Extremal Properties of Degree Sequences: Potential Functions for Subgraphs and Forbidden Subgraphs

by

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Abstract

Any pair of graphs with the same degree sequence have the same number of edges, but they may not have the same subgraphs. In 1991 Erdős, Jacobson, and Lehel, introduced the concept of the 'potential function' of a graph H: the least number of edges in a graph on n vertices for which some other graph with the same degree sequence contains a copy of a fixed graph H. They gave a conjecture for the value of the potential function for the case when H is complete that has since been shown to be true when n is sufficiently large in terms of the order of H. This thesis gives a survey of these results and the techniques used to prove them. For arbitrary graphs H, this thesis also provides asymptotic results about the potential function along with some properties of sequences without such realizations. Finally, I present some original results about the maximum number of edges in a graph whose degree sequence has realizations avoiding H. To avoid some trivial cases, the problem is restricted to connected realizations and is solved completely in the cases that either H is complete or a small cycle. I then present a conjecture for all larger cycles along with supporting results.

Acknowledgements

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Besides my advisor, I would like to thank my examining committee, composed of Dr. Shahin Kamali and Dr. Andriy Prymak, for taking the time to provide their thoughts.

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Notation:

Unless otherwise stated, all graphs in this thesis are assumed to being finite simple graphs. That is, graphs without multiple edges between the same pair of vertices and without loops. Moreover, unless otherwise indicated, subgraphs considered are not necessarily induced subgraphs.

- $V(G)$: The set of vertices in the graph G.
- $E(G)$: The set of edges in the graph G.
- K_n: The complete graph on n vertices.
- C_n: The cycle on n vertices.
- $\overline{G}$: The complement of the graph G.
- $G \cup H$: The graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.
- $G + H$: The graph with vertex set $V(G) \cup V(H)$ and edge set
$$E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

- $\deg_G(v)$: The degree of the vertex v in the graph G.
- $\delta(G)$: The minimum degree of the graph G.
- $\Delta(G)$: The maximum degree of the graph G.
- $\chi(G)$: The chromatic number of the graph G.
- $\alpha(G)$: The order of the largest set of pairwise non-adjacent vertices in the graph G, called the independence number of G. A set of pairwise non-adjacent vertices is called an independent set.
- $o(g)$: Given functions $f(n)$ and $g(n)$, say that $f = o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Given a graph G, a degree sequence of G is a finite sequence whose terms are the degrees of G. Generally, the terms of a degree sequence are written in non-increasing order, and this sequence is denoted $\pi(G)$. A finite sequence of integers π is called *graphic* if there is a graph for which π is its degree sequence and any such graph is called a *realization* of π. For ease of notation, if the term b is repeated m times in a row in a sequence, abbreviate the subsequence by replacing the m occurrences of b with the term $(b)^m$. For instance, the degree sequence for K_n can be written as $((n-1)^n)$.

Unless otherwise specified, notation and conventions follow standards used in Bollobás's *Modern Graph Theory* [4].

Chapter 1

Introduction

One of the central problems in extremal graph theory is determining the number of edges a graph on a fixed number of vertices can have while not containing a given subgraph. The maximum such value is called the Turán number for a graph. While all graphs with a given degree sequence have the same number of edges, it is possible for one realization to contain a fixed subgraph, while another realization does not. In 1991, Erdős, Jacobson, and Lehel [8] studied the question of which degree sequences have at least one realization which contains a fixed desired subgraph. In particular, they introduced the concept of the 'potential function', which describes which graphical sequences have realizations containing a given subgraph based on the number of edges present.

For $n > k$, the graph $K_{k-1} + \overline{K_{n-k+1}}$, consisting of an independent set of $n - k + 1$ vertices all adjacent to each vertex of a complete graph of order $k - 1$, contains no clique of order $k + 1$. Moreover, it is the unique graph with degree sequence consisting of $k-1$ entries being $n-1$ and the remaining $n-k+1$ being $k-1$. Erdős, Jacobson, and Lehel conjectured that this is the extremal such case in that if a graph G has n vertices and more edges than $K_{k-1}+\overline{K_{n-k+1}}$, then there is some other graph with the same degree sequence as G which contains a copy of K_{k+1}. More generally they define the following: Given a graph H, the *potential function for H at n* is the minimum s so that for every graphic sequence with entry-sum at least s, there is a realization containing a copy of H. In 1996, Li and Song [19] showed that for some values of n which are small relative to k, Erdős, Jacobson, and Lehel's conjecture about the potential function for complete graphs is false, but for $n \geq 8$, the conjecture is true for K_4. In 1998, the same authors [20] proved the conjecture for K_5 true for $n \geq 10$. In the same year, along with Luo [21] they proved the conjecture for all remaining cliques provided n is sufficiently large and evaluated the potential function for such cliques for some smaller values of n. In Chapter 3, I survey many of the results which lead to these conclusions.

The Erdős-Stone-Simonovits Theorem [9, 10] determines the asymptotic value of the Turán numbers by focusing on a quantity defined in terms of the subgraph's chromatic number. In the same way, Ferrara, Lesaulnier, Moffatt, and Wenger [11] showed that the asymptotic behaviour of the potential function for a given graph can be obtained by focusing on a quantity defined in terms of the graph's independence number. In Chapter 4, I survey the construction of this quantity and describe its utility in this problem. Also in

Chapter 4, I survey results by Erbes, Ferrara, Martin, and Wenger [5] about the 'shape' of degree sequences with no realizations which contain a given subgraph, as well as results by these authors [6] about a couple notions of 'stability' for the potential function.

In Chapter 5, I present my original results involving a new variation on the problem of potential functions relating to avoiding a given subgraph. Turán numbers describe the maximum number of edges a graph can have while not containing a given subgraph. This is equivalent to the dual problem of determining the minimum number of edges a graph is required to have to be such that no matter how the edges appear in the graph, it contains a copy of said subgraph. However, this type of dual problem is not as straightforward for the potential function. That is, methods for determining the maximum sum of a sequence which is potentially H-free do not follow directly from the method of evaluating the potential function. For example, the only realization of the sequence consisting of k entries of $k-1$ and $n-k$ zeroes is a complete graph of order k together with a set of $n-k$ isolated vertices. Clearly, there is no way for this sequence to be potentially K_k-free, but it is also not possible for a realization of this graph to be connected. Insisting on considering only connected realizations yields results similar to those otherwise established. In Chapter 5, I will explore ideas of extending more general results regarding the potential function into the specific case which considers only connected realizations, specifically for subgraphs which are complete or are cycles. In particular, conditions are given to guarantee connected realizations which avoid given complete graphs or avoid given cycles of short length.

Before introducing results about the potential function however, Chapter 2 discusses many useful conditions for the graphicality of degree sequences. In particular, a proof is given for the well-known result by Havel [16] and Hakimi [14] which not only establishes a condition equivalent to graphicality of a sequence, but also introduces a graph operator which preserves degree sequence. This operator, known as edge swapping, as well as some generalizations thereof prove useful throughout this survey. A proof is also given for the Erdős-Gallai Theorem [7], which provides a method of determining graphicality of a sequence based on an inequality involving the terms of the sequence. Several other conditions for graphicality, some of which are used repeated throughout, are also established for the sake of completeness.

Chapter 2

Degree Sequences and Graphicality

2.1 Edge-Swapping Techniques

Given a finite sequence, one could ask does there exist a graph whose degree sequence is this sequence. This question has been answered numerous times. Recall the definitions from the notation section: such sequences are called graphical and such graphs are called realizations thereof. One of the most well known methods for determining whether or not a sequence is graphical was independently authored by Havel in 1955 [16] and Hakimi in 1962 [14]. Their method involves removing terms from the sequence and lowering the value of other terms to obtain a new sequence for which a realization may be simpler to ascertain. The algorithm hinges on Theorem 2.1.1.

Theorem 2.1.1 (Havel [16], Hakimi [14])**.** *The non-increasing sequence* $(d_1, ..., d_n)$

is graphical if and only if the sequence $(d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n)$ *is graphical.*

Before proving Theorem 2.1.1, an exceedingly useful graph operation is introduced which preserves the degree sequence of its affected graph.

Definition 2.1.2. Let G be a graph and let $a, b, c, d \in V(G)$ be such that $ab, cd \in E(G)$, but $ac, bd \notin E(G)$. Define the graph G' by $V(G) = V(G')$ and $E(G') = (E(G) \setminus \{ab, cd\}) \cup \{ac, bd\}$. The process of obtaining G' from G called *swapping* the edges ab and cd for ac and bd. See Figure 2.1.

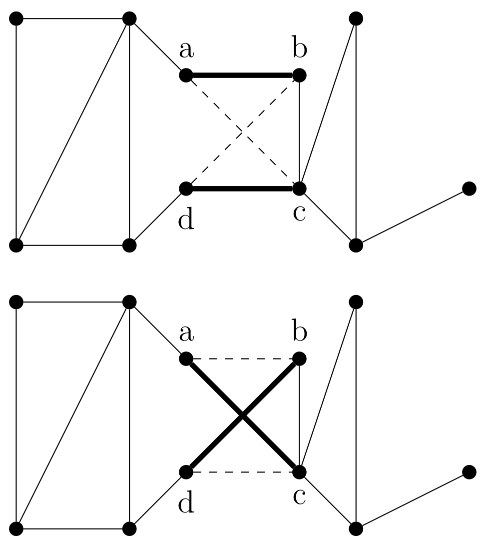


Figure 2.1: Degree-sequence preserving edge swap.

The key fact about swapping edges in a graph in this way is that it does not alter the degree sequence of the resulting graph since all vertices are incident with the same number of edges after the operation is performed. Moreover, this operation can be used to ensure that the neighbourhood of any given vertex is any subset of vertices of the correct order. This can be used to prove the Havel-Hakimi theorem:

Proof of Theorem 2.1.1. To prove one direction of the claim, let $(d_2-1, ..., d_{d_1+1}-1, d_{d_1+2}, ..., d_n)$ be a graphical sequence with realization H. Without loss of generality, let $V(H) = \{v_2, ..., v_n\}$ such that for $2 \leq i \leq d_{d_1+1}$, $\deg_H(v_i) = d_i - 1$ and for $d_{d_1+2} \leq i \leq n$, $\deg_H(v_i) = d_i$. Define the graph H' by $V(H) = \{v_1, ..., v_n\}$ and $E(H') = E(H) \cup \{v_1v_i, ..., v_1v_{d_{d_1+1}}\}$ and note that H' is a realization of $(d_1, ..., d_n)$.

To prove the remaining direction of the claim, let $(d_1, ..., d_n)$ be a graphical sequence and let G be a realization such that $V(G) = \{w_1, ..., w_n\}$, for $1 \leq i \leq n$, $\deg_G(w_i) = d_i$. Assume firstly that there exists $u \in \{w_2, ..., w_{d_{d_1+1}}\} \backslash N_G(w_1)$. Since $\deg_G(w_1) = d_1$, it follows that there exists $v \in \{w_{d_{d_1+2}}, ..., w_n\}$ adjacent to w_1. Thus,

$$|N_G(v) \setminus \{w_1\}| < |N_G(v)| = \deg_G(v) \leq \deg_G(u) = |N_G(u)|$$

and so there exists $w \in N_G(u) \backslash (N_G(v) \backslash \{w_1\}) = N_G(u) \backslash N_G(v)$. Swapping the edges w_1v and uw for w_1u and uv yields a realization of $(d_1, ..., d_n)$ such that u is adjacent to w_1. Repeating this process for every vertex in $\{w_2, ..., w_{d_{d_1+1}}\} \setminus N_G(w_1)$ yields a realization for which $N(w_1) = \{w_2, ..., w_{d_{d_1+1}}\}$ and removing w_1 from this graph yields a realization of $(d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n)$. $\square$

In the proof of Theorem 2.1.1, the vertex of degree d_1 was either removed or added to provide realizations of the mentioned sequences. In 1973, Kleitman and Wang [18] showed that showed that it is not necessary to limit the technique used in this proof to the vertex of maximum degree. They formalized and generalized the definition of the shorter sequence mentioned in Theorem 2.1.1.

Definition 2.1.3 (Kleitman, Wang [18])**.** For a nonincreasing sequence $\pi = (d_1, ..., d_n)$, and $k \in \{1, ..., n\}$, the sequence

$$\pi' = \begin{cases} (d_1 - 1, ..., d_{k-1} - 1, d_{k+1} - 1, ..., d_{d_k+1} - 1, d_{d_k+2}, ..., d_n) & \text{if } d_k \geq k \\ (d_1 - 1, ..., d_k - 1, d_{k+1}, ..., d_{k-1}, d_{k+1}, ..., d_n) & \text{if } d_k < k. \end{cases}$$

is called the *residual sequence obtained by laying off* d_k *from* π.

Upon introducing this definition, the authors used an edge-swapping technique similar to that used in the proof of Theorem 2.1.1 to prove the following generalization.

Theorem 2.1.4 (Kleitman, Wang [18])**.** *For a nonincreasing sequence* $\pi = (d_1, ..., d_n)$ *and any* $k \in \{1, ..., n\}$*, the residual sequence obtained by laying off* d_k *from* π *is graphic if and only if* π *is graphic.*

This process of swapping a pair of edges for a pair of non-edges itself can be generalized. In particular, if one swapped only one edge for some non-edge, the resulting graph would have a different degree sequence. However, there is nothing to prevent an edge swap being performed on a set of more than two edges and non-edges so long as the degree sequence of the resulting graph is unchanged. We introduce a more general notion of swapping edges here to formalize some general categories of transformations on graphs.

Definition 2.1.5. Let G be a graph and let $S \subset E(G)$ and $T \subseteq E(\overline{G})$ be such that for any $x \in V(G)$, the number of edges in S containing x is equal to the number of edges in T containing x. Define G' by $V(G) = V(G')$ and $E(G') = (E(G) \setminus S) \cup T$. The process of obtaining G' from G is called *swapping* the edges S for T.

This more general notion of edge swapping also preserves the degree sequence of a given graph and from it, the following useful special case of this general edge swap can be defined.

Definition 2.1.6. Let G be a graph, $P = (x_1, ..., x_k)$ be a path in G such that x_1 and x_k are not adjacent, and let $e = \{a, b\}$ be an edge in G not in P. Let H be a graph defined as follows, $V(G) = V(H)$ and

$$E(H) = (E(G) \setminus \{x_1x_2, x_{k-1}x_k, e\}) \cup \{x_1x_k, x_2a, x_{k-1}b\}.$$

The process of obtaining the graph H is the *path-edge swap* of P for e in G.

The fact that the path-edge swap preserves degree sequence is used to prove several results in Section 5.3. In particular, this path-edge swap can be used on graphs which contain cycles to ensure that the graph does not contain

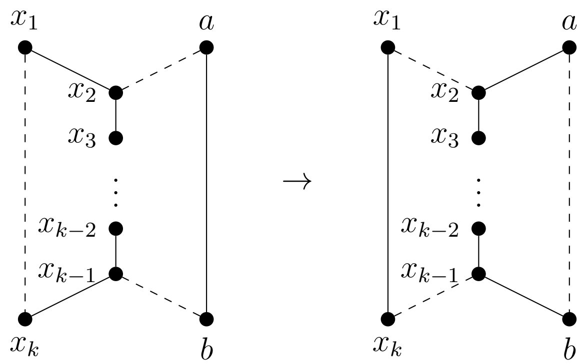


Figure 2.2: The Path-Edge Swap

a cycle of a given length. That said, there is nothing stopping one taking this further and performing such a swap between a path and another path, as demonstrated in Figure 2.3.

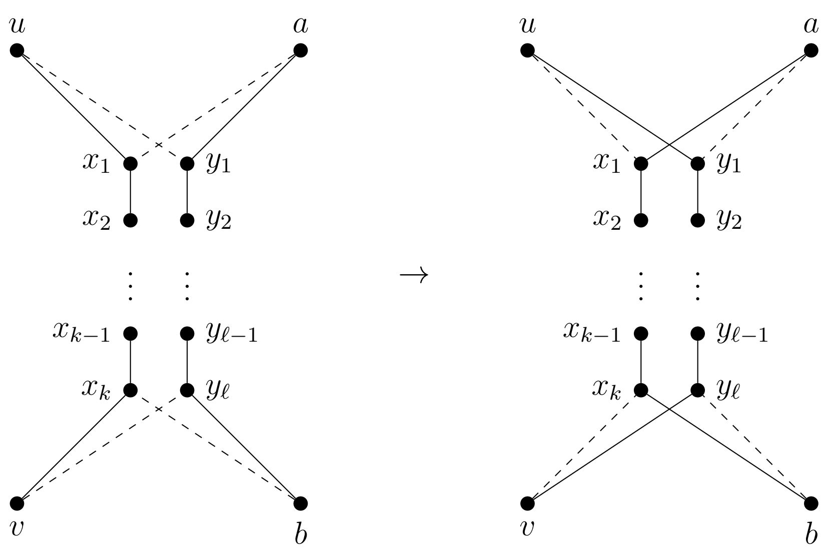


Figure 2.3: The Path-Path Swap as a generalization of the Path-Edge Swap.

2.2 Erdős-Gallai Theorem

There are many criteria which are equivalent to the graphicality of a sequence. One of the most well documented is the Erdős-Gallai theorem. Originally proven by Erdős and Gallai in 1960 [7], many proofs of the result have since been put forward. The proof provided here was given by Tripathi, Venugopalan, and West in 2009 [28].

Definition 2.2.1. Given a finite sequence π, let $\sigma(\pi)$ be the sum of the terms of π.

Theorem 2.2.2 (Erdős, Gallai [7])**.** *A sequence $\pi = (d_1, ..., d_n)$ is graphical if and only if the following two conditions are satisfied:*

1. *$\sigma(\pi)$ is even*

2. *for each $k \in \{1, ..., n\}$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i).$$

Proof. The forward direction follows immediately from the Handshaking Lemma (section I.1 [4]) and the observation that the right side of the equation in the second condition is the number of possible edges incident to the k vertices of the highest degree.

To prove the reverse direction, let a *subrealization* of the sequence $(d_1, ..., d_n)$ be a graph G with $V(G) = \{v_1, ..., v_n\}$ such that for $i \in \{1, ..., r-1\}$, $\deg(v_i) \leq d_i$. The result is proven by constructing a sequence of subrealizations starting from the empty graph on $V(G)$. Given a subrealization G of the sequence $(d_1, ..., d_n)$, the *critical index* (denoted by r) is the largest index such that for $i \in \{1, ..., r-1\}$, $\deg_G(v_i) = d_i$. If each $d_i = 0$, then the empty graph is a realization of $(d_1, ..., d_n)$. Otherwise, the critical index of the empty graph is 1. While $r \leq n$, the following process constructs a new subrealization G' from the subrealization G for which $\deg_G(v_r)$ is greater while not changing the degree of any vertex in $\{v_1, ..., v_{r-1}\}$ and keeping the set $\{v_{r+1}, ..., v_n\}$ independent.

Case 1: If there exists i such that $v_r v_i \notin E(G)$ and $\deg_G(v_i) < d_i$, then let $E(G') = E(G) \cup \{v_r v_i\}$. See Figure 2.4.

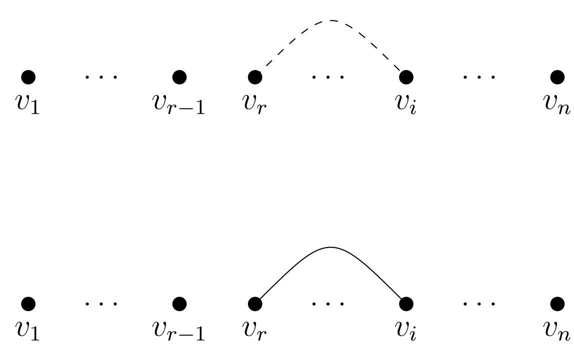


Figure 2.4: Case 1, add this edge.

Case 2: If there exists $i < r$ such that $v_r v_i \notin E(G)$, then since

$$\deg_G(v_i) = d_i \geq d_r > \deg_G(v_r)$$

there exists $u \in N(v_i) \setminus (N(v_r) \cup \{v_r\})$. If $d_r - \deg_G(v_r) \geq 2$, then let $E(G') = (E(G) \setminus \{uv_i\}) \cup \{uv_r, v_i v_r\}$. See Figure 2.5.

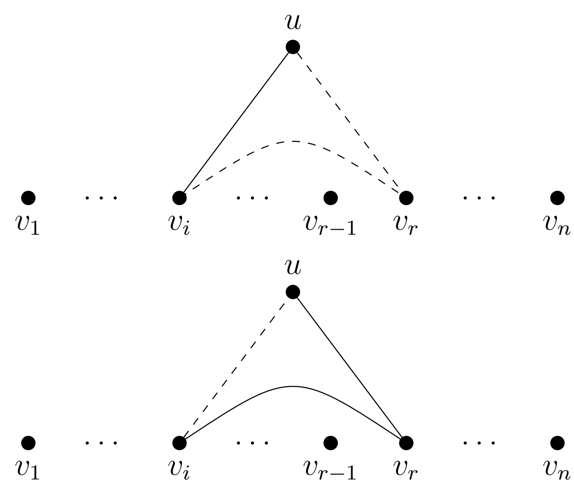


Figure 2.5: Case 2, if $d_r - \deg_G(v_r) \geq 2$, exchange these edges and nonedges.

If $d_r - \deg_G(v_r) = 1$, then note that since, by condition 1, $\sum_{i=1}^{n} d_i - \deg_G(v_i)$ is even, it follows that there exists $k > r$ such that $\deg_G(v_k) < d_k$. If $v_r v_k \notin E(G)$, then case 1 applies. If $v_r v_k \in E(G)$, then let $E(G') = (E(G) \setminus \{v_r v_k, uv_i\}) \cup \{uv_r, v_i v_r\}$. See Figure 2.6

Case 3: Assume that $v_1, ..., v_{r-1} \in N(v_r)$ and there exists $k > r$ such that $\deg_G(v_k) \neq \min(r, d_k)$. Since $\{v_{r+1}, ..., v_n\}$ is independent and $k > r$, $\deg_G(v_k) \leq r$. Since G is a subrealization, $\deg_G(v_k) \leq d_k$. Thus, $\deg_G(v_k) <$

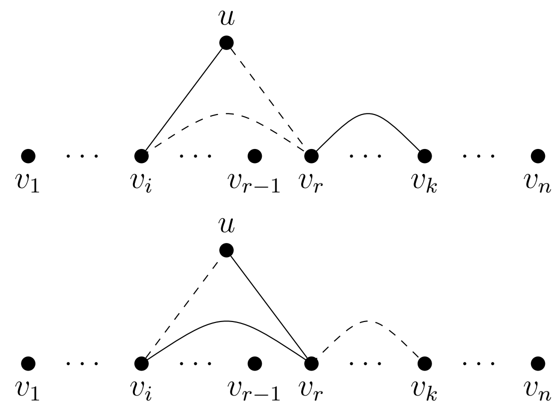


Figure 2.6: Case 2, if $d_r - \deg_G(v_r) = 1$, exchange these edges and nonedges.

$\min(r, d_k)$ and so there exists $i < r$ such that $v_k v_i \notin E(G)$. Moreover,

$$\deg_G(v_i) = d_i \geq d_r > \deg_G(v_r)$$

and so there exists $u \in N(v_i) \setminus (N(v_r), \{v_r\})$. If $v_r v_k \notin E(G)$, then case 1 applies. If $v_r v_k \in E(G)$, let $E(G') = (E(G) \setminus \{uv_i\}) \cup \{uv_r, v_i v_k\}$. See Figure 2.7.

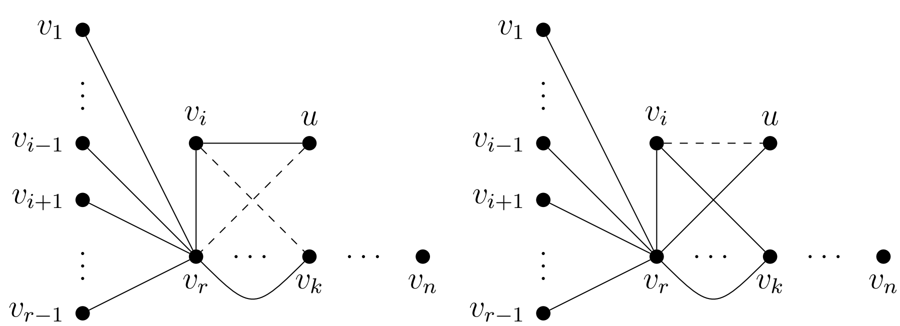


Figure 2.7: Case 3, exchange these edges and nonedges.

Case 4: Assume that $v_1, ..., v_{r-1} \in N(v_r)$ and there exist $i, j \in \{1, ..., r\}$

such that $i < j < r$ and $v_iv_j \notin E(G)$. Note that

$$\deg_G(v_i) = d_i \geq d_j = \deg_G(v_j) \geq d_r > \deg_G(v_r)$$

and so there exists $u \in N(v_i) \setminus (N(v_r) \cup \{v_r\})$ and $w \in N(v_j) \setminus (N(v_r) \cup \{r\})$. Thus, $u, w \notin \{v_1, ..., v_r\}$ and so $u, w \in \{v_{r+1}, ..., v_n\}$. Let $E(G') = (E(G) \setminus \{uv_i, wv_j\}) \cup \{v_iv_j, uv_r\}$. See Figure 2.8.

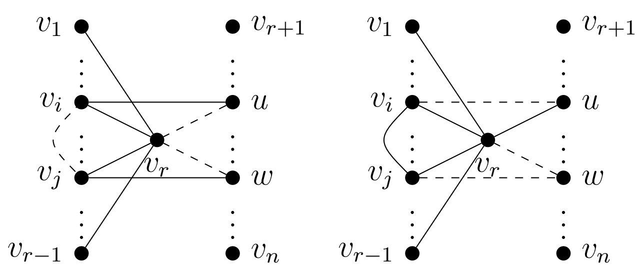


Figure 2.8: Case 4, exchange these edges and nonedges.

(Note that in the above case, it is possible that $u = w$.)

If none of the above cases apply, then the set $\{v_1, ..., v_r\}$ induces a clique, the set $\{v_{r+1}, ..., v_n\}$ is independent, and for every $k > r$, $\deg_G(v_k) = \min(r, d_k)$. That is,

$$\sum_{i=1}^{r} \deg_G(v_i) = r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i)$$

and by condition 2,

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i).$$

Hence,

$$0 \leq d_r - \deg_G(v_r) = \sum_{i=1}^{r} d_i - \sum_{i=1}^{r} \deg_G(v_i) \leq 0$$

and so $d_r = \deg_G(v_r)$, which contradicts the definition of r. At each stage, $\deg_{G'}(v_r) \geq \deg_G(v_r)$ and so r can only increase. When $r = n+1$, the assertion holds. □

2.3 More Conditions Equivalent to Graphicality

As mentioned previously, there are many conditions equivalent to the statement that a given sequence is graphical. In 1991, Sierksma and Hoogeveen [27] compiled a list of seven of the better known such conditions and showed that they were all equivalent, one of which is Theorem 2.2.2. The following are the remaining six such criteria which provide a relatively quick test to prove graphicality of a given sequence which they mention.

