Semilinear Elliptic Problems involving a Parameter

by

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Abstract

We consider

$$\begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. The nonlinear term f is smooth, positive, increasing, convex, superlinear at ∞, and $\lambda > 0$ a parameter. We also consider (1) in case of nonlinearity $\frac{1}{(1-u)^2}$ (MEMS nonlinearity), and with a divergence free advection term, a(x) ($-\Delta u + a(x) \cdot \nabla u = \lambda f(u)$ in Ω and $u = 0$ on $\partial\Omega$). In this thesis, we are interested in talking about the existence of stable minimal solutions to these partial differential equations (pde's). We show, when $\lambda < \lambda^*$ (a critical parameter), there is a minimal stable solution and when $\lambda > \lambda^*$, there exists no solution. Here, stability of solution means nonnegativeness of the first eigenvalue of the linearized operator associated with the pde. This nonnegative inequality can also be viewed as the second variation of energy functional associated with the pde at u. At λ^*, we obtain a unique weak solution which is the limit of minimal solutions ($\lim_{\lambda \nearrow \lambda^*} u_\lambda \nearrow u^*$), we call it extremal solution. Properties of extremal solution depend strongly on Ω, f, N.

For (1), the extremal solution is smooth in $N \leq 9$ with $f(u) = e^u$ while it is singular for $N \geq 10$, $\Omega = B_1$. The best result is by Nedev, which says u^* is bounded for any f and Ω when $N \leq 3$. We discuss the radial case which shows the optimal regularity result for u^* in $N \leq 9$. For the MEMS model, all stable solutions are smooth iff the dimension is $N \leq 7$. For the pde with advection, there is no suitable variational characterization for the stability assumption. To overcome this difficulty, we use a general version of Hardy's inequality to show smoothness of extremal solution in dimension $N \leq 9$ with exponential nonlinearity.

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We consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Such problem arises in a variety of situations, in the theory of nonlinear diffusion generated by nonlinear sources, in the theory of thermal ignition of gases (see [14],[20]), in quantum field theory and mechanical statistics (see [29],[6]), and in the theory of gravitational equilibrium of stars (see [22],[20]). Depending on the nonlinear term $f(u)$, this equation plays important roles in different fields. The equation

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3}$$

known as Gelfand problem (bears many names: Barrenblatt, Bratu, Emden, Fowler, Frank-Kamenetskii, and Liouville are some of the famous scientists to whom the equation (3) has been attributed). Here $\lambda > 0$ a parameter, B_1 unit ball in $\mathbb{R}^N$. In dimension $N = 1, 2, 3$, equation (3) can be derived from the thermal self-ignition model. The full model describes the reaction process in a combustible material during the ignition period. On the left side, $-\Delta$ is a diffusion operator, accounting for the diffusion of heat from the hot reactant to cold boundary and on the right e^u is the reaction term. A solution u represents the temperature inside a cylindrical vessel whose walls are ideally conducting. The solution u is dimensionless when the system has reached an intermediate-asymptotic steady state and the underlying space variable $x \in B_1$ should be thought of as dimensionless (the vessel's size has been normalized). The term e^u models the production of heat induced by the chemical reaction. The diffusion operator and the reaction term compete. In one kind of reaction, the produced heat does not have time to be carried away through the walls of the vessel: either the combustible rarifies and the reaction dies out, or there is so much combustible that a thermal explosion happens. Both way no solution of (3) should be expected. On the opposite, a solution exists in another kind of reaction with a quick occurrence of an equilibrium between produced and diffused heat. The balance between diffusion and reaction is quantified by the parameter $\lambda > 0$: we should expect no solution when λ is large and solution exists when λ small. The original motivation of studying pde (2) is also to understand what happens when a advection term is present ($-\Delta u + a(x) \cdot \nabla u = \lambda f(u)$ in Ω). Of particular interest is to undertsand how the presence of an advection term and its features affect on λ^* (extremal value of λ).

We also discuss the pde (2) with the nonlinear term $f(u) = \frac{1}{(1-u)^2}$,

$$\begin{cases} -\Delta u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

This equation is used to model a simple Micro-Electro-Mechanical device. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a parallel rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate. At present, the variety of MEMS devices and applications are continually increasing, and the advent of MEMS has revolutionized numerous branches of science and industry.

Literature Review

We consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \quad\quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

in a domain $\Omega \subset \mathbb{R}^N$. The nonlinearity $f(u)$ is smooth, positive, convex, increasing and superlinear at ∞.

It is Haïm Brezis who stresses, since mid-nineties, the significance of this problem. It has been shown in the pioneering works of Keener and H.Keller (1974) [21] , Joseph and Lundgren (1972/73) [20], and Crandall and Rabinowitz (1975) [9] that there exists a critical parameter $\lambda^* > 0$, so that (5) admits positive solutions for $0 < \lambda < \lambda^*$, while no positive solution exists for $\lambda > \lambda^*$. Crandal and Rabinowitz proved the boundedness of solution at $\lambda = \lambda^*$ not only for exponential nonlinearities but also for others when $N \leq 9$. Nonexistence of weak solution of (5) for $\lambda > \lambda^*$ was established in a seminal paper by Brezis, Cazenave, Martel, and Ramiandrisoa [2] and Martel's result showed the uniqueness of weak solution at $\lambda = \lambda^*$ in 1996. The linearized problem was studied by Brezis and Vázquez in [3] in great detail. In particular, when the domain is a ball, and for the exponential and power nonlinearities (typical examples: $f(u) = e^u, f(u) = (1+u)^p$ where $p > 1$), the solutions at the critical value are uniformly bounded in dimensions less or equal to $N = 9$ and $N = 10$ respectively, while in higher dimensions they are unbounded. For more general nonlinearities and domain Ω, Nedev [25] established the regularity of solutions at $\lambda = \lambda^*$ in dimensions $N = 2, 3$ in 2000, and in dimension $N = 4$ is established by Cabré [4] in 2010. The radial case, $\Omega = B_1$, was settled by Cabré and Capella in 2006 [5]. Finally, the semilinear equation (5) has also been studied for f blowing up at $u = 1$. This is connected with Micro-Electro-Mechanical devices (MEMS). Experimental work in this area, dates back to 1967 and the work of Nathanson et.el. [24], and in 1968, G.I Taylor [30] . The elliptic problem

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda > 0$ denotes the applied voltage and the nonnegative continuous function $f(x)$ characterizes the varying dielectric permittivity of the elastic membrane. In dimension $N = 2$, this equation models a simple MEMS device. The problem was first studied by Pelesko [27] in (2001/02), where the author focused on lower dimensions $N = 1$ or $N = 2$ and he considered the profile $f(x) \geq C > 0$ or $f(x) = |x|^\alpha$. The study of (6) was then extended by Guo, Pan and Ward [17] in 2005, where the authors considered (6) for a more general profile $f(x)$ which can vanish at somewhere. After that time within 2 years this elliptic problem was further extended and sharpened in [12], [19], [18], [16], [11] and the references therein.

Outline of the thesis

In this thesis, we discuss three different semilinear elliptic pde's. This thesis is divided into 4 chapters. Chapter 1 summarizes the concepts that are required to understand this thesis. Some spaces of functions, weak derivatives, weak solutions, very weak solutions, ultra weak solutions, Sobolev spaces, Sobolev embeddings, regularity results for the weak solutions and ultra weak solutions, and stability of solutions have been discussed here.

In chapter 2, we explore the semilinear pde, $-\Delta u = \lambda f(u)$ in Ω and $u = 0$ on $\partial\Omega$, where f is positive, increasing, convex, superlinear at ∞. We show that for all $0 < \lambda < \lambda^*$, there exists a minimal stable solution and for $\lambda > \lambda^*$, there is no solution, not even in a weak sense. Uniqueness of extremal solution (defined to be the limit of minimal solution) is discussed here. We also discuss the regularity of extremal solution depending on Ω, f, N. This chapter also includes the discussion of the singular extremal solution.

Chapter 3 outlines the same result (when $\lambda < \lambda^*$ there is a stable minimal solution and when $\lambda > \lambda^*$ no solution exists) in case of MEMS type nonlinearity $\frac{1}{(1-u)^2}$. Here we discuss the regularity of extremal solution in $N \leq 7$.

Chapter 4 deals with the mentioned result (when $\lambda < \lambda^*$ there is a stable minimal solution and when $\lambda > \lambda^*$ no solution exists) for the pde with advection, $-\Delta u + a(x) \cdot \nabla u = \lambda f(u)$ in Ω and $u = 0$ on $\partial\Omega$, where $a(x)$ is a smooth divergence free vector field. Here we discuss the proof of this result with a slight modification. In this chapter, we also discuss the regularity of extremal solution in dimension $N \leq 9$ with the help of general Hardy's inequality.

1

Mathematical Background

1.1 Some Function Spaces and some basic Definitions

We begin with some functions spaces. Let Ω denote an open, bounded domain in $\mathbb{R}^N$ with smooth boundary.

1. (Continuous functions). We let $C(\Omega)$ denote the set of continuous functions on Ω.

   Example 1.1. *Let $\Omega = (0,1) \subset \mathbb{R}$. Let $u(x) = |x|$, $v(x) := \frac{1}{|x|}$. Then $u, v \in C(\Omega)$.*

2. (Uniformly continuous functions) Let $C(\overline{\Omega})$ denote the set of uniformly continuous functions on Ω. It can be shown that $(C(\overline{\Omega}), \|\cdot\|)$ is a Banach space where

$$\|u\| := \sup_{x\in\Omega} |u(x)| = \max_{x\in\overline{\Omega}} |u(x)|.$$

   Example 1.2. $\Omega := (0,1)$. *Note that $u(x) = |x| \in C(\overline{\Omega})$ but $v(x) = \frac{1}{|x|} \notin C(\overline{\Omega})$.*

3. (L^p spaces) For $1 \leq p < \infty$ set

$$\|u\|_{L^p(\Omega)} := \|u\|_{L^p} = \|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

   We define

$$L^p(\Omega) := \{u : u \text{ is a function defined on } \Omega \text{ with } \|u\|_{L^p(\Omega)} < \infty \}.$$

   Then $L^p(\Omega)$ is a Banach space.

4. $L^\infty(\Omega)$ is space of bounded functions with sup norm.

5. (Smooth compactly supported functions) We let $C_c^\infty(\Omega)$ denote the set of smooth functions $u : \Omega \to \mathbb{R}$ which are compactly supported in Ω.

6. $C_c^\infty(\Omega \setminus \{k\})$ means infinitely differentiable with compact support take away set $\{k\}$ in Ω means any function belonging to this space is zero near the set $\{k\}$.

Definition 1.3. *(Hilbert space) we call $(X, \|\cdot\|)$ is a real normed linear space provided*

- *X is a vector space over $\mathbb{R}$.*
- *$\|\cdot\| : X \to \mathbb{R}$ is a norm on X means*
  - $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in X$
  - $\|\lambda u\| = |\lambda|\|u\| \ \forall \lambda \in \mathbb{R}\ ,\ u \in X$
  - $\|u\| \geq 0\ \forall u \in X$
  - $\|u\| = 0$ *if and only if* $u = 0$.

The real normed linear space X is called Banach space if X is a complete normed linear space. By complete we mean if every Cauchy sequence in X converges to an *element in X. By this we mean if $\{u_n\}_n \subset X$ such that $\forall \epsilon > 0$ there is some N_ϵ such that $\forall n, m \geq N_\epsilon$ we have $\|u_n - u_m\| < \epsilon$, then there is some $u \in X$ such that $\|u_n - u\| \to 0$ as $n \to \infty$. A real Hilbert space $(H, \|\cdot\|)$ is a Banach space endowed with an inner product which generates the norm. The mapping $(,) : H \times H \to \mathbb{R}$ is called an inner product if*

- $(u, v) = (v, u)\ \forall u, v \in H$
- *For all $v \in H$, $u \to (u, v)$ is a linear mapping from H to $\mathbb{R}$*
- $(u, u) \geq 0\ \forall u \in H$
- $(u, u) = 0$ *if and only if* $u = 0$.

If $(,)$ is an inner product then the norm is given by

$$(u, u) = \|u\|^2 \tag{1.1}$$

By using Cauchy-Schwarz inequality ($|(u, v)| \leq \|u\|\|v\|$) we can easily verify (1.1) satisfies all the properties of normed linear space.

Example 1.4. *The space $L^2(\Omega)$ is a Hilbert space with the inner product given by*

$$(u, v) = \int_\Omega uv\ dx.$$

The Sobolev space H^1 is a Hilbert space with inner product

$$(u, v) = \int_\Omega (uv + \nabla u \nabla v)\ dx.$$

Definition 1.5. *(Hölder space) Suppose $\Omega \subset \mathbb{R}^N$ and $0 < \gamma \leq 1$. Suppose $u : \Omega \to \mathbb{R}$ be a Lipschitz continuous function which by definition satisfies the estimate*

$$|u(x) - u(y)| \leq l|x - y|, \tag{1.2}$$

$\forall x, y \in \Omega$ and for any constant l. So the function is continuous and provides a uniform modulus of continuity over a closed and bounded interval in real line. Let us consider also a function u satisfying variant of (1.2) namely

$$|u(x) - u(y)| \leq l|x - y|^{\gamma},$$

for some constant l. Such a function is said to be Hölder continuous with exponent γ. The Hölder space $C^{k,\gamma}(\overline{\Omega})$ consists of functions $u \in C^k(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{k,\gamma}} = \sum_{|\alpha| \leq k} \|\partial^{\alpha} u\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [\partial^{\alpha} u]_{C^{0,\gamma}(\overline{\Omega})} \text{ is finite,} \tag{1.3}$$

where

$$\sum_{|\alpha|=k} [\partial^{\alpha} u]_{C^{0,\gamma}(\overline{\Omega})} = \sum_{|\alpha|=k} \sup_{x,y \in \Omega, x \neq y} \frac{|\partial^{\alpha} u(x) - \partial^{\alpha} u(y)|}{|x - y|^{\gamma}},$$

is known as γ^{th} Hölder seminorm of order k. If $k = 0, 1, 2$ we have

$$\|u\|_{C^{0,\gamma}} = \|u\|_{C(\overline{\Omega})} + \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\gamma}},$$

$$\|u\|_{C^{1,\gamma}} = \|u\|_{C(\overline{\Omega})} + \|\nabla u\|_{C(\overline{\Omega})} + \sum_{k=1}^{N} \sup_{x,y \in \Omega, x \neq y} \frac{|u_{x_k}(x) - u_{y_k}(y)|}{|x - y|^{\gamma}},$$

$$\|u\|_{C^{2,\gamma}} = \|u\|_{C(\overline{\Omega})} + \|\nabla u\|_{C(\overline{\Omega})} + \|\partial^2 u\|_{C(\overline{\Omega})} + \sum_{i,j=1}^{N} \sup_{x,y \in \Omega, x \neq y} \frac{|u_{x_i x_j}(x) - u_{y_i y_j}(y)|}{|x - y|^{\gamma}},$$

respectively. So the space $C^{k,\gamma}(\overline{\Omega})$ consists of those functions that are k times continuously differentiable and whose k^{th} partial derivatives are Hölder continuous with exponent γ.

Properties of Hölder space

- *For $k = 0, 1, 2$ and $0 < \gamma < 1$ the space of functions $C^{k,\gamma}(\overline{\Omega})$ is a Banach space. That means the norm $\|\cdot\|_{C^{k,\gamma}}$ defined in (1.3) satisfies all the properties of normed linear space and in addition each Cauchy sequence converges with* *respect to this norm.*

- *For* $k = 0, 1, 2$ *and* $0 < \gamma < \beta$ *one has* $C^{k,\beta}(\overline{\Omega}) \subset\subset C^{k,\gamma}(\overline{\Omega})$.

- *For* $i, j = 0, 1, 2$ *with* $i < j$ *one has* $C^{j,\beta}(\overline{\Omega}) \subset\subset C^{i,\gamma}(\overline{\Omega})$, *where* $\gamma, \beta \in (0, 1)$.

Definition 1.6. *(Young's inequality) Let* $p > 1, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. *Then for* $a, b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Definition 1.7. *(Young's inequality with* ε*) For* $a, b > 0$ *and* $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q,$$

where $C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$.

Definition 1.8. *(Hölder's inequality) Let* $p, q \in [0, \infty]$ *are conjugate exponents* $(\frac{1}{p} + \frac{1}{q} = 1)$ *and* f, w *are two measurable functions then*

$$\|fw\|_1 \leq \|f\|_p \|w\|_q.$$

Definition 1.9. *(Minkowski's inequality) Assume* $1 \leq p \leq \infty$ *and* $u, v \in L^p(\Omega)$ *then* $\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$.

Definition 1.10. *(Bounded linear operator) Suppose* X *and* Y *are Banach spaces. A mapping* $A : X \to Y$ *is bounded linear operator if* $\forall u \in X$ *there exists some* $M \geq 0$ *such that*

$$\frac{\|Au\|}{\|u\|} \leq M.$$

The smallest such M *is called the operator norm* $\|A\|_{op}$ *of* A. *A linear operator between normed space is bounded iff it is continuous.*

Example 1.11. *The Laplace operator* $\Delta : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *is bounded.*

Definition 1.12. *(Adjoint operator) Suppose $A : H \to H$ is a bounded linear operator. The operator $A^* : H \to H$ is said to be adjoint of the operator A if for all $x, y \in H$ it satisfies*

$$(Ax, y) = (x, A^*y).$$

Definition 1.13. *(Self adjoint operator) A bounded linear operator defined on H is self adjoint if $A = A^*$. Then $\forall x, y \in H$*

$$(Ax, y) = (x, Ay).$$

We also have a result that says if A is a bounded linear operator on H then it is self adjoint iff (Ax, x) is real $\forall x \in H$. The eigenvalues of a bounded self adjoint linear operator on H are real.

Definition 1.14. *(Rayleigh's quotient) The Rayleigh's quotient of a bounded self adjoint operator is defined as*

$$R(x) := \frac{(Ax, x)}{(x, x)}.$$

Theorem 1. *(Lebesgue's Monotone Convergence Theorem) Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of monotonically increasing nonnegative measurable functions on a measurable set X, and let $f(x) = \lim_{n\to\infty} f_n(x)$ almost everywhere on X. Then $\displaystyle\int_X f = \lim_{n\to\infty} \int_X f_n$.*

Theorem 2. *(Lebesgue's Dominated Convergence Theorem) Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions that converge pointwise to a limit function f almost everywhere on X. Suppose there exists a Lebesgue integrable function g such that $|f_n(x)| \leq g(x)$ almost everywhere on X for each $n \in \mathbb{N}$ and for all points $x \in X$. Then f is Lebesgue integrable on X and $\lim_{n\to\infty} \displaystyle\int_X f_n(x)\, dx = \int_X \lim_{n\to\infty} f_n(x)\, dx = \int_X f(x)\, dx$.*

Theorem 3. *(Fatou's Lemma) Let* $\{f_n(x)\}_{n=1}^{\infty}$ *be a sequence of nonnegative Lebesgue integrable functions on an interval* X *that converges to a function* f *almost everywhere on* X*, and suppose there exists an* $L \in \mathbb{R}$*,* $L > 0$ *such that* $\int_X f_n(x)\,dx \leq L$ *forall* $n \in \mathbb{N}$*. Then* f *is Lebesgue integrable on* X *and* $\int_X f(x)\,dx \leq L$

Theorem 4. *(Riesz representation theorem) Let* H *be a Hilbert space with inner product* $(,)$*. Suppose* H^* *is the dual of* H *and* $H^* := \{f : H \to \mathbb{R}, f$ *is linear and continuous* $\}$*. Given* $f \in H^*, \exists\ !x_f \in H$ *such that*

$$(f, y) = (x_f, y) \quad \forall y \in H.$$

Definition 1.15. *(Weak derivatives) Let* u *be a function in Lebesgue space* $L^1(\Omega)$*. We say* $v \in L^1(\Omega)$ *is a weak derivative of* u *if for* $\phi \in C_c^{\infty}(\Omega)$ *we have*

$$\int_{\Omega} u(x)\phi_{x_i}(x)dx = -\int_{\Omega} v(x)\phi(x)dx.$$

Here we have used integration by parts technique and $\phi = 0$ *on* $\partial\Omega$*. Generalizing to* N *dimensions, if both* u *and* v *are in the space* $L_{1,loc}(\Omega)$ *of locally integrable functions of some open subset* $\Omega \subseteq \mathbb{R}^N$ *and if* α *is a multi-index then we say* v *is* α^{th} *weak derivative of* u *if*

$$\int_{\Omega} u(x)D^{\alpha}\phi(x)dx = (-1)^{|\alpha|}\int_{\Omega} v(x)\phi(x)dx. \tag{1.4}$$

Here $D^{\alpha}\phi$ *is defined as* $D^{\alpha}\phi = \frac{\partial^{|\alpha|}\phi}{\partial_{x_1}^{\alpha_1}, \partial_{x_2}^{\alpha_2}, \partial_{x_3}^{\alpha_3}, \ldots\ldots, \partial_{x_N}^{\alpha_N}}$*. The relation (1.4) eliminates the need of* u *to be differentiable. In other words, if we are given a function* u *and if there happens to exist a function* v *which verifies (1.4), we say* $D^{\alpha}u = v$ *in the weak sense. If there does not exist such* v *then* u *does not posses a weak* α^{th} *partial derivative. A weak* α^{th} *partial derivative, if it exists, is uniquely defined up to a set of measure* 0*.*

Example 1.16. *Define $l \in C(\mathbb{R})$ by*

$$l(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then l is weakly differentiable with $l'(x) = \mathbf{1}_{(0,\infty)}$,

$$l'(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

The choice of $l'(x)$ at $x = 0$ is irrelevent since the weak derivative is only defined up to a set of pointwise almost everywhere equivalence. To check it, for any test function $\phi \in C_c^\infty(\mathbb{R})$ integration by parts formula gives

$$\int l(x)\phi'(x)dx = \int_0^\infty x\phi'(x)dx = -\int_0^\infty \mathbf{1}_{(0,\infty)}\phi(x)dx.$$

Some formulas

- Let $u : \mathbb{R}^N \to \mathbb{R}^N$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ then

$$div[\phi u] = \nabla\phi \cdot u + \phi div(u).$$

- (Green's identities)

$$\int_\Omega u_{x_i} v = -\int_\Omega u v_{x_i} + \int_{\partial\Omega} u v \nu^i, \tag{1.5}$$

where $u, v \in C^1(\overline{\Omega})$ and $\nu^i = (\nu^1 \nu^2 ... \nu^N)$ is the outward pointing normal on $\partial\Omega$.

$$\int_\Omega \nabla u \nabla v = \int_\Omega (-\Delta u)v + \int_{\partial\Omega} v \partial_\nu u, \tag{1.6}$$

where $\partial_\nu u = \nabla u \cdot \nu$. Here $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$. And finally we have

$$\int_\Omega u\Delta v = \int_\Omega v\Delta u + \int_{\partial\Omega} u\partial_\nu v - v\partial_\nu u. \tag{1.7}$$

- (Radial formula) Assume B_R open ball in $\mathbb{R}^N$ centered at the origin and $f(x)$ is radial ($f(x) = f(|x|) = f(r)$). Then

$$\int_{B_R} f(x)dx = C_N \int_0^R f(r)r^{N-1}dr.$$

Maximum principle

Theorem 5. *(The weak maximum principle) Let $\Omega \subset\subset \mathbb{R}^N$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $-\Delta u(x) = f(x) \geq 0$. Then*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

If $-\Delta u(x) = f(x) \leq 0$ then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

The weak maximum principal states that the extreme values muct occur on $\partial\Omega$ but it does not exclude the possibility of occuring those values inside Ω .

