A STUDY OF THE GEOMETRIC AND ALGEBRAIC SEWING OPERATIONS

by

Bryan Penfound

A Thesis submitted to

the Faculty of Graduate Studies

In Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

Department of Mathematics

University of Manitoba

Winnipeg, Manitoba

Abstract

The sewing operation is an integral component of both Geometric Function Theory and Conformal Field Theory and in this thesis we explore the interplay between the two fields. We will first generalize Huang's Geometric Sewing Equation to the quasi-symmetric case. That is, given specific maps $g(z)$ and $f^{-1}(z)$, we show the existence of the sewing maps $F_1(z)$ and $F_2(z)$. Second, we display an algebraic procedure using convergent matrix operations showing that the coefficients of the Conformal Welding Theorem maps $F(z)$ and $G(z)$ are dependent on the coefficients of the map ϕ. We do this for both the analytic and quasi-symmetric cases, and it is done using a special block/vector decomposition of a matrix representation called the power matrix. Lastly, we provide a partial result: given specific maps $g(z)$ and $f^{-1}(z)$ with analytic extensions, as well as a particular analytic map ϕ, it is possible to provide a method of determining the coefficients of the complementary maps.

I would like to acknowlege the Natural Sciences and Engineering Research Council of Canada (N.S.E.R.C.) for their two years of financial support through the Postgraduate Scholarship (P.G.S.) and Alexander Graham Bell Canada Graduate Scholarship (C.G.S.).

Also, many thanks to my supervisor, Dr. Eric Shippers, and my examining committee, for all their experience and guidance during my final year.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation and Statement of Results

The sewing operation, a way of obtaining a Riemann surface with punctures from two Riemann surfaces with punctures, appears in both Geometric Function Theory and Conformal Field Theory. In Conformal Field Theory, it appears when "gluing" together Riemann surfaces [Hua97] [Seg04]. In Geometric Function Theory, it appears in conformal welding, which is one of the basic tools of Teichmüller Theory [Leh87] [FM07]. A link between the sewing operations in Conformal Field Theory and Geometric Function Theory was drawn by [RS06]. For the purpose of this thesis, we intend to explore what sewing in the Geometric Function Theory setting, which tends to be analytic in nature, can tell us about Conformal Field Theory. We also explore what sewing in the Conformal Field Theory setting, which tends to be algebraic in nature, can tell us about Geometric Function Theory. What we wish to do first, is to briefly motivate the sewing operation in the simplest case: the case of two Riemann spheres being sewn together at their respective punctures.

Denote the complex plane as $\mathbb{C}$. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk and $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ be its complement (here, and throughout, we use the "*" notation to denote the complement of the closure of a set in the Riemann sphere $\overline{\mathbb{C}}$). Huang [Hua97] formulates his Sewing Equation in the following manner. Given two one-to-one and holomorphic maps $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \overline{\mathbb{C}}$, find analytic maps $F_1(z)$ and $F_2(z)$ satisfying

$$F_1 = F_2 \circ g \circ f^{-1}. \tag{1.1}$$

We will motivate this further in Section 1.2.

Huang treats equation (1.1) both geometrically and algebraically. Geometrically, he shows the existence of the conformal maps $F_1(z)$ and $F_2(z)$. Huang uses analytic extensions of the boundary parametrizations to formulate the Geometric Sewing Equation, so that the composition of maps "makes sense" on neighbourhoods of the parametrizations [Hua97]. This method is less general than the approach used by Radnell and Schippers, which is to restrict the boundary maps to be quasi-symmetries with quasi-conformal extensions (these extensions need not be analytic). Quasi-symmetries are homeomorphisms which are not necessarily differentiable (see Subsection 1.3.1). The composition of such maps is defined on quasi-circles only [RS09]. The second set of assumptions has the advantage of being the appropriate conditions necessary for utilizing operators on certain Hilbert spaces, as well as being connected to Teichmüller Theory [RS06].

Huang also considers the algebraic properties of (1.1), in the sense that he investigates the algebraic dependence of the coefficients of the maps $F_1(z)$ and $F_2(z)$ on the coefficients of the maps $f^{-1}(z)$ and $g(z)$. Here Huang uses uses rings of formal power series. The main issue when dealing with formal power series is the convergence of infinite sums; and Huang resolves this issue by creating a large enough ring so that convergence of certain infinite sums is not necessary. Doing this, he is able to show that the maps $F_1(z)$ and $F_2(z)$ are indeed algebraically dependent on the coefficients of the maps $f^{-1}(z)$ and $g(z)$ [Hua97]. Instead of formal power series, our main theorems will use convergent matrix operations to analyze algebraic dependence. We will show that the infinite sums in the entries of doubly-infinite matrix products converge. In order to do this, we will assume either that the boundary map is a quasi-symmetry, or that it has an analytic extension.

Throughout the thesis, when discussing the existence of the maps $F_1(z)$ and $F_2(z)$, we will refer to (1.1) as the Geometric Sewing Equation. If we refer to (1.1) as the Algebraic Sewing Equation, it will be understood that we are discussing the algebraic dependence of the coefficients of $F_1(z)$ and $F_2(z)$ on the coefficients of $f^{-1}(z)$ and $g(z)$.

Tied closely to Huang's Geometric and Algebraic Sewing Equations (H.S.E.) are the Conformal Welding Theorem (C.W.T.) and the existence/dependence of the complementary map (C.M.). Denote the unit circle as $S^1 = \{z : |z| = 1\}$. The Conformal Welding Theorem can be briefly stated as follows: given a map $\phi : S^1 \to S^1$, find maps $F(z)$ and $G(z)$ conformal on the interior of the closure of the unit disk $\overline{\mathbb{D}}$ and the closure of the unit disk's complement $\overline{\mathbb{D}^*}$ satisfying

$$G^{-1} \circ F = \phi. \tag{1.2}$$

This will be made precise in Section 2.1. The definition of the complementary map will be given in Section 1.2, but for now we can assume that, in the geometric setting, the existence of the complementary map is given by the Riemann Mapping Theorem. However, when one works in the algebraic setting using formal power series or matrix operations, the dependence of the coefficients of the complementary map on the coefficients of the original map may not be so obvious. More on this issue will be given in Section 2.2.

What we wish to give next are two charts summarizing the known results of the three associated problems, one chart for each method: geometric and algebraic. Along the left side, we write the three closely related theorems; and along the top we write the assumptions utilized. For the geometric approach to sewing, we have the following chart:

Geometric Sewing

	Quasi-symmetric Maps	Analytic Extensions
H.S.E.	Theorem 2.3	solved by Huang in [Hua97]
C.W.T.	solved using function theory	special case of quasi-symmetric
C.M.	existence elementary	existence elementary

We have the following chart for the algebraic approach:

Algebraic Sewing

	Formal Power Series	Convergent Matrix Operations
H.S.E.	solved by Huang in [Hua97]	open problem
C.W.T.	open problem	Theorem 3.4 & Theorem 3.6
C.M.	algebraic dependence open	Theorem 3.7 (partial result)

This stated, we can summarize the main goals of this thesis. First, when dealing with algebraic sewing, we attempt to replace formal power series with convergent matrix operations. Doing this will allow for a better bridge between the algebraic and geometric approaches, as well as a better bridge with geometric function theory. Second, we wish to generalize from the analytic setting to the quasi-symmetric setting as much as possible. This is because in the quasi-symmetric setting, there are nice representations of operators on Hilbert spaces. Also, the quasi-symmetric setting has ties to Teichmüller Theory (see [RS06], [RS09] [RS10]). Lastly, we hope to answer the questions: (1) What does Huang's approach to sewing imply about geometric function theory? and (2) What does geometric function theory imply about Huang's approach to sewing?

Finally, we give a brief summary of our main results. In the geometric sewing setting, we show in Theorem 2.3 that Huang's Geometric Sewing Equation is solvable in the quasi-symmetric setting. That is, given the maps $g(z)$ and $f^{-1}(z)$, we can find unique sewing maps $F_1(z)$ and $F_2(z)$. In the algebraic sewing setting, Theorem 3.4 shows we are able to obtain the coefficients of the welding functions $F(z)$ and $G(z)$ from $\phi(z)$ using convergent matrix operations. Here, we assume the existence of an analytic extension of the map $\phi(z)$. Generalizing this argument to the quasi-symmetric case, we get Theorem 3.6, which gives an algebraic procedure to obtain the coefficients of the welding maps given a quasi-symmetric map $\phi(z)$. Both theorems require the existence of the welding maps to obtain the algebraic dependence of the maps on the data. In our last result, Theorem 3.7, we provide partial results for the dependence of the complementary map. We show that given analytic extensions of the maps $g(z)$ and $f^{-1}(z)$, as well as a particular quasi-symmetry $\phi(z)$, it is possible to provide a method for determining the coefficients of the complementary maps.

1.2 Motivation of Huang's Geometric Sewing Equation

To be able to motivate sewing geometrically, we first need to introduce some special holomorphic maps. Before doing this, however, we briefly discuss how the sewing operation has ties to a particular Lie group. The diffeomorphisms of the circle $Diff(S^1)$ is a Lie group, and its corresponding Lie algebra is $Vect(S^1)$, the set of smooth vector fields on the circle. Complexifying $Vect(S^1)$, we obtain the Witt algebra. However, it is well-known that there is no Lie group whose Lie algebra is the Witt algebra. Thus, there is no Lie group corresponding to the complexification of $Diff(S^1)$ [Sch08]. Segal and Neretin independently defined a particular semigroup which is, in some sense, the complexification of the diffeomophisms of the circle. This semigroup is referred to as the Neretin-Segal semigroup [Seg04] [Ner87] [RS09] and it will be used to motivate Huang's Geometric Sewing Equation.

Definition 1.1. Let $\mathcal{A} = \{(f, g)\}$ where $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \overline{\mathbb{C}}$ are one-to-one holomorphic maps that satisfy

1. $f(z)$ has a quasi-conformal extension $\widetilde{f}^0(z)$ to $\mathbb{C}$ and $g(z)$ has a quasi-conformal extension $\widetilde{g}^\infty(z)$ to $\overline{\mathbb{C}}$ (see Subsection 1.3.1 for the definition of quasi-conformal map and quasi-conformal extension).

2. The images of the maps do not overlap: $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \varnothing$.

3. The normalization for $f(z)$ is $f(0) = 0$.

4. The normalizations for $g(z)$ are $g(\infty) = \infty$ and $g'(\infty) = 1$.

Here, and throughout, we take $g'(\infty) = (h \circ g \circ h)'(0)$, where $h(z) = 1/z$. For the purpose of this thesis, we also introduce the superscript notation (f^0, g^∞). The superscript 0 denotes that the map is holomorphic at zero and the superscript ∞ denotes that the map is meromorphic at infinity. If it is not specified with a superscript, it will be understood that the letter f represents a map centered at zero, while the letter g designates a map centered at infinity.

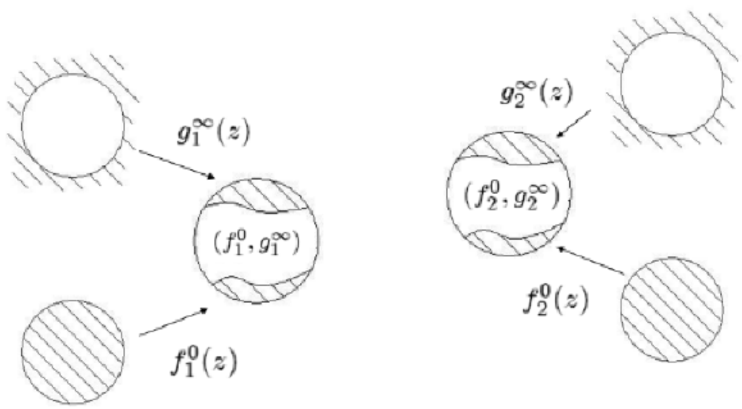
Definition 1.2. Let $f^0 : \mathbb{D} \to \mathbb{C}$ be one-to-one and holomorphic. The complementary map $f^\infty : \mathbb{D}^* \to \overline{\mathbb{C}} \setminus \overline{f^0(\mathbb{D})}$ of $f^0(z)$ is the unique one-to-one and holomorphic map satisfying $f^0(\mathbb{D}) = f^\infty(\mathbb{D}^*)^*$, $f^\infty(\infty) = \infty$ and ${f^\infty}'(\infty) > 0$; here, the complement is taken in the Riemann sphere.

Remark. Similarly, one can define a complementary map of a one-to-one holomorphic map $g^\infty : \mathbb{D}^* \to \overline{\mathbb{C}}$. This map will be denoted as $g^0 : \mathbb{D} \to \overline{\mathbb{C}} \setminus \overline{g^\infty(\mathbb{D}^*)}$ and satisfies the conditions $g^\infty(\mathbb{D}^*) = g^0(\mathbb{D})^*$, $g^0(0) = 0$ and ${g^0}'(0) > 0$; where the complement is taken in the Riemann sphere.

Next, we will describe the sewing process Huang uses to obtain his Geometric Sewing Equation (1.1), as well as describe the semigroup Neretin and Segal independently introduced. The multiplication in this semigroup is a special case of a more general sewing procedure. We will focus on this special case: sewing together two spheres with punctures together. More details on this approach can be found in [Hua97]; although we adopt the complementary map notation used in [RS09].

Consider two elements (f_1^0, g_1^∞) and (f_2^0, g_2^∞) of $\mathcal{A}$, as in Figure 1.1. By restricting

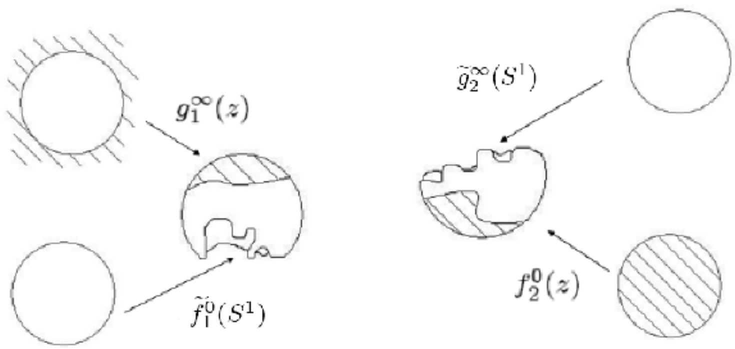
Figure 1.1: Selecting two elements of $\mathcal{A}$.



the analytic extensions $\widetilde{g}_2^\infty(z)$ and $\widetilde{f}_1^0(z)$ to the unit circle S^1, we are able to identify points on the boundaries of the two surfaces via the map $\widetilde{g}_2^\infty \circ (\widetilde{f}_1^0)^{-1}(z)$, as in Figure 1.2. We note that given $x \in \widetilde{f}_1^0(S^1)$ and $y \in \widetilde{g}_2^\infty(S^1)$, $x \sim_1 y$ if $y = \widetilde{g}_2^\infty \circ (\widetilde{f}_1^0)^{-1}(x)$.