Theorem 2.3.1 (Ryser Criterion [26]). *A sequence* $(a_1, ...a_m; b_1, ..., b_n)$ *is called bipartite-graphic if and only if there exists a bipartite graph such that one component has degree sequence* $(a, ..., a_m)$ *and the other component has degree sequence* $(b_1, ..., b_n)$. *Given a sequence* $(d_1, ..., d_n)$, *define* $f = \max\{i : d_i \geq i\}$ *and*

$$\overline{d_i} = \begin{cases} d_i + 1 & \text{if } i \in \{1, ..., f\} \\ d_i & \text{otherwise.} \end{cases}$$

The sequence $(d_1, ..., d_n)$ *is graphic if and only if the sequence*

$$(\overline{d_1}, ..., \overline{d_n}; \overline{d_1}, ..., \overline{d_n})$$

is bipartite-graphic.

Theorem 2.3.2 (Berge Criterion [2])**.** *Given a sequence* $(d_1, ..., d_n)$*, define* $f = \max\{i : d_i \geq i\}$*. Moreover, define* $(\overline{d_1}, ..., \overline{d_n})$ *as follows: For* $k \in \{1, ..., n\}$*,* $\overline{d_k}$ *is the kth column sum of the* $(0, 1)$*-matrix*

$$[a_{ij}] = \begin{cases} 1 & \text{if } i \leq f \text{ and } j \in \{1, ..., d_i + 1\} \setminus \{i\} \\ 1 & \text{if } i > f \text{ and } j \leq d_i \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $(d_1, ..., d_n)$ *is graphic if and only if for every* $k \in \{1, ..., n\}$*,*

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \overline{d_i}.$$

Note that the matrix referenced in Theorem 2.3.2 is often called the left-most off-diagonal matrix (0,1)-matrix of $(d_1, ..., d_n)$.

Theorem 2.3.3 (Fulkerson-Hoffman-McAndrew Criterion [12])**.** *The sequence* $(d_1, ..., d_n)$ *is graphic if and only if for every* $k \in \{1, ..., n\}$ *and every* $m \geq 0$ *such that* $k + m \leq n$*,*

$$\sum_{i=1}^{k} d_i \leq k(n - m - 1) + \sum_{i=n-m+1}^{n} d_i.$$

Theorem 2.3.4 (Bollobás Criterion [3]). *The sequence $(d_1, ..., d_n)$ is graphic if and only if for every $k \in \{1, ..., n\}$,*

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i + \sum_{i=1}^{k} \min(d_i, k-1).$$

Theorem 2.3.5 (Grünbaum Criterion [13]). *The sequence $(d_1, ..., d_n)$ is graphic if and only if for every $k \in \{1, ..., n\}$,*

$$\sum_{i=1}^{k} \max(k-1, d_i) \leq k(k-1) + \sum_{i=k+1}^{n} d_i.$$

Theorem 2.3.6 (Hässelbarth Criterion [15]). *Define $(\overline{d_1}, ..., \overline{d_n})$ as follows: For $i \in \{1, ..., n\}$, $\overline{d_k}$ is the kth column sum of the $(0,1)$-matrix*

$$[a_{ij}] = \begin{cases} 1 & \text{if } i \in \{1, ..., d_i\} \\ 0 & \text{otherwise.} \end{cases}$$

Define $f = \max\{i : d_i \geq i\}$. The sequence $(d_1, ..., d_n)$ is graphic if and only if for every $k \in \{1, ..., f\}$,

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} (\overline{d_i} - 1).$$

Chapter 3

Complete Subgraphs in Realizations

3.1 The Potential Function

Given a graph F and a postive integer n, the Turán number is the maximum number, denoted $ex(F, n)$, of edges in a graph of order n which does not contain a copy of the graph F as a subgraph. The question of determining Turán numbers can be rephrased in terms degree sequences in that given a degree sequence, the number of edges in a realization is simply half the sum of its terms. That is, given a graph F and a positive integer n, the corresponding Turán number also describes a minimum value x such that for every graphical sequence π with $\sigma(\pi) > x$, every realization of π contains the graph F as a subgraph. Erdős, Jacobson, and Lehel posed the related question: given a graph F and a positive integer n, what is the minimum value x for which every graphical sequence π of length n such that $\sigma(\pi) \geq x$ has at least one realization of π contains F as a subgraph.

Definition 3.1.1. Given a graph H, the sequence π is called *potentially H-graphic* if there is a realization of π which contains H as a subgraph.

Definition 3.1.2 (Erdős, Jacobson, Lehel, [8])**.** Given a graph F and integer $n > 0$, the *potential function* $\sigma(F, n)$ is the least value q such that for every graphical sequence π of length n such that $\sigma(\pi) = q$, there is a realization of π containing a copy of F as a subgraph.

Not only did they introduce this notion of potential function, but they also mentioned a more general concept: given a graph property P, what is the least value q for which there exists a graphical sequence π of length n such that $\sigma(\pi) = q$ and there is a realization of π which has the property P. In the paper in which the potential function is introduced, the authors consider the case where the subgraph in question is a complete graph. In particular, they noted that the sequence $((n-1)^{k-1}, (k-1)^{n-k+1})$ has the unique realization $K_{k-1} + \overline{K_{n-k+1}}$. Hence, this sequence is not potentially K_{k+1}-graphic. They conjectured that this is the extremal case in that if a graph G has more edges than $K_{k-1} + \overline{K_{n-k+1}}$, then the degree sequence of G is potentially K_{k+1}-graphic.

Conjecture 3.1.3 (Erdős, Jacobson, Lehel [8])**.** *For $k \geq 2$, $n \geq 6$,*

$$\sigma(K_{k+1}, n) = (k-1)(2n-k) + 2$$

From 1996 through 1999, Li, Song, and Luo, released a series of papers [19–21] which proved Conjecture 3.1.3 for n sufficiently large and detailed some of the cases where it is false. In this chapter, I will provide a path through many of the developments which lead to the proof of this conjecture for $n \geq \binom{k}{2} + 3$, and provide proofs for many of the more interesting results.

Before continuing, some notation is introduced for ease of understanding. Throughout this chapter, the notation for the column sum of the leftmost off diagonal matrix from Theorem 2.3.2 is used. That is, if $\pi = (d_1, ..., d_n)$ is a sequence, then for $1 \leq i \leq n$, $\overline{d_i}$ is the i^{th} column sum of the leftmost off-diagonal $(0, 1)$-matrix. Moreover, given a sequence $\pi = (d_1, ..., d_n)$, for $1 \leq i \leq n$, let $rs(\pi, d_i)$ denote the residual sequence after laying off d_i from π as defined in Definition 2.1.3.

3.2 Triangles in Realizations

In the paper in which Conjecture 3.1.3 is introduced, the authors prove that it holds in the case of $k = 2$. The proof makes use of the fact that not only is the graph K_3 complete, but it is also a cycle.

Theorem 3.2.1 (Erdős, Jacobson, Lehel [8]). *For $n \geq 6$,*

$$\sigma(K_3, n) = 2n$$

Proof. Let π be a graphic sequence of length n such that $\sigma(\pi) \geq 2n$. Let G be a realization of π and note that since $\sigma(\pi) \geq 2n$, it follows that G contains a cycle. Let C be the smallest cycle in G and let $V(C) = \{v_1, ..., v_{|V(C)|}\}$ be such that v_1 and $v_{|V(C)|}$ are adjacent and for $1 \leq i \leq |V(C)| - 1$, v_i and v_{i+1} are adjacent. If $|V(C)| = 3$, then G contains a triangle.

Case i) $|V(C)| = 4$ and C is not an isolated cycle. Since C is not isolated, it follows that there exists a vertex x adjacent to some vertex in C. Without loss of generality, assume that x and v_1 are adjacent and note that since G has no triangles, v_1 is not adjacent to v_3 and x is not adjacent to v_2. Hence, swapping the edges xv_1 and v_2v_3 for the edges xv_2 and v_1v_3 yields a realization of π which contains the triangle $v_1v_3v_4$. See Figure 3.1.

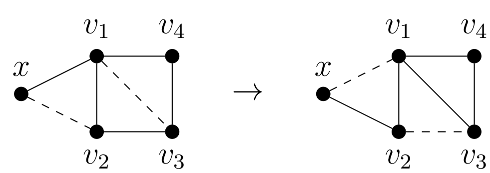


Figure 3.1: Edge swap yielding K_3 from K_4.

Case ii) $|V(C)| = 4$ and C is an isolated cycle. Since G has at least as many edges as vertices and $n \geq 6$, it follows that there exists a pair of adjacent vertices y and z not in C and not adjacent to any vertex in C. In particular, neither y nor z is adjacent to y_2 and since G has no triangles, v_1 is not adjacent to v_3. Hence, swapping the edges yz, v_1v_2, and v_2v_3 for the edges yv_2, zv_2 and v_1v_3 yields a realization of π which contains the triangle $v_1v_3v_4$. See Figure 3.2.

Case iii) $|V(C)| = 5$ and C is not an isolated cycle. Since C is not isolated, it follows that there exists a vertex x adjacent to some vertex in C. Without

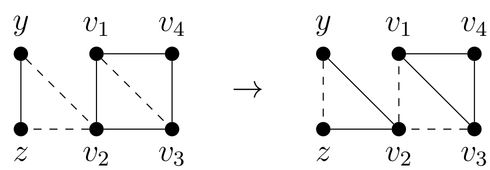


Figure 3.2: Edge swap yielding K_3 from K_4.

loss of generality, assume that x and v_1 are adjacent and note that since G has no cycles of length 4, v_1 is not adjacent to v_4 and x is not adjacent to v_3. Hence, swapping the edges xv_1 and v_3v_4 for the edges xv_3 and v_1v_4 yields a realization of π which contains the triangle $v_1v_4v_5$. See Figure 3.3.

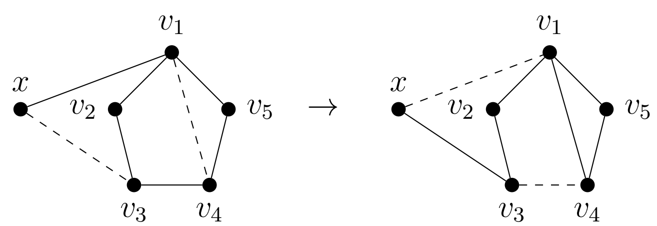


Figure 3.3: Edge swap yielding K_3 from K_5.

Case iv) $|V(C)| = 5$ and C is an isolated cycle. Since G has at least as many edges as vertices and $n \geq 6$, it follows that there exists a pair of adjacent vertices y and z not in C and not adjacent to any vertex in C. In particular, y is not adjacent to v_2 and z is not adjacent to v_3 and since G has no triangles, v_1 is not adjacent to v_4. Hence, swapping the edges yz, v_1v_2, and v_3v_4 for the edges yv_2, zv_3 and v_1v_4 yields a realization of π which contains the triangle $v_1v_4v_5$. See Figure 3.4.

Case v) $|V(C)| \geq 6$. Since C is the smallest cycle in G, v_1 is not adjacent

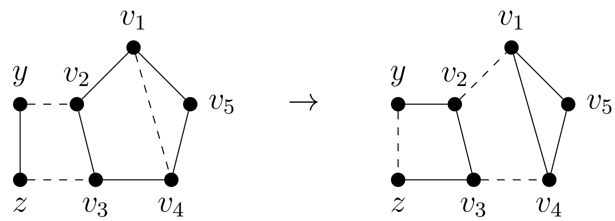


Figure 3.4: Edge swap yielding K_3 from K_5.

to v_3 and $v_{|V(C)|}$ is not adjacent to v_4. Hence, swapping the edges $v_1v_{|V(C)|}$ and v_3v_4 for the edges v_1v_3 and $v_4v_{|V(C)|}$ yields a realization of π which contains the triangle $v_1v_2v_3$. See Figure 3.5.

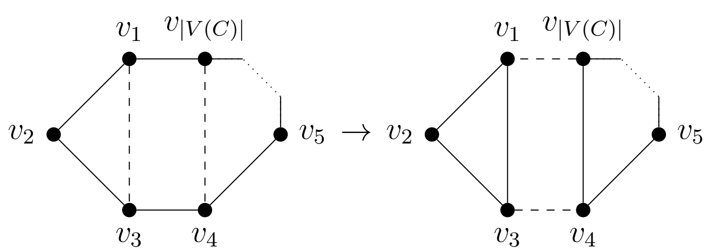


Figure 3.5: Edge swap yielding K_3 from K_6.

$\square$

3.3 Small Complete Subgraphs

In 1996, Li and Song [19] showed that there are values n and k for which Conjecture 3.1.3 is false. Theorems 3.3.1 and 3.3.2 are two results which demonstrate this that for some small values of n and give the actual value in those cases.

Theorem 3.3.1 (Li, Song [19]). *If $k+1 \leq n \leq 2k$, then*

$$\sigma(K_{k+1}, n) = (k-1)(2n-k) + (n-k)(n-k-1) + 2.$$

Proof. Let π be a graphic sequence such that

$$\sigma(\pi) = (k-1)(2n-k) + (n-k)(n-k-1).$$

Let G be a realization of π and note that $2|E(G)| = \sigma(\pi)$. Hence,

$$2\left|E\left(\overline{G}\right)\right| = n(n-1) - \sigma(\pi) = 2(n-k)$$

and so the largest independent set in $\overline{G}$ is of order at most k. That is, the largest possible clique in G is of order k and so π is not potentially K_{k+1}-graphic.

Now let π be a graphic sequence such that

$$\sigma(\pi) \geq (k-1)(2n-k) + (n-k)(n-k-1) + 2.$$

Let G be a realization of π and note that $2|E(G)| = \sigma(\pi)$. Hence,

$$2\left|E\left(\overline{G}\right)\right| = n(n-1) - \sigma(\pi) \leq 2(n-k-1)$$

and so G is obtained by removing at most $n-k-1$ edges from K_n. Thus, any maximal clique in G is of order at least $k+1$ and so π is potentially K_{k+1}-graphic. □

While Theorem 3.3.1 dealt with the cases for $k+1 \leq n \leq 2k$, Theorem 3.3.2 deals with the case where $n = 2k+1$.

Theorem 3.3.2 (Li, Song [19]). *For $k \geq 2$,*

$$\sigma(K_{k+1}, 2k+1) = (k-1)(4k+2) + 2.$$

Proof. Let π_0 be the sequence of length $2k+1$ whose only entry is $2k-1$. Observe that π_0 is graphic and

$$\sigma(\pi_0) = (2k+1)(2k-2) = (k-1)(4k+2).$$

Let G be a realization of π_0 and note that $\pi(\overline{G})$ is a sequence of length $2k+1$ whose only entry is 2. That is, $\overline{G}$ is a disjoint union of cycles and since $\overline{G}$ consists of an odd number of vertices, it follows that at least one of its cycles is of odd length. From this, it follows that no independent set in $\overline{G}$ consists of more than half of the vertices of $\overline{G}$ and so G has no clique of order larger than k and π is not potentially K_{k+1}-graphic.

Suppose now that π is a nonincreasing graphic sequence such that

$$\sigma(\pi) \geq (k-1)(4k+2) + 2$$

and let G be a realization of G. Note that

$$2\left|E\left(\overline{G}\right)\right| \leq 2k(2k+1) - 2|E(G)| = 2k(2k+1) - \sigma(\pi) \leq 4k = 2|V(G) - 1|$$

and so G is obtained by removing at most $2k$ edges from K_{2k+1}. Thus, any maximal clique in G is of order at least $k+1$ and so π is potentially K_{k+1}-graphic. $\square$

3.4 Conditions for K_{k+1}-graphicality

To prove Conjecture 3.1.3, it proves useful to have at least some tools which can be used to determine K_{k+1}-graphicality of a sequence more generally. In this section, several conditions which guarantee K_{k+1}-graphicality are introduced. Firstly, it is useful to understand the following new graph property. In 1979, A. R. Rao [25] introduced the concept of A_k-graphicality.

Definition 3.4.1 (Rao, [24])**.** A graphic sequence $\pi = (d_1, ..., d_n)$ is called *potentially A_k-graphic* if it has a realization G for which $V(G) = \{v_1, ..., v_n\}$, for each $i \in \{1, ..., n\}$, $\deg_G(v_i) = d_i$, and the subgraph of G induced by $\{v_1, ..., v_{k+1}\}$ is complete.

Upon introducing this new concept, Rao proved that a sequence is potentially K_{k+1}-graphic if and only if it is potentially A_k-graphic. This was accomplished using an edge-swapping technique very similar to that used in proof of Theorem 2.1.1, the Havel-Hakimi theorem. Correspondingly, in the same paper, Rao introduced a generalization of Theorem 2.2.2, the Erdős-Gallai theorem, which guarantees A_k-graphicality. Although this result was stated in his 1979 paper, it was not proven there. However Li and Song reference the proof to be from an unpublished note from Rao [24] in their papers proving Conjecture 3.1.3 for the K_4 [19] and K_5 [20] cases. Moreover, they mention that this result was also proven at a conference by Kezdy and Lehel [17], but no proof is given here.

Theorem 3.4.2 (Rao [24]). *Let $\pi = (d_1, ..., d_n)$ be a sequence of non-negative integers in which $d_1 \geq ... \geq d_{k+1}$ and $d_{k+2} \geq ... \geq d_n$. Then π is potentially A_k-graphic if and only if:*

1. *$d_{k+1} \geq k$ and $d_n \geq 0$*
2. *$\sigma(\pi)$ is even*
3. *For any s, t, $0 \leq s \leq k+1$, $0 \leq t \leq n-k-1$*

$$\sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}$$

Theorem 3.4.2 can be applied directly in the proof of Theorem 3.4.3, which is used repeatedly in Sections 3.5 and 3.6 where the case for $k = 3$ and $k = 4$ are proven respectively.

Theorem 3.4.3 (Li, Song [20]). *Let $\pi = (d_1, ..., d_n)$ be a graphic sequence, $n \geq 2k+2$. If*

$$n - 2 \geq d_1 \geq d_k = ... = d_{d_1+2} \geq ... \geq d_n \geq k,$$

then π is potentially K_{k+1}-graphic.

Proof. For $0 \leq s \leq k+1$, $0 \leq t \leq n-k-1$, let

$$L(s,t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j}$$

and

$$R(s,t) = (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.$$

By Theorem 3.4.2, it suffices to prove that for all s, t such that $0 \leq s \leq k+1$, $0 \leq t \leq n-k-1$,

$$L(s,t) \leq R(s,t).$$

Case 1: $d_k \leq s+t-1$ and $s \geq k$. Thus, for $1 \leq i \leq k+1$,

$$d_i - k + s \geq (d_k - k) + s \geq s \geq k$$

and so for $s < i \leq k+1$,

$$\min\{s+t, d_i - k + s\} \geq s \geq k.$$

Moreover,

$$s + t > d_k \geq ... \geq d_n \geq k$$

and so for all s, t such that $0 \leq s \leq k+1$, $0 \leq t \leq n-k-1$,

$$\begin{aligned}
L(s,t) &\leq (k-1)d_1 + (s-k+1)d_k + td_k \\
&\leq (k-1)(n-2) + (s-k+1)(s+t-1) + (s+t-1)t \\
&= (k-1)(n-2) + (s+t-(k-1))(s+t-1) \\
&\leq (k-1)(n-2) + (s+t-(k-1))(s+t-1) \\
&\qquad + (n+k-1-(s+t)) \\
&= (s+t)(s+t-1) + (k-1)(n-2-(s+t-1)+1) \\
&\qquad + n-(s+t) \\
&= (s+t)(s+t-1) + k(n-s-t) \\
&= (s+t)(s+t-1) + k(k+1-s) + k(n-k-1-t) \\
&\leq R(s,t).
\end{aligned}$$

Case 2: $d_k \leq s+t-1$ and $s < k$. Thus, for $1 \leq i \leq k+1$,

$$s + t > d_k \geq ... \geq d_n \geq k > s$$

and so

$$
\begin{aligned}
L(s,t) &\leq sd_1 + td_k \\
&< s(n-1) + t(s+t-1) \\
&= sn - s + ts + t^2 - t \\
&= (s+t)(s+t-1) + (k+1-s)k + (n-k-1-s) \\
&\leq R(s,t).
\end{aligned}
$$

Case 3: $d_k \geq s+t$ and $d_k \geq t+k$. It follows that for $s+1 \leq i \leq k+1$,

$$d_i - k + s \geq d_k - k + s \geq s + t$$

and so

$$
\begin{aligned}
L(s,t) &\leq (s+t)d_1 \\
&\leq (s+t)(d_1+1) \\
&= (s+t)(s+t-1+k+1-s+d_1+1-k-t) \\
&= (s+t)(s+t-1)+(s+t)(k+1-s)+(d_1+1-k-t)(s+t) \\
&= (s+t)(s+t-1)+\sum_{i=s+1}^{k+1}\min\{s+t, d_i-k+s\} \\
&\qquad + \sum_{j=k+2+t}^{d_1+2}\min\{s+t, d_j\} \\
&\leq R(s,t).
\end{aligned}
$$

Case 4: $t+k > d_k \geq s+t$. Thus, $s<k$ and

$$d_k-k+s \leq (t+k-1)+(s-k) \leq s+t-1.$$

Moreover, for $1 \leq i \leq k+1$,

$$d_k-k+s \leq d_i-k+s$$

and so for $1 \leq i \leq k+1$,

$$\min\{s+t, d_i-k-s\} \geq d_k-k+s.$$

Furthermore, note that

$$d_n \geq k \geq k-(d_k-t-s)$$

and that

$$s+t = d_k - (d_k - t - s) \geq k - (d_k - t - s).$$

Hence,

$$\begin{aligned}
L(s,t) &\leq sd_1 + td_k \\
&\leq s(n-2) + td_k \\
&\leq s(n-2) + td_k + (k+t-d_k)(n-2k-2) + s(k+t+1-d_k) \\
&= s(n-1) + td_k + (k+t-d_k)(s+n-2k-2) \\
&= (s+t)(s+t-1) + (k+1-s)(d_k-k+s) \\
&\qquad + (k-(d_k-t-s))(n-t-k-1) \\
&\leq R(s,t).
\end{aligned}$$

Thus, in any case, the result holds, which proves the theorem. □

As mentioned previously, Theorem 3.4.3 is used repeatedly in the proofs of the K_4 and K_5 cases of Conjecture 3.1.3. However, in order to prove the K_6 case, a variation of this result is also used. If the given sequence is one term longer and the k^{th} term is greater than k, then the required lower bound on the final term can be loosened by 1.

Theorem 3.4.4 (Li, Song, Luo [21]). *Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $n \geq 2k+3$ and $d_k \geq k$. If*

$$n-2 \geq d_1 \geq d_k = ... = d_{d_1+2} \geq ... \geq d_n = k-1,$$

then π is potentially K_{k+1}-graphic.

Proof. For $0 \le s \le k+1$, $0 \le t \le n-k-1$, let

$$L(s,t) = \sum_{i=1}^{s} d_i + \sum_{j=1}^{t} d_{k+1+j}$$

and

$$R(s,t) = (s+t)(s+t-1) + \sum_{i=s+1}^{k+1} \min\{s+t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s+t, d_j\}.$$

By Theorem 3.4.2, it suffices to prove that for all s, t such that $0 \le s \le k+1$, $0 \le t \le n-k-1$,

$$L(s,t) \le R(s,t).$$

Case 1: $d_k \le s+t-1$ and $s \ge k$. Thus, for $1 \le i \le k+1$,

$$d_i - k + s \ge (d_k - k) + s \ge s \ge k$$

and so for $s < i \le k+1$,

$$\min\{s+t, d_i - k + s\} \ge s \ge k.$$

Moreover,

$$s + t > d_k \ge ... \ge d_n = k-1$$

and so for all s, t such that $0 \leq s \leq k+1$, $0 \leq t \leq n-k-1$,

$$\begin{aligned}
L(s,t) &\leq (k-1)d_1 + (s-k+1)d_k + td_k \\
&\leq (k-1)(n-2) + (s-k+1)(s+t-1) + (s+t-1)t \\
&= (k-1)(n-2) + (s+t-(k-1))(s+t-1) \\
&\leq (k-1)(n-2) + (s+t-(k-1))(s+t-1) \\
&\qquad + (n+k-1-(s+t)) \\
&= (s+t)(s+t-1) + (k-1)(n-2-(s+t-1)+1) \\
&\qquad + n-(s+t) \\
&= (s+t)(s+t-1) + k(n-s-t) \\
&= (s+t)(s+t-1) + k(k+1-s) + k(n-k-1-t) \\
&\leq R(s,t).
\end{aligned}$$

Case 2: $d_k \leq s+t-1$ and $s < k$. Thus, for $1 \leq i \leq k+1$,

$$s+t > d_k \geq ... \geq d_n = k-1 \geq s$$

and so

$$\begin{aligned}
L(s,t) &\leq sd_1 + td_k \\
&< s(n-1) + t(s+t-1) \\
&= sn - s + ts + t^2 - t \\
&= (s+t)(s+t-1) + (k+1-s)k + (n-k-1-s) \\
&\leq R(s,t).
\end{aligned}$$

Case 3: $d_k \geq s+t$ and $d_k \geq t+k$. It follows that for $s+1 \leq i \leq k+1$,

$$d_i - k + s \geq d_k - k + s \geq s + t$$

and so

$$\begin{aligned}
L(s,t) &\le (s+t)d_1 \\
&\le (s+t)(d_1+1) \\
&= (s+t)(s+t-1+k+1-s+d_1+1-k-t) \\
&= (s+t)(s+t-1)+(s+t)(k+1-s)+(d_1+1-k-t)(s+t) \\
&= (s+t)(s+t-1)+\sum_{i=s+1}^{k+1}\min\{s+t, d_i-k+s\} \\
&\qquad +\sum_{j=k+2+t}^{d_1+2}\min\{s+t, d_j\} \\
&\le R(s,t).
\end{aligned}$$

Case 4: $t+k>d_k$ and $d_k>s+t$. Thus, $s<k$, $d_k-t-s>0$, and

$$d_k-k+s \le (t+k-1)+(s-k) \le s+t-1.$$

Moreover, for $1 \le i \le k+1$,

$$d_k-k+s \le d_i-k+s$$

and so for $1 \le i \le k+1$,

$$\min\{s+t, d_i-k-s\} \ge d_k-k+s.$$

Furthermore, note that

$$d_n = k-1 \ge k-(d_k-t-s)$$

and that

$$s + t = d_k - (d_k - t - s) \geq k - (d_k - t - s).$$

Hence,

$$\begin{aligned}
L(s,t) &\leq sd_1 + td_k \\
&\leq s(n-2) + td_k \\
&\leq s(n-2) + td_k + (k+t-d_k)(n-2k-2) + s(k+t+1-d_k) \\
&= s(n-1) + td_k + (k+t-d_k)(s+n-2k-2) \\
&= (s+t)(s+t-1) + (k+1-s)(d_k-k+s) \\
&\qquad + (k-(d_k-t-s))(n-t-k-1) \\
&\leq R(s,t).
\end{aligned}$$

Case 5: $t + k > d_k$ and $d_k = s + t$. Note that

$$d_{d_1+2} = d_k \geq k > k - 1 = d_n$$

and so $d_1 + 2 < n$ and $d_1 \leq n - 3$. Thus, since $s < k$,

$$\begin{aligned}
L(s,t) &\leq sd_1 + td_k \\
&\leq s(n-3) + td_k \\
&= s(n-3) + t(s+t) \\
&\leq s(n-3) + t(s+t) + (n-2k-3)(k-1-s) \\
&\qquad + s(k-s) + (s+t-2) \\
&= (s+t)(s+t+1) + (s+t-k+s)(k+1-s) \\
&\qquad + (n-k-1-t)(k-1) \\
&= (s+t)(s+t+1) + (d_k-k+s)(k+1-s) \\
&\qquad + (n-k-1-t)(d_n) \\
&\leq R(s,t).
\end{aligned}$$

Thus, in any case then the result holds, which proves the theorem. □

3.5 Complete Subgraphs of Order 4

The proof of the Erdős, Jacobson, Lehel conjecture [Conjecture 3.1.3] in the case $k = 3$ is given by induction on n.

