On Hadwiger Covering Problem in Five- and Six-dimensional Euclidean Spaces

by

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Abstract

We denote by H_n the minimum number such that any convex body in $\mathbb{R}^n$ can be covered by H_n of its smaller homothets. Considering an n-dimensional cube, one can easily see that $H_n \geqslant 2^n$. It is a well-known conjecture that $H_n = 2^n$ for all $n \geqslant 3$. The main result of this thesis is the inequalities $H_5 \leqslant 1002$ and $H_6 \leqslant 14140$. The previously known upper bounds were $H_5 \leqslant 1091$ and $H_6 \leqslant 15373$. Specifically, we apply certain generalizations of an approach by Papadoperakis, which essentially reduces the problem to the study of covering of $(n-2)$-dimensional faces of an n-dimensional cube by parallelepipeds of a particular form. A step in the construction of the required covering uses computer assistance. We also study limitations of this technique and establish some lower bounds on performance of this method.

Acknowledgment Page

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Chapter 1

Introduction

1.1 Covering Conjecture and Illumination Conjecture

Let us introduce some notations and definitions. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For $K \subset \mathbb{R}^n$ and any $\boldsymbol{x}, \boldsymbol{y} \in K$, if the segment $[\boldsymbol{x}, \boldsymbol{y}]$ lies entirely in K, then K is called a convex set. If $K \subset \mathbb{R}^n$ is a compact convex set without empty interior, then K is called a convex body, see Figure 1.1. For $\lambda \in \{\mathbb{R} \setminus 0\}$ and $\boldsymbol{t}, \boldsymbol{x} \in \mathbb{R}^n$, the affine transformation $\boldsymbol{x} \mapsto \boldsymbol{t} + \lambda \boldsymbol{x}$ is called a homothety of $\mathbb{R}^n$. Also, the image $\boldsymbol{t} + \lambda K$ is referred to as a homothetic copy of K (or briefly a homothet of K, see [3]) with λ as its ratio. When $\lambda > 0$, $\boldsymbol{t} + \lambda K$ is called a positive homothet of K. When $\lambda \in (0, 1)$, $\boldsymbol{t} + \lambda K$ is called a smaller homothet of K, and the mapping is called a smaller homothety of K. For some general geometric properties of convex bodies, an interested reader is referred to [22] and [32].

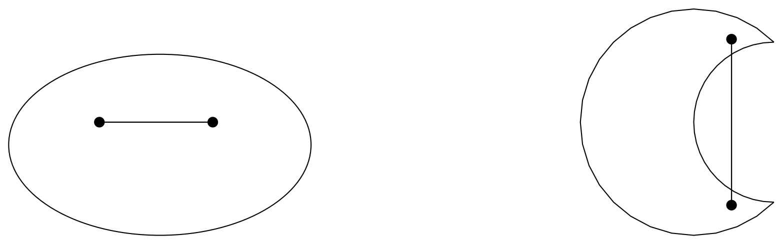

Figure 1.1: A convex set and a non-convex set

In 1957, Hadwiger [15] put forward a question that turned out to be a core issue in discrete geometry and is still unsolved. The question can be stated as follows: "Let N_n be the minimal natural number so that for any convex body $K \subset \mathbb{R}^n$, K can be covered by N_n translates of the interior of K. What is N_n when $n \geqslant 3$?". The planer case was settled earlier, in 1955, by Levi [23]. Later in 1960, Gohberg and Markus [14] stated this question again, but from the perspective of covering by homothets.

For a convex body $K \subset \mathbb{R}^n$, if there exists a positive integer m, some $\lambda_i \in (0,1)$ and $\boldsymbol{t}_i \in \mathbb{R}^n$ such that $K \subset \bigcup_{i=1}^{m}(\boldsymbol{t}_i + \lambda_i K)$, then we say that K is covered by m of its smaller homothets. Define the smallest value of such m as the covering number of K with notation $H(K)$. Moreover, let H_n be the maximum value of $H(K)$ over all convex bodies K in $\mathbb{R}^n$. Recall that an affine transform is a composition of a linear transformation and a translation. An affine transform is said to be non-degenerate if the matrix for linear transformation has full rank. By an affine n-cube, we mean a non-degenerate affine image of the n-cube $[0,1]^n$. Remark that $H(K)$ is invariant under non-degenerate affine transforms, i.e. $H(K) = H(A(K))$ for any non-degenerate affine transform A. The following conjecture is known as Hadwiger Covering Conjecture, Levi-Hadwiger Conjecture, or Gohberg-Markus Covering Conjecture.

Conjecture 1.1.1 (Covering Conjecture)**.** [3] *For a convex body $K \subset \mathbb{R}^n$, $H(K) \leqslant 2^n$ holds. Moreover, $H(K) = 2^n$ if and only if K is an affine n-cube.*

It is a simple matter to verify that Covering Conjecture is valid for $n = 1$. Indeed, for $n = 1$, K is an element of the collection of finite segments, say $K = [a, b]$ where $a < b$. On one hand, the length of any smaller homothet of K is less than $b - a$, so it can not cover a and b at the same time, which means $H_1 \geqslant 2$. On the other hand, K can be covered by $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, which are both smaller homothets of K of ratio $\frac{1}{2}$. In this way, we have $H_1 \leqslant 2$ and therefore $H_1 = 2^1$. For $n = 2$, Levi proved $H_2 = 4$ in [23]. It can be easily seen that an n-cube requires at least 2^n smaller homothets to be covered (see Conjecture 1.1.1), since an n-cube has 2^n vertices while each smaller homothet can not contain two distinct vertices simultaneously, see Figure 1.2 for an illustration of the case $n = 2$.

Now we describe an equivalent formulation of the Covering Conjecture. For a convex body $K \subset \mathbb{R}^n$, denote the boundary of K by bd$K$ and its interior by int$K$. Consider the illumination of K using external points as light sources, where any point $\boldsymbol{p}$ as a light source is outside of the body K. Let

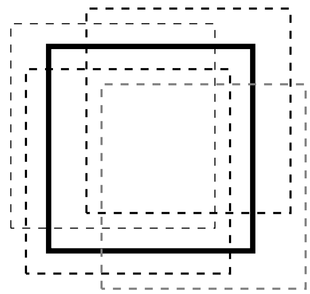

Figure 1.2: Covering of the square with 4 smaller homothets

$l(\boldsymbol{p}, \boldsymbol{x}) \coloneqq \{\boldsymbol{p} + \lambda(\boldsymbol{x} - \boldsymbol{p}) : \lambda \geqslant 0\}$ denote the halfline origining from $\boldsymbol{p}$ and passing through $\boldsymbol{x} \in \mathrm{bd}K$. If $l(\boldsymbol{p}, \boldsymbol{x}) \cap \mathrm{int}K \neq \emptyset$, then such $\boldsymbol{x} \in \mathrm{bd}K$ is said to be illuminated by the light source $\boldsymbol{p}$. Define the least number of light sources needed to illuminate all points in $\mathrm{bd}K$ as $I^{'}(K)$. Another approach to define the illumination number is to illuminate a point $\boldsymbol{x} \in \mathrm{bd}K$ by a given direction $\boldsymbol{l}$. If there exists $\lambda > 0$ satisfying $(\boldsymbol{x} + \lambda\boldsymbol{l}) \in \mathrm{int}K$, then we say the point $\boldsymbol{x}$ is illuminated in the direction $\boldsymbol{l}$. If any $\boldsymbol{x} \in \mathrm{bd}K$ is illuminated in at least one of the directions $\boldsymbol{l}_1, \boldsymbol{l}_2, \dots, \boldsymbol{l}_m$, then we say $\mathrm{bd}K$ is illuminated in the directions $\boldsymbol{l}_1, \boldsymbol{l}_2, \dots, \boldsymbol{l}_m$. Let $I(K)$ denote the smallest m so that $\mathrm{bd}K$ can be illuminated in at least m directions. It was shown that both definitions always give the same result: $I^{'}(K) = I(K)$, see [8, Theorem 34.3]. The notion $I'(K)$ was introduced by Hadwiger [16] and the notion $I(K)$ by Boltyanski [5]. The following statement by Boltyanski is known as Illumination Conjecture.

Conjecture 1.1.2 (Illumination Conjecture)**.** *Given a convex body $K \subset \mathbb{R}^n$, we have $I(K) \leqslant 2^n$, and the equality holds if and only if K is an affine n-cube.*

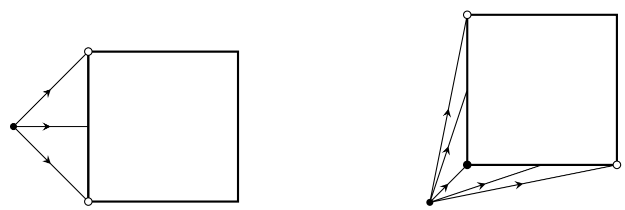

Figure 1.3: Illumination of square by point light sources

Since a light source can not illuminate two distinct vertices of a cube simultaneously (see Figure 1.3 for the 2-dim case), it is not hard to see that, similarly to the covering by smaller homothets, an n-cube requires exactly 2^n external light sources to be illuminated.

Although Conjecture 1.1.1 and Conjecture 1.1.2 seem to be different at the first sight, they are equivalent as $I(K) = H(K)$, see [8, Theorem 34.3] for a detailed proof. Let us give a specific example illustrating how the illumination problem is related to the covering by smaller homothets. If $K = \{(x,y) : x^2 + y^2 \leqslant 1\}$ is a unit disk in $\mathbb{R}^2$, then the light source positioned at the point $(a, 0)$ where $a > 1$ illuminates certain parts of bd$K$, i.e. the arc $\{(x,y) : x^2 + y^2 = 1, x > \frac{1}{a}\}$. This arc can be covered by the smaller disk $\sqrt{1 - \frac{1}{a^2}} \cdot K + \left(\frac{1}{a}, 0\right)$, see Figure 1.4. Now it is not hard to see that $I^{'}(K) = 3$ when K is a disk, see Figure 1.5.

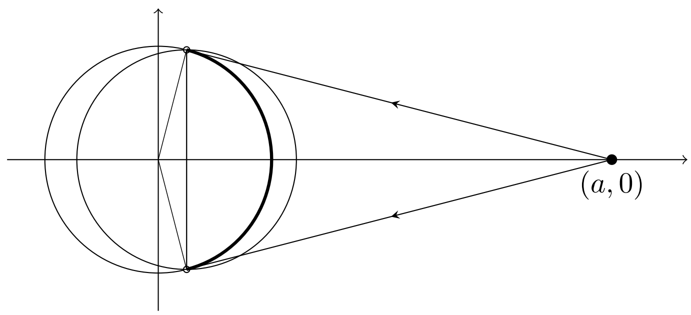

Figure 1.4: Illumination by point light source

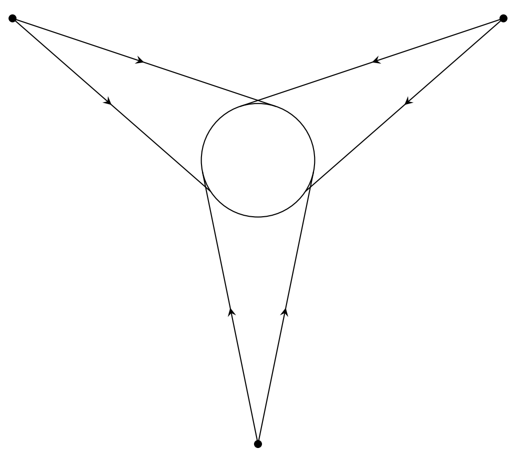

Figure 1.5: $I'(K) = 3$

1.2 Results in $\mathbb{R}^3$

In this subsection, we will present an overview of the known results when $n = 3$. Up until now, the best result for the upper bound on H_3 is given by Papadoperakis.

Theorem 1.2.1. [26] $H_3 \leqslant 16$.

Recall that the main conjecture in 3-dimensional case is $H_3 \leqslant 8$. While still unknown in the general case, now it is confirmed for certain classes of convex bodies. If a convex body $K \subset \mathbb{R}^n$ has a point of symmetry, then K is said to be centrally symmetric. Lassak showed the following result:

Theorem 1.2.2. [19] *For any centrally symmetric convex body $K \subset \mathbb{R}^3$, we have $H(K) \leqslant 8$.*

Dekster considered the case of planar symmetry. A convex body K is symmetric about a plane if K is unchanged after applying the reflection about some plane.

Theorem 1.2.3. [11] *If a convex body $K \subset \mathbb{R}^3$ is symmetric about a plane, then $H(K) \leqslant 8$.*

In the following theorem, Bezdek confirmed the conjecture for a class of convex polyhedra of affine symmetry.

Theorem 1.2.4. [1] *Let $K \subset \mathbb{R}^3$ be a convex polyhedron such that there exists a non-singular affine transform A of $\mathbb{R}^3$ different from the identity with the property $A(K) = K$. Then $H(K) \leqslant 8$.*

Here is an example to illustrate the existence of a polyhedron for which Theorem 1.2.4 is applicable while neither Theorem 1.2.2 nor Theorem 1.2.3 works. Set the affine transform $A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} \\ 3 & 0 & 0 \end{bmatrix}$, and $K \subset \mathbb{R}^3$ can be the convex hull of $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$. It is not hard to see that K is neither centrally symmetric nor symmetric about a plane, and we have $A(K) = K$.

A convex body $K \subset \mathbb{R}^3$ is said to have constant width if the distances between any pair of parallel supporting hyperplanes of K are always the same. There is a sharper upper bound of $H(K)$ given by Lassak in the following theorem:

Theorem 1.2.5. [21] *If a convex body $K \subset \mathbb{R}^3$ has constant width, then $H(K) \leqslant 6$.*

There are other partial results in $\mathbb{R}^3$ which are more complicated to state. For example in [3], Bezdek had a conjecture for convex bodies with constant width. In [6], Boltyanski showed $H(K) \leqslant 6$ for convex body K satisfying $md(K) = 2$ where md is a certain geometric characteristics of K. In [2], Bezdek showed $H(K) = 4$ where K is the intersection of the closed unit balls in $\mathbb{R}^3$ which satisfy certain conditions.

1.3 Results in high dimensions

The best-known upper bound of H_n for large n follows from the results of Rogers and Shephard [29], [30]:

$$H_n \leq \binom{2n}{n} n(\ln n + \ln \ln n + 5), \tag{1.1}$$

where some slight improvements of the constant 5 are possible for sufficiently large n, see [3] [13]. Recently, a new asymptotic upper bound was obtained by Huang, Slomka, Tkocz and Vritsiou in [18]:

$$H_n \leq \binom{2n}{n} e^{-c\sqrt{n}}, \tag{1.2}$$

where $c > 0$ and $n \geqslant 3$, c is still unspecified.

Remark 1.3.1. *In the above two upper bounds (1.1) and (1.2), $\binom{2n}{n}$ has the order of 4^n as n becomes large, while in Covering Conjecture the upper bound is 2^n (1.1.1).*

For centrally symmetric convex bodies, a much better asymptotic estimate is known, which is much closer to the conjectured value of 2^n when n is large.

Theorem 1.3.2. [3] [29] [30] *For any centrally symmetric convex body $K \subset \mathbb{R}^n$ where $n \geqslant 2$, we have*

$$H(K) \leqslant 2^n \cdot n \cdot (\ln n + \ln \ln n + 5).$$

For a convex body $K \subset \mathbb{R}^n$, a point $\boldsymbol{a} \in \mathrm{bd}K$ is said to be regular if there is only one supporting hyperplane of K passing through $\boldsymbol{a}$, and K is said to be smooth if any point $\boldsymbol{x} \in \mathrm{bd}K$ is regular. The following related conclusion is given by Bolytanski, which provides an exact value for $H(K)$.

Theorem 1.3.3. [6] *For a smooth convex body $K \subset \mathbb{R}^n$, the equality $H(K) = n+1$ holds.*

In [7], Boltyanski and Martini obtained a more general conclusion:

Theorem 1.3.4. *For a compact convex body $K \subset \mathbb{R}^n$, if the number of non-regular boundary points does not exceed n, then we have $H(K) = n+1$.*

A convex polytope $B \subset \mathbb{R}^n$ is called a belt polytope [3] if for each 2-face and any edge included in that 2-face, we can find an opposite parallel edge in the same 2-face. Martini proved the upper bound of $H(B)$ in [25]:

Theorem 1.3.5. *For any belt polytope $B \subset \mathbb{R}^n$ which is not a paralleltope, we have $H(B) \leqslant 3 \cdot 2^{n-2}$.*

There are also other related results for high dimensional convex bodies satisfying certain additional conditions. In [24], Martini showed the upper bound $H(K)$ where K is a belt polytope. In particular, for convex bodies of constant width, the conjecture has been confirmed, see [4], [9] and [31].

