\Omega(X) \succcurlyeq 0 \text{ and } L_{\overline{\Omega}}(X) \succcurlyeq 0 \iff L_{\Omega\oplus\overline{\Omega}}(X) \overset{\mathrm{u}}{\equiv} L_\Omega(X) \oplus L_{\overline{\Omega}}(X) \succcurlyeq 0.$$

Thus $\mathcal{D}_\Omega^{\mathbb{C}} = \mathcal{D}_{\Omega\oplus\overline{\Omega}}^{\mathbb{C}}$.

We can write $\Omega = A + iB$, where $A = (\Omega + \overline{\Omega})/2$ and $B = (\Omega - \overline{\Omega})/2i$. Then A and B are tuples of real matrices. Define

$$\Psi := \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix} \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix} \overset{\mathrm{u}}{\equiv} \begin{bmatrix} \Omega & 0 \\ 0 & \overline{\Omega} \end{bmatrix}.$$

Therefore $\mathcal{D}_\Psi^{\mathbb{C}} = \mathcal{D}_{\Omega\oplus\overline{\Omega}}^{\mathbb{C}} = \mathcal{D}_\Omega^{\mathbb{C}}$. Since $\Psi \in (\mathbb{S}^{\mathbb{R}})^g$, we are done.

Conversely, suppose we have $\Psi \in (\mathbb{S}^{\mathbb{R}})^g$ with $\mathcal{D}_\Psi^{\mathbb{C}} = \mathcal{D}_\Omega^{\mathbb{C}}$. If $L_\Omega(X) \succcurlyeq 0$, then $L_\Psi(X) \succcurlyeq 0$, and since Ψ is real we have $L_\Psi(\overline{X}) = \overline{L_{\overline{\Psi}}(X)} = \overline{L_\Psi(X)} \succcurlyeq 0$. Hence $L_\Psi(\overline{X}) \succcurlyeq 0$, implying $L_\Omega(\overline{X}) \succcurlyeq 0$. Thus $\mathcal{D}_\Omega^{\mathbb{C}}$ is closed under complex conjugation. $\square$

Consider that a real matrix tuple $\Omega \in (\mathbb{S}^{\mathbb{R}})^g$ defines both $\mathcal{D}_\Omega^{\mathbb{R}}$ and $\mathcal{D}_\Omega^{\mathbb{C}}$, and that $\mathcal{D}_\Omega^{\mathbb{R}} \subseteq \mathcal{D}_\Omega^{\mathbb{C}}$. It is a convenient fact about free spectrahedra defined by real tuples that any element of $\mathcal{D}_\Omega^{\mathbb{C}}$ is a compression of an element of $\mathcal{D}_\Omega^{\mathbb{R}}$.

Lemma 3.4.2. *Let $\Omega \in (\mathbb{S}_d^{\mathbb{R}})^g$. If $X \in \mathcal{D}_\Omega^{\mathbb{C}}$, then X is a compression of some $Y \in \mathcal{D}_\Omega^{\mathbb{R}}$.*

Proof. Let $X \in \mathcal{D}_\Omega^{\mathbb{C}}$. Then $X = A + iB$, where $A = (X + \overline{X})/2$ and $B = (X - \overline{X})/2i$. Note that A and B have real entries. Since $\mathcal{D}_\Omega^{\mathbb{C}}$ is closed under complex conjugation by Lemma 3.4.1, $A - iB = \overline{X} \in \mathcal{D}_\Omega^{\mathbb{C}}$ too. Then

$$Y := \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix} \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix} \overset{\text{u}}{\equiv} X \oplus \overline{X} \in \mathcal{D}_\Omega^{\mathbb{C}}.$$

Since Y has real entries, we in fact have $Y \in \mathcal{D}_\Omega^{\mathbb{R}}$. $\square$

The next result shows that the Arveson boundary points of the real free spectrahedra are precisely the real Arveson boundary points of the complex free spectrahedra.

Lemma 3.4.3. *Let $\Omega \in (\mathbb{S}_d^{\mathbb{R}})^g$ and $X \in \mathcal{D}_\Omega^{\mathbb{R}}$. Then*

$$X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{R}} \iff X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{C}}.$$

Proof. If $X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{C}}$, then any real 1-dilation of $\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{R}}$ is also a complex 1-dilation of X in $\mathcal{D}_\Omega^{\mathbb{C}}$. Hence $\beta = 0$, and $X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{R}}$.

Conversely, suppose that $X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{R}}$, and consider a complex 1-dilation

$$\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{C}}.$$

Since Ω is real, $\mathcal{D}_\Omega^{\mathbb{C}}$ is necessarily closed under complex conjugation by Lemma 3.4.1. Hence

$$\begin{bmatrix} X & \overline{\beta} \\ \overline{\beta}^* & \gamma \end{bmatrix} = \begin{bmatrix} \overline{X} & \overline{\beta} \\ \overline{\beta}^* & \overline{\gamma} \end{bmatrix} = \overline{\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix}} \in \mathcal{D}_\Omega^{\mathbb{C}}.$$

Let $\alpha = (\beta + \overline{\beta})/2$. By convexity,

$$\begin{bmatrix} X & \alpha \\ \alpha^* & \gamma \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} + \begin{bmatrix} X & \overline{\beta} \\ \overline{\beta}^* & \gamma \end{bmatrix} \right) \in \mathcal{D}_\Omega^{\mathbb{C}}.$$

The leftmost matrix has real entries, and so $\begin{bmatrix} X & \alpha \\ \alpha^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{R}}$. Since X is assumed to be an Arveson boundary point of $\mathcal{D}_\Omega^{\mathbb{R}}$, it must be that $\alpha = 0$. Hence $\beta = -\overline{\beta}$, which implies that $i\beta$ is real. Note that

$$\begin{bmatrix} X & i\beta \\ (i\beta)^* & \gamma \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -iI \end{bmatrix} \begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & iI \end{bmatrix} \in \mathcal{D}_\Omega^{\mathbb{C}},$$

and in fact it belongs to $\mathcal{D}_\Omega^{\mathbb{R}}$ since the leftmost matrix is real. Again, since $X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{R}}$ we have that $i\beta = 0$, and so $\beta = 0$. Hence $X \in \partial^{\text{Arv}} \mathcal{D}_\Omega^{\mathbb{C}}$. □

We now put all these lemmas together with Theorem 3.3.2 to prove the main result.

Theorem 3.4.4. *Let $\Omega \in (\mathbb{S}^{\mathbb{R}})^g$, and suppose the complex free spectrahedron $\mathcal{D}_\Omega^{\mathbb{C}}$ is compact. Then $\mathcal{D}_\Omega^{\mathbb{C}} = \operatorname{mco} \partial^{\text{abs}} \mathcal{D}_\Omega^{\mathbb{C}}$.*

Proof. Let $X \in \mathcal{D}_\Omega^{\mathbb{C}}$. By Lemma 3.4.2, there is $Y \in \mathcal{D}_\Omega^{\mathbb{R}}$ that is a dilation of X, say $X = V^*YV$ for an isometry V. By Theorem 3.3.3, $Y \in \operatorname{mco} \partial^{\text{abs}} \mathcal{D}_\Omega^{\mathbb{R}}$. So there are $Y_1, \ldots, Y_t \in \partial^{\text{abs}} \mathcal{D}_\Omega^{\mathbb{R}}$ and an isometry W such that $Y = W^*(Y_1 \oplus \cdots \oplus Y_t)W$. By Lemma 3.4.3, each $Y_i \in \partial^{\text{abs}} \mathcal{D}_\Omega^{\mathbb{C}}$. We now have

$$X = V^*YV = (WV)^*(Y_1 \oplus \cdots \oplus Y_t)(WV) \in \operatorname{mco} \partial^{\text{abs}} \mathcal{D}_\Omega^{\mathbb{C}}.$$

□

It is important to know that the assumption of being generated by a real matrix tuple is strictly necessary. Passer [11] studies the failure of the Krein-Milman-type result obtained by dropping this assumption.

Combining Theorem 3.4.4, Theorem 3.3.3, and the minimality property of Theorem 3.0.2 into one statement, we have the following.

Theorem 3.4.5. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, $\Omega \in (\mathbb{S}^{\mathbb{R}})^g$, and suppose that the free spectrahedron $\mathcal{D}_\Omega^{\mathbb{K}}$ is compact. Then*

$$\mathcal{D}_\Omega^{\mathbb{K}} = \operatorname{mco} \partial^{\mathrm{abs}} \mathcal{D}_\Omega^{\mathbb{K}}.$$

Moreover, for any matrix convex set $K \subseteq (\mathbb{S}^{\mathbb{K}})^g$, and subset $E \subseteq K$ of irreducible matrix tuples which satisfies $K = \operatorname{mco} E$, we have $\partial^{\mathrm{abs}} K \subseteq E$ up to unitary equivalence.

