Group Embeddings of (n, k) Configurations

by

Eric James Loepp Ens

A thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements of the degree of

Master of Science

Department of Mathematics
University of Manitoba
Winnipeg

Abstract

An (n, k) configuration is a set of n “points” and n “lines” such that each point lies on k lines and each line contains k points. Motivated by the geometric definition of a group law on non-singular cubic curves, we define the concept of group embeddability of (n, k) configuration C as a mapping g of C into an abelian group G such that a set of k points $\{P_1, P_2, \ldots, P_k\}$ are collinear in the configuration C if and only if $\sum g(P_i) = 0$ in the group G. Here we classify the set of all $(n, 3)$ configurations for $n \leq 11$ as well as some other notable configurations which can be embedded into abelian groups.

Here we use the notation introduced by Branko Grünbaum [2]. The following theorems are proved in this thesis:

n	$(n, 3)$	group
7	Fano Plane	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
8	(8, 3)	$\mathbb{Z}_3 \times \mathbb{Z}_3$
9	Of the three configurations, two are embeddable in groups.	
10	Of the 10 configurations, five are embeddable in groups.	
11	Of the 31 configurations, 9 have group embeddings.	

But for the first three examples (n = 7, 8 and the Pappus configuration), all other embeddability theorems proved here are new. In doing so we develop several different techniques for finding a group embedding or proving that no such embedding exists. Some ideas in this thesis were inspired by the late Professor N. S. Mendelsohn. For example, group embeddings can be thought of as extensions of configurations to Mendelsohn Triple Systems (see [8], [10]). In fact, configurations naturally give rise to partial quasigroups and adding the “missing triples” including the so-called "tangential relations" are the essential ideas behind the Mendelsohn triple Systems [8].

Acknowledgements

Thank you first to my supervisor, Dr. R. Padmanabhan, for giving me something to do for the last 3 years. Thank you to Dr. Michael Doob, Dr. Ben Li and Dr. Craig Platt for the helpful suggestions and comments. Thank you to the Faculty of Science and the Department of Mathematics at the University of Manitoba for keeping me afloat during my studies. Finally, thank you to my family for supporting me and my decision to pursue graduate studies.

Contents

Table of Figures

Introduction

> "...it might be mentioned here that there was a time when the study of configurations was considered the most important branch of all geometry..."
>
> – David Hilbert and S. Cohn-Vossen.

In this thesis, we study representing abstract configurations by concrete groups. A configuration is a finite set of elements (which we will call points to provide an analogy to plane geometry) and a finite set of blocks (called lines for the same reason) such that each point is incident with the same number of lines and each line is incident with same number of points. This topic was popularized by Hilbert and Cohn-Vossen in their book Anschauliche Geometry (reprinted in English as Geometry and Imagination [5]). We use the language of geometry to aid as a guiding principal. So, here $\{P_1, P_2, \ldots, P_k\}$ in a configuration, C, is collinear simply means means that $\{P_1, P_2, \ldots, P_k\}$ is a block in C.

Definition: An (n, k) configuration is a set of n points and n lines where each point is incident to exactly k lines and each line is incident to exactly k points.

There are many familiar such configurations; the Fano plane is a (7, 3) configuration and the Desargues and Pappus theorems are well-known configuration theorems.

We seek to embed such configurations into abelian groups by adapting the geometric definition of the group law on cubic curves (see chapter 1).

Definition: A *group embedding* is a one-to-one mapping, f, from an (n, k) configuration, C, into an abelian group, G, such that a set of k points $\{P_1, P_2, \ldots, P_k\}$ in C is collinear if and only if $\sum f(P_i) = 0$ in the group G.

Definition: We say that a configuration, *C,* is *geometrically realizable* if it can be mapped into a projective plane of order *k*. That is, that the points in *C* are the points in the projective plane and the blocks (or lines) in *C* are straight lines in the plane. This terminology is adapted from that used by Branko Grünbaum.

There are two well known examples of group embeddings for combinatorial structures. These are:

(i) the 7-point Fano Plane (7, 3) embedded in the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and

(ii) the famous Hesse (9, 3) configuration (or *Wendephunkts-configuration*) of 9 inflection points of a nonsingular plane cubic curve realized as the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

These are examples of the so-called extended triple systems (now famously known as Mendelsohn triple systems). Here the triples are of the form $(P, Q, P*Q)$. In other words, the group embedding extends the original configuration into a full binary algebra satisfying the identities $x*y = y*x$ and $x*(y*x) = y$. These are, also, the defining identities of a commutative Mendelsohn triple system ([8]).

Apart from these examples, no other group realizations of configurations are known. In this thesis we classify the set of all (*n,* 3) configuration for n ≤ 11 as well as several other interesting examples with n > 12 along with accompanying proofs. The proofs and counter-examples draw from a variety of fields: classical geometry, combinatorics, commutative algebra, number theory and equational logic (with computers). These proofs will take a few different forms using different techniques. Each such proof will be introduced as needed. All diagrams for (*n*, 3) configurations are from [2].

Chapter 1: The Group Law on a Cubic Curve

Our notion of embeddability of $(n, 3)$ configurations is motivated by the geometric definition of the group law on nonsingular cubic curves. We take this opportunity to review this group law.

Let C be a nonsingular cubic curve and let the points P, Q and R lie on C. We say that $R = P * Q$ if the straight line connecting P and Q intersects C again at the unique point R. If P and Q are the same point, then the line connecting them is a tangent line and the point is counted twice. If this tangent line intersects C in a point distinct from P, then we call this point R, and say that $R = P * P$. If this line does not intersect C in any place then R is said the be the point "at infinity" (see [7], [10]).

Thus, the points on C form an abelian group under an addition operation. We define this group operation "+" where $P + Q + R = 0$ when P, Q and R lie on the same line, and infinity is counted as 0. Then $P + Q = -R$, where -R is the intersection of C and the line connecting R and infinity (since $R + (-R) + 0 = 0$). The mechanics of " $*$" and "+" are both easily understood with figure 1.1.

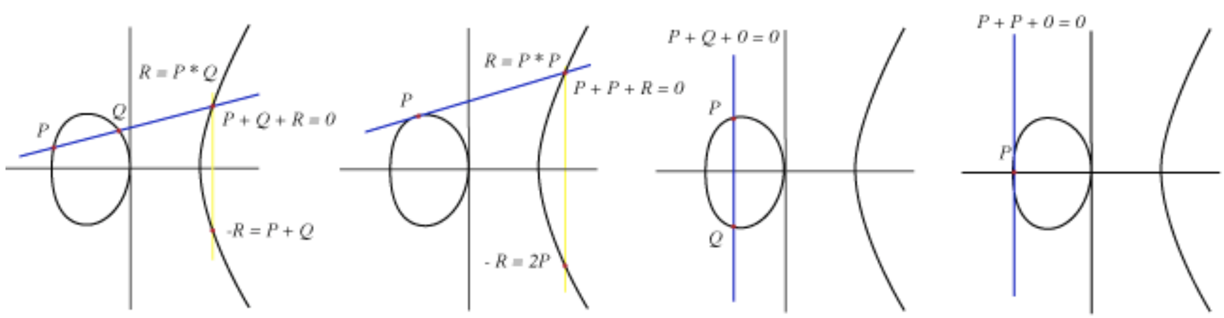


Figure 1.1

Defining the point at infinity to be the identity element, it is well-known that this addition defines an abelian group on the points of the curve. Furthermore, if *x, y, z* are points on a cubic curve, then “*” defined as above simply gets interpreted as *-x -y* in the group. Hence it is easy to see that *x * y* satisfies the following laws:

$x * y = y * x$	commutativity
$x * (y * x) = y$	Steiner law
$(x * y) * (z * u) = (x * z) * (y * u)$	median law

A consequence of the above is that $x * (y * (z * u)) = z * (y * (x * u))$ which will occasionally be used in proofs in place of the median law.

Theorem 1.1: $x * (y * (z * u)) = z * (y * (x * u))$.

Proof:

$$
\begin{aligned}
x * (y * (z * u)) &= (u * (u * x)) * (y * (z * u)) && \text{by commutativity and the steiner law} \\
&= (u * y) * ((u * x) * (z * u)) && \text{by the median law} \\
&= (u * y) * ((u * z) * (x * u)) && \text{by the median law} \\
&= (u * (u * z)) * (y * (x * u)) && \text{by associativity} \\
&= z * (y * (x * u)) && \text{by the Steiner law and commutativity.}
\end{aligned}
$$

Q.E.D.

Chapter 2: Embedding an $(n, 3)$ configuration into an abelian group

We now seek to adapt the ideas in Chapter 1 for use with $(n, 3)$ configurations. The points on an $(n, 3)$ configuration may or may not form an abelian group, instead, we seek to find a group embedding as defined in the introduction.

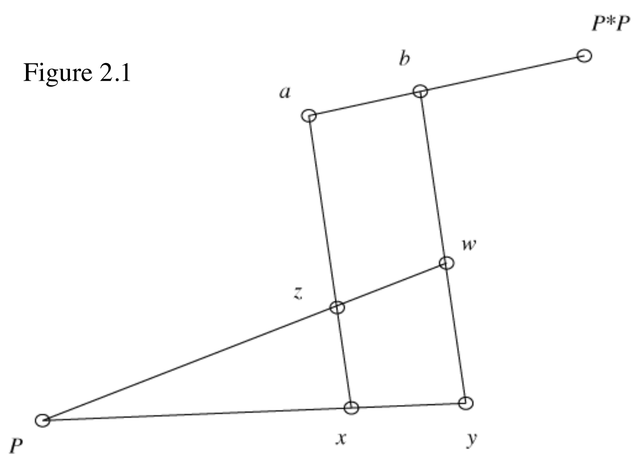
If A, B and C are points in an $(n, 3)$ configuration and f is a group embedding then $f(A)+f(B)+f(C)=0$ in the group, if and only if A, B and C are collinear. We now seek to adapt " $*$" for use in an $(n, 3)$ configuration. If K is an $(n, 3)$ configuration, and P, Q, and R are distinct points in K, we say that $P*Q=R$ if P, Q, and R are all on the same line. Then, since they are all on the same line and if f is a group embedding from K to a group G, $f(P)+f(Q)+f(R)=0$. Furthermore, we can say that $f(P*Q)=f(R)=-f(P)-f(Q)$.

One primary technique used in this thesis is examining tangential relations, i.e. $P*P$. The point $P*P$ is not immediately apparent within a configuration, but under a group embedding, it has a mapping. In fact, we can look for what $P*P$ maps into, $f(P*P)=-(f(P))+-(f(P))=-2f(P)$. In otherwords, we extend the $*$, $P*Q$, operation to include the case where $P=Q$.

If we interpret the three laws from Chapter 1 we can see that,

$P*Q=Q*P$	since the group is abelian
$P*(Q*P)=Q$	since $f(P*(Q*P))=f(Q)=-f(P)-(-f(Q)-f(P))=f(Q)$
$(P*Q)*(R*S)=(P*R)*(Q*S)$	since both sides reduce to $f(P)+f(Q)+f(R)+f(S)$.

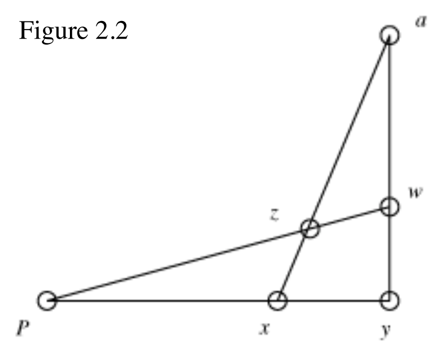
Figure 2.1



We can also use the properties of "*" from cubic curves to find a point on a configuration which could make sense of $P * P$. In figure 2.1 we note that $P = x * y$ and that $P = z * w$. Thus $P * P = (x * y) * (z * w)$. From the median law we have that $P * P = (x * z) * (y * w)$. Then we can see that $a = (x * z)$ and $b = (w * y)$, therefore $P * P = a * b$. If however, $a = b$, as in figure 2.2, then $-2f(P) = -2f(a)$ even if this point does not exist in the configuration.

Different applications of commutativity, the Steiner law and the median law can lead to different results for $P * P$. For instance, in figure 2.1, instead of interpreting $P * P = (x * z) * (y * w)$, it could be written as $P * P = (x * w) * (y * z)$, which may be meaningless in the context of the original configuration but is clearly meaningful in the target group (as $-2f(P)$). As a result, $P * P$ may be one point, multiple points or may not exist at all. These different computations determine the type of group that we obtain for embedding a given configuration, if such a group exists.

Figure 2.2



If a configuration K cannot be embedded into a group then we should not be able to find an abelian group G that satisfies the condition that $f(A) + f(B) + f(C) = 0$ for three points A, B and C that are collinear in K. One way to prove this is to show that the cancellation property of "*" fails ($a * b = a * c \to -f(a) - f(b) = -f(a) - f(c) \to f(b) = f(c)$). For three distinct points, P, Q and R in a configuration K, if we can show that $P * Q = P * R$ in any potential target group containing K but $Q \neq R$. Thus, any mapping from K to a group G cannot be one-to-one. This is how all non-embeddability theorems are proved.

Chapter 3: (7,3) Configurations

The first, and smallest (n,3) configuration is the only (7,3) configuration [2], better known as the Fano Plane. Since the Fano plane *is* the projective plane PG(2,2), finding a group embedding is a trivial exercise.

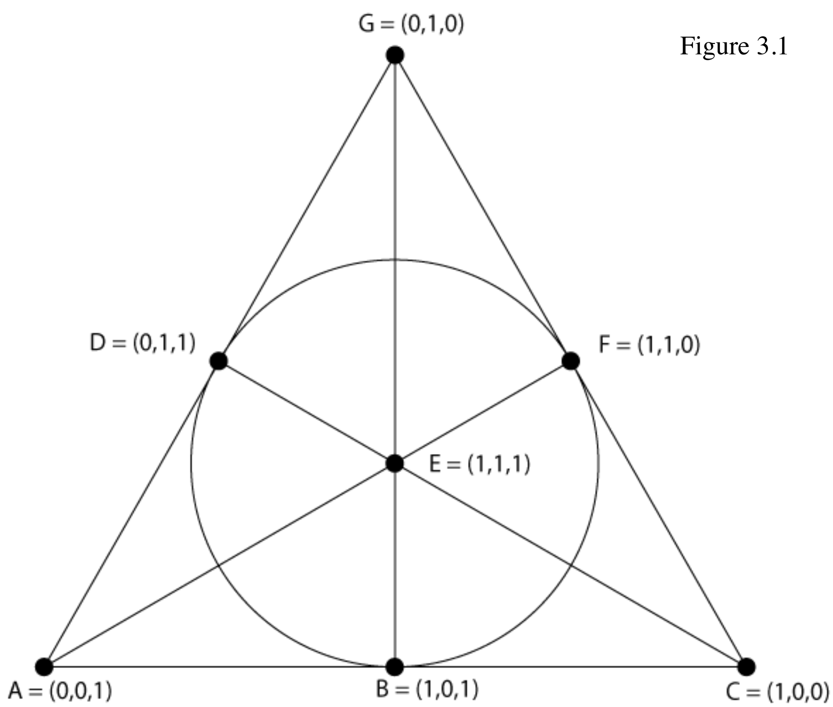


Figure 3.1

The usual homogeneous coordinatization provides the embedding and mapping of the Fano Plane. The points fall into $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the condition that $a * b * c = 0$ holds when a, b and c are collinear. Incidentally, we can see that the lines are indeed given by linear equations. For example, the line (B,D,F) is given by the homogeneous equation $x + y + z = 0$.

Chapter 4: (8,3) Configurations

§4.1

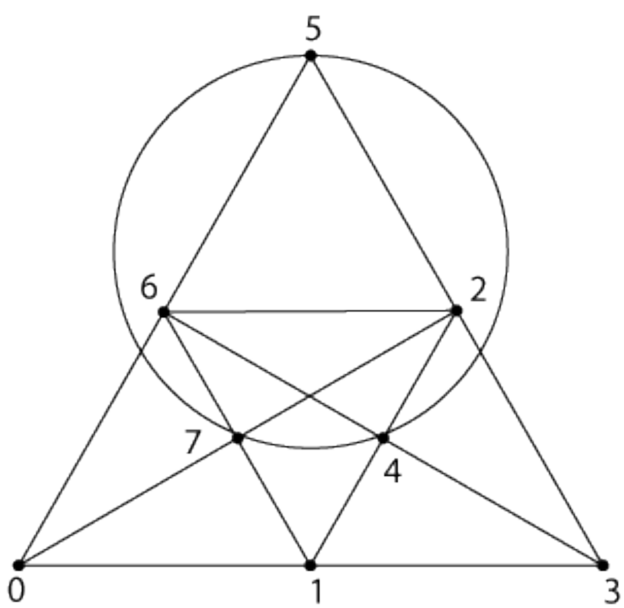
There is only one combinatorial configuration possible with 8 points and 8 lines [2], which is called the Möbius-Kantor configuration. We will label the points with the numbers 1 through 8. We describe this configuration as cyclic with the description {(0,1,3) **mod** 8} (or $C(8)$ using Grunbaum's notation, [2] page 18). This notation defines the lines in the configuration.
We will establish a minimal group embedding in $\mathbb{Z}_3 \times \mathbb{Z}_3$. using two different methods of proof, one by parameterization and the other by examining idempotent elements.

§4.2 Embedding by parameterization.

In order to find an embedding we will give parameters to the points in $C(8)$.
First, let $f(0)=a, f(1)=b$ and $f(2)=c$. Note that (0,1,2) is not a line in the configuration.

Figure 4.1

$$0*1=3$$
$$1*2=4$$
$$2*3=5$$
$$3*4=6$$
$$4*5=7$$
$$5*6=0$$
$$6*7=1$$
$$7*0=2$$



This then forces the following:

$$
\begin{aligned}
&f(3) = -a - b && \text{from } (0,1,3) \\
&f(4) = -b - c && \text{from } (1,2,4) \\
&f(5) = a + b - c && \text{from } (2,3,5) \\
&f(6) = a + 2b + c && \text{from } (3,4,6) \\
&f(7) = b + c - a - b + c = -a + 2c && \text{from } (4,5,7)
\end{aligned}
$$

There are three lines remaining in the configurations. These lines give us restrictions on a, b and c.

$$
\begin{aligned}
(5,6,0) &\to a + a + b - c + a + 2b + c = 0 \\
&\to 3a + 3b = 0. \\
(6,7,1) &\to b + a + 2b + c - a + 2c = 0 \\
&\to 3b + 3c = 0. \\
(7,0,2) &\to a + c - a + 2c = 0 \\
&\to 3c = 0 \\
&\to 3b = 0 \text{ and hence } 3a = 0.
\end{aligned}
$$

This tells us that all 8 points must be of order 3 in our target group. We try the smallest group with 8 elements of order three, namely, $\mathbb{Z}_3 \times \mathbb{Z}_3$. This method of finding the group also finds the mapping at the same time.

Theorem 4.2.1: {(0,1,3) **mod** 8} can be embedded in $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof: If we let a = (0,1), b = (1,0) and c = (1,1) (once more, note that these choices for *a, b* and *c* are not collinear in $\mathbb{Z}_3 \times \mathbb{Z}_3$), Then we get the following mapping, $f : C(8) \to \mathbb{Z}_3 \times \mathbb{Z}_3$.

$$
\begin{aligned}
f(0) &= (0,1) \\
f(1) &= (1,0) \\
f(2) &= (1,1) \\
f(3) &= -(0,1)-(1,0) \equiv (2,2) \\
f(4) &= -(1,0)-(1,1) \equiv (1,2) \\
f(5) &= (0,1)+(1,0)-(1,1) \equiv (0,0) \\
f(6) &= (0,1)+2(1,0)+(1,1) \equiv (0,2) \\
f(7) &= -(0,1)+2(1,1) \equiv (2,0)
\end{aligned}
$$

Since this mapping comes directly from the above process, the other lines ((5,6,0), (6,7,1), (7,0,2)) will also sum to zero in the target group, as per our requirement from the introduction. Thus this is a mapping for $C(8)$ into $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Q.E.D.

§4.3 Embedding by tangential relations.

To find information about the target group, we can look for "tangents" of elements using the group law. This involves calculating $P * P$ for any P in the configuration. If we find that $P * P = Q$ for some point Q then we note that $-2f(P) = f(Q)$ in our group, where f is a group embedding. This is analogous to the classical group law on cubic curves.

$$
\begin{aligned}
1*1 &= (6*7)*(4*2) \\
&= (6*4)*(7*2) \\
&= 3*8 = 1
\end{aligned}
$$

Since this is a cyclic configuration, the above demonstration shows that $x * x = x$ for all x in $C(8)$. Thus, from the group law, we have that $f(x) + f(x) + f(x) = 3f(x) = 0$. So we need to find a group with 8 elements of order 3, and we arrive at $\mathbb{Z}_3 \times \mathbb{Z}_3$ just as before. Finding a mapping involves assigning arbitrary points in $\mathbb{Z}_3 \times \mathbb{Z}_3$ to the points 0, 1 and 2 (taking care that the three points in $\mathbb{Z}_3 \times \mathbb{Z}_3$ are not all collinear) and generating the remaining points in much the same manner as in §4.2, then verifying that all the lines are valid.

Chapter 5: (9,3) Configurations

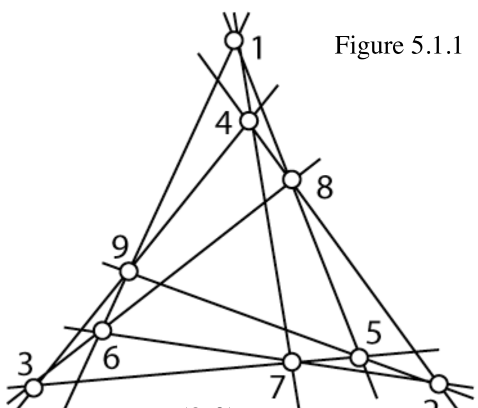
§5.0

There are three distinct (9,3) configurations, one of which occurs in the familiar Pappus' Theorem. Two of the three have group embeddings, and the other does not. We will use a different method of proof for each configuration. In particular, we will introduce computer proofs for non-embeddability.

§5.1 Group embedding of $(9,3)_1$, Pappus' configuration.

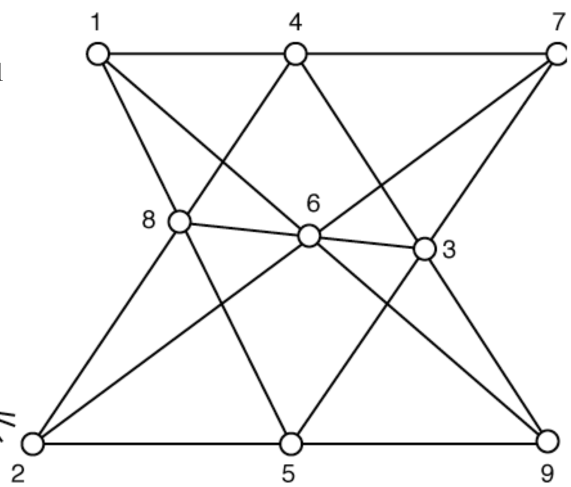
We define this (9,3) configuration by the lines {(1,4,7), (1,5,8), (1,6,9), (2,4,8), (2,5,9), (2,6,7), (3,4,9), (3,5,7), (3,6,8)}. From the diagram on the right in figure 5.1.1, it is clear that this $(9,3)_1$ is

(1,4,7)
(1,5,8)
(1,6,9)
(2,4,8)
(2,5,9)
(2,6,7)
(3,4,9)
(3,5,7)
(3,6,8)



$(9,3)_1$

Figure 5.1.1



just the Pappus configuration.

This 9-point configuration occurs as the celebrated configuration of the 9 inflexion points of a non-singular cubic curve in the complex projective plane and hence is well-known to be isomorphic to the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Also, we can embed this in the cyclic group $\mathbb{Z}_9$ as follows:

Points	1	2	3	4	5	6	7	8	9
f	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
group $\mathbb{Z}_9$	1	4	7	2	5	8	6	3	0

Figure 5.1.2 is the configuration redrawn with group labels inserted. It is easy to verify that three points *P, Q* and *R* are collinear then $f(P) + f(Q) + f(R) = 0$ in the cyclic group $\mathbb{Z}_9$.

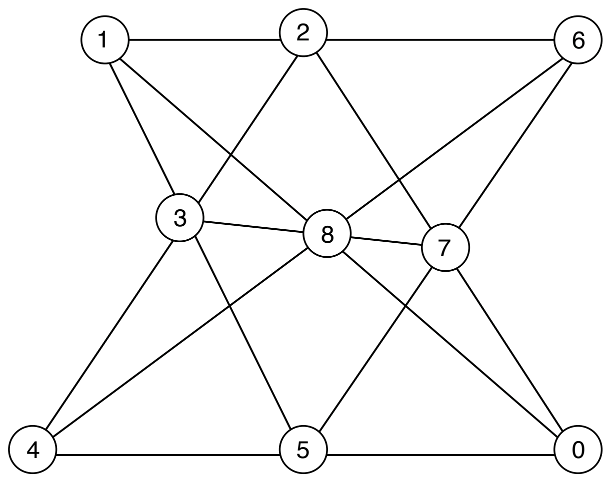
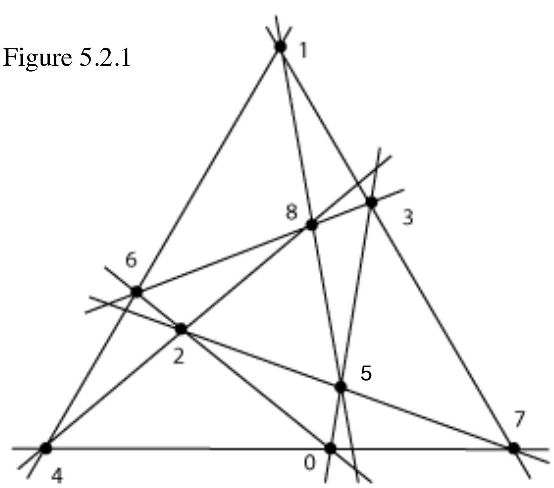


Figure 5.1.2

§5.2 Group embedding of $(9,3)_2$

We define this second (9,3) configuration by the lines {(0,4,7), (0,2,6), (0,3,5), (1,3,7), (1,4,6), (1,5,8), (2,4,8), (2,5,7), (3,6,8)}, seen in figure 4.2.1.

Figure 5.2.1



(0,4,7)
(0,2,6)
(0,3,5)
(1,3,7)
(1,4,6)
(1,5,8)
(2,4,8)
(2,5,7)
(3,6,8)

From this we can start looking for tangential relations. Note that for any point P, $f(P*P) = -2f(P)$ from the group law. Then, $f((P*P)*(P*P)) = -2f(P*P) = 4f(P)$. We start with the point 1.

$$\begin{aligned} 1*1 &= (3*7)*(5*8) \\ &= (3*8)*(5*7) \\ &= 6*2 \\ &= 0 \end{aligned}$$

Similarly we find that,

$$\begin{aligned} 0*0 &= 8 \\ 8*8 &= 7 \\ 7*7 &= 6 \\ 6*6 &= 5 \\ 5*5 &= 4 \\ 3*3 &= 2 \\ 2*2 &= 1 \end{aligned}$$

Thus, for any point in $(9,3)_2$, $-512f(P) = f(P)$, and hence $513f(P) = 0$ or that $513 \equiv 0$ in our target group, since $513 = 27 \times 19$. So we choose $\mathbb{Z}_{27}$ as our target group since $513 \equiv 0 \bmod 27$.

Theorem 5.2.1: $(9,3)_2$ can be embedded in $\mathbb{Z}_{27}$.

Proof: We take the coset of all elements of $\mathbb{Z}_{27}$ that $\equiv 1 \bmod 3$. Namely {1, 4, 7, 10, 13, 16, 19, 22, 25}.

Then if we call a mapping from the points in $(9,3)_2$ to $\mathbb{Z}_{27}$, f, then starting at mapping the point 0 to 1 in $\mathbb{Z}_{27}$, we find that then 8 maps to 25 (since $f(8) = -2f(0)$). Similarly,

$$
\begin{aligned}
&f(0)=1\\
&f(8)=25\\
&f(7)=4\\
&f(6)=19\\
&f(5)=16\\
&f(4)=22\\
&f(3)=10\\
&f(2)=7\\
&f(1)=13.
\end{aligned}
$$

To verify that this mapping is, indeed, valid we make sure all the lines sum to zero, ie that if (*A*,*B*,*C*) is a line, then $f(A)+f(B)+f(C)=0$.

$$
\begin{aligned}
&(0,2,6)\to f(0)+f(2)+f(6)=1+7+19\equiv 0 \bmod 27\\
&(1,3,7)\to f(1)+f(3)+f(7)=13+10+4\equiv 0 \bmod 27\\
&(2,4,8)\to f(2)+f(4)+f(8)=7+22+25\equiv 0 \bmod 27\\
&(3,5,0)\to f(3)+f5+f(0)=10+16+1\equiv 0 \bmod 27\\
&(4,6,1)\to f(4)+f(6)+f(1)=22+19+13\equiv 0 \bmod 27\\
&(5,7,2)\to f(5)+f(7)+f(2)=16+4+7\equiv 0 \bmod 27\\
&(6,8,3)\to f(6)+f(8)+f(3)=19+25+10\equiv 0 \bmod 27\\
&(7,0,4)\to f(7)+f(0)+f4=4+1+22\equiv 0 \bmod 27\\
&(8,1,5)\to f(8)+f(1)+f(5)=25+13+16\equiv 0 \bmod 27
\end{aligned}
$$

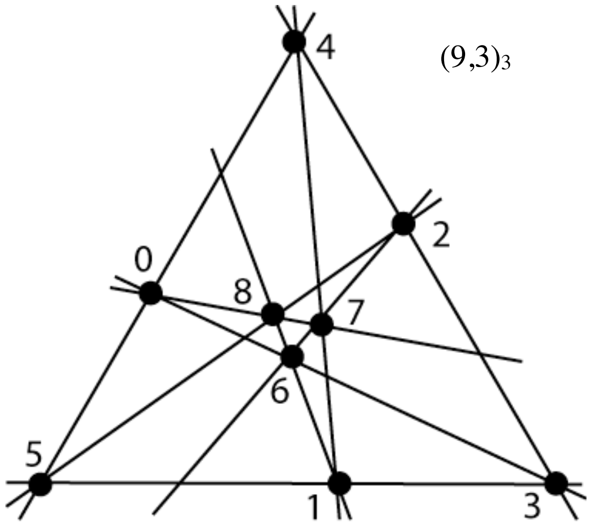
Thus, $(9,3)_2$ can be embedded into the group $\mathbb{Z}_{27}$.

Q.E.D.

§5.3 Group embedding of $(9,3)_3$

We define the lines in $(9,3)_3$ as {(0,3,6), (0,4,5), (0,7,8), (1,3,5), (1,4,7), (1,6,8), (2,3,4), (2,5,8), (2,6,7)}. As seen in figure 4.3.1.

$(9,3)_3$



(0,3,6)
(0,4,5)
(0,7,8)
(1,3,5)
(1,4,7)
(1,6,8)
(2,3,4)
(2,5,8)
(2,6,7)

Figure 5.3.1

In this section we will use the computer program Prover9 to prove that there is no group embedding for this configuration. When given the principle of the operator "*" in the configuration and all of the collinearities, this program will find that the points 8 = 5.

Theorem 5.3.1: $(9,3)_3$ cannot be embedded in any group.

Proof: Here we use the first order theorem proving program Prover9, [9]. From it we can extrapolate a more "human" proof.

```
1 7 = 4 # label(non_clause) # label(goal).  [goal].

2 x * y = y * x.  [assumption].

3 x * (y * x) = y.  [assumption].

4 x * (y * (z * u)) = z * (y * (x * u)).  [assumption].

5 0 * 3 = 6.  [assumption].

6 0 * 4 = 5.  [assumption].

7 0 * 7 = 8.  [assumption].

8 1 * 3 = 5.  [assumption].

9 1 * 4 = 7.  [assumption].

10 1 * 6 = 9.  [assumption].

11 2 * 3 = 4.  [assumption].

12 2 * 5 = 8.  [assumption].

13 2 * 6 = 7.  [assumption].

14 7 != 4.  [deny(1)].

15 x * (x * y) = y.  [para(2(a,1),3(a,1,2))].
```

24 0 * (x * (y * 3)) = y * (x * 6). [para(5(a,1),4(a,1,2,2)),flip(a)].

44 8 * 0 = 7. [para(7(a,1),15(a,1,2)),rewrite([2(3)])].

46 1 * 7 = 4. [para(9(a,1),15(a,1,2))].

353 1 * (x * 6) = 0 * (x * 5). [para(8(a,1),24(a,1,2,2)),flip(a)].

389 7 = 4. [para(12(a,1),353(a,2,2)),rewrite([13(4),46(3),12(4),44(4)]),flip(a)].

390 $F. [resolve(389,a,14,a)].

Clause 1 defines the goal the of the proof.
Clauses 2 through 13 repeat the laws that are given to the computer by the user.
Clause 14 is the negation of the goal, setting up a proof by contradiction.
Clause 15 applies or "paramodulates" Clause 2 to Clause 3 with (a,1) and (a,1,2,2) referring to the positions to obtain x * (x * y) = y.
Clause 24 applies Clause 5 to Clause 4 with x = 0, u = 3 and x * u = 0 * 3 = 6. It then flips the equality sign. This yields 0 * (x * (y * 3)) = y * (x * 6)
Clause 44 applies Clause 7 to Clause 15, to get 0 * 8 = 7. It then uses Clause 2 to obtain 8 * 0 = 7.
Clause 46 does the same thing as Clause 44, except to Clause 9 to obtain 1 * 7 = 4.
Clause 353 applies Clause 8 to Clause 24 substituting 1 for y to obtain 1 * (x * 6) = 0 * (x * 5).
Clause 389 applies Clause 12 to Clause 353 substituting 3 for x to get 1 * (2 * 6) = 0 * (2 * 5). It the rewrites this using Clause 13 to get 1 * 7 = 0 * (2 * 5), then rewrites this using Clause 46 to get 4 = 0 * (2 * 5), then rewrites using Clause 12 to get 4 = 0 * 8, then rewrites using Clause 44 to get 4 = 7. It then flips the equality to get 7 = 4.
Clause 390 demonstrates the contradiction.

This proof can be represented in a more "human" format:

$$
\begin{aligned}
7 &= 0 * 8 = 0 * (2 * 5) \\
&= 0 * (2 * (1 * 3)) \\
&= 1 * (2 * (0 * 3)) \\
&= 1 * (2 * 6) \\
&= 1 * 7 = 4.
\end{aligned}
$$

Thus, any embedding of this configuration into a group would have to map both 8 and 5 to the same element. Thus this configuration cannot be embedded into any group.

Q.E.D.

Chapter 6: (10,3) Configurations

§6.0

There are 10 distinct (10,3) configurations, one of which forms Desargues theorem. Half of these configurations do not have group representations, for those configurations we will use Prover9 [9] generated proofs. We will use Grünbaum's notation to identify each configuration. [2]

The table below shows the results of group embeddings for all 10 configurations.

Configuration	Group	Remarks
$(10,3)_1$	$\mathbb{Z}_2^4$	Desargues Theorem
$(10,3)_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	
$(10,3)_3$	$\mathbb{Z}_{16}$	
$(10,3)_4$	$\mathbb{Z}_4^3$	
$(10,3)_5$	none	proved by Prover9
$(10,3)_6$	none	proved by Prover9
$(10,3)_7$	none	proved by Prover9
$(10,3)_8$	none	proved by Prover9
$(10,3)_9$	none	proved by Prover9
$(10,3)_{10}$	$\mathbb{Z}_{11}$	

§6.1: $(10,3)_1$ Desargues theorem.

We define the lines in $(10,3)_1$ as {(1,2,3), (1,4,5), (1,6,7), (2,4,8), (2,6,9), (3,5,8), (3,7,9), (4,6,0), (5,7,0), (8,9,0)}. Figures 6.1.1 and 6.1.2 show how this configuration is the familiar Desargues theorem.

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,8)
(2,6,9)
(3,5,8)
(3,7,9)
(4,6,0)
(5,7,0)
(8,9,0)

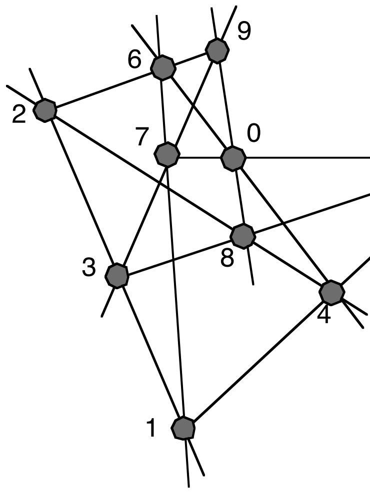


Figure 6.1.1

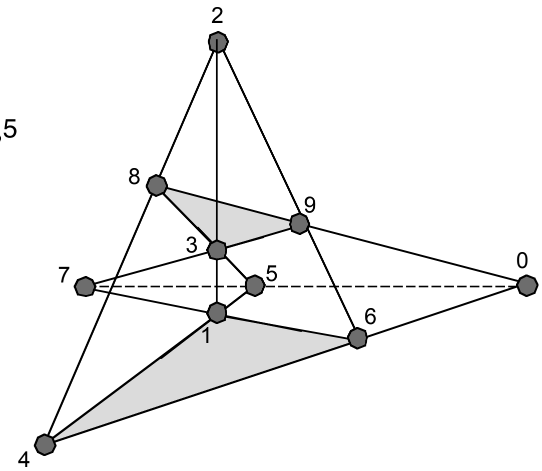


Figure 6.1.2

We first prove that $3*3=0*0$, this is easy, since $3*3=(5*8)*(7*9)=(5*7)*(8*9)=0*0$.

Now, since 0 is the unique point on the axis of perspectivity which is not connected to 3, it is easy to see with figure 6.1.2 that $8*8=7*7$, since 7 is the one point on the axis of perspectivity not connected to 8. So, by symmetry, $6*6=5*5$, and $9*9=5*5$, $1*1=0*0$ etc.

Finally, $2*2=(1*3)*(6*9)=(1*6)*(3*9)=7*7$. Similarly, $2*2=5*5$ and $2*2=0*0$. Therefore, $x*x=y*y$ for all *x* and *y* in the configuration $(10,3)_1$. That is, $2x=2y$ for all *x* and *y*, $2(x-y)=e$, the identity of the desired group *G*.

So we need to look for a group with at least 10 elements of order 2. One such group is $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. We will find our mapping, *f*, by assigning 4 points (no three collinear) to four independent elements of $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. We then follow the collinearities to get the other points.

Theorem 6.1.2: $(10,3)_1$ can be embedded in $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

Proof: We start with the points 1, 2, 4 and 0 in the configuration and map them to (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1) respectively.

$$f:(10,3)_1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\begin{array}{llll} 1 \to (1,0,0,0) & 2 \to (0,1,0,0) & 3 \to (1,1,0,0) & 4 \to (0,0,1,0) \\ 5 \to (1,0,1,0) & 6 \to (0,0,1,1) & 7 \to (1,0,1,1) & 8 \to (0,1,1,0) \\ & 9 \to (0,1,1,1) & 0 \to (0,0,0,1) & \end{array}$$

To verify this we check all 10 collinearities.

$$\begin{aligned}
f(1)+f(2)+f(3) &= (1,0,0,0)+(0,1,0,0)+(1,1,0,0) = (0,0,0,0) \\
f(1)+f(4)+f(5) &= (1,0,0,0)+(0,0,1,0)+(1,0,1,0) = (0,0,0,0) \\
f(1)+f(6)+f(7) &= (1,0,0,0)+(0,0,1,1)+(1,0,1,1) = (0,0,0,0) \\
f(2)+f(4)+f(8) &= (0,1,0,0)+(0,0,1,0)+(0,1,1,0) = (0,0,0,0) \\
f(2)+f(6)+f(9) &= (0,1,0,0)+(0,0,1,1)+(0,1,1,1) = (0,0,0,0) \\
f(3)+f(5)+f(8) &= (1,1,0,0)+(1,0,1,0)+(0,1,1,0) = (0,0,0,0) \\
f(3)+f(7)+f(9) &= (1,1,0,0)+(1,0,1,1)+(0,1,1,1) = (0,0,0,0) \\
f(4)+f(6)+f(0) &= (0,0,1,0)+(0,0,1,1)+(0,0,0,1) = (0,0,0,0) \\
f(5)+f(7)+f(0) &= (1,0,1,0)+(1,0,1,1)+(0,0,0,1) = (0,0,0,0) \\
f(8)+f(9)+f(0) &= (0,1,1,0)+(0,1,1,1)+(0,0,0,1) = (0,0,0,0).
\end{aligned}$$

So f is a valid mapping and $(10,3)_1$ can be embedded into $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Q.E.D.

§6.2: $(10,3)_2$

The lines in $(10,3)_2$ are {(1,2,3), (1,4,5), (1,6,7), (2,4,8), (2,6,9), (3,5,9), (3,7,8), (0,8,9), (0,4,6), (0,5,7)}, shown in figure 5.2.1.

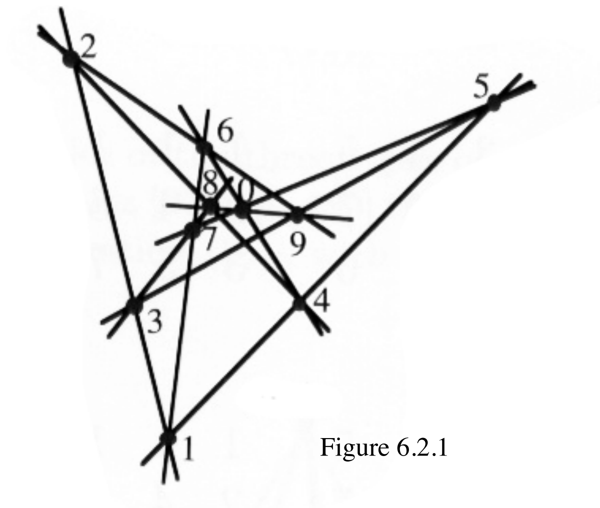


Figure 6.2.1

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,8)
(2,6,9)
(3,5,9)
(3,7,8)
(0,8,9)
(0,4,6)
(0,5,7)

We use the method of tangents to give us information on a group embedding. The following tangents were found using Prover9. They are, however, easy to find by hand.

Prover9 gives us the following tangential relations (page 64, P_1, P_2, P_3, P_4),

$0*0=0$ (hence "0" is a point of order 3 in the target group),

$1*1=0$ (hence the point "1" has an order of 6 in the group),

$2*2=0$,

$3*3=0$ (hence 1, 2 and 3 must be points of order 6 in the target group),

$7*7=4*4=9*9$,

$5*5=6*6=8*8$.

Our target group must therefore have at least 10 elements with one element of order 3 and at least 3 elements of order 6. The smallest such group is $\mathbb{Z}_2\times\mathbb{Z}_6$.

Theorem 6.2.1: $(10,3)_2$ can be embedded in $\mathbb{Z}_2\times\mathbb{Z}_6$.