1.4 Results in $\mathbb{R}^n$, $n = 4, 5, 6$

Lassak [20] found an upper bound for H_n as

$$H_n \leq (n+1)n^{n-1} - (n-1)(n-2)^{n-1},$$

which is better than the estimate (1.1) given by Rogers and Shepard under the condition $n \leqslant 5$.

Recently, Prymak and Shepelska obtained a significant improvement for the upper bound of H_n where $n = 4, 5, 6$.

Theorem 1.4.1. [27] *$H_4 \leqslant 96$, $H_5 \leqslant 1091$, $H_6 \leqslant 15373$.*

The results in Theorem 1.4.1 are two-thirds less than the previous best results. Let us provide a table to compare the different upper bounds on H_n in dimensions $n = 4, 5, 6$, see Table 1.1.

1.5 Purpose and result of the thesis

In this thesis, we improve the upper bound on H_n for $n = 5$ and $n = 6$ by applying certain generalizations of the method given by Prymak and Shepelska in [27], which is, in turn, a generalization of the method of Papadoperakis [26]. Thus for an arbitrary convex body K in $\mathbb{R}^5$ and $\mathbb{R}^6$, we improve the upper bounds on the number of smaller homothetic copies required to cover K, or equivalently $I(K)$. Our main result is:

n	Theorem 1.5.1	[27]	[20]	[29][30]
4		96	296	1879
5	1002	1091	3426	8927
6	14140	15373	49312	40886

Table 1.1: Upper bounds on H_n for $n = 5, 6$.

Theorem 1.5.1. *$H_5 \leqslant 1002$, $H_6 \leqslant 14140$.*

In our result, the new upper bounds on H_5 and H_6 are 8% less than the best-known results so far. For the upper bounds on H_n for $n = 5, 6$, see Table 1.1.

1.6 Structure of the thesis

In Chapter 2, we introduce several tools that will be used later in the thesis. In Chapter 3, we shall build certain constructions to find the upper bounds of H_n for $n = 5, 6$. More precisely, we show that $C_5 \leqslant 1002$ and $C_6 \leqslant 14140$, where C_5 and C_6 are certain related numbers defined in Section 3.1 for which one has $H_n \leqslant C_n$. Finally, in Chapter 4 we will show the lower bounds on C_n for $n = 5, 6$.

Chapter 2

Tools

In this chapter, we will introduce some tools which play significant roles in this thesis, i.e. the independent set algorithm in Sagemath, inequality between means, and Karamata's inequality.

2.1 Independent Set Algorithm

In this thesis, we consider simple graphs without loops or multiple edges. In a graph, an independent set [10] is a collection of vertices in which any two of them are non-adjacent. The independent set algorithm [33] in Sagemath can be used to list all independent sets of a given graph, count the cardinality of certain independent sets, find the maximal independent sets in a graph and so on. For example, in Section 3.2, when we explore the upper bound for $n = 5$, the "vertices" in a graph are the pairs of indexes of all opposite 2-dimensional faces. The "edges" represent certain relations between two vertices, for example, two vertices are in the same 3-cube. By applying the independent set algorithm, we can avoid certain edges, thus forbidding certain properties we try to avoid.

The algorithm of listing all the independent sets works in a straightforward way: Begin by listing all independent sets of size 1, which are simply vertices. For an independent set of size k, $k \geqslant 1$, try all possible ways to add a new vertex to it to make it a larger independent set of size $(k+1)$ until it can not find one more vertex to make a larger independent set. The effectiveness of the algorithm relies on its implementation, which requires special data structures in the form of multi-way tree. Each vertex in graph corresponds to a node in multi-way tree. As more vertices are added to the tree, the run time increases more than linearly. Finally, for all independent sets found in the graph, return the largest one as the output.

In the problem of finding the upper bound of H_6, we need to find an independent set of 60 vertices in a graph of 240 vertices. This specific problem turned out to be too difficult to be directly solved by the independent set algorithm. We applied the following approach which was successful: choose a random subgraph from 240 vertices and run the algorithm for that subgraph. If the maximal independents set has 60 vertices or more, we are done. Otherwise, repeat it for a different random subgraph. Such randomized approach made the problem computationally feasible.

2.2 Inequality Between Means

The second tool is the classical inequality between means, which plays an important role in Chapter 4. For $1 \leqslant r \leqslant n$ and non-negative a_i where $1 \leqslant i \leqslant n$, denote the sum of a_i by $S = \sum_{i=1}^{n} a_i$. Let $S_n^r = S_r(a_1, ..., a_n) = \sum_{1 \leqslant i_1, i_2, ..., i_r \leqslant n} a_{i_1} a_{i_2} ... a_{i_r}$ be the sum of all possible products of r distinct elements in $\{a_1, a_2, ..., a_n\}$. Denote $p_r = \frac{S_n^r}{\binom{n}{r}}$, then we have:

Theorem 2.2.1. [17, Theorem 52] *The inequality* $p_1 \geqslant p_2^{\frac{1}{2}} \geqslant p_3^{\frac{1}{3}} \geqslant ... \geqslant p_n^{\frac{1}{n}}$ *holds. The equality is achieved when all* a_i *are equal for* $1 \leqslant i \leqslant n$.

Corollary 2.2.2. *For* $1 \leqslant r \leqslant n$*, by the inequality* $p_r^{\frac{1}{r}} \leqslant p_1$*, we obtain* $S_n^r \leqslant \left(\frac{S}{n}\right)^r \cdot \binom{n}{r}$.

Here is an example for Theorem 2.2.1. Let $n = 4$, then we have:

$$p_1 = \frac{a_1 + a_2 + a_3 + a_4}{\binom{4}{1}} = \frac{S_4^1}{4},$$

$$p_2 = \frac{a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4}{\binom{4}{2}} = \frac{S_4^2}{6},$$

$$p_3 = \frac{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4}{\binom{4}{3}} = \frac{S_4^3}{4},$$

$$p_4 = \frac{a_1a_2a_3a_4}{\binom{4}{4}} = S_4^4,$$

and the following inequality holds:

$$\frac{S_4^1}{4} \geqslant \sqrt{\frac{S_4^2}{6}} \geqslant \sqrt[3]{\frac{S_4^3}{4}} \geqslant \sqrt[4]{S_4^4}.$$

The equality holds if and only if $a_1 = a_2 = a_3 = a_4$.

2.3 Karamata's inequality

The third tool is Karamata's inequality. If the n-tuples $(x_1, ..., x_n) \in \mathbb{R}^n$ and $(y_1, ..., y_n) \in \mathbb{R}^n$ satisfy:

(1) $x_1 \geqslant x_2 \geqslant ... \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant ... \geqslant y_n$,

(2) $x_1 + ... + x_i \geqslant y_1 + ... + y_i$ for every $i \in \{1, ..., n-1\}$,

(3) $x_1 + ... + x_n = y_1 + ... + y_n$,

then we say $(x_1, ..., x_n)$ majorizes $(y_1, ..., y_n)$ with the notation $(x_1, ..., x_n) \succ (y_1, ..., y_n)$.

The following Karamata's Inequality will be used later in Chapter 4, more details can be found in [12].

Theorem 2.3.1. *(Karamata's Inequality) Let f be a convex function defined on $I \subset \mathbb{R}$. If $x_1, ..., x_n, y_1, ..., y_n \in I$ satisfy $(x_1, ..., x_n) \succ (y_1, ..., y_n)$, then*

$$f(x_1) + f(x_2) + ... + f(x_n) \geqslant f(y_1) + f(y_2) + ... + f(y_n).$$

Chapter 3

Upper bounds

In this chapter, we prove Theorem 1.5.1. In Section 3.1, we introduce some notations related to high-dimensional cubes, and then present generalizations of Papadoperakis' approach into higher dimensions. In each of Section 3.2 and Section 3.3, we shall show the construction of an appropriate covering, and then use Sagemath to prove the existence of certain independent sets required to construct such covering.

3.1 Papadoperakis' approach and its generalization

For a compact convex set $K \subset \mathbb{R}^n$, let H be a supporting hyperplane of K and let $F = K \cap H$. If $F \neq \emptyset$ and $F \neq K$, then F is called a proper face of K, or a face of K in our terminology. For more details, see [22, Definition 20.1]. If the dimension of F is k where $k \in \{0, 1, ..., n\}$, then F is called a k-dimensional face of K, or simply a k-face of K. For $K = [0,1]^n$, we give a description of k-face of K in terms of coordinates: k of the coordinates range over $[0,1]$, while the remaining $(n-k)$ coordinates take a fixed value in $\{0,1\}$. For $K = [0,1]^n$, the number of k-faces is $\binom{n}{k} \cdot 2^{n-k}$. For example in $K = [0,1]^5$, each 0-face is a vertex of the $[0,1]^5$, and the total number of 0-faces is $2^5 = 32$. 1-faces are all edges in $[0,1]^5$ and the total number of 1-faces of $[0,1]^5$ is $\binom{5}{1} \cdot 2^4 = 80$. The number of 2-faces of $[0,1]^5$ is $\binom{5}{2} \cdot 2^3 = 80$, and each 2-face is shared by three 3-faces. Let $k \in \{0, 1, ..., n\}$, then in $[0,1]^n$, we call the the union of all k-faces as the k-skeleton of $[0,1]^n$ and denote it by $B_{k,n}$. Moreover, a point in $B_{k,n}$ has at most k coordinates in the interval $(0,1)$ while the rest coordinates are in $\{0,1\}$.

Now let $\mathcal{A}, \mathcal{B}$ be collections of subsets of $\mathbb{R}^n$. For $A \in \mathcal{A}$, we use $C(A, \mathcal{B})$ to denote the minimum number of the translates of elements of $\mathcal{B}$ needed to cover A, i.e. $C(A, \mathcal{B}) = \min\{m : \exists t_1, ..., t_m \in \mathbb{R}^n, \exists B_1, ..., B_m \in \mathcal{B}, A \subset \bigcup_{i=1}^{m} (B_i + t_i)\}$. We define $C(\mathcal{A}, \mathcal{B})$ as the largest value of $C(A, \mathcal{B})$ over $A \in \mathcal{A}$, so that we can always cover any element in $\mathcal{A}$ by $C(\mathcal{A}, \mathcal{B})$ translates of elements of $\mathcal{B}$, i.e. $C(\mathcal{A}, \mathcal{B}) = \max_{A \in \mathcal{A}} C(A, \mathcal{B})$. For $B \in \mathcal{B}$, define $\mathcal{A} \cup B = \big\{ A \cup B : A \in \mathcal{A} \big\}$.

Next we introduce a few more notations which may be seemingly unrelated, but will ultimately be combined in a useful definition. Let $\boldsymbol{e}_i$ be the vector whose ith entry is 1 and the remaining coordinates are 0, i.e. the i-th basic unit vector. Suppose $\boldsymbol{a}_i$ has i-th coordinate equal to 0, i.e. $\boldsymbol{a}_i$ belongs to the corresponding $(n-1)$-face of $[0,1]^n$, then clearly $\boldsymbol{a}_i + \boldsymbol{e}_i$ is on the opposite $(n-1)$-face with the i-th coordinate as 1. Define $\mathcal{A}_n$ as the collection of all unions of n pairs of such points on each opposite pair of $(n-1)$-faces, i.e.

$$\mathcal{A}_n \coloneqq \left\{ \bigcup_{i=1}^{n} \{\boldsymbol{a}_i, \boldsymbol{a}_i + \boldsymbol{e}_i\} : \boldsymbol{a}_i, \boldsymbol{a}_i + \boldsymbol{e}_i \in [0,1]^n \right\}.$$

We introduce two kinds of n-dimensional boxes with constrained size:

$$\mathcal{P}_n \coloneqq \left\{ \prod_{i=1}^{n} [x_i, x_i + \delta_i] : x_i \in \mathbb{R}, \delta_i \geqslant 0, \sum_{i=1}^{n} \delta_i < 1 \right\},$$

$$\mathcal{P}_n^* \coloneqq \left\{ \prod_{i=1}^{n} [x_i, x_i + \delta_i] : x_i \in \mathbb{R}, \delta_i \geqslant 0, \sum_{i=1}^{n} \delta_i \leqslant 1 \right\}.$$

Note that some of these boxes may be degenerate as $\delta_i = 0$ is allowed. Using the above notations, the covering number C_n is defined as

$$C_n \coloneqq C\left(\mathcal{A}_n \cup B_{n-2,n}, \mathcal{P}_n\right).$$

The covering number C_n is the minimum number of boxes from $\mathcal{P}_n$ needed to cover the $(n-2)$-skeleton of $[0,1]^n$ and an arbitrary configuration of n pairs of points from $\mathcal{A}_n$. It turns out that this covering number C_n can be used to obtain an upper bound of H_n.

Theorem 3.1.1. *$H_n \leqslant C_n$.*

This approach is due to Papadoperakis [26], where he used it for the case $n = 3$. Prymak and Shepelska [28] observed its applicability to $n \geqslant 4$. Here we followed the notations of [28]. Now let us provide a brief outline of the proof of Theorem 3.1.1. Consider the smallest parallelotope P containing K. Due to the affine invariance of the problem, without loss of generality, we can assume that $P = [0,1]^n$. Let $F_1, \dots, F_{2n}$ be the $(n-1)$-faces of P. Since P has the smallest volume among paralleltopes containing K, then $F_i \cap P \neq \emptyset$ for each $i = 1, 2, ..., 2n$. Furthermore, [26, Lemma 3] shows that using the minimality of the volume, $\text{bd}K \cap P$ contains some configuration A_n of $2n$ points from the collection $\mathcal{A}_n$. [26, Lemma 4] shows that if there exists $P_1, P_2, ..., P_m \in \mathcal{P}_n$ such that

$$\left(A_n \cup \left(\bigcup_{i=1}^{2n} \text{relbd} F_i\right)\right) \subset \bigcup_{j=1}^{m} P_j,$$

then $H(K) \leqslant m$. The union of all relative boundaries of F_i of $[0,1]^n$ is $B_{n-2,n}$, which implies $H_n \leqslant C_n$.

Recently, Prymak and Shepelska found an upper bound for C_n.

Theorem 3.1.2. [28] *For any $n \geqslant 5$, we have:*

$$C_n \leqslant 2n(n-1)(n-2)^{n-2} + 2n + 1.$$

By a delicate construction, they found a much stronger upper bound for C_4 and showed this is almost the best possible result through such a method.

Theorem 3.1.3. [28] $95 \leqslant C_4 \leqslant 96$.

In the following two sections, we obtain new upper bounds on C_5 and C_6, thus proving the upper bounds in Theorem 1.5.1.

3.2 Upper Bound for $n = 5$

In Subsection 3.2.1, we show a construction of covering $B_{3,5}$ using 992 boxes from $\mathcal{P}_5$. This construction relies on the existence of certain pairs of 2-faces of 5-cube which is shown in Subsection 3.2.2. In turn, this is done by reducing the problem to the existence of certain independent set in a graph (see Section 2.1) which was confirmed using computer assistance. Our main result Theorem 1.5.1 follows directly from Corollary 3.2.5.

3.2.1 The Covering Construction Using Boxes in $\mathcal{P}_5$

In this section, we work with faces of several dimensions in $[0,1]^5$. Recall that in $[0,1]^5$, there are eighty 2-faces and forty 3-faces. Let $S_1, ..., S_{80}$ be an enumeration of all 2-faces of $[0,1]^5$. Next we give some definitions and notations that will be used later. In what follows, for a box $[a_1, b_1] \times ... \times [a_n, b_n]$, we write its dimensions as a product $(b_1 - a_1) \cdot ... \cdot (b_n - a_n)$, and we power it if some of the lengths are equal. For example, the dimensions of $[0,1]^n$ can be written as 1^n.

Definition 3.2.1. *If there exists a* 3*-face* C *containing both* S_i *and* S_j *while* $S_i \cap S_j = \emptyset$*, then* (S_i, S_j) *is called a pair of opposite* 2*-faces.*

Definition 3.2.2. *Two pairs of opposite* 2*-faces* (S_i, S_j) *and* $(S_{i'}, S_{j'})$ *are non-overlapping if all four* 2*-faces* $\{S_i, S_j, S_{i'}, S_{j'}\}$ *are distinct.*

Before showing the construction, let us introduce a useful result in Lemma 3.2.3 first, which will be applied in the construction in Theorem 3.2.4.

Lemma 3.2.3. *It is possible to choose exactly one pair of opposite* 2*-faces in each* 3*-face so that:*

(a) any two of the forty chosen pairs are non-overlapping;

(b) each 2*-face belongs to exactly one chosen pair.*

We will show the proof of Lemma 3.2.3 in Subsection 3.2.2. In the next theorem, we show the construction for our new estimate.