3.5 Examples of Absolute Extreme Points

The previous material was largely oriented towards presenting the proof of the main result, Theorem 3.4.5, and as such, we did not look at very many examples of absolute extreme points or Arveson boundary points. We would like to do so now, by mentioning some examples that have appeared in the recent literature. They have the additional benefit of being relevant to the discussion of strict matrix convexity in the next chapter.

3.5.1 The Matrix Interval

For real numbers $a < 0 < b$, consider the matrix interval

$$[aI, bI] = \left\{ X \in \mathbb{S}^1 : aI \preccurlyeq X \preccurlyeq bI \right\},$$

which we saw first in Example 2.3.8 and then again in Example 3.1.7. We saw that $[aI, bI]$ is a free spectrahedron with defining matrix $\begin{bmatrix} b^{-1} & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathbb{S}_2^{\mathbb{R}}$. Evidently $[aI, bI]$ is bounded, and is therefore a compact free spectrahedron with real defining tuple. By Theorem 3.4.5, $[aI, bI] = \operatorname{mco} \partial^{\mathrm{abs}}[aI, bI]$.

We will show now that

$$\partial^{\text{abs}}[aI, bI] = \{a, b\}\,.$$

Firstly, any absolute extreme point must be irreducible. Since $g = 1$ in this example, our "tuples" of self-adjoint matrices are just single self-adjoint matrices. As every self-adjoint matrix is diagonalizable, it follows that the irreducible elements of $[aI, bI]$ are precisely the elements of the first matrix level $[aI, bI](1)$. Note that $[aI, bI](1)$ is exactly the closed interval $[a, b] \subseteq \mathbb{R}$. Since absolute extreme points also need to be Euclidean extreme points of their respective matrix level (Proposition 2.5.5), the endpoints a and b are now the only candidates for absolute extreme points of $[aI, bI]$.

To see that a is, in fact, absolute extreme, suppose $\begin{bmatrix} a & \beta \\ \beta^* & \gamma \end{bmatrix} \in [aI, bI]$ for some $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. Then

$$\begin{bmatrix} 0 & -\beta \\ -\beta^* & a - \gamma \end{bmatrix} = aI_2 - \begin{bmatrix} a & \beta \\ \beta^* & \gamma \end{bmatrix} \geqslant 0.$$

It follows that $\beta = 0$, and so $a \in \partial^{\text{abs}}[aI, bI]$. Similarly $b \in \partial^{\text{abs}}[aI, bI]$.

3.5.2 The Free Cube

The *free cube* is the matrix convex set

$$\mathcal{C} := \{X \in \mathbb{S}^g : -I \leqslant X_i \leqslant I \text{ for all } 1 \leqslant i \leqslant g\}\,.$$

Evidently $\mathcal{C}(1)$ is the g-dimensional cube of width 2 centered at the origin. As can be found in [5, Section 7.1], the free cube is a free spectrahedron having a real defining tuple. Boundedness is an immediate consequence of its definition, and so Theorem 3.4.5 says that $\mathcal{C} = \operatorname{mco} \partial^{\text{abs}} \mathcal{C}$.

In [5, Section 7.1] it was shown that the Arveson boundary points of $\mathcal{C}$ are the g-tuples of *symmetries*[1], which, since our tuples are self-adjoint, means that

$$\partial^{\mathrm{Arv}} \mathcal{C} = \left\{ X \in \mathbb{S}^g : X_i^2 = I \text{ for } 1 \leqslant i \leqslant g \right\}.$$

In particular, the absolute extreme points of the first level are g-tuples of 1's and -1's, which are precisely the Euclidean extreme points (vertices) of the g-dimensional cube. The same paper in fact proves that the Arveson boundary of any matrix level of the free cube coincides with the Euclidean extreme points.

In case $g = 1$, the free cube is the matrix interval $[-I, I]$, a special case of Example 3.5.1.

3.5.3 The Spin Disk

The *spin disk* is defined as

$$B_2^{\mathrm{spin}} := \left\{ X = (X_1, X_2) \in \mathbb{S}^2 : \begin{bmatrix} I - X_1 & -X_2 \\ -X_2 & I + X_1 \end{bmatrix} \succcurlyeq 0 \right\}.$$

This is a free spectrahedron with defining tuple $\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \in (\mathbb{S}_2^{\mathbb{R}})^2$.

Proposition 3.5.1. *B_2^{spin} is bounded.*

Proof. Let $X \in B_2^{\mathrm{spin}}(n)$. It is a routine computation to show that $-\Lambda_\Omega(X) \overset{\mathrm{u}}{\equiv} \Lambda_\Omega(X)$ via the unitary $U = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. Hence $\pm\Lambda_\Omega(X) \preccurlyeq I$. This implies that $\Lambda_\Omega(X)^2 \preccurlyeq I$,

[1] A square matrix P is a symmetry if and only if $P^* = P$ and $P^2 = I$.

and we compute

$$\Lambda_\Omega(X)^2 = \begin{bmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{bmatrix}^2 = \begin{bmatrix} X_1^2 + X_2^2 & X_1X_2 - X_2X_1 \\ X_2X_1 - X_1X_2 & X_1^2 + X_2^2 \end{bmatrix} \preccurlyeq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Hence $X_1^2 + X_2^2 \preccurlyeq I$. This holds for all $X \in B_2^{\mathrm{spin}}$, and so B_2^{spin} is bounded. □

We now know that B_2^{spin} is a compact free spectrahedron with a real defining tuple, and so Theorem 3.4.5 gives $B_2^{\mathrm{spin}} = \operatorname{mco} \partial^{\mathrm{abs}} B_2^{\mathrm{spin}}$. This also follows in a more ad-hoc manner from the following discussion.

We say that $X \in \mathbb{S}^g$ *commutes* if $X_iX_j = X_jX_i$ for all $i, j = 1, \ldots, g$. Equivalently [9, Theorem 4.5.15], X commutes if and only if the component matrices can be diagonalized simultaneously, which is to say that $X \overset{\mathrm{u}}{\equiv} x_1 \oplus \cdots \oplus x_t$ for some $x_1, \ldots, x_t \in \mathbb{R}^g$.

In ([7, Proposition 14.14]) it was shown that the matrix pair (X_1, X_2) belongs to the spin disk if and only if it dilates to a commuting pair of self-adjoint matrices having joint spectrum in the unit disk. This means that $X \in B_2^{\mathrm{spin}}(n)$ if and only if there are $x_1, \ldots, x_t \in \mathbb{B}_2$ and an isometry $V : \mathbb{C}^n \to \mathbb{C}^t$ such that $X = V^*(x_1 \oplus \cdots \oplus x_t)V$. In light of Proposition 2.4.1, we have $B_2^{\mathrm{spin}} = \operatorname{mco} \mathbb{B}_2$. We can refine this to $B_2^{\mathrm{spin}} = \operatorname{mco} \partial^{\mathrm{top}} \mathbb{B}_2$, since the interior points of $\mathbb{B}_2$ are Euclidean convex combinations of the topological boundary points. This lets us characterize the absolute extreme points of the spin disk, and our proof, as well as this paragraph leading up to it, has essentially the same content as the proof of Proposition 7.5 from [5].

Proposition 3.5.2. $\partial^{\mathrm{abs}} B_2^{\mathrm{spin}} = \partial^{\mathrm{top}} \mathbb{B}_2$.

Proof. Since $\partial^{\mathrm{top}} \mathbb{B}_2$ is a set of irreducible matrix tuples that is closed under unitary equivalence and satisfies $B_2^{\mathrm{spin}} = \operatorname{mco} \partial^{\mathrm{top}} \mathbb{B}_2$, it follows from Theorem 3.0.2 that $\partial^{\mathrm{abs}} B_2^{\mathrm{spin}} \subseteq \partial^{\mathrm{top}} \mathbb{B}_2$.

Conversely, suppose that $x \in \partial^{\text{top}} \mathbb{B}_2$, that is, $x_1^2 + x_2^2 = 1$. Consider a 1-dilation

$$Y = \begin{bmatrix} x & \beta \\ \beta^* & \gamma \end{bmatrix} \in B_2^{\text{spin}}.$$

From our proof of the boundedness of the spin disk (Proposition 3.5.1), we saw that $Y \in B_2^{\text{spin}}$ implies $Y_1^2 + Y_2^2 \preceq I$. Expanding this out, we get

$$Y_1^2 + Y_2^2 = \begin{bmatrix} x_1^2 + x_2^2 + \beta_1^*\beta_1 + \beta_2^*\beta_2 & A \\ A^* & B \end{bmatrix} = \begin{bmatrix} 1 + \beta_1^*\beta_1 + \beta_2^*\beta_2 & A \\ A^* & B \end{bmatrix},$$

where the exact form of A and B does not matter here. Since $Y_1^2 + Y_2^2 \preceq I$, it follows that $1 + \beta_1^*\beta_1 + \beta_2^*\beta_2 \preceq 1$. It can only be that $\beta_1 = \beta_2 = 0$, which proves that $x \in \partial^{\text{abs}} B_2^{\text{spin}}$. □

It needs to be mentioned that Proposition 7.5 of [5] is the statement that the Arveson boundary points of the spin disk are those matrix pairs (X_1, X_2) for which X_1 and X_2 commute and $X_1^2 + X_2^2 = I$, with the corollary that $\partial^{\text{Euc}} B_2^{\text{spin}} = \partial^{\text{Arv}} B_2^{\text{spin}}$. Let us summarize these findings.

