DERIVATIONS, MULTIPLIERS AND TOPOLOGICAL CENTERS OF CERTAIN BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

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ABSTRACT

We introduce certain Banach algebras related to locally compact groups and study their properties. Specifically, we prove that $L^1(G)$ is an ideal of $L^\infty_{\sigma 0}(G)^*$ if and only if G is compact. We also demonstrate that the left topological centers of $L^\infty_{\sigma 0}(G)^*$and $(M(G)^*_{\sigma 0})^*$ are $L^1(G)$ and $M(G)$ respectively. Next, we turn our attention to various derivation and left multiplier problems. Specifically, we show that for every weak-star continuous derivation $D : L^1(G)^{**} \longrightarrow L^1(G)^{**}$ there is $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$. We also prove that every derivation from $L^\infty_0(G)^*$ into $L^1(G)$ is inner. Next, we focus on weakly compact derivations and left multipliers and show that for every weakly compact derivation D on $M(G)$ there is $f \in L^1(G)$ such that $D = \mathrm{ad}_f$. We also prove that there exists a non-zero weakly compact derivation on $L^1(G)$ (or $L^\infty_0(G)^*$ for the special case where there is a unique right invariant mean on $L^\infty(G)$) if and only if G is a non-abelian compact group. We present necessary and sufficient conditions for the existence of non-zero weakly compact left multiplier on $L^\infty_0(G)^*$. We also show that for the special case where there is a unique right invariant mean on $L^\infty(G)$, every weakly compact derivation D on $L^1(G)^{**}$ is of the form ad_h where h is in $L^1(G)$. We introduce the concepts of quasi-Arens regularity, quasi topological center and quasi-weakly almost periodic functionals and show that $\lambda \in QWAP(A)$ if and only if ad_λ is weakly compact. Finally, for a particular G, we construct a continuous non-weakly compact derivation $D : L^1(G) \longrightarrow L^\infty(G)$ such that $D(L^1(G)) \subseteq WAP(G)$.

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My Parents: Ali and Effat

And To

My Siblings: Maryam, Leila and Reza

INTRODUCTION

The question of whether every *derivation* on a group algebra is inner in measure algebra had been one of the long-standing open questions which has motivated a great deal of research. The question attracted the attention of Johnson as an example of his theory of cohomology in Banach algebra and was settled in the affirmative for several special cases in [33] and [34]. Further results were obtained in [24]. Finally, Losert in [41] answered the question, for general locally compact groups, in the affirmative. Later on, in [5], Bader et al. provided a shorter solution to the problem. In [47], Sakai answered the question in the affirmative for the derivations on von Neumann algebras.

The study of the second dual of a Banach algebra, as a Banach algebra, goes back to R. Arens who introduced the *Arens products* in [3] and [4]. For the special case of the second dual of a group algebra, extensive research was conducted in [8], [9], [25], [51], [56], [26], [52] and [53]. For instance Young, in [56], proved that $L^1(G)$ is *Arens regular* if and only if G is finite and S. Watanabe, in [52] and [53], proved that $L^1(G)$ is an ideal of its second dual if and only if G is compact. A survey of all results, obtained until 1979, can be found in [14]. In [29] Isik, Pym and Ulger proved, among other results, that when G is compact then the *topological center* of $L^1(G)^{**}$ is precisely $L^1(G)$. This work motivated other mathematicians to conduct further research which led to the publication of [37], [20], [19] and [39]. For example Lau and Pym, in

[39], introduced $L_0^\infty(G)^*$ as the dual of a subspace of $L^\infty(G)$ consisting of functions that vanish at infinity almost everywhere (equipped with a product resembling the Arens product) and then showed that it can be identified with a closed subalgebra of $L^1(G)^{**}$. Furthermore, the authors showed that most of the results which had been proven for $L^\infty(G)^*$, in the case of compact groups G, can be recovered for all locally compact groups if we replace $L^\infty(G)^*$ by $L_0^\infty(G)^*$. For instance the topological center of $L_0^\infty(G)^*$ is again $L^1(G)$. In [37], Lau and Losert demonstrated that in fact, the topological center of $L^1(G)^{**}$ is also $L^1(G)$ for all locally compact groups. Therefore the algebraic center of $L^1(G)^{**}$ is $L^1(G)$ when G is an abelian group. It was conjectured in [21] by Ghahramani and Lau that the topological center of $M(G)^{**}$ should be $M(G)$. In [45], Neufang's proof confirmed the conjecture partially for some specific groups. Later, Losert et al., in [40], proved that the Ghahramani-Lau conjecture holds for all locally compact groups.

During the last 50 years several authors have investigated the question of when *multipliers* or derivations on various classes of Banach algebras are *(weakly) compact*. For example Charles Akemann in [1] studied when the group algebra $L^1(G)$ has a non-zero (weakly) compact multiplier, and, in [2] he and Steve Wright, characterized (weakly) compact derivations of C^*-algebras. In [21] and [22], the (weakly) compact right multipliers of $L^1(G)^{**}$ have been studied. However, since $L^1(G)$ is not Arens regular (unless G is finite), the left multipliers behave differently and only partial results has been established in [23]. In [42], Losert proved that there exists a non-zero weakly compact left multiplier on $L^1(G)^{**}$ or $M(G)^{**}$ if and only if G is compact. When G is compact, he was also able to characterize all non-zero weakly compact left multipliers for the case where $L^\infty(G)$ has a unique right invariant mean.

In this thesis, we study derivations and multipliers on certain Banach algebras related to locally compact groups and investigate the (weak) compactness of these operators. We also introduce some new Banach algebras related to locally compact groups and study their multipliers and derivations.

In chapter 1, we have compiled some preliminary definitions and basic results that we have used in the following chapters.

In chapter 2, we introduce new commutative C^*-algebras $L^\infty_{\sigma 0}(G)$, $M(G)^*_0$, $M(G)^*_{\sigma 0}$ and we show that their duals can be made into Banach algebras if we equip them with multiplications resembling the Arens products. In addition to various results and generalizations pertaining to these Banach algebras, in Theorem 2.65 we demonstrate that $L^1(G)$ is an ideal of $L^\infty_{\sigma 0}(G)^*$ if and only if G is compact.

In chapter 3, we prove that the left topological centers of $L^\infty_{\sigma 0}(G)^*$ and $(M(G)^*_{\sigma 0})^*$ are $L^1(G)$ and $M(G)$ respectively.

In chapter 4, among other results, we show in Theorem 4.7 that every weak-star continuous derivation D on $L^1(G)^{**}$ is of the form ad_μ for some $\mu \in M(G)$. In Theorem 4.10, we prove that every derivation from $L^\infty_0(G)^*$ into $L^1(G)$ is inner. Next, we demonstrate in Theorem 4.20 that for every weakly compact derivation D on $M(G)$ there is $f \in L^1(G)$ such that $D = \mathrm{ad}_f$. We also prove in Theorems 4.21 and 4.26, respectively, that there exists a non-zero weakly compact derivation on $L^1(G)$ (or $L_0^\infty(G)^*$ for the case where there is a unique right invariant mean on $L^\infty(G)$) if and only if G is a non-abelian compact group. In Theorem 4.22, we present necessary and sufficient conditions for the existence of non-zero weakly compact left multipliers on $L_0^\infty(G)^*$. For the special case where there is a unique right invariant mean on $L^\infty(G)$, in Theorem 4.25, we prove that every weakly compact derivation D on $L^1(G)^{**}$ is of the form ad_h for some $h \in L^1(G)$. In section 4.3, we introduce the concepts of quasi-Arens regularity and quasi topological center for a general Banach algebra $\mathcal{A}$. Also the space of all quasi-weakly almost periodic functionals as a closed submodule of $\mathcal{A}^*$ is defined and its relationship with weakly compact derivations is investigated in Proposition 4.37. Theorem 4.46 shows that the converse of part (b) of Proposition 4.43 does not hold.

Finally, in chapter 5, we conclude this thesis with a short list of selective open problems that can be the subject of further studies.

TABLE OF CONTENTS

Chapter 1

PRELIMINARIES

In this chapter we review some basic concepts and results that are used throughout this thesis. All of the definitions and theorems can be found in the literature. However, for the sake of completeness we will give a brief proof for some of the theorems.

1.1 Banach algebras and Banach bimodules

Let $\mathcal{X}$ be a normed vector space over the field $\mathbb{F}$ (which is equal to $\mathbb{C}$ or $\mathbb{R}$, but more often $\mathbb{C}$). We say that $\mathcal{X}$ is a *Banach space* if its norm is complete. An *algebra* over $\mathbb{F}$ is a vector space $\mathcal{A}$ over $\mathbb{F}$ that also has a multiplication defined on it that makes $\mathcal{A}$ into a ring such that if $\alpha \in \mathbb{F}$ and $a, b \in \mathcal{A}$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Definition 1.1. *A* Banach algebra *is an algebra $\mathcal{A}$ over $\mathbb{F}$ that has a norm $\|\cdot\|$ relative to which $\mathcal{A}$ is a Banach space and such that for all $a, b \in \mathcal{A}$,*

$$\|ab\| \leq \|a\| \, \|b\| .$$

For a Banach algebra $\mathcal{A}$, the space of *right annihilators* of $\mathcal{A}$ is denoted by $ran(\mathcal{A})$ and is defined to be

$$ran(\mathcal{A}) = \{a \in \mathcal{A} : ba = 0, \text{ for all } b \in \mathcal{A}\} .$$

Let $\mathcal{A}$ be an algebra over $\mathbb{F}$. A *left $\mathcal{A}$-module* is a vector space $\mathcal{X}$ over $\mathbb{F}$ with a bilinear map $(a,x) \longmapsto a \cdot x$, $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$, such that

$$a \cdot (b \cdot x) = (ab) \cdot x \quad (a, b \in \mathcal{A}, x \in \mathcal{X}).$$

A *right $\mathcal{A}$-module* is a vector space $\mathcal{X}$ over $\mathbb{F}$ with a bilinear map $(a,x) \longmapsto x \cdot a$, $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ such that

$$(x \cdot a) \cdot b = x \cdot (ab) \quad (a, b \in \mathcal{A}, x \in \mathcal{X}).$$

An *$\mathcal{A}$-bimodule* is a vector space $\mathcal{X}$ over $\mathbb{F}$ that is a left and right $\mathcal{A}$-module and such that

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in \mathcal{A}, x \in \mathcal{X}).$$

Definition 1.2. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach space that is also an $\mathcal{A}$-bimodule. We say that $\mathcal{X}$ is a* Banach $\mathcal{A}$-bimodule *if there is a constant c such that*

$$\|a \cdot x\| \leq c \, \|a\| \, \|x\| \, , \|x \cdot a\| \leq c \, \|x\| \, \|a\| \quad (a \in \mathcal{A}, x \in \mathcal{X}).$$

The Banach left and right $\mathcal{A}$-modules can be defined similarly.

Example 1.3. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule where we consider the product on $\mathcal{A}$ as the module action as well.*

Definition 1.4. *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. We say that $\mathcal{X}$ is an essential Banach $\mathcal{A}$-bimodule if $\overline{\mathcal{A} \cdot \mathcal{X}} = \overline{\mathcal{X} \cdot \mathcal{A}} = \mathcal{X}$, where $\mathcal{A} \cdot \mathcal{X} = \{a \cdot x : a \in \mathcal{A}, x \in \mathcal{X}\}$ and similarly $\mathcal{X} \cdot \mathcal{A} = \{x \cdot a : x \in \mathcal{X}, a \in \mathcal{A}\}$.*

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach left $\mathcal{A}$-module. A *left bounded approximate identity* in $\mathcal{A}$ for $\mathcal{X}$ is a bounded net $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ such that

$$e_\alpha \cdot x \to x \quad (x \in \mathcal{X});$$

where convergence is in the norm-topology of $\mathcal{X}$. The *right bounded approximate* and *bounded approximate identity* can be defined similarly. We have the following factorization theorem that is useful throughout this thesis.

Proposition 1.5. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach left $\mathcal{A}$-module. Suppose that $\mathcal{A}$ has a bounded left approximate identity for $\mathcal{X}$. Then, for each $z \in \mathcal{X}$ and $\delta > 0$, there exists $a \in \mathcal{A}$ and $x \in \mathcal{X}$ such that $z = a \cdot x$ and $\|z - x\| \leq \delta$.*

Proof. See [7], page 61. □

The focus of Chapter 4 of this thesis is on derivations and multipliers of Banach algebras related to locally compact groups. Here we define these concepts in general setting:

Definition 1.6. *Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A linear map $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a* derivation *if*

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).$$

For example take $x \in \mathcal{X}$ and define the map $\mathrm{ad}_x : \mathcal{A} \longrightarrow \mathcal{X}$ by setting

$$\mathrm{ad}_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

It is easy to verify that ad_x is a derivation. These particular type of derivations are called *inner derivations*.

Definition 1.7. *Let $\mathcal{A}$ be an algebra and $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A linear map $T : \mathcal{A} \longrightarrow \mathcal{X}$ is called a* left multiplier *if*

$$T(ab) = T(a) \cdot b \quad (a, b \in \mathcal{A}).$$

Similarly a linear map $R : \mathcal{A} \longrightarrow \mathcal{X}$ is called a right multiplier *if*

$$R(ab) = a \cdot R(b) \quad (a, b \in \mathcal{A}).$$

1.2 Second dual of Banach algebras

Let us recall from functional analysis that if $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then the space of all continuous linear transformations from $\mathcal{X}$ to $\mathcal{Y}$ is a normed space, denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, where the norm is defined by

$$\|T\| = \sup_{x \in \mathcal{X}, \|x\| \leq 1} |T(x)| \quad (T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})).$$

When $\mathcal{Y}$ is a Banach space, $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a Banach space too. In particular $\mathcal{B}(\mathcal{X}, \mathbb{C})$, the space of all bounded functionals on $\mathcal{X}$, is a Banach space, denoted by $\mathcal{X}^*$.

Beside the norm-topology we use two weaker topologies defined as follows:

Let $\mathcal{X}$ be a normed space. For each $x^* \in \mathcal{X}^*$, define $p_{x^*}(x) = |x^*(x)|$. Then p_{x^*} is a semi-norm and the set $\{p_{x^*}, x^* \in \mathcal{X}^*\}$ defines a locally convex topology on $\mathcal{X}$ that is called the *weak topology* and it is denoted by $\sigma(\mathcal{X}, \mathcal{X}^*)$. From now on by $x_\alpha \xrightarrow{\text{wk}} x$, we mean that the net $(x_\alpha)_{\alpha \in I}$ converges to $x \in \mathcal{X}$ with respect to the weak topology. The next proposition will be useful in the future.

Proposition 1.8. *Let $\mathcal{X}$ be a normed space. Suppose that S is a convex subset of $\mathcal{X}$. Then the closures of S with respect to the norm-topology and the weak topology on $\mathcal{X}$ are the same.*

Proof. This is Theorem 1.4. in chapter 5 of [10]. $\square$

Now let $q_x : \mathcal{X}^* \longrightarrow [0, \infty)$ to be defined by $q_x(x^*) = |x^*(x)|$. Then the set of semi-norms $\{q_x : x \in \mathcal{X}\}$ defines a locally convex topology on $\mathcal{X}^*$ that is called the *weak-star topology* and it is denoted by $\sigma(\mathcal{X}^*, \mathcal{X})$. From now on, when a net $(x^*_\alpha)_{\alpha \in I}$ in $\mathcal{X}^*$ converges to $x^* \in \mathcal{X}^*$ in weak-star topology, we indicate this by $x^*_\alpha \xrightarrow{\text{wk}^*} x^*$. Let us agree to denote the set $\{x \in \mathcal{X} : \|x\| \leq 1\}$, in a normed space $\mathcal{X}$, by $(\mathcal{X})_1$ and call it the unit ball of $\mathcal{X}$.

Definition 1.9. *Suppose that $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. We say that T is* (weakly) compact *if the closure of $T((\mathcal{X})_1)$ is (weakly) compact.*

Definition 1.10. *Let $\mathcal{A}$ be a Banach algebra. We say that $a \in \mathcal{A}$ is a* left weakly completely continuous element *if $T_a : \mathcal{A} \longrightarrow \mathcal{A}$, $b \longmapsto ab$ $(\forall b \in \mathcal{A})$ is weakly compact.*

We have the following proposition regarding the weakly compact continuous linear transformations:

Proposition 1.11. *Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is weakly compact. Assume that $R \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{B}(\mathcal{W}, \mathcal{X})$, then $R \circ T$ and $T \circ S$ are weakly compact.*

Proof. For a proof see [10], page 183. □

We also remind the reader of some classic results in functional analysis that will be useful throughout this thesis:

Proposition 1.12. ***Banach-Alaoglu's theorem.*** *Let $\mathcal{X}$ be a normed space. Then $(\mathcal{X}^*)_1$ is weak-star compact.*

Proof. See [10] page 130. □

Proposition 1.13. ***Goldstine's theorem.*** *Let $\mathcal{X}$ be a normed space. Then $(\mathcal{X})_1$ is $\sigma(\mathcal{X}^{**}, \mathcal{X}^*)$ dense in $(\mathcal{X}^{**})_1$.*

Proof. See [10] page 131. □

Definition 1.14. *For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, the adjoint of T is denoted by T^* and $T^* : \mathcal{Y}^* \longrightarrow \mathcal{X}^*$ is defined by*

$$\langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle \quad \textit{for every } y^* \in Y^*, x \in X.$$

Proposition 1.15. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then the following are equivalent.*

(a) T is weakly compact;

(b) T^* is continuous with respect to the weak-star topology on $\mathcal{Y}^*$ and weak topology on $\mathcal{X}^*$;

*(c) $T^{**}(\mathcal{X}^{**}) \subseteq \mathcal{Y}$ **(Gantmacher)**;*

(d) T^* is weakly compact.

Proof. The fact that (a), (c) and (d) are equivalent has been proved in Theorem 5.5 in chapter 6 of [10]. It is also easy to verify that (b) is equivalent to other clauses. □

We also have the following proposition.

Proposition 1.16. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. For a linear transformation $T : \mathcal{X} \longrightarrow \mathcal{Y}$, the following conditions are equivalent:*

(a) T is bounded;

(b) T is weakly continuous.

Proof. See Theorem 1.1. in chapter 6 of [10]. □

Proposition 1.17. *Let $\mathcal{A}$ be an abelian Banach algebra with identity and $h : \mathcal{A} \longrightarrow \mathbb{C}$ be a non-zero multiplicative functional. Then $\|h\| = h(1) = 1$.*

Proof. This is Proposition 8.4 in chapter 7 of [10]. $\square$

In this thesis we discuss the second dual of Banach algebras endowed with the Arens products. Let $\mathcal{A}$ be a Banach algebra. The second dual of $\mathcal{A}$ is already a Banach space. We aim to make it into a Banach algebra. For $a, b \in \mathcal{A}$ and $\lambda \in \mathcal{A}^*$, we define:

$$\langle \lambda \cdot a, b \rangle = \langle \lambda, ab \rangle \qquad \langle a \cdot \lambda, b \rangle = \langle \lambda, ba \rangle .$$

Note that with the above definitions $\mathcal{A}^*$ becomes a Banach $\mathcal{A}$-bimodule.

Remark 1.18. *More generally and in a similar way as above, we can make $\mathcal{X}^*$ into a Banach $\mathcal{A}$-bimodule, where $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule.*

For $F \in \mathcal{A}^{**}$ $\lambda \in \mathcal{A}^*$ and $a \in \mathcal{A}$ we define:

$$\langle F \cdot \lambda, a \rangle = \langle F, \lambda \cdot a \rangle \qquad \langle \lambda \cdot F, a \rangle = \langle F, a \cdot \lambda \rangle .$$

Finally, for $F, G \in \mathcal{A}^{**}$ and $\lambda \in \mathcal{A}^*$, we define the following two products on $\mathcal{A}^{**}$:

$$\langle F \Box G, \lambda \rangle = \langle F, G \cdot \lambda \rangle \qquad \langle F \diamond G, \lambda \rangle = \langle G, \lambda \cdot F \rangle .$$

It is easy to verify that $\mathcal{A}^{**}$ with respect to both $\Box$ and $\diamond$ is a Banach algebra. The product $\Box$ is called the *first Arens product* and $\diamond$ is called *the second Arens product.* Generally these two products do not coincide and when they do $\mathcal{A}$ is called *Arens regular*. In this thesis, most of the time we will work with the first Arens product.

Therefore for the sake of simplicity we suppress the symbol $\Box$ in our equations unless we want to emphasize or distinguish it from other products. We have the following proposition:

Proposition 1.19. *Suppose that $\mathcal{I}$ is a closed two-sided ideal of a Banach algebra $\mathcal{A}$. Then $\mathcal{I}^{**}$ can be identified with a closed two-sided ideal of $\mathcal{A}^{**}$, where we consider the first Arens product for both algebras.*

Proof. Let $i : \mathcal{I} \longrightarrow \mathcal{A}$ be the inclusion map. Then easy calculations show that $i^{**}(\mathcal{I})$ is a closed two-sided ideal in $\mathcal{A}^{**}$. $\Box$

The following proposition is immediate from the definitions.

Proposition 1.20. *Let $\mathcal{A}$ be a Banach algebra and suppose that F is in $\mathcal{A}^{**}$. Then*

(a) The map $G \longmapsto G \Box F$, $\mathcal{A}^{**} \longrightarrow \mathcal{A}^{**}$ is weak-star continuous.

(b) The map $G \longmapsto F \diamond G$, $\mathcal{A}^{**} \longrightarrow \mathcal{A}^{**}$ is weak-star continuous.

The *left* and *right topological centers* of $\mathcal{A}^{**}$ are defined respectively by

$$Z_t^l(\mathcal{A}^{**}) = \{F \in \mathcal{A}^{**} : \text{the mapping } G \mapsto F \Box G \text{ is weak-star continuous on } \mathcal{A}^{**}\}$$

$$Z_t^r(\mathcal{A}^{**}) = \{F \in \mathcal{A}^{**} : \text{the mapping } G \mapsto G \diamond F \text{ is weak-star continuous on } \mathcal{A}^{**}\}\,.$$

Obviously, $\mathcal{A} \subseteq Z_t^l(\mathcal{A}^{**}) \bigcap Z_r^t(\mathcal{A}^{**})$. We say that $\mathcal{A}$ is *strongly Arens irregular* if $\mathcal{A} = Z_t^l(\mathcal{A}^{**}) = Z_r^t(\mathcal{A}^{**})$. When $\mathcal{A}$ is Arens regular we have $\mathcal{A}^{**} = Z_t^l(\mathcal{A}^{**}) = Z_r^t(\mathcal{A}^{**})$.

Using the Goldstine's theorem, it is an easy observation that

$$Z_t^l(\mathcal{A}^{**}) = \{F \in \mathcal{A}^{**} : (\forall G \in \mathcal{A}^{**}) F \square G = F \diamond G\}$$

$$Z_t^r(\mathcal{A}^{**}) = \{F \in \mathcal{A}^{**} : (\forall G \in \mathcal{A}^{**}) G \square F = G \diamond F\}.$$

1.3 Measure theory

Let X be a Hausdorff locally compact topological space. We denote the set of open, compact and Borel subsets of X by $\mathcal{U}_X$, $\mathcal{K}_X$ and $\mathcal{B}_X$ respectively. A measure is called a *Borel measure* if its domain is the set of all Borel subsets of X. A positive Borel measure μ is said to be *regular* if:

(a) $\mu(W) = \inf\{\mu(U) : U \in \mathcal{U}_X, U \supseteq W\}$ for every $W \in \mathcal{B}_X$ (outer regularity).

(b) $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}_X, K \subseteq U\}$ for every $U \in \mathcal{U}_X$ (inner regularity).

For a Borel measure μ, we define $|\mu|(V) = \sup\{\sum_{i=1}^{\infty} |\mu(V_i)|\}$ where $\{V_i \in \mathcal{B}_X\}$ is a partition of $V \in \mathcal{B}_X$. It can be proved that $|\mu|$ is a positive Borel measure. We define the total variation of a Borel measure μ to be $\|\mu\| = |\mu|(X)$. A Borel measure μ is called *regular* if $|\mu|$ is regular. We reserve the term measure for the set of all regular measures and we denote the space of all regular Borel measures with finite total variations by $M(X)$. Obviously $(M(X), \|\cdot\|)$ is a Banach space.

Definition 1.21. *The* support *of $\mu \in M(X)$, $\operatorname{supp}\mu$, is the complement of the maximal open subset U of X such that $|\mu|(U) = 0$.*

For $x \in X$ and a Borel set E, we define

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E; \\ \\ 0 & \text{if } x \notin E. \end{cases}$$

The measure δ_x is in $M(X)$ and it is called the *point-mass* or *Dirac* measure at x. Suppose that μ is a positive Borel measure on X. We recall that the positive measure $\nu \in M(X)$ is said to be singular with respect to μ, written $\mu \perp \nu$, if there is a Borel set V such that $\mu(V) = 0$ and $\nu(V^c) = 0$. The measure ν is said to be absolutely continuous with respect to μ, written $\nu \ll \mu$, if $\mu(V) = 0$ implies $\nu(V) = 0$ for each Borel set V. For general Borel measures $\mu, \nu \in M(G)$, $\mu \perp \nu$ means $|\mu| \perp |\nu|$ and $\nu \ll \mu$ means $|\nu| \ll |\mu|$. We also have the following important theorem.

Theorem 1.22. ***Dieudonne's theorem*** *Let X be a Hausdorff locally compact topological space. Suppose that A is a bounded set of $M(X)$. Then the closure of A is weakly compact in $M(X)$ if and only if both following conditions hold:*

(a) For each compact set $K \subseteq X$ and each $\epsilon > 0$, there exists an open set $U \supseteq K$ such that $|\mu|\,(U \cap K^c) < \epsilon$ for all $\mu \in A$.

(b) For each $\epsilon > 0$, there exists a compact set $K \subseteq X$ such that $|\mu|\,(X \cap K^c) < \epsilon$ for all $\mu \in A$.

Proof. This is part 4 of Theorem 4.22.1 in [15]. □

1.4 C^*-algebras

Definition 1.23. *Let $\mathcal{A}$ be a complex Banach algebra. We say that $\mathcal{A}$ is a $*$-* algebra *if there is an isometric mapping $* : \mathcal{A} \to \mathcal{A}$ that satisfies the following conditions.*

(a) $(x+y)^* = x^* + y^*$ *for all* $x, y \in \mathcal{A}$.

(b) $(\alpha x)^* = \overline{\alpha} x^*$ *for all* $x \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$.

(c) $(xy)^* = y^* x^*$ *for all* $x, y \in \mathcal{A}$.

(d) $(x^*)^* = x$ *for all* $x \in \mathcal{A}$.

We say that $\mathcal{A}$ is a C^*-algebra *if in addition to (a)-(c) above it satisfies $\|xx^*\| = \|x\|^2$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be* positive *if there exists $y \in \mathcal{A}$ such that $x = yy^*$. We denote the set of all positive elements of $\mathcal{A}$ by $\mathcal{A}^+$.*

Theorem 1.24. *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is Arens regular.*

Proof. See Corollary 3.2.37 of [12]. □

We also have the following theorems and propositions which will be useful in next chapters.

Theorem 1.25. *Let $\mathcal{A}$ be a commutative C^*-algebra. Then $\mathcal{A}^{**}$ is a commutative C^*-algebra. Moreover $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{A}^{**}$ when $\mathcal{A}$ is embedded in $\mathcal{A}^{**}$ canonically.*

Proof. From Theorem 1.24, we know that $\mathcal{A}$ is Arens regular. It is easy to verify that the second dual of a commutative Arens regular Banach algebra is commutative too. For the proof of the rest of the statement, see Theorem 1.17.2 of [47]. $\square$

Proposition 1.26. *Each element x of a C^*-algebra $\mathcal{A}$ is a linear combination of at most four elements of $\mathcal{A}^+$.*

Proof. This is Corollary 4.2.4 of [35]. $\square$

Theorem 1.27. *Suppose that $\mathcal{A}$ is a C^*-algebra. Then $x \in \mathcal{A}$ is a positive element if and only if there exists a positive $y \in \mathcal{A}$ such that $y^2 = x$.*

Proof. See Theorem 4.2.6 of [35]. $\square$

Definition 1.28. *Suppose that $\mathcal{A}$ is a C^*-algebra. A linear functional $\phi \in \mathcal{A}^*$ is said to be* positive *if $\phi(x) \geq 0$ whenever $x \in \mathcal{A}$ is positive. A positive linear functional ϕ on a C^*-algebra with identity 1 is called a* state *if $\phi(1) = 1$.*

We list some more propositions and theorems which will be useful later on:

Theorem 1.29. *Let $\mathcal{A}$ be a commutative C^*-algebra. Then there exists an isometric isomorphism from $\mathcal{A}^*$ onto $M(X)$ where X is a locally compact space. Moreover a positive linear functional on A corresponds to a positive measure.*

Proof. See Theorem 2.1, Corollary 2.2 in chapter 8 and also C.17, C.18 of [10]. $\square$

Theorem 1.30. *Let $\mathcal{A}$ be a C^*-algebra with identity. Then a linear functional ϕ is positive if and only if ϕ is bounded and $\|\phi\| = \phi(1)$.*

Proof. See proposition 4.3.2 of [35]. □

Proposition 1.31. *Let $\mathcal{A}$ be a commutative C^*-algebra with a bounded approximate identity (e_i). Suppose that $m \in \mathcal{A}^*$ is positive. Then $\|m\| = \lim_i \langle m, e_i \rangle$.*

Proof. Let E be a cluster point of (e_i) with respect to the weak-star topology in $\mathcal{A}^{**}$. We have $\langle E, m \rangle = \lim_i \langle e_i, m \rangle$. Furthermore, it is easy to verify that E is the identity for $\mathcal{A}^{**}$. If we consider m as a positive linear functional in $\mathcal{A}^{***}$, from Theorem 1.30, we have $\|m\| = \langle m, E \rangle$. Therefore

$$\|m\| = \langle m, E \rangle = \langle E, m \rangle = \lim_i \langle e_i, m \rangle = \lim_i \langle m, e_i \rangle .$$

□

Proposition 1.32. *Suppose that $\mathcal{A}$ is a C^*-algebra. Then every bounded linear functional on $\mathcal{A}$ is the linear combination of at most four positive linear functionals.*

Proof. This is Corollary 4.3.7 of [35]. □

1.5 Topological groups

Let G be a group endowed with a Hausdorff topology τ. We say that (G, τ) is a *topological group* if the multiplication map $(s, t) \longmapsto st, G \times G \longrightarrow G$, and the inversion $s \longmapsto s^{-1}, G \longrightarrow G$ are both continuous where we consider the product topology on $G \times G$. Furthermore if G is a topological group such that (G, τ) is a locally compact topological space, then we say that G is a *locally compact group.* A locally compact group G is called an *[IN] group* if G contains a compact invariant (under inner automorphisms) neighborhood of the identity, and a *[SIN] group* if it contains a fundamental family of compact invariant neighborhoods of the identity.

