Faber and Grunsky Operators on Bordered Riemann Surfaces of Arbitrary Genus and the Schiffer Isomorphism

by

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Abstract

Let Σ be a bordered Riemann surface of genus $\mathfrak{g} > 0$ with $n \geq 1$ borders $\Gamma_1, \ldots, \Gamma_n$, each one homeomorphic to the unit circle $\mathbb{S}^1$. This surface can be described as a compact Riemann surface $\mathfrak{R}$ of the same genus with finitely many simply connected domains $\Omega_1^+, \ldots, \Omega_n^+$ removed. That is, $\Sigma = \mathfrak{R} \setminus \cup_{k=1}^n \mathrm{cl}(\Omega_k^+)$, $\partial\Omega_k^+ = \Gamma_k$.

Let f_k be a conformal map from the unit disc $\mathbb{D}$ in $\mathbb{C}$ onto Ω_k^+ for each $k = 1, \ldots, n$. We first generalize the classical *Faber* and *Grunsky* operators, to operators on Σ associated to the maps $\mathbf{f} = (f_1, \ldots, f_n)$. These two operators are used to characterize the holomorphic Dirichlet space $\mathcal{D}(\Sigma)$. More precisely, we show that the pull-back of the conformally-non-tangential boundary values of functions in $\mathcal{D}(\Sigma)$ under $\mathbf{f}$ is the graph of the Grunsky operator. We also show that the Grunsky operator is a bounded operator of norm strictly less than one. So far this had only be proven for the genus zero case.

The central problem is to prove that the Faber operator is a bounded isomorphism. This is done by using the *Schiffer operators*, which we generalize it for Σ in this work. We characterize the function space on which the Schiffer operator is a bounded isomorphism. This characterization depends on the topology of the surface. The condition that the boundary curves are quasicircles plays a vital role in this proof. This is also a generalization of the genus zero case.

The Grunsky operator is used to define a map, say $\Pi_{\mathfrak{g}}$, on the Teichmüller space of Σ. The map $\Pi_{\mathfrak{g}}$ has some analogies with the classical period map defined for compact surfaces. We conclude the thesis with a conjecture regarding the holomorphicity of this map for $\mathfrak{g}$, which was an important source of motivation for the

results of the thesis.

Declaration of Authorship

I, Mohammad SHIRAZI, declare that this thesis titled, “Faber and Grunsky Operators on Bordered Riemann Surfaces of Arbitrary Genus and the Schiffer Isomorphism” and the work presented in it are my own.

Signed:

Date:

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Proposition. Every list of people in any "acknowledgement" is incomplete.

Introduction

In this thesis, we start with a bordered Riemann surface Σ of type $(\mathfrak{g}, n)$. That is, let Σ be a bordered Riemann surface of genus $\mathfrak{g} \geq 0$ with $n \geq 1$ boundary curves such that each boundary curve is homeomorphic to the unit circle $\mathbb{S}^1$. This surface can be obtained from a compact Riemann surface $\mathfrak{R}$ with n simply connected domains $\Omega_1^+, \ldots, \Omega_n^+$ removed. Fix conformal maps f_k from $\mathbb{D}$, the unit disc in complex plane $\mathbb{C}$, onto Ω_k^+ admitting a quasiconformal extension on an open neighbourhood of $\mathrm{cl}(\mathbb{D})$ for each $k = 1, \ldots, n$. The thesis answers the following two problems/questions about the surface Σ.

• **Problem 1.** What does the holomorphic Dirichlet space of Σ look like? In other words, how can one characterize the pull-back of $\mathcal{D}(\Sigma)$ under the conformal maps f_k? This requires characterizing the boundary values of functions in $\mathcal{D}(\Sigma)$.

• **Problem 2.** How can one define a period map on the Teichmüller space of Σ which is similar to the classical period map defined for compact Riemann surfaces? In other words, how can one find a map, say $\Pi_\mathfrak{g}$, on the Teichmüller space of Σ into some Banach space of bounded operators (on some Hilbert space) such that $\Pi_\mathfrak{g}$ is symmetric, satisfies $I - \Pi_\mathfrak{g}\overline{\Pi}_\mathfrak{g} > 0$ (positive definite) and also $\Pi_\mathfrak{g}$ is holomorphic?

These two questions have both already been answered in the case $\mathfrak{g} = 0$.

The second question for the one boundary curve case ($n = 1$) was answered by L. A. Takhtajan and L.-P. Teo [76] (with some roots in A. A. Kirillov and D. Yur'ev [28], S. Nag and D. Sullivan [39]), and independently by Y. L. Shen [67]. The first and second questions for the case of zero genus and many boundary curves ($n > 1$) were answered by D. Radnell, E. Schippers, and W. Staubach [46, 47].

The solutions for the genus zero case have roots in classical complex analysis, univalent function theory including the Faber, Grunsky and Schiffer operators on planar domains, the jump problem, the Dirichlet and Riemann boundary value problems, and the L. Bers' construction of the universal Teichmüller space. To construct a similar machinery to solve these problems, we first generalize these three operators from the planar case to surfaces such as Σ of type $(\mathfrak{g}, n)$ where $\mathfrak{g} > 0$. As expected, some topological obstacles appear when we move to higher genus and we address them in the thesis.

Another technical issue is that we are required to deal with some boundary value problems on Riemann surfaces; in particular, the jump problem for quasicircles. We use some recent results of E. Schippers and W. Staubach [63] to overcome this issue.

We show that the Faber operator corresponding to Σ is a bounded isomorphism when all the boundary curves are quasicircles. This generalizes some results in the planar case; see Chapter 2. For this choice of boundary curves, we show that the Grunsky operator corresponding to Σ is a bounded operator of norm strictly less than one. This generalizes a result of Ch. Pommerenke and R. Kühnau in the planar case; see e.g. Ch. Pommerenke [41].

The proof that the Faber operator is a bounded isomorphism, aside from the boundary regularity, relies on the properties of the Schiffer operator. We find a subspace of L^2 holomorphic 1-forms, depending on the topology of the surface, on which the Schiffer operator is a bounded isomorphism. The adjoints of these operators are also calculated. This is based on the results of E. Schippers and W. Staubach [62] for the one boundary curve case (surfaces of type $(\mathfrak{g}, 1), \mathfrak{g} \geq 1$) including the first calculation of the adjoint formula; also M. Schiffer [55] and S. Bergman and M. Schiffer [10] for the planar case.

To prove that the norm of the Grunsky operator is strictly less than one when all the boundary curves are quasicircles, we require some density theorems. They were stated and proven by E. Schippers, W. Staubach and the author in a joint paper [59].

The above two important results (the Faber operator is a bounded isomorphism and the norm of the Grunsky operator is strictly less than one) are the cornerstones for characterizing the pull-back of $\mathcal{D}(\Sigma)$; that is the first problem above. We show that this pull-back is the graph of the Grunsky operator, generalizing the work done by D. Radnell, E. Schippers and W. Staubach [46] for surfaces of type $(0, n), n \geq 1$. This part of the thesis answers the first question above.

The Grunsky operator is used to define a period map on the Teichmüller space of Σ. The properties proven for the Grunsky map are enough to show that this map satisfies the positive definiteness property mentioned above. The holomorphicity, however, is left as a conjecture. Therefore, the second question above, except the holomorphicity part, is answered in the thesis. Using Grunsky operator to define period maps on Teichmüller spaces has some roots in recent works of Takhtajan and Teo [76], Shen [67], and Radnell, Schippers and Staubach [46, 47] and some others on period maps on

Teichmüller spaces. Finally, we also discuss some open problems related to above results.

Here is a brief outline of the thesis.

In chapter 1, we review all the main definitions and theorems that we need in the rest of the thesis. Some notation is also introduced.

Chapter 2 is devoted to historical aspects and literature review of Faber polynomials, Grunsky coefficients and their connections to Teichmüller spaces. A more comprehensive history of period maps, however, is given in Chapter 4. We review some important results relevant to the thesis problems. This chapter is divided into two parts: Faber polynomials, series or operator; and Grunsky coefficients, inequality, or operator.

Chapter 3 includes the main results of the thesis. This chapter is divided into two parts, surfaces with one boundary curve and surfaces with more than one boundary curve. The classical Faber, Grunsky, and Schiffer operators are generalized to bordered Riemann surfaces of type $(\mathfrak{g}, n)$. We characterize some function spaces on which these operators behave nicely. Finally, the pull-back of Dirichlet holomorphic space of Σ is described as the graph of the Grunsky operator.

Chapter 4 is about the second problem mentioned above, namely the definition of a period map on the Teichmüller space of Σ via the Grunsky operator and the proof of its holomorphicity. A short history of the problem is provided. Some connection to the Grunsky operator, defined in Chapters 2 and 3, is explained in different cases of genus and boundary curves. We close this chapter, and the whole thesis, by a conjecture concerning the holomorphicity of the period map defined on the Teichmüller space of Σ.

Contents

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List of Symbols

$\mathbb{A}$	An Annulus in the Complex Plane
A^c	The Set Theoretic Complement of The Set A
$A_{harm}(\Sigma)$	The Space L^2 Integrable Harmonic 1-forms on region Σ
$A_{harm}(\Sigma)_e$	The Space L^2 Integrable Harmonic Exact 1-forms on region Σ
$A(\Sigma)$	The Space of L^2 Integrable Holomorphic 1-forms on region Σ
$A(\Sigma)_e$	The Space of L^2 Integrable Holomorphic Exact 1-forms on region Σ
$A(\overline{\Sigma})$	The Space of Functions Holomorphic on a Region Σ and Continuous on $\mathrm{cl}(\Sigma)$
B_p	The Besov Space
$B(X)$	The (Banach) Space of Bounded Operators on a Banach Space X
$B > 0$	The Matrix B is Positive Definite
B^t	The Transpose of a Matrix B
C^n	The Set of Functions Having n Continuous Derivatives
C^∞	The Set of Functions Having Infinity Many Continuous Derivatives
$\mathcal{C}_f$	The Composition Operator with a Map f
$\overline{\mathbb{C}}$	The Riemann Sphere
$cl(A)$	The Closure of a Set A
$\mathbb{C}$	The Complex Plane, The Set of Complex Numbers
$C^\infty(U)$	The Set of Smooth Functions on a Set U

χ_A	The Characteristic Function of a Set A
$\mathcal{D}_{harm}(\Sigma)$	Dirichlet Harmonic Space of a Region Σ
$\mathcal{D}(\Sigma)$	The Dirichlet Space of Domain Σ
$\mathfrak{O}(\Sigma_i, \Sigma_j)$	The Transmission Operator From Subsurface Σ_i to Subsurface Σ_j
$\mathbb{D}$	The Unit Disc in the Complex
$\emptyset$	The Empty Set
Eq	An Abbreviation for Equation
$F^\star$	The Adjoint of an Operator F on a Given Hilbert Space
Γ	A Curve in General
$\mathfrak{g}$	Genus of a Riemann Surface
$g(w; z, q)$	The Green's Function
$g(w, w_0; z, q)$	The Green's Function with Zero at w_0
$graph(F)$	The Graph of an Operator F.
$\mathbf{Gr}_f, \mathbf{Gr}_\mathbf{f}$	The Grunsky Operator Correspondence to a Map f, $\mathbf{f} = (f_1, \ldots, f_n)$
Gr_f	The Classical Grunsky Operator Correspondence to a Map f
$\mathbf{Gr}$	The Classical Grunsky Map on the Universal Teichmüller Space
$\mathbb{H}$	The Upper Half Plane
$Hol(A)$	The set of all Holomorphic Functions on set A
$Har(A)$	The set of all Harmonic Functions on set A
$\overline{H}$	$\overline{H}(z) := \overline{H(z)}$ for a Complex Value Function H
H_p	The Hardy Space
$h\|_A$	The Restriction of the Domain of a Function h to a Set A
$\mathbf{I}_f$	The Faber Operator Correspondence to a Map f
$Int(A)$	The Interior of a Set A
$Im(f)$	The Image of a Map f
$Im(z), Re(z)$	The Imaginary (Real) Part of a Complex Number z

$J_q(\Gamma, \Sigma)$	The Jump Operator with respect to Curve Γ when the Limiting curves are in Σ
$J_q(\Gamma)_\Sigma$	The Jump Operator with respect to Curve Γ, Output Restricted on the Surface Σ
$K_{\mathfrak{R}}(w, z)$	The Bergman Kernel of $\mathfrak{R}$
$\ell^2(\mathbb{C})$	The Space of Complex Square Integrable Sequences
$\\|.\\|_{L^\infty}$	Essential Sup Norm
$\Pi, \Pi_{\mathfrak{g}}$	Period Map on Teichmüller Spaces, Teichmüller Space of Surface of Genus $\mathfrak{g}$
$\widetilde{\Pi}, \widetilde{\Pi}_{\mathfrak{g}}$	Period Map on Rigged Teichmüller Spaces, Rigged Teichmüller Space of Surface of Genus $\mathfrak{g}$
$P_G(\overline{P}_G)$	The Projection Operator of a Harmonic Function on G to Its Holomorphic (Anti-holomorphic) Part
$P(G)(\overline{P}(G))$	The Projection Operator of a Harmonic 1-Form on G to Its Holomorphic (Anti-holomorphic) part
∂K	The Boundary of a Set K
$\partial f, \overline{\partial} f$	The Wirtinger Derivatives of f
$\Phi_n(f), \Phi_n$	The Faber polynomials or Functions Correspondence to a Map f
$\mathfrak{R}, \mathfrak{S}$	A Riemann Surface
$\mathfrak{R}^B, \mathfrak{R}^P$	A Bordered, Punctured Riemann Surface
$\mathfrak{R}^B_{\mathfrak{g}}, \mathfrak{R}^P_{\mathfrak{g}}$	A Bordered, Punctured Riemann Surface of Genus $\mathfrak{g}$
$\mathbb{R}$	The Real Line, The Set of Real Numbers
$\mathcal{R}_\Gamma$	The Reflection Operator with Respect to a Curve Γ
$Res(\Sigma)$	The Restriction Operator to the Subsurface Σ
Σ	A Surface or Subsurface
$\mathfrak{G}(A, \Sigma)$	The Extension Operator from A to the Subsurface Σ

	in the Complex Plane
$<,>_{A_{harm}(\Sigma)}$	The Inner Product on $A_{harm}(\Sigma)$
$(x_n)_n, \{x_n\}_{n=1}^{\infty}$	A Sequence with Terms x_n
(x_{mn})	A Matrix with Entry x_{mn}
$\mathbb{Z}$	The Set of Integer Numbers

Dedicated to brilliant minds who deprive of the right to education …

Chapter 1

Preliminaries

In this chapter we review some basic and important definitions, theorems and notation that will be used in the thesis. We try to keep it as minimal as possible. Some references are provided for further studies. We assume the reader is familiar with the concept of a Riemann surface.

1.1 Basic Notation

Let $\mathbb{C}$ and $\overline{\mathbb{C}}$ denote the complex plane and the Riemann sphere, respectively. We have $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc in $\mathbb{C}$, and $\mathbb{D}^- = \{z \in \mathbb{C} : |z| > 1\}$ the complement of its closure in $\mathbb{C}$. $\mathbb{H} = \{z \in \mathbb{C} : Im\ z > 0\}$ denotes the upper half plane. $\mathrm{cl}(G)$ is used to indicate the closure of a set G in a topological space.

We use the notation $(a_k)_{k=1}^n$ to denote the n-tuple $(a_1, \ldots, a_n)$. As a linear operator I_A denotes the identity operator on a normed space A.

For a Riemann surface $\mathfrak{R}$, open or compact, $\mathfrak{g} \geq 0$ is used to denote its **genus** i.e. the number of handles. By a **domain** G in a Riemann surface $\mathfrak{R}$ we mean an open, connected subsurface of $\mathfrak{R}$ such that the inclusion map is holomorphic.

When we say a **bordered** Riemann surface we mean it is bordered in the sense of Ahlfors and Sario [4, II. 3A].

Suppose $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are some Banach spaces. $B(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y. We also have $B(X) := B(X,X)$.

1.2 Some Function Spaces

We briefly recall the definition of some function spaces here which will be used later in the thesis.

1.2.1 Spaces of Univalent Functions

The following two spaces are seen frequently in literature (see e.g. P. L. Duren [14]).

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \,:\, f \textit{ is one-to-one and holomorphic}, f(0) = 0, f'(0) = 1\}$$

$$\mathbf{\Sigma} = \left\{g : \mathbb{D}^- \to \mathbb{C} \,:\, g \textit{ is one-to-one and holomorphic, } g(\infty) = \infty \textit{ with residue } 1\right\}.$$

It is also customary to represent a univalent (i.e. one-to-one) function by its Taylor expansion about zero by

$$f(z) = z + a_2 z^2 + \dots,$$

for elements in $\mathcal{S}$ or about ∞ by

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots,$$

for elements in $\mathbf{\Sigma}$.

$\mathbf{\Sigma}_0$ is used to indicate those functions in $\mathbf{\Sigma}$ with $b_0 = 0$. Finally, we need the following important theorem [3].

Theorem 1.2.1 (Riemann Mapping Theorem)**.** *Let G be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in G. Then there exists a unique function f which maps G conformally onto the unit disc and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

The map f in above theorem may be called the **Riemann map** of the domain G.

1.2.2 Harmonic Functions and Forms and Dirichlet Spaces

Here we review some standard definitions on a Riemann surface $\mathfrak{R}$ with atlas $\{(U, \phi)\}$ of holomorphic charts. For this subsection, we mostly use H. L. Royden [52] or H. M. Farkas and I. Kra [17] as references.

For a chart (U, ϕ) the map $z = \phi(w), w \in U$ may be called a **uniformizer**.

Let Σ be a domain of $\mathfrak{R}$, which could be $\mathfrak{R}$. A **differential** 1**-form** or simply a 1-form α on Σ is assigning to each holomorphic chart $(U, z = x+iy)$ of Σ a pair of functions a and b (i.e. $a\ dx + b\ dy$) such that if $U_k \cap U_j \neq \emptyset$, then

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} = \frac{\partial(x_j, y_j)}{\partial(x_k, y_k)} \begin{bmatrix} a_j \\ b_j \end{bmatrix} \qquad \text{on } U_k \cap U_j.$$

If $z = x + iy$ is a local holomorphic parameter, the **dual** of 1-form α is defined by

$$\star\alpha = \star(a\ dx + b\ dy) = a\ dy - b\ dx.$$

The **conjugate** of α is defined by

$$\bar{\alpha} = \bar{a}\,dx + \bar{b}\,dy.$$

A 1-form α is called **harmonic** on $\mathfrak{R}$ if both α and $\star\alpha$ are closed 1-forms. That is $d\alpha = d\star\alpha = 0$.

Fix a holomorphic chart (U, z) on $\mathfrak{R}$. The operators ∂ and $\overline{\partial}$ denote the **Wirtinger derivatives** where the output is understood as 1-form. That is, for a function $f(z) = f(x+iy)$ with continuous first order partial derivatives in U, one has

$$\begin{aligned} \frac{\partial f}{\partial z} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), & \partial f &:= \frac{\partial f}{\partial z}\,dz, \\ \frac{\partial f}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), & \overline{\partial} f &:= \frac{\partial f}{\partial \bar{z}}\,d\bar{z}. \end{aligned}$$

We may use ∂_z or $\overline{\partial}_z$ when derivatives are taken with respect to a specific variable z. Also d denotes $\partial + \overline{\partial}$.

A function u is said to be **harmonic** on Σ if at each point ζ of Σ it is harmonic function of a holomorphic chart (U, z) of Σ about ζ for every holomorphic chart (U, z) about ζ. That is, u is C^2 in U and $d\star du = 0$ (the Laplace equation).

On the vector space of all harmonic 1-forms on Σ, we define the following inner product and subspaces.

$$< \alpha, \beta >_{A_{harm}(\Sigma)} = \frac{1}{2}\iint_{\Sigma} \alpha \wedge \star\overline{\beta}, \quad \alpha, \beta \textit{ are harmonic 1-forms on } \Sigma,$$

$$A_{harm}(\Sigma) := \{\alpha : \alpha \textit{ is a harmonic 1-form on } \Sigma \textit{ and } < \alpha, \alpha >_{A_{harm}(\Sigma)} < \infty\},$$

$$\mathcal{D}_{harm}(\Sigma) := \{h : \Sigma \to \mathbb{C} : h \textit{ is harmonic and } dh = \partial h + \bar{\partial} h \in A_{harm}(\Sigma)\}.$$

We call $A_{harm}(\Sigma)$ the Dirichlet space of harmonic 1-forms on Σ and $\mathcal{D}_{harm}(\Sigma)$ the Dirichlet space of harmonic functions on Σ, or simply the harmonic Dirichlet space of Σ.

Define $A(\Sigma)$ the subspace of $A_{harm}(\Sigma)$ which contains all the holomorphic 1-forms in $A_{harm}(\Sigma)$. Similarly $\overline{A(\Sigma)}$ contains all the anti-holomorphic ones. We naturally have

$$A_{harm}(\Sigma) = A(\Sigma) \oplus \overline{A(\Sigma)}.$$

When Σ is simply connected, a similar decomposition for $\mathcal{D}_{harm}(\Sigma)$ may be written.

We may use $\mathcal{D}_{harm}(\Sigma)_q$ to indicate the set of functions in $\mathcal{D}_{harm}(\Sigma)$ which vanish at $q \in \Sigma$. $\mathcal{D}(\Sigma)_q$ and $\overline{\mathcal{D}(\Sigma)}_q$ may be defined similarly. By $A_{harm}(\Sigma)_e$ we mean the exact elements in $A_{harm}(\Sigma)$. That is, if $\alpha \in A_{harm}(\Sigma)_e$, then there exists $h \in \mathcal{D}_{harm}(\Sigma)$ such that $dh = \alpha$. $A(\Sigma)_e$ and $\overline{A(\Sigma)}_e$ are defined in the same way.