Proposition 3.5.3.

1. *$B_2^{\text{spin}} = \operatorname{mco} \mathbb{B}_2 = \operatorname{mco} \partial^{\text{top}} \mathbb{B}_2$;*

2. *$\partial^{\text{Euc}} B_2^{\text{spin}} = \partial^{\text{Arv}} B_2^{\text{spin}} = \{(X_1, X_2) \in \mathbb{S}^g : X_1X_2 = X_2X_1 \text{ and } X_1^2 + X_2^2 = I\}$;*

3. *$\partial^{\text{abs}} B_2^{\text{spin}} = \partial^{\text{top}} \mathbb{B}_2$.*

3.5.4 The OH-ball

The *OH-ball*, seen before in Example 2.3.9 and again in Example 3.1.8, is given as

$$B_g^{\text{OH}} := \left\{ X \in \mathbb{S}^g : \sum_{i=1}^{g} X_i^2 \preceq I \right\}.$$

We saw that B_g^{OH} is a free spectrahedron with a real defining tuple, and it is bounded by definition. Hence Theorem 3.4.5 implies that $B_g^{\mathrm{OH}} = \operatorname{mco} \partial^{\mathrm{abs}} B_g^{\mathrm{OH}}$. The Arveson boundary of B_g^{OH} at least includes the set $\partial^{\mathrm{van}} B_g^{\mathrm{OH}} = \{X \in \mathbb{S}^g : \sum_{i=1}^g X_i^2 = I\}$, which the paper [5] calls the *vanishing boundary*. Compare the following proof to the proof of Proposition 3.5.2.

Proposition 3.5.4. *Let $X \in \mathbb{S}_n^g$. If $\sum_{i=1}^g X_i^2 = I_n$, then $X \in \partial^{\mathrm{Arv}} B_g^{\mathrm{OH}}$. That is, $\partial^{\mathrm{van}} B_g^{\mathrm{OH}} \subseteq \partial^{\mathrm{Arv}} B_g^{\mathrm{OH}}$.*

Proof. Suppose $\sum_{i=1}^g X_i^2 = I_n$, and suppose we have a 1-dilation

$$Y := \begin{bmatrix} X & \beta \\ \beta^* & \gamma \end{bmatrix} \in B_g^{\mathrm{OH}}.$$

Then $\sum_{i=1}^g Y_i^2 \preceq I_{n+1}$, and we compute

$$\sum_{i=1}^g Y_i^2 = \begin{bmatrix} \sum_{i=1}^g X_i^2 + \sum_{i=1}^g \beta_i\beta_i^* & A \\ A^* & B \end{bmatrix} = \begin{bmatrix} I_n + \sum_{i=1}^g \beta_i\beta_i^* & A \\ A^* & B \end{bmatrix},$$

where the precise form of A and B is of no consequence to this proof. It follows that

$$I_n = \begin{bmatrix} I_n & 0 \end{bmatrix} I_{n+1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \succeq \begin{bmatrix} I_n & 0 \end{bmatrix} (\sum_{i=1}^g Y_i^2) \begin{bmatrix} I_n \\ 0 \end{bmatrix} = I_n + \sum_{i=1}^g \beta_i\beta_i^*.$$

Hence $\sum_{i=1}^g \beta_i\beta_i^* \preceq 0$, but since $\sum_{i=1}^g \beta_i\beta_i^* \succeq 0$, it follows that $\sum_{i=1}^g \beta_i\beta_i^* = 0$. Hence $\beta_1 = \cdots = \beta_g = 0$. This proves that X is an Arveson boundary point of B_g^{OH}. □

Whereas Theorem 3.4.5 implies that $B_g^{\mathrm{OH}} = \operatorname{mco} \partial^{\mathrm{abs}} B_g^{\mathrm{OH}}$, it is *not* true in general that $B_g^{\mathrm{OH}} = \operatorname{mco} \partial^{\mathrm{van}} B_g^{\mathrm{OH}}$. As described in Subsection 7.2.1 of [5], at least in the case of $g = 2$, it is known that there exist Arveson boundary points of B_2^{OH} that are *not* in the vanishing boundary. Any such point fails to be in the matrix convex hull of the vanishing boundary. It is unknown whether B_2^{OH} is the *closed* matrix convex hull of its vanishing boundary.

It is easy to construct matrix g-tuples X of all sizes for which $\sum_{i=1}^{g} X_i^2 = I$: simply take X to be the direct sum of any finite number of points from $\mathbb{B}_g$. Since each direct summand will satisfy the relation, so will their direct sum. While this gives a plethora of Arveson boundary points, only the irreducible ones are absolute extreme points, so this procedure doesn't yield any absolute extreme points of matrix level 2 or higher.

Another procedure is to pick any tuple $X \in \mathbb{S}^g$ such that $X_i^2 = I$ for each $i = 1, \ldots, g$. Then evidently $Y := \frac{1}{\sqrt{g}} X$ satisfies $\sum_{i=1}^{g} Y_i^2 = I$, and so Y is an Arveson boundary point. If we ensure that at least one Y_i is a diagonal matrix and another is not, then Y is irreducible, whereby $Y \in \partial^{\mathrm{abs}} B_g^{\mathrm{OH}}$.

As a specific example in the case of $g = 2$, consider the pair

$$\Omega = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

which, incidentally, is the defining tuple for the spin disk. By the argument just given, $\frac{1}{\sqrt{2}} \Omega \in \partial^{\mathrm{abs}} B_2^{\mathrm{OH}}$.

The following result, Proposition 7.4 of [5], is noteworthy. It implies that for any $X \in B_2^{\mathrm{OH}}(n)$ for which $I - X_1^2 - X_2^2$ has $(n-1)$-dimensional kernel, either X or some 1-dilation of X is an Arveson boundary point of B_2^{OH}.

Proposition 3.5.5 (See [5], Proposition 7.4)**.** *Let $X \in B_2^{\mathrm{OH}}(n)$. If* $\mathrm{Ker}(I - X_1^2 - X_2^2)$ *has dimension $n-1$, then either $X \in \partial^{\mathrm{Arv}} B_2^{\mathrm{OH}}$ or X dilates to a pair $Y \in \mathbb{S}^2(n+1)$ such that $Y_1^2 + Y_2^2 = I$.*

Another noteworthy fact is that $\partial^{\mathrm{top}} B_g^{\mathrm{OH}}(1) \subseteq \partial^{\mathrm{abs}} B_g^{\mathrm{OH}}$. This follows both from [5, Corollary 6.1] and, at least in the case $g = 2$, from our forthcoming Theorem 4.3.3.

4

Strict Matrix Convexity

A matrix convex set has several different notions of extremal subset to consider. We have seen the topological boundary, the Euclidean/matrix/absolute extreme points, and the Arveson boundary. Proposition 2.5.5 gave a relationship between these sets for any matrix convex set $K \subseteq \mathbb{S}^g$, namely

$$\partial^{\text{abs}} K \subseteq \partial^{\text{mat}} K \subseteq \partial^{\text{Euc}} K \subseteq \partial^{\text{top}} K.$$

It is natural to ask if any of these containments can be equalities for some K. More generally, what are the exact relationships between these different kinds of extremal sets, beyond the aforementioned sequence of set inclusions? In classical convexity, a closed convex set K for which $\partial^{\text{top}} K \subseteq \partial^{\text{Euc}} K$ is called *strictly convex*. In this chapter we will study the relationship between $\partial^{\text{top}} K$ and $\partial^{\text{abs}} K$ for a closed matrix convex set K, by posing several tentative definitions of strict matrix convexity, each of which extend the classical notion of strictly convex sets. Our own Theorem 4.3.3, currently only proven in the $g = 2$ case, gives a sufficient condition for the first matrix level of a free spectrahedron to be strictly matrix convex by one of these definitions.

We will find that each of our proposed definitions lack some nice properties that we would naively expect strictly matrix convex sets to enjoy. This suggests that we may not yet have identified a property worthy of the name "strict matrix convexity".

4.1 Seeking a Definition of Strict Matrix Convexity

A closed convex subset $K \subseteq V$, where V is a topological vector space, is called *strictly convex* if it satisfies the equivalent properties in the following proposition. Recall that $\operatorname{Int} K$ denotes the topological interior of K.