Maximum principle for more general elliptic equation. Assume Ω is bounded in $\mathbb{R}^N$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $-\Delta u(x) + a(x) \cdot \nabla u = f(x) \geq 0$. Then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

If $-\Delta u(x) + a(x) \cdot \nabla u = f(x) \leq 0$ then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Theorem 6. *(The strong maximum principle) Suppose $\Omega \subset \mathbb{R}^N$ connected, bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\Delta u = 0$ in Ω. Then if u attains its $\max / \min$ over $\overline{\Omega}$ at some point $x_0 \in \Omega$, then $u(x) = constant$.*

Strong maximum principle states every non-constant harmonic function must achieve its maximum or minimum value on the boundary. It does not allow the minimum or maximum value inside Ω. If it occurs then it is flat. That is why we need Ω to be connected. Here connected means path connected that connects any two points by finite at overlapping balls in Ω.

Mean value formula. Suppose $\Omega \subset \mathbb{R}^N$ open, bounded. Let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Mean value formula declares that, provided $B(x,r) \subset \Omega$, $u(x)$ is equal to both the average of u over the sphere $\partial B(x,r)$ and average of u over the entire ball $B(x,r)$. That is

$$u(x) = \int_{B(x,r)} u dy = \int_{\partial B(x,r)} u ds.$$

If $u \in C(\Omega)$ and satisfies mean value formula then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

1.2 Distributions

The utility of distribution arises from the fact that they are generalized functions which allows for operations such as differentiation and convolution on objects that fails to be function. A distribution is a linear form on the space of smooth function with compact support satisfying some continuity property.

Definition 1.17. *(Convergence in $C_c^\infty(\Omega)$) For smooth compactly supported functions the notion for convergence is that of uniform convergence since the uniform limit of a sequence of continuous functions is continuous. First we define the uniform norm on $\Omega \subset \mathbb{R}^N$ by $|f|_u = |f|_{L^\infty} = \sup_{x\in\Omega} |f(x)|$. For norm on $C_c^\infty(\Omega)$ we apply the uniform norm on all partial derivatives with respect to multi-index α i.e. $\|\phi\|_{[\alpha]} = \|\partial^\alpha \phi\|_u = \|\partial^\alpha \phi\|_{L^\infty} = \sup_{x\in\Omega} |\partial^\alpha \phi(x)|$.*

Suppose ϕ_j be a sequence in $C_c^\infty(\Omega)$ and $\phi \in C_c^\infty(\Omega)$ and we say ϕ_j converges to ϕ in $C_c^\infty(\Omega)$ provided

- *There exists a compact set $K \subset \Omega$ such that for all j, supp(ϕ_j), supp$(\phi) \subset K$.*
- *For all $k \in \mathbb{N}, ||\phi_j^{(k)} - \phi^{(k)}||_{L^\infty(K)} = \sup|\phi_j^{(k)} - \phi^{(k)}| \to 0$ as $j \to \infty$.*

Thus we say ϕ_j converges to ϕ in C_c^∞ if and only if $\partial^\alpha \phi_j$ converges uniformly to $\partial^\alpha \phi$ in the compact set for all α. This notion of convergence is extremely strong.

Definition 1.18. *(Distributions) A linear form $T : C_c^\infty(\Omega) \to \mathbb{R}$ (i.e. $T(au+bv) = aT(u) + bT(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in C_c^\infty(\Omega)$) is said to be a distribution on Ω, in notation $T \in D'(\Omega)$ if T is continuous with respect to the above notion of convergence i.e.*

$$\phi_j \to \phi \quad \text{in } C_c^\infty(\Omega) \qquad \Longrightarrow \qquad T(\phi_j) \to T(\phi).$$

It is conventional to write $\langle T, \phi\rangle = T(\phi)$.

Example 1.19. *(Distribution associated with a function or signed measure)*

- *For $\Omega \subset \mathbb{R}^N$, let us define a real valued function in Ω. If f is integrable on every compact subset $K \subset \Omega$ then $T_f : \phi \to \int_\Omega f(x)\phi(x)dx$ is a distribution. Clearly T_f is linear in $C_c^\infty(\Omega)$. So for T_f to be a distribution it is sufficient that $f \in L^1_{loc}(\Omega)$. Let $\phi_j \to \phi$ in $C_c^\infty(\Omega)$ then*

  $$|T(\phi_j) - T(\phi)| \leq \int_\Omega |f||\phi_j - \phi| = \int_K |f||\phi_j - \phi| \leq \|f\|_{L^1(K)} \|\phi_j^{(k)} - \phi^{(k)}\|_{L^\infty(K)},$$

  where last inequality is coming from Hölder's inequality and is 0 since $\|\phi_j^{(k)} - \phi^{(k)}\|_{L^\infty(K)} \to 0$ and we also have $\int_K |f| = C_K < \infty$. Hence $T(\phi_j) \to T(\phi)$ and T is a distribution. This is called regular distribution.

- *If μ is a signed measure with finite total variation on each compact set $K \subset \Omega$ we associate the distribution*

$$\langle \mu, \phi \rangle := \langle T_\mu, \phi \rangle := \int_\Omega \phi(x) d\mu(x).$$

We can show that μ is continuous with respect to uniform convergence.

Definition 1.20. *(Derivative of a distribution) $T \in D'(\Omega)$ and α is a multi-index, we define the distributional derivative $\partial^\alpha T$ by*

$$\langle \partial^\alpha T, \phi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle .$$

Example 1.21. *A distribution does not see the boundary $\partial\Omega$. Let $f \in C^\infty(\overline{\Omega})$ and set*

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega \\ h(x) & x \in \partial\Omega. \end{cases}$$

Then f and $\tilde{f}$ are the equal as distributions (i.e. $\langle T_f, \phi \rangle = \left\langle T_{\tilde{f}}, \phi \right\rangle$ for all $\phi \in C_c^\infty(\Omega)$.) This is clear since the test functions ϕ are zero near the boundary of Ω.

Definition 1.22.

- *We say distribution T is zero on an open set $G \subset \Omega$ provided $\langle T, \phi \rangle = 0$ for all $\phi \in C_c^\infty(G)$. Let G denote the maximal open set in Ω with $T = 0$ on G and define the support of T by $\overline{\Omega} \backslash G$ (which is compact provided Ω is bounded).*
- *Given $T \in \mathcal{D}'(\Omega)$ we say $T \geq 0$ provided $\langle T, \phi \rangle \geq 0$ for all $0 \leq \phi \in C_c^\infty(\Omega)$.*

Example 1.23. *A distribution which is not given by a signed measure. Consider $\Omega = (-1, 1)$ and set $T := \delta_0$ the Dirac mass at the origin (so T is a measure). Now consider $T_x \in \mathcal{D}'(\Omega)$ and consider the linear operator*

$$\langle T_x, \phi \rangle = -\langle T, \phi_x \rangle = -\phi'(0)$$

and note that we can find a sequence of $\phi_m \to 0$ uniformly on $[-1,1]$ but $\phi_m'(0) = 1$. So this would show that the mapping $\phi \mapsto \langle T_x, \phi\rangle$ is not continuous with respect to uniform convergence. So we must have that the distribution T_x is not given by a signed measure.

Theorem 7. *Suppose $0 \leq T \in \mathcal{D}'(\Omega)$. Then T is given by a measure with local finite total variation.*

1.3 Sobolev space

Let $1 \leq p \leq \infty$ and k is a nonnegative integer. We define Sobolev space to be the set of functions whose members have derivatives of different orders lying in different L^p spaces. We denote Sobolev space by $W^{k,p}$ which consists of functions $u \in L^p$ with $\partial^\alpha u \in L^p$ (distributional derivative) for all multi-indices α i.e.

$$W^{k,p} = \{u \in L^p \ : \ \partial^\alpha u \in L^p \text{ for all multi-indices } \alpha, \ |\alpha| \leq k\}.$$

The norm of the Sobolev space is defined by

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^p} & 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |\partial^\alpha u| & p = \infty, \end{cases} \tag{1.8}$$

where the sum is over all multi-indices of order less than or equal k including when $|\alpha| = 0$.

Note in the case where $k = 1$ we have

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p} + \sum_{i=1}^{N} \|u_{x_i}\|_{L^p},$$

where, of course, the derivatives are in the distributional sense.

Special case. When $p = 2$ we use a special notation $H^k(\Omega) = W^{k,2}(\Omega)$ ($k =$

$0, 1, 2, \ldots)$. Then the space $H^k(\Omega)$ turns out to be a Hilbert space endowed with the inner product

$$(u, v) := \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha u \partial^\alpha v,$$

which generates the norm (1.8). Note that when $k = 0$ then $H^0 = L^2$ is a Hilbert space with the inner product $(u, v) = \int_\Omega uv dx$ and when $k = 1$ then H^1 is a Hilbert space with the inner product $(u, v) = \int_\Omega uv + \partial^\alpha u \cdot \partial^\alpha v$.

Remark 1.24. *(The completion method) ($W^{k,p}(\Omega)$) Let $1 \leq k$ an integer, $1 \leq p < \infty$ and let $\|u\|_{W^{k,p}(\Omega)}$ for $u \in C^\infty(\overline{\Omega})$ be as defined above. We then let $W^{k,p}(\Omega)$ denotes the completion of $C^\infty(\overline{\Omega})$ with respect to this norm.*

Definition 1.25. *Let u_n be a sequence of functions in $W^{k,p}(\Omega)$ and $u \in W^{k,p}(\Omega)$. We say u_n converges to u in $W^{k,p}(\Omega)$ if*

$$\lim_{n \to \infty} \|u_n - u\|_{W^{k,p}(\Omega)} = 0.$$

Definition 1.26. *(Sobolev space $W_0^{k,p}(\Omega)$) Let $1 \leq k$ is an integer, $1 \leq p < \infty$ and let $\|u\|_{W^{k,p}(\Omega)}$ for $u \in C_c^\infty(\Omega)$ be as defined above. We then define $W_0^{k,p}(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ with respect to this norm. Thus $u \in W_0^{k,p}(\Omega)$ iff there exists a sequence of functions $u_n \in C_c^\infty(\Omega)$ such that $\lim_{n\to\infty} \|u_n - u\|_{W^{k,p}(\Omega)} = 0$. So we can interpret $W_0^{k,p}(\Omega)$ consisting of those functions u in $W^{k,p}(\Omega)$ such that*

$$\partial^\alpha u = 0 \quad \text{on } \partial\Omega \ \forall |\alpha| \leq k - 1.$$

When $p = 2$ we denote $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

Theorem 1.27. *(Sobolev spaces as function spaces) For each $k \geq 1$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space with respect to the above norm.*

1.3.1 Sobolev embeddings

There is a class of Sobolev inequalities relating norms including those of Sobolev spaces. These inequalities are used to prove Sobolev embedding theorem. Sobolev embedding theorem establishes the relation between different Sobolev spaces. In this section we will discuss embeddings of various Sobolev spaces into others. See Evans book (chapter 5 of [13]) for details and proof of the Theorems below.

Theorem 8. *(The classical Poincaré inequality) Suppose* $1 \leq p < \infty$ *and* $\Omega \subset \mathbb{R}^N$ *is bounded in one direction. Then there exists a constant* $C(\Omega, p)$ *such that*

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p} \quad \forall u \in H_0^1(\Omega).$$

Gagliardo-Nirenberg-Sobolev inequality

Definition 1.28. *If* $1 \leq p < N$ *then Sobolev conjugate of* p *is*

$$p^* = \frac{Np}{N-p}.$$

Also note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}, \ p^* > p.$$

Theorem 9. *(Gagliardo-Nirenberg-Sobolev inequality) Suppose* u *is continuously differentiable function with compact support. Assume* $1 \leq p < N$*. There exists a constant* C*, depending only on* p *and* N *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|Du\|_{L^p(\mathbb{R}^N)} \quad \forall u \in C_c^1(\mathbb{R}^N).$$

Theorem 10. *(Estimate for* $W^{1,p}, 1 \leq p < N$*) Suppose* $\Omega \subset \mathbb{R}^N$ *open, bounded. Suppose* $\partial\Omega$ *is* C^1*. Assume* $1 \leq p < N$ *and* $u \in W^{1,p}(\Omega)$*. Then* $u \in L^{p^*}(\Omega)$ *with the* *estimate*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

C depends only on p, N, Ω.

Theorem 11. *(Estimate for $W_0^{1,p}, 1 \leq p < N$) $\Omega \subset \mathbb{R}^N$ open, bounded. Suppose $u \in W_0^{1,p}$ then*

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

for each $q \in [1, p^*]$. The constant C depends on p, q, N, Ω.

This inequality is sometimes called Poincaré inequality.

Remark 1.29.
- *If we consider the borderline case $N = 1$, then $W^{1,N} \subseteq L^\infty$. However, when $N > 1$ this fails.*
- *The difference between Theorem 10 and 11 is that on Theorem 11 first derivative of u appears on right hand side. In view of Theorem 11 on $W_0^{1,p}$, $\|Du\|_{L^p}$ is equivalent to $\|u\|_{W^{1,p}}$ if Ω is bounded. The theorem below shows this for the case $p = 2$.*

Theorem 12. *Suppose Ω is bounded in one direction. Then $\|u\|_{H^1} := \|u\|_{L^2} + \|\nabla u\|_{L^2}$ (the usual norm on H_0^1) and $\|u\|_{H_0^1} := \|\nabla u\|_{L^2}$ are equivalent norms.*

Proof. Normally one introduces $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with the norm of $H^1(\Omega)$. Thus H_0^1 inherits the topology of H^1 i.e. it is a Hilbert space with the scalar product given by

$$(u, u)_{H^1} = \int_\Omega u \cdot u + \int_\Omega \nabla u \cdot \nabla u.$$

So it is indeed make sense to have this scalar product in H_0^1 and this one also canonical one in H_0^1. Now by Poincaré inequality there exists some $C > 0$ such that

$$\|\nabla u\|_{L^2}^2 \leq \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (C+1)\|\nabla u\|_{L^2}^2 \quad \forall u \in H_0^1.$$

Hence we define $\|u\|_{H_0^1} := \|\nabla u\|_{L^2}$ which is equivalent to $\|u\|_{H^1}$ on $H_0^1(\Omega)$. Further it induces the inner product

$$(u,u)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla u \quad \forall u \in H_0^1.$$

$\square$

Morrey's inequality

Morrey's inequality shows that every function in $W^{1,p}(\Omega)$ for $p > N$ has a $1 - \frac{N}{p}$ Hölder continuous representative up to a set of measure zero.

Theorem 13. *(Morrey's inequality) Assume $N < p \leq \infty$. Then there exists a constant $C(p,N)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)} \quad \forall u \in C^1(\mathbb{R}^N),$$

where $\gamma := 1 - \frac{N}{p}$.

Definition 1.30. *We say u^* is a version of a given function u provided*

$$u = u^* \quad a.e.$$

Theorem 14. *(Estimate for $W^{1,p}, N < p \leq \infty$). $\Omega \subset \mathbb{R}^N$ open, bounded and $N < p \leq \infty$. Assume $\partial\Omega$ is C^1 and $u \in W^{1,p}(\Omega)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{\Omega})$, for $\gamma = 1 - \frac{N}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)},$$

where the constant C depends on p, N and Ω.

This version of inequality follows from Theorem 13 by applying norm preserving extension of $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^N)$.

Theorem 15. *(General Sobolev inequality) Let $\Omega \subset \mathbb{R}^N$ open, bounded with a C^1 boundary. Assume $u \in W^{k,p}$ then we consider two cases.*

- *If $k < \frac{N}{p}$ then $u \in L^q$, where $\frac{1}{p} - \frac{1}{q} = \frac{k}{N}$ and we have in addition the estimate*

$$\|u\|_{L^q} \leq C(k,p,N,\Omega)\|u\|_{W^{k,p}}.$$

- *If $k > \frac{N}{p}$ then $u \in C^{k-[\frac{N}{p}]-1,\gamma}(\overline{\Omega})$, where*

$$\gamma = \begin{cases} [\frac{N}{p}] + 1 - \frac{N}{p} & \text{if } \frac{N}{p} \text{ is not an integer} \\ \text{any positive number} < 1 & \text{if } \frac{N}{p} \text{ is an integer.} \end{cases}$$

  We have in addition the estimate

$$\|u\|_{C^{k-[\frac{N}{p}]-1,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)},$$

  the constant C depends on k, p, N, Ω, γ.

Sobolev embedding theorem. Let $W^{k,p}(\mathbb{R}^N)$ denotes the Sobolev space and $k \geq 0$ an integer, $1 \leq p < \infty$. The first part of the Sobolev embedding theorem states that if $k > l$ and $1 \leq p < \infty$, $1 \leq q < \infty$ are two real numbers such that

$$\frac{1}{p} - \frac{k}{N} = \frac{1}{q} - \frac{l}{N}.$$

Then

$$W^{k,p}(\mathbb{R}^N) \subseteq W^{l,q}(\mathbb{R}^N).$$

This embedding is continuous. In the special case $k = 1, l = 0$ we have $W^{1,p}(\mathbb{R}^N) \subseteq L^{p^*}(\mathbb{R}^N)$(embeddings in other orders on $\mathbb{R}^N$ are obtained by suitable iteration) which is the consequence of Gagliardo-Nirenberg-Sobolev inequality.

The second part of the Sobolev embedding theorem establishes the embedding of Sobolev space into the Hölder space $C^{r,\gamma}(\mathbb{R}^N)$. If $\frac{N}{p} < k$ and $\frac{N}{p} - k = -(r+\gamma)$ with $\gamma \in (0,1]$ then

$$W^{k,p}(\mathbb{R}^N) \subset C^{r,\gamma}(\mathbb{R}^N).$$

This embedding is direct consequence of Morrey's inequality. This inclusion expresses the fact that the existance of sufficiently many weak derivatives implies some continuity of the classical derivatives.

Scaling argument

Here we use scaling argument to see these are the best results (at least for the case $p = 2$) one can hope for: for $N \geq 3$, $H^1(\Omega) \subset\subset L^q(\Omega) \quad \forall 1 \leq q < 2^*$ (compact embedding) and $H^1(\Omega) \subseteq L^{2^*}(\Omega)$ (continuous but not compact).

Theorem 16. $\Omega \subset \mathbb{R}^N$ *open, bounded. When* $q > 2^*$, *then* $H^1(\Omega)$ *is not continuously embedded into* $L^q(\Omega)$.

Proof. Assume $B_1 \subset \Omega$. Fix $u \geq 0$ and $u \in C_c^\infty(B_1), u \not\equiv 0$. For $\lambda > 0$ define the rescaled function

$$u_\lambda(x) = u(\frac{x}{\lambda}).$$

Then $\nabla u_\lambda(x) = \frac{1}{\lambda}\nabla u(\frac{x}{\lambda})$. Also $u_\lambda \in C_c^\infty(B_\lambda)$, $|\frac{x}{\lambda}| < 1$. $u_\lambda = 0$ when $|x| \geq \lambda$. Suppose there exists $C > 0$ such that

$$\|\phi\|_{L^q(\Omega)} \leq C\|\nabla\phi\|_{L^2(\Omega)} \quad \forall \phi \in C_c^\infty(\Omega). \tag{1.9}$$

Also we take $\nabla\phi = 0$ on ∂B_1. Set $\phi(x) = u(\frac{x}{\lambda})$. Then $\forall\ 0 < \lambda < 1$ (we can't take $\lambda > 1$ since then $\phi \neq 0$ on ∂B_1) we have

$$\|u_\lambda\|_{L^q(\Omega)} \leq C\|\nabla u_\lambda\|_{L^2(\Omega)}. \tag{1.10}$$

We now estimate both right and left hand side terms of (1.10).

$$\begin{aligned}\|u_\lambda\|^q_{L^q(\Omega)} &= \int_\Omega |u(\frac{x}{\lambda})|^q dx \\ &= \lambda^N \int_{|y|<1} |u(y)|^q dy \\ &= \lambda^N \|u\|^q_{L^q(B_1)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_\lambda\|^2_{L^2(\Omega)} &= \int_\Omega |\frac{1}{\lambda}\nabla u(\frac{x}{\lambda})|^2 dx \\ &= \lambda^{N-2} \int_{|y|<1} |\nabla u(y)|^2 dy \\ &= \lambda^{N-2} \|\nabla u(y)\|^2_{L^2(B_1)},\end{aligned}$$

Writing out (1.10) gives

$$\|u\|_{L^q(B_1)} \leq C\lambda^{\frac{N-2}{2}-\frac{N}{q}} \|\nabla u\|_{L^2(B_1)}. \tag{1.11}$$

Now we can send λ to 0 or 1. By sending λ to 0, inequality (1.9) to hold we must need $\frac{N-2}{2} - \frac{N}{q} \leq 0$. That is $q \leq \frac{2N}{N-2} = p^*$. □

The same proof works when domain is $\mathbb{R}^N$ until we get inequality (1.11) and then since we are considering the whole region we can think of u in a large ball and we can send λ to 0 or ∞. So in this case to hold the inequality (1.9), forall $u \in C_c^\infty(\mathbb{R}^N)$, we must need $q = p^*$.