Utilizing the complementary maps and the domains given in Figure 1.3, we have a "sewn" surface $\widehat{S}$ with "upper half" $f_1^\infty(\mathbb{D}^*)$ and "lower half" $g_2^0(\mathbb{D})$. We write $\widehat{S} = g_2^0(\mathbb{D})\sharp_{\widetilde{g}_2^\infty \circ (\widetilde{f}_1^0)^{-1}} f_1^\infty(\mathbb{D}^*)$, where the $\sharp$ notation represents the union of the surfaces modulo the $\sim_1$ relation. This sewn surface is bi-holomorphic to the

Figure 1.2: Restricting two maps to the unit circle.

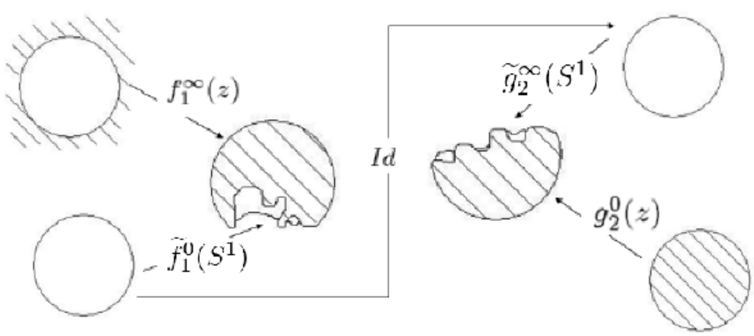


Riemann sphere (one must use existence of solutions to the Beltrami differential equation [RS09]), and the existence of $\widehat{H}(z)$

$$\widehat{H}(z) = \begin{cases} F_2(z), z \in g_2^0(\mathbb{D}) \subset \widehat{S} \\ F_1(z), z \in f_1^{\infty}(\mathbb{D}^*) \subset \widehat{S} \end{cases}, \tag{1.3}$$

defines the sewing maps $F_1(z)$ and $F_2(z)$ in terms of $\widehat{H}(z)$. The sewn surface $\widehat{S}$ and its bi-holomorphism $\widehat{H}(z)$ are shown in of Figure 1.4.

Figure 1.3: The complementary maps and their domains.

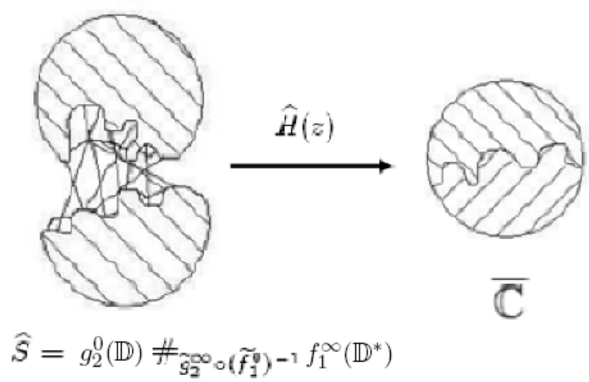


The fact that the two boundaries of the sewn surfaces are mapped to the same place on the new sphere yields the Geometric Sewing Equation

$$\widetilde{F}_1 = \widetilde{F}_2 \circ \widetilde{g}_2^{\infty} \circ (\widetilde{f}_1^0)^{-1}, \tag{1.4}$$

where $\widetilde{F}_1(z)$ and $\widetilde{F}_2(z)$ are analytic extensions of $F_1(z)$ and $F_2(z)$. This corresponds to the continuity of $\widehat{H}(z)$.

Figure 1.4: Biholomorphism between the sewn surface $\widehat{S}$ and the Riemann sphere.



We note that this sewing operation can be viewed as multiplication in the Neretin-Segal semigroup, provided we carry the remaining data through. The mutiplication arises from the boundary identifications used and the new surfaces that are produced. Initially, we start with four pieces of data (two pairs from $\mathcal{A}$): the elements (f_1^0, g_1^{∞}) and (f_2^0, g_2^{∞}). After the identification and sewing, the two remaining maps $g_1^{\infty}(z)$ and $f_2^0(z)$, are used to create the new element of the semigroup (or a new pair in $\mathcal{A}$). We define the product $(f_1^0, g_1^{\infty}) \cdot (f_2^0, g_2^{\infty})$ to be the semigroup element:

$$\left(F_2 \circ f_2^0, F_1 \circ g_1^\infty\right). \tag{1.5}$$

While the multiplication is interesting, we will not make use of this in our thesis.

1.3 Preliminaries

Before stating the main results of the thesis, we introduce the three main theorems in a more precise way. To do this, we need to establish the required machinery. In this section, we introduce the reader to quasi-symmetric maps and quasi-conformal extensions, as well as some of the nice properties associated to these maps. In Chapter 2, we will see that each quasi-symmetry can be decomposed into a composition of special holomorphic functions: one defined on $\mathbb{D}$, and one defined on $\mathbb{D}^*$. We will introduce these holomorphic maps, as well as show the complementary property that they have. Finally, to aid in the algebraic approach to sewing, we show how the holomorphic functions centered at zero or at infinity can be displayed as formal power series, and we give a matrix representation of these series. This representation is called the power matrix, and some elementary properties of these matrices will be introduced.

1.3.1 Quasi-Symmetric Maps and Quasi-Conformal Extensions

First, we define quasi-symmetries and their properties. Denote the extended real line as $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Definition 1.3. An orientation-preserving homeomorphism $\phi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is called k-quasi-symmetric if there exists a constant $0 < k < 1$ satisfying

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k \tag{1.6}$$

for all x and t in $\mathbb{R}$. If k exists, but is unspecified, we simply say $\phi(z)$ is quasi-symmetric.

We note that it is more convenient to work on S^1 rather than on the extended real line. Consider the map

$$T(z) = \frac{i(1+z)}{1-z} \tag{1.7}$$

sending the unit circle to $\overline{\mathbb{R}}$ such that $T(1) = \infty$. We can define a homeomorphism of the unit circle to be a quasi-symmetry in the following way:

Definition 1.4. A homeomorphism $\phi : S^1 \to S^1$ is a quasi-symmetry if $e^{i\theta}$ is chosen so that $e^{i\theta}\phi(1) = 1$ and $T \circ e^{i\theta}\phi \circ T^{-1}$ is quasi-symmetric (according to Definition 1.3).

The set of all quasi-symmetries acting on the unit circle will be denoted as $QS(S^1)$. Next, we state some useful theorems about quasi-symmetries that we make use of in this thesis. See both [Leh87] and [FM07] for more information.

Theorem 1.5. *The composition of two quasi-symmetries is quasi-symmetric.*

Theorem 1.6. *The inverse of a quasi-symmetry is quasi-symmetric.*

Finally, we define quasi-conformal maps and quasi-conformal extensions. Here, we quote an important theorem stating that the composition of quasi-conformal maps is quasi-conformal. A proof of this theorem can be found in [Leh87].

Definition 1.7. Let D be an open and connected set in $\mathbb{C}$. A map $f(z)$ is absolutely continuous on lines (A.C.L.) in D if for every line $y = y_0$ and $x = x_0$ in D, $f(z)$ restricted to $z = x_0 + iy$ and $z = x + iy_0$ is absolutely continuous.

Definition 1.8. Let D be an open and connected set in the plane. An orientation-preserving homeomorphism $f : D \to f(D)$ is k-quasi-conformal if

1. $f(z)$ is A.C.L. in D.
2. There exists $0 \le k < 1$ such that $\left|\frac{\partial f}{\partial \bar{z}}\right| \le k \left|\frac{\partial f}{\partial z}\right|$ almost everywhere.

Theorem 1.9. *The composition of two quasi-conformal maps is quasi-conformal.*

Theorem 1.10. *The inverse of a quasi-conformal map is quasi-conformal.*

Remark. Let $f_1(z)$ be a k_1-quasi-conformal map and $f_2(z)$ be a k_2-quasi-conformal map. [Ahl66] and [FM07] state that $f_1 \circ f_2(z)$ must be a k_1k_2-quasi-conformal map. It is also shown that $f_1^{-1}(z)$ must be a k_1-quasi-conformal map.

Many of the theorems we present require that certain maps have quasi-conformal extensions to the plane $\mathbb{C}$, the sphere $\overline{\mathbb{C}}$ or the unit disk $\mathbb{D}$. We discuss briefly what this means. For a proof of the Ahlfors-Beurling Theorem, see [Leh87]. We also mention the relation between analytic maps and quasi-conformal maps.

Definition 1.11. A holomorphic map $f : \mathbb{D} \to \mathbb{C}$ has a quasi-conformal extension to $\mathbb{C}$ if there exists a map $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ such that $\left.\widetilde{f}(z)\right|_{\mathbb{D}} = f(z)$ and $\widetilde{f}(z)$ is quasi-conformal according to Definition 1.8.

Theorem 1.12. (Ahlfors-Beurling Theorem) *Let $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Then there exists a quasi-conformal extension $\widetilde{f} : H \to H$ of $f(z)$ taking the upper half plane to itself if and only if the boundary correspondence mapping $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a quasi-symmetry.*

The following corollary states that a quasi-symmetry $f : S^1 \to S^1$ has a quasi-conformal extension to the unit disk $\mathbb{D}$ or, similarly, to the complement of the unit disk $\mathbb{D}^*$ (here one must use the transformation $z \mapsto \frac{1}{z}$). Also, we mention that any quasi-conformal map onto the interior of a quasi-circle must have a quasi-conformal extension to the plane [LV73]. A quasi-circle is simply the image of a circle under a quasi-conformal map.

Corollary 1.13. *An orientation preserving homeomorphism $f : S^1 \to S^1$ is a quasi-symmetry if and only if it has a quasi-conformal extension to the unit disk.*

Theorem 1.14. *Let D be the interior of a quasi-circle in the complex plane. Let $f : \mathbb{D} \to D$ be a quasi-conformal map onto the interior of a quasi-circle. Then the map $f(z)$ has a quasi-conformal extension $\widetilde{f}(z)$ to $\mathbb{C}$.*

Remark. There is a similar theroem for a map $g(z)$ that maps into the exterior of a quasi-circle. It is possible to show the map $g(z)$ has a quasi-conformal extension $\widetilde{g}(z)$ to $\overline{\mathbb{C}}$.

Often, we will use the fact that an analytic map is quasi-conformal or that an analytic map $\phi : S^1 \to S^1$ is a quasi-symmetry. These results are stated below; proofs are found in [LV73]. Also, provided one has an analytic extension in a neighbourhood of the closed unit disk (or the closure of the unit disk's complement) this implies the existence of a quasi-conformal extension to the plane or sphere. A proof of this theorem is given in [LV73].

Corollary 1.15. *Let $f(z)$ be an analytic map in its domain. Then $f(z)$ is quasi-conformal.*

Remark. In fact, the map $f(z)$ in Corollary 1.15 is 0-quasi-conformal.

Theorem 1.16. *Let $\phi : S^1 \to S^1$ be analytic. Then $\phi(z)$ is a quasi-symmetry.*

Theorem 1.17. *Let $D \subset \mathbb{C}$ be a simply-connected open set. Suppose $f : D \to \mathbb{C}$ has a quasi-conformal extension $\widetilde{f}_1(z)$ to an open neighbourhood around D. Then the map $f(z)$ has a quasi-conformal extension $\widetilde{f}_2(z)$ to the entire plane $\mathbb{C}$.*

Remark. Theorem 1.17 is given in its greatest generality. Often we will use Corollary 1.15, as well as this theorem to show that a map with an analytic extension to an open neighbourhood of the unit disk has a quasi-conformal extension to the complex plane.

Finally, we explain briefly what is meant by a quasi-conformal extension to the Riemann sphere.

Definition 1.18. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an orientation-preserving homeomorphism. Then

$$\|\mu(f)\|_\infty \leq k = \frac{\frac{\partial f}{\partial \bar{z}}}{\frac{\partial f}{\partial z}}. \tag{1.8}$$

Definition 1.19. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an orientation-preserving homeomorphism. Let $h(z) = 1/z$. Then $\|\mu(f)\|_\infty \leq k$ has the following meaning on open subsets U of $\overline{\mathbb{C}}$:

1. if $\infty \notin U$ and $\infty \notin f(U)$ then $\|\mu(f)\|_\infty \leq k$.
2. if $\infty \notin U$ and $\infty \in f(U)$ then $\|\mu(h \circ f)\|_\infty \leq k$.
3. if $\infty \in U$ and $\infty \notin f(U)$ then $\|\mu(f \circ h)\|_\infty \leq k$.
4. if $\infty \in U$ and $\infty \in f(U)$ then $\|\mu(h \circ f \circ h)\|_\infty \leq k$.

Remark. We note that $\|\mu(f \circ h)\|_\infty = \|\mu(f)\|_\infty$ for the transformation $h(z) = 1/z$. In fact, it can be shown that

$$\|\mu(f \circ T)\|_\infty = \|\mu(T \circ f)\|_\infty = \|\mu(f)\|_\infty \tag{1.9}$$

for any Möbius transformation $T(z)$.

Using a generalization of the removeable singularity theorem, one can extend a quasi-conformal map of the plane to a quasi-conformal map of the sphere. Consider a quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$. If $h(z) = 1/z$, we write $g = h \circ f \circ h$ such that $g : \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$. The map $g(z)$ is quasi-conformal by Theorem 1.9. By a generalization of the removeable singularity theorem, $g(z)$ extends to a quasi-conformal map $\widetilde{g}(z) : \mathbb{D} \to \mathbb{D}$ such that $\widetilde{g}(0) = 0$. This implies that the map $f(z)$ extends to a map $\widetilde{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\widetilde{f}(\infty) = \infty$.

1.3.2 Introduction to Formal Power Series and the Power Matrix

In this section we will first give a representation of he holomorphic and one-to-one maps of $\mathbb{D}$ and $\mathbb{D}^*$ defined in Section 1.2 as formal power series. Next, we will introduce the power matrix, which is a matrix representation of a formal power series. Finally, we will show that there exist natural group isomorphisms between the spaces of formal power series and the spaces of power matrices.