Definition 1.33. *A left* Haar *measure on G is a positive regular measure m, defined on a σ-algebra containing the σ-algebra of Borel sets on G, such that*

(a) $m(K) < \infty$ *for all* $K \in \mathcal{K}_G$.

(b) $m(U) > 0$ *for all* $U \in \mathcal{U}_G$.

(c) $m(xE) = m(E)$ *for all* $E \in \mathcal{B}_G$ *and* $x \in G$.

Proposition 1.34. *Let G be a locally compact group. There exists a left Haar measure m on G. Moreover m is unique up to a positive multiple.*

Proof. See page 37 of [17]. □

Example 1.35. *Consider the set of real numbers $\mathbb{R}$ with the ordinary topology and addition. It is easy to verify that $\mathbb{R}$ is a locally compact group. The Lebesgue measure on $\mathbb{R}$ restricted to Borel measurable sets is a left Haar measure.*

For each $a \in G$ the function $E \longrightarrow m(Ea)$, $\mathcal{B}_G \longrightarrow [0, \infty]$, is also a left Haar measure on G. Therefore there is $\Delta(a) > 0$ such that $m(Ea) = \Delta(a)m(E)$ for all $E \in \mathcal{B}_G$. The function $\Delta_G : G \longrightarrow [0, \infty]$ defined in this way is called the *modular function*. It is easy to prove that Δ is continuous and $\Delta_G(ab) = \Delta_G(a)\Delta_G(b)$ for all $a, b \in G$. We say that G is *unimodular* if Δ_G is equal to the constant function 1. We have the following proposition:

Proposition 1.36. *Let G be a locally compact group. Suppose that G is abelian, discrete or compact. Then it is unimodular.*

Proof. The proof for discrete and abelian case is trivial. For the compact case see [17], page 47. □

Theorem 1.37. *Let G be a locally compact group. Suppose that U is a symmetric neighborhood of e. Then $H = \bigcup_{n \in \mathbb{N}} U^n$ is an open and closed subgroup of G. Moreover if U has a compact (or σ-compact) closure then H is σ-compact.*

Proof. The proof of the first part of theorem can be found in Theorem 5.7 of [28]. For the second part suppose that $\overline{U}$ is compact (σ-compact). Note that $U^n \subset (\overline{U})^n \subset \overline{(U^n)}$ and $(\overline{U})^n$ is compact (σ-compact) for each $n \in \mathbb{N}$. Also since H is closed and $U^n \subset H$ we have $\overline{(U^n)} \subset H$. Hence:

$$H = \bigcup_{n \in \mathbb{N}} U^n \subset \bigcup_{n \in \mathbb{N}} (\overline{U})^n \subset \bigcup_{n \in \mathbb{N}} \overline{(U^n)} \subset H.$$

Therefore $H = \bigcup_{n \in \mathbb{N}} (\overline{U})^n$ is σ-compact. □

1.6 Some Banach algebras related to locally compact groups

Definition 1.38. *Let G be a locally compact group and m be a fixed Haar measure on G.*

(a) *The space of all extended complex-valued functions that are continuous on G is denoted by $C(G)$. The subspace of all bounded functions in $C(G)$ is denoted by $C_b(G)$. If $f \in C_b(G)$, we define the norm of f to be*

$$\|f\| = \sup_{x \in G} |f(x)| .$$

The space $C_b(G)$ with this norm and natural addition and scalar multiplication is a Banach space. Considering the pointwise multiplication of two functions in $C_b(G)$, it is easy to verify that $C_b(G)$ is in fact a Banach algebra. For $f \in C_b(G)$, we define the left and right translation of f, $l_x f$ and $r_x f$, respectively by $l_x f(y) = f(xy)$ and $r_x f(y) = f(yx)$ for all $y \in G$.

(b) *The space of all functions in $C(G)$ with compact support is denoted by $C_c(G)$, where the* support *of $f \in C(G)$ is defined to be the closure of the set $\{x \in G : f(x) \neq 0\}$.*

(c) *We shall use the notation $C_0(G)$ for the space of all functions in $C(G)$ that vanishes at infinity. We recall that a function $f \in C(G)$ vanishes at infinity if for each $\epsilon > 0$ there exists a compact set K such that $|f(x)| \leq \epsilon$ for all $x \in K^c$. Obviously $C_0(G)$ is a closed subalgebra of $C_b(G)$ and therefore it is a Banach* *algebra too. It is also easy to prove that $C_c(G)$ is dense in $C_0(G)$.*

(d) *A complex-valued function f on G is said to be essentially bounded if the set $\{x \in G : |f(x)| > \alpha\}$ is locally null for some $\alpha > 0$. Recall that a measurable set N is called* locally null *if $m(N \cap K) = 0$ for all $K \in \mathcal{K}_G$. The space of all essentially bounded measurable functions on G is denoted by $L^\infty(G)$. Given $f \in L^\infty(G)$ we define:*

$$\|f\|_\infty = \inf \left\{ \sup_{x \in N^c} |f(x)| : N \quad \textit{is locally null} \quad \right\}.$$

By considering the natural addition, scalar multiplication and pointwise multiplication, $L^\infty(G)$ is a Banach algebra.

(e) *For a measurable set $U \subseteq G$ and $f \in L^\infty(G)$ write*

$$\operatorname{ess\,sup} \{|f(x)| : x \in U\} = \inf \left\{ \sup_{x \in U \setminus N} |f(x)| : N \subseteq U \quad \textit{is locally null} \quad \right\}.$$

We say that a functions $f \in L^\infty(G)$ vanishes at infinity if for every $\epsilon > 0$, there exists a compact subset $K \subseteq G$ such that $\operatorname{ess\,sup} \{|f(x)| : x \in G \setminus K\} < \epsilon$. The space of all $f \in L^\infty(G)$ that vanish at infinity will be denoted by $L_0^\infty(G)$. Recall that $C_b(G)$, $L_0^\infty(G)$ and $C_0(G)$ are all Banach subalgebras of $L^\infty(G)$.

(f) *A* bounded left uniformly continuous function *is a function $f \in C_b(G)$ for which the map $x \longmapsto l_x f$, from G to $C_b(G)$, is continuous. The set of all bounded left uniformly continuous functions is denoted by $LUC(G)$. The space $LUC(G)$ is a closed subalgebra of $C_b(G)$ and therefore it is itself a Banach algebra.*

(g) *A* bounded right uniformly continuous function *is a function* $f \in C_b(G)$ *for which the map* $x \longmapsto r_x f$*, from* G *to* $C_b(G)$*, is continuous. The set of all bounded right uniformly continuous functions is denoted by* $RUC(G)$*. It is easy to verify that* $RUC(G)$ *is a closed subalgebra of* $C_b(G)$ *and therefore it is a Banach algebra itself. One can see that* $C_0(G) \subseteq LUC(G) \cap RUC(G)$*. The space* $LUC(G) \cap RUC(G)$ *is called the space of* uniformly continuous functions *and denoted by* $UC(G)$*.*

(h) *A function* $f \in C_b(G)$ *is called* almost periodic *if the set* $\{l_x f : x \in G\}$ *has a compact closure in the norm-topology of* $C_b(G)$*. The space of all almost periodic functions is denoted by* $AP(G)$*.*

(i) *A function* $f \in C_b(G)$ *is called* weakly almost periodic *if the set* $\{l_x f : x \in G\}$ *has a weakly compact closure in the weak topology of* $C_b(G)$*. The space of all weakly almost periodic functions is denoted by* $WAP(G)$*. Again it can be verified that* $WAP(G)$ *is a closed subalgebra of* $C_b(G)$ *and therefore it is a Banach algebra itself. It can be also verified that* $C_0(G) \subseteq AP(G) \subseteq WAP(G) \subseteq LUC(G)$*. Moreover if* G *is compact then* $C_0(G) = AP(G) = WAP(G) = LUC(G) = RUC(G) = C_b(G)$*.*

(j) *A closed subspace* X *of* $C_b(G)$ *is called* left introverted *if* $l_x X \subseteq X$ *and the function* nf *defined by* $nf(t) = \langle n, l_t f \rangle$*,* $t \in G$*, is in* X *for all* $f \in X$ *and* $n \in X^*$*. In this case for* $m, n \in X^*$*, we define* $mn \in X^*$ *by* $\langle mn, f \rangle = \langle m, nf \rangle$ *where*

$f \in X$. It is easy to check that this product makes X^* into a Banach algebra. It can be proved that $C_0(G)$, $LUC(G)$ and $WAP(G)$ are left introverted.

For a real number $p \geq 1$, the space of all extended complex-valued functions on G, measurable with respect to m and satisfying $\int_G |f(x)|^p \, dm < \infty$ is denoted by $L^p(G)$. We define

$$\|f\|_p = \left(\int_G |f(x)|^p \, dm \right)^{1/p}.$$

It is a classical result of functional analysis that when $p \in [1, \infty)$, $L^p(G)$, with respect to the natural addition and scalar multiplication of functions, is a Banach space. We have the following classic result about the dual of these spaces where we suppose that $\frac{1}{\infty} = 0$:

Proposition 1.39. *Let G be a locally compact group. Then $L^p(G)^* = L^q(G)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \in [1, \infty)$*

Proof. See [10] page 375. □

The convolution of two function f and g in $L^1(G)$ is denoted by $f \star g$ and is defined by

$$f \star g(s) = \int_G f(t) g(t^{-1}s) dm(t).$$

This product makes $L^1(G)$ a Banach algebra. The Banach algebra $L^1(G)$ is called the *group algebra* of G.

Proposition 1.40. *Let G be a locally compact group. Then $L^1(G)$ has a two-sided approximate identity bounded by* 1.

Proof. See Proposition 2.42 of [17]. □

Corollary 1.41. *Let G be a locally compact group and suppose that $f \in L^1(G)$. Then there are the functions $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$.*

Proof. This is a special case of a more general result that every Banach algebra with a bounded approximate identity factorizes (see Proposition 1.5). □

Corollary 1.42. *Let G be a locally compact group. Then $L^1(G)$ is an essential Banach $L^1(G)$-bimodule.*

We also have the following equivalent conditions for [IN] and [SIN] groups.

Proposition 1.43. *Let G be a locally compact group.*

(a) G is an [IN] group if and only if $L^1(G)$ has a non-trivial center.

(b) G is a [SIN] group if and only if $L^1(G)$ has a bounded approximate identity consisting of central functions.

Proof. See the Proposition in [44]. □

Proposition 1.44. *Let G be a locally compact group. Then $L^1(G)$ is strongly irregular.*

Proof. See Theorem 1 of [37]. □

We also have the following propositions.

Proposition 1.45. *Let G be a locally compact group. Suppose that G is compact and $T : L^1(G) \longrightarrow L^1(G)$ is a bounded left multiplier (resp. right multiplier). Then the following conditions are equivalent:*

(a) T is compact;

(b) T is weakly compact;

(c) There exists $f \in L^1(G)$ such that $T(g) = f \star g$ (resp. $T(g) = g \star f$) for each $g \in L^1(G)$.

Proof. See Theorem 4 in [1]. □

Corollary 1.46. *Let G be a locally compact group. Then the following conditions are equivalent:*

(a) G is compact;

(b) There exists a non-zero weakly compact left multiplier (resp. right multiplier) on $L^1(G)$.

Proof. If G is compact, part (c) of previous proposition provides us with a non-zero weakly compact left multiplier. For the proof of converse, see Theorem 1 in [48]. □

The Banach space $L^\infty(G)$ as the dual space of $L^1(G)$ is an $L^1(G)$-bimodule. Note that for $f \in L^\infty(G)$ and $\phi, \varphi \in L^1(G)$,

$$\begin{aligned}\int_G (\phi \cdot f)(x)\varphi(x)\,dx &= \langle \phi \cdot f, \varphi\rangle = \langle f, \varphi \star \phi\rangle = \int_G f(t)(\varphi \star \phi)(t)dt \\ &= \int_G \int_G f(t)\varphi(x)\phi(x^{-1}t)dxdt = \int_G \int_G f(xt)\phi(t)\varphi(x)dtdx.\end{aligned}$$

Therefore $(\phi \cdot f)(x) = \int_G f(xt)\phi(t)dt$ for all $x \in G$. One can similarly show that $(f \cdot \phi)(x) = \int_G f(tx)\phi(t)dt$, for $x \in G$. We have the following proposition:

Proposition 1.47. *Let G be a locally compact group. Then*

(a) $L^\infty(G) \cdot L^1(G) = LUC(G)$.

(b) $L^1(G) \cdot L^\infty(G) = RUC(G)$.

(c) $L^1(G) \cdot C_0(G) = C_0(G) \cdot L^1(G) = C_0(G)$.

(d) $L_0^\infty(G) \cdot L^1(G) = L^1(G) \cdot L_0^\infty(G) = C_0(G)$.

Proof. For the proof of part (a), (b) and (c), see Theorem 32.44 and 32.45 of [28]. For the proof of part (d), note that $C_0(G) \subseteq L^1(G) \cdot C_0(G) \subseteq L^1(G) \cdot L_0^\infty(G)$. For the converse, first let $f \in L^1(G)$ and $\phi \in L_0^\infty(G)$ have compact supports. Then $f \cdot \phi$ is a continuous function with compact support and hence it is in $C_0(G)$. Now suppose that $f \in L^1(G)$ and $\phi \in L_0^\infty(G)$. Since elements with compact support of $L^1(G)$ are dense in $L^1(G)$ and elements with compact support of $L_0^\infty(G)$ are dense in $L_0^\infty(G)$ there exist sequences $(f_n) \subset L^1(G)$ and $(\phi_n) \subset L_0^\infty(G)$ such that $f_n \to f$

in $L^1(G)$ and $\phi_n \to \phi$ in $L_0^\infty(G)$. Then by joint continuity of module actions and the fact that $C_0(G)$ is closed in $L_0^\infty(G)$ we have $f \cdot \phi = \lim_n f_n \cdot \phi_n \in C_0(G)$. Hence $C_0(G) = L^1(G) \cdot L_0^\infty(G)$. We can similarly prove that $L_0^\infty(G) \cdot L^1(G) = C_0(G)$. $\square$

We have already introduced the Banach space $M(X)$ where X is a topological space. In this thesis we are concerned with the case when our topological space is a locally compact group. Here we review some classic results about $M(G)$ where G is a locally compact group. As an application of Riesz representation theorem we have:

Proposition 1.48. *Let G be a locally compact group. Then $C_0(G)^* = M(G)$.*

Proof. See [10] page 383. $\square$

For $\mu, \nu \in M(G)$ and $f \in C_0(G)$, we define

$$\langle \mu \star \nu, f \rangle = \int_G \int_G f(st) d\mu(s) d\nu(t).$$

We already know that $M(G)$ is a Banach space. The space $M(G)$ endowed with the above convolution becomes a unital Banach algebra with identity δ_e. A measure $\mu \in M(G)$ is said to be *discrete* if there is a countable subset E of G such that $|\mu|(E^c) = 0$. The measure $\mu \in M(G)$ is said to be *continuous* if $\mu(\{x\}) = 0$ for all $x \in G$. A continuous measure μ is said to be *singular* if μ is singular with respect to the Haar measure. A continuous measure μ is said to be *absolutely continuous* if μ is absolutely continuous with respect to the Haar measure. The set of discrete, continuous, singular and absolutely continuous measures in $M(G)$ are denoted by $M_d(G), M_c(G), M_s(G)$ and $M_a(G)$ respectively.

Proposition 1.49. *Let G be a locally compact group and m be the Haar measure on G.*

(a) Every discrete measure can be represented uniquely as $\sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ where $\alpha_i \in \mathbb{C}$, $x_i \in G$ and such that $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. Moreover the set of discrete measures $M_d(G)$ is a closed subalgebra of $M(G)$ and G is discrete if and only if $M_d(G) = M(G) = M_a(G)$ and $M_s(G) = \{0\}$.

(b) The set $M_s(G)$ is a closed linear subspace of $M(G)$.

(c) The sets $M_a(G)$ and $M_c(G)$ are closed two-sided ideals in $M(G)$.

(d) There is an isometric isomorphism $i : L^1(G) \longrightarrow M_a(G)$, $f \longmapsto \mu_f$ such that $d\mu_f = f dm$.

(e) For every $\mu \in M(G)$, there is a net (μ_i) of linear combination of point-masses such that for every $f \in L^1(G)$, $(\mu_i \star f)$ and $(f \star \mu_i)$ converge to $\mu \star f$ and $f \star \mu$ respectively with respect to the norm-topology in $L^1(G)$.

Proof. For parts $(a), (b)$ and (c), see Theorem. 19.15 of [28]. For part (d), see Theorem 19.18 of [28]. $\square$

We say that a net (μ_i) in $M(G)$ converges to $\mu \in M(G)$ with respect to *strong operator* topology if for every $f \in L^1(G)$, $(\mu_i \star f)$ and $(f \star \mu_i)$ converge to $\mu \star f$ and $f \star \mu$ respectively with respect to the norm-topology in $L^1(G)$. Therefore part (e) of Proposition 1.49 implies that the linear combination of point-masses are dense in $M(G)$ with respect to the strong operator topology on $M(G)$.

Proposition 1.50. *Let G be a locally compact group. Then $L^1(G)^{**}$ is a two-sided ideal of $M(G)^{**}$, where both of the algebras are endowed with the first Arens product.*

Proof. Use Proposition 1.19 and parts (c), (d) of Proposition 1.49 □

We also have the following propositions:

Proposition 1.51. *Let G be a locally compact group. Then*

(a) The algebra $M(G)$ can be decomposed as $M(G) = M_d(G) \oplus M_a(G) \oplus M_s(G)$.

(b) Suppose that $\mu \in M(G)$ is in the form $\mu = \mu_d + \mu_a + \mu_s$ where $\mu_d \in M_d(G)$, $\mu_a \in M_a(G)$ and $\mu_s \in M_s(G)$. Then $|\mu| = |\mu_d| + |\mu_a| + |\mu_s|$ and $\|\mu\| = \|\mu_d\| + \|\mu_a\| + \|\mu_s\|$.

Proof. See Theorem 19.20 of [28] □

Proposition 1.52. *Let G be a locally compact group. Then the following conditions are equivalent:*

(a) G is abelian;

(b) $L^1(G)$ *is abelian;*

(c) $M(G)$ *is abelian.*

Proof. See Theorem 20.24 of [28]. $\square$

Let $Z(\mathcal{A})$ denote the center of an algebra $\mathcal{A}$. The following lemma will be useful in the future.

Lemma 1.53. *Let G be a locally compact group. Then $Z(M(G)) \cap L^1(G) = Z(L^1(G))$.*

Proof. Obviously $Z(M(G)) \cap L^1(G) \subseteq Z(L^1(G))$. Let f be an element of $Z(L^1(G))$ and (e_i) be an approximate identity for $L^1(G)$. For an arbitrary $\mu \in M(G)$, we can write

$$\mu \star f = \lim_i (\mu \star (e_i \star f)) = \lim_i ((\mu \star e_i) \star f) = \lim_i (f \star (\mu \star e_i)) = \lim_i ((f \star e_i) \star \mu) = f \star \mu.$$

This shows that $f \in Z(M(G)) \cap L^1(G)$. Therefore $Z(L^1(G)) \subseteq Z(M(G)) \cap L^1(G)$. $\square$

The following propositions will be useful in the next chapter:

Proposition 1.54. *Let G be a locally compact group. Then $M(G)$ is strongly irregular.*

Proof. See Theorem 3 of [40]. $\square$

Proposition 1.55. *Let X be a left introverted subspace of $C_b(G)$ containing $C_0(G)$. Then $\mu \longmapsto \Gamma_\mu$ from $M(G)$ into X^*, where $\Gamma_\mu(f) = \int_G f d\mu$ $(f \in X)$, defines an* *isometric embedding of $M(G)$ into X^*. Moreover $C_0(G)^\perp$ is a two-sided closed ideal of X^* and $X^* = M(G) \oplus C_0(G)^\perp$. Furthermore if $\mu \in M(G)$ and $n \in C_0(G)^\perp$, then $\|n + \mu\| = \|n\| + \|\mu\|$.*

Proof. See Theorem 1.5.5 of [16]. $\square$

Proposition 1.56. *Let G be a locally compact group. Then*

$$WAP(G)^{**} = L^\infty(G) \oplus_\infty M_d(G)^* \oplus_\infty M_s(G)^* \oplus_\infty (C_0(G)^\perp)^*.$$

Proof. Recall that $WAP(G)$ is a left introverted subspace of $C_b(G)$ that contains $C_0(G)$. From Proposition 1.55 we have that $WAP(G)^* = M(G) \oplus_1 C_0(G)^\perp$. Therefore $WAP(G)^{**} = M(G)^* \oplus_\infty (C_0(G)^\perp)^*$. Using part (a) of Proposition 1.51, part (d) of Proposition 1.49 and Proposition 1.39, we can conclude the final statement. $\square$

Chapter 2

ARENS PRODUCTS ON BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

Let G be a locally compact group. In this chapter we will discuss some properties of $M(G)^{**}$, $L^1(G)^{**}$ and some other Banach algebras that are either the dual of a subspace of $L^\infty(G)$ or $M(G)^*$. Many of the techniques we have used are similar to the ones used in [19], [29] and [39]. We would also like to thank Dr. Nico Spronk for bringing to our attention that there is another approach to model the spaces $M(G)^*_0$, $M(G)^*_{\sigma 0}$ (see [30], [50], [55]).

2.1 Some Banach subalgebras of $M(G)^{**}$ and $L^1(G)^{**}$ and their properties

We start off with some definitions:

Definition 2.1. *We let $M(G)^*_0$ (resp. $M(G)^*_{\sigma 0}$) to be the space of all functionals $\lambda \in M(G)^*$ with the property that for every $\epsilon > 0$, there exists a compact (resp. σ-compact) set K such that if $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$ then $|\langle \lambda, \mu \rangle| < \epsilon$.*

Definition 2.2. *We say that $\lambda \in M(G)^*$ has a* compact *(resp.σ-*compact*)* support *if there exists a compact (resp. σ-compact) set K such that, for $\mu \in M(G)$, if $|\mu|(K) = 0$ then $\langle \lambda, \mu \rangle = 0$.*

In this thesis we denote the set of all elements of $M(G)^*$ with compact (resp. σ-compact) support by $M(G)^*_c$ (resp. $M(G)^*_{\sigma c}$). Similarly, we have the following definitions:

Definition 2.3. *We let $L^\infty_0(G)$ (resp. $L^\infty_{\sigma 0}(G)$) to be the space of all functions $g \in L^\infty(G)$ with the property that for each $\epsilon > 0$ there exists a compact (resp. σ-compact) set K such that if $f \in L^1(G)$, $|f|\,|_K = 0$ and $\|f\| = 1$ then $\left|\int_G f(x)g(x)dm(x)\right| < \epsilon$.*

Definition 2.4. *We say that $g \in L^\infty(G)$ has a* compact (resp. σ-compact) support *if there exists a compact (resp. σ-compact) set K such that if $f \in L^1(G)$ and $|f|\,|_K = 0$ then $\int_G f(x)g(x)dm(x) = 0$.*

We denote the space of all functions in $L^\infty(G)$ with compact (resp. σ-compact) support by $L^\infty_c(G)$ (resp. $L^\infty_{\sigma c}(G)$).

Remark 2.5. *Note that the way we defined $L^\infty_0(G)$ in Definition 2.3 is equivalent to the one we provided in part (e) of Definition 1.38. One could also have defined $L^\infty_c(G)$ (resp. $L^\infty_{\sigma c}(G)$) in a similar fashion as we defined $C_c(G)$ in part (b) of Definition 1.38. This way of defining $L^\infty_c(G)$ (resp. $L^\infty_{\sigma c}(G)$) would be equivalent to the one we gave in Definition 2.4.*

We have the following proposition:

Proposition 2.6. *Let G be a locally compact group. Then $\overline{M(G)^*_c}$ (resp. $\overline{M(G)^*_{\sigma c}}$) is $M(G)^*_0$ (resp. $M(G)^*_{\sigma 0}$) when we take the closure with respect to the norm-topology* *in* $M(G)^*$.

Proof. We prove the proposition for $M(G)_0^*$. The proof for $M(G)_{\sigma 0}^*$ is similar. Take $\lambda \in M(G)_0^*$ and $\epsilon > 0$. Then there is a compact set K such that if $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$ then $|\langle \lambda, \mu \rangle| < \epsilon$. For an arbitrary $\nu \in M(G)$, we define ν_K and ν_{K^c} to be

$$\nu_K(E) = \nu(E \cap K), \qquad \nu_{K^c}(E) = \nu(E \cap K^c), \ \ (E \in \mathcal{B}_G). \tag{2.1}$$

Obviously $\nu_K, \nu_{K^c} \in M(G)$, $|\nu_K|(K^c) = 0$, $|\nu_{K^c}|(K) = 0$ and $\nu = \nu_K + \nu_{K^c}$. We define $\left\langle \lambda', \nu \right\rangle = \langle \lambda, \nu_K \rangle$ for $\nu \in M(G)$. Obviously $\lambda' \in M(G)^*$ has the compact support K and for an arbitrary $\nu \in M(G)$ with $\|\nu\| = 1$, we have

$$\left| \left\langle \lambda' - \lambda, \nu \right\rangle \right| = \left| \left\langle \lambda' - \lambda, \nu_K \right\rangle + \left\langle \lambda' - \lambda, \nu_{K^c} \right\rangle \right| = |0 + \langle \lambda, \nu_{K^c} \rangle| < \epsilon.$$

Conversely, suppose that $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence in $M(G)_c^*$ that converges to λ. Take $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that $\|\lambda_N - \lambda\| < \epsilon$. Because λ_N has compact support there is a compact subset K such that if $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$ then $\langle \lambda_N, \mu \rangle = 0$. This implies that for each $\epsilon > 0$ there exists a compact set K such that if $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$ then $|\langle \lambda, \mu \rangle| \leq \epsilon$. Therefore we have $\lambda \in M(G)_0^*$. □

Similarly we have the following proposition for $L_0^\infty(G)$ and $L_{\sigma 0}^\infty(G)$:

Proposition 2.7. *Let G be a locally compact group. Then $\overline{L_c^\infty(G)}$ (resp. $\overline{L_{\sigma c}^\infty(G)}$) is $L_0^\infty(G)$ (resp. $L_{\sigma 0}^\infty(G)$) when we take the closure with respect to the norm-topology in $L^\infty(G)$.*

Proof. The proof for $L_0^\infty(G)$ has been given in [39] and for $L_{\sigma 0}^\infty(G)$ is similar to that of Proposition 2.6. □

Remark 2.8. *We know that $M(G)^* = C_0(G)^{**}$ is a Banach algebra with respect to both Arens products. Note that since $C_0(G)$ is a commutative C^*-algebra, by Theorem 1.25, the Banach algebra $M(G)^*$ is in fact a commutative C^*-algebra. The involution of an element $\lambda \in M(G)^*$, λ^*, is defined by $\langle \lambda^*, \nu \rangle = \overline{\langle \lambda, \overline{\nu} \rangle}$, where $\overline{\nu}(E) = \overline{\nu(E)}$ for $\nu \in M(G)$ and $E \in \mathcal{B}_G$. Also note that, by Theorem 1.24, both Arens products coincide. Therefore we simply write $\lambda_1\lambda_2$ for both Arens products of $\lambda_1, \lambda_2 \in M(G)^*$.*

We have the following proposition:

Proposition 2.9. *Let $\lambda_1, \lambda_2 \in M(G)^*$ have compact (resp. σ-compact) supports. Then $\lambda_1\lambda_2$ has a compact (resp. σ-compact) support too.*

Proof. Assume that λ_1 and λ_2 have compact supports. Therefore there exists a compact set K_1 such that, for $\nu \in M(G)$, if $|\nu|(K_1) = 0$ then $\langle \lambda_1, \nu \rangle = 0$. Similarly, there is a compact set K_2 such that, for $\eta \in M(G)$, if $|\eta|(K_2) = 0$ then $\langle \lambda_2, \eta \rangle = 0$. Take $\mu \in M(G)$ such that $|\mu|(K_1 \cap K_2) = 0$. We aim to prove that $\langle \lambda_1\lambda_2, \mu \rangle = 0$.

Since $\langle \lambda_1\lambda_2, \mu\rangle = \langle \lambda_1, \lambda_2 \cdot \mu\rangle$, it is enough to prove that the intersection of the support of $\lambda_2 \cdot \mu$ and K_1 is empty. To show this, let $W_1 \subseteq K_1$ and we will prove that $\lambda_2 \cdot \mu(W_1) = \langle \lambda_2, \mu \cdot 1_{W_1}\rangle = 0$. This is true if we can prove that the intersection of the support of $\mu \cdot 1_{W_1}$ and K_2 is empty. To prove the latter statement, let $W_2 \subseteq K_2$. Then $\mu \cdot 1_{W_1}(W_2) = \mu(W_1 \cap W_2) = 0$, because $|\mu|(K_1 \cap K_2) = 0$. Note that $K_1 \cap K_2$ is compact and we just showed that, for $\mu \in M(G)$ with $|\mu|(K_1 \cap K_2) = 0$, we have $\langle \lambda_1\lambda_2, \mu\rangle = 0$. Therefore $\lambda_1\lambda_2$ has compact support. A similar argument is valid if we replace compact everywhere by σ-compact. □

Similarly we have the following proposition:

Proposition 2.10. *Let $g_1, g_2 \in L^\infty(G)$ have compact (resp. σ-compact) supports. Then g_1g_2 has compact (resp. σ-compact) support too.*

Proof. The proof is similar to that of Proposition 2.9. □

We have the following proposition:

Proposition 2.11. *Let G be a locally compact group. Then $M(G)^*_0$ (resp. $M(G)^*_{\sigma 0}$) is a commutative C^*-subalgebra of $M(G)^*$. Moreover $M(G)^*_0$ is a C^*-subalgebra of $M(G)^*_{\sigma 0}$.*

Proof. We prove that $M(G)^*_0$ is a C*-algebra. A similar argument can be used to show that $M(G)^*_{\sigma 0}$ is a C*-algebra too. First of all, by Proposition 2.6, $M(G)^*_0$ is closed with respect to the norm-topology of $M(G)^*$. Therefore $M(G)^*_0$ is a Banach subspace of $M(G)^*$. The fact that $M(G)^*_0$ is closed under multiplication follows from Propositions 2.6 and Proposition 2.9. To show that the Banach algebra $M(G)^*_0$ is closed under involution, take $\lambda \in M(G)^*_0$. For $\epsilon > 0$ there exists a compact set K such that if $\mu \in M(G)$, $|\mu|(K) = 0$ and $\|\mu\| = 1$, then $|\langle \lambda, \mu \rangle| \le \epsilon$. Therefore $\lambda^* \in M(G)^*_0$. Because, given $\epsilon > 0$, there exists the compact set K such that if $\nu \in M(G)$, $\|\nu\| = 1$ and $|\nu|(K) = 0$ we have:

$$|\langle \lambda^*, \nu \rangle| = \left| \overline{\langle \lambda, \overline{\nu} \rangle} \right| = |\langle \lambda, \overline{\nu} \rangle| \le \epsilon.$$

Note that $\|\nu\| = 1$ and $|\nu|(K) = 0$ implies that $\|\overline{\nu}\| = 1$ and $|\overline{\nu}|(K) = 0$. To prove that $M(G)^*_0$ is a C*-subalgebra of $M(G)^*_{\sigma 0}$, note that every compact set is σ-compact and use the definitions of $M(G)^*_0$ and $M(G)^*_{\sigma 0}$. □

As we mentioned in Definition 1.38, $L^\infty(G) = L^1(G)^*$ is a Banach algebra. The conjugate of a function $f \in L^\infty(G)$, $\overline{f}$, is an involution on $L^\infty(G)$ that makes it into a C*-algebra. We have the following proposition:

Proposition 2.12. *Let G be a locally compact group. Then $L^\infty_0(G)$ (resp. $L^\infty_{\sigma 0}(G)$) is a commutative C*-subalgebra of $L^\infty(G)$. Moreover $L^\infty_0(G)$ is a commutative C*-subalgebra of $L^\infty_{\sigma 0}(G)$.*

Proof. The proof is similar to that of Proposition 2.11. □

Similar to Definition 2.1, one can define $M_d(G)^*_0$, $M_d(G)^*_{\sigma 0}$, $M_s(G)^*_0$ and $M_s(G)^*_{\sigma 0}$. We have the following proposition:

Proposition 2.13. *Let G be a locally compact group. Then*

(a) We have $M(G)^* = M_d(G)^* \oplus_\infty L^\infty(G) \oplus_\infty M_s(G)^*$.