Scaling to see feasibility of Hölder embedding

Suppose $\Omega \subset \mathbb{R}^N$ open, bounded and $N < p < \infty$. Then $W^{1,p}(\Omega) \subset C^{0,1-\frac{N}{p}}(\overline{\Omega})$. We use scaling to see the exponent $\gamma := 1 - \frac{N}{p}$ is the correct exponent.

Assume $W^{1,p}(\Omega) \subset C^{0,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$. Let $B_1 \subset \Omega$ and $0 \leq u \in C_c^\infty(B_1)$.

Put

$$u_\lambda(x) = u(\frac{x}{\lambda}),$$

for $0 < \lambda < 1$. Suppose there exists a constant $C > 0$ such that

$$\|\phi\|_{C^{0,\gamma}} \leq C\|\nabla \phi\|_{L^p} \quad \forall \phi \in C_c^\infty(\Omega).$$

Then for all $0 < \lambda < 1$ we have

$$\|u_\lambda\|_{C^{0,\gamma}} \leq C\|\nabla u_\lambda\|_{L^p}. \tag{1.12}$$

Now using the similar way we did before we get from the right side term

$$\|\nabla u_\lambda\|_{L^p(\Omega)} = \lambda^{\frac{N-p}{p}} \|\nabla u\|_{L^p(\Omega)}.$$

and the left term gives

$$\begin{aligned}
\|u_\lambda\|_{C^{0,\gamma}} &= \|u_\lambda\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x-y|^\gamma} \\
&= \|u_\lambda\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|u(\frac{x}{\lambda}) - u(\frac{y}{\lambda})|}{|x-y|^\gamma} \\
&= \|u_\lambda\|_{L^\infty} + \sup_{x,y\in B_\lambda} \frac{|u(\frac{x}{\lambda}) - u(\frac{y}{\lambda})|}{|x-y|^\gamma} \\
&= \|u_\lambda\|_{L^\infty} + \sup_{\hat{x},\hat{y}\in B_1} \frac{|u(\hat{x}) - u(\hat{y})|}{\lambda^\gamma |\hat{x}-\hat{y}|^\gamma} \\
&= \frac{1}{\lambda^\gamma}[u_\lambda]_\gamma.
\end{aligned}$$

So we are assuming there exists $C > 0$ such that

$$\frac{1}{\lambda^\gamma}[\phi]_\gamma \leq C\|\nabla \phi\|_{L^p} \quad \forall \phi \in C_c^\infty(\Omega)$$

and

$$\frac{1}{\lambda^{\gamma}}[u]_{\gamma} \leq C\|\nabla u\|_{L^p}.$$

So writting out (1.12) gives

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{\lambda^{\gamma}|x-y|^{\gamma}} \leq C\lambda^{\frac{N-p}{p}}\|\nabla u\|_{L^p(\Omega)},$$

which implies we must need $\gamma \leq \frac{p-N}{p}$.

1.3.2 Dual space of Sobolev space

Consider the Hilbert space $H_0^1(\Omega)$ with inner product

$$(u,v) := \int_{\Omega} \nabla u \cdot \nabla v dx.$$

Now suppose f is some smooth function which is zero on $\partial\Omega$. We want to view f as an element in the dual of H_0^1 and so we need to specify how it acts on $\phi \in H_0^1$, i.e. we need to define (f,ϕ). There are two obvious ways to do this:

1.
$$(f,\phi) := \int_{\Omega} \nabla f \cdot \nabla \phi dx = (f,\phi) = (f,\phi)_{H_0^1}$$

   which is the way one would do it if they are thinking of Riesz representation in H_0^1.

2. The other option would be to define

$$(f,\phi) := \int_{\Omega} f(x)\phi(x)dx$$

   i.e. using Riesz on L^2 and not on H_0^1.

What functions are in $(H_0^1(\Omega))^* := H^{-1}(\Omega)$?

To be more precise, let $f(x)$ be a function defined on Ω and define T_f (the distribution) by $(T_f, \phi) := \int_\Omega f(x)\phi(x)dx$. When is $T_f \in H^{-1}$. We give a partial answer in terms of L^p spaces.

$$|(T_f, \phi)| = |\int_\Omega f(x)\phi(x)dx| \leq \|f\|_{L^{2^{*\prime}}} \|\phi\|_{L^{2^*}} \leq \|f\|_{L^{2^{*\prime}}} C(\Omega) \|\nabla\phi\|_{L^2},$$

after using Hölders inequality and the Sobolev imbedding theorem. So we need $f \in L^{2^{*\prime}}$ for $T_f \in H^{-1}$; and doing the computation shows we need $f \in L^{\frac{2N}{N+2}}$. So we can write this in symbols by

$$L^{\frac{2N}{N+2}} \subset H^{-1}.$$

Another way to see this is to recall that $H_0^1 \subset L^{2^*}$ and hence, symbolically we have

$$(L^{2^*})^* \subset (H_0^1)^* := H^{-1}.$$

1.3.3 Weak solution of Dirichlet problem

Consider the Dirichlet problem

$$\begin{cases} -\Delta u &= f(x) & \Omega \\ u &= 0 & \partial\Omega. \end{cases} \tag{1.13}$$

$f(x)$ is some function and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

Suppose $u(x)$ is a smooth solution of (1.13). Multiply (1.13) by smooth test function ϕ which is zero on the boundary we get

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\phi,$$

after considering the fact that $\phi = 0$ on $\partial\Omega$.

Definition 1.31. *(Weak solution; $H_0^1(\Omega)$) We say $u \in H_0^1$ is weak solution of (1.13) provided*

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\phi \qquad \forall \phi \in H_0^1(\Omega).$$

Remark 1.32. *(Boundary condition) Note that in the Dirichlet problem that the boundary $u = 0$ is obtained from the fact that $u \in H_0^1$ (i.e. the boundary condition is imposed by the function space).*

Theorem 17. *(Existance of weak solution of (1.13)) Suppose Ω is a bounded domain in $\mathbb{R}^N$. Let $f(x)$ be a function with $f \in H^{-1}$ (i.e. $T_f \in H^{-1}$). Then there is a unique $u \in H_0^1(\Omega)$ which satisfies (1.13).*

Proof. Consider the pde (1.13). Since $f \in H^{-1}$ there exists a unique $u \in H_0^1$ (by Riesz) such that $(u, \phi) = (f, \phi)$ for all $\phi \in H_0^1$ (as above (u, ϕ) is the inner product on H_0^1). But this is precisely the definition of a weak solution. □

Weak solution of more general symmetric Dirichlet problems

Here we consider the more general versions of the Dirichlet problems given by

$$\begin{cases} -\Delta u + a(x)u &= f(x) \quad \Omega \\ u &= 0 \qquad \partial\Omega, \end{cases} \tag{1.14}$$

where $f(x)$ is some function, $\Omega \subset \mathbb{R}^N$ is a smooth bounded set. We assume $a(x) \geq 0$ is smooth and bounded.

Definition 1.33. *$u \in H_0^1$ is a weak solution of (1.14) provided*

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega a(x)u\phi = \int_\Omega f\phi \qquad \forall \phi \in H_0^1(\Omega).$$

Theorem 18. *(Existence of weak solutions of (1.14) .*

Proof. One uses the exact same proof but instead of using the inner product $(u, \phi) = \int_\Omega \nabla u \cdot \nabla \phi$ on H_0^1 one uses the inner product

$$(u, \phi)_{a(x)} := \int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega a(x) u \phi.$$

H_0^1 is infact a complete space under this inner product and we want to show that the norm induced by a is equivalent to H_0^1 norm. So we show there is some $C > 0$ such that

$$C \|\nabla \phi\|_{L^2} \leq \|\phi\|_a \leq \frac{1}{C} \|\nabla \phi\|_{L^2} \quad \forall \phi \in H_0^1(\Omega).$$

It's enough to prove it for $\phi \in C_c^\infty(\Omega)$. Clearly $\|\nabla \phi\|_{L^2} \leq \|\phi\|_a$ since $a(x) \geq 0$. For the other direction

$$\begin{aligned}
\|\phi\|_a^2 &= \int_\Omega |\nabla \phi|^2 dx + \int_\Omega a(x) \phi^2 dx \\
&\leq \int_\Omega |\nabla \phi|^2 dx + \|a\|_{L^\infty} \int_\Omega \phi^2 dx \\
&\leq \int_\Omega |\nabla \phi|^2 dx + \frac{\|a\|_{L^\infty}}{\lambda_1} \int_\Omega |\nabla \phi|^2 dx \\
&= \left(1 + \frac{\|a\|_{L^\infty}}{\lambda_1}\right) \|\nabla \phi\|_{L^2}^2,
\end{aligned}$$

where $\lambda_1 = \lambda_1(-\Delta, \Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. □

1.4 Regularity and Weak solution

Some specific partial differential equation (Laplace equation) can be solved in a classical sense but most of the pde cannot be solved in a classical way. For those kind of pde's it's wise to first try to find a weak or generalized solution. Because at the outset if we require our solution is classical or smooth enough then we would have very hard time finding them as our proof must then necessarily include possibly intricate demonstrations that the function we are building are infact smooth enough.

So it is often convenient to aim at proving well-posedness (well posedness of a pde means the problem has infact a solution, the solution is unique and the solution depends continuously on the data given in the problem) in some appropriate class of weak solution. For various pde's this is the best can be done. For other equations we can hope that our weak solutions may turn out afterall to be smooth enough to qualify as a classical solution. This leads to the question of regularity of weak solution.

Regularity Theory

Consider the Dirichlet problem

$$\begin{cases} -\Delta u &= f(x) \quad \Omega \\ u &= 0 \qquad \partial\Omega, \end{cases} \tag{1.15}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and where $f \in H^{-1}$. Then we know there exists a unique solution $u \in H_0^1$.

Theorem 19. *(Regularity results)*

1. *(L^2 regularity theory) If $f \in H^k(\Omega)$ ($k \geq 0$ an integer; $H^0 := L^2$) then $u \in H^{k+2}(\Omega)$. In particular if $f \in L^2$ then $u \in H^2$ and one has the estimate*

$$\|u\|_{H^{k+2}} \leq C(\Omega, k)\|f\|_{H^k}.$$

2. *(L^p regularity theory). If $f \in L^p(\Omega)$ where $1 < p < \infty$ then $u \in W^{2,p}(\Omega)$. Moreover there is some $C = C(p, \Omega)$ such that*

$$\|u\|_{W^{2,p}} \leq C(p, \Omega)\|f\|_{L^p}.$$

Remark 1.34. *This result still holds if L^p is not a subset of H^{-1} (i.e. if p to* *close to 1), but then u is not an H_0^1 solution and one uses a weaker notion of a solution.*

3. *(Hölder space regularity) If $f \in C^{0,\alpha}(\Omega)$ where $0 < \alpha < 1$ then $u \in C^{2,\alpha}(\Omega)$. More over there is some $C(\alpha, \Omega)$ such that*

$$\|u\|_{C^{2,\alpha}} \leq C(\alpha, \Omega)\|f\|_{C^{0,\alpha}}.$$

4. *Suppose $u \in H_0^1(\Omega)$ is a weak solution of $-\Delta u + C(x)u = f(x)$ in Ω, and $u = 0$ on $\partial\Omega$. If $f, C \in L^p(\Omega)$ where $p > \frac{N}{2}$ then $u \in L^\infty(\Omega)$. If $p = \frac{N}{2}$ the result fails.*

The above regularity results are referred to as "boundary regularity results" since we are obtaining regularity results on the solution u over the full space Ω (i.e. all the way to the boundary). These results are much harder to prove than "interior regularity results". See chapter 6 of [13]. An example of an interior regularity L^2 result would be an apriori estimates where we assume u, f sufficiently smooth to perform calculations and we obtain some estimates.

Apriori estimates

$$\begin{cases} -\Delta u &= f(x) \quad \Omega, \\ u &= 0 \quad\quad \partial\Omega, \end{cases} \tag{1.16}$$

where Ω a smooth bounded domain in $\mathbb{R}^N$.

Theorem 20. *(Interior H^2 estimate) Suppose u is a smooth solution of (1.16) and $\Omega_0 \subset\subset \Omega$. Then there is some $C = C(\Omega_0)$ such that*

$$\|u\|_{H^2(\Omega_0)} \leq C(\Omega_0)\|f\|_{L^2(\Omega)}.$$

Proof. Let $\Omega_0 \subset\subset \Omega$ and let $0 \leq \phi \leq 1$ be a smooth cut-off function which is supported in Ω and which is 1 on Ω_0. Take a derivative of (1.15) to obtain $-\Delta u_{x_k} = f_{x_k}$. Now multiply this by $u_{x_k}\phi^2$ and integrate by parts to obtain

$$\int |\nabla u_{x_k}|^2 \phi^2 = \int f_{x_k} u_{x_k} \phi^2 - 2 \int \nabla u_{x_k} u_{x_k} \phi \nabla \phi \tag{1.17}$$

and then note we have

$$\int f_{x_k} u_{x_k} \phi^2 = -\int f u_{x_k x_k} \phi^2 - 2 \int f u_{x_k} \phi \phi_{x_k}$$

and then by Young's inequality we obtain

$$\int f_{x_k} u_{x_k} \phi^2 \leq -\int f u_{x_k x_k} \phi^2 + 2\varepsilon \int u_{x_k}^2 \phi^2 + \frac{1}{2\varepsilon} \int f^2 \phi_{x_k}^2$$

and summing in k from 1 to N gives

$$\sum_{k=1}^{N} \int f_{x_k} u_{x_k} \phi^2 \leq \int f^2 \phi^2 + 2\varepsilon \int |\nabla u|^2 \phi^2 + \frac{1}{2\varepsilon} \int f^2 |\nabla \phi|^2.$$

Also

$$\begin{aligned} -2 \int \nabla u_{x_k} u_{x_k} \phi \nabla \phi &\leq 2 \int |\nabla u_{x_k}| \phi |u_{x_k}| |\nabla \phi| \\ &\leq 2\varepsilon \int |\nabla u_{x_k}|^2 \phi^2 + \frac{1}{2\varepsilon} \int |u_{x_k}|^2 |\nabla \phi|^2 \end{aligned}$$

and then we sum in k in (1.17) to see

$$\begin{aligned} (1 - 2\varepsilon) \int |D^2 u|^2 \phi^2 &\leq \int f^2 \phi^2 + \frac{1}{2\varepsilon} \int f^2 |\nabla \phi|^2 \\ &\quad + 2\varepsilon \int |\nabla u|^2 \phi^2 + \frac{1}{2\varepsilon} \int |\nabla u|^2 |\nabla \phi|^2. \end{aligned}$$

Fix $0 < \varepsilon < \frac{1}{4}$ and then there is some $C = C(\Omega_0)$ such that

$$\int_{\Omega_0} |D^2 u|^2 \leq C \int_\Omega f^2 + C \int_\Omega |\nabla u|^2.$$

Using the pde we get

$$\|\nabla u\|_{L^2} \leq C_1(\Omega)\|f\|_{L^2} \qquad \text{over } \Omega. \tag{1.18}$$

By combining with the previous estimate we get

$$\int_{\Omega_0} |D^2 u|^2 \leq C_3 \int_\Omega f^2.$$

$\square$

Weaker notions of solution

Definition 1.35. *(Distance function $\delta(x) := dist(x, \partial\Omega)$.) For Ω a bounded open set in $\mathbb{R}^N$ we define*

$$\delta(x) := \inf_{y \in \partial\Omega} |y - x| = \min_{y \in \partial\Omega} |y - x|.$$

Consider a smooth solution u of (1.13) and multiply the pde by $\phi \in C^\infty(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$. So we have

$$\begin{aligned}
\int_\Omega f\phi &= \int_\Omega (-\Delta u)\phi \\
&= \int_\Omega \nabla u \cdot \nabla \phi - \int_{\partial\Omega} \phi \partial_\nu u \\
&= \int_\Omega \nabla u \cdot \nabla \phi \\
&= \int_\Omega (-\Delta \phi) u + \int_{\partial\Omega} u \partial_\nu \phi \\
&= \int_\Omega (-\Delta \phi) u.
\end{aligned}$$

Definition 1.36. *Set* $X := \{\phi \in C^\infty(\overline{\Omega}) : \phi = 0 \quad \partial\Omega\}$. *We say* $u \in L^1(\Omega)$ *is a very weak solution of (1.13) provided*

$$\int_\Omega (-\Delta\phi)u = \int_\Omega f\phi \qquad \forall \phi \in X.$$

Remark 1.37. • *Note this definition of a very weak solution is almost the distributional definition other than the fact that out test functions are not compactly supported. It should be noted that this definition actually encodes* $u = 0$ *on* $\partial\Omega$ *into the definition. For instance suppose* $u \in C^2(\overline{\Omega})$ *and satisfies the above notion of a weak solution (No assumtions on the value of* u *on* $\partial\Omega$*) then actually* $u = 0$ *on* $\partial\Omega$.

Definition 1.38. *(Weaker than* L^1 *solution) We call* u *an ultra weak solution of (1.13) if* $\delta u, \delta f \in L^1(\Omega)$ *and*

$$\int_\Omega (-\Delta\phi)u = \int_\Omega f\phi \quad \forall \phi \in X,$$

where $\delta(x) := dist(x, \partial\Omega)$ *and* $X := \{\phi \in C^\infty(\overline{\Omega}) : \phi = 0 \quad \partial\Omega\}$.

Theorem 21. *(Existence and regularity of very weak solutions;* $f \in L^1(\Omega)$*). Let* $f \in L^1(\Omega)$. *Then there exists a very weak solution* u *of (1.13) and* $u \in L^p$ *for all* $1 \leq p < \frac{N}{N-2}$.

1.5 Stability and variation of Energy

A system is in a stable state if it recovers from perturbations, a small change will not affect the system to return to equilibrium. Energy of a system is a quantity that the system tends to minimize that varies with respect to a physical parameter. The energy that lies inside the neighbourhood of a local minimum of energy unable to convert to another type of energy and this state is called stable state of the system. For elaborate discussion in stability analysis of solution see [10].

Definition 1.39. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a smooth energy function of class C^2. A point $t_0 \in \mathbb{R}$ is called a stable critical point of g if $g'(t_0) = 0$ and $g''(t_0) \geq 0$.*

Here we are going to discuss stable state of a pde by looking at the variation of energy associated to it.

Theorem 22. *Consider*

$$\begin{cases} -\Delta u &= f(u) \quad \Omega \\ u &= 0 \qquad \partial\Omega. \end{cases} \tag{1.19}$$

Consider the energy given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(u)dx \quad \forall u \in X,$$

where $X = \{u \in C^2(\overline{\Omega}) | u = 0 \text{ on } \partial\Omega\}$ and $F : \mathbb{R} \to \mathbb{R}$ is a class of C^2 function which is antiderivative of the given function $f \in C^1$. Suppose there is some $u \in X$ such that

$$E(u) = min_{v \in X} E(v).$$

Then the minimizer u satisfies (1.19).

Proof. Fix $u \in X$. We study the variation of the energy E along a given direction $\phi \in X \setminus \{0\}$, that is, we consider the function $g : \mathbb{R} \to \mathbb{R}$ defined for $t \in \mathbb{R}$ by

$$g(t) := E(u + t\phi),$$

where $u + t\phi \in X$. Since E is minimized at u so $g'(0) = 0$ i.e. 0 is a stable critical point of g. So first we calculate the difference quotient of g at 0 and then we will pass to the limit as $t \to 0$.

$$\frac{g(t) - g(0)}{t} = \int_\Omega \nabla u \nabla \phi dx + \frac{t}{2}\int_\Omega |\nabla \phi|^2 dx - \int_\Omega \frac{F(u + t\phi) - F(u)}{t} dx. \tag{1.20}$$

Taking $|t| \leq 1$ and writting $b = \|u\|_{L^\infty} + \|\phi\|_{L^\infty}$, from the mean value theorem we get $\frac{F(u+t\phi)-F(u)}{t\phi} = F'(c) = f(c)$, where $c \in (u, u + t\phi)$ which implies

$$|\frac{F(u + t\phi) - F(u)}{t\phi}| = |f(c)| \leq \|f\|_{L^\infty[-b,b]}.$$

Now by Dominated Convergence Theorem we can pass to the limit in (1.20) as $t \to 0$ such that $g'(0) = E'(u) = 0$ if and only if

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega F'(u)\phi dx = \int_\Omega f(u)\phi dx.$$

Integration by parts gives

$$\int_\Omega (-\Delta u - f(u))\phi dx = 0.$$

Since this is true for arbitrary $\phi \in X$ so we can conclude u solves

$$\begin{cases} -\Delta u &= f(u) \quad \Omega \\ \quad u &= 0 \qquad \partial\Omega. \end{cases}$$

$u = 0$ on $\partial\Omega$ is coming from function space. □

We have shown minimizer u solves the pde. This holds in particular if u is a critical point of $E(u)$ and the condition for the solution u to be stable we require

$g''(0) \geq 0$. So first we compute the difference quotient,

$$
\begin{aligned}
\frac{g'(t) - g'(0)}{t} &= \frac{DE(u + t\phi) \cdot \phi}{t} \\
&= \frac{1}{t} \int_\Omega \nabla u \nabla \phi dx + \int_\Omega |\nabla \phi|^2 dx - \int_\Omega \frac{f(u + t\phi)}{t} \phi dx \\
&= \int_\Omega |\nabla \phi|^2 dx - \int_\Omega \frac{f(u + t\phi) - f(u)}{t} \phi dx.
\end{aligned}
$$

By Dominated Convergence Theorem $g''(0) = \int_\Omega |\nabla \phi|^2 dx - \int_\Omega f'(u)\phi^2 dx$. So we deduce the following definition:

Definition 1.40. *Let $f \in C^1(\mathbb{R})$ and $\Omega \in \mathbb{R}^N$ is open, bounded. A solution $u \in C^2(\overline{\Omega})$ of (1.19) is stable if*

$$
\int_\Omega |\nabla \phi|^2 dx \geq \int_\Omega f'(u)\phi^2 dx \quad \forall \phi \in C_c^1(\Omega). \tag{1.21}
$$

Equation (1.21) is called the second variation of the energy functional associated with the pde at u.