Theorem 3.5.1 (Li, Song [19])**.** $\sigma(K_4, 8) = 28$.

Proof. Let $\pi = (d_1, ..., d_8)$ be a graphic sequence such that $d_1 \geq ... \geq d_8 \geq 1$ and $\sigma(\pi) \geq 28$. It suffices to show that π is potentially K_4-graphic. Note that

$$8d_1 \geq \sigma(\pi) \geq 28$$

and so $d_1 \geq 4$. Claim now that $d_4 \geq 3$. Indeed, if $d_4 \leq 2$, then $\overline{d_3} \leq 3$ and since $d_1 \geq ... \geq d_8 \geq 1$, it follows that $\overline{d_1} = 7 \geq \overline{d_2}$ and so

$$\overline{d_1} + \overline{d_2} + \overline{d_3} \leq 17.$$

However if $d_4 \leq 2$, then

$$28 \leq \sigma(\pi) \leq d_1 + d_2 + d_3 + 5 \times 2$$

and so

$$d_1 + d_2 + d_3 \geq 18 > 17 \geq \overline{d_1} + \overline{d_2} + \overline{d_3},$$

which contradicts Theorem 2.3.2 and π is not graphic. Hence, $d_4 \geq 3$.

Claim now that $d_6 \geq 2$. Indeed if $d_6 = 1$, then for $2 \leq i \leq 5$, $\overline{d_i} \leq 4$ and so

$$d_1 + d_2 + d_3 + d_4 + d_5 = \sigma(\pi) - 3 \geq 25$$

$$> 23 \geq \overline{d_1} + \overline{d_2} + \overline{d_3} + \overline{d_4} + \overline{d_5},$$

which contradicts Theorem 2.3.2 and so π is not graphic. Hence, $d_6 \geq 2$.

Let $rs(\pi, d_1) = (d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_8)$ be the residual sequence after laying off d_1 from π. If $rs(\pi, d_1)$ contains a zero term, then $6 \leq d_1 \leq 7$ and there are at most 2 zero terms.

Case 1: $rs(\pi, d_1)$ has exactly 1 zero term. It follows that

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 28 - 14 = 14 = \sigma(K_3, 7) > \sigma(K_3, 6).$$

Thus, $rs(\pi, d_1)$ is potentially K_3-graphic. Adding a vertex adjacent to all vertices is a realization of $rs(\pi, d_1)$ yields a realization of π which contains a K_4 and so π is potentially K_4-graphic.

Case 2: $rs(\pi, d_1)$ has exactly 2 zero terms. This is only possible if $d_1 = 7$, $d_7 = d_8 = 1$. By Theorem 2.3.2, $d_2 \leq \overline{d_2} \leq 5$ and since $\sigma(\pi) \geq 28$,

$$5d_2 \geq d_2 + d_3 + d_4 + d_5 + d_6 = \sigma(\pi) - d_1 - d_7 - d_8 \geq 19$$

and so $d_2 \geq 4$.

If $d_2 = 4$, then $\pi = (7, 4, 4, 4, 4, 3, 1, 1)$, which is potentially K_4-graphical as shown in Figure 3.6.

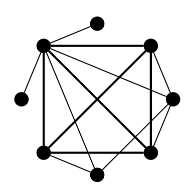


Figure 3.6: A realization of $(7, 4, 4, 4, 4, 3, 1, 1)$ containing K_4.

If $d_2 = 5$, then let $rs(\pi, d_2) = (d_1 - 1, d_3 - 1, ..., d_6 - 1, d_7, d_8)$ be the residual sequence after laying off d_2 and note that

$$\sigma(rs(\pi, d_2)) = \sigma(\pi) - 2d_2 \geq 15 > \sigma(K_3, 7)$$

and so $rs(\pi, d_2)$ is potentially K_3-graphic. Similarly to the case 1, where d_1 was laid off, it follows that π is potentially K_4-graphic.

Case 3: $rs(\pi, d_1)$ has no zero terms. In this case,

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 14 = \sigma(K_3, 7)$$

and so $rs(\pi, d_1)$ is potentially K_3-graphic and hence potentially A_2-graphic. If there exists $t \in \{4, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_3 in some realization of π guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_2-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_3-graphic and hence K_4-graphic. Thus, it suffices to assume that

$$d_1 \geq d_2 \geq d_3 \geq d_4 = ... = d_{d_1+2} \geq ... \geq d_8.$$

Let $\ell = d_{d_1+2}$ and let $rs(\pi, \ell)$ be the residual sequence after laying off ℓ from π. Note that

$$rs(\pi, \ell) = (d_1 - 1, ..., d_\ell - 1, d_{\ell+1}, ..., d_{d_1+1}, d_{d_1+3}, ..., d_8)$$

and so

$$\sigma(rs(\pi, \ell)) = \sigma(\pi) - 2\ell \geq \sigma(\pi) - 2d_1 \geq 14 = \sigma(K_3, 7).$$

Hence, $rs(\pi, \ell)$ is potentially K_3-graphic and so potentially A_2-graphic. If $d_3 > d_4$, then it follows that the K_3 in some realization of π guaranteed by the fact that $rs(\pi, \ell)$ is potentially A_2-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially A_3-graphic and hence K_4-graphic. Thus, it suffices to assume that

$$d_1 \geq d_2 \geq d_3 = d_4 = ... = d_{d_1+2} \geq ... \geq d_8.$$

If $d_8 \geq 3$, then by Theorem 3.4.3, π is potentially K_4-graphic and so it suffices to assume that $d_8 \leq 2$. If $6 \leq d_1 \leq 7$, then $d_4 = ... = d_8 \geq 3$ which contradicts the assumption and so it suffices to assume $4 \leq d_1 \leq 5$.

Subcase 3a: $d_8 = 2$. Note that

$$5d_3 + 12 \geq 2d_1 + 5d_3 + d_8 \geq \sigma(\pi) \geq 28$$

and so $d_3 \geq 4$. If $d_3 = 5$, then $5 \geq d_1 \geq d_3 = 5$ and so $d_1 = ... = d_7 = 5$. However, $(5, 5, 5, 5, 5, 5, 5, 2)$ is not graphic and so in this case $d_3 = 4$. Furthermore, if $d_1 = 5$, then

$$\pi = (5, 5, 4, 4, 4, 4, 4, 2),$$

which is potentially K_4-graphic as demonstrated in Figure 3.7.

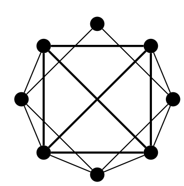


Figure 3.7: A realization of $(5, 5, 4, 4, 4, 4, 4, 2)$ containing K_4.

Moreover, if $d_1 = 4$, then either

$$\pi = (4, 4, 4, 4, 4, 4, 4, 2),$$

which is potentially K_4-graphic as demonstrated in Figure 3.8

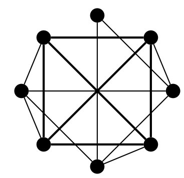


Figure 3.8: A realization of $(4, 4, 4, 4, 4, 4, 4, 2)$ containing K_4.

or

$$\pi = (4, 4, 4, 4, 4, 4, 4, 2, 2),$$

which is potentially K_4-graphic as demonstrated in Figure 3.9.

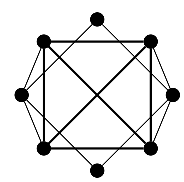


Figure 3.9: A realization of $(4, 4, 4, 4, 4, 4, 2, 2)$ containing K_4.

Subcase 3b: $d_8 = 1$. Note that

$$5d_3 + 11 \geq 2d_1 + 5d_3 + d_8 \geq \sigma(\pi) \geq 28$$

and so $d_3 \geq 4$. If $d_3 = 5$, then $5 \geq d_1 \geq d_3 = 5$ and so $d_1 = ... = d_7 = 5$ and

$$\pi = (5, 5, 5, 5, 5, 5, 5, 1)$$

which is potentially K_4-graphic as demonstrated in Figure 3.10.

If $d_3 = 4$ and $d_1 = 5$, then

$$\pi = (5, 4, 4, 4, 4, 4, 4, 1)$$

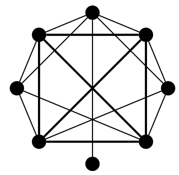


Figure 3.10: A realization of $(5,5,5,5,5,5,5,1)$ containing K_4.

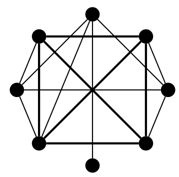


Figure 3.11: A realization of $(5,4,4,4,4,4,4,1)$ containing K_4.

which is potentially K_4-graphic as demonstrated in Figure 3.11.

If $d_3 = 4$ and $d_1 = 4$, then

$$\pi = (4,4,4,4,4,4,3,1)$$

which is potentially K_4-graphic as demonstrated in Figure 3.12.

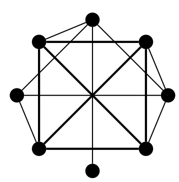


Figure 3.12: A realization of $(4,4,4,4,4,4,3,1)$ containing K_4.

Hence, in any case, π is potentially potentially K_4-graphic and

$$\sigma(K_4, 8) = 28.$$

□

Theorem 3.5.1 is the base case to the proof of Theorem 3.5.2, which proves Conjecture 3.1.3 in the case where $k = 3$ and $n \geq 8$.

Theorem 3.5.2 (Li, Song [19])**.** *For* $n \geq 8$, $\sigma(K_4, n) = 4n - 4$.

Proof. Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $d_1 \geq ... \geq d_n \geq 1$ and $\sigma(\pi) \geq 4n - 4$. It suffices to show that π is potentially K_4-graphic. The proof of this is by induction on $n \geq 8$. Theorem 3.5.1 shows that the statement is true for $n = 8$, and so it can be assumed that $n > 8$. Moreover, assume that if $\tilde{\pi}$ is a graphic sequence of length $n - 1$ such that $\sigma(\pi) \geq 4(n - 1) - 4$, then $\tilde{\pi}$ is potentially K_4-graphic.

Firstly, note that if $d_n \leq 2$, then by Theorem 2.1.4, $rs(\pi, d_n)$ is graphic. Moreover,

$$\sigma(rs(\pi, d_n)) = \sigma(\pi) - 2d_n \geq 4n - 4 - 4 = 4(n - 1) - 4$$

and so $rs(\pi, d_n)$ is potentially K_4-graphic and hence so is π. Thus, it suffices to assume that $d_n \geq 3$.

If $d_1 = n - 1$, then $rs(\pi, d_1) = (d_2 - 1, ..., d_n - 1)$ is graphic,

$$d_2 - 1 \geq ... \geq d_n - 1 \geq 2$$

and $\sigma(rs(\pi, d_1)) \geq 2(n - 1)$. Thus, by Theorem 3.2.1, $rs(\pi, d_1)$ is potentially K_3-graphic and so π is potentially K_4-graphic and it suffices to assume that $d_1 \leq n - 2$. Note that

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 4n - 4 - 2(n - 2) = 2n > \sigma(K_3, n - 1)$$

and so $rs(\pi, d_1)$ is potentially K_3-graphic and so potentially A_2-graphic. If there exists $t \in \{4, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_3 in some realization of π guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_2-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_3-graphic and hence potentially K_4-graphic. So assume

$$n - 2 \geq d_1 \geq d_2 \geq d_3 \geq d_4 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 3.$$

Let $\ell = d_{d_1+2}$ and note that

$$\sigma(rs(\pi, \ell)) = \sigma(\pi) - 2\ell \geq 4n - 4 - 2(n - 2) = 2n > \sigma(K_3, n - 1)$$

and so $rs(\pi, \ell)$ is potentially K_3-graphic and so potentially A_2-graphic. If $d_3 > d_4$, then the K_3 in some realization of $rs(\pi, \ell)$ guaranteed by the fact that $rs(\pi, \ell)$ is potentially A_2-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially A_3-graphic and hence potentially K_4-graphic. So, it suffices to assume

$$n - 2 \geq d_1 \geq d_2 \geq d_3 = d_4 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 3.$$

Thus, π is potentially K_4-graphic by Theorem 3.4.3. $\square$

3.6 Complete Subgraphs of Order 5

The proof of Conjecture 3.1.3 for the case of K_5 is very similar to the case for K_4. As with Section 3.5, the proof is given by induction on n.

Theorem 3.6.1 (Li, Song [20]). $\sigma(K_5, 10) = 50$

Proof. Let $\pi = (d_1, ..., d_{10})$ be a graphic sequence such that $d_1 \geq ... \geq d_{10} \geq 1$ and $\sigma(\pi) \geq 50$. It suffices to show that π is potentially K_5-graphic. Firstly, note that

$$50 \leq \sigma(\pi) \leq 10d_1$$

and so $d_1 \geq 5$. Claim that $d_5 \geq 4$. Indeed, if $d_5 \leq 3$, then

$$50 \leq \sigma(\pi) \leq d_1 + d_2 + d_3 + d_4 + 6d_5 \leq d_1 + d_2 + d_3 + d_4 + 18$$

and so

$$d_1 + d_2 + d_3 + d_4 \geq 32.$$

Observe that $\overline{d_1} \leq 9$, $\overline{d_2} \leq 9$, and $\overline{d_3} \leq 9$ and if $d_5 \leq 3$, then $\overline{d_4} \leq 3$. That is,

$$d_1 + d_2 + d_3 + d_4 \geq 32 > 30 \geq \overline{d_1} + \overline{d_2} + \overline{d_3} + \overline{d_4},$$

which contradicts Theorem 2.3.2 and so π is not graphic. Thus, $d_5 \geq 4$.

Moreover, claim that $d_8 \geq 2$. Observe that $\overline{d_1} \leq 9$ and if $d_8 = 1$, then for $2 \leq i \leq 7$, $\overline{d_i} \leq 6$. That is,

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 \geq \sigma(\pi) - 3d_8 \geq 45$$

$$= 9 + 6 \times 6 \geq \overline{d_1} + \overline{d_2} + \overline{d_3} + \overline{d_4} + \overline{d_5} + \overline{d_6} + \overline{d_7},$$

which contradicts Theorem 2.3.2 and so π is not graphic. Thus, it is sufficient to assume $d_8 \geq 2$.

Note that since $d_8 \geq 2$, any realization of $rs(\pi, d_1)$ has at most 2 of its 9 vertices isolated. By Theorem 3.5.2 and Theorem 3.3.2, it follows that

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 50 - 18 = 32 = \sigma(K_4, 9)$$

$$> 30 = \sigma(K_4, 7) > 28 = \sigma(K_4, 8)$$

and so $rs(\pi, d_1)$ is potentially K_4-graphic and so potentially A_3-graphic. If $rs(\pi, d_1)$ has zero terms, then since $d_8 \geq 2$, it follows that $d_1 \geq 8$ and that $d_{d_1+2} = 1$. Moreover,

$$d_2 - 1 \geq ... \geq d_{d_1+1} - 1$$

and since $d_5 \geq 4$,

$$d_2 - 1 \geq ... \geq d_5 - 1 \geq 3 > 1 = d_{d_1+2} \geq ... \geq d_{10}.$$

Thus, since $rs(\pi, d_1)$ is potentially A_3-graphic, there is a realization of $rs(\pi, d_1)$ in which the vertices of degree $d_2 - 1$, $d_3 - 1$, $d_4 - 1$, and $d_5 - 1$ are all adjacent and so there is a realization of π in which the vertices of degree d_1, d_2, d_3, d_4, and d_5 are all adjacent and π is potentially K_5-graphic.

Hence, it suffices to assume that $rs(\pi, d_1)$ has no zero terms. If there exists $t \in \{5, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_4 in some realization of $rs(\pi, d_1)$ guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_3-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_4-graphic and hence K_5-graphic. Thus, it suffices to assume that

$$d_1 \geq ... \geq d_4 \geq d_4 = ... = d_{d_1+2} \geq ... \geq d_{10}.$$

Let $\ell = d_{d_1+1}$ and note that

$$\sigma(rs(\pi, \ell)) = \sigma(\pi) - 2\ell \geq \sigma(\pi) - 2d_1 \geq 32 = \sigma(K_4, 9).$$

Hence, $rs(\pi, \ell)$ is potentially K_4-graphic and so potentially A_3-graphic. If $d_4 > d_5$, then it follows that the K_4 in some realization of $rs(\pi, \ell)$ guaranteed by the fact that $rs(\pi, \ell)$ is potentially A_3-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially K_5-graphic. Thus, it suffices to assume that

$$d_1 \geq ... \geq d_4 = d_5 = ... = d_{d_1+2} \geq ... \geq d_{10}.$$

If further $d_1 = 9$, then $rs(\pi, d_1) = (d_2 - 1, ..., d_{10} - 1)$ and since $rs(\pi, d_1)$ is K_4-graphic, it follows that there is a realization of $rs(\pi, d_1)$ which contains a K_4 and so adding a vertex adjacent to all other vertices yields a realization of π which contains a K_5 and π is potentially K_5-graphic. Thus, it suffices to assume that $d_1 \leq 8$. If further $d_{10} \geq 4$, then by Theorem 3.4.3, π is potentially K_5-graphic and so it suffices to assume that $d_{10} \leq 3$. Note that $d_{d_1+2} = d_5 \geq 4 > d_{10}$ and so $d_1 \leq 7$. Moreover,

$$47 \leq \sigma(\pi) - 3 \leq 9d_1 + d_{10} - 3 \leq 9d_1$$

and so $d_1 \geq 6$. Thus, $6 \leq d_1 \leq 7$ and $5 \leq d_4 \leq d_1 \leq 7$.

Case 1: $d_{10} = 1$. If $d_1 = 7$ and $d_4 = 7$, then $d_4 = ... = d_9 = 7$ and so

$$\pi = (7, 7, 7, 7, 7, 7, 7, 7, 7, 1)$$

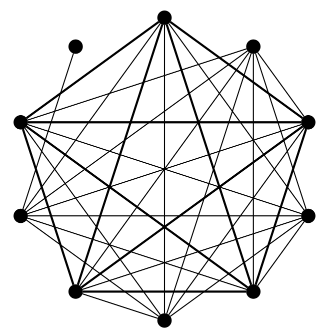


Figure 3.13: A realization of $(7,7,7,7,7,7,7,7,7,7,1)$ containing a K_5

which is potentially K_5-graphic as demonstrated in Figure 3.13.

If $d_1 = 7$ and $d_4 = 6$, then $d_4 = ... = d_9 = 6$ and since

$$6 = d_4 \leq d_3 \leq d_2 \leq d_1 = 7,$$

it follows that

$$12 \leq d_2 + d_3 \leq 14.$$

Hence, since $\sigma(\pi)$ is even, either

$$\pi = (7,7,7,6,6,6,6,6,6,6,1)$$

or

$$\pi = (7,6,6,6,6,6,6,6,6,6,1)$$

each of which is potentially K_5-graphic as demonstrated in Figure 3.14.

If $d_1 = 7$ and $d_4 = 5$, then $d_4 = ... = d_9 = 5$ and since $d_3 \leq d_2 \leq d_1 = 7$ and

$$50 \leq \sigma(\pi) \leq 7 + d_2 + d_3 + 6 \times 5 + 1$$

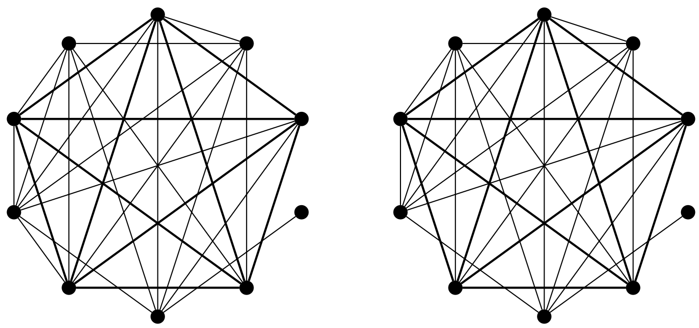


Figure 3.14: Realizations of $(7, 7, 7, 6, 6, 6, 6, 6, 6, 6, 1)$ and $(7, 6, 6, 6, 6, 6, 6, 6, 6, 6, 1)$, each containing K_5.

it follows that

$$12 \leq d_2 + d_3 \leq 14.$$

Hence, since $\sigma(\pi)$ is even, either

$$\pi = (7, 6, 6, 5, 5, 5, 5, 5, 5, 5, 1)$$

or

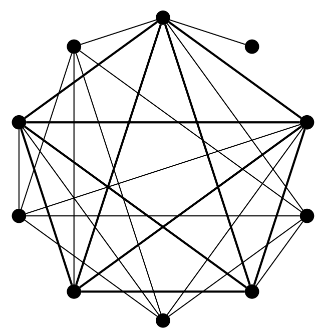
$$\pi = (7, 7, 7, 5, 5, 5, 5, 5, 5, 5, 1),$$

each of which is potentially K_5-graphic as demonstrated in Figure 3.15.

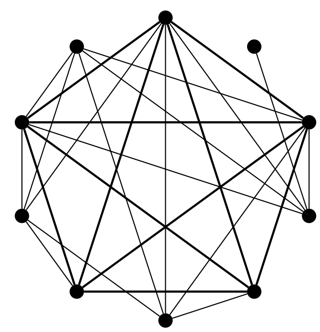
If $d_1 = 6$, then $d_4 = ... = d_8$ and since

$$50 \leq \sigma(\pi) \leq 3d_1 + 6d_4 + d_{10} = 19 + 6d_4,$$

it follows that $6 \leq d_4 \leq d_1 = 6$ and so $d_4 = 6$. Since $\sigma(\pi)$ is even, it follows



$(7, 6, 6, 5, 5, 5, 5, 5, 5, 1)$



$(7, 7, 7, 5, 5, 5, 5, 5, 5, 1)$

Figure 3.15: Realizations of $(7, 6, 6, 5, 5, 5, 5, 5, 5, 1)$ and $(7, 7, 7, 5, 5, 5, 5, 5, 5, 1)$ each containing K_5.

that either

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 1, 1),$$

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 3, 1),$$

or

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 5, 1),$$

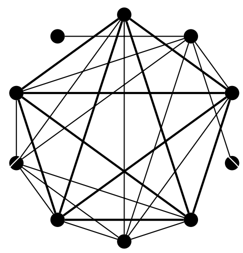
each of which is potentially K_5-graphic as demonstrated in Figure 3.16.

Case 2: $d_{10} = 2$. If $d_1 = 7$ and $d_4 = 7$, then $d_4 = ... = d_9$ and

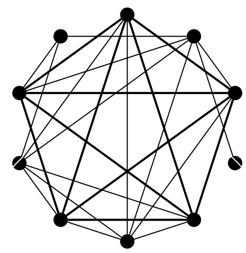
$$\pi = (7, 7, 7, 7, 7, 7, 7, 7, 7, 2),$$

which is not graphical. If $d_1 = 7$ and $d_4 = 6$, then $d_4 = ... = d_9$ and since

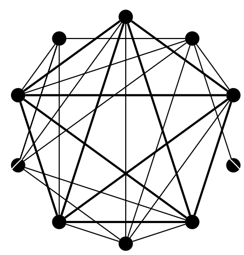
$$6 = d_4 \leq d_3 \leq d_2 \leq d_1 = 7$$



$(6, 6, 6, 6, 6, 6, 6, 6, 1, 1)$



$(6, 6, 6, 6, 6, 6, 6, 6, 3, 1)$



$(6, 6, 6, 6, 6, 6, 6, 6, 5, 1)$

Figure 3.16: Realizations of $(6, 6, 6, 6, 6, 6, 6, 6, 1, 1)$, $(6, 6, 6, 6, 6, 6, 6, 6, 3, 1)$, and $(6, 6, 6, 6, 6, 6, 6, 6, 5, 1)$, each containing K_5.

it follows that

$$12 \leq d_2 + d_3 \leq 14.$$

Hence, since $\sigma(\pi)$ is even,

$$\pi = (7, 7, 6, 6, 6, 6, 6, 6, 6, 2),$$

which is potentially K_5-graphic as demonstrated in Figure 3.17.

If $d_1 = 7$ and $d_4 = 5$, then $d_4 = ... = d_9 = 5$ and since $d_3 \leq d_2 \leq d_1 = 7$

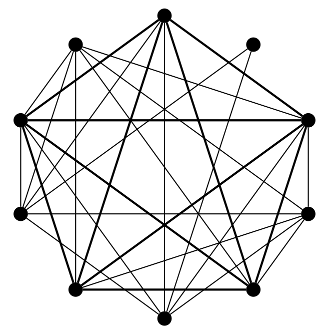


Figure 3.17: A realization of $(7, 7, 6, 6, 6, 6, 6, 6, 6, 2)$ containing a K_5.

and

$$50 \leq \sigma(\pi) \leq 7 + d_2 + d_3 + 6 \times 5 + 2$$

it follows that

$$11 \leq d_2 + d_3 \leq 14.$$

Hence, since $\sigma(\pi)$ is even, either

$$\pi = (7, 7, 6, 5, 5, 5, 5, 5, 5, 2)$$

or

$$\pi = (7, 6, 5, 5, 5, 5, 5, 5, 5, 2),$$

each which is potentially K_5-graphic as demonstrated in Figure 3.18.