Let $\mathbb{C}$ be the ring of complex numbers. We denote $\mathbb{C}[z]$ as the ring of polynomials in the indeterminate z with coefficients in $\mathbb{C}$. By extending the polynomials in $\mathbb{C}[z]$ to have an infinite number of terms, and by defining multiplication of elements using the Cauchy product, we obtain the ring of formal power series $\mathbb{C}[[z]]$ (addition is defined in the usual way) [MS02]. We define two specific spaces of formal power series as follows:

Definition 1.20. For formal power series centered at 0, we define

$$\mathbb{C}_1[[z]] = \left\{ f(z) = \sum_{n=1}^{\infty} a_n z^n;\ a_n \in \mathbb{C}, a_1 \neq 0 \right\}, \tag{1.10}$$

and for formal power series centered at ∞, we define

$$\mathbb{C}_1[[z^{-1}]] = \left\{ g(z) = \sum_{n=-\infty}^{1} a_n z^n;\ a_n \in \mathbb{C}, a_1 \neq 0 \right\}. \tag{1.11}$$

The spaces $\mathbb{C}_1[[z]]$ and $\mathbb{C}_1[[z^{-1}]]$ are the main spaces of interest. The space $\mathbb{C}_1[[z]]$ contains the formal power series corresponding to the functions holomorphic in a neighbourhood of zero. The space $\mathbb{C}_1[[z^{-1}]]$ contains the formal power series corresponding to the functions meromorphic in a neighbourhood of infinity. But, of course, not every formal power series represents a holomorphic function.

Remark. Since $\mathbb{C}$ is a field, and the coefficient of the z term is non-zero, the space $\mathbb{C}_1[[z]]$ is a group under composition. That is, one can find an explicit formula for the coefficients of the inverse of the formal power series $f(z) \in \mathbb{C}_1[[z]]$. Similarly, one can do this for $g(z) \in \mathbb{C}_1[[z^{-1}]]$.

We define a nice matrix representation of formal power series called the power matrix.

Definition 1.21. Let $f(z) \in \mathbb{C}_1[[z]]$. The power matrix at 0 of $f(z)$, denoted as $[f]$, is the matrix whose entry in the mth row and nth column ($m, n \in \mathbb{Z}$) satisfies the following equation:

$$(f(z))^m = \sum_{n=m}^{\infty} [f]_n^m z^n. \tag{1.12}$$

Note that the power matrices of functions in the space $\mathbb{C}_1[[z]]$ are upper triangular. We denote the set of all power matrices $[f]$ as

$$\mathcal{M}(0) = \{[f] : f(z) \in \mathbb{C}_1[[z]]\}. \tag{1.13}$$

Definition 1.22. Let $g(z) \in \mathbb{C}_1[[z^{-1}]]$. The power matrix at ∞ of $g(z)$, denoted as $[g]$, is the matrix whose entry in the mth row and nth column ($m, n \in \mathbb{Z}$) satisfies the following equation:

$$(g(z))^m = \sum_{n=-\infty}^{m} [g]_n^m z^n. \tag{1.14}$$

Note that the power matrices of functions in the space $\mathbb{C}_1[[z^{-1}]]$ are lower triangular. We denote the set of all power matrices $[g]$ as

$$\mathcal{M}(\infty) = \left\{[g] : g(z) \in \mathbb{C}_1[[z^{-1}]]\right\}. \tag{1.15}$$

A natural question now arises: does composition of functions in $\mathbb{C}_1[[z]]$ or $\mathbb{C}_1[[z^{-1}]]$ relate to some matrix operation in $\mathcal{M}(0)$ and $\mathcal{M}(\infty)$? It turns out that composition of functions corresponds to matrix multiplication [Jab53].

Lemma 1.23. *Let $f_1(z)$ and $f_2(z) \in \mathbb{C}_1[[z]]$, then*

$$[f_2 \circ f_1] = [f_2][f_1]. \tag{1.16}$$

Similarly, for $g_1(z)$ and $g_2(z) \in \mathbb{C}_1[[z^{-1}]]$, we have

$$[g_2 \circ g_1] = [g_2][g_1]. \tag{1.17}$$

Proof. Consider the power series of $f_2 \circ f_1(z)$:

$$(f_2 \circ f_1)^m(z) = \sum_{n=m}^{\infty} [f_2]^m_n f_1(z)^n \tag{1.18}$$

$$= \sum_{n=m}^{\infty} [f_2]^m_n \sum_{l=n}^{\infty} [f_1]^n_l z^l \tag{1.19}$$

$$= \sum_{l=m}^{\infty} \left(\sum_{n=m}^{l} [f_2]^m_n [f_1]^n_l \right) z^l. \tag{1.20}$$

Similarly, for the power series of $g_2 \circ g_1(z)$, we have

$$(g_2 \circ g_1)^m(z) = \sum_{n=-\infty}^{m} [g_2]^m_n g_1(z)^n \tag{1.21}$$

$$= \sum_{n=-\infty}^{m} [g_2]^m_n \sum_{l=-\infty}^{n} [g_1]^n_l z^l \tag{1.22}$$

$$= \sum_{l=-\infty}^{m} \left(\sum_{n=m}^{l} [g_2]^m_n [g_1]^n_l \right) z^l. \tag{1.23}$$

□

There is a natural isomorphism of groups between the space of power matrices and the formal power series centered at zero. Partial proofs of the following proposition are given in [Jab53] [Sch10b]; we expand on these below.

Proposition 1.24. *$\mathcal{M}(0)$ is a group under matrix multiplication. The map φ_0*

given by

$$\varphi_0 : \mathbb{C}_1[[z]] \to \mathcal{M}(0) \tag{1.24}$$

$$f \to [f], \tag{1.25}$$

is a group isomorphism. Similarly, $\mathcal{M}(\infty)$ is a group under matrix multiplication and the map φ_∞ given by

$$\varphi_\infty : \mathbb{C}_1[[z^{-1}]] \to \mathcal{M}(\infty) \tag{1.26}$$

$$g \to [g], \tag{1.27}$$

is also a group isomorphism.

Proof. Lemma 1.23 shows that both $\mathcal{M}(0)$ and $\mathcal{M}(\infty)$ are closed under multiplication. Associativity clearly holds in both sets, as their respective elements are matrices. The identity element of both spaces is $[z]$. Finally, for any element $[f]$ or $[g]$, we can find an inverse $[f]^{-1}$ or $[g]^{-1}$ since both coefficients f_1 and g_1 are non-zero (see remark above). Thus, we conclude that $\mathcal{M}(0)$ and $\mathcal{M}(\infty)$ are both groups.

Next, we show that φ_0 is an isomorphism. The computations for φ_∞ are similar, and are left to the reader. First, we prove the map φ_0 is one-to-one. Let $f_1(z)$ and $f_2(z)$ be elements of $\mathbb{C}_1[[z]]$ such that $\varphi_0(f_1(z)) = \varphi_0(f_2(z))$. Then $[f_1] = [f_2]$. For these two matrices to be equal, we have $[f_1]^1_n = [f_2]^1_n$ for $1 \le n < \infty$, or that $f_1(z) = f_2(z)$.

Next, we show that φ_0 is onto. Let $[f] \in \mathcal{M}(0)$. Then for $m = 1$,

$$f(z) = \sum_{n=1}^{\infty} [f]_n^1 z^n, \tag{1.28}$$

for some $f(z) \in \mathbb{C}_1[[z]]$, and we are done.

Finally, φ_0 is operation preserving:

$$\begin{aligned}
\varphi_0(f_1(z) \circ f_2(z)) &= [f_1 \circ f_2] &(1.29)\\
&= [f_1][f_2] &(1.30)\\
&= \varphi_0(f_1(z))\varphi_0(f_2(z)) &(1.31)
\end{aligned}$$

by Lemma 1.23. Thus, we have that φ_0 is indeed a group isomorphism.

□

Remark. Provided that $f^{-1}(z)$ is the inverse of $f(z)$ in a neighbourhood of zero, we have $[f^{-1}] = [f]^{-1}$. There is a similar result for the power matrices at infinity [Sch10b].

Chapter 2

Geometric Sewing

The purpose of this chapter is prove that the Geometric Sewing Equation (1.1) is solvable in the quasi-symmetric setting. Before showing this, we will introduce the Conformal Welding Theorem. This theorem allows us to view a quasi-symmetry as a decomposition of two holomorphic maps: one centered at zero and one centered at infinity. Tied closely to this theorem is the existence of the complementary maps; we will explain this idea more thoroughly in this chapter as well.

Theorem 2.3 is the main result in this section, and we will use complementary maps and the Conformal Welding Theorem to prove our claim. In [Hua97], Huang solves this problem in the analytic setting. He assumes that $\widetilde{g}^{\infty} \circ (\widetilde{f}^{0})^{-1}(z)$ is analytic on S^1 so that this composition has an analytic extension to a neighbourhood of the unit circle. We have generalized his argument to the quasi-symmetric case: we require our composition to be a quasi-symmetry which must have a quasi-conformal extension to the sphere. Note that the quasi-conformal extension may not necessarily be an analytic extension.

2.1 The Conformal Welding Theorem

First, we give a statement of the Conformal Welding Theorem. The normalizations we have used below are taken from [RS09] and are used throughout for convenience purposes.

Theorem 2.1. (Conformal Welding Theorem) *Fix $a \in \mathbb{C} \setminus \{0\}$. Let $\phi : S^1 \to S^1$ be a quasi-symmetry. Then there exists a unique pair of one-to-one holomorphic functions (F, G) satisfying:*

1. *$F : \mathbb{D} \to \mathbb{C}$ and $G : \mathbb{D}^* \to \overline{\mathbb{C}}$.*

2. *$F(z)$ and $G(z)$ have quasi-conformal extensions $\widetilde{F}(z)$ and $\widetilde{G}(z)$ to $\mathbb{C}$ and $\overline{\mathbb{C}}$, respectively.*

3. *As sets, we have $\widetilde{F}(S^1) = \widetilde{G}(S^1)$.*

4. *$F(0) = 0$, $G(\infty) = \infty$, and $G'(\infty) = a$.*

5. *$\widetilde{G}^{-1} \circ \widetilde{F} \mid_{S^1} = \phi$.*

If $\phi(z)$ is analytic on S^1, then there are quasi-conformal extension of $F(z)$ and $G(z)$ satisfying the above conditions, which, furthermore, are analytic on neighbourhoods of the closures of $\mathbb{D}$ and $\mathbb{D}^*$, respectively.

Proof. Let $w_\mu : \mathbb{D}^* \to \mathbb{D}^*$ be a quasi-conformal extension of $\phi(z)$, in the sense that it extends homeomorphically to $\overline{\mathbb{D}^*}$ and $w_\mu \mid_{S^1} = \phi$. Let μ be the complex dilatation of this extension (for a definition of complex dilatation see [Leh87]). This quasi-conformal extension exists by Corollary 1.13; this extension is not unique.

Let $w^\mu : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the unique quasi-conformal map with dilatation μ on $\mathbb{D}^*$ and 0 on $\mathbb{D}$ satisfying $w^\mu(0) = 0$, $w^\mu \circ w_\mu^{-1}(\infty) = \infty$ and $(w^\mu \circ w_\mu^{-1})'(\infty) = a$ ($w^\mu(z)$ is unique in the sense that it is fixed by these normalizations and the dilatation μ).

Set

$$F = w^\mu \mid_{\mathbb{D}} \tag{2.1}$$

$$G = w^\mu \circ w_\mu^{-1} \mid_{\mathbb{D}^*} . \tag{2.2}$$

$w^\mu(z)$ has dilatation 0 on $\mathbb{D}$, so $F(z)$ is conformal on its domain. Since $w^\mu(z)$ and $w_\mu(z)$ both have the same dilatation on the fixed domain $\mathbb{D}^*$, the composition $w^\mu \circ w_\mu^{-1}(z)$ is holomorphic [LV73], and $G(z)$ is conformal on its domain. Both $F(z)$ and $G(z)$ have quasi-conformal extensions to $\overline{\mathbb{C}}$; this follows from the definitions of $w^\mu(z)$ and $w_\mu(z)$. We have $\widetilde{G}^{-1} \circ \widetilde{F} \mid_{S^1} = \phi$, where $\widetilde{G}(z)$ and $\widetilde{F}(z)$ are the quasi-conformal extensions of $G(z)$ and $F(z)$. Finally, it is clear the maps satisfy the desired normalizations.

Now we show (F, G) are uniquely determined by $\phi(z)$. Given $\phi(z) \in QS(S^1)$, define a map $\mathcal{T}(\phi) = (F, G)$, where $F(z)$ and $G(z)$ are given by the above procedure. First, we show $\mathcal{T}(\phi)$ is well-defined.

Given two different quasi-conformal extensions $w_\mu(z)$ and $w_\nu(z)$ of $\phi(z)$, let

$$S(z) = \begin{cases} w^\mu \circ (w^\nu)^{-1} & z \in w^\nu(\mathbb{D}) \\ w^\mu \circ w_\mu^{-1} \circ w_\nu \circ (w^\nu)^{-1} & z \in w^\nu(\mathbb{D}^*) \end{cases} \tag{2.3}$$

$S(z)$ is quasi-conformal on $\overline{\mathbb{C}} \setminus w^\nu(S^1)$, and extends to a one-to-one continuous map of $\overline{\mathbb{C}}$ since $w_\mu^{-1} \circ w_\nu \mid_{S^1} = \phi^{-1} \circ \phi = Id$. Thus, $S(z)$ is quasi-conformal on $\overline{\mathbb{C}}$ by removeability of quasi-circles [LV73]. $S(z)$ is conformal on $w^\nu(\mathbb{D})$ since $w^\mu(z)$ and $w^\nu(z)$ both have dilatation 0 on $\mathbb{D}$. $S(z)$ is conformal on $w^\nu(\mathbb{D}^*)$ since $w^\mu \circ w_\mu^{-1}(z)$ and $w^\nu \circ w_\nu^{-1}(z)$ both have dilatation 0 on $\mathbb{D}^*$. This means $S(z)$ is conformal.

One can check that $S(0) = 0$, $S(\infty) = \infty$ and $S'(\infty) = 1$. So $S(z) = z$, or $w^\mu = w^\nu$ on $\mathbb{D}$ and $w^\mu \circ w_\mu^{-1} = w^\nu \circ w_\nu^{-1}$ on $\mathbb{D}^*$.