Remark 1.41. *By using density argument and Levi's theorem the above inequality holds for all $\phi \in H_0^1(\Omega)$ i.e.*

$$
\int_\Omega |\nabla \phi|^2 dx \geq \int_\Omega f'(u)\phi^2 dx \quad \forall \phi \in H_0^1(\Omega).
$$

Linearized stability

Suppose Ω is a smooth bounded domain. Consider

$$
\begin{cases}
-\Delta u &= f(u) \quad \Omega \\
u &= 0 \qquad \partial\Omega.
\end{cases} \tag{1.22}
$$

Suppose u is a classical solution of (1.22). Consider the linearized operator

$$L(\phi) := -\Delta\phi - f'(u)\phi.$$

Since L is self adjoint there is variational characterization of the eigenvalues. Assume the linearized operator has eigenvalues which are ordered $0 < \lambda_1 < \lambda_2 \leq .... \leq \lambda_k$ and $\lambda_k \to \infty$ as $k \to \infty$. The first eigenvalue λ_1 is simple and occurs at say $0 < \phi_1 \in H_0^1$ such that $L(\phi_1) = \lambda_1\phi_1$ and

$$\lambda_1 = \min\left\{\frac{\int_\Omega |\nabla\phi|^2 - f'(u)\phi^2 dx}{\int_\Omega \phi^2 dx} : \phi \in H_0^1(\Omega)\backslash\{0\}\right\}.$$

The 2nd eigenvalue λ_2 will occur let say at ϕ_2 and this ϕ_2 will minimize the Rayleigh's quotient such that

$$\lambda_2 = \min\left\{\frac{\int_\Omega |\nabla\phi|^2 - f'(u)\phi^2 dx}{\int_\Omega \phi^2 dx} : \phi \in H_0^1(\Omega)\backslash\{0\}, \int_\Omega \phi\phi_1 = 0\right\},$$

and $L(\phi_2) = \lambda_2\phi_2$. The k^{th} eigenvalue λ_k will occur let say at ϕ_k and

$$\lambda_k = \min_{\phi \in H_{k-1}}\left\{\frac{\int_\Omega |\nabla\phi|^2 - f'(u)\phi^2 dx}{\int_\Omega \phi^2 dx} : \phi \in H_0^1(\Omega)\backslash\{0\}\right\},$$

where $H_{k-1} = \{\phi \in H_0^1 : (\phi, \phi_j) = \int_\Omega \phi\phi_j = 0, j = 1, ..., k-1\}$. Then $L(\phi_k) = \lambda_k\phi_k$. We define the Morse index of u as the number of negative eigenvalues of L. If $\lambda_k \geq 0$ then we say u is semistable and if $\lambda_k > 0$ we say u is a stable solution.

We call λ_1 the principal eigenvalue of L. The theorem below shows the variational principal for the principal eigenvalue.

Theorem 23. *(Principal eigenvalue) Define*

$$E(\phi) := \frac{\displaystyle\int_\Omega |\nabla \phi|^2 - f'(u)\phi^2 dx}{\displaystyle\int_\Omega \phi^2 dx}.$$

The smallest eigenvalue λ_1 *is simple and*

$$\lambda_1 = \min_{\phi \in H_0^1 \setminus \{0\}} E(\phi).$$

This minimum is obtained at say $\phi_1 > 0$ *and* ϕ_1 *satisfies*

$$\begin{cases} L(\phi_1) & = & \lambda_1 \phi_1 & \Omega \\ \phi_1 & = & 0 & \partial\Omega. \end{cases}$$

Proof. Suppose $\phi_1 \in H_0^1$ is the minimizer of $E(\phi)$ that is

$$E(\phi_1) = \min_{\phi \in H_0^1} E(\phi).$$

Fix $\psi \in H_0^1$ and note $\phi + t\psi \in H_0^1$ for all $t > 0$. We consider a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(t) = E(\phi_1 + t\psi).$$

Since ϕ_1 minimizes $E(\phi)$, so g must satisfy $g'(0) = 0$. By taking the derivative of $g(t)$ and by setting $g'(t)|_{t=0}$ we get

$$g'(0) = \frac{\displaystyle\int_\Omega \phi_1^2 dx \cdot \left[\int_\Omega 2|\nabla\phi_1||\nabla\psi| dx - \int_\Omega 2\phi_1\psi f'(u) dx\right] - \left[\int_\Omega |\nabla\phi_1|^2 - f'(u)\phi_1^2 dx\right] \cdot \int_\Omega 2\phi_1\psi dx}{\displaystyle[\int_\Omega \phi_1^2 dx]^2}.$$

Now $g'(0) = 0$ implies

$$\left[\int_\Omega (-\Delta\phi_1) - \int_\Omega \phi_1 f'(u) - \int_\Omega \lambda_1\phi_1\right] \psi dx = 0.$$

Since this is true forall $\psi \in H_0^1$, so we conclude that

$$\begin{cases} L(\phi_1) &= \lambda_1 \phi_1 \quad \Omega \\ \phi_1 &= 0 \qquad \partial\Omega. \end{cases}$$

The boundary condition is coming from the function space. $\square$

2

A Nonlinear Eigenvalue problem

2.1 Existence of Stable Minimal solution

Consider

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where f satisfies

(A) smooth, increasing, convex, $f(0) = 1$ and superlinear at ∞, ($i.e.$ $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$).

Here Ω a bounded domain in $\mathbb{R}^N$ and $\lambda > 0$ is a parameter.

Definition 2.1. *We say a solution u of $(P)_\lambda$ is* ***minimal*** *provided any other solution v of $(P)_\lambda$ satisfies $u \leq v$ a.e. in Ω.*

Lemma 2.2. *(Sub/Supersolution approach to (1.22)) Suppose there is some $\underline{u}, \overline{u}$ smooth such that $\underline{u} \leq 0, \overline{u} \geq 0$ on $\partial\Omega$ and*

$$-\Delta \underline{u} \leq f(\underline{u}) \qquad \Omega,$$

$$-\Delta \overline{u} \geq f(\overline{u}) \qquad \Omega.$$

In addition, assume f is smooth and increasing on $\mathbb{R}$ and $\underline{u} \leq \overline{u}$ in Ω. Then there is some smooth u such that $\underline{u} \leq u \leq \overline{u}$ in Ω and u is a solution of (1.22).

Proof. One uses an iteration argument to find an increasing sequence of functions. Define $u_0 := \underline{u}$ and define u_{n+1} by

$$\begin{cases} -\Delta u_{n+1} = f(u_n) & \text{in } \Omega \\ u_{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

for $n \geq 0$ (note that at each stage we are solving a linear equation). f is increasing so note that

$$\begin{cases} -\Delta u_1 = f(\underline{u}) & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta(u_1 - \underline{u}) \geq f(\underline{u}) - f(\underline{u}) = 0 & \text{in } \Omega \\ u_1 - \underline{u} \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and so by maximum principle $0 \leq u_1 - \underline{u} = u_1 - u_0$. Now note that

$$\begin{cases} -\Delta(u_2 - u_1) = f(u_1) - f(u_0) & \text{in } \Omega \\ u_2 - u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and since $u_1 \geq u_0$ and since f increasing we have $f(u_1) - f(u_0) \geq 0$ in Ω. So by maximum principle we have $u_2 \geq u_1$. One can continue on like this to show that $u_0 \leq u_1 \leq u_2 \leq u_3 \leq ......$ in Ω.

We now show these are all bounded above by $\overline{u}$. First note that

$$\begin{cases} -\Delta(\overline{u} - u_1) \geq f(\overline{u}) - f(u_0) & \text{in } \Omega \\ \overline{u} - u_1 \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and since f is increasing and by assumption we have $f(\overline{u}) - f(u_0) = f(\overline{u}) - f(\underline{u}) \geq 0$ in Ω. So the maximum principle shows that $u_1 \leq \overline{u}$. We continue on with the iteration to show that $u_k \leq \overline{u}$ for all k. So u_k is an increasing sequence bounded above by $\overline{u}$. So it has a pointwise limit, say $u(x) := \lim_{n\to\infty} u_{n+1}(x)$. So we just need to pass to the limit in the equation. There are various ways we can try and do that. The easiest will be to show that u is a 'very weak solution' of the associated pde. Multiply (2.2) by $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$ and by using Green's formula we get

$$\int_\Omega -\Delta\phi u_{n+1} = \int_\Omega f(u_n)\phi.$$

Then by Dominated Convergence Theorem

$$\int_\Omega -\Delta\phi u = \int_\Omega f(u)\phi.$$

Instead lets try for a bit more. Recall that u_{n+1} is a classical solution of $-\Delta u_{n+1} = f(u_n)$ in Ω with zero Dirichlet boundary conditions and $u_k(x) \nearrow u(x)$ pointwise. We first show that $u_{n+1} \in C^{1,\alpha}$ for some fixed $0 < \alpha < 1$. Fix $N < p < \infty$ big enough such that $W^{2,p} \subset C^{1,\alpha}$. Then for $n > m$ we have

$$-\Delta(u_{n+1} - u_{m+1}) = f(u_n) - f(u_m).$$

But we have $f(u_k)$ is Cauchy in L^p by dominated convergence. Hence $\{u_{n+1}\}_n$ is Cauchy in $W^{2,p}$ and hence Cauchy in $C^{1,\alpha}$. But then $\{f(u_n)\}_n$ is Cauchy in $C^{0,\alpha}$ and hence u_{n+1} is Cauchy in $C^{2,\alpha}$. This is enough to pass to the limit in the equation. Note if f wasn't smooth we couldn't do the last step but we could still pass to the limit in the definition of a weak solution.

□

Theorem 24. *Let f, Ω be as above.*

1. *There exists some $\lambda^* \in (0,\infty)$ (the extremal parameter) such that for all $0 < \lambda < \lambda^*$ there exists a smooth solution u of $(P)_\lambda$. For all $\lambda > \lambda^*$ there are no*

smooth solutions of $(P)_\lambda$.

2. *For all* $0 < \lambda < \lambda^*$ *there exists a smooth minimal solution* u_λ *of* $(P)_\lambda$. *For all* $x \in \Omega$, *one has that* $u_\lambda(x)$ *is increasing in* λ *on* $(0, \lambda^*)$.

3. *For all* $0 < \lambda < \lambda^*$ *the minimal solution is stable.*

4. *Define* $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ *(well defined by monotonicity except possibly infinite). Then* u^* *(which we call the extremal solution) is a weak solution of* $(P)_{\lambda^*}$ *and is the unique ultra weak solution of* $(P)_{\lambda^*}$.

Proof. 1. Define

$$\lambda^* := \sup\left\{0 \leq \lambda : (P)_\lambda \text{ has a smooth solution}\right\}.$$

First we will prove λ^* is finite. Let λ_1 is the first eigenvalue of the operator $-\Delta$ that occurs at say $0 < \phi \in H_0^1$ i.e. $-\Delta\phi = \lambda_1\phi$. Let u be smooth solution of $(P)_\lambda$. Multiply $(P)_\lambda$ by ϕ and integrate by parts to get

$$\int_\Omega \left(\lambda f(u) - \lambda_1 u\right)\phi(x)dx = 0.$$

Now since $\phi > 0$ there exists $x \in \Omega$ such that

$$\lambda f(u(x)) - \lambda_1 u(x) \leq 0 \implies \lambda \leq \frac{\lambda_1 u(x)}{f(u(x))}. \tag{2.3}$$

Since $\lambda f(u(x)) \geq 0$ in Ω, so from (2.1) by maximum principle $u(x) \geq 0$ in Ω. So there exists some $\hat{x} \in \Omega$ such that $u(\hat{x}) \geq 0$ such that $\lambda f(u(\hat{x})) - \lambda_1 u(\hat{x}) = 0$. So by using (2.3) we get

$$\lambda = \frac{\lambda_1 u(\hat{x})}{f(u(\hat{x}))} \leq \lambda_1 \frac{u(x)}{f(u(x))} \leq \lambda_1 \sup_{x\in\Omega} \frac{u(x)}{f(u(x))} \leq \lambda_1 \sup_{t\geq 0} \frac{t}{f(t)}.$$

Hence $\lambda^* \leq \lambda_1 * C_f < \infty$ by considering the fact that the upper bound of λ depends on f (f is superlinear at ∞).

By sub/supersolution approach we now show that $\lambda^* > 0$. Clearly $\underline{u} = 0$ is a subsolution. We just need to find a supersolution $\overline{u} \geq 0$. Let $-\Delta\overline{u} = 1$ in Ω with $\overline{u} = 0$ on $\partial\Omega$. Then $\overline{u} > 0$ in Ω and let $M := \max_\Omega \overline{u}$. Then $\overline{u}$ a supersolution provided $1 \geq \lambda f(\overline{u})$ in Ω and it is sufficient that $1 \geq \lambda f(M)$. But taking $\lambda > 0$ small enough ensures this.

Now we will show for all $\lambda \in (0, \lambda^*)$ there exists a smooth solution. Fix $0 < \lambda < \lambda^*$ and let $\lambda \leq \ t \leq \lambda^*$ such that v is a smooth solution of $(P)_t$ (this exists by the definition of λ^* and supremum). Then to solve $(P)_\lambda$ we apply the sub/supersolution lemma with $\underline{u} = 0$ and $\overline{u} = v$ to find a smooth solution $0 \leq u \leq v$ in Ω of $(P)_\lambda$. Let's check that smooth solution $\overline{u} := v$ of $(P)_t$ is indeed a supersolution for $(P)_\lambda$. Its nonnegative on $\partial\Omega$ and

$$\begin{cases} -\Delta v &= tf(v) \geq \lambda f(\overline{u}) \quad \Omega \\ \quad v &\geq 0 \qquad\qquad\qquad\quad \partial\Omega. \end{cases}$$

So it is a supersolution. For $\lambda > \lambda^*$ there is no smooth solution by definition.

2. To construct the minimal solution u_λ we basically did this in the previous part when showing existence. Let $0 < \lambda < \lambda^*$ and set $\underline{u} = 0$. As before there is some $\lambda < \gamma \leq \lambda^*$ and some smooth solution v of $(P)_\lambda$. So we use the iteration procedure from Lemma 1 with $\underline{u} = 0$ and $\overline{u} = v$. **Define** $u_\lambda(x) := \lim_n u_{n+1}(x)$, where we are using the notation from Lemma 1. We now show that u_λ is minimal. The first thing to note is that we used the existence of a supersolution $\overline{u}$ to prove the limit u_λ exists. But the limit does not depend on $\overline{u}$. So suppose v is a solution of $(P)_\lambda$. Then $v \geq 0$ by the maximum principle and so we can apply the iteration procedure again with $\overline{u} = v$ to see that $u_k \leq v, \forall k \geq 0$ and

hence we will have $u_k \leq u_{k+1}$ in Ω and by passing to the limit $u_\lambda \leq v$ in Ω.

We now show the minimal solutions are increasing in λ. Let $0 < \lambda < \lambda^*$ and let $\varepsilon > 0$ be small enough such that $\lambda + \varepsilon < \lambda^*$. Then $u_{\lambda+\varepsilon}$ is a supersolution of $(P)_\lambda$. To check that first we write out the pde for $u_{\lambda+\varepsilon}$,

$$\begin{cases} -\Delta u_{\lambda+\varepsilon} &= (\lambda+\varepsilon) f(u_{\lambda+\varepsilon}) \quad \text{in } \Omega \\ u_{\lambda+\varepsilon} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since f is positive and $\varepsilon > 0$ we get

$$\begin{cases} -\Delta u_{\lambda+\varepsilon} &\geq \lambda f(u_{\lambda+\varepsilon}) \quad \text{in } \Omega \\ u_{\lambda+\varepsilon} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

So $u_{\lambda+\varepsilon}$ is supersolution. Now we do the induction procedure by taking $\underline{u} = 0$ and $\overline{u} = u_{\lambda+\varepsilon}$ to have $u_\lambda \leq u_{\lambda+\varepsilon}$. To see that it is increasing (and not just non-decreasing) note that

$$-\Delta(u_{\lambda+\varepsilon} - u_\lambda) = (\lambda+\varepsilon) f(u_{\lambda+\varepsilon}) - \lambda f(u_\lambda) = \lambda[f(u_{\lambda+\varepsilon} - f(u_\lambda)] + \varepsilon f(u_{\lambda+\varepsilon}) > 0 \text{ in } \Omega,$$

since $u_{\lambda+\varepsilon} \geq u_\lambda$, f is increasing, positive on $[0, \infty)$ and $f(0) = 1$. By strong maximum principle $u_{\lambda+\varepsilon} > u_\lambda$. This means if $\partial_\lambda u_\lambda(x)$ exists then $\partial_\lambda u_\lambda(x) > 0$. To show $\partial_\lambda u_\lambda(x) > 0$ note first that the minimal solution u_λ satisfies (2.1) and taking a derivative in λ gives

$$\begin{cases} -\Delta \partial_\lambda u_\lambda - \lambda f'(u_\lambda) \partial_\lambda u_\lambda &= f(u_\lambda) \quad \Omega \\ \partial_\lambda u_\lambda &= 0 \quad \partial\Omega. \end{cases} \tag{2.4}$$

Put $\partial_\lambda u_\lambda = v$. f is positive and increasing, we get $-\Delta v - \lambda f'(u_\lambda) v > 0$. By maximum principle $v \geq 0$ and by strong maximum principle $v > 0$.

3. To prove minimal solution is stable we need to prove the first eigenvalue of the linearized operator $L := -\Delta - \lambda f'(u_\lambda)$ is positive. We can prove it in two different ways:

   - (First approach) Recall $\partial_\lambda u_\lambda = v > 0$ in Ω. For any $\psi \in C_c^\infty(\Omega)$ (test function) multiply both sides of (2.4) by $\frac{\psi^2}{v}$ and integrate on both sides we get

$$\begin{aligned}
\int_\Omega f(u_\lambda)\frac{\psi^2}{v} &= \int_\Omega -\Delta v \frac{\psi^2}{v} - \int_\Omega \lambda f'(u_\lambda) v \frac{\psi^2}{v} \\
&= \int_\Omega (-\Delta v)\psi^2 v^{-1} - \int_\Omega \lambda f'(u_\lambda)\psi^2 \\
&= \int_\Omega \nabla v \cdot \nabla(\psi^2 v^{-1}) - \int_\Omega \lambda f'(u_\lambda)\psi^2 \\
&= \int_\Omega \nabla v \cdot \left[\frac{2\psi\nabla\psi}{v} - \frac{\psi^2 \nabla v}{v^2}\right] - \int_\Omega \lambda f'(u_\lambda)\psi^2 \\
&\leq 2\int_\Omega \frac{|\nabla v|\,|\psi|}{v}\,|\nabla\psi| - \int_\Omega \frac{\psi^2\,|\nabla v|^2}{v^2} - \int_\Omega \lambda f'(u_\lambda)\psi^2 \\
&\leq 2\left[\frac{1}{2}\int_\Omega \frac{\psi^2\,|\nabla v|^2}{v^2} + \frac{1}{2}\int_\Omega \left|\nabla\psi^2\right|\right] - \int_\Omega \frac{\psi^2\,|\nabla v|^2}{v^2} - \int_\Omega \lambda f'(u_\lambda)\psi^2 \\
&= \int_\Omega |\nabla\psi|^2 - \int_\Omega \lambda f'(u_\lambda)\psi^2,
\end{aligned}$$

   which implies $\int_\Omega |\nabla\psi|^2 - \int_\Omega \lambda f'(u_\lambda)\psi^2 \geq 0$. Hence minimal solution u_λ is stable $\forall\lambda \in (0, \lambda^*)$.

   - (Second approach) Let $\phi > 0$ be the first eigenfunction and μ is the first eigenvalue such that $L(\phi) = \mu\phi$ and ϕ satisfies

$$\begin{cases} -\Delta\phi - \lambda f'(u_\lambda)\phi &= \mu\phi \quad \Omega \\ \phi &= 0 \qquad \partial\Omega. \end{cases}$$

Multiply both sides of $L(v) = f$ by ϕ and integrate by parts,

$$\begin{aligned}\int_\Omega f\phi &= \int_\Omega L(v)\phi \\ &= \int_\Omega vL(\phi) \\ &= \int_\Omega v[\mu\phi] \\ &= \mu \int_\Omega v\phi.\end{aligned}$$

Hence $\mu > 0$.

4. For the regularity of extremal solution see Theorem 26 and for the uniqueness see Section 2.3.

□

2.2 Regularity of Extremal solution

Extremal solution is so called the limit of minimal solution ($\lim_{\lambda \nearrow \lambda^*} u_\lambda = u^*$). The extremal solution can be either classical or singular. It's regularity strongly depends on domain Ω, nonlinearity f and dimension N.

Theorem 25. *(Crandall and Rabinowitz) Assume $f(u) = e^u$ and $N \leq 9$ then the extremal solution associated with $(P)_\lambda$ is smooth.*

Idea: To obtain the regularity of u^* we first establish an uniform $W^{2,p}$ bound on u_λ that is independent of λ. Then we want to pass to the limit to the minimal solution as $\lambda \nearrow \lambda^*$ to obtain the same regularity of u^*.