If $d_1 = 6$ and $d_4 = 6$, then $d_1 = ... = d_8$ and since $\sigma(\pi)$ is even, either

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 6, 2),$$

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 4, 2),$$

or

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 2, 2),$$

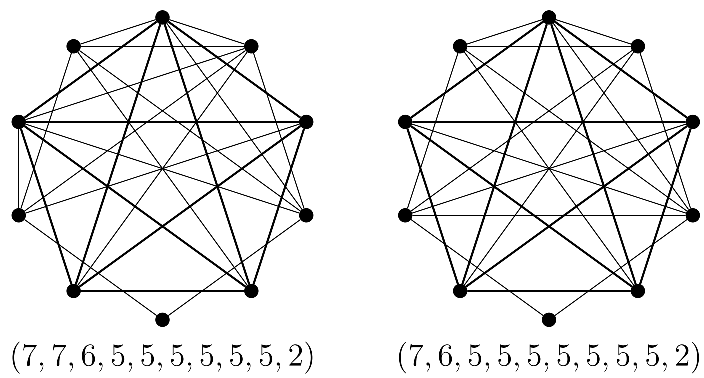


Figure 3.18: Realizations of $(7, 7, 6, 5, 5, 5, 5, 5, 5, 2)$ and $(7, 6, 5, 5, 5, 5, 5, 5, 5, 2)$ each containing K_5.

each of which is potentially K_5-graphic as demonstrated in Figure 3.19.

If $d_1 = 6$ and $d_4 = 5$, then $d_4 = ... = d_8 = 5$ and

$$50 \leq \sigma(\pi) \leq 3d_1 + 5d_4 + d_9 + d_{10} \leq 18 + 25 + d_9 + 2.$$

That is, $5 \leq d_9 \leq d_4 = 5$ and so $d_9 = 5$. Moreover

$$50 \leq \sigma(\pi) \leq 6 + d_2 + d_3 + 6(5) + 2$$

and so

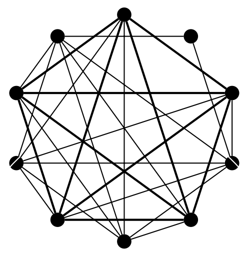
$$d_2 + d_3 \geq 12.$$

However,

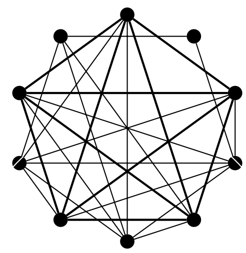
$$6 \geq d_1 \geq d_2 \geq d_3$$

and hence,

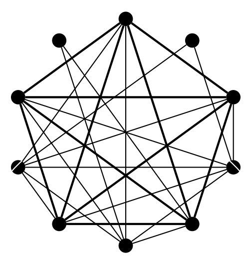
$$\pi = (6, 6, 6, 5, 5, 5, 5, 5, 5, 2)$$



$(6, 6, 6, 6, 6, 6, 6, 6, 6, 2)$



$(6, 6, 6, 6, 6, 6, 6, 6, 4, 2)$



$(6, 6, 6, 6, 6, 6, 6, 6, 2, 2)$

Figure 3.19: Realizations of $(6, 6, 6, 6, 6, 6, 6, 6, 6, 2)$, $(6, 6, 6, 6, 6, 6, 6, 6, 4, 2)$, and $(6, 6, 6, 6, 6, 6, 6, 6, 2, 2)$, each containing K_5.

which is potentially K_5-graphic as demonstrated in Figure 3.20

Case 3: $d_{10} = 3$. If $d_1 = 7$ and $d_4 = 7$, then $d_1 = ... = d_9$ and

$$\pi = (7, 7, 7, 7, 7, 7, 7, 7, 7, 3),$$

which is potentially K_5-graphic as demonstrated in Figure 3.21.

If $d_1 = 7$ and $d_4 = 6$, then $d_4 = ... = d_9 = 6$ and since

$$6 = d_4 \leq d_3 \leq d_2 \leq d_1 = 7,$$

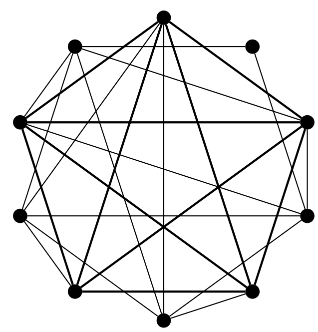


Figure 3.20: A realization of $(6,6,6,5,5,5,5,5,5,5,2)$ containing a K_5.

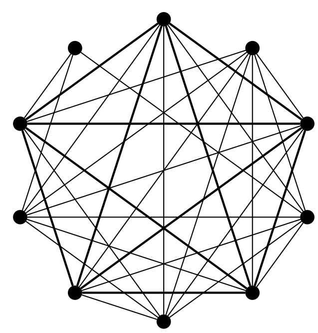


Figure 3.21: A realization of $(7,7,7,7,7,7,7,7,7,7,3)$ containing a K_5.

it follows that

$$12 \leq d_2 + d_3 \leq 14.$$

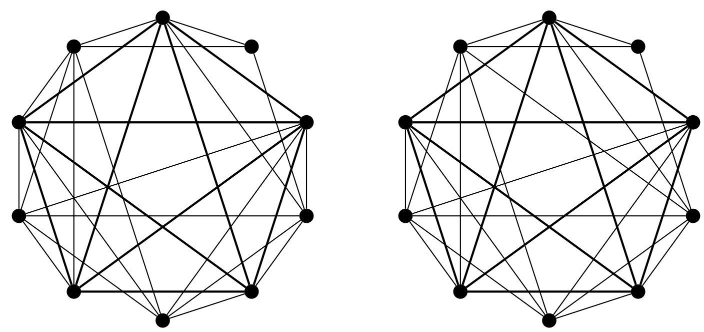
Hence, since $\sigma(\pi)$ is even, either

$$\pi = (7,7,7,6,6,6,6,6,6,6,3)$$

or

$$\pi = (7,6,6,6,6,6,6,6,6,6,3),$$

each of which is potentially K_5-graphic as demonstrated in Figure 3.22.



$(7,7,7,6,6,6,6,6,6,3)$ $(7,6,6,6,6,6,6,6,6,3)$

Figure 3.22: Realizations of $(7,7,7,6,6,6,6,6,6,3)$ and $(7,6,6,6,6,6,6,6,6,3)$, each containing K_5.

If $d_1 = 7$ and $d_4 = 5$, then $d_4 = ... = d_9 = 5$ and since

$$5 = d_4 \leq d_3 \leq d_2 \leq d_1 = 7,$$

it follows that

$$10 \leq d_2 + d_3 \leq 14.$$

Hence, since $\sigma(\pi)$ is even, either

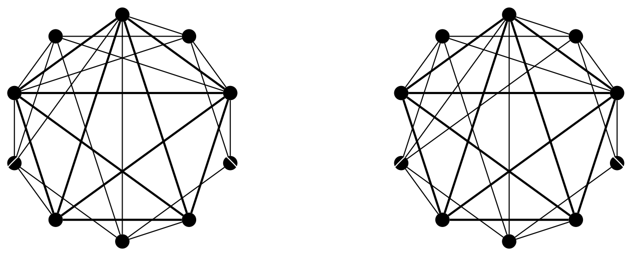
$$\pi = (7,7,7,5,5,5,5,5,5,3),$$

$$\pi = (7,7,5,5,5,5,5,5,5,3),$$

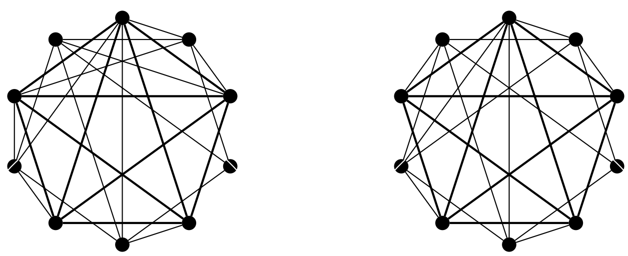
$$\pi = (7,6,6,5,5,5,5,5,5,3),$$

or

$$\pi = (7,5,5,5,5,5,5,5,5,3),$$



$(7, 7, 7, 5, 5, 5, 5, 5, 5, 5, 3)$ $(7, 7, 5, 5, 5, 5, 5, 5, 5, 5, 3)$



$(7, 6, 6, 5, 5, 5, 5, 5, 5, 5, 3)$ $(7, 5, 5, 5, 5, 5, 5, 5, 5, 5, 3)$

Figure 3.23: Realizations of $(7, 6, 6, 5, 5, 5, 5, 5, 5, 3)$ and $(7, 5, 5, 5, 5, 5, 5, 5, 5, 3)$, each containing K_5.

each of which is potentially K_5-graphic as demonstrated in Figure 3.23.

If $d_1 = 6$ and $d_4 = 6$, then $d_1 = ... = d_8 = 6$ and since

$$3 = d_{10} \leq d_9 \leq d_8 = 6$$

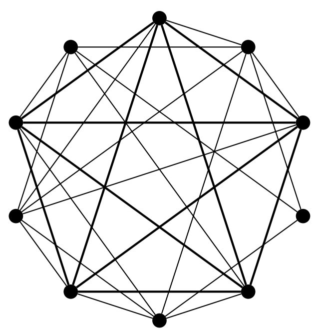
and $\sigma(\pi)$ is even, it follows that either

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 5, 3),$$

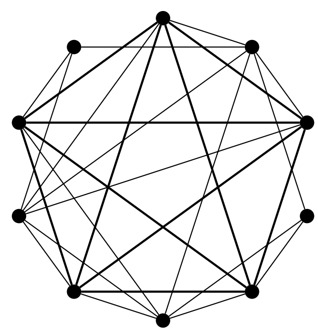
or

$$\pi = (6, 6, 6, 6, 6, 6, 6, 6, 3, 3),$$

each of which is potentially K_5-graphic as demonstrated in Figure 3.24.



$(6, 6, 6, 6, 6, 6, 6, 6, 5, 3)$



$(6, 6, 6, 6, 6, 6, 6, 6, 3, 3)$

Figure 3.24: Realizations of $(6, 6, 6, 6, 6, 6, 6, 6, 5, 3)$ and $(6, 6, 6, 6, 6, 6, 6, 6, 3, 3)$, each containing K_5.

If $d_1 = 6$ and $d_4 = 5$, then $d_4 = ... = d_8 = 5$ and

$$50 \leq \sigma(\pi) \leq 3d_1 + 5d_4 + d_9 + d_{10} \leq 18 + 25 + d_9 + 3$$

and so $4 \leq d_9 \leq d_4 = 5$. Moreover

$$5 = d_4 \leq d_3 \leq d_2 \leq d_1 = 6$$

and hence, since $\sigma(\pi)$ is even, either

$$\pi = (6, 6, 6, 5, 5, 5, 5, 5, 4, 3)$$

or

$$\pi = (6, 6, 5, 5, 5, 5, 5, 5, 5, 3),$$

each of which is potentially K_5-graphic as in Figure 3.25.

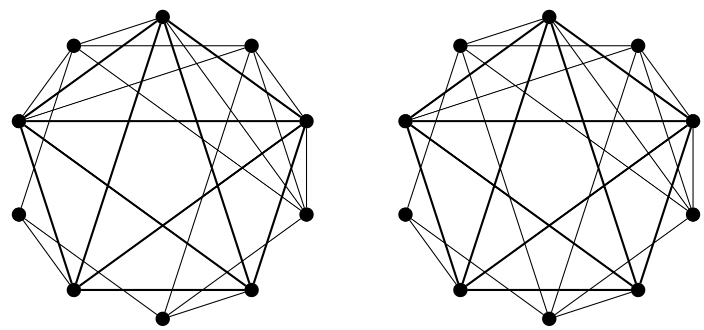


Figure 3.25: Realizations of $(6, 6, 6, 5, 5, 5, 5, 5, 4, 3)$ and $(6, 6, 5, 5, 5, 5, 5, 5, 5, 3)$, each containing K_5.

Thus, in any case π is potentially K_5-graphic and so

$$\sigma(K_5, 10) = 50.$$

□

Theorem 3.6.1 is the base case for the inductive proof of Theorem 3.6.2, which proves Conjecture 3.1.3 in the case where $k = 4$ and $n \geq 10$.

Theorem 3.6.2 (Li, Song [20]). *For $n \geq 10$, $\sigma(K_5, n) = 6n - 10$.*

Proof. Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $d_1 \geq ... \geq d_n \geq 1$ and $\sigma(\pi) \geq 6n - 10$. It suffices to show that π is potentially K_5-graphic. The proof is by induction on $n \geq 10$. Theorem 3.6.1 shows that the statement is true for $n = 10$, and so it can be assumed that $n > 10$. Moreover, assume that if $\tilde{\pi}$ is a graphic sequence of length $n - 1$ such that $\sigma(\tilde{(\pi)}) \geq 6(n-1) - 10$, then $\tilde{\pi}$ is potentially K_5-graphic.

Firstly, note that if $d_n \leq 3$, then by Theorem 2.1.4,

$$rs(\pi, d_n) = (d_1 - 1, ..., d_{d_n} - 1, d_{d_n+1}, ..., d_{n-1})$$

is graphic and has no zero terms. Moreover,

$$\sigma(rs(\pi, d_n)) = \sigma(\pi) - 2d_n \geq 6n - 10 - 6 \geq 6(n-1) - 10$$

and so $rs(\pi, d_n)$ is potentially K_5-graphic and hence so is π. Thus, it suffices to assume that $d_n \geq 4$.

If $d_1 = n - 1$, then $rs(\pi, d_1) = (d_2 - 1, ..., d_n - 1)$ is graphic, $d_2 - 1 \geq ... \geq d_n - 1 \geq 2$ and $\sigma(rs(\pi, d_1)) \geq 4n - 4$. Thus, by Theorem 3.5.2, $rs(\pi, d_1)$ is potentially K_4-graphic and so π is potentially K_5-graphic and it suffices to assume that $d_1 \leq n - 2$. Note that

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 6n - 10 - 2(n-2) \geq 4(n-1) - 4 = \sigma(K_4, n-1)$$

and so $rs(\pi, d_1)$ is potentially K_4-graphic and so potentially A_3-graphic. If there exists $t \in \{5, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_4 in some realization of π guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_3-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_4-graphic and hence potentially K_5-graphic. So it suffices to assume

$$n-2 \geq d_1 \geq ... \geq d_4 \geq d_5 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 4.$$

Let $\ell = d_{d_1+2}$ and note that

$$\sigma(rs(\pi,\ell)) = \sigma(\pi) - 2\ell \geq 6n - 10 - 2(n-2) \geq 4(n-1) - 4 = \sigma(K_4, n-1)$$

and so $rs(\pi, \ell)$ is potentially K_4-graphic and so potentially A_3-graphic. If $d_4 > d_5$, then the K_4 in some realization of $rs(\pi, \ell)$ guaranteed by the fact that $rs(\pi, \ell)$ is potentially A_3-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially K_4-graphic. So it suffices to assume

$$n-2 \geq d_1 \geq ... \geq d_4 = d_5 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 4.$$

Thus, π is potentially K_5-graphic by Theorem 3.4.3. □

3.7 Complete Subgraphs of Order at Least 6

To prove Conjecture 3.1.3 for $k \geq 5$, a key technical result is used. The proof of this key lemma, Lemma 3.7.3, is omitted here but makes use of the following pair of results. The following is merely a way to check if a sequence is graphical.

Theorem 3.7.1 (Li, Song, Luo [21]). *Let $n \geq 2k+2$ and $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $d_{k+1} \geq k$, $1 \leq d_n \leq k-1$, $d_1 + d_n \geq n-2$, and*

$$n-3 \geq d_1 \geq ... \geq d_k = d_{k+1} = ... = d_{d_1+2} = ... = d_p > d_{p+1} \geq ... \geq d_n.$$

If $s \geq k+1$, and

$$\pi' = (d_2 - 1, ..., d_s - 1, d_{s+1} - 2, ..., d_p - 2, d_{p+1} - 1, ..., d_{n-1} - 1)$$

then π' is graphic.

The proof of Theorem 3.7.1 follows from Theorem 2.3.2 while the proof of Lemma 3.7.2 involves checking cases depending on the last term of a sequence.

Lemma 3.7.2 (Li, Song, Luo [21]). *If $k \geq 5$ and $\pi = (d_1, ..., d_{2k+2})$ is a nonincreasing graphic sequence such that $\sigma(\pi) \geq 4k^2 - 4k$, then $d_{k+1} \geq k$ and $d_{2k} \geq 2$.*

Proof. Suppose that $d_{k+1} \leq k-1$ and note that if $\overline{d_i}$ is the i^{th} column sum of the leftmost off-diagonal matrix of π, then $\overline{d_k} \leq k-1$ and for $1 \leq i \leq k-1$,

$\overline{d_i} \leq 2k+1$. Hence, since π is graphic, Theorem 2.3.2 gives that

$$\begin{aligned}
4k^2 - 4k &\leq \sigma(\pi) \\
&= d_1 + ...d_k + d_{k+1} + ... + d_{2k+2} \\
&\leq \overline{d_1} + ... + \overline{d_k} + d_{k+1} + ... + d_{2k+2} \\
&\leq (k-1)(2k+1) + (k-1) + (k+2)(k-1) \\
&= (k-1)(3k+4) \\
&= 3k^2 + k - 4,
\end{aligned}$$

which is a contradiction and so $d_{k+1} \geq k$.

Now suppose that $d_{2k} = 1$ and note that $\overline{d_1} = 2k+1$ and for $2 \leq i \leq 2k+2$, $\overline{d_i} \leq 2k-2$. Hence, since π is graphic, Theorem 2.3.2 gives that

$$\begin{aligned}
4k^2 - 4k &\leq \sigma(\pi) \\
&= d_1 + ... + d_{2k-1} + d_{2k} + d_{2k+1} + d_{2k+2} \\
&\leq \overline{d_1} + ... + \overline{d_{2k-1}} + d_{2k} + d_{2k+1} + d_{2k+2} \\
&= \overline{d_1} + ... + \overline{d_{2k-1}} + 3 \\
&\leq 2k + 1 + (2k-2)(2k-2) + 3 \\
&= 4k^2 - 6k + 8,
\end{aligned}$$

which is a contradict and so $d_{2k} \geq 2$. $\square$

The key lemma used here to prove the remainder of Conjecture 3.1.3 is provided without proof.

Lemma 3.7.3 (Li, Song, Luo [21]). *Let $\pi = (d_1, ..., d_{2k+2})$ be a nonincreasing graphic sequence such that $d_{2k+2} \geq 1$ and $\sigma(\pi) \geq 4k^2 - 4k$. If $k = 5$, then π is potentially K_{k+1}-graphic. If $k \geq 6$, $\sigma(K_k, 2k) \leq 4k^2 - 12k + 8$, and $\sigma(K_k, 2k+1) \leq 4k^2 - 10k + 2$, then π is potentially K_{k+1}-graphic.*

As stated prior, the proof of Lemma 3.7.3 is omitted here but the proof as stated in [21] is structured somewhat similarly to the proofs of Theorems 3.5.2 and 3.6.2. Theorems 3.2.1 and 3.6.2 are used along with Lemma 3.7.2 to show that the sequence $\pi = (d_1, ..., d_{2k+2})$ is potentially K_{k+1}-graphical unless

$$2k \geq d_1 \geq ... \geq d_k = ...d_{d_1+2} \geq ... \geq d_{2k+2}.$$

Theorem 3.7.1 is then used to show that in the case where $d_1 + d_{2k+2} \geq 2k$, a given residual sequence can be shown to be potentially K_k-graphical and Theorem 3.6.2 is used to show that this means π is potentially K_{k+1}-graphical. In the case where $d_1 + d_{2k+2} \leq 2k - 1$, several bounds are found for the entries of π and cases are checked to prove the result.

Considering $k = 5$ in Lemma 3.7.3 yields the following corollary.

Corollary 3.7.4 (Li, Song, Luo [21]). $\sigma(K_6, 12) \leq 80$

Corollary 3.7.4 is used directly to prove Conjecture 3.1.3 for $k = 5$, $n = 13$.

Theorem 3.7.5 (Li, Song, Luo [21])**.** $\sigma(K_6, 13) \leq 86$

Proof. Let $\pi = (d_1, ..., d_{13})$ be a nonincreasing, graphic sequence with no zero terms such that $\sigma(\pi) \geq 86$. Note that if $d_{13} \leq 3$, then

$$\sigma(rs(\pi, d_{13})) = \sigma(\pi) - 2d_{13} \geq 80 \geq \sigma(K_6, 12)$$

and so $rs(\pi, d_{13})$ is potentially K_6-graphic by Corollary 3.7.4 and hence, so is π. Thus, it suffices to assume that $d_{13} \geq 4$.

Note that $rs(\pi, d_1)$ is graphical with no zeroes. Moreover,

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 86 - 2(13 - 1) = 62 = \sigma(K_5, 12)$$

and so $rs(\pi, d_1)$ is potentially K_5-graphic and so potentially A_4-graphic. If $d_1 = 12$, then

$$rs(\pi, d_1) = (d_2 - 1, ..., d_{13} - 1)$$

and so adding a vertex adjacent to all other vertices to any realization of $rs(\pi, d_1)$ which contains a K_5 yields a realization of π which contains a K_6. Thus, it suffices to assume that $d_1 \leq 11$. If there exists $t \in \{6, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_5 in some realization of $rs(\pi, d_1)$ guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_4-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_5-graphic and hence potentially K_6-graphic. So it suffices to assume

$$11 \geq d_1 \geq ... \geq d_5 \geq d_6 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 4.$$

Let $\ell = d_{d_1+2}$ and note that

$$\sigma(rs(\pi,\ell)) = \sigma(\pi) - 2\ell \geq 86 - 2(12) = 62 = \sigma(K_5, 12)$$

and so $rs(\pi,\ell)$ is potentially K_5-graphic and so potentially A_4-graphic. If $d_5 > d_6$, then the K_5 in some realization of $rs(\pi,\ell)$ guaranteed by the fact that $rs(\pi,\ell)$ is potentially A_4-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially K_6-graphic. So it suffices to assume

$$n-2 \geq d_1 \geq ... \geq d_5 = d_6 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 4.$$

Claim that $d_5 \geq 5$. Indeed if $d_5 \leq 4$, then by Theorem 2.3.2,

$$\begin{aligned} 86 &\leq \sigma(\pi) \\ &\leq \overline{d_1} + ... + \overline{d_4} + \overline{d_5} + d_6 + ... + d_13 \\ &\leq 4(13-1) + 4 + 4(8) \\ &= 84. \end{aligned}$$

This is a contradiction, and so $d_5 \geq 5$. Thus, π is potentially K_6-graphic by Theorem 3.4.4. $\square$

Perhaps computer-assisted proofs of Theorem 3.7.5 and similar results could be possible.

As with the previous proofs of cases of Conjecture 3.1.3, the case for $k = 5$ is given by induction on $n \geq 13$. Theorem 3.7.5 acts as the base step for this argument.

Theorem 3.7.6 (Li, Song, Luo [21])**.** *For $n \geq 13$, $\sigma(K_6, n) = 8n - 18$.*

Proof. Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $d_1 \geq ... \geq d_n \geq 1$ and $\sigma(\pi) \geq 8n - 18$. It suffices to show that π is potentially K_6-graphic. The proof of this is by induction on $n \geq 13$. Theorem 3.7.5 shows that the statement is true for $n = 13$, and so it can be assumed that $n > 13$. Moreover, assume that if $\tilde{\pi}$ is a graphic sequence of length $n - 1$ such that $\sigma(\pi) \geq 8(n-1) - 18$, then $\tilde{\pi}$ is potentially K_6-graphic.

Firstly, note that if $d_n \leq 4$, then by Theorem 2.1.4,

$$rs(\pi, d_n) = (d_1 - 1, d_2, ..., d_{n-1})$$

is graphic and has no zero terms. Moreover,

$$\sigma(rs(\pi, d_n)) = \sigma(\pi) - 2d_n \geq 8n - 18 - 8 \geq 8(n-1) - 18$$

and so $rs(\pi, d_n)$ is potentially K_6-graphic and hence so is π. Thus, it suffices to assume that $d_n \geq 5$.

If $d_1 = n - 1$, then $rs(\pi, d_1) = (d_2 - 1, ..., d_n - 1)$ is graphic, $d_2 - 1 \geq ... \geq d_n - 1 \geq 4$ and $\sigma(rs(\pi, d_1)) \geq 6n - 10$. Thus, by Theorem 3.6.2, $rs(\pi, d_1)$ potentially K_5-graphic and so π is potentially K_6-graphic and it suffices to assume that $d_1 \leq n-2$. Note that

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq 8n - 18 - 2(n-2) \geq 6(n-1) - 10 = \sigma(K_5, n-1)$$

and so $rs(\pi, d_1)$ is potentially K_5-graphic and so potentially A_4-graphic. If there exists $t \in \{6, ..., d_1 + 1\}$ such that $d_t > d_{t+1}$, then the K_5 in some realization of π guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_4-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_5-graphic and hence potentially K_6-graphic. So it suffices to assume

$$n-2 \geq d_1 \geq ... \geq d_5 \geq d_6 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 5.$$

Let $\ell = d_{d_1+2}$ and note that

$$\sigma(rs(\pi, \ell)) = \sigma(\pi) - 2\ell \geq 8n - 18 - 2(n-2) = 6(n-1) - 10 = \sigma(K_5, n-1)$$

and so $rs(\pi, \ell)$ is potentially K_5-graphic and so potentially A_4-graphic. If $d_5 > d_6$, then the K_5 in some realization of $rs(\pi, \ell)$ guaranteed by the fact that $rs(\pi, \ell)$ is potentially A_4-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially K_6-graphic. So it suffices to assume

$$n-2 \geq d_1 \geq ... \geq d_5 = d_6 = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq 5.$$

Thus, π is potentially K_6-graphic by Theorem 3.4.3. $\square$

Now that Conjecture 3.1.3 has been proven for $2 \leq k \leq 5$, the remaining values of k are dealt with inductively all at the same time. Moreover, a bound on the potential function is established for some values of n for which the conjecture is not proven.

Theorem 3.7.7 (Li, Song, Luo [21]). *For $k \geq 5$,*

$$\sigma(K_{k+1}, n) \leq \begin{cases} 2n(k-2)+8 & \text{if } 2k+2 \leq n \leq \binom{k}{2}+3 \\ (k-1)(2n-k)+2 & \text{if } n \geq \binom{k}{2}+3. \end{cases}$$

Proof. The proof of this result is by induction on $k \geq 5$. Note that by Corollary 3.7.4 and Theorem 3.7.5, the result holds for $k = 5$. Fix $k \geq 6$ and assume that

$$\sigma(K_k, n) \leq \begin{cases} 2n(k-3)+8 & \text{if } 2k \leq n \leq \binom{k-1}{2}+3 \\ (k-2)(2n-k+1) & \text{if } \binom{k-1}{2}+3. \end{cases} \tag{3.1}$$

It remains to prove the result holds for the given choice of k and this is proved by induction on $n \geq 2k+2$.

If $n = 2k+2$, then note that by (3.1),

$$\sigma(K_k, 2k) \leq 2(2k)(k-3)+8 = 4k^2 - 12k + 8$$

and

$$\sigma(K_k, 2k+1) \leq 2(2k+1)(k-3)+8 = 4k^2 - 10k + 2.$$

Thus, since $k \geq 6$, by Lemma 3.7.3, π is potentially K_{k+1}-graphic and so the result holds for $n = 2k+2$.