(b) We have $M(G)^*_0 = M_d(G)^*_0 \oplus_\infty L^\infty_0(G) \oplus_\infty M_s(G)^*_0$.

(c) We have $M(G)^*_{\sigma 0} = M_d(G)^*_{\sigma 0} \oplus_\infty L^\infty_{\sigma 0}(G) \oplus_\infty M_s(G)^*_{\sigma 0}$.

(d) The Banach algebra $LUC(G)$ can be isometrically embedded into $M(G)^*$.

(e) The Banach algebra $C_0(G)$ can be isometrically embedded into C^*-subalgebra of $M(G)^*_0$.

(f) The space $M(G)^*$ is a Banach $L^1(G)$-bimodule and $M(G)^* \cdot L^1(G) = LUC(G)$.

(g) The space $M(G)^*_0$ is a Banach $L^1(G)$-bimodule. Moreover, we have

$$M(G)^*_0 \cdot L^1(G) = L^1(G) \cdot M(G)^*_0 = C_0(G).$$

(h) The space $M(G)^*_{\sigma 0}$ is a Banach $L^1(G)$-bimodule.

(i) $(M(G)^*_0)^* = (M_d(G)^*_0)^* \oplus_1 L^\infty_0(G)^* \oplus_1 (M_s(G)^*_0)^*$.

(j) $(M(G)^*_{\sigma 0})^* = (M_d(G)^*_{\sigma 0})^* \oplus_1 L^\infty_{\sigma 0}(G)^* \oplus_1 (M_s(G)^*_{\sigma 0})^*$.

Proof. (a) This follows from Proposition 1.51 part (a).

(b) This follows from part (a), Definition 2.1 and part (b) of Proposition 1.51.

(c) This follows from part (a), Definition 2.1 and part (b) of Proposition 1.51.

(d) Given $f \in LUC(G)$, we can define a functional Λ_f on $M(G)$ by

$$\Lambda_f(\mu) = \int_G f d\mu \ \ (\mu \in M(G)).$$

We define the linear mapping $\Lambda : LUC(G) \longrightarrow M(G)^*$, $f \longmapsto \Lambda_f$ for every $f \in LUC(G)$. For every $\mu \in M(G)$, $f \in LUC(G)$, we have

$$|\Lambda_f(\mu)| \leq \left| \int_G f d\mu \right| \leq \int_G |f| \, d\, |\mu| \leq \|f\| \, \|\mu\| \, .$$

Therefore $\|\Lambda_f\| \leq \|f\|$. On the other hand, given $f \in LUC(G)$, for every $\epsilon > 0$ there exists $h \in L^1(G)$ with $\|h\|_1 = 1$ such that $|\langle f, h \rangle| \geq \|f\| - \epsilon$. Therefore, by using part (d) of Proposition 1.49, we can write

$$\|\Lambda_f\| \geq |\Lambda_f(h)| = \left| \int_G f(x) dh \right| = \left| \int_G f(x)h(x)dm(x) \right| = |\langle f, h \rangle| \geq \|f\| - \epsilon.$$

Therefore $\|\Lambda_f\| \geq \|f\|$. Hence Λ is an isometry. To prove that Λ preserves the multiplication, note that for each $\mu \in M(G)$ and $h, k \in C_0(G)$,

$$\int_G k(x) d(\mu \cdot h) = \langle \mu \cdot h, k \rangle = \langle \mu, hk \rangle = \int_G k(x)h(x)d\mu.$$

Therefore for each $\mu \in M(G)$, $h \in C_0(G)$, we have $d(\mu \cdot h) = hd\mu$. This implies that $d(\Lambda_g \cdot \mu) = gd\mu$ for each $g \in LUC(G)$ and $\mu \in M(G)$ because for every

$h \in C_0(G)$, we have

$$\int_G h(x)d(\Lambda_g \cdot \mu) = \langle \Lambda_g \cdot \mu, h \rangle = \langle \Lambda_g, \mu \cdot h \rangle = \int_G g(x)d(\mu \cdot h) = \int_G g(x)h(x)d\mu.$$

Hence for each $f, g \in LUC(G)$ and $\mu \in M(G)$, we can write

$$\langle \Lambda_{fg}, \mu \rangle = \int_G f(x)g(x)d\mu = \int_G f(x)d(\Lambda_g \cdot \mu) = \langle \Lambda_f, \Lambda_g \cdot \mu \rangle = \langle \Lambda_f \Lambda_g, \mu \rangle .$$

From the last equation we conclude that Λ is multiplicative.

(e) We know that $C_0(G)$ can be embedded isometrically as a Banach algebra into its second dual $C_0(G)^{**} = M(G)^*$. We prove that in fact $C_0(G) \subseteq M(G)_0^*$. Let $\epsilon > 0$ and $f \in C_0(G)$. Therefore there exists a compact set K such that $|f(x)| < \epsilon$ for $x \in K^c$. If $\mu \in M(G)$ such that $\|\mu\| = 1$ and $|\mu|(K) = 0$, then $\langle f, \mu \rangle = \int_G fd(\mu) = \int_{K^c} fd(\mu) < \epsilon$. Therefore $f \in M(G)_0^*$.

(f) We know that $M(G)$ is a Banach $L^1(G)$-bimodule and therefore its dual $M(G)^*$ is also a Banach $L^1(G)$-bimodule. If $\lambda \in M(G)^*$ and $f \in L^1(G)$, using part (a), we can write $\lambda = \lambda_d + g + \lambda_s$ where λ_d, g, λ_s are in $M_d(G)^*, L^\infty(G)$ and $M_s(G)^*$ respectively. From part (c) of Proposition 1.49, for each $\mu \in M(G)$ we have $f\mu \in L^1(G)$ and therefore we can write

$$\langle \lambda \cdot f, \mu \rangle = \langle \lambda_d \cdot f + g \cdot f + \lambda_s \cdot f, \mu \rangle = \langle \lambda_d, f\mu \rangle + \langle g \cdot f, \mu \rangle + \langle \lambda_s, f\mu \rangle = \langle g \cdot f, \mu \rangle .$$

Hence $\lambda \cdot f \in LUC(G)$. The fact that $LUC(G) \subseteq M(G)^* \cdot L^1(G)$ is immediate from part (a) above and part (a) of Proposition 1.47.

(g) This is a similar argument. Use part (b) and note that, by Proposition 1.47, $L_0^\infty(G) \cdot L^1(G) = L^1(G) \cdot L_0^\infty(G) = C_0(G)$.

(h) This can be easily derived from part (c).

(i) This is immediate from part (b).

(j) This can be easily derived from (c).

□

Lemma 2.14. *Suppose that (h_i) is a net in $L_{\sigma 0}^\infty(G)^*$. Then (h_i) converges to $h \in L_{\sigma 0}^\infty(G)^*$ with respect to the weak-star topology in $L_{\sigma 0}^\infty(G)^*$ if and only if (h_i) converges to h with respect to the weak-star topology in $(M(G)_{\sigma 0}^*)^*$.*

Proof. By part (c) of Proposition 2.13 an arbitrary $\lambda \in M(G)_{\sigma 0}^*$ can be decomposed as $\lambda = \lambda_d + \lambda_a + \lambda_s$, where $\lambda_d, \lambda_a, \lambda_s$ are in $M_d(G)_{\sigma 0}^*, L_{\sigma 0}^\infty(G), M_s(G)_{\sigma 0}^*$ respectively. Note that $\langle h_i, \lambda_d + \lambda_s \rangle = \langle h, \lambda_d + \lambda_s \rangle = 0$. Therefore the net (h_i) converges to h with respect to the weak-star topology in $L_{\sigma 0}^\infty(G)^*$ if and only if it converges to h with respect to the weak-star topology in $(M(G)_{\sigma 0}^*)^*$. □

Remark 2.15. *The Lemma 2.14 remains valid if we substitute $L_{\sigma 0}^\infty(G)^*$ and $(M(G)_{\sigma 0}^*)^*$ by $L^1(G)^{**}$ and $M(G)^{**}$ or $L_0^\infty(G)^*$ and $(M(G)_0^*)^*$.*

In [39] Lemma 2.2, the authors introduce the homomorphism $\pi : L^1(G)^{**} \longrightarrow LUC(G)^*$ that sends every element in $L^1(G)^{**}$ to its restriction on $LUC(G)$. Viewed differently, π is the adjoint map to the inclusion map $\varphi : LUC(G) \longrightarrow L^\infty(G)$. Hence it is weak-star continuous. In the proposition below we extend the definition of π to $M(G)^{**}$. For $F \in M(G)^{**}$, we consider $\overline{\pi}(F)$ to be the restriction of F to $LUC(G)$ and we state similar results.

Proposition 2.16. *Let $\overline{\pi} : M(G)^{**} \longrightarrow LUC(G)^*$ be as above. Then*

(a) The map $\overline{\pi}$ is identity on $M(G)$.

(b) The map $\overline{\pi}$ is weak-star continuous.

(c) The set $\mathrm{Ker}(\overline{\pi})$ is weak-star closed.

(d) The map $\overline{\pi}$ is a homomorphism.

(e) The restriction of $\overline{\pi}$ to $L^1(G)^{**}$ is the same as π.

Proof. (a) Note that $\overline{\pi} : M(G)^{**} \longrightarrow LUC(G)^*$ is the adjoint of the map $\Lambda : LUC(G) \longrightarrow M(G)^*$ which we introduced in the proof of Proposition 2.13 part (d). We also note that $LUC(G)$ is a left introverted subspace of $C_b(G)$ containing $C_0(G)$. Therefore from Proposition 1.55, the mapping $\mu \longmapsto \Gamma_\mu$ from $M(G)$ into $LUC(G)^*$, where $\Gamma_\mu(f) = \int_G f d\mu$ $(f \in LUC(G))$, defines an isometric embedding of $M(G)$ into $LUC(G)^*$. For $\mu \in M(G)$ and $f \in LUC(G)$, we have

$$\langle \overline{\pi}(\mu), f \rangle = \langle \mu, \Lambda_f \rangle = \int_G f d(\mu) = \langle \Gamma_\mu, f \rangle .$$

(b) In the proof of Proposition 2.13 part (d), we showed that $\Lambda : LUC(G) \longrightarrow M(G)^*$ is an isometric isomorphism. Therefore $\overline{\pi}$, as the adjoint of Λ, is weak-star continuous.

(c) The kernel of a weak-star continuous map is weak-star closed.

(d) Let $F, H \in M(G)^{**}$. Suppose that the nets (μ_i) and (ν_j) in $M(G)$ converge to F and H, respectively, in weak-star topology of $M(G)^{**}$. Then using the fact that $\overline{\pi}$ is identity on $M(G)$, we have

$$\begin{aligned}\overline{\pi}(FH) &= \overline{\pi}\left(\text{wk*}\lim_i \text{wk*}\lim_j \mu_i\nu_j\right) = \text{wk*}\lim_i \text{wk*}\lim_j \overline{\pi}\left(\mu_i\nu_j\right) \\ &= \text{wk*}\lim_i \text{wk*}\lim_j \overline{\pi}(\mu_i)\overline{\pi}(\nu_j) = \overline{\pi}\left(F\right)\overline{\pi}\left(H\right).\end{aligned}$$

(e) Let (f_i) be a net in $L^1(G)$ that converges to F with respect to the weak-star topology in $L^1(G)^{**}$. Then by Remark 2.15, the net (f_i) converges to F with respect to the weak-star topology in $M(G)^{**}$ as well. Therefore for each $h \in LUC(G)$ we can write

$$\langle \overline{\pi}(F), h\rangle = \lim_i \langle \overline{\pi}(f_i), h\rangle = \lim_i \langle f_i, h\rangle = \lim_i \langle \pi(f_i), h\rangle = \langle \pi(F), h\rangle .$$

$\square$

Definition 2.17. *Suppose that $F \in L^1(G)^{**}$ and $H \in LUC(G)^*$. For each $\lambda \in M(G)^*$, we define $FH \in M(G)^{**}$ as $\langle FH, \lambda\rangle = \langle F, H\lambda\rangle$ where $H\lambda \in L^\infty(G)$ is defined as $\langle H\lambda, f\rangle = \langle H, \lambda \cdot f\rangle$ for every $f \in L^1(G)$.*

Remark 2.18. *This is a generalization of the definition which has been given in equation (2.1) of [39]. One may ask why we did not attempt to extend the above definition by taking $F \in M(G)^{**}$ and $f \in M(G)$. That is because in that case there would be no guarantee that $\lambda \cdot f$ lies in $LUC(G)$.*

Proposition 2.19. *Suppose that $F \in L^1(G)^{**}$, $H, E \in M(G)^{**}$ and $\overline{\pi}(E) = \delta_e$. Then $FH = F\overline{\pi}(H)$. Moreover E a right identity of $L^1(G)^{**}$ with respect to the first Arens product in $M(G)^{**}$. Furthermore, if $E \in M(G)^{**}$ is a right identity for $L^1(G)^{**}$ with respect to the first Arens product in $M(G)^{**}$ then $\overline{\pi}(E) = \delta_e$.*

Proof. Let (f_i) be a net in $L^1(G)$ that converges to F with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$(they are equivalent by Remark 2.15). For $\lambda \in M(G)^*$, by part (f) of Proposition 2.13 and Definition 2.17, we have

$$\begin{aligned} \langle FH, \lambda \rangle &= \lim_i \langle H, \lambda \cdot f_i \rangle = \lim_i \langle \overline{\pi}(H), \lambda \cdot f_i \rangle = \lim_i \langle \overline{\pi}(H)\lambda, f_i \rangle \\ &= \lim_i \langle f_i, \overline{\pi}(H)\lambda \rangle = \langle F, \overline{\pi}(H)\lambda \rangle = \langle F\overline{\pi}(H), \lambda \rangle . \end{aligned}$$

Hence we have $FH = F\overline{\pi}(H)$. Consequently,

$$FE = F\overline{\pi}(E) = F\delta_e = F$$

This proves that E is a right identity for $L^1(G)^{**}$ with respect to the first Arens product in $M(G)^{**}$. Next, we prove the last part of the proposition. Assume that

E is a right identity of $L^1(G)^{**}$ with respect to the first Arens product in $M(G)^{**}$. Then if (e_i) is an approximate identity for $L^1(G)$, we have $e_iE = e_i$. Therefore

$$e_i\overline{\pi}(E) = \overline{\pi}(e_i)\overline{\pi}(E) = \overline{\pi}(e_iE) = \overline{\pi}(e_i) = e_i.$$

Take the limit of both sides with respect to the weak-star topology in $LUC(G)^*$ and note that $\text{wk}^*\lim_i e_i = \delta_e$. We conclude that $\overline{\pi}(E) = \delta_e$. $\square$

Proposition 2.20. *Let G be a locally compact group. Then (1_{K_α}), the net of characteristic functions of all compact (resp. σ-compact) subsets of G ordered by upward inclusion, is a bounded approximate identity for $M(G)_0^*$ (resp. $M(G)_{\sigma 0}^*$).*

Proof. We prove that the net (1_{K_α}) of all characteristic functions of σ-compact subsets of G is a bounded approximate identity for $M(G)_{\sigma 0}^*$. The proof of other statement is similar. Suppose that $\lambda \in M(G)_{\sigma 0}^*$. Given $\epsilon > 0$ there exists a σ-compact subset K_β such that if $\nu \in M(G)$, $\|\nu\| = 1$ and $|\nu|(K_\beta) = 0$ then $|\langle \lambda, \nu\rangle| \leq \epsilon$. If $K_\gamma \supseteq K_\beta$ then for each $\mu \in M(G)$ with $\|\mu\| = 1$, we have

$$\left|\left\langle \lambda 1_{K_\gamma} - \lambda, \mu\right\rangle\right| = \left|\left\langle \lambda, 1_{K_\gamma}\cdot\mu - \mu\right\rangle\right| = \left|\left\langle \lambda, \mu_{K_\gamma} - \mu\right\rangle\right| \leq \epsilon.$$

Finally, note that $M(G)_{\sigma 0}^*$ is commutative and 1_{K_α} is bounded for each K_α. $\square$

In Proposition 2.21, we generalize Lemma 2.5 in [39] that was proved for $L_0^\infty(G)^*$. Note that this can not be generalized for $L_{\sigma 0}^\infty(G)^*$ or $(M(G)_{\sigma o}^*)^*$. In fact Corollary 2.64 presents a counter-example.

Proposition 2.21. *Assume that $m \in (M(G)_0^*)^*$ is positive. Let μ denote the restriction of m to $C_0(G)$. Then $\|m\| = \|\mu\|$.*

Proof. On one hand we have $\|\mu\| \leq \|m\|$. On the other hand, by Proposition 1.31 and Proposition 2.20, $\|m\| = \lim_i \langle m, 1_{K_\alpha}\rangle$, where (1_{K_α}) is the net of characteristic functions of all compact subsets of G. Therefore for each $\epsilon > 0$ there is K_β such that $\|m\| - \epsilon \leq \left\langle m, 1_{K_\beta}\right\rangle$. Consider $g \in C_0(G)$ such that $0 \leq g \leq 1$ and g is one on K_β. Obviously $g \geq 1_{K_\beta}$. This implies that

$$\|m\| - \epsilon \leq \left\langle m, 1_{K_\beta}\right\rangle \leq \langle m, g\rangle = \langle \mu, g\rangle \leq \|\mu\| .$$

This proves that $\|m\| = \|\mu\|$. □

Next, we have the following definition.

Definition 2.22. *Suppose that $F \in (M(G)_0^*)^*$ (resp. $F \in (M(G)_{\sigma 0}^*)^*$). We say that F has a compact (resp. σ-compact)* carrier *if there is a compact (resp. σ-compact) set K such that $\langle F, \lambda\rangle = \langle F, 1_K\lambda\rangle$ for every $\lambda \in (M(G)_0^*$ (resp. $\lambda \in M(G)_{\sigma 0}^*$).*

The following proposition gives us an equivalent condition for the above definition.

Proposition 2.23. *An element $F \in (M(G)_0^*)^*$ (resp. $F \in (M(G)_{\sigma 0}^*)^*$) has a compact (resp. σ-compact) carrier if and only if there exists a compact (resp. σ-compact) set K such that $\langle F, \lambda\rangle = 0$ for each $\lambda \in M(G)_0^*$ with $\lambda|_{M(K)} = 0$.*

Proof. We prove the proposition for $(M(G)_0^*)^*$. The proof for $(M(G)_{\sigma 0}^*)^*$ is similar. Suppose that F has a compact carrier. Therefore, there exists a compact set K such that $\left\langle F, \lambda' \right\rangle = \left\langle F, 1_K \lambda' \right\rangle$ for every $\lambda' \in M(G)_0^*$. Given $\lambda \in M(G)_0^*$ with $\lambda|_{M(K)} = 0$, for every $\mu \in M(G)$, we have

$$\langle 1_K \lambda, \mu \rangle = \langle \lambda, 1_K \cdot \mu \rangle = \langle \lambda, \mu_K \rangle = 0.$$

Therefore for $\lambda \in M(G)_0^*$ with $\lambda|_{M(K)} = 0$, we have $\langle F, \lambda \rangle = \langle F, 1_K \lambda \rangle = \langle F, 0 \rangle = 0$. Conversely, suppose that there exists a compact set K such that $\langle F, \lambda \rangle = 0$ for each $\lambda \in M(G)_0^*$ with $\lambda|_{M(K)} = 0$. For every $\lambda' \in M(G)_0^*$, since $(1_{K^c} \lambda')|_{M(K)} = 0$, we have

$$\langle F, \lambda' \rangle = \langle F, 1_K \lambda' + 1_{K^c} \lambda' \rangle = \langle F, 1_K \lambda' \rangle.$$

Hence F has a compact carrier. □

Definition 2.22 is the generalization of concept of elements in $L_0^\infty(G)^*$ with compact carrier introduced in page 449 of [39]. Similarly, one can define the concept of elements in $L_{\sigma 0}^\infty(G)^*$ with σ-compact carrier and verify that Proposition 2.23 still holds. In Proposition 2.6 of [39], the authors prove that elements in $L_0^\infty(G)^*$ with compact carrier are dense in $L_0^\infty(G)^*$. Similarly, we have the following proposition.

Proposition 2.24. *Functionals in $(M(G)_0^*)^*$ with compact carrier are dense in $(M(G)_0^*)^*$.*

Proof. Take $m \in (M(G)_0^*)^*$. Since by Proposition 1.32 every bounded linear functional can be written as linear combination of positive linear functionals, without loss of generality, we may assume that $m \geq 0$. Let μ denote the restriction of m to $C_0(G)$. For $\epsilon > 0$, by regularity of μ there exists a compact set K such that $|\mu|(K^c) < \epsilon$. Define m_K by $\langle m_K, \lambda \rangle = \langle m, 1_K \lambda \rangle$ for each $\lambda \in M(G)_0^*$. Then $0 \leq m_K \leq m$. By applying Proposition 2.21 for $_K$, we have $\|m - m_K\| = |\mu|(K^c) \leq \epsilon$. $\square$

As we already mentioned Proposition 2.21 is not necessarily valid for $L^\infty_{\sigma 0}(G)^*$ or $(M(G)^*_{\sigma o})^*$. Therefore for these Banach algebras we can not follow the argument in Proposition 2.24. However it is true that functionals in $(M(G)^*_{\sigma 0})^*$ (resp. $L^\infty_{\sigma 0}(G)^*$) with σ-compact carrier are dense in $(M(G)^*_{\sigma 0})^*$ (resp. $L^\infty_{\sigma 0}(G)^*$).

Proposition 2.25. *Let G be a locally compact group. Then the space of elements in $(M(G)^*_{\sigma 0})^*$ (resp. $L^\infty_{\sigma 0}(G)^*$) with σ-compact carrier is dense in $(M(G)^*_{\sigma 0})^*$ (resp. $L^\infty_{\sigma 0}(G)^*$).*

Proof. We give the proof for $(M(G)^*_{\sigma 0})^*$. A similar argument can be given for $L^\infty_{\sigma 0}(G)^*$. Take $m \in (M(G)^*_{\sigma 0})^*$. Again without loss of generality we can assume that m is positive. By Propositions 2.11 and 2.20, we know that $M(G)^*_{\sigma 0}$ is a commutative C^*-algebra with a bounded approximate identity (1_{K_α}). By Theorem 1.25, $(M(G)^*_{\sigma 0})^{**}$ is a commutative C^*-algebra. Moreover it is easy to verify that $(M(G)^*_{\sigma 0})^{**}$ has an identity E which is a weak-star cluster point of (1_{K_α}). We have in particular

$$\langle m, E \rangle = \lim_{K_\alpha \uparrow G} \langle m, 1_{K_\alpha} \rangle .$$

Therefore given $\epsilon > 0$, there exists a σ-compact set K_β such that:

$$|\left\langle m, E - 1_{K_\beta} \right\rangle| \leq \epsilon.$$

Since $m - 1_{K_\beta} \cdot m$ is a positive linear functional in $(M(G)^*_{\sigma 0})^*$, by Theorem 1.30 and above equation we have:

$$\begin{aligned} \|m - 1_{K_\beta} \cdot m\| &= |\left\langle m - 1_{K_\beta} \cdot m, E \right\rangle| = |\langle m, E \rangle - \left\langle m, 1_{K_\beta} E \right\rangle| \\ &= |\left\langle m, E - 1_{K_\beta} \right\rangle| \leq \epsilon. \end{aligned}$$

□

Corollary 2.26. *Let G be a locally compact group. Then for each $m \in (M(G)^*_{\sigma 0})^*$ (resp. $m \in L^\infty_{\sigma 0}(G)^*$), there exists a set Γ of clopen σ-compact subgroups of G ordered by upward inclusion such that*

$$m = \lim_{H \in \Gamma, H \uparrow G} (1_H \cdot m).$$

Proof. Again, without loss of generality we can assume that m is positive. We proved in Proposition 2.25 that m can be approximated by a net $(1_{K_\beta} \cdot m)$ where (K_β) is a net of σ-compact subsets of G ordered by upward inclusion. We can also assume that each K_β is a symmetric neighborhood of e. Now apply Theorem 1.37. □

In [39] the authors define the concept of *topologically introverted* for subspaces of $L^\infty(G)$. We have the following similar definition for $M(G)^*$.

Definition 2.27. *Let X be a closed subspace of $M(G)^*$. Then X is said to be a topologically introverted subspace of $M(G)^*$ if:*

(a) For any $\lambda \in X$ and $\mu \in M(G)$, $\lambda\mu \in X$;

(b) For $F \in X^*$ and $\lambda \in X$, $F\lambda \in X$.

We have the following proposition:

Proposition 2.28. *The space $M(G)_0^*$ (resp. $M(G)_{\sigma 0}^*$) is an introverted subspace of $M(G)^*$.*

Proof. We prove the proposition for $M(G)_0^*$. The proof for $(M(G)_{\sigma 0}^*)^*$ follows exactly the same lines. We already know from proposition 2.11 that $M(G)_0^*$ is a closed subspace of $M(G)^*$. Take $\lambda \in M(G)_0^*$ and $\mu \in M(G)$. By Proposition 2.6, we can suppose that λ has a compact support. Therefore there exists a compact set A such that if $\eta \in M(G)$ and $|\eta|(A) = 0$ then $\langle \lambda, \eta \rangle = 0$. We can also assume that μ has compact support B. Consider the compact set $B^{-1}A$ and take $\nu \in M(G)$ with support C contained in $(B^{-1}A)^c$. Then $\langle \lambda\mu, \nu \rangle = \langle \lambda, \mu\nu \rangle = 0$, because $A \cap BC = \emptyset$. This shows that for $\nu \in M(G)$ with $|\nu|(B^{-1}A) = 0$, we have $\langle \lambda\mu, \nu \rangle = 0$. Therefore $\lambda\mu$ has a compact support. Take $F \in (M(G)_0^*)^*$ and $\lambda \in M(G)_0^*$. By Proposition 2.24, we can assume F has a compact carrier and by using Proposition 2.6, we can suppose that λ has a compact support. By Proposition 2.23, there exists a compact set A such that $\langle F, \lambda' \rangle = 0$ for each $\lambda' \in M(G)_0^*$ with $\lambda'|_{M(A)} = 0$. Also there exists a compact set B such that if $\eta \in M(G)$ with $|\eta|(B) = 0$ then $\langle \lambda, \eta \rangle = 0$. Consider the compact set BA^{-1}. Take $\mu \in M(G)$ whose support C is contained in $(BA^{-1})^c$. Then $\langle F\lambda, \mu \rangle = \langle F, \lambda\mu \rangle = 0$, because $\lambda\mu|_{M(A)} = 0$. The reason for the latter is that if we take $\nu \in M(G)$ whose support is contained in A, then the fact that $\operatorname{supp}(\mu\nu) \cap B = (BA^{-1})^c A \cap B = \emptyset$ implies that $|\mu\nu|(B) = 0$ and therefore $\langle \lambda\mu, \nu \rangle = \langle \lambda, \mu\nu \rangle = 0$. □

In Proposition 2.7 of [39], the authors prove that $L_0^\infty(G)$ is a topologically introverted subspace of $L^\infty(G)$. We can similarly prove that $L_{\sigma 0}^\infty(G)$ is a topologically introverted subspace of $L^\infty(G)$.

Definition 2.29. *For $F, H \in (M(G)_0^*)^*$ we define $FH \in (M(G)_0^*)^*$ by*

$$\langle FH, \lambda \rangle = \langle F, H\lambda \rangle \quad (\lambda \in M(G)_0^*),$$

where $H\lambda \in M(G)_0^*$ is defined by

$$\langle H\lambda, \mu \rangle = \langle H, \lambda\mu \rangle \quad (\mu \in M(G)).$$

Similar definitions of products can be given for $(M(G)_{\sigma 0}^*)^*$ and $L_{\sigma 0}^\infty(G)^*$.

Because $M(G)_0^*$ and $M(G)_{\sigma 0}^*$ are introverted subspaces of $M(G)^*$ and also $L_{\sigma 0}^\infty(G)$ is an introverted subspace of $L^\infty(G)$ the above definitions make sense and we have the following propositions:

Proposition 2.30. *For G a locally compact group $(M(G)_0^*)^*$, $(M(G)_{\sigma 0}^*)^*$ and $L_{\sigma 0}^\infty(G)^*$ are Banach algebras with the products defined above.*

Remark 2.31. *One can similarly define a second product on $(M(G)_0^*)^*$, $(M(G)_{\sigma 0}^*)^*$ and $L_{\sigma 0}^\infty(G)^*$ that makes these spaces into Banach algebras. All the following results, concerning the first products defined above, can be stated for second products in a symmetric way.*

The following proposition is generalization of Proposition 2.7 in [39] stated for elements in $L_0^\infty(G)^*$.