Denote the potential inverse of $\mathcal{T}(\phi)$ by $\mathcal{S}(F,G) = \widetilde{G}^{-1} \circ \widetilde{F}$. Surjectivity of $\mathcal{S}(F,G)$ comes from the first paragraph of the proof. We must show injectivity; we will show $\mathcal{T} \circ \mathcal{S} = Id$. Given (F,G) satisfying the desired normalizations, let $\widetilde{F}(z)$ be a quasi-conformal extension of $F(z)$ to the Riemann sphere with complex dilatation μ. By definition, $w^\mu = \widetilde{F}$. Also note that $w_\mu = \widetilde{G}^{-1} \circ \widetilde{F}$ is the quasi-conformal extension of $\widetilde{G}^{-1} \circ \widetilde{F}$ to $\mathbb{D}^*$.

We have

$$\mathcal{T} \circ \mathcal{S}(F,G) = \mathcal{T}(\widetilde{G}^{-1} \circ \widetilde{F}) \tag{2.4}$$

$$= (w^\mu \mid_{\mathbb{D}}, w^\mu \circ w_\mu^{-1} \mid_{\mathbb{D}^*}) \tag{2.5}$$

$$= (F, \widetilde{F} \circ \widetilde{F}^{-1} \circ G) \tag{2.6}$$

$$= (F,G). \tag{2.7}$$

Now assume that $\phi(z)$ is analytic on S^1. We claim there exists a quasi-conformal extension $\widehat{w}_\mu(z)$ of $\phi(z)$ to a disk $\{z : |z| > r_1\} \cup \{\infty\}$ which is analytic on $\{z : r_1 < |z| < r_2\}$ for $r_1 < 1$ and $r_2 > 1$. To see this, let $\widehat{\phi}(z)$ be an analytic extension to an annulus $\{z : R_1 < |z| < R_2\}$ for $R_1 < 1$ and $R_2 > r_2$, which exists by assumption. Choose r_1 and r_2 such that $R_1 < r_1 < 1$ and $1 < r_2 < R_2$. By rescaling by a factor of r_2, and applying the preceeding proof, we can show there exists a quasi-conformal extension $\widetilde{w}_\mu(z)$ of $\widetilde{\phi}\mid_{\{z:|z|=r_2\}}$ to $\{z : |z| > r_2\} \cup \{\infty\}$. The map

$$\widehat{w}_\mu(z) = \begin{cases} \widetilde{\phi}(z) & r_1 < |z| \leq r_2 \\ \widetilde{w}_\mu(z) & |z| > r_2,\ z = \infty \end{cases} \tag{2.8}$$

is conformal on $\{z : r_1 < |z| \leq r_2\}$ and quasi-conformal on $\{z : |z| > r_2\} \cup \{\infty\}$. It is continuous by definition and quasi-conformal on its domain by removeability of quasi-circles [LV73].

Now, set $w_\mu = \widehat{w}_\mu \mid_{\mathbb{D}^*}$ and apply the proof above. It is clear that the resulting map $w^\mu(z)$ is, in fact, conformal on $\{z : |z| < r_2\}$ so that $F = w^\mu \mid_{\mathbb{D}}$ has an analytic extension to $\{z : |z| < r_2\}$. Similarly, $w^\mu \circ \widehat{w}_\mu(z)$ is conformal on $\{z : |z| > r_1\} \cup \{\infty\}$ and agrees with $G(z)$ on $\mathbb{D}$. This proves the claim.

□

Of course, one may wonder if the reverse direction is true. Provided that the functions $F(z)$ and $G(z)$ satisfy (1)-(3) of Theorem 2.1, it has been shown that the composition $\widetilde{G}^{-1} \circ \widetilde{F}\Big|_{S^1}$ is a quasi-symmetry. For a proof of this theorem, see [Leh87].

Theorem 2.2. *Let $F(z)$ and $G(z)$ be a pair of one-to-one holomorphic functions satisfying the following:*

1. *$F : \mathbb{D} \to \mathbb{C}$ and $G : \mathbb{D}^* \to \overline{\mathbb{C}}$.*

2. *$F(z)$ and $G(z)$ both have quasi-conformal extensions $\widetilde{F}(z)$ and $\widetilde{G}(z)$ to $\overline{\mathbb{C}}$.*

3. *As sets, we have $\widetilde{F}(S^1) = \widetilde{G}(S^1)$.*

Then $\widetilde{G}^{-1} \circ \widetilde{F} : S^1 \to S^1$ is a quasi-symmetry.

2.2 The Complementary Map Problem

Given a pair of maps $f^0(z)$ and $g^\infty(z)$, there exist complementary maps $f^\infty(z)$ and $g^0(z)$ (as stated in Section 1.2). The complementary pair $f^0(z)$ and $f^\infty(z)$ (as well as the pair $g^\infty(z)$ and $g^0(z)$) satisfy the properties of Theorem 2.2. This means that $(\widetilde{f}^\infty)^{-1} \circ \widetilde{f}^0 = \phi$, where $\phi(z)$ is a quasi-symmetry, the equation is restricted to the unit circle, and the tilde notation represents the quasi-conformal extension of the map (of course, the same is true for the composition $(\widetilde{g}^\infty)^{-1} \circ \widetilde{g}^0(z)$). So we see that a complementary pair is inherently tied together by a quasi-symmetry.

The existence of the complementary map can be stated as: "Knowing one map of the complementary pair, can we find the other map?" In the geometric sewing setting, the Riemann Mapping Theorem immediately guarantees a positive answer to this question without needing the quasi-symmetry $\phi(z)$. However, in the algebraic sewing setting, the answer may not be so obvious. That is, we are searching for an algebraic method of determining the coefficients of the complementary maps $f^\infty(z)$ and $g^0(z)$ knowing only the coefficients of $f^0(z)$ and $g^\infty(z)$, and the Riemann Mapping Theorem does not give us any information regarding the coefficients of these maps - only the existence. It turns out that knowing a particular quasi-symmetry (not the $\phi(z)$ given in the previous paragraph), it is possible to determine the coefficients of the complementary maps; this is certainly not as strong as we would like, but it is a nice result. The proof of this is given in Chapter 3.

2.3 The Geometric Sewing Equation of Huang

Next, we show the main result of this chapter. Here we will make use of two one-to-one and holomorphic maps $f^0 : \mathbb{D} \to \mathbb{C}$ and $g^\infty : \mathbb{D}^* \to \overline{\mathbb{C}}$ that have quasi-conformal extensions to $\mathbb{C}$ and $\overline{\mathbb{C}}$, respectively. Their complementary maps will be denoted as $f^\infty(z)$ and $g^0(z)$. We take the normalizations given in Subsection 1.2.2. We will see that by composing with the appropriate complementary maps, the Geometric Sewing Equation is reduced to applying the Conformal Welding Theorem, which we do in the quasi-symmetric setting.

Theorem 2.3. *Let $f^0 : \mathbb{D} \to \mathbb{C}$ be a one-to-one and holomorphic map with normalization $f^0(0) = 0$ such that $f^0(z)$ has a quasi-conformal extension $\widetilde{f}^0(z)$ to $\mathbb{C}$. Let $g^\infty : \mathbb{D}^* \to \overline{\mathbb{C}}$ be a one-to-one and holomorphic map with normalizations $g^\infty(\infty) = \infty$ and ${g^\infty}'(\infty) = 1$ such that $g^\infty(z)$ has a quasi-conformal extension $\widetilde{g}^\infty(z)$ to $\overline{\mathbb{C}}$. Then there exist two one-to-one and holomorphic maps*

$$F_1 : f^\infty\left(\mathbb{D}^*\right) \to \overline{\mathbb{C}} \tag{2.9}$$

$$F_2 : g^0\left(\mathbb{D}\right) \to \mathbb{C} \tag{2.10}$$

with quasi-conformal extensions $\widetilde{F}_1(z)$ and $\widetilde{F}_2(z)$ to $\overline{\mathbb{C}}$ and $\mathbb{C}$, respectively, such that $\widetilde{F}_1(z)$ and $\widetilde{F}_2(z)$ satisfy the Geometric Sewing Equation

$$\widetilde{F}_1 = \widetilde{F}_2 \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \tag{2.11}$$

restricted to $\widetilde{f}^0\left(S^1\right)$. Furthermore, under the normalizations $F_2(0) = 0$, $F_1(\infty) = \infty$ and $F_1'(\infty) = 1$, the maps $F_1(z)$ and $F_2(z)$ are unique.

Proof. Consider the complementary pair (g^0, g^∞). We have $g^0 : \mathbb{D} \to \mathbb{C}$ and $g^\infty : \mathbb{D}^* \to \overline{\mathbb{C}}$. Since $g^\infty(z)$ has a quasi-conformal extension $\widetilde{g}^\infty(z)$ to $\overline{\mathbb{C}}$, $\widetilde{g}^\infty(S^1)$ is a quasi-circle and $g^0(z)$ maps into the exterior of this quasi-circle. By Theorem 1.14, $g^0(z)$ has a quasi-conformal extension $\widetilde{g}^0(z)$ to $\mathbb{C}$. Finally, as sets, we have $\widetilde{g}^0(S^1) = \widetilde{g}^\infty(S^1)$; this follows from the continuity of the quasi-conformal extensions of $g^0(z)$ and $g^\infty(z)$. Thus, (g^0, g^∞) satisfy the conditions of Theorem 2.2. A similar argument holds for the pair of maps (f^0, f^∞).

To motivate the proof, we compose (2.11) by complementary maps. Note that the first equality is restricted to $\widetilde{f}^0\,(S^1)$, and the second equality is restricted to the unit circle.

$$\widetilde{F}_1 = \widetilde{F}_2 \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \tag{2.12}$$

$$\widetilde{F}_1 \circ \widetilde{f}^\infty = \widetilde{F}_2 \circ \widetilde{g}^0 \circ (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty \tag{2.13}$$

We will show $(\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty$ is a quasi-symmetry and then apply the Conformal Welding Theorem.

We define

$$(\phi_g)^{-1} = (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \tag{2.14}$$

$$(\phi_f)^{-1} = (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty \tag{2.15}$$

as the quasi-symmetries restricted to S^1 obtained by Theorem 2.2. Since a quasi-symmetry is generally given as the inverse of a map centered at infinity composed with a map centered at zero, we have used the inverses of a quasisymmetries $\phi_g(z)$ and $\phi_f(z)$ to denote this subtle change; this is not a concern, since Theorem 1.6

guarantees $(\phi_g)^{-1}(z)$ and $(\phi_f)^{-1}(z)$ are indeed quasi-symmetries.

Next, Theorem 1.5 allows us to compose the quasi-symmetries $(\phi_g)^{-1}(z)$ and $(\phi_f)^{-1}(z)$ to obtain

$$(\Phi)^{-1} = (\phi_g)^{-1} \circ (\phi_f)^{-1} \tag{2.16}$$

as a quasi-symmetry restricted to the unit circle. We now apply the Conformal Welding Theorem 2.1. Given $\Phi(z)$, there exist maps $G(z)$ and $F(z)$ with quasi-conformal extensions $\widetilde{G}(z)$ and $\widetilde{F}(z)$ to $\overline{\mathbb{C}}$ such that $G'(\infty) = {f^{\infty}}'(\infty)$ and the equation

$$\widetilde{G}^{-1} \circ \widetilde{F} = \Phi. \tag{2.17}$$

restricted to S^1 is satisfied.

Define $F_1 = G \circ (f^{\infty})^{-1}$ and $F_2 = F \circ (g^0)^{-1}$. Consider the compositions of quasi-conformal maps: $\widetilde{G} \circ (\widetilde{f}^{\infty})^{-1}$ and $\widetilde{F} \circ (\widetilde{g}^0)^{-1}$. By Theorem 1.9, these compositions must be quasi-conformal extensions of $F_1(z)$ and $F_2(z)$, respectively. We denote

$$\widetilde{F}_1 = \widetilde{G} \circ (\widetilde{f}^{\infty})^{-1} \tag{2.18}$$

$$\widetilde{F}_2 = \widetilde{F} \circ (\widetilde{g}^0)^{-1} \tag{2.19}$$

as these quasi-conformal extensions. A simple substitution shows that the extension maps $\widetilde{F}_1(z)$ and $\widetilde{F}_2(z)$ satisfy the Geometric Sewing Equation (2.11). One can check that the normalizations of $F_1(z)$ and $F_2(z)$ are $F_2(0) = 0$, $F_1(\infty) = \infty$ and $F_1'(\infty) = 1$.

□

Chapter 3

Algebraic Sewing

The purpose of this chapter is to show that the welding maps $F(z)$ and $G(z)$ display an algebraic dependence on the coefficients of the map $\phi(z)$ in the analytic setting, as well as the quasi-conformal setting. There are two separate sets of assumptions of the data used when dealing with algebraic sewing. The first set of assumptions is to assume the data are simply formal power series - this is the approach used by Huang. The second set of assumptions on the data is to assume that the maps are either analytic with analytic extensions, or quasi-symmetric with quasi-conformal extensions; one would then use convergent matrix operations to display an algebraic procedure of finding certain coefficients. By "algebraic procedure" we mean that coefficients can be obtained by infinite, but convergent, matrix multiplication. This is the approach we utilize in this chapter.

In this chapter we first discuss briefly a formal power series approach to solving the Conformal Welding Theorem. Then, we will turn our attention to solving the three problems with convergent matrix operations. In our main theorems, we use special block and vector decompositions of power matrices, as well as invertibility of certain blocks in our main theorems. This new material is discussed in detail in Subsection 3.2.1. In Theorem 3.4 and Theorem 3.6, we obtain an algebraic procedure of finding the coefficients of the conformal welding pair $F(z)$ and $G(z)$ from the coefficients of $\phi(z)$. In the first theorem we use analytic maps, and in the second we use quasi-symmetric maps. In Theorem 3.7, we display an algebraic procedure to solve for the complementary map. Here, we require three pieces of data, including the composition $\phi = (g^0)^{-1} \circ g^\infty \circ (f^0)^{-1} \circ f^\infty$.

3.1 The Formal Power Series Approach

In this section, we briefly outline Huang's approach to solving the Algebraic Sewing Equation using formal power series. We also briefly outline our unsuccessful attempt to apply this procedure in obtaining the coefficients of the welding maps $F(z)$ and $G(z)$ in a formal power series setting.