Proof: There is a degree of freedom in assigning points to elements in $\mathbb{Z}_2\times\mathbb{Z}_6$. The point 0 can be mapped to either (0,0) or (0,4). We choose (0,0) and then assign (1,0), (0,3) and (1,3) to the points 1, 2 and 3, respectively. The point 4 is a free choice, so we assign it to the point (0,1). This then generates the remaining points. The following table summarizes the mapping.

Points in $(10,3)_2$	Mapping to $\mathbb{Z}_2 \times \mathbb{Z}_6$	Remark
0	$(0,0)$	idempotent
1	$(1,0)$	$1*1=0$
2	$(0,3)$	$2*2=0$
3	$(1,3)$	$3*3=0$
4	$(0,1)$	free choice
5	$(1,5)$	$1*4=5$
6	$(0,5)$	$0*4=6$
7	$(1,1)$	$1*6=7$
8	$(0,2)$	$2*4=8$
9	$(0,4)$	$3*5=9$

Since 5 lines were used to generate the remaining points, there are only 5 remaining lines to verify in order to complete the mapping. These are (1,2,3), (2,6,9), (3,7,8), (0,8,9), (0,5,7).

$$(1,2,3) \to (1,0)+(0,3)+(1,3)=(0,0)$$
$$(2,6,9) \to (0,3)+(0,5)+(0,4)=(0,0)$$
$$(3,7,9) \to (1,3)+(1,1)+(0,4)=(0,0)$$
$$(0,8,9) \to (0,0)+(0,2)+(0,4)=(0,0)$$
$$(0,5,7) \to (0,0)+(1,5)+(1,1)=(0,0)$$

Thus, $(10,3)_2$ can be embedded in $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Q.E.D.

§6.3: $(10,3)_3$

The lines in $(10,3)_3$ are {(1,2,3), (1,4,5), (1,6,7), (2,4,8), (2,7,9), (3,5,9), (3,6,8), (0,8,9), (0,4,6), (0,5,7)}, shown in figure 5.3.1.

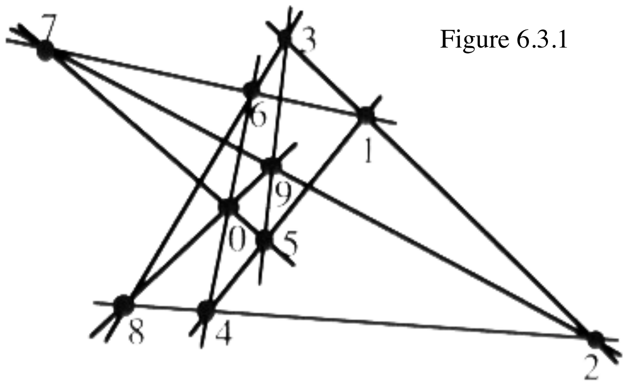


Figure 6.3.1

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,8)
(2,7,9)
(3,5,9)
(3,6,8)
(0,8,9)
(0,4,6)
(0,5,7)

To find a group, G, into which $(10,3)_3$ can be embedded we examine tangential relations and find that (page 64, P_5, P_6, P_7, P_8):

$0*0=0$	thus, 0 is of order 3
$1*1=0$	the point 1 is of order 2
$8*8=9*9=1$	the points 8 and 9 have order 4
$5*5=6*6$.	

The minimal such group is the cyclic group $\mathbb{Z}_{16}$. So we will try to find an embedding, *f,* into this group.

Theorem 6.3.1: The configuration, $(10,3)_3$, can be embedded into $\mathbb{Z}_{16}$.

Proof: As with the previous proof,

$f(0) = 0$ since 0 is idempotent.
$f(1) = 8$ since 8 is the only element of order 2 in $\mathbb{Z}_{16}$.
$f(8) = 4$ $f(8*8) = -2(4) = -8 = 8 = f(1)$
$f(9) = 12$ $f(9) + f(0) + f(8) = 12 + 0 + 4 = 16 = 0$
$f(9*9) = -2(12) = 4 + 4 = 8 = f(1)$.

As before, we use these to generate the group labels for the remaining points.

First, let $f(4) = a$.
$f(5) + f(1) + f(4) = 0$, thus, $f(5) = -8 - a = 8 - a$.
$f(3) + f(5) + f(9) = 0$, thus, $f(3) = -8 + a - 12 = 12 + a$.
$f(2) + f(4) + f(8) = 0$, thus, $f(2) - a - 4$.
$f(1) + f(2) + f(3) = 0$, thus, $8 - a - 4 + 12 + a = 0$. Which is true
$f(7) + f(2) + f(9) = 0$, thus, $f(7) = a + 4 - 12 = a - 8$
$f(7) + f(0) + f(5) = 0$, thus, $f(7) = 0 - (8 - a) = a - 8$
$f(6) + f(0) + f(4) = 0 - a = -a$
$f(6) + f(1) + f(7) = -8 - (a - 8) = -a$
$f(6) + f(3) + f(8) = -(12 + a) - 4 = -a$.

Then, taking $a = 1$, or $f(4) = 1$, we arrive at the following embedding.

$(10,3)_3$	$\mathbb{Z}_{16}$.	Remarks
1	8	order 2
2	11	
3	13	
4	1	
5	7	
6	15	

$(10,3)_3$	$\mathbb{Z}_{16}$.	Remarks
7	9	
8	4	order 4
9	12	order 4
0	0	idempotent

§6.4: $(10,3)_4$

Definition: The configuration $(10,3)_4$ is defined by the lines {(1,2,3), (1,4,5), (1,6,7), (2,4,8), (2,5,9), (3,6,8), (3,7,9), (4,6,0), (5,7,0), (8,9,0)} shown in figure 6.4.1.

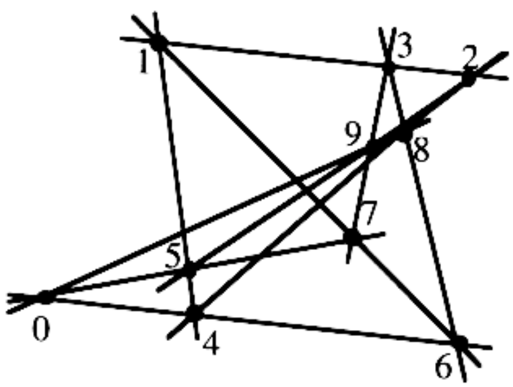


Figure 6.4.1

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,8)
(2,5,9)
(3,6,8)
(3,7,9)
(4,6,0)
(5,7,0)
(8,9,0)

We examine tangential relations and note that this places the points of $(10,3)_4$ into two distinct classes.

The first set of tangential relations define the first class. These relations are $0*0=1*1=4*4=5*5=6*6=7*7$. This makes the first class {0,1,4,5,6,7}.

The second class is formed similarly. From $2*2=3*3=8*8=9*9$ we find the second class to be the remaining elements, {2,3,8,9} (page 66, P_9, P_{10}, P_{11}).

Then, $P*P=Q*Q$ for any P or Q both in the same class. Thus for any embedding, f, $-2f(P)=-2f(Q)$ or $2(P-Q)=e$. Thus, any group that $(10,3)_4$ can be embedded into must have at least 10 elements of order 2. $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ is such a group.

Theorem 6.4.1: The configuration $(10,3)_4$ can be embedded in the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

Proof: Note that each of the six points in the first class can be obtained by drawing a line through 2 of the points in the second class (e.g. $0=8*9$). So, we assign elements of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ to the points 2, 3, 8 and 9 in $(10,3)_4$.

$f(2)=(3,1,1)$, $f(3)=(1,1,1)$, $f(8)=(1,1,3)$, $f(9)=(1,3,1)$

We use this to generate the remaining points.

$(10,3)_4$	Mapping to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	line
1	$(0,2,2)$	123
2	$(3,1,1)$	assigned
3	$(1,1,1)$	assigned
4	$(0,2,0)$	248
5	$(0,0,2)$	259
6	$(2,2,0)$	368
7	$(2,0,2)$	379
8	$(1,1,3)$	assigned
9	$(1,3,1)$	assigned
0	$(2,0,0)$	890

That leaves 4 lines remaining to be verified, the lines (1,6,7), (1,4,5), (5,7,0), (4,6,0). So we now test them.

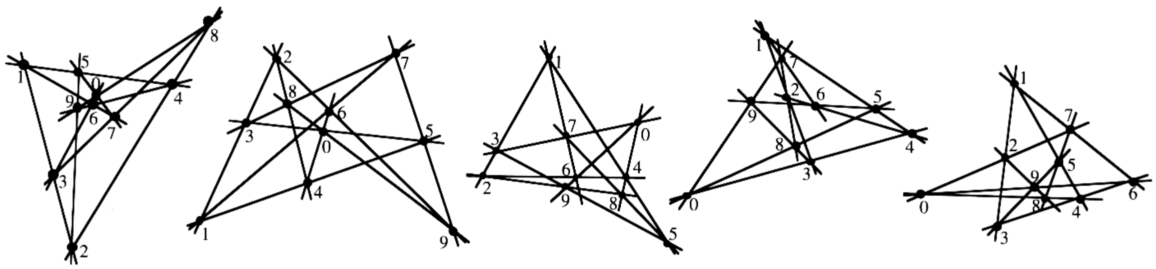
$$f(1)+f(6)+f(7)=(0,2,2)+(2,2,0)+(2,0,2)=(0,0,0)$$
$$f(1)+f(4)+f(5)=(0,2,2)+(0,2,0)+(0,0,2)=(0,0,0)$$
$$f(5)+f(7)+f(0)=(0,0,2)+(2,0,2)+(2,0,0)=(0,0,0)$$
$$f(4)+f(6)+f(0)=(0,2,0)+(2,2,0)+(2,0,0)=(0,0,0)$$

Thus the embedding is complete.

Q.E.D.

§6.5 Configurations $(10,3)_5$, $(10,3)_6$, $(10,3)_7$, $(10,3)_8$ and $(10,3)_9$

These configurations are shown in figure 6.5.1. None of these configurations can be embedded into a group. The collinearities for each configuration are as follows:



$(10,3)_5$ $(10,3)_6$ $(10,3)_7$ $(10,3)_8$ $(10,3)_9$

Figure 6.5.1

$(10,3)_5$	$(10,3)_6$	$(10,3)_7$	$(10,3)_8$	$(10,3)_9$
(1,2,3)	(1,2,3)	(1,2,3)	(1,2,3)	(1,2,3)
(1,4,5)	(1,4,5)	(1,4,5)	(1,4,5)	(1,4,5)
(1,6,7)	(1,6,7)	(1,6,7)	(1,6,7)	(1,6,7)
(2,4,8)	(2,4,8)	(2,4,6)	(2,4,6)	(2,7,0)
(2,5,9)	(2,6,9)	(2,8,9)	(2,7,8)	(2,8,9)
(3,6,0)	(3,5,0)	(3,5,9)	(3,4,0)	(3,4,6)
(3,7,8)	(3,7,8)	(3,7,0)	(3,8,9)	(3,5,9)
(4,6,9)	(4,6,0)	(4,8,0)	(5,6,9)	(4,8,0)
(5,7,0)	(5,7,0)	(5,8,7)	(5,8,0)	(5,7,8)
(8,9,0)	(8,9,0)	(6,9,0)	(7,9,0)	(6,9,0)

Theorem 6.5.1: $(10,3)_5$ has no group embedding.

Proof: Using the operations, *, and its properties it can be shown that the points 9 = 8. A complete proof of this, generated by Prover9, is given on page 67.

Theorem 6.5.2: $(10,3)_6$ has no group embedding.

Proof: Using the operations, *, and its properties it can be shown that the points 7 = 5. A complete proof of this, generated by Prover9, is given on page 67.

Theorem 6.5.3: $(10,3)_7$ has no group embedding.

Proof: Using the operations, *, and its properties it can be shown that the points 9 = 8. A complete proof of this, generated by Prover9, is given on page 68.

Theorem 6.5.4: $(10,3)_8$ has no group embedding.

Proof: Using the operations, *, and its properties it can be shown that the points 9 = 5. A complete proof of this, generated by Prover9, is given on page 68.

Theorem 6.5.5: $(10,3)_9$ has no group embedding.

Proof: Using the operations, *, and its properties it can be shown that the points 9 = 0. A complete proof of this, generated by Prover9, is given on page 68.

§6.6 $(10,3)_{10}$

This configuration is defined by the lines {(1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,8,0), (3,4,0), (3,8,9), (5,6,9), (5,7,8), (7,9,0)}. This is shown in figure 6.6.1.

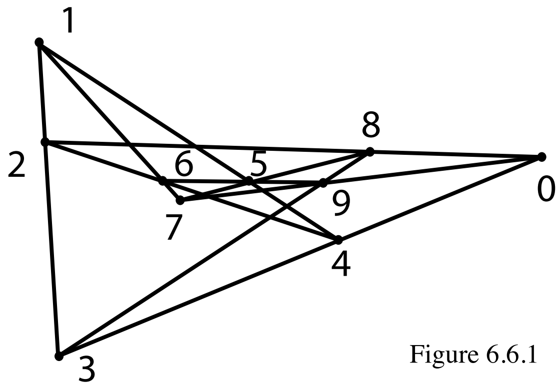


Figure 6.6.1

(1,2,3)
(1,4,5)
(1,6,7)
(2,4,6)
(2,8,0)
(3,4,0)
(3,8,9)
(5,6,9)
(5,7,8)
(7,9,0)

We examine tangential relations to find a group embedding, *f,* from $(10,3)_{10}$ to $\mathbb{Z}_{11}$. These tangential relations are:

$0*0=5$	$3*3=7$
$5*5=2$	$7*7=4$
$2*2=9$	$4*4=8$
$9*9=1$	$8*8=6$
$1*1=0$	$6*6=3$

Thus, for all P in $(10,3)_{10}$ $-32f(P)=f(P)$, or the order of every element in the target group must divide 33. $\mathbb{Z}_{11}$ is the smallest group with 10 elements whose order divides 33 (namely, 11). So we will try to embed this configuration in $\mathbb{Z}_{11}$.

Theorem 6.6.1: $(10,3)_{10}$ can be embedded in the group $\mathbb{Z}_{11}$.

Proof: We start by assigning $f(0)=1$. Then $f(5)=f(0*0)=-1-1=-2\equiv 9 \bmod 11$. Similarly, $f(2)=4, f(9)=3, f(1)=5, f(0)=1$. Next, we do the same for the other 5 points by starting with $f(3)=2$. The final mapping is given the below chart.

$(10,3)_{10}$	$\mathbb{Z}_{11}$.
1	5
2	4
3	2
4	8
5	9
6	10
7	7
8	6
9	3
0	1

Next, we check the collinearities to verify that this embedding is valid.

$$f(1)+f(2)+f(3)=5+4+2\equiv 0 \bmod 11$$
$$f(1)+f(4)+f(5)=5+8+9\equiv 0 \bmod 11$$
$$f(1)+f(6)+f(7)=5+10+7\equiv 0 \bmod 11$$
$$f(2)+f(4)+f(6)=4+8+10\equiv 0 \bmod 11$$
$$f(2)+f(8)+f(0)=4+6+1\equiv 0 \bmod 11$$
$$f(3)+f(4)+f(0)=2+8+1\equiv 0 \bmod 11$$
$$f(3)+f(8)+f(9)=2+6+3\equiv 0 \bmod 11$$
$$f(5)+f(6)+f(9)=9+10+3\equiv 0 \bmod 11$$
$$f(5)+f(7)+f(8)=9+7+6\equiv 0 \bmod 11$$
$$f(7)+f(9)+f(0)=7+3+1\equiv 0 \bmod 11$$

Thus, this embedding is valid.

Q.E.D.

Definition: Tight embedding.

We call an embedding, *f*, from a configuration *C* to a group *G* “tight” if *f* is either a surjection or *f* is a function from *C* onto $\{G \setminus e\}$.

Thus, the above mapping from $(10,3)_{10}$ to $\mathbb{Z}_{11}$ is a tight embedding.

§6.7 Other representations

We can represent $(10,3)_{10}$ in a different way, using the cycle notation seen in chapter 3. We define the lines as {(0,1,3) mod 10}. Thus the lines are (0,1,3), (1,2,4), etc. The diagram (figure 5.7.1) is also different from figure 5.6.1. So it would seem the two configurations are different and that there are, in fact, 11 (10,3) configurations, which is false. Thus, we aim to show that these two configurations are the same by way of showing that their group embedding is the same.

We also demonstrate a new technique for finding a group embedding. This technique works when using a cyclic configuration.

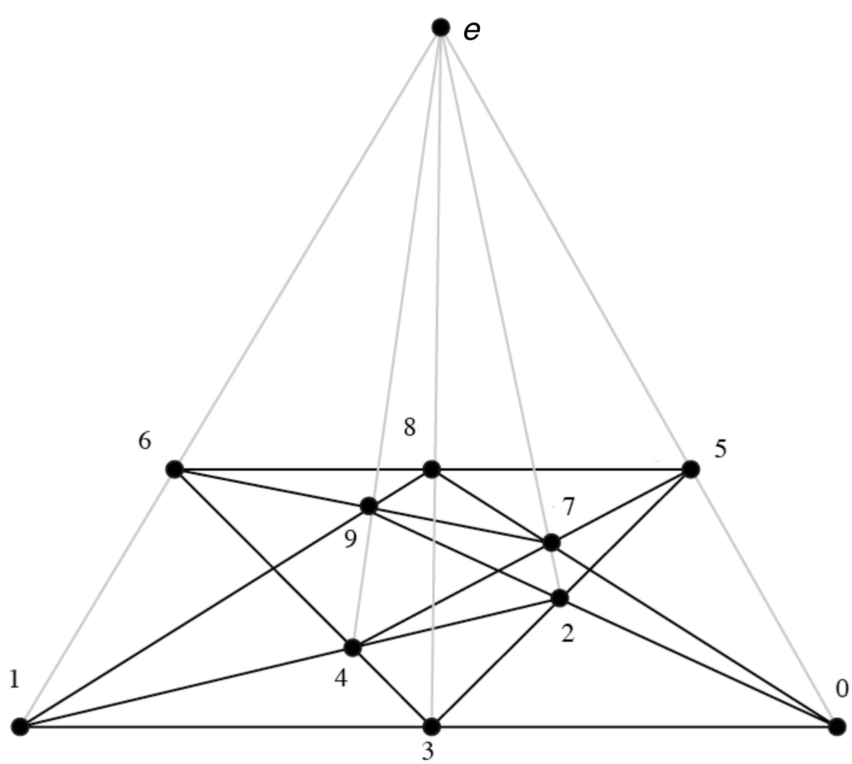


Figure 6.7.1

Theorem 6.7.1: The configuration {(0,1,3) mod 10} can be embedded into $\mathbb{Z}_{11}$.

Proof: Let $f(x) = p^x$, where p is a primitive root modulo 11. Thus if (0,1,3) is a line in the configuration, then we want to show that $p^0 + p^1 + p^3 = 0$ in $\mathbb{Z}_{11}$. We choose p to be 2, making $f(x) = 2^x \bmod 11$. We test the line (0,1,3), $f(0) + f(1) + f(3) = 1 + 2 + 8 \equiv 0 \bmod 11$. We do not need to test the other lines, since this configuration is cyclic.

Q.E.D.

Theorem 6.7.2: The configuration {(0,1,3) mod 10} is isomorphic to $(10,3)_{10}$.

Proof: Let us start from the above mapping $i \to 2^i \bmod 11$. Then take the reverse mapping from the previous section to obtain a one-one correspondence between these two (10, 3)'s.

{(0,1,3) mod 10}	Mapping to $(10,3)_{10}$	Lines in $(10,3)_{10}$	Mapping to {(0,1,3) mod 10}
$0 = 2^0 \equiv 1 \bmod 11$	0	$(1,2,3)$	$(1,2,4)$
$1 = 2^1 \equiv 2 \bmod 11$	3	$(1,4,5)$	$(3,4,6)$
$2 = 2^2 \equiv 4 \bmod 11$	2	$(1,6,7)$	$(4,5,7)$
$3 = 2^3 \equiv 8 \bmod 11$	4	$(2,4,6)$	$(2,3,5)$
$4 = 2^4 \equiv 5 \bmod 11$	1	$(2,8,0)$	$(9,0,2)$
$5 = 2^5 \equiv 10 \bmod 11$	6	$(3,4,0)$	$(0,1,3)$
$6 = 2^6 \equiv 9 \bmod 11$	5	$(3,8,9)$	$(8,9,1)$
$7 = 2^7 \equiv 7 \bmod 11$	7	$(5,6,9)$	$(5,6,8)$
$8 = 2^8 \equiv 3 \bmod 11$	9	$(5,7,8)$	$(6,7,9)$
$9 = 2^9 \equiv 6 \bmod 11$	8	$(7,9,0)$	$(7,8,0)$

Q.E.D.

Chapter 7: (11,3) Configurations

§7.0

There are 31 distinct (11,3) configurations most of which have no group embedding. All negative proofs will be provided in Prover9 section along with each of the defined lines of the configuration, given in the opening clauses. As before, we will use Grünbaum's notation to identify each configuration. [2] The table below shows the results of group embeddings for all (11,3) configurations.

Configuration	Group	Configuration	Group
$(11,3)_1$	none (page 68)	$(11,3)_{17}$	$\mathbb{Z}_{11}$
$(11,3)_2$	none (page 69)	$(11,3)_{18}$	$\mathbb{Z}_{15}$
$(11,3)_3$	none (page 69)	$(11,3)_{19}$	$\mathbb{Z}_{21}$
$(11,3)_4$	none (page 69)	$(11,3)_{20}$	none (page 75)
$(11,3)_5$	none (page 70)	$(11,3)_{21}$	none (page 75)
$(11,3)_6$	none (page 70)	$(11,3)_{22}$	$\mathbb{Z}_4 \times \mathbb{Z}_{12}$
$(11,3)_7$	none (page 71)	$(11,3)_{23}$	none (page 75)
$(11,3)_8$	none (page 71)	$(11,3)_{24}$	none (page 75-76)
$(11,3)_9$	none (page 71)	$(11,3)_{25}$	none (page 76)
$(11,3)_{10}$	none (page 72)	$(11,3)_{26}$	none (page 76)
$(11,3)_{11}$	none (page 72)	$(11,3)_{27}$	$\mathbb{Z}_{15}$
$(11,3)_{12}$	none (page 73)	$(11,3)_{28}$	$\mathbb{Z}_3 \times \mathbb{Z}_6$
$(11,3)_{13}$	none (page 73)	$(11,3)_{29}$	$\mathbb{Z}_3 \times \mathbb{Z}_6$
$(11,3)_{14}$	none (page 73)	$(11,3)_{30}$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
$(11,3)_{15}$	none (page 74)	$(11,3)_{31}$	$\mathbb{Z}_{23}$
$(11,3)_{16}$	none (page 74)		

§7.1 $(11,3)_{17}$

We define the lines in $(11,3)_{17}$ as {(1,2,3), (1,4,5), (1,6,7), (2,4,9), (2,8,10), (3,6,10), (3,7,0), (4,7,8), (5,6,9), (5,0,10), (8,9,0)}

In order to find a group, we will use a parameter method. Let *f* be a mapping from $(11,3)_{17}$ to some group *G*. We let $f(1) = a, f(2) = b$, and $f(4) = c$. We then use each of the configuration's lines to develop restrictions on *a, b* and *c*.

$f(3) = -a - b$	from $(1,2,3)$
$f(5) = -a - c$	from $(1,4,5)$
$f(9) = -b - c$	from $(2,4,9)$
$f(6) = a + b + 2c$	from $(5,6,9)$
$f(7) = -2a - b - 2c$	from $(1,6,7)$
$f(0) = 3a + 2b + 2c$	from $(3,7,0)$
$f(10) = -2c$	from $(3,6,10)$
$f(8) = -b + 2c$	from $(2,8,10)$
$3c + 3a = 0,\ a = -c$	from $(8,9,0)$
$3c = 2b$	from $(4,7,8)$
$3c = 2b$	from $(5,0,10)$

Any choice for *a, b* or *c* will give an embedding into $\mathbb{Z}$. So *G* will be $\mathbb{Z}$.

Theorem 7.1.1: $(11,3)_{17}$ can be embedded into $\mathbb{Z}$.

Proof: We use the above calculations to find an embedding in $\mathbb{Z}$. By letting *a* = -2, that makes *b* = 3 and *c* = 2 and gives the following mapping.

Points in $(11,3)_{17}$	Mapping to Free Abelian Group	Lines
0	4	$(1,2,3) \to -2+3-1=0$
1	-2	$(1,4,5) \to -2+2+0=0$
2	3	$(1,6,7) \to -2+5=3=0$
3	-1	$(2,4,9) \to 3+2-5=0$
4	2	$(2,8,10) \to 3+1-4=0$
5	0	$(3,6,10) \to -1+5-4=0$
6	5	$(3,7,0) \to -1+3-4=0$
7	-3	$(4,7,8) \to 2-3+1=0$
8	1	$(5,6,9) \to 0+5-5=0$
9	-5	$(5,0,10) \to 0+4-4=0$
10	-4	$(8,9,0) \to 1-5+4=0$

The lines on the right show that this embedding is valid.

Q.E.D.

Theorem 7.1.2: $(11,3)_{17}$ can be embedded into $\mathbb{Z}_{11}$.

Proof: From the mapping to the free abelian group, it can easily be seen that this same mapping with a = -2, b = 3 and c = 2 also yields a mapping into $\mathbb{Z}_{11}$. All 11 lines will still hold for the same reasons.

§7.2 $(11,3)_{18}$

This configuration is defined by the following lines: {(1,2,3), (1,4,5), (1,6,7), (2,5,0), (2,9,10), (3,7,0), (3,8,10), (4,6,10), (4,7,8), (5,6,9), (8,9,0)}.

To find a group in which we can embed this configuration we will use both the parameter method and some Tangential relations. Prover9 verifies that (page 76, P_{11}, P_{12}):

$5*5=5$
$7*7=7$
$1*1=0$
$2*2=8$
$3*3=9$
$6*6=3$
$10*10=1$.

Let f be a mapping from $(11,3)_{18}$ to some group G, and let $f(1)=a, f(2)=b$ and $f(4)=c$. We can then generate the following mapping.

$f(3)=-a-b$	from $(1,2,3)$
$f(5)=-a-c$	from $(1,4,5)$
$f(0)=-b+a+c$	from $(2,5,0)$
$f(7)=2b-c$	from $(3,7,0)$
$f(6)=-a-2b+c$	from $(1,6,7)$
$f(8)=-2b$	from $(4,7,8)$ also $f(2*2)=-2f(2)=-2b$
$f(9)=2a+2b$	from $(5,6,9)$
$f(10)=a+3b$	from $(3,8,10)$

We still have not used the lines (2,9,10), (4,6,10) and (8,9,0). Instead we will use the fact that 5 and 7 are idempotent to learn about G. Since 7 is idempotent, $-2b+c-2b+c=2b-c$ or $6b=3c$. Also, from the idempotency of 5 we get $a+c+a+c=-a-c$ or $3a+3c=0$. From $1*1=0$ we get $-2a=-b+a+c$ or $3a+c=b$. Putting the above together we get that $b=-2c$ and thus $-12c=3c$ or $15c=0$. So a possible group for G is $\mathbb{Z}_{15}$.

Theorem 7.2.1: $(11,3)_{18}$ can be embedded in $\mathbb{Z}_{15}$.

Proof: If we assign $b = 1$, then $c = 7$ and $a = 3$. And we get the following mapping.

Points in $(11,3)_{18}$	Mapping to $\mathbb{Z}_{15}$	Lines
0	9	$f(1)+f(2)+f(3)=3+1+11=15$
1	3	$f(1)+f(4)+f(5)=3+7+5=15$
2	1	$f(1)+f(6)+f(7)=3+2+10=15$
3	11	$f(2)+f(5)+f(0)=1+5+9=15$
4	7	$f(2)+f(9)+f(10)=1+8+6=15$
5	5	$f(3)+f(7)+f(0)=11+10+9=30$
6	2	$f(3)+f(8)+f(10)=11+13+6=30$
7	10	$f(4)+f(6)+f(10)=7+2+6=15$
8	13	$f(4)+f(7)+f(8)=7+10+13=30$
9	8	$f(5)+f(6)+f(9)=5+2+8=15$
10	6	$f(8)+f(9)+f(0)=13+8+9=30$

Thus, the mapping is valid.

Q.E.D.

§7.3 $(11,3)_{19}$

This configuration is defined by the following lines: {(1,2,3), (1,4,5), (1,6,7), (2,4,9), (2,5,0), (3,6,8), (3,7,0), (4,6,10), (5,9,10), (7,8,10), (8,9,0)}.

To find a group in which we can embed this configuration we will use the parameter method. Let f be a mapping from $(11,3)_{19}$ to some group G. We start with $f(1)=a$, $f(2)=b$ and $f(4)=c$ as with $(11,3)_{17}$ and work out restrictions.

$f(3) = -a - b$ from $(1,2,3)$
$f(5) = -a - c$ from $(1,4,5)$
$f(9) = -b - c$ from $(2,4,9)$
$f(0) = -b + a + c$ from $(2,5,0)$
$f(7) = 2b - c$ from $(3,7,0)$
$f(6) = c - a - 2b$ from $(1,6,7)$
$f(8) = 2a + 3b - c$ from $(3,6,8)$
$f(10) = a + 2b - 2c$ from $(4,6,10)$
$b = 4c$ from $(5,9,10)$
$3a + 24c = 0$ from $(7,8,10)$
$3a + 3c = 0$ from $(8,9,0)$

Thus, the last two lines give us that $21c = 0$ so we choose G to be $\mathbb{Z}_{21}$.

Theorem 7.3.1: $(11,3)_{19}$ can be embedded in $\mathbb{Z}_{21}$.

Proof: We use the above parameterization to come up with an embedding and show that it is valid. We start by setting c to 1 in $\mathbb{Z}_{21}$ since it must have order 21. Then b is 4 and a is 6. This yields the following mapping:

Points in $(11,3)_{19}$	Mapping to $\mathbb{Z}_{21}$	Lines
0	3	$(1,2,3) = 6 + 4 + 11 = 21$
1	6	$(1,4,5) = 6 + 1 + 14 = 21$
2	4	$(1,6,7) = 6 + 8 + 7 = 21$
3	11	$(2,4,9) = 4 + 1 + 16 = 21$
4	1	$(2,5,0) = 4 + 14 + 3 = 21$

Points in $(11,3)_{19}$	Mapping to $\mathbb{Z}_{21}$	Lines
5	14	$(3,6,8)=11+8+2=21$
6	8	$(3,7,0)=11+7+3=21$
7	7	$(4,6,10)=1+8+12=21$
8	2	$(5,9,10)=14+16+12=42$
9	16	$(7,8,10)=7+2+12=21$
10	12	$(8,9,0)=2+16+3=21$

The fact that all the lines sum to zero shows this mapping to be valid.

Q.E.D.

§7.4 $(11,3)_{22}$

The lines in $(11,3)_{22}$ are defined as {(1,2,3), (1,4,5), (1,6,7), (2,4,9), (2,8,10), (3,6,8), (3,9,10), (4,7,0), (5,6,0), (5,7,10), (8,9,0)}.

To find a group embedding, *f*, we first look for a group *G* by examining tangential relations derived using Prover9. The first, and most important relation is that $(10*10)*(10*10)=10*10$. In other words, $4f(10)=-2f(10)$ or that 10 has an order that divides 6. Now, Prover9 also shows us that $0*0=1*1=4*4=6*6=10$ (page 77, P_{13} - P_{20}). So the order of each of those points must divide 12. We surmise that *G* is $\mathbb{Z}_4\times\mathbb{Z}_{12}$.

Theorem 7.4.1: $(11,3)_{22}$ can be embedded in $\mathbb{Z}_4\times\mathbb{Z}_{12}$.

Proof: Since $-2(2,0)=(-4,0)=(0,0)$ we let $f(10)=(2,0)$. From this we can find values for 0, 1, 4 and 6. This then gives us the points 5 and 7. With further help from Prover9 generating the relations, $2*2=8*8=7$, we get the remaining points. The mapping and its verification are included in the below table.

Points in $(11,3)_{22}$	Mapping to $\mathbb{Z}_4 \times \mathbb{Z}_{12}$	Lines
0	$(1,0)$	$f(1)+f(2)+f(3)=(3,0)+(1,3)+(0,9)=(0,0)$
1	$(3,0)$	$f(1)+f(4)+f(5)=(3,0)+(1,6)+(0,6)=(0,0)$
2	$(1,3)$	$f(1)+f(6)+f(7)=(3,0)+(3,6)+(2,6)=(0,0)$
3	$(0,9)$	$f(2)+f(4)+f(9)=(1,3)+(1,6)+(2,3)=(0,0)$
4	$(1,6)$	$f(2)+f(8)+f(10)=(1,3)+(1,9)+(2,0)=(0,0)$
5	$(0,6)$	$f(3)+f(6)+f(8)=(0,9)+(3,6)+(1,9)=(0,0)$
6	$(3,6)$	$f(3)+f(9)+f(10)=(0,9)+(2,3)+(2,0)=(0,0)$
7	$(2,6)$	$f(4)+f(7)+f(0)=(1,6)+(2,6)+(1,0)=(0,0)$
8	$(1,9)$	$f(5)+f(6)+f(0)=(0,6)+(3,6)+(1,0)=(0,0)$
9	$(2,3)$	$f(5)+f(7)+f(10)=(0,6)+(2,6)+(2,0)=(0,0)$
10	$(2,0)$	$f(8)+f(9)+f(0)=(1,9)+(2,3)+(1,0)=(0,0)$

Q.E.D.

§7.5 $(11,3)_{27}$

The lines in $(11,3)_{27}$ are defined as {(1,4,5), (1,6,7), (1,0,10), (2,3,10), (2,4,9), (2,8,0), (3,6,8), (3,9,0), (5,6,9), (5,8,10), (9,0,10)}.

To find a group embedding we first look for a group *G* by examining tangential relations. Using Prover9 we find the following relations (page 82, P_{21}, P_{22}):

$10 * 10 = 10$
$2 * 2 = 7$
$7 * 7 = 3$
$3 * 3 = 5$
$5 * 5 = 2.$

From the last 4 relations we can see that for the points 2, 7, 3 and 5, $16P = P$ or $15P = 0$. So G must have at least 4 elements of order 15 and at least one idempotent element. $\mathbb{Z}_{15}$ is such a group.

Theorem 7.5.1: $(11,3)_{28}$ can be embedded in $\mathbb{Z}_{15}$.

Proof: Assigning $f(2) = 1$ generates 7, 3, 5 and then also 10. Then assigning $f(0) = 2$ and $f(1) = 3$ will generate the remaining points.

Points in $(11,3)_{28}$	Mapping to $\mathbb{Z}_{15}$	Lines
0	2	$f(1)+f(4)+f(5)=3+5+7=15$
1	3	$f(1)+f(6)+f(7)=3+14+13=30$
2	1	$f(1)+f(0)+f(10)=3+2+10=15$
3	4	$f(2)+f(3)+f(10)=1+4+10=15$
4	5	$f(2)+f(4)+f(9)=1+5+9=15$
5	7	$f(2)+f(8)+f(0)=1+12+2=15$
6	14	$f(3)+f(6)+f(8)=4+14+12=30$
7	13	$f(3)+f(9)+f(0)=4+9+2=15$
8	12	$f(4)+f(7)+f(8)=5+13+12=30$
9	9	$f(5)+f(6)+f(9)=7+14+9=30$
10	10	$f(5)+f(7)+f(10)=7+13+10=30$

Q.E.D.

§7.6 $(11,3)_{28}$

The lines in $(11,3)_{28}$ are defined as {(1,2,3), (1,4,5), (1,6,7), (2,6,0), (2,8,9), (3,4,8), (3,7,0), (4,7,10), (5,6,9), (5,8,10), (9,0,10)}

To find a group embedding we first look for a group *G* by examining tangential relations. Using Prover9 we find that there are seven idempotent elements. Namely, $\{0, 2, 5, 6, 8, 9, 10\}$. Prover9 also finds the following tangential relations (page 82, P_{23} - P_{29}):

$$3 * 3 = 6$$
$$1 * 1 = 0$$
$$7 * 7 = 2.$$

Thus *G* must have at least 9 elements of order 3 and also contain $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a subgroup. The smallest such group with at least 11 elements is the group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_6$. So we let *G* be $\mathbb{Z}_3 \times \mathbb{Z}_6$.

Theorem 7.6.1: $(11,3)_{28}$ can be embedded in the group $\mathbb{Z}_3 \times \mathbb{Z}_6$.

Proof: Start by noting that the idempotent elements in $\mathbb{Z}_3 \times \mathbb{Z}_6$ are (0,0), (0,2), (0,4), (1,0), (1,2), (1,4), (2,0), (2,2), (2,4). We seek a mapping, *f,* from $(11,3)_{28}$ to $\mathbb{Z}_3 \times \mathbb{Z}_6$. We start by assigning $f(0) = (0,0)$. Then, since $1 * 1 = 0,$ we can find $f(1)$ by taking $f(1 * 1) = (-2,-2) = (0,0)$, so $f(1) = (0,3)$.

Next we assign $f(6) = (0,4)$. Thus, $f(3) = (0,1)$ since $3 * 3 = 6$. Then, from the line (1,2,3), we get $f(2) = (0,2)$, which satisfies the idempotent condition. Then, since $7 * 7 = 2$, $f(7) = (0,5)$. We next assign $f(5) = (1,4)$, and then, from the line (1,4,5) we get $f(4) = (2,5)$. Then the remaining collinearities generate the rest of the mapping. The following table summarizes the mapping and collinearities, proving the theorem.

Points in $(11,3)_{28}$	Mapping to $\mathbb{Z}_3 \times \mathbb{Z}_6$.	idempotent	Lines
0	$(0,0)$	yes	$(1,2,3) \to (0,3)+(0,2)+(0,1)=(0,0)$
1	$(0,3)$	no	$(1,4,5) \to (0,3)+(2,5)+(1,4)=(0,0)$
2	$(0,2)$	yes	$(1,6,7) \to (0,3)+(0,4)+(0,5)=(0,0)$
3	$(0,1)$	no	$(2,6,0) \to (0,2)+(0,4)+(0,0)=(0,0)$
4	$(2,5)$	no	$(2,8,9) \to (0,2)+(1,0)+(2,4)=(0,0)$
5	$(1,4)$	yes	$(3,4,8) \to (0,1)+(2,5)+(1,0)=(0,0)$
6	$(0,4)$	yes	$(3,7,0) \to (0,1)+(0,5)+(0,0)=(0,0)$
7	$(0,5)$	no	$(4,7,10) \to (2,5)+(0,5)+(1,2)=(0,0)$
8	$(1,0)$	yes	$(5,6,9) \to (1,4)+(0,4)+(2,4)=(0,0)$
9	$(2,4)$	yes	$(5,8,10) \to (1,4)+(1,0)+(1,2)=(0,0)$
10	$(1,2)$	yes	$(9,0,10) \to (2,4)+(0,0)+(1,2)=(0,0)$

Q.E.D.

§7.7 $(11,3)_{29}$

The lines in $(11,3)_{29}$ are defined as {(1,2,3), (1,4,5), (1,6,7), (2,6,0), (2,8,9), (3,7,0), (3,9,10), (4,8,10), (4,9,0), (5,6,10), (5,7,0)}. To find a group embedding, *f*, we first look for a group *G* by using a few tangential relations. The only tangential relations we have to work with are $5*5=5$, $9*9=5$, $6*6=3*3$ and $0*0=1*1$ (page 86, P_{30} - P_{34}). So we need at least one element of order 3 and at least one of order 6. We guess at $\mathbb{Z}_3 \times \mathbb{Z}_6$ and try to find a group embedding.

Theorem 7.7.1: $(11,3)_{29}$ can be embedded in $\mathbb{Z}_3 \times \mathbb{Z}_6$.

Proof: If we choose $f(5)=(1,4)$ and $f(9)=(a,b)$. Then, since $9*9=5$, we have

$(-2a,-2b)=(1,4)$. Thus we have two solutions: $a=1,\ b=1$ or $a=1,\ b=4$.

Since $f(5)=(1,4)$ we have a unique choice for 9, i.e. $f(9)=(1,1)$.

Consider the relation $1*9=0*5$, proved by Prover9.

Let $f(1)=(a,b)$ and $f(0)=(u,v)$.
$1*9=0*5$ implies that $(a+1,b+1)=(u+1,v+4)$.
Or, that $a=u$ and $b=v+3$.
Also, $f(4)=f(1*5)=(-u-1,-v-7)=(-u-1,-v-1)$.
We check the validity of the relation $5*9=0$.
$f(4*9)=(u+1-1,v+1-1)=(u,v)=0$.
We check the validity of the relation $0*0=1*1$.
$f(0*0)=(-2u,-2v)$,
$f(1*1)=(-2u,-2v-6)=(-2u,-2v)$.

Now let $f(3)=(x,y)$.

Then $f(7)=f(3*0)=(-x-u,-y-v)$.
We have $f(8)=f(5*7)=(x+u+2,y+v+2)$.
Now $f(2)=f(8*9)=(-x-u,y+v+3)$.
Use the collinearity of $(1,2,3)$ i.e. $1*2=3$.
$f(3)=f(1*2)=(x,-v-3-y-v-3)=(x,-y-2v)$.
So $(x,y)=(x,-y-2v)$ or $2y=4v \bmod 6$.
Also, $f(6)=f(2*0)=(x+u-u,-y-v-3-v)=(x,y-2v-3)$.
We also have that $6*6=3*3$, i.e. that,
$(-2x,-2y+4v)=(-2x,-2y)$. In particular, $4v=0$.
Thus, $v=0$ or 3.
Finally, $f(10)=f(3*9)=(-x-1,-y-1)=(-x+2,-y+5)$.

All that is left is to make a choice for *x, y, u* and *v* ($2y = 2v = 0 \bmod 6$). So we take

$x = 2,\ u = 1,\ \mathrm{y} = 0,\ v = 3$. This yields the following mapping.

Points in $(11,3)_{29}$	Mapping to $\mathbb{Z}_3 \times \mathbb{Z}_6$	Lines
0	$(1,3)$	$f(1)+f(2)+f(3)=(1,0)+(0,0)+(2,0)=(0,0)$
1	$(1,0)$	$f(1)+f(4)+f(5)=(1,0)+(1,2)+(1,4)=(0,0)$
2	$(0,0)$	$f(1)+f(6)+f(7)=(1,0)+(2,3)+(0,3)=(0,0)$
3	$(2,0)$	$f(2)+f(6)+f(0)=(0,0)+(2,3)+(1,3)=(0,0)$
4	$(1,2)$	$f(2)+f(8)+f(9)=(0,0)+(2,5)+(1,1)=(0,0)$
5	$(1,4)$	$f(3)+f(7)+f(0)=(2,0)+(0,3)+(1,3)=(0,0)$
6	$(2,3)$	$f(3)+f(9)+f(10)=(2,0)+(1,1)+(0,5)=(0,0)$
7	$(0,3)$	$f(4)+f(8)+f(10)=(1,2)+(2,5)+(0,5)=(0,0)$
8	$(2,5)$	$f(4)+f(9)+f(0)=(1,2)+(2,3)+(1,3)=(0,0)$
9	$(1,1)$	$f(5)+f(6)+f(10)=(1,4)+(2,3)+(0,5)=(0,0)$
10	$(0,5)$	$f(5)+f(7)+f(8)=(1,4)+(0,3)+(2,5)=(0,0)$

Q.E.D.