If α is an element in $A_{harm}(\Sigma)$ the Dirichlet semi-norm of α is defined by

$$\|\alpha\|^2_{A_{harm}(\Sigma)} :=< \alpha, \alpha >_{A_{harm}(\Sigma)} .$$

Similarly for an element $h \in \mathcal{D}_{harm}(\Sigma)$, we have

$$\|h\|_{\mathcal{D}_{harm}(\Sigma)} := \|dh\|_{A_{harm}(\Sigma)}$$

as it is known that the differential of a harmonic function on Σ is a harmonic 1-form on Σ. This implies that both operators

$$\partial : \mathcal{D}(\Sigma) \to A(\Sigma)_e$$
$$h \to \partial h$$

and

$$\overline{\partial} : \overline{\mathcal{D}(\Sigma)} \to \overline{A(\Sigma)}_e$$
$$\bar{h} \to \overline{\partial}\bar{h}$$

are isometries. They, however, are not isomorphisms, in general.

Elements in $A_{harm}(\Sigma)$ ($\mathcal{D}_{harm}(\Sigma)$) may be called L^2 bounded harmonic 1-forms (functions) on Σ.

Remark 1.2.2. *One way to make ∂ and $\overline{\partial}$ isomorphisms is to define them on $\mathcal{D}(\Sigma)_q$ and $\overline{\mathcal{D}(\Sigma)}_q$ for some $q \in \Sigma$, respectively.*

If there is no risk of confusion, we usually drop the index in $< \alpha, \beta >_{A_{harm}(\Sigma)}$ or in the semi-norm expressions.

Suppose A and B are two regions of some Riemann surfaces and $\phi : A \to B$ is a biholomorphism. It is easy to show that $h \in \mathcal{D}_{harm}(B)$ if and only if $h \circ \phi \in \mathcal{D}_{harm}(A)$. In particular, the composition with ϕ preserves the Dirichlet semi-norm. That is

$$\|h\|_{\mathcal{D}_{harm}(B)} = \|h \circ \phi\|_{\mathcal{D}_{harm}(A)}.$$

In other words, the composition operator defined by

$$\mathcal{C}_\phi : \mathcal{D}_{harm}(B) \to \mathcal{D}_{harm}(A)$$
$$h \to h \circ \phi$$

is an isometry with respect to Dirichlet semi-norm.

Every harmonic function $h \in \mathcal{D}_{harm}(G)$ on a simply connected domain G (on a Riemann surface), can be decomposed as $h_1 + \overline{h}_2$ for $h_1, h_2 \in \mathcal{D}(G)$ where h_1 and h_2 are determined uniquely by the condition that $h_1(p) = 0$ for fixed $p \in G$. Define the **projection operator**

$$P_G : \mathcal{D}_{harm}(G) \to \mathcal{D}(G)_p$$

taking h to h_1. Similarly, define the projection

$$\overline{P}_G : \mathcal{D}_{harm}(G) \to \overline{\mathcal{D}(G)}$$

taking h to $\bar{h}_2$. In particular, by $P_{\mathbb{D}}$ and $\overline{P}_{\mathbb{D}}$ ($P_{\mathbb{D}^-}$ and $\overline{P}_{\mathbb{D}^-}$) we mean the projection operators for $\mathbb{D}$ ($\mathbb{D}^-$) in $\mathbb{C}$ where $p = 0$ ($p = \infty$). Note that $P_G(a) = 0$, and $\overline{P}_G(a) = a$, for any constant $a \in \mathbb{C}$.

We may also need to project a harmonic 1-form on a given region G (not necessarily simply connected) to its holomorphic and anti-holomorphic parts. By $P(G)$ and $\overline{P}(G)$ we mean the projection operators of harmonic 1-forms on G to the holomorphic and the anti-holomorphic parts, respectively.

1.2.3 Bergman Space

For a domain G of $\mathbb{C}$, the Bergman space $A^2(G)$ is defined as the space of all holomorphic square integrable functions on G with respect to the usual planar Lebesgue measure dA_z. That is

$$A^2(G) = Hol(G) \cap L^2(G).$$

It is known that $A^2(G)$ is Hilbert space equipped with the inner product

$$< \phi, \psi >= \iint_G \phi(z)\, \overline{\psi(z)}\, dA_z.$$

1.3 Quasiconformal Maps, Quasicircles, and Quasidiscs

Among all different (equivalent) ways to define a quasiconformal map, let us start with the following analytic one; see [41].

Definition 1.3.1 (Quasiconformal Map)**.** *Let G be a domain in $\mathbb{C}$ and $0 \leq k < 1$. A sense (orientation) preserving homeomorphism h on G is called a k-quasiconformal map if h satisfies the following two conditions.*

- *On each finite rectangle in G, $h(x + iy)$ is absolutely continuous in x for almost all y and absolutely continuous in y for almost all x. This property is called absolutely continuous on lines (ACL).*
- *For almost all $z \in G$ one has*

$$\left|\frac{\partial h}{\partial \bar{z}}(z)\right| \leq k \left|\frac{\partial h}{\partial z}(z)\right|. \tag{1.1}$$

The second condition above may also be written as

$$\left\|\frac{\partial h}{\partial \bar{z}}\right\|_{L^\infty} \leq k \left\|\frac{\partial h}{\partial z}\right\|_{L^\infty},$$

where $\|.\|_{L^\infty}$ is the essential sup norm.

We say a function $h \in \Sigma$ admits a k-quasiconformal extension to $\mathbb{C}$ if it has a homeomorphic extension to $\mathbb{C}$ which satisfies Definition 1.3.1 for $G = \mathbb{D}$ and k. A similar formulation can be given for elements in $\mathcal{S}$. These will be used in Subsection 2.2.1, and Section 4.3.

Example 1.3.2. *For fixed $0 \leq k < 1$, the map $h(z) = z + k\bar{z}$ is a k-quasiconformal map.*

Definition 1.3.3 (Quasidisc)**.** *A domain G in $\mathbb{C}$ is called a quasidisc if $G = h(\mathbb{D})$ for some quasiconformal map h.*

Definition 1.3.4 (Quasicircle)**.** *A Jordan curve Γ in $\mathbb{C}$ is called quasicircle if $\Gamma = h(\mathbb{S}^1)$ for some quasiconformal map h.*

It worth noting that not every quasicircle is rectifiable (i.e. of a finite length) and not every rectifiable curve is necessarily a quasicircle [2]. Here are some other important properties of quasiconformal maps and quasicircles.

Theorem 1.3.5.

1. *The composition of two quasiconformal maps is a quasiconformal map.*
2. *Quasicircles have planar Lebesgue measure zero.*
3. *Quasiconformal maps preserves sets of logarithmic capacity zero.*
4. *A Quasiconformal maps small circles to small ellipses of bounded eccentricity.*

Proof. See [19], [32], [41], and references therein. □

Here is the definition of quasiconformal maps between Riemann surfaces.

Definition 1.3.6. *A homeomorphism h between two Riemann surfaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$ is called k-quasiconformal if for any holomorphic chart (U_k, ϕ_k) of $\mathfrak{R}_k$, $k = 1, 2$, the mapping $\phi_1 \circ h \circ \phi_2^{-1}$ is k-quasiconformal map in the sense of Definition 1.3.1.*

This definition is one of the key definitions that one need to define the analytic Teichmüller space of a Riemann surface, see Section 1.9. We may apply this to quasiconformal maps between open subsets of Riemann surfaces.

1.4 Transmission Operator and CNT Limits

Definitions and notations in this section are the same as Schippers and Staubach [62, 63]. Let $\mathfrak{R}$ be a compact Riemann surface. Then we have the following definitions.

• A Jordan curve in $\mathfrak{R}$ is a homeomorphic image of $\mathbb{S}^1$.

• A subset G is called a **doubly connected neighbourhood** of a Jordan curve Γ in $\mathfrak{R}$ if G is an open connected subset of $\mathfrak{R}$ containing Γ and bounded by two non-intersecting Jordan curves, each one homotopic to Γ within the closure of G.

• The pair (G, ϕ) where G is as above and $\phi : G \to \mathbb{A}$ is a biholomorphism to some annulus $\mathbb{A}$ in $\mathbb{C}$ is called a **doubly connected chart** for Γ.

• Every Jordan curve Γ in $\mathfrak{R}$ which has a doubly connected chart is called a **strip-cutting** Jordan curve.

• A **collar neighbourhood** of a Jordan curve Γ in $\mathfrak{R}$ is an open connected subset A of $\mathfrak{R}$ bordered by Γ and Γ', where Γ' is a Jordan curve in $\mathfrak{R}$ which is homotopic to Γ from within the closure of A and such that $\Gamma \cap \Gamma'$ is empty.

• A **collar chart** for a Jordan curve Γ in $\mathfrak{R}$ is a collar neighbourhood A together with a biholomorphism $\phi : A \to \mathbb{A}$, for some annulus $\mathbb{A}$ in $\mathbb{C}$.

• A Jordan curve Γ on a Riemann surface $\mathfrak{R}$ is called **quasicircle** if there exists a biholomorphism $\phi : A \to B$, where A is a doubly connected neighbourhood of Γ in $\mathfrak{R}$ and B is doubly connected region in $\mathbb{C}$, such that $\phi(\Gamma)$ is a quasicircle in $\mathbb{C}$.

We define the following conformally non-tangential (abbreviated by **CNT**) boundary limits for harmonic functions defined on simply connected regions of a compact Riemann surface $\mathfrak{R}$. This is like the well-known notion of non-tangential limit of harmonic functions defined on $\mathbb{D}$ in $\mathbb{C}$.

Definition 1.4.1 (CNT Boundary Limit)**.** *Let $\mathfrak{R}$ be as above. Let Ω be a simply connected domain of compact Riemann surface $\mathfrak{R}$ bounded by a strip-cutting Jordan curve Γ. Let also $s \in \Gamma$ and $f : \mathbb{D} \to \Omega$ a conformal map. We say that $h : \Omega \to \mathbb{C}$ has CNT limit at s if $h \circ f$ has non-tangential limit at $f^{-1}(s)$.*

Suppose Γ is a strip-cutting Jordan curve dividing $\mathfrak{R}$ into two connected subsurfaces Σ_1 and Σ_2. Suppose also that $q \in \Sigma_1$ and Γ is positively oriented with respect to Σ_1. Two problems regarding the existence of CNT limits need to be addressed here.

Problem 1. We need a notion of potential-theoretically small sets on Γ.

Using the Green's function of Σ_1 with singularity at q, it was shown in [62] that there exists A, a collar neighbourhood of Γ in Σ_1, with a biholomorphism

$$\phi_0 : A \to \mathbb{A}$$

where $\mathbb{A}$ is an annulus in $\mathbb{C}$, such that $\phi_0(\Gamma) = \mathbb{S}^1$. ϕ_0 is called the **canonical collar chart** with respect to q and Σ_1. Now we have the following definition.

Definition 1.4.2 (Null Set). *A Borel set I in Γ is called a null set (with respect to Σ_1 and q) if $\phi_0(I)$ has logarithmic capacity zero in $\mathbb{S}^1$.*

It can be shown that this definition is independent of the choice of the point q. Moreover, when Γ is a quasicircle the definition is also independent of the choice of $\Sigma_1 \in \Sigma_1$. That is, I is null set with respect to Σ_1 if and only if it is a null set with respect to Σ_2.

Lemma 1.4.3. *Let $\mathfrak{R}$, Γ, Σ_1, Σ_2 and ϕ_0 be as above. Then a finite union of null sets in Γ is a null set in Γ.*

Proof. This was proven in E. Schippers and W. Staubach paper [60, Theorem 2.14]. Briefly, they proved it by using the the sub-additivity of the outer capacity under countable unions and Choquet's theorem, which says that for bounded Borel sets in $\mathbb{S}^1$ the outer capacity is the capacity. □

Having defined the above, here is a theorem [15], rephrased conformally invariantly in [63], showing the existence and uniqueness of the CNT boundary values for harmonic functions on simply connected domains of $\mathfrak{R}$.

Theorem 1.4.4 (Beurling-Zygmund). *Let $\mathfrak{R}$, Γ, Ω, and f be the same as Definition 1.4.1. Then for every $h \in \mathcal{D}_{harm}(\Omega)$, h has CNT limit at p for all p except possibly on a null set in Γ with respect to Ω. If $h_1, h_2 \in \mathcal{D}_{harm}(\Omega)$ have the same CNT boundary values except possibly on a null set in Γ, then $h_1 = h_2$ on Ω.*

Schippers and Staubach proved a stronger result. Namely, for the existence of the CNT boundary values of a function h on Ω, under some conditions, h is only required to be defined on a collar neighbourhood of Γ in Ω. Here is their theorem [63, Theorem 3.17].

Theorem 1.4.5. *Let $\mathfrak{R}, \Gamma, \Sigma_1$ and Σ_2 be as above. Let A be a collar neighbourhood of Γ in Σ_1. Then for any $h \in \mathcal{D}_{harm}(A)$, h has conformally non-tangentially boundary values on Γ except possibly on a null set in Γ. Furthermore, there exists a unique $H \in \mathcal{D}_{harm}(\Sigma_1)$ whose CNT boundary values agrees with those of h except possibly on a null set in Γ.*

Problem 2. Suppose $h_1 \in \mathcal{D}_{harm}(\Sigma_1)$ with CNT boundary value function H_1 on Γ. Then a question naturally arises here: Is there any function $h_2 \in \mathcal{D}_{harm}(\Sigma_2)$ with CNT boundary value function H_2 on Γ such that $H_1 = H_2$ on Γ except possibly on a null set in Γ? It is good to mention that if one has a Jordan curve, the notion of null set changes from one side to the other.

The notation $\mathfrak{O}(\Sigma_1, \Sigma_2)h_1$ is being used for h_2 if such h_2 exists and we will say h_2 is the transmission of h_1 through the Jordan curve Γ. $\mathfrak{O}(\Sigma_1, \Sigma_2)$ is clearly linear on $\mathcal{D}_{harm}(\Sigma_1)$ for which the transmission exists.

Example 1.4.6. *If $\mathfrak{R}$ is the Riemann sphere and $\Gamma = \mathbb{S}^1$, then such a transmission can be written explicitly. That is, for every $h_1 \in \mathcal{D}_{harm}(\mathbb{D}^-)$ we have*

$$[\mathfrak{O}(\mathbb{D}^-, \mathbb{D})h_1](z) = h_1(\frac{1}{\bar{z}}).$$

In particular, $\mathfrak{O}(\mathbb{D}^-, \mathbb{D})(\frac{1}{z^n}) = \bar{z}^n$, where by $\frac{1}{z^n}$ we mean the harmonic function $h_1(z) = \frac{1}{z^n}, n \geq 0$. The operator $\mathfrak{O}(\mathbb{D}, \mathbb{D}^-)$ on $\mathcal{D}_{harm}(\mathbb{D})$ can be written similarly.

Remark 1.4.7. *For a polynomial $h \in \mathcal{D}_{harm}(\mathbb{D})$, the above example shows that $\mathfrak{O}(\mathbb{D}, \mathbb{D}^-)h$ admits a holomorphic extension to every annulus $\mathbb{A}$ with outer boundary equal to $\mathbb{S}^1$.*

Schippers and Staubach [63, Theorem 3.29] showed the existence and the boundedness of the transmission operator for quasicircles on compact Riemann surfaces.

Theorem 1.4.8 (Schippers and Staubach)**.** *Let $\mathfrak{R}, \Gamma, \Sigma_1$ and Σ_2 be as above, where Γ is a quasicircle. $\mathfrak{O}(\Sigma_1, \Sigma_2)$ is a bounded linear operator from $\mathcal{D}_{harm}(\Sigma_1)$ onto $\mathcal{D}_{harm}(\Sigma_2)$ with respect to the Dirichlet semi-norm.*

In other words, they proved that for every $h_1 \in \mathcal{D}_{harm}(\Sigma_1)$ there exists a unique $h_2 \in \mathcal{D}_{harm}(\Sigma_2)$ such that both h_1 and h_2 have the same CNT boundary values on Γ except possibly on a null-set in Γ. A similar result for $\mathfrak{O}(\Sigma_2, \Sigma_1)$ is clearly valid due to the symmetry of the definition.

1.5 $\mathfrak{G}$ Operator

Let $\mathfrak{R}$, Γ, Σ_1, and Σ_2 be as the previous section. For a given harmonic function h on a collar neighbourhood A of Γ in Σ_1, Theorem 1.4.5 ensures that the following operator is well defined and is bounded.

$$\mathfrak{G}(A, \Sigma_1) : \mathcal{D}_{harm}(A) \to \mathcal{D}_{harm}(\Sigma_1)$$
$$h \to \tilde{h}$$

where $\tilde{h}$ and h have the same CNT boundary values except possibly on a null set in Γ.

It was shown that the condition that Γ is a quasicircle is sufficient for $\mathfrak{G}(A,\Sigma)$ to be bounded [63, Theorem 3.22].

1.6 Cauchy Kernel and Green's Function on Riemann Surfaces

The well-known Cauchy kernel

$$\frac{1}{2\pi i}\frac{d\zeta}{\zeta - z},$$

for $z \in \mathbb{C}$, has a simple pole of residue 1 at z and a simple pole of residue -1 at ∞ (after composition with holomorphic chart $\zeta \to \frac{1}{\zeta}$) on the Riemann sphere. This property is essential and we would like to have it on arbitrary compact Riemann surfaces $\mathfrak{R}$ of genus $\mathfrak{g}$. The point ∞, however, has to be changed by a fixed point, say $q \neq z$ on $\mathfrak{R}$. This generalization is classical, for example in H. L. Royden's paper [54, Proposition 1] the existence of the **Green's function** for $\mathfrak{R}$ is proved and its derivative plays a similar role. Here, we recall his definition and the existence theorem.

Definition 1.6.1 (Logarithmic Pole). *A function f on $\mathfrak{R}$ is said to have a logarithmic pole of mass m at the point $q \in \mathfrak{R}$ if in some holomorphic chart (U,ϕ) about q one has*

$$f(p) = -m \log|\phi(p) - \phi(q)| + \psi(p),$$

where $\psi \in C^\infty(U)$.

Definition 1.6.2 (Green's Function)**.** *A function g on $\mathfrak{R}$ which is harmonic on $\mathfrak{R}$ except at the points q and q_0 and has logarithmic poles there with masses $+1$ and -1, respectively, is called the Green's function for $\mathfrak{R}$ with poles at q and q_0.*

Royden indicates that if such a function exists, then it is unique up to additive constants. He, therefore, suggested the notation $g(p, p_0; q, q_0)$ which indicates the normalization in which g is zero at p_0. Here is [54, Proposition 1].

Theorem 1.6.3 (Royden)**.** *If $p_0 \neq q, q_0$, the Green's function $g(p, p_0; q, q_0)$ exists.*

Here we have changed Royden's notation a little bit. That is, by Green's function we mean $g(w, w_0; z, q)$ where g is as above. He showed the following essential properties of the Green's function [54, Proposition 2]:

$$g(w, w_1; z, q) = g(w, w_0; z, q) - g(w_1, w_0; z, q),$$

$$g(w_0, w; z, q) = -g(w, w_0; z, q),$$

$$g(z, q; w, w_0) = g(w, w_0; z, q).$$

As we mentioned we are interested in forming a kernel analogous to the Cauchy kernel from the derivative of g. The first equation above indicates that $\partial_w g$ is independent of the choice of w_0, so we do not write w_0 for simplicity. That is,

$$g(w; z, q) := g(w, w_0; z, q).$$

The last equation shows the harmonicity of g in z variable when we are away from w, w_0. Royden showed that $\partial_w g$ is a differential 1-form whose coefficients with respect to a coordinate system at w are harmonic functions of z with dipole singularities at w.

For a domain Σ in $\mathfrak{R}$ the Green's function can be defined like the planar cases. That is, $g_\Sigma(\, . \, , z)$ is called the Green's function of Σ with singularity at $z \in \Sigma$ if

- $g_\Sigma(w, z)$ is harmonic function in w everywhere on $\Sigma \backslash \{z\}$.
- If (U, ϕ) is holomorphic chart containing z in Σ, then $g_\Sigma(w, z) + \log|\phi(z) - \phi(w)|$ is harmonic in w for $w \in U$.
- $\lim_{w \to w_0} g_\Sigma(w, z) = 0$ for every $w_0 \in \partial\Sigma$, when the limit is taken from within Σ.

The existence of such a function is not trivial; see e.g. L. V. Ahlfors and L. Sario book [4] for a proof of its existence in the case that $\mathfrak{R}$ is compact and no boundary curve of Σ reduces to a point in $\mathfrak{R}$.

1.7 RBVP, Cauchy-Type Integral Operators

The Riemann boundary value problem (RBVP) can be stated as:

Suppose Ω is domain in $\mathbb{C}$ bounded by a curve Γ, $\Sigma = \mathbb{C} \backslash (\Omega \cup \Gamma)$ and H is a function on Γ. Are there any holomorphic functions h_Ω on Ω and h_Σ on Σ such that

$$H = H_\Sigma - H_\Omega,$$

on Γ, where H_Ω and H_Σ are the boundary value functions of h_Ω and h_Σ on Γ, respectively.

It is customary to call the pair h_Ω and h_Σ a **jump decomposition** of H. This problem can be formulated on Riemann surfaces (compact or open) and it has extensive literature both in the complex plane and in Riemann surfaces; see Y. L. Rodin [49], B. A. Kats and D. B. Katz [27], E. Schippers and W. Staubach [60, 64] (the quasicircle case), and references therein.