Huang, in [Hua97], uses the ring of formal power series $\mathbb{C}[[\mathcal{F}, \mathcal{G}]][[z^{-1}, z]]$, where $\mathcal{F} = \{f_1, f_2, f_3 ...\}$ and $\mathcal{G} = \{g_1, g_0, g_{-1}, ...\}$ are infinite sequences of formal variables representing the known coefficients of the functions $g^\infty(z)$ and $(f^0)^{-1}(z)$, to give an algebraic procedure to find the coefficients of the functions $F_1(z)$ and $F_2(z)$ in the Algebraic Sewing Equation (1.1). This large ring avoids convergence issues of the infinite sums and he obtains $F_1(z)$ and $F_2(z)$ as functions of the coefficients of $g^\infty(z)$ and $(f^0)^{-1}(z)$. The crucial step involved is noticing that $F_1(z)$ and $F_2(z)$ only have coefficients in one direction (we use a similar procedure in the theorems presented in this chapter). This theorem establishes the existence of the maps $F_1(z)$ and $F_2(z)$, as well as their algebraic dependence on $g^\infty(z)$ and $(f^0)^{-1}(z)$. Huang, in a separate theorem, shows that when $g^\infty(z)$ and $(f^0)^{-1}(z)$ (considered as geometric maps) satisfy certain properties, then the algebraically determined series $F_1(z)$ and $F_2(z)$ must converge and be equal to the geometrically determined maps $F_1(z)$ and $F_2(z)$.

We tried to use a similar approach to obtain the formal power series expansions of $F(z)$ and $G(z)$ in the Conformal Welding Theorem. However, this attempt gave no useful information since the problem degenerates. See Section 5.2.1 for a more in-depth explanation.

3.2 The Convergent Matrix Operations Approach

3.2.1 The Conformal Welding Theorem: Analytic Case

The purpose of this section is to give an algebraic procedure to find the coefficients of the maps $F(z)$ and $G(z)$ in the case the map $\phi(z)$ is analytic on the annulus and maps the unit circle to itself. To do this, we must first define a power matrix for such an analytic $\phi(z)$.

Definition 3.1. Let $\phi(z)$ be analytic and non-zero on the annulus $A = \{r < |z| < R\}$, where $r < 1$ and $R > 1$. We write $[\phi]_n^m$ as the nth coefficient of the Laurent series of $(\phi(z))^m$ (for $n, m \in \mathbb{Z}$).

What we wish to do now is introduce the block structure of the power matrix representation for an analytic $\phi(z)$. Given $[\phi]$ as in Definition 3.1, we denote

$$[\phi] = \begin{bmatrix} \phi_{-,-} & \vec{\phi}_{c_1} & \phi_{-,+} \\ \vec{0}_r & 1 & \vec{0}_r \\ \phi_{+,-} & \vec{\phi}_{c_2} & \phi_{+,+} \end{bmatrix}, \tag{3.1}$$

where $\phi_{-,-} = [\phi]_n^m$ for $m, n < 0$, $\phi_{+,+} = [\phi]_n^m$ for $m, n > 0$, $\phi_{-,+} = [\phi]_n^m$ for $m < 0$ and $n > 0$, $\phi_{+,-} = [\phi]_n^m$ for $m > 0$ and $n < 0$, $\vec{\phi}_{c_1} = [\phi]_n^m$ for $m < 0$ and $n = 0$, and $\vec{\phi}_{c_2} = [\phi]_n^m$ for $m > 0$ and $n = 0$. We denote a full row of zeros by $\vec{0}_r$.

Remark. In [TT06], it has been shown that the block $\phi_{+,+}$ is invertible. We use this fact in Theorem 3.4 and Theorem 3.6.

Next, we explore an application of this matrix representation. Consider the Conformal Welding Theorem and the welding pair (F, G). Given the power matrix of an analytic $\phi : S^1 \to S^1$, we show it is possible to determine an algebraic procedure to find the coefficients of both $F(z)$ and $G(z)$ using convergent matrix operations. Before we give the proof, we need to introduce some new notation. Also, recall that, given a quasi-symmetric $\phi(z)$, there exist maps $F(z)$ and $G(z)$ satisfying $\widetilde{G}^{-1} \circ \widetilde{F} = \phi$ (where the tilde notation represents the quasi-conformal extension map).

For the next theorem, we require a power matrix version of the equation $\widetilde{G}^{-1} \circ \widetilde{F} = \phi$, which we will give in Lemma 3.3. One can rearrange this equation to obtain

$$\widetilde{F} \circ \phi^{-1} = \widetilde{G}. \tag{3.2}$$

Since $\phi^{-1}(z)$ is a quasi-symmetry by Theorem 1.6, we have that

$$\widetilde{F}' \circ \phi = \widetilde{G}', \tag{3.3}$$

where $F'(z)$ and $G'(z)$ are the welding maps obtained by the Conformal Welding

Theorem using $\phi^{-1}(z)$ as the quasi-symmetry. We will write (3.3) as

$$\widetilde{F} \circ \phi = \widetilde{G}, \tag{3.4}$$

and it will be understood that the welding maps $F(z)$ and $G(z)$ are obtained by the Conformal Welding Theorem using $\phi^{-1}(z)$ as the quasi-symmetry. This is the notation used by [NS95].

Definition 3.2. Let $f(z) \in \mathbb{C}_1[[z]]$ and $g(z) \in \mathbb{C}_1[[z^{-1}]]$. We will use the following notation for the power matrix restricted to the first row:

$$(f) = \begin{pmatrix} \dots & 0 & 0 & f_1 & f_2 & f_3 & \dots \end{pmatrix} \tag{3.5}$$

$$(g) = \begin{pmatrix} \dots & g_{-1} & g_0 & g_1 & 0 & 0 & \dots \end{pmatrix}. \tag{3.6}$$

It will also be beneficial to further break down these vectors. We will denote the positive coefficients of $f(z)$ as

$$(f)_+ = \begin{pmatrix} f_1 & f_2 & f_3 & \dots \end{pmatrix}, \tag{3.7}$$

and the negative coefficients (including the constant term) of $g(z)$ as

$$(g)_- = \begin{pmatrix} \dots & g_{-2} & g_{-1} & g_0 \end{pmatrix}. \tag{3.8}$$

The remaining coefficient of $g(z)$ will be represented as

$$(g)_1 = \begin{pmatrix} g_1 & 0 & 0 & \dots \end{pmatrix}. \tag{3.9}$$

Next, we look at how we can find the coefficients of the welding pair given the doubly-infinite matrix $[\phi]$, but first, we need the following lemma.

Lemma 3.3. *Let $A = \{r < |z| < R\}$, where $r < 1$ and $R > 1$. Let $f(z)$ and $g(z)$ be non-zero and analytic on A. Let $\phi(z)$ be one-to-one, non-zero and analytic on A such that $\phi : S^1 \to S^1$. Assume $f \circ \phi = g$. Then the power matrices satisfy*

$$[f][\phi] = [g] \tag{3.10}$$

and the left hand side of (3.10) converges in the sense that each infinite sum corresponding to each individual entry of the matrix product converges.

Proof. There exists a sub-annulus A' satisfying the following properties:

1. The closure of A' is contained in A: $\overline{A'} \subset A$.

2. $\overline{\phi\left(A'\right)} \subset A$.

3. The maps $f(z)$ and $g(z)$ have a lower bound on $\overline{\phi\left(A'\right)}$ and $\overline{A'}$, respectively. That is, $|f| \geq p > 0$ on $\overline{\phi\left(A'\right)}$ for some p and $|g| \geq q > 0$ on $\overline{A'}$ for some q.

We have that the Laurent series of both $g(z)^m$ and $f(z)^m$ converge uniformly on A' and $\phi\left(A'\right)$, respectively, for $m \in \mathbb{Z}$ so that, on A', we have that the right hand side

$$g(z)^m = \sum_{n=-\infty}^{\infty} [f]_n^m \left(\phi(z)\right)^n \tag{3.11}$$

converges uniformly on A'. Computing the Laurent coefficients, we see that

$$[g]_k^m = \frac{1}{2\pi i}\int_{S^1} \frac{g(z)^m}{z^{k+1}} dz \tag{3.12}$$

$$= \frac{1}{2\pi i}\int_{S^1} \sum_{n=-\infty}^{\infty} \frac{[f]_n^m\,(\phi(z))^n}{z^{k+1}} dz \tag{3.13}$$

$$= \sum_{n=-\infty}^{\infty} [f]_n^m \frac{1}{2\pi i}\int_{S^1} \frac{\phi(z)^n}{z^{k+1}} dz \tag{3.14}$$

$$= \sum_{n=-\infty}^{\infty} [f]_n^m [\phi]_k^n, \tag{3.15}$$

or that (3.15) converges to $[g]_k^m$. Here we have used the uniform convergence of the sum in (3.11).

□

Now, we are able to state one of the main results of this section. In the following theorem, we require a conformal extension of the map $\phi : S^1 \to S^1$. Mathematically, this means there exists a map $\widetilde{\phi}(z)$ taking an annulus (or doubly-connected domain) into the plane such that $\widetilde{\phi}(z)$ is conformal and restricts to $\phi(z)$ on S^1.

Theorem 3.4. *Let $\phi(z) : S^1 \to S^1$ have a one-to-one, non-zero and analytic extension to a neighbourhood of the unit circle. Let $[\phi]$ be as in (3.1). Let $F(z)$ and $G(z)$ be the welding maps obtained by the Conformal Welding Theorem, such that $G'(\infty) = a$. Then the coefficients of the welding maps are given by*

$$(F)_+ = (G)_1\, \phi_{+,+}^{-1} \tag{3.16}$$

and

$$(G)_- = \left(\; (F)_+\, \phi_{+,-} \quad (F)_+\, \vec{\phi}_{c_2} \;\right). \tag{3.17}$$

Proof. Choose quasi-conformal extensions of $F(z)$ and $G(z)$ with analytic extensions to a neighbourhood of the unit circle; this can be done using the Conformal Welding Theorem 2.1. Note that $\phi(z)$ is non-zero, analytic and one-to-one on a neighbourhood of the unit circle. The map $F(z)$ is one-to-one and $F(0) = 0$, so that its extension is non-zero on S^1. Since the map $G(z)$ is one-to-one, fixes infinity and satisfies $\widetilde{F}(S^1) = \widetilde{G}(S^1)$, its extension is also non-zero on S^1. Thus, we can use Lemma 3.3. This implies the equation $[F][\phi] = [G]$ converges (in the sense that the sums of the entries of the matrix product converge). Now, restricting the power matrices of $F(z)$ and $G(z)$ to the first row, we have

$$(F)\,[\phi] = (G)\,. \tag{3.18}$$

Writing (3.18) as two block multiplications, we obtain:

$$\begin{pmatrix} f_1 & f_2 & f_3 & \ldots \end{pmatrix} \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \ldots \\ \phi_{2,1} & \phi_{2,2} & \ldots \\ \vdots & \vdots & \end{bmatrix} = \begin{pmatrix} g_1 & 0 & 0 & \ldots \end{pmatrix} \tag{3.19}$$

and

$$\begin{pmatrix} f_1 & f_2 & f_3 & \ldots \end{pmatrix} \begin{bmatrix} \ldots & \phi_{1,-2} & \phi_{1,-1} & \phi_{1,0} \\ \ldots & \phi_{2,-2} & \phi_{2,-1} & \phi_{2,0} \\ & \vdots & \vdots & \vdots \end{bmatrix} = \begin{pmatrix} \ldots & g_{-2} & g_{-1} & g_0 \end{pmatrix}. \tag{3.20}$$

Writing (3.19) as $(F)_+\,\phi_{+,+} = (G)_1$, we see we can solve for the components of

$(F)_+$ by using the fact that $\phi_{+,+}$ is invertible [TT06]:

$$(F)_+ = (G)_1 \, \phi_{+,+}^{-1}. \tag{3.21}$$

Here, $(F)_+$ depends only on a and the multiplication on the right hand side converges since all sums are finite.

Finally, writing (3.20) as $\begin{pmatrix} (F)_+ \, \phi_{+,-} & (F)_+ \, \vec{\phi}_{c_2} \end{pmatrix} = (G)_-$, we solve for the components of $(G)_-$ by simply performing the multiplications on the left-hand side.

$\square$

Remark. If we had a formal power series $F(z) \in \mathbb{C}_1[[z]]$, such that the multiplication $(F)[\phi]$ converged (in the sense that all the infinite sums of each entry converged), then the resulting formal power series $F(z)$ and $G(z) \in \mathbb{C}_1[[z^{-1}]]$ obtained from the above procedure must be the welding maps. This is because, using the above procedure, there would only be one unique solution to (3.18). Since the geometric solution of the Conformal Welding Theorem also satisfies (3.18), any other solution must be exactly this solution.

3.2.2 The Conformal Welding Theorem: Quasi-Symmetric Case

What we intend to do in this section, is to generalize Theorem 3.4 to the case of quasi-symmetric maps. We note that these are analytic maps with quasi-conformal extensions, which do not necessarily have analytic extensions. To do this, we require a substitute for the power matrix equation obtained in Lemma 3.3. We can view the power matrix of the doubly-infinite formal power series $\phi(z)$ as a bounded linear operator on the Hilbert space $\mathcal{H}^{\mathbb{C}}$; this is what we discuss next.

Consider the following Hilbert space

$$\mathcal{H}^{\mathbb{C}} = \left\{ f(e^{i\theta}) = {\sum_{n=-\infty}^{\infty}}' f_n e^{in\theta};\ {\sum_{n=-\infty}^{\infty}}' |n| \, |f_n|^2 < \infty,\ f : S^1 \to \mathbb{C} \right\}, \tag{3.22}$$

where the prime notation denotes that the sum does not contain the $n = 0$ term. The elements in $\mathcal{H}^{\mathbb{C}}$ are boundary values of a certain class of complex harmonic functions [NS95]. It can be shown that, given $\phi(z) \in QS(S^1)$ (the set of quasi-symmetries), the set of all linear operators $\left\{ \widehat{C}_\phi \right\}$ on $\mathcal{H}^{\mathbb{C}}$ defined by

$$\widehat{C}_\phi : \mathcal{H}^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}} \tag{3.23}$$

$$\widehat{C}_\phi \circ f = f \circ \phi - \frac{1}{2\pi} \int_0^{2\pi} f \circ \phi \, d\theta, \tag{3.24}$$

are bounded, and as such, have a doubly-infinite matrix representation, which we denote as $[\phi]$, in the orthogonal basis $e_n = \{z^n, n \in \mathbb{Z} \setminus \{0\}\}$ [NS95] [TT06]. We note that the constant term in the composition $\widehat{C}_\phi \circ f$ is removed by the subtraction of the integral. This effectively removes the constant column and row in the matrix representation, or the row where $m = 0$ and the column where $n = 0$.