Fix $n \geq 2k+3$ and assume that

$$\sigma(K_{k+1}, n-1) \leq \begin{cases} 2(n-1)(k-2)+8 & \text{if } 2k+2 \leq n-1 \leq \binom{k}{2}+3 \\ (k-1)(2(n-1)-k)+2 & \text{if } n-1 \geq \binom{k}{2}+3. \end{cases} \tag{3.2}$$

It remains to show the result holds for the given choice of n. Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $d_1 \geq ... \geq d_n \geq 1$ and

$$\sigma(\pi) \geq \begin{cases} 2n(k-2)+8 & \text{if } 2k+2 < n \leq \binom{k}{2}+3 \\ (k-1)(2n-k)+2 & \text{if } n \geq \binom{k}{3}+8. \end{cases} \tag{3.3}$$

It suffices to show that π is potentially K_{k+1}-graphic.

Case 1: $d_n \leq k-2$. By Theorem 2.1.4,

$$rs(\pi, d_n) = (d_1 - 1, ..., d_{d_n} - 1, d_{d_n+1}, ..., d_{n-1})$$

is graphic and has no zero terms. Moreover,

$$\sigma(rs(\pi, d_n)) = \sigma(\pi) - 2d_n \geq \sigma(\pi) - 2(k-2).$$

If $2k+2 < n \leq \binom{k}{2}+3$, then

$$\sigma(rs(\pi, d_n)) \geq 2n(k-2)+8-2(k-2) \geq 2(n-1)(k-2)+8$$

and by the inductive hypothesis in (3.2), $\sigma(rs(\pi, d_n)) \geq \sigma(K_{k+1}, n)$. If $n \geq \binom{k}{2}+3$, then

$$\sigma(rs(\pi, d_n)) \geq (k-1)(2n-k)+2-2(k-2) > (k-1)(2(n-1)-k)+2$$

and by the inductive hypothesis in (3.2), $\sigma(rs(\pi, d_n)) \geq \sigma(K_{k+1}, n)$. Hence, in any case, $rs(\pi, d_n)$ is potentially K_{k+1}-graphic, and so is π.

Case 2: $d_n \geq k-1$. By Theorem 2.1.4, $rs(\pi, d_1)$ is graphic and has no zero terms and

$$\sigma(rs(\pi, d_1)) = \sigma(\pi) - 2d_1 \geq \sigma(\pi) - 2(n-1).$$

If $2k+2 < n \leq \binom{k}{2} + 3$, then

$$2k+2 \leq n-1 \leq \binom{k}{2} + 2 = \binom{k-1}{2} + k + 1$$

and by (3.3),

$$\sigma(rs(\pi, d_1)) \geq 2n(k-2) + 8 - 2(n-1) = 2n(k-3) + 10.$$

If $2k+2 \leq n-1 \leq \binom{k-1}{2} + 3$, then by the induction hypothesis in (3.2),

$$\begin{aligned}
\sigma(rs(\pi, d_1)) &\geq 2n(k-3) + 10 \\
&= 2(n-1)(k-3) + 2(k+2) \\
&> 2(n-1)(k-3) + 8 \\
&\geq \sigma(K_k, n-1).
\end{aligned}$$

If $\binom{k-1}{2} + 3 \leq n-1 \leq \binom{k-1}{2} + k + 1$, then

$$\begin{aligned}
\sigma(rs(\pi, d_1)) &\geq 2n(k-3) + 10 \\
&\geq (k-2)(2(n-1) - k + 1) + 2 \\
&\qquad + 2\left(\binom{k-1}{2} + k + 1 - (n-1)\right) \\
&\geq (k-2)(2(n-1) - k + 1) + 2 \\
&= \sigma(K_k, n-1).
\end{aligned}$$

That is, if $2k+2 < n \leq \binom{k}{2}+3$, then $\sigma(rs(\pi, d_1)) \geq \sigma(K_k, n-1)$. If $n \geq \binom{k}{2}+3$, then

$$n - 1 \geq \binom{k}{2} + 2 = \binom{k-1}{2} + k + 1 \geq \binom{k-1}{2} + 3$$

and

$$\begin{aligned}
\sigma(\pi'') &\geq \sigma(\pi) - 2d_1 \\
&\geq (k-1)(2n-k) + 2 - 2(n-1) \\
&= (k-2)(2(n-1) - k + 1) + 2 \\
&\geq \sigma(K_k, n-1).
\end{aligned}$$

Hence, in any case, $\sigma(rs(\pi, d_1)) \geq \sigma(K_k, n-1)$ and so $rs(\pi, d_1)$ is potentially K_k-graphic and so potentially A_{k-1}-graphic. If $d_1 = n-1$, then adding a vertex adjacent to all vertices in a realization of $rs(\pi, d_1)$ which contains a copy of K_k yields a realization of π which contains a copy of K_{k+1} and so π is potentially K_{k+1}-graphic. Thus, it suffices to assume that $d_1 \leq n-2$.

If there exists $t \in \{k+1, ..., d_1+1\}$ such that $d_t > d_{t+1}$, then the K_k in some realization of $rs(\pi, d_1)$ guaranteed by the fact that $rs(\pi, d_1)$ is potentially A_{k-1}-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree d_1 yield a realization of π. That is, π is potentially A_k-graphic and hence potentially K_{k+1}-graphic. So it suffices to assume that

$$n-2 \geq d_1 \geq ... \geq d_k \geq d_{k+1} = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq k-1.$$

Let $\ell = d_{d_1+1}$ and note that by Theorem 2.1.4, $rs(\pi, \ell)$ is graphic and has no zero terms and

$$\sigma(rs(\pi, \ell)) = \sigma(\pi) - 2\ell \geq \sigma(\pi) - 2(n-1).$$

If $2k+2 < n \leq \binom{k}{2} + 3$, then

$$2k+2 \leq n-1 \leq \binom{k}{2} + 2 = \binom{k-1}{2} + k + 1$$

and by (3.3),

$$\sigma(rs(\pi, \ell)) \geq 2n(k-2) + 8 - 2(n-1) = 2n(k-3) + 10.$$

If $2k+2 \leq n-1 \leq \binom{k-1}{2}+3$, then by the induction hypothesis in (3.2),

$$\begin{aligned}
\sigma(rs(\pi,\ell)) &\geq 2n(k-3)+10 \\
&= 2(n-1)(k-3)+2(k+2) \\
&> 2(n-1)(k-3)+8 \\
&\geq \sigma(K_k, n-1).
\end{aligned}$$

If $\binom{k-1}{2}+3 \leq n-1 \leq \binom{k-1}{2}+k+1$, then

$$\begin{aligned}
\sigma(rs(\pi,\ell)) &\geq 2n(k-3)+10 \\
&= (k-2)(2(n-1)-k+1)+2 \\
&\qquad + 2\left(\binom{k-1}{2}+k+1-(n-1)\right) \\
&\geq (k-2)(2(n-1)-k+1)+2 \\
&= \sigma(K_k, n-1).
\end{aligned}$$

That is, if $2k+2 < n \leq \binom{k}{2}+3$, then $\sigma(rs(\pi,\ell)) \geq \sigma(K_k, n-1)$. If $n \geq \binom{k}{2}+3$, then

$$n-1 \geq \binom{k}{2}+2 = \binom{k-1}{2}+k+1 \geq \binom{k-1}{2}+3$$

and

$$\begin{aligned}
\sigma(rs(\pi,\ell)) &\geq \sigma(\pi) - 2d_1 \\
&\geq (k-1)(2n-k) + 2 - 2(n-1) \\
&= (k-2)(2(n-1) - k + 1) + 2 \\
&\geq \sigma(K_k, n-1).
\end{aligned}$$

Hence, in any case, $\sigma(rs(\pi,\ell)) \geq \sigma(K_k, n-1)$ and so $rs(\pi,\ell)$ is potentially K_k-graphic and so potentially A_{k-1}-graphic. If $d_k > d_{k+1}$, then the K_k in some realization of $rs(\pi,\ell)$ guaranteed by the fact that $rs(\pi,\ell)$ is potentially A_{k-1}-graphic occurs amongst the vertices which when made adjacent to a new vertex of degree ℓ yield a realization of π. That is, π is potentially K_{k+1}-graphic. So it suffices to assume

$$n - 2 \geq d_1 \geq ... \geq d_k = d_{k+1} = ... = d_{d_1+2} \geq d_{d_1+3}, ..., d_n \geq k - 1.$$

Finally, claim that $d_{k+1} \geq k$. Indeed if $d_{k+1} \leq k-1$, then by Theorem 2.3.2,

$$\begin{aligned}
(k-1)(2n-k) + 2 &\leq \sigma(\pi) \\
&\leq \sum_{i=1}^{k-1} \overline{d_i} + \sum_{i=k}^{n} d_i \\
&\leq (k-1)(n-1) + (n-k+1)(k-1) \\
&= (k-1)(2n-k),
\end{aligned}$$

Which is a contradiction. That is, $d_{k+1} \geq k$, and so π is potentially K_{k+1}-graphic by Theorem 3.4.4.

Thus, for any $n \geq 2k+2$,

$$\sigma(K_{k+1}, n) \leq \begin{cases} 2n(k-2)+8 & \text{if } 2k+2 \leq n \leq \binom{k}{2}+3 \\ (k-1)(2n-k)+2 & \text{if } n \geq \binom{k}{2}+3. \end{cases}$$

Hence, for any $k \geq 5$, the assertion holds. □

The latter portion of Theorem 3.7.7 proves the Erdős, Jacobson, and Lehel conjecture, Conjecture 3.1.3, for $n \geq \binom{k}{2}+3$.

Corollary 3.7.8. *For $k \geq 5$ and $n \geq \binom{k}{2}+3$,*

$$\sigma(K_{k+1}, n) = (k-1)(2n-k)+2.$$

Chapter 4

Arbitrary Graphs in Realizations

4.1 Realizations with Given Subgraphs

In this chapter, I survey asymptotic results pertaining to the potential function. Turán gave values for $ex(H,n)$ when H is complete, but his techniques do not generally extend to non-complete graphs. However, asymptotic behaviour of $ex(H,n)$ for a given non-complete graph H has been studied. In particular, the Erdős-Stone-Simonovits Theorem [9, 10] states that

$$ex(H,n) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

As Chapter 3 determined the values of the potential function for complete graphs, this section will set out to determine the asymptotic behaviour of the potential function for graphs H in terms of their independence number $\alpha(H)$.

Ferrara, LeSaulnier, Moffatt, and Wenger collectively deciphered an analogue to the Erdős-Stone-Simonovits Theorem for the potential function in terms of the maximum degree, denoted Δ, of induced subgraphs.

Definition 4.1.1 (Ferrara, LeSaulnier, Moffatt, Wenger [11]). For a graph H and a set $S \subseteq V(H)$, let $H[S]$ denote the subgraph of H induced by the vertex set S. For a graph H and $i \in \{1, ..., k\}$, define

$$\nabla_i(H) = \min\{\Delta(H[S]) : S \subseteq V(H),\ |S| = i\}.$$

Set $q = (k-i)(n-1) + (n-k+i)(k-i+\nabla_i(H)-1)$. If q is even, let

$$\tilde{\pi}_i(H,n) = ((n-1)^{k-i}, (k-i+\nabla_i(H)-1)^{n-k+i})$$

and if q is odd, let

$$\tilde{\pi}_i(H,n) = ((n-1)^{k-i}, (k-i+\nabla_i(H)-1)^{n-k+i-1}, k-i+\nabla_i(H)-2).$$

Note that if $i \leq \alpha(H)$, then after considering a subgraph which is empty, it is clear that $\nabla_i(H) = 0$. Moreover, observe that in any realization F of $\tilde{\pi}_i(H,n)$, there is a graph F' of order $n-k+i$ such that $\Delta(F') = \nabla_i(H) - 1$ and the remaining $k-i$ vertices of F are adjacent to all other vertices of F. This leads to the following lower bound on the potential function.

Lemma 4.1.2 (Ferrara, LeSaulnier, Moffatt, Wenger [11]). *Let $0 < k \leq n$ and let H be a graph of order k. For all $i \in \{\alpha(H)+1, ..., k\}$,*

$$\sigma(H,n) > \sigma(\tilde{\pi}_i(H,n)).$$

Proof. To prove this, it suffices to show that no realization of $\tilde{\pi}(H,n)$ contains H as a subgraph. As observed, if G is a realization of $\tilde{\pi}_i(H,n)$, then there is a graph G_i of order $n-k+i$ such that $\Delta(G_i) = \nabla_i(H) - 1$ and G is a set of $k-i$ pairwise adjacent vertices which are also adjacent to every vertex of G_i. By construction, any induced subgraph of H of order i has maximum degree at least $\nabla_i(H)$. Moreover, any subgraph of G of order k contains at least i vertices from G_i. Thus, if H is a subgraph of G, then there is a subgraph of H of order i contained in G_i which has maximum degree $\nabla_i(H)$, which contradicts $\Delta(G_i) = \nabla_i(H) - 1$. Thus, no realization of $\tilde{\pi}(H,n)$ contains H as a subgraph. $\square$

This lower bound can be used to evaluate the asymptotic behaviour of the potential function. It suffices to consider the behaviour of the leading term.

Definition 4.1.3 (Ferrara, LeSaulnier, Moffatt, Wenger [11])**.** For a graph H and $n \geq 1$, let

$$\tilde{\sigma}_i(H) = 2(k-i) + \nabla_i(H) - 1$$

and let

$$\tilde{\sigma}(H) = \max_{\alpha(H)+1 \leq i \leq |V(H)|} \tilde{\sigma}_i(H).$$

Note that $\tilde{\sigma}_i(H,n)$ is the coefficient of n in the expression of $\sigma(\tilde{\pi}_i(H,n))$. Using this fact, the following can be shown.

Theorem 4.1.4 (Ferrara, LeSaulnier, Moffatt, Wenger [11]). *For $n, k \geq 1$ and a graph H of order k,*

$$\sigma(H, n) = \tilde{\sigma}(H)n + o(n).$$

In order to prove this result, several preliminary results are useful.

Theorem 4.1.5 (Li, Yin [22]). *Let $\pi = (d_1, ..., d_n)$ be a nonincreasing graphical sequence and let $k \geq 1$.*

1. *If $d_k \geq k - 1$ and for all $i \in \{1, ..., k\}$, $d_i \geq 2(k-1) - i$, then π is potentially K_k-graphic.*
2. *If $d_k \geq k - 1$ and $d_{2k} \geq k - 2$, then π is potentially K_k-graphic.*

The proof for Theorem 4.1.5 as well as that for Theorem 4.1.6 both rely on an edge swapping trick similar to that used in the proof of Theorem 2.1.1.

Theorem 4.1.6 (Yin [30]). *If π is a potentially $K_r \vee \overline{K_s}$-graphical sequence, then π has a realization in which the r vertices of highest degree are pairwise adjacent and the next s vertices of highest degree are not adjacent to each other.*

Theorem 4.1.7 makes use of Theorem 2.1.4 and so also uses the same terminology.

Lemma 4.1.7 (Ferrara, LeSaulnier, Moffatt, Wenger [11]). *Let $m, k \geq 1$ and let $\omega = \omega(n)$ be an increasing function tending to infinity. There exists $N = N(m, k, \omega)$ such that for all $n \geq N$, if π is a graphic sequence of length n such that*

$$\sigma(\pi) \geq mn + \omega$$

then one of the two following conditions hold:

1. *The sequence π is potentially K_k-graphic.*

2. *Iteratively applying the Kleitman-Wang algorithm (Theorem 2.1.4) by laying off the minimum term in each sequence yields a sequence π' of length n' such that*

$$n' \geq \frac{\omega(n)}{2(k-2)-m},$$

$$\sigma(\pi') \geq mn' + \omega(n'),$$

and the minimum term of π' is greater than $m/2$.

Proof. The result is trivial if $k \leq 2$, so assume that $k \geq 3$. Let $N \geq 0$ be such that if $n \geq N$, then

$$\frac{\omega(n)}{2(k-2)-m} \geq \binom{k}{2} + 3.$$

Let $\pi_0 = \pi$ and for $i \in \{1, ..., n\}$, define π_i to be the sequence obtained by laying off the lowest term of π_{i-1} and rearranging the remaining terms into nonincreasing order. Let $\pi' = \pi_j$ where j is the lowest index for which the lowest term is greater than $m/2$. Observe that for $i \in \{0, ..., j\}$, by construction

$$\sigma(\pi_{i+1}) \geq \sigma(\pi_i) - m$$

and so arguing inductively, it follows that for every $i \in \{0, ..., j\}$,

$$\sigma(\pi_{i+1}) \geq m(n - i - 1) + \omega(n).$$

By Theorem 3.7.7, it follows that for $n \geq \binom{k}{2} + 3$,

$$\sigma(K_k, n) = (k - 2)(2n - k + 1) + 2.$$

If $m \geq 2(k - 2)$, then since $k \geq 3$, it follows that

$$\begin{aligned}
\sigma(\pi) &\geq mn + \omega(n) \\
&\geq 2(k-2)n + \omega \\
&\geq 2(k-2)n \\
&\geq 2(k-2)n - (k-1)(k-2) + 2
\end{aligned}$$

and so π is potentially K_k-graphic. Thus, it suffices to consider $m < 2(k - 2)$.

In this case, if $j \geq n - \frac{\omega(n)}{2(k-2)-m}$, then

$$\begin{aligned}
&\sigma(\pi') - 2(k-2)(n-j) - (k-1)(k-2) + 2 \\
&\geq m(n-j) + \omega(n) - 2(k-2)(n-j) - (k-1)(k-2) + 2 \\
&\geq m(n-j) + \omega(n) - 2(k-2)(n-j) \\
&\geq (m - 2(k-2))(n-j) + \omega(n) \\
&\geq (m - 2(k-2))\left(\frac{\omega(n)}{2(k-2)-m}\right) + \omega(n) \\
&= 0.
\end{aligned}$$

That is, since $\frac{\omega(n)}{2(k-2)-m} \geq \binom{k}{3} + 2$, if $j \geq n - \frac{\omega(n)}{2(k-2)-m}$, then π_j, and hence, π is K_k-graphic. If π is not K_k-graphic, then it follows that $j < n - \frac{\omega(n)}{2(k-2)-m}$ and so if n' is the length of π', then

$$n' = n - j \geq \frac{\omega(n)}{2(k-2)-m}.$$

Moreover, since ω is increasing and $n' \leq n$,

$$\sigma(\pi') \geq m(n-j) + \omega(n) = mn' + \omega(n) \geq mn' + \omega(n')$$

and by construction, the minimum term of π' is greater than $m/2$. $\square$

The next result is worthy of study on its own, and is used repeatedly in the proof of the main result of this section. It requires the following terminology.

Definition 4.1.8. If H is a graph with degree sequence $(h_1, ..., h_k)$, then the graphic sequence $(d_1, ..., d_n)$ is *degree sufficient* for H if for $i \in \{1, ..., k\}$, $d_i \geq h_i$.

Theorem 4.1.9 (Bounded Maximum Degree Theorem)**.** *Let H be a graph of order k. There exists a function $f(\alpha(H), k)$ such that, for $n \geq f(\alpha(H), k)$, if $\pi = (d_1, ..., d_n)$ is a graphic sequence such that π is degree sufficient for H, $d_n \geq k - \alpha(H)$, and $d_1 \leq n - f(\alpha(H), k)$, then π is potentially H-graphic.*

Proof. Let $V(H) = \{u_1, ..., u_k\}$ be such that the sequence $(\deg_H(u_1), ..., \deg_H(u_n))$ is nonincreasing. Let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that π is degree sufficient for H, $d_n \geq k - \alpha(H)$, and π is not potentially H-graphic and let G be a realization of π. Let $S = \{v_1, ..., v_k\} \subseteq V(G)$ be such that for $i \in \{1, ..., k\}$, $\deg_G(v_i) = d_i$ and let H_S be the graph such that $V(H_S) = S$ and

$$E(H_S) = \{\{v_i, v_j\} : \{u_i, u_j\} \in E(H)\}.$$

That is, H_S is isomorphic to H and so if $E(H_S) \subseteq E(G)$, then G has a subgraph isomorphic to H and so π is potentially H-graphic. Thus, without loss of generality, assume that G is such that $|E(G) \cap E(H_S)|$ is maximal, but $E(H_S)$ is not contained in $E(G)$.

Let

$$g(\alpha(H), k) = \binom{k}{k - \alpha(H)} \left[2\binom{k - \alpha(H)}{2} + \alpha(H) - 1\right]$$

and claim that there are at most $g(\alpha(H), k)$ vertices in $V(G) \setminus S$ whose neighbourhoods are contained in S.

To prove the claim, assume there are $g(\alpha(H), k) + 1$ vertices in $V(G) \setminus S$ whose neighbourhoods are contained in S. Since $d_n \geq k - \alpha(H)$, it follows that every vertex in G has degree at least $k - \alpha(H)$ neighbours. For every $w \in V(G) \setminus S$ such that $N(w) \subseteq S$, let $S_w \subseteq N(w)$ be such that $|S_w| = k - \alpha(H)$. Let $\ell = 2\binom{k-\alpha(H)}{2} + \alpha(H)$ and note that since it is assumed that there are at least $g(\alpha(H), k) + 1$ vertices in $V(G) \setminus S$ whose neighbourhoods are contained in S, by the pigeonhole principle, there exists an independent set $\{w_1, ..., w_\ell\} \subseteq V(G) \setminus S$ such that $S_{w_1} = ... = S_\ell$. Let $\hat{S} \subseteq S$ be such that $S_{w_1} = ... = S_\ell = \hat{S}$.

Let $\mathcal{P} = \{w_1 w_2, w_3 w_4, ..., w_{\ell-\alpha(H)-1} w_{\ell-\alpha(H)}\}$ and note that $|\mathcal{P}| = \binom{k-\alpha(H)}{2}$. Thus, let P be a bijection from the set of unordered pairs in $\hat{S}$ to $\mathcal{P}$. If $v_r, v_s \in \hat{S}$ are not adjacent in G and $t \geq 0$ is such that $P(v_r v_s) = w_t w_{t+1}$, then let G' be the graph obtained from G by swapping the edges $v_r w_t$ and $v_s w_{t+1}$ for $v_r v_s$ and $w_t w_{t+1}$. Note that G' and G have the same degree sequence and so repeating this process for every non-adjacent pair in $\hat{S}$ yields a realization of π in which the vertices of $\hat{S}$ form a complete subgraph, each vertex of which is adjacent to every vertex in the independent set $\{w_{\ell-\alpha(H)+1}, ..., w_\ell\}$. That is, there exists a realization of π which contains a subgraph isomorphic to $K_{k-\alpha(H)} \vee \overline{K_{\alpha(H)}}$, which itself contains a subgraph isomorphic to H. This contradicts the assumption that π is not potentially H-graphic and so it follows that there are at most $g(\alpha(H), k)$ vertices in $V(G) \setminus S$ whose neighbourhoods are contained in S.

Fix $i, j \in \{1, ..., k\}$ such that $v_i v_j \in E(H_S) \setminus E(G)$. The remainder of the proof consists of using the previous claim to perform an edge swap which adds the edge $v_i v_j$, and preserves the other edges of H_S. Since π is degree sufficient for H, it follows that $\deg_G(v_i) \geq \deg_H(u_i) = d_{H_S}(v_i)$ and $\deg_G(v_j) \geq \deg_H(u_j) = d_{H_S}(v_j)$ and so there exist $a_i, a_j \in V(G)$ such that $v_i a_i, v_j a_j \in E(G) \setminus E(H_S)$. Note that it is possible that a_i and a_j need not be distinct. Let $G - S$ be the subgraph of G induced by $V(G) \setminus S$ and define the sets

$$X_i = \{v \in V(G - S) \setminus N_{G-S}(a_i) : \deg_{G-S}(v) > 0\}$$

and

$$X_j = \{v \in V(G - S) \setminus N_{G-S}(a_j) : \deg_{G-S}(v) > 0\}.$$

Let

$$f(\alpha(H), k) = g(\alpha(H), k) + 4k^2 + k - 1 \tag{4.1}$$

and suppose that $d_1 \leq n-1-f(\alpha(H), k)$. Thus, $\deg_G(a_i) \leq n-1-f(\alpha(H), k)$ and so

$$|V(G - S) \setminus N_{G-S}(a_i)| \geq (n - k) - (n - 1 - f(\alpha(H), k)) = f(\alpha(H), k) + 1 - k.$$

It follows from the previous claim that at most $g(\alpha(H), k)$ vertices in $V(G - S) \setminus N_{G-S}(a_i)$ have neighbourhoods contained in S and so

$$|X_i| \geq f(\alpha(H), k) + 1 - k - g(\alpha(H), k) \geq 4k^2.$$

Similarly, $|X_j| \geq 4k^2$. Let

$$Y_i = \bigcup_{v \in X_i} N_{G-S}(v)$$

and

$$Y_j = \bigcup_{v \in X_j} N_{G-S}(v).$$

By construction, the vertices of S have degree at least $\Delta(G-S)$. Thus, if $\Delta(G-S) \geq 2k-3$, then it follows that for $m \in \{1, ..., k\}$, $d_m \geq 2k-3 \geq 2(k-1)-m$ and so by Theorem 4.1.5, G is potentially K_k-graphic. The graph K_k contains a subgraph isomorphic to H and so this would contradict the assumption that π is not potentially H-graphic. Thus, every vertex in $G-S$ has degree at most $2k-4$. Fix $y_i \in Y_i$ and let $x_i \in N_{G-S}(y_i) \cap X_i$. The goal now is to find two more vertices to involve in the edge swap. Let

$$A = X_j \setminus N_{G-S}(y_i)$$

and

$$B = \{w \in Y_j \setminus \{y_i\} : N_{G-S} \cap X_j \neq \{x_i\}\}$$

and note that $|A| \geq 4k^2 - 2k + 4$. Moreover, note that by definition of X_j, every vertex in A is contained in an edge whose other endpoint is in Y_j and since this vertex is in A, it follows that this other endpoint need not be y_i. By construction, any such edge has one endpoint in A and the other in B. Since every vertex in B has degree at most $2k-3$, it follows that

$$\begin{aligned} 2k|B| &\geq (2k-3)|B| \\ &\geq |\{(v,w) : v \in A,\, w \in B,\, \{v,w\} \in E(G-S)\}| \\ &\geq |A| \\ &\geq 4k^2 - 2k + 4 \end{aligned}$$

and so

$$|B| \geq \frac{4k^2 - 2k + 4}{2k} = 2k - 1 + \frac{2}{k} \geq 2k.$$

Since $\deg_{G-S}(y_i) < 2k$, it follows that there exists $y_j \in B \setminus N_{G-S}(y_i)$ and since $y_j \in B$, there exists $x_j \in N_{G-S}(y_i) \setminus \{x_i\}$. Let $\tilde{G}$ be the graph obtained from G by swapping the edges $v_i a_i$, $v_j a_j$, $x_i y_i$, and $x_j y_j$ for $v_i v_j$, $a_i x_i$, $a_j x_i$, and $y_i y_j$. Note that the degree sequence of $\tilde{G}$ is π and that $|E(\tilde{G}) \cap E(H_S)| = |E(G) \cap E(H_S)| + 1$, which contradicts the choice of G such that $|E(G) \cap E(H_S)|$ is maximal. □

In the paper [11], the function f as defined in (4.1) is

$$f(\alpha(H), k) = g(\alpha(H), k) + 4k^2 + k + 1$$

rather than

$$f(\alpha(H), k) = g(\alpha(H), k) + 4k^2 + k - 1$$

as it is defined here. Only a small adjustment in estimates in the proof was needed to obtain this tighter bound.