Theorem 3.2.4. $C(B_{3,5}, \mathcal{P}_5) \leqslant 992$*.*

Proof. Here we show a construction of the covering of $B_{3,5}$ by 992 boxes from $\mathcal{P}_5$. Let $0 \leqslant a \leqslant b \leqslant 1$. We start by covering the vertices of $[0,1]^n$. Consider an arbitrary vertex of $[0,1]^5$, without loss of generality say $(0,0,0,0,0)$. For sufficiently small $\varepsilon > 0$, we use the box $[0, \frac{1}{6} + \varepsilon]^5$ to cover $(0,0,0,0,0)$. If $\varepsilon < \frac{1}{30}$, this box belongs to $\mathcal{P}_5$. Then we cover all other vertices of $[0,1]^5$ in the same way using the boxes of dimensions $(\frac{1}{6}+\varepsilon)^5$ containing the vertex and located inside $[0,1]^5$. In total, we have used 32 boxes so far.

Next, we proceed to covering of the edges of $[0,1]^5$. Consider the edge $[(0,0,0,0,0),(1,0,0,0,0)]$, the uncovered portion is from $(\frac{1}{6}+\varepsilon,0,0,0,0)$ to $(\frac{5}{6}-\varepsilon,0,0,0,0)$. We use the box $[\frac{1}{6}+\varepsilon,\frac{1}{2}]\times[0,\frac{1}{6}+\frac{\varepsilon}{5}]^4$ to cover the segment $[(\frac{1}{6}+\varepsilon,0,0,0,0),\ (\frac{1}{2},0,0,0,0)]$, and similarly use the box $[\frac{1}{2},\frac{5}{6}-\varepsilon]\times[0,\frac{1}{6}+\frac{\varepsilon}{5}]^4$ to cover the segment $[(\frac{1}{2},0,0,0,0),(\frac{5}{6}-\varepsilon,0,0,0,0)]$. For $\varepsilon\in(0,\frac{1}{3}]$, these two boxes with dimensions $(\frac{1}{3}-\varepsilon)\cdot(\frac{1}{6}+\frac{\varepsilon}{5})^4$ belong to $\mathcal{P}_5$. Since all the vertices have already been covered, the considered edge $[(0,0,0,0,0),(1,0,0,0,0)]$ is completely covered by the two boxes of dimensions $(\frac{1}{3}-\varepsilon)\cdot(\frac{1}{6}+\frac{\varepsilon}{5})^4$ we just constructed and the two boxes corresponding to vertices. Now we use the boxes of dimensions $(\frac{1}{3}-\varepsilon)\cdot(\frac{1}{6}+\frac{\varepsilon}{5})^4$ to cover the remaining 79 edges in the same manner. Therefore, covering all edges needs $2\cdot 80=160$ boxes from $\mathcal{P}_5$ in total. Up to now, we have constructed $32+160=192$ boxes.

Next, we turn to cover the 2-faces of $[0,1]^5$. As is shown in Figure 3.1, for each 2-face, the uncovered region is contained in a square of dimensions $(\frac{2}{3}-\frac{2}{5}\varepsilon)^2$. For example, the uncovered region of the 2-face $[0,1]^2\times\{0\}^3$ is contained in the square $R_1=[\frac{1}{6}+\frac{\varepsilon}{5},\frac{5}{6}-\frac{\varepsilon}{5}]^2\times\{0\}^3$. Let $\mathcal{M}_5$ be the 40 pairs of 2-faces provided by Lemma 3.2.3. Note that for each 2-face S_i, there exists exactly one S_j such that (S_i,S_j) is an element of $\mathcal{M}_5$. For each S_i, we denote the 3-face containing (S_i,S_j) as A_i. There are two more 3-faces containing

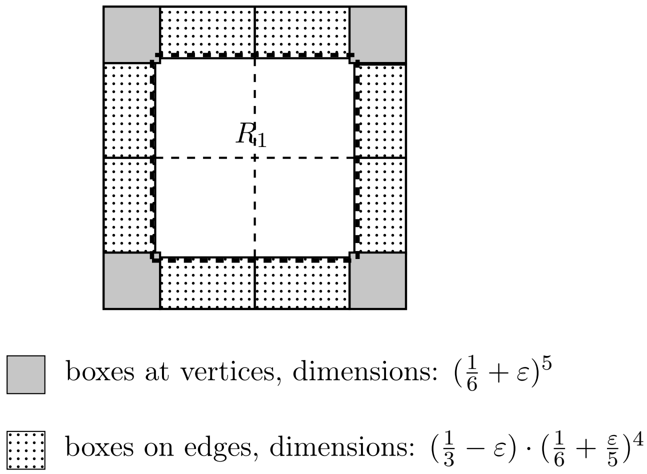

Figure 3.1: Covering of S_1

S_i, which we denote by B_i and C_i respectively. Without loss of generality, by applying a symmetry if necessary, we consider a 2-face $S_1 = [0,1]^2 \times \{0\}^3$ and assume $A_1 = [0,1]^2 \times \{0\}^2 \times [0,1]$. We are going to cover the square $[\frac{1}{6}+\frac{\varepsilon}{5}, \frac{5}{6}-\frac{\varepsilon}{5}]^2 \times \{0\}^3 \subset S_1$. Passing through the centre of S_1, i.e. $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$, we sketch two perpendicular segments which divide the square R_1 into 4 congruent smaller squares (see Figure 3.1). To describe how to cover these 4 smaller squares, consider one of them, for example $R_1' = [\frac{1}{6}+\frac{\varepsilon}{5}, \frac{1}{2}]^2 \times \{0\}^3$. We use the box $D_1 = [\frac{1}{6}+\frac{\varepsilon}{5}, \frac{1}{2}]^2 \times [0, \frac{1}{6}+\frac{\varepsilon}{10}]^2 \times \{0\}$ to cover R_1'. Note that this box also covers certain parts of B_1 and C_1. Each of these parts is a box of dimensions $(\frac{1}{3}-\frac{\varepsilon}{5})^2 \cdot (\frac{1}{6}+\frac{\varepsilon}{10}) \cdot 0^2$. Note that D_1 does not cover interior of A_1. If $\varepsilon \in (0, \frac{5}{3})$, then the boxes of dimensions $(\frac{1}{3}-\frac{\varepsilon}{5})^2 \cdot (\frac{1}{6}+\frac{\varepsilon}{10})^2 \cdot 0$ belong to $\mathcal{P}_5$. Apply this procedure to the other 3 smaller squares in $[\frac{1}{6}+\frac{\varepsilon}{5}, \frac{5}{6}-\frac{\varepsilon}{5}]^2 \times \{0\}^3$ using the boxes of dimensions $(\frac{1}{3} - \frac{\varepsilon}{5})^2 \cdot (\frac{1}{6} + \frac{\varepsilon}{10})^2 \cdot 0$, then the square $[\frac{1}{6} + \frac{\varepsilon}{5}, \frac{5}{6} - \frac{\varepsilon}{5}]^2 \times \{0\}^3$ is covered by 4 boxes, so the 2-face S_1 is covered. Likewise, we cover the remaining 2-faces S_i where $i = 2, ..., 80$. Hence, covering all 2-faces requires $4 \cdot 80 = 320$ boxes of dimensions $(\frac{1}{3} - \frac{\varepsilon}{5})^2 \cdot (\frac{1}{6} + \frac{\varepsilon}{10})^2 \cdot 0$ in total. Until now, we have used $192 + 320 = 512$ boxes.

Finally, we focus on covering the forty 3-faces defining $B_{3,5}$. Without loss of generality, consider the 3-face $A_1 = [0, 1]^2 \times \{0\}^2 \times [0, 1]$, which is located in a 3-dimensional space generated by the 1st, 2nd, and 5th coordinates. Recall that for $S_1 = [0, 1]^2 \times \{0\}^2 \times \{0\}$ and $S_1' = [0, 1]^2 \times \{0\}^2 \times \{1\}$, (S_1, S_1') is an element of $\mathcal{M}_5$. We consider the cross-sections of the cube $[0, 1]^3 \times \{0\} \times [0, 1]$, where the 5th coordinate x_5 is a fixed number in $[0, 1]$. According to the value of x_5, all such cross-sections can be sorted into 3 cases, see Figure 3.2. By observation, the uncovered portion of A_1 is contained in the 3-dimensional rectangular solid $E_1 = [\frac{1}{6} + \frac{\varepsilon}{10}, \frac{5}{6} - \frac{\varepsilon}{10}]^2 \times \{0\}^2 \times [0, 1]$. Note that the largest width of the uncovered portion along the 1st and 2nd coordinates is $(\frac{2}{3} - \frac{\varepsilon}{5})$. We denote the base of E_1 as the square $Q_1 = [\frac{1}{6} + \frac{\varepsilon}{10}, \frac{5}{6} - \frac{\varepsilon}{10}]^2$ (see Figure 3.3) with dimensions $(\frac{2}{3} - \frac{\varepsilon}{5})^2$, while the height of E_1 equals to 1. Then split E_1 into 2 equal parts along the 1st and 2nd coordinate, and into 3 equal parts along the 5th coordinate. This produces $2 \cdot 2 \cdot 3 = 12$ equal boxes with each having dimensions $(\frac{1}{3} - \frac{\varepsilon}{10})^2 \cdot \frac{1}{3}$. It is evident that all such boxes belong to $\mathcal{P}_5$, and they will cover E_1 and hence A_1 is completely covered. Then we cover the remaining 39 3-faces in the same way. Overall, this requires $12 \cdot 40 = 480$ boxes

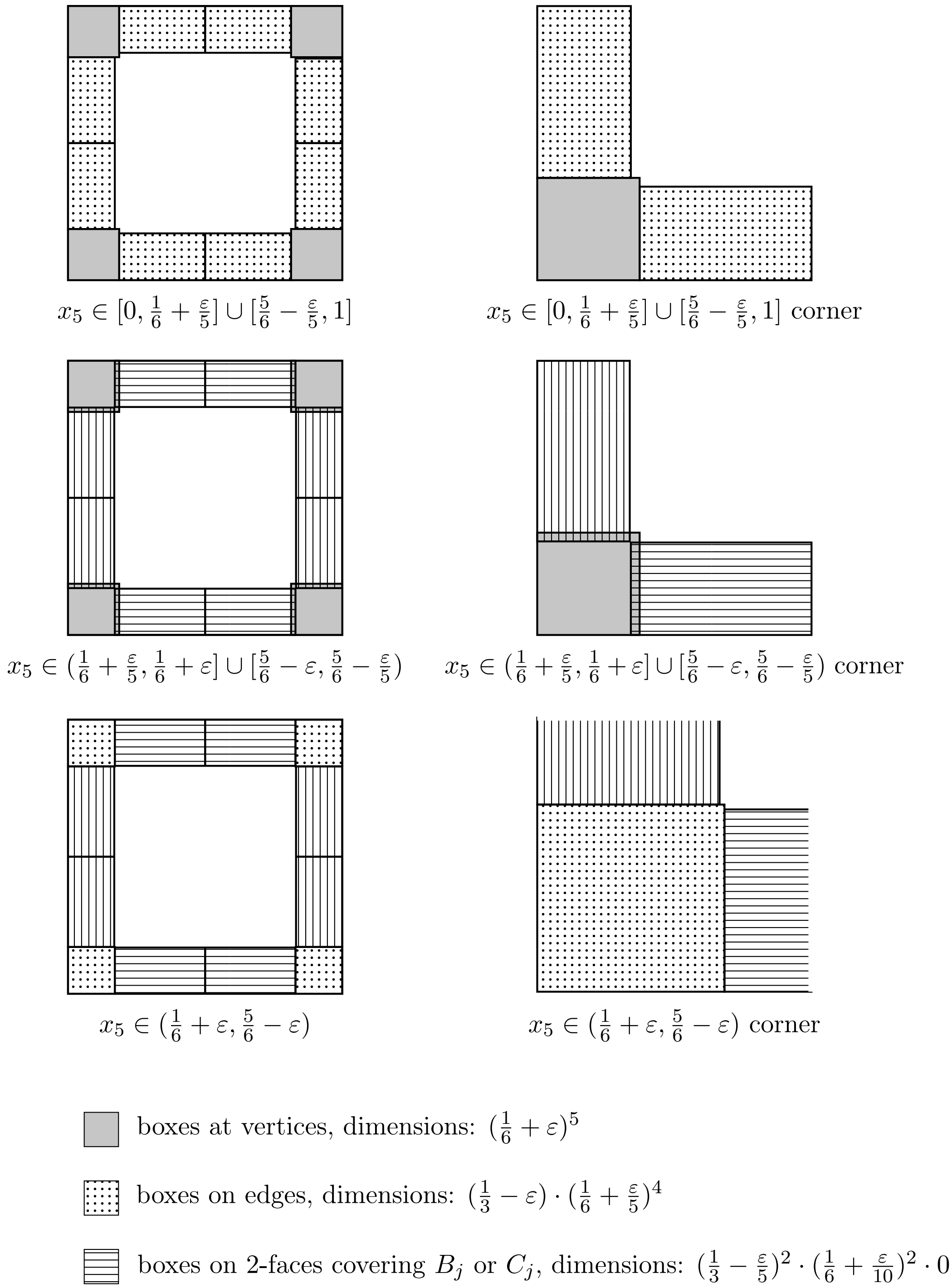

Figure 3.2: Cross-sections of $[0,1]^3 \times \{0\} \times [0,1]$

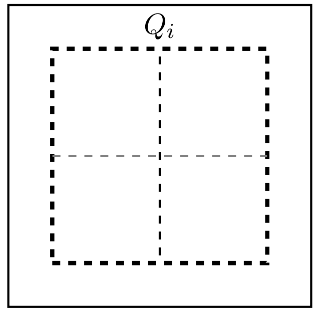

Figure 3.3: Cross-sections of E_1

of dimensions $(\frac{1}{3} - \frac{\varepsilon}{10})^2 \cdot \frac{1}{3} \cdot 0^2$.

In summary, $B_{3,5}$ can be covered by $512+480 = 992$ boxes from $\mathcal{P}_5$ following the described construction with $\varepsilon \in (0, \frac{1}{30})$. □

Based on Theorem 3.2.4, the new upper bound of H_5 follows easily:

Corollary 3.2.5. *$H_5 \leqslant C_5 \leqslant 1002$.*

Proof. Since $C_5 = C(\mathcal{A}_5 \cup B_{5,3}, \mathcal{P}_5)$, any configuration $A_5 \in \mathcal{A}_5$ contains 10 points, and each point can be covered by an element of $\mathcal{P}_5$, we get $C_5 \leqslant C(B_{3,5}, \mathcal{P}_5) + 10 \leqslant 1002$ and hence $H_5 \leqslant 1002$. □

3.2.2 Appropriate Selection of Non-overlapping Pairs of 2-faces in $[0,1]^5$

In this subsection, we show how to deduce Lemma 3.2.3 from a graph theoretical problem of the existence of an independent set of 40 vertices in given certain graph G_5. In the end, we give a description of the Sagemath code in Appendix A used to show the existence of such an independent set in G_5.

Now we describe the vertex set V_5 and edge set E_5 of $G_5 = (V_5, E_5)$. In G_5, V_5 consists of all $3 \cdot 40 = 120$ pairs of opposite 2-faces in $[0,1]^5$ (see Definition 3.2.1). Two vertices in G_5 are adjacent, if the corresponding pairs of opposite 2-faces satisfy any of the following conditions:

(i) the pairs are located in the same 3-face of $[0,1]^5$;

(ii) the pairs are located in different 3-faces but share a 2-face in common.

Recall that an independent set in graph G_5 is a collection of vertices, where any two vertices are non-adjacent. The existence of sufficiently large independent set in G_5 can be used to prove Lemma 3.2.3.

Proposition 3.2.6. *If there exists an independent set of* 40 *vertices in* G_5*, then Lemma 3.2.3 holds.*

Proof. Let $U_5 \subset V_5$ be an independent set in G_5 with $|U_5| = 40$. Let $\mathcal{M}_5$ be the 40 pairs of opposite 2-faces corresponding to the vertices in U_5. We will show that they justify the conditions of Lemma 3.2.3. Due to the adjacency condition (i), no two pairs from $\mathcal{M}_5$ can belong to the same 3-face. Since there are forty 3-faces, each 3-face contains exactly one such pair. (This also shows that G_5 does not have an independent set of more than 40 vertices.) As a result of the non-validity of the adjacency condition (ii), any two pairs from $\mathcal{M}_5$ are non-overlapping (see Definition 3.2.2), so the condition (a) of Lemma 3.2.3 is proved. Moreover, by the non-validity of adjacency condition (ii), the 40 pairs from $\mathcal{M}_5$ contain eighty distinct 2-faces, while $[0,1]^5$ has eighty 2-faces in total. Hence, each 2-face of $[0,1]^5$ must belong to exactly one pair from $\mathcal{M}_5$. This establishes condition (b) of Lemma 3.2.3. □

We use computer assistance to generate graph G_5 and compute the cardinality of the largest independent set in G_5 (see Section 2.1). The Sagemath [34] code we used is given in Appendix A. Now let us provide some remarks about the implementation of this code. The running time of our script is under a second on a modern personal computer.