Proposition 2.32. *Suppose that $F, H \in (M(G)_0^*)^*$ (resp. $F, H \in L_0^\infty(G)^*$) have compact carriers. Then FH has a compact carrier too.*

Proof. Since F has a compact carrier, by Proposition 2.23, there is a compact set A such that $\langle F, \lambda' \rangle = 0$ for $\lambda' \in M(G)_0^*$ with $\lambda'|_{M(A)} = 0$. Similarly, there exists a compact set B such that $\langle H, \lambda' \rangle = 0$ for $\lambda' \in M(G)_0^*$ with $\lambda'|_{M(B)} = 0$. Consider the compact set AB and suppose that $\lambda \in M(G)_0^*$ with $\lambda|_{M(AB)} = 0$. Take $\mu \in M(G)$ whose support is contained in A. For each ν whose support is contained in B, we have $\langle \lambda\mu, \nu \rangle = \langle \lambda, \mu\nu \rangle = 0$, because $\operatorname{supp}(\mu\nu) \subseteq AB$. This proves that $\lambda\mu|_{M(B)} = 0$. Therefore $\langle H\lambda, \mu \rangle = \langle H, \lambda\mu \rangle = 0$. This proves that $H\lambda|_{M(A)} = 0$. Hence $\langle FH, \lambda \rangle = \langle F, H\lambda \rangle = 0$. Therefore Proposition 2.23 implies that FH has a compact carrier. $\square$

Similarly we have the following proposition for $(M(G)_{\sigma 0}^*)^*$ and $L_{\sigma 0}^\infty(G)^*$.

Proposition 2.33. *Suppose that $F, H \in (M(G)_{\sigma 0}^*)^*$ (resp. $F, H \in L_{\sigma 0}^\infty(G)^*$) have σ-compact carriers. Then FH has a σ-compact carrier too.*

Proof. The proof is similar to that of Proposition 2.32. □

We can define the concept of compact (resp. σ-compact) carrier for $F \in M(G)^{**}$ as we did for elements of $(M(G)_0^*)^*$ and $(M(G)_{\sigma 0}^*)^*$ in Definition 2.22. Let M_G (resp. $M_{\sigma G}$) be the closure of all elements in $M(G)^{**}$ with compact (resp. σ-compact) carrier. Similar to Proposition 2.32 and Proposition 2.33, one can prove that M_G (resp. $M_{\sigma G}$) is also a closed subalgebra of $M(G)^{**}$. Also note that the analogue of Proposition 2.23 holds for elements in M_G (resp. $M_{\sigma G}$) with compact (resp. σ-compact) carrier. In the next propositions we show that in fact M_G (resp. $M_{\sigma G}$) is isometrically isomorphic with $(M(G)_0^*)^*$ (resp. $(M(G)_{\sigma 0}^*)^*$).

Proposition 2.34. *Let G be a locally compact group. Then there is an isometric isomorphism from M_G onto $(M(G)_0^*)^*$.*

Proof. Let $\tau : M_G \to (M(G)_0^*)^*$ be the restriction map to $M(G)_0^*$. First we prove that τ is an isometry. Consider $F \in M_G$ with a compact carrier. There exists a compact set K such that if $\lambda' \in M(G)^*$ and $\lambda'|_{M(K)} = 0$ then $\left\langle F, \lambda' \right\rangle = 0$. Hence $\left\langle F, 1_{K^c}\lambda' \right\rangle = 0$ for every $\lambda' \in M(G)^*$. Moreover for each $\epsilon > 0$ there is $\lambda \in M(G)^*$ with $\|\lambda\| = 1$ such that $\|F\| - \epsilon \leq |\langle F, \lambda\rangle|$. Note that $1_K\lambda \in M(G)_c^*$, because for each $\mu \in M(G)$ with $|\mu|(K) = 0$ we have $\langle 1_K\lambda, \mu\rangle = \langle \lambda, 1_K\mu\rangle = 0$. Therefore we have

$$\|F\| - \epsilon < |\langle F, \lambda\rangle| = \langle F, 1_K\lambda + 1_{K^c}\lambda\rangle = |\langle F, 1_K\lambda\rangle| = |\langle \tau(F), 1_K\lambda\rangle| \leq \|\tau(F)\|.$$

Therefore $\|F\| \leq \|\tau(F)\|$. Hence $\|\tau(F)\| = \|F\|$ for each $F \in M_G$ with a compact carrier. Since elements in M_G with compact carrier are dense in M_G, we conclude that τ is an isometry on M_G. To prove that it is an algebra isomorphism we just check that it preserves multiplication. The preservation of addition and scalar multiplication is trivial. Take $F, H \in M_G$ and assume that (μ_i) converges to F in the weak-star topology of $M(G)^{**}$. Then for $\lambda \in M(G)_0^*$ we have

$$\begin{aligned} \langle \tau(F)\tau(F), \lambda \rangle &= \langle \tau(F), \tau(H)\lambda \rangle = \langle F, \tau(H)\lambda \rangle = \lim_i \langle \mu_i, \tau(H)\lambda \rangle = \lim_i \langle \tau(H), \lambda\mu_i \rangle \\ &= \lim_i \langle H, \lambda\mu_i \rangle = \lim_i \langle \mu_i H, \lambda \rangle = \langle FH, \lambda \rangle = \langle \tau(FH), \lambda \rangle . \end{aligned}$$

Hence $\tau(FH) = \tau(F)\tau(H)$. To prove that τ is surjective, by Proposition 2.24 and the fact that τ is an isometry on M_G, it is enough to show that for each $H \in (M(G)_0^*)^*$ with compact carrier there exists $F \in M_G$ such that $\tau(F) = H$. Take $H \in (M(G)_0^*)^*$ with compact carrier. There exists a compact set K such that if $\lambda' \in M(G)_0^*$ and $\lambda'|_{M(K)} = 0$ then $\langle H, \lambda' \rangle = 0$. Define $F \in M(G)^{**}$ by $\langle F, \lambda \rangle = \langle H, 1_K\lambda \rangle$ for each $\lambda \in M(G)^*$. Obviously $F \in M_G$. Moreover, for each $\lambda'' \in M(G)_0^*$, we have $1_{K^c}\lambda''|_{M(K)} = 0$. Therefore

$$\left\langle \tau(F), \lambda'' \right\rangle = \left\langle F, \lambda'' \right\rangle = \left\langle H, 1_K\lambda'' \right\rangle = \left\langle H, 1_K\lambda'' + 1_{K^c}\lambda'' \right\rangle = \left\langle H, \lambda'' \right\rangle .$$

Hence $\tau(F) = H$. Therefore we can conclude that M_G is isometrically isomorphic with $(M(G)_0^*)^*$. $\square$

Proposition 2.35. *Let G be a locally compact group. Then there is an isometric isomorphism from $M_{\sigma G}$ onto $(M(G)_{\sigma 0}^*)^*$.*

Proof. The proof is similar to that of Proposition 2.34. $\square$

Corollary 2.36. *Let G be a locally compact group. The Banach algebra $M(G)$ can be isometrically embedded into both $(M(G)^*_{\sigma 0})^*$ and $(M(G)^*_0)^*$.*

Proof. To prove this corollary it suffices to demonstrate that $M(G) \subseteq M_G$. To this end assume that $\mu \in M(G)$ and for each compact subset K of G define $\mu_K \in M(G)$ to be

$$\mu_K(E) = \mu(E \cap K), \qquad (E \in \mathcal{B}_G).$$

As we have pointed out before Proposition 2.34, a result similar to Proposition 2.23 holds for elements in $M(G)^{**}$ with compact carrier. Therefore it is straightforward to show that μ_K is in M_G. On the other hand if Γ is the set of all compact subsets of G ordered by upward inclusion then we have

$$\lim_{K \in \Gamma, K \uparrow G} (\mu_K) = \mu.$$

Since M_G is closed with respect to the norm-topology in $M(G)^{**}$, we conclude that μ is in M_G too. $\square$

The concept of compact carrier for elements in $L^\infty(G)^*$ has been introduced in [39]. Similarly, one can define the concept of σ-compact carrier for $F \in L^\infty(G)^*$. Let L_G (resp. $L_{\sigma G}$) be the closure of all elements in $L^\infty(G)^*$ with compact (resp. σ-compact) carrier. Similar to Proposition 2.32 and 2.33 one can prove that L_G (resp.

$L_{\sigma G}$) is also a closed subalgebra of $L^\infty(G)^*$. We have the following propositions that was already stated in [39]:

Proposition 2.37. *Let G be a locally compact group. Then there is an isometric isomorphism from L_G onto $L_0^\infty(G)^*$.*

Proof. The proof is similar to that of Proposition 2.34. □

Proposition 2.38. *Let G be a locally compact group. Then there is an isometric isomorphism from $L_{\sigma G}$ onto $L_{\sigma 0}^\infty(G)^*$.*

Proof. The proof is similar to that of Proposition 2.34. □

We have the following proposition:

Proposition 2.39. *Let G be a locally compact group. Then $(M(G)_0^*)^*$ is isometrically isomorphic with a subalgebra of $(M(G)_{\sigma 0}^*)^*$.*

Proof. By using the fact that every compact set is σ-compact and based on definition of M_G and $M_{\sigma G}$, we can conclude that $M_G \subseteq M_{\sigma G}$. Therefore Propositions 2.34 and 2.35 imply that $(M(G)_0^*)^*$ can be isometrically identified with a subalgebra of $(M(G)_{\sigma 0}^*)^*$. □

One can similarly prove the following:

Proposition 2.40. *Let G be a locally compact group. Then $L_0^\infty(G)^*$ is isometrically isomorphic with a subalgebra of $L_{\sigma 0}^\infty(G)^*$.*

In [39] the authors prove the following proposition by using the next lemma.

Lemma 2.41. *Let $\mu \in C_0(G)^*$. Then μ has a unique norm preserving extension to a continuous linear functional on $C_b(G)$.*

Proof. This is Lemma 1 in [38]. $\square$

Proposition 2.42. *The Banach algebra $L^1(G)^{**}$ is the Banach space direct-sum*

$$(L_0^\infty(G))^* \oplus (L_0^\infty(G))^\perp$$

of a norm-closed subalgebra $L_0^\infty(G)^*$ and a weak-star closed ideal $(L_0^\infty(G))^\perp$ such that if $F \in L^1(G)^{**}$ and $F = F_1 + F_2$, $F_1 \in L_0^\infty(G)^*$ and $F_2 \in (L_0^\infty(G))^\perp$ then

(a) $\|F\| = \|F_1\| + \|F_2\|$, and

(b) $F \geq 0 \Leftrightarrow F_1 \geq 0, F_2 \geq 0$.

Furthermore $\pi(L_0^\infty(G)^*) = M(G)$ and $\pi((L_0^\infty(G))^\perp) = C_0(G)^\perp$.

Proof. This is Theorem 2.8 in [39]. $\square$

Similar to Proposition 2.42, we have following proposition that decomposes $M(G)^{**}$:

Proposition 2.43. *The algebra $M(G)^{**}$ is the Banach space direct-sum*

$$(M(G)_0^*)^* \oplus (M(G)_0^*)^\perp$$

of a norm-closed subalgebra $(M(G)_0^*)^*$ and a weak-star closed ideal $(M(G)_0^*)^\perp$ such that if $F \in M(G)^{**}$ and $F = F_1 + F_2$, $F_1 \in (M(G)_0^*)^*$ and $F_2 \in (M(G)_0^*)^\perp$ then *(a)* $\|F\| = \|F_1\| + \|F_2\|$, *and*

(b) $F \geq 0 \Leftrightarrow F_1 \geq 0, F_2 \geq 0$.

Furthermore $\overline{\pi}((M(G)_0^*)^*) = M(G)$ *and* $\overline{\pi}((M(G)_0^*)^\perp) = C_0(G)^\perp$.

Proof. From Proposition 2.34, $(M(G)_0^*)^*$ is isometrically isomorphic with the closed subalgebra M_G of $M(G)^{**}$. Also we can easily verify that $(M(G)_0^*)^* \cap (M(G)_0^*)^\perp = \{0\}$ and $(M(G)_0^*)^\perp$ is weak-star closed. Suppose that $F \in M(G)^{**}$. Let F_1 be the element corresponding to the restriction of F to $M(G)_0^*$. Consider $F_2 = F - F_1$. Obviously $F_2 \in (M(G)_0^*)^\perp$. Therefore every $F \in M(G)^{**}$ can be written uniquely as sum of $F_1 \in (M(G)_0^*)^*$ and $F_2 \in (M(G)_0^*)^\perp$. To see that $(M(G)_0^*)^\perp$ is an ideal, suppose that $F \in M(G)^{**}$, $H \in (M(G)_0^*)^\perp$ and $\lambda \in M(G)_0^*$. For each $\mu \in M(G)$, $\lambda\mu$ is in $M(G)_0^*$, because $M(G)_0^*$ is an introverted subspace of $M(G)^*$. Therefore $\langle H\lambda, \mu\rangle = \langle H, \lambda\mu\rangle = 0$. This implies that $\langle FH, \lambda\rangle = \langle F, H\lambda\rangle = 0$. Therefore $FH \in (M(G)_0^*)^\perp$. This proves that $(M(G)_0^*)^\perp$ is a left ideal of $M(G)^{**}$. To prove that it is also a right ideal, recall that $F = F_1 + F_2$ for $F_1 \in (M(G)_0^*)^*$ and $F_2 \in (M(G)_0^*)^\perp$. Therefore $F\lambda = F_1\lambda + F_2\lambda = F_1\lambda$. Again since $M(G)_0^*$ is an introverted subspace of $M(G)^*$, $F_1\lambda$ is in $M(G)_0^*$. Therefore $F\lambda \in M(G)_0^*$. Hence $\langle HF, \lambda\rangle = \langle H, F\lambda\rangle = 0$. Therefore $HF \in (M(G)_0^*)^\perp$. Next for $F \in M(G)^{**}$, and a measurable set K we define

$$\langle F_K, \lambda\rangle = \langle F, 1_K\lambda\rangle,$$

where $\lambda \in M(G)^*$. Obviously $F_K, F_{K^c} \in M(G)^{**}$ and $F = F_K + F_{K^c}$. For each

$\epsilon > 0$ there are $\lambda_1, \lambda_2 \in (M(G)^*)_1$ such that $\langle F, 1_K \lambda_1 \rangle = \langle F_K, \lambda_1 \rangle \geq \|F_K\| - \epsilon$ and $\langle F, 1_{K^c} \lambda_2 \rangle = \langle F_{K^c}, \lambda_2 \rangle \geq \|F_{K^c}\| - \epsilon$. Since $\|1_K \lambda_1 + 1_{K^c} \lambda_2\| \leq 1$,

$$\|F\| \geq \langle F, 1_K \lambda_1 + 1_{K^c} \lambda_2 \rangle = \langle F_K, \lambda_1 \rangle + \langle F_{K^c}, \lambda_2 \rangle \geq \|F_K\| + \|F_{K^c}\| - 2\epsilon.$$

Therefore we have $\|F\| = \|F_K\| + \|F_{K^c}\|$. On the other hand we know that M_G is the norm closure of elements in $M(G)^{**}$ with compact support. Therefore there exists a sequence (K_n) of compact sets such that $F_{K_n} \to F_1$. Therefore $F_{{K_n}^c} \to F_2$ too. This implies that

$$\|F\| = \|F_1\| + \|F_2\|.$$

If $F \geq 0$ then $F_K, F_{K^c} \geq 0$. This implies that F_1, F_2 are positive too. Conversely if $F_1, F_2 \geq 0$ then obviously $F \geq 0$. If $m \in (M(G)_0^*)^*$ is positive and $\mu \in M(G)$ is the restriction of m to $C_0(G)$ then, by Proposition 2.21, $\|m\| = \|\mu\|$. We have $\|\mu\| \leq \|\overline{\pi}(m)\| \leq \|m\| \leq \|\mu\|$. Therefore $\|\overline{\pi}(m)\| = \|\mu\|$. By using Lemma 2.41 and Proposition 1.55, we have $\overline{\pi}(m) = \mu$. Since each $m \in (M(G)_0^*)^*$ is linear combination of positive elements, we have $\overline{\pi}((M(G)_0^*)^*) \subseteq M(G)$. On the other hand if $\mu \in M(G)$ then there exists $m \in (M(G)_0^*)^*$ that extends μ and such that $\|m\| = \|\mu\|$. Hence

$$\|\mu\| \leq \|\overline{\pi}(m)\| \leq \|m\| \leq \|\mu\|.$$

Therefore $\|\overline{\pi}(m)\| = \|\mu\|$. Again by using Lemma 2.41 and Proposition 1.55, we have $\mu = \overline{\pi}(m)$. This proves that $\overline{\pi}((M(G)_0^*)^*) = M(G)$. It is clear that $\overline{\pi}((M(G)_0^*)^\perp) \subseteq C_0(G)^\perp$. For $n \in C_0(G)^\perp$, let $F \in M(G)^{**}$ be an extension of n to $M(G)^*$. We have

$F = F_1 + F_2$ for $F_1 \in (M(G)_0^*)^*$ and $F_2 \in (M(G)_0^*)^\perp$. Then $n = \overline{\pi}(F) = \overline{\pi}(F_1) + \overline{\pi}(F_2)$ where $\overline{\pi}(F_1) \in M(G)$ and $\overline{\pi}(F_2) \in C_0(G)^\perp$. Note that $\overline{\pi}(F_1) = n - \overline{\pi}(F_2)$. Therefore $\overline{\pi}(F_1) \in C_0(G)^\perp \cap M(G) = \{0\}$. Hence $\overline{\pi}(F_1) = 0$. Consequently, we have $n = \overline{\pi}(F_2) \in \overline{\pi}((M(G)_0^*)^\perp)$. Therefore $\overline{\pi}((M(G)_0^*)^\perp) = C_0(G)^\perp$. □

Note that in the proof of Proposition 2.43, we used Proposition 2.21 that—as we have already mentioned —is not necessarily valid for $(M(G)_{\sigma 0}^*)^*$ or $L_{\sigma 0}^\infty(G)^*$. However we have the following propositions:

Proposition 2.44. *The algebra $M(G)^{**}$ is the Banach space direct-sum*

$$(M(G)_{\sigma 0}^*)^* \oplus (M(G)_{\sigma 0}^*)^\perp$$

of a norm-closed subalgebra $(M(G)_{\sigma 0}^*)^*$ and a weak-star closed ideal $(M(G)_{\sigma 0}^*)^\perp$ such that if $F \in M(G)^{**}$ and $F = F_1 + F_2$, $F_1 \in (M(G)_{\sigma 0}^*)^*$ and $F_2 \in (M(G)_{\sigma 0}^*)^\perp$ then

(a) $\|F\| = \|F_1\| + \|F_2\|$, and

(b) $F \geq 0 \Leftrightarrow F_1 \geq 0, F_2 \geq 0$

Moreover $M(G) \subseteq \overline{\pi}((M(G)_{\sigma 0}^*)^*)$ and $\overline{\pi}((M(G)_{\sigma 0}^*)^\perp) \subseteq C_0(G)^\perp$.

Proof. The proof of all parts is similar to that of Proposition 2.43. For proving that $M(G) \subseteq \overline{\pi}((M(G)_{\sigma 0}^*)^*)$, note that $(M(G)_0^*)^* \subseteq (M(G)_{\sigma 0}^*)^*$ and use the fact that $\overline{\pi}((M(G)_0^*)^*) = M(G)$. For proving that $\overline{\pi}((M(G)_{\sigma 0}^*)^\perp) \subseteq C_0(G)^\perp$, note that $(M(G)_{\sigma 0}^*)^\perp \subseteq (M(G)_0^*)^\perp$ and use the fact that $\overline{\pi}((M(G)_0^*)^\perp) = C_0(G)^\perp$. □

Similarly, we have

Proposition 2.45. *The algebra $L^1(G)^{**}$ is the Banach space direct-sum*

$$(L^\infty_{\sigma 0}(G))^* \oplus (L^\infty_{\sigma 0}(G))^\perp$$

of a norm-closed subalgebra $L^\infty_{\sigma 0}(G)^*$ and a weak-star closed ideal $(L^\infty_{\sigma 0}(G))^\perp$ such that if $F \in L^1(G)^{**}$ and $F = F_1 + F_2$, $F_1 \in L^\infty_{\sigma 0}(G)^*$ and $F_2 \in (L^\infty_{\sigma 0}(G))^\perp$ then

(a) $\|F\| = \|F_1\| + \|F_2\|$, and

(b) $F \geq 0 \Leftrightarrow F_1 \geq 0, F_2 \geq 0$.

Moreover $M(G) \subseteq \overline{\pi}((L^\infty_{\sigma 0}(G))^*)$ and $\overline{\pi}(L^\infty_{\sigma 0}(G)^\perp) \subseteq C_0(G)^\perp$.

Proof. The proof is similar to that of Proposition 2.44. □

Remark 2.46. *Note that for example if G is σ-compact and non-compact then*

$$M(G) \subset LUC(G)^* = \overline{\pi}((L^\infty_{\sigma 0}(G))^*) = \overline{\pi}((M(G)^*_{\sigma 0})^*),$$

$$\overline{\pi}((L^\infty_{\sigma 0}(G))^\perp) = \overline{\pi}((M(G)^*_{\sigma 0})^\perp) = \{0\} \subset C_0(G)^\perp.$$

Corollary 2.47. *Let $F, H \in M(G)^{**}$ such that $\|F\| = \|H\| = \|FH\| = 1$. If $FH \in (M(G)^*_0)^*$ (resp. $FH \in (M(G)^*_{\sigma 0})^*$) then F and H are in $(M(G)^*_0)^*$ (resp. $(M(G)^*_{\sigma 0})^*$).*

Proof. We prove the statement for $FH \in (M(G)^*_0)^*$. The other case can be proved similarly. Using Proposition 2.43, we have $F = F_1 + F_2$ and $H = H_1 + H_2$ for $F_1, H_1 \in$

$(M(G)_0^*)^*$ and $F_2, H_2 \in (M(G)_0^*)^\perp$. Hence $FH = F_1H_1 + F_1H_2 + F_2H_1 + F_2H_2$. Since $F_1H_1 \in (M(G)_0^*)^*$ and $F_1H_2 + F_2H_1 + F_2H_2 \in (M(G)_0^*)^\perp$, we have $FH = F_1H_1$. Therefore $\|F_1H_1\| = 1$. Consequently, $\|F_1\| = \|H_1\| = 1$. By using Proposition 2.43 again, we have $F_2 = H_2 = 0$. Therefore $F, H \in (M(G)_0^*)^*$. □

Corollary 2.48. *Let $F, H \in M(G)^{**}$ be non-zero multiplicative linear functionals on $M(G)^*$. If $FH \in (M(G)_0^*)^*$ (resp. $FH \in (M(G)_{\sigma 0}^*)^*$), then $F, H \in (M(G)_0^*)^*$ (resp. $F, H \in (M(G)_0^*)^*$).*

Proof. We prove the statement for $FH \in (M(G)_0^*)^*$. The other clause can be proved similarly. Recall that $M(G)^*$ is a unital abelian C*-algebra. Proposition 1.17 implies that $\|F\| = \|H\| = F(1) = H(1) = 1$. Also note that for $\mu \in M(G)$, we have

$$\langle H \cdot 1, \mu \rangle = \langle H, 1 \cdot \mu \rangle = \langle H, \langle 1, \mu \rangle 1 \rangle = \langle 1, \mu \rangle .$$

Hence $\langle FH, 1 \rangle = \langle F, H \cdot 1 \rangle = \langle F, 1 \rangle = 1$. Therefore $\|FH\| = 1$ too. By using the previous corollary, we conclude that $F, H \in (M(G)_0^*)^*$. □

The following corollaries are generalizations of Corollary 2.9 and Corollary 2.10 in [39]. The proofs are similar to those of previous corollaries.

Corollary 2.49. *Let $F, H \in L^\infty(G)^*$ such that $\|F\| = \|H\| = \|FH\| = 1$. If $FH \in L_0^\infty(G)^*$ (resp. $FH \in L_{\sigma 0}^\infty(G)^*$) then F and H are in $L_0^\infty(G)^*$ (resp. $FH \in L_{\sigma 0}^\infty(G)^*$).*

Corollary 2.50. *Let F, H be elements in the spectrum of $L^\infty(G)$. If $FH \in L_0^\infty(G)^*$ (resp. $FH \in L_{\sigma 0}^\infty(G)^*$), then $F, H \in L_0^\infty(G)^*$ (resp. $F, H \in L_{\sigma 0}^\infty(G)^*$).*

2.2 Some preliminary results

Lemma 2.51. *Let G be a locally compact group. Then*

(a) Take $\lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$ and $f \in L^1(G)$. Then $\lambda \cdot f = 0$.

(b) Take $H \in (M(G)_0^*)^*$ and $\lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$. Then $H \cdot \lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$.

(c) Take $\lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$ and $\mu \in M(G)$. Then $\lambda \cdot \mu \in M_d(G)_0^* \oplus M_s(G)_0^*$.

(d) Take $F \in L_0^\infty(G)^*$ and $\lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$. Then $F \cdot \lambda = 0$

Proof. (a) For $\mu \in M(G)$, $\langle \lambda \cdot f, \mu \rangle = \langle \lambda, f\mu \rangle = 0$, because $f\mu \in L^1(G)$.

(b) By part (a), for each $f \in L^1(G)$, we have $\langle H \cdot \lambda, f \rangle = \langle H, \lambda \cdot f \rangle = 0$. Therefore $H \cdot \lambda \in M_d(G)_0^* \oplus M_s(G)_0^*$.

(c) For $f \in L^1(G)$, we have $\langle \lambda \cdot \mu, f \rangle = \langle \lambda, \mu f \rangle = 0$. Therefore $\lambda \cdot \mu \in M_d(G)_0^* \oplus M_s(G)_0^*$.

(d) $\langle F \cdot \lambda, \mu \rangle = \langle F, \lambda \cdot \mu \rangle = 0$, because $\lambda \cdot \mu \in M_d(G)_0^* \oplus M_s(G)_0^*$ by part (c).

$\square$

Similarly one can prove the following lemma.

Lemma 2.52. *Let G be a locally compact group. Then*

(a) If $\lambda \in M_d(G)_{\sigma 0}^* \oplus M_s(G)_{\sigma 0}^*$ and $f \in L^1(G)$, then $\lambda \cdot f = 0$.

(b) If $H \in (M(G)^*_{\sigma 0})^*$ *and* $\lambda \in M_d(G)^*_{\sigma 0} \oplus M_s(G)^*_{\sigma 0}$*, then* $H \cdot \lambda \in M_d(G)^*_{\sigma 0} \oplus M_s(G)^*_{\sigma 0}$.

(c) If $\lambda \in M_d(G)^*_{\sigma 0} \oplus M_s(G)^*_{\sigma 0}$ *and* $\mu \in M(G)$*, then* $\lambda \cdot \mu \in M_d(G)^*_{\sigma 0} \oplus M_s(G)^*_{\sigma 0}$.

(d) If $F \in L^\infty_{\sigma 0}(G)^*$ *and* $\lambda \in M_d(G)^*_{\sigma 0} \oplus M_s(G)^*_{\sigma 0}$*, then* $F \cdot \lambda = 0$

Part (i) and part (j) of Proposition 2.13 assure that $L^\infty_0(G)^*$ and $L^\infty_{\sigma 0}(G)^*$ can be identified with Banach subspaces of $(M(G)^*_0)^*$ and $(M(G)^*_{\sigma 0})^*$ respectively. In fact, we can assert the following:

Proposition 2.53. *Let* G *be a locally compact group. Then* $L^\infty_0(G)^*$ *can be identified with an ideal of* $(M(G)^*_0)^*$.

Proof. Suppose that $F \in L^\infty_0(G)^*$, $H \in (M(G)^*_0)^*$. Then by part (b) of Lemma 2.51, for $\lambda \in (M_d(G)^*)_0 \oplus (M_s(G)^*)_0$, we have $\langle FH, \lambda \rangle = \langle F, H \cdot \lambda \rangle = 0$. This shows that $FH \in L^\infty_0(G)^*$. On the other hand by part (d) of Lemma 2.51, for $\lambda \in (M_d(G)^*)_0 \oplus (M_s(G)^*)_0$, we have $\langle HF, \lambda \rangle = \langle H, F \cdot \lambda \rangle = 0$. This shows that $HF \in L^\infty_0(G)^*$. □

Proposition 2.54. *Let* G *be a locally compact group. Then* $L^\infty_{\sigma 0}(G)^*$ *can be identified with an ideal of* $(M(G)^*_{\sigma 0})^*$.

Proof. The proof is similar to that of 2.53. □

Lemma 2.55. *Let* G *be a locally compact group. Then*

(a) The algebra $L^1(G)$ *is an ideal of* $L^\infty_0(G)^*$.

(b) The algebra $L^1(G)$ is an ideal (left, right or two-sided) of $L^1(G)^{**}$ if and only if G is compact.

Proof. (a) See Theorem 2.11 of [39].

(b) See the Corollaries 4.3 and 4.4 of [18].

□

Proposition 2.56. *Let G be a locally compact group. Then*

(a) The algebra $L^1(G)$ is an ideal of $(M(G)^*_0)^*$.

(b) The algebra $L^1(G)$ is an ideal of $M(G)^{**}$ if and only if G is compact.

Proof. (a) Take $f \in L^1(G)$ and $F \in (M(G)^*_0)^*$. Using Corollary 1.41, there exist functions $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$. Using Lemma 2.55 and Proposition 2.53, we conclude that $fF = (f_1 \star f_2)F = f_1(f_2F)$ is in $L^1(G)$. Similarly, $Ff \in L^1(G)$.

(b) If $L^1(G)$ is an ideal of $M(G)^{**}$, then it will be an ideal of $L^1(G)^{**}$ too. Therefore by using Lemma 2.55, we conclude that G is compact. On the other hand suppose that G is compact. Consider $f \in L^1(G)$ and $F \in M(G^{**})$. Using Corollary 1.41 there exist functions $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$. By Lemma 2.55 and the fact that $L^1(G)^{**}$ is an ideal of $M(G)^{**}$, we conclude that $fF = (f_1 \star f_2)F = f_1(f_2F)$ is in $L^1(G)$. Similarly, we can prove that $Ff \in L^1(G)$.

□

Proposition 2.57. *Let G be a locally compact group. Then*

(a) The algebra $(M(G)_0^*)^*$ is an ideal of $M(G)^{**}$ if and only if G is compact.

(b) The algebra $L_0^\infty(G)^*$ is an ideal of $L^1(G)^{**}$ if and only if G is compact.

(c) The algebra $L_0^\infty(G)^*$ is an ideal of $M(G)^{**}$ if and only if G is compact.

Proof. (a) The proof in one direction is trivial. Suppose that $(M(G)_0^*)^*$ is an ideal of $M(G)^{**}$. Because $M(G) \subseteq (M(G)_0^*)^*$ and $M(G)$ has an identity, we conclude that $(M(G)_0^*)^* = M(G)^{**}$. Then by Proposition 2.43, $(M(G)_0^*)^\perp = \{0\}$. This implies that $M(G)^* = M(G)_0^*$. Hence G is compact.