Proof. Let $\frac{\lambda^*}{2} \leq \lambda < \lambda^*$ and $u = u_\lambda$ denotes the minimal solution for $(P)_\lambda$. By stability u must satisfies the condition,

$$\int_\Omega |\nabla\phi|^2 \geq \lambda \int_\Omega e^u \phi^2 \quad \forall \phi \in H_0^1. \tag{2.5}$$

Take $\phi = e^{tu} - 1$ ($t > 0$ we can choose later) which is a smooth function and 0 on the boundary. Substitute ϕ in (2.5) we get

$$\lambda \int e^u (e^{tu} - 1)^2 \leq t^2 \int e^{2tu} |\nabla u|^2. \tag{2.6}$$

Now the idea is to multiply the equation by an appropriate function and integrate by parts so we get exactly the right hand side of the above inequality; this procedure we call "closing the estimate". Multiply $(P)_\lambda$ by $e^{2tu} - 1$ and integrate by parts to get

$$\begin{aligned}
\int \lambda e^u (e^{2tu} - 1) &= \int (-\Delta u)(e^{2tu} - 1) \\
&= \int_\Omega \nabla u \cdot \nabla (e^{2tu} - 1) - \int_{\partial\Omega} (e^{2tu} - 1) \partial_\nu u \\
&= \int_\Omega \nabla u \cdot 2te^{2tu} \cdot \nabla u - 0 \\
&= \ 2t \int e^{2tu} |\nabla u|^2.
\end{aligned} \tag{2.7}$$

The right hand side of (2.7) almost same as the right hand side of (2.6). By combining (2.6) and (2.7) we get

$$\lambda \int e^u (e^{tu} - 1)^2 \leq \frac{\lambda t}{2} \int e^u (e^{2tu} - 1).$$

By expanding, regrouping and dropping a positive term we get

$$\begin{aligned}
(1 - \frac{t}{2}) \int_\Omega e^{(2t+1)u} dx &\leq 2 \int_\Omega e^{(t+1)u} dx \\
&= 2 \int_\Omega 1 \cdot e^{(t+1)u} dx \\
&\leq 2 \left(\int_\Omega e^{\tau(t+1)u} dx \right)^{\frac{1}{\tau}} \left(\int_\Omega 1^{\tau'} dx \right)^{\frac{1}{\tau'}} \\
&= 2 \left(\int_\Omega e^{(2t+1)u} dx \right)^{\frac{1}{\tau}} |\Omega|^{\frac{1}{\tau'}}.
\end{aligned}$$

Here we used Hölder inequality on the right side. we took τ such that $\tau(t+1) = 2t+1$ and τ' denotes its conjugate. We now fix $t < 2$ but close and we get

$$\int e^{(2t+1)u} \leq (1 - \frac{t}{2})^{-\tau'} 2^{\tau'} |\Omega|,$$

where $\tau' - \frac{\tau'}{\tau} = 1$. So for all $0 < t < 2$ there is some $C_t > 0$ (independent of λ) such that for all $\frac{\lambda^*}{2} \leq \lambda < \lambda^*$ we have the bound,

$$\int_\Omega e^{(2t+1)u_\lambda} dx \leq C_t.$$

Now note that since we can take any $t < 2$ and since $-\Delta u_\lambda = \lambda e^{u_\lambda}$ we obtain a uniform (in λ) $W^{2,p}$ bound on u_λ for any $p < 5$. This shows that (independent on the dimension N) that $u^* \in W^{2,p}$ for all $p < 5$ (we are omitting the limiting procedure here). Now note that if $N \leq 9$ that $W^{2,p} \subset L^\infty$ provided p is chosen close enough to 5. So we see that for $N \leq 9$ that u^* is bounded and hence smooth. $\square$

Regularity of the Extremal solution on general domains

Lemma 2.3. *Let f satisfies (A). Then there is some $C > 0$ such that*

$$\int_\Omega f(u^*) f'(u^*) dx \leq C. \tag{2.8}$$

Proof. We start the proof by defining $H(t) := \int_0^t [f(\tau) - 1] f''(\tau)\, d\tau$ for any number $t > 0$. Then

$$H'(t) = [f(t) - 1] f''(t). \tag{2.9}$$

Fix $0 < T < \infty$ large such that we can get an estimate $H(t) \geq 2f'(t)$ forall $t > T$.

To see that,

$$
\begin{aligned}
H(t) = \int_0^t [f(\tau)-1]f''(\tau)d\tau &= \int_0^T [f(\tau)-1]f''(\tau)d\tau + \int_T^t [f(\tau)-1]f''(\tau)d\tau \\
&\geq \int_T^t [f(\tau)-1]f''(\tau)d\tau \\
&\geq [f(T)-1][f'(t)-f'(T)],
\end{aligned}
$$

which implies

$$\frac{H(t)}{f'(t)} \geq [f(T)-1][1-\frac{f'(T)}{f'(t)}].$$

Since $\lim_{t\to\infty} f'(t) \geq \lim_{t\to\infty} \frac{f(t)-1}{t} = \infty$ (by convexity of f on $[0,t]$), we have $\lim_{t\to\infty} \frac{f'(T)}{f'(t)} = 0$. Hence $\lim_{t\to\infty} \frac{H(t)}{f'(t)} \geq [f(T)-1]$. T is fixed so $f(T)-1$ is some number and we can conclude that $H(t) \geq 2f'(t)$.

Assume $\frac{\lambda^*}{2} < \lambda < \lambda^*$. By stability of $u = u_\lambda$ we have

$$
\begin{aligned}
\lambda \int_\Omega f'(u)\phi^2 &\leq \int_\Omega |\nabla\phi|^2 \qquad \forall \phi \in H_0^1(\Omega) \\
&= \int_\Omega \nabla\phi \cdot \nabla\phi \\
&= \int_\Omega (-\Delta\phi)\phi. \qquad (2.10)
\end{aligned}
$$

Put $\phi = f(u)-1$ in (2.10) we get

$$
\begin{aligned}
\lambda \int_\Omega f'(u)[f(u)-1]^2 &\leq \int_\Omega [-\Delta f(u)|\nabla u|^2 + \nabla f(u)(-\Delta u)][f(u)-1] \\
&= \int_\Omega [-\Delta f(u)|\nabla u|^2 + \nabla f(u)(\lambda f(u))][f(u)-1].
\end{aligned}
$$

By expanding, regrouping, cancelling and dropping a positive term we get

$$\int_\Omega [[f(u)-1]f''(u)|\nabla u|] \cdot \nabla u \leq \lambda \int_\Omega f'(u)f(u). \qquad (2.11)$$

By using (2.9) and the fact $\nabla H(u(x)) = H'(u(x)) \cdot \nabla u(x)$ we get from (2.11)

$$\int_\Omega \nabla H(u) \cdot \nabla u \leq \lambda \int_\Omega f'(u) f(u).$$

Which follows

$$\int_\Omega f(u)H(u) \leq \int_\Omega f(u)f'(u). \tag{2.12}$$

We are going to split the domain because we want to know the bound for $\int f(u)f'(u)$ when $u > T$(much bigger). So we write (2.12) as

$$\begin{aligned}\int_{u>T} f(u)H(u) &\leq \int_{u<T} f(u)f'(u) + \int_{u\geq T} f(u)f'(u)\\ &\leq f(T)f'(T)|\Omega| + \int_{u\geq T} f(u)f'(u).\end{aligned}$$

Now using the fact $\lim_{t\to\infty} H(t) \geq 2f'(t)$ we get

$$\int_{u\geq T} f(u)f'(u) \leq f(T)f'(T)|\Omega|. \tag{2.13}$$

So for all $u > T$ (fixed large number) we have from (2.13)

$$\int_{u\geq T} f(u)f'(u) \leq C.$$

So we got $\int_{u_\lambda \geq T} f(u_\lambda)f'(u_\lambda)$ is bounded by a constant that is independent of λ. Now we can pass to the limit. By Monotone Convergence Theorem as $\lambda \nearrow \lambda^*$ we get $u_\lambda \nearrow u^*$. Hence

$$\begin{aligned}&\int_{u\geq T} f(u^*)f'(u^*) \leq C\\ \implies &\int_\Omega f(u^*)f'(u^*) \leq C_2.\end{aligned}$$

$\square$

Theorem 26. *Let f satisfies (A). Then the extremal solution u^* associated with $(P)_{\lambda^*}$ satisfies $u^* \in W^{2,p}(\Omega)$ for all $p < \frac{N}{N-2}$. In particular by elliptic regularity this proves that $u^* \in L^\infty(\Omega)$ for $N \leq 3$, which is a result of [25]; but our proof is different.*

Proof. Let $\frac{\lambda^*}{2} < \lambda < \lambda^*$ and $u = u_\lambda$. We are trying to obtain the regularity of u^*. Set $v_\lambda(x) := f(u) - 1$ and then $v_\lambda(x)$ satisfies

$$\begin{cases} -\Delta v_\lambda = \lambda f'(u)f(u) - f''(u)|\nabla u|^2 & \Omega \\ v_\lambda = 0 & \partial\Omega. \end{cases} \tag{2.14}$$

Consider the pde

$$\begin{cases} -\Delta w_\lambda = \lambda f'(u)f(u) & \Omega \\ w_\lambda = 0 & \partial\Omega. \end{cases} \tag{2.15}$$

Also we have

$$\begin{cases} -\Delta(w_\lambda - v_\lambda) = f''(u)|\nabla u|^2 \geq 0 & \Omega \\ w_\lambda - v_\lambda = 0 & \partial\Omega. \end{cases} \tag{2.16}$$

Since $f''(u)|\nabla u|^2 \geq 0$, by maximum principle $w_\lambda \geq v_\lambda \geq 0$ in Ω. $w_\lambda \in L^1(\Omega)$ is an ultra weak solution of (2.15) with $\lambda f(u)f'(u) \in L^1(\Omega)$ (by Lemma 2.3) in the sense of

$$\int_\Omega (-\Delta\phi)w_\lambda = \int_\Omega \lambda f(u)f'(u)\phi \quad \phi \in H_0^1.$$

Now by applying L^1 regularity theory on (2.15) we get $\forall 1 < p < \frac{N}{N-2}$,

$$\|w_\lambda\|_{L^p} \leq C_p\lambda\|f(u)f'(u)\|_{L^1} \leq C.$$

Which follows

$$\|f(u) - 1\|_{L^p} = \|v_\lambda\|_{L^p} \leq \|w_\lambda\|_{L^p} \leq C.$$

By applying L^p regularity on (2.1) we can conclude $\forall 1 < p < \frac{N}{N-2}$ and $\forall \frac{\lambda^*}{2} < \lambda < \lambda^*$,

$$\|u_\lambda\|_{W^{2,p}} \leq C_p \|f(u_\lambda)\|_{L^p} \leq C_1,$$

where C_1 is independent of λ. By Monotone Convergence Theorem we pass to the limit as $\lambda \nearrow \lambda^*, u_\lambda \nearrow u^*$. Hence

$$\|u^*\|_{W^{2,p}} \leq C_p \|f(u^*)\|_{L^p} \leq C_1.$$

By Sobolev imbedding $W^{2,p} \subset L^\infty$ provided $p < \frac{N}{N-2}$. Hence u^* smooth. $N \leq 3$ is coming from the fact that by elliptic regularity $u^* \in L^\infty$ provided $p > \frac{N}{2}$ and $\frac{N}{2}$ is the dual exponent of $\frac{N}{N-2}$.

□

Singular Extremal solution

The following Proposition is important since it allows one to recognize when some singular solution is in fact the extremal solution.

Proposition 2.4. *(Brezis and Vasquez; 1997) Consider the problem $(P)_\lambda$. Suppose $v \in H_0^1(\Omega)$ is a singular weak solution of $(P)_\lambda$. Then the following are equivalent*

1. *$f(v)\delta \in L^1(\Omega)$ and*
$$\int_\Omega |\nabla \phi|^2 - \lambda \int_\Omega f'(v)\phi^2 \geq 0, \tag{2.17}$$
holds for all $\phi \in C_c^1(\Omega)$.

2. *$\lambda = \lambda^*, v = u^*$.*

Proof. • (2 $\Longrightarrow$ 1). The extremal solution u^* is in the increasing limit of minimal solution u_λ as $\lambda \nearrow \lambda^*$ and u_λ satisfies $\int_\Omega |\nabla\phi|^2 - \lambda \int_\Omega f'(u_\lambda)\phi^2 \geq 0$ $\forall \phi \in C_c^1(\Omega)$, which in the limit gives (2.17) (by Fatou's lemma).

- (1 $\Longrightarrow$ 2) Since no solution exists for $\lambda > \lambda^*$ so we have $\lambda \leq \lambda^*$. Assume the contrary that $\lambda < \lambda^*$. By the density argument and the fact that $v, u_\lambda \in H_0^1$, we can take the test function $\phi = v - u_\lambda \in H_0^1$. By the minimality of u_λ, we see $v - u_\lambda \geq 0$ in Ω, and the assumption $v \notin L^\infty$ implies that $v - u_\lambda \neq 0$, since u_λ bounded for $\lambda < \lambda^*$. Combining the equation satisfied by (2.17) we obtain

$$\lambda \int (v - u_\lambda)[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)] \geq 0 \qquad \forall \phi \in H_0^1(\Omega).$$

  Since f is convex the integrand is nonpositive. So the inequality is only possible if

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \quad \text{ a.e. in } \Omega.$$

  Again the strict convexity of f implies $v = u_\lambda$ a.e. in Ω, which is a contradiction. Thus we must have $\lambda = \lambda^*$. Since $u_\lambda \leq v \quad \forall \lambda < \lambda^*$ and $u_\lambda \in H_0^1$, then in the limit we have $u^* \leq v$ and $u^* \in H_0^1$. Now $v = u^*$ follows from Martel result [23] which we will discuss in Section 2.3.

□

The inequality (2.17) is the version of Hardy inequality with weight, which says that for a certain weight function $W(x) \in L^1_{loc}(\Omega)$, $w \geq 0$ we have

$$\int_\Omega W\phi^2 dx \leq \int_\Omega |\nabla\phi|^2 dx \quad \forall \phi \in C_c^1(\Omega).$$

We have the classical Hardy inequality which occurs for a weight function of the form $W(x) = \frac{C}{|x|^2}$ when $N \geq 3$.

Theorem 27. *[26](Classical Hardy inequality) Suppose* $0 \in \Omega \subset \mathbb{R}^N$ *with* $N \geq 3$.

Then

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\phi^2}{|x|^2}dx \le \int_\Omega |\nabla\phi|^2 dx \qquad \forall \phi \in H_0^1(\Omega). \tag{2.18}$$

Proof. Let $\phi \in C_c^\infty(\Omega)$ and $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$. We are going to use $div(\frac{x}{|x|^t}) = \frac{N-t}{|x|^t}$ to prove the inequality. So then

$$\begin{aligned}(N-2)\int_{\Omega_\varepsilon}\frac{\phi^2}{|x|^2}dx &= \int_{\Omega_\varepsilon} div(\frac{x}{|x|^2})\phi^2 dx\\ &= -\int_{\Omega_\varepsilon}\frac{x}{|x|^2}2\phi\nabla\phi + \int_{\partial\Omega_\varepsilon}\phi^2\frac{(x\cdot\nu)}{|x|^2}\\ &= -\int_{\Omega_\varepsilon}\frac{x}{|x|^2}2\phi\nabla\phi + I_\varepsilon\\ &\le \int_{\Omega_\varepsilon}\frac{2|\phi|}{|x|}|\nabla\phi| + I_\varepsilon. \end{aligned}\tag{2.19}$$

We compute the last term I_ε to see

$$\begin{aligned} I_\varepsilon &= \int_{\partial\Omega_\varepsilon}\phi^2\frac{(x\cdot\nu)}{|x|^2}dx\\ &= \int_{\partial\Omega}\phi^2\frac{(x\cdot\nu)}{|x|^2}dx + \int_{|x|=\varepsilon}\frac{x\cdot\frac{-x}{\varepsilon}\phi^2}{|x|^2}dx\\ &= 0 + \int_{|x|=\varepsilon}\frac{x\cdot\frac{-x}{\varepsilon}\phi^2}{|x|^2}dx\\ &\le \frac{1}{\varepsilon}\int_{|x|=\varepsilon}\phi^2 dx\\ &\le \frac{\sup\phi^2}{\varepsilon}\int_{|x|=\varepsilon}ds(x)\\ &\le \sup\phi^2 D_N\varepsilon^{N-2}\to 0 \qquad ifN\ge 3. \end{aligned}$$

From (2.19) we get

$$\begin{aligned}(N-2)\int_{\Omega_\varepsilon}\frac{\phi^2}{|x|^2}dx &\le \int_{\Omega_\varepsilon}\frac{2|\phi|}{|x|}|\nabla\phi|dx\\ &\le 2\left(\int_\Omega\frac{\phi^2}{|x|^2}dx\right)^{\frac12}\left(\int_\Omega|\nabla\phi|^2dx\right)^{\frac12}. \end{aligned}$$

Thus for $N \geq 3$,

$$\frac{(N-2)^2}{4} \int_\Omega \frac{\phi^2}{|x|^2} dx \leq \int_\Omega |\nabla \phi|^2 dx \qquad \forall \phi \in C_c^\infty(\Omega).$$

$\square$

Theorem 28. *(Joseph and Lundgren) For $N \geq 10$ the extremal solution associated with $(P)_\lambda$ in the case of $f(u) = e^u$ and $\Omega = B_1$ is singular.*

Remark 2.5. *For $1 \leq N \leq 9$ the solution curve turns around λ^* where we have $\lambda_1(-\Delta - \lambda^* f'(u), B_1) = 0$. All solutions lying above the stable brunch are unstable, so their Morse index is at least equal to one. For $3 \leq N \leq 9$ the solution curve exhibits infinitely many turning points accumulating towards $\lambda_N := 2(N-2), v(x) := -2ln(|x|)$. At each of these points, the Morse index of solutions increases by one unit. When $N \geq 10$ then $\lambda^* = \lambda_N$ and for each $\lambda \in (0, \lambda^*)$ there is unique solution and v stays at the end of curve of stable solution. Note that $v(x)$ is a solution to (2.1)(in the sense of distribution) only for $N \geq 3$. The picture below makes the idea clear.*

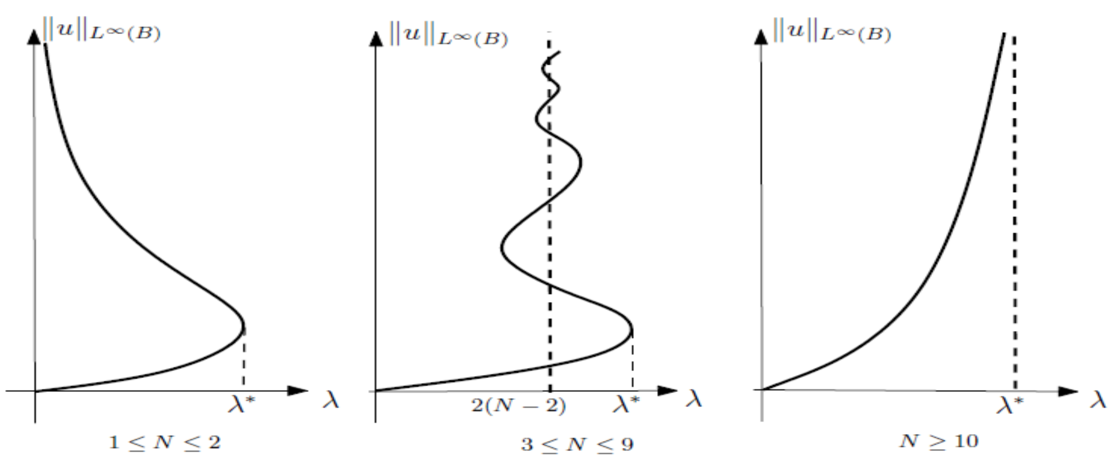

Figure 2.1: Bifurcation diagrams for the Gelfand problem

Proof of Theorem 28. v and λ_N are defined above. For $N \geq 3$, $v \in H_0^1(B_1)$. We want to show v is a distributional solution of $-\Delta v = \lambda_N e^v$ in B_1, means $\forall \psi \in C_c^\infty(B_1)$ we will have

$$\int_{B_1} -\Delta\psi v = \int_{B_1} \lambda_N e^v \psi. \tag{2.20}$$

A computation shows that

$$-\Delta v = 2\nabla \cdot \frac{x_i}{|x|^2} = \lambda_N e^v \qquad B_1 \setminus \{0\},$$

where $\lambda_N > 0$ and clearly $v = 0$ on ∂B_1. So v is a classical solution of the pde away from the origin. This v would be a distributional solution provided the test function is compactly supported away from the origin. So for $\varepsilon > 0$ small, let $\psi = \psi\gamma_\varepsilon$, where γ_ε is a smooth cut-off function ($\gamma_\varepsilon = 0$ inside B_ε and $\gamma_\varepsilon = 1$ outside $B_{2\varepsilon}$). Replace ψ in (2.20) by $\psi\gamma_\varepsilon$ we get

$$\int_{B_1} -\Delta\psi\gamma_\varepsilon v - 2\nabla\psi \cdot \nabla\gamma_\varepsilon v + \psi(-\Delta\gamma_\varepsilon)v = \int_{B_1} \lambda_N e^v \psi\gamma_\varepsilon v. \tag{2.21}$$

Note that $\gamma_\varepsilon = 0$ if $|\frac{x}{\varepsilon}| < 1$ and $\gamma_\varepsilon = 1$ if $|\frac{x}{\varepsilon}| > 2$ and we have $\nabla\gamma_\varepsilon(x) = \frac{\nabla\gamma(\frac{x}{\varepsilon})}{\varepsilon}$. Also

$$\begin{aligned}
\sup_{\varepsilon<|x|<2\varepsilon} |\nabla\gamma_\varepsilon(x)| &= \frac{1}{\varepsilon} \sup_{\varepsilon<|x|<2\varepsilon} |\nabla\gamma(\frac{x}{\varepsilon})| \\
&= \frac{1}{\varepsilon} \sup_{1<|\frac{x}{\varepsilon}|<2} |\nabla\gamma(\frac{x}{\varepsilon})| \\
&= \frac{1}{\varepsilon} \sup_{1<|y|<2} |\nabla\gamma(y)| \\
&= \frac{c}{\varepsilon},
\end{aligned}$$

where c is independent of ε. Now we estimate the last two terms of left side of (2.21) to see

$$\begin{aligned}
\int_{B_1} |\nabla\psi||\nabla\gamma_\varepsilon||v| &= c_1 \int_{\varepsilon<|x|<2\varepsilon} |\nabla\gamma_\varepsilon||v| \\
&\leq \frac{c_1 c}{\varepsilon} \int_{\varepsilon<|x|<2\varepsilon} |-2ln(|x|)|\,dx \\
&= \frac{d_1}{\varepsilon} C_N \int_{\varepsilon<r<2\varepsilon} |-lnr| r^{N-1}\,dr \\
&\leq \frac{d_1}{\varepsilon} C_N |-ln\varepsilon| \varepsilon^{N-1}\varepsilon \\
&= d_N |-ln\varepsilon| \varepsilon^{N-1} \\
&< d_N \varepsilon^{\frac{-1}{2}} \varepsilon^{N-1},\ N \geq 3
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
\int_{B_1} |\psi||\Delta\gamma_\varepsilon||v| &\leq \frac{c_2 c}{\varepsilon^2} \int_{\varepsilon<|x|<2\varepsilon} |v| \\
&= \frac{d_2}{\varepsilon^2} C_N \int_{\varepsilon<r<2\varepsilon} |-lnr| r^{N-1}\,dr \\
&\leq \frac{d_2}{\varepsilon^2} C_N |-ln\varepsilon| \varepsilon^{N-1}\varepsilon \\
&= e_N |-ln\varepsilon| \varepsilon^{N-2} \\
&< e_N \varepsilon^{\frac{-1}{2}} \varepsilon^{N-2},\ N \geq 3.
\end{aligned} \tag{2.23}$$

Here we used $|-ln\varepsilon| < \varepsilon^{\frac{-1}{2}}$ for ε sufficiently small which follows from the fact that $\lim_{\varepsilon\to 0}(-ln\varepsilon)\varepsilon^{\frac{1}{2p}} = 0$ (by L'Hospital's rule). Now we see that when $\varepsilon \to 0$ both terms on (2.22) and (2.23) go to zero. Also $\gamma_\varepsilon \to 1$ in B_1. Hence from (2.21) we obtain (2.20). This proves $v \in H_0^1(B_1)$ is a distributional solution of the pde $-\Delta v = \lambda_N e^v$ in B_1 and $v = 0$ on ∂B_1 for $N \geq 3$.