In general, the existence of a jump decomposition depends on the regularity of the curve Γ and of the function H on Γ. We particularly interested in the case that H is the boundary values of a function in $\mathcal{D}_{harm}(\Omega)$ (or $\mathcal{D}_{harm}(\Sigma)$) and the curve Γ is analytic or quasicircle.

Example 1.7.1. *Let $\mathfrak{R}$ be a compact Riemann surface and Γ be a quasicircle in $\mathfrak{R}$ which divides $\mathfrak{R}$ into two connected subsurfaces Ω and Σ. It is clear that if H is the boundary values of an element $h \in \mathcal{D}(\Omega)$ on Γ except on a null set in Γ, then a jump decomposition is given by*

$$h_\Omega = -h + a, \qquad h_\Sigma = a,$$

where a is any constant complex value.

This example also shows that to solve the Riemann boundary value problem for the boundary values arising from $\mathcal{D}_{harm}(\Omega)$, when Ω is simply connected, it suffices to take care of elements in $\overline{\mathcal{D}(\Omega)}$. The reason is that since Ω is simply connected domain, every harmonic function on Ω can be decomposed to a holomorphic part plus an anti-holomorphic part, and the problem is trivial for the boundary values of the holomorphic part. That is, elements in $\mathcal{D}(\Omega)$.

The following formulation of the RBVP is what we frequently need in the thesis.

Let Γ be a strip-cutting Jordan curve (quasicircle, or smooth curve in the thesis) on a compact Riemann surface $\mathfrak{R}$ dividing that into two subsurfaces Ω and Σ. Let H

be a function on Γ. The question is are there functions $h_\Omega \in \mathcal{D}(\Omega)$ and $h_\Sigma \in \mathcal{D}(\Sigma)$ (not just holomorphic but of finite Dirichlet semi-norm) with CNT boundary values H_Ω and H_Σ on Γ, respectively, such that

$$H = H_\Sigma - H_\Omega,$$

except possibly on a null set in Γ?

A certain Cauchy-type integral operator can be used to solve the above. The approach that is used here is the one suggested by Royden [54], one of the several approaches to this problem. We start with recalling the definition of the Cauchy-type integral operator after the following remark.

Remark 1.7.2. *In many cases in the thesis we are dealing with a domain G with quasicircle boundary curve Γ. Since quasicircles might not be rectifiable we can not calculate a line integral by traversing through Γ. Therefore, we formulated the integral operators in this case by using the analytic level curves*

$$\Gamma_\epsilon^p = \{w \in G : g_G(w,p) = \epsilon\}$$

of the Green's function of G with singularity at fixed $p \in G$. These are indeed analytic curves for ϵ sufficiently small. These curves approach Γ from within G as $\epsilon > 0$ tends to zero.

Definition 1.7.3 (Cauchy-Type Integral Operator). *Let $\mathfrak{R}$ be a compact Riemann surface and Γ be a Jordan curve dividing $\mathfrak{R}$ into two connected subsurfaces Ω and Σ. Let $g_\Omega(.,p)$ be the Green's function of Ω with singularity at $p \in \Omega$ and Γ_ϵ^p*

be the level curves of $g_\Omega(.,p)$. For $q \in \mathfrak{R}\backslash\Gamma$ fixed, the operator

$$\begin{aligned} J_q(\Gamma) : \mathcal{D}_{harm}(\Omega) &\to \mathcal{D}_{harm}(\Omega \cup \Sigma) \\ h &\to - \lim_{\epsilon \to 0^+} \frac{1}{\pi i} \int_{\Gamma_\epsilon^p} \partial_w g(w; z, q)\, h(w), \end{aligned}$$

where $z \in \mathfrak{R}\backslash\Gamma$, is called the Cauchy-type integral operator with respect to the curve Γ.

Remark 1.7.4. *For a proof of the existence of the above limiting integral, its independence of the choice of $p \in \Omega$, and its boundedness see [62, Section 4.1].*

The output of $J_q(\Gamma)$ is a harmonic function on $\mathfrak{R}\backslash\Gamma = \Omega \cup \Sigma$ and is clearly zero at $z = q$ by the properties of the Green's function $g(w; z, q)$. We may use $\mathcal{D}(\Omega \cup \Sigma)_q$ to indicate this vanishing property at q. The Cauchy-type integral operator for other component of $\mathfrak{R}\backslash\Gamma$ is defined similarly; see [62] for more detail.

In the case that both points q and z are in Σ, by applying Stokes' theorem the operator $J_q(\Gamma)$ can be rewritten as a double integral on Ω. That is,

$$[J_q(\Gamma)h](z) = -\frac{1}{\pi i} \iint_\Omega \partial_w g(w; z, q) \wedge \overline{\partial} h(w), \;\; z \in \Sigma.$$

We may use the notation $[J_q(\Gamma)h]_A$ or simply $J_q(\Gamma)_A h$ (or both once we have a double restriction) to indicate that the output of this operator is restricted on a set A.

A restriction of the Cauchy-type integral operator for holomorphic functions in a collar neighbourhood of a curve Γ in $\mathfrak{R}$ is given in [62, Section 4.3]. Here is the definition and some of its properties.

Definition 1.7.5. *Let $\mathfrak{R}$, Γ, Σ, Ω, and Γ^p_ϵ be as Definition 1.7.3. Let A be a collar neighbourhood of Γ in Ω. Define*

$$[J'_q(\Gamma)h](z) = -\lim_{\epsilon\to 0^+} \frac{1}{\pi i} \int_{\Gamma^p_\epsilon} \partial_w g(w; z, q)\, h(w),$$

where $h \in \mathcal{D}(A)$ and $q \in \Sigma$.

The above Cauchy-type integral operator is defined for holomorphic functions in A whereas the Cauchy-type integral operator in Definition 1.7.3 is defined for harmonic functions in Ω. By the holomorphicity of the integrand the integral is independent of the choice of the analytic curve Γ^p_ϵ provided that ϵ is small enough such that $\Gamma^p_\epsilon \in A$. The following shows a relationship between these two Cauchy-type integral operators [62, Theorem 4.9].

Theorem 1.7.6 (Schippers-Staubach)**.** *Let $\mathfrak{R}$, Γ, Σ, Ω, and Γ^p_ϵ be as Definition 1.7.3. Let A be a collar neighbourhood of Γ in Ω. If Γ is a quasicircle, then*

$$[J'_q(\Gamma)h](z) = [J_q(\Gamma)\mathfrak{G}(A,\Omega)h](z)$$

for every $z, q \in \mathfrak{R}\backslash\Gamma$.

1.8 Kernel Functions on Riemann Surfaces

Consider a finite multiply-connected domain G in $\mathbb{C}$ bounded by n pairwise disjoint closed analytic curves with the Green's function g. Schiffer [55] and

Bergman and Schiffer [10] defined the following two important kernel functions, which we call **kernel functions** of the first and second kind,

$$K(z,\bar{\zeta}) = -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\bar{\zeta}},$$
$$L(z,\zeta) = -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\zeta}.$$

These kernels are analytic functions in each variable on G. The paper of Bergman and Schiffer is devoted to discovering the most important properties of the above two kernel functions. In particular, the decomposition

$$L(z,\zeta) = \frac{1}{\pi(z-\zeta)^2} - l(z,\zeta)$$

for the L kernel, where l for fixed ζ is an analytic function on G, was given. They defined l-transform by

$$Tf(z) = \iint_G l(z,\zeta)\overline{f(\zeta)}\,dA_\zeta$$

where $f \in A^2(G)$, the Bergman space of G. It was shown that Tf is analytic everywhere in $\mathbb{C}$ except on the boundary of G.

The above kernel functions may be extended to compact Riemann surfaces with small changes. For a compact Riemann surface $\mathfrak{R}$ with the Green's function $g(w;z,q)$ having singularities at z and q, the following two bi-differential forms, also called kernel functions, are well-known. The **Schiffer kernel** is defined by

$$L_{\mathfrak{R}}(z,w) = \frac{1}{\pi i}\partial_z\partial_w g(w;z,q),$$

and the **Bergman kernel** is defined by

$$K_{\mathfrak{R}}(z,w) = -\frac{1}{\pi i}\partial_z\overline{\partial}_w g(w;z,q).$$

The following theorem summarizes some important properties of these kernels; see M. Schiffer and D. Spencer [58, Chapter 4] or [62, Proposition 3.3] for a proof.

Theorem 1.8.1. *Let $\mathfrak{R}$, $g(w;z,q)$, $L_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ be as above. Then*

1. *$L_{\mathfrak{R}}$ and $K_{\mathfrak{R}}$ are independent of the choice of the points q and w_0.*
2. *$L_{\mathfrak{R}}(z,w)$ is holomorphic in both variables w and z, except for a pole of order two when $w = z$.*
3. *$K_{\mathfrak{R}}(z,w)$ is holomorphic in z for fixed w, and is anti-holomorphic in w for fixed z.*
4. *$L_{\mathfrak{R}}(w,z) = L_{\mathfrak{R}}(z,w)$.*
5. *$K_{\mathfrak{R}}(w,z) = -\overline{K_{\mathfrak{R}}(z,w)}$.*

If Σ is a bordered Riemann surface with Green's function $g_\Sigma(w,z)$ the above two kernels can be defined simply by substituting g_Σ instead of g. Then Theorem 1.8.1, except its first item, is valid for $K_\Sigma(z,w)$ and $L_\Sigma(z,w)$. Moreover, we have two more identities for these kernels explained here.

The reproducing property of the Bergman kernel on Σ can be written as

$$\iint_\Sigma K_\Sigma(z,w) \wedge \alpha(w) = \alpha(z)$$

for all $\alpha \in A(\Sigma)$.

The following identity concerning the Schiffer kernel was stated by Schiffer for finite domains in $\mathbb{C}$ bounded by analytic curves [55, Equation 17]. Schippers and Staubach [62, Theorem 3.7] generalized it to domains on Riemann surfaces bordered by a strip-cutting Jordan curve. That is,

$$\iint_{\Sigma} L_{\Sigma}(z,w) \wedge \overline{\alpha(w)} = 0$$

for all $\overline{\alpha} \in \overline{A(\Sigma)}$.

1.9 Teichmüller Space

Let $\mathfrak{R}$ be a Riemann surface with no boundary. Consider the set of all quasiconformal mappings f of $\mathfrak{R}$ onto other Riemann surfaces (see Section 1.3 for a definition). Two quasiconformal mappings f_1 and f_2 are said to be equivalent whenever $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $f_1(\mathfrak{R})$ onto $f_2(\mathfrak{R})$. We will use $f_1 \sim f_2$ to indicate this relation.

In the case that $\mathfrak{R}$ has boundary, there is a definition for the Teichmüller space of $\mathfrak{R}$ in the literature. Nevertheless, in this thesis we restrict to some type of surfaces with boundary, namely bordered Riemann surfaces in the sense of Ahlfors and Sario [4, II. 3A].

On this surfaces, the only change to the above definition is $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $f_1(\mathfrak{R})$ onto $f_2(\mathfrak{R})$ “rel boundary". “Homotopic rel boundary" means the homotopy is constant on the boundary of $\mathfrak{R}$.

Definition 1.9.1 (Teichmüller Space)**.** *The Teichmüller space $T_{\mathfrak{R}}$ of the Riemann surface $\mathfrak{R}$ is the set of all the equivalence classes of quasiconformal mappings of $\mathfrak{R}$.*

In other words, consider the set of all $(\mathfrak{R}, f_1, \mathfrak{R}_1)$, where $\mathfrak{R}_1 = f_1(\mathfrak{R})$ for quasiconformal map f_1 on $\mathfrak{R}$. Two $(\mathfrak{R}, f_1, \mathfrak{R}_1)$ and $(\mathfrak{R}, f_2, \mathfrak{R}_2)$ are equivalent if f_1 and f_2 satisfy the above mentioned equivalence relation. $[\mathfrak{R}, f_1, \mathfrak{R}_1]$ may be used to indicate the equivalence class $(\mathfrak{R}, f_1, \mathfrak{R}_1)$ in $T_{\mathfrak{R}}$. Therefore, one symbolically has

$$\begin{aligned} T_{\mathfrak{R}} &= \{(\mathfrak{R}, f_1, \mathfrak{R}_1)\} / \sim \\ &= \{[\mathfrak{R}, f_1, \mathfrak{R}_1] : f_1 \textit{ is a quasiconformal map from } \mathfrak{R} \textit{ onto } \mathfrak{R}_1\}. \end{aligned}$$

As a special case of this definition, the universal Teichmüller space, modelled on $\mathbb{D}^-$ or equivalently on the upper half plane $\mathbb{H}$, can be defined.

Definition 1.9.2 (Universal Teichmüller Space)**.** *If* $\mathfrak{R} = \mathbb{D}^-$, *then* $T(1) := T_{\mathfrak{R}}$ *is called the universal Teichmüller space.*

The universal Teichmüller space was first introduced by L. Bers; see [76], also O. Lehto [31, Chap V, Sec 2.1]. There are other equivalent models for the universal Teichmüller space. We explain one of them here. First note that by Caratheódory's theorem, a quasiconformal map can be extended homeomorphically to its boundary if the boundary is a quasicircle.

Theorem 1.9.3. *The space* $T(1)$ *can be identified with the following space*

$$\{f : \mathbb{H} \to \mathbb{H} \ : \ f \textit{ is a quasiconformal map and fixed } 0, 1, \infty\} / \sim .$$

where $f_1 \sim f_2$ *if and only if* $f_1|_{\mathbb{R}} = f_2|_{\mathbb{R}}$.

There are some other (equivalent) ways to define the universal Teichmüller space. In Chapter 4 we will talk more about them. Each Teichmüller space is a complex manifold; for compact Riemann surfaces they are of finite dimension; see [31, 37]. It is well-known that the universal Teichmüller space contains all the Teichmüller spaces as complex sub-manifolds.

In Chapter 4 we will see another type of Teichmüller spaces, namely the rigged Teichmüller space, corresponding to bordered surfaces for which the borders have parametrization using conformal maps.

1.10 Marking and Period Matrices

Let $\mathfrak{R}$ be a compact Riemann surface of genus $\mathfrak{g}$. Consider a canonical homology basis $\{a_1, \ldots, a_{\mathfrak{g}}, b_1, \ldots, b_{\mathfrak{g}}\}$ on the surface $\mathfrak{R}$. This basis may also be called a **marking** of $\mathfrak{R}$. By a canonical basis we intuitively mean a_k and a_j (similarly b_k and b_j) do not intersect if $k \neq j$; a_j interests b_k if and only if $j = k$. See Farkas and Kra [17, Subsection III.1] for a precise definition.

It is known that the set of all holomorphic 1-forms on $\mathfrak{R}$ is a complex vector space of dimension $\mathfrak{g}$ [17, Proposition III.2.7]. In particular, there is no holomorphic 1-form on the Riemann sphere.

Now we have the following important theorem [17, Proposition II.2.8].

Theorem 1.10.1. *Let $\mathfrak{R}$ be as above. There exists a unique basis $\{\alpha_1, \ldots, \alpha_{\mathfrak{g}}\}$ for the space of holomorphic 1-forms with the property*

$$\int_{a_j} \alpha_k = \delta_{jk}.$$

Furthermore, for this basis, the matrix $\pi = (\pi_{jk})$ with

$$\pi_{jk} := \int_{b_j} \alpha_k,$$

is symmetric with positive imaginary part.

The above theorem says that one can assign a $\mathfrak{g} \times \mathfrak{g}$ matrix π to each compact Riemann surface of genus $\mathfrak{g}$ called the **period matrix**, such that

$$\pi = \pi^t, \qquad Im(\pi) = \frac{1}{2i}(\pi - \overline{\pi}) > 0.$$

By t we mean the transpose operator, and by > 0 we mean the positive definiteness. These are consequences of bilinear relations of Riemann. This assignment may be called the **classical period map** in the thesis.

We may use $\pi_{\mathfrak{g}}$ to include the genus of $\mathfrak{R}$ when it is necessary. It can be shown that this assignment is invariant under the equivalence relation defined in Section 1.9 for Teichmüller space, see S. Nag [37, Section 4.1]. Therefore, period matrices can be considered as operators on the Teichmüller space of $\mathfrak{R}$, which maps the Teichmüller space into a space called the **Siegel upper half space** $D_{\mathfrak{g}}$ of genus $\mathfrak{g}$. The Siegel upper half space is a complex manifold, in fact it is an open subset of $\mathfrak{g} \times \mathfrak{g}$ symmetric matrices with entries in $\mathbb{C}$ [17, Subsection VI. 1.1].

Chapter 2

Grunsky Coefficients and Faber Polynomials, Old and New

It is well-known in function theory that to each simply connected domain G with the Riemann map ϕ, one can assign a set of polynomials called the Faber operators. These polynomials can be used to approximate holomorphic functions on $\mathrm{cl}(G)^c$, the complement of $\mathrm{cl}(G)$. They can also be used to define the Grunsky coefficients corresponding to the map ϕ. These coefficients, and some inequalities corresponding to them, have had many applications in classical complex analysis and the analytic theory of Teichmüller spaces. In this chapter, we review the history and the literature of both the Faber polynomials (and operator) and the Grunsky coefficients (and operator). We recall some of their definitions for different choices of domains and function spaces. We see how they were used to define a period maps on the universal Teichmüller space. Additionally, some generalizations of these polynomials and coefficients to Riemann surfaces, other than the Riemann sphere, are mentioned. At the end of the chapter, we talk more about the problem of the thesis in this context.

2.1 Faber Polynomials, Series, and Operator

We give a historical overview of the subject and also provide a brief summary of the more recent papers relevant to my research problem. Of course, we do not claim to be comprehensive. Before defining the Faber polynomials, we recall that by the celebrated Riemann mapping theorem there is a correspondence between the set of all normalized univalent functions defined on $\mathbb{D}$ and the set of all simply connected proper subsets of $\mathbb{C}$. This means that each univalent function on $\mathbb{D}$ can be viewed as a simply connected proper subset of $\mathbb{C}$ and vice versa [14, Chp3].

There are many ways to define Faber Polynomials associated to a conformal map f. These definitions are often equivalent but sometimes with subtle differences. The reader may find the following general comments helpful.

To define Faber polynomials on the complex plane or the Riemann sphere, we deal with three objects:

• A subset G, which itself could be a simply connected domain in $\overline{\mathbb{C}}$, such that G (or $\mathrm{cl}(G)$) has simply connected complement. The Faber polynomials of the domain G are associated to the Riemann map of the *complement* of G.

• A simple closed curve Γ, usually the boundary of G which is positively oriented with respect to G^c.

• A conformal map ϕ (or $f, \Phi, \Psi, \dots$) which either ϕ or ϕ^{-1} maps the simply connected domain $\mathrm{cl}(G)^c$ with boundary Γ conformally onto $\mathbb{D}$ (or $\mathbb{D}^-$). The map ϕ may be called the generating map for the Faber polynomials.

In the literature, the Faber polynomials are variously said to be associated to a conformal map or to a domain. On the other hand, it is known that

G. Faber himself defined them as being associated with an analytic curve [16]. That is, Γ is the boundary of $Im(\phi)$ (or $Dom(\phi)$), see J. L. Ullman [82] or P. K. Suetin [75] for more detail.

We briefly talk about some of the aspects of the Faber polynomials and their importance and show some other ways to define them. Let us start with the following definition. Then, in sequence, two other definitions of these polynomials will be provided.

Definition 2.1.1 (Faber Polynomials I)**.** *Let f be a holomorphic function defined on a neighbourhood of zero, such that $f'(0) \neq 0$. For integers $n \geq 0$, let*

$$(f^{-1}(z))^{-n} = \sum_{k=-n}^{\infty} \alpha_k^{(n)} z^k$$

be the Laurent series of $(f^{-1})^{-n}$ in the disc $B(0, r)$ for $r > 0$ sufficiently small. Then the n-th Faber polynomial of f, $\Phi_n(f)$, is defined by

$$\Phi_n(f)(z) = \sum_{k=-n}^{-1} \alpha_k^{(n)} z^k.$$

Remark 2.1.2. *The (n) on top $\alpha_k^{(n)}$ indicates the coefficients $\alpha_k^{(n)}$ are correspondence to the n-th power of $\frac{1}{f^{-1}}$. The notation $\Phi_n(f)$ should not be confused with the composition of Φ_n with f. This is used to indicate the dependence of Φ_n's on the map f.*

Faber polynomials provide us a systematic way to define and calculate the Grunsky coefficients of a univalent function (defined shortly here and with more detail in the next section). To see this we can recall for example I. Schur [65] work here. He introduced the following definition of Faber polynomials.

Definition 2.1.3 (Faber Polynomials II)**.** *Let*

$$f(z) = z + a_1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots = z \sum_{k=0}^{\infty} a_k z^{-k} = zg\left(\frac{1}{z}\right), \qquad a_0 = 1,$$

be a formal power series (no convergence assumption is made). There is a unique polynomial P_m of degree $m, m = 1, 2, \ldots,$ such that

$$(P_m \circ f)(z) = z^m + \sum_{k=1}^{\infty} c_{mk} \frac{1}{z^k}. \tag{2.1}$$

P_m is called the m-th Faber polynomial of f.

The existence and the uniqueness can be proven by recursion on m assuming $P_0 \equiv 1$. Shur could show c_{mk} in (2.1) (called the Grunsky's coefficients) are polynomials of a_m with non-negative integer coefficients; he provided a complicated algebraic formula for c_{mk} in terms of these coefficients. He also proved that

$$k\, c_{mk} = m\, c_{km}$$

for each m, k, which is known as the Grunsky's identity (or the Grunsky's theorem).