(b) If G is compact then $L_0^\infty(G) = L^\infty(G)$. Therefore $L_0^\infty(G)^* = L^1(G)^{**}$ and obviously $L_0^\infty(G)^*$ is an ideal of itself. Conversely suppose that $L_0^\infty(G)^*$ is an ideal of $L^1(G)^{**}$. Consider $f \in L^1(G)$ and $F \in L^1(G)^{**}$. Using Corollary 1.41 there exist functions $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$. By part (a) of Lemma 2.55 and the assumption that $L_0^\infty(G)^*$ is an ideal of $L^1(G)^{**}$, we conclude that $fF = (f_1 \star f_2)F = f_1(f_2F)$ is in $L^1(G)$. Similarly, we can prove that $Ff \in L^1(G)$. Hence $L^1(G)$ is an ideal of $L^1(G)^{**}$. By part (b) of Lemma 2.55, we conclude that G is compact.

(c) If G is compact then $L_0^\infty(G)^* = L^1(G)^{**}$. Since $L^1(G)^{**}$ is an ideal of $M(G)^{**}$, we conclude that $L_0^\infty(G)^*$ is an ideal of $M(G)^{**}$. Conversely suppose that $L_0^\infty(G)^*$

is an ideal of $M(G)^{**}$. Consider $f \in L^1(G)$ and $F \in M(G)^{**}$. Similar to part (b), we can prove that $Ff \in L^1(G)$ and $Ff \in L^1(G)$. Hence $L^1(G)$ is an ideal of $M(G)^{**}$. By part (b) of Lemma 2.56, we conclude that G is compact.

□

Proposition 2.58. *Suppose that G is a locally compact group such that $C_c(G) = C_0(G)$. Then G is compact.*

Proof. See part (e) of 11.43 in [28]. □

Corollary 2.59. *Suppose that G is a locally compact group such that every Borel measurable symmetric neighborhood of e that is σ-compact is contained in a compact subset of G. Then G is compact.*

Proof. We know that $C_c(G) \subseteq C_0(G)$. We aim to prove that $C_0(G) \subseteq C_c(G)$. Take $f \in C_0(G)$. For $\epsilon = \frac{1}{n}$ $(n = 1, 2, ...)$, there exists a compact set K_n such that $|f(x)| \leq \frac{1}{n}$ for $x \in {K_n}^c$. Therefore $|f(x)| = 0$ if $x \in K^c$, where $K = \bigcup_{n\in\mathbb{N}} K_n$ is σ-compact. We can also assume that K is a symmetric neighborhood of e. Since K is contained in a compact set, we conclude that $f \in C_c(G)$. Therefore $C_0(G) = C_c(G)$. From this and Proposition 2.58, it follows that G is compact. □

Corollary 2.60. *Suppose that G is a locally compact group such that $L_0^\infty(G) = L_{\sigma 0}^\infty(G)$. Then G is compact.*

Proof. Suppose that $L_0^\infty(G) = L_{\sigma 0}^\infty(G)$ and G is not compact. Then by Corollary 2.59, there exists K, a σ-compact subset of G, that is not contained in a compact set. Note that the characteristic function of K is in $L_{\sigma 0}^\infty(G)$ but it is not in $L_0^\infty(G)$. This contradicts the assumption that $L_0^\infty(G) = L_{\sigma 0}^\infty(G)$. $\square$

Lemma 2.61. *Suppose that G is a non-compact locally compact group. Then G has a non-compact, clopen, σ-compact subgroup H.*

Proof. Since G is not compact, by Corollary 2.59, there exists U, a symmetric σ-compact neighborhood of e, that is not contained in a compact set. Put $H = \bigcup_{n\in\mathbb{N}} U^n$. By Theorem 1.37, H is open, closed and σ-compact. We observe that H is non-compact because if it were compact, since $\overline{U} \subseteq H$, then $\overline{U}$ is contained in a compact set that is a contradiction. $\square$

Lemma 2.62. *Suppose that G is a non-compact group. Then there exists a non-zero $f \in L^\infty(G)$ such that $f \in LUC(G) \cap L_{\sigma 0}^\infty(G)$ and $f \notin L_0^\infty(G)$.*

Proof. Suppose that G is not compact. By Lemma 2.61, there is a non-compact, closed, open σ-compact subgroup H of G. Consider 1_H, the characteristic function of H. This function is in $LUC(G)$, because H is a clopen subgroup. It is also in $L_{\sigma 0}^\infty(G)$, because H is σ-compact. Note that H not only is non-compact but because it is a closed non-compact set, it can not even be contained in a compact set. Therefore $1_H \notin L_0^\infty(G)$. $\square$

Lemma 2.63. *Suppose that G is not compact. Then there is a positive non-zero F in $L^{\infty}_{\sigma 0}(G)^*$ such that F is non-zero on $LUC(G)$ and it is zero on $L^{\infty}_{0}(G)$.*

Proof. Let f be the function as obtained in Lemma 2.62. Let $\mathcal{M}$ be the subspace of $L^{\infty}(G)$ generated by f. Using the Hann-Banach theorem, we can find an F' in $L^{\infty}_{\sigma 0}(G)^*$ such that F' is non-zero on $LUC(G)$ and it is zero on $L^{\infty}_{0}(G)$. Then $F = \left|F'\right|$ satisfies the desired properties. $\square$

As we had promised the reader, we prove in following corollary that Proposition 2.21 does not hold for elements in $L^{\infty}_{\sigma 0}(G)^*$ or $(M(G)^*_{\sigma o})^*$.

Corollary 2.64. *Suppose that G is not compact. Then there is a positive non-zero F in $L^{\infty}_{\sigma 0}(G)^*$ such that $\|F\| \neq \|\mu\|$, where $\mu \in M(G)$ is the restriction of F to $C_0(G)$.*

Proof. Let F be the element in $L^{\infty}_{\sigma 0}(G)^*$ obtained in previous lemma. Obviously F is in $(L^{\infty}_{0}(G))^{\perp}$. Using Proposition 2.42, we conclude that $\|\mu\| = 0 \neq \|F\|$. $\square$

Theorem 2.65. *Let G be a locally compact group. Then $L^1(G)$ is an ideal of $L^{\infty}_{\sigma 0}(G)^*$ if and only if G is compact.*

Proof. Suppose that G is compact. Then $L^{\infty}_{\sigma 0}(G)^* = L^{\infty}(G)^* = L^{\infty}_{0}(G)^*$ and so $L^1(G)$ is an ideal of $L^{\infty}_{\sigma 0}(G)^*$ by Lemma 2.55. Conversely suppose that $L^1(G)$ is an ideal of $L^{\infty}_{\sigma 0}(G)^*$ and towards a contradiction assume that G is non-compact. Then by Lemma 2.63, we get an $F \in L^{\infty}_{\sigma 0}(G)^*$ such that F is non-zero on $LUC(G)$ and is zero on $L_0^\infty(G)$. From the assumption, for each $f \in L^1(G)$, we have $fF \in L^1(G)$. Now, by using Lemma 2.2 and Theorem 2.8 of [39], and Proposition 1.55, we have $fF = f\pi(F) \in C_0(G)^\perp$. Therefore again by Proposition 1.55 we conclude that $fF = 0$. Therefore $F \in \mathrm{ran}(L^1(G)^{**})$. This implies that F restricted to $LUC(G)$ is zero, that is a contradiction. $\square$

Proposition 2.66. *Let G be a locally compact group then the following are equivalent*

(a) G is compact;

(b) The algebra $L^1(G)$ is an ideal of $(M(G)^*_{\sigma 0})^*$;

(c) The algebra $L_0^\infty(G)^*$ is an ideal of $L_{\sigma 0}^\infty(G)^*$;

(d) The algebra $L_0^\infty(G)^*$ is an ideal of $(M(G)^*_{\sigma 0})^*$;

(e) The algebra $L_{\sigma 0}^\infty(G)^*$ is an ideal of $(M(G)^*_0)^*$;

(f) The algebra $(M(G)^*_0)^*$ is an ideal of $(M(G)^*_{\sigma 0})^*$;

(g) The algebra $(M(G)^*_0)^*$ is an ideal of $M(G)^{**}$.

Proof. Obviously (a) implies all other statements. It suffices to show that other statements imply (a). Suppose that (b) is true. Theorem 2.65 and the fact that $L_{\sigma 0}^\infty(G)^* \subseteq (M(G)^*_{\sigma 0})^*$ implies (a). Suppose that (c) is true. An argument similar to the one in part (b) of Proposition 2.57 proves that $L^1(G)$ is an ideal of $L_{\sigma 0}^\infty(G)^*$. Hence by Theorem 2.65, G is compact. Suppose that (d) is true. Then an argument similar to one in part (b) of Proposition 2.57 proves that $L^1(G)$ is an ideal of $(M(G)^*_{\sigma 0})^*$. But this is exactly statement (b) which we have proved that it implies (a). Next assume that (e) is true. Then G is compact because if G is not compact then by Lemma 2.63, there is an $F \in L^\infty_{\sigma 0}(G)^*$ such that F is non-zero on $LUC(G)$ and it is zero on $L^\infty_0(G)$. Since $F \in L^\infty_{\sigma 0}(G)^* \subseteq (M(G)^*_0)^*$, by Proposition 2.43, we have $\overline{\pi}(F) \in M(G)$. On the other hand since F is zero on $L^\infty_0(G)$, F is zero on $C_0(G)$. This implies that $\overline{\pi}(F) = 0$ which contradicts the fact that F is non-zero on $LUC(G)$. Therefore G is compact. Next suppose that (f) is true. By an argument similar to the one in part (b) of Proposition 2.57 and by using part (a) of Proposition 2.56, we conclude that $L^1(G)$ is an ideal of $(M(G)^*_{\sigma 0})^*$. But this is exactly statement (b) which we have proved that it implies (a). Next suppose that (g) holds. By part (a) of Proposition 2.56 and an argument similar to one in part (b) of Proposition 2.57, we can conclude that $L^1(G)$ is an ideal of $M(G)^{**}$. Then part (b) of Proposition 2.56 implies that G is compact. $\square$

2.3 Right identities of $L^1(G)^{**}$ in $L^1(G)^{**}$ and in $M(G)^{**}$

In this section, we study elements in $M(G)^{**}$ whose right action on $L^1(G)^{**}$ is the identity operator. We call such elements *right identities of* $L^1(G)^{**}$ *in* $M(G)^{**}$. In doing so, we are motivated by Proposition 2.1 of [19] where a characterization of right identities of $L^1(G)^{**}$ is given. It is important to note that, if G is a non-discrete group, there are nontrivial examples of such elements. For instance all elements in the form of $\delta_e + r$ is a right identity of $L^1(G)^{**}$ in $M(G)^{**}$ when $r \in L^1(G)^{**}$ is a right annihilator of $L^1(G)^{**}$. Note that for such an $r \in \mathrm{ran}(L^1(G)^{**})$, when G is non-discrete, $\delta_e + r \notin L^1(G)^{**}$ is a right identity for $L^1(G)^{**}$.

Proposition 2.67. *Let G be a locally compact group and $E \in M(G)^{**}$. Then the following are equivalent.*

(a) $\|E\| = 1$ and for every bounded approximate identity (e_i) in $L^1(G)$, we have $\lim_i \langle e_i, f\rangle = \langle E, f\rangle$ for every $f \in LUC(G)$;

(b) $\|E\| = 1$ and $\langle E, f\rangle = f(e)$ for all $f \in LUC(G)$;

(c) $E \geq 0$ and $\phi = \phi E$ for all $\phi \in L^1(G)$;

(d) $\|E\| = 1$ and E is a right identity for $L^1(G)^{**}$.

Proof. We first prove that (a) implies (b). Let $f = g \cdot h$ be an arbitrary element in $LUC(G)$ for $g \in L^\infty(G)$ and $h \in L^1(G)$. Then

$$\begin{aligned}\langle E, f\rangle &= \lim_i \langle e_i, f\rangle = \lim_i \langle f, e_i\rangle = \lim_i \langle g\cdot h, e_i\rangle = \lim_i \langle g, h * e_i\rangle = \langle g, h\rangle \\ &= \int_G h(x)g(x)dx = g\cdot h(e) = f(e).\end{aligned}$$

Next, we want to show that (b) implies (c). Take the constant function 1 in $LUC(G)$. Therefore $\langle E, 1\rangle = 1(e) = 1 = \|E\|$. Hence by using Theorem 1.30, we can conclude that $E \geq 0$. Suppose that $\phi \in L^1(G)$. By part (a) of Proposition 2.13, each $\lambda \in M(G)^*$ can be written as $\lambda = g + \lambda_d + \lambda_s$ for $g \in L^\infty(G)$, $\lambda_d \in M_d(G)^*$ and $\lambda_s \in M_s(G)^*$. In the proof of Proposition 2.13 part (f), we showed that $\lambda \cdot \phi = g \cdot \phi$. Therefore we have

$$\begin{aligned}\langle \phi E, \lambda \rangle &= \langle E, \lambda \cdot \phi \rangle = \langle E, g \cdot \phi \rangle = g \cdot \phi(e) = \int_G \phi(x) g(x) dx \\ &= \langle g, \phi \rangle = \langle g + \lambda_d + \lambda_s, \phi \rangle = \langle \lambda, \phi \rangle = \langle \phi, \lambda \rangle\end{aligned}$$

This proves that $\phi E = \phi$. To prove that (c) implies (d), first note that for $\phi \in L^1(G)$ with $\phi \geq 0$ and $\|\phi\| = 1$, we have $1 \cdot \phi = 1$ because for each $\mu \in M(G)$, we can write

$$\langle 1 \cdot \phi, \mu \rangle = \langle 1, \phi\mu \rangle = \int_G d(\phi\mu) = \int_G \int_G d(\phi) d(\mu) = \int_G \int_G \phi(x) dx d(\mu) = \int_G d(\mu) = \langle 1, \mu \rangle .$$

Hence we can write

$$\langle E, 1 \rangle = \langle E, 1 \cdot \phi \rangle = \langle \phi E, 1 \rangle = \langle \phi, 1 \rangle = 1.$$

By using Theorem 1.30, we conclude that $\|E\| = 1$. For a bounded approximate identity (e_i), knowing that $e_i E = e_i$, we can write

$$e_i \overline{\pi}(E) = \overline{\pi}(e_i E) = \overline{\pi}(e_i) = e_i.$$

By taking limit of both sides with respect to the weak-star topology in $LUC(G)^*$, we conclude that $\delta_e \overline{\pi}(E) = \delta_e$. Therefore $\overline{\pi}(E) = \delta_e$. By Proposition 2.19, E is a right identity for $L^1(G)^{**}$.

Next, we prove that (d) implies (a). Since E is right identity of $L^1(G)^{**}$, by Proposition 2.19, $\overline{\pi}(E) = \delta_e$. Hence for $f \in LUC(G)$, we have

$$\langle E, f\rangle = \langle \overline{\pi}(E), f\rangle = \langle \delta_e, f\rangle = \lim_i \langle e_i, f\rangle .$$

$\square$

Definition 2.68. *The set of all elements in $M(G)^{**}$ that satisfy one of the above equivalent conditions is denoted by $\xi_1^{'}(G)$. We also agree to use the notation $\xi_1(G)$ for the set of all right identities of $L^1(G)^{**}$ in itself with norm 1. We also denote the set of all right identities (without restriction on norm) of $L^1(G)^{**}$ in $L^1(G)^{**}$ and $M(G)^{**}$ by $\xi(G)$ and $\xi^{'}(G)$ respectively.*

It is worth mentioning that in Proposition 2.1 of [19], the authors prove that $E \in \xi_1(G)$ if and only if E is a cluster point (with respect to the weak-star topology in $L^1(G)^{**}$) of an approximate identity, bounded by 1, in $L^1(G)$. With regard to that we have the following proposition.

Proposition 2.69. *Let G be a locally compact group. Then*

(a) $E \in \xi_1(G)$ if and only if E is a cluster point (with respect to the weak-star topology in $L^1(G)^{**}$) of an approximate identity, bounded by 1, in $L^1(G)$.

(b) If $E \in \xi_1^{'}(G) \cap \xi_1(G)^c$ then E is not a cluster point (with respect to the weak-star topology in $M(G)^{**}$) of an approximate identity, bounded by 1, in $L^1(G)$.

Proof. For a proof of (a), see Proposition 2.1 of [19]. To prove (b), suppose that $E \in \xi_1'(G) \cap \xi_1(G)^c$ is a cluster point of (e_i) with respect to the weak-star topology in $M(G)^{**}$, where (e_i) is an approximate identity, bounded by 1, in $L^1(G)$. By considering a subnet of (e_i) (we denote this subnet by (e_i) again.), we can assume that (e_i) converges to E with respect to the weak-star topology in $M(G)^{**}$. By Goldstine's theorem, there exists $F \in L^1(G)^{**}$ such that a subnet of (e_i) (we denote this subnet by (e_i) again.) converges to F with respect to the weak-star topology in $L^1(G)^{**}$. Part (a) implies that $F \in \xi_1(G)$. By Remark 2.15, (e_i) converges to F with respect to the weak-star topology in $M(G)^{**}$ too. Therefore $E = F$. This is a contradiction, because $E \notin \xi_1(G)$ by assumption. $\square$

In [39], the authors prove the following proposition.

Proposition 2.70. *Let G be a locally compact group. Then $\xi_1(G) \subseteq L_0^\infty(G)^*$.*

Proof. See part (i) of Theorem 2.11 in [39]. $\square$

Similarly, we have

Proposition 2.71. *Let G be a locally compact group. Then $\xi_1'(G) \subseteq (M(G)_0^*)^*$.*

Proof. Take $E \in \xi_1'(G)$. By Proposition 2.67, $\|E\| = 1$ and for $\phi \in L^1(G)$ such that $\|\phi\| = 1$, we have $\phi E = \phi \in L^1(G) \subseteq (M(G)_0^*)^*$. By Corollary 2.47, we conclude that $E \in (M(G)_0^*)^*$. $\square$

Moreover we can generalize Propositions 2.70 and 2.71 as follows:

Proposition 2.72. *Let G be a locally compact group. Then $\xi^{'}(G) \subseteq (M(G)_0^*)^*$.*

Proof. Suppose that $\xi^{'}(G) \nsubseteq (M(G)_0^*)^*$. Then by Proposition 2.43 there exists a non-zero $E \in ((M(G)_0^*)^{\perp} \cap \xi^{'}(G))$. From Proposition 2.43 we also know that $\overline{\pi}(E) \in C_0(G)^{\perp}$. But by Proposition 2.19, we know that $\overline{\pi}(E) = \delta_e$. That is a contradiction.

□

Proposition 2.73. *Let G be a locally compact group. Then $\xi(G) \subseteq L_0^{\infty}(G)^*$.*

Proof. The proof is similar to that of Proposition 2.72. □

Corollary 2.74. *Let G be a locally compact group and $E \in \xi^{'}(G)$. Then $E\phi = \phi$ for all $\phi \in L^1(G)$.*

Proof. By using Proposition 2.72 and part (a) of Proposition 2.56, $E\phi \in L^1(G)$. By part (a) of Proposition 2.16 and Proposition 2.19, we have

$$0 = \phi - \phi = \delta_e\phi - \phi = \overline{\pi}(E)\overline{\pi}(\phi) - \overline{\pi}(\phi) = \overline{\pi}(E\phi - \phi) = E\phi - \phi.$$

□

Lemma 2.75. *Let G be a locally compact group. Then*

(a) $\bigcap_{E\in\xi_1(G)} E\Box M(G) = L^1(G)$.

(b) $\bigcap_{E\in\xi_1(G)} M(G) \diamond E = L^1(G)$.

(c) $\bigcap_{E\in\xi_1(G)} M(G)\square E = L^1(G)$.

(d) $\bigcap_{E\in\xi_1'(G)} E\square M(G) = L^1(G)$.

(e) $\bigcap_{E\in\xi_1'(G)} M(G)\square E = L^1(G)$.

Proof. Part (a) is Proposition 2.7 of [19]. By symmetry and using part (a), we have proof of part (b). To prove part (c), use part (b) and Proposition 1.54. To prove part (d), note that by Corollary 2.74 and the fact that $L^1(G)$ is an ideal of $M(G)$, we have $L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} E\square M(G)$. On the other hand by part (a), we have

$$\bigcap_{E\in\xi_1'(G)} E\square M(G) \subseteq \bigcap_{E\in\xi_1(G)} E\square M(G) = L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} E\square M(G).$$

The proof of part (e) is similar to that of part (d). $\square$

Proposition 2.76. *Let G be a locally compact group. Then*

$$\bigcap_{E\in\xi_1'(G)} E(M(G)_0^*)^* = \bigcap_{E\in\xi_1(G)} E(M(G)_0^*)^* = L^1(G).$$

Proof. By Proposition 2.19 and Proposition 2.43, for $E \in \xi_1(G)$

$$E(M(G)_0^*)^* = E\overline{\pi}((M(G)_0^*)^*) = EM(G).$$

From Lemma 2.75, we conclude that

$$\bigcap_{E\in\xi_1(G)} E(M(G)_0^*)^* = \bigcap_{E\in\xi_1(G)} EM(G) = L^1(G).$$

To prove the other equality, note that by Corollary 2.74 and part (a) of Proposition 2.56 , we have $L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} E(M(G)_0^*)^*$. Therefore

$$\bigcap_{E\in\xi_1'(G)} E(M(G)_0^*)^* \subseteq \bigcap_{E\in\xi_1(G)} E(M(G)_0^*)^* = L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} E(M(G)_0^*)^*.$$

□

Proposition 2.77. *Let G be a locally compact group. Then*

$$\bigcap_{E\in\xi_1(G)} EL_0^\infty(G)^* = \bigcap_{E\in\xi_1'(G)} EL_0^\infty(G)^* = L^1(G).$$

Proof. The fact that $\bigcap_{E\in\xi_1(G)} EL_0^\infty(G)^* = L^1(G)$ is part (iii) of Theorem 2.11 in [39]. By Corollary 2.74 and part (a) of Lemma 2.55, we know that $L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} EL_0^\infty(G)^*$. Therefore

$$\bigcap_{E\in\xi_1'(G)} EL_0^\infty(G)^* \subseteq \bigcap_{E\in\xi_1(G)} EL_0^\infty(G)^* = L^1(G) \subseteq \bigcap_{E\in\xi_1'(G)} EL_0^\infty(G)^*.$$

□

The following proposition generalizes Proposition 2.6 in [19].

Proposition 2.78. *Let G be a locally compact group and*

$$\Delta(G) = \bigcap_{E\in\xi_1'(G)} EL^1(G)^{**}.$$

Then $\Delta(G)$ is a right ideal of $M(G)^{**}$ containing $L^1(G)$. Moreover $L^1(G)$ is an ideal of $\Delta(G)$ if and only if G is compact, in which case $\Delta(G) = L^1(G)$.

Proof. We prove that for each $E \in \xi_1'(G)$, $EL^1(G)^{**}$ is a right ideal of $M(G)^{**}$ and from that the intersection Δ is a right ideal of $M(G)^{**}$ too. Take $F \in M(G)^{**}$ and $Em \in EL^1(G)^{**}$ for some $m \in L^1(G)^{**}$. Then $(Em)F = E(mF) \in EL^1(G)^{**}$, because $L^1(G)^{**}$ is an ideal of $M(G)^{**}$. By Corollary 2.74, $EL^1(G) = L^1(G)$. Therefore $\Delta(G)$ contains $L^1(G)$.

For proving the second part of the proposition, Suppose that G is compact. Then by Proposition 2.77 and because $L_0^\infty(G)^* = L^1(G)^{**}$, we conclude that $\Delta(G) = L^1(G)$, so that $L^1(G)$ is an ideal of $\Delta(G)$. Conversely, suppose that $L^1(G)$ is an ideal of $\Delta(G)$. We aim to prove that $L^1(G)$ is a right ideal of $L^1(G)^{**}$. Take $\phi \in L^1(G)$ and $m \in L^1(G)^{**}$. Then $\phi m \in \Delta(G)$, because $\Delta(G)$ contains $L^1(G)$ and is a right ideal of $M(G)^{**}$. Let the net (e_i) be an arbitrary approximate identity of $L^1(G)$. Then obviously $(e_i\phi)$ converges to ϕ with respect to the norm-topology in $L^1(G)$. The net $(e_i\phi m)$ converges to ϕm with respect to the norm-topology in $L^1(G)^{**}$. Since $\phi m \in \Delta(G)$, $e_i \in L^1(G)$ and $L^1(G)$ is an ideal of $\Delta(G)$, we can conclude that $e_i\phi m \in L^1(G)$. Moreover since $L^1(G)$ is a closed subalgebra of $L^1(G)^{**}$, we conclude that $\phi m \in L^1(G)$. Hence $L^1(G)$ is a right ideal of $L^1(G)^{**}$. This implies that G is compact, by part (b) of Lemma 2.55. $\square$

Remark 2.79. *Note that $\Delta(G)$ is not necessarily closed, because E^2 is not necessarily equal to E in $M(G)^{**}$, as E only acts as a right identity on $L^1(G)^{**}$.*

Given $\mu \in M(G)$, we define the continuous linear transformation $\rho_\mu : M(G) \longrightarrow$

$M(G)$ to be $\rho_\mu(\nu) = \nu \star \mu$ for all $\nu \in M(G)$. It is easy to check that $\rho_\mu^* : M(G)^* \longrightarrow M(G)^*$ will be given by the formula $\rho_\mu^*(\lambda) = \mu \cdot \lambda$ for each $\lambda \in M(G)^*$. Also we can verify that $\rho_\mu^{**} : M(G)^{**} \longrightarrow M(G)^{**}$ is given by the formula $\rho_\mu^{**}(F) = F \Box \mu$ for every $F \in M(G)^{**}$. For $E \in \xi^{'}(G)$, we define $\Gamma_E : M(G) \longrightarrow M(G)^{**}$ by $\Gamma_E(\mu) = \rho_\mu^{**}(E)$. We have the following proposition which is similar to Proposition 2.3 in [19].

Proposition 2.80. *Let G be a locally compact group, $E \in \xi^{'}(G)$ and Γ_E be defined as above. Then*

(a) $\langle \Gamma_E(\mu), f \rangle = \int_G f d\mu$ *for all* $f \in LUC(G)$ *and* $\mu \in M(G)$.

(b) $\Gamma_E(\mu) = \mu$ *for all* $\mu \in L^1(G)$.

(c) $\langle \Gamma_E(\mu) \cdot \lambda, \phi \rangle = \langle \lambda, \phi\mu \rangle$ *for all* $\lambda \in M(G)^*$, $\mu \in M(G)$ *and* $\phi \in L^1(G)$. *In particular for* $f \in L^\infty(G)$,

$$\langle \Gamma_E(\mu) \cdot f, \phi \rangle = \langle \mu, \tilde{\phi} \star f \rangle$$

(d) If $E \in \xi_1^{'}(G)$ *then* Γ_E *is an isometric linear transformation from* $M(G)$ *into* $(M(G)_0^*)^*$ *that extends the natural embedding of* $L^1(G)$ *into* $(M(G)_0^*)^*$.

Proof. (a) Take $\mu \in M(G)$ and $f \in LUC(G)$. By using Proposition 2.16, we have:

$$\langle \Gamma_E(\mu), f \rangle = \langle E\mu, f \rangle = \langle \overline{\pi}(E\mu), f \rangle = \langle \overline{\pi}(E)\overline{\pi}(\mu), f \rangle = \langle \delta_e \mu, f \rangle = \int_G f d\mu.$$

(b) For $\mu \in L^1(G)$, by using Corollary 2.74, $\Gamma_E(\mu) = E\mu = \mu$.

(c) By part (d) and (f) of Proposition 2.13 and part (a), we have

$$\langle \Gamma_E(\mu) \cdot \lambda, \phi \rangle = \langle \Gamma_E(\mu), \lambda \cdot \phi \rangle = \int_G (\lambda \cdot \phi)\, d\mu = \langle \mu, \lambda \cdot \phi \rangle = \langle \lambda, \phi\mu \rangle.$$

In particular for $f \in L^\infty(G)$, from the definition $\tilde{\phi}(x) = \Delta(x^{-1})\phi(x^{-1})$ we have

$$\langle \Gamma_E(\mu) \cdot f, \phi \rangle = \langle \mu, f \cdot \phi \rangle = \langle \mu, \tilde{\phi} \star f \rangle.$$

(d) The fact that Γ_E is a linear transformation is obvious. It is also clear that it is equal to the identity on $L^1(G)$, by Corollary 2.74. We just show that it is an isometry (here we need E to be of norm 1). On the one hand

$$\|\Gamma_E(\mu)\| = \|\rho_\mu^{**}(E)\| \leq \|\rho_\mu^{**}\| \|E\| \leq \|\mu\| \|E\| = \|\mu\|.$$

This implies that $\|\Gamma_E\| \leq 1$. On the other hand if $\mu \in L^1(G)$ and $\|\mu\| = 1$, then by part (b), $\|\Gamma_E(\mu)\| = \|\mu\|$. Therefore $\|\Gamma_E\| \geq 1$. Hence $\|\Gamma_E\| = 1$.

□

Proposition 2.81. *A measure μ belongs to $L^1(G)$ if and only if $\Gamma_{E_1}(\mu) = \Gamma_{E_2}(\mu)$ for all $E_1, E_2 \in \xi_1'(G)$.*

Proof. This is immediate from part (ii) of Proposition 2.4 of [19]. □

The following proposition has been proved in 2.5 of [19]. We will give a different proof of it.

Proposition 2.82. *Let $m \in L^1(G)^{**}$ and $E \in \xi_1(G)$. Then the followings are equivalent.*

(a) $m = \Gamma_E(\mu)$ *for some* $\mu \in M(G)$*;*

(b) *As a functional on* $LUC(G)$*,* m *is an extension of* $\mu \in M(G)$ *such that* $\|m\| = \|\mu\|$ *and* $Em = m$*.*

Proof. Suppose that $m = \Gamma_E(\mu)$. Then $m = E\mu$. Therefore by Proposition 2.16 and Proposition 2.19

$$\pi(m) = \overline{\pi}(m) = \overline{\pi}(E\mu) = \overline{\pi}(E)\overline{\pi}(\mu) = \delta_e\mu = \mu.$$

This shows that m is an extension of μ. So $\|\mu|\| \leq \|m\|$. On the other hand

$$\|m\| = \|E\mu\| \leq \|E\|\|\mu\| \leq \|\mu\|.$$

Therefore $\|m\| = \|\mu\|$. Moreover by multiplying both sides of $m = E\mu$ by E, we have $Em = EE\mu = E\mu = m$. Conversely, suppose that (b) holds. From the fact that $Em = m$ and Proposition 2.19, we have

$$m = Em = E\overline{\pi}(m) = E\mu.$$

□

The following proposition is a generalization of Proposition 2.8 in [19].