Since v is singular, if we can show that v is a stable solution then we'd have that $(\lambda_N, v) = (\lambda^*, u^*)$ and hence we'd have u^* singular. So to show that v is stable we need to show that

$$\int \lambda_N e^v \phi^2 \leq \int |\nabla\phi|^2 \qquad \forall\phi \in H_0^1(B).$$

Putting in the values λ_N and v shows that we need

$$2(N-2)\int_B \frac{\phi^2}{|x|^2} \leq \int_B |\nabla\phi|^2 \qquad \forall\phi \in H_0^1(B). \tag{2.24}$$

Now we have the Hardy inequality

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\phi^2}{|x|^2}dx \leq \int_\Omega |\nabla\phi|^2 dx, \qquad \forall\phi \in H_0^1(\Omega).$$

So if $\frac{(N-2)^2}{4} \geq 2(N-2)$ (which is exactly $N \geq 10$) then (2.24) holds and hence v is a stable solution of $(P)_{\lambda_N}$. $\square$

2.3 Uniqueness of Extremal solution

Theorem 29. *Let $\lambda = \lambda^*$. Suppose there exists a weak super solution $z \in L^1, z \geq 0$ of*

$$\begin{cases} -\Delta z = \lambda f(z) & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.25}$$

in the following sense

$$f(z)\delta \in L^1,$$

and

$$-\int_\Omega z\Delta\phi \geq \lambda \int_\Omega f(z)\phi,$$

for all $\phi \in C^\infty(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$. Then $z = u^*$ is the extremal solution.

Corollary 2.6. *(Yvan Martel) Under the assumption of Theorem 29 there is atmost one weak solution of (2.25) at $\lambda = \lambda^*$.*

We need a lemma dealing with some ode results.

Lemma 2.7. *Let f satisfies **(A)**, let $\lambda > 0$ and $\varepsilon > 0$ and consider the ode given by*

$$y'(t) = \frac{\lambda f(y(t)) + \frac{\varepsilon}{2}}{\lambda f(t) + \varepsilon}, \quad t \in (0, T_{max}) \tag{2.26}$$

where T_{max} is the maximal time of existence. Then $T_{max} = \infty$, and for $t \in (0, T_{max})$ we have $y''(t) < 0, 0 < y(t) < t$ and $\sup_{t>0} y(t) < \infty$.

Proof. Since

$$\int_0^\infty \frac{1}{\lambda f(s) + \varepsilon} ds < \int_0^\infty \frac{1}{\lambda f(s) + \frac{\varepsilon}{2}} ds, \tag{2.27}$$

we see for all $t > 0$ there is some $y(t) > 0$ such that

$$\int_0^t \frac{1}{\lambda f(s) + \varepsilon} ds = \int_0^{y(t)} \frac{1}{\lambda f(s) + \frac{\varepsilon}{2}} ds, \tag{2.28}$$

and so $y(t)$ solves (2.26). From this we see that $T_{max} = \infty$ and also we see that $\sup_{t\geq 0} y(t) < \infty$. For $t > 0$, $y(t)$ is the unique real number such that (2.28) holds. Now when t goes to ∞, $y(t)$ goes to ∞ as well since $y(t)$ is increasing. So from (2.28) we have

$$\int_0^\infty \frac{1}{\lambda f(s) + \varepsilon} ds = \int_0^\infty \frac{1}{\lambda f(s) + \frac{\varepsilon}{2}} ds,$$

which contradicts (2.27). So (2.28) to hold we must need $\sup(y(t)) < \infty$ when $t \to \infty$.

We now claim that $y(t) < t$ for all $t > 0$. To see this we can use last point argument or we can get this directly from the formula for $y(t)$. Note that

$$\frac{1}{\lambda f(s) + \varepsilon} < \frac{1}{\lambda f(s) + \frac{\varepsilon}{2}},$$

for all $s \geq 0$ and hence we must have $y(t) < t$. Also $y''(t) = \frac{y'(t)[\lambda f'(y(t)) - \lambda f'(t)]}{\lambda f(t) + \varepsilon} < 0$ since f is convex and $y(t) < t$. So $y(t)$ concave. In particular, $0 < y'(t) \leq y'(0)$ $\forall t \geq 0$, that is $y'(t)$ bounded. $\square$

.

Lemma 2.8. *Suppose $w(x)$ is a weak solution of*

$$\begin{cases} -\Delta w = f(w) + \varepsilon & \textit{in } \Omega \\ w = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.29}$$

for $\varepsilon > 0$. Then there exists a classical solution $u \in C^2(\overline{\Omega})$ of

$$\begin{cases} -\Delta u = (1+\alpha) f(u) & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.30}$$

for some $\alpha > 0$.

Proof. Suppose $y(w(x))$ satisfies the pde

$$\begin{cases} -\Delta y(w) \geq f(y(w)) + \frac{\varepsilon}{2} & \text{in } \Omega \\ y(w) \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{2.31}$$

where $y(w)$ satisfies properties of Lemma 2.7. Then by sub/supersolution approach there is a classical solution $0 \leq l \leq y(w) < \infty$ of

$$\begin{cases} -\Delta l = f(l) + \frac{\varepsilon}{2} & \text{in } \Omega \\ l = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the function χ satisfying the pde

$$\begin{cases} -\Delta \chi = 1 & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.32}$$

Note that $\chi \leq l$ by maximum principle and there exists $\alpha > 0$ small such that $\alpha[f(l) + \frac{\varepsilon}{2}] - \frac{\varepsilon}{2} \geq 0$ holds, which implies $2\alpha l \leq \varepsilon\chi$. Now we are going to construct a supersolution of (2.30) using l and χ. Set

$$z = l + \alpha l - \frac{\varepsilon}{2}\chi.$$

By maximum principle $v, \chi \geq 0$. Hence its clear $0 < z \leq l$ and z satisfies

$$\begin{cases} -\Delta z &= (1+\alpha)f(l) + \frac{\varepsilon}{2}(\alpha+2) \geq (1+\alpha)f(z) \quad \text{in } \Omega \\ z &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Hence z is a bounded supersolution of (2.30). By taking 0 as subsolution we can show by iteration argument there exists a classical solution u of (2.30) between sub and supersolution. □

Proof of Theorem 29. Suppose $\lambda^* = 1$. u^* denotes the extremal solution of $(P)^*_\lambda$ which is the minimal weak solution at λ^*. Suppose v is also a weak solution of $(P)_{\lambda^*}$ and $v \neq u^*$. We will show $v = u^*$.

$$\text{Set } \Omega_0 := \{x \in \Omega | u^*(x) \neq v(x), u^*(x) < v(x) < \infty\}.$$

Note that $|\Omega_0| > 0$. Define

$$h(x) := \begin{cases} \frac{f(u^*)+f(v)}{2} - f(\frac{u^*(x)+v(x)}{2}) & x \in \Omega_0 \\ 0 & \textit{otherwise}. \end{cases}$$

By strict convexity of f (by hypothesis), we have $h(x) \geq 0$ in Ω and $h(x) > 0$ in Ω_0. Also note that $h(x) \in L^1(\Omega)$. Define $z := \frac{u^*+v}{2}$. Since u^* and v are weak solutions of $(S)_{\lambda^*}$ so z is a weak solution of

$$\begin{cases} -\Delta z &= f(z) + h(x) \quad \text{in } \Omega \\ z &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Consider

$$\begin{cases} -\Delta\psi &= h(x) \quad \text{in } \Omega \\ \psi &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $\psi \in L^1(\Omega)$ verifies $h(x)\delta \in L^1(\Omega)$. Such solution exists and unique in the sense that

$$-\int_\Omega \Delta\phi\psi dx = \int_\Omega h(x)\phi dx \quad \forall\phi \in C_c^\infty(\overline{\Omega}).$$

Also consider the function χ satisfies the pde

$$\begin{cases} -\Delta\chi &= 1 \quad \text{in } \Omega \\ \chi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

By maximum principle $\psi, \chi \geq 0$ in Ω. For $\varepsilon > 0$ it follows from the properties of laplace equation that $\varepsilon\chi \leq \psi$. Set $\tau = z + \varepsilon\chi - \psi$. It is clear that $0 < \tau \leq z$ and τ is a weak solution of

$$\begin{cases} -\Delta\tau &= f(z) + \varepsilon \quad \text{in } \Omega \\ \tau &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

By maximum principle $\tau \geq 0$ in Ω. But from the fact that $\tau \leq z$ we get $-\Delta\tau = f(z) + \varepsilon \geq f(\tau) + \varepsilon$ in Ω and $\tau \geq 0$ on $\partial\Omega$. By iteration procedure there is an ultra weak solution $w(x)$ of

$$\begin{cases} -\Delta w &= f(w) + \varepsilon \quad \text{in } \Omega \\ w &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$-\int_\Omega \Delta\phi w_{m+1} = \int_\Omega (f(w_m) + \varepsilon)\phi \quad \forall m \geq 0,$$

and by pointwise convergence

$$-\int_{\Omega} \Delta\phi w = \int_{\Omega} (f(w) + \varepsilon)\phi \quad \forall m \geq 0,$$

where $0 \leq w_1 \leq w_2 \ldots\ldots \leq w_m \leq w_{m+1} \leq \tau$ and $\lim_{m\to\infty} w_{m+1} = w(x)$.

Now by Lemma 2.8 we get a contradiction at $\lambda^* = 1$. This completes the proof.

$\square$

2.4 Radial solution in a Ball

Theorem 30. *(Positive solutions on ball are radial) Suppose f satisfies (A) and $0 \leq u$ is a solution of $(P)_\lambda$ where Ω is a ball centered at the origin. Then u is a radial solution and $u'(r) < 0$.*

Consider

$$(P')_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) \quad B_1 \\ u &= 0 \qquad \partial B_1, \end{cases} \tag{2.33}$$

where B_1 is the unit ball in $\mathbb{R}^N$.

Theorem 31. *[5] Suppose f satisfies (A) and $N \leq 9$. Then the extremal solution associated with $(P')_\lambda$ is smooth.*

Note: λ is skipped throughout the whole proof. It will not affect on the boundedness of u. The constant that will bound u is independent of λ.

Proof. Consider

$$\begin{cases} -\Delta u &= f(u) \quad B_1 \\ u &= 0 \qquad \partial B_1. \end{cases} \tag{2.34}$$

Assume $\frac{\lambda^*}{2} < \lambda < \lambda^*$. We consider $u \in H_0^1(B_1 \setminus \{0\})$ is a radial solution of (P'_λ). Let say $u_\lambda = u$. By stability of u we have

$$\int_{B_1} f'(u)\phi^2 \leq \int_{B_1} |\nabla\phi|^2 \quad \forall\phi \in H_0^1(B_1 \setminus \{0\}).$$

Take $\phi = u_r\psi$ where $\psi \in H_0^1(B_1)$ is a smooth cut-off function which has compact support in $B_1 \setminus \{0\}$ and we get

$$\begin{aligned}
\int_{B_1} f'(u)u_r^2\psi^2 &\leq \int_{B_1} |\nabla(u_r\psi)|^2 \\
&= \int_{B_1} |\nabla u_r\psi + u_r\nabla\psi|^2 \\
&= \int_{B_1} |\nabla u_r|^2\psi^2 + 2\nabla u_r u_r \nabla\psi\psi + u_r^2|\nabla\psi|^2. \qquad (2.35)
\end{aligned}$$

We are going to estimate the middle term say I,

$$\begin{aligned}
I &= \int_{B_1} 2\nabla u_r u_r \nabla\psi\psi dx \\
&= \int_{B_1} (\nabla u_r)u_r\nabla(\psi^2)dx \\
&= -\int_{B_1} div[(\nabla u_r)u_r]\psi^2 + \int_{\partial B_1} \psi^2[(\nabla u_r)u_r]\cdot\nu \\
&= -\int_{B_1} div[(\nabla u_r)u_r]\psi^2 + 0 \\
&= -\int_{B_1} div(\nabla u_r)u_r\psi^2 - \int_{B_1} |\nabla u_r|^2\psi^2 \\
&= \int_{B_1} (-\Delta u_r)u_r\psi^2 - \int_{B_1} |\nabla u_r|^2\psi^2.
\end{aligned}$$

Differentiating (2.34) with respect to r to get

$$-\Delta u_r = f'(u)u_r - \frac{N-1}{r^2}u_r,$$

where $u_r \in (H^2_{loc} \cap L^\infty_{loc})(B_1 \setminus \{0\})$(by local elliptic regularity) which is supported in $B_1 \setminus \{0\}$. Thus $I = \int_{B_1} \psi^2 u_r \left(f'(u)u_r - \frac{N-1}{r^2}u_r\right) - \int_{B_1} |\nabla u_r|^2\psi^2$. By substituting this in (2.35) and after cancellation of some terms we get

$$\int_{B_1} \psi^2 u_r^2 \frac{N-1}{r^2} \leq \int_{B_1} |\nabla\psi|^2 u_r^2. \qquad (2.36)$$

Now we are going to build an estimate for the L^2 norm of $u_r r^{-1-\alpha}$ for certain positive exponent α which depends on the dimension N. This is the key ingredient to prove u bounded.

For $\varepsilon > 0$ small we define the cut-off function

$$\psi_\varepsilon(x) := \begin{cases} \frac{1}{|x|^\alpha} - 1 & \text{if} \quad \varepsilon < |x| < 1 \\ \frac{1}{\varepsilon^\alpha} - 1 & \text{if} \quad |x| < \varepsilon. \end{cases}$$

Assume $0 < \alpha < \sqrt{N-1}$. By applying $\psi = \psi_\varepsilon$ in (2.36) we get

$$\int_{B_1 \setminus B_\varepsilon} \left(\frac{1}{|x|^\alpha} - 1 \right)^2 u_r^2 \frac{N-1}{r^2} \leq \int_{B_1 \setminus B_\varepsilon} |\nabla \left(\frac{1}{|x|^\alpha} - 1 \right) |^2 u_r^2 + 0. \tag{2.37}$$

Here we have cancelled a term from the left and we have used the fact $|\nabla \psi| = 0$ on B_ε. By expanding, regrouping, cancelling some terms and substituting $r = |x|$ we get from (2.37)

$$(N-1-\alpha^2) \int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^{2\alpha+2}} \leq \int_{B_1 \setminus B_\varepsilon} 2 \frac{u_r^2}{|x|^{\alpha+2}} (N-1).$$

We use Hölder on the right side,

$$(N-1-\alpha^2) \int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^{2\alpha+2}} \leq 2(N-1) \int_{B_1 \setminus B_\varepsilon} \left(\frac{u_r^2}{|x|^{2\alpha+2}} \right)^{\frac{1}{2}} \left(\frac{u_r^2}{|x|^2} \right)^{\frac{1}{2}},$$

which in turns implies

$$\int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^{2+2\alpha}} \leq \frac{4(N-1)^2}{(N-1-\alpha^2)^2} \int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^2}$$

$$= C_{\alpha,N} \int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^2},$$

where $C_{\alpha,N} := \frac{4(N-1)^2}{(N-1-\alpha^2)^2}$ (independent of ε). We need to show $\displaystyle\int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^{2+2\alpha}}$ is bounded. First we need to estimate $\displaystyle\int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^2}$. For this we choose an explicit

$\psi(x) = 1$ inside $B_{\frac{1}{2}}$ and $\psi(x) = 0$ outside of $B_{\frac{3}{4}}$ and from (2.36) we get

$$\int_{B_{\frac{1}{2}}} \psi^2 u_r^2 \frac{N-1}{r^2} + \int_{\frac{1}{2}<|x|<\frac{3}{4}} \psi^2 u_r^2 \frac{N-1}{r^2} \leq \int_{B_{\frac{1}{2}}} |\nabla \psi|^2 u_r^2 + \int_{\frac{1}{2}<|x|<\frac{3}{4}} |\nabla \psi|^2 u_r^2.$$

Dropping a term from left side and using the fact $\nabla \psi = 0$ on $B_{\frac{1}{2}}$ and $|\nabla \psi|$ is bounded on $B_{\frac{3}{4}} \setminus B_{\frac{1}{2}}$ we get

$$\int_{B_{\frac{1}{2}}} \frac{u_r^2}{|x|^2} \leq c \int_{\frac{1}{2}<|x|<\frac{3}{4}} u_r^2. \tag{2.38}$$

Now we need a bound on $\int_{\frac{1}{2}<|x|<\frac{3}{4}} u_r^2$. For $0 < R_1 < R_2 \leq 1$ we will find a bound on $u_r(|x|)$. By using (2.34) we get

$$\int_{R_1}^{R_2} \frac{d}{dr}(r^{N-1}) u_r(r) = -\int r^{N-1} f(u),$$

which implies $R_2^{N-1} u_r(R_2) - R_1^{N-1} u_r(R_1) \leq 0$. Now put $R_1 = R$ and $R_2 = 1$ and $\forall R \in (\frac{1}{4}, 1)$ $\exists D > 0$ such that $-u_r(R) \leq D(-1)u_r(1)$. Here $D := \frac{1}{R^{N-1}}$. Since $u(r)$ is a decreasing function for all $r \in (0,1)$, so $\forall R \in (\frac{1}{4}, 1)$ we have $0 \leq -u_r(R) \leq D(-1)u_r(1)$. By integrating the pde $-\Delta u = f(u)$ over the unit ball we get

$$-\int_{\partial B_1} \partial_\nu u = \int_{B_1} f(u).$$

For $x_0 \in \partial B_1$, $\partial_\nu u(x_0) = \nabla u(x_0) \cdot \nu(x_0) = u_r(1) \frac{x_0}{|x_0|} \cdot \frac{x_0}{|x_0|} = u_r(1)$. Hence $-u_r(1)|\partial B_1| = \int_{B_1} f(u) = \|f(u)\|_{L^1(B_1)}$. So for all $\frac{1}{4} < R < 1$ we got

$$0 \leq -u_r(R) \leq D_1 \|f(u)\|_{L^1(B_1)}. \tag{2.39}$$

Now

$$\begin{aligned}
\int_{B_1\setminus B_\varepsilon} \frac{u_r^2}{|x|^2} &\le \int_{B_{\frac{1}{2}}\setminus B_\varepsilon} \frac{u_r^2}{|x|^2} + \int_{B_1\setminus B_{\frac{1}{2}}} \frac{u_r^2}{|x|^2} \\
&\le c\int_{B_{\frac{3}{4}}\setminus B_{\frac{1}{2}}} u_r^2 + c\int_{B_1\setminus B_{\frac{1}{2}}} u_r^2 \qquad by (2.38) \\
&\le c_1 \int_{\frac{1}{2}<|x|<1} u_r^2 \\
&\le D_2 \|f(u)\|^2_{L^1(B_1)}.
\end{aligned}$$

We have shown that for all $0 < \alpha < \sqrt{N-1}$ and $\frac{\lambda^*}{2} < \lambda < \lambda^*$ there exists $C'_{\alpha,N} > 0$ such that

$$\int_{B_1\setminus B_\varepsilon} \frac{u_r^2}{|x|^{2+2\alpha}} \le C'_{\alpha,N}\|f(u)\|^2_{L^1(B_1)}. \tag{2.40}$$

We need to estimate the right hand side term in a ball. Our goal is to show $f(u)$ is bounded by a constant independent of λ. Fix $\gamma > 0$ (big) $\exists T_\gamma > 0$ such that $\forall u \ge T_\gamma$ we have $\frac{f(u)}{u} \ge \gamma$ (by superlinearity of $f(u)$). Suppose λ_1 is the first eigenvalue of the linearized operator $-\Delta$ and $\zeta > 0$ is the associated eigenfunction. $\sup_{B_1} \zeta = 1$ and ζ satisfies

$$\begin{cases} -\Delta\zeta = \lambda_1\zeta & \text{in } B_1 \\ \zeta = 0 & \text{on } \partial B_1, \end{cases} \tag{2.41}$$

Multiply (2.34) by ζ and integrate by parts to get

$$\begin{aligned}
\int_{B_1} f(u)\zeta &= \int_{B_1} \zeta(-\Delta u) \\
&= \int_{B_1} \lambda_1 \zeta u.
\end{aligned} \tag{2.42}$$