Another Matrix-Algebraic approach to the Faber polynomials, can be found in E. Jabotinsky [25]. It takes advantage of the representation of a holomorphic function $f(z)$ by an infinite matrix (f_{nm}) defined as follows. Let $\rho > 0$ and f be a holomorphic function on $|z| < \rho$ with the expansion

$$f(z) = \sum_{n=1}^{\infty} f_n z^n, \quad f_1 \neq 0.$$

By raising f to integer powers m we generate the coefficients f_{mn} as follows

$$[f(z)]^m = \sum_{n=-\infty}^{\infty} f_{mn} z^n$$

which is valid on $|z| < \rho'$, for some $\rho' > 0$. It is clear that $f_{mn} = 0$ for $n < m$. Jabotinsky provided an explicit formula for the n-th Faber polynomial of $f(z)$ based on this infinite matrix. Here is his theorem [25, Theorem III].

Theorem 2.1.4 (Jabotinsky). *The m-th Faber polynomial of the map $f(z)$ is*

$$F_m(t) = \phi_{-m0} + \sum_{p=1}^{m} \frac{m}{p} f_{pm} t^p,$$

where (ϕ_{mn}) is the matrix that represents the inverse of the function f.

Remark 2.1.5. *By the help of an identity relating the coefficients of ϕ and f, it can be shown that the above equation is equivalent to Definition 2.1.1 of the Faber polynomials.*

Another important aspect of Faber polynomials is their role in approximation theory. One may approximate holomorphic function g on a domain G in terms of Faber polynomials generated by the Riemann map of the complement of G. The analytic properties of this approximation, in general, depend on the regularity of the boundary of G and the analytic and the boundary behaviour of the function g on G and ∂G, respectively. See e.g. H. Tietz [79] when Γ is an analytic curve and also P. K. Suetin [74] for more varieties of the curve Γ and the function g.

Despite the approximation property they do not form an orthogonal subset of polynomials on their domain of definition.

H. Tietz in his illuminating paper [79] by the method of "Laurent Decomposition" compared a Faber series

$$\sum_{n=0}^{\infty} a_n F_n$$

with the power series (in ϕ)

$$\sum_{n=0}^{\infty} a_n \phi^n,$$

having the same coefficients.

More precisely, let Γ be an analytic curve in $\mathbb{C}$ dividing that into $I(\Gamma)$ and $A(\Gamma)$ to which Γ is positively and negatively oriented, respectively. Let ϕ be the Riemann map, mapping $A(\Gamma)$ conformally onto $\{z \in \mathbb{C} : |z| > k\}$ for some $k > 0$, with $\phi(\infty) = \infty$. He defined the Faber polynomial F_n relative to the curve Γ as follows.

Definition 2.1.6 (Faber Polynomial III)**.** *Let Γ and ϕ be as above. For $n \geq 0$ the n-th Faber polynomial of Γ is defined as*

$$F_n = L(\phi^n),$$

where by L we mean the continuous linear operator

$$[L(\psi)](z) = \frac{1}{2\pi i} \int_\Gamma \frac{\psi(\zeta)}{\zeta - z}\, d\zeta, \quad z \in I(\Gamma)$$

for every holomorphic function ψ on (a collar neighbourhood of) Γ.

By "Faber series $\sum_{n=0}^{\infty} a_n F_n$" here we mean the series is convergent uniformly on compact subsets of $I(\Gamma)$. Tietz showed the following relationship between these two series [79, Theorem 2].

Theorem 2.1.7 (Tietz). *Let $\sum_{n=0}^{\infty} a_n F_n$ be a Faber series, let $F(z)$ denote its sum in $I(\Gamma)$, and let $g(z) = \sum_{n=0}^{\infty} a_n \phi^n$. Then the function $F(z) - g(z)$ can be continued analytically throughout $A(\Gamma) \cup \Gamma$ and vanishes at ∞.*

This shows that when the boundary curve Γ is analytic, the Faber series behaves similar to power series in ϕ. Given F analytic on $I(\Gamma)$, the paper also shows that F has a unique convergent Faber series for $z \in I(\Gamma)$ and a Laurent expansion in ϕ for z close to Γ [79, Theorem 1].

T. Kövari and Ch. Pommerenke [29] tried to answer the question of when the Faber series of f converges to f uniformly on G, in the case that G is a continuum with simply connected complement in $\overline{\mathbb{C}}$. Here is one of their results when G is a Jordan domain.

Theorem 2.1.8. *(Kövari and Pommerenke) Let G be a closed Jordan domain, whose boundary Γ is of bounded rotation and has no zero exterior angles. Suppose that f is analytic in the interior of G, continuous on G, and moreover satisfies Dini's condition*

$$\int_0^h \frac{\omega_f(x)}{x}\, dx < \infty$$

for some $h > 0$, where ω_f is the module of the continuity of f. Then the Faber (series) expansion of f converges uniformly on G to f.

For a similar result but with different choices of the regularity of the boundary curve Γ, less smooth but still rectifiable, see F. D. Lesley, V. S. Vinge and S. E. Warschawski [33].

There are some connections between Faber series and Fourier series. J. H. Curtiss [13] shows some interesting connections of this kind. He pointed out the difficulties that may arise to prove convergence theorems concerning Faber series to its generating function on G when neither f nor ϕ can be extended analytically pass the boundary curve Γ. He also (by investigating S. Ya A'lper and V. V. Ivanov [7] and Kövari and Pommerenke [29]) showed that the behaviour of Faber series of f and the Fourier series of $f(\phi(e^{i\theta}))$ are tightly related and is a tool in investigating different regularities for the curve Γ and the function f.

The operator theoretic point of view of Faber polynomials shows another aspect of them. The Faber operator, like the Faber polynomials, has many different definitions and settings. Putting analytical assumptions aside temporarily, a definition of the Faber operator can be given as follows.

Definition 2.1.9 (Faber Operator I). *If Ψ is the Riemann map of $\mathbb{D}^-$ onto $\mathbb{C}\backslash K$ for K a compact subset of $\mathbb{C}$ with simply connected interior, then the Faber operator is defined by*

$$(Tf)(z) = \frac{1}{2\pi i} \int_{|w|=1} f(w) \frac{\Psi'(w)}{\Psi(w) - z} \, dw. \tag{2.2}$$

The properties of the Faber operator clearly depend on the function space on which it operates and also the regularity of the boundary curve $\Gamma = \partial K$. We now go through some examples of the Faber operator and a short history of that.

J. M. Anderson [8] discussed about the boundedness of the Faber operator and its inverse on many function spaces including the disc algebra. He showed some relationships between the Faber operator and the approximation by polynomials or rational functions on compact subsets of $\mathbb{C}$. In particular, he posted the following conjecture

Let G be closed Jordan domain with rectifiable boundary in $\mathbb{C}$. Then the Faber operator T is an isomorphism of B_p onto $B_p(G)$ for $1 < p < \infty$, where B_p and

$B_p(G)$ are the Besov spaces of analytic functions on the closed unit disc and on G, respectively.

Later on H. Y. Wei, M. L. Wang and Y. Hu [85] disproved Anderson's conjecture for the case $p = 2$. They assumed the boundary of G is rectifiable and $p = 2$. By these assumptions, they proved that the Faber operator is an isomorphism if and only if ∂G is a rectifiable quasicircle.

A rather complete book on Faber polynomials, Faber series and Faber operator is the one was written by Suetin [75]. Chapter VII of the book, seems to be one of the first systematic treatments towards the Faber operator.

Remark 2.1.10. *The book of Suetin [75] was originally published in Russian in 1984. The translation, published in 1998, has an extensively updated bibliography in addition to its original one.*

Definition 2.1.11 (Faber Operator II)**.** *Let G be a finite domain bounded by rectifiable Jordan curve Γ and let $\varphi(t)$ be an analytic function in $\mathbb{D}$ and have angular boundary values almost everywhere on $\mathbb{S}^1$. Suppose Φ maps $\mathbb{D}^-$ to $\mathrm{cl}(G)^c$ and $\psi = \Phi^{-1}$. If $\psi'(t) \in H_2$ in $\mathbb{D}^-$, then*

$$\begin{aligned} F_0 : H_2 &\longrightarrow Hol(G) \\ \varphi &\longrightarrow \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\Phi(\zeta))}{\zeta - z}\, d\zeta, \end{aligned}$$

for $z \in G$, is called the Faber operator corresponding to the domain G (or the curve Γ).

As was pointed out in [75], all we need to define the above operator is conditions on Γ and φ such that those conditions ensure the function $\varphi \circ \Phi$

is integrable on Γ. The choice of $\psi'(t) \in H_2$ in $\mathbb{D}^-$ and H_2 as the domain is just one of the many possibilities.

Then the book provides some examples of function spaces such that the Faber operator would be an isomorphism if it is defined on them and a few sufficient conditions which make the Faber operator a bounded isomorphism. It, however, does not include anything concerning Faber operator (also Faber series) for domains bounded by quasicircles; see [75, Page 285].

The work of A. Çavuş [11] concerned approximation of functions in the Bergman space $A^2(G)$ of a finite domain G in $\mathbb{C}$ with a quasicircle boundary, using what was called the generalized Faber series. By generalized Faber series there he means a series in F'_m, the derivative of Faber polynomials corresponding to G.

Let $0 \in G$, and ϕ be the conformal map which maps $\mathbb{C} \backslash \mathrm{cl}(G)$ onto $\mathbb{D}^-$ with

$$\phi(\infty) = \infty \textit{ and } \lim_{z \to \infty} \frac{\phi(z)}{z} > 0;$$

let also Ψ be the inverse of ϕ.

To each element $f \in A^2(G)$ a generalized Faber series $\sum_{m=1}^{\infty} a_m(f) F'_m(z)$ was assigned. The coefficient $a_m(f)$, called the generalized Faber coefficient, is defined by

$$a_m(f) = -\frac{1}{\pi} \iint_{\mathbb{D}^-} \frac{f(\mathcal{R}_\Gamma(\Psi(w)))\, \overline{\Psi'}(w)}{w^{m+1}} \frac{\partial \mathcal{R}_\Gamma}{\partial \bar{\zeta}}(\Psi(w))\, dA_w, \quad m = 1, 2, \ldots .$$

where $\mathcal{R}_\Gamma$ (y in the paper's notation) is a differentiable k-quasiconformal reflection with respect to Γ and $\zeta = \Psi(w)$.

Theorem 2.1.12 (Çavuş). *Let G, ϕ, and f be as above. If*

$$\sum_{m=1}^{\infty} a_m(f) F'_m(z)$$

is the generalized Faber series of f, then $\sum_{m=1}^{\infty} a_m(f) F'_m(z)$ converges uniformly to f on every compact subset of G.

We emphasize that ∂G is quasicircle in above theorem, making the reflection well-defined. In other words, if Ψ admit a quasiconformal extension to $\mathbb{C}$, then each element in $A^2(G)$ can be expanded in a generalized Faber series, converging uniformly and absolutely on compact subsets of G. Nothing was mentioned about the converse of this theorem. The converse was later proven by Y. L. Shen [68]. Another important point is the independence of the expansion on the choice of the reflection $\mathcal{R}_\Gamma$ [11, page 33].

D. Gaier [18] defined a Faber operator operating like equation (2.2) but from $A(\overline{\mathbb{D}})$ to $A(\overline{G})$ where $A(\overline{G})$ is the Banach space of functions holomorphic in G and continuous on $\mathrm{cl}(G)$ for any Jordan domain G in $\mathbb{C}$. He could show that if $\Gamma = \partial G$ is piecewise Dini-smooth the Faber operator is bounded and gave an example of a rectifiable quasicircle Γ for which the operator is unbounded.

J. Müller [35] extends the Faber operator to $C(\mathbb{S}^1)$. That is, the space of continuous functions on $\mathbb{S}^1$, rather than defined on $A(\overline{\mathbb{D}})$. Assume G is a compact subset of $\mathbb{C}$ (having at least two points) with simply connected complement in $\overline{\mathbb{C}}$ generating the Faber polynomials. His operator maps $C(\mathbb{S}^1)$ to *Har*(G), where *Har*(G) is the Banach space of functions ϕ harmonic on $int(G)$ and continuous on G, endowed with sup-norm

$$\|\phi\|_G = \sup_{z \in G} |\phi(z)| .$$

He showed that if Γ is of bounded boundary secant variation, then the Faber operator is a bounded operator.

Shen [68] defined an operator on $\ell^2(\mathbb{C})$, the space of complex square summable sequences, into a function space of holomorphic functions. To be more precise, we have to recall his definitions here. Let us start with the following definition of the Faber polynomials given in [41, Chapter 3] with slightly different formulation.

Definition 2.1.13 (Faber Polynomials IV). *Let $g \in \Sigma$ where Σ is the one in Section 1.2.1. Consider the expansion*

$$\log \frac{g(\zeta) - w}{\zeta} = -\sum_{n=1}^{\infty} \frac{1}{n} F_n(w)\, \zeta^{-n}$$

around the point ∞ for $w \in \mathbb{C}$. Then $F_n(w)$ is called the n-th Faber polynomial of g ($F_0(w) \equiv 1$).

Remark 2.1.14. *In the above definition, the function g need not be analytic on $\mathbb{D}^-$ nor be in Σ_0 (defined in Section 1.2.1). All we need is g to be analytic in a neighbourhood of the point ∞ and univalent near that point.*

For $f \in \Sigma_0$ define $D^* = Im(f)$ and $E = \overline{\mathbb{C}} \backslash D^*$ and D is the interior of E. D could be empty but throughout his paper he assumed it is not. $\mathcal{AD}(D)$ in his paper is $\mathcal{D}(D)$ in the thesis notation. Here is his definition of what we will call the **sequential Faber operator**.

Definition 2.1.15 (sequential Faber operator)**.** *The operator P is defined by*

$$P : \ell^2(\mathbb{C}) \to \mathcal{AD}(D)$$
$$x \to P_x$$

where

$$x = (x_n), \quad P_x(w) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n}} F_n(w), \quad w \in D,$$

and $F_n(w)$ is the n-th Faber polynomials of f.

Shen did not call P the Faber operator. It's worth mentioning that by defining the operator on $\ell^2(\mathbb{C})$ (rather than as an integral over the boundary) he avoids making any assumptions on the regularity of the boundary.

It was shown that P is a bounded operator and $\|P\| \leq 1$. Finally, he gave the following beautiful theorem which reveals more connections between the boundary behaviour of D and the properties of the operator P.

Theorem 2.1.16 (Shen)**.** *The operator $P : \ell^2(\mathbb{C}) \to \mathcal{AD}(D)$ is a bounded isomorphism if and only if the map f can be extended to a quasiconformal map into the unit disk $\mathbb{D}$.*

In other words, P is a bounded isomorphism if and only if ∂D is a quasicircle or equivalently D is a quasidisc. Shen's result gives a converse to Çavuş' in a sense.

V. V. Napalkov, Jr. and R. S. Yulmukhametov [84] defined a Hilbert transform H mapping $A^2(G)$ into $A^2(\mathbb{C}\backslash \mathrm{cl}(G))$, where G is an open, simply connected subset of $\overline{\mathbb{C}}$ whose boundary is a Jordan curve. With some small changes in the domain, the operator H is exactly the Schiffer operator formulated in [64]. They showed that H is bounded and injective operator; H

is surjective if and only if the boundary of G is quasicircle. This is the first proof showing that the Schiffer operator is a bounded isomorphism if and only if the boundary of G is a quasicircle.

Y. E. Yidirir and R. Çetintaş [86] defined the Faber operator F_0 from the Hardy-Orlicz space $H_M(\mathbb{D})$ into the Smirnov-Orlicz space $E_M(G)$. Here G is a bounded domain in $\mathbb{C}$ with Dini-smooth boundary Γ. They also defined the inverse Faber operator and proved that under above condition on G, both F_0 and its inverse are bounded operators. That is, F_0 is an isomorphism.

E. Schippers and W. Staubach [64] formulated the Faber operator as an operator between the normalized Dirichlet spaces $\mathcal{D}_*(\mathbb{D}^-)$ and $\mathcal{D}_*(\Omega^-)$. More precisely, let Γ be a Jordan curve not passing through ∞ dividing the Riemann sphere into two simply connected domains $\Omega^\pm$ with $\infty \in \Omega^-$. They defined $\mathcal{H}_\pm(\Gamma)$ to be the set of boundary values of functions in $\mathcal{D}_{harm}(\Omega^\pm)$ in a sense that can be shown to be like the CNT boundary limits, see Section 1.4 for the definition. They used a family of curves approaching the possibly non-rectifiable boundary or an equivalent L^2 integral. Here is one of their results [64, Theorem 2.5].

Theorem 2.1.17 (Schippers and Staubach)**.** *Let Γ and $\Omega^\pm$ be as above. If Γ is a quasicircle, then $\mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$.*

That is if $h_1 \in \mathcal{D}_{harm}(\Omega^+)$ with CNT boundary value function H_1, then there exists $h_2 \in \mathcal{D}_{harm}(\Omega^-)$ with CNT boundary value function H_2 such that $H_1 = H_2$ except possibly on a set of logarithmic capacity zero. Therefore for quasicircles $\mathcal{H}(\Gamma) := \mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$ definition makes sense. As was mentioned in Section 1.4, they generalized their result, see [62, Theorem 4.10] and the note after that in the paper.

Remark 2.1.18. *Schippers and Staubach used different setting for the boundary limits in [64]. That is, in the Osborn sense. The CNT boundary values obtained are equivalent to the boundary values obtained in their original formulation.*

They showed that the Faber operator, defined on the Dirichlet space, is a bounded isomorphism precisely for quasicircles.

Remark 2.1.19. *Assume we define the Faber polynomials $\Phi_n(f)$ corresponding to the map $f : \mathbb{D} \to \Omega^+$ by Definition 2.1.1. It was shown in [64] that $\Phi_n(f)$ maps $\{z^{-n} : n > 0\}$, which is a dense subset of harmonic functions on $\mathbb{D}^-$, to Faber polynomials. In another word,*

$$\Phi_n(f) = \mathbf{I}_f(z^{-n}) \quad ; \quad n > 0.$$

Later on, Radnell, Schippers and Staubach [46] generalized the Faber operator to the case of arbitrary n ($n \geq 1$) conformal maps $f_k : \mathbb{D} \to \Omega_k^+$ where Ω_k^+ are non-overlapping domains of the Riemann sphere, i.e. a genus zero Riemann surface.

Very recently, M. Müller in his thesis [36] investigated a harmonic Faber operators correspondence to G, a simply connected domain in $\mathbb{C}$. Müller tried to find a set of harmonic polynomials in G approximating harmonic functions in G by using the approximation property of

$$\mathcal{T} = \{z^n\}_{n \geq 0} \cup \{\bar{z}^n\}_{n>0},$$

for the harmonic functions in $\mathbb{D}$. This is like the Faber idea about approximation of elements in *Hol*(G). He showed that when ∂G is an analytic Jordan curve, the harmonic Faber operator, primarily defined on $\mathcal{T}$, has continuous extension. It will be an isomorphism with respect to the locally uniformly convergent topology between the space of the harmonic functions in $\mathbb{D}$ onto the space of harmonic functions in G. Similar results for the case that ∂G is a Dini-continuous curve were proven.

We have seen some examples of how to define the Faber operator and the space it acts on. We now talk about a point which is common in all the approaches. Assume a Faber operator $\mathbf{I}_f$ is defined between the function space A and B (e.g. Dirichlet space, Bergman space, ...) each one corresponding to a domain on some Riemann surfaces $\mathfrak{R}$ and $\mathfrak{S}$. Assume for some analytic condition, this operator is a bounded isomorphism. Then this isomorphism maps a dense subset in A to a dense subset of B; therefore approximation by meromorphic functions $\mathbf{I}_f(A)$ makes sense. In another word, the role of "$\mathbf{I}_f$ is an isomorphism" is to transfer a dense subset of functions in A to a dense subset of functions in B. This is the main reason that we are looking for conditions which makes our Faber operator a bounded isomorphism. In other words, surjectivity is approximability, and injectivity is uniqueness of the approximation.

In the next section, once we defined the Grunsky coefficients of a conformal map, we will see more applications of the Faber polynomials and/or operators.

2.2 Grunsky Coefficient, Grunsky Inequality and Grunsky Operator

Grunsky coefficients provide much information about holomorphic functions. For example, they can be used to state a necessary and sufficient condition for a given holomorphic function on a domain to be univalent. There is no room to talk about all the known important aspect of these coefficients and their generalizations here. We, therefore, try to focus on those aspects that are more relevant to the thesis results; see [14, Chap 4], [41, Chap 3, 4] and [20] for more detail.

Like the Faber polynomials, the Grunsky coefficients or matrix of a holomorphic function f can be defined in different settings. The original formulation of H. Grunsky [21] for what later called the Grunsky inequality can be summarized as follows. Let

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

be a holomorphic function on a multiply-connected subset $\mathfrak{B}$ of $\overline{\mathbb{C}}$ containing the point ∞. Let also the Grunsky coefficients c_{mn} of f be defined by the equation (2.1). He first proved that for every $m \geq 1$ there exist functions A_m and B_m on $\mathfrak{B}$ with the following series expansions about the point ∞;

$$A_m(z) = z^m + \frac{a_{m1}}{z} + \dots,$$
$$B_m(z) = \frac{b_{m1}}{z} + \frac{b_{m2}}{z^2} + \dots,$$

such that on each boundary curve Γ_k, $k = 1, 2, \ldots, n$, of $\mathfrak{B}$ there exists a unique complex constant d_{mk} satisfying

$$B_m = \overline{A}_m + d_{mk}.$$

He then showed that both matrices $\{n\, a_{mn}\}$ and $\{n\, b_{mn}\}$ are symmetric. Here is the original Grunsky inequality theorem [21, Section II, 4].