Proposition 2.83. *Let $E \in \xi'(G)$. Then $\overline{\pi}^{-1}(\mu) = \Gamma_E(\mu)$ for all $\mu \in M(G)$.*

Proof. By Proposition 2.16 and Proposition 2.19, we have

$$\overline{\pi}(E\mu) = \overline{\pi}(E)\overline{\pi}(\mu) = \delta_e\mu = \mu.$$

$\square$

2.4 Further preliminary properties

When $E \in \xi(G)$ we can easily prove that $EL^1(G)^{**}$ is equal to $EM(G)^{**}$. However if $E \in \xi^{'}(G) \bigcap \xi(G)^c$, then $EL^1(G)^{**}$ is not necessarily equal to $EM(G)^{**}$.

Example 2.84. *Consider $E = \delta_e + r$ where $r \in L^1(G)^{**}$ is a right annihilator for $L^1(G)^{**}$. We claim that $EL^1(G)^{**} \neq EM(G)^{**}$. For example consider $E \in EM(G)^{**}$. This is not in $EL^1(G)^{**}$ because if $E = EF$ for some $F \in L^1(G)^{**}$ then E will be in $L^1(G)^{**}$. That is a contradiction.*

Theorem 2.3 of [39] can be extended as follows:

Proposition 2.85. *Suppose that G is a locally compact group and $E \in \xi(G)$. Then $M(G)^{**}$ can be written as the algebraic direct-sum of the norm-closed right ideal $EM(G)^{**}$ and a weak-star closed ideal of $M(G)^{**}$. Moreover $EM(G)^{**} = EL^1(G)^{**}$ is isomorphic to $LUC(G)^*$ and if $\|E\| = 1$ the isomorphism is isometric.*

Proof. We first show that $(\delta_e - E)M(G)^{**} = \mathrm{Ker}(\overline{\pi})$. For $F \in M(G)^{**}$, we have

$$\overline{\pi}((\delta_e - E)F) = \overline{\pi}(F) - \overline{\pi}(E)\overline{\pi}(F) = \overline{\pi}(F) - \overline{\pi}(F) = 0.$$

On the other hand if $F \in \mathrm{Ker}(\overline{\pi})$, we have $EF = E\overline{\pi}(F) = 0$. Hence $F = \delta_e F - EF = (\delta_e - E)F$. Therefore $\mathrm{Ker}(\overline{\pi}) = (\delta_e - E)M(G)^{**}$. This shows that $(\delta_e - E)M(G)^{**}$ is a weak-star closed ideal of $M(G)^{**}$. Obviously every $F \in M(G)^{**}$ can be written as $F = EF + (\delta_e - E)F$. If $H \in EM(G)^{**} \bigcap (\delta_e - E)M(G)^{**}$. Then $H = EI = (\delta_e - E)F$ for some F and I in $M(G)^{**}$. By multiplying both sides by E and using the fact that $E \in L^1(G)^{**}$, we have $EI = EEI = E(F - EF) = EF - EF = 0$. Therefore $H = 0$. This shows that the decomposition is direct-sum. The norm-closeness of $EM(G)^{**}$ is a consequence of $E^2 = E$. The rest has been already proved in [39] (including the fact that the mapping $\Lambda : EL^\infty(G)^* \longrightarrow M(G)$, $EF \longmapsto \pi(F)$ $(\forall F \in L^\infty(G)^*)$ is an isometric isomorphism when $\|E\| = 1$). $\square$

Corollary 2.86. *Suppose that G is a locally compact group and $E \in \xi_1(G)$. Then the Banach algebra $EL_0^\infty(G)^*$ is isometrically isomorphic to $M(G)$.*

Proof. This is Theorem 2.11 (ii) in [39]. From Proposition 2.42 and Proposition 2.85, the mapping $\Lambda : EL_0^\infty(G)^* \longrightarrow M(G)$, $EF \longmapsto \pi(F)$ $(\forall F \in L_0^\infty(G)^*)$ is an isometric isomorphism. $\square$

Remark 2.87. *Note that in the above proposition, if we consider E as a right identity of $L^1(G)^{**}$ that is not in $L^1(G)^{**}$, then $EM(G)^{**} \cap (\delta_e - E)M(G)^{**} \neq \{0\}$. For example consider $E = \delta_e + r$ as in Example 2.84 for $r \neq 0$. We write*

$$(\delta_e + r)r = (\delta_e - (\delta_e + r))(-\delta_e) = r \neq 0 \in EM(G)^{**} \cap (\delta_e - E)M(G)^{**}.$$

Also $EM(G)^{**}$ will not necessarily be norm-closed. Moreover, $(\delta_e - E)M(G)^{**} \subset \mathrm{Ker}(\overline{\pi})$ and we do not have equality necessarily.

Proposition 2.88. *Suppose that G is a locally compact group. Then*

(a) The $\mathrm{ran}(L^1(G)^{**})$ in $M(G)^{**}$ is equal to $\mathrm{Ker}(\overline{\pi})$.

(b) The $\mathrm{ran}(L^1(G)^{**})$ in $L^1(G)^{**}$ is equal to $\mathrm{Ker}(\pi)$.

(c) If G is non-discrete, then the $\mathrm{ran}(L^1(G)^{**})$ in $M(G)^{**}$ is different from the $\mathrm{ran}(L^1(G)^{**})$ in $L^1(G)^{**}$.

Proof. (a) Let $H \in \mathrm{Ker}(\overline{\pi})$. Then for each $F \in L^1(G)^{**}$, $FH = F\overline{\pi}(H) = F0 = 0$. Therefore H is in $\mathrm{ran}(L^1(G)^{**})$ in $M(G)^{**}$. On the other hand suppose that $H \in M(G)^{**}$ is a right annihilator for $L^1(G)^{**}$. Then if (e_i) is an approximate identity of $L^1(G)$, we have $e_i H = 0$. This implies that

$$e_i\overline{\pi}(H) = \overline{\pi}(e_i)\overline{\pi}(H) = \overline{\pi}(e_i H) = 0$$

By taking the weak-star limit of both sides in $LUC(G)^*$ we conclude that $\overline{\pi}(H) = 0$.

(b) The proof is similar to that of part (a) with the difference that we work with π instead of $\overline{\pi}$.

(c) Consider $\delta_e - E$ where E is in $\xi(G)$. If G is non-discrete, this is a right annihilator that is not in $L^1(G)^{**}$.

$\square$

Remark 2.89. *Similar results can be stated for* $\operatorname{ran}(L^\infty_{\sigma 0}(G)^*)$ *or* $\operatorname{ran}(L^\infty_0(G)^*)$ *in* $M(G)^{**}$ *and* $\operatorname{ran}(L^\infty_{\sigma 0}(G)^*)$ *and* $\operatorname{ran}(L^\infty_0(G)^*)$ *in* $L^1(G)^{**}$. *Therefore* $\operatorname{ran}(L^\infty_{\sigma 0}(G)^*)$, $\operatorname{ran}(L^\infty_0(G)^*)$ *and* $\operatorname{ran}(L^1(G)^{**})$ *in* $L^1(G)^{**}$ *are all equal to* $\operatorname{Ker}(\pi)$. *Also* $\operatorname{ran}(L^\infty_{\sigma 0}(G)^*)$, $\operatorname{ran}(L^\infty_0(G)^*)$ *and* $\operatorname{ran}(L^1(G)^{**})$ *in* $M(G)^{**}$ *are all equal to* $\operatorname{Ker}(\overline{\pi})$.

Proposition 2.90. *Let G be a locally compact group. Then*

$$\operatorname{Ker}(\overline{\pi}) \square L^1(G) = L^1(G) \square \operatorname{Ker}(\overline{\pi}) = \{0\}.$$

Proof. From part (a) of Proposition 2.88, we have $L^1(G)\square\operatorname{Ker}(\overline{\pi}) = \{0\}$. Symmetrically, we can prove that $\operatorname{Ker}(\overline{\pi}) \diamond L^1(G) = \{0\}$. By Proposition 1.54,

$$\operatorname{Ker}(\overline{\pi}) \diamond L^1(G) = \operatorname{Ker}(\overline{\pi}) \square L^1(G).$$

Therefore $\operatorname{Ker}(\overline{\pi})\square L^1(G) = 0$. $\square$

Chapter 3

TOPOLOGICAL CENTERS OF $L^\infty_{\sigma 0}(G)^*$ AND $(M(G)^*_{\sigma 0})^*$

In section 1.2 we defined $Z_t^r(\mathcal{A}^{**})$ and $Z_t^l(\mathcal{A}^{**})$, the right and left topological center of the second dual of a Banach algebra and we stated that $L^1(G)$ and $M(G)$ are strongly irregular. As we mentioned in Proposition 2.30 and Remark 2.31, since $L^\infty_{\sigma 0}(G)$ (resp. $(M(G)^*_{\sigma 0})$) is a topologically introverted subspace of $L^\infty(G)$ (resp. $M(G)^*$), both Arens products are well defined on $L^\infty_{\sigma 0}(G)^*$ (resp. $(M(G)^*_{\sigma 0})^*$). These Arens products coincide with the ones $L^\infty_{\sigma 0}(G)^*$ (resp. $(M(G)^*_{\sigma 0})^*$) inherits from $L^1(G)^{**}$ (resp. $M(G)^{**}$) if we consider it as (see Proposition 2.38 and Proposition 2.35) a Banach subalgebra of $L^1(G)^{**}$ (resp. $M(G)^{**}$). We define the left topological center of $L^\infty_{\sigma 0}(G)^*$ to be

$$Z_t^l(L^\infty_{\sigma 0}(G)^*) = \{m \in L^\infty_{\sigma 0}(G)^* : (\forall n \in L^\infty_{\sigma 0}(G)^*) \;\; m \Box n = m \diamond n\}\,.$$

Similarly, one can define the left topological center of $(M(G)^*_{\sigma 0})^*$ and the right topological centers for these Banach algebras. In this chapter we prove that the left topological centers of $L^\infty_{\sigma 0}(G)^*$ and $(M(G)^*_{\sigma 0})^*$ are $L^1(G)$ and $M(G)$ respectively. We shall focus on left topological center. Noting that $Z_t^r(\mathcal{A}^{**}) = Z_t^l((\mathcal{A}^{op})^{**}), L^1(G^{op}) = L^1(G)^{op}$ and $M(G^{op}) = M(G)^{op}$, it can be verified that $Z_t^r(L^\infty_{\sigma 0}(G)^*) = Z_t^l(L^\infty_{\sigma 0}(G^{op})^*)$, $Z_t^r((M(G)^*_{\sigma 0})^*) = Z_t^l((M(G^{op})^*_{\sigma 0})^*)$, $Z_t^r(L^\infty(G)^*) = Z_t^l(L^\infty(G^{op})^*)$ and $Z_t^r(M(G)^{**}) =$

$Z_t^l(M(G^{op})^{**})$. Therefore we can extend the results concerning left topological centers to right topological centers.

3.1 Left topological center of $(M(G)^*_{\sigma 0})^*$

Let G be a locally compact group. For a closed subgroup H of G, H will be a locally compact topological group with respect to the topology induced from the topology of G. We define $T_H : M(H) \longrightarrow M(G)$ and $R_H : M(G) \longrightarrow M(H)$ respectively by:

$$T_H(\mu)(E) = \mu(E \cap H), \ \ (\mu \in M(H), E \in \mathcal{B}_G)\,.$$

$$R_H(\mu)(E) = \mu(E), \ \ (\mu \in M(G), E \in \mathcal{B}_H)\,.$$

It is easy to check that T_H and R_H are elements of $\mathcal{B}(M(H), M(G))$ and $\mathcal{B}(M(G), M(H))$ respectively. We have the following proposition:

Proposition 3.1. *Let G be a locally compact group and H be a clopen subgroup of G. Then*

(a) $R_H(T_H(\nu)) = \nu$ for every $\nu \in M(H)$, i.e. R_H is a left inverse for T_H.

(b) $R_H(\mu \star T_H(\nu)) = R_H(T_H(R_H(\mu)) \star T_H(\nu))$ for $\mu \in M(G)$, $\nu \in M(H)$.

(c) $T_H(\nu_1 \star \nu_2) = T_H(\nu_1) \star T_H(\nu_2)$ for $\nu_1, \nu_2 \in M(H)$.

(d) $T_H(R_H(\mu)) = 1_H \cdot \mu = \mu \cdot 1_H$ for each $\mu \in M(G)$.

Proof. Noting that T_H and R_H are weak-star continuous, the convolution map is separately weak-star continuous and the discrete measures are weak-star dense in measure algebra, it is enough to prove these statements for point-mass measures. For $a \in H$, we have:

$$R_H(T_H(\delta_a)) = R_H(\delta_a) = \delta_a.$$

This proves part (a). To prove the second statement, suppose that $a \in H$ and $b \in G$. Two cases may occur:

Case 1: When $a \in H$, $b \in G$ and $b \notin H$. Then $ba \notin H$ and we have:

$$R_H(\delta_b \star T_H(\delta_a)) = R_H(\delta_b \star \delta_a)) = R_H(\delta_{ba}) = 0 = R_H(T_H(0) \star \delta_a) = R_H(T_H(R_H(\delta_b)) \star T_H(\delta_a)).$$

Case 2: When $a \in H$ and $b \in H$. Then $ba \in H$ and we have:

$$R_H(T_H(R_H(\delta_b)) \star T_H(\delta_a)) = R_H(T_H(\delta_b) \star \delta_a) = R_H(\delta_b \star \delta_a) = R_H(\delta_b \star T_H(\delta_a)).$$

For part (c), suppose that $a, b \in H$. Then $ab \in H$ and we have:

$$T_H(\delta_a \star \delta_b) = T_H(\delta_{ab}) = \delta_{ab} = \delta_a \star \delta_b = T_H(\delta_a) \star T_H(\delta_b).$$

Part (d) is trivial. □

Proposition 3.2. *Let G be a locally compact group and H be a clopen σ-compact subgroup of G. Then the restriction of T_H^* to $M(G)^*_{\sigma 0}$ is a surjection onto $M(H)^*$.*

Proof. Take an arbitrary element $F \in M(H)^*$. We let $J = R_H^*(F) \in M(G)^*$. We prove that J is in fact in $M(G)^*_{\sigma 0}$. Take an arbitrary $\epsilon > 0$. If $\mu \in M(G)$ and $|\mu| = 0$ on H then we have

$$|\langle J, \mu \rangle| = |\langle F, R_H(\mu) \rangle| = |\langle F, 0 \rangle| = 0 < \epsilon.$$

Therefore $J \in M(G)^*_{\sigma 0}$. Moreover, from part (a) of Proposition 3.1, $F = T_H^*(J)$. Hence the restriction of T_H^* to $M(G)^*_{\sigma 0}$ is surjective. □

We also have the following proposition:

Proposition 3.3. *Suppose that G is a locally compact group and H is a clopen σ-compact subgroup of G. Assume that $F \in M(H)^*$ and $K = R_H^*(F) \in M(G)^*_{\sigma 0}$. If $\lambda \in M(G)^*_{\sigma 0}$ with $T_H^*(\lambda) = F$ then $\lambda 1_H = K$.*

Proof. For each $\mu \in M(G)$, by part (d) of Proposition 3.1, we have:

$$\langle K, \mu \rangle = \langle F, R_H(\mu) \rangle = \langle T_H^*(\lambda), R_H(\mu) \rangle = \langle \lambda, T_H(R_H(\mu)) \rangle = \langle \lambda, 1_H \cdot \mu \rangle = \langle \lambda 1_H, \mu \rangle.$$

□

Remark 3.4. *From now on we use the notation Θ_H for the restriction of adjoint of T to $M(G)^*_{\sigma 0}$. Therefore we can assume that Θ_H^* maps $M(H)^{**}$ into $(M(G)^*_{\sigma 0})^*$.*

We have the following propositions:

Proposition 3.5. *Suppose that G is a locally compact group and H is a clopen σ-compact subgroup of G. Assume that $n \in (M(G)^*_{\sigma 0})^*$ and $m \in M(H)^{**}$. Let $n_H \in M(H)^{**}$ be defined as*

$$\langle n_H, F\rangle = \langle n, R_H^*(F)\rangle\,,\ \ (F \in M(H)^*)\,.$$

Then

$$\langle \Theta_H^*(n_H)\Box\Theta_H^*(m), R_H^*(F)\rangle = \langle n\Box\Theta_H^*(m), R_H^*(F)\rangle\,,$$

$$\langle \Theta_H^*(n_H)\diamond\Theta_H^*(m), R_H^*(F)\rangle = \langle n\diamond\Theta_H^*(m), R_H^*(F)\rangle\,.$$

Proof. We prove the first of the two identities above. The proof of the second one follows the same lines with some minor modifications.

Step 1: First of all we prove that if $\mu \in M(G)$ then $\Theta_H(R_H^*(F)\cdot\mu) = \Theta_H(R_H^*(F)\cdot T_H(R_H(\mu)))$. To this end, for each $\nu \in M(H)$, by using part (b) of Proposition 3.1 we have:

$$\begin{aligned}\langle \Theta_H(R_H^*(F)\cdot\mu), \nu\rangle &= \langle R_H^*(F)\cdot\mu, T_H(\nu)\rangle = \langle R_H^*(F), \mu\star T_H(\nu)\rangle = \langle F, R_H(\mu\star T_H(\nu))\rangle \\ &= \langle F, R_H(T_H(R_H(\mu))\star T_H(\nu))\rangle = \langle R_H^*(F), T_H(R_H(\mu))\star T_H(\nu)\rangle \\ &= \langle R_H^*(F)\cdot T_H(R_H(\mu)), T_H(\nu)\rangle = \langle \Theta_H(R_H^*(F)\cdot T_H(R_H(\mu))), \nu\rangle\,.\end{aligned}$$

Step 2: Next we prove that

$$\langle \Theta_H^*(m)\cdot R_H^*(F), \mu\rangle = \langle \Theta_H(\Theta_H^*(m)\cdot R_H^*(F)), R_H(\mu)\rangle\ \ (\forall \mu \in M(G))\,.$$

This is true because by using the result of step 1, we have:

$$\begin{aligned}\langle \Theta_H^*(m) \cdot R_H^*(F), \mu\rangle &= \langle \Theta_H^*(m), R_H^*(F) \cdot \mu\rangle = \langle m, \Theta_H(R_H^*(F) \cdot \mu)\rangle \\ &= \langle m, \Theta_H(R_H^*(F) \cdot T_H(R_H(\mu)))\rangle = \langle \Theta_H^*(m), R_H^*(F) \cdot T_H(R_H(\mu))\rangle \\ &= \langle \Theta_H^*(m) \cdot R_H^*(F), T_H(R_H(\mu))\rangle = \langle \Theta_H(\Theta_H^*(m) \cdot R_H^*(F)), R_H(\mu)\rangle .\end{aligned}$$

Step 3: Note that $\Theta_H(\Theta_H^*(m) \cdot R_H^*(F))$ is in $M(H)^*$. Let

$$L = R_H^*(\Theta_H(\Theta_H^*(m) \cdot R_H^*(F))) \in M(G)_{\sigma 0}^*$$

so that $\langle L, \mu\rangle = \langle \Theta_H(\Theta_H^*(m) \cdot R_H^*(F)), R_H(\mu)\rangle$ for every $\mu \in M(G)$. By using the definition of n_H in the statement of the proposition, and the definition of L, we have

$$\langle n_H, \Theta_H(\Theta_H^*(m) \cdot R_H^*(F))\rangle = \langle n, L\rangle . \tag{3.1}$$

Moreover, by using the result of step 2, for each $\mu \in M(G)$ we have

$$\langle L, \mu\rangle = \langle \Theta_H(\Theta_H^*(m) \cdot R_H^*(F)), R_H(\mu)\rangle = \langle \Theta_H^*(m) \cdot R_H^*(F), \mu\rangle .$$

Therefore

$$L = \Theta_H^*(m) \cdot R_H^*(F). \tag{3.2}$$

Step 4: Finally by using equations (3.1), (3.2), we have:

$$\begin{aligned}\langle \Theta_H^*(n_H) \square \Theta_H^*(m), R_H^*(F)\rangle &= \langle \Theta_H^*(n_H), \Theta_H^*(m) \cdot R_H^*(F)\rangle = \langle n_H, \Theta_H(\Theta_H^*(m) \cdot R_H^*(F))\rangle \\ &= \langle n, L\rangle = \langle n, \Theta_H^*(m) \cdot R_H^*(F)\rangle = \langle n \square \Theta_H^*(m), R_H^*(F)\rangle .\end{aligned}$$

$\square$

Proposition 3.6. *Suppose that G is a locally compact group and H is a clopen σ-compact subgroup of G. Assume that $n \in (M(G)^*_{\sigma 0})^*$ and $n_H \in M(H)^{**}$ be as defined in the preceding proposition. Then $\Theta_H^*(n_H) = 1_H \cdot n$, where 1_H is considered as an element of $(M(G)^*_{\sigma 0})^{**}$.*

Proof. Take $\lambda \in M(G)^*_{\sigma 0}$. Suppose that $K = R_H^*(\Theta_H(\lambda))$. By using Proposition 3.3, we can write $K = 1_H\lambda$. Also by using the definition of n_H, we have:

$$\langle \Theta_H^*(n_H), \lambda \rangle = \langle n_H, \Theta_H(\lambda) \rangle = \langle n, K \rangle = \langle n, \lambda 1_H \rangle = \langle 1_H \cdot n, \lambda \rangle .$$

Hence $\Theta_H^*(n_H) = 1_H \cdot n$. $\square$

Proposition 3.7. *Suppose that G is a locally compact group and H is a clopen σ-compact subgroup of G. Suppose that $m_1, m_2 \in M(H)^{**}$. Then*

$$\Theta_H^*(m_1 \square m_2) = \Theta_H^*(m_1) \square \Theta_H^*(m_2),$$

$$\Theta_H^*(m_1 \diamond m_2) = \Theta_H^*(m_1) \diamond \Theta_H^*(m_2).$$

Proof. Let (μ_i), (ν_i) be nets in $M(H)$ such that $\mu_i \xrightarrow{\text{wk}^*} m_1$ and $\nu_i \xrightarrow{\text{wk}^*} m_2$ with respect to the weak-star topology in $M(H)^{**}$. Using part (c) of Proposition 3.1, we have:

$$\begin{aligned}\Theta_H^*(m_1 \square m_2) &= \Theta_H^*(\text{wk}^* \lim_i \text{wk}^* \lim_j \mu_i \star \nu_j) = \text{wk}^* \lim_i \text{wk}^* \lim_j \Theta_H^*(\mu_i \star \nu_j) \\ &= \text{wk}^* \lim_i \text{wk}^* \lim_j T_H(\mu_i \star \nu_j) = \text{wk}^* \lim_i \text{wk}^* \lim_j T_H(\mu_i) \star T_H(\nu_j) \\ &= \text{wk}^* \lim_i \text{wk}^* \lim_j \Theta_H^*(\mu_i) \square \Theta_H^*(\nu_j) = \Theta_H^*(m_1) \square \Theta_H^*(m_2).\end{aligned}$$

The proof of $\Theta_H^*(m_1 \diamond m_2) = \Theta_H^*(m_1) \diamond \Theta_H^*(m_2)$ follows similar lines. $\square$

Remark 3.8. *The proof of the preceding proposition is the same as the proof of the result that if T is a continuous algebra homomorphism from the Banach algebra A into the Banach algebra B, then $T^{**} : (A^{**}, \square) \longrightarrow (B^{**}, \square)$ is an algebra homomorphism.*

Theorem 3.9. *Let G be a locally compact group. Then the left topological center of $(M(G)^*_{\sigma 0})^*$ is $M(G)$.*

Proof. It is easy to check that $M(G)$ is contained in the left topological center. Our objective is to prove that $Z_t((M(G)^*_{\sigma 0})^*) \subseteq M(G)$. Take $n \in Z_t((M(G)^*_{\sigma 0})^*)$. Let H be a clopen σ-compact subgroup of G and suppose that n_H is defined as in Proposition 3.5. Take an arbitrary $m \in M(H)^{**}$ and $F \in M(H)^*$. Let $K = R_H^*(F) \in M(G)^*_{\sigma 0}$. By using the fact that $n \in Z_t((M(G)^*_{\sigma 0})^*)$ together with Proposition 3.5 and Proposition 3.7 , we have:

$$\begin{aligned}\langle n_H \square m, F\rangle &= \langle n_H \square m, \Theta_H(K)\rangle = \langle \Theta_H^*(n_H \square m), K\rangle = \langle \Theta_H^*(n_H) \square \Theta_H^*(m), K\rangle \\ &= \langle n \square \Theta_H^*(m), K\rangle = \langle n \diamond \Theta_H^*(m), K\rangle = \langle \Theta_H^*(n_H) \diamond \Theta_H^*(m), K\rangle \\ &= \langle \Theta_H^*(n_H \diamond m), K\rangle = \langle n_H \diamond m, \Theta_H(K)\rangle = \langle n_H \diamond m, F\rangle .\end{aligned}$$

Since $F \in M(H)^*$ is arbitrary, this implies that $n_H \Box m = n_H \diamond m$. Since $m \in M(H)^{**}$ is arbitrary, we conclude that $n_H \in Z_t(M(H)^{**})$. By using Proposition 1.54 for $M(H)$, we conclude that $n_H \in M(H)$. Therefore $\Theta_H^*(n_H) = T_H(n_H) \in M(G)$.

Proposition 3.6 and the fact that $\Theta_H^*(n_H)$ is in $M(G)$ implies that $1_H \cdot n \in M(G)$. On the other hand by using Corollary 2.26, we have $n = \lim_{H \in \Gamma}(1_H \cdot n)$. This implies that $n \in M(G)$ because $M(G)$ is closed with respect to the norm-topology and each $1_H \cdot n$ is in $M(G)$. Since n is an arbitrary element of $Z_t((M(G)^*_{\sigma 0})^*)$, we conclude that $Z_t((M(G)^*_{\sigma 0})^*) \subseteq M(G)$. $\Box$

3.2 Left topological center of $L^\infty_{\sigma 0}(G)^*$

All the propositions in the preceding section remain valid if we replace $M(G)$, $M(H)$, $M(G)^{**}$, $M(G)^*_{\sigma 0}$, $(M(G)^*_{\sigma 0})^*$, $M(H)^{**}$ by $L^1(G)$, $L^1(H)$, $L^1(G)^{**}$, $L^\infty_{\sigma 0}(G)$, $L^\infty_{\sigma 0}(G)^*$, $L^1(H)^{**}$. Therefore one can similarly prove that the left topological center of $L^\infty_{\sigma 0}(G)^*$ is $L^1(G)$. However here we aim to prove this fact in a different way. First, we will prove a lemma.

Lemma 3.10. *Suppose that $m \in L^1(G)^{**}$ and assume that $\mu \in M(G)$ is the restriction of m to $C_0(G)$. If $mf \in L^1(G)$ for every $f \in L^1(G)$ then $mf = \mu f$ for every $f \in L^1(G)$.*

Proof. Recall from the preceding chapter that $\overline{\pi}$ maps $M(G)^{**}$ onto $LUC(G)^*$. From Proposition 1.55, there exists $n \in C_0(G)^\perp$ such that $\overline{\pi}(m) = \mu + n$. For every

$f \in L^1(G)$, by using Proposition 2.16, we can write

$$mf = \overline{\pi}(mf) = \overline{\pi}(m)\overline{\pi}(f) = \overline{\pi}(m)f = \mu f + nf.$$

The fact that $nf = mf - \mu f \in L^1(G) \cap C_0(G)^{\perp}$ implies that $nf = 0$ for each $f \in L^1(G)$. Therefore $mf = \mu f$ for every $f \in L^1(G)$. $\square$

Theorem 3.11. *Let G be a locally compact group. The left topological center of $L^{\infty}_{\sigma 0}(G)^*$ is $L^1(G)$.*

Proof. It is easy to check that $L^1(G) \subseteq Z_t^l(L^{\infty}_{\sigma 0}(G)^*)$. Therefore it suffices to show that $L^1(G)$ includes $Z_t^l(L^{\infty}_{\sigma 0}(G)^*)$. Take an arbitrary element $m \in Z_t^l(L^{\infty}_{\sigma 0}(G)^*)$.
Step 1: We first aim to prove that $mf \in L^1(G)$ for every $f \in L^1(G)$. Consider $n \in (M(G)^*_{\sigma 0})^*$ and a net (n_i) in $(M(G)^*_{\sigma 0})^*$ such that n_i converges to n with respect to the weak-star topology in $(M(G)^*_{\sigma 0})^*$. Then (fn_i) converges to fn with respect to the weak-star topology in $(M(G)^*_{\sigma 0})^*$. Moreover Proposition 2.54 implies that (fn_i) and fn are in $L^{\infty}_{\sigma 0}(G)^*$. Hence, by Lemma 2.14, (fn_i) converges to fn with respect to the weak-star topology in $L^{\infty}_{\sigma 0}(G)^*$ too. Since $m \in Z_t^l(L^{\infty}_{\sigma 0}(G)^*)$ we can conclude that $m(fn_i)$ converges to $m(fn)$ with respect to the weak-star topology of $L^{\infty}_{\sigma 0}(G)^*$. Note that Proposition 2.54 implies that (fn_i) and (fn) are in $L^{\infty}_{\sigma 0}(G)^*$. Therefore $m(fn_i)$ and $m(fn)$ are in $L^{\infty}_{\sigma 0}(G)^*$ too and again by Lemma 2.14, we can conclude that $(mf)n_i$ converges to $(mf)n$ with respect to the weak-star topology in $(M(G)^*_{\sigma 0})^*$ too. Therefore mf is in $Z_t^l((M(G)^*_{\sigma 0})^*) = M(G)$. Moreover by Corollary 1.41, we

can find $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$. Therefore $mf = (mf_1)f_2$ is in $L^1(G)$, because $mf_1 \in M(G)$.

Step 2: By part (a) of Proposition 2.69, every $E \in \xi_1(G)$ is a cluster point (with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$) of a net (e_i), an approximate identity of $L^1(G)$ bounded by 1. Let $\mu \in M(G)$ be the restriction of m to $C_0(G)$. From Lemma 3.10 and the result of step 1, $me_i = \mu e_i$. Therefore, by using Proposition 1.54, we have

$$m = mE = \text{wk*} \lim_i (me_i) = \text{wk*} \lim_i (\mu e_i) = \mu E.$$

Where the limit is with respect to the weak-star topology on $L^1(G)^{**}$ or equivalently with respect to the weak-star topology on $M(G)^{**}$ (see Remark 2.15). Finally, Part (c) of Lemma 2.75 implies that $m \in L^1(G)$. □

Chapter 4

DERIVATIONS AND LEFT MULTIPLIERS ON SOME BANACH ALGEBRAS RELATED TO A LOCALLY COMPACT GROUP

The first section of this chapter is concerned with addressing various derivation and left multiplier questions related to a locally compact group. In the second section we focus on some questions regarding weakly compact derivation and left multipliers. In section 4.3 we introduce the concept of quasi-Arens regularity, topological quasi-center and quasi-weakly almost periodic functionals for a general Banach algebra $\mathcal{A}$ and prove some results about derivations from $\mathcal{A}$ to its dual. Finally, In section 4.4, we focus on a special case where $\mathcal{A} = L^1(G)$.