§7.8 $(11,3)_{30}$

The lines in $(11,3)_{30}$ are defined as {(1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,8,9), (3,8,10), (3,9,0), (4,0,10), (5,6,10), (5,7,9), (7,8,0)}

To find a group embedding, *f*, we first look for a group *G* by examining tangential relations. Prover 9 gives us the following relations (page 87, P_{35} - P_{44}):

$4*4=8$
$9*9=6$
$3*3=6$
$6*6=6$
$5*5=8$
$1*1=8$
$8*8=8$
$0*0=6.$

Thus, we have 2 idempotent elements and 8 elements who's order divides 6. Such a group is $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Theorem 7.8.1: $(11,3)_{30}$ can be embedded in $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Proof: First we need that $f(6*6)=-2f(6)$ and $f(8*8)=-2f(8)$. The points (0,2) and (0,4) satisfy these expressions so we let $f(6)=(0,2)$ and $f(8)=(0,4)$ and generate the remaining points from these.

Points in $(11,3)_{30}$	Mapping to $\mathbb{Z}_2 \times \mathbb{Z}_6$	Lines
0	$(0,5)$	$f(1)+f(2)+f(3)=(0,1)+(1,3)+(1,2)=(0,0)$
1	$(0,1)$	$f(1)+f(4)+f(5)=(0,1)+(1,1)+(1,4)=(0,0)$
2	$(1,3)$	$f(1)+f(6)+f(7)=(0,1)+(0,2)+(0,3)=(0,0)$
3	$(1,2)$	$f(2)+f(4)+f(6)=(1,3)+(1,1)+(0,2)=(0,0)$
4	$(1,1)$	$f(2)+f(8)+f(9)=(1,3)+(0,4)+(1,5)=(0,0)$
5	$(1,4)$	$f(3)+f(8)+f(10)=(1,2)+(0,4)+(1,0)=(0,0)$
6	$(0,2)$	$f(3)+f(9)+f(0)=(1,2)+(1,5)+(0,5)=(0,0)$
7	$(0,3)$	$f(4)+f(0)+f(10)=(1,1)+(0,5)+(1,0)=(0,0)$

Points in $(11,3)_{30}$	Mapping to $\mathbb{Z}_2 \times \mathbb{Z}_6$	Lines
8	$(0,4)$	$f(5)+f(6)+f(10)=(1,4)+(0,2)+(1,0)=(0,0)$
9	$(1,5)$	$f(5)+f(7)+f(9)=(1,4)+(0,3)+(1,5)=(0,0)$
10	$(1,0)$	$f(7)+f(8)+f(0)=(0,3)+(0,4)+(0,5)=(0,0)$

Q.E.D.

§7.9: $(11,3)_{31}$

The lines in $(11,3)_{30}$ are defined as {(1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,8,9), (3,4,8), (3,9,0), (5,6,10), (5,7,10), (7,9,10), (8,0,10)}

To find a group embedding, f, we first look for a group G by the parameter method. We start with $f(1)=a$, $f(2)=b$ and $f(4)=c$. This generates,

$f(3)=-a-b$ from $(1,2,3)$,
$f(5)=-a-c$ from $(1,4,5)$,
$f(6)=-b-c$ from $(2,4,6)$,
$f(7)=-a+b+c$ from $(1,6,7)$,
$f(8)=a+b-c$ from $(3,4,8)$,
$f(7)=2b-c$ from $(3,7,0)$,
$f(10)=a+b+2c$ from $(5,6,10)$,
$f(9)=-a-2b+c$ from $(2,8,9)$,
$f(0)=2a-3$ from $(5,7,0)$,
$c=4b$ from $(3,9,0)$,
$a=16b$ from $(7,9,10)$,
$4a+b+c=69b=0$ from $(8,0,10)$.

$\mathbb{Z}_{69}$ would work, but for a smaller embedding, we will try $\mathbb{Z}_{23}$.

Theorem 7.9.1: $(11,3)_{31}$ can be embedded in $\mathbb{Z}_{23}$.

Proof: We start by using the above parameters and assigning $b = 1$. This gives us the following mapping. We do not need to work out the lines, since each point was generated by them.

Points in $(11,3)_{31}$	Mapping to $\mathbb{Z}_{23}$.
0	8
1	16
2	1
3	6
4	4
5	3
6	18
7	12
8	13
9	9
10	2

Q.E.D.

§7.10 {(0,1,3) mod 11}

Here we have another cyclic configuration and we will use a similar method as in chapter 5 to work out a group embedding and prove that this configuration is isomorphic to one of the previous configurations.

Theorem 7.10.1: {(0,1,3) mod 11} can be embedded in $\mathbb{Z}_{23}$. Furthermore, {(0,1,3) mod 11} is isomorphic to $(11,3)_{31}$.

Proof: Let $f(x) = 4^x \bmod 23$. We will show that this will cover every line. We look first at the line (0,1,3). $f(0)+f(1)+f(3) = 4^0 + 4^1 + 4^3 = 1+4+64 = 69 \equiv 0 \bmod 23$. Now consider any line in this configuration. It will take the form $(m, m+1, m+3)$. That line under our mapping is, $4^m + 4^{m+1} + 4^{m+3} = 4^m\left(f(0)+f(1)+f(3)\right) \equiv 0 \bmod 23$. Thus every line sums to zero under *f*. Thus *f* embeds {(0,1,3) mod 11} into $\mathbb{Z}_{23}$.

(Remark: how we arrived at $f(x) = 4^x \bmod 23$ will be dealt with in more detail in chapter 8.2)

Next we look at the mapping in it's entirety and notice that this mapping maps {(0,1,3) mod 11} to the same subset of $\mathbb{Z}_{23}$ as the mapping in theorem 7.9.1. Thus, there is a one to one mapping from {(0,1,3) mod 11} to $(11,3)_{31}$ that preserves lines, and so the two configurations are isomorphic.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	4	16	18	3	12	2	8	9	13	6

Lines in {(0,1,3) mod 11}	Lines in $(11,3)_{31}$
$(0,1,3)$	$(2,4,6)$
$(1,2,4)$	$(1,4,5)$
$(2,3,5)$	$(1,6,7)$
$(3,4,6)$	$(5,6,10)$
$(4,5,7)$	$(5,7,0)$
$(5,6,8)$	$(7,9,10)$
$(6,7,9)$	$(8,10,0)$

Lines in {(0,1,3) mod 11}	Lines in $(11,3)_{31}$
$(7,8,10)$	$(3,9,0)$
$(8,9,0)$	$(2,8,9)$
$(9,10,1)$	$(3,4,8)$
$(10,0,2)$	$(1,2,3)$

Q.E.D.

Chapter 8: Other notable configurations

§8.0

In this last chapter, we present some numerical techniques to decide the group embeddability of some $(n, 3)$ configurations with $n > 11$.

§8.1 (12,3) #1

This configuration is defined by the lines {(1,2,5), (1,3,6), (1,4,7), (2,3,4), (2,9,0), (3,8,0), (4,10,11), (5,6,10), (5,8,11), (6,9,11), (7,8,9), (7,0,10)}.

Theorem 8.1.1: This (12, 3)-configuration is not group embeddable.

Proof:

$$\begin{aligned}1 &= 2*5\\ &= 2*(8*11)\\ &= 2*(8*(10*4))\\ &= 10*(8*(2*4)) \quad \text{we are using the identity } x*(y*(z*u)) = z*(y*(x*u)).\\ &= 10*(8*3)\\ &= 10*0\\ &= 7.\end{aligned}$$

Thus, in any group embedding, 1 and 7 will have the same image and hence not one-to-one.

§8.2 Cyclic configurations.

Now we demonstrate the technique used in chapter 7.10 in greater detail and how this technique can be used on a number cyclic configurations. Using the Singer polynomial and primitive roots we can represent many cyclic configurations. We mention some three examples [12]

Theorem 8.2.1: The cyclic (12,3) configuration given by {(0,1,3) mod 12} can be embedded in the cyclic group $\mathbb{Z}_{13}$.

Proof. Map the point *i* to $7^i \bmod 13$ to get a group representation. It is now easy to see that this does the job. But why number 7 and why does this work?

Let us associate the cubic polynomial $1+x+x^3$ with the cyclic difference set generated by the base (0, 1, 3). This is called the Singer polynomial. The number 7 happens to be a primitive root of 13 and it also satisfies the Singer equation $1+x+x^3 = 0 \bmod 13$. By the very cyclic nature of the configuration, we see that three points *P, Q* and *R* are collinear in {(0,1,3) mod 12} if and only if $f(P)+f(Q)+f(R)=0$ in the cyclic group $\mathbb{Z}_{13}$.

P	0	1	2	3	4	5	6	7	8	9	10	11
f(*P*)	1	7	10	5	9	11	12	6	3	8	4	2

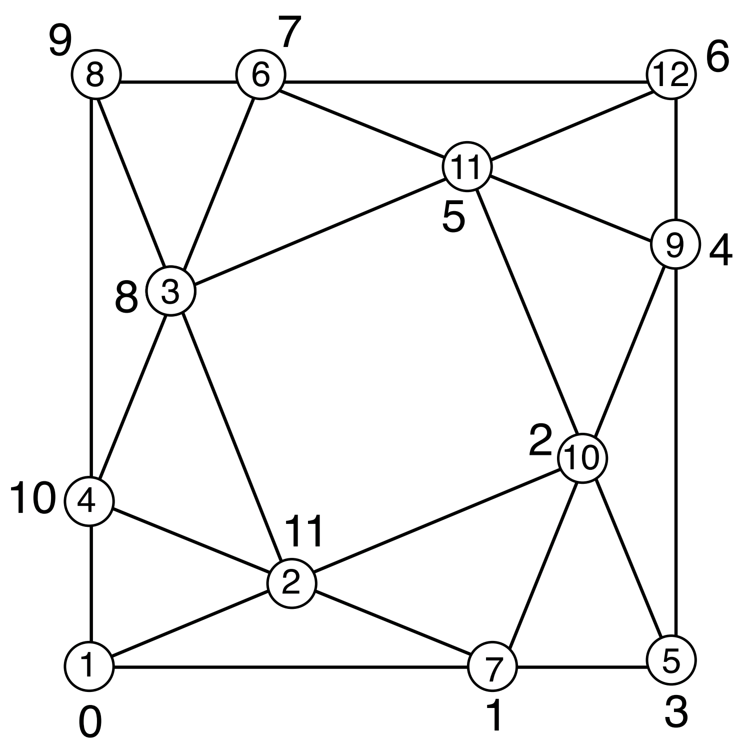


Figure 8.2.1: group labels are shown inside the circles corresponding to the points

This technique was used extensively by the late Professor N.S. Mendelsohn and a large number of such representable configurations were reported in [10].

§8.3 Two of Daublebsky's configurations.

The collinearities defining the (12, 3) configuration D#226 are {(1,2,5), (1,3,6), (1,4,7), (2,3,8), (2,7,11), (3,10,12), (4,5,12), (4,9,11), (5,10,11), (6,7,8), (6,9,12), (8,9,10)} [3], seen in figure 8.3.1.

In order to find a group, we will use the parameter method. Let f be a mapping from D#226 to some group G. We let $f(1) = a, f(2) = b$, and $f(3) = c$. We then use each of the configuration's lines to develop restrictions on a, b and c.

point	parameter	collinearity	value in Z[15]
1	a		1
2	b		2
3	c		3
4	$-2a - b - 2c$	1, 4, 7	-2-2-6 = -10 = 5
5	$-a - b$	1, 2, 5	-1-2 = -3 = 12
6	$-a-c$	1, 3, 6	-1-3 = -4 = 11
7	$a + b + 2c$	6, 7, 8	1+2+6 = 9
8	$-b -c$	2, 3, 8	-2-3 = -5 = 10
9	$3a + 3b + 4c$	4, 9, 11	3+6+12 = 6
10	$-3a - 2b - 3c$	3, 10, 12	-3-4-9 = 14
11	$-a -2b - 2c$	2, 7, 11	-1-4-6 = -11 = 4
12	$3a + 2b + 2c$	4, 5, 12	3+4+6 = 13

Thus, using the given collinearities, we find the connection among the three parameters *a, b* and c. It is easy to see that the above scheme will work if $5(a+b+c)=0$ in the target group. Now assign $a = 1$, $b = 2$, $c = 3$ and calculate the values of other points (mod 15). This should give us a desired group embedding for the Daublebsky's configuration (12, 3)-D#226 into the group $\mathbb{Z}_{15}$.

Figure 8.3.1

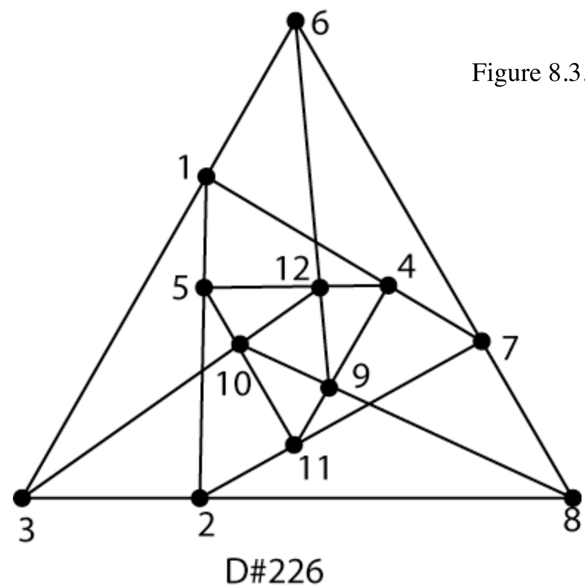
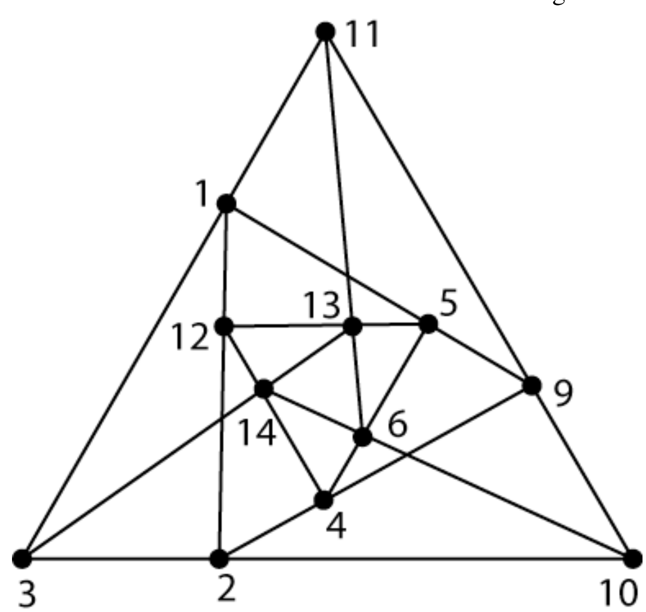


Figure 8.3.2



D#226 embedded in the cyclic group $\mathbb{Z}_{15}$.

If three points are collinear then their sum = 0 in the group $\mathbb{Z}_{15}$.

The next of Daublebsky's configurations is labeled D#222 and is defined by the lines {(1,2,5), (1,3,6), (1,4,5), (2,3,8), (2,7,12), (3,10,11), (4,5,10), (4,11,12), (5,9,12), (6,7,8), (6,9,11), (8,9,10)} [3]. However, this configuration is isomorphic to {(0,3,7) mod 13}. Figure 7.3.3 shows the relabelling of D#222 as {(0,1,3) mod 12}.

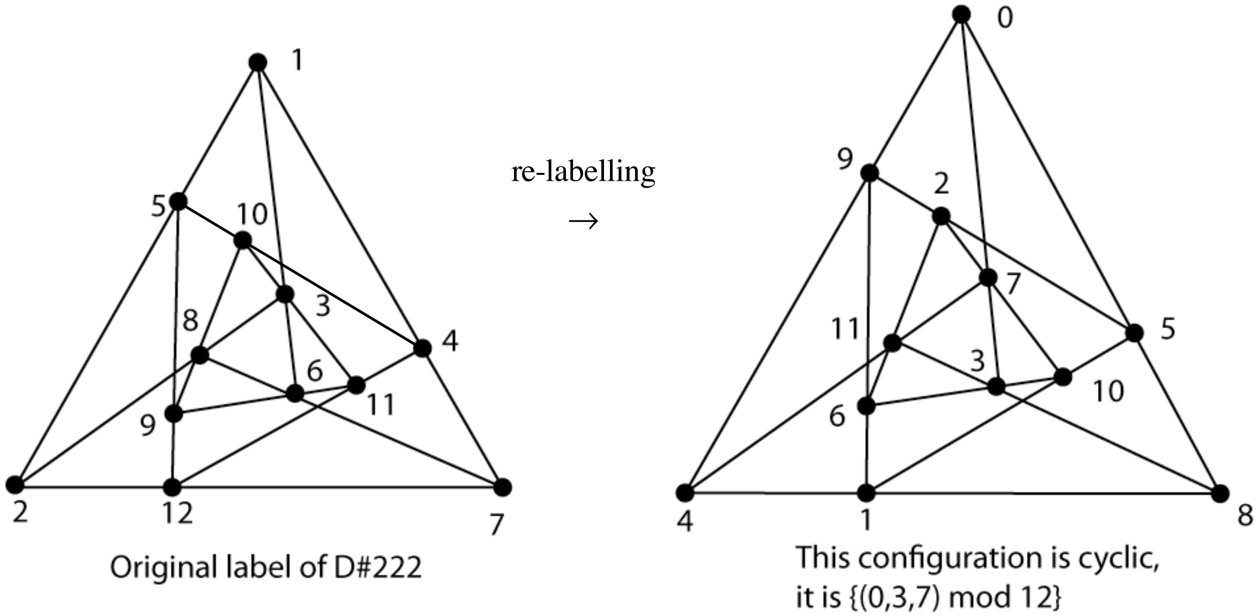


Figure 8.3.3

Theorem 8.3.1: The two configurations D#222 and D#226 have non-isomorphic group embeddings.

Proof: The configuration D#222 satisfies 12 tangential relations including

$$1*1=10,\ 10*10=6,\ 6*6=2,\ 2*2=4, \text{ etc.}$$

In fact, renaming the points as 0, 1 = 0*0, 2 = 1*1 etc. we realize that the configuration D#222 is isomorphic to the cyclic configuration given by {(0, 3, 7) mod 12}. As shown above.

Once we obtain the tangential relations and consequently recognize the cyclic nature of the configuration, it is rather easy to find a group embedding. Recall that $f(a*a)=-2f(a)$ in the target group, if such a group exists. Chasing the tangent cycle, we see that the target group must be of order m dividing the number 4095 $(= 4096 - 1 = 2^{12} - 1)$. Now $4095 = 91 \times 45$. We try the abelian group $\mathbb{Z}_{45}$. The mapping is given by $f(a) = (-2)^a = 43^a \bmod 45$.

a	0	1	2	3	4	5	6	7	8	9	10	11
$f(a)$	1	43	4	37	16	13	19	7	31	28	34	22

Check, for example, the basic collinearity: (0,3,7): $f(0) + f(3) + f(7) = 1 + 37 + 7 = 45 = 0$ in $\mathbb{Z}_{45}$. Since this group is cyclic, all other lines are automatically satisfied. Since the first configuration D#226 does not satisfy such cyclic tangential relations (we have compete Prover9 proofs of all tangential relations for D#226), these two do not share the same group properties.

Q.E.D.

§8.4 {(0,1,3) mod 14}

In this section, we construct a group embedding for the cyclic design {(0,1,3) mod 14}.

Mapping of {(0,1,3), mod 14} into the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$

0, 1, 3
1, 2, 4
2, 3, 5
3, 4, 6
4, 5, 7
5, 6, 8
6, 7, 9
7, 8, 10
8, 9, 11
9, 10, 12
10,11,13
11,12,0
12,13,1
13, 0, 2

Point	**Map f**
0	(1, 0, 0)
1	(0, 1, 0)
2	(0, 0, 1)
3	(3, 3, 0)
4	(0, 3, 3)
5	(1, 1, 3)
6	(1, 2, 1)
7	(3, 0, 2)
8	(2, 1, 0)
9	(0, 2, 1)
10	(3, 3, 2)
11	(2, 1, 3)
12	(1, 3, 1)
13	(3, 0, 3)

Let us verify all the 14 collinearities:

$$f(0)+f(1)+f(3)=(1,0,0)+(0,1,0)+(3,3,0)=(0,0,0).$$
$$f(1)+f(2)+f(4)=(0,1,0)+(0,0,1)+(0,3,3)=(0,0,0).$$
$$f(2)+f(3)+f(5)=(0,0,1)+(3,3,0)+(1,1,3)=(0,0,0).$$
$$f(3)+f(4)+f(6)=(3,3,0)+(0,3,3)+(1,2,1)=(0,0,0).$$
$$f(4)+f(5)+f(7)=(0,3,3,)+(1,1,3)+(3,0,2)=(0,0,0).$$
$$f(5)+f(6)+f(8)=(1,1,3)+(1,2,1)+(2,1,0)=(0,0,0).$$
$$f(6)+f(7)+f(9)=(1,2,1)+(3,0,2)+(0,2,1)=(0,0,0).$$
$$f(7)+f(8)+f(10)=(3,0,2)+(2,1,0)+(3,3,2)=(0,0,0).$$
$$f(8)+f(9)+f(11)=(2,1,0)+(0,2,1)+(2,1,3)=(0,0,0).$$
$$f(9)+f(10)+f(12)=(0,2,1)+(3,3,2)+(1,3,1)=(0,0,0).$$
$$f(10)+f(11)+f(13)=(3,3,2)+(2,1,3)+(3,0,3)=(0,0,0).$$
$$f(11)+f(12)+f(0)=(2,1,3)+(1,3,1)+(1,0,0,)=(0,0,0).$$
$$f(12)+f(13)+f(1)=(1,3,1)+(3,0,3)+(0,1,0)=(0,0,0).$$
$$f(12)+f(13)+f(2)=(3,0,3)+(1,0,0)+(0,0,1)=(0,0,0).$$

This completes the proof that the cyclic configuration {(0,1,3) mod 14} is embeddable in the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. In fact, the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ contains four copies of {(0,1,3) mod 14} and one copy of the 7-element Fano plane {(0,1,3) mod 7} as shown below ($4 \times 14 + 7 + 1 = 56 + 7 + 1 = 64$ is OK). We exhibit explicitly the mappings.

Mappings of (0, 1, 3) mod 14

Point	**Map 1**	**Map 2**	**Map 3**	**Map 4**
0	(1, 0, 0)	(3, 0, 0)	(1, 1, 1)	(3, 3, 3)
1	(0, 1, 0)	(0, 3, 0)	(3, 0, 1)	(1, 0, 3)
2	(0, 0, 1)	(0, 0, 3)	(3, 2, 0)	(1, 2, 0)
3	(3, 3, 0)	(1, 1, 0)	(0, 3, 2)	(0, 1, 2)
4	(0, 3, 3)	(0, 1, 1)	(2, 2, 3)	(2, 2, 1)
5	(1, 1, 3)	(3, 3, 1)	(1, 3, 2)	(3, 1, 2)

6	(1, 2, 1)	(3, 2, 3)	(2, 3, 3)	(2, 1, 1)
7	(3, 0, 2)	(1, 0, 2)	(1, 3, 3)	(3, 1, 1)
8	(2, 1, 0)	(2, 3, 0)	(1, 2, 3)	(3, 2, 1)
9	(0, 2, 1)	(0, 2, 3)	(1, 2, 2)	(3, 2, 2)
10	(3, 3, 2)	(1, 1, 2)	(2, 3, 2)	(2, 1, 2)
11	(2, 1, 3)	(2, 3, 1)	(2, 0, 3)	(2, 0, 1)
12	(1, 3, 1)	(3, 1, 3)	(1, 3, 0)	(3, 1, 0)
13	(3, 0, 3)	(1, 0, 1)	(0, 1, 3)	(0, 3, 1)

Point	**Map**
0	(2, 0, 0)
1	(0, 2, 0)
2	(0, 0, 2)
3	(2, 2, 0)
4	(0, 2, 2)
5	(2, 2, 2)
6	(2, 0, 2)

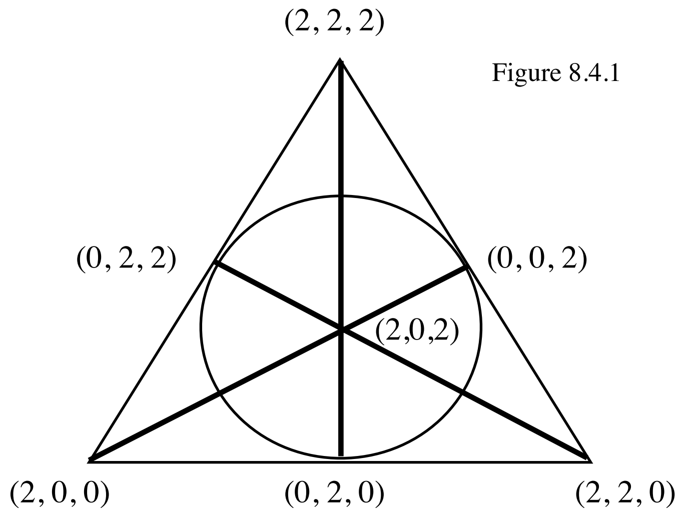


Figure 8.4.1

Corollary. This (14, 3) configuration cannot be drawn in the real or complex plane.
Proof. The Fano plane exist in a OG(2,k) only if there is an element a $\neq 0$ in k such that $a + a = 0$. As seen in Figure 8.4.1.

This clearly demonstrates that the Mendelsohn triple systems motivated by the original tangential relations and the group law on non-singular cubic curves offer new techniques and new insights into the problems on the classical topic of planar configurations. [8]

§8.5 An (n, 4) configuration.
J.A. Singer has shown that the projective plane PG(2,3) over the field $\mathbb{Z}_3$ can be defined as the cyclic difference set {(0, 1, 3, 9); mod 13} [12]. Recall from Chapter 3 that the unique (8,3) gets

embedded in the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ and, in fact, manifests itself as a subset of the affine plane $AG(2,3)$. Since this projective plane of order 3 contains the affine AG(2,3), hindsight suggests that all the elements in PG(2,3) will be of order 3 in the target if a group realization exists. First we prove this is indeed the case, since all the elements are of order 3 in any potential group embedding (see page 98 for an automated proof obtained by Prover9, P_{45}). Thus the target group, if it exists, must contain at least 13 points of order 3 and hence a minimal such group is $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. In particular, the minimum rank for such a group must be at least 3.

Once this is realized, it is not difficult to find an actual mapping of the projective plane into $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. In fact, the group will contain two copies of the projective plane and the zero element. (13+13+1 = 27 = 3^3 is OK).

Theorem 8.4.1: The desarguesian projective plane PG(2,3) over the three-element field can embedded in the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ such that if (*P, Q, R, S*) is a straight line in the plane PG(2,3), then $f(P)+f(Q)+f(R)+f(S)=0$ in the group.

Proof: Here is a mapping of PG(2,3) into the group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Points in PG(2,3)	Mapping to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	Lines
0	$(1,0,0)$	$f(0)+f(1)+f(3)+f(9)=(1,0,0)+(0,1,0)+(1,2,2)+(1,0,1)=(0,0,0)$
1	$(0,1,0)$	$f(1)+f(2)+f(4)+f(10)=(0,1,0)+(0,0,1)+(2,2,0)+(1,0,2)=(0,0,0)$
2	$(0,0,1)$	$f(2)+f(3)+f(5)+f(11)=(0,0,1)+(1,2,2)+(0,2,2)+(2,2,1)=(0,0,0)$
3	$(1,2,2)$	$f(3)+f(4)+f(6)+f(12)=(1,2,2)+(2,2,0)+(2,1,0)+(1,1,1)=(0,0,0)$
4	$(2,2,0)$	$f(4)+f(5)+f(7)+f(0)=(2,2,0)+(0,2,2)+(0,2,1)+(0,2,1)=(0,0,0)$
5	$(0,2,2)$	$f(5)+f(6)+f(8)+f(1)=(0,2,2)+(2,1,0)+(1,2,1)+(0,1,0)=(0,0,0)$
6	$(2,1,0)$	$f(6)+f(7)+f(9)+f(2)=(2,1,0)+(0,2,1)+(1,0,1)+(0,0,1)=(0,0,0)$

Points in PG(2,3)	Mapping to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	Lines
7	$(0,2,1)$	$f(7)+f(8)+f(10)+f(3)=(0,2,1)+(1,2,1)+(1,0,2)+(1,2,2)=(0,0,0)$
8	$(1,2,1)$	$f(8)+f(9)+f(11)+f(4)=(1,2,1)+(1,0,1)+(2,2,1)+(2,2,0)=(0,0,0)$
9	$(1,0,1)$	$f(9)+f(10)+f(12)+f(5)=(1,0,1)+(1,0,2)+(1,1,1)+(0,2,2)=(0,0,0)$
10	$(1,0,2)$	$f(10)+f(11)+f(0)+f(6)=(1,0,2)+(2,2,1)+(1,0,0)+(2,1,0)=(0,0,0)$
11	$(2,2,1)$	$f(11)+f(12)+f(1)+f(7)=(2,2,1)+(1,1,1,)+(0,1,0)+(0,2,1)=(0,0,0)$
12	$(1,1,1)$	$f(12)+f(0)+f(2)+f(8)=(1,1,1)+(1,0,0)+(0,0,2)+(1,2,1)=(0,0,0)$

Q.E.D.

Conclusion

We conclude with a few open problems in the area of embedding configuration into algebraic groups. The first and most obvious next step is to find a group embedding for all (12,3)'s. Though, there is an enormous number of these. This concept can also be extended to (n,4) configurations as we did at the end of chapter 7.

In chapter 5.7 we introduced a method of finding a group embedding for cyclic configurations. We make a conjecture that all such cyclic configurations have a group embedding and such an embedding can be found with this method. Here the basic technique was originally developed in 1987 by Dr. N. S. Mendelsohn, R. Padmanabhan and Barry Wolk, using Singer polynomials and primitive roots. [10]

As we mentioned in the beginning, configurations arose historically in geometry. The Pappus configuration, the Desargues configuration and the Fano configuration all arose in the context of classical geometry of projective planes. Hence the converse problem of geometric realizability of abstract configurations is a natural problem in this area. In fact, the famous of Grunbaum is closely related to this problem: every (n,3) configuration representable over the real plane is, indeed, representable over the rational plane. In [13], Strumfels and White prove that all (11,3)'s and (12,3)'s are rational. It is still an open problem for n = 14 onwards. In this thesis we get some of these results as a mere byproduct of our group embedding (see [7]). For example, all the (11,3) configurations which have group embeddings, do have geometric realizations over the complex field because they are subgroups of the direct product of two copies of the circle group. In fact, all the configurations for which have group realizations happen to exist in projective planes over some division rings. Hence the following conjecture is plausible.

If a configuration C has a group realization, then it is geometrically consistent in the sense that there exists a division ring k such that C can be drawn in PG(2,k), the projective plane over k where the blocks in C are straight lines in the projective plane.

We have demonstrated the non-group realizability of several (n, 3)'s. Here we assumed that the target group is abelian. More generally, the target algebra could be a non-abelian group or even a non-associative Moufang loop. No such realizations over other algebraic systems are known so far.

Prover9 proofs.

Proofs from Chapter 6.

Tangential relations in §6.2.

P_1

================ PROOF
4 3 * 3 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
10 1 * 6 = 7. [assumption].
12 2 * 4 = 8. [assumption].
13 3 * 7 = 8. [assumption].
16 0 * 4 = 6. [assumption].
21 3 * 3 != 0. [deny(4)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
46 7 * 8 = 3. [para(13(a,1),6(a,1,2))].
55 4 * 6 = 0. [para(16(a,1),6(a,1,2))].
66 1 * 7 = 6. [para(10(a,1),22(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
691 **3 * 3 = 0**. [para(46(a,1),32(a,2,2)),rewrite([66(3),68(4),5(3),55(3)]),flip(a)].
692 $F. [resolve(691,a,21,a)].
============= end of proof ==========================

P_2

============= PROOF
2 1 * 1 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
15 3 * 5 = 9. [assumption].
19 1 * 1 != 0. [deny(2)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
71 3 * 9 = 5. [para(15(a,1),22(a,1,2))].
678 4 * (0 * 1) = 5. [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),71(8)])].
912 0 * 1 = 1. [para(678(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
922 **1 * 1 = 0.** [para(912(a,1),6(a,1,2))].
923 $F. [resolve(922,a,19,a)].
============== end of proof
==========================

P_3

============= PROOF ============================
3 2 * 2 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
12 2 * 4 = 8. [assumption].
13 3 * 7 = 8. [assumption].
17 0 * 5 = 7. [assumption].
20 2 * 2 != 0. [deny(3)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
43 4 * 8 = 2. [para(12(a,1),6(a,1,2))].
65 1 * 5 = 4. [para(9(a,1),22(a,1,2))].
273 (x * 1) * (y * 5) = 4 * (x * y). [para(65(a,1),7(a,1,2)),rewrite([5(3)]),flip(a)].
682 4 * (0 * 2) = 8. [para(17(a,1),32(a,2,2)),rewrite([5(3),273(7),13(8)])].
1242 0 * 2 = 2. [para(682(a,1),22(a,1,2)),rewrite([43(3)]),flip(a)].
1251 **2 * 2 = 0.** [para(1242(a,1),6(a,1,2))].
1252 $F. [resolve(1251,a,20,a)].
============= end of proof ==========================

P_4

================ PROOF
1 0 * 0 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
10 1 * 6 = 7. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
13 3 * 7 = 8. [assumption].
15 3 * 5 = 9. [assumption].
16 0 * 4 = 6. [assumption].
17 0 * 5 = 7. [assumption].
18 0 * 0 != 0. [deny(1)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
33 (x * 1) * (y * 2) = 3 * (x * y). [para(8(a,1),7(a,1,2)),rewrite([5(3)]),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
43 4 * 8 = 2. [para(12(a,1),6(a,1,2))].
46 7 * 8 = 3. [para(13(a,1),6(a,1,2))].
55 4 * 6 = 0. [para(16(a,1),6(a,1,2))].
65 1 * 5 = 4. [para(9(a,1),22(a,1,2))].
66 1 * 7 = 6. [para(10(a,1),22(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
71 3 * 9 = 5. [para(15(a,1),22(a,1,2))].
273 (x * 1) * (y * 5) = 4 * (x * y). [para(65(a,1),7(a,1,2)),rewrite([5(3)]),flip(a)].
678 4 * (0 * 1) = 5. [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),71(8)])].
682 4 * (0 * 2) = 8. [para(17(a,1),32(a,2,2)),rewrite([5(3),273(7),13(8)])].
691 3 * 3 = 0. [para(46(a,1),32(a,2,2)),rewrite([66(3),68(4),5(3),55(3)]),flip(a)].
912 0 * 1 = 1. [para(678(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
922 1 * 1 = 0. [para(912(a,1),6(a,1,2))].
1242 0 * 2 = 2. [para(682(a,1),22(a,1,2)),rewrite([43(3)]),flip(a)].
1251 2 * 2 = 0. [para(1242(a,1),6(a,1,2))].
1718 **0 * 0 = 0.** [para(8(a,1),33(a,2,2)),rewrite([922(3),1251(4),691(6)])].
1719 $F. [resolve(1718,a,18,a)].
============== end of proof
==========================

Tangential relations in §6.3

P_5

============================== PROOF
-------- Comments from original proof --------
% Proof 1 at 0.29 (+ 0.04) seconds.
% Length of proof is 19.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 49.

2 1 * 1 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].

8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
15 3 * 5 = 9. [assumption].
19 1 * 1 != 0. [deny(2)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
71 3 * 9 = 5. [para(15(a,1),22(a,1,2))].
677 4 * (0 * 1) = 5. [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),71(8)])].
734 0 * 1 = 1. [para(677(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
900 1 * 1 = 0. [para(734(a,1),6(a,1,2))].
901 $F. [resolve(900,a,19,a)].

============================== end of proof ==========================

P_6

============================== PROOF

% -------- Comments from original proof --------
% Proof 2 at 0.30 (+ 0.05) seconds.
% Length of proof is 24.
% Level of proof is 7.
% Maximum clause weight is 15.
% Given clauses 50.

4 9 * 9 = 1 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
14 2 * 7 = 9. [assumption].
15 3 * 5 = 9. [assumption].
17 0 * 5 = 7. [assumption].
21 9 * 9 != 1. [deny(4)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
71 3 * 9 = 5. [para(15(a,1),22(a,1,2))].
73 0 * 7 = 5. [para(17(a,1),22(a,1,2))].
677 4 * (0 * 1) = 5. [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),71(8)])].
713 9 * (0 * 1) = 9. [para(73(a,1),32(a,2,2)),rewrite([5(3),14(6),5(5),15(8)])].
734 0 * 1 = 1. [para(677(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
741 1 * 9 = 9. [back_rewrite(713),rewrite([734(4),5(3)])].
917 9 * 9 = 1. [para(741(a,1),6(a,1,2))].
918 $F. [resolve(917,a,21,a)].

============================== end of proof ==========================

P_7

============================== PROOF

% -------- Comments from original proof --------
% Proof 3 at 0.30 (+ 0.05) seconds.
% Length of proof is 23.
% Level of proof is 7.
% Maximum clause weight is 15.
% Given clauses 51.

3 8 * 8 = 1 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
13 3 * 6 = 8. [assumption].
15 3 * 5 = 9. [assumption].
16 0 * 4 = 6. [assumption].
20 8 * 8 != 1. [deny(3)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].
71 3 * 9 = 5. [para(15(a,1),22(a,1,2))].
677 4 * (0 * 1) = 5. [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),71(8)])].
680 8 * (0 * 1) = 8. [para(16(a,1),32(a,2,2)),rewrite([5(3),12(6),5(5),13(8)])].
734 0 * 1 = 1. [para(677(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
742 1 * 8 = 8. [back_rewrite(680),rewrite([734(4),5(3)])].
936 8 * 8 = 1. [para(742(a,1),6(a,1,2))].
937 $F. [resolve(936,a,20,a)].

============================== end of proof ==========================

P_8

============================== PROOF

% -------- Comments from original proof --------
% Proof 4 at 0.84 (+ 0.08) seconds.
% Length of proof is 26.
% Level of proof is 7.
% Maximum clause weight is 15.
% Given clauses 68.