The following definition provides notation useful in the proof of Theorem 4.1.11, which is the main result of this section.

Definition 4.1.10. Let G be a graph and let $i \in \{0, ..., |V(G)|\}$. Let $\mathcal{D}^{(i)}(G)$ denote the set of induced subgraphs of G of order $|V(G)| - i$.

Theorem 4.1.11 (Ferrara, LeSaulnier, Moffatt, Wenger [11])**.** *Let H be a graph and let $\omega = \omega(n)$ be an increasing function tending to infinity as n increases. There exists $N = N(\omega, H)$ such that if $n \geq N$,*

$$\sigma(H, n) \leq \tilde{\sigma}(H)n + \omega(n).$$

The proof for this theorem proceeds broadly in two parts: laying off entries of a sequence by repeated applications of Lemma 4.1.7 and Theorem 2.1.4 until the resulting sequence satisfies two certain conditions and then proving that these two conditions are sufficient to guarantee that the original sequence is potentially H-graphical. More specifically, laying off a vertex of high degree allows one to deal with the case of searching for a subgraph of H in a realization, which may prove easier and laying off vertices of low degree raises average degree of any realization, which makes it more likely to contain a subgraph of H in a realization.

Proof. As stated, this proof establishes an algorithm for laying off terms from a given sequence. To better understand the term N, it is defined after this algorithm. Let $n \geq N$ and let $\pi = \{d_1, ..., d_n\}$ be a graphic sequence such that

$$\sigma(\pi) \geq \tilde{\sigma}(H)n + \omega(n).$$

It remains to show that π is potentially H-graphic. If $d_{k+1} \geq 2k-3$, then by Theorem 4.1.5, π is potentially K_k-graphic and so potentially H-graphic. Thus, it suffices to assume that $d_{k+1} \leq 2k-4$. Since $\sigma(\pi) \geq \tilde{\sigma}(H)n + \omega(n)$, it follows from Theorem 4.1.7 that either π is potentially K_k-graphic and so potentially H-graphic or there exists a sequence $\pi_0 = (d_{0,1}, ..., d_{0,n_0})$ such that $d_{0,n_0} \geq \tilde{\sigma}(H)/2$,

$$n_0 \geq \frac{\omega(n)}{2(k-2) - \tilde{\sigma}(H)},$$

and

$$\sigma(\pi_0) \geq \tilde{\sigma}(H)n + \omega(n_0).$$

From here, for $i \geq 1$, iteratively construct sequences $\pi_i = (d_{i,1}, ..., d_{i,n_i})$ such that each π_i is nonincreasing, $d_{i,n_i} \geq k - i - \alpha(H)$, and if π_i has a realization in $\mathcal{D}^{(i)}$, then π is potentially H-graphic. The construction of π_{i+1} from π_i only involves laying off vertices using Kleitman Wang algorithm and so if for some $i \geq 0$, π_i is potentially H-graphic, then so is π.

Let $\tilde{G}_i$ be a realization of π_i such that a vertex $v_{i,1}$ of maximum degree is adjacent to the the next $d_{i,1}$ vertices of highest degree. Let $\tilde{G}_i'$ be the subgraph of $\tilde{G}_i$ induced by the set $N_{\tilde{G}_i}(v_{i,1}) \cup \{v_{i,1}\}$. Let π_i' be the degree sequence of $\tilde{G}_i'$ and let n_i' be the length of π_i'. If π_i' is potentially K_k-graphic, then it is also

H-graphic and so is π. Otherwise, let π_i'' be the sequence of length n_i'' which Lemma 4.1.7 guarantees. Let π_{i+1} be the residual sequence obtained from π_i'' by laying off the vertex of highest degree.

To properly define N, let $\ell \geq 0$ the be smallest index i for which either $i = k - \alpha(H)$ or $d_{i,1} < n_i - \binom{k}{\lceil k/2 \rceil}(k^2 - 1)$. Let $N(\omega, H)$ be such that if $n \geq N$, then $n_\ell \geq 3k^2 + 8k$, $n_\ell > f(\alpha(H), k - \ell)$, and for $i \in \{0, ..., \ell\}$,

$$\omega(n_i') \geq 2^{i+1} \binom{k}{\lceil k/2 \rceil} (k^2 - 1)(2k - 4),$$

and

$$n_i'' \geq \frac{\omega(n_i')/2^i}{2(k-2) - \tilde{\sigma}(H)}.$$

The next step is to verify that for $i \in \{0, ..., \ell - 1\}$, the sequences π_i satisfy the conditions specified previously. Namely, it is to be shown that for $i \geq 0$, $d_{i,n_i} \geq k - i - \alpha(H)$, and if π_i has a realization in $\mathcal{D}^{(i)}$, then π is potentially H-graphic. Both of these are proven by induction.

Claim that for $i \in \{0, ..., \ell - 1\}$,

$$\sigma(\pi_i) \geq (\tilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i$$

and $d_{i,n_i} \geq k - i - \alpha(H)$. This is proven by induction. Note that

$$d_{0,n_0} \geq \tilde{\sigma}(H)/2 \geq k - \alpha(H).$$

Moreover, since $i < \ell$,

$$d_{i,1} \geq n_i - \binom{k}{\lceil k/2 \rceil}(k^2-1)(2k-4)$$

it follows that π_i' is obtained from π_i by removing at most $\binom{k}{\lceil k/2 \rceil}(k^2-1)$ vertices, each with degree at most $2k-4$. Thus,

$$\sigma(\pi_i') \geq \sigma(\pi_i) - 2\binom{k}{\lceil k/2 \rceil}(k^2-1)(2k-4)$$

and so by the induction hypothesis,

$$\sigma(\pi_i') \geq (\tilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i - 2\binom{k}{\lceil k/2 \rceil}(k^2-1)(2k-4)$$

Without loss of generality, assume n_i is large enough that

$$\omega(n_i)/2^i \geq 4\binom{k}{\lceil k/2 \rceil}(k^2-1)(2k-4)$$

and so

$$\sigma(\pi_i') \geq (\tilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^{i+1}.$$

Lemma 4.1.7 states that since $\omega(m)/2^{i+1}$ is increasing in m, π_i'' is a sequence of length n such that

$$\sigma(\pi_i'') \geq (\tilde{\sigma}(H) - 2i)n_i'' + \omega(n_i'')/2^{i+1}$$

and that the minimum term of π_i'' is at least $\tilde{\sigma}(H)/2$, which is at least $k - i - \alpha(H)$. Since the minimum term of π_i'' is one greater than the minimum term of π_{i+1}, it follows that

$$d_{i+1,n_{i+1}} \geq k - (i+1) - \alpha(H)$$

and that

$$\begin{aligned}
\sigma(\pi_{i+1}) &= \sigma(\pi_i'') - 2(n_i'' - 1) \\
&\geq (\tilde{\sigma}(H) - 2i)n_i'' + \omega(n_i'')/2^{i+1} - 2(n_i'' - 1) \\
&= (\tilde{\sigma}(H) - 2i)n_i'' + \omega(n_i'')/2^{i+1} - 2(n_{i+1}) \\
&\geq (\tilde{\sigma}(H) - 2i)n_{i+1} + \omega(n_i'')/2^{i+1} - 2(n_{i+1}) \\
&= (\tilde{\sigma}(H) - 2(i+1))n_{i+1} + \omega(n_i'')/2^{i+1} \\
&\geq (\tilde{\sigma}(H) - 2(i+1))n_{i+1} + \omega(n_{i+1})/2^{i+1}.
\end{aligned}$$

Thus, the assertion holds.

Now, to prove that if π_i is has a realization in $\mathcal{D}^{(i)}(H)$, then π is potentially H-graphic, note that $\mathcal{D}^{(0)}(H) = \{H\}$ and so the base case holds trivially. For some fixed $i \in \{0, ..., \ell - 1\}$ assume that if π_i has a realization in $\mathcal{D}^{(i)}$, then π is potentially H-graphic. Since any realization of π_{i+1} can be obtained from a realization of π_i'' by removing a vertex which is adjacent to all others, it follows that π_{i+1} has a realization in $\mathcal{D}^{(i+1)}$ if and only if π_i'' has a realization in $\mathcal{D}^{(i)}$.

If π_i'' has a realization in $\mathcal{D}^{(i)}$, then so does π_i and so it follows that if π_i has a realization in $\mathcal{D}^{(i)}$, then π is potentially H-graphic.

It now remains to show that π_ℓ has a realization in $\mathcal{D}^{(\ell)}(H)$. If $\ell = k-\alpha(H)$, then $\overline{K_{\alpha(H)}} \in \mathcal{D}^{(\ell)}$, and π_ℓ has a realization in $\mathcal{D}^{(\ell)}(H)$. Hence, it suffices to assume that $\ell < k - \alpha(H)$ and $d_{\ell,1} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1)$.

Let $t = \max\{i \in \{0, ..., \ell\} : d_{\ell,i} \geq k-\ell-1\}$ and note that if $t \geq k-\ell-\alpha(H)$, then π_ℓ is degree sufficient for $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}$. Moreover,

$$d_{\ell,n_\ell} \geq k - \ell - \alpha(H)$$

and

$$\begin{aligned} d_{\ell,1} &< n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1) \\ &< n_\ell - f(\alpha(H), k-\ell) \\ &= n_\ell - f(\alpha(K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}), k-\ell). \end{aligned}$$

Thus, by Theorem 4.1.9, π_ℓ is potentially $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}$-graphic. Since $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}} \in \mathcal{D}^{(\ell)}(H)$, it follows that π_ℓ is potentially $\mathcal{D}^{(\ell)}(H)$-graphic and hence, π is potentially H-graphic.

Thus, it suffices to assume that $t < k - \ell - \alpha(H)$. For $i \in \{1, ..., \ell\}$, let F_i be an induced subgraph of H of order i such that $\Delta(F_i) = \nabla_i(H)$ and assume that π_ℓ is not degree sufficient for $K_t \vee F_{k-\ell-t}$. Note that

$$\tilde{\sigma}(H) - 2\ell \geq \tilde{\sigma}_{k-\ell-t}(H) - 2\ell = 2t - \nabla_{k-\ell-t}(H) - 1$$

and so

$$\sigma(\pi_\ell) \geq (2t - \nabla_{k-\ell-t} - 1)n_\ell + \omega(n_\ell)/2^\ell.$$

Moreover, since π_ℓ is not degree sufficient for $K_t \vee F_{k-\ell-t}$, it follows that

$$d_{\ell,k-\ell} < \delta(K_t \vee F_{k-\ell-t}) = t + \nabla_{k-\ell-t}(H)$$

and hence, since by definition, $d_{\ell,t+1} \leq k - \ell - 2$ and $\ell \leq k-1$

$$\sigma(\pi_\ell) < (2t + \nabla_{k-\ell-t}(H) - 1)n_\ell + k^2.$$

This contradicts the previous lower bound and so it follows that π_ℓ is degree sufficient for $K_t \vee F_{k-\ell-t}$.

Since π_ℓ is degree sufficient for $K_t \vee F_{k-\ell-t}$, it follows that it is also degree sufficient for $K_t \vee \overline{K_{k-\ell-t}}$. Moreover, since $d_{\ell,n_\ell} \geq k - \ell - \alpha(H)$ and

$$d_{\ell,1} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1) < n_\ell - f(\alpha(\overline{K_{k-\ell-t}}), k-\ell),$$

it follows from Theorem 4.1.9 that π_ℓ is potentially $K_t \vee \overline{K_{k-\ell-t}}$-graphic. Let G_ℓ be a realization of π_ℓ such that $V(G_\ell) = \{v_1, ..., v_{n_\ell}\}$, for $j \in \{1, ..., n_\ell\}$, $\deg_{G_\ell}(v_j) = d_{i,\ell}$, the vertices $v_1, ..., v_t$ are all adjacent to each of $v_1, ..., v_{k-\ell}$, and $\{v_{t+1}, ..., v_{k-\ell}\}$ is an independent set. Let $\tilde{G}_\ell$ be the subgraph of G_ℓ induced by $\{v_{t+1}, ..., v_{n_\ell}\}$ and let $\tilde{\pi}_\ell$ be the degree sequence of $\tilde{G}_\ell$. Since $d_{\ell,n_\ell} \geq k - \ell - \alpha(H) > t$, it follows that the minimum term of $\tilde{\pi}_\ell$ is at least 1. All that remains to show is that $\tilde{\pi}_\ell$ is potentially $F_{k-\ell-t}$-graphic.

Let $V(F_{k-\ell-t}) = \{w_1, ..., w_{k-\ell-t}\}$ such that for $i \leq j$, $\deg_{F_{k-\ell-t}}(w_i) \geq \deg_{F_{k-\ell-t}}(w_j)$. For $i \in \{t+1, ..., k-\ell\}$, let $b_i = \deg_{\tilde{G}_\ell}(v_i) - \deg_{F_{k-\ell-t}}(w_i)$ and let $b = \sum_{i=t+1}^{k-\ell} b_i$. For ease of notation later, also let $b_t = 0$. Since $n \geq N$, it follows that $n_\ell \geq 3k^2 + 8k$. Thus, since $\ell < k$,

$$\begin{aligned}
n_\ell &\geq 2k + k^2 + k^2 + 3k + k^2 + 3k \\
&\geq 2k + k^2 + \ell^2 + 3\ell + k(k - \ell - \alpha(H)) + 3(k - \ell - \alpha(H)) \\
&\geq 2k + k^2 + \ell^2 + 3\ell + t\ell + 3t \\
&\geq 2k + k^2 + \ell^2 - \ell k - 3k - \ell k + 3\ell - tk + t\ell + 3t \\
&= 2k + (k - \ell - t)(k - \ell - 3) \\
&\geq (k - \ell) + (k - \ell - t)(k - \ell - 3)
\end{aligned}$$

and

$$k - \ell - 2 \leq n_\ell - (k - \ell) - 2 - (k - \ell - t)(k - \ell - 3).$$

Furthermore, note that $F_{k-\ell-t}$ is a graph of order $k - \ell - t$ and maximum degree at most $k - \ell - 3$ and so

$$b \leq (k - \ell - t)(k - \ell - 3).$$

Hence, $n_\ell \geq b$. Let

$$\hat{\pi}_\ell = (\deg_{\tilde{G}_\ell}(v_{k-\ell+1}), ..., \deg_{\tilde{G}_\ell}(v_{n_\ell - b}), \deg_{\tilde{G}_\ell}(v_{n_\ell - b+1}) - 1, ..., \deg_{\tilde{G}_\ell}(v_{n_\ell}) - 1)$$

and note that every term of $\hat{\pi}_\ell$ is at most $k-\ell-2$. Moreover, since π_ℓ is degree sufficient for $K_t \vee F_{k-\ell-t}$, it follows that for $i \in \{t+1, ..., k-\ell\}$, $b_i \geq 0$ and since every term of $\tilde{\pi}_\ell$ is at least 1, it follows that $\hat{\pi}_\ell$ is a nonnegative, nonincreasing sequence. Thus, for $s \in \{1, ..., n_\ell - (k-\ell+1)\}$, the sum of the first s terms of $\hat{\pi}_\ell$ is at most $s(k-\ell-2)$ and the sum of the mimima of the remaining terms or s is at least $s[n_\ell-(k-\ell)-(k-\ell-t)(k-\ell-3)]$. That is, $\hat{\pi}_\ell$ satisfies the Erdős Gallai criterion and so is graphic by Theorem 2.2.2. Let F be a realization of $\hat{\pi}_\ell$ and consider the graph with vertex set $V(F_{k-\ell-t}) \cup V(F)$ and edge set

$$E(F_{k-\ell-t}) \cup E(F) \cup \bigcup_{i=t+1}^{k-\ell} \{\{w_i, v_{n_\ell-j}\} : b_{i-1} < j \leq b_i\}.$$

This graph is a realization of $\tilde{\pi}_\ell$ which contains a subgraph isomorphic to $F_{k-\ell-t}$. Hence, $\tilde{\pi}_\ell$ is potentially $F_{k-\ell-t}$-graphic, π_ℓ is potentially $K_t \vee F_{k-\ell-t}$-graphic, and so π is potentially H-graphic. $\square$

4.2 Avoiding Given Subgraphs in their Realizations

As Section 4.1 deals with degree sequences whose realizations are potentially H-graphic, this section describes the 'shape' of sequences whose realizations are not potentially H-graphic. In a pair of papers by Erbes, Ferrara, Martin, and Wenger [5] [6], a few notions of shape of a sequence were developed. The main results from these papers describe conditions on graphical sequences which do not contain given subgraphs in any of their realizations. The first of these papers give descriptions based on whether or not the given sequence is or is not degree sufficient for the desired graph. Firstly, some definitions and notation are useful to understand what is meant by the shape of a degree sequence.

Definition 4.2.1. If $S_1 = (x_1, ..., x_n)$ and $S_2 = (y_1, ..., y_n)$ are two sequences such that for every $i \in \{1, ..., n\}$, $x_i \geq y_i$, then we say that S_1 *majorizes* S_2 and write $S_1 \succeq S_2$.

Two sequences can be said to be close if altering a few terms in one of them switches the order of which majorizes the other. This is formalized in the following definition.

Definition 4.2.2. Let $\pi_1 = (d_1, ..., d_n)$ and π_2 be n-term sequences and let a_1, a_2, and b be nonnegative integers such that $a_1 \leq a_2$. Let

$$S_1 = (d_1, ..., d_{a_1-1}, \underbrace{d_{a_2}, ..., d_{a_2}}_{a_2-a_1+1\, times}, d_{a_2+1}, ..., d_n)$$

The sequence π_1 is said to be $([a_1, a_2], b)$-close to π_2 if there is a (not necessarily graphic) sequence π_1' obtained from S_1 by subtracting a total of at most b from the terms of S_1 such that $\pi_2 \succeq \pi_1'$. The process of obtaining the sequence S_1 is called *levelling off terms* a_1 *to* a_2 *of* π and the remaining step is called *editing* the sequence.

With $\tilde{\pi}_i(H,n)$ as defined in Definition 4.1.1, Theorem 4.2.3 is true. It describes the shape of a graphic sequence which is not degree sufficient for H and hence is not potentially H-graphic.

Theorem 4.2.3 (Erbes, Ferrara, Martin, Wenger [5]). *Let H be a graph with $\pi(H) = (h_1, ..., h_k)$, let $\pi = (d_1, ..., d_n)$ be a graphic sequence which is not degree sufficient for H, and let j be the largest integer for which $d_{k-j+1} < h_{k-j+1}$. If $j > \alpha(H)$, then $\tilde{\pi}_j(H,n) \succeq \pi$. If $j \leq \alpha(H)$, then π is $([k-\alpha(H), k-j+1], 0)$-close to $\tilde{\pi}_{\alpha(H)+1}(H,n)$.*

Proof. Note that for $i > \alpha(H)$, if $h_{k-i+1} > k - i + \nabla_i(H)$, then every subgraph of H of order i has maximum degree greater than $\nabla_i(H)$, which is a contradiction. Thus, for $i > \alpha(H)$, $h_{k-i+1} \leq k - i + \nabla_i(H)$.

Case 1: $j > \alpha(H)$. For $\ell \in \{1, ..., k-j\}$, $d_\ell \leq n-1$. Since

$$d_{k-j+1} < h_{k-j+1} \leq k - j + \nabla_j(H),$$

it follows that for $\ell \in \{k-j+1, ..., k\}$, $d_\ell \leq k - j + \nabla_j(H) - 1$.

Moreover, for $i \leq \alpha(H)$, if $h_{k-i+1} > k - \alpha(H)$, then there are at most $i-1$ vertices of degree at most $k-\alpha(H)$, which contradicts the fact that there are at least $\alpha(H) > i-1$ vertices of degree at most $k-\alpha(H)$. Thus, for $i \leq \alpha(H)$, $h_{k-i+1} \leq k - \alpha(H)$.

Case 2: $j \leq \alpha(H)$. For $\ell \in \{1, ..., k-\alpha(H)-1\}$, $d_\ell \leq n-1$. Note that

$$d_{k-j+1} < h_{k-j+1} \leq k - \alpha(H),$$

and so

$$d_{k-j+1} \le k - \alpha(H) - 1 \le k - (\alpha(H)+1) - \nabla_{\alpha(H)+1} - 1.$$

Thus, levelling off the terms $d_{k-\alpha(H)}$ through d_{k-j+1} results in a sequence which is majorized by $\tilde{\pi}_{\alpha(H)+1}(H, n)$. $\square$

This result is best possible as demonstrated by considering the following example: for $k, r \ge 2$, let $H = K_{k-r} \vee \overline{K_r}$. Note that $\alpha(H) = r$ and for $n \ge 1$,

$$\tilde{\pi}_{\alpha(H)+1}(H, n) = ((n-1)^{k-r-1}, (k-r-1)^{n-k+r+1}).$$

For $n \ge (k-1)(k-r+1)$ divisible by $(k+1)$ and $j \le r$, let $\frac{n(k-j)}{k+1} + (k-r-1)(n-k+j) = q$ and

$$\pi_j = \begin{cases} \left(\left(\frac{n}{k+1}\right)^{k-j}, (k-r-1)^{n-k+j}\right) & \text{if } q \text{ is even} \\ \left(\left(\frac{n}{k+1}\right)^{k-j}, (k-r-1)^{n-k+j-1}, (k-r-2)\right) & \text{if } q \text{ is odd.} \end{cases}$$

By Theorem 2.2.2, π_j is graphic. Moreover, π_j is not degree sufficient for H and by choice of n, it follows that only the first $k-r-1$ and the last $n-k+j$ terms of $\tilde{\pi}_{\alpha(H)+1}(H, n)$ dominate π_j. Thus, to obtain a sequence dominated by $\tilde{\pi}_{\alpha(H)+1}(H, n)$ from π_j, the terms $d_{k-\alpha(H)}, \ldots, d_{k-j}$ must be reduced to $d_{k-j+1} = k-r-1$.

As Theorem 4.2.3 describes the shape of a sequence which is not potentially H-graphic since it is not degree sufficient, Theorem 4.2.5 describes the shape of a graphic sequence which is not potentially H-graphic, despite being degree sufficient for H. To do this, the following notation is required.

Definition 4.2.4. Define

$$i^*(H) = \min\{i \in \{\alpha(H), ..., k\} : \tilde{\sigma}_i(H) = \tilde{\sigma}(H)\}.$$

Theorem 4.2.5 (Erbes, Ferrara, Martin, Wenger [5]). *Let H be a graph of order k with at least one non-trivial component and let π be a graphic sequence of length n that is degree sufficient for H. If π is not potentially H-graphic, then π is $([k - i^* + 1, k], (6\alpha(H) + 3)k^2 + (\alpha(H))^3 k)$-close to $\tilde{\pi}_{i^*}(H, n)$.*

Theorem 4.2.5 can be viewed as an Erdős-Stone-Simonovits type result for the Erdős-Jacobson-Lehel problem. Before addressing the proof of Theorem 4.2.5, it proves illuminating to consider the following example which shows that the result can be obtained with at most $O(\alpha(H)k^2)$ editing. In general, the authors point out that the sharpness of Theorem 4.2.5 is still an open question.

Let $k \geq 2$ and consider $H = K_k$ and

$$\pi = (n - 1, (2k - 5)^{2k-3}, (1)^{n-2k+2}).$$

By Theorem 2.2.2, π is graphic and note that π is degree sufficient for H. By Theorem 2.1.1, $\pi' = ((2k - 6)^{2k-3})$ is graphic and since any realization of π' is the complement of a 2-regular graph, it follows that π is not potentially

K_k-graphic. Moreover, note that $i^*(K_k) = 2$ and so

$$\tilde{\pi}_{i^*}(K_k, n) = ((n-1)^{k-2}, (k-2)^{n-k+3}).$$

Leveling off the $k - i^* + 1$ through k terms does not alter the sequence, but in order to obtain a sequence majorized by $\tilde{\pi}_{i^*}(K_k, n)$, the $k-1$ through $2k-2$ terms must be each reduced by $k-3$. That is, this can be achieved with $k(k-3)$, which is $O(\alpha k^2)$, total editing.

The proof of Theorem 4.2.5 is omitted here, but it follows from a more technical result, the proof of which is outlined here with use of the following concept.

If $0 \leq r \leq k$ and $\pi = (d_1, ..., n)$ is a graphic sequence which is degree sufficient for $K_r \vee \overline{K}_{k-r}$, then construct a new sequence π^w known as the want sequence of π for $K_r \vee \overline{K}_{k-r}$. To do this, let G be a realization of π such that $V(G) = \{v_1, ..., v_n\}$ and for $i \in \{1, ..., n\}$, $\deg_G(v_i) = d_i$ which maximizes the sum of the number of edges amongst the vertices $\{v_1, ..., v_r\}$ and the number of edges with one endpoint in $\{v_1, ..., v_r\}$ and the other in $\{v_{r+1}, ..., v_k\}$. Since π is degree sufficient for $K_r \vee \overline{K}_{k-r}$, it follows that any vertex in $\{v_1, ..., v_k\}$ has at least as many neighbours in $\{v_{k+1}, ..., v_n\}$ as it does nonneighbours in $\{v_1, ..., v_k\}$ and so for $i \in \{1, ..., k\}$,

$$|N_G(v_i) \cap \{v_{k+1}, ..., v_n\}| \geq k - 1 - \deg_{G[v_1,...,v_k]}(v_i).$$

For each $i \in \{1, ..., k\}$, let $W_i \subseteq N_G(v_i) \cap \{v_{k+1}, ..., v_n\}$ such that $|W_i| =$

$k-1-\deg_{G[v_1,\ldots,v_k]}(v_i)$. For $j \in \{r+1, \ldots, n\}$, let

$$w_j = \begin{cases} \deg_G(v_j) + |\{W_i : v_j \in W_i\}| - r & \text{if } r < j \leq k \\ \deg_{G[v_{r+1},\ldots,v_n]}(v_j) + |\{W_i : v_j \in W_i\}| & \text{if } j > k \end{cases}$$

and call $\pi^w = (w_{r+1}, \ldots, w_n)$ the want sequence of π for $K_r \vee \overline{K}_{k-r}$. By construction, this sequence is nonnegative with at most r terms equal to 0.