Proposition 4.1.1. *For a topological vector space V, and a closed convex subset $K \subseteq V$, the following two properties are equivalent.*

1. *For any pair of distinct points $x, y \in K$ and any scalar $0 < t < 1$, we have $tx + (1-t)y \in \operatorname{Int} E$.*

2. *$\partial^{\text{top}} K \subseteq \partial^{\text{Euc}} K$.*

Proof. Since K is a closed subset of a topological space, we have $K = \partial^{\text{top}} K \cup \operatorname{Int} K$ and $\varnothing = \partial^{\text{top}} K \cap \operatorname{Int} K$. Hence $\partial^{\text{top}} K \subseteq \partial^{\text{Euc}} K$ if and only if $K \backslash \partial^{\text{Euc}} K \subseteq \operatorname{Int} K$. But $K \backslash \partial^{\text{Euc}} K$ is precisely the set of points $z \in K$ which can be represented as a convex combination $z = tx + (1-t)y$ for some distinct points $x, y \in K$ and $0 < t < 1$. $\square$

Geometrically, there are no non-degenerate line segments to be found on the topological boundary of a strictly convex set.

It is a well-known fact that an extreme point is necessarily a topological boundary point (for, any non-degenerate line segment containing an extreme point of a set must have one endpoint outside the set), and so strict convexity is really the property that the extreme points and topological boundary coincide.

A simple example is the unit ball $\mathbb{B}_g$ in $\mathbb{R}^g$, which has the unit sphere as its set of topological boundary points and its set of Euclidean extreme points. An example of a convex set that is **not** strictly convex is a (filled) triangle in the plane, whose extreme points are the three vertices whereas the boundary points consist of the three vertices as well as all other points on the three edges.

How may a definition of strict matrix convexity be chosen which suitably extends the classical notion? Although this thesis does not claim to provide a definitive answer to this question, we do collect here some facts suggestive of what strict matrix convexity might and might not be.

Is one of the two equivalent conditions of Proposition 4.1.1 more amenable to our task? Recall that the definition of absolute extreme point (Definition 2.5.3) is stated in a way that uses matrix convex combinations of any number of points. The classical definition of extreme point is commonly stated using convex combinations with just two points. We saw in Proposition 2.5.2 that, in fact, the classical two-point criterion is equivalent to one involving convex combinations of any number of points. Thus it would seem that a definition of strict matrix convexity in the style of condition (1.) of Proposition 4.1.1 would be quite messy. Due to the relative simplicity of condition (2.), we choose it as our prototype, and attempt to adapt it into a definition for strict matrix convexity. Afterwards, we will try to extract an equivalent condition in the style of (1.).

The naive attempt at a definition is to say that strict matrix convexity of a closed matrix convex set $K \subseteq \mathbb{S}^g$ is the property that $\partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{abs}} K$. We will see, however, that this definition is never satisfied (Proposition 4.1.3).

What characteristics should we *expect* the definition to have? We feel confident in stating that the definition should involve topological boundary points and absolute extreme points in some central way. What follows are a list of properties that seem consistent with the idea that strict matrix convexity should extend strict Euclidean convexity, though none of them should be thought of as truly necessary.

1. If $K \subseteq \mathbb{S}^g$ is strictly matrix convex, then $K(1)$ is strictly Euclidean convex.

2. If $K \subseteq \mathbb{S}^g$ is strictly matrix convex, then $K(n)$ is strictly Euclidean convex for all $n \geqslant 1$.

3. If $E \subseteq \mathbb{R}^g$ is strictly Euclidean convex, then $\operatorname{mco} E$ is strictly matrix convex.

4. If $K \subseteq \mathbb{S}^g$ is strictly matrix convex, then $\partial^{\mathrm{top}} K(1) \subseteq \partial^{\mathrm{abs}} K$.

5. There exist strictly matrix convex sets with non-empty interior.

Fundamentally, strict convexity is a property that combines the topological structure and the convex structure of a set. As such, it seems prudent to have more topological knowledge of matrix convex sets than we have hitherto established. First, we prove that dilations of topological boundary points are again topological boundary points.

Proposition 4.1.2. *Let $K \subseteq \mathbb{S}^g$ be a closed matrix convex set. If $X \in \partial^{\mathrm{top}} K$ and $Y \in K$ is a dilation of X, then $Y \in \partial^{\mathrm{top}} K$.*

Proof. We write Y in matrix form as $Y = \begin{bmatrix} X & A \\ A^* & B \end{bmatrix}$. We want to show that $Y \in \partial^{\mathrm{top}} K$, which is the intersection of the closure of K and the closure of $\mathbb{S}^g \backslash K$. Since $Y \in K$, then already Y is in the closure of K. Now, we have that X is in the closure of $\mathbb{S}^g \backslash K$, so X is the norm-limit of a sequence outside of K:

$$X = \lim_{k \geqslant 1} X_k, \quad X_k \in \mathbb{S}^g \backslash K.$$

For each $k \geqslant 1$, we have $\begin{bmatrix} X_k & A \\ A^* & B \end{bmatrix} \in \mathbb{S}^g \backslash K$, since membership in K would imply that $X_k \in K$. By continuity, we have

$$Y = \begin{bmatrix} X & A \\ A^* & B \end{bmatrix} = \lim_{k \geqslant 1} \begin{bmatrix} X_k & A \\ A^* & B \end{bmatrix},$$

whence Y is in the closure of $\mathbb{S}^g \backslash K$. Hence $Y \in \partial^{\mathrm{top}} K$. □

As a consequence of this, we can easily see that there do not exist closed matrix convex sets K with the property that $\partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{abs}} K$. For, supposing K has this property and choosing $X \in \partial^{\mathrm{top}} K$, we also have $X \oplus X \in \partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{abs}} K$. This contradicts the irreducibility of absolute extreme points.

With more work, we can exclude $\partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{Arv}} K$ is a viable criterion for strict matrix convexity.

Proposition 4.1.3. *For a closed matrix convex set $K \subseteq \mathbb{S}^g$, we have $\partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{Arv}} K$ if and only if $K = \mathbb{S}^g$ or $K = \mathrm{mco}\,\{x\}$ for some $x \in \mathbb{R}^g$.*

Proof. If $K = \mathbb{S}^g$ then $\partial^{\mathrm{top}} K = \varnothing \subseteq \partial^{\mathrm{Arv}} K$. So we will proceed under the assumption that $K \neq \mathbb{S}^g$. This guarantees that $\partial^{\mathrm{top}} K \neq \varnothing$.

Suppose that $\partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{Arv}} K$, and fix $X \in \partial^{\mathrm{top}} K$. Then for all $Z \in K$, Proposition 4.1.2 guarantees $X \oplus Z \in \partial^{\mathrm{top}} K \subseteq \partial^{\mathrm{Arv}} K$. Since direct summands of Arveson boundary points are again on the Arveson boundary (Lemma 2.6.7), we have $Z \in \partial^{\mathrm{Arv}} K$. Hence $K = \partial^{\mathrm{Arv}} K$. This combined with Lemma 2.5.5 implies that $K = \partial^{\mathrm{Euc}} K$.

Suppose $X, Y \in K(n)$ for some $n \geqslant 1$. Then $(X+Y)/2 \in K(n) = \partial^{\mathrm{Euc}} K(n)$. Therefore $X = (X+Y)/2 = Y$, which shows that $K(n)$ consists of a single element. In particular, $K(1) = \{x\}$ for some $x \in \mathbb{R}^g$, and it follows that $K(n) = \{\bigoplus_{i=1}^n x\}$ for all $n \geqslant 1$. That is, $K = \mathrm{mco}\,\{x\}$.

Conversely, if $K = \mathrm{mco}\,\{x\}$ for some $x \in \mathbb{R}^g$, then $K(n) = \{\bigoplus_{i=1}^n x\}$ for all

$n \geqslant 1$. Hence $\partial^{\text{top}} K(n) = \{\bigoplus_{i=1}^{n} x\}$. If $\begin{bmatrix} \bigoplus_{i=1}^{n} x & \beta \\ \beta^* & \gamma \end{bmatrix} \in K(n+1)$, then, since $K(n+1) = \left\{\bigoplus_{i=1}^{n+1} x\right\}$, we have that $\beta = 0$. This shows that $\partial^{\text{top}} K(n) \subseteq \partial^{\text{Arv}} K(n)$ for all $n \geqslant 1$. □

Since strict Euclidean convexity is abundant with examples of strictly convex sets with more than one point, we expect a definition of strict matrix convexity to allow for this as well. The previous result therefore suggests that the criterion $\partial^{\text{top}} K \subseteq \partial^{\text{Arv}} K$ is not a good definition for strict matrix convexity.

To lead into another possible definition, consider that in the classical setting, the convex hull of a single point is the singleton set containing just that point. However, as Corollary 2.4.2 shows, the convex hull of a single matrix tuple X consists of all compressions of direct sums of X with itself. Thus every point comes equipped with a set of points which are very closely related to it in a manner that collapses to the singleton set in the classical case. Morally speaking, definitions which are intended to extend classical convexity notions to the matrix convexity setting may be justified in replacing single points with the matrix convex hulls of single points.

What we propose, then, is that instead of every topological boundary point being itself an absolute extreme point or Arveson boundary point (which we argued would not work), the definition of strict matrix convexity might say that boundary points have some absolute extreme point in their matrix convex hull. The following lemma demonstrates that we don't actually need the full strength of this.