Remark 3.2.7. *We use the centre of a* 2*-face to represent this* 2*-face. Then we build a list of centres of all* 2*-faces in* $[0,1]^5$ *and denote it by* F *with* $|F| = 80$. *The elements of* F *have* 2 *of the* 5 *coordinates equal to* $\frac{1}{2}$*, while the remaining* 3 *coordinates are either* 0 *or* 1*. Each time we need to refer to a* 2*-face of* $[0,1]^5$*, we only use the index of its centre in the list* F*. For the corresponding code, see the lines 1-18.*

Two 2*-faces are opposite (see Definition 3.2.1) if and only if the corresponding centres differ in exactly* 1 *coordinate. See the code in lines 19-27.*

Then we generate W*, a list of pairs* (i, j) *with* $i < j$ *of indexes of elements of* F *which correspond to opposite* 2*-faces.* W *is in one-to-one correspondence with the vertex set* V_5 *of our graph. See the code in lines 28-32.*

Next, we check if two elements of W correspond to a pair of adjacent vertices in V_5. Adjacency condition (ii) is verified in lines 36, 37. Adjacency condition (i) is verified in lines 38-41. Observe that two pairs of opposite 2-faces belong to the same 3-face if and only if they have the same midpoints of the segment joining their centres (which would coincide with the centre of the 3-face).

Finally, we generate the graph G_5 using its adjacency matrix, and use the independent set algorithm to find $U_5 \subset V_5$. The algorithm outputs all elements of U_5 together with the cardinality of U_5. See the code in lines 42-45.

3.3 Upper bound for $n = 6$

In this section, we follow and extend the methods used in the previous Section 3.2. In Subsection 3.3.1, we will present the structure of the covering of $B_{4,6}$ by 14128 boxes from $\mathcal{P}_6$. In Subsection 3.3.2, we will show the construction depending on the existence of certain pairs of opposite 3-faces in $[0,1]^6$. Likewise, this covering problem can be converted into the existence of certain independent set (see Section 2.1) in a graph, which has been verified by computer. Our main result for H_6 in Theorem 1.5.1 comes straight from Corollary 3.3.5.

3.3.1 The Covering Construction Using Boxes in $\mathcal{P}_6$

In this subsection, we focus on the k-faces of $[0,1]^6$ where $k \in \{0, 1, ..., 4\}$, i.e. 64 vertices, 192 edges, 240 2-faces, 160 3-faces and 60 4-faces. Write all 3-faces of $[0,1]^6$ as $T_1, ..., T_{160}$. In this section, we will use k-face to denote a k-face of $[0,1]^6$.

Definition 3.3.1. *(T_i, T_j) is a pair of opposite 3-faces if T_i and T_j are in the same 4-face while $T_i \cap T_j = \emptyset$.*

Definition 3.3.2. *Two pairs of opposite 3-faces (T_i, T_j) and $(T_{i'}, T_{j'})$ are non-overlapping if all four 3-faces $\{T_i, T_j, T_{i'}, T_{j'}\}$ are distinct.*

Next, we introduce a choice of certain pairs of non-overlapping 4-faces before the construction of covering, which will be useful in the latter construction in Theorem 3.3.4.

Lemma 3.3.3. *One can choose exactly one pair of opposite 3-faces in each 4-face of $[0,1]^6$ so that any two of the 60 chosen pairs are non-overlapping.*

To make a comment, unlike Lemma 3.2.3 in 5-dimensional case, not all 3-faces of $[0,1]^6$ will be chosen in Lemma 3.3.3. Denote the set of these 60 pairs of non-overlapping 3-faces of $[0,1]^6$ by $\mathcal{M}_6$. We will show the validity of Lemma 3.3.3 in Subsection 3.2.2. Now we are going to show the construction of covering $B_{4,6}$ in the following Theorem 3.3.4.

Theorem 3.3.4. *$C(B_{4,6}, \mathcal{P}_6) \leqslant 14128$.*

Proof. Here we describe the structure of the covering of $B_{4,6}$ using boxes from $\mathcal{P}_6$. To cover a vertex $(0,0,0,0,0,0)$, we use the box $[0, \frac{1}{8}+\varepsilon]^6$ where $\varepsilon < \frac{1}{24}$ so that $[0, \frac{1}{8}+\varepsilon]^6 \in \mathcal{P}_6$. Use 63 boxes of dimensions $(\frac{1}{8}+\varepsilon)^6$ to cover the remaining 63 vertices. Thus we need 64 boxes from $\mathcal{P}_6$ of dimensions $(\frac{1}{8}+\varepsilon)^6$ in total to get all vertices of $[0,1]^6$ covered.

Next, we move on to covering all edges of $[0,1]^6$. Consider a specific edge, say $[(0,0,0,0,0,0),(1,0,0,0,0,0)]$. The uncovered portion is from $(\frac{1}{8}+\varepsilon,0,0,0,0,0)$ to $(\frac{7}{8}-\varepsilon,0,0,0,0,0)$. We use the boxes $[\frac{1}{8}+\varepsilon,\frac{1}{2}]\times[0,\frac{1}{8}+\frac{\varepsilon}{6}]^5$ and $[\frac{1}{2},\frac{7}{8}-\varepsilon]\times[0,\frac{1}{8}+\frac{\varepsilon}{6}]^5$ to cover the segments $[(\frac{1}{8}+\varepsilon,0,0,0,0,0),(\frac{1}{2},0,0,0,0,0)]$ and $[(\frac{1}{2},0,0,0,0,0),(\frac{7}{8}-\varepsilon,0,0,0,0,0)]$. For $\varepsilon\in(0,\frac{3}{8}]$, these two boxes with dimensions $(\frac{3}{8}-\varepsilon)\cdot(\frac{1}{8}+\frac{\varepsilon}{6})^5$ belong to $\mathcal{P}_6$. At present, the edge $[(0,0,0,0,0,0),(1,0,0,0,0,0)]$ has been covered by two boxes of dimensions $(\frac{3}{8}-\varepsilon)\cdot(\frac{1}{8}+\frac{\varepsilon}{6})^5$ corresponding to the edges and two boxes corresponding to the vertices. In this way, to cover all 192 edges, we need $2\cdot 192 = 384$ boxes of dimensions $(\frac{3}{8}-\varepsilon)\cdot(\frac{1}{8}+\frac{\varepsilon}{6})^5$ from $\mathcal{P}_6$. Up until the present moment, we have chosen $64+384=448$ boxes.

Next we are going to cover the 2-faces of $[0,1]^6$. For each 2-face, the uncovered region is contained in a square of dimensions $(\frac{3}{4}-\frac{\varepsilon}{3})^2\cdot 0^4$. For example, the uncovered region of the 2-face $S_1=[0,1]^2\times\{0\}^4$ is contained in the square $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{7}{8}-\frac{\varepsilon}{6}]^2\times\{0\}^4$. We divide the square $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{7}{8}-\frac{\varepsilon}{6}]^2\times\{0\}^4$ into 3 equal parts along the 1st and 2nd coordinates. Then the square is divided into $3\cdot 3=9$ congruent smaller squares of dimensions $(\frac{1}{4}-\frac{\varepsilon}{9})^2$. Let us describe how to cover these 9 smaller squares by considering one smaller square $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{3}{8}+\frac{\varepsilon}{18}]^2\times\{0\}^4$. We use the box $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{3}{8}+\frac{\varepsilon}{18}]^2\times[0,\frac{1}{8}+\frac{\varepsilon}{20}]^4$ to cover it. If $\varepsilon\in(0,\frac{9}{4})$, then the boxes with dimensions $(\frac{1}{4}-\frac{\varepsilon}{9})^2\cdot(\frac{1}{8}+\frac{\varepsilon}{20})^4$ belong to $\mathcal{P}_6$. Repeat this procedure on the other 8 smaller squares in $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{7}{8}-\frac{\varepsilon}{6}]^2\times\{0\}^4$ using the boxes of dimensions $(\frac{1}{4}-\frac{\varepsilon}{9})^2\cdot(\frac{1}{8}+\frac{\varepsilon}{20})^4$ likewise. Hence the square $[\frac{1}{8}+\frac{\varepsilon}{6},\frac{7}{8}-\frac{\varepsilon}{6}]^2\times\{0\}^4$ needs 9 boxes to be covered altogether. Then, we cover the remaining 2-faces in the same manner. To sum up, covering all 2-faces needs $9\cdot 240=2160$ boxes of dimensions $(\frac{1}{4}-\frac{\varepsilon}{9})^2\cdot(\frac{1}{8}+\frac{\varepsilon}{20})^4$. Up to now, we have already used $448+2160=2608$ boxes.

At last, we describe the construction for the uncovered portions in 4-faces. Take a 4-face $L_1=[0,1]^4\times\{0\}^2$ for example. Right now in L_1, the uncovered portion is contained in the union of the following two kinds of 4-dimensional boxes:

1) "Central" box $[\frac{1}{8}+\frac{\varepsilon}{40},\frac{7}{8}-\frac{\varepsilon}{40}]^4\times\{0\}^2$;

2) "Flank" boxes of dimensions $(\frac{3}{4}-\frac{\varepsilon}{10})^3\cdot(\frac{1}{8}+\frac{\varepsilon}{40})$. One 3-face of a flank box is on the boundary of $[0,1]^6$, while the opposite 3-face is contained in a 3-face of the central box. For example, the flank box $[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^3\times[0,\frac{1}{8}+\frac{\varepsilon}{40}]\times\{0\}^2\subset L_1$ has a 3-face $[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^3\times\{\frac{1}{8}+\frac{\varepsilon}{40}\}\times\{0\}^2$ which is contained in a 3-face of the

central box $[\frac{1}{8}+\frac{\varepsilon}{40},\frac{7}{8}-\frac{\varepsilon}{40}]^4\times\{0\}^2\subset L_1$, while the 3-face $[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^3\times\{0\}^3$ is on the boundary of $[0,1]^6$. Each 4-face contains 8 flank boxes.

Since $[0,1]^6$ contains sixty 4-faces, there are $60\cdot 8=480$ flank boxes in total. There are 160 3-faces in $[0,1]^6$, each of which is shared by 3 flank boxes. Now we classify 160 3-faces into 2 types: the first type is the collection I with $60\cdot 2=120$ 3-faces in $\mathcal{M}_6$, while the second type J consists of the remaining $160-120=40$ 3-faces which are not in $\mathcal{M}_6$. Next, we cover all flank boxes in different ways according to the type of the 3-faces it contains.

To describe the covering for the forty 3-faces in J, take the 3-face $T_1=[0,1]\times\{0\}\times[0,1]^2\times\{0\}^2\subset L_1$ for example. The uncovered part of T_1 is contained in the 3-dimensional box $T_1^*=[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]\times\{0\}\times[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^2\times\{0\}^2$. There are 3 flank boxes sharing the 3-dimensional T_1^* at the same time, i.e. $F_{1,1}=[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]\times[0,\frac{1}{8}+\frac{\varepsilon}{40}]\times[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^2\times\{0\}^2\subset L_1$, $F_{1,2}=[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]\times\{0\}\times[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^2\times[0,\frac{1}{8}+\frac{\varepsilon}{40}]\times\{0\}$, and $F_{1,3}=[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]\times\{0\}\times[\frac{1}{8}+\frac{\varepsilon}{20},\frac{7}{8}-\frac{\varepsilon}{20}]^2\times\{0\}\times[0,\frac{1}{8}+\frac{\varepsilon}{40}]$. The 3-dimensional T_1^* can also be viewed as a base of $F_{1,1},F_{1,2},F_{1,3}$. We fix an arbitrary flank box in $F_{1,1},F_{1,2},F_{1,3}$, without loss of generality, say $F_{1,1}\subset L_1$. To cover $F_{1,1}$, we divide the 3-dimensional base T_1^* into $3\cdot 3\cdot 2=18$ boxes of dimensions $(\frac{1}{4}-\frac{\varepsilon}{30})^2\cdot(\frac{3}{8}-\frac{\varepsilon}{20})$ by cutting each coordinate of T_1^* into 3 equal parts along the 1st and 3rd coordinates and 2 equal parts along the 4th coordinate. Then $F_{1,1}$ can be covered by 18 boxes of dimensions $(\frac{1}{4}-\frac{\varepsilon}{30})^2\cdot(\frac{3}{8}-\frac{\varepsilon}{20})\cdot(\frac{1}{8}+\frac{\varepsilon}{40})$

which belong to $\mathcal{P}_6$. For example, one of such 18 boxes covering $F_{1,1}$ is the box $[\frac{1}{8}+\frac{\varepsilon}{20}, \frac{3}{8}+\frac{\varepsilon}{60}] \times [0, \frac{1}{8}+\frac{\varepsilon}{40}] \times [\frac{1}{8}+\frac{\varepsilon}{20}, \frac{3}{8}+\frac{\varepsilon}{60}] \times [\frac{1}{8}+\frac{\varepsilon}{20}, \frac{1}{2}] \times \{0\}^2$. As for $F_{1,2}$ and $F_{1,3}$, they share the same 3-dimensional base T_1^*. We can cover both $F_{1,2}$ and $F_{1,3}$ using one box from $\mathcal{P}_6$ at the same time. By dividing the 3-dimensional base T_1^* into 3 equal parts along all coordinates, T_1^* can be partitioned into $3 \cdot 3 \cdot 3 = 27$ boxes of dimensions $(\frac{1}{4} - \frac{\varepsilon}{30})^3$. Therefore $F_{1,2}$ and $F_{1,3}$ can be covered by 27 boxes from $\mathcal{P}_6$ of dimensions $(\frac{1}{4}-\frac{\varepsilon}{30})^3 \cdot (\frac{1}{8}+\frac{\varepsilon}{40})^2$. Overall, covering the 3 flank boxes sharing the 3-face T_1 needs $18+27 = 45$ boxes. Since $|J| = 40$, covering all $3 \cdot 40 = 120$ flank boxes with 3-dimensional bases from J requires $45 \cdot 40 = 1800$ boxes from $\mathcal{P}_6$. Now we have selected $2608 + 1800 = 4408$ boxes.

Next we consider the 3-faces in I where $|I| = 120$. For L_1, assume the 3-faces $T_2 = [0,1]^3 \times \{0\}^3$ and $T_3 = [0,1]^3 \times \{0\}^2 \times \{1\}$ make a pair of opposite 3-faces in $\mathcal{M}_6$. Define a 3-dimensional base $Q_2 = [\frac{1}{8}+\frac{\varepsilon}{20}, \frac{7}{8}-\frac{\varepsilon}{20}]^3 \times \{0\}^3 \subset T_2$. Consider the two 4-faces containing T_2 that are different from L_1. We cover the two flank boxes in these 4-faces having Q_2 as a base in the same way as $F_{1,2}$ and $F_{1,3}$ were covered. Namely, we partition Q_2 into $3 \cdot 3 \cdot 3 = 27$ boxes by dividing the three coordinates of Q_2 into 3 equal parts, and use boxes of dimensions $(\frac{1}{4} - \frac{\varepsilon}{30})^3 \cdot (\frac{1}{8} + \frac{\varepsilon}{40})^2 \cdot 0$, where the base of dimensions $(\frac{1}{4} - \frac{\varepsilon}{30})^3$ is in T_2, and two sides of length $(\frac{1}{8} + \frac{\varepsilon}{40})$ are in 5th and 6th coordinates. For example, the box $[\frac{1}{8}+\frac{\varepsilon}{20}, \frac{3}{8}+\frac{\varepsilon}{60}]^3 \times \{0\} \times [0, \frac{1}{8}+\frac{\varepsilon}{40}]^2$ covers $T_{2,1}$ and $T_{2,2}$ at the same time. Regarding the covering in terms of $Q_2 \subset T_2$, we need 27 boxes from $\mathcal{P}_6$ of dimensions $(\frac{1}{4} - \frac{\varepsilon}{30})^3 \cdot (\frac{1}{8} + \frac{\varepsilon}{40})^2 \cdot 0$. For all 3-faces in I, it requires $120 \cdot 27 = 3240$ boxes. Up to now, we have selected $4408 + 3240 = 7648$ boxes. Also in this step, we covered another $120 \cdot 2 = 240$ flank boxes, leaving $480 - 120 - 240 = 120$ flank boxes uncovered.