4.1 Some derivation and left multiplier problems

Assume that (e_i) is a bounded approximate identity of $L^1(G)$ bounded by 1. As a special case of Johnson's theorem (see Theorem 2.9.51 of [12]), every essential Banach $L^1(G)$-bimodule $\mathcal{X}$ can be made into a Banach $M(G)$-bimodule by defining left and right module actions as

$$\mu \cdot x = \lim_i ((\mu \star e_i) \cdot x), \quad x \cdot \mu = \lim_i (x \cdot (e_i \star \mu)), \ \ (\mu \in M(G), x \in \mathcal{X}).$$

Note that, since $\mathcal{X}$ is an essential Banach $L^1(G)$-bimodule, both above limits exist. The following theorem will be useful in several occasions later on in this chapter.

Theorem 4.1. *Let G be a locally compact group and $\mathcal{X}$ be an essential Banach $L^1(G)$-bimodule. Suppose that (e_i) is a bounded approximate identity of $L^1(G)$ bounded by 1 and $D : L^1(G) \longrightarrow \mathcal{X}^*$ is a continuous derivation. Then there is a unique derivation $\widetilde{D} : M(G) \longrightarrow \mathcal{X}^*$, $\mu \longmapsto \mathrm{wk}^* \lim_i D(\mu \star e_i)$ $(\forall \mu \in M(G))$ such that $\widetilde{D}|_{L^1(G)} = D$; in the case where D is inner, $\widetilde{D}$ is also inner. Furthermore $\widetilde{D} : (M(G), so) \longrightarrow (\mathcal{X}^*, \sigma(\mathcal{X}^*, \mathcal{X}))$ and $\widetilde{D} : M(G) \longrightarrow \mathcal{X}^*$ are both continuous and $\left\|\widetilde{D}\right\| = \|D\|$.*

Proof. See Theorem 2.9.53 and Theorem 3.3.40 of [12]. □

We also have the following propositions.

Proposition 4.2. *Let G be a locally compact group and $D : L^1(G) \longrightarrow L^1(G)$ be a derivation. Then D is automatically continuous.*

Proof. The Banach algebra $L^1(G)$ is semi-simple. Hence Theorem 2.2 of [32] applies. □

Theorem 4.3. *Let G be a locally compact group and suppose that $D : L^1(G) \longrightarrow M(G)$ is a derivation. Then, there exists $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$.*

Proof. See Corollary 1.2 of [41] or Corollary B in [5]. □

The following Corollaries are pertinent to Theorem 4.3.

Corollary 4.4. *Let G be a locally compact group and suppose that $D : L^1(G) \longrightarrow L^1(G)$ is a derivation. Then, there exists $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$.*

Proof. This is immediate from Theorem 4.3. $\square$

Corollary 4.5. *Let G be a locally compact group and suppose that $D : M(G) \longrightarrow M(G)$ is a derivation. Then D is inner.*

Proof. Part (c) of Proposition 1.47 implies that $C_0(G)$ is an essential Banach $L^1(G)$-bimodule. Apply Theorem 4.3 and Theorem 4.1. $\square$

As a special case of Corollary 4.5, we have the following proposition.

Proposition 4.6. *Suppose that G is a locally compact group. Let $D : M(G) \longrightarrow L^1(G)$ be a derivation. Then D is inner.*

Proof. Corollary 4.5 implies that there is $\mu \in M(G)$ such that $D(\nu) = \mu \star \nu \ - \nu \star \mu$ for every $\nu \in M(G)$. We write $\mu = \mu_d + \mu_a + \mu_s$, where μ_d, μ_a and μ_s are respectively elements of $M_d(G)$, $L^1(G)$ and $M_s(G)$. On the other hand, for $x \in G$, we have

$$D(\delta_x) = \delta_x \star \mu - \mu \star \delta_x \quad = \quad \delta_x \star \mu_d - \mu_d \star \delta_x + \delta_x \star \mu_a - \mu_a \star \delta_x + \delta_x \star \mu_s - \mu_s \star \delta_x.$$

Noting that $D(\delta_x)$ is in $L^1(G)$ and the translation of elements of $M_d(G)$, $L^1(G)$ and $M_s(G)$ is again in $M_d(G)$, $L^1(G)$ and $M_s(G)$ respectively, we conclude that

$$\delta_x \star \mu_d - \mu_d \star \delta_x + \delta_x \star \mu_s - \mu_s \star \delta_x = 0.$$

Therefore $D(\delta_x) = \delta_x \star \mu_a - \mu_a \star \delta_x$. For every $\nu \in M(G)$, by part (e) of Proposition 1.49, there exists a net (ν_i), consisting of linear combination of point-masses, that converges to ν with respect to the strong operator topology in $M(G)$. We use the last part of Theorem 4.1 and write

$$D(\nu) = \text{wk*}\lim_i D(\nu_i) = \text{wk*}\lim_i (\nu_i \star \mu_a - \mu_a \star \nu_i) = \nu \star \mu_a - \mu_a \star \nu.$$

This shows that D is indeed inner in $L^1(G)$. □

Although our attempt to characterize all derivations on $L^1(G)^{**}$ has not been successful but for the special case of weak-star continuous derivations on $L^1(G)^{**}$, we have the following interesting theorem.

Theorem 4.7. *Suppose that G is a locally compact group and $D : L^1(G)^{**} \to L^1(G)^{**}$ is a derivation that is weak-star continuous. Then there is $\mu \in M(G)$ such that $D(n) = n\mu - \mu n$ for all $n \in L^1(G)^{**}$.*

Proof. For $n \in L^1(G)^{**}$ and $f \in L^1(G)$, we have $D(fn) = D(f)n + fD(n)$. Therefore $D(f)n = D(fn) - fD(n)$. For a fixed $f \in L^1(G)$, by weak-star continuity of D we can conclude that $n \longmapsto D(fn) - fD(n) = D(f)n$ is weak-star continuous. This implies that $D(f) \in L^1(G)$, because $Z_t^1(L^1(G)^{**}) = L^1(G)$. By Theorem 4.3, there exists $\mu \in M(G)$ such that $D(f) = f * \mu - \mu * f$. Therefore by Goldstine's theorem, the fact that $M(G) = Z_t^1(M(G)^{**})$ and weak-star continuity of D, we conclude that $D(n) = n\mu - \mu n$. □

Remark 4.8. *In Theorem 4.7, one can replace* $L^1(G)^{**}$ *by* $L_0^\infty(G)^*$, $L_{\sigma 0}^\infty(G)^*$, $(M(G)_{\sigma 0}^*)^*$ *or* $M(G)^{**}$ *and the result will still be valid. The proof follows exactly the same lines. However we can not assure this proof works for* $(M(G)_0^*)^*$, *because we have not been able to prove that* $M(G) = Z_t((M(G)_0^*)^*)$.

We know from part (a) of Lemma 2.55 that $L^1(G)$ is an ideal of $L_0^\infty(G)^*$. Therefore if we consider the algebra product in $L_0^\infty(G)^*$ as module action, then $L^1(G)$ is a Banach $L_0^\infty(G)^*$-bimodule. Hence $L^1(G)$ is a Banach $EL_0^\infty(G)^*$-bimodule for $E \in \xi_1(G)$. We have the following proposition.

Proposition 4.9. *Suppose that* G *is a locally compact group. Assume that* $E \in \xi_1(G)$ *and* $D : EL_0^\infty(G)^* \longrightarrow L^1(G)$ *is a derivation. Then* D *is inner.*

Proof. According to Corollary 2.86, the mapping $\Lambda : EF \longmapsto \pi(F)$ $(\forall F \in L_0^\infty(G)^*)$ is an isometric algebra isomorphism from $EL_0^\infty(G)^*$ onto $M(G)$. It is straightforward to verify that for $\mu \in M(G)$ and $f \in L^1(G)$, we have $\Lambda^{-1}(\mu) \cdot f = \mu \star f$ and $f \cdot \Lambda^{-1}(\mu) = f \star \mu$. Hence if we define $\Delta : M(G) \longrightarrow L^1(G)$ by $\Delta(\mu) = D(\Lambda^{-1}(\mu)), (\mu \in M(G))$, then Δ is a derivation from $M(G)$ into $L^1(G)$. Hence by Proposition 4.6, there exists $h \in L^1(G)$ such that $\Delta(\mu) = \mu \star h - h \star \mu$. Therefore for each $F \in L_0^\infty(G)^*$, we have

$$\begin{aligned} D(EF) &= D(\Lambda^{-1}(\Lambda(EF))) = \Delta(\Lambda(EF)) = \Lambda(EF) \star h - h \star \Lambda(EF) \\ &= \pi(F) \star h - h \star \pi(F) = (EF) \cdot h - h \cdot (EF). \end{aligned}$$

□

Theorem 4.10. *Suppose that G is a locally compact group and $D : L_0^\infty(G)^* \longrightarrow L^1(G)$ is a derivation. Then D is inner.*

Proof. Assume that (e_i) is a bounded approximate identity in $L^1(G)$ that converges to $E \in \xi_1(G)$ with respect to weak-star topology. For each $n \in L_0^\infty(G)^*$, since D maps into $L^1(G)$, we can write

$$D(n) = \lim_i (e_i \star D(n)) = \lim_i [D(e_i n) - D(e_i)n]. \tag{4.1}$$

If we consider the restriction of D to $L^1(G)$, from Corollary 4.4, there is $\mu \in M(G)$ such that $D(f) = f \star \mu - \mu \star f$. Thus from (4.1)

$$D(n) = \lim_i [(e_i n) \star \mu - \mu \star (e_i n) - (e_i \star \mu - \mu \star e_i)n] = \lim_i [e_i (n\mu - \mu n)]. \tag{4.2}$$

From (4.2), it is obvious that if $n \in L_0^\infty(G)^*$ is a right annihilator for $L_0^\infty(G)^*$ then $D(n) = 0$. It is also easy to verify that $L_0^\infty(G)^*$ is the direct-sum of $EL_0^\infty(G)^*$ and $\mathrm{ran}(L_0^\infty(G)^*) \cap L_0^\infty(G)^*$. Therefore using Proposition 4.9, there exists $f \in L^1(G)$ such that for every $n \in L_0^\infty(G)^*$ we have

$$D(n) = D(En - (n - En)) = D(En) - 0 = Enf - fEn. \tag{4.3}$$

Note that $n - En$ is a right annihilator for $L_0^\infty(G)^*$ and therefore, as mentioned above, $D(n - En) = 0$. Since $nf \in L^1(G)$, we have $Enf = nf$. Also $fE = f$. Hence from (4.3), we have $D(n) = n \cdot f - f \cdot n$. $\square$

The following corollary is immediate.

Corollary 4.11. *Let G be a compact locally compact group and $D : L^1(G)^{**} \longrightarrow L^1(G)$ is a derivation. Then D is inner.*

Proposition 4.12. *Let G be a locally compact group and $E \in \xi_1(G)$ be the cluster point (with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$) of a central bounded approximate identity (e_i) in $L^1(G)$. Assume that $D^{'} : EL_0^\infty(G)^* \to L_0^\infty(G)^*$ is a derivation. Then there is $p \in (M(G)_0^*)^*$ such that $D^{'} = \mathrm{ad}_p$.*

Proof. Define $D_1 : EL_0^\infty(G)^* \to L_0^\infty(G)^*$ and $D_2 : EL_0^\infty(G)^* \to L_0^\infty(G)^*$ respectively by $D_1(Em) = ED^{'}(Em)$ and $D_2(Em) = D^{'}(Em) - ED^{'}(Em)$ for every $m \in L_0^\infty(G)^*$. Obviously, $D^{'} = D_1 + D_2$. For $m_1, m_2 \in L_0^\infty(G)^*$, we have

$$\begin{aligned} D_1(Em_1Em_2) &= ED^{'}(Em_1Em_2) = E\left[D^{'}(Em_1)Em_2 + Em_1D^{'}(Em_2)\right] \\ &= D_1(Em_1)Em_2 + Em_1ED^{'}(Em_2) = D_1(Em_1)Em_2 + Em_1D_1(Em_2). \end{aligned}$$

Therefore D_1 is a derivation. Moreover, since for each $f \in L^1(G)$, there are $f_1, f_2 \in L^1(G)$ such that $f = f_1 \star f_2$, we have

$$D_1(f) = D_1(f_1 \star f_2) = D_1(f_1)f_2 + f_1D_1(f_2).$$

Since $L^1(G)$ is an ideal of $L_0^\infty(G)^*$, the equation above implies that $D_1(L^1(G)) \subseteq L^1(G)$. By using Theorem 4.3, there is $\nu \in M(G)$ such that $D_1(f) = f \star \nu - \nu \star f$ for each $f \in L^1(G)$. Without loss of generality, we may assume that (e_i) converges to E with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$. For each $m \in L_0^\infty(G)^*$, we have

$$\begin{aligned} D_1(Em) &= ED_1(Em) = \text{wk* }\lim_i (e_i D_1(Em)) = \text{wk* }\lim_i (D_1(e_i Em) - D_1(e_i)Em) \\ &= \text{wk* }\lim_i (D_1(e_i m) - D_1(e_i)m) \qquad (4.4) \\ &= \text{wk* }\lim_i (e_i m\nu - (\nu \star e_i)m - (e_i \star \nu - \nu \star e_i)m). \end{aligned}$$

By using Lemma 1.53, we have $e_i \star \nu = \nu \star e_i$. Since $m\nu \in L^1(G)^{**}$ we have wk* $\lim_i(e_i m\nu) = Em\nu$. Moreover, the fact that ν is in the topological center of $M(G)^{**}$ and $(e_i m)$ converges to Em with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$ implies that wk* $\lim_i(\nu \star e_i m) = \nu Em$. Combining these observations with (4.4), we have:

$$D_1(Em) = \text{wk* }\lim_i (e_i m\nu - \nu e_i m) = Em\nu - \nu Em.$$

Note that $D^{'}(E) = D^{'}(EE) = D^{'}(E) + ED^{'}(E)$. Therefore $ED^{'}(E) = 0$. Hence $D^{'}(E)$ is in $\text{ran}(L_0^\infty(G)^*)$. We also have

$$\begin{aligned} D_2(Em) &= D^{'}(Em) - ED^{'}(Em) \qquad (4.5) \\ &= D^{'}(E)Em + ED^{'}(Em) - ED^{'}(Em) = D^{'}(E)m. \end{aligned}$$

Therefore D_2 maps into $\text{ran}(L_0^\infty(G)^*)$. Also for every $m_1, m_2 \in L_0^\infty(G)^*$, from equation (4.5) we have

$$D_2(Em_1Em_2) = D^{'}(E)(m_1Em_2) = (D^{'}(E)m_1)Em_2 = D_2(Em_1)Em_2.$$

Hence D_2 is a left multiplier from $EL_0^\infty(G)^*$ into $\mathrm{ran}(L_0^\infty(G)^*)$ and $D_2(Em) = D^{'}(E)Em = D^{'}(E)Em - EmD^{'}(E)$. For $p = \nu - D^{'}(E)$, we have $D^{'} = \mathrm{ad}_p$. Obviously, p is in $(M(G)_0^*)^*$. $\square$

Proposition 4.13. *Let G be a [SIN] group. Suppose that $D : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a derivation. Then there is $p \in (M(G)_0^*)^*$ and a left multiplier $\Delta : L_0^\infty(G)^* \to$ $\mathrm{ran}(L_0^\infty(G)^*)$ such that $D = \mathrm{ad}_p + \Delta$.*

Proof. From part (b) of Proposition 1.43, $L^1(G)$ has a central approximate identity (e_i) bounded by 1. Assume that $E \in \xi_1(G)$ be the cluster point of (without loss of generality we can assume it is the limit of) (e_i) with respect to the weak-star topology in $L^1(G)^{**}$ or $M(G)^{**}$. Let $D^{'}$ be the restriction of D to $EL_0^\infty(G)^*$. Obviously $D^{'}$ is a derivation. Hence from Proposition 4.12 there is $p \in (M(G)_0^*)^*$ such that $D'(Em) = pEm - Emp$ for each $m \in L_0^\infty(G)^*$. Put $\Delta = D - \mathrm{ad}_p$. We will prove that Δ maps $L_0^\infty(G)^*$ into $\mathrm{ran}(L_0^\infty(G)^*)$ and consequently it is a left multiplier. Obviously $\Delta(Em) = D(Em) - D^{'}(Em) = 0$ for each $m \in L_0^\infty(G)^*$. Therefore $\Delta(m) = \Delta(Em) + \Delta(m - Em) = \Delta(m - Em)$ for each $m \in L_0^\infty(G)^*$. On the other hand, $n\Delta(m - Em) = \Delta(n(m - Em)) - \Delta(n)(m - Em) = 0$, for each $n \in L^1(G)^{**}$. Hence, for each $m \in L_0^\infty(G)^*$, $\Delta(m) \in \mathrm{ran}(L_0^\infty(G)^*)$. $\square$

Note that every locally compact group that is compact or abelian is a [SIN] group. Therefore the following corollaries are immediate from Proposition 4.13.

Corollary 4.14. *Let G be an abelian locally compact group. Suppose that $D : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a derivation. Then there is $p \in (M(G)_0^*)^*$ and a left multiplier $\Delta : L_0^\infty(G)^* \to \operatorname{ran}(L_0^\infty(G)^*)$ such that $D = \operatorname{ad}_p + \Delta$.*

Corollary 4.15. *Let G be a compact locally compact group and $D : L^1(G)^{**} \to L^1(G)^{**}$ is a derivation. Then there is $p \in M(G)^{**}$ and a left multiplier $\Delta : L^1(G)^{**} \to \operatorname{ran}(L^1(G)^{**})$ such that $D = \operatorname{ad}_p + \Delta$.*

Since $L_0^\infty(G)^*$ is an ideal of $(M(G)_0^*)^*$, the following proposition can be stated and proved.

Proposition 4.16. *Let G be a [SIN] group. Suppose that $D : (M(G)_0^*)^* \to L_0^\infty(G)^*$ is a derivation. Then there is $p \in (M(G)_0^*)^*$ and a left multiplier $\Delta : (M(G)_0^*)^* \to \operatorname{ran}(L_0^\infty(G)^*)$ such that $D = \operatorname{ad}_p + \Delta$.*

Proof. Note that for $E \in \xi(G)$, $EL_0^\infty(G)^* = E\,(M(G)_0^*)^*$, because $\xi(G) \subseteq L_0^\infty(G)^*$. Therefore Proposition 4.12 remains valid. One can follow an argument similar to that of Proposition 4.13 and prove the statement. □

In a similar way one can extend Corollary 4.14 and Corollary 4.15. We also have the following proposition.

Proposition 4.17. *Suppose that G is a locally compact group and $m \in L_0^\infty(G)^*$. Then $\operatorname{ad}_m : L_0^\infty(G)^* \longrightarrow L_0^\infty(G)^*,\ n \longmapsto nm - mn\ \ (\forall n \in L_0^\infty(G)^*)$ maps into $\operatorname{ran}(L_0^\infty(G)^*)$ in $L^1(G)^{**}$ if and only if $\pi(m) \in Z(M(G))$.*

Proof. Suppose that $nm - mn \in \operatorname{ran}(L_0^\infty(G)^*)$ for every $n \in L_0^\infty(G)^*$. By Proposition 2.42, for every $\nu \in M(G)$ there exists $p \in L_0^\infty(G)^*$ such that $\nu = \pi(p)$. Together, Propositions 2.16 and 2.88 imply that $\pi(m)\nu - \nu\pi(m) = \pi(mp - pm) = 0$. Hence $\pi(m) \in Z(M(G))$. For the converse, suppose that $\pi(m) \in Z(M(G))$. Then for each $n \in L_0^\infty(G)^*$, $\pi(nm - mn) = \pi(n)\pi(m) - \pi(m)\pi(n) = 0$. From this and Proposition 2.88 we have $nm - mn \in \operatorname{ran}(L_0^\infty(G)^*)$. $\square$

Similarly, one can prove the following proposition.

Proposition 4.18. *Suppose that G is a locally compact group and $m \in (M(G)_0^*)^*$. Then $\operatorname{ad}_m : (M(G)_0^*)^* \longrightarrow (M(G)_0^*)^*$, $n \longmapsto nm - mn$ $(\forall n \in (M(G)_0^*)^*)$ maps into $\operatorname{ran}(L_0^\infty(G)^*)$ in $M(G)^{**}$ if and only if $\overline{\pi}(m) \in Z(M(G))$.*

Remark 4.19. *Proposition 4.18 will remain valid if we substitute $Z(M(G))$ by the center of $M(G)$ in $LUC(G)^*$ and we replace $(M(G)_0^*)^*$ by $(M(G)_{\sigma 0}^*)^*$ or $M(G)^{**}$.*

4.2 Weakly compact derivations and multipliers

Next, we focus on some problems related to weakly compact derivations and left multipliers.

Theorem 4.20. *Let G be a locally compact group. Suppose that $D : M(G) \longrightarrow M(G)$ is a weakly compact derivation. Then there is $f \in L^1(G)$ such that $D = \operatorname{ad}_f$.*

Proof. From Corollary 4.5 we infer that there exists $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$. We aim to prove that $D(M(G)) \subseteq L^1(G)$. Take $\nu \in M(G)$. Let (e_i) be a bounded approximate identity in $L^1(G)$. Knowing that $e_i \xrightarrow{\text{wk*}} \delta_e$ with respect to the weak-star topology in $M(G)$, we can write

$$\begin{aligned} D(\nu) &= \nu \star \mu - \mu \star \nu = \delta_e \star \nu \star \mu - \mu \star \delta_e \star \nu = \text{wk*}\lim_i \left[e_i \star \nu \star \mu - \mu \star e_i \star \nu\right] \\ &= \text{wk*}\lim_i D(e_i \star \nu). \end{aligned} \tag{4.6}$$

On the other hand, since D is weakly compact and $(e_i \star \nu)$ is bounded, there exists a subnet of $(e_i \star \nu)$ (we denote it by $(e_i \star \nu)$ again) and $\eta \in M(G)$ such that $e_i \star \nu \star \mu - \mu \star e_i \star \nu \xrightarrow{\text{wk}} \eta$. Since $L^1(G)$ is closed with respect to the weak topology and $(e_i \star \nu \star \mu - \mu \star e_i \star \nu)$ is in $L^1(G)$, we conclude that $\eta \in L^1(G)$. From this and equation (4.6) we deduce that $D(\nu) = \eta \in L^1(G)$. Therefore by Proposition 4.6, there is $f \in L^1(G)$ such that $D = \mathrm{ad}_f$. □

Next we give a necessary and sufficient condition for the existence of a non-zero weakly compact derivation on $L^1(G)$.

Theorem 4.21. *Let G be a locally compact group. Then the following conditions are equivalent:*

(a) G is a non-abelian compact group;

(b) There exists a non-zero weakly compact derivation on $L^1(G)$.

Moreover, if $D : L^1(G) \longrightarrow L^1(G)$ is a weakly compact derivation, then there exists $h \in L^1(G)$ such that $D = \mathrm{ad}_h$.

Proof. Suppose that G is compact and non-abelian. Then Proposition 1.52 implies that there exists $h \in L^1(G)$ that is not in the center of $L^1(G)$. Therefore $D = \mathrm{ad}_h$ is non-zero. Proposition 1.45, implies that D, as the difference of a weakly compact right and a weakly compact left multiplier, is weakly compact. Conversely, suppose that D is a non-zero weakly compact derivation. Then there exists $f \in L^1(G)$ with $D(f) \neq 0$. Define $T_{D(f)} : L^1(G) \longrightarrow L^1(G)$, $g \longmapsto D(f) \star g \;\; (\forall g \in L^1(G))$. We can write

$$T_{D(f)}(g) = D(f) \star g = D(f \star g) - f \star D(g) = D \circ T_f(g) - T_f \circ D(g), \;\; \left(g \in L^1(G)\right).$$

Therefore $T_{D(f)}$ is weakly compact by Proposition 1.11 and the fact that D is weakly compact. Moreover $T_{D(f)}$ is non-zero. Suppose to the contrary that it is zero and let (e_i) be a bounded approximate identity for $L^1(G)$. Then $D(f) = \lim_i(D(f) \star e_i) = 0$, that is a contradiction. Therefore $T_{D(f)}$ is a non-zero weakly compact left multiplier on $L^1(G)$. Then from Corollary 1.46 we have that G is compact. Now we prove that G is non-abelian. Suppose to the contrary that G is abelian. Using Proposition 4.4, there exists $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$. But Proposition 1.52 implies that $\mathrm{ad}_\mu = 0$, which is a contradiction. To prove the last clause of the proposition, assume that D is a non-zero weakly compact derivation on $L^1(G)$. Theorem 4.1 implies that there exists $\widetilde{D} : M(G) \longrightarrow M(G)$ that extends D and $\widetilde{D}(\mu) = \text{wk}^* \lim_i D(\mu \star e_i)$ for each $\mu \in M(G)$. Using the assumption that D is weakly compact and the fact that the net $(\mu \star e_i)$ is a bounded net in $L^1(G)$, there exists a subnet of $(\mu \star e_i)$ (we denote it by $(\mu \star e_i)$ again) such that

$$\tilde{D}(\mu) = \text{wk}^* \lim_i \left(D(\mu \star e_i)\right) = \text{wk} \lim_i \left(D(\mu \star e_i)\right). \tag{4.7}$$

By using the equation (4.7), we have

$$\left\{\tilde{D}(\mu) : \mu \in M(G), \|\mu\| \leq 1\right\} \subseteq \left\{D(f) : f \in L^1(G), \|f\| \leq 1\right\}^{\text{wk-closure}}.$$

The fact that D is weakly compact implies that $\{D(f) : f \in L^1(G), \|f\| \leq 1\}^{\text{wk-closure}}$ is weakly compact. Therefore $\left\{\tilde{D}(\mu) : \mu \in M(G), \|\mu\| \leq 1\right\}^{\text{wk-closure}}$ is weakly compact. Hence $\tilde{D}$ is weakly compact too. Theorem 4.20 implies that there is $f \in L^1(G)$ such that $\widetilde{D} = \text{ad}_f$. From Theorem 4.1 we conclude that $D = \text{ad}_f$ too. $\square$

The following proposition has been proved in Theorem 3.1 of [43]. Below we give a different proof which can be replicated for $(M(G)_0^*)^*$.

Theorem 4.22. *Let G be a locally compact group. Then the following conditions are equivalent.*

(a) G is compact;

(b) $L_0^\infty(G)^*$ has a non-zero left weakly completely continuous element;

(c) There is a non-zero weakly compact left multiplier on $L_0^\infty(G)^*$.

Proof. Suppose that G is compact. Hence by Proposition 1.45, for every $f \in L^1(G)$, $\lambda_f : g \longmapsto f \star g \ \ (\forall g \in L^1(G))$ is weakly compact. Since $\lambda_f^{**}(n) = f \square n, (\forall n \in L^1(G)^{**})$, multiplication by f is a weakly compact operator on $L^1(G)^{**}$. Hence (b) follows from (a). The fact that part (b) implies part (c) is trivial. We prove that (b) implies (a). Suppose that there is a non-zero left weakly completely continuous element $n \in L_0^\infty(G)^*$ and G is not compact. Without loss of generality, we can assume that $\|n\| = 1$. Take $E \in \xi(G)$. Since the set $\{n\delta_x : x \in G\}$ is bounded and we are assuming that λ_n is weakly compact, the set $\{nE\delta_x : x \in G\}$ has a weakly compact closure in $L_0^\infty(G)^*$. Since $L_0^\infty(G)$ is a commutative C*-algebra with regard to pointwise multiplication, by Theorem 1.29, we can view $L_0^\infty(G)^*$ as $M(X)$, where X is a locally compact space. By Theorem 1.22, the set $\{|nE\delta_x| : x \in G\}$ has weakly compact closure in $L_0^\infty(G)^*$. But

$$\{|nE\delta_x| : x \in G\} = \{|nE|\,\delta_x : x \in G\} = \{|n|\,\delta_x : x \in G\}.$$

Therefore $\{|n|\,\delta_x : x \in G\}$ has a weakly compact closure in $L_0^\infty(G)^*$. Hence if we let $f \in L^1(G)$, $f > 0$, then since $m \longmapsto fm$ $(m \in L_0^\infty(G)^*)$ is weakly continuous on $L_0^\infty(G)^*$ and $f\,|n| \in L^1(G)$, the set $\{f\,|n|\,\delta_x : x \in G\}$ has a weakly compact closure in $L^1(G)$. The collection $\mathcal{K}$ of all compact subsets of G is a directed set for upward inclusion: $K_1 < K_2 \Leftrightarrow K_1 \subseteq K_2$. Since G is not compact for every $K \in \mathcal{K}$, there exists

$g_K \in K^c$. Thus we have a net $\{\delta_{g_K} : K \in \mathcal{K}\}$. The set $\{(f\,|n|) \star \delta_{g_K} : K \in \mathcal{K}\}$ has weakly compact closure, since it is contained in the set $\{(f\,|n|) \star \delta_x : x \in G\}$. Hence there exists a subnet $((f\,|n|)\star\delta_{g_{K_i}})$ and some $h \in L^1(G)$ such that $((f\,|n|)\star\delta_{g_{K_i}}) \longrightarrow h$, weakly. On the other hand, from the definition of (g_K) it can be seen that $\delta_{g_K} \longrightarrow 0$ with respect to the weak-star topology of $M(G)$. Hence $((f\,|n|) \star \delta_{g_{K_i}}) \longrightarrow 0$ weakly. In particular

$$\|f\|\,\|n\| = \langle f, 1\rangle\langle |n|\,, 1\rangle\langle \delta_{g_{K_i}}, 1\rangle = \langle f\,|n|\,\delta_{g_{K_i}}, 1\rangle \longrightarrow 0,$$

and we have a contradiction. Now we prove that (c) implies (b). Suppose that T is a non-zero weakly compact left multiplier on $L_0^\infty(G)^*$. We pick $n \in L_0^\infty(G)^*$ such that $T(n) \neq 0$. Then since $T(n)m = T(nm)$ $(\forall m \in L_0^\infty(G)^*)$ and T is weakly compact, the set $\{T(n)m : |\,\|m\| \leq 1\} = \{T(nm) : |\,\|m\| \leq 1\}$ has weakly compact closure. Thus $m \longmapsto T(n)m$ is a weakly compact operator. $\square$

We now concentrate on the case when G is compact. Note that when G is compact, $L^\infty(G) = L_0^\infty(G)$. Let's recall from Remark 1.18 that $L^\infty(G)$ is an $M(G)$-bimodule. We have the following definition.