Lemma 4.1.4. *Let $K \subseteq \mathbb{S}^g$ be a matrix convex set, $X \in K$ and $Y \in \partial^{\text{abs}} K$. Then the following are equivalent:*

1. *$Y \in \operatorname{mco} X$;*

2. *Y is a compression of X;*

3. *Y is a direct summand of X.*

Proof. Suppose $Y \in \operatorname{mco} X$. Then $Y = \sum_{i=1}^{k} C_i^* X C_i$, a matrix convex combination. Without loss of generality we can take this matrix convex combination to have non-zero coefficients. Applying the definition of an absolute extreme point to Y, we deduce that Y is a direct summand of X (including the possibility that $Y \overset{\mathrm{u}}{\equiv} X$). Conversely, if Y is a direct summand of X, then Y is a compression of X and therefore $Y \in \operatorname{mco} X$. This establishes that (1) and (3) are equivalent.

That (2) implies (3) follows from the dilation-theoretic characterization of absolute extreme points, Theorem 2.6.4. That (3) implies (2) follows from the fact that a direct summand is a compression. □

With this in mind, we now formally state our proposed definition of strict matrix convexity.

4.2 A Proposed Definition: Property A

Definition 4.2.1 (Property A)**.** We say that a closed matrix convex set $K \subseteq \mathbb{S}^g$ has *Property A* provided that every topological boundary point has a compression which is an absolute extreme point. Equivalently (Lemma 4.1.4), every topological boundary point has an absolute extreme point as a direct summand.

This definition is easily seen to have the property that if $K \in \mathbb{S}^g$ has Property A, then $\partial^{\mathrm{top}} K(1) \subseteq \partial^{\mathrm{abs}} K$. In particular, $K(1)$ is strictly Euclidean convex. For, if $x \in \partial^{\mathrm{top}} K(1)$, then the only direct summand of x is x itself. Hence x is an absolute extreme point, which is itself a type of Euclidean extreme point by Proposition 2.5.5. This fact can be leveraged to quickly exclude many sets from consideration in the search for examples satisfying Property A, by simply observing that their first level is not strictly convex in the classical sense. For instance, the free cube of Section 3.5.2, $\{X \in \mathbb{S}^g : I \leqslant X_i \leqslant I \text{ for all } i = 1, \ldots, g\}$ has a g-dimensional cube as its first level, so cannot satisfy Property A when $g \geqslant 2$. Likewise, the matrix convex hull of any non-strictly Euclidean convex subset of $\mathbb{R}^g$ does not have Property A.

When seeking examples of matrix convex sets which **do** satisfy Property A, we are naturally drawn towards matrix convex sets for which the absolute extreme points and the topological boundary are known in detail. In Section 3.5 we provided some examples for which the absolute extreme points have already been studied. All of them are free spectrahedra.

4.3 Property A for Free Spectrahedra

When considering Property A for free spectrahedra, the following well-known characterization of the topological boundary (as stated, for example, in [8]) is available to us.

Proposition 4.3.1. *Let $\Omega \in \mathbb{S}^g$ and $X \in \mathcal{D}_\Omega(n)$. Then $X \in \partial^{\mathrm{top}} \mathcal{D}_\Omega(n)$ if and only if $L_\Omega(X)$ has non-trivial kernel.*

Proof. Suppose that $\operatorname{Ker} L_\Omega(X)$ is trivial. Since $L_\Omega(X) \geqslant 0$, there exists $\delta > 0$ such that $L_\Omega(X) \geqslant \delta I$. Consider any $Y \in \mathbb{S}_n^g$ such that $\|\Omega_i\| \, \|Y_i\| \leqslant \delta/2g$ for each $i = 1, \ldots, g$. Observe that

$$L_\Omega(X+Y) = L_\Omega(X) - \Lambda_\Omega(Y) \geqslant \delta I - \Lambda_\Omega(Y),$$

and that

$$\Lambda_\Omega(Y) = \sum_{i=1}^{g} \Omega_i \otimes Y_i \leqslant \sum_{i=1}^{g} \|\Omega_i\| \, \|Y_i\| \, I \leqslant (\delta/2)I.$$

Hence

$$L_\Omega(X+Y) \geqslant (\delta/2)I > 0.$$

Since this holds for all Y in an open neighbourhood around X, we see that X is not a topological boundary point of $\mathcal{D}_\Omega(n)$.

Note that $L_\Omega(0) = I$ has trivial kernel, so we now know that 0 is in the interior of $\mathcal{D}_\Omega(n)$ for all $n \geqslant 1$.

Conversely, suppose $\operatorname{Ker} L_\Omega(X)$ is non-trivial, and consider a unit-norm vector v therein. Then $\Lambda_\Omega(X)v = v$. For all $t > 0$ we have

$$v^* L_\Omega((1+t)X)v = v^* L_\Omega(X)v - tv^*\Lambda_\Omega(X)v = -t.$$

Hence $(1+t)X \notin \mathcal{D}_\Omega(n)$ for all $t > 0$. Since $\mathcal{D}_\Omega(n)$ is a convex set with 0 in the interior, this suffices to prove that $X \in \partial^{\mathrm{top}} \mathcal{D}_\Omega(n)$.

□

As we noted in the proof, 0 is an interior point of every matrix level of a free spectrahedron.

Corollary 4.3.2. *Let $\Omega \in \mathbb{S}^g$. Then $0 \in \operatorname{Int}(\mathcal{D}_\Omega(n))$ for all $n \geqslant 1$. In particular,* $\operatorname{Int} \mathcal{D}_\Omega \neq \varnothing$*.*

We saw that, as an immediate consequence of the definition, a necessary condition for K to have Property A is that $\partial^{\mathrm{top}} K(1) \subseteq \partial^{\mathrm{abs}} K$. The following result shows that this necessary condition holds for any two-variable free spectrahedron defined by a tuple Ω having no joint eigenvector, meaning that Ω is *not* unitarily equivalent to a tuple of the form $X \oplus Y$, where $X \in \mathcal{D}_\Omega(1)$. In particular, this result applies when a defining tuple is irreducible.

Theorem 4.3.3 (Clouâtre-Stephens)**.** *Let $\Omega \in \mathbb{S}^2$ be a pair of self-adjoint matrices admitting no joint eigenvector. Then $\partial^{\mathrm{top}} \mathcal{D}_\Omega(1) \subseteq \partial^{\mathrm{abs}} \mathcal{D}_\Omega$.*

Proof. Let $x = (x_1, x_2) \in \partial^{\mathrm{top}} \mathcal{D}_\Omega(1)$. By Proposition 4.3.1 there is $v \neq 0$ such that $L_\Omega(x)v = 0$. Hence $(I - x_1\Omega_1 - x_2\Omega_2)v = 0$, or equivalently

$$x_1\Omega_1 v + x_2\Omega_2 v = v. \tag{1}$$

To show that x is an absolute extreme point of $\mathcal{D}_\Omega$, suppose that we have a 1-dilation $\begin{bmatrix} x & \beta \\ \beta^* & \gamma \end{bmatrix} \in \mathcal{D}_\Omega$. By Proposition 3.2.7, this implies that $\operatorname{Ker} L_\Omega(x) \subseteq \operatorname{Ker} \Lambda_\Omega(\beta)^*$, whence $\Lambda_\Omega(\beta^*)v = \Lambda_\Omega(\beta)^* v = 0$. Expanding this out we obtain

$$\overline{\beta_1}\Omega_1 v + \overline{\beta_2}\Omega_2 v = 0. \tag{2}$$

Now our goal is to show that $\beta_1 = \beta_2 = 0$. Suppose that $\beta_1 \neq 0$. Then equation (2) can be rewritten as $\Omega_1 v = (-\overline{\beta_2}/\overline{\beta_1})\Omega_2 v$. Substituting this into equation (1), we obtain

$$v = x_1(-\overline{\beta_2}/\overline{\beta_1})\Omega_2 v + x_2\Omega_2 v = \left(x_1(-\overline{\beta_2}/\overline{\beta_1}) + x_2\right)\Omega_2 v.$$

From this we conclude that v is an eigenvector of Ω_2. By (2), v is also an eigenvector of Ω_1. This contradiction means that in fact $\beta_1 = 0$. Similar reasoning shows that $\beta_2 = 0$, from which it follows that x is an absolute extreme point of $\mathcal{D}_\Omega$. $\square$

It is not known by us at this time whether the result can be extended to $\mathbb{S}^g$ for $g \geqslant 3$, or, even in the case of $g = 2$, whether some similar property holds for the topological boundary points of $\mathcal{D}_\Omega(n)$, $n \geqslant 2$.

4.4 Seeking Examples with Property A

We already know the free cube does not have Property A, by virtue of its first matrix level failing to be strictly Euclidean convex. Other examples, namely the matrix interval, spin disk, and OH-ball, are strictly Euclidean convex on their first matrix level, and so they require a closer look.