Fix $\gamma \gg 1$ such that $1 - \frac{\lambda_1}{\gamma} = \frac{1}{2}$ and T_γ as above and we have from (2.42)

$$
\begin{aligned}
\int_{u>T_\gamma} f(u)\zeta\, dx &\leq \int_{u<T_\gamma} \lambda_1 \zeta u\, dx + \int_{u>T_\gamma} \lambda_1 \zeta u\, dx \\
&\leq \lambda_1 \max_{B_1} \zeta \int_{u<T_\gamma} T_\gamma\, dx + \int_{u>T_\gamma} \lambda_1 \zeta u\, dx \\
&\leq \lambda_1 T_\gamma |B_1| + \int_{u>T_\gamma} \lambda_1 \zeta \frac{f(u)}{\gamma}\, dx \\
&\leq C_2 + \int_{u>T_\gamma} \lambda_1 \zeta \frac{f(u)}{\gamma}\, dx.
\end{aligned}
$$

Hence we get

$$
\frac{1}{2} \int_{u>T_\gamma} f(u)\zeta\, dx \leq C_2 \quad \text{implies}
$$

$$
\int_{\{x \in B_{\frac{1}{2}} : u(x) > T_\gamma\}} f(u)\zeta\, dx \leq 2C_2.
$$

So

$$
\begin{aligned}
\int_{B_{\frac{1}{2}}} f(u)\zeta\, dx &= \int_{\{x \in B_{\frac{1}{2}} : u(x) > T_\gamma\}} f(u)\zeta\, dx + \int_{\{x \in B_{\frac{1}{2}} : u(x) < T_\gamma\}} f(u)\zeta\, dx \\
&\leq 2C_2 + C_3,
\end{aligned}
$$

where $C_3 := f(T_\gamma) \cdot 1 \cdot |B_{\frac{1}{2}}|$ and we get $\int_{B_{\frac{1}{2}}} f(u) \leq C_4 \quad \forall \lambda \in (\frac{\lambda^*}{2}, \lambda^*)$. But $u_r(|x|) < 0$ and $f'(u(|x|)) > 0$. Hence we can write the average value of $f(u(r))$ in a full ball is less than the average value of $f(u(r))$ in half ball,

$$
⨍_{B_1} f(u)\, dx \leq ⨍_{B_{\frac{1}{2}}} f(u)\, dx.
$$

Which implies $\int_{B_1} f(u)\, dx \leq \dfrac{|B_1| C_4}{|B_{\frac{1}{2}}|} := C_5$ (uniform in $\frac{\lambda^*}{2} < \lambda < \lambda^*$). Hence we get from (2.40)

$$
\int_{B_1 \setminus B_\varepsilon} \frac{u_r^2}{|x|^{2+2\alpha}} \leq D_{\alpha,N}. \tag{2.43}
$$

Finally we are about to show for $N \leq 9$, the minimal solution ($u_\lambda = u$) is bounded. We start with

$$
\begin{aligned}
u(R) =& u(1) + \int_R^1 -u_r r^{\frac{-2\alpha-2+N-1}{2}} r^{\frac{2\alpha+2-N+1}{2}}\, dr \\
&\leq u(1) + c_N \left(\int_{B_1 \setminus B_\varepsilon} u_r^2 |x|^{-2\alpha-2}\, dx \right)^{\frac{1}{2}} \left(\int_R^1 r^{2\alpha+2-N+1}\, dr \right)^{\frac{1}{2}} \\
&\leq u(1) + c_N D'_{\alpha,N} \left(\int_R^1 r^{2\alpha+2-N+1}\, dr \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.44}
$$

Here $u(1) = 0$ and the integral on the right is finite with $R = 0$ if we take $2\alpha + 3 - N > -1$ i.e.

$$\alpha > \frac{N-4}{2}. \tag{2.45}$$

Since $N \leq 9$, then $\frac{N-4}{2} < \sqrt{N-1}$ and we can choose α satisfying (2.45) and $\alpha < \sqrt{N-1}$ such that (2.40) holds. Since u is radially decreasing, (2.44) is the desired L^∞ estimate and passing to the limit as $\lambda \nearrow \lambda^*$ gives $u^* \in L^\infty$ provided $N \leq 9$. Hence smooth.

$\square$

3

MEMS model

Consider

$$(M)_\lambda \qquad \begin{cases} -\Delta u &= \frac{\lambda}{(1-u)^2} \quad \Omega \\ u &= 0 \qquad \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. The above equation is often been used to model electronic MEMS device which roughly consists of a dielectric elastic membrane that is attached to the boundary of Ω, whose upper surface has a thin conducting film. At a distance of 1 above the undeflected membrane sits on a grounded plate, i.e. a plate held at zero voltage. When a voltage $v > 0$ is applied to the thin film of the membrane, it deflects towards the ground plate. There is a maximum voltage v^* beyond which, there is a snap through at a finite time, the steady state of the elastic membrane is lost. Here $\lambda > 0$ is proportional to applied voltage and $0 < u(x) < 1$ is the dynamic deflection of the membrane. See [27],[28] in regards to the model and [15],[17],[12] for mathematical aspects of $(M)_\lambda$.

Theorem 32. *1. There exists some $\lambda^* \in (0, \infty)$ (finite pull in voltage) such that for all $0 < \lambda < \lambda^*$ there exists a smooth solution u of $(M)_\lambda$. For all $\lambda > \lambda^*$ there are no smooth solutions of $(M)_\lambda$.*

2. *For all* $0 < \lambda < \lambda^*$ *there exists a smooth minimal solution* u_λ *of* $(S)_\lambda$. *For all* $x \in \Omega$, *one has that* $u_\lambda(x)$ *is increasing in* λ *on* $(0, \lambda^*)$.

3. *For all* $0 < \lambda < \lambda^*$ *the minimal solution is stable.*

4. $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ *is called the extremal solution is the unique weak solution of* $(M)_\lambda$.

Proof. 1. Define

$$\lambda^* := \sup\left\{0 \leq \lambda : (M)_\lambda \text{ has a smooth solution}\right\}.$$

Suppose $0 < u(x) < 1$ be a solution of $(M)_\lambda$. To show finiteness of λ we multiply (3.1) by the first eigenfunction of $-\Delta$ and integrate by parts as the same way we did for (2.1) (just we replace $f(u)$ by $\frac{1}{(1-u)^2}$) and at last we deduce the following upper bound,

$$\lambda \leq \lambda_1 \frac{u}{\frac{1}{(1-u)^2}} \leq \lambda_1 \sup \frac{u}{\frac{1}{(1-u)^2}}.$$

Since $0 < u(x) < 1$ we conclude that $\lambda^* < \infty$. We now show $\lambda^* > 0$. Clearly $\underline{u} = 0$ is a strict subsolution since $-\Delta \underline{u} = 0 \leq \frac{\lambda}{(1-\underline{u})}$ in Ω and $\underline{u} \leq 0$ in $\partial\Omega$. Consider $-\Delta \overline{u} = 1$ in Ω and $\overline{u} = 0$ in $\partial\Omega$. Then by strong maximum principle $\overline{u} > 0$ in Ω. Suppose $m := max\{\overline{u}(x)\}$ $\forall x \in \Omega$. For $\overline{u}$ to be supersolution of $(M)_\lambda$ we need $1 \geq \frac{\lambda}{(1-\overline{u})^2}$. So it is sufficient that $\lambda \leq (1-m)^2$ for all $0 < \overline{u}(x) < 1$. But $\lambda > 0$ small enough ensures this.

We now show $\forall \lambda \in (0, \lambda^*)$ there is a smooth solution of $(M)_\lambda$. By using the definition of λ^* pick $\hat{\lambda} \in (\lambda, \lambda^*)$ such that $(M)_{\hat{\lambda}}$ has a smooth solution $u_{\hat{\lambda}}(x) = v$ which satisfies

$$-\Delta v = \frac{\hat{\lambda}}{(1-v)^2} \quad \text{in } \Omega,\ 0 < v < 1 \quad \text{in } \Omega,\ v = 0 \quad \text{on } \partial\Omega.$$

But in particular, $-\Delta v = \frac{\hat{\lambda}}{(1-v)^2} \geq \frac{\lambda}{(1-v)^2}$. It implies v is a supersolution of $(M)_\lambda$. Since $\underline{u} = 0$ is subsolution then by sub/supersolution approach there is a smooth solution u of $(M)_\lambda$ for every $\lambda \in (0, \lambda^*)$. There is no smooth solution of $(M)_\lambda$ for $\lambda > \lambda^*$ (by definition).

2. For any $0 < \lambda < \lambda^*$ there exists a minimal positive smooth solution. It is obtained as the limit of increasing sequence $\{u_n(x)\}_{n\geq 0}$ that is constructed recursively as follows: subsolution $\underline{u} = u_0(x) = 0$ in Ω and for $n \geq 1$ we define the linear equation

$$\begin{cases} -\Delta u_{n+1} & = & \frac{\lambda}{(1-u_n)^2} & \Omega \\ 0 & \leq & u_n < 1 & \Omega \\ u_{n+1} & = & 0 & \partial\Omega. \end{cases} \tag{3.2}$$

Suppose $u(x)$ is a smooth solution of $(M)_\lambda$ (we proved in previous part for all $0 < \lambda < \lambda^*$ there will be a smooth solution). To construct the minimal solution we start with $u_0(x) = 0$ in (3.2) and by using maximum principle several times (as in Lemma (1)) we can obtain a sequence $u_n(x)_{n\geq 0}$ such that $0 \leq u_n(x) \leq u(x) < 1 \ \forall n \geq 0$. So here we can see $u_n(x)$ is constructed by using subsolution so its independent of $u(x)$. We consider $u(x)$ here just to show $u_n(x)$ is bounded. Since $u_n(x)$ is bounded it converges to a positive solution $u_\lambda(x)$ which satisfies $u_\lambda(x) \leq u(x)$ in Ω. Since this inequality holds for any solution u of $(M)_\lambda$, so u_λ is a minimal positive solution of $(M)_\lambda$.

To show the minimal solution is increasing in λ we follow the same procedure as we did for (2.1) and we conclude $\delta_\lambda u_\lambda > 0$ by strong maximum principle.

3. To show u_λ stable for $0 < \lambda < \lambda^*$, we require first eigenvalue of $-\Delta - \frac{2\lambda}{(1-u_\lambda)^3}$

is positive. This property can be expressed variationally as follows

$$\int_{\Omega} |\nabla \psi|^2 \geq \int_{\Omega} \frac{2\lambda}{(1-u_\lambda)^3} \psi^2 \quad \forall \psi \in H_0^1. \tag{3.3}$$

We follow the same procedure of stability part of (2.1) to obtain this inequality, which can be viewed as nonnegativeness of second variation of energy functional associated with $(M)_\lambda$ at u_λ.

4. See Theorem 29 and [8]

□

Lemma 3.1. *For* $\frac{\lambda^*}{2} < \lambda < \lambda^*$ *and for* $t < 2 + \sqrt{6}$ *we have the following estimate:*

$$\int_{\Omega} \frac{1}{(1-u_\lambda)^{3+2t}} \leq C_t,$$

where C_t *a constant independent of* λ.

Proof. Fix $\frac{\lambda^*}{2} < \lambda < \lambda^*$. Let $t > 0$ and $u = u_\lambda$ denote the minimal solution of $(M)_\lambda$. We shall use inequality (3.3) with $\psi = (1-u)^{-t} - 1$ for $t > 0$ to obtain

$$\int_{\Omega} \frac{|\nabla u|^2}{(1-u)^{2t+2}} \geq \frac{2\lambda}{t^2} \int_{\Omega} \frac{1}{(1-u)^{2t+3}} - \frac{4\lambda}{t^2} \int_{\Omega} \frac{1}{(1-u)^{t+3}}. \tag{3.4}$$

Multiply $(M)_\lambda$ by an appropriate test function $\phi = (1-u)^{-2t-1}$ and integrate by parts to get

$$\int_{\Omega} \frac{|\nabla u|^2}{(1-u)^{2t+2}} \approx \frac{1}{2t+1} \int_{\Omega} \frac{\lambda}{(1-u)^{2t+3}}. \tag{3.5}$$

By using (3.5) and (3.4) we get

$$(\frac{2}{t^2} - \frac{1}{2t+1}) \int_{\Omega} \frac{1}{(1-u)^{2t+3}} \leq \frac{4}{t^2} \int_{\Omega} \frac{1}{(1-u)^{t+3}}. \tag{3.6}$$

Now by applying Hölder on the right and by taking $\tau = \frac{2t+3}{t+3}$ (its conjugate $\tau' = \frac{2t+3}{t}$)

we obtain the following inequality

$$(\frac{1}{2}-\frac{t^2}{4(2t+1)})^{\tau'}\int_\Omega \frac{1}{(1-u)^{2t+3}} \le |\Omega|.$$

By ensuring $(\frac{1}{2}-\frac{t^2}{4(2t+1)})>0$ we get our required estimate. $\square$

Theorem 33. *Suppose $N \le 7$. Then the extremal solution u^* associated with $(M)_\lambda$, satisfies* $\sup_\Omega u^* < 1$.

Proof. Suppose for $\tau, \varepsilon > 0$ small and for some $t < 2+\sqrt{6}$ the estimate

$$(\tau+3)(\frac{N}{2}+\varepsilon) < 3+2t \quad \text{holds.} \tag{3.7}$$

Define $v_\lambda := (1-u_\lambda)^{-\tau} - 1$ and v_λ satisfies

$$\begin{cases} -\Delta v_\lambda &= \frac{\lambda\tau}{(1-u_\lambda)^{\tau+3}} - \frac{\tau(\tau+1)|\nabla u_\lambda|^2}{(1-u_\lambda)^{\tau+2}} \quad \Omega \\ v_\lambda &= 0 \quad \partial\Omega. \end{cases}$$

Consider

$$\begin{cases} -\Delta w_\lambda &= \frac{\lambda\tau}{(1-u_\lambda)^{\tau+3}} \quad \Omega \\ w_\lambda &= 0 \quad \partial\Omega. \end{cases} \tag{3.8}$$

Then

$$\begin{cases} -\Delta (w_\lambda - v_\lambda) &= \frac{\tau(\tau+1)|\nabla u_\lambda|^2}{(1-u_\lambda)^{\tau+2}} \quad \Omega \\ v_\lambda &= 0 \quad \partial\Omega. \end{cases} \tag{3.9}$$

Since $\tau > 0$ and $0 < u_\lambda < 1$, by maximum principle $w_\lambda \ge v_\lambda$ in Ω.

By applying elliptic regularity to (3.8) (and the Sobolev imbedding theorem) there is some $C > 0$ such that

$$\sup_\Omega w_\lambda \le C \Big\| \frac{\lambda\tau}{(1-u_\lambda)^{\tau+3}} \Big\|_{L^{\frac{N}{2}+\varepsilon}},$$

for all $\frac{\lambda^*}{2} < \lambda < \lambda^*$. Now $v_\lambda \leq w_\lambda$ and $\|w_\lambda\|_{L^\infty} \leq D$(a constant). Hence v_λ is bounded by a constant independent of λ. Therefore we can pass to the limit in v_λ and we conclude $\sup_\Omega u^* < 1$.

Now by using the estimate in Lemma 3.1 and (3.7) we get

$$\int_\Omega \frac{\lambda}{(1-u_\lambda)^{2(\frac{3}{2}+t)}} \leq \int_\Omega \frac{\lambda\tau}{(1-u_\lambda)^{(\tau+3)(\frac{N}{2}+\varepsilon)}} \leq C.$$

$\int_\Omega \frac{1}{(1-u_\lambda)^2}$ is bounded in L^p and by elliptic regularity we get uniform $W^{2,p}$ bound on u_λ provided $P < \frac{7}{2} + \sqrt{6}$. After passing to the limit we get the same uniform $W^{2,p}$ bound on u^* and for $N \leq 7$, u^* doesn't touch 1 provided p close to $\frac{7}{2} + \sqrt{6}$. Now note that if $N \leq 7$, $W^{2,p} \subset L^\infty$ provided p is chosen close enough to 5.94. So we see that for $N \leq 7$ that u^* is bounded and hence smooth iff $\sup_\Omega u^* < 1$.

$\square$

4

Nonlinear Eigenvalue problem with Advection term

In this chapter, we consider more general Dirichlet problem given by

$$(S)_\lambda \qquad \begin{cases} -\Delta u + a(x)\cdot\nabla u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where f satisfies

(A1) is smooth, increasing, convex, $f(0) = 1$ and f is superlinear at ∞.

Here Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is a parameter and a is a smooth and bounded mapping from Ω to $\mathbb{R}^N$ which we assume is divergence free.

Definition 4.1. *(Adjoint operator) Consider an operator $L(u) := -\Delta u + a(x)\cdot\nabla u$ associated with the pde (4.1). We say $L^*(u)$ is adjoint operator of $L(u)$ if $\forall \phi, \psi \in C^2(\overline{\Omega})$ and $\phi, \psi = 0$ on $\partial\Omega$ it satisfies*

$$\int_\Omega L(\phi)\psi = \int_\Omega \phi L^*(\psi).$$

Theorem 34. *Let f, Ω be as above.*

1. *There exists some $\lambda^* \in (0, \infty)$ (the extremal parameter) such that for all $0 < \lambda < \lambda^*$ there exists a smooth solution u of $(S)_\lambda$. For all $\lambda > \lambda^*$ there are no smooth solutions of $(S)_\lambda$.*

2. *For all $0 < \lambda < \lambda^*$ there exists a smooth minimal solution u_λ of $(S)_\lambda$. For all $x \in \Omega$, one has that $u_\lambda(x)$ is increasing in λ on $(0, \lambda^*)$.*

3. *For all $0 < \lambda < \lambda^*$ the minimal solution is stable.*

4. *Define $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$. Then u^* (which we call the extremal solution) is a weak solution of $(S)_{\lambda^*}$.*

Proof. 1. Define

$$\lambda^* := \sup \{0 \le \lambda : (S)_\lambda \text{ has a smooth solution}\}.$$

First we will prove λ^* is finite. Consider the operator $L := -\Delta + a(x) \cdot \nabla$. Here we are looking for first eigenvalue and eigenfunction of the adjoint operator L^* because first eigenpair of L does not help us to show λ^* finite. First we use the definition (4.1) to get

$$\int_\Omega -\Delta\phi\psi + \int_\Omega (a(x) \cdot \nabla\phi)\psi = \int_\Omega \phi L^*(\psi),$$

which turns into (use Green's formula)

$$\int_\Omega -\Delta\psi\phi - \int_\Omega \nabla\psi \cdot (\phi a(x)) + \int_{\partial\Omega} \psi((\phi a(x)) \cdot \nu) - \int_\Omega \phi div(a(x))\psi = \int_\Omega \phi L^*(\psi).$$

Hence $L^*(\psi) = -\Delta\psi - \nabla\psi \cdot a(x)$.

Now let λ_1 is the first eigenvalue of the operator $L^*(\psi)$ and $0 < \phi_1 \in H_0^1$ is the first eigenfunction and ϕ_1 satisfies

$$\begin{cases} L^*(\phi_1) &= \lambda_1 \phi_1 \quad \text{in } \Omega \\ \phi_1 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Let u be smooth solution of $(S)_\lambda$. Multiply $(S)_\lambda$ by ϕ_1

$$\begin{aligned} \int_\Omega \lambda f(u)\phi_1 &= \int_\Omega L(u)\phi_1 \\ &= \int_\Omega uL^*(\phi_1) \\ &= \int_\Omega u\lambda_1\phi_1. \end{aligned}$$

Therefore $\int_\Omega (\lambda f(u) - \lambda_1 u)\, \phi_1(x)dx = 0$. Now we follow the same procedure as we did for the pde $(P)_\lambda$ to obtain λ^* finite, and $\lambda^* > 0$.

2. See proof of Theorem 24 part 2.

3. Since minimal solution is increasing in λ (we get it from Theorem 24 part 2), for $\varepsilon > 0$ we get $u_{\lambda+\varepsilon} > u_\lambda$. So we are assuming if $\partial_\lambda u_\lambda$ exists then $\partial_\lambda u_\lambda \geq 0$. Take a derivative of u_λ in λ in $(S)_\lambda$ we get

$$\begin{cases} -\Delta v + a(x)\cdot\nabla v &= \lambda f'(u_\lambda)v + f(u_\lambda) \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where we put $v = \partial_\lambda u_\lambda$. By strong maximum principle $\partial_\lambda u_\lambda > 0$. To prove minimal solution is stable we show the first eigenvalue of the linearized operator $L_u := -\Delta + a(x)\nabla - \lambda f'(u_\lambda)$ of the Dirichlet problem (4.3) is positive. Take $div(a) = 0$, then $L_u^* := -\Delta - \nabla\cdot a(x) - \lambda f'(u_\lambda)$. Suppose $\hat{\phi}_1$ and $\hat{\psi}_1$ are the first eigenfunctions and μ_1 and μ_2 are the eigenvalues of L_u and L_u^* respectively such that

$$
\begin{cases} L_u(\hat{\phi}_1) &= \mu_1\hat{\phi}_1 \quad \text{in } \Omega \\ \hat{\phi}_1 &= 0 \qquad \text{on } \partial\Omega \\ \hat{\phi}_1 &> 0 \qquad \text{on } \Omega. \end{cases} \tag{4.4}
$$

$$
\begin{cases} L_u^*(\hat{\psi}_1) &= \mu_2\hat{\psi}_1 \quad \text{in } \Omega \\ \hat{\psi}_1 &= 0 \qquad \text{on } \partial\Omega \\ \hat{\psi}_1 &> 0 \qquad \text{on } \Omega. \end{cases} \tag{4.5}
$$

Multiply (4.4) by $\hat{\psi}_1$ and integrate by parts to see

$$
\begin{aligned}
\int_\Omega \mu_1\hat{\phi}_1\hat{\psi}_1 &= \int_\Omega L_u(\hat{\phi}_1)\hat{\psi}_1 \\
&= \int_\Omega \hat{\phi}_1 L_u^*(\hat{\psi}_1) \\
&= \int_\Omega \hat{\phi}_1\mu_2\hat{\psi}_1.
\end{aligned}
$$

Both $\hat{\psi}_1, \hat{\phi}_1 > 0$ so we can conclude $\mu_1 = \mu_2$. We need this result for our proof. Now we return to our goal to show $\mu_1 > 0$. We multiply $L_u(v) = f(u_\lambda)$ by $\hat{\phi}_1$ and integrate by parts to get

$$
\begin{aligned}
\int_\Omega f\hat{\phi}_1 &= \int_\Omega L_u(v)\hat{\phi}_1 \\
&= \int_\Omega vL_u^*(\hat{\phi}_1) \\
&= \int_\Omega \hat{\phi}_1\mu_1\hat{\psi}_1.
\end{aligned}
$$

Here $f > 0$ and $\hat{\phi}_1, \hat{\psi}_1 > 0$. Hence u_λ stable.