Definition 3.5. In the standard orthogonal basis e_n, the power matrix representation of a quasi-symmetry $\phi(z)$ is defined to be the matrix $[\phi]$ associated with the bounded linear operator $\widehat{C}_\phi$

$$[\phi] = \begin{bmatrix} \phi_{-,-} & \phi_{-,+} \\ \phi_{+,-} & \phi_{+,+} \end{bmatrix}. \tag{3.25}$$

Now we are able to generalize our argument in Theorem 3.4 to the quasi-symmetric case.

Theorem 3.6. *Let $\phi : S^1 \to S^1$ be a quasi-symmetry. Let $F(z)$ and $G(z)$ be the welding maps obtained by the Conformal Welding Theorem, such that $G'(\infty) = a$. Then the coefficients of the welding maps (with constant terms removed) are given by*

$$(F)_+ = (G)_1 \, \phi^{-1}_{+,+} \tag{3.26}$$

and

$$(G)_- = (F)_+ \, \phi_{+,-}. \tag{3.27}$$

Proof. Before we begin, we make a couple of remarks. First, we note that the constant terms have been removed throughout. Second, it is important to assure that map $F(z)$ corresponds to a map that is indeed inside the Hilbert space $\mathcal{H}^{\mathbb{C}}$; if $F(z)$ is in $\mathcal{H}^{\mathbb{C}}$, then the matrix $[\phi]$, given in (3.25), will act on the power matrix of $F(z)$. This will guarantee convergence of the infinite sums of the product $[F][\phi]$. For $F(z)$, we have

$$\sum_{n=1}^{\infty} n \left| f_n \right|^2 = \iint_{\mathbb{D}} \left| F' \right|^2 dA = A\left(F\left(\mathbb{D}\right)\right) < \infty, \tag{3.28}$$

where $A\left(F\left(\mathbb{D}\right)\right)$ is the area of the domain $F\left(\mathbb{D}\right)$. The boundedness of the area follows from the fact that $F(z)$ has a quasi-conformal extension. The domain $F\left(\mathbb{D}\right)$ is bounded by a Jordan curve not containing infinity since $G(\infty) = \infty$ and $\widetilde{G}\left(S^1\right) = \widetilde{F}\left(S^1\right)$.

Now, we begin the proof. Since $\phi(z)$ is a quasi-symmetry, we can write $[\phi]$ as in (3.25). Since the map $F(z)$ is in the Hilbert space $\mathcal{H}^{\mathbb{C}}$, the matrix product $[F][\phi]$ converges in the sense that all infinite sums corresponding to the individual entries converge. We set the product equal to $[G]$, the power matrix of the map $G(z)$. Consider the equation

$$(F)\,[\phi] = (G)\,. \tag{3.29}$$

Writing (3.29) as two block multiplications, we obtain:

$$\begin{pmatrix} f_1 & f_2 & f_3 & \dots \end{pmatrix} \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \dots \\ \phi_{2,1} & \phi_{2,2} & \dots \\ \vdots & \vdots & \end{bmatrix} = \begin{pmatrix} g_1 & 0 & 0 & \dots \end{pmatrix} \tag{3.30}$$

and

$$\begin{pmatrix} f_1 & f_2 & f_3 & \dots \end{pmatrix} \begin{bmatrix} \dots & \phi_{1,-2} & \phi_{1,-1} \\ \dots & \phi_{2,-2} & \phi_{2,-1} \\ & \vdots & \vdots \end{bmatrix} = \begin{pmatrix} \dots & g_{-2} & g_{-1} \end{pmatrix}. \tag{3.31}$$

Writing (3.30) as $(F)_+\,\phi_{+,+} = (G)_1$, we see we can solve for the components of $(F)_+$ by using the fact that $\phi_{+,+}$ is invertible [TT06]:

$$(F)_+ = (G)_1\,\phi_{+,+}^{-1}. \tag{3.32}$$

Here, $(F)_+$ depends only on a and the multiplication on the right hand side converges since all sums are finite.

Finally, writing (3.31) as $(F)_+\,\phi_{+,-} = (G)_-$, we solve for the components of $(G)_-$ by simply performing the multiplications on the left-hand side. □

Compared to Theorem 3.4, Theorem 3.6 is more general since the quasi-symmetry need not have an analytic extension. The other main difference is the fact that the constant terms have been removed. This is due to the fact that the Hilbert space $\mathcal{H}^{\mathbb{C}}$ effectively removes the constant terms. To be able to include the constant term in this calculation, we would require a natural representation of the quasi-symmetries on a one-dimensional extension of $\mathcal{H}^{\mathbb{C}}$. It is not immediately clear how to do this.

3.2.3 The Dependence of the Complementary Map

Lastly, recall that the Riemann Mapping Theorem only determines the existence and normalizations of the complementary maps in the geometric setting, and that it does not help in the algebraic setting (in the sense that it yields no information about the coefficients of the complementary maps). What we will do here is determine a partial result: we will give an algebraic procedure using convergent matrix operations to solve for the coefficients of the complementary map given that the assumptions of the Geometric Sewing Equation are satisfied, and that the analytic map $\phi = (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty$ restricted to S^1 is known (here, and in the theorem, the tilde notation denotes the analytic extension of the map unless otherwise stated). Thus, three pieces of data are required: the maps $g^\infty(z)$ and $(f^0)^{-1}(z)$, as well as $\phi(z)$.

Theorem 3.7. *Let $f^0 : \mathbb{D} \to \mathbb{C}$ be a one-to-one and holomorphic map with normalization $f^0(0) = 0$ such that $f^0(z)$ has a one-to-one analytic extension $\widetilde{f}^0(z)$ to an open neighbourhood $\{|z| < R\}$ for some $R > 1$. Let $g^\infty : \mathbb{D}^* \to \overline{\mathbb{C}}$ be a one-to-one and holomorphic map with normalizations $g^\infty(\infty) = \infty$ and ${g^\infty}'(\infty) = 1$ such that $g^\infty(z)$ has a one-to-one analytic extension $\widetilde{g}^\infty(z)$ to an open neighbourhood $\{|z| > r\} \cup \{\infty\}$ for some $r < 1$. Let $f^\infty(z)$ and $g^0(z)$ be the complementary* *maps associated to $f^0(z)$ and $g^\infty(z)$, respectively, satisfying $g^0(0) = 0$, ${g^0}'(\infty) > 0$, $f^\infty(\infty) = \infty$, and ${f^\infty}'(\infty) > 0$. Finally, let $\phi = (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty$. Then the coefficients of the complementary maps $f^\infty(z)$ and $g^0(z)$ are determined by the entries of the power matrix of $\phi(z)$, the power matrices of the maps $F_1(z)$ and $F_2(z)$ satisfying Huang's Geometric Sewing Equation, and the value ${f^\infty}'(\infty)$, through convergent matrix operations.*

Proof. Let $F_1(z)$ and $F_2(z)$ be the solution to Huang's Geometric Sewing Equation in Theorem 2.3 such that $F_1(\infty) = \infty$, $F_1^{'}(\infty) = 1$ and $F_2(0) = 0$. Note that we must use the fact that an analytic extension in a neighbourhood implies a quasi-conformal extension to the plane; this follows from Corollary 1.15 and Theorem 1.17. Denote the quasi-conformal extensions of $F_1(z)$ and $F_2(z)$ to $\overline{\mathbb{C}}$ and $\mathbb{C}$ as $\widetilde{F}_1(z)$ and $\widetilde{F}_2(z)$.

Let $F(z)$ and $G(z)$ be the welding maps obtained by the Conformal Welding Theorem 2.1 such that $\phi^{-1} = \widetilde{G}^{-1} \circ \widetilde{F}\Big|_{S^1}$, $F(0) = 0$, $G(\infty) = \infty$ and $G'(\infty) = {f^\infty}'(\infty)$. Denote the quasi-conformal extensions with analytic extensions to the closure of $\mathbb{D}$ and $\mathbb{D}^*$ of $F(z)$ and $G(z)$ as $\widetilde{F}(z)$ and $\widetilde{G}(z)$. Here, we note that since $\phi(z)$ is analytic, it must also be quasi-symmetric by Corollary 1.16. We verify that

$$f^\infty = F_1^{-1} \circ G \tag{3.33}$$

$$g^0 = F_2^{-1} \circ F \tag{3.34}$$

both hold. This result will follow from the uniqueness of the welding maps.

Set

$$\widehat{G} = F_1 \circ f^{\infty} \tag{3.35}$$

$$\widehat{F} = F_2 \circ g^0. \tag{3.36}$$

Following the normalizations, we have

$$F_1 \circ f^{\infty}(\infty) = F_1(\infty) = \infty \tag{3.37}$$

and

$$\left(F_1 \circ f^{\infty}\right)'(\infty) = F_1'(\infty) \cdot f^{\infty\prime}(\infty) \tag{3.38}$$

$$= f^{\infty\prime}(\infty), \tag{3.39}$$

as well as

$$F_2 \circ g^0(0) = 0. \tag{3.40}$$

Furthermore, the domain of $F_1(z)$ is $f^{\infty}(\mathbb{D}^*)$ and the domain of $F_2(z)$ is $g^0(\mathbb{D})$ (see Figure 1.4 in Subsection 1.2.3). This means

$$F_1 \circ f^{\infty} : \mathbb{D}^* \to \overline{\mathbb{C}} \tag{3.41}$$

$$F_2 \circ g^0 : \mathbb{D} \to \mathbb{C}. \tag{3.42}$$

Finally, restricting to S^1, and denoting the quasi-conformal extensions of $\widehat{F}(z)$

and $\widehat{G}(z)$ as $\widetilde{\widehat{F}}(z)$ and $\widetilde{\widehat{G}}(z)$, we have

$$\widetilde{\widehat{F}} \circ \phi = \widetilde{F}_2 \circ \widetilde{g}^0 \circ (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty \tag{3.43}$$

$$= \widetilde{F}_2 \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty \tag{3.44}$$

$$= \widetilde{F}_1 \circ \widetilde{f}^\infty \tag{3.45}$$

$$= \widetilde{\widehat{G}} \tag{3.46}$$

by the definition of $\phi(z)$ and using the Geometric Sewing Equation (2.11). Thus, by the uniqueness of the conformal welding maps in Theorem 2.1, we have that

$$F = \widehat{F} = F_2 \circ g^0 \tag{3.47}$$

$$G = \widehat{G} = F_1 \circ f^\infty. \tag{3.48}$$

In $\widetilde{F}_1 = \widetilde{F}_2 \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1}$, we have used the fact that an analytic extension is also a quasi-conformal extension restricted to a neighbourhood.

Next, we discuss the existence of analytic extensions of the complementary maps $f^\infty(z)$ and $g^0(z)$ used above. Denote $D = f^0(\mathbb{D})$ where D is an open and connected subset of the plane. Since $\widetilde{f}^0(S^1)$ is an analytic curve, we have that $f^\infty(\mathbb{D}^*) = D^*$ is bounded by an analytic curve. By the Schwarz Reflection Principle, $f^\infty(z)$ has an analytic extension $\widetilde{f}^\infty(z)$ to a neighbourhood of $\overline{\mathbb{D}^*}$. A similar argument shows there exists an analytic extension $\widetilde{g}^0(z)$ of $g^0(z)$ to a neighbourhood of the unit disk. See [Dav74] for more details.

To complete the proof, we note that the coefficients of the maps $F(z)$ and $G(z)$ are completely determined by convergent matrix operations given $G'(\infty) = {f^\infty}'(\infty)$

and $[\phi]$; this is the result of Theorem 3.4. We apply the power matrix notation, use Lemma 1.23 to write the power matrix of the composition as two separate power matrices, and use Proposition 1.24 to invert the power matrices of $F_1(z)$ and $F_2(z)$. This gives the desired result:

$$[F] = [F_2 \circ g^0] \Rightarrow [F] = [F_2][g^0] \Rightarrow [g^0] = [F_2^{-1}][F] \tag{3.49}$$

$$[G] = [f_1 \circ f^\infty] \Rightarrow [G] = [f_1][f^\infty] \Rightarrow [f^\infty] = [F_1^{-1}][G]. \tag{3.50}$$

$\square$

Remark. Note that there is only one real degree of freedom when choosing ${f^\infty}'(\infty)$. Once the domain is set, there are three real degrees of freedom. The map $f^\infty(z)$ fixes infinity, so that only one degree of freedom remains. We can alter ${f^\infty}'(\infty)$ by composing with a Möbius transformation $T(z) = e^{i\theta}z$ (a rotation to be exact). Thus, the input ${f^\infty}'(\infty)$ into the theorem is not completely arbitrary.

The theorem also determines a complementary $g^0(z)$ and specifies the real degree of freedom ${g^0}'(0) > 0$. If one desired a different normalization ${g^0}'(0)$ one could use convergent matrix operations $[\widehat{g}^0] = [g^0][e^{i\theta}z]$.

Chapter 4

Further Findings

In this chapter we give some of the further findings that arose while attempting to prove the main theorems. First, we briefly introduce the generalized Faber polynomials and the generalized Grunsky matrix of a disjoint normalized pair of maps $f(z)$ and $g(z)$. We will then give two interesting results: one that ties the matrix representation $[\phi]$ to the generalized Grunksy matrix; and one that ties Faber polynomial coefficients to entries of the matrix representation $[\phi]$. After this, we state a new theorem that relates blocks of a power matrix $[f]$ to blocks of its inverse $[f^{-1}]$. Lastly, we give a new proof of a known result relating the entries of a power matrix $[f]$ to the entries of its inverse $[f^{-1}]$.

4.1 Faber Polynomials

Throughout the course of the project, we felt that Faber polynomials could be used to help answer some of the important questions that arose. As will be explained in this chapter, Faber polynomials are the basis of choice when dealing with a simply-connected and bounded domain D since an analytic function $h(z)$ can be expressed as a Faber Polynomial sum, and this infinite sum converges to $h(z)$ on compact subsets of D. This is quite stronger in general than working in the standard orthogonal basis $e_n = \{z^n, n \in \mathbb{Z}\}$, since a power series expansion of a univalent map in the basis e_n only converges to that univalent map on disks. See Section 5.2.2 for more details regarding the project and Faber polynomials.