4.4.1 The Unit Matrix Interval

The matrix interval $[-I, I] = \{X \in \mathbb{S}^1 : -I \leqslant X \leqslant I\}$ has Property A. We know from Section 3.5.1 that $\partial^{\text{abs}}[-I, I] = \{-1, 1\}$, and we will now proceed to show that

$$\partial^{\text{top}}[-I, I] = \{X \in [-I, I] : -1 \in \sigma(X) \text{ or } 1 \in \sigma(X)\},$$

where $\sigma(X)$ denotes the spectrum of X (since X is a matrix, this is the set of eigenvalues of X).

This argument makes use of the boundary characterization of Proposition 4.3.1, by noticing that the matrix interval is the free spectrahedron $\mathcal{D}_\Omega$ with

$$\Omega = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_\Omega(X) = \begin{bmatrix} I + X & 0 \\ 0 & I - X \end{bmatrix}.$$

A self-adjoint matrix X is a topological boundary point of $[-I, I]$ if and only if $L_\Omega(X)$ has non-trivial kernel, and this is equivalent to either $I + X$ or $I - X$ having non-trivial kernel. This is then equivalent to the existence of some vector $v \neq 0$ such that either $Xv = -v$ or $Xv = v$, which, in turn, is equivalent to either 1 or -1 being an eigenvalue of X.

We can now proceed to prove that the matrix interval has Property A, and in fact we have done most of the work already. If $X \in \partial^{\text{top}}[-I, I]$, then either 1 or -1 is an eigenvalue of X. Let λ be that eigenvalue, so that $\lambda \in \partial^{\text{abs}}[-I, I]$. Then, $X \overset{\text{u}}{\equiv} \begin{bmatrix} \lambda & 0 \\ 0 & Y \end{bmatrix}$ for some $Y \in [-I, I]$, and so λ, an absolute extreme point, is a compression of X. Therefore $[-I, I]$ has Property A.

4.4.2 The Spin Disk

We begin to consider the question of Property A for a selection of *matrix balls*, meaning matrix convex sets having the unit ball $\mathbb{B}_g$ as matrix level 1.

To begin, consider the spin disk B_2^{spin} of Section 3.5.3. Recall that $B_2^{\text{spin}} = \mathcal{D}_\Omega$, for the defining tuple

$$\Omega = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

We will show that the spin disk does *not* have Property A.

We saw in Section 3.5.3 that $\partial^{\text{abs}} B_2^{\text{spin}} = \partial^{\text{top}} \mathbb{B}_2$. Therefore, the statement that the spin disk fails to satisfy Property A is the statement that there exists some topological boundary point of B_2^{spin} that does not have a compression on the unit circle.

Proposition 4.4.1. $\frac{1}{2}\Omega \in \partial^{\text{top}} B_2^{\text{spin}}$.

Proof. First we need to prove that $\frac{1}{2}\Omega \in B_2^{\text{spin}}$, and to this end, note that

$$\Lambda_\Omega(\frac{1}{2}\Omega) = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

and so

$$L_\Omega(\frac{1}{2}\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Now $L_\Omega(\frac{1}{2}\Omega) \succcurlyeq 0$ if and only if $2L_\Omega(\frac{1}{2}\Omega) \succcurlyeq 0$. Let us conjugate by a vector

$\begin{bmatrix} x & y & z & w \end{bmatrix}^T \in \mathbb{C}^4$:

$$\begin{bmatrix} \overline{x} & \overline{y} & \overline{z} & \overline{w} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \overline{x} & \overline{y} & \overline{z} & \overline{w} \end{bmatrix} \begin{bmatrix} x - w \\ 3y - z \\ -y + 3z \\ -x + w \end{bmatrix}$$

$$= |x|^2 - \overline{x}w + 3|y|^2 - \overline{y}z - \overline{z}y + 3|z|^2 - \overline{w}x + |w|^2$$

$$= |x - w|^2 + 2|y|^2 + 2|z|^2 + |y - z|^2 \geqslant 0.$$

Therefore $\frac{1}{2}\Omega \in B_2^{\text{spin}}$. To see that it is a topological boundary point, note that

$$L_\Omega(\frac{1}{2}\Omega) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore $\operatorname{Ker} L_\Omega(\frac{1}{2}\Omega) \neq \{0\}$, and by Proposition 4.3.1 we have $\frac{1}{2}\Omega \in \partial^{\text{top}} B_2^{\text{spin}}$. $\square$

Proposition 4.4.2. *The spin disk B_2^{spin} does not have Property A.*

Proof. We will show that $\frac{1}{2}\Omega$, which is a topological boundary point, does not compress to the unit circle. Suppose we have an isometry $w \in \mathbb{M}_{2,1}$. Then w has the form $w = \begin{bmatrix} u \\ v \end{bmatrix}$ for some $u, v \in \mathbb{C}$ satisfying $|u|^2 + |v|^2 = 1$. We compute

$$w^* \frac{1}{2}\Omega w = \frac{1}{2} \left(\begin{bmatrix} \overline{u} & \overline{v} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad \begin{bmatrix} \overline{u} & \overline{v} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right)$$

$$= \frac{1}{2} \left(|u|^2 - |v|^2, \ \overline{u}v + \overline{v}u \right).$$

Now observe that

$$
\begin{aligned}
\left\|(|u|^2 - |v|^2, \overline{u}v + \overline{v}u)\right\|^2 &= (|u|^2 - |v|^2)^2 + (\overline{u}v + \overline{v}u)^2 \\
&= (|u|^2)^2 - 2|u|^2|v|^2 + (|v|^2)^2 + (\overline{u}v)^2 + 2|u|^2|v|^2 + (\overline{v}u)^2 \\
&= \left|u^2\right|^2 + \overline{u^2}v^2 + \overline{v^2}u^2 + \left|v^2\right|^2 \\
&= \left|u^2 + v^2\right|^2 \\
&\leqslant (|u|^2 + |v|^2)^2 \\
&= 1.
\end{aligned}
$$

Hence $\left\|w^* \frac{1}{2}\Omega w\right\| \leqslant \frac{1}{2}$. This means that no compression of $\frac{1}{2}\Omega$ has norm 1, and therefore we have exhibited a topological boundary point of B_2^{spin} that does not compress to an absolute extreme point. □

As we mentioned in Section 3.5.3, the spin disk B_2^{spin} is the matrix convex hull of unit disk $\mathbb{B}_2$. So we have found an example of a strictly Euclidean convex subset of $\mathbb{R}^2$ whose matrix convex hull does not satisfy Property A.

Moreover, since $\Omega_1\Omega_2 = -\Omega_2\Omega_1 \neq \Omega_2\Omega_1$, it follows that $\frac{1}{2}\Omega$ is not a Euclidean extreme point of the spin disk: we saw in Section 3.5.3 that Euclidean extreme points of the spin disk must commute. Therefore $B_2^{\mathrm{spin}}(2)$ is not strictly Euclidean convex, which also may be taken as further evidence that Property A is not a good definition of strict matrix convexity.

4.4.3 The OH-ball

We do not know if the OH-ball $B_g^{\mathrm{OH}} = \{X \in \mathbb{S}^g : \sum_{i=1}^g X_i^2 \leqslant I\}$ has Property A. Some properties of its absolute extreme points were seen in Section 3.5.4. As for the topological boundary points, we have the following.

Proposition 4.4.3. $\partial^{\mathrm{top}} B_g^{\mathrm{OH}} = \{X \in \mathbb{S}^g : \operatorname{Ker}(I - \sum_{i=1}^g X_i^2) \neq \{0\}\}$.

Proof. Suppose $X \in \partial^{\text{top}} B_g^{\text{OH}}(n)$. Then there is a non-zero $u \in \operatorname{Ker} L_\Omega(X)$. Recalling from Section 3.5.4 that

$$L_\Omega(X) = \begin{bmatrix} I & -X_1 & -X_2 & \cdots & -X_g \\ -X_1 & I & 0 & \cdots & 0 \\ -X_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_g & 0 & 0 & \cdots & I \end{bmatrix},$$

we can write u in the form $\begin{bmatrix} v_0 & v_1 & v_2 & \cdots & v_g \end{bmatrix}^T$ where each $v_i \in \mathbb{C}^n$ and some v_i is non-zero. Using the fact that $L_\Omega(X)u = 0$, we obtain the relations

$$v_0 = X_1 v_1 + X_2 v_2 + \cdots X_g v_g, \tag{1}$$

and

$$v_i = X_i v_0, \quad \forall i = 1, \ldots, g. \tag{2}$$

Plugging the equalities from (2) into equation (1) gives

$$v_0 = (X_1^2 + X_2^2 + \cdots + X_g^2)v_0,$$

so that $v_0 \in \operatorname{Ker}(I - \sum_{i=1}^g X_i^2)$. Note that if $v_0 = 0$, then the equations in (2) imply that each $v_i = 0$. But this implies $u = 0$, which we know to be false. Hence $v_0 \neq 0$, and thus $\operatorname{Ker}(I - \sum_{i=1}^g X_i^2) \neq \{0\}$.