If the want sequence π^w is graphic, then let G^w be a realization of it with $V(G^w) = \{v_{r+1}, \ldots, v_n\}$ and define the graph G' by $V(G') = \{v_1, \ldots, v_n\}$ and

$$E(G') = \{\{v_i, v_j\} : i \leq r,\, j \leq k\} \cup E(G^w) \cup$$

$$\bigcup_{i=1}^{r} \{\{v_i, x\} : x \in N_G(v_i) \cap \{v_{k+1}, \ldots, v_n\} \setminus W_i\}.$$

Note that G' is a realization of π for which the subgraph induced by the vertices $\{v_1, \ldots, v_k\}$ is isomorphic to the graph $K_r \vee \overline{K}_{k-r}$.

Theorem 4.2.6 (Erbes, Ferrara, Martin, Wenger [5])**.** *Let H be a fixed graph of order k with at least one non-trivial component. If π is a graphic sequence of length n that is degree sufficient for H but not potentially H-graphic and $\nabla_{i^*}(H) = \nabla_{i^*}$, then π is*

$$([k-i^*+1, k], k^2 + ki^* + 2 + (6k^2 + (i^* - \nabla_{i^*})^2 k + \nabla_{i^*})(i^* - \nabla_{i^*} - 2)) - close$$

to $\tilde{\pi}_{i^*}(H, n)$.

Theorem 4.2.5 follows from Theorem 4.2.6 which itself follows from the fact that

$$2i^* - \nabla_{i^*}(H) \leq 2(\alpha(H)+1) - \nabla_{\alpha(H)+1}(H) \leq 2\alpha(H)+1$$

and $i^* \geq \alpha(H)+1$ that $i^* - \nabla_{i^*}(H) \leq \alpha(H)$. The proof of Theorem 4.2.6 is broken up into several cases. In some cases, it is shown that there is a limit to the amount of editing required to show the result, whereas the remaining cases show that no editing is required, but a specific want sequence is determined to exist and be graphic which guarantees the result.

Another way to look at the closeness of sequences is known as σ-stability. Some notation is useful to understand this concept.

Definition 4.2.7 (Erbes, Ferrara, Martin, Wenger [6])**.** For a graph H and an integer n,

$$\mathcal{P}(H,n) = \{\tilde{\pi}_i(H,n) : \tilde{\sigma}_i(H) = \tilde{\sigma}(H)\}.$$

Definition 4.2.8. For $m \leq n$ and graphic sequences $\pi_1 = (x_1, ..., x_n)$ and $\pi_2 = (y_1, ..., y_m)$, define

$$\|\pi_1 - \pi_2\| = \sum_{i=1}^{m} |x_i - y_i| + \begin{cases} \sum_{j=m+1}^{n} |x_j| & \text{if } m < n \\ 0 & \text{if } m = n. \end{cases}$$

Definition 4.2.9 (Erbes, Ferrara, Martin, Wenger [6])**.** A graph H is called *σ-stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 = n_0(\varepsilon, H)$ such that for any graphic sequence π of length $n \geq n_0$ that is not potentially H-graphic but satisfies

$$\sigma(\pi) \geq \sigma(H, n) - \delta n,$$

there is some $\pi' \in \mathcal{P}(H, n)$ such that $\|\pi - \pi'\| < \varepsilon n$.

To better understand σ-stability, it proves useful to note that complete graphs are not σ-stable. To see this, note that the only sequence in $\mathcal{P}(K_k, n)$ is $((n-1)^k, (k)^{n-1})$, whose unique realization is a copy of K_k, all of whose vertices are adjacent to all other vertices. However the sequence $\pi = ((n-1)^{k-1}, \lceil n/2 \rceil + k - 1, \lfloor n/2 \rfloor + k - 1, (k)^{n-2})$ is such that

$$\sigma(\pi) = \sigma(K_k, n) + k^2 - 4k - 1 - 4n,$$

its only realization is not potentially K_k-graphic, and for any $\pi' \in \mathcal{P}(K_k, n)$,

$$\|\pi - \pi'\| = \lfloor n/2 \rfloor - 1 + |\lfloor n/2 \rfloor - k|.$$

That is, K_k is not σ-stable.

Theorems 4.2.10 and 4.2.11 describe many graphs which are σ-stable. For the sake of notation, let $S_{x,y}$ denote the double star graph with vertices of degree $x+1$, $y+1$, and 1.

Theorem 4.2.10 (Erbes, Ferrara, Martin, Wenger [6])**.** *If H is a graph such that $2i^*(H) - \nabla_{i^*}(H) \leq 2\alpha(H)$, then H is σ-stable.*

Theorem 4.2.11 (Erbes, Ferrara, Martin, Wenger [6])**.** *Let H be a graph of order k such that*

$$2i^*(H) - \nabla_{i^*}(H) = 2\alpha(H) + 1$$

and H has a set X of $\alpha(H)+1$ vertices such that the subgraph of H induced by the set X has exactly one edge. The graph H is σ-stable if and only if H there exist integers x and y such that is a subgraph of $K_{k-\alpha(H)-2} \vee S_{x,y}$ and $x+y=\alpha(H)$.

Theorem 4.2.10 follows directly from Lemma 4.2.12.

Lemma 4.2.12 (Erbes, Ferrara, Martin, Wenger [6])**.** *Let H be a graph of order k and let $0 < \varepsilon < 1/2$. There exists $n_0 = n_0(\varepsilon, H)$ and $\delta > 0$ such that for any graphic sequence π of length $n \geq n_0$ satisfying $\sigma(H,n) \geq \sigma(\pi) \geq \sigma(H,n) - \delta n$, one of the following is true:*

1. *π is potentially H-graphic,*
2. *there exists $\pi' \in \mathcal{P}(H,n)$ such that $\|\pi - \pi'\| < \varepsilon n$, or*
3. *π is potentially $(K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1})$-graphic.*

The proof of Lemma 4.2.12 is omitted yet outlined here. It proceeds in a very similar way to Theorem 4.1.11 in that an algorithm is established which deletes specific vertices from a specific realization of a given sequence and then laying off small terms from the degree sequence of the remaining graph.

If the leading term of the given sequence is too large to satisfy the condition on d_1 in Theorem 4.1.9, then the algorithm says to delete all vertices not adjacent to the vertex of highest degree in a canonical realization of the sequence. Then, delete the dominating vertex from this remaining graph. Consider now the degree sequence of this new graph and begin laying off the smallest terms until the condition on the smallest term in Theorem 4.1.9 is satisfied. If enough terms are laid off at this step, it is possible to show that π is potentially H-graphic. In the case that this does not happen, perform another iteration of the algorithm to find a new sequence.

If the algorithm hits its halting condition without first allowing it to be shown that π is potentially H-graphic, then consider the possible realizations of the resultant sequence. A consequence of the algorithm is that in a realization of the original sequence, there is graph of the form $K_r \vee \overline{K}_t$, the vertices of the clique of which are exactly the dominating vertices deleted by the algorithm. Thus, it is now sufficient to alter this realization of the original sequence to find a $(k-r)-$vertex induced subgraph of H within the vertices of the independent set in the aforementioned split graph. If this is not possible, then it is possible to show that one of the two remaining conditions in the lemma are satisfied.

When H is a graph such that $2i^*(H) - \nabla_{i^*}(H) \leq 2\alpha(H)$, the proof of Lemma 4.2.12 shows that one of the first two conclusions of the lemma are satisfied, which proves Theorem 4.2.12. However, if H is a graph such that $2i^*(H) - \nabla_{i^*}(H) = 2\alpha(H) + 1$, Lemma 4.2.12 may only guarantee a realization which containing a graph of the form $K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1}$. In order to prove this remaining case, Theorem 4.2.13 gives a condition under which such a graph is not σ-stable and the remainder of the proof (which is omitted here) consists of constructing a sequence which demonstrates the result.

Theorem 4.2.13 (Erbes, Ferrara, Martin, Wenger [6])**.** *If H is a graph of order k such that $2i^*(H) - \nabla_{i^*}(H) = 2\alpha(H) + 1$ and for any x, y with $x + y = \alpha(H)$, H is not a subgraph of $K_{k-\alpha(H)-2} \vee S_{x,y}$, then H is not σ-stable.*

Note that the example given earlier that K_3 (as well as K_k for $k \geq 3$) satisfies the conditions of Theorem 4.2.13. However, though these graphs are not σ-stable, they need not be written off entirely.

Definition 4.2.14 (Erbes, Ferrara, Martin, Wenger [6])**.** A graph H is called *weakly σ-stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 = n_0(\varepsilon, H)$ such that for any graphic sequence π of length $n \geq n_0$ that is degree sufficient for H, not potentially H-graphic, and satisfies

$$\sigma(\pi) \geq \sigma(H, n) - \delta n,$$

there is some $\pi' \in \mathcal{P}(H, n)$ such that $\|\pi - \pi'\| < \varepsilon n$.

Theorem 4.2.15 (Erbes, Ferrara, Martin, Wenger [6])**.** *For $k \geq 3$, K_k is weakly σ-stable.*

Proof. Fix $k \geq 3$ and let $\pi = (d_1, ..., d_n)$ be a nonincreasing sequence which is degree sufficient for K_k. Note that $d_k \geq k - 1$ and so it follows from Lemma 4.1.5 that $d_{2k} < 2k - 3$ and there exists some $i \leq k - 2$ such that

$d_i < 2(k-1) - i$. Thus, if π is not potentially K_k-graphic, then it these bounds on the terms of π give that

$$\sigma(\pi) \leq 2(k-3)n - k^2 + 7k - 2.$$

From Chapter 3, it is known that

$$\sigma(K_k, n) = 2(k-2)n - k^2 + 3k$$

and so

$$\sigma(K_k, n) - \sigma(\pi) \geq 2n - 4k + 2.$$

That is, there are no sequences π which are degree sufficient for K_k, not potentially K_k-graphic, and are such that $\sigma(\pi) - \sigma(K_k, n) - \delta n$ for $\delta < 2$. Thus, for all $\varepsilon > 0$, K_k satisfies the conditions of being weakly σ-stable with $n_0 = 2k$ and $\delta = 1$. □

Chapter 5

Connected Realizations

5.1 Avoiding Subgraphs in Realizations

Turán numbers are the maximum number of edges in a graph with a fixed number of vertices which does not contain a copy of a given subgraph H. It is equivalent to determine the minimum number of edges necessary to guarantee a graph with a fixed number of vertices contains a copy of H. It is not apparent that a similar duality holds for the potential function, and indeed it does not. The potential function asks the minimum sum for a graphical sequence for which some realization of it has a copy of a given subgraph, however it is possible for a graphical sequence to have realizations which do contain copies of H and other realizations which do not contain H. For example consider the two realizations of $(3, 3, 2, 2, 2, 1, 1)$ in Figure 5.1.

One of these realizations contains a copy of K_3 while the other does not and similarly, one realization contains a copy of C_4 while the other does not.

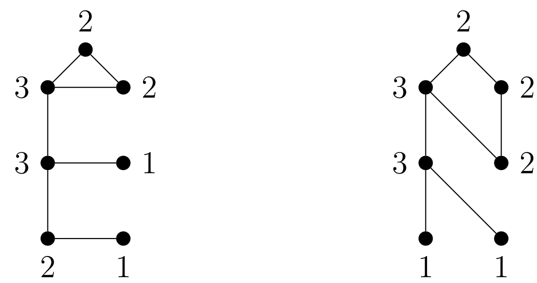


Figure 5.1: Two realizations of $(3, 3, 2, 2, 2, 1, 1)$.

That is, more information is needed to determine if a given graphical sequence has a realization not containing a copy of H. Clearly if the sequence is not degree sufficient for H, it is not potentially H-graphic. One way to deal with sequences which are degree sufficient for H but not potentially H-graphic was detailed in Section 4.2. In this chapter, I propose another way: restricting to connected realizations.

Definition 5.1.1. Given a graph H and an integer $n \geq 1$, let $\sigma_c(H, n)$ be the least value q such that for every graphical sequence π of length n for which $\sigma(\pi) < q$, either π has no connected realization, or π has a connected realization which does not contain a copy of F as a subgraph.

To better understand this, consider the question of which graphical sequences have triangle-free realizations. The only realization of a sequence π such that $\sigma(\pi) = n(n-1)$ is a complete graph, which contains a triangle (and any complete graph of order less than n) when $n \geq 3$ and the only realization of a sequence π such that $\sigma(\pi) = 0$ is the empty graph, which is triangle-free.

Moreover, Mantel's theorem [23] states that if a graph of order n is triangle-free, then it contains no more than $\frac{n^2}{4}$ edges. However, this does not mean that there is some minimum value the terms of π must sum to to have a realization containing a triangle, while all other sequences do not. For instance, consider the sequence $(2, 2, 2, 0, ..., 0)$ of length $n > 3$. Regardless of length n of the sequence, $\sigma(2, 2, 2, 0, ..., 0) = 6$, but the only realization (up to isomorphism) is a triangle and a set of $n - 3$ isolated vertices. That is, any realization of this sequence contains a triangle even though the sum of the elements is fixed at a low value. Between this and Mantel's theorem, it follows that for a given sequence π, if $\sigma(\pi) < 6$, then no realization will contain a triangle and if $\sigma(\pi) > \frac{n^2}{4}$, then every realization contains a triangle. For sequences π such that $6 \leq \sigma(\pi) \leq \frac{n^2}{4}$, if $\sigma(\pi) \geq \sigma(K_3, n)$, then at least one realization contains a triangle and if $\sigma(\pi) < \sigma(K_3, n)$ then it is possible that no realization contains a triangle.

The question addressed here restricts the problem of finding triangle-free realizations to connected realizations. If a graph on n vertices has fewer than $n-1$ edges, then it is disconnected. Thus, if π is a graphic sequence of length n such that $\sigma(\pi) < 2n - 2$, then any realization of π is disconnected. Moreover, any connected graph with n vertices and $n-1$ edges is a tree and in particular, is triangle-free. That is, if $\sigma(\pi) \leq 2n - 2$, then any connected realization of π is a tree and so is triangle-free. This is indeed best possible to guarantee the existence of a connected triangle-free realization of a sequence. That is, there are sequences π such that $\sigma(\pi) = 2n$ with connected realizations, all of which contain a triangle. For instance, consider the sequence $(a, b, c, 1, ..., 1)$ of length n where $a + b + c = n + 3$. Up to isomorphism, the only connected realization of this sequence is a triangle whose three vertices are incident to $a - 2$, $b - 2$, and $c - 2$ pendant edges respectively, as demonstrated in Figure 5.2.

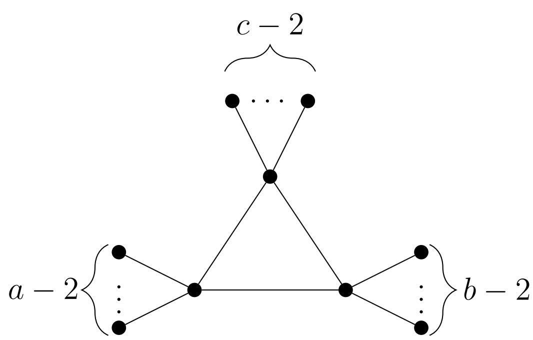


Figure 5.2: The only connected realization of $(a - 2, b - 2, c - 2, 1, ..., 1)$.

5.2 Realizations without given Cliques

This particular example of triangle-free realizations can be generalized for K_r-free for $r \geq 3$.

Lemma 5.2.1. *For any $r \geq 3$ and any integers $x_1 \geq ... \geq x_r \geq 0$ such that $\sum_{i=1}^{r} x_i = n - r$, the sequence*

$$\pi = (x_1 + r - 1, ..., x_r + r - 1, 1, ..., 1)$$

of length n has a unique connected realization which contains a copy of K_r.

Proof. Let G be a realization of π and let $v_1, ..., v_r \in V(G)$ be such that for $1 \leq i \leq r$, $\deg_G(v_i) = x_i + r - 1$. If there exists $i \in \{1, ..., r\}$ such that v_i is adjacent to fewer than x_i vertices of degree one, it follows that v_i is adjacent to at least r vertices of degree at least two, which is not possible. That is, for $1 \leq i \leq r$, v_i is adjacent to at least x_i vertices of degree one. If there exists $i \in \{1, ..., r\}$ such that v_i is adjacent to more than x_i vertices of degree one, then as there are $n - r = \sum_{i=1}^{r} x_i$ vertices of degree one, it follows that there exists $j \in \{1, ..., r\}$ such that v_j is adjacent to fewer than x_j vertices of degree one. This was just shown to be impossible and so for $1 \leq i \leq r$, v_i is adjacent to exactly x_i vertices of degree one. There are only r vertices in G which are not of degree one, and so

$$V(G) = \{v_1, ..., v_r\} \cup \bigcup_{k=1}^{r} \{w_{k,1}, ..., w_{k,x_k}\}$$

and

$$E(G) = \{v_i v_j : 1 \leq i < j \leq r\} \cup \bigcup_{k=1}^{r} \{v_k w_{k,1}, ..., v_k w_{k,x_k}\}$$

as demonstrated in Figure 5.3.

□

Theorem5.2.1 establishes that for $r \geq 3$, $\sigma_c(K_r, n) \leq 2n + r^2 - 3r$. This bound is tight as Theorem 5.2.2 demonstrates.

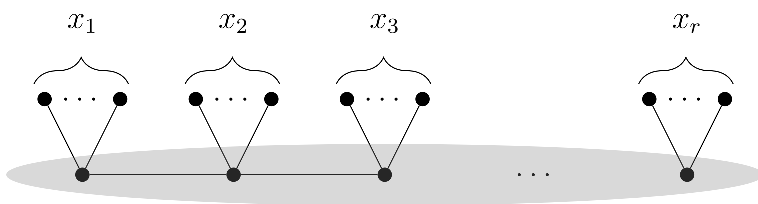


Figure 5.3: The unique realization of $(x_1 + r - 1, ..., x_n + r - 1, 1..., 1)$.

Theorem 5.2.2. *For a sequence π of length n and $r \geq 3$, if $\sigma(\pi) \leq 2n + r^2 - 3r - 2$, then every connected realization of π is K_r-free.*

Proof. Let π be a sequence of length n such that $\sigma(\pi) \leq 2n + r^2 - 3r - 2$ and let G be a connected realization of π. Let H be a subgraph of G of order r and let T be a spanning subtree of G. Note that $|E(T)| = n - 1$ and so the number of edges in G which are not in T is

$$|E(G) \setminus E(T)| = |E(G)| - |E(T)| = \frac{\sigma(\pi)}{2} - n + 1.$$

Moreover, since T is a tree and $|V(H)| = r$, it follows that

$$|E(H) \cap E(T)| \leq r - 1.$$

Thus,

$$\begin{aligned}
|E(H)| &= |E(H) \cap E(T)| + |E(H) \setminus E(T)| \\
&\leq r - 1 + |E(H) \setminus E(T)| \\
&\leq r - 1 + |E(G) \setminus E(T)| \\
&= r - 1 + \frac{\sigma(\pi)}{2} - n - 1 \\
&\leq r - 1 + n + \frac{r^2}{2} - \frac{3r}{2} - 1 - n + 1 \\
&= \frac{r^2 - r}{2} - 1 \\
&< \binom{r}{2}
\end{aligned}$$

and so H is not a clique. Since H was arbitrary, G is K_r-free. □

Lemma 5.2.1 demonstrates that this is best possible since the sum of the elements of the sequence is $2n + r^2 - 3r$. These results yield the following corollary.

Corollary 5.2.3. *For $r \geq 3$, $\sigma_c(K_r, n) = 2n + r^2 - 3r$.*

Note that this is in contrast to the potential function for a complete graph, since Theorem 3.7.7 states that for $r \geq 4$ and $n \geq \binom{r-1}{2} + 3$, $\sigma(K_r, n) = (2r-4)n - r^2 - 3r$.

5.3 Cycle-Free Realizations

Not only are triangles complete graphs, they are also cycles. As Section 5.2 generalized the results about triangles to cliques, this section will generalize such results to cycles. A tool which proves quite useful in this context is the path-edge swapping operation established in Definition 2.1.6. This operation is useful since it can be used to alter the length of a cycle in a given realization without changing the degree sequence.

If G is a graph containing exactly one cycle of length greater than 3 and at least one other edge, then swapping a path from the cycle for the other edge will shrink the cycle. Similarly, if there is an edge not contained in a cycle, then swapping it for a path in the cycle will grow the cycle. Thus, one can avoid any given cycle length by either growing or shrinking the cycle. See Figure 5.4.

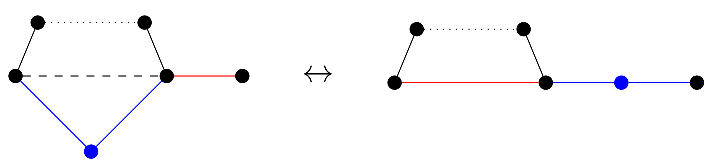


Figure 5.4: Path-edge swapping alters the length of cycles.

Theorem 5.3.1. *Let $n > r \geq 3$ and let $\pi = (d_1, ..., d_n)$ be a graphic sequence such that $\sigma(\pi) = 2n$, $d_r > 1$, and $d_{r+1} = 1 = d_n$. There are connected realizations $G_3, ..., G_r$ of π such that for each $3 \leq i \leq r$, G_i contains a cycle of length i.*

Proof. Let G be a connected realization of π and note that since $\sigma(\pi) = 2n$, G is unicyclic. Assume, without loss of generality, that no realization of π contains a cycle larger than the cycle in G. Claim that G contains a cycle of length r. Indeed, if C is the cycle in G and C is of length shorter than r, then there exists a vertex v of degree greater than 2 outside of C since $d_r > 1$. Swapping a path of length two whose centre vertex is v into C would result in a realization of π with a cycle longer than that of G, which is a contradiction. Thus, G contains a cycle of length r, and so let $G = G_r$. Since $n > r$, G contains at least one leaf. Without loss of generality, there exists a path of length two whose centre vertex is not the vertex in C of minimal distance to this leaf. For $3 \leq i \leq r-1$, obtain G_i from G_{i+1} by swapping such a path of length two in the cycle of G_{i+1} for the edge incident with a leaf in G_{i+1}. This process results in a realization of π with a cycle which contains 1 fewer vertices than its predecessor. That is, $G_3, ..., G_r$ are realizations of π such that for each $3 \leq i \leq r$, G_i contains a cycle of length i. □

Any realization of such a sequence contains exactly one cycle and so a realization containing a given cycle avoids all other lengths of cycle. This leads to the following corollary.

Corollary 5.3.2. *Let $n > r > 3$ and let $\pi = (d_1, ..., d_n)$ be a graphic sequence with connected realizations such that $\sigma(\pi) = 2n$, $d_r > 1$, and $d_{r+1} = 1 = d_n$. There are connected realizations of π which avoid any given cycle length.*

This unicyclic case can be used to find C_4-free realizations.

Lemma 5.3.3. *Let π be a graphical sequence of length $n \geq 5$ such that $\sigma(\pi) \leq 2n$. If π has a connected realization, then it has a C_4-free connected realization.*

Proof. Note that since $n \geq 5$, if a realization of π contains a C_4, then there is another edge in the graph outside of the C_4. If $\sigma(\pi) < 2n - 2$, then π has no connected realization. If $\sigma(\pi) = 2n - 2$, then any connected realization of π is a tree and so is C_4-free. If $\sigma(\pi) = 2n$, then given a connected realization of π which contains a C_4, this is the only cycle in the graph and so a path-edge swap can be performed which swaps a path of length 2 in the cycle for an edge elsewhere in the graph. The result of this path-edge swap is a realization of π which is C_4-free. □

This bound is tight as there is a sequence for which every realization has 2 more edges than vertices and a cycle of length 4.

Lemma 5.3.4. *For any $n \geq 4$, the sequence*

$$\pi = (n - 1, 3, 2, 2, 1, ..., 1)$$

of length n has a unique connected realization which contains a copy of C_4.

Proof. In any realization of π, there is a vertex adjacent to all other vertices and a set of $n-4$ leaf vertices. If these vertices are removed from the realization, all that remains are two vertices of degree 1 and one vertex of degree 2. The only way for these remaining three vertices to have the necessary degrees is for

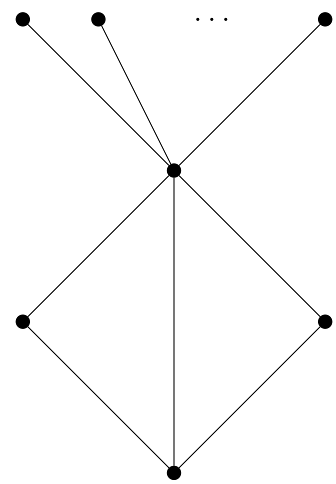


Figure 5.5: The unique realization of $(n-1, 3, 2, 2, 1, ..., 1)$.

them to form a path of length 2. Thus, the unique realization of π is described in Figure 5.5. This realization contains a C_4 on its vertices of highest degree.

$\square$

Lemmas 5.3.3 and 5.3.4 prove Theorem 5.3.5.

Theorem 5.3.5. *For $n \geq 5$, $\sigma_c(C_4, n) = 2n + 2$.*

For the C_5-free case, one more edge can be added to the realization.

Lemma 5.3.6. *Let π be a graphical sequence of length $n \geq 6$ such that $\sigma(\pi) \leq 2n+2$. If π has a connected realization, then it has a C_5-free connected realization.*

Rather than prove Lemma 5.3.6, a generalization is proven in Lemma 5.3.9. However, as with the C_4 case, this bound is tight is there is a sequence for which every realization has 3 more edges than vertices and a cycle of length 5.

Lemma 5.3.7. *For any $n \geq 5$, the sequence*

$$\pi = (n-1, 3, 3, 2, 2, 1, ..., 1)$$

of length n has a unique connected realization which contains a copy of C_5.

Proof. In any realization of π, there is a vertex adjacent to all other vertices and a set of $n-5$ leaf vertices. If these vertices are removed from the realization, all that remains are two vertices of degree 1 and two vertices of degree 2. The only way for the remaining four vertices to have the necessary degrees is for them to form a path of length 3. Thus, the unique realization of π is shown in Figure 5.6. This realization contains a C_5 on its vertices of highest degree.

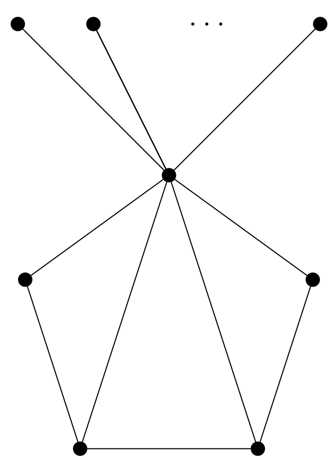


Figure 5.6: The unique realization of $(n-1, 3, 3, 2, 2, 1, ..., 1)$.

□

Lemmas 5.3.6 and 5.3.7 prove Theorem 5.3.8.