Definition 4.23. *A state ϕ on $L^\infty(G)$ is called a* right invariant mean *if $\phi \bullet \delta_x = \phi$ for each $x \in G$ where $\langle \phi \bullet \delta_x, g\rangle = \langle \phi, \delta_x \cdot g\rangle$, for every $g \in L^\infty(G)$.*

Proposition 4.24. *Suppose that G is compact and $L^\infty(G)$ has a unique right invariant mean. Then for every weakly compact left multiplier $T : L^1(G)^{**} \to L^1(G)^{**}$ there* *exists* $f \in L^1(G)$ *such that* $T(F) = fF$ *for all* $F \in L^1(G)^{**}$.

Proof. This has been proved by V. Losert, see the corollary in [42]. $\square$

Theorem 4.25. *Let* G *be a compact group such that there is a unique right invariant mean on* $L^\infty(G)$. *Suppose that* $D : L^1(G)^{**} \to L^1(G)^{**}$ *is a weakly compact derivation. Then there is* $h \in L^1(G)$ *such that* $D(m) = hm - mh$ *for each* $m \in L^1(G)^{**}$.

Proof. Note that every compact group is a [SIN] group. In order to skip the duplication, let us revisit the proof of Proposition 4.13 and Proposition 4.12 and reconsider ν, p, E, $D^{'}$, D_1, D_2 and Δ as defined before. When D is weakly compact then $D^{'}$ is weakly compact too. Consequently, it can easily be verified that D_1 and D_2 are also weakly compact. We have proved that $D_1(L^1(G)) \subseteq L^1(G)$. Hence, from Theorem 4.21, we conclude that $\nu \in L^1(G)$. On other hand, let us define $D_3 : L^1(G)^{**} \to L^1(G)^{**}$ by $D_3(n) = D_2(En) = D^{'}(E)n$. Since D_2 is weakly compact, D_3 is weakly compact too. Therefore D_3 is a weakly compact left multiplier on $L^1(G)^{**}$. From this and Proposition 4.24 we have $D_3(n) = fn$ for some $f \in L^1(G)$. This means $(D^{'}(E) - f)n = 0$ for all $n \in L^1(G)^{**}$. In particular, $(D^{'}(E)-f) = (D^{'}(E)-f)E = 0$. Hence $D^{'}(E) \in L^1(G)$. We have already proved that $D^{'}(E) \in \mathrm{ran}(L^1(G)^{**})$. Therefore $D^{'}(E) = 0$. This implies that $p = \nu - D^{'}(E) = \nu$. Hence $D = \mathrm{ad}_\nu + \Delta$ where $\nu \in L^1(G)$. We also know that Δ is a weakly compact left multiplier that maps $L^1(G)^{**}$ into $\mathrm{ran}(L^1(G)^{**})$. Therefore again, using Proposition

4.24, there is $g \in L^1(G)$ such that $\Delta(n) = gn$. Therefore Δ maps $L^1(G)^{**}$ into $L^1(G)$ too. Hence $\Delta = 0$. This implies that $D = \mathrm{ad}_\nu$ for $\nu \in L^1(G)$. $\square$

We also have the following theorem:

Theorem 4.26. *Let G be a locally compact group. Suppose that there is a unique right invariant mean on $L^\infty(G)$. There is a non-zero weakly compact derivation on $L_0^\infty(G)^*$ if and only if G is compact and non-abelian.*

Proof. We recall that when G is compact $L_0^\infty(G) = L^\infty(G)$. Suppose that $D : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a non-zero weakly compact derivation. Then there is $n \in L_0^\infty(G)^*$ such that $D(n)$ is non-zero. We aim to show that the left multiplier $T_{D(n)} : L_0^\infty(G)^* \to L_0^\infty(G)^*$, $m \mapsto D(n)m$ is weakly compact. Note that

$$T_{D(n)}(m) = D(n)m = D(nm) - nD(m) = D \circ T_n(m) - T_n \circ D(m).$$

Hence $T_{D(n)} = D \circ T_n - T_n \circ D$. Therefore $T_{D(n)}$ is weakly compact by Proposition 1.11 and the fact that D is weakly compact. Moreover $T_{D(n)}(E) = D(n)E = D(n) \neq 0$. Hence $T_{D(n)}$ is non-zero. By Proposition 4.22, we conclude that G is compact. We aim to prove that G is also non-abelian. To the contrary suppose G is abelian. Using Theorem 4.25, there is $h \in L^1(G)$ such that $D = \mathrm{ad}_h$. But because G is abelian, we have $L^1(G) \subseteq Z(L^1(G)^{**})$. Therefore $D = \mathrm{ad}_h = 0$, that is a contradiction. Hence G is not abelian. For proving the converse, suppose that G is compact and non-abelian. Take $f \in L^1(G)$ such that $f \notin Z(L^1(G))$. There exists such an f because G is not abelian. The derivation $D = \mathrm{ad}_f$ is non-zero because $f \notin Z(L^1(G))$. It is also weakly compact by Theorem 1.45. $\square$

We also have the following result:

Proposition 4.27. *Let G be a compact group. Then every (weakly compact) derivation from $L^1(G)^{**}$ into $L^1(G)^{**}$ is inner in $M(G)^{**}$ if and only if every (weakly compact) left multiplier from $L^1(G)^{**}$ into $L^1(G)^{**}$ is inner in $M(G)^{**}$.*

Proof. Suppose that every (weakly compact) left multiplier from $L^1(G)^{**}$ into $L^1(G)^{**}$ is inner. Then by using Proposition 4.13, we can conclude that every (weakly compact) derivation from $L^1(G)^{**}$ into $L^1(G)^{**}$ is inner in $M(G)^{**}$. Conversely, suppose that every (weakly compact) derivation from $L^1(G)^{**}$ into $L^1(G)^{**}$ is inner and $T : L^1(G)^{**} \to L^1(G)^{**}$ is a (weakly compact) left multiplier. We can write $T = T_1 + T_2$ where $T_1(n) = T(E)n$ and $T_2(n) = T(n - En)$. Obviously T_1 and T_2 are (weakly compact) left multipliers. For every $f \in L^1(G)$, $T(n - En)f = T(nf - Enf) = T(nf - nf) = 0$, since by compactness of G, $L^1(G)$ is an ideal of $L^1(G)^{**}$. Consequently, for every $\phi \in L^\infty(G)$, we have

$$\langle T(n - En), f\phi\rangle = \langle T(n - En)f, \phi\rangle = 0.$$

Considering the fact that for G compact $LUC(G) = RUC(G) = C_b(G)$ and from Proposition 1.47, we conclude that $T(n - En) \in \mathrm{ran}(L^1(G)^{**})$. Therefore T_2 maps $L^1(G)^{**}$ into $\mathrm{ran}(L^1(G)^{**})$ and so it is a (weakly compact) derivation. By using the assumption there is $m \in M(G)^{**}$ such that $T_2(n) = nm - mn \in \operatorname{ran}(L^1(G)^{**})$. Proposition 4.17 implies that $\pi(m) \in Z(M(G))$ so that $\pi(m)F = F\pi(m)$ for every $F \in M(G)^{**}$. Hence, by Proposition 2.19,

$$T_2(n) = nm - mn = n\pi(m) - mn = \pi(m)n - mn = (\pi(m) - m)n.$$

Therefore $T(n) = (T(E) + \pi(m) - m)n$. This means that T is inner in $M(G)^{**}$. □

4.3 Weakly compact derivations from $\mathcal{A}$ into $\mathcal{A}^*$

Definition 4.28. *Let $\mathcal{A}$ be a Banach algebra. The functional $\lambda \in \mathcal{A}^*$ is said to be a weakly almost periodic functional if the map $a \longrightarrow a \cdot \lambda$ from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact. The set of all weakly almost periodic functionals is denoted by $WAP(\mathcal{A})$.*

We have already mentioned in chapter 1 that $\mathcal{A}^*$ is a Banach $\mathcal{A}$-bimodule. It can be proved that $WAP(\mathcal{A})$ is a closed submodule of $\mathcal{A}^*$ and therefore it is a Banach $\mathcal{A}$-bimodule itself. We also have the following propositions.

Proposition 4.29. *Let $\mathcal{A}$ be a Banach algebra and $\lambda \in \mathcal{A}^*$. Then $\lambda \in WAP(\mathcal{A})$ if and only if $\langle F \square G, \lambda \rangle = \langle G \diamond F, \lambda \rangle$ for each $F, G \in \mathcal{A}^{**}$.*

Proof. See Proposition 3.3 of [36]. □

Proposition 4.30. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}) = \mathcal{A}^*$.*

Proof. See Theorem 2.6.17 of [12] □

The next proposition lists several equivalent conditions.

Proposition 4.31. *Let $\mathcal{A}$ be a Banach algebra. Then the following statements are equivalent.*

(a) For every $G \in \mathcal{A}^{**}$, the map $F \mapsto F \diamond G - G \square F, \mathcal{A}^{**} \longrightarrow \mathcal{A}^{**}$ is weak-star continuous;

(b) $F \diamond G - G \square F = F \square G - G \diamond F$ for every $G, F \in \mathcal{A}^{**}$;

(c) $\mathrm{wk}^* \lim_i \mathrm{wk}^* \lim_j (a_i b_j + b_j a_i) = \mathrm{wk}^* \lim_j \mathrm{wk}^* \lim_i (a_i b_j + b_j a_i)$ whenever the limits exist and $(a_i), (b_j)$ are bounded nets in $\mathcal{A}$.

Proof. The proof is trivial. □

The previous proposition gives rise to the following definition.

Definition 4.32. *Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is* quasi-Arens regular *if it satisfies any and hence all of the statements in Proposition 4.31.*

Remark 4.33. *It is obvious that Arens regularity implies quasi-Arens regularity but the converse is not true. For example, every commutative Banach algebra is quasi-Arens regular but for an infinite abelian locally compact group G, $L^1(G)$ is not Arens regular.*

Similarly, we can generalize the concept of topological center as follows.

Definition 4.34. *Let $\mathcal{A}$ be a Banach algebra. The* topological quasi-center *of $\mathcal{A}$ is denoted by $Z_{tq}(\mathcal{A})$ and is defined as*

$$Z_{tq}(\mathcal{A}) = \{G \in A^{**} : F\square G + G\square F = F \diamond G + G \diamond F \ \textit{for all} \ F \in A^{**}\}.$$

We can also generalize the concept of weakly almost periodic functions.

Definition 4.35. *Let $\mathcal{A}$ be a Banach algebra. A functional $\lambda \in \mathcal{A}^*$ is said to be* quasi-weakly almost periodic *if $\langle F\square G + G\square F, \lambda\rangle = \langle F \diamond G + G \diamond F, \lambda\rangle$ for all $F, G \in \mathcal{A}^{**}$. The space of all quasi-weakly almost periodic functionals is denoted by $QWAP(\mathcal{A})$.*

Proposition 4.36. *Let $\mathcal{A}$ be a Banach algebra. Then $QWAP(\mathcal{A})$ is a closed submodule of $\mathcal{A}^*$.*

Proof. The proof of $QWAP(\mathcal{A})$ being a subspace of $\mathcal{A}^*$ is trivial from the definition of quasi-weakly almost periodic functions. Suppose that (λ_n) is a sequence in $QWAP(\mathcal{A})$ that converges to $\lambda \in \mathcal{A}^*$ with respect to the norm-topology in $\mathcal{A}^*$. This implies that (λ_n) converges to $\lambda \in \mathcal{A}^*$ with respect to the weak-topology in $\mathcal{A}^*$ as well. Hence for every $F, G \in \mathcal{A}^{**}$ we have:

$$\langle F\square G + G\square F, \lambda\rangle = \lim_n \langle F\square G + G\square F, \lambda_n\rangle = \lim_n \langle F \diamond G + G \diamond F, \lambda_n\rangle = \langle F \diamond G + G \diamond F, \lambda\rangle.$$

This proves that $\lambda \in QWAP(\mathcal{A})$. Therefore $QWAP(\mathcal{A})$ is a closed subspace of $\mathcal{A}^*$. To prove that $QWAP(\mathcal{A})$ is closed under module actions of $\mathcal{A}$ on $\mathcal{A}^*$, take $a \in \mathcal{A}$ and

$\lambda \in QWAP(\mathcal{A})$. Then $a \cdot \lambda \in QWAP(\mathcal{A})$ because for every $F, G \in \mathcal{A}^{**}$ we have:

$$\begin{aligned}\langle F\square G + G\square F, a\cdot\lambda\rangle &= \langle F\square(G\square a) + G\square(F\square a), \lambda\rangle = \langle F\diamond(G\diamond a) + G\diamond(F\diamond a), \lambda\rangle \\ &= \langle (F\diamond G)\diamond a + (G\diamond F)\diamond a, \lambda\rangle = \langle F\diamond G + G\diamond F, a\cdot\lambda\rangle .\end{aligned}$$

Similarly, it can be shown that $\lambda \cdot a \in QWAP(\mathcal{A})$. The rest of the proof is trivial. $\square$

The next two propositions provide us with equivalent conditions to determine when a functional is quasi-weakly almost periodic. We also have a characterization of weakly compact derivations.

Proposition 4.37. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\lambda \in \mathcal{A}^*$. The derivation $\mathrm{ad}_\lambda : \mathcal{A} \longrightarrow \mathcal{A}^*$ is weakly compact if and only if $\lambda \in QWAP(\mathcal{A})$.*

Proof. Suppose that $\lambda \in QWAP(\mathcal{A})$. We aim to prove that ad_λ is weakly compact. Take a bounded net (a_i) in $\mathcal{A}$. Then (a_i) has a subnet (again we denote it by (a_i)) that converges with respect to the weak-star topology in $\mathcal{A}^{**}$. Suppose that $\mathrm{wk}^*\lim_i (a_i) = F$ for some $F \in \mathcal{A}^{**}$. Given that $\lambda \in QWAP(\mathcal{A})$, for each $G \in \mathcal{A}^{**}$ we have

$$\begin{aligned}\lim_i \langle G, a_i\cdot\lambda - \lambda\cdot a_i\rangle &= \lim_i \langle \lambda\cdot G - G\cdot\lambda, a_i\rangle = \lim_i \langle a_i, \lambda\cdot G - G\cdot\lambda\rangle \\ &= \langle F, \lambda\cdot G - G\cdot\lambda\rangle = \langle G\diamond F - F\square G, \lambda\rangle \\ &= \langle G\square F - F\diamond G, \lambda\rangle = \langle G, F\cdot\lambda - \lambda\cdot F\rangle .\end{aligned}$$

This implies that $\mathrm{ad}_\lambda(a_i) = a_i\cdot\lambda - \lambda\cdot a_i$ converges to $F\cdot\lambda - \lambda\cdot F$ with respect to the weak topology on $\mathcal{A}^*$. Therefore ad_λ is weakly compact. For the converse, suppose that $D = \mathrm{ad}_\lambda$ is weakly compact. Take $F, G \in \mathcal{A}^{**}$. Suppose that (a_i) is a bounded net in $\mathcal{A}$ that converges to F with respect to the weak-star topology in $\mathcal{A}^{**}$. From part (b) of Proposition 1.15 we know that $D^* : \mathcal{A}^{**} \longrightarrow \mathcal{A}^*$ is continuous if we consider weak-star topology on $\mathcal{A}^{**}$ and weak topology on $\mathcal{A}^*$. Therefore $D^*(a_i)$ converges to $D^*(F)$ with respect to the weak topology on $\mathcal{A}^*$. Therefore we have

$$\begin{aligned} \langle F \square G - G \diamond F, \lambda \rangle &= \lim_i \langle a_i \square G - G \diamond a_i, \lambda \rangle = \lim_i \langle a_i \diamond G - G \square a_i, \lambda \rangle \\ &= \lim_i \langle G, \lambda \cdot a_i - a_i \cdot \lambda \rangle = \langle G, \lambda \cdot F - F \cdot \lambda \rangle \\ &= \langle F \diamond G - G \square F, \lambda \rangle . \end{aligned}$$

Hence $\lambda \in QWAP(\mathcal{A})$. $\square$

Throughout the next proposition the right action of an element $H \in (\mathcal{A}^{**}, \square)$ on $\phi \in \mathcal{A}^{***}$ is denoted by $\cdot^{\square}$. Similar notations are used for left action when we regard $H \in (\mathcal{A}^{**}, \diamond)$

Proposition 4.38. *Let $\mathcal{A}$ be a Banach algebra and $\lambda \in \mathcal{A}^*$. Then the derivation $D = \mathrm{ad}_\lambda$ is weakly compact if and only if $H \cdot^{\diamond} \hat{\lambda} - \hat{\lambda} \cdot^{\square} H$ is in $\mathcal{A}^*$ for every $H \in \mathcal{A}^{**}$ where $\hat{\lambda}$ is the image of λ in $\mathcal{A}^{***}$ through the canonical embedding of $\mathcal{A}^*$ into $\mathcal{A}^{***}$.*

Proof. Let $D^* : \mathcal{A}^{**} \longrightarrow \mathcal{A}^*$ and $D^{**} : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{***}$ be respectively the first and the second adjoint of D. Then for every $F, H \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$ we have:

$$\langle D^*(F), a \rangle = \langle F, D(a) \rangle = \langle F, a \cdot \lambda - \lambda \cdot a \rangle = \langle \lambda \cdot F - F \cdot \lambda, a \rangle .$$

$$\begin{aligned}\langle D^{**}(H), F\rangle &= \langle H, D^*(F)\rangle = \langle H, \lambda\cdot F - F\cdot\lambda\rangle = \langle F\diamond H - H\square F, \lambda\rangle \\ &= \left\langle \hat{\lambda}, F\diamond H - H\square F\right\rangle = \left\langle H\cdot^{\diamond}\hat{\lambda} - \hat{\lambda}\cdot^{\square} H, F\right\rangle.\end{aligned}$$

Therefore $D^{**}(H) = H\cdot^{\diamond}\hat{\lambda} - \hat{\lambda}\cdot^{\square} H$, for every $H\in\mathcal{A}^{**}$. Using Proposition 1.15, we conclude that $D = \mathrm{ad}_\lambda$ is weakly compact if and only if $H\cdot^{\diamond}\hat{\lambda} - \hat{\lambda}\cdot^{\square} H$ is in $\mathcal{A}^*$, for every $H\in\mathcal{A}^{**}$. $\square$

4.4 Weakly compact derivations from $L^1(G)$ into $L^\infty(G)$

Now we focus on the special case of $L^\infty(G)$ as a Banach $L^1(G)$- bimodule. As we mentioned in Remark 1.18, $L^\infty(G)$ is also an $M(G)$-bimodule. More precisely, the module actions are defined as:

$$\langle g\cdot\mu, f\rangle = \langle g, \mu\star f\rangle \qquad \langle \mu\cdot g, f\rangle = \langle g, f\star\mu\rangle.$$

for $f\in L^1(G), \mu\in M(G)$ and $g\in L^\infty(G)$. Obviously the latter module actions coincide with the first one for $\mu\in L^1(G)$. Also note that $\delta_x\cdot g = l_x g$ and $g\cdot\delta_x = r_x g$. The next proposition guarantees that the concept of almost periodicity defined in Definition 4.28 coincides with the one we gave in Definition 1.38 part (i).

Proposition 4.39. *Let G be a locally compact group. Then a functional $\phi\in L^\infty(G)$ is in $WAP(L^1(G))$ if and only if $\phi\in WAP(G)$.*

Proof. See Theorem 4 of [51]. $\square$

Therefore from now on we interchangeably use $WAP(G)$ or $WAP(L^1(G))$ for the space of weakly almost periodic functions. We also have the following Corollary which is immediate from Proposition 4.37 and Proposition 4.38.

Corollary 4.40. *Let G be a locally compact group. Suppose that $\phi \in L^\infty(G)$ and $D = \mathrm{ad}_\phi : L^1(G) \longrightarrow L^\infty(G)$ is a derivation. Then the following statements are equivalent.*

(a) $D = \mathrm{ad}_\phi$ is weakly compact;

(b) $n \cdot^\diamond \hat{\phi} - \hat{\phi} \cdot^\Box n$ is in $L^\infty(G)$ for every $n \in L^1(G)^{**}$ where $\hat{\phi}$ is the natural image of ϕ in $L^\infty(G)^{**}$;

(c) $\phi \in QWAP(L^1(G))$.

The next theorem was proved by B. E. Johnson in [31]. A simplified proof can be found in [13].

Theorem 4.41. *Suppose that G is a locally compact group. Then every continuous derivation $D : L^1(G) \longrightarrow L^\infty(G)$ is inner.*

Proposition 4.42. *Suppose that G is a locally compact group. A continuous linear map $D : L^1(G) \longrightarrow WAP(G)$ is a derivation if and only if there exists a bounded net $(\phi_i) \subset WAP(G)$ such that $D(f) = \lim_i(f \cdot \phi_i - \phi_i \cdot f)$, $(f \in L^1(G))$.*

Proof. Suppose that $D : L^1(G) \longrightarrow WAP(G)$ is a continuous derivation. From Theorem 4.41, there exists $\phi \in L^\infty(G)$ such that

$$D(f) = f \cdot \phi - \phi \cdot f, \ \ (f \in L^1(G)).$$

From Proposition 1.56, we have $\phi \in WAP(G)^{**}$. Hence there is a net $(\phi_i) \subset WAP(G)$ such that (ϕ_i) converges to ϕ with respect to the weak-star topology of $WAP(G)^{**}$. So we have

$$D(f) = \text{wk}^* \lim_i (f \cdot \phi_i - \phi_i \cdot f), \ \ (f \in L^1(G));$$

here again the limit is with respect to the weak-star topology of $WAP(G)^{**}$. Since $D(f) \in WAP(G)$, we have

$$D(f) = \text{wk} \lim_i (f \cdot \phi_i - \phi_i \cdot f), \ \ (f \in L^1(G)).$$

Now it is standard to find (a possibly different) net $(\psi_i) \subset WAP(G)$ such that

$$D(f) = \lim_i (f \cdot \psi_i - \psi_i \cdot f), \ \ (f \in L^1(G)).$$

The converse is clear, since a map as defined above is a derivation. □

The next proposition lists certain properties of continuous derivations from $L^1(G)$ into its dual (recall from Theorem 4.41 all such derivations are inner).

Proposition 4.43. *Let G be a locally compact group. Suppose that $D = \mathrm{ad}_\phi : L^1(G) \longrightarrow L^\infty(G)$ is a continuous derivation. Then*

(a) *There is a unique derivation* $\tilde{D} = \mathrm{ad}_\phi : M(G) \longrightarrow L^\infty(G)$ *that is defined by* $\tilde{D}(\mu) = \mathrm{wk}^* \lim_i(D(\mu e_i)) = \mu \cdot \phi - \phi \cdot \mu$ *for every* $\mu \in M(G)$ *where* (e_i) *is a two-sided bounded approximate identity for* $L^1(G)$*, bounded by* 1*. Moreover* $\tilde{D}$ *extends* D *and* $\|D\| = \left\|\tilde{D}\right\|$*. The derivation* $\tilde{D}$ *is also continuous when we consider strong operator topology on* $M(G)$ *and weak-star topology on* $L^\infty(G)$*.*

(b) *If* D *is weakly compact then* $D(L^1(G)) \subseteq WAP(G)$*.*

(c) *If* D *is weakly compact then* $\tilde{D}$ *is weakly compact too and* $\tilde{D}(M(G)) \subseteq WAP(G)$*.*

(d) *If* D *is weakly compact then there exists a net* (ϕ_i) *in* $WAP(G)$ *such that* $\tilde{D}(\mu) = \lim_i(\mu \cdot \phi_i - \phi_i \cdot \mu)$ *for every* $\mu \in M(G)$*.*

Proof. (a) According to Corollary 1.42, $L^1(G)$ is an essential Banach $L^1(G)$-bimodule. The statement follows from Theorem 4.1 and Theorem 4.41. See also Theorem 2.9.53 of [12].

(b) Take an arbitrary $f \in L^1(G)$. Define $T : L^1(G) \longrightarrow L^\infty(G)$ by $T(g) = g \cdot D(f)$ for each $g \in L^1(G)$. From Proposition 4.39 in order to prove that $D(f) \in WAP(G)$, it suffices to show that T is weakly compact. To prove this, note that

$$T(g) = g \cdot D(f) = D(gf) - D(g) \cdot f = D \circ T_1(g) - T_2 \circ D(g)$$

where $T_1 : L^1(G) \longrightarrow L^1(G)$ is defined by $T(h) = h \cdot f$ for each $h \in L^1(G)$ and $T_2 : L^\infty(G) \longrightarrow L^\infty(G)$ is defined by $T_2(\phi) = \phi \cdot f$ for each $\phi \in L^\infty(G)$. Therefore

$T = D \circ T_1 + T_2 \circ D$. Using Proposition 1.11 and the fact that D is weakly compact, we conclude that T is weakly compact. Hence $D(f) \in WAP(G)$.

(c) Suppose that D is weakly compact. Let $\mu \in M(G)$. Then using part (a) and the fact that the net $(\mu \star e_i)$ is bounded in $L^1(G)$, there exists a subnet of $(\mu \star e_i)$ (we denote it by $(\mu \star e_i)$ again) such that $\mathrm{wk}\lim_i D(\mu \star e_i)$ exists. Hence

$$\tilde{D}(\mu) = \mathrm{wk}^* \lim_i \left(D(\mu \star e_i)\right) = \mathrm{wk} \lim_i \left(D(\mu \star e_i)\right). \tag{4.8}$$

By using the equation (4.8), we have

$$\left\{\tilde{D}(\mu) : \mu \in M(G), \|\mu\| \leq 1\right\} \subseteq \left\{D(f) : f \in L^1(G), \|f\| \leq 1\right\}^{\text{wk-closure}}.$$

The fact that D is weakly compact implies that $\{D(f) : f \in L^1(G), \|f\| \leq 1\}^{\text{wk-closure}}$ is weakly compact. Therefore $\left\{\tilde{D}(\mu) : \mu \in M(G), \|\mu\| \leq 1\right\}^{\text{wk-closure}}$ is weakly compact. Hence $\tilde{D}$ is weakly compact too. Moreover from equation (4.8) and part (b), we conclude that $\tilde{D}(M(G)) \subseteq WAP(G)$.

(d) From part (a), there exists $\phi \in L^\infty(G)$ such that $\tilde{D}(\mu) = \mu \cdot \phi - \phi \cdot \mu$, $(\mu \in M(G))$. Now the argument in the preceding proposition gives the result.

$\square$

In part (b) of Proposition 4.43 we proved that for every derivation $D : L^1(G) \longrightarrow L^\infty(G)$, if D is weakly compact then $D(L^1(G)) \subseteq WAP(G)$. We aim to show that the converse is not correct. To prove this we need some lemmas.

Lemma 4.44. *Suppose that G is a locally compact group and $xy \neq yx$ for some $x, y \in G$. Then there is an open neighborhood V of y such that $xV \cap Vx = \emptyset$.*

Proof. Since $xy \neq yx$ we can find disjoint open sets U and W such that $xy \in U$ and $yx \in W$. Suppose that e is the identity of group G. Since U and W are open sets we can find open neighborhoods of e say V_1 and V_2 such that $xyV_1 \subseteq U$ and $V_2yx \subseteq W$. Then $V = yV_1 \cap V_2y$ is an open neighborhood of y such that $xV \subseteq xyV_1 \subseteq U$ and $Vx \subseteq V_2yx \subseteq W$. Therefore V is an open neighborhood of y such $xV \cap Vx = U \cap W = \emptyset$. □

Lemma 4.45. *Let G be a non-discrete locally compact group. Suppose that O is a non-empty open subset of G. Then there is a $g \in L^\infty(O)$ such that g is not almost everywhere equal to a continuous function on O.*

Proof. See lemma 2.1 of [49]. □

Theorem 4.46. *Let G be a non-abelian infinite compact group. There is a continuous non-weakly compact derivation $D : L^1(G) \longrightarrow L^\infty(G)$ such that $D(L^1(G)) \subseteq WAP(G)$.*

Proof. The group G is not abelian. Therefore we can find $x, y \in G$ such that $xy \neq yx$. By Lemma 4.44, we can find an open neighborhood V of y such that $xV \cap Vx = \emptyset$. Since G is infinite and compact it should be non-discrete. For the open set xV, by Lemma 4.45, we can find $f \in L^\infty(xV)$ that is not continuous almost everywhere. We can extend $f \in L^\infty(xV)$ to $g \in L^\infty(G)$ by defining g equal to zero outside of xV. Obviously g is not almost everywhere continuous on xV and it is continuous on Vx. Therefore $g \cdot \delta_{x^{-1}}$ is not almost everywhere continuous on V and $\delta_{x^{-1}} \cdot g$ is continuous on V. Hence $\delta_{x^{-1}} \cdot g - g \cdot \delta_{x^{-1}}$ is not almost everywhere continuous on V. Define $D = \mathrm{ad}_g$. For each $h \in L^1(G)$, $D(h) = h \cdot g - g \cdot h$ where $h \cdot g \in RUC(G)$ and $g \cdot h \in LUC(G)$ by Proposition 1.47. Since G is compact $C_0(G) = WAP(G) = LUC(G) = RUC(G)$. Therefore $D(L^1(G)) \subseteq WAP(G) = C(G)$. We claim that D is not weakly compact. We prove our claim by contradiction. Suppose that D is weakly compact. By part (c) of Proposition 4.43, we know that $\tilde{D}(M(G)) \subseteq WAP(G) = C(G)$. This contradicts the fact that $\tilde{D}(\delta_{x^{-1}}) = \delta_{x^{-1}} \cdot g - g \cdot \delta_{x^{-1}}$ is not in $C(G)$. $\square$

Chapter 5

SOME OPEN PROBLEMS

We conclude this thesis with a short list of questions that can be the subject of future investigations. Let G be a locally compact group.

Question 1: Is it true that the left (right) topological center of $(M(G)_0^*)^*$ is $M(G)$? We are inclined to think that the answer should be affirmative.

Question 2: In [11], the authors prove that there is a non-inner derivation from $M(G)$ to its dual. Suppose that G is non-discrete and $D : M(G) \longrightarrow M(G)^{**}$ is a continuous derivation. Is D inner?

Question 3: For X, a left introverted subspace of $C_b(G)$, are all derivations on $\mathcal{X}^*$ inner?

Question 4: Is every continuous derivation $D : L^1(G)^{**} \longrightarrow L^1(G)^{**}$ inner in $M(G)^{**}$?

Question 5: Suppose that G is compact and $L : M(G)^{**} \longrightarrow M(G)^{**}$ is a non-zero weakly compact left multiplier. Does there exist F in $L^1(G)$ or $L^1(G)^{**}$ such that $L = T_F$?

Question 6: Can we drop the condition of existence of unique right invariant mean in Proposition 4.24?

Question 7: Suppose that $D : M(G)^{**} \longrightarrow L^1(G)^{**}$ is a (weakly compact) derivation. Is D inner?

Question 8: Suppose that G is compact and $D : M(G)^{**} \longrightarrow L^1(G)$ is a (weakly) compact derivation. Is D inner?

Question 9: What are the necessary and sufficient condition for $Z_m(L^\infty(G))$ to be a submodule of $L^\infty(G)$ as an $L^1(G)$-bimodule?

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