In the end, for each 4-face, the remaining uncovered portion consists of a central box and a pair of flank boxes with the 3-dimensional bases contained in a pair of opposite 3-faces from $\mathcal{M}_6$. The uncovered region is contained in a rectangular solid of dimensions $(\frac{3}{4} - \frac{\varepsilon}{20})^2 \cdot 1 \cdot 0^3$. For example in L_1, the rectangular solid is $X_1 = [\frac{1}{8} + \frac{\varepsilon}{40}, \frac{7}{8} - \frac{\varepsilon}{40}]^3 \times [0,1] \times \{0\}^2$. Dissect X_1 equally into 3 parts along the 1st, 2nd and 3rd coordinates, and into 4 equal parts along the 4th coordinate. Thus covering the rectangular solid X_1 requires $3 \cdot 3 \cdot 3 \cdot 4 = 108$ boxes of dimensions $(\frac{1}{4} - \frac{\varepsilon}{60})^3 \cdot \frac{1}{4} \cdot 0^2$ from $\mathcal{P}_6$. So covering all 60 rectangular solids needs $108 \cdot 60 = 6480$ boxes. In this step, we covered all the remaining $60 \cdot 2 = 120$ flank boxes of $[0,1]^6$. Now the coverage of $[0,1]^6$ is completed. We have selected $7648 + 6480 = 14128$ boxes from $\mathcal{P}_6$ in total provided $\varepsilon < \frac{1}{24}$. □

Based on Theorem 3.3.1, the upper bound of H_6 comes naturally:

Corollary 3.3.5. *$H_6 \leqslant C_6 \leqslant 14140$.*

Proof. Since there are 12 points in any configuration $A_6 \in \mathcal{A}_6$, each of which can be covered by a box from $\mathcal{P}_6$, then we have $C_6 = C(\mathcal{A}_6 \cup B_{4,6}, \mathcal{P}_6) \leqslant C(B_{4,6}, \mathcal{P}_6) + 12 \leqslant 14140$. Therefore we have $H_6 \leqslant C_6 \leqslant 14140$. □

3.3.2 Appropriate Choice of Independent Pairs of 3-faces in $[0, 1]^6$

In this subsection, we will reformulate Lemma 3.3.3 into the problem of the existence of an independent set U_6 in graph G_6 with U_6 containing 60 vertices. Then we will prove the existence of such independent set U_6 in G_6 using computer assistance by running the Sagemath code given in Appendix B. Here we provide some brief comments highlighting the differences from the case $n = 5$.

Now we introduce the graph $G_6 = (V_6, E_6)$ consisting of vertex set V_6 and edge set E_6. In G_6, V_6 is the collection of all $4 \cdot 60 = 240$ pairs of opposite 3-faces of $[0, 1]^6$ (see Definition 3.3.1). Similarly to the conditions (i) and (ii) for G_5, two vertices in graph G_6 are adjacent if the corresponding pairs of opposite 3-faces satisfy any of the following conditions:

(1) two pairs are within the same 4-face of $[0, 1]^6$;

(2) two pairs are located in different 4-faces but share a 3-face.

The existence of an independent set U_6 with $|U_6| = 60$ in G_6 infers the validity of Lemma 3.3.3.

Proposition 3.3.6. *If there exists an independent set of 60 vertices in G_6, then Lemma 3.3.3 is true.*

Proof. Let $U_6 \subset V_6$ be an independent set of G_6 with cardinality 60. Denote the 60 pairs of opposite 3-faces corresponding to the vertices in $\mathcal{M}_6$ by U_6. We will verify that the elements in $\mathcal{M}_6$ satisfy the conditions in Lemma 3.3.3. Consider the adjacency condition (1), two pairs of 3-faces from $\mathcal{M}_6$ can not be located in the same 4-face. Seeing that $[0,1]^6$ has sixty 4-faces, each 4-face contains exactly one such pair. Due to the impossibility of the adjacency condition (2), any two pairs from $\mathcal{M}_6$ are non-overlapping (see Definition 3.3.2). Therefore, the validity of Lemma 3.3.3 is proved. □

In Sagemath code (see Appendix B), we generate the graph G_6 using the adjacency matrix. Then we look for an independent set U_6 in G_6. The code in lines 1-43 is handled in much the same way as we did in Appendix A. The slight difference is the elements of F have 6 coordinates instead of 5. However, if we attempt to find the largest independent set in G_6 similarly to what was done in G_5, it will take more than 1 week to run (we interrupted the computation after one week). The key difference is the larger size of the graph and of the required independent set. To solve it, instead of finding U_6 using all 240 vertices in G_6, we randomly selected 160 vertices of G_6 (see lines 44-45), and ran the program several times to look for an independent set U_6 within these 160 vertices instead of all 240 vertices. While to complete the proof, it suffices to find at least one independent set of cardinality 60, and the randomized approach allows to do this consistently. Namely, we got 24 successes in a test of running the program 101 times on a modern personal computer. On average, each run took around

7 minutes. In other words, such independent sets in G_6 are not rare.

Remark 3.3.7. *Dr. Karen Gunderson has kindly informed us of an alternative proof of existence of the required independent sets in G_5 and G_6, as well as in their generalizations to higher dimensions. The proof does not rely on computer assistance and is based on an application of Hall's theorem.*

Chapter 4

Lower bounds on C_5 and C_6

In this chapter, we find some lower bounds on C_5 and C_6. Remark that we only have the inequality $H_n \leqslant C_n$ while we do not have its converse. Therefore, a lower bound on C_n is not a lower bound on H_n. In other words, the results in this chapter show the limitations of the technique by Papadoperakis of obtaining the upper bound on H_n. In Section 4.1, we show some recent results and our new lower bounds on C_5 and C_6. In Section 4.2 and Section 4.3, we present the estimates for C_5 and C_6, respectively.

4.1 Results for lower bounds on C_n, $n = 4, 5, 6$

In this section, we will show some new results on the lower bound of C_n. Recently, Prymak and Shepelska found a universal lower bound on C_n for $n \geqslant 5$:

Theorem 4.1.1. [28, Theorem 1.1] *For $n \geqslant 5$, we have $C_n \geqslant 4n^{n-2} + 2n$.*

n	Theorem 4.1.1	Theorem 4.1.3
5	510	717
6	5196	7177

Table 4.1: Lower bounds of C_5 and C_6

Also, they achieved an almost sharp lower bound on C_4 (recall that in the other direction they showed $C_4 \leqslant 96$):

Theorem 4.1.2. [28, Theorem 1.2] $C_4 \geqslant 95$.

Here are our new results about the lower bounds on C_5 and C_6:

Theorem 4.1.3. *$C_5 \geqslant 717$, $C_6 \geqslant 7177$.*

Table 4.1 compares the lower bounds of C_5 and C_6 given by Theorem 4.1.1 with our new results: We admit that these lower bounds on C_5 and C_6 are still quite far from the upper bounds $C_5 \leqslant 1002$ and $C_6 \leqslant 14140$ established in Chapter 3.

4.2 Lower bound on C_5

For a measurable $A \subset \mathbb{R}^n$, define $\lambda_{n-2}(A)$ as the sum of $(n-2)$-dimensional volumes of $A \cap F$ taken over all $(n-2)$-faces F of $[0,1]^n$. Sometimes it will be convenient to work with boxes from the larger set $\mathcal{P}_n^*$ allowing the sum of dimensions to be 1. The main method in this section is estimating the λ_{n-2} that various boxes from $\mathcal{P}_n$ or their unions can cover.

4.2.1 Main result on C_5

In this subsection, we obtain $C_5 \geqslant 717$ as a consequence of several lemmas. All the proofs of these lemmas are postponed till Subsection 4.2.2. In [28, Lemma 4.1], Prymak and Shepelska found an upper bound of $\lambda_{n-2}(P)$ for a single box $P \in \mathcal{P}_n$:

Lemma 4.2.1. *For any $P \in \mathcal{P}_n$, we have the inequality $\lambda_{n-2}(P) < \frac{n-1}{2n^{n-3}}$.*

By Lemma 4.2.1, let $n = 5$ and assume $\mathbf{0} \in P$, then we have:

Corollary 4.2.2. *For P satisfying $\mathbf{0} \in P \in \mathcal{P}_5$, the inequality $\lambda_3(P) < \frac{2}{25}$ holds.*

In [28, Lemma 4.2], an upper bound for λ_{n-2} of $P \in \mathcal{P}_n$ containing a point from A_n^* was found, where $A_n^* \in \mathcal{A}_n$ is the configuration of $2n$ centres of $(n-1)$-faces:

Lemma 4.2.3. *For any $P \in \mathcal{P}_n$ containing an element of A_n^*, the inequality $\lambda_{n-2}(P) < \frac{1}{(2n-4)^{n-2}}$ holds.*

In Lemma 4.2.3, let $n = 5$, and then we have:

Corollary 4.2.4. *If $P \in \mathcal{P}_5$ contains an element of A_5^*, then $\lambda_3(P) < \frac{1}{216}$.*

From now on, we will distinguish 4 kinds of boxes from $\mathcal{P}_5$ that can be used to cover $B_{3,5}$: 1) the boxes containing the vertices of $[0,1]^5$; 2) the boxes containing the centres of 2-faces of $[0,1]^5$; 3) for each point of A_5^*, arbitrarily choose one box covering that point; 4) all the remaining boxes. Remark that a box from $\mathcal{P}_5$ can not contain two vertices, or two centres, or a vertex and a centre at the same time. For the first type of boxes, i.e. boxes containing vertices, without loss of generality, we consider a box P containing $\mathbf{0} = (0,0,0,0,0)$. For $P \in \mathcal{P}_5^*$ containing the vertex $\mathbf{0}$, we get the following lemma:

Lemma 4.2.5. *If $\mathbf{0} \in P \in \mathcal{P}_5$, then we have $\lambda_3\left(P \setminus \left[0, \frac{1}{5}\right]^5\right) < \frac{411}{10000}$.*

For λ_3 evaluated at the union of 2 such boxes, we have:

Lemma 4.2.6. *If $\mathbf{0} \in P_1, P_2 \in \mathcal{P}_5$, then the inequality $\lambda_3(P_1 \cup P_2) < \frac{19161}{125000}$ holds.*

By the combination of Corollary 4.2.2, Lemma 4.2.5 and Lemma 4.2.6, we can obtain an upper bound on λ_3 for the union of k boxes from $\mathcal{P}_5$ containing $\mathbf{0}$ as a linear function of k as follows:

Lemma 4.2.7. *Suppose $\mathbf{0} \in P_i \in \mathcal{P}_5$, $1 \leqslant i \leqslant k$, then we have*

$$\lambda_3\left(\bigcup_{i=1}^{k} P_i\right) < \frac{1}{18}k + \frac{47449}{1125000}.$$

Next we are devoted to the second type of boxes, i.e. the boxes from $\mathcal{P}_5$ containing centres of 2-faces. Note that we can always enlarge a box from $\mathcal{P}_5$ to make them to be from $\mathcal{P}_5^*$. Our goal is to estimate the value of λ_3 on the union of several such boxes from $\mathcal{P}_5^*$ containing the centre of one face, while the approach is similar to the boxes of the first type and the details are more complicated. Without loss of generality, consider the 2-face $S_1 = [0,1]^2 \times \{0\}^4$ and its centre $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. The role of $\left[0, \frac{1}{5}\right]^5$ will now be played by the box $B = \left[\frac{1}{2} - \frac{1}{7}, \frac{1}{2} + \frac{1}{7}\right]^2 \times \left[0, \frac{1}{7}\right]^3 = \left[\frac{5}{14}, \frac{9}{14}\right]^2 \times \left[0, \frac{1}{7}\right]^3$. Then we get an upper bound for λ_3 of parts of P located outside of B:

Lemma 4.2.8. *For $B = \left[\frac{5}{14}, \frac{9}{14}\right]^2 \times \left[0, \frac{1}{7}\right]^3$ and P satisfying $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \in P \in \mathcal{P}_5^*$, we have $\lambda_3 \left(P \setminus B\right) \leq \frac{316}{9261}$.*

For a single box from $\mathcal{P}_5^*$ containing $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, we have:

Lemma 4.2.9. *If $P \in \mathcal{P}_5^*$ contains $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, then we have $\lambda_3(P) \leqslant \frac{1}{27}$.*

With the result of Lemma 4.2.8, we can get an upper bound of the value of λ_3 on the union of a collection of boxes from $\mathcal{P}_5^*$ containing the centre $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$:

Lemma 4.2.10. *If $P_i \in \mathcal{P}_5^*$ contains $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, $1 \leqslant i \leqslant k$, then we have $\lambda_3 \left(\bigcup_{i=1}^{k} P_i\right) < \frac{1}{18}k - \frac{73}{9261}$.*

For the last type of boxes, the following lemma holds:

Lemma 4.2.11. *If $P \in \mathcal{P}_5^*$ does not contain any vertex, then $\lambda_3 \left(P\right) \leqslant \frac{1}{18}$.*

Using the above lemmas for 4 kinds of boxes, we can derive a lower bound for C_5:

Theorem 4.2.12. $C_5 \geq 717$.

Proof. Recall that there are forty vertices, eighty 2-faces and forty 3-faces in $[0,1]^5$ and 10 elements in A_5^*. We are going to cover $B_{3,5}$ by 4 types of boxes mentioned above. Note that each box of the covering belongs to only one of the four described types, and a box of the third type which contains the centre of a 4-face can not belong to the first or the second type of boxes. Let u_i, $1 \leqslant i \leqslant 32$, be the number of the first type of boxes covering the i-th vertex. Let v_j, $1 \leqslant j \leqslant 80$, be the quantity of the second type of boxes covering the centre of j-th 2-face. We have 10 boxes of the third type such that each box contains an element of A_5. Finally, let w denote the total number of boxes in the covering so that there are $\left(w - 10 - \sum\limits_{i=1}^{32} u_i - \sum\limits_{j=1}^{80} v_j\right)$ boxes of the forth type. Then the value of λ_3 on the union of all these boxes is no less than 40. Applying Corollary 4.2.4, Lemma 4.2.7, Lemma 4.2.10 and Lemma 4.2.11, we have:

$$10 \cdot \frac{1}{216} + \sum_{i=1}^{32}\left(\frac{1}{18}u_i + \frac{47449}{1125000}\right) + \sum_{j=1}^{80}\left(\frac{1}{18}v_j - \frac{73}{9261}\right)$$

$$+ \frac{1}{18}\left(w - 10 - \sum_{i=1}^{32} u_i - \sum_{j=1}^{80} v_j\right) \geqslant 40,$$

$$\frac{1}{18}w \geqslant 40 - \frac{47449}{1125000} \cdot 32 + \frac{73 \cdot 80}{9261} - 10 \cdot \frac{1}{216} + \frac{10}{18},$$

implying $w \geqslant 716.22360472$, therefore we have $C_5 \geqslant 717$. $\square$

4.2.2 Proofs of lemmas

We repeat the proofs of Lemma 4.1 and Lemma 4.2 from [28] here for completeness. Note that both proofs use the inequalities between means from Section 2.2 (see Theorem 2.2.1). For convenience, we repeat the statements of the lemmas here.

Lemma 4.2.1. *For any $P \in \mathcal{P}_n$, we have the inequality $\lambda_{n-2}(P) < \frac{n-1}{2n^{n-3}}$.*

Proof of Lemma 4.2.1. Suppose $P = \prod_{i=1}^{n} [x_i, x_i + \delta_i]$ with $\sum_{i=1}^{n} \delta_i \in (0, 1)$. Prymak and Shepelska applied the inequality between means [17, Theorem 52](see also Section 2.2) to prove Lemma 4.2.1 as:

$$\lambda_{n-2}(P) \leqslant \sum_{1 \leqslant i < j \leqslant n} \prod_{k \in \{1,\ldots,n\} \setminus \{i,j\}} \delta_k \leqslant \frac{n-1}{2n^{n-3}} \left(\sum_{i=1}^{n} \delta_i \right)^{n-2} < \frac{n-1}{2n^{n-3}}.$$

□

Lemma 4.2.3. *For any $P \in \mathcal{P}_n$ containing an element of A_n^*, the inequality $\lambda_{n-2}(P) < \frac{1}{(2n-4)^{n-2}}$ holds.*

Proof of Lemma 4.2.3. For $(0, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in P = \prod_{i=1}^{n} [x_i, x_i + \delta_i]$ where $\sum_{i=1}^{n} \delta_i < 1$, P can intersect at most one $(n-2)$-face of $[0, 1]^n$, without loss of generality say

$\{0\}^2 \times [0,1]^{n-2}$. If P intersects with $\{0\}^2 \times [0,1]^{n-2}$, then we have $\delta_2 \geqslant \frac{1}{2}$ and:

$$\lambda_{n-2}(P) \leqslant \prod_{i=3}^{n} \delta_i \leqslant \frac{1}{(n-2)^{n-2}} \left(\sum_{i=3}^{n} \delta_i \right)^{n-2} < \frac{1}{(2n-4)^{n-2}}.$$

$\square$

Lemma 4.2.5. *If $\mathbf{0} \in P \in \mathcal{P}_5$, then we have $\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right) < \frac{411}{10000}$.*

Proof of Lemma 4.2.5. Recall that $S_r(a_1, ..., a_n) = \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} a_{i_1} a_{i_2} ... a_{i_r}$ represents the sum of all possible products of r different elements in $\{a_1, a_2, ..., a_n\}$. The result of Corollary 2.2.2, i.e. the inequality $S_n^r \leqslant \left(\frac{S}{n}\right)^r \cdot \binom{n}{r}$, will be frequently used below. For convenience, we consider a box $P = \prod_{i=1}^{5} [x_i, x_i + \delta_i] \in \mathcal{P}_5^*$ instead of $\mathcal{P}_5$ where $x_i \leq 0 \leq x_i + \delta_i$. To get the maximum value of $\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right)$, suppose $x_i = 0$, $i \in \{1, ..., 5\}$, which will not decrease $\lambda_3(P)$. Assume that $\delta_1 \geqslant \delta_2 \geqslant \delta_3 \geqslant \delta_4 \geqslant \delta_5$ with $\sum_{i=1}^{5} \delta_i = 1$, and let $m \geqslant 1$ be the largest index of δ so that $\delta_m \geqslant \frac{1}{5}$. We get a general formula for $\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right)$, where

$1 \leqslant m \leqslant 5$:

$$\begin{aligned}\lambda_3\left(P \setminus \left[0, \frac{1}{5}\right]^5\right) &= \sum_{1\le i<j<k\le 5}\left(\delta_i\delta_j\delta_k - \min\left\{\delta_i, \frac{1}{5}\right\}\min\left\{\delta_j, \frac{1}{5}\right\}\min\left\{\delta_k, \frac{1}{5}\right\}\right)\\ &= \sum_{1\le i<j<k\le m}\left(\delta_i\delta_j\delta_k - \frac{1}{125}\right) + \sum_{1\le i<j\le m<k\le 5}\left(\delta_i\delta_j - \frac{1}{25}\right)\delta_k\\ &\quad + \sum_{1\le i\le m<j<k\le 5}\left(\delta_i - \frac{1}{5}\right)\delta_j\delta_k.\end{aligned} \tag{4.1}$$

Now we let m range from 5 to 1 to find all possible volumes. For $m = 5$, which means $P = \left[0, \frac{1}{5}\right]^5$, it is a simple matter to see $\lambda_3\left(P \setminus \left[0, \frac{1}{5}\right]^5\right) = 0$.