Theorem 2.2.1 (Grunsky). *Let f, $\mathfrak{B}$, a_{mn}, b_{mn} and c_{mn} be as above. Then the function f is univalent in $\mathfrak{B}$ if and only if*

$$\left| \sum_{\mu,\nu=1}^{m} \nu \left(c_{\mu\nu} - a_{\mu\nu} \right) x_\mu \, x_\nu \right| \le \sum_{\mu,\nu=1}^{m} \nu b_{\mu\nu} \, x_\mu \, \overline{x}_\nu; \quad m = 1, 2, \ldots,$$

for any finite arbitrary complex variables x_μ. In particular, when $\mathfrak{B}$ is $\mathbb{D}^-$ we have

$$\left| \sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu} \, x_\mu \, x_\nu \right| \le \sum_{\mu,\nu=1}^{m} \nu \left| x_\mu \right|^2.$$

Finding the Grunsky coefficients of holomorphic map is, in general, a difficult task. A method was given to find an explicit formula for these coefficients. After Shur [65] and J. A. Hummel [23], P. G. Todorov [81] provides a less complicated formula to explicitly calculate the Grunsky coefficients of a function f when its Taylor expansion is given. Both Hummel and Todorov used the Grunsky coefficients of the function $f(z^p)^{\frac{1}{p}}, p \ge 1$ in their calculations. On the other hand, some machine computations, to find finitely many of the coefficients, were also done e.g. A. R. Miller [34]. However, the analytic methods showed their advantages and also the complicated nature of these coefficients.

M. Schiffer [56] provided a generating function for the Grunsky coefficients of a holomorphic function and derived the Grunsky inequality via his variational method. Here is the definition given by Schiffer, taken from the Pommerenke's book [41, Chap 3].

Definition 2.2.2 (Grunsky Coefficients)**.** *Let*

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$$

be an analytic function in some neighbourhood of ∞ *and univalent for* $|z| > R$*, for some* $R > 0$*. This ensures that the left hand side of the following equality is a holomorphic functions on* $|z|, |\zeta| > R$*. Therefore, the left hand side has an expansion*

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = -\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl}\, z^{-k} \zeta^{-l}. \tag{2.3}$$

The coefficients $\{c_{kl}\}$ *are called the Grunsky coefficients or matrix of* g*.*

These coefficients are independent of b_0, satisfying $c_{kl} = c_{lk}$ and $c_{k1} = b_k$ for $k, l \geq 1$.

The so called Grunsky inequalities for a function $g \in \Sigma$, then can be read by

$$\sum_{k=1}^{\infty} k \left| \sum_{l=1}^{\infty} c_{kl} \lambda_l \right|^2 \leq \sum_{k=1}^{\infty} \frac{|\lambda_k|^2}{k},$$

$$\left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl}\, \lambda_k \lambda_l \right| \leq \sum_{k=1}^{\infty} \frac{|\lambda_k|^2}{k},$$

for arbitrary complex variables λ_k's, provided that the right hand side series converges.

The **Grunsky operator**, corresponding to the map g, is defined by

$$\begin{aligned} Gr_g &: \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C}) \\ (x_k) &\to \left(\sum_{l=1}^{\infty} \sqrt{kl}\, c_{kl}\, x_l \right). \end{aligned} \tag{2.4}$$

Pommerenke showed that the Grunsky inequalities have the following operator theoretic forms

$$\|Gr_g x\| \leq \|x\|\,, \qquad |< x, Gr_g x >| \leq \|x\|^2\,.$$

Therefore the Grunsky operator is a bounded operator of norm less than or equal to one for $g \in \mathbf{\Sigma}$. It has been known that when the domain $g(\mathbb{D}^-)$ is a quasidisc the Grunsky operator norm is strictly less than one; see the next subsection for more explanation.

Remark 2.2.3. *A similar definition of the Grunsky coefficients and operator can be written for holomorphic functions defined on a neighbourhood of zero.*

S. Bergman and M. Schiffer in their fundamental paper [10], by using the kernel functions, derived a different formulation of the Grunsky inequalities which are equivalent to the Grunsky's. Namely [10, Eq 9.9] which are sometimes called the Bergman and Schiffer inequalities or condition. They showed that they in totality form a sufficient condition for a holomorphic map f on B to be univalent where B is a finite multiply-connected domain with n non-overlapping analytic closed boundary curves.

V. Singh [73] worked on the same domain $\mathfrak{B}$ but with the extra assumption that the map f is bounded holomorphic. He could derive similar Grunsky inequalities.

A nice modification of the Grunsky inequalities for bounded univalent functions on a neighbourhood of zero in $\mathbb{C}$ was given by Z. Nehari [40].

A more general version, including the result of Singh, was given by Yu. E. Alenitsyn [5, 6]. His result concerns a system of $n \geq 1$ holomorphic and univalent functions f_k defined on a multiply-connected finite domain B of $\mathbb{C}$. $\mathfrak{B}$ is bounded by a finite number of closed analytic curves with mutually disjoint images. He found an inequality like that of Grunsky, which includes the Bergman kernel functions of the first and second kind. The Bergman and Schiffer inequalities can be derived from his inequalities.

D. Aharonov introduced the concept of a pair which after J. A. Hummel's paper [24] is now called the Aharonov pairs of analytic functions [1]. If $F(z)$ and $G(z)$ are analytic and univalent functions on $\mathbb{D}$, then we call them an Aharonov pair if

$$F(z)\,G(\zeta) \neq 1 \quad \textit{for every } z, \zeta \in \mathbb{D}.$$

Hummel proved a matrix form of the Grunsky inequalities for such a pair of analytic functions which generalizes Aharonov's result. His work can be considered as deriving Grunsky inequalities for a pair of non-overlapping functions in the genus zero Riemann surface case.

G. L. Jones [26] by considering the derivatives of Equation (2.3) as an integral operator kernel, defined the operator Γ_g, called the Grunsky operator, on the Bergman space of $\mathbb{D}$. Here g is the conformal mapping which maps $\mathbb{D}$ into a finite domain G bounded by a Jordan curve Γ. More precisely,

$$[\Gamma_g f](z) = \int \left(\frac{1}{(z-\zeta)^2} - \frac{g'(z)\,g'(\zeta)}{(g(z)-g(\zeta))^2} \right) f(\frac{1}{\zeta})\, dA(\zeta), \quad f \in A^2(\mathbb{D}). \tag{2.5}$$

This integral operator was also mentioned in Bergman and Schiffer [10]. One of the two main results of the paper concerning the operator Γ_g is

Theorem 2.2.4 (Jones). *The Grunsky operator lies in the p-th Schatten ideal ($p \geq 1$) of operators on the Dirichlet space if and only if $\log g' \in B_p$, the Besov space.*

The condition $\log g' \in B_p$ can be replaced by "g admits a quasiconformal extension G to $\mathbb{C}$ with

$$\mu_G \in L^p(d\lambda)$$

where $d\lambda = dA(z)/(1-|z|^2)^2$ is the hyperbolic area density". Jones' theorem introduces another characterization of the quasiconformal extendibility. The author also used the idea of projection on holomorphic and anti-holomorphic parts and considering G as a region in Riemann sphere.

A. Baranov and H. Hedenmalm [9] defined the Grunsky operator corresponding to a map $\phi \in \mathcal{S}$ by

$$\mathfrak{B}_\phi[f](z) = p.v. \iint_{\mathbb{D}} \frac{\phi'(z)\phi'(w)}{(\phi(z)-\phi(w))^2} f(w)\, dA(w), \;\; z \in \mathbb{D}.$$

As we see, in comparison to the Jones' definition, they removed the factor $\frac{1}{(z-w)^2}$ in the kernel and changed the domain of the integration.

As was claimed, using the unitary properties of the Beurling operator on $L^2(\mathbb{C})$, they obtained an operator identity. Namely,

$$\mathfrak{B}_\phi - \mathfrak{B}_{id} = \mathfrak{P}\mathfrak{B}_\phi,$$

where $\mathfrak{P}$ is the orthogonal projection on the analytic functions in $L^2(\mathbb{D})$. That is, $P_{\mathbb{D}}$ in the thesis' notation, see Subsection 1.2.2 . They showed that this operator identity implies the Grunsky inequalities. Later on H. Hedenmalm

[22] extended this result to the weighted Hilbert space with the weight function $|z|^{2\theta}$, for $0 \leq \theta \leq 1$, and found a reformulation of generalized Grunsky inequalities for the weighted $L^2_\theta(\mathbb{D})$ space.

As was mentioned, the Grunsky coefficients have had many applications in complex analysis. In the next section, we will talk about some special applications of these coefficients to Teichmüller spaces defined in Section 1.9.

2.2.1 Grunsky Coefficients and Teichmüller Spaces

Connections discovered between the spaces $\mathcal{S}$ (or equivalently Σ) and the Teichmüller spaces reveal other aspects of the Grunsky coefficients and inequalities. Some applications of the Grunsky inequalities in the Teichmüller spaces date back to 1980s. The paper of I. V. Žuravlev [83] seems to be one of the first rigorous works in the subject. The key relation is the following definition and theorem.

Let $g \in \mathbf{\Sigma}$ and $0 \leq k \leq 1$. We say $g \in \mathbf{\Sigma}(k)$ if

$$\left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl} \lambda_k \lambda_l \right| \leq k \sum_{k=1}^{\infty} \frac{|\lambda_k|^2}{k} \tag{2.6}$$

where λ_k are arbitrary complex numbers and c_{kl} are the Grunsky coefficients of g defined by Equation 2.3.

Here is an important theorem regarding this set of holomorphic functions in $\mathbf{\Sigma}$.

Theorem 2.2.5 (Kühnau). *If $g \in \Sigma$ and has k-quasiconformal extension $(0 \leq k < 1)$ to $\mathbb{C}$, then $g \in \mathbf{\Sigma}(k)$.*

M. Schiffer and G. Schober claimed that the above theorem was first proved by R. Kühnau, see [57], [41] and the references therein for more detail. Pommerenke [41, Theorem 9.12] and later on Žuravlev proved the converse. That is,

Theorem 2.2.6 (Pommerenke and Žuravlev). *If $g \in \Sigma(k)$, then g admits a k'-quasiconformal extension to $\overline{\mathbb{C}}$ such that $k' \geq k$.*

H. Shiga [69] investigated the boundary of Teichmüller spaces by using the Grunsky inequalities and other function-theoretic tools. It was claimed that the method is motivated by Žuravlev paper; see also H. Shiga [70].

H. Shiga and H. Tanigawa [71] improved a result of S. Krushkal regarding the equality of Kobayashi and Carateódory metrics on Teichmüller space. Grunsky inequalities were used in some of their proofs.

More connections between the Grunsky map, which maps each element $[\mu]$ in a Teichmüller space to the Grunsky operator of the homomorphic map corresponding to $[\mu]$, and Teichmüller spaces were recently found. See S. L. Krushkal's survey [30] and the references therein.

Takhtajan and Teo [76, Remark 2.11] defined a map called "the universal period mapping" on the universal Teichmüller space. It was defined via the Grunsky coefficients of a conformal map from $\mathbb{D}$ into $\overline{\mathbb{C}}$. They showed that the universal period mapping is holomorphic; also this map coincides with the map that was defined by Kirillov and Yur'ev [28], Nag and Sullivan [39], defined on the universal Teichmüller space. They called this map KYNS period mapping of $T(1)$. In our context, i.e. surfaces with boundaries, they worked on the $\mathfrak{g} = 0$ and $n = 1$ case, where the boundary curve is a quasicircle.

Shen [67], taking the same definition as the one given by (2.4) for the Grunsky operator associated to $f \in \Sigma_0$, showed that the Grunsky operator is a holomorphic map on the universal Teichmüller space $T(1)$. To define the Grunsky operator on $T(1)$ for each $[\mu] \in T(1)$ we have $\mathbf{Gr}([\mu]) := \mathbf{Gr}_f$ where $f = f_\mu|_{\mathbb{D}^-}$. He mentioned some previous similar attempts at this result: S. L. Krushkal (1985), H. Shiga-H. Tanigawa (1993).

Schippers and Staubach [64], using their Faber operator, defined the Grunsky operator on $\mathcal{D}_*(\mathbb{D}^-)$ into $\mathcal{D}(\mathbb{D})$. They showed that the Grunsky operator is a bounded operator with norm less than or equal to one. The norm is strictly less than one if and only if Γ, the boundary of Ω^+, is a quasicircle [64, Theorem 3.13]. The paper also provides an integral representation of the Grunsky operator which is similar to Equation (2.5).

In Chapter 4, we will explain the connection between the Teichmüller spaces and the Grunsky coefficients and/or inequalities in more detail. In the next section we will review some work corresponding to Faber polynomials and Grunsky coefficients on Riemann surfaces.

2.3 Faber Polynomials and Grunsky Coefficients on Riemann Surfaces

All the objects in the previous two sections involve only the complex plane or the Riemann sphere ($\mathfrak{g} = 0$). Let us write a few words about some work that has been done on Riemann surfaces of arbitrary genus $\mathfrak{g} > 0$. This seems not to be as much investigated as the planar case.

H. Tietz [77] constructs a system of functions and a system of 1-forms (later called the Faber-Tietz functions or differentials) on a closed Riemann surface $\mathfrak{R}$. This system plays the same approximation role on surfaces that of the Faber polynomials' in the plane. That is, assume $G \subset \mathfrak{R}$ is a domain bounded by an analytic curve whose complement is simply connected. He assumed the complement of G contains a non Weierstrass point, say p. Then he showed that every holomorphic function (1-form) on G can be expanded by functions (1-forms) in this system. These functions were rational functions with only singularity at p. This expansion was later called the Faber-Tietz expansion. See also H. Tietz [78] and H. Röhrl [50] for removing the Weierstrass point restriction.

Tietz, in another paper [80], using the Laurent decomposition analogue on Riemann surfaces extends this results to open Riemann surfaces. In particular, he derived an expansion in Faber functions for analytic functions on some subdomains of an open Riemann surface.

H. Röhrl [51] generalized the Faber-Tietz expansion (in the sense of [78] and [50]) on open Riemann surfaces of finite genus. He used a theorem proved by S. Bochner and L. Sario stating that any open Riemann surface of finite genus can be embbeded in a closed Riemann surface. That is, for an open Riemann surface $\mathfrak{R}^*$ of finite genus, there exists a closed Riemann surface $\mathfrak{R}$ such that $\mathfrak{R}^*$ is a connected subsurface of $\mathfrak{R}$.

Generalization of the Grunsky coefficients of conformal maps into Riemann surfaces and their Grunsky inequalities have been occasionally obtained. K. Reimer and E. Schippers [48] consider the Grunsky coefficients for a conformal map f from a neighbourhood of zero into a torus, which is a Riemann surface of genus one. Their work was based on the Faber-Tietz meromorphic functions and 1-forms defined by Tietz [77]. They showed a generalized Grunsky inequality for these coefficients. A few first Grunsky coefficients of f, using the Weierstrass $\wp$ function on torus, were also calculated explicitly by them.

Now the reader is familiar with Faber polynomials and Grunsky coefficients corresponding to a conformal map, and some generalizations of them in different senses. To conclude this chapter we would like to state two important results. As was mentioned the Faber and Grunsky operators corresponding to a set of conformal maps $(f_1, \ldots, f_n)$ with non-overlapping images Ω_k^+, were defined on the Riemann sphere [46],. It was shown that under some boundary condition (the boundary curves are quasicircles) the Faber operator is a bounded isomorphism and the Grunsky operator has norm strictly less than one. It was also shown that the pull-back of $\mathcal{D}(\Sigma)$, where $\Sigma = \overline{\mathbb{C}} \setminus \cup_{k=1}^{n} \Omega_k^+$, under the map $\mathbf{f} = (f_1, \ldots, f_n)$ is the graph of the Grunsky operator corresponding to $\mathbf{f}$. In the next chapter, we will talk more about these problems and generalize them on some bordered Riemann surfaces of arbitrary genus.

Chapter 3

Faber Isomorphism and Grunsky Inequality on Bordered Surfaces

In Chapter 2, we reviewed some generalizations of the Faber polynomials, or operator, and the Grunsky coefficients, or operator, on the complex plane and Riemann surfaces. As was mentioned in the introduction, the first main problem of the thesis is to characterize the holomorphic Dirichlet space of Σ, where Σ is a bordered Riemann surface of genus $\mathfrak{g} > 0$ with $n \geq 1$ boundary curves homeomorphic to $\mathbb{S}^1$ in $\mathbb{C}$. This chapter is about this problem. In addition to that, the results here will be used in Chapter 4 to solve the second problem mentioned in the introduction.

This chapter is divided into two parts. The first part is about surfaces with one boundary curve (i.e. $n = 1$) and the second part is about surfaces with more than one boundary curve (i.e. $n > 1$). In each part, by using a Cauchy-type integral operator, the Faber operator corresponding to Σ is defined. When all the boundary curves are quasicircles, the Grunsky operator is also defined. This definition is based on the Faber operator and a transmission operator which was recently developed for quasicircles on Riemann surfaces by Schippers and Staubach [63]. We will see that the Grunsky operator has norm strictly less than one in this case. In addition to the boundary regularity, the Grunsky's norm proof requires a density theorem on open Riemann surfaces. This work, joint with Schippers and Staubach is now submitted. [59].

The Schiffer operator is defined and we find its adjoint operator. A short history of this operator, including its adjoint calculation in genus zero, is provided. We show that on which function space, depending on the topology of the surface, this operator is a bounded isomorphism. The Schiffer operator is used to show that in the case that all the boundary curves are quasicircles, the Faber operator is a bounded isomorphism.

Finally, we obtain a characterization of $\mathcal{D}(\Sigma)$ in terms of the generalizations of the Faber and Grunsky operators for Σ. We show that when all the boundary curves are quasicircles, the pull-back of $\mathcal{D}(\Sigma)$ under the conformal maps parametrizing the boundary of Σ, is the graph of the generalized Grunsky operator.

3.1 Surfaces with One Border

In this section, unless otherwise mentioned, we assume that $\mathfrak{R}$ is a compact Riemann surface of genus $\mathfrak{g} > 0$, and Γ is a strip-cutting Jordan curve on $\mathfrak{R}$ (see Section 1.4 for definition) which separates $\mathfrak{R}$ into two subsurfaces Ω and Σ. Note that not every strip-cutting Jordan curve separates. We assume that Γ is positively oriented with respect to Ω. In the case that Ω is a simply connected subsurface, we fix a point $p \in \Omega$, and assume that there exists a biholomorphism

$$f_p : \mathbb{D} \to \Omega,$$

such that $f_p(0) = p$. We usually drop the point p in f_p notation if there will be no risk of confusion. The conformal map f may be called a **uniformizing map** (or a **boundary parametrization**) of Ω (of Γ).

Now, the surface $\Sigma = \mathfrak{R}\backslash cl(\Omega) = \mathfrak{R}\backslash cl(f(\mathbb{D}))$ is a Riemann surface of genus at most $\mathfrak{g}$ with one boundary curve homeomorphic to $\mathbb{S}^1$. We aim to characterize the holomorphic Dirichlet space of Σ in terms of pull back under f.

Remark 3.1.1. *The results of this section, surfaces with one boundary, was submitted as a part of a joint paper with Radnell, Schippers and Staubach. See D. Radnell, E. Schippers and W. Staubach [45, Section 3] for more detail.*

3.1.1 Schiffer Operator for Surfaces with One Boundary

We first give the definition of the Schiffer operator and some of its important properties from [62]. We then write a few words about the history of the Schiffer operator.

Let $\mathfrak{R}$, Γ, Σ and Ω (not necessarily simply connected) be as above. The integral operators

$$\begin{aligned} T(\Omega;\Sigma) : \overline{A(\Omega)} &\to A(\Sigma) \\ \overline{\alpha} &\to \frac{1}{\pi i} \iint_{\Omega,w} \partial_z \partial_w g(w;z,q) \wedge \overline{\alpha(w)}, \end{aligned}$$

where $z \in \Sigma$, and

$$
\begin{aligned}
S(\Omega) : A(\Omega) &\to A(\mathfrak{R}) \\
\alpha &\to -\frac{1}{\pi i} \iint_{\Omega} \partial_z \overline{\partial}_w g(w; z, q) \wedge \alpha(w),
\end{aligned}
$$

are called the Schiffer (or Schiffer comparison) operators.

In other words, in terms of the kernel functions defined in Section 1.8, the Schiffer operator T maps $\overline{\alpha}$ to

$$\iint_{\Omega} L_{\mathfrak{R}}(z, w) \wedge \overline{\alpha(w)},$$

and the Schiffer operator S maps α to

$$\iint_{\Omega} K_{\mathfrak{R}}(z, w) \wedge \alpha(w).$$

Note that the assumption $z \in \Sigma$ makes the kernel of the integration in T operator a non-singular one, see Theorem 1.8.1. The operator $T(\Sigma; \Omega)$ may be defined by exchanging the role of Σ and Ω in the definition above.

The integration for the case $T(\Omega; \Omega)$; that is, when z is in Ω, is understood in the principal value sense. In this case, and similarly for $T(\Sigma; \Sigma)$, the Schiffer operator can be shown to be equal to

$$
\begin{aligned}
T(\Omega; \Omega) : \overline{A(\Omega)} &\to A(\Omega) \\
\overline{\alpha} &\to \iint_{\Omega, w} \left(L_{\mathfrak{R}}(z, w) - L_{\Omega}(z, w) \right) \wedge \overline{\alpha}(w),
\end{aligned}
$$

where $z \in \Omega$.

Schippers and Staubach showed that the output of above operator is in

$A(\Omega)$. They proved that the Schiffer operator, in all the above cases, is independent of the choice of holomorphic chart. Last but not the least, they showed that this operator is bounded operator [62, Theorem 3.9]. See e.g. R. Seeley [66] for the theory of singular integral operators on compact manifolds for more detail.