Definition 4.1. A normalized disjoint pair of functions $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \overline{\mathbb{C}}$ are a pair of one-to-one and holomorphic maps satisfying $f(0) = 0$, $f'(0) = 1$, $g(\infty) = \infty$ and $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \varnothing$.

Remark. A normalized disjoint pair of maps is not necessarily a complementary pair of maps. This is because $f(\mathbb{D})^*$ may not equal $g(\mathbb{D}^*)$. However, we note that all complementary pairs of maps must be a normalized disjoint pair since $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \varnothing$ will be satisfied and we can obtain $f'(0) = 1 > 0$ by the Riemann mapping theorem.

Definition 4.2. We define the generalized Faber polynomials of a normalized disjoint pair $f(z)$ and $g(z)$, respectively, as

$$\log \frac{w - f(z)}{w} = \log \frac{f(z)}{z} - \sum_{n=1}^{\infty} \frac{\Phi_n(f)(w)}{n} z^n \tag{4.1}$$

$$\log \frac{g(z) - w}{bz} = -\sum_{n=1}^{\infty} \frac{\Phi_n(g)(w)}{n} z^{-n}. \tag{4.2}$$

[Sue98] writes that if a simply connected domain G has a closure $\overline{G}$ that is bounded by a rectifiable Jordan curve γ, then an analytic function $f(z)$ in G that is continuous in $\overline{G}$ and has bounded variation on γ can be expressed in terms of some

Faber polynomial series

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \Phi_n(z). \tag{4.3}$$

Here, $\Phi(z)$ is the one-to-one and onto conformal map taking the simply connected domain $D = G^*$ to $\mathbb{D}^*$ under the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. $\Phi_n(z)$ is defined to be the sum of the terms of non-negative degree in z in the Laurent expansion of $\Phi^n(z)$. The Faber series (4.3) converges uniformly on all closed subsets of the domain G and the coefficients α_n are given by

$$\alpha_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)\Phi'(w)}{\Phi^{n+1}(w)} dw. \tag{4.4}$$

We can relate this to the inverses of a complementary pair (f^0, f^∞). Since the inverses $(f^0)^{-1}(z)$ and $(f^\infty)^{-1}(z)$ satisfy all the conditions stated above, we can express $(f^0)^{-1}(z)$ as a Faber polynomial sum using the coefficients of $\Phi(z) = (f^\infty)^{-1}(z)$

$$(f^0)^{-1}(z) = \sum_{n=0}^{\infty} \alpha_n (f^\infty)_n^{-1}(z). \tag{4.5}$$

Here, $(f^\infty)_n^{-1}(z)$ are the sums of the terms of non-negative degree in z in the Laurent expansion of $\Phi^n(z) = ((f^\infty)^{-1}(z))^n$

4.2 The Grunsky Matrix

The Faber polynomials can be used to define the generalized Grunsky matrix. The generalized Grunsky matrix $\mathbf{B}$ can be thought of as a bounded linear operator on the Hilbert space $\ell^2 \oplus \ell^2$, with entries that arise from Faber polynomial compositions. For more details, see [Pom75] [TT06].

Definition 4.3. Let $f(z)$ and $g(z)$ be a disjoint normalized pair. The generalized Grunsky matrix $\mathbf{B}$ is defined to be the block matrix

$$\mathbf{B} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \tag{4.6}$$

where the entries of the semi-infinite block matrices are

$$(B_1)_{mn} = \sqrt{mn} b_{-m,-n} \tag{4.7}$$

$$(B_2)_{mn} = \sqrt{mn} b_{-m,n} \tag{4.8}$$

$$(B_3)_{mn} = \sqrt{mn} b_{m,-n} \tag{4.9}$$

$$(B_4)_{mn} = \sqrt{mn} b_{mn}, \tag{4.10}$$

and the b_{mn}, $m, n \in \mathbb{Z}$ are given by

$$\Phi_n(g)(g(z)) = z^n + n \sum_{m=1}^{\infty} b_{mn} z^{-m} \tag{4.11}$$

$$\Phi_n(g)(f(z)) = n b_{n,0} + n \sum_{m=0}^{\infty} b_{n,-m} z^m \tag{4.12}$$

$$\Phi_n(f)(f(z)) = z^{-n} + n \sum_{m=1}^{\infty} b_{-n,-m} z^m \tag{4.13}$$

$$\Phi_n(f)(g(z)) = -n b_{-n,0} + n \sum_{m=1}^{\infty} b_{m,-n} z^{-m}. \tag{4.14}$$

4.3 Some Corollaries

Corollary 4.4. *Suppose that $F(z)$ and $G(z)$ are a normalized disjoint pair satisfying $G'(\infty) = 1$. Suppose further that $F(z)$ and $G(z)$ have quasi-conformal extensions*

$\widetilde{F}(z)$ and $\widetilde{G}(z)$ to $\overline{\mathbb{C}}$. Then $(F)_+ = [\phi^{-1}]^1_m$ and $\left(B_3^T\right)^m_1 = (B_2)^1_m = \left(\overline{\phi^{-1}_{+,+}}\right)^1_m$ (for $m \geq 1$); where $\phi = \widetilde{G}^{-1} \circ \widetilde{F}$ restricted to S^1 is a quasi-symmetry, and B_2 and B_3 are generalized Grunsky matrices associated with $F(z)$ and $G(z)$.

Proof. Write $[\phi]$ as in (3.25). Using Theorem 3.6, we have $(F)_+ = (G)_1 \, \phi^{-1}_{+,+}$. Since $g_1 = 1$, the first row of $\phi^{-1}_{+,+}$ is exactly the coefficients of $F(z)$. Using the relations $B_2 = \overline{\phi^{-1}_{+,+}}$ and $B_2 = B_3^T$, given in [TT06], completes the proof. $\square$

Corollary 4.5. *Let $f^0(z)$ and $f^\infty(z)$ be a normalized disjoint pair. Assume $\partial f^0(\mathbb{D})$ is an analytic curve. Assume further that $(f^0)^{-1}(z)$ is analytic on $G = f^0(\mathbb{D})$, continuous on $\overline{G}$ and has bounded variation on γ. Then we have*

$$\alpha^m_n = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{(\phi^{-1}(z))^m}{z^n} dz, \tag{4.15}$$

where α^m_n is the nth coefficient of the Faber polynomial series of the mth power of $(f^0)^{-1}(z)$ and ϕ^{-1} is the map given by $\phi^{-1} = f^0 \circ (f^\infty)^{-1}$.

Proof. The result follows from performing a change of variables, $\zeta = f^\infty(z)$, in expression for the Faber Polynomial coefficients (see (4.4) and (4.5)):

$$\begin{align}
\alpha^m_n &= \frac{1}{2\pi i} \int_{\partial f^0(\mathbb{D})} \frac{(f^0)^{-1}(\zeta)^m (f^\infty)^{-1'}(\zeta)}{((f^\infty)^{-1}(\zeta))^{n+1}} d\zeta \tag{4.16}\\
&= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{((f^0)^{-1} \circ f^\infty(z))^m}{z^n} dz \tag{4.17}\\
&= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{(\phi^{-1}(z))^m}{z^n} dz. \tag{4.18}
\end{align}$$

$\square$

Remark. We note that if the map $\phi(z)$ is analytic on the annulus and maps the unit circle to itself, then we have

$$\alpha_n^m = [\phi^{-1}]_n^m \tag{4.19}$$

by (3.1), where $[\phi^{-1}]_n^m$ is the nth coefficient of the mth power of $\phi^{-1} = f^0 \circ (f^\infty)^{-1}$.

4.4 The Block Structure of the Power Matrix

In this section, we will show that, given $f(z) \in \mathbb{C}_1[[z]]$ and $g(z) \in \mathbb{C}_1[[z^{-1}]]$, certain blocks of the matrices $[f]$ and $[g]$ are invertible and that the off-diagonal blocks of $[f^{-1}]$ and $[g^{-1}]$ are simply a multiplication of blocks of $[f]$ or $[g]$.

We will make use of the following block and vector decomposition of the power matrix $[f]$: write $f_{-,-} = [f]_n^m$ for $m, n < 0$, $f_{+,+} = [f]_n^m$ for $m, n > 0$, $f_{-,+} = [f]_n^m$ for $m < 0$ and $n > 0$, and $\vec{f_c} = [f]_n^m$ for $m > 0$ and $n = 0$. Similarly, for $[g]$, we have $g_{-,-} = [g]_n^m$ for $m, n < 0$, $g_{+,+} = [g]_n^m$ for $m, n > 0$, $g_{+,-} = [g]_n^m$ for $m > 0$ and $n < 0$, and $\vec{g_c} = [g]_n^m$ for $m < 0$ and $n = 0$. The subscripts c and r distinguish between column and row vectors, respectively. We denote a full row of zeros by $\vec{0}_r$ and a full column of zeros by $\vec{0}_c$.

Lemma 4.6. *Write*

$$[f] = \begin{bmatrix} f_{-,-} & \vec{f_c} & f_{-,+} \\ \vec{0}_r & 1 & \vec{0}_r \\ 0 & \vec{0}_c & f_{+,+} \end{bmatrix}, \tag{4.20}$$

and let the inverse $[f^{-1}]$ have the form

$$[f^{-1}] = \begin{bmatrix} A & \vec{v_c} & B \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & C \end{bmatrix}. \tag{4.21}$$

Suppose that the coefficient $f_1 = [f]_1^1$ *of* $f(z)$ *is non-zero. Then* $A = f_{-,-}^{-1}$, $B = -f_{-,-}^{-1} f_{-,+} f_{+,+}^{-1}$, $C = f_{+,+}^{-1}$ *and* $\vec{v_c} = -f_{-,-}^{-1}\ \vec{f_c}$, *where* $f_{+,+}^{-1}$ *denotes the multiplicative inverse of* $f_{+,+}$ *and* $f_{-,-}^{-1}$ *denotes the multiplicative inverse of* $f_{-,-}$.

Proof. Since the power matrices $[f]$ and $[f^{-1}]$ are multiplicative inverses in $\mathcal{M}(0)$, we have $[f^{-1}][f] = [f][f^{-1}] = I$. This gives

$$\begin{bmatrix} f_{-,-} & \vec{f_c} & f_{-,+} \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & f_{+,+} \end{bmatrix} \begin{bmatrix} A & \vec{v_c} & B \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & C \end{bmatrix} = \begin{bmatrix} I & \vec{0_c} & 0 \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & I \end{bmatrix} \tag{4.22}$$

$$= \begin{bmatrix} A & \vec{v_c} & B \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & C \end{bmatrix} \begin{bmatrix} f_{-,-} & \vec{f_c} & f_{-,+} \\ \vec{0_r} & 1 & \vec{0_r} \\ 0 & \vec{0_c} & f_{+,+} \end{bmatrix}. \tag{4.23}$$

Multiplying out the blocks gives $f_{-,-}A = I = Af_{-,-}$ and $f_{+,+}C = I = Cf_{+,+}$, which implies $A = f_{-,-}^{-1}$ and $C = f_{+,+}^{-1}$ as inverses. Utilizing these inverses gives the other desired results.

□

Corollary 4.7. *Write*

$$[g] = \begin{bmatrix} g_{-,-} & \vec{0_c} & 0 \\ \vec{0_r} & 1 & \vec{0_r} \\ g_{+,-} & \vec{g_c} & g_{+,+} \end{bmatrix}, \tag{4.24}$$

and let the inverse $[g^{-1}]$ *have the form*

$$[g^{-1}] = \begin{bmatrix} A & \vec{0}_c & 0 \\ \vec{0}_r & 1 & \vec{0}_r \\ B & \vec{v}_c & C \end{bmatrix}. \tag{4.25}$$

Suppose that the coefficient $g_1 = [g]_1^1$ *of* $g(z)$ *is non-zero. Then* $A = g_{-,-}^{-1}$, $B = -g_{+,+}^{-1} g_{+,-} g_{-,-}^{-1}$, $C = g_{+,+}^{-1}$ *and* $\vec{v}_c = -g_{+,+}^{-1}\ \vec{g}_c$, *where* $g_{+,+}^{-1}$ *denotes the multiplicative inverse of* $g_{+,+}$ *and* $g_{-,-}^{-1}$ *denotes the multiplicative inverse of* $g_{-,-}$.

Proof. The result follows from the method of Lemma 4.6. □

4.5 The Relation Between a Power Matrix and its Inverse

Often, it is useful to relate a power matrix $[f]$ with its inverse $[f]^{-1}$. This relation appears in much of the literature directly and indirectly [Jab53] [Pom75] [TT06] [Sch10b], but here we will give a new proof using the properties of the power matrix under a specific adjoint given in [Sch10a]. We also make use of the matrix exponential and some of the elementary properties of this map. First, we must define two matrices N and N': $N = [n_{ij}]$, where

$$n_{ij} = \begin{cases} i, \text{ if } i = j \\ 0, \text{ otherwise} \end{cases} \tag{4.26}$$

and $N' = [n'_{ij}]$, where

$$n'_{ij} = \begin{cases} \frac{1}{i}, \text{ if } i = j \\ 0, \text{ otherwise} \end{cases} \tag{4.27}$$

For a formal power series $f(z) = \sum_{k=1}^{\infty} f_k z^k$, [Hua97] showed that

$$f(z) = \exp\left(h(z)\frac{\partial}{\partial z}\right) z = \sum_{k=1}^{\infty} \frac{\left(h(z)\frac{\partial}{\partial z}\right)^k}{k!} z \tag{4.28}$$

for some formal power series $h(z) = \sum_{k=1}^{\infty} h_k z^k$. Schippers, in [Sch10b], proved that, in the power matrix setting, the relation

$$[f] = \exp\langle h\rangle \tag{4.29}$$

is equivalent to (4.28). Here, the angle brackets are defined to be the matrix with entry in the mth row and nth column is mh_{n-m+1}, and exp is the matrix exponential $(m, n \in \mathbb{Z})$. Of course, there is an analagous result for power matrices centered at infinity. It can easily be shown that the following lemma holds.