Our approach is to obtain an estimate in terms of a function of one variable s, and then use standard calculus. We denote the initial sum of i terms from $\delta_i, ..., \delta_n$ as $s_i = \sum_{j=1}^{i}\delta_j$ and the tail-end sum of i terms as $t_i = \sum_{j=n-i+1}^{n}\delta_j$ where $n \in \{5, 6\}$. For $m = 4$, we have $\delta_5 \in \left[0, \frac{1}{5}\right)$ and denote $s_4 = \sum_{i=1}^{4}\delta_i \in \left(\frac{4}{5}, 1\right)$.

With the conclusion in Corollary 4.2.2 and the volume formula (4.1), we obtain:

$$
\begin{aligned}
\lambda_3\left(P \setminus \left[0, \frac{1}{5}\right]^5\right) &= \sum_{1\leq i<j<k\leq 4}\left(\delta_i\delta_j\delta_k - \frac{1}{125}\right) + \sum_{1\leq i<j\leq 4}\left(\delta_i\delta_j - \frac{1}{25}\right)\delta_5 \\
&= S_3(\delta_1, ..., \delta_4) - \frac{4}{125} + \sum_{1\leq i<j\leq 4}\left(\delta_i\delta_j - \frac{1}{25}\right)\delta_5 \\
&\leqslant \left(\frac{s_4}{4}\right)^3 \cdot 4 - \frac{4}{125} + \left(\left(\frac{s_4}{4}\right)^2 \cdot 6 - \frac{1}{25}\right)(1 - s_4) \\
&=: f(s_4) \leqslant f\left(\frac{2\cdot\left(15+\sqrt{285}\right)}{75}\right) \approx 0.04102345 < \frac{411}{10000}.
\end{aligned}
$$

For the case $m = 3$, i.e. $\delta_1, \delta_2, \delta_3 \in \left[\frac{1}{5}, 1\right)$, we denote $t_2 = \delta_4 + \delta_5 \in \left[0, \frac{2}{5}\right)$, then we have:

$$
\begin{aligned}
&\lambda_3\left(P \setminus \left[0,\frac{1}{5}\right]^5\right)\\
=&\delta_1\delta_2\delta_3-\frac{1}{125}+\sum_{1\leq i<j\leq 3}\left(\delta_i\delta_j-\frac{1}{25}\right)(\delta_4+\delta_5)+\sum_{1\leqslant i\leq 3}\left(\delta_i-\frac{1}{5}\right)\delta_4\delta_5\\
=&\delta_1\delta_2\delta_3-\frac{1}{125}+\left(S_2(\delta_1,\delta_2,\delta_3)-\frac{3}{25}\right)(\delta_4+\delta_5)+\left(\sum_{i=1}^{3}\delta_i-\frac{3}{5}\right)\delta_4\delta_5\\
\leqslant&\left(\frac{\sum_{i=1}^{3}\delta_i}{3}\right)^3-\frac{1}{125}+\left(\frac{\left(\sum_{i=1}^{3}\delta_i\right)^2}{3}-\frac{3}{25}\right)(\delta_4+\delta_5)+\left(\sum_{i=1}^{3}\delta_i-\frac{3}{5}\right)\delta_4\delta_5\\
\leqslant&\left(\frac{1-t_2}{3}\right)^3+\left(\frac{(1-t_2)^2}{3}-\frac{3}{25}\right)t_2+\left(\frac{2}{5}-t_2\right)\left(\frac{t_2}{2}\right)^2-\frac{1}{125}\\
=:&f(t_2)\leqslant f\left(\frac{82-6\sqrt{174}}{25}\right)\approx 0.034838575<\frac{411}{10000}.
\end{aligned}
$$

If $m = 2$, letting $t_3 = \sum_{i=3}^{5} \delta_i \in \left[0, \frac{3}{5}\right)$, we have:

$$\begin{aligned}
&\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right) \\
&= \left(\delta_1\delta_2 - \frac{1}{25} \right) (\delta_3 + \delta_4 + \delta_5) + (\delta_3\delta_4 + \delta_3\delta_5 + \delta_4\delta_5) \left(\delta_1 + \delta_2 - \frac{2}{5} \right) \\
&\le \left(\left(\frac{\delta_1 + \delta_2}{2} \right)^2 - \frac{1}{25} \right) (\delta_3 + \delta_4 + \delta_5) + \frac{(\delta_3 + \delta_4 + \delta_5)^2}{3} \left(\delta_1 + \delta_2 - \frac{2}{5} \right) \\
&\le \left(\frac{(1 - t_3)^2}{4} - \frac{1}{25} \right) t_3 + \frac{t_3^2}{3} \left(\frac{2}{5} - t_3 \right) \\
&=: f(t_3) < f\left(\frac{-6 + \sqrt{57}}{5} \right) \approx 0.03379 < \frac{411}{10000}.
\end{aligned}$$

If $m = 1$, by Section 2.2, we let $s = \sum_{i=2}^{5} \delta_i \in \left[0, \frac{4}{5}\right)$, then we have:

$$\begin{aligned}
\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right) &= \sum_{2 \le j < k \le 5} \left(\delta_1 - \frac{1}{5} \right) \delta_j\delta_k = \left(\delta_1 - \frac{1}{5} \right) \cdot S_2(\delta_2, \delta_3, \delta_4, \delta_5) \\
&\le \frac{3}{8} \cdot \left(\delta_1 - \frac{1}{5} \right) \left(\sum_{i=2}^{5} \delta_i \right)^2 \le \frac{3}{8} \cdot \left(\frac{4}{5} - s \right) s^2 \\
&=: f(s) \le f\left(\frac{8}{15} \right) = \frac{32}{1125} < 0.02844445 < \frac{411}{10000}.
\end{aligned}$$

Overall, we have the upper bound as $\lambda_3 \left(P \setminus \left[0, \frac{1}{5}\right]^5 \right) < \frac{411}{10000}$. $\square$

Lemma 4.2.6. *If $\mathbf{0} \in P_1, P_2 \in \mathcal{P}_5$, then the inequality $\lambda_3(P_1 \cup P_2) < \frac{19161}{125000}$ holds.*

Proof of Lemma 4.2.6. Suppose $P_1 = \prod_{i=1}^{5}[0, \gamma_i] \in \mathcal{P}_5^*$ and $P_2 = \prod_{i=1}^{5}[0, \delta_i] \in \mathcal{P}_5^*$. After the rearrangement of coordinates, we use $m \geq 0$ to denote the largest number of coordinates so that for each $1 \leqslant i \leqslant m$, we have $\gamma_i \geq \delta_i$. For the case $m = 4$, we have:

$$
\begin{aligned}
\lambda_3\left(P_1 \cup P_2\right) &= \lambda_3\left(P_2\right) + \lambda_3\left(P_1 \setminus P_2\right) \\
&= S_3(\delta_1, ..., \delta_5) + \left(S_2(\gamma_1, \gamma_2, \gamma_3, \gamma_4) - S_2(\delta_1, \delta_2, \delta_3, \delta_4)\right)\gamma_5 \\
&\leqslant \frac{2}{25} + \left(S_2(\gamma_1, \gamma_2, \gamma_3, \gamma_4) - S_2(\delta_1, \delta_2, \delta_3, \delta_4)\right)\gamma_5 \\
&= \frac{2}{25} + \left(\left(\frac{1-\gamma_5}{4}\right)^2 \cdot 6 - \left(\frac{1-\delta_5}{4}\right)^2 \cdot 6\right) \cdot \gamma_5
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{2}{25} + \frac{3}{8} \cdot \gamma_5 \cdot (1-\gamma_5)^2 \\
&=: f(\gamma_5) \leq f\left(\frac{1}{3}\right) = \frac{61}{450} \approx 0.13555556 < \frac{19161}{125000}.
\end{aligned}
$$

Now we consider the case $m = 3$. Let $s_3 = \sum\limits_{i=1}^{3} \delta_i \in (0,1]$, then we have:

$$
\begin{aligned}
\lambda_3\left(P_1 \cup P_2\right) =& \lambda_3(P_1) + \lambda_3\left(P_2 \setminus P_1\right) \\
=& S_3(\gamma_1, ..., \gamma_5) + S_2(\delta_1, \delta_2, \delta_3) \cdot ((\delta_4 + \delta_5) - (\gamma_4 + \gamma_5)) \\
& + \sum_{i=1}^{3} \delta_i \cdot (\delta_4\delta_5 - \gamma_4\gamma_5) \\
\leq& \frac{2}{25} + S_2(\delta_1, \delta_2, \delta_3) \cdot ((\delta_4 + \delta_5) - (\gamma_4 + \gamma_5)) + \sum_{i=1}^{3} \delta_i \cdot (\delta_4\delta_5 - \gamma_4\gamma_5) \\
\leqslant& \frac{2}{25} + S_2(\delta_1, \delta_2, \delta_3)(\delta_4 + \delta_5) + \sum_{i=1}^{3} \delta_i \cdot \delta_4\delta_5 \\
\leq& \frac{2}{25} + \frac{\left(\sum\limits_{i=1}^{3} \delta_i\right)^2}{3} \cdot (\delta_4 + \delta_5) + \left(\sum_{i=1}^{3} \delta_i\right) \cdot \left(\frac{\delta_4 + \delta_5}{2}\right)^2 \\
=& \frac{2}{25} + \frac{s_3^2}{3} \cdot (1 - s_3) + s_3 \cdot \left(\frac{1 - s_3}{2}\right)^2 =: f(s_3) \\
\leq& f\left(\frac{\sqrt{13} - 2}{3}\right) \approx 0.15328498 < \frac{19161}{125000}.
\end{aligned}
$$

The case $m = 1$ is completely the same with the case $m = 4$, and the case $m = 3$ is also the same with the case $m = 2$. So we have $\lambda_3\left(P_1 \cup P_2\right) < \frac{19161}{125000}$. □

Lemma 4.2.7. *Suppose* $\mathbf{0} \in P_i \in \mathcal{P}_5$, $1 \leqslant i \leqslant k$, *then we have*

$\lambda_3\left(\bigcup\limits_{i=1}^{k} P_i\right) < \frac{1}{18}k + \frac{47449}{1125000}$.

Proof of Lemma 4.2.7. Consider boxes $P_i \in \mathcal{P}_5^*$, $1 \leqslant i \leqslant k$. If $k = 1$, by Corollary 4.2.2, we have $\lambda_3(P) \leq \frac{2}{25} \leq \frac{1}{18} \cdot 1 + \frac{47449}{1125000}$. If $k = 2$, by Lemma 4.2.6, we have $\lambda_3 (P_1 \cup P_2) < \frac{19161}{125000} \leqslant \frac{1}{18} \cdot 2 + \frac{47449}{1125000}$. If $k \geqslant 3$, by Corollary 4.2.2 and Lemma 4.2.5, we get:

$$\lambda_3\left(\bigcup_{i=1}^{k} P_i\right) = \lambda_3\left(\left[0, \frac{1}{5}\right]^5\right) + \lambda_3\left(\left(\bigcup_{i=1}^{k} P_i\right) \setminus \left[0, \frac{1}{5}\right]^5\right)$$
$$< \frac{2}{25} + \frac{411}{10000}k \leqslant \frac{1}{18}k + \frac{47449}{1125000},$$

where it can be easily checked that the last inequality is equivalent to

$k \geqslant \frac{85102}{32525} > 2.61651038$. Therefore we get $\lambda_3\left(\bigcup_{n=1}^{k} P_i\right) < \frac{1}{18}k + \frac{47449}{1125000}$. $\square$

Lemma 4.2.8. *For $B = \left[\frac{5}{14}, \frac{9}{14}\right]^2 \times \left[0, \frac{1}{7}\right]^3$ and P satisfying $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \in P \in \mathcal{P}_5^*$, we have $\lambda_3 (P \setminus B) \leq \frac{316}{9261}$.*

Proof of Lemma 4.2.8. Recall the box $B = \left[\frac{1}{2} - \frac{1}{7}, \frac{1}{2} + \frac{1}{7}\right]^2 \times \left[0, \frac{1}{7}\right]^3 = \left[\frac{5}{14}, \frac{9}{14}\right]^2 \times \left[0, \frac{1}{7}\right]^3$, and consider a box P from $\mathcal{P}_5^*$ containing $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. Suppose $P = \left[\frac{1}{2} - a_2, \frac{1}{2} + a_1\right] \times \left[\frac{1}{2} - b_2, \frac{1}{2} + b_1\right] \times [0, c_1] \times [0, c_2] \times [0, c_3]$, where for $i, j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$, the non-negative numbers a_i, b_j, c_k satisfy $\sum_{i=1}^{2} a_i + \sum_{j=1}^{2} b_j + \sum_{k=1}^{3} c_k = 1$. Here we denote $a = a_1 + a_2$, $b = b_1 + b_2$ and $c = c_1 + c_2 + c_3$. Note that for $x, y, z \in [0, 1]$, the 3-dimensional volume

$\lambda_3\left(([0,x]\times[0,y]\times[0,z])\setminus\left[0,\frac{1}{7}\right]^3\right)$ is:

$$F(x,y,z)=xyz-\min\left\{x,\frac{1}{7}\right\}\min\left\{y,\frac{1}{7}\right\}\min\left\{z,\frac{1}{7}\right\}.$$

Now by symmetry, i.e. moving the origin to $\left(\frac{1}{2},\frac{1}{2},0,0,0\right)$ and splitting each 3-dimensional subspace into octants, we get $\lambda_3\left(P\setminus B\right)=\sum\limits_{k=1}^{3}\sum\limits_{j=1}^{2}\sum\limits_{i=1}^{2}F\left(a_i,b_j,c_k\right)$.

Fix any two variables from x,y,z, without loss of generality say x and y, leaving z as a single independent variable. Observe that $F\left(x,y,z\right)$ is a continuous piecewise linear function of z, $z\in[0,1]$. When $z\in\left[0,\frac{1}{7}\right]$, the slope of $F\left(x,y,z\right)$ is $\left(xy-\min\left\{x,\frac{1}{7}\right\}\min\left\{y,\frac{1}{7}\right\}\right)$. When $z\in\left[\frac{1}{7},1\right]$, the slope of $F\left(x,y,z\right)$ is xy, which is greater than the slope on the interval $\left[0,\frac{1}{7}\right]$. So $F\left(x,y,z\right)$ is always a convex function in terms of any independent variable with the remaining two variables fixed.

Now we apply Karamata's inequality in Section 2.3. Without loss of generality, we can assume that $c_1\geq c_2\geq c_3$. Then $(c_1+c_2+c_3,0,0)\succ(c_1,c_2,c_3)$ (The relation "$\succ$" is defined from Section 2.3), and by Karamata's inequality we get:

$$\begin{aligned}
&F\left(x,y,c_1\right)+F\left(x,y,c_2\right)+F\left(x,y,c_3\right)\\
\leq&F\left(x,y,c_1+c_2+c_3\right)+F\left(x,y,0\right)+F\left(x,y,0\right)\\
=&F\left(x,y,c_1+c_2+c_3\right)\\
=&F\left(x,y,c\right).
\end{aligned}$$

So we get $\lambda_3(P\setminus B)=\sum\limits_{k=1}^{3}\sum\limits_{j=1}^{2}\sum\limits_{i=1}^{2}F(a_i,b_j,c_k)\leq\sum\limits_{j=1}^{2}\sum\limits_{i=1}^{2}F(a_i,b_j,c)$.