Conversely, suppose there is $v_0 \in \operatorname{Ker}(I - \sum_{i=1}^g X_i^2)$ with $v_0 \neq 0$. Define $v_i := X_i v_0$

for all $i = 1, \ldots, g$, and $u := \begin{bmatrix} v_0 & v_1 & v_2 & \cdots & v_g \end{bmatrix}^T$. Then $u \neq 0$, and

$$L_\Omega(X)u = \begin{bmatrix} I & -X_1 & -X_2 & \cdots & -X_g \\ -X_1 & I & 0 & \cdots & 0 \\ -X_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_g & 0 & 0 & \cdots & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_g \end{bmatrix}$$

$$= \begin{bmatrix} v_0 - \sum_{i=1}^g X_i v_i \\ v_1 - X_1 v_0 \\ v_2 - X_2 v_0 \\ \vdots \\ v_g - X_g v_0 \end{bmatrix} = \begin{bmatrix} v_0 - \sum_{i=1}^g X_i^2 v_0 \\ v_1 - X_1 v_0 \\ v_2 - X_2 v_0 \\ \vdots \\ v_g - X_g v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $\operatorname{Ker} L_\Omega(X) \neq \{0\}$, whereby $X \in \partial^{\text{top}} B_g^{\text{OH}}$. □

In terms of the dimension of $\operatorname{Ker}(I - \sum_{i=1}^g X_i^2)$, we have now seen the following set of results.

Proposition 4.4.4. *Let $X \in B_g^{\text{OH}}(n)$, and $P(X) := \operatorname{Ker}(I_n - \sum_{i=1}^g X_i^2)$.*

1. $\dim \operatorname{Ker} P(X) = 0$ *if and only if* $X \in \operatorname{Int} B_g^{\text{OH}}$.
2. $\dim \operatorname{Ker} P(X) \geqslant 1$ *if and only if* $X \in \partial^{\text{top}} B_g^{\text{OH}}$.
3. *If* $\dim \operatorname{Ker} P(X) = n - 1$, *then either* $X \in \partial^{\text{Arv}} B_g^{\text{OH}}$ *or* $Y \in \partial^{\text{Arv}} B_g^{\text{OH}}$ *for some* 1*-dilation Y of X. [5, Proposition 7.4]*
4. *If* $\dim \operatorname{Ker} P(X) = n$, *then* $X \in \partial^{\text{Arv}} B_g^{\text{OH}}$. *(Proposition 3.5.4)*

As for whether the OH-ball satisfies Proposition A, that remains an open question.

4.5 Interior Point Characterization of Property A

For our final observation, recall that in the classical setting, there are two common characterizations of strict convexity for closed convex sets: one saying that all boundary points are extreme, and the other saying that proper convex combinations of distinct points lie in the interior. Our definition of Property A was meant to generalize the former, so what becomes of the latter? We prove here an equivalence of this kind. One direction always works, but the other only applies when the matrix convex set in question is generated by its absolute extreme points.

Theorem 4.5.1. *Let $K \subseteq \mathbb{S}^g$ be a closed matrix convex set. If K is has Property A, then following property holds:*

- *(*) For all $X \in K$, if X can be expressed as a matrix convex combination $X = \sum_{i=1}^k C_i^* Y_i C_i$ such that each $C_i \neq 0$, each $Y_i \in \partial^{\text{abs}} K$, and such that $Y_i \notin \operatorname{mco} X$ for all i, then $X \in \operatorname{Int} K$.*

If $K = \operatorname{mco} \partial^{\text{abs}} K$, then (*) implies Property A.

Proof. Suppose that K has Property A. Let $X \in \partial^{\text{top}} K = K \setminus \operatorname{Int} K$, and consider any matrix convex combination $X = \sum_{i=1}^k C_i^* Y_i C_i$ with each $C_i \neq 0$ and each $Y_i \in \partial^{\text{abs}} K$. Since X is a topological boundary point, some compression of X, say $V^* X V$, is an absolute extreme point of K.

We can write

$$V^* X V = \sum_{i=1}^k (C_i V)^* Y_i (C_i V).$$

This is a matrix convex combination since $\sum_{i=1}^k (C_i V)^* (C_i V) = \sum_{i=1}^k V^* C_i^* C_i V = V^* V = I$. Let $Z = \{i : C_i V \neq 0\}$. Since $V^* X V \neq 0$, Z is non-empty. We can now express $V^* X V$ as a convex combination with non-zero coefficients, namely

$$V^* X V = \sum_{i \in Z} (C_i V)^* Y_i (C_i V).$$

Since V^*XV and Y_i $(i \in Z)$ are all absolute extreme points, it must be that $Y_i \overset{\text{u}}{\equiv} V^*XV$ for all X. This contradicts the assumption that $Y_i \notin \operatorname{mco} X$.

Now we will prove the converse under the additional assumption that $K = \operatorname{mco} \partial^{\text{abs}} K$. Suppose that K fails to satisfy Property A. Then there exists a topological boundary point $X \in \partial^{\text{top}} K$ which cannot be compressed to an absolute extreme point of K. In particular, X itself is not an absolute extreme point. Hence we can write $X = \sum_{i=1}^{k} C_i^* Y_i C_i$ where $C_i \neq 0$, and without loss of generality[1] each $Y_i \in \partial^{\text{abs}} K$.

We claim that for each i, $Y_i \notin \operatorname{mco} X$. For if we had $Y_i \in \operatorname{mco} X$, then by Lemma 4.1.4 Y_i is a direct summand of X. But this means Y_i is a compression of X, despite our knowledge that no compression of X is an absolute extreme point. Hence, indeed, $Y_i \notin \operatorname{mco} X$ for all i. □

We thus have a characterization of Property A in the style of Proposition 4.1.1, at least for matrix convex sets K satisfying $K = \operatorname{mco} \partial^{\text{abs}} K$. For instance, by Theorem 3.4.5, this Property A characterization holds for compact free spectrahedra with a real defining tuple.

[1]This is where we use the assumption that $K = \operatorname{mco} \partial^{\text{abs}} K$.

5

Conclusion

The recurring theme throughout this thesis has been the matricial notion of extreme point: points in a matrix convex set that can only be represented as matrix convex combinations in some degenerate way. We considered two such notions: the matrix extreme points, and the more restrictive absolute extreme points, the difference between them being which matrix convex combinations count as "degenerate".

The Krein-Milman theorem of classical convexity extends nicely to compact matrix convex sets [13], provided that the extreme points being considered are the *matrix* extreme points rather than merely the absolute extreme points.

In Chapter 3 the main result, Theorem 3.4.5, gives a class of matrix convex sets for which a Krein-Milman-type theorem holds with respect to absolute extreme points: compact free spectrahedra defined by a tuple of self-adjoint matrices over $\mathbb{R}$. The absolute extreme points of such matrix convex sets are abundant enough to generate the entire set, but it is not true that *every* compact complex free spectrahedron has this property, and work has been done elsewhere to study this failure [11].

In Chapter 4, we searched for a notion of strict matrix convexity. After discarding the naive definitions as untenable, we proposed Property A (Definition 4.2.1) as another candidate. We saw one especially nice free spectrahedron that fails to satisfy

Property A: the matrix convex hull of the unit disk. The unit disk is the prototypical strictly convex set in the classical convexity setting, so if that doesn't generate a strictly matrix convex set, then do we truly have a good definition of strict matrix convexity? It remains to be seen whether some alternative notion of strict matrix convexity produces fewer unexpected counterexamples. Despite our lack of a definite answer on this issue, we still performed some analysis of the topological boundary and absolute extreme points of some specific matrix convex sets, since these points likely play a central role in any good definition of strict matrix convexity.

On the subject of strict matrix convexity of free spectrahedra, our own original Theorem 4.3.3 gave a condition under which the first matrix level of a two-variable free spectrahedron satisfies Property A. It is unknown whether the theorem can be strengthened to imply Property A with regards to all matrix levels, nor do we know if a similar result holds for g variables, with $g \geqslant 3$.

Our proposed definition of strict matrix convexity was defined in terms of absolute extreme points. The notion of strict matrix convexity with respect to *matrix* extreme points has yet to be studied.

Appendix A

The Moore-Penrose Pseudoinverse and Positive Semidefiniteness

Definition A.0.1 (Moore-Penrose Pseudoinverse)**.** Let $X \in \mathbb{M}_{m,n}$. A *Moore-Penrose pseudoinverse*, or simply *pseudoinverse*, of X is a matrix $X^\dagger \in \mathbb{M}_{n,m}$ with the following properties:

1. $XX^\dagger X = X$;
2. $X^\dagger X X^\dagger = X^\dagger$;
3. $(XX^\dagger)^* = XX^\dagger$;
4. $(X^\dagger X)^* = X^\dagger X$.

Details regarding the existence and uniqueness, and other properties can be found in [9]. For our purposes, however, we only need to consider pseudoinverses of self-adjoint square matrices. In this case the pseudoinverse is quite simple to construct. Let $X \in \mathbb{S}_n$ be a self-adjoint square matrix. If X is invertible, define $X^\dagger := X^{-1}$. If not, let $m = \dim \operatorname{Ker} X \geqslant 1$. Then we can write $X = U^*(D \oplus 0_m)U$ for some invertible matrix $D \in \mathbb{S}_{n-m}$ and unitary matrix U. Then we define $X^\dagger := U^*(D^{-1} \oplus 0_m)U$.