Lemma 4.8. *Given $\langle h\rangle$ satisfying (4.29), we have*

$$N\langle h\rangle^T N' = \left\langle z^2 h\left(\frac{1}{z}\right)\right\rangle. \tag{4.30}$$

Here, we note that the action of conjugation by N and N' and then transposing displays the natural map between the functions centered at zero and the functions centered at infinity. In [Hua97] and [Sch10b], it is noted that, given an $f(z) \in \mathbb{C}_1[[z]]$, the transformation $T = \frac{1}{f\left(\frac{1}{z}\right)}$ yields a formal power series in $\mathbb{C}_1[[z^{-1}]]$. There is also a corresponding infinitesimal version for the elements $h(z)$.

The transformations can be summed up in the following diagram (we denote the transformations as T and T').

$$\begin{array}{ccccc} & & \exp & & \\ & h(z) & \rightarrow & f(z) & \\ T' & \downarrow & & \downarrow & T \\ & -z^2 h\left(\frac{1}{z}\right) & \rightarrow & \frac{1}{f\left(\frac{1}{z}\right)} & \\ & & \exp & & \end{array} \tag{4.31}$$

Lastly, before we are able to relate $[f]$ and $[f^{-1}]$, we need how this transformation T acts on the entries of power matrix $[f]$.

Lemma 4.9. *Let $[f] \in M(0)$. Then we have*

$$T\left([f]_n^m\right) = \left[\frac{1}{f\left(\frac{1}{z}\right)}\right]_n^m = [f]_{-n}^{-m}. \tag{4.32}$$

Now, we are able to give a new proof of Jabotinsky's result [Jab53].

Proposition 4.10. *Let $[f] \in \mathcal{M}(0)$. Then we have*

$$\frac{m}{n}[f]_m^n = [f^{-1}]_{-n}^{-m}, \tag{4.33}$$

for $m, n \in \mathbb{Z}$, $n \neq 0$.

Proof. We have

$$N\overline{[f]^T}N' = N\overline{exp\,\langle h\rangle^T}N' \tag{4.34}$$

$$N[f]^T N' = N exp\,\langle h\rangle^T N' \tag{4.35}$$

$$\frac{m}{n}([f]_n^m)^T = exp(N\,\langle h\rangle^T N') \tag{4.36}$$

$$\frac{m}{n}[f]_m^n = exp\left\langle z^2 h(\frac{1}{z})\right\rangle \tag{4.37}$$

$$\frac{m}{n}[f]_m^n = exp\left\langle T\left([f^{-1}]_n^m\right)\right\rangle \tag{4.38}$$

$$\frac{m}{n}[f]_m^n = [f^{-1}]_{-n}^{-m} \tag{4.39}$$

by Lemma 4.8 and Lemma 4.9. □

Remark. We note that conjugation by N and N' results in taking the adjoint in the Dirichlet space, as outlined in [Sch10a].

Remark. There is an identical relation for power matrices centered at infinity. This can be derived using (4.39) and the transformations $g(z) = \frac{1}{f(\frac{1}{z})}$ and $g^{-1}(z) = \frac{1}{f^{-1}(\frac{1}{z})}$.

Chapter 5

Summary of Results and Open Questions

In this concluding chapter we first give a brief summary of our main results from this thesis. Following this, we discuss a few open questions that arose throughout the course of the project and give some background and possible direction for future research.

5.1 Summary of Results

What we hoped to do throughout this thesis was to display connections between Conformal Field Theory (C.F.T.) and Geometric Function Theory (G.F.T.) through the sewing operation in a special case. In Theorem 3.4 and Theorem 3.6, we have shown that the coefficients of the welding functions $F(z)$ and $G(z)$ are dependent on the coefficients of the data, in both the analytic and quasi-symmetric settings. This means that given an analytic $\phi(z)$ with analytic extension, or a quasi-symmetric

$\phi(z)$ with quasi-conformal extension, we can find an infinite algebraic procedure, using convergent matrix operations, to obtain the coefficients of the welding maps. By doing this, we have not only replaced the formal power series notation with convergent matrix operations, but have also begun to generalize the analytic setting to the more general quasi-symmetric setting.

These two theorems display our first connection between C.F.T. and G.F.T. Obtaining the coefficients of the sewing maps using an algebraic procedure is an idea used in Conformal Field Theory by Huang in [Hua97]. However, we have utilized this idea to obtain coefficients of the welding maps in the Conformal Welding Theorem, which is a result in G.F.T. While not using Huang's theorem directly, our proofs share a common idea: special decompositions of particular formal power series. A second connection between the two theories is displayed in Theorem 2.3. Here, we have shown that the Geometric Sewing Equation (2.11) is solvable in the quasi-symmetric setting. That is, assuming $f^0(z)$ and g^∞ are one-to-one and holomorphic maps with quasi-conformal extensions, it is possible to find unique sewing maps $F_1(z)$ and $F_2(z)$. We use complementary maps and the Conformal Welding Theorem, which is a G.F.T. result, to do this. Thus, G.F.T. has been used to give a C.F.T. result.

Finally, in Theorem 3.7, we show that given analytic extensions of the maps $g(z)$ and $f^{-1}(z)$, as well as the analytic map $\phi = (\widetilde{g}^0)^{-1} \circ \widetilde{g}^\infty \circ (\widetilde{f}^0)^{-1} \circ \widetilde{f}^\infty$, it is possible to provide a method of determining the coefficients of the complementary maps. In the G.F.T. setting, the Riemann Mapping Theorem guarantees the existence of the complementary map; however, in the algebraic setting of C.F.T., the algebraic dependence of the complementary map on the original map is not so obvious, which is why this result is quite interesting. Utilizing the Riemann Mapping Theorem in G.F.T. to obtain the complementary map begs for a similar result in the algebraic setting of C.F.T.; this is exactly our partial result in Theorem 3.7, which is our final connection between G.F.T and C.F.T.

5.2 Open Questions

Here we discuss a few interesting questions that came up while working on the thesis. First, we will explain how some attempts at obtaining an algebraic dependence on the data in the Conformal Welding Theorem using formal power series in the spirit of Huang's proof did not work. Then, we turn our attention to the Algebraic Sewing Equation: it seems highly possible to obtain an algebraic procedure to find the coefficients, however, a stronger lemma is required and we leave this as an open question. Finally, we ask whether or not it is possible to strengthen Theorem 3.7 so that the quasi-symmetry $\phi(z)$ is not required.

5.2.1 A Formal Power Series Approach to the Conformal Welding Theorem

Since the Conformal Welding Theorem is closely tied to the Algebraic Sewing Equation, we attempted to show dependence of the Conformal Welding Theorem maps $F(z)$ and $G(z)$ on the data $\phi(z)$ using formal power series in a similar fashion to Huang's approach. However, when expressing the known formal power series $\phi(z)$ as a composition of formal automorphisms, the problem degenerates and gives no useful information. This is because when we know the exponential generators $\exp\left(h_F(z)\frac{\partial}{\partial z}\right) z$ and $\exp\left(z^2 h_G\left(\frac{1}{z}\right)\frac{\partial}{\partial z}\right) z$ of the formal power series of $F(z)$ and $G^{-1}(z)$ (this is the natural way to decompose $\phi(z)$), then the functions are automatically known by a version of (4.29). To see this more clearly, consider the decomposition of $\phi(z)$ into its exponential generators:

$$G^{-1} \circ F = \phi \tag{5.1}$$

$$G^{-1} \circ F = \exp\left(z^2 h_G\left(\frac{1}{z}\right)\frac{\partial}{\partial z}\right) z \circ \exp\left(h_F(z)\frac{\partial}{\partial z}\right) z \tag{5.2}$$

$$G^{-1} \circ F = G^{-1} \circ F \tag{5.3}$$

Yet, in Huang's approach, knowing the exponential generators $\exp\left(-z^2 h_g\left(\frac{1}{z}\right)\frac{\partial}{\partial z}\right) z$ and $\exp\left(-h_f(z)\frac{\partial}{\partial z}\right) z$ of $g^\infty(z)$ and $(f^0)^{-1}(z)$ does not immediately imply anything about the functions $F_1(z)$ and $F_2(z)$:

$$F_1 = F_2 \circ g^\infty \circ (f^0)^{-1} \tag{5.4}$$

$$F_1 = F_2 \circ \exp\left(-z^2 h_g\left(\frac{1}{z}\right)\frac{\partial}{\partial z}\right) z \circ \exp\left(-h_f(z)\frac{\partial}{\partial z}\right) z \tag{5.5}$$

Heuristically, Huang's Formal Sewing Equation is a "complexification" of the conformal welding theorem, which is why his method may not work in the restricted Conformal Welding Theorem setting.

5.2.2 Solving the Algebraic Sewing Equation Using Convergent Matrix Operations

We would like to obtain an algebraic dependence of the sewing maps $F_1(z)$ and $F_2(z)$ on the data in the Algebraic Sewing Equation (1.1). The algebraic procedure would utilize convergent matrix operations with the assumption that the data are either analytic maps with analytic extensions, or quasi-symmetric maps with quasi-conformal extensions. In order to do this, we would need to make sense of certain power matrix multiplications. Consider the matrix equation

$$[F_1] = [F_2][g^\infty][\left(f^0\right)^{-1}]. \tag{5.6}$$

If it was possible to show that the multiplication $[g^\infty][\left(f^0\right)^{-1}]$ converged to $[g^\infty \circ \left(f^0\right)^{-1}]$, then it may be possible to obtain such an algebraic procedure using the special block and vector decompositions of these power matrices as outlined in Chapter 3.

Uniform convergence of the power series of the functions $\left(g^\infty \circ \left(f^0\right)^{-1}\right)^m$ for any m on an annulus would be enough to prove the interchangeability of the summations; this is not obvious, though. One reason for this may be that the power matrix representations of the univalent maps $\left(f^0\right)^{-1}(z)$ and $g^\infty(z)$ are given in the standard orthogonal basis $e_n = \{z^n, n \in \mathbb{Z}\}$, so that the power series expansions of $\left(f^0\right)^{-1}(z)$ and $g^\infty(z)$ converge on compact disks for all powers; however, we require that the power series expansion of the composition $\left(g^\infty \circ \left(f^0\right)^{-1}\right)^m$ converge on compact domains that are not necessarily disks. It may be that we are working in an "unsuitable" basis to do this and that Faber polynomials may solve the convergence issues.

5.2.3 Strengthening the Existence of the Complementary Maps Proof using Convergent Matrix Operations

We would like to be able to strengthen the result of Theorem 3.7 by being able to drop the assumption that the quasi-symmetry $\phi(z)$ needs to be known. While working on solving the complementary map problem in the algebraic case, we noticed that not enough information was given - there always seemed to be more unknowns than knowns in the linear equations, even when using the special form of the power matrices. To explain further, consider the matrix equation

$$[f^0][\phi] = [f^\infty]. \tag{5.7}$$

Provided one could find the coefficients of the power matrix of the complementary map $[f^\infty]$, given the coefficients of the power matrix $[f^0]$, Theorem 3.7 could be strengthened. However, when we attempted to solve this posed question, we could only obtain one row or one column of the block $\phi_{+,+}$. If we were able to obtain the whole block $\phi_{+,+}$ from simply one row or column, this would answer the question positively. Perhaps using the relations between the quasi-symmetry $[\phi]$ and the generalized Grunsky matrix given in [TT06] might be of assistance.

Bibliography

[Ahl66] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Mathematical Studies, vol. 10, D. Van Nostrand Company, Princeton, 1966.

[Dav74] Philip J. Davis, *The Schwarz function and its applications*, Carus Mathematical Monographs, vol. 17, Mathematical Association of America, 1974.

[FM07] Alastair Fletcher and Vladimir Markovic, *Quasiconformal maps and Teichmüller theory*, Oxford Graduate Texts in Mathematics, vol. 11, Oxford University Press, New York, 2007.

[Hua97] Yi-Zhi Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, vol. 148, Birkhäuser, Boston, 1997.

[Jab53] Eri Jabotinsky, *Representation of functions by matrices. Application to Faber polynomials*, Proc. Amer. Math. Soc. **4** (1953), 546–553.

[Leh87] Olli Lehto, *Univalent functions and Teichmüller spaces*, Graduate Texts in Mathematics, vol. 109, Springer-Verlag, New York, 1987.

[LV73] Olli Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.

[MS02] César P. Milies and Sudarshan K. Sehgal, *An introduction to group rings*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2002.

[Ner87] Y.A. Neretin, *On a complex semigroup containing the group of diffeomorphisms of the circle*, Functional Anal. Appl. **21** (1987), no. 2, 160–161.

[NS95] Subhashis Nag and Dennis Sullivan, *Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ on the circle*, Osaka J. Math. **32(1)** (1995), 1–34.

[Pom75] Christian Pommerenke, *Univalent functions*, With a chapter on quadratic differentials by Gerd Jensen. Studia Mathematica/Mathematische Lehrbücher, Band XXV, Vandenhoeck and Ruprecht, Göttingen, 1975.

[RS06] David Radnell and Eric Schippers, *Quasisymmetric sewing in rigged Teichmüller space*, Commun. Contemp. Math. **8(4)** (2006), 481–534.

[RS09] ———, *The semigroup of rigged annuli and the Teichmüller space of the annulus*, submitted (2009).

[RS10] ———, *Fiber structure and local coordinates for the Teichmüeller space of a bordered Riemann surface*, Conformal Geometry and Dynamics **14** (2010), 14–34.

[Sch08] Martin Schottenloher, *A mathematical introduction to conformal field theory*, second ed., Springer-Verlag, Berlin, 2008.

[Sch10a] Eric Schippers, *The derivative of the Nehari functional*, Annal. Acad. Scient. Fenn. **35** (2010), 291–307.

[Sch10b] ———, *A power matrix approach to the Witt algebra and Loewner equations*, Computational Methods and Function Theory **10(1)** (2010), 399–420.

[Seg04] G. Segal, *The definition of conformal field theory*, Topology, Geometry and Quantum Field Theory (U. Tillman, ed.), London Mathematical Society Lecture Note Series, vol. 308, Cambridge University Press, New York, 2004.

[Sue98] P.K. Suetin, *Series of Faber polynomials*, Translated from 1984 Russian original by E.V. Pankratiev, Analytical Methods and Special Functions, Gordon and Breach Science Publishers, Amsterdam, 1998.

[TT06] Leon A. Takhtajan and Lee-Peng Teo, *Weil-Petersson metric on the universal Teichmüller space II. Kähler potential and period mapping*, Mem. Amer. Math. Soc. **183** (2006).

Index