By applying the same arguments for the second and the first variables, we have:

$$\lambda_3(P\setminus B)\leqslant\sum_{j=1}^{2}\sum_{i=1}^{2}F(a_i,b_j,c)\leqslant\sum_{i=1}^{2}F(a_i,b,c)\leqslant F(a,b,c)\,.$$

Since $F(x,y,z)$ is symmetric with respect to the interchange of the order of variables, to obtain an upper bound on $F(a,b,c)$, we assume that $a\geqslant b\geqslant c$ in what follows. If $a\geq\frac{1}{7}$ and $0<b,c<\frac{1}{7}$, with slight abuse of notation, we let $s=b+c\in\left(0,\frac{2}{7}\right)$ then:

$$F(a,b,c)=\left(a-\frac{1}{7}\right)\cdot bc\leq\left(\frac{6}{7}-(b+c)\right)\cdot bc\leq\left(\frac{6}{7}-(b+c)\right)\cdot\left(\frac{b+c}{2}\right)^2$$

$$=\frac{s^2}{4}\cdot\left(\frac{6}{7}-s\right)=:f(s)<f\left(\frac{2}{7}\right)=\frac{4}{343}\approx 0.01166180<\frac{316}{9261}.$$

If $a,b\geq\frac{1}{7}$ and $0<c<\frac{1}{7}$, then we have:

$$F(a,b,c)=\left(ab-\frac{1}{49}\right)\cdot c\leq\left(\left(\frac{1-c}{2}\right)^2-\frac{1}{49}\right)\cdot c$$

$$=:f(c)<f\left(\frac{1}{7}\right)\approx 0.02332362<\frac{316}{9261}.$$

If $a,b,c\in\left[\frac{1}{7},1\right)$, then:

$$F(a,b,c)=\left(abc-\frac{1}{7^3}\right)\leq\left(\left(\frac{a+b+c}{3}\right)^3-\frac{1}{7^3}\right)=\left(\frac{1}{3}\right)^3-\left(\frac{1}{7}\right)^3=\frac{316}{9261}.$$

Note that when $a = b = c = \frac{1}{3}$, $F(a, b, c)$ reaches its maximum $\frac{316}{9261}$. $\square$

Lemma 4.2.9. *If $P \in \mathcal{P}_5^*$ contains $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, then we have $\lambda_3(P) \leqslant \frac{1}{27}$.*

Proof of Lemma 4.2.9. Suppose $P = \prod_{i=1}^{n} [x_i, x_i + \delta_i]$ contains $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$ where $\sum_{i=1}^{n} \delta_i = 1$. By letting $s_1 = \delta_1 + \delta_2 \in (0, 1)$, we can get

$$\lambda_3(P) \leqslant \delta_1\delta_2 \left(\sum_{i=3}^{5} \delta_i\right) \leqslant \left(\frac{\delta_1 + \delta_2}{2}\right)^2 (1 - (\delta_1 + \delta_2))$$

$$= \left(\frac{s_1}{2}\right)^2 (1 - s_1) =: f(s_1) \leqslant f\left(\frac{2}{3}\right) = \frac{1}{27}.$$

$\square$

Lemma 4.2.10. *If $P_i \in \mathcal{P}_5^*$ contains $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, $1 \leqslant i \leqslant k$, then we have $\lambda_3\left(\bigcup_{i=1}^{k} P_i\right) < \frac{1}{18}k - \frac{73}{9261}$.*

Proof of Lemma 4.2.10. Consider $P_i \in \mathcal{P}_5^*$ covering $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$ where $1 \leqslant i \leqslant k$. For $k = 1$, by Lemma 4.2.9, we have $\lambda_3(P) \leqslant \frac{1}{27} \leqslant \frac{1}{18} \cdot 1 - \frac{73}{9261}$. For $k \geq 2$, by Lemma 4.2.8, we get:

$$\lambda_3\left(\bigcup_{i=1}^{k} P_i\right) = \lambda_3(B) + \lambda_3\left(\left(\bigcup_{i=1}^{k} P_i\right) \setminus B\right) < \frac{12}{343} + \frac{316}{9261}k \leqslant \frac{1}{18}k - \frac{73}{9261},$$

where it can be easily checked that the last inequality is equivalent to $k \geqslant 2$. Therefore, we get the upper bound $\lambda_3\left(\bigcup_{i=1}^{k} P_i\right) < \frac{1}{18}k - \frac{73}{9261}$. $\square$

Lemma 4.2.11. *If $P \in \mathcal{P}_5^*$ does not contain any vertex, then $\lambda_3(P) \leqslant \frac{1}{18}$.*

Proof of Lemma 4.2.11. Suppose $P = \prod_{i=1}^{5} [x_i, x_i + \delta_i]$. We denote by $m \geqslant 0$ the number of indexes i so that $[x_i, x_i + \delta_i] \cap \{0, 1\} = \emptyset$. Without loss of generality, we assume that for each $i > m$, $0 \notin [x_i, x_i + \delta_i]$. Seeing that there is no vertex in P, then $1 \leqslant m \leqslant 5$. If $m = 1$, we have:

$$\lambda_3(P) \leqslant \delta_1 \cdot S_2(\delta_2, \delta_3, \delta_4, \delta_5) \leqslant \delta_1 \cdot \left(\frac{\sum_{i=2}^{5} \delta_i}{4} \right)^2 \cdot 6 = \frac{3}{8} \cdot \delta_1 (1 - \delta_1)^2$$

$$=: f(\delta_1) \leqslant f\left(\frac{1}{3}\right) = \frac{1}{18}.$$

If $m = 2$, the consideration is the same as in Lemma 4.2.9, so we get $\lambda_3(P) \leq \frac{1}{27}$.

For $m = 3$, there is only one 3-dimensional box contained in both P and $B_{3,5}$, so we have:

$$\lambda_3(P) = \delta_1 \delta_2 \delta_3 \leqslant \left(\frac{\sum_{i=1}^{3} \delta_i}{3} \right)^3 = \frac{1}{27}.$$

At last for $m = 4, 5$, it is obvious that $\lambda_3(P) = 0$. Overall, for any $m \in \{1, ..., 5\}$, we have $\lambda_3(P) \leqslant \frac{1}{18}$. $\square$

4.3 Lower bound on C_6

In this section, we will use different boxes from $\mathcal{P}_6$ and $\mathcal{P}_6^*$ to cover certain parts of $[0,1]^6$, and estimate the λ_4 of these boxes. With these estimates, we look for an upper bound on C_6 in the end.

4.3.1 Main results on C_6

In this subsection, we get $C_6 \geqslant 7177$ using a combination of several lemmas. The proof for each of the following lemmas is in Subsection 4.3.2.

Let $A_6^* \in \mathcal{A}_6$ be the configuration of 12 centres of 5-faces of $[0,1]^6$. For a collection of boxes from $\mathcal{P}_6$ which get $[0,1]^6$ covered, we classify them into 4 types similarly to what we did in Subsection 4.2.1: (i) the boxes containing the vertices of $[0,1]^6$; (ii) the boxes containing the centres of 3-faces of $[0,1]^6$; (iii) for each point of A_6^*, choose an arbitrary box covering that point; (iv) all the remaining boxes. Note that a box from $\mathcal{P}_6$ can not contain two vertices, or two centres of 3-faces, or a vertex and a centre of a 3-face at the same time. For a box P in the first type which contains a vertex, without loss of generality, we assume $\mathbf{0} \in P$. By letting $n = 6$ in Lemma 4.2.1, we get the following result:

Corollary 4.3.1. *For a single box P satisfying $\mathbf{0} \in P \in \mathcal{P}_6$, the inequality $\lambda_4(P) < \frac{5}{432}$ holds.*

For P of the first type, we also have the following conclusion:

Lemma 4.3.2. *If $\mathbf{0} \in P \in \mathcal{P}_6$, then* $\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) \leq \frac{237.946}{6^6}$.

By Corollary 4.3.1 and Lemma 4.3.2, we get an upper bound as a linear function for the λ_4 of a collection of k boxes from $\mathcal{P}_6$ with each box containing a vertex:

Lemma 4.3.3. *For P_i satisfying $\mathbf{0} \in P_i \in \mathcal{P}_6$, $1 \leqslant i \leqslant k$, we have*

$\lambda_4\left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k + \frac{228.572}{6^6}$.

Now we consider the second type of boxes, where each of them is a box in $\mathcal{P}_6$ containing a centre of a 3-face. As we did for any box $P \in \mathcal{P}_5$, one can always enlarge a box $P \in \mathcal{P}_6$ into an element of $\mathcal{P}_6^*$. Without loss of generality, we consider the 3-face $[0,1]^3 \times \{0\}^3$ with its centre $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. Define $B = [\frac{1}{2} - \frac{1}{9}, \frac{1}{2} + \frac{1}{9}]^3 \times \{0\}^3 = \left[\frac{7}{18}, \frac{11}{18}\right]^3 \times [0, \frac{1}{9}]^3$. For a single box $P \in \mathcal{P}_6^*$, assume P is located in $[0,1]^3 \times \{0\}^3$ and contains $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. We find an upper bound for the λ_4 of the region of P located outside of B as:

Lemma 4.3.4. *For P satisfying $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \in P \in \mathcal{P}_6^*$, the inequality $\lambda_4 (P \setminus B) < \frac{161.668}{3^{10}}$ holds.*

By the conclusion of Lemma 4.3.4, we also get an upper bound of λ_4 for a union of k boxes from $\mathcal{P}_6^*$ with each box containing $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$ as follows:

Lemma 4.3.5. *If each $P_i \in \mathcal{P}_6^*$ contains $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ where $1 \leqslant i \leqslant k$, then we have* $\lambda_4\left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k - \frac{222.99332}{6^6}$.

For a box of the third type, let $n = 6$ in Lemma 4.2.3 and then we have:

Corollary 4.3.6. *For any $P \in \mathcal{P}_6^*$ containing an element of A_6^*, we have $\lambda_4(P) < \frac{1}{8^4}$.*

For the last type of boxes, the following lemma holds:

Lemma 4.3.7. *For $P \in \mathcal{P}_6^*$ not containing any vertex, we have $\lambda_4(P) \leq \frac{393.66}{6^6}$.*

By the conclusions in terms of 4 types of boxes, we derive a lower bound on C_6:

Theorem 4.3.8. *$C_6 \geqslant 7177$.*

Proof. Recall that $[0,1]^6$ has 64 vertices, 160 3-faces, 12 elements in A_6^*, and sixty 4-faces. We will cover $B_{4,6}$ by the 4 kinds of boxes mentioned above. Note that each box in the covering has to belong to exactly one of these 4 types, and any box of the third type do not belong to the first or the second type. Let u_i, $1 \leqslant i \leqslant 64$, be the number of the first type of boxes covering the i-th vertex. Let v_k, $1 \leqslant k \leqslant 160$, be the quantity of the second type of boxes covering the centres of k-th 3-face. There are 12 boxes of the third type with each of them containing an element of A_6^*. Let w be the number of boxes required to cover $[0,1]^6$, then the number of boxes of the last type is $\left(w - 12 - \sum\limits_{i=1}^{64} u_i - \sum\limits_{j=1}^{160} v_j\right)$. Since the λ_4 of 4 types of boxes is at least 60, then by Lemma 4.3.3, Lemma 4.3.5 and Corollary 4.3.6, we have:

$$\sum_{i=1}^{64}\left(\frac{393.66}{6^6}u_i+\frac{228.572}{6^6}\right)+\sum_{j=1}^{160}\left(\frac{393.66}{6^6}v_j-\frac{222.99332}{6^6}\right)+\frac{12}{8^4}$$

$$+\frac{393.66}{6^6}\cdot\left(w-12-\sum_{i=1}^{64}u_i-\sum_{j=1}^{160}v_j\right)\geqslant 60,$$

$$w\geqslant 7176.23726,$$

implying $w \geqslant 7177$. Therefore we have $C_6 \geqslant 7177$. □

4.3.2 Proofs of lemmas

Lemma 4.3.2. *If* $\mathbf{0} \in P \in \mathcal{P}_6$, *then* $\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) \leq \frac{237.946}{6^6}$.

Proof of Lemma 4.3.2. Consider $P = \prod_{i=1}^{6}[x_i, x_i+\delta_i] \in \mathcal{P}_6^*$, where $x_i \leq 0 \leq x_i+\delta_i$, we assume that $x_i = 0$, $i = 1, ..., 6$. Suppose $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 \geq \delta_5 \geq \delta_6$ and use $m \geq 1$ to denote the largest index satisfying $\delta_m \geq \frac{1}{6}$. Since $\sum_{i=1}^{6}\delta_i = 1$, for $1 \leq m \leq 6$, we compute

$$\begin{aligned}
&\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) \\
&= \sum_{1\leq i<j<k<l\leq 6} \left(\delta_i\delta_j\delta_k\delta_l - \min\left\{\delta_i, \frac{1}{6}\right\}\min\left\{\delta_j, \frac{1}{6}\right\}\min\left\{\delta_k, \frac{1}{6}\right\}\min\left\{\delta_l, \frac{1}{6}\right\}\right) \\
&= \sum_{1\leq i<j<k<l\leq m\leq 6} \left(\delta_i\delta_j\delta_k\delta_l - \frac{1}{6^4}\right) + \sum_{1\leq i<j<k\leq m<l\leq 6} \left(\delta_i\delta_j\delta_k - \frac{1}{6^3}\right)\delta_l \\
&\quad + \sum_{1\leq i<j\leq m<k<l\leq 6} \left(\delta_i\delta_j - \frac{1}{6^2}\right)\delta_k\delta_l + \sum_{1\leq i\leq m<j<k<l\leq 6} \left(\delta_i - \frac{1}{6}\right)\delta_j\delta_k\delta_l.
\end{aligned}$$

Therefore, if $m = 6$, which means $P = \left[0, \frac{1}{6}\right]^6$, then $\lambda_4\left(P \setminus [0, \frac{1}{6}]^6\right) = 0$.

If $m = 5$, we have $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \in \left[\frac{1}{6}, 1\right]$ and $s_5 = \sum_{i=1}^{5}\delta_i \in \left(\frac{5}{6}, 1\right]$, then we obtain:

$$\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) = S_4\left(\delta_1, ..., \delta_5\right) - \frac{5}{6^4} + \left(S_3\left(\delta_1, ..., \delta_5\right) - \frac{10}{6^3}\right)\cdot\delta_6$$

$$\begin{aligned}
&\leq \left(\frac{s_5}{5}\right)^4 \cdot 5 - \frac{5}{6^4} + \left(\left(\frac{s_5}{5}\right)^3 \cdot 10 - \frac{10}{6^3}\right)\cdot(1 - s_5) \\
&=: f(s_5) < f(0.9955308824066) \\
&< 0.0041457730803 < \frac{193.5}{6^6}.
\end{aligned}$$

If $m = 4$, then $\delta_1, \delta_2, \delta_3, \delta_4 \in \left[\frac{1}{6}, 1\right]$ and $s_4 = \sum_{i=1}^{4} \delta_i \in \left(\frac{2}{3}, 1\right]$, we have:

$$\begin{aligned}
\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) &= \delta_1\delta_2\delta_3\delta_4 - \frac{1}{6^4} + \left(S_3\left(\delta_1, \delta_2, \delta_3, \delta_4\right) - \frac{4}{6^3}\right)(\delta_5 + \delta_6) \\
&\quad + \left(S_2\left(\delta_1, \delta_2, \delta_3, \delta_4\right) - \frac{6}{6^2}\right)\delta_5\delta_6 \\
&\leq \left(\frac{s_4}{4}\right)^4 - \frac{1}{6^4} + \left(\left(\frac{s_4}{4}\right)^3 \cdot 4 - \frac{4}{6^3}\right)(1 - s_4) \\
&\quad + \left(\left(\frac{s_4}{4}\right)^2 \cdot 6 - \frac{6}{6^2}\right)\left(\frac{1 - s_4}{2}\right)^2 \\
&=: f(s_4) < f(0.8754193758508) < 0.0049076164042 < \frac{228.970}{6^6}.
\end{aligned}$$