In the case that $\mathfrak{R}$ is the sphere, the Schiffer operator is the restriction of the Beurling transformation on Ω to the space of all L^2 anti-holomorphic 1-forms on Ω. It can also be considered as a special case of Calderón-Zygmund integral operator on manifolds.

Schiffer [55] and then Bergman and Schiffer [10] defined this operator on planar domains bounded by analytic curves. Schiffer and Spencer extended the operator to Riemann surfaces. Schiffer considered the case of nested open domains in the sphere in a chapter in the book of R. Courant [12]. See E. Schippers and W. Staubach [61] for more historical detail.

Napalkov, Jr. and Yulmukhametov [84] formulated the Schiffer operator as a Hilbert transform on the dual space of the Bergman space of a subdomain G in $\mathbb{C}$. They showed that this operator is one-to-one. It is an onto operator if and only if the boundary of G is a quasicircle, that is, if and only if G is a quasidisc.

Schippers and Staubach [64] named this operator the Schiffer comparison operator and extended the definition to domains bounded by quasicircles, i.e. quasidiscs. They also make a change to the domain of this operator. That is Schiffer defined the above operator on the space of L^2 holomorphic 1-forms on the given domain which makes the operator anti-complex linear, whereas the above operator is a complex linear one. One important result of their work is providing a formula for the adjoint of the Schiffer operator.

Other results will be discussed ahead.

Later on, in Section 3.2 we will generalize the Schiffer operator to the case of many boundary curves for the first time. That is, we generalize the Schiffer operator to Riemann surfaces bounded by $n > 1$ non-overlapping strip-cutting Jordan curves $\Gamma_k, k = 1, \ldots, n$.

We conclude this subsection by recalling a theorem [62, Theorem 4.2] which shows the relationship between the derivatives of the Cauchy-type integral operator defined in Section 1.7 and the Schiffer operator.

Theorem 3.1.2. *Let $\mathfrak{R}$, Γ, Σ and Ω be as Section 3.1. For all $h \in \mathcal{D}_{harm}(\Sigma)$ and any $q \in \mathfrak{R} \backslash \Gamma$, we have*

$$\begin{aligned}
\partial[J_q(\Gamma)h](z) &= -T(\Omega,\Sigma)\overline{\partial}h(z), && z \in \Sigma\\
\partial[J_q(\Gamma)h](z) &= \partial h(z) - T(\Omega,\Omega)\overline{\partial}h(z), && z \in \Omega\\
\overline{\partial}[J_q(\Gamma)h](z) &= \overline{S}(\Omega)\overline{\partial}h(z), && z \in \Omega \cup \Sigma.
\end{aligned}$$

This theorem will be generalized in Subsection 3.2.4 for surfaces with more than one boundary curve, see Theorem 3.2.37.

In the next subsection we introduce some spaces of harmonic functions and forms which will be used to define Faber and Grunsky operators corresponding to Σ.

3.1.2 Some Subspaces of Functions and 1-Forms

In this subsection, we will introduce some spaces of harmonic functions or harmonic 1-forms corresponding to subsurfaces of $\mathfrak{R}$. Let start with the following definition

$$\mathcal{W}_\Omega = \left\{ h \in \mathcal{D}_{harm}(\Omega) \ : \ \lim_{\epsilon \to 0^+} \int_{\Gamma_\epsilon^p} h\alpha = 0 \, , \, \forall \alpha \in A(\mathfrak{R}) \right\}. \tag{3.1}$$

Here the curves Γ_ϵ^p are as Remark 1.7.2.

Remark 3.1.3. *We know that $A(\mathfrak{R})$ is a vector space of dimension $\mathfrak{g}$ over the field $\mathbb{C}$. Therefore, the integral condition used in (3.1) for each function $h \in \mathcal{W}_\Omega$ imposes $\mathfrak{g}$ linear integral conditions. This is where the topology of the surface plays a role.*

We could define the same space for the other subsurface Σ. Here is a consequence of this definition. See [62] for a proof.

Theorem 3.1.4. *Let $\mathfrak{R}$, Γ, Σ and Ω be as in Section 3.1. Suppose that the domain of the Cauchy-type integral operator $J_q(\Gamma)$ (Definition 1.7.3) is restricted to $\mathcal{W}_\Omega$. Then the output of $J_q(\Gamma)$ is a holomorphic function on Σ.*

Here is a theorem [62, Theorem 4.13] that will be frequently used in the thesis. We call it the **transmitted-jump** formula.

Theorem 3.1.5 (Schippers-Staubach). *Let $\mathfrak{R}$, Γ, Σ, Ω and $q \in \mathfrak{R} \backslash \Gamma$ be as Section 3.1. If $h \in \mathcal{W}_\Omega$ and Γ is a quasicircle, then*

$$-\mathfrak{O}(\Sigma_2, \Sigma_1) J_q(\Gamma)_\Sigma h = h - J_q(\Gamma)_\Omega .$$

The following theorem, which is a consequence of [62, Corollary 4.28], shows another aspect of this definition when the curve Γ is a quasicircle.

Theorem 3.1.6. *Let $\mathfrak{R}$, Γ, Σ, and Ω be as Section 3.1. Suppose Γ is a quasicircle. If $h \in \mathcal{W}_\Omega$, then the CNT boundary values of h on Γ, say H, has a jump decomposition. The decomposition is unique up to additive constants.*

They showed that the jump decomposition for $h \in \mathcal{W}_\Omega$ is given by

$$h_\Omega = J_q(\Gamma)_\Omega h, \ \ h_\Sigma = J_q(\Gamma)_\Sigma h.$$

in the sense that

$$H = H_\Sigma - H_\Omega,$$

except on a null set in Γ, where the H_Σ and H_Ω denote the boundary values of h_Σ and h_Ω, respectively.

It worth noting that there are some other approaches other than the Cauchy-type integral operator to find the components of a jump decomposition. This approach was suggested by Royden [54]. What Schippers and Staubach did was to weaken the analytic condition on the curve and the boundary function on the curve.

As we mentioned in Section 1.7, the jump problem for holomorphic functions in $\mathcal{W}_\Omega$ is trivial; therefore, we are particularly interested in anti-holomorphic functions in $\mathcal{W}_\Omega$, namely $\mathcal{W}_\Omega \cap \overline{\mathcal{D}(\Omega)}$. We also need to apply the normalization of vanishing at a fixed point $p \in \Omega$. Now define $\mathcal{W}'_\Omega = \mathcal{W}_\Omega \cap \overline{\mathcal{D}(\Omega)}_p$.

To work on the space of holomorphic 1-forms we define the following space

$$\begin{aligned}\mathcal{V}_\Omega &= \left\{ \overline{\alpha} \in \overline{A(\Omega)} \ : \ \iint_\Omega \overline{\alpha} \wedge \beta = 0, \ \ \forall \overline{\beta} \in \overline{A(\mathfrak{R})} \right\} \\ &= \left\{ \overline{\alpha} \in \overline{A(\Omega)} \ : < \overline{\alpha}, \overline{\beta} >_{A_{harm}(\Omega)} = 0, \ \ \forall \overline{\beta} \in \overline{A(\mathfrak{R})} \right\} \\ &= \overline{A(\Omega)} \cap \overline{A(\mathfrak{R})}^{\perp}.\end{aligned} \tag{3.2}$$

$\mathcal{V}_\Sigma$ may be defined similarly. These spaces are the ones that meet the necessary conditions for the existence of a jump.

Lemma 3.1.7. *Let $\mathfrak{R}$, Γ, Σ and Ω be as Section 3.1. Then*

1. *For every function $\bar{h} \in \mathcal{W}'_\Omega$ we have $\bar{\partial}\bar{h} \in \mathcal{V}_\Omega$.*

2. *If Ω is simply connected domain, then*

$$\begin{aligned} \overline{\partial} : \mathcal{W}'_\Omega &\to \mathcal{V}_\Omega \\ \bar{h} &\to \bar{\partial}\bar{h} \end{aligned}$$

is an isometric isomorphism with respect to Dirichlet norm.

Proof. If $\overline{h} \in \mathcal{W}'_\Omega$, then clearly $\overline{\partial h} \in \overline{A(\Omega)}$. Now for $\overline{\beta} \in \overline{A(\mathfrak{R})}$ we have the following

$$d(\overline{h}\beta) = d\overline{h} \wedge \beta + \overline{h} d\beta = \overline{\partial h} \wedge \beta + 0 = \overline{\partial h} \wedge \beta.$$

Therefore, Stokes' theorem implies

$$< \overline{\partial h}, \overline{\beta} >_{A_{harm}(\Omega)} = \frac{1}{2} \iint_\Omega \overline{\partial h} \wedge \star\overline{(\overline{\beta})} = -\frac{i}{2} \lim_{\epsilon \to 0^+} \int_{\Gamma_\epsilon^p} \overline{h}\beta.$$

the last integral is zero because of our choice of $\overline{h}$. Therefore $\overline{\partial h}$ is in $\mathcal{V}_\Omega$.

The first part proves that the operator $\overline{\partial}|_{\mathcal{W}'_\Omega}$ is well-defined. Now we show that it is one-to-one, onto and (semi) norm preserving.

Let $\overline{\alpha} \in \mathcal{V}_\Omega$. There exists a unique $\overline{h} \in \overline{\mathcal{D}(\Omega)}_p$ $(h(p) = 0)$ such that $\overline{\partial h} = \overline{\alpha}$ as Ω is simply connected. We now show that $\overline{h}$ is in $\mathcal{W}_\Omega$ by calculations like the one above. If $\beta \in A(\mathfrak{R})$, then

$$\lim_{\epsilon \to 0^+} \int_{\Gamma_\epsilon^p} \overline{h}\beta = 2i < \overline{\partial h}, \overline{\beta} >_{A_{harm}(\Omega)} = 2i < \overline{\alpha}, \overline{\beta} >_{A_{harm}(\Omega)} = 0$$

since $\overline{\alpha}$ was in $\mathcal{V}_\Omega$. This implies that $\overline{h} \in \mathcal{W}_\Omega$. Therefore, for every $\overline{\alpha} \in \mathcal{V}_\Omega$ there exists a unique $\overline{h} \in \mathcal{W}'_\Omega$ such that $\overline{\partial h} = \alpha$. Clearly

$$\|\overline{h}\|_{\mathcal{D}_{harm}(\Omega)} = \|\overline{\partial h}\|_{A_{harm}(\Omega)} = \|\overline{\alpha}\|_{A_{harm}(\Omega)},$$

by its definition. The choice of normalization makes the Dirichlet seminorm a norm. This completes the proof of the second part. □

In the case that Ω is simply connected, we are interested in the pull back of those functions in $\mathcal{W}'_\Omega$ under the map f. Thus we make the following definition.

Definition 3.1.8. *Let $p \in \Omega$ be fixed and $f = f_p$ be a uniformizing map of Ω. Define*

$$\overline{\mathcal{D}(\mathbb{D})}_v = \mathcal{C}_f \mathcal{W}'_\Omega = \left\{ \overline{H} \in \overline{\mathcal{D}(\mathbb{D})} : \overline{H} \circ f^{-1} \in \mathcal{W}'_\Omega \right\}.$$

The subindex v is used to indicate the relation to the space $\mathcal{V}_\Omega$. In other words, for $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ we have $\overline{\partial}\left(\mathcal{C}_{f^{-1}}\overline{H}\right) \in \mathcal{V}_\Omega$. The p normalization ensures that every $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ vanishes at zero; therefore, the only constant function in $\overline{\mathcal{D}(\mathbb{D})}_v$ is the zero function.

Now we are ready for the next subsection in which we define the Faber and Grunsky operators.

3.1.3 Faber and Grunsky Operators

In this subsection, we assume $\mathfrak{R}$, Γ, Σ and Ω are defined as in Section 3.1 with the extra assumption that Ω is a simply connected domain. Thus we have the uniformizing map f for Ω. We will define two operators corresponding to the map f or equivalently the bordered surface Σ. These definitions are motivated by their definitions in the complex plane or the Riemann sphere case reviewed in Section 2.1.

Definition 3.1.9. *Let $\mathfrak{R}$, Γ, Ω, Σ, f and q be as above. The operator*

$$\begin{aligned} \mathbf{I}_f : \overline{\mathcal{D}(\mathbb{D})}_v &\to \mathcal{D}(\Sigma)_q \\ \overline{H} &\to -[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Sigma \end{aligned}$$

is called the Faber operator corresponding to f (or Σ).

Definition 3.1.10. *Let $\mathfrak{R}$, Γ, Ω, Σ, f and q be as above. Assume that Γ is a quasicircle. The operator*

$$\begin{aligned} \mathbf{Gr}_f : \overline{\mathcal{D}(\mathbb{D})}_v &\to \mathcal{D}(\mathbb{D}) \\ \overline{H} &\to P_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H} \end{aligned}$$

is called the Grunsky operator corresponding to f (or Σ).

It should be noted that to define the Faber operator we did not assume that Γ is a quasicircle. However, the definition of the Grunsky operator, because of the transmission operator, requires the curve Γ to be a quasicircle. Theorem 3.1.4 and Definition 3.1.8 ensure that the output of the the Faber operator is in $\mathcal{D}(\Sigma)_q$. Similarly Theorem 1.4.8 and the projection operator defined in Section 1.2.2 ensure that the output of the Grunsky operator is in $\mathcal{D}(\mathbb{D})$.

Remark 3.1.11. *We will say "Faber (Grunsky) operator" rather than "Faber (Grunsky) operator corresponding to f" when the map f is clear from context.*

Here is a generalization of some classical theorems concerning the isomorphism property of the Faber operator. See Definition 2.1.9 and the references afterwards.

Theorem 3.1.12. *Let $\mathfrak{R}$, Γ, Ω, Σ, f and q be as above. Assume Γ is a quasicircle. Then the operator $\mathbf{I}_f$ is a bounded isomorphism.*

Proof. The operators $\partial : \mathcal{D}(\Sigma)_q \to A(\Sigma)_e$ and $\overline{\partial} : \mathcal{W}'_\Omega \to \mathcal{V}_\Omega$ are bounded isomorphisms by Remark 1.2.2 and Lemma 3.1.7, respectively. When Γ is a $quasicircle$, then the Schiffer operator $T(\Omega,\Omega)|_{\mathcal{V}_\Omega}$ is also a bounded isomorphism by [62, Theorem 4.20]. Furthermore, by Theorem 3.1.2 one has

$$T(\Omega,\Omega)\overline{\partial} = \partial J_q(\Gamma),$$

Therefore, $J_q(\Gamma)|_{\mathcal{W}'_\Omega} : \mathcal{W}'_\Omega \to \mathcal{D}(\Sigma)_q$ is a bounded isomorphism.

On the other hand, $\mathcal{C}_{f^{-1}}|_{\overline{\mathcal{D}(\mathbb{D})}_v} : \overline{\mathcal{D}(\mathbb{D})}_v \to \mathcal{W}'_\Omega$ is an isomorphism by the composition map properties and Definition 3.1.8.

Therefore, by the the definition of the Faber operator, $\mathbf{I}_f$ is a bounded isomorphism from $\overline{\mathcal{D}(\mathbb{D})}_v$ onto $\mathcal{D}(\Sigma)_q$. □

We will show that the transmission part in above definition can be simplified provided that the boundary curve is a quasicircle.

Lemma 3.1.13. *Let $\mathfrak{R}$, Γ, Ω, Σ, q and f be as the above. If Γ is a quasicircle, then*

$$\mathbf{Gr}_f(\overline{H}) = -\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega,$$

for all $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$.

Proof. Since Γ is a quasicircle, by the transmitted-jump formula (Theorem 3.1.5) for every $h \in \mathcal{W}_\Omega$ we have

$$-\mathfrak{O}(\Sigma,\Omega)[J_q(\Gamma)h]_\Sigma = h - [J_q(\Gamma)h]_\Omega.$$

Now let $h = \mathcal{C}_{f^{-1}}H$. Therefore, by the definitions of the Grunsky and Faber operators we have

$$\begin{aligned}
\mathbf{Gr}_f(\overline{H}) &= P_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H} \\
&= P_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)[-J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Sigma \\
&= P_{\mathbb{D}}\mathcal{C}_f\left(\mathcal{C}_{f^{-1}}\overline{H} - [J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega\right) \\
&= P_{\mathbb{D}}\overline{H} - P_{\mathbb{D}}\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega \\
&= -\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega.
\end{aligned}$$

In the last identity, $P_{\mathbb{D}}\overline{H}$ is zero since $\overline{H}$ is an anti-holomorphic function. Also

$$P_{\mathbb{D}}\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega = \mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega$$

since $\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega$ is a holomorphic map on $\mathbb{D}$. □

As we saw, this new formula is independent of both the transmission and Faber operators. The following theorem is an analog to Theorem 2.2.5 for compact Riemann surfaces.

Theorem 3.1.14. *Let $\mathfrak{R}$, Γ, Σ, Ω, q and f be as above. If Γ is a quasicircle, then the Grunsky operator $\mathbf{Gr}_f$ is a bounded operator of norm less than one.*

Proof. By Lemma 3.1.13 we have $\mathbf{Gr}_f(\overline{H}) = -\mathcal{C}_f[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega$ for all $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$. By applying the inverse of $\mathcal{C}_f$ on both sides and invoking Theorem

3.1.2 with $\Sigma_1 = \Omega$ and $\Sigma_2 = \Sigma$ we have

$$\mathcal{C}_{f^{-1}}\mathbf{Gr}_f\overline{H} = -[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega,$$

which implies

$$\begin{aligned}\partial\,\mathcal{C}_{f^{-1}}\mathbf{Gr}_f\overline{H} &= -\partial\,[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega \\ &= -\partial\,\mathcal{C}_{f^{-1}}\overline{H} - T(\Omega;\Omega)\overline{\partial}\,(\mathcal{C}_{f^{-1}}\overline{H}) \\ &= 0 - T(\Omega;\Omega)\overline{\partial}\,(\mathcal{C}_{f^{-1}}\overline{H}).\end{aligned}$$

Therefore,

$$\left\|\partial\,\mathcal{C}_{f^{-1}}\mathbf{Gr}_f\overline{H}\right\|_{\mathcal{D}_{harm}(\Omega)} = \left\|T(\Omega;\Omega)\overline{\partial}\,\mathcal{C}_{f^{-1}}\overline{H}\right\|_{\mathcal{D}_{harm}(\Omega)}$$

for all $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$.

On the other hand, for $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ one has $\overline{\partial}\mathcal{C}_{f^{-1}}\overline{H}$ is in $\mathcal{V}_\Omega$. Then [62, Theorem 4.20] ($\mathcal{V}_\Omega$ here is V_1 in the theorem) implies that

$$\|\partial\,\mathcal{C}_{f^{-1}}\mathbf{Gr}_f\|_{\overline{\mathcal{D}(\mathbb{D})}_v \to A(\Omega)} = \|T(\Omega;\Omega)|_{\mathcal{V}_\Omega}\|_{\mathcal{V}_\Omega \to A(\Omega)} < 1.$$

By the isometric properties of ∂ and $\mathcal{C}_{f^{-1}}$, see Section 1.2.2, the left hand side of the first identity above is $\|\mathbf{Gr}_f\|_{\overline{\mathcal{D}(\mathbb{D})}_v \to \mathcal{D}(\mathbb{D})}$ which completes the proof. □

We showed that when Γ is a quasicircle, the Faber operator is an isomorphism and in particular, an one-to-one operator. Now a left inverse of this operator is provided and is shown to be bounded.

Theorem 3.1.15. *Let $\mathfrak{R}$, Γ, Σ, Ω and f be as above. If Γ is a quasicircle, then $\overline{P}_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)$ is a left inverse of $\mathbf{I}_f$.*

Proof. Since Γ is a quasicircle by applying the transmitted-jump formula for $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ we have

$$\begin{aligned}\overline{P}_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f(\overline{H}) &= \overline{P}_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)[-J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Sigma \\ &= \overline{P}_{\mathbb{D}}\mathcal{C}_f\left(\mathcal{C}_{f^{-1}}\overline{H} - [J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega\right) \\ &= \overline{H} - 0 = \overline{H}.\end{aligned}$$

□

Remark 3.1.16. *Even though we consider $\mathfrak{g} > 0$, in the case of the Riemann sphere a left inverse for the Faber operator exists see [60, Lemma 3.14]. This inverse in the case of Riemann sphere connects to a classical identity regarding the Faber polynomials that we recall it here. It is an easy consequence of Definition 2.1.1 that*

$$\left(f^{-1}(z)\right)^{-n} = \Phi_n(f)(z) + \sum_{k=0}^{\infty} a_k^{(n)}\, z^k.$$

Thus one has the well-known identity

$$\Phi_n(f)\left(f(\zeta)\right) = \zeta^{-n} + \sum_{k=0}^{\infty} c_{nk}\zeta^k$$

for some c_{nk} which as we know they are called the Grunsky coefficients of f. By taking the projection $P_{\mathbb{D}^-}$ from both sides we have

$$P_{\mathbb{D}^-}\Phi_n(f)\left(f(\zeta)\right) = \zeta^{-n}$$

or equivalently $[P_{\mathbb{D}}\text{-}\mathcal{C}_f \Phi_n(f)](\zeta) = \zeta^{-n}$. *Finally, the identity* $\Phi_n(f) = \mathbf{I}_f(\zeta^{-n})$ *(where* ζ^{-n} *denotes the map* $\zeta \to \zeta^{-n}$*) shows that*

$$P_{\mathbb{D}}\text{-}\mathcal{C}_f \mathbf{I}_f(\zeta^{-n}) = \zeta^{-n}.$$

We should say that the projection operator used in [60] maps constants to zero and is a bit different from the one used in this thesis. We temporarily changed our definition of $P_{\mathbb{D}}$ *here to send constants to zero.*

Therefore, the operator $P_{\mathbb{D}}\text{-}\mathcal{C}_f$ *is a left inverse for monomials. By the boundedness of the operators and the density of these monomials in* $\mathcal{D}(\mathbb{D}^-)$ *this is true for every element in* $\mathcal{D}(\mathbb{D}^-)$ *vanishing at infinity.*

A similar identity for a compact Riemann surface of genus $\mathfrak{g}$ *can be found in the paper of Tietz [77, Satz 4] corresponding to his normalization of the meromorphic* 1*-form* α*, see Section 2.3.*

Here is one of the most important results of the thesis for surfaces with one boundary curve.