Theorem 5.3.8. *For $n \geq 6$, $\sigma_c(C_5, n) = 2n + 4$.*

To find connected C_k-free realizations, the following provides a lower bound.

Lemma 5.3.9. *Let π be a graphical sequence of length $n \geq k + 1 \geq 6$ such that $\sigma(\pi) \leq 2n + 2$. If π has a connected realization, then it has a connected C_k-free realization.*

Proof. If $\sigma(\pi) < 2n-2$, then π has no connected realization. If $\sigma(\pi) = 2n-2$, then any connected realization of π is a tree and so is C_k-free. If $\sigma(\pi) = 2n$, then given a connected realization G of π which contains a C_k, this is the only cycle in the graph. Since $n \geq k + 1$, there exists an edge e which is not contained in the cycle. Let x, y, and z be vertices in this cycle in G such that y is adjacent to both x and z and $y \notin e$. Note that swapping the edge e for the path x, y, z yields a realization of π which is C_k-free.

Assume $\sigma(\pi) = 2n + 2$, and let G be a connected realization of π which contains a C_k. Note that if there is a triple of edges whose removal yields a connected graph, then $E(G) - 3 \geq V(G) - 1$, which is a contradiction. Thus, any set of three edges in G form an edge-cut-set (that is, a set of edges whose removal disconnects the graph). Without loss of generality, let C be a subgraph of G such that

$$V(C) = \{v_1, v_2, ..., v_k\}$$

and

$$E(C) = \{v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv_1\}.$$

Moreover, since G contains $n+1$ edges, it follows that G contains at least one cycle other than C. Let C' be a smallest such cycle and let

$$V(C') = \{x_1x_2, x_2x_3, ..., x_\ell x_1\}$$

and

$$E(C') = \{x_1x_2, x_2x_3, ..., x_{\ell-1}x_\ell, x_\ell x_1\}.$$

Let $p = |E(C) \cap E(C')|$.

Case 1: $p = 0$.

Let A be a path connecting C to C'. Without loss of generality, assume that A runs from v_1 to x_1 and only intersects C at v_1 and C' at x_1. I claim that this is the only path which connects C to C'. (Note that it is possible for this path to be a single vertex.) Indeed, if A and B are distinct paths connecting C to C', then the set of edges v_1v_2, x_1x_2, and any edge in B (or the edge v_1v_k if B has no edges) do not form an edge-cut-set. Moreover, if either of C or C' contain a chord, then the edges v_1v_k, x_1x_ℓ, and that chord do not form an edge-cut-set. That is, in particular, $x_1x_{\ell-1}, v_1v_{k-1} \notin E(G)$.

Subcase 1a: $\ell = k - 1$.

Let G' be the result of swapping the path $x_1x_\ell x_{\ell-1}$ for the edge v_1v_k. See Figure 5.7. That is, G' is also a connected realization of π. The only cycles whose edges are contained in

$$S_1 = \{v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kx_\ell, x_\ell v_1, x_1x_2, x_2x_3, ..., x_{\ell-1}x_1\}$$

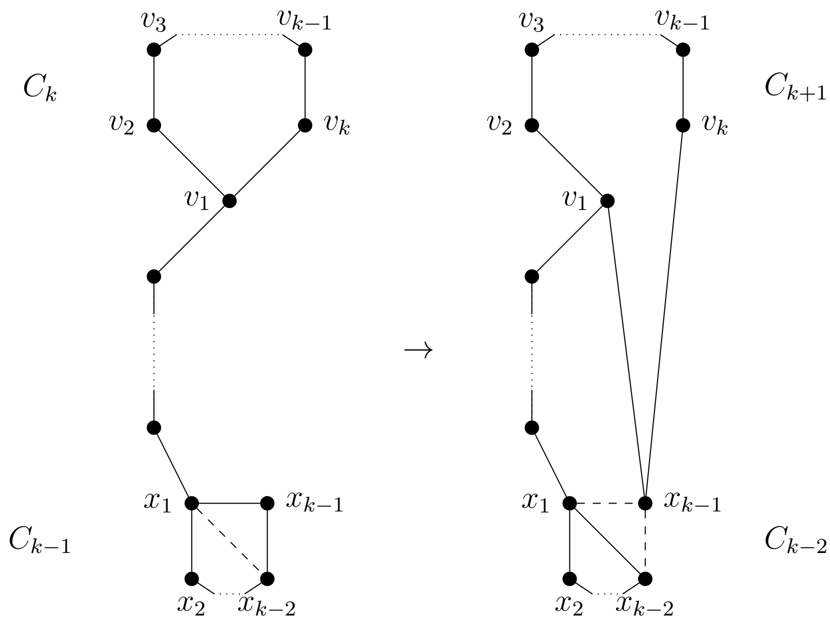


Figure 5.7: The Path-Edge Swap to perform on C_k and C_{k-1} sharing no edges.

are of length $\ell-1$ or $k+1$. G' contains a cycle with an edge e not in S_1, then the edges e, v_1v_2, and x_1x_2 do not form an edge-cut-set, which is a contradiction. That is, G' is a connected, C_k-free realization of π.

Subcase 1b: $\ell \neq k-1$.

Let G' be the result of swapping the path $v_1v_kv_{k-1}$ for the edge x_1x_ℓ. See Figure 5.8. That is, G' is also a connected realization of π. The only cycles whose edges are contained in

$$S_2 = \{v_1v_2, v_2v_3, ..., v_{k-1}v_1, x_1x_2, x_2x_3, ..., x_{\ell-1}x_\ell, x_\ell v_k, v_kx_1\}$$

are of length $\ell+1$ or $k-1$. If G' contains a cycle with an edge e not in S_2, then the edges e, v_1v_2, and x_1x_2 do not form an edge-cut-set, which is a

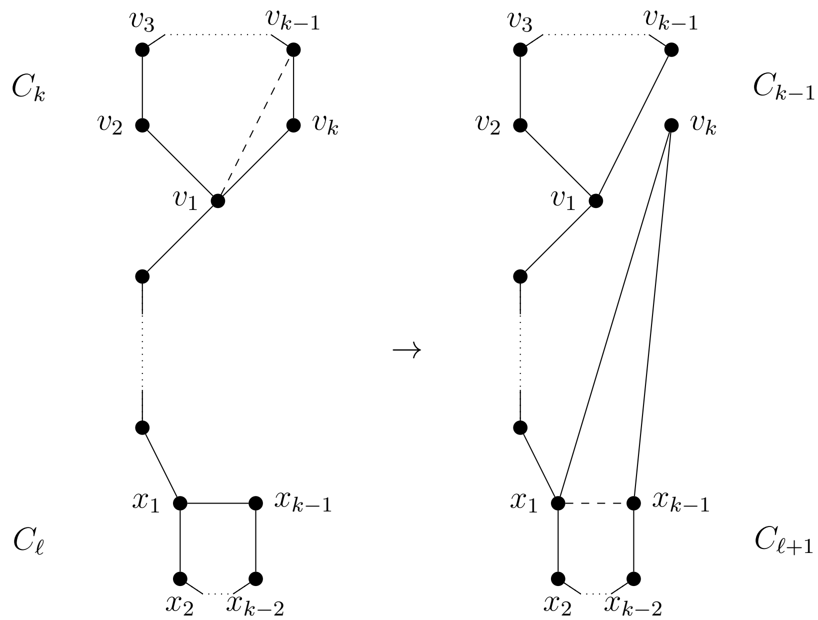


Figure 5.8: The Path-Edge Swap to perform on C_k and C_ℓ sharing no edges.

contradiction. That is, G' is a connected, C_k-free realization of π.

Case 2: $p \geq 1$.

I claim that the edges of $E(C) \cap E(C')$ form a path. Indeed, if the edges of $E(C) \cap E(C')$ form more than one path, let $j \geq 1$ be the number of vertices in a longest path and without loss of generality, assume that for $i \in \{1, ..., j\}$, $v_i = x_i$. Since $v_1...v_j$ is not the only path shared between C and C', there exist $s, t \geq 1$ such that $v_{j+s} = x_{j+t}$. Note that the edges v_1v_k, x_1x_ℓ, and x_jx_{j+1} do not form an edge-cut-set, which is a contradiction. That is, the edges of $E(C) \cap E(C')$ form a path. Thus, suppose without loss of generality that for $i \in \{1, ..., |V(C) \cap V(C')|\}$, $v_i = x_i$. If $p = 1$, let $G'' = G$. If $p \geq 2$, obtain G'' from G by swapping the path $v_1 \ldots v_{p+1}$ and the edge $v_1 v_k$. See Figure 5.9.

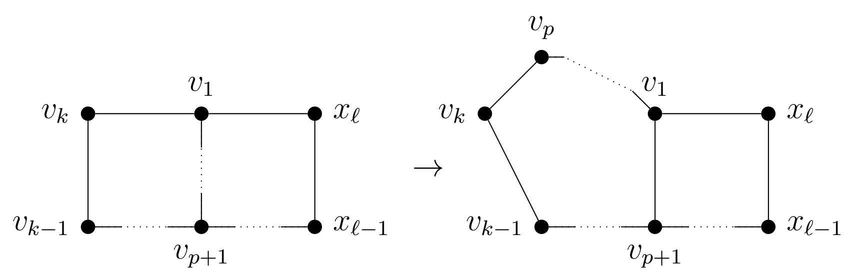


Figure 5.9: The Path-Edge Swap to perform to force the intersection of the cycles to have one edge.

In either case, G'' is also a connected realization of π. Note that if $v_1 x_{\ell-1} \in E(G'')$, then the edges $v_1 x_{\ell-1}$, $x_1 x_\ell$, and $v_1 v_2$ do not form an edge-cut-set. That is, $v_1 x_{\ell-1} \notin E(G'')$ and by a similar argument, $v_{k-1} v_p \notin E(G'')$.

Subcase 2a: $\ell - p + 1 = k - 1$.

Let G''' be the result of swapping the path $v_1 x_\ell x_{\ell-1}$ and the edge $v_k v_{k-1}$ in G''. See Figure 5.10.

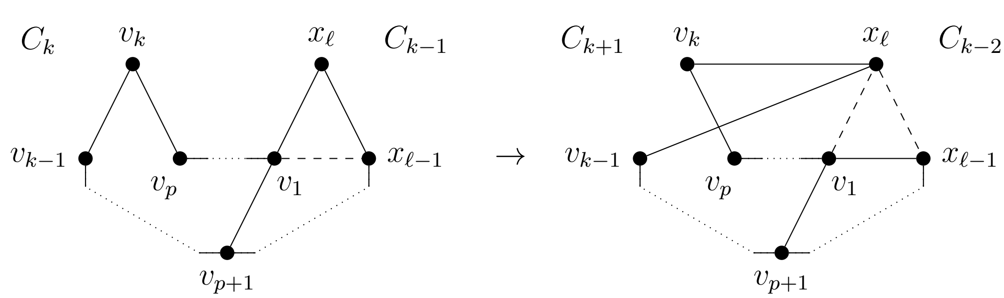


Figure 5.10: The Path-Edge Swap to perform on C_k and C_{k-1} sharing edges.

Note that G''' is also a connected realization of π. Moreover, the only cycles in G''' whose edges are contained in

$$S_3 = \{v_1v_2, v_2v_3, ..., v_{p-1}v_p, v_pv_k, v_kx_\ell, x_\ell v_{k-1}, v_{k-1}v_{k-2}, ..., v_{p+2}v_{p+1},$$

$$v_{p+1}v_1, x_1x_{\ell-1}, x_{\ell-1}x_{\ell-2}, ..., x_{p+2}x_{p+1}\}$$

are of length $\ell - p$, $k+1$, or $k+\ell$. If G' contains a cycle with an edge e not in S_3, then the edges e, v_1v_{p+1}, and v_kv_p do not form an edge-cut-set, which is a contradiction. That is, G''' is a connected, C_k-free realization of π.

Subcase 2b: $\ell - p + 1 \neq k - 1$.

Let G''' be the result of swapping the path $v_{k-1}v_kv_p$ for the edge x_1x_ℓ. See Figure 5.11.

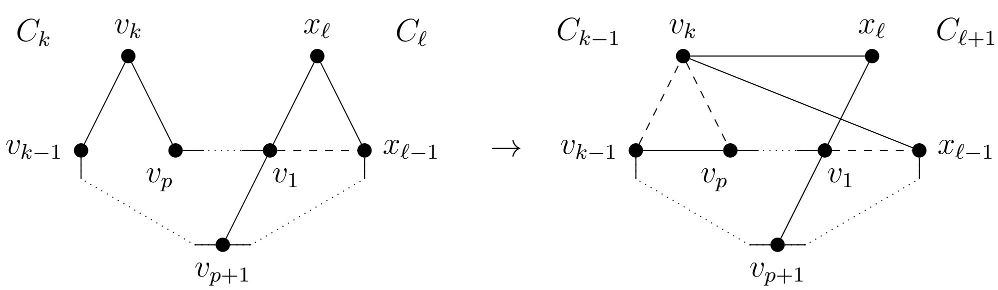


Figure 5.11: The Path-Edge Swap to perform on C_k and C_ℓ sharing vertices.

That is, G''' is also a connected realization of π. The only cycles of G''' whose edges are contained in

$$S_4 = \{v_1v_2, v_2v_3, ..., v_{p-1}v_p, v_pv_{k-1}, v_{k-1}v_{k-2}, ..., v_{p+2}v_{p+1},$$

$$v_{p+1}v_1, v_1v_k, v_kx_\ell, x_\ell x_{\ell-1}, x_{\ell-1}x_{\ell-2}, ..., x_{p+2}x_{p+1}\}$$

are of length $\ell - p + 2$, $k-1$, or $k+\ell$. If G' contains a cycle with an edge e not in S_3, then the edges e, v_1v_{p+1}, and v_1v_k do not form an edge-cut-set, which is a contradiction. That is, G''' is a connected, C_k-free realization of π. □

Going from the C_4-free case to the C_5-free case, the bound increases by 2. For n sufficiently large, this pattern continues: for every edge added, the order of the unavoidable cycle increases by one.

Theorem 5.3.10. *Let $k \geq 6$ and let π be a graphical sequence of length $n \geq (2^{k-3}-3)k+8$ such that $\sigma(\pi) \leq 2(n+k-4)$. If π has a connected realization, then it has a connected, C_k-free realization.*

To prove Theorem 5.3.10, note that Lemmas 5.3.11 and 5.3.13 below provide two cases which together prove the result.

Lemma 5.3.11. *Let π be a graphical sequence of length $n \geq k+1$ such that $\sigma(\pi) \leq 2(n+k-4)$. If π has a connected realization which has an edge not contained in any cycle, then it has a connected, C_k-free realization.*

Proof. Let G be a connected realization of π which has an edge e not contained in any cycle, and assume C is a subgraph of G isomorphic to C_k. Let $\{v, x_0, x_1, ..., x_{k-2}\} = V(C)$ and $\{x_{k-2}v, vx_0, x_0x_1, ..., x_{k-3}x_{k-2}\} = E(G)$ and if e has an endpoint in C, then without loss of generality, let $\{v\} = V(C) \cap e$.

Let G' be the graph $G \setminus \{x_0x_1, ..., x_{k-3}x_{k-2}\}$. Define X to be the set of vertices in $\{x_1, ..., x_{k-3}\}$ in the same connected component as v in G' and define $Y = \{x_i : 1 \leq i \leq k-3, x_{i-1}x_{i+1} \in E(G)\}$. For each $x_i \in X$, let X_i be a shortest path in G' from x_i to v and let e_i be the edge in X_i incident to x_i. Note that removing the edges $\{e_i : x_i \in X\} \cup \{x_{i-1}x_{i+1} : x_i \in Y\}$ from G yields a connected graph with at least one cycle, namely C. Thus,

$$|E(G)| - |X \cup Y| = |E(G)| - |\{e_i : x_i \in X\} \cup \{x_{i-1}x_{i+1} : x_i \in Y\}| \geq V(G),$$

and so $|X \cup Y| \leq k - 4$. Therefore, there exists $j \in \{1, ..., k-3\}$ such that $x_jv, x_{j-1}x_{j+1} \notin E(G)$. Thus, swapping the edge e for the path $x_{j-1}x_jx_{j+1}$ yields a connected realization of π with one fewer copies of C_k as subgraphs. Repeating this process for each copy of C_k provides a connected, C_k-free realization of π. □

In 1897, Ahrens [1] showed the following.

Lemma 5.3.12. *A graph with n vertices, m edges, and r components has between $m - n + r$ and $2^{m-n+r} - 1$ cycles.*

Lemma 5.3.12 is not proved here, but it is useful in the proof of Lemma 5.3.13.

Lemma 5.3.13. *Let $k \geq 6$ and let π be a graphical sequence of length $n \geq (2^{k-3} - 3)k + 8$ such that $\sigma(\pi) \leq 2(n + k - 4)$. If π has a connected realization in which every edge is contained in some cycle, then it has a connected, C_k-free realization.*

Proof. Note: in intermediate steps of this proof, graph operations are performed which may potentially yield a multigraph, but this is resolved by the end.

Without loss of generality, assume $\pi = (d_1, ..., d_n)$ is in nonincreasing order. Let G be a connected realization of π in which every edge is contained in some cycle. Let X be the set of vertices of degree 2 in G and note that by the Handshaking Lemma,

$$\begin{aligned} 2n + 2k - 8 &= 2|E(G)| \\ &= \sum_{\deg_G(v) \geq 3} \deg_G(v) + 2|X| \\ &\geq 3(n - |X|) + 2|X| \end{aligned}$$

and so $|X| \geq n + 2k - 8$. Moreover, since $n \geq (2^{k-3} - 3)k + 8$, $|X| \geq k(2^{k-3} - 1)$ and note that by Lemma 5.3.12, G has at most $2^{k-3} - 1$ cycles. Given $x \in V(G)$ such that $N(x) = \{y, z\}$, note that the graph with vertex set $V(G) \setminus \{x\}$ and edge set $(E(G) \setminus \{xy, xz\}) \cup \{yz\}$ is a connected realization of $(d_1, ..., d_{n-1})$ which has the same number of cycles as G. It is possible that this graph has multiple edges between the vertices y and z. Let $F_0 = G$ and for $i \in \{1, ..., |X|\}$, let F_i be the result of removing a vertex of degree 2 from F_{i-1} and adding an edge between its neighbours. Note that for $i \in \{0, ..., |X|\}$, F_i is a connected realization of $(d_1, ..., d_{n-i})$ with the same number of cycles as G, possibly with more than one edge between any given pair of vertices. In particular, $F_{|X|}$ is a connected realization of $(d_1, ..., d_{n-|X|})$ with the same number of cycles as G and minimum degree 3. Let $G_0 = F_{|X|}$ and for $i \geq 1$, let G_i be obtained from G_{i-1} by replacing an edge in a smallest cycle of G_{i-1} with a path of length 2. (In the cases where G_{i-1} is a multigraph, a cycle of length 2 will be the shortest cycle.) By construction, since G_0 has at most $2^{k-3} - 1$ cycles, it follows that for $i \geq 1$, the length of a shortest cycle in $G_{i(2^{k-3}-1)}$ is greater than the length of a shortest cycle in $G_{(i-1)(2^{k-3}-1)}$. Thus, since the maximum length of a cycle in G_0 is 2 and $|X| \geq k(2^{k-3} - 1)$, every cycle in $G_{|X|}$ has length greater than k. That is, $G_{|X|}$ is a connected realization of π which contains no copy of C_k. □

Theorem 5.3.10 provides a lower bound for $\sigma_c(C_k, n)$ for sufficiently large n, but says nothing as to an upper bound. Such an upper bound is now addressed. The realizations in Lemmas 5.3.4 and 5.3.7 are graphs which contain some vertex adjacent to all others, some set of leaves, and some set of vertices which form a path and hence form a cycle with the vertex of highest degree. This may not work to guarantee cycles of longer length since it is possible that the vertices which must form a path may not. In particular, the graphs in Figure 5.12 is a connected C_6-free realization of the length n sequence $(n-1, 3, 3, 3, 2, 2, 1, ..., 1)$, but only one of them contains a copy of C_6.

In order to guarantee a cycle, a path must be guaranteed on these specific vertices. That is, a Hamiltonian path must be guaranteed in the set of vertices not of degree 1 or $n-1$. The following pair of results provide a way to guarantee this.

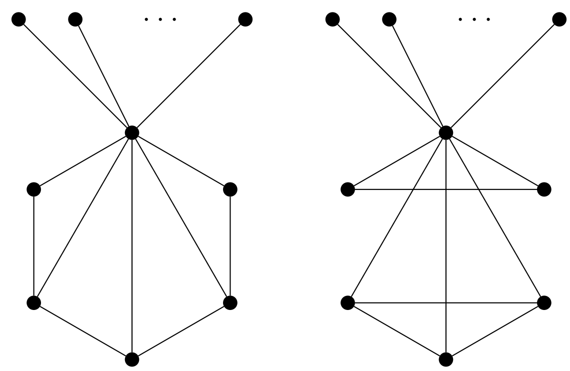


Figure 5.12: Two realizations of $(n-1, 3, 3, 3, 2, 2, 1, ..., 1)$.

Lemma 5.3.14. *For any $k \geq 6$, any realization of*

$$\pi = (k-2, (k-4)^{k-4}, 2, 2)$$

has a Hamiltonian path.

Proof. Let G be a realization of π and let $V(G) = \{x, a, b, x_1, ..., x_{k-4}\}$. Without loss of generality, assume $\deg_G(x) = k-2$ and $\deg_G(a) = 2 = \deg_G(b)$, and let $H = G - x$. If $ab \in E(H)$, then the remaining vertices of H are of degree $k-5$ in H and so form a clique of order $k-4$. Thus, $a, b, x, x_1, ..., x_{k-4}$ is a Hamiltonian path in G. See Figure 5.13.

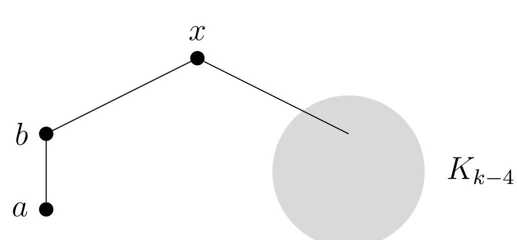


Figure 5.13: Hamiltonian Path in this graph.

Assume $ab \notin E(H)$, and note that there exists $i, j \in \{1, ..., k-4\}$ such that $\{x_i\} = N_H(a)$ and $\{x_j\} = N_H(b)$. Claim that $i \neq j$. Indeed, if $i = j$, let $y \in V(H) \setminus \{a, b, x_i\}$ and note that $\deg_H(y) = k-5$ and $N_H(y) \subseteq V(H) \setminus \{a, b, y\}$. That is, $N_H(y) = V(H) \setminus \{a, b, y\}$ and in particular, $x_i y \in E(H)$. Since y was arbitrary, it follows that $N_H(x_i) \supseteq V(H) \setminus \{a, b, x_i\}$ and so $N_H(x_i) = V(H) \setminus \{x_i\}$. This contradicts the fact that $\deg_G(x_i) = k-4$ and so $i \neq j$. Without loss of generality, assume $i < j$.

For $y \in V(H) \setminus \{a, b, x_i, x_j\}$, $\deg_H(y) = k-5$ and so $N_H(y) = V(H) \setminus \{a, b\}$. Thus,

$$E(H) = \{ax_i, bx_j\} \cup (\{x_s x_t : 1 \leq s < t \leq k-4\} \setminus \{x_i x_j\})$$

and so

$$x_1, ...x_i, a, x, b, x_j, x_{j+1}, ..., x_{k-4}, x_{i+1}, ..., x_{j-1}$$

is a Hamiltonian path in G. See Figure 5.14.

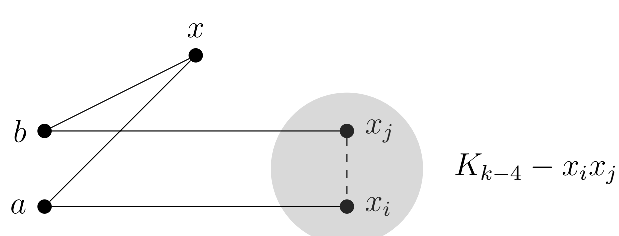


Figure 5.14: Hamiltonian Path in this graph.

□

Theorem 5.3.15. *For any $n \geq k+1 \geq 7$, any connected realization of the sequence*

$$\pi = (n-1, k-1, (k-3)^{k-4}, 3, 3, (1)^{n-k})$$

contains a copy of C_k.

Proof. Let G be a connected realization of π such that $v, w_1, ..., w_{n-k} \in V(G)$ and $deg_G(v) = n-1$ and $\deg_G(w_1) = ... = \deg_G(w_{n-k}) = 1$. Let H be the subgraph of G induced by $V(G) \setminus \{v, w_1, ..., w_{n-k}\}$ and note that H is a realization of the sequence $(k-2, (k-4)^{k-4}, 2, 2)$. By Lemma 5.3.14, H has a Hamiltonian path $v_1, ..., v_{k-1}$ and so the vertices $v, v_1, ..., v_{k-1}, v$ form a cycle of length k in G. See Figure 5.15.

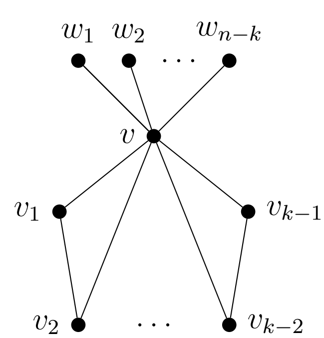


Figure 5.15: Subgraph of the realization which contains a C_k.

□

Theorem 5.3.15 leads to the following corollary.

Corollary 5.3.16. *For $n \geq k+1 \geq 7$, $\sigma_c(C_k, n) \leq 2n + k^2 - 7k + 16$.*

Corollary 5.3.16 and Theorem 5.3.10 establish a set of bounds of $\sigma_c(C_k, n)$ for n sufficiently large.

Corollary 5.3.17. *For $k \geq 6$ and $n \geq (2^{k-3} - 3)k + 8$,*

$$2(n - k + 4) \leq \sigma_c(C_k, n) \leq 2n + k^2 - 7k + 16.$$

The techniques used in the proof of Lemma 5.3.13 make use of the fact that the realizations in question have a sufficiently large number of vertices of degree 2. The thought that this could be proven in a manner more similar to the of the path-edge swapping technique leads to Conjecture 5.3.18 below.

Conjecture 5.3.18. *For $n \geq k + 1 \geq 7$,*

$$2(n - k + 4) \leq \sigma_c(C_k, n) \leq 2n + k^2 - 7k + 16.$$

Moreover, the bounds themselves have not been shown here to be tight. Perhaps a more illuminating counter-example can be found which lowers the upper bound. Even further, it is possible that results similar to those noted in Chapter 4 may be found which could describe the asymptotic behaviour of $\sigma_c(C_k, n)$. Only two classes of graphs, cliques and cycles, were considered in this chapter. Perhaps path-edge swapping techniques could be used to obtain similar results for other classes of graphs.

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