1 0 * 0 = 0 # label(non_clause) # label(goal). [goal].
5 x * y = y * x. [assumption].
6 x * (y * x) = y. [assumption].
7 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
8 1 * 2 = 3. [assumption].
9 1 * 4 = 5. [assumption].
10 1 * 6 = 7. [assumption].
11 0 * 8 = 9. [assumption].
12 2 * 4 = 8. [assumption].
15 3 * 5 = 9. [assumption].
16 0 * 4 = 6. [assumption].
17 0 * 5 = 7. [assumption].
18 0 * 0 != 0. [deny(1)].
22 x * (x * y) = y. [para(5(a,1),6(a,1,2))].
32 (1 * x) * (2 * y) = 3 * (x * y). [para(8(a,1),7(a,1,1)),flip(a)].
34 4 * 5 = 1. [para(9(a,1),6(a,1,2))].
36 (x * 1) * (y * 4) = 5 * (x * y). [para(9(a,1),7(a,1,2)),rewrite([5(3)]),flip(a)].
55 4 * 6 = 0. [para(16(a,1),6(a,1,2))].
58 5 * 7 = 0. [para(17(a,1),6(a,1,2))].
68 2 * 8 = 4. [para(12(a,1),22(a,1,2))].

```
71 3 * 9 = 5.  [para(15(a,1),22(a,1,2))].
677 4 * (0 * 1) = 5.  [para(11(a,1),32(a,2,2)),rewrite([5(3),68(6),5(5),
71(8)])].
734 0 * 1 = 1.  [para(677(a,1),22(a,1,2)),rewrite([34(3)]),flip(a)].
900 1 * 1 = 0.  [para(734(a,1),6(a,1,2))].
1910 0 * 0 = 0.  [para(10(a,1),36(a,2,2)),rewrite([900(3),5(4),55(4),
58(6)])].
1911 $F.  [resolve(1910,a,18,a)].

============================== end of proof
==========================
```

Tangential relations in §6.4

P_9

```
============================== PROOF
1 1 * 1 = 4 * 4 # label(non_clause) # label(goal).  [goal].
3 x * (y * x) = y.  [assumption].   4 x * y = y * x.  [assumption].
5 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].

Collinear relations defining the configuration (10, 3)-4.
6           0 * 4 = 6.  [assumption].                  7
            0 * 5 = 7.  [assumption].
8           0 * 8 = 9.  [assumption].                  9
            1 * 2 = 3.  [assumption].
10          1 * 4 = 5.  [assumption].                  11
            1 * 6 = 7.  [assumption].
12          2 * 4 = 8.  [assumption].                  13
            2 * 5 = 9.  [assumption].
14          3 * 6 = 8.  [assumption].                  15
            3 * 7 = 9.  [assumption].
16          1 * 1 != 4 * 4.  [deny(1)].

17 4 * 4 != 1 * 1.  [copy(16),flip(a)].
19 x * (x * y) = y.  [para(3(a,1),3(a,1,2)),rewrite([4(2)])].
29 (0 * x) * (4 * y) = 6 * (x * y).  [para(6(a,1),5(a,1,1)),flip(a)].
30 (x * 0) * (y * 4) = 6 * (x * y).  [para(6(a,1),5(a,
1,2)),rewrite([4(3)]),flip(a)].
33 (x * 0) * (y * 5) = 7 * (x * y).  [para(7(a,1),5(a,
1,2)),rewrite([4(3)]),flip(a)].
40 4 * 5 = 1.  [para(10(a,1),3(a,1,2))].
43 6 * 7 = 1.  [para(11(a,1),3(a,1,2))].
46 4 * 8 = 2.  [para(12(a,1),3(a,1,2))].
55 7 * 9 = 3.  [para(15(a,1),3(a,1,2))].
61 0 * 6 = 4.  [para(6(a,1),19(a,1,2))].
62 0 * 7 = 5.  [para(7(a,1),19(a,1,2))].
65 1 * 5 = 4.  [para(10(a,1),19(a,1,2))].
66 1 * 7 = 6.  [para(11(a,1),19(a,1,2))].
68 2 * 9 = 5.  [para(13(a,1),19(a,1,2))].
69 3 * 8 = 6.  [para(14(a,1),19(a,1,2))].

734 1 * (0 * 0) = 1.  [para(7(a,1),29(a,2,2)),rewrite([40(6),4(5),
43(8)])].
737 8 * (0 * 1) = 8.  [para(9(a,1),29(a,2,2)),rewrite([4(6),12(6),4(5),
4(8),14(8)])].
754 5 * (4 * 9) = 8.  [para(55(a,1),29(a,2,2)),rewrite([62(3),4(8),
14(8)])].
769 6 * 6 = 2 * (0 * 3).  [para(69(a,1),29(a,2,2)),rewrite([46(6),
4(5)]),flip(a)].
795 1 * 1 = 0 * 0.  [para(734(a,1),19(a,1,2))].
805 4 * 4 != 0 * 0.  [back_rewrite(17),rewrite([795(6)])].
922 8 * 8 = 0 * 1.  [para(737(a,1),19(a,1,2))].
960 5 * 8 = 4 * 9.  [para(754(a,1),19(a,1,2))].
1808 6 * (x * 1) = 5 * (x * 0).  [para(10(a,1),30(a,
1,2)),rewrite([4(4)]),flip(a)].
1836 6 * (0 * 0) = 5 * (0 * 1).  [para(795(a,1),30(a,
2,2)),rewrite([4(3),10(6),4(5)]),flip(a)].
1845 5 * (0 * 0) = 5.  [para(922(a,1),30(a,2,2)),rewrite([4(3),8(3),
4(4),46(4),4(3),68(3),1808(6)]),flip(a)].
1873 5 * 5 = 0 * 0.  [para(1845(a,1),19(a,1,2))].
1952 5 * (0 * 1) = 6.  [para(1873(a,1),30(a,2,2)),rewrite([4(3),7(3),
4(4),40(4),4(3),66(3),1836(6)]),flip(a)].
1953 6 * (0 * 0) = 6.  [back_rewrite(1836),rewrite([1952(10)])].
1978 7 * (x * 1) = 4 * (x * 0).  [para(65(a,1),33(a,
1,2)),rewrite([4(4)]),flip(a)].
2015 4 * (0 * 0) = 4.  [para(922(a,1),33(a,2,2)),rewrite([4(3),8(3),
4(4),960(4),3(5),1978(6)]),flip(a)].
2070 2 * (0 * 3) = 0 * 0.  [para(1953(a,1),19(a,
1,2)),rewrite([769(3)])].
2087 4 * 4 = 0 * 0.  [para(1953(a,1),30(a,2,2)),rewrite([4(3),61(3),
4(6),2015(6),769(6),2070(8)])].
2088 $F.  [resolve(2087,a,805,a)].
============================== end of proof
==========================
```

P_{10}

```
============================== PROOF
1 2 * 2 = 8 * 8 # label(non_clause) # label(goal).  [goal].
2 x * (y * x) = y.  [assumption].
3 x * y = y * x.  [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].
5           0 * 4 = 6.  [assumption].
7           0 * 8 = 9.  [assumption].
8           1 * 2 = 3.  [assumption].
9           1 * 4 = 5.  [assumption].
11          2 * 4 = 8.  [assumption].
12          2 * 5 = 9.  [assumption].
13          3 * 6 = 8.  [assumption].
15          8 * 8 != 2 * 2.  [deny(1)].
16 x * (x * y) = y.  [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
26 (0 * x) * (4 * y) = 6 * (x * y).  [para(5(a,1),4(a,1,1)),flip(a)].
33 (x * 0) * (y * 8) = 9 * (x * y).  [para(7(a,1),4(a,
1,2)),rewrite([3(3)]),flip(a)].
43 4 * 8 = 2.  [para(11(a,1),2(a,1,2))].
46 5 * 9 = 2.  [para(12(a,1),2(a,1,2))].
734 8 * (0 * 1) = 8.  [para(8(a,1),26(a,2,2)),rewrite([3(6),11(6),3(5),
3(8),13(8)])].
918 8 * 8 = 0 * 1.  [para(734(a,1),16(a,1,2))].
932 2 * 2 != 0 * 1.  [back_rewrite(15),rewrite([918(3)]),flip(a)].
2155 2 * (0 * 1) = 2.  [para(9(a,1),33(a,2,2)),rewrite([3(3),43(6),3(5),
3(8),46(8)])].
4289 2 * 2 = 0 * 1.  [para(2155(a,1),16(a,1,2))].
4290 $F.  [resolve(4289,a,932,a)].
============================== end of proof
==========================
```

P_{11}

```
============================== PROOF
1 3 * 3 = 9 * 9 # label(non_clause) # label(goal).  [goal].
2 x * (y * x) = y.  [assumption].
3 x * y = y * x.  [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].
5                       0 * 4 = 6.  [assumption].
6           0 * 5 = 7.  [assumption].
7           0 * 8 = 9.  [assumption].
8           1 * 2 = 3.  [assumption].
9           1 * 4 = 5.  [assumption].
10          1 * 6 = 7.  [assumption].
11          2 * 4 = 8.  [assumption].
12          2 * 5 = 9.  [assumption].
13          3 * 6 = 8.  [assumption].
14          3 * 7 = 9.  [assumption].
15          3 * 3 != 9 * 9.  [deny(1)].
16          9 * 9 != 3 * 3.  [copy(15),flip(a)].
17 x * (x * y) = y.  [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (0 * x) * (4 * y) = 6 * (x * y).  [para(5(a,1),4(a,1,1)),flip(a)].
28 (x * 0) * (y * 4) = 6 * (x * y).  [para(5(a,1),4(a,
1,2)),rewrite([3(3)]),flip(a)].
31 (x * 0) * (y * 5) = 7 * (x * y).  [para(6(a,1),4(a,
1,2)),rewrite([3(3)]),flip(a)].
36 (1 * x) * (2 * y) = 3 * (x * y).  [para(8(a,1),4(a,1,1)),flip(a)].
38 4 * 5 = 1.  [para(9(a,1),2(a,1,2))].
```

41 6 * 7 = 1. [para(10(a,1),2(a,1,2))].
44 4 * 8 = 2. [para(11(a,1),2(a,1,2))].
50 6 * 8 = 3. [para(13(a,1),2(a,1,2))].
53 7 * 9 = 3. [para(14(a,1),2(a,1,2))].
60 0 * 7 = 5. [para(6(a,1),17(a,1,2))].
64 1 * 7 = 6. [para(10(a,1),17(a,1,2))].
65 2 * 8 = 4. [para(11(a,1),17(a,1,2))].
66 2 * 9 = 5. [para(12(a,1),17(a,1,2))].
474 (x * 2) * (y * 9) = 5 * (x * y). [para(66(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
732 1 * (0 * 0) = 1. [para(6(a,1),27(a,2,2)),rewrite([38(6),3(5),41(8)])].
735 8 * (0 * 1) = 8. [para(8(a,1),27(a,2,2)),rewrite([3(6),11(6),3(5),3(8),13(8)])].
752 5 * (4 * 9) = 8. [para(53(a,1),27(a,2,2)),rewrite([60(3),3(8),13(8)])].
793 1 * 1 = 0 * 0. [para(732(a,1),17(a,1,2))].
919 8 * 8 = 0 * 1. [para(735(a,1),17(a,1,2))].
957 5 * 8 = 4 * 9. [para(752(a,1),17(a,1,2))].
1805 6 * (x * 1) = 5 * (x * 0). [para(9(a,1),28(a,1,2)),rewrite([3(4)]),flip(a)].
1833 6 * (0 * 0) = 5 * (0 * 1). [para(793(a,1),28(a,2,2)),rewrite([3(3),9(6),3(5)]),flip(a)].
1842 5 * (0 * 0) = 5. [para(919(a,1),28(a,2,2)),rewrite([3(3),7(3),3(4),44(4),3(3),66(3),1805(6)]),flip(a)].
1870 5 * 5 = 0 * 0. [para(1842(a,1),17(a,1,2))].
1949 5 * (0 * 1) = 6. [para(1870(a,1),28(a,2,2)),rewrite([3(3),6(3),3(4),38(4),3(3),64(3),1833(6)]),flip(a)].
1953 9 * (0 * 1) = 9. [para(8(a,1),31(a,2,2)),rewrite([3(3),12(6),3(5),3(8),14(8)])].
1983 5 * 6 = 4 * 7. [para(65(a,1),31(a,2,2)),rewrite([3(3),3(6),957(6),474(7),5(4),3(6)])].
2029 4 * 7 = 0 * 1. [para(1949(a,1),17(a,1,2)),rewrite([1983(3)])].
2111 9 * 9 = 0 * 1. [para(1953(a,1),17(a,1,2))].
2135 3 * 3 != 0 * 1. [back_rewrite(16),rewrite([2111(3)]),flip(a)].
2448 3 * 3 = 0 * 1. [para(50(a,1),36(a,2,2)),rewrite([10(3),65(4),3(3),2029(3)]),flip(a)].
2449 $F. [resolve(2448,a,2135,a)].
============================== end of proof ==========================

Theorem 6.5.1:

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.24 (+ 0.04) seconds.
% Length of proof is 33.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 43.

1 9 = 0 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (y * x) = y. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 8 = 9. [assumption].
9 0 * 4 = 8. [assumption].
10 0 * 6 = 9. [assumption].
11 5 * 7 = 8. [assumption].
12 3 * 5 = 9. [assumption].
13 0 * 3 = 7. [assumption].
15 0 != 9. [deny(1)].
16 9 != 0. [copy(15),flip(a)].
17 x * (x * y) = y. [para(2(a,1),3(a,1,2))].
26 2 * 3 = 1. [para(5(a,1),3(a,1,2))].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
38 4 * 8 = 0. [para(9(a,1),3(a,1,2))].
41 6 * 9 = 0. [para(10(a,1),3(a,1,2))].
44 7 * 8 = 5. [para(11(a,1),3(a,1,2))].
47 5 * 9 = 3. [para(12(a,1),3(a,1,2))].
59 1 * 3 = 2. [para(5(a,1),17(a,1,2))].
60 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
62 2 * 9 = 8. [para(8(a,1),17(a,1,2))].
676 4 * (2 * 7) = 3 * 8. [para(11(a,1),27(a,2,2)),rewrite([60(3)])].
685 7 = 3. [para(38(a,1),27(a,2,2)),rewrite([6(3),8(4),47(3),2(4),13(4)]),flip(a)].
686 3 * 8 = 3. [para(41(a,1),27(a,2,2)),rewrite([7(3),685(1),62(4),2(6),13(6),685(4)])].
687 9 = 8. [para(44(a,1),27(a,2,2)),rewrite([685(2),59(3),8(4),62(3),12(4)]),flip(a)].
688 3 * 3 = 0. [para(47(a,1),27(a,2,2)),rewrite([60(3),687(3),8(4),687(2),38(3)]),flip(a)].
728 5 = 3. [back_rewrite(676),rewrite([685(3),26(4),2(3),6(3),686(4)])].
768 8 = 0. [back_rewrite(11),rewrite([728(1),685(2),688(3)]),flip(a)].
792 $F. [back_rewrite(16),rewrite([687(1),768(1)]),xx(a)].

============================== end of proof ==========================

Theorem 6.5.2:

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.26 (+ 0.04) seconds.
% Length of proof is 33.
% Level of proof is 8.
% Maximum clause weight is 15.
% Given clauses 43.

1 9 = 0 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (y * x) = y. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
9 2 * 4 = 8. [assumption].
10 3 * 7 = 8. [assumption].
11 2 * 6 = 9. [assumption].
12 7 * 5 = 9. [assumption].
13 5 * 7 = 9. [copy(12),rewrite([2(3)])].
14 0 * 3 = 5. [assumption].
15 0 * 4 = 6. [assumption].
16 9 != 0. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),3(a,1,2))].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
38 4 * 8 = 2. [para(9(a,1),3(a,1,2))].
50 3 * 5 = 0. [para(14(a,1),3(a,1,2))].
53 4 * 6 = 0. [para(15(a,1),3(a,1,2))].
59 1 * 3 = 2. [para(5(a,1),17(a,1,2))].
60 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
61 1 * 7 = 6. [para(7(a,1),17(a,1,2))].
63 2 * 8 = 4. [para(9(a,1),17(a,1,2))].
64 3 * 8 = 7. [para(10(a,1),17(a,1,2))].
66 5 * 9 = 7. [para(13(a,1),17(a,1,2))].
694 7 = 5. [para(53(a,1),27(a,2,2)),rewrite([6(3),11(4),66(3),2(4),14(4)])].
705 8 = 0. [para(64(a,1),27(a,2,2)),rewrite([59(3),63(4),9(3),694(3),50(4)])].
765 6 = 4. [back_rewrite(61),rewrite([694(2),60(3)]),flip(a)].
806 4 = 2. [back_rewrite(38),rewrite([705(2),2(3),15(3),765(1)])].
807 2 * 2 = 0. [back_rewrite(9),rewrite([806(2),705(4)])].

837 9 = 0. [back_rewrite(11),rewrite([765(2),806(2),
807(3)]),flip(a)].
838 $F. [resolve(837,a,16,a)].

============================== end of proof
==========================

Theorem 6.5.3:

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.06 (+ 0.01) seconds.
% Length of proof is 20.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 47.

1 9 = 8 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (y * x) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
6 1 * 2 = 3. [assumption].
7 1 * 4 = 5. [assumption].
10 2 * 8 = 9. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 8 = 0. [assumption].
14 5 * 7 = 8. [assumption].
16 9 != 8. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),3(a,1,2))].
39 1 * (x * (y * 2)) = y * (x * 3). [para(6(a,1),4(a,1,2,2)),flip(a)].
83 1 * 5 = 4. [para(7(a,1),17(a,1,2))].
88 0 * 3 = 7. [para(12(a,1),17(a,1,2)),rewrite([2(3)])].
89 0 * 4 = 8. [para(13(a,1),17(a,1,2)),rewrite([2(3)])].
686 1 * (0 * (x * 2)) = x * 7. [para(88(a,1),39(a,2,2))].
722 1 * (0 * (2 * 5)) = 8. [para(686(a,2),14(a,1)),rewrite([2(5)])].
849 9 = 8. [para(722(a,1),4(a,1)),rewrite([83(6),89(5),
10(4)]),flip(a)].
850 $F. [resolve(849,a,16,a)].

============================== end of proof
==========================

Theorem 6.5.4:

============================== PROOF
=================================
% -------- Comments from original proof --------
% Proof 1 at 0.23 (+ 0.04) seconds.
% Length of proof is 17.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 46.

1 9 = 5 # label(non_clause) # label(goal). [goal].
4 x * y = y * x. [assumption].
5 x * (y * x) = y. [assumption].
6 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
7 0 * 3 = 4. [assumption].
9 0 * 7 = 9. [assumption].
13 2 * 4 = 6. [assumption].
14 2 * 7 = 8. [assumption].
15 3 * 8 = 9. [assumption].
16 5 * 6 = 9. [assumption].
17 9 != 5. [deny(1)].
31 (0 * x) * (3 * y) = 4 * (x * y). [para(7(a,1),6(a,1,1)),flip(a)].
51 7 * 8 = 2. [para(14(a,1),5(a,1,2))].
57 6 * 9 = 5. [para(16(a,1),5(a,1,2))].
683 9 * 9 = 6. [para(51(a,1),31(a,2,2)),rewrite([9(3),15(4),4(6),
13(6)])].
722 9 = 5. [para(683(a,1),5(a,1,2)),rewrite([4(3),57(3)]),flip(a)].
723 $F. [resolve(722,a,17,a)].
============================== end of proof
==========================

Theorem 6.5.5:

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.21 (+ 0.04) seconds.
% Length of proof is 28.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 43.

1 9 = 0 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (y * x) = y. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 8 = 9. [assumption].
9 0 * 4 = 8. [assumption].
10 0 * 6 = 9. [assumption].
11 5 * 7 = 8. [assumption].
12 3 * 5 = 9. [assumption].
15 0 != 9. [deny(1)].
16 9 != 0. [copy(15),flip(a)].
17 x * (x * y) = y. [para(2(a,1),3(a,1,2))].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
38 4 * 8 = 0. [para(9(a,1),3(a,1,2))].
41 6 * 9 = 0. [para(10(a,1),3(a,1,2))].
44 7 * 8 = 5. [para(11(a,1),3(a,1,2))].
47 5 * 9 = 3. [para(12(a,1),3(a,1,2))].
59 1 * 3 = 2. [para(5(a,1),17(a,1,2))].
61 1 * 7 = 6. [para(7(a,1),17(a,1,2))].
62 2 * 9 = 8. [para(8(a,1),17(a,1,2))].
681 0 * 3 = 3. [para(38(a,1),27(a,2,2)),rewrite([6(3),8(4),47(3),
2(4)]),flip(a)].
682 5 = 3. [para(41(a,1),27(a,2,2)),rewrite([7(3),62(4),44(3),2(4),
681(4)])].
683 3 * 3 = 0. [para(44(a,1),27(a,2,2)),rewrite([61(3),8(4),41(3),
682(3)]),flip(a)].
684 9 = 0. [para(47(a,1),27(a,2,2)),rewrite([682(2),59(3),62(4),8(3),
683(4)])].
685 $F. [resolve(684,a,16,a)].

============================== end of proof
==========================

Proofs from Chapter 7.
Negative proofs from §7.0
(11,3)1

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.05 (+ 0.01) seconds.
% Length of proof is 20.
% Level of proof is 5.
% Maximum clause weight is 15.

% Given clauses 47.

1 3 = 2 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 4 = 5. [assumption].
6 1 * 6 = 7. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 3 = 10. [assumption].
9 2 * 4 = 9. [assumption].
10 2 * 5 = 0. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 7 = 10. [assumption].
14 5 * 6 = 9. [assumption].
15 8 * 9 = 0. [assumption].
16 3 != 2. [deny(1)].
17 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
25 1 * (x * (y * 4)) = y * (x * 5). [para(5(a,1),4(a,1,2,2)),flip(a)].
30 10 * 2 = 3. [para(8(a,1),3(a,1,2)),rewrite([2(3)])].
53 0 * 5 = 2. [para(10(a,1),17(a,1,2)),rewrite([2(3)])].
58 0 * 9 = 8. [para(15(a,1),17(a,1,2)),rewrite([2(3)])].
525 1 * (x * 9) = 2 * (x * 5). [para(9(a,1),25(a,1,2,2))].
592 2 * 2 = 10. [para(53(a,1),525(a,2,2)),rewrite([58(4),
7(3)]),flip(a)].
673 3 = 2. [para(592(a,1),3(a,1,2)),rewrite([2(3),30(3)])].
674 $F. [resolve(673,a,16,a)].

============================== end of proof
==========================

(11,3)2

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.05 (+ 0.01) seconds.
% Length of proof is 22.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 48.

1 3 = 2 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 6 = 7. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 3 * 6 = 10. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 5 = 10. [assumption].
13 4 * 7 = 8. [assumption].
14 5 * 6 = 9. [assumption].
15 8 * 9 = 0. [assumption].
16 3 != 2. [deny(1)].
17 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
25 1 * (x * (y * 2)) = y * (x * 3). [para(5(a,1),4(a,1,2,2)),flip(a)].
28 1 * 10 = 8. [para(7(a,1),3(a,1,2))].
32 0 * 2 = 5. [para(9(a,1),3(a,1,2)),rewrite([2(3)])].
44 0 * 8 = 9. [para(15(a,1),3(a,1,2)),rewrite([2(3)])].
51 4 * 9 = 2. [para(8(a,1),17(a,1,2))].
526 1 * (x * 5) = 0 * (x * 3). [para(32(a,1),25(a,1,2,2))].
563 0 * (3 * 4) = 8. [para(12(a,1),526(a,1,2)),rewrite([28(3),
2(5)]),flip(a)].
651 3 * 4 = 9. [para(563(a,1),3(a,1,2)),rewrite([44(3)]),flip(a)].
662 3 = 2. [para(651(a,1),17(a,1,2)),rewrite([51(3)]),flip(a)].
663 $F. [resolve(662,a,16,a)].

============================== end of proof
==========================

(11,3)3

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.09 (+ 0.01) seconds.
% Length of proof is 31.
% Level of proof is 8.
% Maximum clause weight is 15.
% Given clauses 64.

1 7 = 4 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 4 = 5. [assumption].
6 1 * 6 = 7. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 3 = 10. [assumption].
9 2 * 4 = 9. [assumption].
10 2 * 5 = 0. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 7 = 8. [assumption].
14 5 * 6 = 9. [assumption].
15 9 * 0 = 10. [assumption].
16 0 * 9 = 10. [copy(15),rewrite([2(3)])].
17 7 != 4. [deny(1)].
18 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
25 1 * 5 = 4. [para(5(a,1),3(a,1,2))].
26 1 * (x * (y * 4)) = y * (x * 5). [para(5(a,1),4(a,1,2,2)),flip(a)].
29 1 * 10 = 8. [para(7(a,1),3(a,1,2))].
37 3 * 8 = 6. [para(11(a,1),3(a,1,2))].
43 5 * 9 = 6. [para(14(a,1),3(a,1,2))].
54 0 * 5 = 2. [para(10(a,1),18(a,1,2)),rewrite([2(3)])].
56 0 * 7 = 3. [para(12(a,1),18(a,1,2)),rewrite([2(3)])].
526 1 * (x * 9) = 2 * (x * 5). [para(9(a,1),26(a,1,2,2))].
568 2 * 2 = 8. [para(16(a,1),526(a,1,2)),rewrite([29(3),
54(5)]),flip(a)].
579 2 * (5 * 5) = 7. [para(43(a,1),526(a,1,2)),rewrite([6(3)]),flip(a)].
691 5 * (2 * (x * 5)) = x * 7. [para(579(a,1),4(a,1,2)),flip(a)].
1106 5 * 8 = 3. [para(54(a,1),691(a,1,2,2)),rewrite([568(4),56(6)])].
1124 6 = 5. [para(1106(a,1),18(a,1,2)),rewrite([2(3),37(3)])].
1234 7 = 4. [back_rewrite(6),rewrite([1124(2),25(3)]),flip(a)].
1235 $F. [resolve(1234,a,17,a)].

============================== end of proof
==========================

(11,3)4

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.

% Length of proof is 31.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 68.

1 6 = 5 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 6 = 7. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 3 * 6 = 8. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 5 = 10. [assumption].
13 4 * 7 = 8. [assumption].
14 5 * 6 = 9. [assumption].
15 9 * 0 = 10. [assumption].
16 0 * 9 = 10. [copy(15),rewrite([2(3)])].
17 5 != 6. [deny(1)].
18 6 != 5. [copy(17),flip(a)].
19 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
27 1 * (x * (y * 2)) = y * (x * 3). [para(5(a,1),4(a,1,2,2)),flip(a)].
30 1 * 10 = 8. [para(7(a,1),3(a,1,2))].
34 0 * 2 = 5. [para(9(a,1),3(a,1,2)),rewrite([2(3)])].
36 3 * 8 = 6. [para(10(a,1),3(a,1,2))].
44 5 * 9 = 6. [para(14(a,1),3(a,1,2))].
56 0 * 7 = 3. [para(11(a,1),19(a,1,2)),rewrite([2(3)])].
528 1 * (x * 5) = 0 * (x * 3). [para(34(a,1),27(a,1,2,2))].
561 0 * (x * 3) = 1 * (5 * x). [para(2(a,1),528(a,1,2)),flip(a)].
562 1 * (0 * (x * 3)) = x * 5. [para(528(a,1),3(a,1,2))].
819 0 * (3 * 9) = 7. [para(44(a,1),561(a,2,2)),rewrite([2(4),6(8)])].
910 3 * 9 = 3. [para(819(a,1),3(a,1,2)),rewrite([56(3)]),flip(a)].
917 3 * 3 = 9. [para(910(a,1),3(a,1,2))].
1107 3 * 5 = 8. [para(917(a,1),562(a,1,2,2)),rewrite([16(4),
30(3)]),flip(a)].
1109 6 = 5. [para(1107(a,1),3(a,1,2)),rewrite([36(3)])].
1110 $F. [resolve(1109,a,18,a)].

============================== end of proof
==========================

$(11,3)_5$

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 31.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 65.

1 7 = 4 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 3 * 6 = 8. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 7 = 8. [assumption].
13 5 * 6 = 9. [assumption].
14 6 * 7 = 10. [assumption].
15 9 * 0 = 10. [assumption].
16 0 * 9 = 10. [copy(15),rewrite([2(3)])].
17 7 != 4. [deny(1)].
18 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
26 1 * (x * (y * 2)) = y * (x * 3). [para(5(a,1),4(a,1,2,2)),flip(a)].
30 1 * (x * (y * 8)) = y * (x * 10). [para(7(a,1),4(a,1,2,2)),flip(a)].
33 0 * 2 = 5. [para(9(a,1),3(a,1,2)),rewrite([2(3)])].
37 0 * 3 = 7. [para(11(a,1),3(a,1,2)),rewrite([2(3)])].
50 4 * 5 = 1. [para(6(a,1),18(a,1,2))].
59 10 * 9 = 0. [para(16(a,1),18(a,1,2)),rewrite([2(3)])].
527 1 * (x * 5) = 0 * (x * 3). [para(33(a,1),26(a,1,2,2))].
560 0 * (x * 3) = 1 * (5 * x). [para(2(a,1),527(a,1,2)),flip(a)].
572 0 * (3 * 4) = 1 * 1. [para(50(a,1),527(a,
1,2)),rewrite([2(7)]),flip(a)].
650 3 * 4 = 0 * (1 * 1). [para(572(a,1),3(a,1,2)),flip(a)].
901 1 * 9 = 0 * 8. [para(13(a,1),560(a,2,2)),rewrite([2(4),
10(4)]),flip(a)].
996 9 * (0 * 8) = 1. [para(901(a,1),18(a,1,2))].
1037 1 * 1 = 0 * 0. [para(996(a,1),30(a,1,2)),rewrite([2(7),59(7)])].
1061 3 * 4 = 0. [back_rewrite(650),rewrite([1037(7),3(8)])].
1063 7 = 4. [para(1061(a,1),3(a,1,2)),rewrite([2(3),37(3)])].
1064 $F. [resolve(1063,a,17,a)].
============================== end of proof
==========================

$(11,3)_6$

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 20.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 47.

1 4 = 2 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 6 = 7. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 0 = 10. [assumption].
8 0 * 1 = 10. [copy(7),rewrite([3(3)])].
9 2 * 3 = 10. [assumption].
10 2 * 4 = 9. [assumption].
11 2 * 5 = 0. [assumption].
12 3 * 6 = 8. [assumption].
13 3 * 7 = 0. [assumption].
14 4 * 7 = 8. [assumption].
15 5 * 6 = 9. [assumption].
16 8 * 9 = 10. [assumption].
17 4 != 2. [deny(1)].
28 (1 * x) * (6 * y) = 7 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
33 1 * 10 = 0. [para(8(a,1),2(a,1,2))].
42 0 * 5 = 2. [para(11(a,1),2(a,1,2)),rewrite([3(3)])].
51 7 * 8 = 4. [para(14(a,1),2(a,1,2))].
54 6 * 9 = 5. [para(15(a,1),2(a,1,2))].
57 10 * 9 = 8. [para(16(a,1),2(a,1,2)),rewrite([3(3)])].
896 4 = 2. [para(57(a,1),28(a,2,2)),rewrite([33(3),54(4),42(3),
51(4)]),flip(a)].

897 $F. [resolve(896,a,17,a)].

============================== end of proof ==========================

(11,3)7

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 20.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 47.

1 8 = 6 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 3 = 10. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 6 = 10. [assumption].
9 2 * 4 = 9. [assumption].
10 2 * 5 = 0. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 7 = 8. [assumption].
14 5 * 9 = 10. [assumption].
15 8 * 9 = 0. [assumption].
16 8 != 6. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (1 * x) * (3 * y) = 10 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
41 0 * 5 = 2. [para(10(a,1),2(a,1,2)),rewrite([3(3)])].
63 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
65 10 * 2 = 6. [para(8(a,1),17(a,1,2)),rewrite([3(3)])].
69 0 * 3 = 7. [para(12(a,1),17(a,1,2)),rewrite([3(3)])].
355 (x * 1) * (y * 5) = 4 * (x * y). [para(63(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
884 8 = 6. [para(41(a,1),27(a,2,2)),rewrite([3(3),355(7),69(4),13(3),65(4)])].
885 $F. [resolve(884,a,16,a)].

============================== end of proof ==========================

(11,3)8

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.13 (+ 0.01) seconds.
% Length of proof is 33.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 53.

1 3 = 1 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 7 = 10. [assumption].
8 2 * 6 = 10. [assumption].
9 8 * 9 = 0. [assumption].
10 2 * 5 = 0. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 7 = 8. [assumption].
14 4 * 9 = 10. [assumption].
15 8 * 9 = 0. [assumption].
16 3 != 1. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
32 10 * 7 = 1. [para(7(a,1),2(a,1,2)),rewrite([3(3)])].
47 0 * 7 = 3. [para(12(a,1),2(a,1,2)),rewrite([3(3)])].
50 7 * 8 = 4. [para(13(a,1),2(a,1,2))].
63 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
64 1 * 10 = 7. [para(7(a,1),17(a,1,2))].
65 10 * 2 = 6. [para(8(a,1),17(a,1,2)),rewrite([3(3)])].
889 3 * 4 = 10 * (2 * 8). [para(50(a,1),27(a,2,2)),rewrite([7(3)]),flip(a)].
898 10 * (2 * 8) = 0 * (1 * 1). [para(63(a,1),27(a,2,2)),rewrite([10(6),3(5),889(8)]),flip(a)].
900 6 * (1 * 1) = 0. [para(64(a,1),27(a,2,2)),rewrite([3(6),65(6),3(5),12(8)])].
934 3 * 4 = 0 * (1 * 1). [back_rewrite(889),rewrite([898(8)])].
1281 1 * 1 = 0 * 6. [para(900(a,1),17(a,1,2)),rewrite([3(3)]),flip(a)].
1290 3 * 4 = 6. [back_rewrite(934),rewrite([1281(7),17(8)])].
1295 8 = 4. [para(1290(a,1),17(a,1,2)),rewrite([11(3)])].
1389 10 = 0. [back_rewrite(9),rewrite([1295(1),14(3)])].
1456 3 = 1. [back_rewrite(32),rewrite([1389(1),47(3)])].
1457 $F. [resolve(1456,a,16,a)].

============================== end of proof ==========================

(11,3)9

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 33.
% Level of proof is 8.
% Maximum clause weight is 15.
% Given clauses 49.

1 4 = 7 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 3 * 8 = 10. [assumption].
9 8 * 9 = 0. [assumption].
10 2 * 5 = 0. [assumption].
11 3 * 7 = 0. [assumption].
12 3 * 8 = 10. [assumption].
13 4 * 9 = 10. [assumption].
14 5 * 6 = 9. [assumption].
15 2 * 6 = 10. [assumption].
16 7 != 4. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
38 0 * 9 = 8. [para(9(a,1),2(a,1,2)),rewrite([3(3)])].
44 0 * 7 = 3. [para(11(a,1),2(a,1,2)),rewrite([3(3)])].
50 10 * 9 = 4. [para(13(a,1),2(a,1,2)),rewrite([3(3)])].

63 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
64 1 * 7 = 6. [para(7(a,1),17(a,1,2))].
65 10 * 3 = 8. [para(8(a,1),17(a,1,2)),rewrite([3(3)])].
67 0 * 2 = 5. [para(10(a,1),17(a,1,2)),rewrite([3(3)])].
68 0 * 3 = 7. [para(11(a,1),17(a,1,2)),rewrite([3(3)])].
70 10 * 4 = 9. [para(13(a,1),17(a,1,2)),rewrite([3(3)])].
367 (x * 1) * (y * 7) = 6 * (x * y). [para(64(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
872 3 * 9 = 9. [para(14(a,1),27(a,2,2)),rewrite([63(3),15(4),3(3),70(3)]),flip(a)].
880 3 * 3 = 9. [para(44(a,1),27(a,2,2)),rewrite([3(3),367(7),67(4),3(3),14(3)]),flip(a)].
947 9 = 3. [para(880(a,1),2(a,1,2)),rewrite([872(3)])].
1026 8 = 4. [back_rewrite(50),rewrite([947(2),65(3)])].
1027 7 = 4. [back_rewrite(38),rewrite([947(2),68(3),1026(2)])].
1028 $F. [resolve(1027,a,16,a)].

============================== end of proof ==========================

(11,3)10

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 32.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 62.

1 9 = 2 # label(non_clause) # label(goal). [goal].
2 x * y = y * x. [assumption].
3 x * (x * y) = y. [assumption].
4 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 7 = 10. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 6 = 10. [assumption].
10 3 * 7 = 0. [assumption].
11 3 * 6 = 8. [assumption].
12 4 * 7 = 8. [assumption].
13 5 * 6 = 9. [assumption].
14 5 * 0 = 10. [assumption].
16 8 * 9 = 0. [assumption].
17 9 != 2. [deny(1)].
18 x * (y * x) = y. [para(2(a,1),3(a,1,2))].
22 x * (y * (z * (x * u))) = z * (y * u). [para(4(a,1),3(a,1,2))].
25 1 * 3 = 2. [para(5(a,1),3(a,1,2))].
26 1 * (x * (y * 2)) = y * (x * 3). [para(5(a,1),4(a,1,2,2)),flip(a)].
33 10 * 2 = 6. [para(9(a,1),3(a,1,2)),rewrite([2(3)])].
35 0 * 3 = 7. [para(10(a,1),3(a,1,2)),rewrite([2(3)])].
45 0 * 8 = 9. [para(16(a,1),3(a,1,2)),rewrite([2(3)])].
51 10 * 7 = 1. [para(7(a,1),18(a,1,2)),rewrite([2(3)])].
59 0 * 9 = 8. [para(16(a,1),18(a,1,2)),rewrite([2(3)])].
527 10 * (x * 3) = 1 * (x * 6). [para(33(a,1),26(a,1,2,2)),flip(a)].
528 1 * (0 * (x * 2)) = x * 7. [para(35(a,1),26(a,2,2))].
567 10 * (3 * 3) = 1 * 8. [para(11(a,1),527(a,2,2))].
571 1 * (0 * 6) = 1. [para(35(a,1),527(a,1,2)),rewrite([51(3)]),flip(a)].
682 1 * (x * 6) = 0 * (x * 1). [para(571(a,1),22(a,1,2,2)),flip(a)].
708 10 * (x * 3) = 0 * (x * 1). [back_rewrite(527),rewrite([682(8)])].
709 1 * 8 = 0 * 2. [back_rewrite(567),rewrite([708(5),2(4),25(4)]),flip(a)].
945 0 * 2 = 8. [para(528(a,2),12(a,1)),rewrite([2(5),8(5),59(4),709(3)])].
1068 9 = 2. [para(945(a,1),3(a,1,2)),rewrite([45(3)])].
1069 $F. [resolve(1068,a,17,a)]].

============================== end of proof ==========================

(11,3)11

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 39.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 48.

1 3 = 9 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 6 = 10. [assumption].
10 3 * 8 = 10. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 7 = 8. [assumption].
13 5 * 6 = 9. [assumption].
14 5 * 0 = 10. [assumption].
15 0 * 5 = 10. [copy(14),rewrite([3(3)])].
16 8 * 9 = 0. [assumption].
17 9 != 3. [deny(1)].
18 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
28 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
39 10 * 6 = 2. [para(9(a,1),2(a,1,2)),rewrite([3(3)])].
45 0 * 7 = 3. [para(11(a,1),2(a,1,2)),rewrite([3(3)])].
51 6 * 9 = 5. [para(13(a,1),2(a,1,2))].
57 0 * 9 = 8. [para(16(a,1),2(a,1,2)),rewrite([3(3)])].
64 1 * 5 = 4. [para(6(a,1),18(a,1,2))].
66 2 * 9 = 4. [para(8(a,1),18(a,1,2))].
68 10 * 3 = 8. [para(10(a,1),18(a,1,2)),rewrite([3(3)])].
70 4 * 8 = 7. [para(12(a,1),18(a,1,2))].
73 0 * 8 = 9. [para(16(a,1),18(a,1,2)),rewrite([3(3)])].
355 (x * 1) * (y * 5) = 4 * (x * y). [para(64(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
877 4 * (0 * 2) = 8. [para(15(a,1),28(a,2,2)),rewrite([3(3),355(7),3(8),68(8)])].
893 3 * 5 = 8. [para(51(a,1),28(a,2,2)),rewrite([7(3),66(4),3(3),12(3)]),flip(a)].
941 5 = 10. [para(893(a,1),18(a,1,2)),rewrite([10(3)]),flip(a)].
1013 9 = 2. [back_rewrite(13),rewrite([941(1),39(3)]),flip(a)].
1064 0 * 8 = 2. [back_rewrite(73),rewrite([1013(4)])].
1067 0 * 2 = 8. [back_rewrite(57),rewrite([1013(2)])].
1068 3 != 2. [back_rewrite(17),rewrite([1013(1)]),flip(a)].
1072 8 = 7. [back_rewrite(877),rewrite([1067(4),70(3)]),flip(a)].
1078 3 = 2. [back_rewrite(1064),rewrite([1072(2),45(3)])].
1079 $F. [resolve(1078,a,1068,a)].

============================== end of proof ==========================

(11,3)12

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.19 (+ 0.01) seconds.
% Length of proof is 40.
% Level of proof is 8.
% Maximum clause weight is 15.
% Given clauses 61.

1 9 = 6 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 3 = 10. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 2 * 7 = 10. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 8 = 10. [assumption].
14 5 * 6 = 9. [assumption].
15 8 * 9 = 0. [assumption].
16 9 != 6. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
20 (x * y) * (z * (u * y)) = (x * z) * u. [para(2(a,1),4(a,1,2)),flip(a)].
27 (1 * x) * (3 * y) = 10 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
28 (x * 1) * (y * 3) = 10 * (x * y). [para(5(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
38 0 * 5 = 2. [para(9(a,1),2(a,1,2)),rewrite([3(3)])].
62 1 * 10 = 3. [para(5(a,1),17(a,1,2))].
63 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
64 1 * 7 = 6. [para(7(a,1),17(a,1,2))].
67 10 * 2 = 7. [para(10(a,1),17(a,1,2)),rewrite([3(3)])].
68 3 * 8 = 6. [para(11(a,1),17(a,1,2))].
69 0 * 3 = 7. [para(12(a,1),17(a,1,2)),rewrite([3(3)])].
70 10 * 4 = 8. [para(13(a,1),17(a,1,2)),rewrite([3(3)])].
72 0 * 8 = 9. [para(15(a,1),17(a,1,2)),rewrite([3(3)])].
354 (x * 1) * (y * 5) = 4 * (x * y). [para(63(a,1),4(a,1,2)),rewrite([3(3)]),flip(a)].
875 10 * 9 = 10. [para(14(a,1),27(a,2,2)),rewrite([63(3),11(4),13(3)]),flip(a)].
882 4 * 7 = 7. [para(38(a,1),27(a,2,2)),rewrite([3(3),354(7),69(4),67(6)])].
897 10 * 6 = 0 * (1 * 1). [para(64(a,1),27(a,2,2)),rewrite([12(6),3(5)]),flip(a)].
911 6 * (0 * 1) = 10. [para(72(a,1),27(a,2,2)),rewrite([3(3),68(6),3(5),875(8)])].
946 7 * 7 = 4. [para(882(a,1),2(a,1,2))].
1729 0 * (1 * 1) = 0 * 1. [para(911(a,1),17(a,1,2)),rewrite([3(3),897(3)])].
1738 0 * (x * 6) = 10 * (x * 1). [para(911(a,1),20(a,1,2)),rewrite([3(4),3(8)]),flip(a)].
1753 10 * 6 = 0 * 1. [back_rewrite(897),rewrite([1729(8)])].
1921 0 * 6 = 8. [para(946(a,1),28(a,2,2)),rewrite([3(3),64(3),3(4),12(4),3(3),70(6)])].
1931 9 = 6. [para(1753(a,1),28(a,2,2)),rewrite([3(3),62(3),3(4),11(4),68(3),1738(6,R),1921(5),72(4)]),flip(a)].
1932 $F. [resolve(1931,a,16,a)].

============================== end of proof ==========================

(11,3)13

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 18.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 48.

1 9 = 10 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 8 = 10. [assumption].
8 2 * 5 = 0. [assumption].
9 2 * 7 = 10. [assumption].
10 3 * 6 = 8. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 6 = 7. [assumption].
13 4 * 9 = 10. [assumption].
14 8 * 9 = 5. [assumption].
15 6 * 9 = 0. [assumption].
16 9 != 10. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
69 4 * 7 = 6. [para(12(a,1),17(a,1,2))].
71 5 * 8 = 9. [para(14(a,1),17(a,1,2)),rewrite([3(3)])].
905 10 * 5 = 8. [para(69(a,1),27(a,2,2)),rewrite([6(3),9(4),3(3),10(6)])].
935 9 = 10. [para(905(a,1),2(a,1,2)),rewrite([71(3)])].
936 $F. [resolve(935,a,16,a)].

============================== end of proof ==========================

(11,3)14

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.09 (+ 0.01) seconds.
% Length of proof is 42.
% Level of proof is 11.
% Maximum clause weight is 15.
% Given clauses 47.

1 9 = 10 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 10. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 7 = 10. [assumption].
10 3 * 6 = 8. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 7 = 8. [assumption].

13 5 * 6 = 9. [assumption].
14 5 * 0 = 10. [assumption].
15 0 * 5 = 10. [copy(14),rewrite([3(3)])].
16 8 * 9 = 0. [assumption].
17 9 != 10. [deny(1)].
18 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 2 * 3 = 1. [para(5(a,1),2(a,1,2))].
28 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
39 10 * 7 = 2. [para(9(a,1),2(a,1,2)),rewrite([3(3)])].
42 6 * 8 = 3. [para(10(a,1),2(a,1,2))].
51 6 * 9 = 5. [para(13(a,1),2(a,1,2))].
54 10 * 5 = 0. [para(15(a,1),2(a,1,2)),rewrite([3(3)])].
57 0 * 9 = 8. [para(16(a,1),2(a,1,2)),rewrite([3(3)])].
63 1 * 3 = 2. [para(5(a,1),18(a,1,2))].
65 1 * 10 = 6. [para(7(a,1),18(a,1,2))].
66 2 * 9 = 4. [para(8(a,1),18(a,1,2))].
68 3 * 8 = 6. [para(10(a,1),18(a,1,2))].
69 0 * 3 = 7. [para(11(a,1),18(a,1,2)),rewrite([3(3)])].
72 0 * 10 = 5. [para(15(a,1),18(a,1,2))].
73 0 * 8 = 9. [para(16(a,1),18(a,1,2)),rewrite([3(3)])].
878 6 = 0. [para(12(a,1),28(a,2,2)),rewrite([6(3),9(4),3(3),54(3),
68(4)]),flip(a)].
888 5 = 1. [para(39(a,1),28(a,2,2)),rewrite([65(3),878(1),9(4),72(3),
3(4),27(4)])].
897 4 * (0 * 1) = 2. [para(51(a,1),28(a,2,2)),rewrite([878(2),3(3),
66(6),3(5),888(7),3(8),63(8)])].
898 2 = 0. [para(57(a,1),28(a,2,2)),rewrite([3(3),66(6),3(5),897(5),
68(4),878(2)])].
910 7 = 1. [para(65(a,1),28(a,2,2)),rewrite([898(4),72(6),888(4),
3(5),18(5),878(3),3(4),69(4)]),flip(a)].
919 8 = 1. [para(68(a,1),28(a,2,2)),rewrite([63(3),898(1),898(2),
73(4),57(3),878(3),3(4),69(4),910(2)])].
981 0 * 1 = 3. [back_rewrite(42),rewrite([878(1),919(2)])].
982 9 = 3. [back_rewrite(13),rewrite([888(1),878(2),3(3),
981(3)]),flip(a)].
983 3 = 10. [back_rewrite(7),rewrite([878(2),3(3),981(3)])].
1082 $F. [back_rewrite(17),rewrite([982(1),983(1)]),xx(a)].