$\square$

For more details see [1].

Lemma 4.2. *(A general Hardy's inequality)[7] Suppose E is a smooth positive function on Ω and fix a constant α with $0 < \alpha < 2$. Then for all $\psi \in H_0^1$ we have*

$$\int_\Omega |\nabla\psi|^2 \geq \frac{\alpha(2-\alpha)}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}\psi^2 + \frac{\alpha}{2}\int_\Omega \frac{-\Delta E}{E}\psi^2. \tag{4.6}$$

Proof. Let E_0 denote a smooth positive function defined in Ω and let $\psi \in C_c^\infty$. set $v := \frac{\psi}{\sqrt{E_0}}$. Then

$$\begin{aligned}
|\nabla\psi|^2 &= E_0|\nabla v|^2 + v^2|\nabla\sqrt{E_0}|^2 + 2\sqrt{E_0}\nabla v \cdot v\nabla\sqrt{E_0} \\
&= E_0|\nabla v|^2 + \psi^2\frac{|\nabla E_0|^2}{4E_0^2} + \frac{\psi}{E_0}\nabla v \cdot \nabla E_0 \\
&= E_0|\nabla v|^2 + \psi^2\frac{|\nabla E_0|^2}{4E_0^2} + v\nabla v \cdot \nabla E_0.
\end{aligned} \tag{4.7}$$

Integrating the last term by parts

$$\begin{aligned}
\int_\Omega v\nabla v \cdot \nabla E_0 &= \frac{1}{2}\int_\Omega \nabla\frac{\psi^2}{E_0} \cdot \nabla E_0 \\
&= \frac{1}{2}\int_\Omega \frac{-\Delta E_0}{E_0}\psi^2.
\end{aligned}$$

From (4.7) we have

$$\begin{aligned}
\int_\Omega |\nabla\psi|^2 &= \int_\Omega E_0|\nabla v|^2 + \int_\Omega \psi^2\frac{|\nabla E_0|^2}{4E_0^2} + \frac{1}{2}\int_\Omega \frac{-\Delta E_0}{E_0}\psi^2 \\
&\geq \int_\Omega \psi^2\frac{|\nabla E_0|^2}{4E_0^2} + \frac{1}{2}\int_\Omega \frac{-\Delta E_0}{E_0}\psi^2.
\end{aligned} \tag{4.8}$$

By setting $\alpha = 1$ in (4.6) we can get the above inequality. In the case where

$\alpha \neq 1$, we put $E_0 = E^\alpha$ in (4.8) to get

$$
\begin{aligned}
\int_\Omega |\nabla \psi|^2 &\geq \int_\Omega \psi^2 \frac{|\nabla E^\alpha|^2}{4E^{2\alpha}} + \frac{1}{2}\int_\Omega \frac{-div(\nabla E^\alpha)}{E^\alpha}\psi^2 \\
&= \frac{\alpha^2}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}\psi^2 - \frac{\alpha}{2}\int_\Omega \frac{div(E^{\alpha-1}\nabla E)}{E^\alpha}\psi^2 \\
&= \frac{\alpha^2}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}\psi^2 - \frac{\alpha}{2}\int_\Omega \frac{\nabla(E^{\alpha-1})\cdot\nabla E + E^{\alpha-1}div(\nabla E)}{E^\alpha}\psi^2 \\
&= \frac{\alpha^2}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}\psi^2 - \frac{\alpha}{2}\int_\Omega (\alpha-1)\frac{|\nabla E|^2}{E^2}\psi^2 + \frac{\Delta E}{E}\psi^2 \\
&= \frac{\alpha(2-\alpha)}{4}\int_\Omega \frac{|\nabla E|^2}{E^2}\psi^2 + \frac{\alpha}{2}\int_\Omega \frac{-\Delta E}{E}\psi^2.
\end{aligned}
$$

□

Theorem 35. *If $N \leq 9$ and $f(u) = e^u$ then u^* the extremal solution associated with $(S)_\lambda$ is bounded.*

Remark 4.3. *Recall we used inequality (2.5) (which is second variation of the energy functional associated with $(P)_\lambda$) in conjunction with $(P)_\lambda$ to obtain uniform L^p estimates on e^{u_λ} when u_λ is minimal. Unlike the previous advection free ($a = 0$) model here we are not getting any energy inequality since our linearized operator associated with $(S)_\lambda$ is not self adjoint. For u^* associated with $(S)_\lambda$ to be bounded we need the Hardy inequality to overcome this difficulty.*

Proof of Theorem 35. Let $u = u_\lambda$ be the minimal solution. Suppose $E = E_\lambda > 0$ $\exists \mu_\lambda > 0$ such that

$$
\begin{cases} -\Delta E + a(x)\cdot\nabla E = \lambda e^u E + \mu_\lambda E & \text{in } \Omega \\ E = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}
$$

Divide (4.9) by E and integrate by parts to get

$$
\int_\Omega \frac{-\Delta E}{E} + \int_\Omega a(x)\cdot\frac{\nabla E}{E} = \lambda\int_\Omega e^u + \int_\Omega \mu_\lambda. \tag{4.10}
$$

Use equation (4.10) along with (4.6) we have for $\psi \in H_0^1$,

$$\int_\Omega |\nabla \psi|^2 \geq \frac{\alpha(2-\alpha)}{4} \int_\Omega \frac{|\nabla E|^2}{E^2} \psi^2 + \frac{\alpha}{2} \int_\Omega \lambda e^u \psi^2 - \frac{\alpha}{2} \int_\Omega a(x) \cdot \frac{\nabla E}{E} \psi^2. \qquad (4.11)$$

Here we have dropped a nonnegative term involving μ_λ. Fix $0 < \alpha < 2$, let $t > 0$. Let u denote the minimal solution associated with $(S)_\lambda$. Put $\psi = e^{tu} - 1$ in (4.11) to get

$$\begin{aligned} \int_\Omega t^2 e^{2tu} |\nabla u|^2 \;\; & \geq \;\; \frac{\alpha}{2} \int_\Omega \lambda e^u (e^{tu} - 1)^2 \\ & + \frac{\alpha}{2} \int_\Omega \left(\frac{(2-\alpha)}{2} \frac{|\nabla E|^2}{E^2} - a(x) \cdot \frac{\nabla E}{E} \right) (e^{tu} - 1)^2. \quad (4.12) \end{aligned}$$

Now we are going to use the procedure "closing the estimate". Multiply $(S)_\lambda$ by $e^{2tu} - 1$ and integrate by parts to obtain

$$\int_\Omega \lambda e^u (e^{2tu} - 1) = \int_\Omega \nabla u \cdot \nabla (e^{2tu} - 1) + \int_\Omega a \cdot \nabla u (e^{2tu} - 1). \qquad (4.13)$$

The last term is zero after considering the fact that for an appropriately chosen function F with $F(0) = 0$ we have $\int_\Omega a \cdot \nabla u (e^{2tu} - 1) = \int_\Omega a \cdot \nabla F(u) = 0$. So we get from (4.13)

$$\int_\Omega \lambda e^u (e^{2tu} - 1) = 2t \int_\Omega e^{2tu} |\nabla u|^2. \qquad (4.14)$$

Now by combining (4.12) and (4.14) and then by expanding, regrouping and dropping some positive terms we get the inequality

$$\begin{aligned} \lambda(\alpha - t) \int_\Omega e^{(2t+1)u} \;\; & \leq \;\; 2\alpha\lambda \int_\Omega e^{(t+1)u} \\ & + \alpha \int_\Omega \left(\frac{(2-\alpha)}{2} \frac{|\nabla E|^2}{E^2} + a(x) \cdot \frac{\nabla E}{E} \right) (e^{tu} - 1)^2. \, (4.15) \end{aligned}$$

We now estimate the term $\frac{(2-\alpha)}{2} \frac{|\nabla E|^2}{E^2} + a(x) \cdot \frac{\nabla E}{E}$. By using Cauchy's inequality with

$\varepsilon > 0$ small we get

$$\frac{(\alpha-2)}{2}\left[\frac{4}{2(\alpha-2)}a(x)\cdot\frac{\nabla E}{E}-\frac{|\nabla E|^2}{E^2}\right] \le 2(\alpha-2)\left[\varepsilon\frac{a^2}{4(\alpha-2)^2}+\frac{1}{4\varepsilon}\frac{|\nabla E|^2}{E^2}-\frac{1}{4}\frac{|\nabla E|^2}{E^2}\right]$$
$$\le \varepsilon\frac{|a|^2}{2(\alpha-2)}$$
$$\le \varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}.$$

(4.15) becomes

$$\lambda(\alpha-t)\int_\Omega e^{(2t+1)u} \le 2\alpha\lambda\int_\Omega e^{(t+1)u}+\alpha\int_\Omega \varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}e^{2tu}+\alpha\int_\Omega\varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}$$
$$\le 2\alpha\lambda^*\int_\Omega e^{(t+1)u}+\alpha\int_\Omega \varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}e^{2tu}+\alpha\int_\Omega\varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}.$$

We now apply Hölder for each of the terms on right hand side. Take τ such that $\tau(t+1) = 2t+1$ and $2t\tau = 2t+1$ we get

$$\begin{aligned}\lambda(\alpha-t)\int_\Omega e^{(2t+1)u} \quad &\le \quad 2\alpha\lambda^*\left(\int_\Omega e^{(2t+1)u}\right)^{\frac{t+1}{2t+1}}|\Omega|^{\frac{2t+1}{t+1}} \\ &\quad +C_\alpha\left(\int_\Omega e^{(2t+1)u}\right)^{\frac{2t}{2t+1}}|\Omega|^{\frac{2t+1}{2t}}+C_\alpha|\Omega|\left(\int_\Omega e^{(2t+1)u}\right)^0,\end{aligned}$$

where $C_\alpha = \alpha\varepsilon\frac{||a||^2_{L^\infty}}{2(\alpha-2)}$. For $t > 0$ we may assume $\int_\Omega e^{(2t+1)u} > 1$, because on the opposite, we have $\int_\Omega e^{(2t+1)u} \le 1$ and the estimate is independent of $\lambda \in (0,\lambda^*)$ and we are done. In this case if $(\alpha-t) > 0$ and $\frac{2t}{2t+1} > \frac{t+1}{2t+1}$ that is if $2 > \alpha > t > 1$ then we have

$$\begin{aligned}\lambda(\alpha-t)\int_\Omega e^{(2t+1)u} \quad &\le \quad 2\alpha\lambda^*\left(\int_\Omega e^{(2t+1)u}\right)^{\frac{2t}{2t+1}}|\Omega|^{\frac{2t+1}{t+1}} \\ &\quad +C_\alpha\left(\int_\Omega e^{(2t+1)u}\right)^{\frac{2t}{2t+1}}|\Omega|^{\frac{2t+1}{2t}}+C_\alpha|\Omega|\left(\int_\Omega e^{(2t+1)u}\right)^{\frac{2t}{2t+1}}.\end{aligned}$$

Which in turns implies

$$\lambda \int_\Omega e^{(2t+1)u} \leq (\alpha - t)^{-2t-1} \left(2\alpha\lambda^* |\Omega|^{\frac{2t+1}{t+1}} + C_\alpha |\Omega|^{\frac{2t+1}{2t}} + C_\alpha |\Omega|\right)^{2t+1}$$
$$=: C(t, |\Omega|).$$

Now we can conclude that $\int_\Omega (e^{u_\lambda})^{2t+1}$ is uniformly bounded above by C which is independent of $\lambda \in (\frac{\lambda^*}{2}, \lambda^*)$. Now standard elliptic estimates and sobolev embedding imply that $\|u_\lambda\|_{L^\infty(\Omega)} \leq C$ is uniform in λ if $2t + 1 > \frac{N}{2}$. Since we took $t \in (0, 2)$ so we obtain uniform L^∞ bound on u_λ when $N \leq 9$. Now we pass to the limit and get our desired estimate for the extremal solution. □

Lemma 4.4. *Suppose f satisfies (A1) and there is some $\varepsilon > 0$ and $T > 0$ such that*

$$f'(t)t \geq (1 + \varepsilon) f(t), \tag{4.16}$$

for all $t \geq T$. Then there is some $C > 0$ such that $\int_\Omega f'(u_\lambda) u_\lambda^2 dx \leq C$ and $\int_\Omega f(u_\lambda) u_\lambda \leq C \quad \forall \frac{\lambda^*}{2} < \lambda < \lambda^*$.

Proof. Let $u = u_\lambda$. Suppose $\varepsilon > 0$ and $T > 0$ fixed. Let $0 < \alpha < 2$ (close) and $0 < \delta << 1$ such that $(1-\delta)\frac{\alpha}{2} - \frac{1}{1+\varepsilon} > 0$. Define $C_1 := \frac{\alpha \|a\|^2_{L^\infty}}{8\delta_1} > 0$, then note by taking T larger if necessary we can assume that

$$\left[(1-\delta)\frac{\alpha}{2} - \frac{1}{1+\varepsilon}\right] f'(t) - C_1 \geq \frac{(1-\delta)\frac{\alpha}{2} - \frac{1}{1+\varepsilon}}{2} f'(t), \tag{4.17}$$

for all $t \geq T$. Now by substituting $\psi = u$ in (4.11) and considering general nonlinearity f and $div(a) = 0$ we obtain

$$C_\alpha \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 + \frac{\alpha}{2} \int_\Omega \lambda f'(u) u^2 \leq \frac{\alpha}{2} \int_\Omega a(x) \cdot \frac{\nabla E}{E} u^2 + \int_\Omega \lambda f(u) u, \tag{4.18}$$

where $C_\alpha := \frac{\alpha(2-\alpha)}{4}$. By applying Cauchy's inequality on the first term on right side we get

$$C_\alpha \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 + \frac{\alpha}{2} \int_\Omega \lambda f'(u) u^2 \le \frac{\alpha}{2} \int_\Omega \frac{|a(x)|^2 u^2}{4\delta_1} + \frac{\alpha \delta_1}{2} \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 + \int_\Omega \lambda f(u) u,$$

which can be rewritten as

$$\frac{\delta\alpha}{2} \int_{u>T} \lambda f'(u) u^2 + (1-\delta)\frac{\alpha}{2} \int_{u>T} \lambda f'(u) u^2 \ \le C_1 \int_{u>T} u^2 + \frac{1}{1+\varepsilon} \int_{u>T} \lambda f'(u) u^2 + D_T,$$

where

$$D_T := C_1 \int_{u \le T} u^2 \, dx + \int_{u \le T} \lambda f(u) u \, dx \le |\Omega| \left(C_1 T^2 + \lambda f(T) T \right).$$

Here we dropped some positive terms on the left side and D is a constant which is coming from the terms involving the integral when $u \le T$. By regrouping we get

$$\frac{\delta\alpha}{2} \int_{u>T} \lambda f'(u) u^2 + \int_{u>T} \left[[(1-\delta)\frac{\alpha}{2} - \frac{1}{1+\varepsilon}] \lambda f'(u) - C_1 \right] u^2 \le D. \tag{4.19}$$

We use the fact in (4.17) and we can get rid of C_1 in (4.19) to obtain

$$\frac{\delta\alpha}{2} \int_{u>T} \lambda f'(u) u^2 + \int_{u>T_1} [(1-\delta)\frac{\alpha}{2} - \frac{1}{1+\varepsilon}] \lambda f'(u) u^2 \le D.$$

Which in turns implies

$$\int_\Omega f'(u) u^2 \le C.$$

So for $\frac{\lambda^*}{2} < \lambda < \lambda^*$ there exists a C such that

$$\int_\Omega f'(u_\lambda) u_\lambda^2 \le C.$$

Also from the assumption on f (4.16) we get the estimate

$$\int_\Omega f(u_\lambda)u_\lambda \leq C.$$

$\square$

Proof of Theorem 34 part 4 (under extra hypothesis). Here we prove $u^* \in H_0^1(\Omega)$ under the additional hypothesis (4.16).

Multiply the pde (4.1) by $u = u_\lambda$ and integrate by parts to get

$$\begin{aligned}\int_\Omega |\nabla u_\lambda|^2 dx &= \lambda \int_\Omega f(u_\lambda)u_\lambda \\ &\leq C,\end{aligned}$$

where $div(a) = 0$. Hence $u_\lambda \in H_0^1$ and C is independent of λ. consequently $u^* \in H_0^1$ after passing to the limit as $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$. u^* is pointwise limit of minimal solution of (4.1), is also an ultra weak solution of (4.1) since $\forall \phi \in C^\infty$ (test function) and $\phi = 0$ on $\partial\Omega$ we have

$$\int_\Omega (-\Delta\phi)u_\lambda + \int_\Omega \nabla\phi \cdot u_\lambda a = \lambda \int_\Omega f(u_\lambda)\phi.$$

Which allows us to pass to the limit as $\lambda \nearrow \lambda^*$.

Bibliography

[1] Berestycki, H., Kiselev, A., Novikov, A., and Ryzhik, L. The explosion problem in a flow. *J. Anal. Math. 110* (2010), 31–65.

[2] Brezis, H., Cazenave, T., Martel, Y., and Ramiandrisoa, A. Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differential Equations 1*, 1 (1996), 73–90.

[3] Brezis, H., and Vázquez, J. L. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid 10*, 2 (1997), 443–469.

[4] Cabré, X. Regularity of radial extremal solutions of semilinear elliptic equations.

[5] Cabré, X., and Capella, A. Regularity of radial minimizers and extremal solutions of semilinear elliptic equations. *J. Funct. Anal. 238*, 2 (2006), 709–733.

[6] Coleman, S., Glaser, V., and Martin, A. Action minima among solutions to a class of Euclidean scalar field equations. *Comm. Math. Phys. 58*, 2 (1978), 211–221.

[7] Cowan, C. Hardy inequalities for general elliptic operators with improvements. *arXiv preprint arXiv:0805.0610* (2008).

[8] Cowan, C., and Fazly, M. Uniqueness of solutions for a nonlocal elliptic eigenvalue problem. *arXiv preprint arXiv:1109.5146* (2011).

[9] Crandall, M. G., and Rabinowitz, P. H. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal. 58*, 3 (1975), 207–218.

[10] Dupaigne, L. *Stable solutions of elliptic partial differential equations*, vol. 143 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2011.

[11] Esposito, P. Compactness of a nonlinear eigenvalue problem with a singular nonlinearity. *Communications in Contemporary Mathematics 10*, 01 (2008), 17–45.

[12] Esposito, P., Ghoussoub, N., and Guo, Y. Compactness along the branch of semistable and unstable solutions for an elliptic problem with a singular nonlinearity. *Comm. Pure Appl. Math. 60*, 12 (2007), 1731–1768.

[13] Evans, L. C. *Partial differential equations*, second ed., vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010.

[14] Gelfand, I. M. Some problems in the theory of quasilinear equations. *Amer. Math. Soc. Transl. (2) 29* (1963), 295–381.

[15] Ghoussoub, N., and Guo, Y. On the partial differential equations of electrostatic MEMS devices: stationary case. *SIAM J. Math. Anal. 38*, 5 (2006/07), 1423–1449.

[16] Ghoussoub, N., and Guo, Y. On the partial differential equations of electrostatic mems devices: stationary case. *SIAM Journal on Mathematical Analysis 38*, 5 (2007), 1423–1449.

[17] Guo, Y., Pan, Z., and Ward, M. J. Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. *SIAM J. Appl. Math. 66*, 1 (2005), 309–338.

[18] Guo, Z., and Wei, J. On an elliptic problem with a singular nonlinearity. *preprint* (2006).

[19] Guo, Z., and Wei, J. Infinitely many turning points for an elliptic problem with a singular non-linearity. *Journal of the London Mathematical Society 78*, 1 (2008), 21–35.

[20] Joseph, D. D., and Lundgren, T. S. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal. 49* (1972/73), 241–269.

[21] Keener, J. P., and Keller, H. B. Positive solutions of convex nonlinear eigenvalue problems. *J. Differential Equations 16* (1974), 103–125.

[22] Lions, P.-L. Minimization problems in $L^1(\mathbf{R}^3)$. *J. Functional Analysis 41*, 2 (1981), 236–275.

[23] Martel, Y. Uniqueness of weak extremal solutions for nonlinear elliptic problems. In *Houston J. Math* (1997), Citeseer.

[24] Nathanson, H. C., Newell, W. E., Wickstrom, R. A., and Davis, J. R. The resonant gate transistor. *IEEE Transactions on Electron Devices 14*, 3 (1967), 117–133.

[25] Nedev, G. Regularity of the extremal solution of semilinear elliptic equations. *C. R. Acad. Sci. Paris Sér. I Math. 330*, 11 (2000), 997–1002.

[26] Opic, B., and Kufner, A. Hardy type inequalities, pitman research notes in mathematics, vol. 219, 1990.

[27] Pelesko, J. A. Mathematical modeling of electrostatic MEMS with tailored dielectric properties. *SIAM J. Appl. Math. 62*, 3 (2001/02), 888–908.

[28] Pelesko, J. A., and Bernstein, D. H. *Modeling MEMS and NEMS*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[29] Strauss, W. A. Existence of solitary waves in higher dimensions. *Comm. Math. Phys. 55*, 2 (1977), 149–162.

[30] Taylor, G. I. The coalescence of closely spaced drops when they are at different electric potentials. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 306*, 1487 (1968), 423–434.