Proposition A.0.2. *If $D \oplus 0_m = U^*(E \oplus 0_m)U$ for some unitary U and invertible $D, E \in \mathbb{S}_{n-m}$, then $D^{-1} \oplus 0 = U^*(E^{-1} \oplus 0_m)U$.*

Proof. Writing $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, we obtain $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{11}^* E U_{11} & U_{11}^* E U_{21} \\ U_{12}^* E U_{11} & U_{12}^* E U_{21} \end{bmatrix}$. It follows that $D = U_{11}^* E U_{11}$. Hence U_{11} and E are invertible. We have $0 = U_{11}^* E U_{21}$, implying $U_{21} = 0$, and $U_{12}^* E U_{11} = 0$, implying $U_{12} = 0$. Hence $U = U_{11} \oplus U_{22}$, which implies that U_{11} and U_{22} are unitary. We have $D \oplus 0 = U_{11}^* E U_{11} \oplus 0$, whence

$$D^{-1} \oplus 0 = U_{11}^{-1} E^{-1} (U_{11}^*)^{-1} \oplus 0 = U_{11}^* E^{-1} U_{11} \oplus 0 = U^*(E^{-1} \oplus 0)U.$$

□

Therefore our definition of $X^\dagger$ is well-defined, depending not on the choice of D and U.

Proposition A.0.3. *Let $X \in \mathbb{S}_n$. Then*

1. *$XX^\dagger X = X$ and $X^\dagger X X^\dagger = X^\dagger$;*

2. *$(XX^\dagger)^* = XX^\dagger$ and $(X^\dagger X)^* = X^\dagger X$;*

3. *$(X^\dagger)^* = X^\dagger$;*

4. *$\mathrm{Ker}(X^\dagger) = \mathrm{Ker}(X)$.*

Proof. Let $X = U^*(D \oplus 0)U$ where D is invertible. We have

$$XX^\dagger X = (U^*(D \oplus 0)U)(U^*(D^{-1} \oplus 0)U)(U^*(D \oplus 0)U) = U^*(D \oplus 0)U = X,$$

$$X^\dagger X X^\dagger = (U^*(D^{-1} \oplus 0)U)(U^*(D \oplus 0)U)(U^*(D^{-1} \oplus 0)U) = U^*(D^{-1} \oplus 0)U = X^\dagger,$$

$$(XX^\dagger)^* = (U^*(D \oplus 0)UU^*(D^{-1} \oplus 0)U)^* = (U^*(I \oplus 0)U)^* = U^*(I \oplus 0)U = XX^\dagger,$$

$$(X^\dagger X)^* = (U^*(D^{-1} \oplus 0)UU^*(D \oplus 0)U)^* = (U^*(I \oplus 0)U)^* = U^*(I \oplus 0)U = X^\dagger X,$$

$$(X^\dagger)^* = (U^*(D^{-1} \oplus 0)U)^* = U^*((D^{-1})^* \oplus 0)U = U^*((D^*)^{-1} \oplus 0)U = (X^*)^\dagger.$$

Finally, for a vector h we have

$$\begin{aligned} X^{\dagger}h = 0 &\iff U^*(D^{-1} \oplus 0)Uh = 0 \\ &\iff (D^{-1} \oplus 0)Uh = 0 \\ &\iff (D \oplus 0)Uh = (D \oplus 0)(D \oplus 0)(D^{-1} \oplus 0)Uh = 0 \\ &\iff U^*(D \oplus 0)Uh = 0 \\ &\iff Xh = 0. \end{aligned}$$

Hence $\mathrm{Ker}(X^{\dagger}) = \mathrm{Ker}(X)$. □

Proposition A.0.4. *Let $X \in \mathbb{S}_m$, $Z \in \mathbb{S}_n$, $Y \in \mathbb{M}_{m,n}$. Then $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geqslant 0$ if and only if all of the following properties hold:*

1. *$X \geqslant 0$,*
2. *$Z - Y^*X^{\dagger}Y \geqslant 0$, and*
3. *$(I - XX^{\dagger})Y = 0$,*

Proof. Suppose that $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geqslant 0$. Since X is a compression, we have $X \geqslant 0$ too.

To see that $Z - Y^*X^{\dagger}Y \geqslant 0$, first note that $\mathrm{Ker}(X^{\dagger}) = \mathrm{Ker}(X)$. We now have, for all $v \in \mathbb{C}^n$,

$$\begin{aligned} 0 &\leqslant \begin{bmatrix} (X^{\dagger}Yv)^* & (-v)^* \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} X^{\dagger}Yv \\ -v \end{bmatrix} \\ &= \begin{bmatrix} v^*Y^*X^{\dagger} & (-v)^* \end{bmatrix} \begin{bmatrix} XX^{\dagger}Yv + Y(-v) \\ Y^*X^{\dagger}Yv + Z(-v) \end{bmatrix} \\ &= (v^*Y^*X^{\dagger}XX^{\dagger}Yv + v^*Y^*X^{\dagger}Y(-v)) + ((-v)^*Y^*X^{\dagger}Yv + (-v)^*Z(-v)) \end{aligned}$$

$$= v^*Y^*X^\dagger Yv - v^*Y^*X^\dagger Yv - v^*Y^*X^\dagger Yv + v^*Zv$$
$$= v^*Zv - v^*Y^*X^\dagger Yv$$
$$= v^*(Z - Y^*X^\dagger Y)v.$$

Hence $Z - Y^*X^\dagger Y \geqslant 0$.

By Lemma 3.2.3 we have $\operatorname{Ker} X \subseteq \operatorname{Ker} Y^*$. Hence

$$\operatorname{Ran} Y = (\operatorname{Ker} Y^*)^\perp \subseteq (\operatorname{Ker} X)^\perp = \operatorname{Ran} X.$$

So for all v, we can write $Yv = Xu$ for some u, and then

$$XX^\dagger Yv = XX^\dagger Xu = Xu = Yv,$$

whence $(I - XX^\dagger)Y = 0$.

Conversely, suppose the three conditions hold. Then for all u and v of appropriate size, we have

$$\begin{bmatrix} u^* & v^* \end{bmatrix} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$
$$= u^*Xu + 2\operatorname{Re}(u^*Yv) + v^*Zv$$
$$= u^*XX^\dagger Xu + 2\operatorname{Re}(u^*XX^\dagger Yv) + v^*Zv$$
$$\geqslant u^*XX^\dagger Xu + 2\operatorname{Re}(u^*XX^\dagger Yv) + v^*Y^*X^\dagger Yv$$
$$= (Xu)^*X^\dagger(Xu) + (Xu)^*X^\dagger(Yv) + (Yv)^*X^\dagger(Xu) + (Yv)^*X^\dagger(Yv)$$
$$= (Xu + Yv)^*X^\dagger(Xu + Yv) \geqslant 0,$$

whence $\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geqslant 0$. □

Sometimes, but not always, we will be fortunate enough that X or Z is the identity. Then the three properties that characterize positive semidefiniteness collapse to just one property. We record this special case as its own result.

Corollary A.0.5. *Let $X \in \mathbb{S}_m$, and $Y \in \mathbb{M}_{m,n}$. Then $\begin{bmatrix} X & Y \\ Y^* & I \end{bmatrix} \geqslant 0$ if and only if $X - YY^* \geqslant 0$.*

We end this section with a technical lemma which is an essential piece of the proof of the main result.

Proposition A.0.6. *Suppose that*

$$W := \begin{bmatrix} P & Q & R \\ Q^* & S & T \\ R^* & T^* & U \end{bmatrix} \geqslant 0.$$

Then $\operatorname{Ker}(S - Q^* P^\dagger Q) \subseteq \operatorname{Ker}(T^* - R^* P^\dagger Q)$.

Proof. Considering W as a 2×2 block matrix

$$\begin{bmatrix} \begin{pmatrix} P \end{pmatrix} & \begin{pmatrix} Q & R \end{pmatrix} \\ \begin{pmatrix} Q^* \\ R^* \end{pmatrix} & \begin{pmatrix} S & T \\ T^* & U \end{pmatrix} \end{bmatrix},$$

we apply the earlier Proposition A.0.4 to deduce that

$$\begin{bmatrix} S & T \\ T^* & U \end{bmatrix} - \begin{bmatrix} Q^* \\ R^* \end{bmatrix} P^\dagger \begin{bmatrix} Q & R \end{bmatrix} \geqslant 0.$$

But this can be rewritten as

$$\begin{bmatrix} S - Q^* P^\dagger Q & T - Q^* P^\dagger R \\ T^* - R^* P^\dagger Q & U - R^* P^\dagger R \end{bmatrix} \geqslant 0.$$

Now apply Lemma 3.2.3 to reach the desired conclusion. $\square$

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