============================== end of proof
==========================

(11,3)15

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 20.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 47.

1 6 = 10 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 3 = 10. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 7 = 8. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 2 * 6 = 10. [assumption].
11 3 * 6 = 8. [assumption].
12 3 * 7 = 0. [assumption].
13 4 * 7 = 10. [assumption].
14 5 * 8 = 9. [assumption].
15 6 * 9 = 0. [assumption].
16 6 != 10. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
27 (1 * x) * (3 * y) = 10 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
38 0 * 5 = 2. [para(9(a,1),2(a,1,2)),rewrite([3(3)])].
63 1 * 5 = 4. [para(6(a,1),17(a,1,2))].
67 10 * 2 = 6. [para(10(a,1),17(a,1,2)),rewrite([3(3)])].
69 0 * 3 = 7. [para(12(a,1),17(a,1,2)),rewrite([3(3)])].
354 (x * 1) * (y * 5) = 4 * (x * y). [para(63(a,1),4(a,
1,2)),rewrite([3(3)]),flip(a)].
882 6 = 10. [para(38(a,1),27(a,2,2)),rewrite([3(3),354(7),69(4),
13(3),67(4)]),flip(a)].
883 $F. [resolve(882,a,16,a)].

============================== end of proof
==========================

(11,3)16

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 32.
% Level of proof is 7.
% Maximum clause weight is 15.
% Given clauses 49.

1 2 = 10 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 5 = 0. [assumption].
9 2 * 8 = 10. [assumption].
10 3 * 6 = 10. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 7 = 8. [assumption].
13 4 * 9 = 10. [assumption].
14 6 * 9 = 5. [assumption].
15 8 * 9 = 0. [assumption].
16 10 != 2. [deny(1)].
17 2 != 10. [copy(16),flip(a)].
18 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
28 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
42 10 * 6 = 3. [para(10(a,1),2(a,1,2)),rewrite([3(3)])].
45 0 * 7 = 3. [para(11(a,1),2(a,1,2)),rewrite([3(3)])].
48 7 * 8 = 4. [para(12(a,1),2(a,1,2))].
54 5 * 9 = 6. [para(14(a,1),2(a,1,2)),rewrite([3(3)])].
64 1 * 5 = 4. [para(6(a,1),18(a,1,2))].
65 1 * 7 = 6. [para(7(a,1),18(a,1,2))].
66 0 * 2 = 5. [para(8(a,1),18(a,1,2)),rewrite([3(3)])].
72 5 * 6 = 9. [para(14(a,1),18(a,1,2)),rewrite([3(3)])].
371 (x * 1) * (y * 7) = 6 * (x * y). [para(65(a,1),4(a,
1,2)),rewrite([3(3)]),flip(a)].
877 3 * 3 = 9. [para(45(a,1),28(a,2,2)),rewrite([3(3),371(7),66(4),
3(3),72(3)]),flip(a)].
878 3 * 4 = 3. [para(48(a,1),28(a,2,2)),rewrite([65(3),9(4),3(3),
42(3)]),flip(a)].
885 4 * (2 * 9) = 10. [para(54(a,1),28(a,2,2)),rewrite([64(3),10(8)])].
1002 9 = 4. [para(878(a,1),18(a,1,2)),rewrite([877(3)])].
1014 2 = 10. [back_rewrite(885),rewrite([1002(3),2(5)])].

1015 $F. [resolve(1014,a,17,a)].

============================== end of proof ==========================

(11,3)20

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 19.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 49.

1 3 = 10 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 0. [assumption].
10 3 * 6 = 10. [assumption].
11 3 * 7 = 0. [assumption].
12 4 * 7 = 8. [assumption].
13 5 * 6 = 9. [assumption].
14 5 * 8 = 10. [assumption].
15 8 * 9 = 0. [assumption].
16 10 != 3. [deny(1)].
17 3 != 10. [copy(16),flip(a)].
18 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
28 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
51 6 * 9 = 5. [para(13(a,1),2(a,1,2))].
66 2 * 9 = 4. [para(8(a,1),18(a,1,2))].
892 3 * 5 = 8. [para(51(a,1),28(a,2,2)),rewrite([7(3),66(4),3(3),12(3)]),flip(a)].
1010 3 = 10. [para(892(a,1),2(a,1,2)),rewrite([14(3)]),flip(a)].
1011 $F. [resolve(1010,a,17,a)].

============================== end of proof ==========================

(11,3)21

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 23.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 48.

1 1 = 4 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 5 = 8. [assumption].
10 3 * 6 = 8. [assumption].
11 3 * 9 = 10. [assumption].
12 4 * 7 = 0. [assumption].
13 5 * 6 = 0. [assumption].
14 5 * 7 = 10. [assumption].
15 8 * 9 = 0. [assumption].
16 4 != 1. [deny(1)].
17 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
26 2 * 3 = 1. [para(5(a,1),2(a,1,2))].
27 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
41 6 * 8 = 3. [para(10(a,1),2(a,1,2))].
65 2 * 9 = 4. [para(8(a,1),17(a,1,2))].
66 2 * 8 = 5. [para(9(a,1),17(a,1,2))].
68 10 * 3 = 9. [para(11(a,1),17(a,1,2)),rewrite([3(3)])].
880 3 * 3 = 10. [para(41(a,1),27(a,2,2)),rewrite([7(3),66(4),3(3),14(3)]),flip(a)].
938 9 = 3. [para(880(a,1),2(a,1,2)),rewrite([3(3),68(3)])].
1000 4 = 1. [back_rewrite(65),rewrite([938(2),26(3)]),flip(a)].
1001 $F. [resolve(1000,a,16,a)].

============================== end of proof ==========================

(11,3)23

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 19.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 47.

1 9 = 4 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 5 = 10. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 0 = 8. [assumption].
10 0 * 2 = 8. [copy(9),rewrite([3(3)])].
11 3 * 4 = 10. [assumption].
12 3 * 5 = 0. [assumption].
13 4 * 7 = 8. [assumption].
14 6 * 8 = 10. [assumption].
15 7 * 9 = 0. [assumption].
16 5 * 6 = 9. [assumption].
17 9 != 4. [deny(1)].
18 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
28 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
39 2 * 8 = 0. [para(10(a,1),2(a,1,2))].
68 10 * 3 = 4. [para(11(a,1),18(a,1,2)),rewrite([3(3)])].
72 0 * 7 = 9. [para(15(a,1),18(a,1,2)),rewrite([3(3)])].
875 9 = 4. [para(14(a,1),28(a,2,2)),rewrite([7(3),39(4),3(3),72(3),3(4),68(4)])].
876 $F. [resolve(875,a,17,a)].

============================== end of proof ==========================

(11,3)24

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 19.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 47.

1 9 = 4 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 8 = 0. [assumption].
9 2 * 9 = 10. [assumption].
10 2 * 0 = 8. [assumption].
11 0 * 2 = 8. [copy(10),rewrite([3(3)])].
12 3 * 4 = 10. [assumption].
13 3 * 5 = 0. [assumption].
14 4 * 7 = 8. [assumption].
15 6 * 8 = 10. [assumption].
16 7 * 9 = 0. [assumption].
17 5 * 6 = 9. [assumption].
18 9 != 4. [deny(1)].
19 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
29 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
40 2 * 8 = 0. [para(11(a,1),2(a,1,2))].
69 10 * 3 = 4. [para(12(a,1),19(a,1,2)),rewrite([3(3)])].
73 0 * 7 = 9. [para(16(a,1),19(a,1,2)),rewrite([3(3)])].
875 9 = 4. [para(15(a,1),29(a,2,2)),rewrite([7(3),40(4),3(3),73(3),
3(4),69(4)])].
876 $F. [resolve(875,a,18,a)].

============================== end of proof
==========================

(11,3)25

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 21.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 49.

1 6 = 7 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 8 =0 . [assumption].
10 3 * 6 = 8. [assumption].
11 5 * 6 = 9. [assumption].
12 3 * 10 = 0. [assumption].
13 10 * 3 = 0. [copy(12),rewrite([3(3)])].
14 4 * 7 = 10. [assumption].
15 5 * 8 = 10. [assumption].
17 7 * 9 = 0. [assumption].
18 7 != 6. [deny(1)].
19 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
29 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
46 0 * 3 = 10. [para(13(a,1),2(a,1,2)),rewrite([3(3)])].
66 1 * 7 = 6. [para(7(a,1),19(a,1,2))].
67 2 * 9 = 4. [para(8(a,1),19(a,1,2))].
71 10 * 4 = 7. [para(14(a,1),19(a,1,2)),rewrite([3(3)])].
878 4 * 6 = 10. [para(17(a,1),29(a,2,2)),rewrite([66(3),67(4),3(3),
3(6),46(6)])].
955 7 = 6. [para(878(a,1),19(a,1,2)),rewrite([3(3),71(3)])].
956 $F. [resolve(955,a,18,a)].

============================== end of proof
==========================

(11,3)26

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 23.
% Level of proof is 6.
% Maximum clause weight is 15.
% Given clauses 48.

1 1 = 6 # label(non_clause) # label(goal). [goal].
2 x * (y * x) = y. [assumption].
3 x * y = y * x. [assumption].
4 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
5 1 * 2 = 3. [assumption].
6 1 * 4 = 5. [assumption].
7 1 * 6 = 7. [assumption].
8 2 * 4 = 9. [assumption].
9 2 * 0 = 10. [assumption].
10 3 * 6 = 8 . [assumption].
11 4 * 7 = 8. [assumption].
12 3 * 5 = 0. [assumption].
13 8 * 9 = 10. [assumption].
14 5 * 6 = 10. [assumption].
17 7 * 9 = 0. [assumption].
18 6 != 1. [deny(1)].
19 x * (x * y) = y. [para(2(a,1),2(a,1,2)),rewrite([3(2)])].
29 (1 * x) * (2 * y) = 3 * (x * y). [para(5(a,1),4(a,1,1)),flip(a)].
31 4 * 5 = 1. [para(6(a,1),2(a,1,2))].
66 1 * 7 = 6. [para(7(a,1),19(a,1,2))].
67 2 * 9 = 4. [para(8(a,1),19(a,1,2))].
70 0 * 3 = 5. [para(12(a,1),19(a,1,2)),rewrite([3(3)])].
72 10 * 5 = 6. [para(14(a,1),19(a,1,2)),rewrite([3(3)])].
877 4 * 6 = 5. [para(17(a,1),29(a,2,2)),rewrite([66(3),67(4),3(3),
3(6),70(6)])].
936 4 = 10. [para(877(a,1),2(a,1,2)),rewrite([3(3),14(3)]),flip(a)].
1011 6 = 1. [back_rewrite(31),rewrite([936(1),72(3)])].
1012 $F. [resolve(1011,a,18,a)].

============================== end of proof
==========================

Tangential relations from §7.2

P12

============== PROOF
1 7 * 7 = 7 # label(non_clause) # label(goal). [goal].
3 x * y = y * x. [assumption].
4 x * (y * x) = y. [assumption].
5 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
6 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
7 1 * 2 = 3. [assumption].

```
9 1 * 6 = 7.  [assumption].
11 2 * 9 = 10.  [assumption].
12 3 * 7 = 0.  [assumption].
14 4 * 6 = 10.  [assumption].
15 4 * 7 = 8.  [assumption].
17 8 * 9 = 0.  [assumption].
18 7 * 7 != 7.  [deny(1)].
20 x * (x * y) = y.  [para(3(a,1),4(a,1,2))].
44 (1 * x) * (2 * y) = 3 * (x * y).  [para(7(a,1),5(a,1,1)),flip(a)].
75 7 * 8 = 4.  [para(15(a,1),4(a,1,2))].
95 0 * 3 = 7.  [para(12(a,1),20(a,1,2)),rewrite([3(3)])].
97 10 * 4 = 6.  [para(14(a,1),20(a,1,2)),rewrite([3(3)])].
512 10 * (1 * 8) = 7.  [para(17(a,1),44(a,2,2)),rewrite([11(6),3(5),
3(8),95(8)])].
588 1 * (10 * (x * 8)) = x * 7.  [para(512(a,1),6(a,1,2)),flip(a)].
3269 7 * 7 = 7.  [para(75(a,1),588(a,1,2,2)),rewrite([97(4),
9(3)]),flip(a)].
3270 $F.  [resolve(3269,a,18,a)].
============================== end of proof
==========================
```

P12

```
================= PROOF ========================
2 5 * 5 = 5 # label(non_clause) # label(goal).  [goal].
3 x * y = y * x.  [assumption].
4 x * (y * x) = y.  [assumption].
5 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].
7 1 * 2 = 3.  [assumption].
8 1 * 4 = 5.  [assumption].
9 1 * 6 = 7.  [assumption].
10 2 * 5 = 0.  [assumption].
11 2 * 9 = 10.  [assumption].
12 3 * 7 = 0.  [assumption].
13 3 * 8 = 10.  [assumption].
14 4 * 6 = 10.  [assumption].
15 4 * 7 = 8.  [assumption].
16 5 * 6 = 9.  [assumption].
17 8 * 9 = 0.  [assumption].
19 5 * 5 != 5.  [deny(2)].
20 x * (x * y) = y.  [para(3(a,1),4(a,1,2))].
22 (x * y) * (z * u) = (z * x) * (y * u).  [para(3(a,1),5(a,1,1)),flip(a)].
26 (x * y) * (z * (u * y)) = (x * z) * u.  [para(4(a,1),5(a,1,2)),flip(a)].
43 2 * 3 = 1.  [para(7(a,1),4(a,1,2))].
44 (1 * x) * (2 * y) = 3 * (x * y).  [para(7(a,1),5(a,1,1)),flip(a)].
48 (1 * x) * (4 * y) = 5 * (x * y).  [para(8(a,1),5(a,1,1)),flip(a)].
55 0 * 5 = 2.  [para(10(a,1),4(a,1,2)),rewrite([3(3)])].
75 7 * 8 = 4.  [para(15(a,1),4(a,1,2))].
79 6 * 9 = 5.  [para(16(a,1),4(a,1,2))].
91 1 * 5 = 4.  [para(8(a,1),20(a,1,2))].
92 1 * 7 = 6.  [para(9(a,1),20(a,1,2))].
93 0 * 2 = 5.  [para(10(a,1),20(a,1,2)),rewrite([3(3)])].
94 10 * 2 = 9.  [para(11(a,1),20(a,1,2)),rewrite([3(3)])].
95 0 * 3 = 7.  [para(12(a,1),20(a,1,2)),rewrite([3(3)])].
96 10 * 3 = 8.  [para(13(a,1),20(a,1,2)),rewrite([3(3)])].
98 4 * 8 = 7.  [para(15(a,1),20(a,1,2))].
362 (x * 10) * (2 * y) = 9 * (x * y).  [para(94(a,1),22(a,1,1)),flip(a)].
507 5 * (2 * x) = 3 * (4 * x).  [para(8(a,1),44(a,1,1))].
508 7 * (2 * x) = 3 * (6 * x).  [para(9(a,1),44(a,1,1))].
512 10 * (1 * 8) = 7.  [para(17(a,1),44(a,2,2)),rewrite([11(6),3(5),
3(8),95(8)])].
526 0 * (1 * 1) = 3 * 4.  [para(91(a,1),44(a,2,2)),rewrite([10(6),
3(5)])].
589 10 * 7 = 1 * 8.  [para(512(a,1),20(a,1,2))].
777 3 * (1 * 8) = 5.  [para(589(a,1),44(a,2,2)),rewrite([362(7),92(4),
3(3),79(3)]),flip(a)].
930 5 * (x * 8) = 1 * (x * 3).  [para(777(a,1),26(a,1,2)),rewrite([3(4),
3(8)])].
983 1 * 1 = 0.  [para(98(a,1),507(a,2,2)),rewrite([930(5),43(4),
12(6)])].
1018 3 * 4 = 0 * 0.  [back_rewrite(526),rewrite([983(4)]),flip(a)].
1046 3 * (1 * x) = 0 * (2 * x).  [para(983(a,1),44(a,1,1)),flip(a)].
1069 0 * (2 * 8) = 5.  [back_rewrite(777),rewrite([1046(5)])].
1384 4 * (0 * 0) = 3.  [para(1018(a,1),4(a,1,2))].
1419 2 * 8 = 2.  [para(1069(a,1),20(a,1,2)),rewrite([55(3)]),flip(a)].
1430 2 * 2 = 8.  [para(1419(a,1),20(a,1,2))].
1778 3 * (x * 0) = 0 * (x * 4).  [para(1384(a,1),26(a,
1,2)),rewrite([3(4),3(8)])].
1842 3 * (2 * 6) = 4.  [para(1430(a,1),508(a,1,2)),rewrite([75(3),
3(5)]),flip(a)].
1852 2 * 6 = 0 * 0.  [para(1842(a,1),20(a,
1,2)),rewrite([1018(3)]),flip(a)].
2108 5 * (0 * 0) = 8.  [para(1852(a,1),507(a,1,2)),rewrite([14(9),3(8),
96(8)])].
2150 5 * 8 = 0 * 0.  [para(2108(a,1),20(a,1,2))].
2482 2 * 4 = 4.  [para(2150(a,1),44(a,2,2)),rewrite([91(3),1419(4),
3(3),1778(8),20(8)])].
2501 4 * 4 = 2.  [para(2482(a,1),4(a,1,2))].
4026 5 * 5 = 5.  [para(8(a,1),48(a,2,2)),rewrite([983(3),2501(4),
93(3)]),flip(a)].
4027 $F.  [resolve(4026,a,19,a)].
============================== end of proof
==========================
```

Tangential relations from §7.4

P13

```
=============== PROOF
7 3 * 3 = 5 # label(non_clause) # label(goal).  [goal].
9 x * y = y * x.  [assumption].
10 x * (y * x) = y.  [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].
13 1 * 2 = 3.  [assumption].
15 1 * 6 = 7.  [assumption].
17 2 * 8 = 10.  [assumption].
18 3 * 6 = 8.  [assumption].
22 5 * 7 = 10.  [assumption].
30 3 * 3 != 5.  [deny(7)].
56 (1 * x) * (2 * y) = 3 * (x * y).  [para(13(a,1),11(a,1,1)),flip(a)].
75 6 * 8 = 3.  [para(18(a,1),10(a,1,2))].
91 10 * 7 = 5.  [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
530 3 * 3 = 5.  [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),
91(3)]),flip(a)].
531 $F.  [resolve(530,a,30,a)].
============================== end of proof
====================
```

P14

```
============== PROOF
6 8 * 8 = 7 # label(non_clause) # label(goal).  [goal].
9 x * y = y * x.  [assumption].
10 x * (y * x) = y.  [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u).  [assumption].
13 1 * 2 = 3.  [assumption].
15 1 * 6 = 7.  [assumption].
16 2 * 4 = 9.  [assumption].
17 2 * 8 = 10.  [assumption].
18 3 * 6 = 8.  [assumption].
19 3 * 9 = 10.  [assumption].
20 4 * 7 = 0.  [assumption].
21 5 * 6 = 0.  [assumption].
22 5 * 7 = 10.  [assumption].
23 8 * 9 = 0.  [assumption].
29 8 * 8 != 7.  [deny(6)].
32 x * (x * y) = y.  [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y).  [para(11(a,1),9(a,1)),flip(a)].
34 (x * y) * (z * u) = (z * x) * (y * u).  [para(9(a,1),11(a,
1,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u.  [para(10(a,1),11(a,
1,2)),flip(a)].
```

56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].
106 10 * 2 = 8. [para(17(a,1),32(a,1,2)),rewrite([9(3)])].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
374 (x * 10) * (2 * y) = 8 * (x * y). [para(106(a,1),34(a,1,1)),flip(a)].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
623 10 * 10 = 0 * 1. [para(551(a,1),32(a,1,2))].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].
850 8 * (1 * 10) = 3 * (0 * 1). [para(623(a,1),56(a,2,2)),rewrite([9(6),106(6),9(5)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].
1062 8 * (1 * 10) = 2 * 6. [back_rewrite(850),rewrite([1047(10)])].
1080 7 * 8 = 8. [para(1062(a,1),10(a,1,2)),rewrite([374(7),15(4),9(3)])].
1101 8 * 8 = 7. [para(1080(a,1),10(a,1,2))].
1102 $F. [resolve(1101,a,29,a)].
============================== end of proof =================

P_{15}

================ PROOF =================================
4 6 * 6 = 10 # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
13 1 * 2 = 3. [assumption].
15 1 * 6 = 7. [assumption].
16 2 * 4 = 9. [assumption].
17 2 * 8 = 10. [assumption].
18 3 * 6 = 8. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
21 5 * 6 = 0. [assumption].
22 5 * 7 = 10. [assumption].
23 8 * 9 = 0. [assumption].
27 6 * 6 != 10. [deny(4)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y). [para(11(a,1),9(a,1)),flip(a)].
34 (x * y) * (z * u) = (z * x) * (y * u). [para(9(a,1),11(a,1,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].
106 10 * 2 = 8. [para(17(a,1),32(a,1,2)),rewrite([9(3)])].
107 3 * 8 = 6. [para(18(a,1),32(a,1,2))].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
374 (x * 10) * (2 * y) = 8 * (x * y). [para(106(a,1),34(a,1,1)),flip(a)].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
623 10 * 10 = 0 * 1. [para(551(a,1),32(a,1,2))].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].
850 8 * (1 * 10) = 3 * (0 * 1). [para(623(a,1),56(a,2,2)),rewrite([9(6),106(6),9(5)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].
1062 8 * (1 * 10) = 2 * 6. [back_rewrite(850),rewrite([1047(10)])].
1080 7 * 8 = 8. [para(1062(a,1),10(a,1,2)),rewrite([374(7),15(4),9(3)])].
1113 10 * 6 = 6. [para(1080(a,1),56(a,2,2)),rewrite([104(3),17(4),9(3),107(6)])].
1434 6 * 6 = 10. [para(1113(a,1),10(a,1,2))].
1435 $F. [resolve(1434,a,27,a)].
============================== end of proof ==========================

P_{16}

============== PROOF ================================
2 1 * 1 = 10 # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
12 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
13 1 * 2 = 3. [assumption].

14 1 * 4 = 5. [assumption].
15 1 * 6 = 7. [assumption].
16 2 * 4 = 9. [assumption].
17 2 * 8 = 10. [assumption].
18 3 * 6 = 8. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
21 5 * 6 = 0. [assumption].
22 5 * 7 = 10. [assumption].
23 8 * 9 = 0. [assumption].
25 1 * 1 != 10. [deny(2)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y). [para(11(a,1),9(a,1)),flip(a)].
34 (x * y) * (z * u) = (z * x) * (y * u). [para(9(a,1),11(a,1,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
45 x * (y * (z * (u * x))) = z * (y * u). [para(10(a,1),12(a,1,2,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
63 6 * 7 = 1. [para(15(a,1),10(a,1,2))].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
95 0 * 9 = 8. [para(23(a,1),10(a,1,2)),rewrite([9(3)])].
103 1 * 5 = 4. [para(14(a,1),32(a,1,2))].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].
105 2 * 9 = 4. [para(16(a,1),32(a,1,2))].
106 10 * 2 = 8. [para(17(a,1),32(a,1,2)),rewrite([9(3)])].
107 3 * 8 = 6. [para(18(a,1),32(a,1,2))].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
110 0 * 5 = 6. [para(21(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
349 (x * 1) * (y * 5) = 4 * (x * y). [para(103(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
374 (x * 10) * (2 * y) = 8 * (x * y). [para(106(a,1),34(a,1,1)),flip(a)].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
539 4 * (0 * 1) = 6. [para(95(a,1),56(a,2,2)),rewrite([9(3),105(6),9(5),107(8)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
548 4 * (0 * 2) = 8. [para(110(a,1),56(a,2,2)),rewrite([9(3),349(7),18(8)])].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
623 10 * 10 = 0 * 1. [para(551(a,1),32(a,1,2))].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
788 3 * (x * 2) = 0 * (x * 7). [para(563(a,1),38(a,1,2)),rewrite([9(4),9(8)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].
850 8 * (1 * 10) = 3 * (0 * 1). [para(623(a,1),56(a,2,2)),rewrite([9(6),106(6),9(5)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1036 4 * (x * 0) = 1 * (x * 6). [para(539(a,1),45(a,1,2,2)),flip(a)].
1037 2 * (x * 8) = 1 * (x * 6). [para(548(a,1),45(a,1,2,2)),rewrite([1036(8)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].
1062 8 * (1 * 10) = 2 * 6. [back_rewrite(850),rewrite([1047(10)])].
1080 7 * 8 = 8. [para(1062(a,1),10(a,1,2)),rewrite([374(7),15(4),9(3)])].
1101 8 * 8 = 7. [para(1080(a,1),10(a,1,2))].
1429 10 * (1 * 8) = 0 * 2. [para(1101(a,1),56(a,2,2)),rewrite([17(6),9(5),779(8)])].
1472 6 * (1 * 10) = 7. [para(1429(a,1),56(a,2,2)),rewrite([1037(8),15(7),104(6),9(5),788(10),83(9),109(8)])].
1811 1 * 10 = 1. [para(1472(a,1),32(a,1,2)),rewrite([63(3)]),flip(a)].
1887 1 * 1 = 10. [para(1811(a,1),32(a,1,2))].
1888 $F. [resolve(1887,a,25,a)].
============================== end of proof ==========================

P_{17}

============== PROOF =================================
8 10 * 10 = (10 * 10) * (10 * 10) # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
12 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
13 1 * 2 = 3. [assumption].
14 1 * 4 = 5. [assumption].
15 1 * 6 = 7. [assumption].
16 2 * 4 = 9. [assumption].
17 2 * 8 = 10. [assumption].
18 3 * 6 = 8. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
21 5 * 6 = 0. [assumption].
22 5 * 7 = 10. [assumption].
23 8 * 9 = 0. [assumption].
31 (10 * 10) * (10 * 10) != 10 * 10. [deny(8)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y). [para(11(a,1),9(a,1)),flip(a)].
34 (x * y) * (z * u) = (z * x) * (y * u). [para(9(a,1),11(a,1,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
45 x * (y * (z * (u * x))) = z * (y * u). [para(10(a,1),12(a,1,2,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
63 6 * 7 = 1. [para(15(a,1),10(a,1,2))].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
95 0 * 9 = 8. [para(23(a,1),10(a,1,2)),rewrite([9(3)])].
103 1 * 5 = 4. [para(14(a,1),32(a,1,2))].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].

105 2 * 9 = 4. [para(16(a,1),32(a,1,2))].
106 10 * 2 = 8. [para(17(a,1),32(a,1,2)),rewrite([9(3)])].
107 3 * 8 = 6. [para(18(a,1),32(a,1,2))].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
110 0 * 5 = 6. [para(21(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
156 (x * 8) * (y * 2) = 10 * (y * x). [para(17(a,1),33(a,2,2)),rewrite([9(8)])].
349 (x * 1) * (y * 5) = 4 * (x * y). [para(103(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
374 (x * 10) * (2 * y) = 8 * (x * y). [para(106(a,1),34(a,1,1)),flip(a)].
521 7 * (2 * x) = 3 * (6 * x). [para(15(a,1),56(a,1,1))].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
539 4 * (0 * 1) = 6. [para(95(a,1),56(a,2,2)),rewrite([9(3),105(6),9(5),107(8)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
548 4 * (0 * 2) = 8. [para(110(a,1),56(a,2,2)),rewrite([9(3),349(7),18(8)])].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
623 10 * 10 = 0 * 1. [para(551(a,1),32(a,1,2))].
630 (0 * 1) * (0 * 1) != 0 * 1. [back_rewrite(31),rewrite([623(3),623(6),623(10)])].
773 (0 * 0) * (1 * 1) != 0 * 1. [para(11(a,1),630(a,1))].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
788 3 * (x * 2) = 0 * (x * 7). [para(563(a,1),38(a,1,2)),rewrite([9(4),9(8)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].
850 8 * (1 * 10) = 3 * (0 * 1). [para(623(a,1),56(a,2,2)),rewrite([9(6),106(6),9(5)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1036 4 * (x * 0) = 1 * (x * 6). [para(539(a,1),45(a,1,2,2)),flip(a)].
1037 2 * (x * 8) = 1 * (x * 6). [para(548(a,1),45(a,1,2,2)),rewrite([1036(8)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].
1062 8 * (1 * 10) = 2 * 6. [back_rewrite(850),rewrite([1047(10)])].
1072 7 * (x * 2) = 0 * (x * 3). [para(1047(a,1),38(a,1,2)),rewrite([65(6),9(8)])].
1080 7 * 8 = 8. [para(1062(a,1),10(a,1,2)),rewrite([374(7),15(4),9(3)])].
1101 8 * 8 = 7. [para(1080(a,1),10(a,1,2))].
1429 10 * (1 * 8) = 0 * 2. [para(1101(a,1),56(a,2,2)),rewrite([17(6),9(5),779(8)])].
1471 10 * (0 * x) = 1 * (x * 10). [para(1429(a,1),38(a,1,2)),rewrite([156(6),9(8)])].
1472 6 * (1 * 10) = 7. [para(1429(a,1),56(a,2,2)),rewrite([1037(8),15(7),104(6),9(5),788(10),83(9),109(8)])].
1811 1 * 10 = 1. [para(1472(a,1),32(a,1,2)),rewrite([63(3)]),flip(a)].
1887 1 * 1 = 10. [para(1811(a,1),32(a,1,2))].
1903 1 * (0 * 10) != 0 * 1. [back_rewrite(773),rewrite([1887(6),9(5),1471(5)])].
2570 3 * (6 * x) = 0 * (x * 3). [para(9(a,1),521(a,1,2)),rewrite([1072(4)]),flip(a)].
2586 0 * 10 = 0. [para(105(a,1),521(a,1,2)),rewrite([9(3),20(3),2570(6),9(5),19(5)]),flip(a)].
2625 $F. [back_rewrite(1903),rewrite([2586(4),9(3)]),xx(a)].
============================== end of proof ==========================

P_{18}

============== PROOF =================================
1 0 * 0 = 10 # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
13 1 * 2 = 3. [assumption].
15 1 * 6 = 7. [assumption].
16 2 * 4 = 9. [assumption].
17 2 * 8 = 10. [assumption].
18 3 * 6 = 8. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
21 5 * 6 = 0. [assumption].
22 5 * 7 = 10. [assumption].
24 0 * 0 != 10. [deny(1)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y). [para(11(a,1),9(a,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].
105 2 * 9 = 4. [para(16(a,1),32(a,1,2))].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
521 7 * (2 * x) = 3 * (6 * x). [para(15(a,1),56(a,1,1))].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].

1072 7 * (x * 2) = 0 * (x * 3). [para(1047(a,1),38(a,1,2)),rewrite([65(6),9(8)])].
2570 3 * (6 * x) = 0 * (x * 3). [para(9(a,1),521(a,1,2)),rewrite([1072(4)]),flip(a)].
2586 0 * 10 = 0. [para(105(a,1),521(a,1,2)),rewrite([9(3),20(3),2570(6),9(5),19(5)]),flip(a)].
2634 0 * 0 = 10. [para(2586(a,1),32(a,1,2))].
2635 $F. [resolve(2634,a,24,a)].

============================== end of proof ==========================

P19

================ PROOF =================================

3 4 * 4 = 10 # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
13 1 * 2 = 3. [assumption].
14 1 * 4 = 5. [assumption].
17 2 * 8 = 10. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
22 5 * 7 = 10. [assumption].
23 8 * 9 = 0. [assumption].
26 4 * 4 != 10. [deny(3)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
103 1 * 5 = 4. [para(14(a,1),32(a,1,2))].
111 10 * 5 = 7. [para(22(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
523 3 * (x * 8) = 10 * (1 * x). [para(17(a,1),56(a,1,2)),rewrite([9(4)]),flip(a)].
541 4 * (2 * x) = 3 * (5 * x). [para(103(a,1),56(a,1,1))].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
629 10 * (x * 1) = 0 * (x * 10). [para(551(a,1),38(a,1,2)),rewrite([9(4),9(8)])].
3397 10 * (1 * x) = 0 * (x * 10). [para(9(a,1),523(a,2,2)),rewrite([523(4),629(8)])].
3453 3 * (x * 8) = 0 * (x * 10). [back_rewrite(523),rewrite([3397(8)])].
3467 10 * 4 = 4. [para(17(a,1),541(a,1,2)),rewrite([9(3),3453(8),9(7),111(7),83(6)])].
3514 4 * 4 = 10. [para(3467(a,1),10(a,1,2))].
3515 $F. [resolve(3514,a,26,a)].
============================== end of proof ==========================

P20

=============== PROOF

5 2 * 2 = 7 # label(non_clause) # label(goal). [goal].
9 x * y = y * x. [assumption].
10 x * (y * x) = y. [assumption].
11 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
12 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
13 1 * 2 = 3. [assumption].
14 1 * 4 = 5. [assumption].
15 1 * 6 = 7. [assumption].
16 2 * 4 = 9. [assumption].
17 2 * 8 = 10. [assumption].
18 3 * 6 = 8. [assumption].
19 3 * 9 = 10. [assumption].
20 4 * 7 = 0. [assumption].
21 5 * 6 = 0. [assumption].
22 5 * 7 = 10. [assumption].
23 8 * 9 = 0. [assumption].
28 2 * 2 != 7. [deny(5)].
32 x * (x * y) = y. [para(9(a,1),10(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (u * y). [para(11(a,1),9(a,1)),flip(a)].
34 (x * y) * (z * u) = (z * x) * (y * u). [para(9(a,1),11(a,1,1)),flip(a)].
38 (x * y) * (z * (u * y)) = (x * z) * u. [para(10(a,1),11(a,1,2)),flip(a)].
45 x * (y * (z * (u * x))) = z * (y * u). [para(10(a,1),12(a,1,2,2)),flip(a)].
56 (1 * x) * (2 * y) = 3 * (x * y). [para(13(a,1),11(a,1,1)),flip(a)].
57 (x * 1) * (y * 2) = 3 * (x * y). [para(13(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
63 6 * 7 = 1. [para(15(a,1),10(a,1,2))].
65 (x * 1) * (y * 6) = 7 * (x * y). [para(15(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
75 6 * 8 = 3. [para(18(a,1),10(a,1,2))].
83 0 * 7 = 4. [para(20(a,1),10(a,1,2)),rewrite([9(3)])].
87 0 * 6 = 5. [para(21(a,1),10(a,1,2)),rewrite([9(3)])].
91 10 * 7 = 5. [para(22(a,1),10(a,1,2)),rewrite([9(3)])].
95 0 * 9 = 8. [para(23(a,1),10(a,1,2)),rewrite([9(3)])].
103 1 * 5 = 4. [para(14(a,1),32(a,1,2))].
104 1 * 7 = 6. [para(15(a,1),32(a,1,2))].
105 2 * 9 = 4. [para(16(a,1),32(a,1,2))].
106 10 * 2 = 8. [para(17(a,1),32(a,1,2)),rewrite([9(3)])].
107 3 * 8 = 6. [para(18(a,1),32(a,1,2))].
109 0 * 4 = 7. [para(20(a,1),32(a,1,2)),rewrite([9(3)])].
110 0 * 5 = 6. [para(21(a,1),32(a,1,2)),rewrite([9(3)])].
111 10 * 5 = 7. [para(22(a,1),32(a,1,2)),rewrite([9(3)])].
112 0 * 8 = 9. [para(23(a,1),32(a,1,2)),rewrite([9(3)])].
154 (x * 4) * (y * 2) = 9 * (y * x). [para(16(a,1),33(a,2,2)),rewrite([9(8)])].
349 (x * 1) * (y * 5) = 4 * (x * y). [para(103(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
356 (x * 1) * (y * 7) = 6 * (x * y). [para(104(a,1),11(a,1,2)),rewrite([9(3)]),flip(a)].
374 (x * 10) * (2 * y) = 8 * (x * y). [para(106(a,1),34(a,1,1)),flip(a)].
530 3 * 3 = 5. [para(75(a,1),56(a,2,2)),rewrite([15(3),17(4),9(3),91(3)]),flip(a)].
533 6 * (0 * 2) = 3 * 4. [para(83(a,1),56(a,2,2)),rewrite([9(3),356(7)])].
538 7 * (0 * 2) = 3 * 5. [para(87(a,1),56(a,2,2)),rewrite([9(3),65(7)])].
539 4 * (0 * 1) = 6. [para(95(a,1),56(a,2,2)),rewrite([9(3),105(6),9(5),107(8)])].
547 9 * (0 * 1) = 3 * 7. [para(109(a,1),56(a,2,2)),rewrite([9(3),16(6),9(5)])].
548 4 * (0 * 2) = 8. [para(110(a,1),56(a,2,2)),rewrite([9(3),349(7),18(8)])].
551 10 * (0 * 1) = 10. [para(112(a,1),56(a,2,2)),rewrite([9(3),17(6),9(5),19(8)])].
553 3 * 5 = 3. [para(530(a,1),10(a,1,2))].
563 7 * (0 * 2) = 3. [back_rewrite(538),rewrite([553(8)])].
623 10 * 10 = 0 * 1. [para(551(a,1),32(a,1,2))].
779 3 * 7 = 0 * 2. [para(563(a,1),32(a,1,2)),rewrite([9(3)])].
788 3 * (x * 2) = 0 * (x * 7). [para(563(a,1),38(a,1,2)),rewrite([9(4),9(8)])].
792 9 * (0 * 1) = 0 * 2. [back_rewrite(547),rewrite([779(8)])].

850 8 * (1 * 10) = 3 * (0 * 1). [para(623(a,1),56(a,2,2)),rewrite([9(6),106(6),9(5)])].
978 6 * (3 * 4) = 0 * 2. [para(533(a,1),32(a,1,2))].
996 9 * (0 * 2) = 0 * 1. [para(792(a,1),32(a,1,2))].
1022 9 * (0 * x) = 3 * (x * 6). [para(978(a,1),38(a,1,2)),rewrite([154(6),9(8)])].
1028 3 * (2 * 6) = 0 * 1. [back_rewrite(996),rewrite([1022(5)])].
1036 4 * (x * 0) = 1 * (x * 6). [para(539(a,1),45(a,1,2,2)),flip(a)].
1037 2 * (x * 8) = 1 * (x * 6). [para(548(a,1),45(a,1,2,2)),rewrite([1036(8)])].
1047 3 * (0 * 1) = 2 * 6. [para(1028(a,1),32(a,1,2))].
1062 8 * (1 * 10) = 2 * 6. [back_rewrite(850),rewrite([1047(10)])].
1080 7 * 8 = 8. [para(1062(a,1),10(a,1,2)),rewrite([374(7),15(4),9(3)])].
1101 8 * 8 = 7. [para(1080(a,1),10(a,1,2))].
1429 10 * (1 * 8) = 0 * 2. [para(1101(a,1),56(a,2,2)),rewrite([17(6),9(5),779(8)])].
1472 6 * (1 * 10) = 7. [para(1429(a,1),56(a,2,2)),rewrite([1037(8),15(7),104(6),9(5),788(10),83(9),109(8)])].
1811 1 * 10 = 1. [para(1472(a,1),32(a,1,2)),rewrite([63(3)]),flip(a)].
1887 1 * 1 = 10. [para(1811(a,1),32(a,1,2))].
3782 10 * (2 * 2) = 5. [para(13(a,1),57(a,2,2)),rewrite([1887(3),530(8)])].
3822 2 * 2 = 7. [para(3782(a,1),32(a,1,2)),rewrite([111(3)]),flip(a)].
3823 $F. [resolve(3822,a,28,a)].
============================== end of proof ==========================

Tangential relations from §7.5

P_{21}

============================== PROOF

% -------- Comments from original proof --------
% Proof 1 at 0.06 (+ 0.01) seconds.
% Length of proof is 25.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 50.

1 2 * 2 = 7 # label(non_clause) # label(goal). [goal].
3 x * y = y * x. [assumption].
4 x * (y * x) = y. [assumption].
5 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
7 1 * 4 = 5. [assumption].
8 1 * 6 = 7. [assumption].
9 1 * 0 = 10. [assumption].
10 0 * 1 = 10. [copy(9),rewrite([3(3)])].
12 2 * 4 = 9. [assumption].
13 2 * 8 = 0. [assumption].
16 4 * 7 = 8. [assumption].
17 5 * 6 = 9. [assumption].
18 5 * 7 = 10. [assumption].
19 2 * 2 != 7. [deny(1)].
21 x * (x * y) = y. [para(3(a,1),4(a,1,2))].
45 (1 * x) * (4 * y) = 5 * (x * y). [para(7(a,1),5(a,1,1)),flip(a)].
52 1 * 10 = 0. [para(10(a,1),4(a,1,2))].
60 4 * 9 = 2. [para(12(a,1),4(a,1,2))].
64 0 * 8 = 2. [para(13(a,1),4(a,1,2)),rewrite([3(3)])].
80 6 * 9 = 5. [para(17(a,1),4(a,1,2))].
84 10 * 7 = 5. [para(18(a,1),4(a,1,2)),rewrite([3(3)])].
526 5 * 5 = 2 * 7. [para(80(a,1),45(a,2,2)),rewrite([8(3),60(4),3(3)]),flip(a)].
527 2 * 7 = 2. [para(84(a,1),45(a,2,2)),rewrite([52(3),16(4),64(3),526(4)]),flip(a)].
584 2 * 2 = 7. [para(527(a,1),21(a,1,2))].
585 $F. [resolve(584,a,19,a)].

============================== end of proof ==========================

P_{22}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 2 at 0.62 (+ 0.02) seconds.
% Length of proof is 37.
% Level of proof is 9.
% Maximum clause weight is 15.
% Given clauses 103.