If $m = 3$, we have $\delta_1, \delta_2, \delta_3 \in \left[\frac{1}{6}, 1\right]$ and $s_3 = \delta_1 + \delta_2 + \delta_3 \in \left(\frac{1}{2}, 1\right]$, then we have:

$$\begin{aligned}
&\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) \\
&= \left(\delta_1\delta_2\delta_3 - \frac{1}{6^3}\right)(\delta_4 + \delta_5 + \delta_6) + \left(S_2\left(\delta_1, \delta_2, \delta_3\right) - \frac{3}{6^2}\right) \cdot S_2\left(\delta_4, \delta_5, \delta_6\right) \\
&\quad + \left(\delta_1 + \delta_2 + \delta_3 - \frac{3}{6}\right)\delta_4\delta_5\delta_6 \\
&\leq \left(\left(\frac{s_3}{3}\right)^3 - \frac{1}{6^3}\right)(1 - s_3) + \left(3\left(\frac{s_3}{3}\right)^2 - \frac{3}{6^2}\right) \cdot 3\left(\frac{1 - s_3}{3}\right)^2 + \left(s_3 - \frac{1}{2}\right)\left(\frac{1 - s_3}{3}\right)^3 \\
&=: f(s_3) < f(0.7295762244889) < 0.0050993455562 < \frac{237.946}{6^6}.
\end{aligned}$$

If $m = 2$, which means $\delta_1, \delta_2 \in [\frac{1}{6}, 1]$ and $s_2 = \delta_1 + \delta_2 \in \left(\frac{1}{3}, 1\right)$, we have:

$$\begin{aligned}
&\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) \\
&= \left(\delta_1 + \delta_2 - \frac{2}{6}\right) \cdot S_3\left(\delta_3, \delta_4, \delta_5, \delta_6\right) + \left(\delta_1\delta_2 - \frac{1}{6^2}\right) \cdot S_2\left(\delta_3, \delta_4, \delta_5, \delta_6\right) \\
&\leq \left(\delta_1 + \delta_2 - \frac{1}{3}\right)\left(\frac{1-(\delta_1+\delta_2)}{4}\right)^3 \cdot 4 + \left(\delta_1\delta_2 - \frac{1}{36}\right)\left(\frac{1-(\delta_1+\delta_2)}{4}\right)^2 \cdot 6 \\
&\leq \left(s_2 - \frac{1}{3}\right) \cdot \frac{1}{16} \cdot (1-s_2)^3 + \left(\left(\frac{s_2}{2}\right)^2 - \frac{1}{36}\right)(1-s_2)^2 \cdot \frac{3}{8} \\
&=: f(s_2) < f(0.5649778198383) < 0.0048837297423 < \frac{227.870}{6^6}.
\end{aligned}$$

If $m = 1$, which means $\delta_1 \in \left(\frac{1}{6}, 1\right]$, then we have:

$$\begin{aligned}
\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) &= \left(\delta_1 - \frac{1}{6}\right) \cdot S_3\left(\delta_2, ..., \delta_6\right) \\
&\leq \left(\delta_1 - \frac{1}{6}\right) \cdot 10 \cdot \left(\frac{1-\delta_1}{5}\right)^3 \\
&=: f\left(\delta_1\right) \leq f\left(\frac{3}{8}\right) < 0.0040690104168 < \frac{189.890}{6^6}.
\end{aligned}$$

Overall, $\lambda_4\left(P \setminus \left[0, \frac{1}{6}\right]^6\right) < \frac{237.946}{6^6}$. $\square$

Lemma 4.3.3. *For P_i satisfying $\mathbf{0} \in P_i \in \mathcal{P}_6$, $1 \leqslant i \leqslant k$, we have*

$\lambda_4\left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k + \frac{228.572}{6^6}$.

Proof of Lemma 4.3.3. Consider $P_i \in \mathcal{P}_5^*$ containing $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$ where $1 \leqslant i \leqslant k$. For $k = 1$, by Corollary 4.3.1, we have $\lambda_4\left(P\right) = \frac{5}{432} \leqslant \frac{393.66}{6^6} \cdot 1 + \frac{228.572}{6^6}$. For $k \geq 2$, we get:

$$\begin{aligned}\lambda_4\left(\bigcup_{i=1}^{k} P_i\right) &= \lambda_4\left(\left[0, \frac{1}{6}\right]^6\right) + \lambda_4\left(\left(\bigcup_{i=1}^{k} P_i\right) \setminus \left[0, \frac{1}{6}\right]^6\right) \\ &< \frac{5}{432} + \frac{237.946}{6^6}k \leqslant \frac{393.66}{6^6}k + \frac{228.572}{6^6},\end{aligned}$$

where it can be easily checked that the last inequality is equivalent to $k \geqslant 2$. Therefore, we get the upper bound $\lambda_4\left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k + \frac{228.572}{6^6}$. $\square$

Lemma 4.3.4. *For P satisfying $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \in P \in \mathcal{P}_6^*$, the inequality $\lambda_4\left(P \setminus B\right) < \frac{161.668}{3^{10}}$ holds.*

Proof of Lemma 4.3.4. Recall the box $B = \left[\frac{1}{2} - \frac{1}{9}, \frac{1}{2} + \frac{1}{9}\right]^3 \times \left[0, \frac{1}{9}\right]^3 = \left[\frac{7}{18}, \frac{11}{18}\right]^3 \times \left[0, \frac{1}{9}\right]^3$. Define the a box from $\mathcal{P}_6^*$ as $P = \left[\frac{1}{2} - a_1, \frac{1}{2} + a_2\right] \times \left[\frac{1}{2} - b_1, \frac{1}{2} + b_2\right] \times \left[\frac{1}{2} - c_1, \frac{1}{2} + c_2\right] \times [0, d_1] \times [0, d_2] \times [0, d_3]$, where $0 \leqslant a_i, b_i, c_i, d_j \leqslant \frac{1}{2}$, $i, j, k \in \{1, 2\}$, $l \in \{1, 2, 3\}$, and $\sum\limits_{i=1}^{2} a_i + \sum\limits_{j=1}^{2} b_j + \sum\limits_{k=1}^{2} c_k + \sum\limits_{l=1}^{3} d_l = 1$. Define $a = a_1 + a_2$, $b = b_1 + b_2$, $c = c_1 + c_2$ and $d = d_1 + d_2 + d_3$. Now for $a, b, c, d \in [0, 1]$, we find the 4-dimensional volume for the parts of P located out of B as:

$$F(x, y, z, w) = xyzw - \min\left\{x, \frac{1}{9}\right\} \min\left\{y, \frac{1}{9}\right\} \min\left\{z, \frac{1}{9}\right\} \min\left\{w, \frac{1}{9}\right\}.$$

For $P \in \mathcal{P}_6^*$ containing the centre of a 3-face, by symmetry, we have $\lambda_4(P \setminus B) = \sum_{l=1}^{3}\sum_{k=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2} F(a_i, b_j, c_k, d_l)$. Now we are going to find an upper bound for this sum. Similarly to what we have done previously, $F(x, y, z, w)$ is a convex function in each variable, then we proceed the same way in the proof of Lemma 4.2.8 using Karamata's inequality, and obtain $\lambda_4 (P \setminus B) = \sum_{l=1}^{3}\sum_{k=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2} F(a_i, b_j, c_k, d_l) \leqslant F(a, b, c, d)$. Next we look for an upper bound for $F(a, b, c, d)$. Arrange a, b, c, d in order so that $a \geq b \geq c \geq d$ with the sum of them as 1. For the first case, i.e. $a \in \left(\frac{1}{9}, 1\right)$ and $b, c, d \in \left[0, \frac{1}{9}\right]$, with slight abuse of notation s, we let $s = b + c + d \in \left[0, \frac{1}{3}\right]$, we have:

$$F(a, b, c, d) = \left(a - \frac{1}{9}\right) \cdot bcd \leqslant \left(\frac{8}{9} - s\right)\left(\frac{s}{3}\right)^3 =: f(s) < f\left(\frac{1}{3}\right)$$
$$= \frac{5}{3^8} < 0.00076207896 < \frac{161.668}{3^{10}}.$$

For the second case, i.e. $a, b \in \left(\frac{1}{9}, 1\right)$ and $c, d \in \left[0, \frac{1}{9}\right]$, let $s = c + d \in \left[0, \frac{2}{9}\right]$, we have:

$$F(a, b, c, d) = \left(ab - \frac{1}{9^2}\right) \cdot cd \leq \left(\left(\frac{1-s}{2}\right)^2 - \frac{1}{81}\right) \cdot \left(\frac{s}{2}\right)^2$$
$$=: f(s) < f\left(\frac{2}{9}\right) = \frac{5}{2916} < \frac{161.668}{3^{10}}.$$

For the third case, i.e. $a, b, c \in \left(\frac{1}{9}, 1\right)$ and $d \in \left[0, \frac{1}{9}\right]$, we have:

$$F(a,b,c,d) = \left(abc - \frac{1}{9^3}\right) d \leqslant \left(\left(\frac{1-d}{3}\right)^3 - \frac{1}{9^3}\right) \cdot d$$

$$=: f(d) < f\left(\frac{1}{9}\right) < 0.0027378391958 < \frac{161.668}{3^{10}}.$$

Overall, for the box P satisfying $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \in P \in \mathcal{P}_6^*$, we have $\lambda_4 \left(P \setminus B\right) < \frac{161.668}{3^{10}}$. □

Lemma 4.3.5. *If each $P_i \in \mathcal{P}_6^*$ contains $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ where $1 \leqslant i \leqslant k$, then we have $\lambda_4 \left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k - \frac{222.99332}{6^6}$.*

Proof of Lemma 4.3.5. Consider $P_i \in \mathcal{P}_6^*$ containing $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$, where $1 \leqslant i \leqslant k$. Note that $\lambda_4(B) = \left(\frac{2}{9}\right)^3 \cdot \frac{1}{9} \cdot 3 = \frac{24}{9^4}$. For $k = 1$, we have $\lambda_4 \left(P\right) = \frac{24}{9^4} \leqslant \frac{393.66}{6^6} \cdot 1 - \frac{222.99332}{6^6}$. For $k \geq 2$, we get:

$$\lambda_4 \left(\bigcup_{i=1}^{k} P_i\right) = \lambda_4 \left(B\right) + \lambda_4 \left(\left(\bigcup_{i=1}^{k} P_i\right) \setminus B\right) < \frac{24}{9^4} + \frac{161.668}{3^{10}}k$$

$$\leqslant \frac{393.66}{6^6}k - \frac{222.99332}{6^6},$$

where it can be easily checked that the last inequality is equivalent to $k \geqslant 1.480358$. Therefore, we get the upper bound $\lambda_4 \left(\bigcup_{i=1}^{k} P_i\right) < \frac{393.66}{6^6}k - \frac{222.99332}{6^6}$.

□

Lemma 4.3.7. *For $P \in \mathcal{P}_6^*$ not containing any vertex, we have $\lambda_4(P) \leq \frac{393.66}{6^6}$.*

Proof of Lemma 4.3.7. Suppose $P = \prod_{i=1}^{6}[x_i, x_i + \delta_i] \in \mathcal{P}_6^*$. Let $m \geq 0$ be the number of indexes i such that $[x_i, x_i+\delta_i] \cap \{0,1\} = \emptyset$. Without loss of generality, suppose that $x_i > 0$ for $i = 1, \dots, m$. Since P does not contain any vertex, then $m \geq 1$. If $m = 1$, we have:

$$\lambda_4(P) \leqslant \delta_1 \cdot S_3(\delta_2, \delta_3, \delta_4, \delta_5, \delta_6) \leqslant \delta_1 \cdot \left(\frac{1-\delta_1}{5}\right)^3 \cdot 10$$

$$= \frac{2}{25} \cdot \delta_1 \left(1-\delta_1\right)^3 =: f(\delta_1) \leqslant f\left(\frac{1}{4}\right) = \frac{393.66}{6^6}.$$

If $m = 2$, let $s_2 = \delta_1 + \delta_2 \in [0,1]$ and then:

$$\lambda_4(P) \leqslant \delta_1\delta_2 \cdot S_2\left(\delta_3, \delta_4, \delta_5, \delta_6\right) \leq \left(\frac{s_2}{2}\right)^2 \cdot \left(\frac{1-s_2}{4}\right)^2 \cdot 6$$

$$=: f(s_2) \leqslant f\left(\frac{1}{2}\right) = \frac{3}{512} = \frac{273.375}{6^6}.$$

If $m = 3$, let $s_3 = \delta_1 + \delta_2 + \delta_3 \in [0,1]$ and we have:

$$\lambda_4(P) = \delta_1\delta_2\delta_3 \cdot \sum_{i=4}^{6} \delta_i \leqslant \left(\frac{s_3}{3}\right)^3 (1-s_3) =: f(s_3) \leqslant f\left(\frac{3}{4}\right) = \frac{182.25}{6^6}.$$

If $m = 4$, then $\lambda_4(P) = \delta_1\delta_2\delta_3\delta_4 \leq \left(\frac{\sum_{i=1}^{4}\delta_i}{4}\right)^4 = \frac{182.25}{6^6}$. If $m = 5, 6$, we have $\lambda_4(P) = 0$. Overall, if P does not contain any vertex, then $\lambda_4\left(P\right) < \frac{393.66}{6^6}$. $\square$

Appendix A

Sagemath code to show the existence of U_5 in G_5

```
F = [] #centres of 2d faces of 5 cube
for i in range(4):
    for j in range(i+1,5):
        face = [0,0,0,0,0]
        face[i] = 1/2
        face[j] = 1/2
        ind = []
        for k in range(5):
            if face[k]!=1/2:
                ind.append(k)
        for k0 in range(2):
            for k1 in range(2):
                for k2 in range(2):
                    cface = copy(face)
                    cface[ind[0]]=k0
                    cface[ind[1]]=k1
                    cface[ind[2]]=k2
                    F.append(cface)
def are_opposite(f1,f2):
    c = 0
    for i in range(5):
        if f1[i]==f2[i]:
            c += 1
    if c==4:
        return 1
```

```
    else:
        return 0
W = [] #indexes of pairs of "opposite" 2d faces, also vertices of our graph
for i in range(79):
    for j in range(i+1,80):
        if are_opposite(F[i],F[j])==1:
            W.append([i,j])
def is_edge(i,j):
    if i==j:
        return 0
    if W[i][0]== W[j][0] or W[i][0]== W[j][1] or
       W[i][1]== W[j][0] or W[i][1]== W[j][1]:
        return 1
    for k in range(5):
        if F[W[i][0]][k]+ F[W[i][1]][k]- F[W[j][0]][k]- F[W[j][1]][k]!=0:
            return 0
    return 1
g = Graph(Matrix([[is_edge(i,j) for i in range(120)] for j in range(120)]))
ins = g.independent_set()
print(len(ins))
print(ins)

Output:
40
[0, 5, 9, 10, 12, 17, 20, 23, 25, 27, 33, 34, 36, 42, 43, 44, 50, 52, 54,
55, 62, 64, 65, 71, 73, 75, 81, 82, 85, 87, 93, 94, 96, 101, 104, 107, 109,
112, 115, 117]
```

Appendix B

Sagemath code to show the existence of U_6 in G_6

```
F = [] #centres of 3d faces of 6 cube
for i in range(4):
    for j in range(i+1,5):
        for u in range(j+1,6):
            face = [0,0,0,0,0,0]
            face[i] = 1/2
            face[j] = 1/2
            face[u] = 1/2
            ind = []
            for k in range(6):
                if face[k]!=1/2:
                    ind.append(k)
            for k0 in range(2):
                for k1 in range(2):
                    for k2 in range(2):
                        cface = copy(face)
                        cface[ind[0]]=k0
                        cface[ind[1]]=k1
                        cface[ind[2]]=k2
                        F.append(cface)
def are_opposite(f1,f2):
    c = 0
    for i in range(6):
        if f1[i]==f2[i]:
            c += 1
```

```
    if c==5:
        return 1
    else:
        return 0
W = [] #indexes of pairs of "opposite" 3d faces, also vertices of our graph
for i in range(159):
    for j in range(i+1,160):
        if are_opposite(F[i],F[j])==1:
            W.append([i,j])
def is_edge(i,j):
    if i==j:
        return 0
    if W[i][0]== W[j][0] or W[i][0]== W[j][1] or W[i][1]== W[j][0] or W[i][1]== W[j][1]:
        return 1
    for k in range(6):
        if F[W[i][0]][k]+ F[W[i][1]][k]-F[W[j][0]][k]-F[W[j][1]][k]!=0:
            return 0
    return 1
A = Permutations(240).random_element()
g = Graph(Matrix([[is_edge(A[i]-1,A[j]-1) for i in range(160)] for j in range(160)]))
ins = g.independent_set()
print(len(ins))
print([A[i]-1 for i in ins])
```

```
Output:
60
[129, 180, 84, 171, 5, 51, 118, 238, 224, 100, 163, 198, 31, 137,
89, 127, 38, 185, 151, 121, 150, 170, 208, 41, 15, 143, 178, 234,
136, 144, 211, 24, 97, 54, 134, 202, 50, 13, 47, 103, 9, 221, 111,
124, 228, 117, 32, 227, 70, 205, 174, 164, 152, 213, 195, 72, 21,
81, 61, 95]
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