Theorem 3.1.17. *Let* $\mathfrak{R}$ *be a compact Riemann surface of genus* $\mathfrak{g} > 0$ *and* Γ *be a quasicircle curve in* $\mathfrak{R}$ *separating* $\mathfrak{R}$ *into two subsurfaces* Σ *and* Ω. *Assume* Ω *is simply connected and* Γ *is positively oriented with respect to* Ω *and* $q \in \Sigma$ *fixed. Assume also that* $f : \mathbb{D} \to \Omega$ *is a uniformizing map with* $f(0) = p$ *for fixed* $p \in \Omega$. *Then the pull back of the transmissions of the holomorphic functions in* $\mathcal{D}(\Sigma)_q$ *under the conformal map* f *is the graph of the Grunsky operator; that is,* $graph(\mathbf{Gr}_f)$.

Proof. Let $h \in \mathcal{D}(\Sigma)_q$. By Theorem 3.1.12 there exists $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ such that $\mathbf{I}_f\overline{H} = h$. Therefore,

$$\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)h = \mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H} \in \mathcal{D}_{harm}(\mathbb{D}),$$

and we have

$$\begin{aligned}(\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H}) &= [P_{\mathbb{D}} + \overline{P}_{\mathbb{D}}](\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H}) \\ &= P_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H} + \overline{P}_{\mathbb{D}}\mathcal{C}_f\mathfrak{O}(\Sigma,\Omega)\mathbf{I}_f\overline{H} \\ &= \mathbf{Gr}_f(\overline{H}) + \overline{H} \in graph(\mathbf{Gr}_f).\end{aligned}$$

by Theorem 3.1.15 and the definition of the Grunsky operator. □

Remark 3.1.18. *For the genus $\mathfrak{g} = 0$ case, a similar result with slightly different formulation was proven in Radnell, Schippers and Staubach [46]. So we could say that the pull back of functions in $\mathcal{D}(\Sigma)_q$ is the graph of the Grunsky operator for all bordered Riemann surfaces of arbitrary genus $\mathfrak{g} \geq 0$ (with one boundary curve homeomorphic to $\mathbb{S}^1$).*

Theorem 3.1.19. *Let $\mathfrak{R}$, Γ, Σ, Ω and f be as above. If Γ is a quasicircle, and $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$, then*

$$\left\|\mathbf{Gr}_f(\overline{H})\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})} = \left\|\overline{H}\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})} - \left\|\mathbf{I}_f(\overline{H})\right\|^2_{\mathcal{D}_{harm}(\Sigma)}.$$

Proof. Let $\Sigma_1 = \Omega, and \Sigma_2 = \Sigma$. Here is an identity given inside the proof of [62, Theorem 4.20]

$$\|T(\Omega;\Sigma)\overline{\alpha}\|^2_{A_{harm}(\Sigma)} = \|\overline{\alpha}\|^2_{A_{harm}(\Omega)} - \|T(\Omega;\Omega)\overline{\alpha}\|^2_{A_{harm}(\Omega)},$$

for all $\overline{\alpha} \in \mathcal{V}_\Omega$. This implies that for each $\bar{h} \in \mathcal{W}'_\Omega$ we have

$$\left\|T(\Omega;\Sigma)\bar{\partial}\bar{h}\right\|^2_{A_{harm}(\Sigma)} = \left\|\bar{\partial}\bar{h}\right\|^2_{A_{harm}(\Omega)} - \left\|T(\Omega;\Omega)\bar{\partial}\bar{h}\right\|^2_{A_{harm}(\Omega)}.$$

Now by Theorem 3.1.2 one has

$$\begin{aligned}\left\|-\partial[J_q(\Gamma)\bar{h}]_\Sigma\right\|^2_{A_{harm}(\Sigma)} &= \left\|\bar{h}\right\|^2_{\mathcal{D}_{harm}(\Omega)} - \left\|\partial\bar{h} - \partial[J_q(\Gamma)\bar{h}]_\Omega\right\|^2_{A_{harm}(\Omega)} \\ \left\|[J_q(\Gamma)\bar{h}]_\Sigma\right\|^2_{\mathcal{D}_{harm}(\Sigma)} &= \left\|\bar{h}\right\|^2_{\mathcal{D}_{harm}(\Omega)} - \left\|[J_q(\Gamma)\bar{h}]_\Omega\right\|^2_{\mathcal{D}_{harm}(\Omega)}.\end{aligned}$$

by the identity $\|\bar{\partial}\bar{h}\|_{A_{harm}(\Omega)} = \|\bar{h}\|_{\mathcal{D}_{harm}(\Omega)}$.

On the other hand, there exists $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$ such that $\bar{h} = \mathcal{C}_{f^{-1}}\overline{H}$. Therefore we have

$$\begin{aligned}\left\|[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Sigma\right\|^2_{\mathcal{D}_{harm}(\Sigma)} &= \left\|\mathcal{C}_{f^{-1}}\overline{H}\right\|^2_{\mathcal{D}_{harm}(\Omega)} - \left\|[J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_\Omega\right\|^2_{\mathcal{D}_{harm}(\Omega)} \\ \left\|\mathbf{I}_f\overline{H}\right\|^2_{\mathcal{D}_{harm}(\Sigma)} &= \left\|\overline{H}\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})} - \left\|\mathbf{Gr}_f\overline{H}\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})}\end{aligned}$$

by Theorem 3.1.13 and the fact that $\mathcal{C}_f|_{\mathcal{D}(\Omega)} : \mathcal{D}(\Omega) \to \mathcal{D}(\mathbb{D})$ and $\mathcal{C}_f|_{\overline{\mathcal{D}(\Omega)}} : \overline{\mathcal{D}(\Omega)} \to \overline{\mathcal{D}(\mathbb{D})}$ are isometries with respect to the Dirichlet semi-norm. $\square$

Corollary 3.1.20. *Let $\mathfrak{R}$, Γ, Σ, Ω, q and f be as above. If Γ is a quasicircle, then $\|\mathbf{I}_f\| \leq 1$.*

The fact that the Faber operator is an isomorphism when Γ is a quasicircle, is exactly what we need to approximate holomorphic functions on Σ via anti-holomorphic functions in $\overline{H} \in \overline{\mathcal{D}(\mathbb{D})}_v$. This is exactly the idea of Faber for planar domains. That is, to find a set of holomorphic functions corresponding to a simply connected domain G in $\mathbb{C}$ which play the same approximation role for *Hol*(G) as the polynomials $\{z^n : n \geq 0\}$ are playing for *Hol*$(\mathbb{D})$.

Now we can talk about surfaces with more than one boundary curves.

3.2 Surfaces with Finitely Many Borders

In this section, we generalize the operators and theorems from one boundary curve case to many ones. We start with some definitions and introduce some notation.

Throughout the section, unless otherwise mentioned, we use the following notation. Let Σ be a bordered Riemann surface of genus $\mathfrak{g} > 0$ with boundary curves $\Gamma_k, k = 1, \ldots, n$, homeomorphic to $\mathbb{S}^1$. As was mentioned in introduction, Σ can be modelled as follows. We assume there exists a compact Riemann surface $\mathfrak{R}$, of genus $\mathfrak{g}$, and pairwise disjoint simply connected domains $\Omega_k^+, k = 1, \ldots, n$, of $\mathfrak{R}$ such that Σ is biholomorphic to $\mathfrak{R} \setminus \cup_{k=1}^n \mathrm{cl}(\Omega_k^+)$. We assume there are conformal maps $f_k : \mathbb{D} \to \Omega_k^+$. The map f_k may be normalized by assuming $f_k(p_k) = 0$ for fixed point $p_k \in \Omega_k^+$. f_k may be called a uniformizing (or a boundary parametrization) of Ω_k^+ (of Γ_k). It can be assumed that each Γ_k is a strip-cutting Jordan curve. Define $\Omega_k^- = \mathfrak{R} \setminus \mathrm{cl}(\Omega_k^+)$. It is clear that Ω_k^- includes Σ. For simplicity, we sometimes use Ω as an abbreviation for $\cup_{k=1}^n \Omega_k^+$ and Γ for $\cup_{k=1}^n \Gamma_k$.

To illustrate, in Figure 3.1, one can see a sketch of a compact Riemann surface $\mathfrak{R}$ for which $\mathfrak{g} = 2$, and $n = 4$. Also in Figure 3.2 one can see a sketch of the bordered Riemann surface Σ obtained from $\mathfrak{R}$.

Remark 3.2.1. *In Chapter 4 we will see that Σ is a bordered Riemann surface of type $(\mathfrak{g}, n)$, but for now we do not use that terminology; see Definition 4.2.1.*

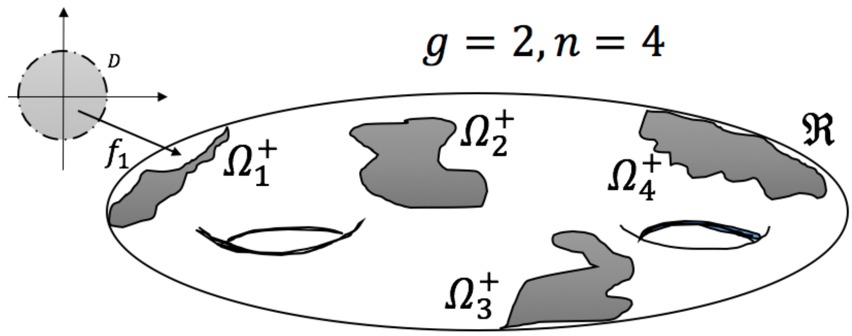

FIGURE 3.1: A Compact Riemann Surface $\mathfrak{R}$

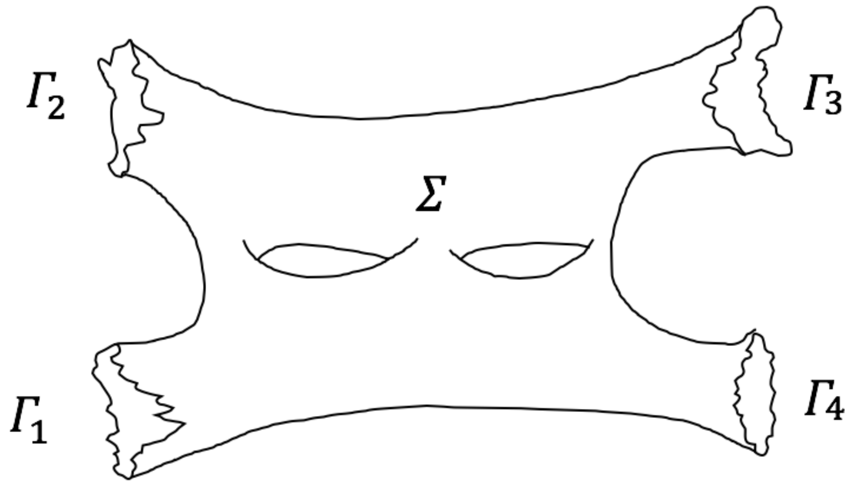

FIGURE 3.2: The Bordered Surface Σ Obtained From $\mathfrak{R}$

By $\Gamma_\epsilon^{p_k}$, we mean the level curves of the Green's function of Ω_k^+ with singularity at $p_k \in \Omega_k^+$. For fixed k and $\epsilon > 0$, these are analytic curves which approach Γ_k from within Ω_k^+ as $\epsilon \to 0^+$, see Remark 1.7.2.

For above set of conformal maps define $\mathbf{f} = (f_1, \ldots, f_n)$. We will assign operators to $\mathbf{f}$ which generalize the Faber and Grunsky operators from the case of one boundary curve to the many ones.

3.2.1 Some Operators and Subsurfaces of Functions and 1-Forms

Definition 3.2.2 (Composition Operator). *The composition operator for a set of conformal maps* $\mathbf{f} = (f_1, \ldots, f_n)$ *is defined by*

$$\widetilde{\mathcal{C}}_{\mathbf{f}} : \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) \longrightarrow \mathcal{D}_{harm}(\mathbb{D})^n$$
$$(g_1, \ldots, g_n) \longrightarrow (\mathcal{C}_{f_1} g_1, \ldots, \mathcal{C}_{f_n} g_n).$$

Similar definitions for the inverse functions, $\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}$, can be given. These operators are clearly bounded isometries with respect to Dirichlet semi-norm.

Definition 3.2.3. *Let* $G_k, k = 1, \ldots, n$*, be a collection of pairwise disjoint simply connected subsets of a Riemann surface* $\mathfrak{R}$*. Let* G *denote* $\cup_{k=1}^{n} G_k$*. Pick a point* $p_k \in G_k$*, for each* $k = 1, \ldots, n$*. Define*

$$\widetilde{P}_G : \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(G_k) \longrightarrow \bigoplus_{k=1}^{n} \mathcal{D}(G_k)$$
$$(h_1, \ldots, h_n) \longrightarrow (P_{G_1} h_1, \ldots, P_{G_n} h_n).$$

where P_{G_k} *projects* $h_k \in \mathcal{D}_{harm}(A_k)$ *to its holomorphic part, as was defined in Section 1.2.2.*

The projection to the anti-holomorphic part $\overline{\widetilde{P}}_G$ is defined similarly. The projections $\tilde{P}(G)$ and $\overline{\tilde{P}}(G)$ on harmonic 1-forms can be defined as well.

One of the operators that was used for the case of one boundary curve was the transmission operator $\mathfrak{O}(\Omega, \Sigma)$ or $\mathfrak{O}(\Sigma, \Omega)$. We need a modification of this transmission for the the case of many boundary curves. The proof of its existence, however, makes use of the one boundary curve case.

For fixed $j = 1, \ldots, n$, we will define $\mathfrak{O}(\Sigma, \Omega_j^+)$; that is, the transmission from Σ to Ω_j^+. We start with B_j a collar neighbourhood of Γ_j in Σ and let $Res(\Sigma, B_j)$ be the restriction from Σ to B_j operator. We have the following definition.

Definition 3.2.4. $\mathfrak{O}(\Sigma, \Omega_j^+) := \mathfrak{O}(\Omega_j^-, \Omega_j^+)\mathfrak{G}(B_j, \Omega_j^-)Res(\Sigma, B_j)$.

Every operator on the right hand side of the above definition is bounded, so $\mathfrak{O}(\Sigma, \Omega_j^+)$ is a bounded operator. We usually drop the restriction operator $Res(\Sigma, B_j)$ it is clear from the context.

We will need the following version of the transmission operator.

Definition 3.2.5 (Transmission Operator)**.** *The transmission operator for* Σ *is defined by*

$$\begin{aligned} \widetilde{\mathfrak{O}}(\Sigma, \Omega) : \mathcal{D}_{harm}(\Sigma) &\longrightarrow \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) \\ h &\longrightarrow \left(\mathfrak{O}(\Sigma, \Omega_1^+)h, \ldots, \mathfrak{O}(\Sigma, \Omega_n^+)h\right). \end{aligned}$$

We generalize the subspace 3.1 to the case of many boundary curves as follows.

Definition 3.2.6. *Define*

$$\mathcal{W} := \left\{ (h_1, \ldots, h_n) \in \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) \;:\; \lim_{\epsilon \to 0^+} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^{p_k}} h_k \alpha = 0 \;\; ; \;\; \forall \alpha \in A(\mathfrak{R}) \right\}.$$

We define the Cauchy-type integral operator for $\Omega_1^+, \ldots, \Omega_n^+$, as follows.

Definition 3.2.7 (Cauchy-Type Integral Operator)**.** *Let $z, q \in \mathfrak{R}\backslash\Gamma$. The Cauchy-type integral operator for domains $\Omega_1^+, \ldots, \Omega_n^+$, is defined by*

$$J_q(\Gamma) : \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) \to \mathcal{D}_{harm}(\mathfrak{R}\backslash\Gamma)_q$$
$$(h_1, \ldots, h_n) \to -\lim_{\epsilon\to 0^+} \sum_{k=1}^{n} \frac{1}{\pi i} \int_{\Gamma_\epsilon^{p_k}} \partial_w g(w; z, q)\, h_k(w).$$

It should be noted that Γ here is understood as $\cup_{k=1}^n \Gamma_k$. Also $\mathcal{D}_{harm}(\mathfrak{R}\backslash\Gamma)_q$ may need clarification. By this we mean the output of the $J_q(\Gamma)$ operator is a harmonic function on Σ if $z \in \Sigma$ and is a harmonic function on Ω_k^+ if $z \in \Omega_k^+$ for some $k = 1, 2, \ldots, n$. The output does not extend to a continuous function on $\mathfrak{R}$.

Theorem 3.2.8. *Let $J_q(\Gamma)$ be defined as above. Then we have the following*

1. *The output of $J_q(\Gamma)$ is in $\mathcal{D}_{harm}(\mathfrak{R}\backslash\Gamma)_q$.*
2. *The operator $[J_q(\Gamma)]_\Sigma$ is a bounded operator from $\bigoplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)$ equipped with norm $\|.\|_{\oplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)}$ to $\mathcal{D}_{harm}(\Sigma)$ equipped with norm $\|.\|_{\mathcal{D}_{harm}(\Sigma)}$. The same result is true for the restriction $[J_q(\Gamma)]_{\Omega_k^+}$ on each Ω_k^+.*
3. *If the domain of $[J_q(\Gamma)]_\Sigma$ is restricted to $\mathcal{W}$, then the output will be a holomorphic function on Σ.*

Proof. We start with proving (1) and (2) together. The Cauchy-type integral operator $J_q(\Gamma)$ can be rewritten as a finite sum of the Cauchy-type integral operators for the one boundary curve case. That is

$$J_q(\Gamma)(h_1, \ldots, h_n) = \sum_{k=1}^{n} J_q(\Gamma_k) h_k. \tag{3.3}$$

Recall that $J_q(\Gamma_k)$ is harmonic everywhere on $\mathfrak{R}$ except on Γ_k and vanishing at q, therefore the right hand side of the above sum is harmonic everywhere on $\mathfrak{R}$ except on Γ and vanishes at q. The boundedness can be proven in the same way.

For the part (3), the proof is similar to the one boundary curve case proven in [62]. $\square$

In the above proof note that even though the sum in the right hand side of (3.3) when is restricted to Σ is a holomorphic function on Σ, not each term in the sum is necessarily a holomorphic function on Σ.

As in the case of one boundary curve, fix $p_k \in \Omega_k^+$ for each k and define

$$\mathcal{W}' = \bigoplus_{k=1}^{n} \overline{\mathcal{D}(\Omega_k^+)}_{p_k} \cap \mathcal{W}.$$

where $\overline{\mathcal{D}(\Omega_k^+)}_{p_k}$ is the set of those anti-holomorphic functions in $\overline{\mathcal{D}(\Omega_k^+)}$ which vanish at $p_k \in \Omega_k^+$. For notational simplicity we define the following operator.

Definition 3.2.9. *Define the operator*

$$K : \mathcal{W}' \to \mathcal{D}(\Sigma)_q$$

by

$$K(\overline{h}_1, \ldots, \overline{h}_n) = [J_q(\Gamma)(\overline{h}_1, \ldots, \overline{h}_n)]_\Sigma = \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_\Sigma.$$

As we see, the operator K is the operator $J_q(\Gamma)$ when we apply some restrictions to both its domain and its output. The boundedness of the operator K is a consequence of part (3) in the above theorem. In the definition of

K, we purposely use $[J_q(\Gamma_k)_{\Omega_k^-}\overline{h}_k]_\Sigma$ to indicates that although the function $J_q(\Gamma_k)\overline{h}_k$ is ultimately restricted to Σ, it carries an extension to the bigger domain Ω_k^-.

To find an appropriate function space on the complex plane for the domain of Faber operator in many boundary curve case, we will pull back functions in $\mathcal{W}'$ to $\overline{\mathcal{D}(\mathbb{D})}^n$, we therefore have this definition

$$\overline{\mathcal{D}_v(\mathbb{D})}^n := \left\{(\overline{H}_1, \ldots, \overline{H}_n) \in \overline{\mathcal{D}(\mathbb{D})}^n : \widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1, \ldots, \overline{H}_n) \in \mathcal{W}'\right\}. \tag{3.4}$$

In particular, $\overline{H_k}(0) = 0$ for each k here.

In the next subsection, we will show that this function space will be the right domain for the Faber operator corresponding to Σ, exactly as in the one boundary curve case.

We continue by extending the $\mathfrak{G}$ operator, defined in Section 1.5, to the many boundary curve case. Let A_k be a collar neighbourhood of Γ_k in Ω_k^+ for each $k = 1, \ldots, n$. Let A denote $\cup A_k$ and Ω denote $\cup\Omega_k^+$. Define $\widetilde{\mathfrak{G}}(A, \Omega)$ by

$$\begin{aligned} \widetilde{\mathfrak{G}}(A, \Omega) : \bigoplus_{k=1}^{n} \mathcal{D}(A_k) &\to \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) \\ (h_1, \ldots, h_n) &\to \left(\mathfrak{G}(A_1, \Omega_1^+)h_1, \ldots, \mathfrak{G}(A_n, \Omega_n^+)h_n\right). \end{aligned}$$

$\widetilde{\mathfrak{G}}(A, \Omega)$ is clearly a bounded operator since for each k, the operator $\mathfrak{G}(A_k, \Omega_k^+)$ is bounded.