2 10 * 10 = 10 # label(non_clause) # label(goal). [goal].
3 x * y = y * x. [assumption].
4 x * (y * x) = y. [assumption].
5 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
7 1 * 4 = 5. [assumption].
8 1 * 6 = 7. [assumption].
9 1 * 0 = 10. [assumption].
10 0 * 1 = 10. [copy(9),rewrite([3(3)])].
11 2 * 3 = 10. [assumption].
13 2 * 8 = 0. [assumption].
15 3 * 9 = 0. [assumption].
16 4 * 7 = 8. [assumption].
17 5 * 6 = 9. [assumption].
18 5 * 7 = 10. [assumption].
20 10 * 10 != 10. [deny(2)].
21 x * (x * y) = y. [para(3(a,1),4(a,1,2))].
45 (1 * x) * (4 * y) = 5 * (x * y). [para(7(a,1),5(a,1,1)),flip(a)].
46 (x * 1) * (y * 4) = 5 * (x * y). [para(7(a,1),5(a,1,2)),rewrite([3(3)]),flip(a)].
52 1 * 10 = 0. [para(10(a,1),4(a,1,2))].
54 (x * 0) * (y * 1) = 10 * (x * y). [para(10(a,1),5(a,1,2)),rewrite([3(3)]),flip(a)].
56 10 * 3 = 2. [para(11(a,1),4(a,1,2)),rewrite([3(3)])].
64 0 * 8 = 2. [para(13(a,1),4(a,1,2)),rewrite([3(3)])].
84 10 * 7 = 5. [para(18(a,1),4(a,1,2)),rewrite([3(3)])].
98 0 * 3 = 9. [para(15(a,1),21(a,1,2)),rewrite([3(3)])].
99 4 * 8 = 7. [para(16(a,1),21(a,1,2))].
100 5 * 9 = 6. [para(17(a,1),21(a,1,2))].
515 5 * (10 * x) = 0 * (4 * x). [para(52(a,1),45(a,1,1)),flip(a)].
516 0 * (3 * 4) = 2 * 5. [para(56(a,1),45(a,2,2)),rewrite([52(3),3(4),3(8)])].
519 2 * 5 = 5. [para(64(a,1),45(a,2,2)),rewrite([3(3),10(3),99(4),84(3),3(4)]),flip(a)].
535 10 * (3 * 4) = 6. [para(98(a,1),45(a,2,2)),rewrite([3(3),10(3),3(4),100(8)])].
539 0 * (3 * 4) = 5. [back_rewrite(516),rewrite([519(8)])].
912 3 * 4 = 10 * 6. [para(535(a,1),21(a,1,2)),flip(a)].
917 0 * (10 * 6) = 5. [back_rewrite(539),rewrite([912(4)])].
1022 0 * (4 * 6) = 0. [para(917(a,1),4(a,1,2)),rewrite([3(5),515(5)])].
1387 4 * 6 = 0 * 0. [para(1022(a,1),21(a,1,2)),flip(a)].
3819 10 * 10 = 10. [para(8(a,1),46(a,2,2)),rewrite([3(6),1387(6),3(7),54(7),10(4),18(6)])].
3820 $F. [resolve(3819,a,20,a)].

============================== end of proof ==========================

Tangential relations from §7.6

P_{23}

============================== PROOF
3 6 * 6 = 6 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
17 3 * 4 = 8. [assumption].
18 3 * 7 = 0. [assumption].
20 5 * 6 = 9. [assumption].
26 6 * 6 != 6. [deny(3)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
74 4 * 8 = 3. [para(17(a,1),9(a,1,2))].
78 0 * 7 = 3. [para(18(a,1),9(a,1,2)),rewrite([8(3)])].
103 1 * 7 = 6. [para(14(a,1),31(a,1,2))].
104 0 * 2 = 6. [para(15(a,1),31(a,1,2)),rewrite([8(3)])].
109 5 * 9 = 6. [para(20(a,1),31(a,1,2))].
355 (x * 1) * (y * 7) = 6 * (x * y). [para(103(a,1),10(a,1,2)),rewrite([8(3)]),flip(a)].
529 3 * 3 = 6. [para(74(a,1),55(a,2,2)),rewrite([13(3),16(4),109(3)]),flip(a)].
530 6 * 6 = 6. [para(78(a,1),55(a,2,2)),rewrite([8(3),355(7),104(4),529(6)])].
531 $F. [resolve(530,a,26,a)].
============================== end of proof ==========================

P_{24}

============================== PROOF =================================
1 0 * 0 = 0 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
18 3 * 7 = 0. [assumption].
19 4 * 7 = 10. [assumption].
20 5 * 6 = 9. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
24 0 * 0 != 0. [deny(1)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
58 4 * 5 = 1. [para(13(a,1),9(a,1,2))].
82 10 * 7 = 4. [para(19(a,1),9(a,1,2)),rewrite([8(3)])].
86 6 * 9 = 5. [para(20(a,1),9(a,1,2))].
105 2 * 9 = 8. [para(16(a,1),31(a,1,2))].
520 0 * (1 * 1) = 0. [para(14(a,1),55(a,2,2)),rewrite([15(6),8(5),18(8)])].
525 8 * (0 * 1) = 10 * 3. [para(23(a,1),55(a,2,2)),rewrite([8(3),105(6),8(5),8(8)])].
536 7 * 8 = 3 * 5. [para(86(a,1),55(a,2,2)),rewrite([14(3),105(4)])].
593 1 * 1 = 0 * 0. [para(520(a,1),31(a,1,2)),flip(a)].
780 8 * (3 * 5) = 7. [para(536(a,1),9(a,1,2))].
971 8 * (10 * 3) = 0 * 1. [para(525(a,1),31(a,1,2))].
1014 8 * (x * 3) = 5 * (x * 7). [para(780(a,1),44(a,1,2,2)),flip(a)].
1018 0 * 1 = 1. [back_rewrite(971),rewrite([1014(5),82(4),8(3),58(3)]),flip(a)].
1071 0 * 0 = 0. [para(1018(a,1),9(a,1,2)),rewrite([593(3)])].
1072 $F. [resolve(1071,a,24,a)].
============================== end of proof ==========================

P_{25}

============================== PROOF =================================
5 8 * 8 = 8 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
16 2 * 8 = 9. [assumption].
17 3 * 4 = 8. [assumption].
19 4 * 7 = 10. [assumption].
20 5 * 6 = 9. [assumption].
21 5 * 8 = 10. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
28 8 * 8 != 8. [deny(5)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
58 4 * 5 = 1. [para(13(a,1),9(a,1,2))].
82 10 * 7 = 4. [para(19(a,1),9(a,1,2)),rewrite([8(3)])].
86 6 * 9 = 5. [para(20(a,1),9(a,1,2))].
102 1 * 5 = 4. [para(13(a,1),31(a,1,2))].
105 2 * 9 = 8. [para(16(a,1),31(a,1,2))].
522 9 * (1 * x) = 3 * (x * 8). [para(16(a,1),55(a,1,2)),rewrite([8(4)])].
524 4 * 9 = 10 * 3. [para(21(a,1),55(a,2,2)),rewrite([102(3),16(4),8(6)])].
525 8 * (0 * 1) = 10 * 3. [para(23(a,1),55(a,2,2)),rewrite([8(3),105(6),8(5),8(8)])].
536 7 * 8 = 3 * 5. [para(86(a,1),55(a,2,2)),rewrite([14(3),105(4)])].
756 9 * (10 * 3) = 4. [para(524(a,1),9(a,1,2))].
780 8 * (3 * 5) = 7. [para(536(a,1),9(a,1,2))].
971 8 * (10 * 3) = 0 * 1. [para(525(a,1),31(a,1,2))].
1014 8 * (x * 3) = 5 * (x * 7). [para(780(a,1),44(a,1,2,2)),flip(a)].
1018 0 * 1 = 1. [back_rewrite(971),rewrite([1014(5),82(4),8(3),58(3)]),flip(a)].
1031 10 * 3 = 1 * 8. [back_rewrite(525),rewrite([1018(4),8(3)]),flip(a)].
1069 3 * (8 * 8) = 4. [back_rewrite(756),rewrite([1031(4),522(5)])].
1432 8 * 8 = 8. [para(1069(a,1),31(a,1,2)),rewrite([17(3)]),flip(a)].
1433 $F. [resolve(1432,a,28,a)].
============================== end of proof ==========================

P_{26}

============================== PROOF =================================
2 2 * 2 = 2 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].

9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
17 3 * 4 = 8. [assumption].
18 3 * 7 = 0. [assumption].
19 4 * 7 = 10. [assumption].
20 5 * 6 = 9. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
25 2 * 2 != 2. [deny(2)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
56 (x * 1) * (y * 2) = 3 * (x * y). [para(12(a,1),10(a,1,2)),rewrite([8(3)]),flip(a)].
58 4 * 5 = 1. [para(13(a,1),9(a,1,2))].
66 0 * 6 = 2. [para(15(a,1),9(a,1,2)),rewrite([8(3)])].
74 4 * 8 = 3. [para(17(a,1),9(a,1,2))].
82 10 * 7 = 4. [para(19(a,1),9(a,1,2)),rewrite([8(3)])].
86 6 * 9 = 5. [para(20(a,1),9(a,1,2))].
105 2 * 9 = 8. [para(16(a,1),31(a,1,2))].
109 5 * 9 = 6. [para(20(a,1),31(a,1,2))].
520 0 * (1 * 1) = 0. [para(14(a,1),55(a,2,2)),rewrite([15(6),8(5),18(8)])].
525 8 * (0 * 1) = 10 * 3. [para(23(a,1),55(a,2,2)),rewrite([8(3),105(6),8(5),8(8)])].
529 3 * 3 = 6. [para(74(a,1),55(a,2,2)),rewrite([13(3),16(4),109(3)]),flip(a)].
536 7 * 8 = 3 * 5. [para(86(a,1),55(a,2,2)),rewrite([14(3),105(4)])].
593 1 * 1 = 0 * 0. [para(520(a,1),31(a,1,2)),flip(a)].
780 8 * (3 * 5) = 7. [para(536(a,1),9(a,1,2))].
971 8 * (10 * 3) = 0 * 1. [para(525(a,1),31(a,1,2))].
1014 8 * (x * 3) = 5 * (x * 7). [para(780(a,1),44(a,1,2,2)),flip(a)].
1018 0 * 1 = 1. [back_rewrite(971),rewrite([1014(5),82(4),8(3),58(3)]),flip(a)].
1071 0 * 0 = 0. [para(1018(a,1),9(a,1,2)),rewrite([593(3)])].
1091 1 * 1 = 0. [back_rewrite(593),rewrite([1071(6)])].
3871 0 * (2 * 2) = 6. [para(12(a,1),56(a,2,2)),rewrite([1091(3),529(8)])].
4031 2 * 2 = 2. [para(3871(a,1),31(a,1,2)),rewrite([66(3)]),flip(a)].
4032 $F. [resolve(4031,a,25,a)].
============================== end of proof ==========================

P_{27}

============================== PROOF =================================
7 10 * 10 = 10 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
17 3 * 4 = 8. [assumption].
18 3 * 7 = 0. [assumption].
19 4 * 7 = 10. [assumption].
20 5 * 6 = 9. [assumption].
21 5 * 8 = 10. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
30 10 * 10 != 10. [deny(7)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (y * u). [para(8(a,1),10(a,1,1)),flip(a)].
41 x * (y * (z * u)) = u * (y * (x * z)). [para(8(a,1),11(a,1,2,2))].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
56 (x * 1) * (y * 2) = 3 * (x * y). [para(12(a,1),10(a,1,2)),rewrite([8(3)]),flip(a)].
57 1 * (x * (y * 2)) = y * (x * 3). [para(12(a,1),11(a,1,2,2)),flip(a)].
58 4 * 5 = 1. [para(13(a,1),9(a,1,2))].
74 4 * 8 = 3. [para(17(a,1),9(a,1,2))].
82 10 * 7 = 4. [para(19(a,1),9(a,1,2)),rewrite([8(3)])].
86 6 * 9 = 5. [para(20(a,1),9(a,1,2))].
102 1 * 5 = 4. [para(13(a,1),31(a,1,2))].
105 2 * 9 = 8. [para(16(a,1),31(a,1,2))].
107 0 * 3 = 7. [para(18(a,1),31(a,1,2)),rewrite([8(3)])].
109 5 * 9 = 6. [para(20(a,1),31(a,1,2))].
245 (x * y) * (z * (x * u)) = (y * z) * u. [para(31(a,1),33(a,1,2)),flip(a)].
520 0 * (1 * 1) = 0. [para(14(a,1),55(a,2,2)),rewrite([15(6),8(5),18(8)])].
525 8 * (0 * 1) = 10 * 3. [para(23(a,1),55(a,2,2)),rewrite([8(3),105(6),8(5),8(8)])].
529 3 * 3 = 6. [para(74(a,1),55(a,2,2)),rewrite([13(3),16(4),109(3)]),flip(a)].
536 7 * 8 = 3 * 5. [para(86(a,1),55(a,2,2)),rewrite([14(3),105(4)])].
539 4 * (2 * x) = 3 * (5 * x). [para(102(a,1),55(a,1,1))].
593 1 * 1 = 0 * 0. [para(520(a,1),31(a,1,2)),flip(a)].
717 9 * (x * (y * 2)) = y * (x * 8). [para(105(a,1),41(a,1,2,2)),flip(a)].
780 8 * (3 * 5) = 7. [para(536(a,1),9(a,1,2))].
971 8 * (10 * 3) = 0 * 1. [para(525(a,1),31(a,1,2))].
1014 8 * (x * 3) = 5 * (x * 7). [para(780(a,1),44(a,1,2,2)),flip(a)].
1018 0 * 1 = 1. [back_rewrite(971),rewrite([1014(5),82(4),8(3),58(3)]),flip(a)].
1031 10 * 3 = 1 * 8. [back_rewrite(525),rewrite([1018(4),8(3)]),flip(a)].
1071 0 * 0 = 0. [para(1018(a,1),9(a,1,2)),rewrite([593(3)])].
1091 1 * 1 = 0. [back_rewrite(593),rewrite([1071(6)])].
1105 10 * (1 * 8) = 3. [para(1031(a,1),31(a,1,2))].
1466 8 * (10 * 2) = 6. [para(1105(a,1),55(a,2,2)),rewrite([245(9),8(5),529(8)])].
1799 6 * 8 = 10 * 2. [para(1466(a,1),31(a,1,2)),rewrite([8(3)])].
1837 3 * (10 * 2) = 7 * 9. [para(1799(a,1),55(a,2,2)),rewrite([14(3),16(4)]),flip(a)].
1854 3 * (7 * 9) = 10 * 2. [para(1837(a,1),31(a,1,2))].
1892 3 * (x * 7) = 10 * (x * 8). [para(1854(a,1),44(a,1,2,2)),rewrite([717(6)]),flip(a)].
3871 0 * (2 * 2) = 6. [para(12(a,1),56(a,2,2)),rewrite([1091(3),529(8)])].
4038 2 * 7 = 7. [para(3871(a,1),57(a,1,2)),rewrite([14(3),107(5)]),flip(a)].
4081 10 * 10 = 10. [para(4038(a,1),539(a,1,2)),rewrite([19(3),1892(6),21(5)]),flip(a)].
4082 $F. [resolve(4081,a,30,a)].
============================== end of proof ==========================

P_{28}

============================== PROOF =================================

4 5 * 5 = 5 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
14 1 * 6 = 7. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
17 3 * 4 = 8. [assumption].
18 3 * 7 = 0. [assumption].
19 4 * 7 = 10. [assumption].
20 5 * 6 = 9. [assumption].
21 5 * 8 = 10. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
27 5 * 5 != 5. [deny(4)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (y * u). [para(8(a,1),10(a,1,1)),flip(a)].
37 (x * y) * (z * (u * y)) = (x * z) * u. [para(9(a,1),10(a,1,2)),flip(a)].
41 x * (y * (z * u)) = u * (y * (x * z)). [para(8(a,1),11(a,1,2,2))].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
58 4 * 5 = 1. [para(13(a,1),9(a,1,2))].
59 (1 * x) * (4 * y) = 5 * (x * y). [para(13(a,1),10(a,1,1)),flip(a)].
74 4 * 8 = 3. [para(17(a,1),9(a,1,2))].
82 10 * 7 = 4. [para(19(a,1),9(a,1,2)),rewrite([8(3)])].
86 6 * 9 = 5. [para(20(a,1),9(a,1,2))].
102 1 * 5 = 4. [para(13(a,1),31(a,1,2))].
105 2 * 9 = 8. [para(16(a,1),31(a,1,2))].
109 5 * 9 = 6. [para(20(a,1),31(a,1,2))].
245 (x * y) * (z * (x * u)) = (y * z) * u. [para(31(a,1),33(a,1,2)),flip(a)].
370 2 * (x * (y * 9)) = y * (x * 8). [para(105(a,1),11(a,1,2,2)),flip(a)].
520 0 * (1 * 1) = 0. [para(14(a,1),55(a,2,2)),rewrite([15(6),8(5),18(8)])].
523 3 * 9 = 0 * 4. [para(20(a,1),55(a,2,2)),rewrite([102(3),15(4),8(3)]),flip(a)].
524 4 * 9 = 10 * 3. [para(21(a,1),55(a,2,2)),rewrite([102(3),16(4),8(6)])].
525 8 * (0 * 1) = 10 * 3. [para(23(a,1),55(a,2,2)),rewrite([8(3),105(6),8(5),8(8)])].
529 3 * 3 = 6. [para(74(a,1),55(a,2,2)),rewrite([13(3),16(4),109(3)]),flip(a)].
536 7 * 8 = 3 * 5. [para(86(a,1),55(a,2,2)),rewrite([14(3),105(4)])].
594 1 * (x * 0) = 0 * (x * 1). [para(520(a,1),37(a,1,2)),rewrite([8(4),8(8)]),flip(a)].
595 9 * (0 * 4) = 3. [para(523(a,1),9(a,1,2))].
646 0 * (x * (9 * y)) = y * (x * 10). [para(23(a,1),41(a,2,2,2))].
780 8 * (3 * 5) = 7. [para(536(a,1),9(a,1,2))].
832 3 * (x * 4) = 0 * (x * 9). [para(595(a,1),37(a,1,2)),rewrite([8(4),8(8)])].
971 8 * (10 * 3) = 0 * 1. [para(525(a,1),31(a,1,2))].
1014 8 * (x * 3) = 5 * (x * 7). [para(780(a,1),44(a,1,2,2)),flip(a)].
1018 0 * 1 = 1. [back_rewrite(971),rewrite([1014(5),82(4),8(3),58(3)]),flip(a)].
1031 10 * 3 = 1 * 8. [back_rewrite(525),rewrite([1018(4),8(3)]),flip(a)].
1070 4 * 9 = 1 * 8. [back_rewrite(524),rewrite([1031(6)])].
1105 10 * (1 * 8) = 3. [para(1031(a,1),31(a,1,2))].
1466 8 * (10 * 2) = 6. [para(1105(a,1),55(a,2,2)),rewrite([245(9),8(5),529(8)])].
1799 6 * 8 = 10 * 2. [para(1466(a,1),31(a,1,2)),rewrite([8(3)])].
1837 3 * (10 * 2) = 7 * 9. [para(1799(a,1),55(a,2,2)),rewrite([14(3),16(4)]),flip(a)].
1868 7 * (x * 8) = 3 * (x * 10). [para(1837(a,1),44(a,1,2,2)),rewrite([370(6)])].
2650 1 * (0 * x) = 0 * (x * 1). [para(8(a,1),594(a,1,2))].
3039 0 * (1 * 9) = 3 * 5. [para(13(a,1),832(a,1,2)),flip(a)].
3088 3 * 5 = 1 * 10. [para(3039(a,1),44(a,2)),rewrite([2650(6),8(5),646(6),31(5)]),flip(a)].
3457 5 * (1 * 10) = 3. [para(3088(a,1),9(a,1,2))].
3517 3 * (x * 10) = 1 * (x * 5). [para(3457(a,1),37(a,1,2)),rewrite([8(4),8(8)])].
3535 7 * (x * 8) = 1 * (x * 5). [back_rewrite(1868),rewrite([3517(8)])].
4108 5 * 5 = 5. [para(86(a,1),59(a,2,2)),rewrite([14(3),1070(4),3535(5),102(4),13(3)]),flip(a)].
4109 $F. [resolve(4108,a,27,a)].

============================== end of proof ==========================

P_{29}

============================== PROOF =================================

6 9 * 9 = 9 # label(non_clause) # label(goal). [goal].
8 x * y = y * x. [assumption].
9 x * (y * x) = y. [assumption].
10 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
11 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
12 1 * 2 = 3. [assumption].
13 1 * 4 = 5. [assumption].
15 2 * 6 = 0. [assumption].
16 2 * 8 = 9. [assumption].
20 5 * 6 = 9. [assumption].
21 5 * 8 = 10. [assumption].
22 9 * 0 = 10. [assumption].
23 0 * 9 = 10. [copy(22),rewrite([8(3)])].
29 9 * 9 != 9. [deny(6)].
31 x * (x * y) = y. [para(8(a,1),9(a,1,2))].
33 (x * y) * (z * u) = (z * x) * (y * u). [para(8(a,1),10(a,1,1)),flip(a)].
37 (x * y) * (z * (u * y)) = (x * z) * u. [para(9(a,1),10(a,1,2)),flip(a)].
44 x * (y * (z * (u * x))) = z * (y * u). [para(9(a,1),11(a,1,2,2)),flip(a)].
55 (1 * x) * (2 * y) = 3 * (x * y). [para(12(a,1),10(a,1,1)),flip(a)].
102 1 * 5 = 4. [para(13(a,1),31(a,1,2))].
111 0 * 10 = 9. [para(23(a,1),31(a,1,2))].
245 (x * y) * (z * (x * u)) = (y * z) * u. [para(31(a,1),33(a,1,2)),flip(a)].
523 3 * 9 = 0 * 4. [para(20(a,1),55(a,2,2)),rewrite([102(3),15(4),8(3)]),flip(a)].
524 4 * 9 = 10 * 3. [para(21(a,1),55(a,2,2)),rewrite([102(3),16(4),8(6)])].
595 9 * (0 * 4) = 3. [para(523(a,1),9(a,1,2))].
756 9 * (10 * 3) = 4. [para(524(a,1),9(a,1,2))].
832 3 * (x * 4) = 0 * (x * 9). [para(595(a,1),37(a,1,2)),rewrite([8(4),8(8)])].
1012 9 * (x * 10) = 0 * (x * 9). [para(756(a,1),44(a,1,2,2)),rewrite([832(4)]),flip(a)].

4903 9 * 9 = 9. [para(1012(a,1),9(a,1,2)),rewrite([245(7),8(3),
111(3)])].
4904 $F. [resolve(4903,a,29,a)].
============================== end of proof
==========================

Tangential relations from §7.7

P_{30}

============================== PROOF

% -------- Comments from original proof --------
% Proof 1 at 0.07 (+ 0.01) seconds.
% Length of proof is 17.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 49.

4 3 * 3 = 6 * 6 # label(non_clause) # label(goal). [goal].
6 x * y = y * x. [assumption].
7 x * (y * x) = y. [assumption].
8 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
10 1 * 2 = 3. [assumption].
12 1 * 6 = 7. [assumption].
13 2 * 6 = 0. [assumption].
15 3 * 7 = 0. [assumption].
25 6 * 6 != 3 * 3. [deny(4)].
27 x * (x * y) = y. [para(6(a,1),7(a,1,2))].
51 (1 * x) * (2 * y) = 3 * (x * y). [para(10(a,1),8(a,1,1)),flip(a)].
70 0 * 7 = 3. [para(15(a,1),7(a,1,2)),rewrite([6(3)])].
99 1 * 7 = 6. [para(12(a,1),27(a,1,2))].
100 0 * 2 = 6. [para(13(a,1),27(a,1,2)),rewrite([6(3)])].
351 (x * 1) * (y * 7) = 6 * (x * y). [para(99(a,1),8(a,
1,2)),rewrite([6(3)]),flip(a)].
524 6 * 6 = 3 * 3. [para(70(a,1),51(a,2,2)),rewrite([6(3),351(7),
100(4)])].
525 $F. [resolve(524,a,25,a)].

============================== end of proof
==========================

P_{31}

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 2 at 0.07 (+ 0.01) seconds.
% Length of proof is 16.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 50.

2 9 * 9 = 5 # label(non_clause) # label(goal). [goal].
6 x * y = y * x. [assumption].
7 x * (y * x) = y. [assumption].
8 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
10 1 * 2 = 3. [assumption].
11 1 * 4 = 5. [assumption].
14 2 * 8 = 9. [assumption].
16 3 * 9 = 10. [assumption].
17 4 * 8 = 10. [assumption].
22 9 * 9 != 5. [deny(2)].
27 x * (x * y) = y. [para(6(a,1),7(a,1,2))].
51 (1 * x) * (2 * y) = 3 * (x * y). [para(10(a,1),8(a,1,1)),flip(a)].
103 10 * 3 = 9. [para(16(a,1),27(a,1,2)),rewrite([6(3)])].
519 5 * 9 = 9. [para(17(a,1),51(a,2,2)),rewrite([11(3),14(4),6(6),
103(6)])].
544 9 * 9 = 5. [para(519(a,1),7(a,1,2))].
545 $F. [resolve(544,a,22,a)].

============================== end of proof
==========================

P_{32}

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 3 at 0.08 (+ 0.01) seconds.
% Length of proof is 15.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 53.

3 0 * 0 = 1 * 1 # label(non_clause) # label(goal). [goal].
6 x * y = y * x. [assumption].
7 x * (y * x) = y. [assumption].
8 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
10 1 * 2 = 3. [assumption].
12 1 * 6 = 7. [assumption].
13 2 * 6 = 0. [assumption].
15 3 * 7 = 0. [assumption].
23 0 * 0 != 1 * 1. [deny(3)].
24 1 * 1 != 0 * 0. [copy(23),flip(a)].
27 x * (x * y) = y. [para(6(a,1),7(a,1,2))].
51 (1 * x) * (2 * y) = 3 * (x * y). [para(10(a,1),8(a,1,1)),flip(a)].
516 0 * (1 * 1) = 0. [para(12(a,1),51(a,2,2)),rewrite([13(6),6(5),
15(8)])].
584 1 * 1 = 0 * 0. [para(516(a,1),27(a,1,2)),flip(a)].
585 $F. [resolve(584,a,24,a)].

============================== end of proof
==========================

P_{33}

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 4 at 0.22 (+ 0.01) seconds.
% Length of proof is 27.
% Level of proof is 7.
% Maximum clause weight is 15.
% Given clauses 72.

5 2 * 5 = 7 * 9 # label(non_clause) # label(goal). [goal].
6 x * y = y * x. [assumption].
7 x * (y * x) = y. [assumption].
8 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
9 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
10 1 * 2 = 3. [assumption].
11 1 * 4 = 5. [assumption].
12 1 * 6 = 7. [assumption].
13 2 * 6 = 0. [assumption].
14 2 * 8 = 9. [assumption].
18 4 * 9 = 0. [assumption].
26 7 * 9 != 2 * 5. [deny(5)].
27 x * (x * y) = y. [para(6(a,1),7(a,1,2))].
40 x * (y * (z * (u * x))) = z * (y * u). [para(7(a,1),9(a,
1,2,2)),flip(a)].

51 (1 * x) * (2 * y) = 3 * (x * y). [para(10(a,1),8(a,1,1)),flip(a)].
82 0 * 9 = 4. [para(18(a,1),7(a,1,2)),rewrite([6(3)])].
95 (x * y) * (z * (y * u)) = (x * z) * u. [para(27(a,1),8(a,
1,2)),flip(a)].
97 1 * 3 = 2. [para(10(a,1),27(a,1,2))].
100 0 * 2 = 6. [para(13(a,1),27(a,1,2)),rewrite([6(3)])].
515 7 * (2 * x) = 3 * (6 * x). [para(12(a,1),51(a,1,1))].
517 3 * (x * 6) = 0 * (1 * x). [para(13(a,1),51(a,
1,2)),rewrite([6(4)]),flip(a)].
1030 3 * (6 * x) = 0 * (1 * x). [para(6(a,1),515(a,
2,2)),rewrite([515(4),517(8)])].
1039 0 * (1 * 8) = 7 * 9. [para(14(a,1),515(a,
1,2)),rewrite([1030(8)]),flip(a)].
1060 2 * (7 * x) = 1 * (0 * x). [para(515(a,1),40(a,
1,2)),rewrite([1030(5),40(6)]),flip(a)].
1076 3 * (7 * 9) = 6 * 8. [para(1039(a,1),51(a,2,2)),rewrite([6(3),
95(9),100(3)]),flip(a)].
1421 7 * 9 = 2 * 5. [para(1076(a,1),51(a,2,2)),rewrite([97(3),
1060(6),82(5),11(4),1030(8),1039(8)]),flip(a)].
1422 $F. [resolve(1421,a,26,a)].

============================== end of proof
==========================

P_{34}

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 5 at 0.82 (+ 0.03) seconds.
% Length of proof is 57.
% Level of proof is 13.
% Maximum clause weight is 15.
% Given clauses 103.

1 5 * 5 = 5 # label(non_clause) # label(goal). [goal].
6 x * y = y * x. [assumption].
7 x * (y * x) = y. [assumption].
8 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
9 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
10 1 * 2 = 3. [assumption].
11 1 * 4 = 5. [assumption].
12 1 * 6 = 7. [assumption].
13 2 * 6 = 0. [assumption].
14 2 * 8 = 9. [assumption].
15 3 * 7 = 0. [assumption].
16 3 * 9 = 10. [assumption].
17 4 * 8 = 10. [assumption].
18 4 * 9 = 0. [assumption].
19 5 * 6 = 10. [assumption].
21 5 * 5 != 5. [deny(1)].
27 x * (x * y) = y. [para(6(a,1),7(a,1,2))].
28 (x * y) * (z * u) = (z * x) * (u * y). [para(8(a,1),6(a,1)),flip(a)].
29 (x * y) * (z * u) = (z * x) * (y * u). [para(6(a,1),8(a,1,1)),flip(a)].
33 (x * y) * (z * (u * y)) = (x * z) * u. [para(7(a,1),8(a,1,2)),flip(a)].
40 x * (y * (z * (u * x))) = z * (y * u). [para(7(a,1),9(a,
1,2,2)),flip(a)].
51 (1 * x) * (2 * y) = 3 * (x * y). [para(10(a,1),8(a,1,1)),flip(a)].
82 0 * 9 = 4. [para(18(a,1),7(a,1,2)),rewrite([6(3)])].
86 10 * 6 = 5. [para(19(a,1),7(a,1,2)),rewrite([6(3)])].
95 (x * y) * (z * (y * u)) = (x * z) * u. [para(27(a,1),8(a,
1,2)),flip(a)].
97 1 * 3 = 2. [para(10(a,1),27(a,1,2))].
100 0 * 2 = 6. [para(13(a,1),27(a,1,2)),rewrite([6(3)])].
101 2 * 9 = 8. [para(14(a,1),27(a,1,2))].
103 10 * 3 = 9. [para(16(a,1),27(a,1,2)),rewrite([6(3)])].
145 (x * 4) * (y * 1) = 5 * (y * x). [para(11(a,1),28(a,
2,2)),rewrite([6(8)])].
289 (x * 10) * (y * 6) = 5 * (x * y). [para(86(a,1),8(a,
1,2)),rewrite([6(3)]),flip(a)].
362 (x * 0) * (2 * y) = 6 * (x * y). [para(100(a,1),29(a,1,1)),flip(a)].
515 7 * (2 * x) = 3 * (6 * x). [para(12(a,1),51(a,1,1))].
516 0 * (1 * 1) = 0. [para(12(a,1),51(a,2,2)),rewrite([13(6),6(5),
15(8)])].
517 3 * (x * 6) = 0 * (1 * x). [para(13(a,1),51(a,
1,2)),rewrite([6(4)]),flip(a)].
518 9 * (1 * x) = 3 * (x * 8). [para(14(a,1),51(a,
1,2)),rewrite([6(4)])].
519 5 * 9 = 9. [para(17(a,1),51(a,2,2)),rewrite([11(3),14(4),6(6),
103(6)])].
531 8 * (0 * 1) = 3 * 4. [para(82(a,1),51(a,2,2)),rewrite([6(3),101(6),
6(5)])].
532 0 * (1 * 10) = 3 * 5. [para(86(a,1),51(a,2,2)),rewrite([13(6),
6(5)])].
584 1 * 1 = 0 * 0. [para(516(a,1),27(a,1,2)),flip(a)].
782 6 * (0 * x) = 3 * (1 * x). [para(584(a,1),51(a,
1,1)),rewrite([362(6)])].
814 8 * (3 * 4) = 0 * 1. [para(531(a,1),27(a,1,2))].
927 0 * (3 * 5) = 1 * 10. [para(532(a,1),27(a,1,2))].
951 5 * (0 * x) = 3 * (x * 8). [para(814(a,1),33(a,
1,2)),rewrite([145(6),6(8)])].
1030 3 * (6 * x) = 0 * (1 * x). [para(6(a,1),515(a,
2,2)),rewrite([515(4),517(8)])].
1039 0 * (1 * 8) = 7 * 9. [para(14(a,1),515(a,
1,2)),rewrite([1030(8)]),flip(a)].
1060 2 * (7 * x) = 1 * (0 * x). [para(515(a,1),40(a,
1,2)),rewrite([1030(5),40(6)]),flip(a)].
1076 3 * (7 * 9) = 6 * 8. [para(1039(a,1),51(a,2,2)),rewrite([6(3),
95(9),100(3)]),flip(a)].
1421 7 * 9 = 2 * 5. [para(1076(a,1),51(a,2,2)),rewrite([97(3),
1060(6),82(5),11(4),1030(8),1039(8)]),flip(a)].
1472 9 * (2 * 5) = 7. [para(1421(a,1),7(a,1,2))].
1513 5 * (1 * 9) = 0. [para(1472(a,1),51(a,2,2)),rewrite([27(8),6(5),
15(8)])].
1528 1 * 9 = 0 * 5. [para(1513(a,1),27(a,
1,2)),rewrite([6(3)]),flip(a)].
2201 8 * (3 * x) = 1 * (9 * x). [para(518(a,1),40(a,
1,2)),rewrite([6(4),40(6)])].
2956 3 * (0 * 5) = 4 * 6. [para(82(a,1),782(a,1,2)),rewrite([6(3),
1528(7)]),flip(a)].
3624 5 * (1 * 10) = 4 * 6. [para(927(a,1),951(a,1,2)),rewrite([6(11),
2201(11),6(10),519(10),1528(9),2956(10)])].
3666 5 * 5 = 5. [para(3624(a,1),7(a,1,2)),rewrite([289(7),11(4)])].
3667 $F. [resolve(3666,a,21,a)].

============================== end of proof
==========================

Tangential relations from §7.8

P_{35}

============================== PROOF

% -------- Comments from original proof --------
% Proof 1 at 0.08 (+ 0.01) seconds.
% Length of proof is 17.
% Level of proof is 4.
% Maximum clause weight is 15.
% Given clauses 58.

4 4 * 4 = 8 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
19 2 * 4 = 6. [assumption].
21 3 * 8 = 10. [assumption].
25 5 * 6 = 10. [assumption].
32 4 * 4 != 8. [deny(4)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6),115(6)])].
997 $F. [resolve(996,a,32,a)].

============================== end of proof ==========================

P_{36}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 2 at 0.08 (+ 0.01) seconds.
% Length of proof is 17.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 60.

9 9 * 9 = 6 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
16 1 * 2 = 3. [assumption].
18 1 * 6 = 7. [assumption].
20 2 * 8 = 9. [assumption].
22 3 * 9 = 0. [assumption].
27 7 * 8 = 0. [assumption].
37 9 * 9 != 6. [deny(9)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
116 0 * 3 = 9. [para(22(a,1),41(a,1,2)),rewrite([12(3)])].
999 6 * 9 = 9. [para(27(a,1),63(a,2,2)),rewrite([112(3),20(4),12(6),116(6)])].
1077 9 * 9 = 6. [para(999(a,1),13(a,1,2))].
1078 $F. [resolve(1077,a,37,a)].

============================== end of proof ==========================

P_{37}

============================== PROOF

% -------- Comments from original proof --------
% Proof 3 at 0.18 (+ 0.01) seconds.
% Length of proof is 21.
% Level of proof is 5.
% Maximum clause weight is 15.
% Given clauses 68.

3 3 * 3 = 6 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
16 1 * 2 = 3. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
31 3 * 3 != 6. [deny(3)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6),12(5),12(8),115(8)])].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3),114(6),12(5)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1813 $F. [resolve(1812,a,31,a)].

============================== end of proof ==========================

P_{38}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 4 at 0.19 (+ 0.01) seconds.
% Length of proof is 34.
% Level of proof is 6.
% Maximum clause weight is 17.
% Given clauses 69.

6 6 * 6 = 6 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
34 6 * 6 != 6. [deny(6)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a,1,2)),flip(a)].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].

995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6), 12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6), 115(6)])].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3), 114(6),12(5)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3), 114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a, 1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a, 2,1)),rewrite([1040(6)]),flip(a)].
1983 $F. [back_rewrite(34),rewrite([1853(3),18(5),1063(4), 19(3)]),xx(a)].

============================== end of proof ==========================

P_{39}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 5 at 0.52 (+ 0.02) seconds.
% Length of proof is 54.
% Level of proof is 11.
% Maximum clause weight is 17.
% Given clauses 79.

2 2 * 2 = 0 * 1 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
29 0 * 1 != 2 * 2. [deny(2)].
30 2 * 2 != 0 * 1. [copy(29),flip(a)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a, 1,1)),flip(a)].
45 (x * y) * ((z * x) * (u * y)) = z * u. [para(14(a,1),13(a,1,2))].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a, 1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
74 4 * 6 = 2. [para(19(a,1),13(a,1,2))].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].
253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6), 12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6), 115(6)])].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3), 114(6),12(5)])].
1015 6 * 8 = 3 * 5. [para(98(a,1),63(a,2,2)),rewrite([112(3), 114(4)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3), 114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a, 1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1095 4 * 7 = 3. [para(1063(a,1),13(a,1,2)),rewrite([12(3)])].
1453 2 * (7 * (1 * x)) = 4 * x. [para(1095(a,1),45(a, 1,2,1)),rewrite([1040(6)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1818 3 * 5 = 0 * 1. [para(995(a,1),41(a,1,2)),rewrite([1015(3)])].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a, 2,1)),rewrite([1040(6)]),flip(a)].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y). [back_rewrite(253),rewrite([1853(2),247(9)])].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))). [back_rewrite(1976),rewrite([2026(6)]),flip(a)].
3183 4 * x = 2 * (3 * (1 * (3 * (1 * x)))). [back_rewrite(1453),rewrite([2253(5)]),flip(a)].
3355 2 * (0 * 1) = 2. [back_rewrite(74),rewrite([3183(3),18(7), 1063(6),17(5),1818(4)])].
3915 2 * 2 = 0 * 1. [para(3355(a,1),41(a,1,2))].
3916 $F. [resolve(3915,a,30,a)].

============================== end of proof ==========================

P_{40}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 6 at 0.62 (+ 0.02) seconds.
% Length of proof is 58.
% Level of proof is 14.
% Maximum clause weight is 19.
% Given clauses 84.

5 5 * 5 = 8 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].

27 7 * 8 = 0. [assumption].
33 5 * 5 != 8. [deny(5)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a,1,1)),flip(a)].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a,1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
120 0 * 7 = 8. [para(27(a,1),41(a,1,2)),rewrite([12(3)])].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].
253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
264 (3 * x) * (y * 9) = 0 * (x * y). [para(22(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
345 (7 * x) * (y * 9) = 5 * (x * y). [para(98(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6),12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6),115(6)])].
998 4 * (2 * 7) = 0. [para(26(a,1),63(a,2,2)),rewrite([111(3),22(8)])].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3),114(6),12(5)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3),114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a,1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1099 3 * 4 = 7. [para(1063(a,1),41(a,1,2))].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a,2,1)),rewrite([1040(6)]),flip(a)].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y). [back_rewrite(253),rewrite([1853(2),247(9)])].
2008 2 * 7 = 10. [para(998(a,1),41(a,1,2)),rewrite([12(3),24(3)]),flip(a)].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))). [back_rewrite(1976),rewrite([2026(6)]),flip(a)].
3424 10 * 2 = 7. [para(2008(a,1),41(a,1,2)),rewrite([12(3)])].
3449 (10 * x) * (2 * y) = 3 * (1 * (3 * (x * y))). [para(3424(a,1),14(a,1,1)),rewrite([2253(3)]),flip(a)].
3462 7 * x = 3 * (1 * (3 * x)). [para(3424(a,1),47(a,2,1)),rewrite([3449(6),13(5)]),flip(a)].
3504 5 * (x * y) = 1 * (3 * (0 * (x * y))). [back_rewrite(345),rewrite([3462(2),2026(9),264(7)]),flip(a)].
3508 3 * (0 * 1) = 5. [back_rewrite(98),rewrite([3462(3),22(5),12(4)])].
3933 (3 * x) * ((0 * 1) * y) = 1 * (3 * (0 * (x * y))). [para(3508(a,1),14(a,1,1)),rewrite([3504(3)]),flip(a)].
3949 5 * x = 1 * (3 * (0 * x)). [para(3508(a,1),47(a,2,1)),rewrite([3933(8),13(5)]),flip(a)].
3991 1 * (3 * (0 * 5)) != 8. [back_rewrite(33),rewrite([3949(3)])].
4435 $F. [para(15(a,1),3991(a,1)),rewrite([111(5),1099(4),120(3)]),xx(a)].

============================== end of proof ==========================

P_{41}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 7 at 0.73 (+ 0.03) seconds.
% Length of proof is 131.
% Level of proof is 28.
% Maximum clause weight is 23.
% Given clauses 91.

1 1 * 1 = 8 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
27 7 * 8 = 0. [assumption].
28 1 * 1 != 8. [deny(1)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a,1,1)),flip(a)].
45 (x * y) * ((z * x) * (u * y)) = z * u. [para(14(a,1),13(a,1,2))].
46 (x * y) * ((z * x) * u) = z * (y * u). [para(13(a,1),14(a,1,1)),flip(a)].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a,1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
62 2 * 3 = 1. [para(16(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
74 4 * 6 = 2. [para(19(a,1),13(a,1,2))].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
110 1 * 3 = 2. [para(16(a,1),41(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
116 0 * 3 = 9. [para(22(a,1),41(a,1,2)),rewrite([12(3)])].
117 0 * 10 = 4. [para(24(a,1),41(a,1,2))].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
120 0 * 7 = 8. [para(27(a,1),41(a,1,2)),rewrite([12(3)])].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].
253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
256 (4 * x) * (2 * y) = 6 * (x * y). [para(19(a,1),43(a,2,1))].
264 (3 * x) * (y * 9) = 0 * (x * y). [para(22(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
274 (7 * x) * (5 * y) = 9 * (x * y). [para(26(a,1),43(a,2,1))].
345 (7 * x) * (y * 9) = 5 * (x * y). [para(98(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].

474 (2 * x) * (y * 9) = 8 * (x * y). [para(114(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
475 (9 * x) * (2 * y) = 8 * (x * y). [para(114(a,1),43(a,2,1))].
557 (0 * x) * (3 * y) = 9 * (x * y). [para(116(a,1),14(a,1,1)),flip(a)].
604 3 * ((x * 1) * y) = x * (2 * y). [para(16(a,1),46(a,1,1))].
607 6 * ((x * 2) * y) = x * (4 * y). [para(19(a,1),46(a,1,1))].
616 (x * y) * ((y * x) * z) = z. [para(46(a,2),41(a,1))].
619 1 * ((x * 2) * y) = x * (3 * y). [para(62(a,1),46(a,1,1))].
668 2 * ((x * 1) * y) = x * (3 * y). [para(110(a,1),46(a,1,1))].
734 (x * y) * (z * (y * x)) = z. [para(12(a,1),616(a,1,2))].
989 5 * (2 * x) = 3 * (4 * x). [para(17(a,1),63(a,1,1))].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6),12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6),115(6)])].
998 4 * (2 * 7) = 0. [para(26(a,1),63(a,2,2)),rewrite([111(3),22(8)])].
999 6 * 9 = 9. [para(27(a,1),63(a,2,2)),rewrite([112(3),20(4),12(6),116(6)])].
1001 (1 * x) * y = 3 * (x * (2 * y)). [para(41(a,1),63(a,1,2))].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3),114(6),12(5)])].
1015 6 * 8 = 3 * 5. [para(98(a,1),63(a,2,2)),rewrite([112(3),114(4)])].
1024 4 * (1 * x) = 3 * (x * 6). [para(113(a,1),63(a,1,2)),rewrite([12(4)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3),114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a,1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1077 9 * 9 = 6. [para(999(a,1),13(a,1,2))].
1085 (6 * x) * (y * 9) = 9 * (x * y). [para(999(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
1095 4 * 7 = 3. [para(1063(a,1),13(a,1,2)),rewrite([12(3)])].
1442 (9 * x) * (y * 9) = 6 * (x * y). [para(1077(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
1453 2 * (7 * (1 * x)) = 4 * x. [para(1095(a,1),45(a,1,2,1)),rewrite([1040(6)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1818 3 * 5 = 0 * 1. [para(995(a,1),41(a,1,2)),rewrite([1015(3)])].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a,2,1)),rewrite([1040(6)]),flip(a)].
1905 (9 * x) * (y * 9) = 2 * (3 * (1 * (x * y))). [back_rewrite(1442),rewrite([1853(8)])].
1928 8 * ((3 * (1 * x)) * y) = 9 * (x * y). [back_rewrite(1085),rewrite([1853(2),474(9)])].
1947 2 * (3 * (x * (3 * y))) = x * (4 * y). [back_rewrite(607),rewrite([1853(5),619(7)])].
1973 (4 * x) * (2 * y) = 2 * (3 * (1 * (x * y))). [back_rewrite(256),rewrite([1853(8)])].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y). [back_rewrite(253),rewrite([1853(2),247(9)])].
2008 2 * 7 = 10. [para(998(a,1),41(a,1,2)),rewrite([12(3),24(3)]),flip(a)].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))). [back_rewrite(1976),rewrite([2026(6)]),flip(a)].
2259 9 * (x * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(1928),rewrite([2026(6)]),flip(a)].
3183 4 * x = 2 * (3 * (1 * (3 * (1 * x)))). [back_rewrite(1453),rewrite([2253(5)]),flip(a)].
3246 3 * (x * 6) = 2 * (3 * (1 * (3 * x))). [back_rewrite(1024),rewrite([3183(4),41(8)]),flip(a)].
3289 (0 * x) * (3 * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(557),rewrite([2259(8)])].
3292 (7 * x) * (5 * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(274),rewrite([2259(8)])].
3326 3 * (2 * (3 * (1 * (3 * (x * y))))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1973),rewrite([3183(2),2026(13),2026(11),1001(9),41(9)])].
3327 2 * (3 * (x * (3 * y))) = x * (2 * (3 * (1 * (3 * (1 * y))))). [back_rewrite(1947),rewrite([3183(9)])].
3339 5 * (2 * x) = 2 * (3 * x). [back_rewrite(989),rewrite([3183(7),3326(16),41(10)])].
3355 2 * (0 * 1) = 2. [back_rewrite(74),rewrite([3183(3),18(7),1063(6),17(5),1818(4)])].
3424 10 * 2 = 7. [para(2008(a,1),41(a,1,2)),rewrite([12(3)])].
3449 (10 * x) * (2 * y) = 3 * (1 * (3 * (x * y))). [para(3424(a,1),14(a,1,1)),rewrite([2253(3)]),flip(a)].
3462 7 * x = 3 * (1 * (3 * x)). [para(3424(a,1),47(a,2,1)),rewrite([3449(6),13(5)]),flip(a)].
3477 1 * (3 * ((3 * x) * (5 * y))) = 8 * (1 * (3 * (x * y))). [back_rewrite(3292),rewrite([3462(2),2026(9)])].
3504 5 * (x * y) = 1 * (3 * (0 * (x * y))). [back_rewrite(345),rewrite([3462(2),2026(9),264(7)]),flip(a)].
3508 3 * (0 * 1) = 5. [back_rewrite(98),rewrite([3462(3),22(5),12(4)])].
3522 1 * (3 * (0 * (2 * x))) = 2 * (3 * x). [back_rewrite(3339),rewrite([3504(4)])].
3648 (3 * x) * (5 * y) = (0 * 1) * (x * y). [para(1818(a,1),14(a,1,1)),flip(a)].
3656 (5 * x) * (3 * y) = (0 * 1) * (x * y). [para(1818(a,1),43(a,2,1))].
3670 8 * (1 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3477),rewrite([3648(7),604(8)]),flip(a)].
3687 (0 * x) * (3 * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(3289),rewrite([3670(12)])].
3714 9 * (x * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(2259),rewrite([3670(10)])].
3915 2 * 2 = 0 * 1. [para(3355(a,1),41(a,1,2))].
3927 0 * (x * 2) = 2 * (x * 1). [para(3355(a,1),47(a,1,2)),rewrite([12(4),12(8)]),flip(a)].
3933 (3 * x) * ((0 * 1) * y) = 1 * (3 * (0 * (x * y))). [para(3508(a,1),14(a,1,1)),rewrite([3504(3)]),flip(a)].
3949 5 * x = 1 * (3 * (0 * x)). [para(3508(a,1),47(a,2,1)),rewrite([3933(8),13(5)]),flip(a)].
3960 (0 * 1) * (x * y) = 3 * (0 * (2 * (x * y))). [back_rewrite(3656),rewrite([3949(2),2026(9),3687(7),41(10)]),flip(a)].
3992 1 * 10 = 9. [back_rewrite(26),rewrite([3949(3),120(5),21(4)])].
3993 1 * (3 * (0 * 6)) = 10. [back_rewrite(25),rewrite([3949(3)])].
4005 3 * (x * (2 * (10 * y))) = 1 * (0 * (2 * (x * y))). [para(3992(a,1),14(a,1,1)),rewrite([3714(3),1001(12)]),flip(a)].
4017 1 * (0 * (x * (3 * y))) = x * (10 * y). [para(3992(a,1),46(a,1,1)),rewrite([3714(5),668(7)])].
4022 9 * x = 1 * (0 * (2 * x)). [para(3992(a,1),47(a,2,1)),rewrite([1001(6),4005(8),13(5)]),flip(a)].
4061 0 * (1 * (8 * (x * y))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1905),rewrite([4022(2),2026(9),474(7)])].
4068 0 * (1 * ((2 * x) * (2 * y))) = 8 * (x * y). [back_rewrite(475),rewrite([4022(2),2026(9)])].
4453 (2 * x) * (2 * y) = 3 * (0 * (2 * (x * y))). [para(3915(a,1),14(a,1,1)),rewrite([3960(5)]),flip(a)].
4466 8 * (x * y) = 0 * (2 * (3 * (x * y))). [back_rewrite(4068),rewrite([4453(7),3522(10)]),flip(a)].
4483 2 * (3 * (1 * (x * y))) = 0 * (2 * (10 * (x * y))). [back_rewrite(4061),rewrite([4466(5),4017(10)]),flip(a)].

4497 2 * (10 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3670),rewrite([4466(7),4483(10),41(11)])].
4614 2 * (3 * (x * (3 * y))) = x * (0 * (1 * (0 * (2 * (1 * y))))). [back_rewrite(3327),rewrite([4483(17),4497(16)])].
4622 3 * (x * 6) = 1 * (0 * (1 * (0 * (2 * (1 * x))))). [back_rewrite(3246),rewrite([4614(12)])].
4624 4 * x = 1 * (0 * (1 * (0 * (2 * x)))). [back_rewrite(3183),rewrite([4614(12),41(11)])].
4785 0 * (1 * (0 * 2)) = 10. [back_rewrite(3993),rewrite([4622(6), 12(9),3927(10,R),41(10),41(11)])].
4791 1 * (0 * (x * (0 * 2))) = x * 10. [para(4785(a,1),15(a, 1,2)),flip(a)].
4792 1 * (0 * 2) = 4. [para(4785(a,1),41(a, 1,2)),rewrite([117(3)]),flip(a)].
4842 0 * 2 = 5. [para(4792(a,1),41(a,1,2)),rewrite([17(3)]),flip(a)].
4850 1 * (0 * (1 * (0 * (2 * 5)))) = 1. [para(4792(a,1),734(a, 1,2)),rewrite([12(3),4842(3),12(3),4624(3)])].
4860 1 * (0 * (x * 5)) = x * 10. [back_rewrite(4791),rewrite([4842(5)])].
4946 1 * 8 = 1. [back_rewrite(4850),rewrite([4860(9),12(5),3424(5), 120(4)])].
5237 1 * 1 = 8. [para(4946(a,1),41(a,1,2))].
5238 $F. [resolve(5237,a,28,a)].

============================== end of proof ==========================

P_{42}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 8 at 0.82 (+ 0.03) seconds.
% Length of proof is 133.
% Level of proof is 30.
% Maximum clause weight is 23.
% Given clauses 95.

8 8 * 8 = 8 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
27 7 * 8 = 0. [assumption].
36 8 * 8 != 8. [deny(8)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a, 1,1)),flip(a)].
45 (x * y) * ((z * x) * (u * y)) = z * u. [para(14(a,1),13(a,1,2))].
46 (x * y) * ((z * x) * u) = z * (y * u). [para(13(a,1),14(a, 1,1)),flip(a)].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a, 1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
62 2 * 3 = 1. [para(16(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
74 4 * 6 = 2. [para(19(a,1),13(a,1,2))].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
110 1 * 3 = 2. [para(16(a,1),41(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
116 0 * 3 = 9. [para(22(a,1),41(a,1,2)),rewrite([12(3)])].
117 0 * 10 = 4. [para(24(a,1),41(a,1,2))].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
120 0 * 7 = 8. [para(27(a,1),41(a,1,2)),rewrite([12(3)])].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].
253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
256 (4 * x) * (2 * y) = 6 * (x * y). [para(19(a,1),43(a,2,1))].
264 (3 * x) * (y * 9) = 0 * (x * y). [para(22(a,1),43(a, 1,2)),rewrite([12(3)]),flip(a)].
274 (7 * x) * (5 * y) = 9 * (x * y). [para(26(a,1),43(a,2,1))].
345 (7 * x) * (y * 9) = 5 * (x * y). [para(98(a,1),43(a, 1,2)),rewrite([12(3)]),flip(a)].
474 (2 * x) * (y * 9) = 8 * (x * y). [para(114(a,1),43(a, 1,2)),rewrite([12(3)]),flip(a)].
475 (9 * x) * (2 * y) = 8 * (x * y). [para(114(a,1),43(a,2,1))].
557 (0 * x) * (3 * y) = 9 * (x * y). [para(116(a,1),14(a,1,1)),flip(a)].
604 3 * ((x * 1) * y) = x * (2 * y). [para(16(a,1),46(a,1,1))].
607 6 * ((x * 2) * y) = x * (4 * y). [para(19(a,1),46(a,1,1))].
616 (x * y) * ((y * x) * z) = z. [para(46(a,2),41(a,1))].
619 1 * ((x * 2) * y) = x * (3 * y). [para(62(a,1),46(a,1,1))].
668 2 * ((x * 1) * y) = x * (3 * y). [para(110(a,1),46(a,1,1))].
734 (x * y) * (z * (y * x)) = z. [para(12(a,1),616(a,1,2))].
989 5 * (2 * x) = 3 * (4 * x). [para(17(a,1),63(a,1,1))].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6), 12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6), 115(6)])].
998 4 * (2 * 7) = 0. [para(26(a,1),63(a,2,2)),rewrite([111(3),22(8)])].
999 6 * 9 = 9. [para(27(a,1),63(a,2,2)),rewrite([112(3),20(4),12(6), 116(6)])].
1001 (1 * x) * y = 3 * (x * (2 * y)). [para(41(a,1),63(a,1,2))].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3), 114(6),12(5)])].
1015 6 * 8 = 3 * 5. [para(98(a,1),63(a,2,2)),rewrite([112(3), 114(4)])].
1024 4 * (1 * x) = 3 * (x * 6). [para(113(a,1),63(a, 1,2)),rewrite([12(4)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3), 114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a, 1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1077 9 * 9 = 6. [para(999(a,1),13(a,1,2))].
1085 (6 * x) * (y * 9) = 9 * (x * y). [para(999(a,1),43(a, 1,2)),rewrite([12(3)]),flip(a)].
1095 4 * 7 = 3. [para(1063(a,1),13(a,1,2)),rewrite([12(3)])].
1442 (9 * x) * (y * 9) = 6 * (x * y). [para(1077(a,1),43(a, 1,2)),rewrite([12(3)]),flip(a)].
1453 2 * (7 * (1 * x)) = 4 * x. [para(1095(a,1),45(a, 1,2,1)),rewrite([1040(6)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1818 3 * 5 = 0 * 1. [para(995(a,1),41(a,1,2)),rewrite([1015(3)])].

1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a,
2,1)),rewrite([1040(6)]),flip(a)].
1905 (9 * x) * (y * 9) = 2 * (3 * (1 * (x * y))).
[back_rewrite(1442),rewrite([1853(8)])].
1928 8 * ((3 * (1 * x)) * y) = 9 * (x * y).
[back_rewrite(1085),rewrite([1853(2),474(9)])].
1947 2 * (3 * (x * (3 * y))) = x * (4 * y).
[back_rewrite(607),rewrite([1853(5),619(7)])].
1973 (4 * x) * (2 * y) = 2 * (3 * (1 * (x * y))).
[back_rewrite(256),rewrite([1853(8)])].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y).
[back_rewrite(253),rewrite([1853(2),247(9)])].
2008 2 * 7 = 10. [para(998(a,1),41(a,1,2)),rewrite([12(3),
24(3)]),flip(a)].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))).
[back_rewrite(1976),rewrite([2026(6)]),flip(a)].
2259 9 * (x * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(1928),rewrite([2026(6)]),flip(a)].
3183 4 * x = 2 * (3 * (1 * (3 * (1 * x)))).
[back_rewrite(1453),rewrite([2253(5)]),flip(a)].
3246 3 * (x * 6) = 2 * (3 * (1 * (3 * x))).
[back_rewrite(1024),rewrite([3183(4),41(8)]),flip(a)].
3289 (0 * x) * (3 * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(557),rewrite([2259(8)])].
3292 (7 * x) * (5 * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(274),rewrite([2259(8)])].
3326 3 * (2 * (3 * (1 * (3 * (x * y))))) = 2 * (3 * (1 * (x * y))).
[back_rewrite(1973),rewrite([3183(2),2026(13),2026(11),1001(9),
41(9)])].
3327 2 * (3 * (x * (3 * y))) = x * (2 * (3 * (1 * (3 * (1 * y))))).
[back_rewrite(1947),rewrite([3183(9)])].
3339 5 * (2 * x) = 2 * (3 * x). [back_rewrite(989),rewrite([3183(7),
3326(16),41(10)])].
3355 2 * (0 * 1) = 2. [back_rewrite(74),rewrite([3183(3),18(7),
1063(6),17(5),1818(4)])].
3424 10 * 2 = 7. [para(2008(a,1),41(a,1,2)),rewrite([12(3)])].
3449 (10 * x) * (2 * y) = 3 * (1 * (3 * (x * y))). [para(3424(a,1),
14(a,1,1)),rewrite([2253(3)]),flip(a)].
3462 7 * x = 3 * (1 * (3 * x)). [para(3424(a,1),47(a,
2,1)),rewrite([3449(6),13(5)]),flip(a)].
3477 1 * (3 * ((3 * x) * (5 * y))) = 8 * (1 * (3 * (x * y))).
[back_rewrite(3292),rewrite([3462(2),2026(9)])].
3504 5 * (x * y) = 1 * (3 * (0 * (x * y))).
[back_rewrite(345),rewrite([3462(2),2026(9),264(7)]),flip(a)].
3508 3 * (0 * 1) = 5. [back_rewrite(98),rewrite([3462(3),22(5),
12(4)])].
3522 1 * (3 * (0 * (2 * x))) = 2 * (3 * x).
[back_rewrite(3339),rewrite([3504(4)])].
3648 (3 * x) * (5 * y) = (0 * 1) * (x * y). [para(1818(a,1),14(a,
1,1)),flip(a)].
3656 (5 * x) * (3 * y) = (0 * 1) * (x * y). [para(1818(a,1),43(a,2,1))].
3670 8 * (1 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))).
[back_rewrite(3477),rewrite([3648(7),604(8)]),flip(a)].
3687 (0 * x) * (3 * y) = 1 * (0 * (2 * (x * y))).
[back_rewrite(3289),rewrite([3670(12)])].
3714 9 * (x * y) = 1 * (0 * (2 * (x * y))).
[back_rewrite(2259),rewrite([3670(10)])].
3915 2 * 2 = 0 * 1. [para(3355(a,1),41(a,1,2))].
3927 0 * (x * 2) = 2 * (x * 1). [para(3355(a,1),47(a,
1,2)),rewrite([12(4),12(8)]),flip(a)].
3933 (3 * x) * ((0 * 1) * y) = 1 * (3 * (0 * (x * y))). [para(3508(a,1),
14(a,1,1)),rewrite([3504(3)]),flip(a)].
3949 5 * x = 1 * (3 * (0 * x)). [para(3508(a,1),47(a,
2,1)),rewrite([3933(8),13(5)]),flip(a)].
3960 (0 * 1) * (x * y) = 3 * (0 * (2 * (x * y))).
[back_rewrite(3656),rewrite([3949(2),2026(9),3687(7),
41(10)]),flip(a)].
3992 1 * 10 = 9. [back_rewrite(26),rewrite([3949(3),120(5),21(4)])].
3993 1 * (3 * (0 * 6)) = 10. [back_rewrite(25),rewrite([3949(3)])].
4005 3 * (x * (2 * (10 * y))) = 1 * (0 * (2 * (x * y))). [para(3992(a,
1),14(a,1,1)),rewrite([3714(3),1001(12)]),flip(a)].
4017 1 * (0 * (x * (3 * y))) = x * (10 * y). [para(3992(a,1),46(a,
1,1)),rewrite([3714(5),668(7)])].
4022 9 * x = 1 * (0 * (2 * x)). [para(3992(a,1),47(a,
2,1)),rewrite([1001(6),4005(8),13(5)]),flip(a)].
4061 0 * (1 * (8 * (x * y))) = 2 * (3 * (1 * (x * y))).
[back_rewrite(1905),rewrite([4022(2),2026(9),474(7)])].
4068 0 * (1 * ((2 * x) * (2 * y))) = 8 * (x * y).
[back_rewrite(475),rewrite([4022(2),2026(9)])].
4453 (2 * x) * (2 * y) = 3 * (0 * (2 * (x * y))). [para(3915(a,1),14(a,
1,1)),rewrite([3960(5)]),flip(a)].
4466 8 * (x * y) = 0 * (2 * (3 * (x * y))).
[back_rewrite(4068),rewrite([4453(7),3522(10)]),flip(a)].
4483 2 * (3 * (1 * (x * y))) = 0 * (2 * (10 * (x * y))).
[back_rewrite(4061),rewrite([4466(5),4017(10)]),flip(a)].
4497 2 * (10 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))).
[back_rewrite(3670),rewrite([4466(7),4483(10),41(11)])].
4614 2 * (3 * (x * (3 * y))) = x * (0 * (1 * (0 * (2 * (1 * y))))).
[back_rewrite(3327),rewrite([4483(17),4497(16)])].
4622 3 * (x * 6) = 1 * (0 * (1 * (0 * (2 * (1 * x))))).
[back_rewrite(3246),rewrite([4614(12)])].
4624 4 * x = 1 * (0 * (1 * (0 * (2 * x)))).
[back_rewrite(3183),rewrite([4614(12),41(11)])].
4785 0 * (1 * (0 * 2)) = 10. [back_rewrite(3993),rewrite([4622(6),
12(9),3927(10,R),41(10),41(11)])].
4791 1 * (0 * (x * (0 * 2))) = x * 10. [para(4785(a,1),15(a,
1,2)),flip(a)].
4792 1 * (0 * 2) = 4. [para(4785(a,1),41(a,
1,2)),rewrite([117(3)]),flip(a)].
4842 0 * 2 = 5. [para(4792(a,1),41(a,1,2)),rewrite([17(3)]),flip(a)].
4850 1 * (0 * (1 * (0 * (2 * 5)))) = 1. [para(4792(a,1),734(a,
1,2)),rewrite([12(3),4842(3),12(3),4624(3)])].
4860 1 * (0 * (x * 5)) = x * 10.
[back_rewrite(4791),rewrite([4842(5)])].
4946 1 * 8 = 1. [back_rewrite(4850),rewrite([4860(9),12(5),3424(5),
120(4)])].
5237 1 * 1 = 8. [para(4946(a,1),41(a,1,2))].
5698 3 * (x * (2 * (1 * y))) = 0 * (2 * (3 * (x * y))). [para(5237(a,1),
14(a,1,1)),rewrite([4466(3),1001(12)]),flip(a)].
5705 8 * x = 0 * (2 * (3 * x)). [para(5237(a,1),47(a,
2,1)),rewrite([1001(6),5698(8),13(5)]),flip(a)].
5745 $F. [back_rewrite(36),rewrite([5705(3),21(5),12(4),3424(4),
120(3)]),xx(a)].

============================== end of proof
==========================

P43

============================== PROOF
=================================

% -------- Comments from original proof --------
% Proof 9 at 0.83 (+ 0.03) seconds.
% Length of proof is 124.
% Level of proof is 26.
% Maximum clause weight is 23.
% Given clauses 97.

11 0 * 0 = 6 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
27 7 * 8 = 0. [assumption].
40 0 * 0 != 6. [deny(11)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a,1,1)),flip(a)].
45 (x * y) * ((z * x) * (u * y)) = z * u. [para(14(a,1),13(a,1,2))].
46 (x * y) * ((z * x) * u) = z * (y * u). [para(13(a,1),14(a,1,1)),flip(a)].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a,1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
51 x * (y * (z * u)) = u * (y * (x * z)). [para(12(a,1),15(a,1,2,2))].
62 2 * 3 = 1. [para(16(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
74 4 * 6 = 2. [para(19(a,1),13(a,1,2))].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
110 1 * 3 = 2. [para(16(a,1),41(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
116 0 * 3 = 9. [para(22(a,1),41(a,1,2)),rewrite([12(3)])].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
120 0 * 7 = 8. [para(27(a,1),41(a,1,2)),rewrite([12(3)])].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].
253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
256 (4 * x) * (2 * y) = 6 * (x * y). [para(19(a,1),43(a,2,1))].
264 (3 * x) * (y * 9) = 0 * (x * y). [para(22(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
274 (7 * x) * (5 * y) = 9 * (x * y). [para(26(a,1),43(a,2,1))].
345 (7 * x) * (y * 9) = 5 * (x * y). [para(98(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
474 (2 * x) * (y * 9) = 8 * (x * y). [para(114(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
475 (9 * x) * (2 * y) = 8 * (x * y). [para(114(a,1),43(a,2,1))].
557 (0 * x) * (3 * y) = 9 * (x * y). [para(116(a,1),14(a,1,1)),flip(a)].
604 3 * ((x * 1) * y) = x * (2 * y). [para(16(a,1),46(a,1,1))].
607 6 * ((x * 2) * y) = x * (4 * y). [para(19(a,1),46(a,1,1))].
619 1 * ((x * 2) * y) = x * (3 * y). [para(62(a,1),46(a,1,1))].
668 2 * ((x * 1) * y) = x * (3 * y). [para(110(a,1),46(a,1,1))].
989 5 * (2 * x) = 3 * (4 * x). [para(17(a,1),63(a,1,1))].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6),12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6),115(6)])].
998 4 * (2 * 7) = 0. [para(26(a,1),63(a,2,2)),rewrite([111(3),22(8)])].
999 6 * 9 = 9. [para(27(a,1),63(a,2,2)),rewrite([112(3),20(4),12(6),116(6)])].
1001 (1 * x) * y = 3 * (x * (2 * y)). [para(41(a,1),63(a,1,2))].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3),114(6),12(5)])].
1015 6 * 8 = 3 * 5. [para(98(a,1),63(a,2,2)),rewrite([112(3),114(4)])].
1024 4 * (1 * x) = 3 * (x * 6). [para(113(a,1),63(a,1,2)),rewrite([12(4)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3),114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a,1,2)),rewrite([12(10)])].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1077 9 * 9 = 6. [para(999(a,1),13(a,1,2))].
1085 (6 * x) * (y * 9) = 9 * (x * y). [para(999(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
1095 4 * 7 = 3. [para(1063(a,1),13(a,1,2)),rewrite([12(3)])].
1442 (9 * x) * (y * 9) = 6 * (x * y). [para(1077(a,1),43(a,1,2)),rewrite([12(3)]),flip(a)].
1453 2 * (7 * (1 * x)) = 4 * x. [para(1095(a,1),45(a,1,2,1)),rewrite([1040(6)])].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1818 3 * 5 = 0 * 1. [para(995(a,1),41(a,1,2)),rewrite([1015(3)])].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a,2,1)),rewrite([1040(6)]),flip(a)].
1905 (9 * x) * (y * 9) = 2 * (3 * (1 * (x * y))). [back_rewrite(1442),rewrite([1853(8)])].
1928 8 * ((3 * (1 * x)) * y) = 9 * (x * y). [back_rewrite(1085),rewrite([1853(2),474(9)])].
1947 2 * (3 * (x * (3 * y))) = x * (4 * y). [back_rewrite(607),rewrite([1853(5),619(7)])].
1973 (4 * x) * (2 * y) = 2 * (3 * (1 * (x * y))). [back_rewrite(256),rewrite([1853(8)])].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y). [back_rewrite(253),rewrite([1853(2),247(9)])].
2008 2 * 7 = 10. [para(998(a,1),41(a,1,2)),rewrite([12(3),24(3)]),flip(a)].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))). [back_rewrite(1976),rewrite([2026(6)]),flip(a)].
2259 9 * (x * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(1928),rewrite([2026(6)]),flip(a)].
3183 4 * x = 2 * (3 * (1 * (3 * (1 * x)))). [back_rewrite(1453),rewrite([2253(5)]),flip(a)].
3246 3 * (x * 6) = 2 * (3 * (1 * (3 * x))). [back_rewrite(1024),rewrite([3183(4),41(8)]),flip(a)].
3289 (0 * x) * (3 * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(557),rewrite([2259(8)])].
3292 (7 * x) * (5 * y) = 8 * (1 * (3 * (x * y))). [back_rewrite(274),rewrite([2259(8)])].
3326 3 * (2 * (3 * (1 * (3 * (x * y))))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1973),rewrite([3183(2),2026(13),2026(11),1001(9),41(9)])].
3327 2 * (3 * (x * (3 * y))) = x * (2 * (3 * (1 * (3 * (1 * y))))). [back_rewrite(1947),rewrite([3183(9)])].
3339 5 * (2 * x) = 2 * (3 * x). [back_rewrite(989),rewrite([3183(7),3326(16),41(10)])].
3355 2 * (0 * 1) = 2. [back_rewrite(74),rewrite([3183(3),18(7),1063(6),17(5),1818(4)])].
3424 10 * 2 = 7. [para(2008(a,1),41(a,1,2)),rewrite([12(3)])].
3449 (10 * x) * (2 * y) = 3 * (1 * (3 * (x * y))). [para(3424(a,1),14(a,1,1)),rewrite([2253(3)]),flip(a)].

3462 7 * x = 3 * (1 * (3 * x)). [para(3424(a,1),47(a,2,1)),rewrite([3449(6),13(5)]),flip(a)].
3477 1 * (3 * ((3 * x) * (5 * y))) = 8 * (1 * (3 * (x * y))). [back_rewrite(3292),rewrite([3462(2),2026(9)])].
3504 5 * (x * y) = 1 * (3 * (0 * (x * y))). [back_rewrite(345),rewrite([3462(2),2026(9),264(7)]),flip(a)].
3508 3 * (0 * 1) = 5. [back_rewrite(98),rewrite([3462(3),22(5),12(4)])].
3522 1 * (3 * (0 * (2 * x))) = 2 * (3 * x). [back_rewrite(3339),rewrite([3504(4)])].
3648 (3 * x) * (5 * y) = (0 * 1) * (x * y). [para(1818(a,1),14(a,1,1)),flip(a)].
3656 (5 * x) * (3 * y) = (0 * 1) * (x * y). [para(1818(a,1),43(a,2,1))].
3670 8 * (1 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3477),rewrite([3648(7),604(8)]),flip(a)].
3687 (0 * x) * (3 * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(3289),rewrite([3670(12)])].
3714 9 * (x * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(2259),rewrite([3670(10)])].
3915 2 * 2 = 0 * 1. [para(3355(a,1),41(a,1,2))].
3927 0 * (x * 2) = 2 * (x * 1). [para(3355(a,1),47(a,1,2)),rewrite([12(4),12(8)]),flip(a)].
3933 (3 * x) * ((0 * 1) * y) = 1 * (3 * (0 * (x * y))). [para(3508(a,1),14(a,1,1)),rewrite([3504(3)]),flip(a)].
3949 5 * x = 1 * (3 * (0 * x)). [para(3508(a,1),47(a,2,1)),rewrite([3933(8),13(5)]),flip(a)].
3960 (0 * 1) * (x * y) = 3 * (0 * (2 * (x * y))). [back_rewrite(3656),rewrite([3949(2),2026(9),3687(7),41(10)]),flip(a)].
3992 1 * 10 = 9. [back_rewrite(26),rewrite([3949(3),120(5),21(4)])].
3993 1 * (3 * (0 * 6)) = 10. [back_rewrite(25),rewrite([3949(3)])].
4005 3 * (x * (2 * (10 * y))) = 1 * (0 * (2 * (x * y))). [para(3992(a,1),14(a,1,1)),rewrite([3714(3),1001(12)]),flip(a)].
4017 1 * (0 * (x * (3 * y))) = x * (10 * y). [para(3992(a,1),46(a,1,1)),rewrite([3714(5),668(7)])].
4022 9 * x = 1 * (0 * (2 * x)). [para(3992(a,1),47(a,2,1)),rewrite([1001(6),4005(8),13(5)]),flip(a)].
4061 0 * (1 * (8 * (x * y))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1905),rewrite([4022(2),2026(9),474(7)])].
4068 0 * (1 * ((2 * x) * (2 * y))) = 8 * (x * y). [back_rewrite(475),rewrite([4022(2),2026(9)])].
4453 (2 * x) * (2 * y) = 3 * (0 * (2 * (x * y))). [para(3915(a,1),14(a,1,1)),rewrite([3960(5)]),flip(a)].
4466 8 * (x * y) = 0 * (2 * (3 * (x * y))). [back_rewrite(4068),rewrite([4453(7),3522(10)]),flip(a)].
4483 2 * (3 * (1 * (x * y))) = 0 * (2 * (10 * (x * y))). [back_rewrite(4061),rewrite([4466(5),4017(10)]),flip(a)].
4497 2 * (10 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3670),rewrite([4466(7),4483(10),41(11)])].
4614 2 * (3 * (x * (3 * y))) = x * (0 * (1 * (0 * (2 * (1 * y))))). [back_rewrite(3327),rewrite([4483(17),4497(16)])].
4622 3 * (x * 6) = 1 * (0 * (1 * (0 * (2 * (1 * x))))). [back_rewrite(3246),rewrite([4614(12)])].
4785 0 * (1 * (0 * 2)) = 10. [back_rewrite(3993),rewrite([4622(6),12(9),3927(10,R),41(10),41(11)])].
4801 2 * (1 * (0 * 0)) = 10. [para(4785(a,1),51(a,1)),flip(a)].
5778 1 * (0 * 0) = 7. [para(4801(a,1),41(a,1,2)),rewrite([12(3),3424(3)]),flip(a)].
5792 0 * 0 = 6. [para(5778(a,1),41(a,1,2)),rewrite([112(3)]),flip(a)].
5793 $F. [resolve(5792,a,40,a)].

============================== end of proof ==========================

P_{44}

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 10 at 0.97 (+ 0.03) seconds.
% Length of proof is 231.
% Level of proof is 35.
% Maximum clause weight is 27.
% Given clauses 99.

10 10 * 10 = 0 * 1 # label(non_clause) # label(goal). [goal].
12 x * y = y * x. [assumption].
13 x * (y * x) = y. [assumption].
14 (x * y) * (z * u) = (x * z) * (y * u). [assumption].
15 x * (y * (z * u)) = z * (y * (x * u)). [assumption].
16 1 * 2 = 3. [assumption].
17 1 * 4 = 5. [assumption].
18 1 * 6 = 7. [assumption].
19 2 * 4 = 6. [assumption].
20 2 * 8 = 9. [assumption].
21 3 * 8 = 10. [assumption].
22 3 * 9 = 0. [assumption].
23 4 * 0 = 10. [assumption].
24 0 * 4 = 10. [copy(23),rewrite([12(3)])].
25 5 * 6 = 10. [assumption].
26 5 * 7 = 9. [assumption].
27 7 * 8 = 0. [assumption].
38 0 * 1 != 10 * 10. [deny(10)].
39 10 * 10 != 0 * 1. [copy(38),flip(a)].
41 x * (x * y) = y. [para(12(a,1),13(a,1,2))].
43 (x * y) * (z * u) = (z * x) * (y * u). [para(12(a,1),14(a,1,1)),flip(a)].
44 (x * y) * (z * u) = (x * u) * (y * z). [para(12(a,1),14(a,1,2))].
45 (x * y) * ((z * x) * (u * y)) = z * u. [para(14(a,1),13(a,1,2))].
46 (x * y) * ((z * x) * u) = z * (y * u). [para(13(a,1),14(a,1,1)),flip(a)].
47 (x * y) * (z * (u * y)) = (x * z) * u. [para(13(a,1),14(a,1,2)),flip(a)].
49 (x * (y * z)) * (u * (w * v5)) = (x * u) * ((y * w) * (z * v5)). [para(14(a,1),14(a,1,2)),flip(a)].
50 (x * (y * z)) * u = y * (x * (u * z)). [para(15(a,1),12(a,1)),flip(a)].
51 x * (y * (z * u)) = u * (y * (x * z)). [para(12(a,1),15(a,1,2,2))].
54 x * (y * (z * (u * x))) = z * (y * u). [para(13(a,1),15(a,1,2,2)),flip(a)].
57 (x * y) * (z * (u * w)) = u * (y * ((x * z) * w)). [para(15(a,1),14(a,1)),flip(a)].
62 2 * 3 = 1. [para(16(a,1),13(a,1,2))].
63 (1 * x) * (2 * y) = 3 * (x * y). [para(16(a,1),14(a,1,1)),flip(a)].
74 4 * 6 = 2. [para(19(a,1),13(a,1,2))].
86 0 * 9 = 3. [para(22(a,1),13(a,1,2)),rewrite([12(3)])].
98 7 * 9 = 5. [para(26(a,1),13(a,1,2))].
102 0 * 8 = 7. [para(27(a,1),13(a,1,2)),rewrite([12(3)])].
110 1 * 3 = 2. [para(16(a,1),41(a,1,2))].
111 1 * 5 = 4. [para(17(a,1),41(a,1,2))].
112 1 * 7 = 6. [para(18(a,1),41(a,1,2))].
113 2 * 6 = 4. [para(19(a,1),41(a,1,2))].
114 2 * 9 = 8. [para(20(a,1),41(a,1,2))].
115 10 * 3 = 8. [para(21(a,1),41(a,1,2)),rewrite([12(3)])].
116 0 * 3 = 9. [para(22(a,1),41(a,1,2)),rewrite([12(3)])].
117 0 * 10 = 4. [para(24(a,1),41(a,1,2))].
118 10 * 5 = 6. [para(25(a,1),41(a,1,2)),rewrite([12(3)])].
119 5 * 9 = 7. [para(26(a,1),41(a,1,2))].
120 0 * 7 = 8. [para(27(a,1),41(a,1,2)),rewrite([12(3)])].
247 (2 * x) * (1 * y) = 3 * (x * y). [para(16(a,1),43(a,2,1))].

253 (6 * x) * (1 * y) = 7 * (x * y). [para(18(a,1),43(a,2,1))].
256 (4 * x) * (2 * y) = 6 * (x * y). [para(19(a,1),43(a,2,1))].
262 (8 * x) * (3 * y) = 10 * (x * y). [para(21(a,1),43(a,2,1))].
264 (3 * x) * (y * 9) = 0 * (x * y). [para(22(a,1),43(a,
1,2)),rewrite([12(3)]),flip(a)].
274 (7 * x) * (5 * y) = 9 * (x * y). [para(26(a,1),43(a,2,1))].
277 (8 * x) * (7 * y) = 0 * (x * y). [para(27(a,1),43(a,2,1))].
286 (3 * x) * (2 * y) = 1 * (x * y). [para(62(a,1),43(a,2,1))].
345 (7 * x) * (y * 9) = 5 * (x * y). [para(98(a,1),43(a,
1,2)),rewrite([12(3)]),flip(a)].
347 (0 * x) * (8 * y) = 7 * (x * y). [para(102(a,1),14(a,1,1)),flip(a)].
432 (x * 9) * (y * 7) = 5 * (x * y). [para(98(a,1),44(a,
1,2)),rewrite([12(3)]),flip(a)].
441 (1 * x) * (5 * y) = 4 * (x * y). [para(111(a,1),14(a,1,1)),flip(a)].
466 (6 * x) * (2 * y) = 4 * (x * y). [para(113(a,1),43(a,2,1))].
469 (x * 2) * (y * 9) = 8 * (x * y). [para(114(a,1),14(a,
1,2)),rewrite([12(3)]),flip(a)].
474 (2 * x) * (y * 9) = 8 * (x * y). [para(114(a,1),43(a,
1,2)),rewrite([12(3)]),flip(a)].
475 (9 * x) * (2 * y) = 8 * (x * y). [para(114(a,1),43(a,2,1))].
556 (x * 3) * (y * 10) = 8 * (x * y). [para(115(a,1),44(a,
1,2)),rewrite([12(3)]),flip(a)].
557 (0 * x) * (3 * y) = 9 * (x * y). [para(116(a,1),14(a,1,1)),flip(a)].
566 (0 * x) * (10 * y) = 4 * (x * y). [para(117(a,1),14(a,1,1)),flip(a)].
576 (x * 10) * (y * 5) = 6 * (x * y). [para(118(a,1),14(a,
1,2)),rewrite([12(3)]),flip(a)].
604 3 * ((x * 1) * y) = x * (2 * y). [para(16(a,1),46(a,1,1))].
607 6 * ((x * 2) * y) = x * (4 * y). [para(19(a,1),46(a,1,1))].
608 9 * ((x * 2) * y) = x * (8 * y). [para(20(a,1),46(a,1,1))].
616 (x * y) * ((y * x) * z) = z. [para(46(a,2),41(a,1))].
619 1 * ((x * 2) * y) = x * (3 * y). [para(62(a,1),46(a,1,1))].
668 2 * ((x * 1) * y) = x * (3 * y). [para(110(a,1),46(a,1,1))].
734 (x * y) * (z * (y * x)) = z. [para(12(a,1),616(a,1,2))].
989 5 * (2 * x) = 3 * (4 * x). [para(17(a,1),63(a,1,1))].
995 6 * (0 * 1) = 8. [para(24(a,1),63(a,2,2)),rewrite([12(3),19(6),
12(5),12(8),115(8)])].
996 4 * 4 = 8. [para(25(a,1),63(a,2,2)),rewrite([111(3),113(4),12(6),
115(6)])].
998 4 * (2 * 7) = 0. [para(26(a,1),63(a,2,2)),rewrite([111(3),22(8)])].
999 6 * 9 = 9. [para(27(a,1),63(a,2,2)),rewrite([112(3),20(4),12(6),
116(6)])].
1001 (1 * x) * y = 3 * (x * (2 * y)). [para(41(a,1),63(a,1,2))].
1007 8 * (0 * 1) = 3 * 3. [para(86(a,1),63(a,2,2)),rewrite([12(3),
114(6),12(5)])].
1015 6 * 8 = 3 * 5. [para(98(a,1),63(a,2,2)),rewrite([112(3),
114(4)])].
1024 4 * (1 * x) = 3 * (x * 6). [para(113(a,1),63(a,
1,2)),rewrite([12(4)])].
1036 4 * 8 = 3 * 7. [para(119(a,1),63(a,2,2)),rewrite([111(3),
114(4)])].
1040 (x * y) * (3 * (z * y)) = 2 * (x * (1 * z)). [para(63(a,1),47(a,
1,2)),rewrite([12(10)])].
1058 4 * (x * y) = 3 * (x * (2 * (5 * y))).
[back_rewrite(441),rewrite([1001(5)]),flip(a)].
1063 3 * 7 = 4. [para(996(a,1),13(a,1,2)),rewrite([1036(3)])].
1077 9 * 9 = 6. [para(999(a,1),13(a,1,2))].
1085 (6 * x) * (y * 9) = 9 * (x * y). [para(999(a,1),43(a,
1,2)),rewrite([12(3)]),flip(a)].
1095 4 * 7 = 3. [para(1063(a,1),13(a,1,2)),rewrite([12(3)])].
1099 3 * 4 = 7. [para(1063(a,1),41(a,1,2))].
1102 (x * 3) * (7 * y) = 4 * (x * y). [para(1063(a,1),43(a,
1,1)),flip(a)].
1442 (9 * x) * (y * 9) = 6 * (x * y). [para(1077(a,1),43(a,
1,2)),rewrite([12(3)]),flip(a)].
1449 (x * 4) * (y * 7) = 3 * (x * y). [para(1095(a,1),14(a,
1,2)),rewrite([12(3)]),flip(a)].
1453 2 * (7 * (1 * x)) = 4 * x. [para(1095(a,1),45(a,
1,2,1)),rewrite([1040(6)])].
1467 (x * 4) * (y * 3) = 7 * (x * y). [para(1099(a,1),44(a,
1,2)),rewrite([12(3)]),flip(a)].
1701 (x * (y * z)) * (y * (x * u)) = z * u. [para(49(a,2),616(a,1))].
1812 3 * 3 = 6. [para(995(a,1),13(a,1,2)),rewrite([12(5),1007(5)])].
1818 3 * 5 = 0 * 1. [para(995(a,1),41(a,1,2)),rewrite([1015(3)])].
1841 3 * 6 = 3. [para(1812(a,1),13(a,1,2))].
1845 (x * 3) * (3 * y) = 6 * (x * y). [para(1812(a,1),43(a,
1,1)),flip(a)].
1853 6 * x = 2 * (3 * (1 * x)). [para(1812(a,1),47(a,
2,1)),rewrite([1040(6)]),flip(a)].
1869 (x * 3) * (3 * y) = 2 * (3 * (1 * (x * y))).
[back_rewrite(1845),rewrite([1853(8)])].
1905 (9 * x) * (y * 9) = 2 * (3 * (1 * (x * y))).
[back_rewrite(1442),rewrite([1853(8)])].
1928 8 * ((3 * (1 * x)) * y) = 9 * (x * y).
[back_rewrite(1085),rewrite([1853(2),474(9)])].
1947 2 * (3 * (x * (3 * y))) = x * (4 * y).
[back_rewrite(607),rewrite([1853(5),619(7)])].
1953 (x * 10) * (y * 5) = 2 * (3 * (1 * (x * y))).
[back_rewrite(576),rewrite([1853(8)])].
1956 (2 * (3 * (1 * x))) * (2 * y) = 4 * (x * y).
[back_rewrite(466),rewrite([1853(2)])].
1973 (4 * x) * (2 * y) = 2 * (3 * (1 * (x * y))).
[back_rewrite(256),rewrite([1853(8)])].
1976 3 * ((3 * (1 * x)) * y) = 7 * (x * y).
[back_rewrite(253),rewrite([1853(2),247(9)])].
1991 (x * 6) * (y * 3) = 3 * (x * y). [para(1841(a,1),44(a,
1,2)),rewrite([12(3)]),flip(a)].
2008 2 * 7 = 10. [para(998(a,1),41(a,1,2)),rewrite([12(3),
24(3)]),flip(a)].
2026 (x * (y * z)) * u = y * (x * (z * u)). [para(12(a,1),50(a,2,2,2))].
2253 7 * (x * y) = 3 * (1 * (3 * (x * y))).
[back_rewrite(1976),rewrite([2026(6)]),flip(a)].
2257 4 * (x * y) = 3 * (2 * (3 * (x * y))).
[back_rewrite(1956),rewrite([2026(9),1001(7),41(7)]),flip(a)].
2259 9 * (x * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(1928),rewrite([2026(6)]),flip(a)].
2405 x * (y * (z * (x * (y * u)))) = z * u.
[back_rewrite(1701),rewrite([2026(5)])].
3177 (x * 4) * (y * 3) = 3 * (1 * (3 * (x * y))).
[back_rewrite(1467),rewrite([2253(8)])].
3183 4 * x = 2 * (3 * (1 * (3 * (1 * x)))).
[back_rewrite(1453),rewrite([2253(5)]),flip(a)].
3204 (0 * x) * (8 * y) = 3 * (1 * (3 * (x * y))).
[back_rewrite(347),rewrite([2253(8)])].
3237 (x * 3) * (7 * y) = 2 * (3 * (1 * (3 * (1 * (x * y))))).
[back_rewrite(1102),rewrite([3183(8)])].
3243 3 * (x * (2 * (5 * y))) = 2 * (3 * (1 * (3 * (1 * (x * y))))).
[back_rewrite(1058),rewrite([3183(3)]),flip(a)].
3246 3 * (x * 6) = 2 * (3 * (1 * (3 * x))).
[back_rewrite(1024),rewrite([3183(4),41(8)]),flip(a)].
3257 (0 * x) * (10 * y) = 2 * (3 * (1 * (3 * (1 * (x * y))))).
[back_rewrite(566),rewrite([3183(8)])].
3283 8 * (1 * (3 * ((x * 2) * y))) = x * (8 * y).
[back_rewrite(608),rewrite([2259(5)])].
3289 (0 * x) * (3 * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(557),rewrite([2259(8)])].
3292 (7 * x) * (5 * y) = 8 * (1 * (3 * (x * y))).
[back_rewrite(274),rewrite([2259(8)])].

3321 2 * (3 * (1 * (3 * (1 * (x * y))))) = 3 * (2 * (3 * (x * y))). [back_rewrite(2257),rewrite([3183(3)])].
3326 3 * (2 * (3 * (1 * (3 * (x * y))))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1973),rewrite([3183(2),2026(13),2026(11),1001(9), 41(9)])].
3327 2 * (3 * (x * (3 * y))) = x * (2 * (3 * (1 * (3 * (1 * y))))). [back_rewrite(1947),rewrite([3183(9)])].
3339 5 * (2 * x) = 2 * (3 * x). [back_rewrite(989),rewrite([3183(7), 3326(16),41(10)])].
3355 2 * (0 * 1) = 2. [back_rewrite(74),rewrite([3183(3),18(7), 1063(6),17(5),1818(4)])].
3366 (0 * x) * (10 * y) = 3 * (2 * (3 * (x * y))). [back_rewrite(3257),rewrite([3321(16)])].
3379 3 * (2 * (3 * (x * y))) = 3 * (x * (2 * (5 * y))). [back_rewrite(3243),rewrite([3321(18)]),flip(a)].
3385 (x * 3) * (7 * y) = 3 * (2 * (3 * (x * y))). [back_rewrite(3237),rewrite([3321(16)])].
3424 10 * 2 = 7. [para(2008(a,1),41(a,1,2)),rewrite([12(3)])].
3429 (7 * x) * (2 * y) = 10 * (x * y). [para(2008(a,1),43(a,2,1))].
3449 (10 * x) * (2 * y) = 3 * (1 * (3 * (x * y))). [para(3424(a,1), 14(a,1,1)),rewrite([2253(3)]),flip(a)].
3462 7 * x = 3 * (1 * (3 * x)). [para(3424(a,1),47(a, 2,1)),rewrite([3449(6),13(5)]),flip(a)].
3468 1 * (3 * (1 * (x * y))) = 10 * (x * y). [back_rewrite(3429),rewrite([3462(2),2026(9),286(7)])].
3472 3 * (2 * (3 * (x * y))) = 2 * (3 * (1 * (x * (1 * (3 * y))))). [back_rewrite(3385),rewrite([3462(4),1869(9)]),flip(a)].
3477 1 * (3 * ((3 * x) * (5 * y))) = 8 * (1 * (3 * (x * y))). [back_rewrite(3292),rewrite([3462(2),2026(9)])].
3504 5 * (x * y) = 1 * (3 * (0 * (x * y))). [back_rewrite(345),rewrite([3462(2),2026(9),264(7)]),flip(a)].
3506 10 * (x * (1 * (3 * y))) = 0 * (x * y). [back_rewrite(277),rewrite([3462(4),262(9)])].
3508 3 * (0 * 1) = 5. [back_rewrite(98),rewrite([3462(3),22(5), 12(4)])].
3511 3 * (2 * (3 * (x * y))) = 2 * (3 * (10 * (x * y))). [back_rewrite(3321),rewrite([3468(9)]),flip(a)].
3522 1 * (3 * (0 * (2 * x))) = 2 * (3 * x). [back_rewrite(3339),rewrite([3504(4)])].
3553 (x * 9) * (y * 7) = 1 * (3 * (0 * (x * y))). [back_rewrite(432),rewrite([3504(8)])].
3565 2 * (3 * (10 * (x * y))) = 2 * (3 * (1 * (x * (1 * (3 * y))))). [back_rewrite(3472),rewrite([3511(7)])].
3594 3 * (x * (2 * (5 * y))) = 2 * (3 * (10 * (x * y))). [back_rewrite(3379),rewrite([3511(7)]),flip(a)].
3605 (0 * x) * (10 * y) = 2 * (3 * (10 * (x * y))). [back_rewrite(3366),rewrite([3511(12)])].
3648 (3 * x) * (5 * y) = (0 * 1) * (x * y). [para(1818(a,1),14(a, 1,1)),flip(a)].
3656 (5 * x) * (3 * y) = (0 * 1) * (x * y). [para(1818(a,1),43(a,2,1))].
3670 8 * (1 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3477),rewrite([3648(7),604(8)]),flip(a)].
3687 (0 * x) * (3 * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(3289),rewrite([3670(12)])].
3693 1 * (0 * (2 * ((x * 2) * y))) = x * (8 * y). [back_rewrite(3283),rewrite([3670(9)])].
3714 9 * (x * y) = 1 * (0 * (2 * (x * y))). [back_rewrite(2259),rewrite([3670(10)])].
3915 2 * 2 = 0 * 1. [para(3355(a,1),41(a,1,2))].
3924 2 * ((x * 2) * y) = x * ((0 * 1) * y). [para(3355(a,1),46(a,1,1))].
3927 0 * (x * 2) = 2 * (x * 1). [para(3355(a,1),47(a, 1,2)),rewrite([12(4),12(8)]),flip(a)].
3933 (3 * x) * ((0 * 1) * y) = 1 * (3 * (0 * (x * y))). [para(3508(a,1), 14(a,1,1)),rewrite([3504(3)]),flip(a)].
3949 5 * x = 1 * (3 * (0 * x)). [para(3508(a,1),47(a, 2,1)),rewrite([3933(8),13(5)]),flip(a)].
3960 (0 * 1) * (x * y) = 3 * (0 * (2 * (x * y))). [back_rewrite(3656),rewrite([3949(2),2026(9),3687(7), 41(10)]),flip(a)].
3969 3 * (x * (2 * (1 * (3 * (0 * y))))) = 2 * (3 * (10 * (x * y))). [back_rewrite(3594),rewrite([3949(4)])].
3992 1 * 10 = 9. [back_rewrite(26),rewrite([3949(3),120(5),21(4)])].
3993 1 * (3 * (0 * 6)) = 10. [back_rewrite(25),rewrite([3949(3)])].
4004 10 * 9 = 1. [para(3992(a,1),13(a,1,2))].
4005 3 * (x * (2 * (10 * y))) = 1 * (0 * (2 * (x * y))). [para(3992(a, 1),14(a,1,1)),rewrite([3714(3),1001(12)]),flip(a)].
4017 1 * (0 * (x * (3 * y))) = x * (10 * y). [para(3992(a,1),46(a, 1,1)),rewrite([3714(5),668(7)])].
4022 9 * x = 1 * (0 * (2 * x)). [para(3992(a,1),47(a, 2,1)),rewrite([1001(6),4005(8),13(5)]),flip(a)].
4061 0 * (1 * (8 * (x * y))) = 2 * (3 * (1 * (x * y))). [back_rewrite(1905),rewrite([4022(2),2026(9),474(7)])].
4068 0 * (1 * ((2 * x) * (2 * y))) = 8 * (x * y). [back_rewrite(475),rewrite([4022(2),2026(9)])].
4083 0 * (1 * (3 * x)) = 10 * x. [para(4004(a,1),45(a, 1,2,1)),rewrite([4022(2),2026(10),247(8),13(5)])].
4453 (2 * x) * (2 * y) = 3 * (0 * (2 * (x * y))). [para(3915(a,1),14(a, 1,1)),rewrite([3960(5)]),flip(a)].
4459 (x * y) * (3 * (0 * (2 * (x * y)))) = 0 * 1. [para(3915(a,1),45(a, 2)),rewrite([4453(6)])].
4462 (0 * 1) * x = 3 * (0 * (2 * x)). [para(3915(a,1),47(a, 2,1)),rewrite([4453(6),13(5)]),flip(a)].
4466 8 * (x * y) = 0 * (2 * (3 * (x * y))). [back_rewrite(4068),rewrite([4453(7),3522(10)]),flip(a)].
4473 2 * ((x * 2) * y) = x * (3 * (0 * (2 * y))). [back_rewrite(3924),rewrite([4462(9)])].
4483 2 * (3 * (1 * (x * y))) = 0 * (2 * (10 * (x * y))). [back_rewrite(4061),rewrite([4466(5),4017(10)]),flip(a)].
4497 2 * (10 * (3 * (x * y))) = 1 * (0 * (2 * (x * y))). [back_rewrite(3670),rewrite([4466(7),4483(10),41(11)])].
4546 (x * 3) * (y * 10) = 0 * (2 * (3 * (x * y))). [back_rewrite(556),rewrite([4466(8)])].
4555 (x * 2) * (y * 9) = 0 * (2 * (3 * (x * y))). [back_rewrite(469),rewrite([4466(8)])].
4558 x * (8 * y) = x * (10 * (0 * (2 * y))). [back_rewrite(3693),rewrite([4473(7),4017(11)]),flip(a)].
4590 2 * (3 * (10 * (x * y))) = 0 * (2 * (0 * (x * y))). [back_rewrite(3565),rewrite([4483(18),3506(16)])].
4614 2 * (3 * (x * (3 * y))) = x * (0 * (1 * (0 * (2 * (1 * y))))). [back_rewrite(3327),rewrite([4483(17),4497(16)])].
4622 3 * (x * 6) = 1 * (0 * (1 * (0 * (2 * (1 * x))))). [back_rewrite(3246),rewrite([4614(12)])].
4624 4 * x = 1 * (0 * (1 * (0 * (2 * x)))). [back_rewrite(3183),rewrite([4614(12),41(11)])].
4641 (x * 10) * (y * 5) = 0 * (2 * (10 * (x * y))). [back_rewrite(1953),rewrite([4483(12)])].
4677 0 * (2 * (0 * (x * (0 * (2 * y))))) = 3 * (1 * (3 * (x * y))). [back_rewrite(3204),rewrite([4558(5),3605(9),4590(11)])].
4702 3 * (x * (2 * (1 * (3 * (0 * y))))) = 0 * (2 * (0 * (x * y))). [back_rewrite(3969),rewrite([4590(18)])].
4785 0 * (1 * (0 * 2)) = 10. [back_rewrite(3993),rewrite([4622(6), 12(9),3927(10,R),41(10),41(11)])].
4791 1 * (0 * (x * (0 * 2))) = x * 10. [para(4785(a,1),15(a, 1,2)),flip(a)].
4792 1 * (0 * 2) = 4. [para(4785(a,1),41(a, 1,2)),rewrite([117(3)]),flip(a)].
4801 2 * (1 * (0 * 0)) = 10. [para(4785(a,1),51(a,1)),flip(a)].
4842 0 * 2 = 5. [para(4792(a,1),41(a,1,2)),rewrite([17(3)]),flip(a)].

4850 1 * (0 * (1 * (0 * (2 * 5)))) = 1. [para(4792(a,1),734(a,1,2)),rewrite([12(3),4842(3),12(3),4624(3)])].
4853 0 * (2 * (0 * x)) = 1 * (0 * (1 * (0 * (2 * x)))). [para(4792(a,1),47(a,2,1)),rewrite([4842(5),3949(5),1001(10),4702(12),13(5),4624(8)])].
4860 1 * (0 * (x * 5)) = x * 10. [back_rewrite(4791),rewrite([4842(5)])].
4942 3 * (1 * (3 * (x * y))) = 1 * (0 * (1 * (x * y))). [back_rewrite(4677),rewrite([4853(11),2405(12)]),flip(a)].
4946 1 * 8 = 1. [back_rewrite(4850),rewrite([4860(9),12(5),3424(5),120(4)])].
4995 (x * 4) * (y * 3) = 1 * (0 * (1 * (x * y))). [back_rewrite(3177),rewrite([4942(12)])].
5024 x * (3 * (y * (10 * (0 * (2 * x))))) = y * 10. [para(21(a,1),54(a,2,2)),rewrite([4558(4)])].
5210 0 * 5 = 2. [para(4842(a,1),41(a,1,2))].
5237 1 * 1 = 8. [para(4946(a,1),41(a,1,2))].
5276 2 * ((x * 0) * y) = x * (1 * (3 * (0 * y))). [para(5210(a,1),46(a,1,1)),rewrite([3949(7)])].
5698 3 * (x * (2 * (1 * y))) = 0 * (2 * (3 * (x * y))). [para(5237(a,1),14(a,1,1)),rewrite([4466(3),1001(12)]),flip(a)].
5703 0 * (2 * (x * (2 * y))) = x * (1 * y). [para(5237(a,1),46(a,1,1)),rewrite([4466(5),604(7)])].
5705 8 * x = 0 * (2 * (3 * x)). [para(5237(a,1),47(a,2,1)),rewrite([1001(6),5698(8),13(5)]),flip(a)].
5737 x * (0 * (2 * (3 * y))) = x * (10 * (0 * (2 * y))). [back_rewrite(4558),rewrite([5705(2)])].
5778 1 * (0 * 0) = 7. [para(4801(a,1),41(a,1,2)),rewrite([12(3),3424(3)]),flip(a)].
5788 3 * (x * (10 * (0 * y))) = 1 * (0 * (1 * (x * y))). [para(5778(a,1),14(a,1,1)),rewrite([3462(3),4942(7),1001(14),5276(14),4083(16)]),flip(a)].
5792 0 * 0 = 6. [para(5778(a,1),41(a,1,2)),rewrite([112(3)]),flip(a)].
5805 x * (1 * (0 * (1 * (y * (2 * x))))) = y * 10. [back_rewrite(5024),rewrite([5788(9)])].
5816 (0 * x) * (0 * y) = 0 * (2 * (10 * (x * y))). [para(5792(a,1),14(a,1,1)),rewrite([1853(3),4483(7)]),flip(a)].
5826 2 * (3 * (1 * x)) = 0 * (2 * (10 * x)). [para(5792(a,1),47(a,2,1)),rewrite([5816(6),13(5),1853(8)]),flip(a)].
5868 6 * x = 0 * (2 * (10 * x)). [back_rewrite(1853),rewrite([5826(8)])].
5881 (x * y) * (z * 3) = 1 * (y * (2 * (x * z))). [para(16(a,1),57(a,1,2,2)),rewrite([12(8)])].
5883 (x * y) * (z * 5) = 1 * (y * (1 * (0 * (1 * (0 * (2 * (x * z))))))). [para(17(a,1),57(a,1,2,2)),rewrite([12(8),4624(8)])].
5885 (x * y) * (z * 7) = 1 * (y * (0 * (2 * (10 * (x * z))))). [para(18(a,1),57(a,1,2,2)),rewrite([12(8),5868(8)])].
5891 (x * y) * (z * 9) = 2 * (y * (10 * (0 * (2 * (x * z))))). [para(20(a,1),57(a,1,2,2)),rewrite([12(8),5705(8),5737(13)])].
5894 (x * y) * (z * 10) = 1 * (0 * (1 * (y * (2 * (x * z))))). [para(21(a,1),57(a,1,2,2)),rewrite([12(8),5705(8),5737(13),5788(14)])].
6190 1 * (0 * (1 * (x * y))) = 0 * (1 * (0 * (x * y))). [back_rewrite(4995),rewrite([5881(5),4624(6),41(10),41(11)]),flip(a)].
6194 3 * (x * y) = 1 * (10 * (1 * (x * y))). [back_rewrite(1991),rewrite([5881(5),5868(6),5703(10)]),flip(a)].
6199 0 * (2 * (10 * (x * y))) = 1 * (10 * (0 * (1 * (2 * (x * y))))). [back_rewrite(4641),rewrite([5883(5),6190(13),41(11)]),flip(a)].
6208 0 * (2 * (1 * (10 * (0 * (1 * (2 * (x * y))))))) = 10 * (1 * (0 * (x * y))). [back_rewrite(3553),rewrite([5885(5),6199(9),4022(14),41(19),6194(21),41(26)])].
6212 1 * (10 * (1 * (x * y))) = 1 * (0 * (10 * (x * y))). [back_rewrite(1449),rewrite([5885(5),6199(9),4624(14),6208(19),6190(14),2405(13),6194(10)]),flip(a)].
6221 0 * (2 * (1 * (0 * (10 * (x * y))))) = 10 * (0 * (2 * (x * y))). [back_rewrite(4555),rewrite([5891(5),41(11),6194(12),6212(16)]),flip(a)].
6226 10 * (0 * (2 * (x * y))) = 1 * (10 * (2 * (x * y))). [back_rewrite(4546),rewrite([5894(5),6194(8),6212(12),41(13),41(10),6194(12),6212(16),6221(18)]),flip(a)].
6316 x * (0 * (1 * (0 * (y * (2 * x))))) = y * 10. [back_rewrite(5805),rewrite([6190(9)])].
6659 10 * 10 = 0 * 1. [back_rewrite(4459),rewrite([6194(8),6212(12),6226(10),6190(12),6316(13)])].
6660 $F. [resolve(6659,a,39,a)].

============================== end of proof ==========================

Proofs from §8.5

P_{45}

============================== PROOF

% -------- Comments from original proof --------
% Proof 1 at 18.31 (+ 0.50) seconds.
% Length of proof is 174.
% Level of proof is 18.
% Maximum clause weight is 19.
% Given clauses 1442.

1 (0 + 0) + 0 = (1 + 1) + 1 # label(non_clause) # label(goal). [goal].
2 (x + y) + z = x + (y + z). [assumption].
3 x + y = y + x. [assumption].
4 x + e = x. [assumption].
5 x + -x = e. [assumption].
6 f(x,y,z) = -x + (-y + -z). [assumption].
7 f(0,1,3) = 9. [assumption].
8 -0 + (-1 + -3) = 9. [copy(7),rewrite([6(4)])].
9 f(1,2,4) = 10. [assumption].
10 -1 + (-2 + -4) = 10. [copy(9),rewrite([6(4)])].
11 f(2,3,5) = 11. [assumption].
12 -2 + (-3 + -5) = 11. [copy(11),rewrite([6(4)])].
13 f(3,4,6) = 12. [assumption].
14 -3 + (-4 + -6) = 12. [copy(13),rewrite([6(4)])].
15 f(4,5,7) = 0. [assumption].
16 -4 + (-5 + -7) = 0. [copy(15),rewrite([6(4)])].
17 f(5,6,8) = 1. [assumption].
18 -5 + (-6 + -8) = 1. [copy(17),rewrite([6(4)])].
21 f(7,8,10) = 3. [assumption].
22 -7 + (-10 + -8) = 3. [copy(21),rewrite([6(4),3(7)])].
23 f(8,9,11) = 4. [assumption].
24 -8 + (-11 + -9) = 4. [copy(23),rewrite([6(4),3(7)])].
27 f(10,11,0) = 6. [assumption].
28 -10 + (-0 + -11) = 6. [copy(27),rewrite([6(4),3(7)])].
29 f(11,12,1) = 7. [assumption].
30 -11 + (-1 + -12) = 7. [copy(29),rewrite([6(4),3(7)])].
31 f(12,0,2) = 8. [assumption].
32 -12 + (-0 + -2) = 8. [copy(31),rewrite([6(4)])].
33 (1 + 1) + 1 != (0 + 0) + 0. [deny(1)].
34 1 + (1 + 1) != 0 + (0 + 0). [copy(33),rewrite([3(5),3(10)])].
35 x + (y + z) = y + (x + z). [para(3(a,1),2(a,1,1)),rewrite([2(2)])].
36 -0 + (-12 + -2) = 8. [back_rewrite(32),rewrite([35(8)])].
37 -1 + (-11 + -12) = 7. [back_rewrite(30),rewrite([35(8)])].
38 -0 + (-10 + -11) = 6. [back_rewrite(28),rewrite([35(8)])].
40 -11 + (-8 + -9) = 4. [back_rewrite(24),rewrite([35(8)])].

41 -10 + (-7 + -8) = 3. [back_rewrite(22),rewrite([35(8)])].
44 e + x = x. [para(4(a,1),3(a,1)),flip(a)].
45 x + (-x + y) = y. [para(5(a,1),2(a,1,1)),rewrite([44(2)]),flip(a)].
47 -x + (y + x) = y. [para(5(a,1),2(a,2,2)),rewrite([3(3),4(5)])].
62 x + (y + -x) = y. [para(5(a,1),35(a,1,2)),rewrite([4(2)]),flip(a)].
72 -x + (y + (x + z)) = y + z. [para(45(a,1),2(a,2,2)),rewrite([35(4),
2(3)])].
73 --x = x. [para(5(a,1),45(a,1,2)),rewrite([4(2)]),flip(a)].
74 -1 + -3 = 0 + 9. [para(8(a,1),45(a,1,2)),flip(a)].
75 -2 + -4 = 1 + 10. [para(10(a,1),45(a,1,2)),flip(a)].
76 -3 + -5 = 11 + 2. [para(12(a,1),45(a,1,2)),rewrite([3(3)]),flip(a)].
77 -4 + -6 = 12 + 3. [para(14(a,1),45(a,1,2)),rewrite([3(3)]),flip(a)].
78 -5 + -7 = 0 + 4. [para(16(a,1),45(a,1,2)),rewrite([3(3)]),flip(a)].
79 -6 + -8 = 1 + 5. [para(18(a,1),45(a,1,2)),rewrite([3(3)]),flip(a)].
90 -12 + -2 = 0 + 8. [para(36(a,1),45(a,1,2)),flip(a)].
91 -x + (y + (z + x)) = y + z. [para(2(a,1),47(a,1,2))].
94 x + -(x + y) = -y. [para(47(a,1),47(a,1,2)),rewrite([3(3)])].
96 x + (y + (z + -x)) = y + z. [para(2(a,1),62(a,1,2))].
97 -1 + (-3 + x) = 0 + (9 + x). [para(74(a,1),2(a,
1,1)),rewrite([2(4)]),flip(a)].
98 -3 + (x + -1) = x + (0 + 9). [para(74(a,1),2(a,
2,2)),rewrite([3(6)])].
100 0 + (1 + 9) = -3. [para(74(a,1),45(a,1,2)),rewrite([35(5)])].
101 0 + (3 + 9) = -1. [para(74(a,1),47(a,1,2)),rewrite([73(3),35(5)])].
102 0 + (1 + (9 + x)) = -3 + x. [para(100(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
103 1 + (9 + (x + 0)) = x + -3. [para(100(a,1),2(a,
2,2)),rewrite([35(6),3(5)])].
109 -11 + -12 = 1 + 7. [para(37(a,1),45(a,1,2)),flip(a)].
113 0 + (x + (3 + 9)) = x + -1. [para(101(a,1),35(a,1,2)),flip(a)].
115 -2 + (-4 + x) = 1 + (10 + x). [para(75(a,1),2(a,
1,1)),rewrite([2(4)]),flip(a)].
116 -4 + (x + -2) = x + (1 + 10). [para(75(a,1),2(a,
2,2)),rewrite([3(6)])].
118 1 + (10 + 2) = -4. [para(75(a,1),45(a,1,2)),rewrite([35(5),3(4)])].
120 1 + (10 + (2 + x)) = -4 + x. [para(118(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
131 -10 + -11 = 0 + 6. [para(38(a,1),45(a,1,2)),flip(a)].
133 -3 + (-5 + x) = 11 + (2 + x). [para(76(a,1),2(a,
1,1)),rewrite([2(4)]),flip(a)].
136 11 + (2 + 3) = -5. [para(76(a,1),45(a,1,2)),rewrite([35(5),3(4)])].
137 11 + (2 + 5) = -3. [para(76(a,1),47(a,1,2)),rewrite([73(3),35(5),
3(4)])].
138 11 + (2 + (3 + x)) = -5 + x. [para(136(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
150 12 + (3 + 6) = -4. [para(77(a,1),47(a,1,2)),rewrite([73(3),35(5),
3(4)])].
160 12 + (3 + (6 + x)) = -4 + x. [para(150(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
161 3 + (6 + (x + 12)) = x + -4. [para(150(a,1),2(a,
2,2)),rewrite([35(6),3(5)])].
167 0 + (4 + 5) = -7. [para(78(a,1),45(a,1,2)),rewrite([35(5),3(4)])].
168 0 + (4 + 7) = -5. [para(78(a,1),47(a,1,2)),rewrite([73(3),35(5),
3(4)])].
176 -8 + -9 = 11 + 4. [para(40(a,1),45(a,1,2)),flip(a)].
185 1 + (5 + 6) = -8. [para(79(a,1),45(a,1,2)),rewrite([35(5),3(4)])].
186 1 + (5 + 8) = -6. [para(79(a,1),47(a,1,2)),rewrite([73(3),35(5),
3(4)])].
187 1 + (5 + (6 + x)) = -8 + x. [para(185(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
198 -7 + -8 = 10 + 3. [para(41(a,1),45(a,1,2)),flip(a)].
216 0 + (12 + 8) = -2. [para(90(a,1),45(a,1,2)),rewrite([35(5)])].
217 0 + (2 + 8) = -12. [para(90(a,1),47(a,1,2)),rewrite([73(3),
35(5)])].
223 12 + (8 + (x + 0)) = x + -2. [para(216(a,1),2(a,
2,2)),rewrite([35(6),3(5)])].
250 -(x + y) = -y + -x. [para(47(a,1),94(a,1,2,1)),flip(a)].
258 -2 + -3 = 11 + 5. [para(136(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
259 -2 + -5 = 11 + 3. [para(137(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
264 -4 + -5 = 0 + 7. [para(167(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
265 -4 + -7 = 0 + 5. [para(168(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
267 -5 + -6 = 1 + 8. [para(185(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
268 -5 + -8 = 1 + 6. [para(186(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
273 -12 + -8 = 0 + 2. [para(216(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
274 -2 + -8 = 0 + 12. [para(217(a,1),94(a,1,2,1)),rewrite([73(4),
250(7),3(8)]),flip(a)].
275 -x + (y + (z + (x + u))) = y + (z + u). [para(2(a,1),72(a,
1,2)),rewrite([2(7)])].
282 3 + (x + (11 + 2)) = x + -5. [para(76(a,1),72(a,
1,2,2)),rewrite([73(3)])].
284 -11 + (x + -3) = x + (2 + 5). [para(137(a,1),72(a,1,2,2))].
303 1 + (11 + 7) = -12. [para(109(a,1),45(a,1,2)),rewrite([35(5)])].
304 1 + (12 + 7) = -11. [para(109(a,1),47(a,1,2)),rewrite([73(3),
35(5)])].
312 1 + (x + (12 + 7)) = x + -11. [para(304(a,1),35(a,1,2)),flip(a)].
317 0 + (10 + 6) = -11. [para(131(a,1),45(a,1,2)),rewrite([35(5)])].
318 0 + (11 + 6) = -10. [para(131(a,1),47(a,1,2)),rewrite([73(3),
35(5)])].
321 0 + (10 + (6 + x)) = -11 + x. [para(317(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
340 -x + (y + (z + (u + x))) = y + (z + u). [para(2(a,1),91(a,1,2,2))].
351 -12 + -7 = 1 + 11. [para(303(a,1),91(a,1,2)),rewrite([3(5)])].
356 -10 + -6 = 0 + 11. [para(318(a,1),91(a,1,2)),rewrite([3(5)])].
369 11 + (4 + 9) = -8. [para(176(a,1),47(a,1,2)),rewrite([73(3),35(5),
3(4)])].
382 -4 + -8 = 11 + 9. [para(369(a,1),72(a,1,2))].
387 10 + (3 + 7) = -8. [para(198(a,1),45(a,1,2)),rewrite([35(5),
3(4)])].
391 10 + (3 + (7 + x)) = -8 + x. [para(387(a,1),2(a,
1,1)),rewrite([2(8)]),flip(a)].
395 -3 + -8 = 10 + 7. [para(387(a,1),72(a,1,2))].
411 -1 + (x + (-3 + y)) = 0 + (9 + (x + y)). [para(35(a,1),97(a,1,2))].
413 11 + (2 + -1) = 0 + (9 + -5). [para(76(a,1),97(a,
1,2)),rewrite([35(6),3(5)])].
441 -3 + (x + (y + -1)) = 0 + (9 + (x + y)). [para(2(a,1),98(a,
1,2)),rewrite([35(12),3(11)])].
442 3 + (x + (y + (0 + 9))) = x + (y + -1). [para(98(a,1),72(a,
1,2,2)),rewrite([73(3)])].
448 -2 + (-3 + x) = 11 + (5 + x). [para(258(a,1),2(a,
1,1)),rewrite([2(4)]),flip(a)].
453 3 + (x + (11 + 5)) = x + -2. [para(258(a,1),91(a,
1,2,2)),rewrite([73(3)])].
462 5 + (x + (11 + 3)) = x + -2. [para(259(a,1),91(a,
1,2,2)),rewrite([73(3)])].
466 0 + (1 + (x + (9 + y))) = -3 + (x + y). [para(35(a,1),102(a,
1,2,2))].
496 1 + (9 + (x + (y + 0))) = -3 + (x + y). [para(2(a,1),103(a,
1,2,2)),rewrite([3(11)])].
525 8 + (x + (1 + 6)) = x + -5. [para(268(a,1),91(a,
1,2,2)),rewrite([73(3)])].

550 -12 + (-8 + x) = 0 + (2 + x). [para(273(a,1),2(a,1,1)),rewrite([2(4)]),flip(a)].
553 0 + (12 + 2) = -8. [para(273(a,1),45(a,1,2)),rewrite([35(5)])].
557 12 + (2 + (x + 0)) = x + -8. [para(553(a,1),2(a,2,2)),rewrite([35(6),3(5)])].
560 -2 + (-8 + x) = 0 + (12 + x). [para(274(a,1),2(a,1,1)),rewrite([2(4)]),flip(a)].
563 2 + (x + (0 + 12)) = x + -8. [para(274(a,1),72(a,1,2,2)),rewrite([73(3)])].
565 -12 + (-7 + x) = 1 + (11 + x). [para(351(a,1),2(a,1,1)),rewrite([2(4)]),flip(a)].
594 0 + (x + (y + (3 + 9))) = -1 + (x + y). [para(2(a,1),113(a,1,2)),rewrite([3(11)])].
619 -2 + (x + (-4 + y)) = 1 + (10 + (x + y)). [para(35(a,1),115(a,1,2))].
623 1 + (10 + (x + (4 + y))) = -2 + (x + y). [para(72(a,1),115(a,1,2)),flip(a)].
646 -3 + (-8 + x) = 10 + (7 + x). [para(395(a,1),2(a,1,1)),rewrite([2(4)]),flip(a)].
660 0 + (1 + (1 + (10 + 9))) = 1 + (10 + -3). [para(116(a,2),102(a,1,2,2)),rewrite([35(9),3(8),75(8),35(7),3(6),35(15),3(14)])].
661 11 + (5 + -4) = 1 + (10 + -3). [para(116(a,2),102(a,2)),rewrite([35(7),3(6),660(9),3(13),258(13),35(12),3(11)]),flip(a)].
688 -1 + -4 = 10 + 2. [para(120(a,1),96(a,1)),rewrite([3(5)])].
697 10 + (2 + 4) = -1. [para(688(a,1),47(a,1,2)),rewrite([73(3),35(5),3(4)])].
703 -1 + -2 = 10 + 4. [para(697(a,1),72(a,1,2)),rewrite([3(5)])].
708 -1 + (-2 + x) = 10 + (4 + x). [para(703(a,1),2(a,1,1)),rewrite([2(4)]),flip(a)].
718 -1 + -7 = 11 + 12. [para(109(a,1),250(a,1,1)),rewrite([250(4),3(5),73(8),73(9),3(8)])].
721 -11 + -4 = 8 + 9. [para(176(a,1),250(a,1,1)),rewrite([250(4),3(5),73(8),73(9),3(8)])].
730 -1 + -8 = 5 + 6. [para(267(a,1),250(a,1,1)),rewrite([250(4),3(5),73(8),73(9),3(8)])].
739 -0 + -11 = 10 + 6. [para(356(a,1),250(a,1,1)),rewrite([250(4),3(5),73(8),73(9),3(8)])].
803 11 + (12 + 7) = -1. [para(718(a,1),47(a,1,2)),rewrite([73(3),35(5),3(4)])].
831 11 + (2 + (x + (5 + y))) = -3 + (x + y). [para(72(a,1),133(a,1,2)),flip(a)].
837 11 + (2 + -8) = 1 + (6 + -3). [para(268(a,1),133(a,1,2)),rewrite([35(6),3(5)]),flip(a)].
846 4 + (8 + 9) = -11. [para(721(a,1),47(a,1,2)),rewrite([73(3)])].
848 4 + (8 + (9 + x)) = -11 + x. [para(846(a,1),2(a,1,1)),rewrite([2(8)]),flip(a)].
991 3 + (6 + (x + (y + 12))) = -4 + (x + y). [para(2(a,1),161(a,1,2,2)),rewrite([3(11)])].
1146 -8 + (x + -2) = 12 + (x + 0). [para(223(a,1),72(a,1,2))].
1485 2 + (5 + -1) = 0 + (9 + -11). [para(74(a,1),284(a,1,2)),rewrite([35(6),3(5),35(12),3(11)]),flip(a)].
1522 12 + (7 + -3) = 0 + (9 + -11). [para(312(a,1),102(a,1,2)),rewrite([35(12),3(11)]),flip(a)].
1552 0 + (x + (10 + (6 + y))) = x + (-11 + y). [para(321(a,1),35(a,1,2)),flip(a)].
1967 10 + (7 + -5) = 1 + (6 + -3). [para(282(a,1),391(a,1,2)),rewrite([35(12),3(11),837(12)])].
2336 12 + (6 + -2) = 1 + (10 + -3). [para(453(a,1),160(a,1,2)),rewrite([35(12),3(11),661(12)])].
2425 11 + (3 + -8) = 1 + (6 + -2). [para(462(a,1),187(a,1,2)),rewrite([35(12),3(11)]),flip(a)].
2758 11 + (2 + (x + -1)) = 0 + (9 + (-5 + x)). [para(441(a,1),133(a,1)),flip(a)].
2860 12 + (2 + (x + (y + 0))) = -8 + (x + y). [para(2(a,1),557(a,1,2,2)),rewrite([3(11)])].
2867 1 + (6 + -2) = 0 + (12 + -5). [para(563(a,1),138(a,1,2)),rewrite([2425(6),35(12),3(11)])].
3672 10 + (6 + -3) = 1 + (9 + -11). [para(739(a,1),496(a,2,2)),rewrite([3(8),35(9),739(8),317(7),35(12),3(11)]),flip(a)].
3997 4 + (9 + -5) = 1 + (6 + -11). [para(525(a,1),848(a,1,2)),rewrite([35(12),3(11)])].
4511 5 + (6 + -12) = 0 + (2 + -1). [para(550(a,1),442(a,1,2)),rewrite([35(7),35(9),35(8),35(7),113(8),3(13),730(13),35(12),3(11)]),flip(a)].
4538 10 + (10 + (4 + (7 + x))) = 0 + (0 + (12 + (9 + x))). [para(560(a,1),411(a,2,2,2)),rewrite([646(10),35(9),35(8),35(10),35(9),708(8),35(7),35(6),35(15),35(14)])].
5102 1 + (1 + (-11 + -11)) = 11 + -1. [para(413(a,1),623(a,2,2)),rewrite([35(9),2758(10),3(8),35(9),3997(9),35(11),35(10),1552(11),35(18),35(17),259(16),35(15),3(14),113(16)])].
5262 11 + (5 + (-8 + x)) = 10 + (7 + (-2 + x)). [para(646(a,1),448(a,1,2)),rewrite([35(7),35(6)]),flip(a)].
5264 10 + (7 + (-2 + x)) = 0 + (12 + (-3 + x)). [para(646(a,1),623(a,2,2)),rewrite([35(10),646(9),35(8),4538(9),35(10),35(9),466(9),35(6),35(14),35(13)]),flip(a)].
5265 11 + (5 + (-8 + x)) = 0 + (12 + (-3 + x)). [back_rewrite(5262),rewrite([5264(14)])].
5285 1 + (1 + (10 + -3)) = 0 + (5 + (5 + -12)). [para(661(a,1),565(a,2,2)),rewrite([35(9),3(8),265(8),35(7),35(8),35(7),3(6)]),flip(a)].
5723 1 + (1 + (1 + (9 + -11))) = 11 + (11 + 9). [para(837(a,1),619(a,2,2,2)),rewrite([35(10),382(9),35(8),35(10),35(9),35(8),3(7),62(8),35(14),3672(13)]),flip(a)].
6675 12 + (-3 + (x + 0)) = 10 + (7 + (x + -2)). [para(1146(a,1),646(a,1,2)),rewrite([35(7)])].
6679 10 + (7 + (x + -2)) = 0 + (12 + (-3 + x)). [para(1146(a,1),831(a,2,2)),rewrite([35(10),1146(9),35(8),35(9),2860(9),35(6),5265(7),35(14),6675(14)]),flip(a)].
9761 0 + (0 + (0 + (9 + -11))) = 11 + (11 + 9). [para(1967(a,1),619(a,2,2,2)),rewrite([35(10),264(9),35(8),35(9),35(10),35(9),35(8),3(7),6679(9),3(7),1522(8),35(19),3672(18),5723(20)])].
12153 11 + (11 + (3 + 9)) = 0 + (0 + (12 + 7)). [para(2867(a,1),991(a,2,2)),rewrite([3(9),2336(9),5285(10),35(11),35(10),35(9),4511(9),35(10),35(9),1485(9),9761(11),35(7),35(6),35(16),35(15),264(14),35(13)])].
16389 11 + (11 + -1) = 1 + (1 + -11). [para(5102(a,1),275(a,1,2)),rewrite([73(3)])].
16404 1 + (1 + (1 + -11)) = 11 + 11. [para(16389(a,1),91(a,1,2)),rewrite([73(3)])].
16431 11 + (11 + 11) = 1 + (1 + 1). [para(16404(a,1),340(a,1,2)),rewrite([73(3)])].
16459 0 + (0 + (0 + -1)) = 1 + 1. [para(16431(a,1),594(a,2,2)),rewrite([35(9),3(8),35(8),3(7),35(9),35(8),12153(9),35(10),35(9),803(8),35(16),35(15),3(14),5(14),4(12)])].
16494 1 + (1 + 1) = 0 + (0 + 0). [para(16459(a,1),340(a,1,2)),rewrite([73(3)])].
16495 $F. [resolve(16494,a,34,a)].

============================== end of proof ==========================

Bibliography

[1] J. Bokowski and B. Sturmfels, *Computational Synthetic Geometry*, Lecture Notes in Math.,vol. 1355, Springer, Heidelberg, 1989.

[2] Branko Gruenbaum, *Configurations of Points and Lines*, Graduate Studies in Mathmatics Volume 103, American mathematical Society, Rhode Island, 2009.

[3] R. Daublebsky, Die Configurationen (11, 3), *Monatshefte math.Phys*. 5 (1894)325-330 + Plate

[4] R. Daublebsky, Die Configurationen (12, 3), *Monatshefte Math.Phys*. 6 (1895)223--255 + Plate

[5] David Hilbert and Cohn-Vossen, *Geometry and Imagination*, Chelsea, New York.
This is a classic in which complete list of configurations (n, 3) for n up to 10 discussed in detail. (another reference proving the uniqueness of (7,3) and (8,3)).

[6] Eric Ens and R. Padmanabhan , *Group Embeddings of (11, 3) Configurations*, to appear

[7] Dale Husemoller, *Elliptic Curves*, Springer-Verlag, New York 1986.
the group structure of an elliptic curve is given here (e.g. the group structure non-singular cubic in the real plane is either S^1 or $\mathbb{Z}_2 \times S^1$)

[8] D. Johnson and N.S. Mendelsohn, Extended triple systems, *Aequationes Mathematicae*, 8 (1972) 291-298
Cubic curves and tangential relations are the motivations for extending the classical Steiner triple systems to "extended" triple systems - we are giving group realizations of extended triple systems by adding the tangential relations to the original configurations. This is the connection here.

[9] W. McCune, Prover9 and Mace4 (http://www.cs.unm.edu/~mccune/prover9/)

[10] N.S. Mendelsohn, R. Padmanabhan and Barry Wolk, Designs embeddable in a plane cubic curve, *Note di Mathematica* 7 (1987) 113-145
This paper gives many examples of tight embeddings using Galois fields and primitive root techniques

[11] N.S. Mendelsohn, R. Padmanabhan and Barry Wolk, Placement of the Desargues configuration on a cubic curve. *Geom. Dedicata* 40 (1991), no. 2, 165–170.

Here it is shown that the Desargues configuration (10,3) can be embedded in the group (Z2)^4. (with a different proof using Galois fields)

[12] Singer, James, A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.* 43 (1938), no. 3, 377–385.

[13] B. Sturmfels and Neil White, All (11,3) and (12, 3) configurations are rational, *Aequationes Mathematicae* 39 (1990) 254-260.
This paper gives the complete list all 31 types of (11, 3)-configurations and all 228 types of (12, 3) -configurations in the combinatorial classification of R. Daublebsky in modern notation. We are using this list for our proofs of group realizations.