The following density theorem is similar to a density theorem proven in [62] to show that the transmitted-jump formula is valid. We want to generalize it from the one boundary curve case to the many ones. This theorem will be used to show the existence of a left inverse for the K operator.

Theorem 3.2.10. *Let A_k be a collar neighbourhood of Γ_k in Ω_k^+ and let Γ'_k be a simple closed analytic curve in A_k, isotopic to Γ_k from within A_k for each $k = 1, 2, \ldots, n$. If*

$$E = \left\{ (h_1, \ldots, h_n) \in \bigoplus_{k=1}^{n} \mathcal{D}(A_k) : \sum_{k=1}^{n} \int_{\Gamma'_k} h_k \alpha = 0 \ ; \ \forall \alpha \in A(\mathfrak{R}) \right\},$$

then $\widetilde{\mathfrak{G}}(A, \Omega)E$ is dense in $\mathcal{W}$.

Proof. Let $\mathcal{P} : \bigoplus_{k=1}^{n} \mathcal{D}(A_k) \to E$ be the orthogonal projection to the subspace E. Fix a basis $\{\alpha_1, \ldots, \alpha_{\mathfrak{g}}\}$ for $A(\mathfrak{R})$, where $\mathfrak{g}$ is the genus of $\mathfrak{R}$. Define the operator Q by

$$\begin{aligned} Q : \bigoplus_{k=1}^{n} \mathcal{D}(A_k) &\to \mathbb{C}^{\mathfrak{g}} \\ (u_1, \ldots, u_n) &\to \left(\sum_{k=1}^{n} \int_{\Gamma'_k} u_k \alpha_1, \ldots, \sum_{k=1}^{n} \int_{\Gamma'_k} u_k \alpha_{\mathfrak{g}} \right). \end{aligned}$$

For fixed j the map $(u_1, \ldots, u_n) \to \sum_{k=1}^{n} \int_{\Gamma'_k} u_k \alpha_j$ is a bounded linear functional on $\bigoplus_{k=1}^{n} \mathcal{D}(A_k)$. By using the Reisz representation theorem and the Gram-Schmidt process, there exists a $C > 0$ such that

$$\|\mathcal{P}(u_1, \ldots, u_n) - (u_1, \ldots, u_n)\|_{\bigoplus_{k=1}^{n} \mathcal{D}_{harm}(A_k)} \leq C \|Q(u_1, \ldots, u_n)\|_{\mathbb{C}^{\mathfrak{g}}}.$$

The operator Q_1 is defined by

$$\begin{aligned} Q_1 : \bigoplus_{k=1}^{n} \mathcal{D}_{harm}(\Omega_k^+) &\to \mathbb{C}^{\mathfrak{g}} \\ (h_1, \ldots, h_n) &\to \left(\lim_{\epsilon \to 0^+} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^{p_k}} h_k \alpha_1, \ldots, \lim_{\epsilon \to 0^+} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^{p_k}} h_k \alpha_{\mathfrak{g}} \right). \end{aligned}$$

By the definition of $\mathcal{W}$, $Q_1(h_1,\ldots,h_n) = 0$ if $(h_1,\ldots,h_n) \in \mathcal{W}$. Similarly there exists $D > 0$, such that

$$\|Q_1(h_1,\ldots,h_n)\|_{\mathbb{C}^{\mathfrak{g}}} \leq D\,\|(h_1,\ldots,h_n)\|_{\bigoplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)}.$$

Furthermore, [62, Theorem 4.9] implies

$$Q(u_1,\ldots,u_n) = Q_1(\mathfrak{G}(A_1,\Omega_1^+)u_1,\ldots,\mathfrak{G}(A_n,\Omega_n^+)u_n) = Q_1\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\ldots,u_n).$$

Now let $(h_1,\ldots,h_n) \in \mathcal{W}$. By the density of $\mathfrak{G}(A_k,\Omega_k^+)\mathcal{D}(A_k)$ in $\mathcal{D}_{harm}(\Omega_k^+)$ [62, Theorem 4.6], for each $h_k \in \mathcal{D}_{harm}(\Omega_k^+)$ there exists $u_k \in \mathcal{D}(A_k)$ such that

$$\left\|\mathfrak{G}(A_k,\Omega_k^+)u_k - h_k\right\|_{\mathcal{D}_{harm}(\Omega_k^+)} \leq \frac{\epsilon}{\sqrt{n}}.$$

Thus, by the Minkowski inequality we have

$$\begin{aligned}
&\left\|\widetilde{\mathfrak{G}}(A,\Omega)\mathcal{P}(u_1,\ldots,u_n) - (h_1,\ldots,h_n)\right\| \\
&\leq \left\|\widetilde{\mathfrak{G}}(A,\Omega)\mathcal{P}(u_1,\ldots,u_n) - \widetilde{\mathfrak{G}}(A,\Omega)(u_1,\ldots,u_n)\right\| \\
&\quad + \left\|\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\ldots,u_n) - (h_1,\ldots,h_n)\right\| \\
&\leq \left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|\left\|\mathcal{P}(u_1,\ldots,u_n) - (u_1,\ldots,u_n)\right\| \\
&\quad + \left\|\left(\mathfrak{G}(A_1,\Omega_1^+)u_1 - h_1,\ldots,\mathfrak{G}(A_n,\Omega_n^+)u_n - h_n\right)\right\|
\end{aligned} \tag{3.5}$$

where all the norms are $\|.\|_{\bigoplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)}$ except the operator norm $\left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|$.

Since $Q_1(h_1,\dots,h_n)=0$ the second part of the first term in (3.5) can be estimated as

$$\begin{aligned}\|\mathcal{P}(u_1,\dots,u_n)-(u_1,\dots,u_n)\| &\le C\,\|Q(u_1,\dots,u_n)\|_{\mathbb{C}^{\mathfrak{g}}}\\ &= C\left\|Q_1\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\dots,u_n)\right\|_{\mathbb{C}^{\mathfrak{g}}}\\ &= C\left\|Q_1\left(\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\dots,u_n)-(h_1,\dots,h_n)\right)\right\|_{\mathbb{C}^{\mathfrak{g}}}\\ &\le CD\left\|\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\dots,u_n)-(h_1,\dots,h_n)\right\|.\end{aligned}$$

By our choice of u_k, for the second term in (3.5) we have

$$\begin{aligned}&\left\|\left(\mathfrak{G}(A_1,\Omega_1^+)u_1-h_1,\dots,\mathfrak{G}(A_n,\Omega_n^+)u_n-h_n\right)\right\|\\ &=\left(\sum_{k=1}^{n}\left\|\mathfrak{G}(A_k,\Omega_k^+)u_k-h_k\right\|^2_{\mathcal{D}_{harm}(\Omega_k^+)}\right)^{\frac{1}{2}}\\ &\le\left(\sum_{k=1}^{n}\frac{\epsilon^2}{n}\right)^{\frac{1}{2}}=\epsilon.\end{aligned}$$

Finally, by inserting the above two inequalities in (3.5) one has

$$\begin{aligned}&\left\|\widetilde{\mathfrak{G}}(A,\Omega)\mathcal{P}(u_1,\dots,u_n)-(h_1,\dots,h_n)\right\|\\ &\le CD\left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|\,\left\|\widetilde{\mathfrak{G}}(A,\Omega)(u_1,\dots,u_n)-(h_1,\dots,h_n)\right\|+\epsilon\\ &\le CD\left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|\epsilon+\epsilon=(CD\left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|+1)\epsilon.\end{aligned}$$

Therefore, for given $\epsilon>0$ and $(h_1,\dots,h_n)\in\mathcal{W}$, there exists $\mathcal{P}(u_1,\dots,u_n)\in E$ such that

$$\left\|\widetilde{\mathfrak{G}}(A,\Omega)\mathcal{P}(u_1,\dots,u_n)-(h_1,\dots,h_n)\right\|\le(\left\|\widetilde{\mathfrak{G}}(A,\Omega)\right\|CD+1)\epsilon.$$

which shows that $\widetilde{\mathfrak{G}}(A,\Omega)E$ is dense in $\mathcal{W}$. $\square$

We need the following lemma to show that the K operator has a left inverse and also for some other points in thesis. We temporary change the definitions of Σ and Ω in the following lemma.

Lemma 3.2.11. *Let Γ be a quasicircle in $\mathfrak{R}$ separating $\mathfrak{R}$ into two subsurfaces Ω and Σ. If A is a collar neighbourhood of the curve Γ in Ω, then for all $h \in \mathcal{D}(cl(\Sigma) \cup A)$ the following is true:*

$$\mathfrak{O}(\Sigma, \Omega)(h|_\Sigma) = \mathfrak{G}(A, \Omega)(h|_A).$$

Proof. By Theorem 1.4.4 we need to show that both sides of the above equality have the same CNT boundary values except possibly on a null set in Γ. The harmonic function $\mathfrak{O}(\Sigma, \Omega)(h|_\Sigma)$ has the same CNT boundary values on Γ as $h|_\Sigma$ has except possibly on a null set in Γ.

On the other hand, $\mathfrak{G}(A, \Omega)(h|_A)$ is the unique element in $\mathcal{D}_{harm}(\Omega)$ for which its CNT limits equal to $h|_A$ except possibly on a null set in Γ. Furthermore, the equality $h|_A = h|_\Sigma$ on Γ is clearly true since h is defined on $cl(\Sigma) \cup A$. Therefore, $\mathfrak{O}(\Sigma, \Omega)(h|_\Sigma) = \mathfrak{G}(A, \Omega)(h|_A)$ except possibly on a null set in Γ by Lemma 1.4.3, and they are equal on Ω by Lemma 1.4.3 and Theorem 1.4.4. □

Theorem 3.2.12. *The operator $-\widetilde{\widetilde{P}}_\Omega \widetilde{\mathfrak{O}}(\Sigma, \Omega)$ is a bounded left inverse of the K operator .*

Proof. For each $k = 1, 2, \ldots, n$, let A_k be a collar neighbourhood of Γ_k in Ω_k^+, let also $h_k \in \mathcal{D}(A_k)$. Define $\mathcal{E} = \cup_{k=1}^n (\Omega_k^+ \backslash A_k)$ and $\mathcal{O} = \cup_{k=1}^n \Omega_k^+$ which are a closed subset and an open subset of $\mathfrak{R}$, respectively. $\Gamma = \cup_{k=1}^n \Gamma_k$ is the boundary of $\mathcal{O}$. We also have $\mathcal{O} \backslash \mathcal{E} = \cup_{k=1}^n A_k$.

We want to first show that for elements in E, the $-\widetilde{\widetilde{P}}_{\Omega}\widetilde{\mathfrak{O}}(\Sigma,\Omega)$ operator is a bounded left inverse of the K operator; then by density of $\widetilde{\mathfrak{G}}(A,\Omega)E$ in $\mathcal{W}$ (Theorem 3.2.10) and the boundedness of the operators here, the claim is true for every element in $\mathcal{W}'$ which is a subset of $\mathcal{W}$.

Let $(h_1,\ldots,h_n) \in E$. Define $h = \sum_{k=1}^{n} h_k\chi_{A_k}$. Here χ_{A_k} is the characteristic function of A_k. By applying H. L. Royden [53, Theorem 4], there exists $H_1 \in \mathcal{D}(\mathcal{O})$ ($F - f$ in the theorem) and $H_2 \in \mathcal{D}(\mathfrak{R}\backslash\mathcal{E}) = \mathcal{D}(cl(\Sigma) \cup (\cup_{k=1}^{n} A_k))$ ($-f$ in the theorem) which satisfy

$$h(z) = H_1(z) - H_2(z) \;\; ; \;\; \forall z \in \mathcal{O}\backslash\mathcal{E} = \cup_{k=1}^{n} A_k.$$

We take the CNT boundary value limits of the above equation when z goes to points on $\partial\mathcal{O} = \cup_{k=1}^{n}\Gamma_k$ from within $\cup_{k=1}^{n} A_k$. Using the same notation h for the boundary value function of h, we have

$$h(z) = H_1(z) - H_2(z) \;\; ; \;\; \forall z \in \partial\mathcal{O}.$$

Note that by restricting to each $A_k, k = 1,\ldots,n$, or equivalently on each Γ_k, the above equation generates in fact n equations. Now we restrict the above equation to Ω_j^+ for a fixed $j = 1,\ldots,n$. By Lemma 3.2.11, the CNT boundary values of the function $\mathfrak{O}(\Sigma,\Omega_j^+)(H_2|_\Sigma)$ on Γ_j is equal to CNT boundary values of the function $\mathfrak{G}(A_j,\Omega_j^+)(H_2|_{A_j})$ or equivalently $H_2|_{A_j}$ except possibly on a null set in Γ. Therefore,

$$
\begin{aligned}
&h_j(z) = h|_{A_j}(z) = H_1|_{A_j}(z) - H_2|_{A_j}(z) \quad ; \quad \forall z \in \Gamma_j, \\
&\mathfrak{G}(A_j, \Omega_j^+)h_j(z) = \mathfrak{G}(A_j, \Omega_j^+)(H_1|_{A_j})(z) - \mathfrak{G}(A_j, \Omega_j^+)(H_2|_{A_j})(z) \;\; ; \;\; \forall z \in \Omega_j^+, \\
&\mathfrak{G}(A_j, \Omega_j^+)h_j(z) = H_1|_{\Omega_j^+}(z) - \mathfrak{O}(\Sigma, \Omega_j^+)(H_2|_{\Sigma})(z) \;\; ; \;\; \forall z \in \Omega_j^+.
\end{aligned}
\tag{3.6}
$$

We proceed by calculating H_1 restricted on Ω_j^+ and H_2 restricted to Σ using the explicit formula that was given in Royden's theorem [53, Theorem 4]. Let $\Gamma' = \cup_{k=1}^n \Gamma'_k$ and $z \in \Omega_j^+ \subset \mathcal{O}$. We then have

$$
\begin{aligned}
H_1|_{\Omega_j^+}(z) = [J'_q(\Gamma')h]_{\mathcal{O}}(z) &= -\frac{1}{\pi i}\int_{\Gamma'} (\sum_{k=1}^n h_k \chi_{A_k})(w)\, \partial_w g(w; z, q) \\
&= -\frac{1}{\pi i}\sum_{k=1}^n \int_{\Gamma'_k} h_k(w)\, \partial_w g(w; z, q) \\
&= \sum_{k \neq j} [J'_q(\Gamma'_k)_{\Omega_k^-} h_k]_{\Omega_j^+}(z) + [J'_q(\Gamma'_j)_{\Omega_j^+} h_j](z) \\
&= \sum_{k \neq j} [J_q(\Gamma_k)_{\Omega_k^-} \mathfrak{G}(A_k, \Omega_k^+) h_k]_{\Omega_j^+}(z) \\
&\quad + [J_q(\Gamma_j)_{\Omega_j^+} \mathfrak{G}(A_j, \Omega_j^+) h_j](z)
\end{aligned}
$$

where the last equality is coming from [62, Theorem 4.9]. On the other hand, if $z \in \Sigma$, we have

$$\begin{aligned} H_2|_\Sigma(z) = [J'_q(\Gamma')h]_\Sigma(z) &= -\frac{1}{\pi i}\int_{\Gamma'} (\sum_{k=1}^{n} h_k \chi_{A_k})(w)\, \partial_w g(w; z, q) \\ &= -\frac{1}{\pi i}\sum_{k=1}^{n}\int_{\Gamma'_k} h_k(w)\, \partial_w g(w; z, q) \\ &= \sum_{k=1}^{n} [J'_q(\Gamma'_k)_{\Omega_k^-} h_k]_\Sigma(z) \\ &= \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} \mathfrak{G}(A_k, \Omega_k^+) h_k]_\Sigma(z). \end{aligned}$$

By these two equalities, the last equation in (3.6) can be written as

$$\begin{aligned} \mathfrak{G}(A_j, \Omega_j^+)h_j(z) = &\sum_{k\neq j} [J_q(\Gamma_k)_{\Omega_k^-} \mathfrak{G}(A_k, \Omega_k^+) h_k]_{\Omega_j^+}(z) + [J_q(\Gamma_j)_{\Omega_j^+} \mathfrak{G}(A_j, \Omega_j^+) h_j](z) \\ &- \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} \mathfrak{G}(A_k, \Omega_k^+) h_k]_\Sigma(z) \quad ; \quad \forall z \in \Omega_j^+. \end{aligned} \tag{3.7}$$

Recall that $H_1 \in \mathcal{D}(\Omega_j^+)$ and $\mathfrak{O}(\Sigma, \Omega_j^+)(H_2|_\Sigma) \in \mathcal{D}_{harm}(\Omega_j^+)$ in the last line of (3.6). By applying the projection operator $\overline{P}_{\Omega_j^+}$ to the both sides of this equation we have

$$\begin{aligned} \overline{P}_{\Omega_j^+} \mathfrak{G}(A_j, \Omega_j^+)h_j(z) &= \overline{P}_{\Omega_j^+} H_1(z) - \overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+)(H_2|_\Sigma)(z) \\ &= 0 - \overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+)(H_2|_\Sigma)(z) \\ &= -\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+)(H_2|_\Sigma)(z) \\ &= -\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^{n} \left[J_q(\Gamma_k)_{\Omega_k^-} \mathfrak{G}(A_k, \Omega_k^+) h_k\right]_\Sigma(z), \end{aligned}$$

for all $z \in \Omega_j^+$. Now by density of $\widetilde{\mathfrak{G}}(A, \Omega)E$ in $\mathcal{W}$ and the boundedness of each operator that we have used, we derive the following important equation,

$$\overline{P}_{\Omega_j^+} h_j(z) = -\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} h_k]_\Sigma(z). \tag{3.8}$$

for all $(h_1, \ldots, h_n) \in \mathcal{W}$ and $z \in \Omega_j^+$. We can simplify this equation further by expanding the transmission operator $\mathfrak{O}(\Sigma, \Omega_j^+)$ to each term

$$\begin{aligned}
\overline{P}_{\Omega_j^+} h_j(z) &= -\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} h_k]_\Sigma(z) \\
&= -\overline{P}_{\Omega_j^+} \sum_{k=1}^{n} \mathfrak{O}(\Sigma, \Omega_j^+) [J_q(\Gamma_k)_{\Omega_k^-} h_k]_\Sigma(z) \\
&= -\overline{P}_{\Omega_j^+} \sum_{k \neq j} \mathfrak{O}(\Sigma, \Omega_j^+) [J_q(\Gamma_k)_{\Omega_k^-} h_k]_\Sigma(z) \\
&\quad - \overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) [J_q(\Gamma_j)_{\Omega_j^-} h_j]_\Sigma(z) \\
&= -\overline{P}_{\Omega_j^+} \sum_{k \neq j} [J_q(\Gamma_k)_{\Omega_k^-} h_k]_{\Omega_j^+}(z) \\
&\quad - \overline{P}_{\Omega_j^+} \mathfrak{O}(\Omega_j^-, \Omega_j^+) [J_q(\Gamma_j)_{\Omega_j^-} h_j](z)
\end{aligned} \tag{3.9}$$

for $z \in \Omega_j^+$, which is again valid for all $(h_1, \ldots, h_n) \in \bigoplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)$.

Finally, if $(\overline{h}_1, \ldots, \overline{h}_n) \in \bigoplus_{k=1}^n \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W}$, then by (3.8) we have

$$\begin{aligned}
\overline{h}_j(z) = \overline{P}_{\Omega_j^+} \overline{h}_j(z) &= -\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_\Sigma(z) \\
&= [-\overline{P}_{\Omega_j^+} \mathfrak{O}(\Sigma, \Omega_j^+) K(\overline{h}_1, \ldots, \overline{h}_n)](z).
\end{aligned} \tag{3.10}$$

Therefore, a left inverse of the operator K is

$$-\widetilde{\overline{P}}_\Omega \widetilde{\mathfrak{O}}(\Sigma, \Omega).$$

□

The above theorem has some consequences for the Faber operator which will be defined soon.

Lemma 3.2.13. *Let $f : \mathbb{D} \to \Omega^+ \subset \mathfrak{R}$ be a conformal map and $\overline{P}$ and $\mathcal{C}_f$ be the projection to the anti-holomorphic part and the composition with f operators, respectively. Then*

$$\overline{P}_{\mathbb{D}}\, \mathcal{C}_f = \mathcal{C}_f \overline{P}_{\Omega^+}.$$

Proof. Given any $h \in \mathcal{D}_{harm}(\Omega^+)$, $h \circ f \in \mathcal{D}_{harm}(\mathbb{D})$ can be written as $H_1 + \overline{H}_2$ for some $H_k \in \mathcal{D}(\mathbb{D}), k = 1, 2$, with $H_1(0) = 0$. Therefore, $\overline{P}_{\mathbb{D}}\, \mathcal{C}_f h = \overline{P}_{\mathbb{D}}(H_1 + \overline{H}_2) = \overline{H}_2$.

The above equality implies also that h can be written as $H_1 \circ f^{-1} + \overline{H}_2 \circ f^{-1}$, thus the right hand side is $\mathcal{C}_f \overline{P}_{\Omega^+} h = \mathcal{C}_f(\overline{H}_2 \circ f^{-1}) = \overline{H}_2$. Since $h \in \mathcal{D}_{harm}(\Omega^+)$ was arbitrary this completes the proof. □

Remark 3.2.14. *The same result is clearly valid for P_{Ω^+} and $P_{\mathbb{D}}$ or the composition with the map f^{-1}.*

Before defining the Faber and Grunsky operators for the many boundary curve case we need another density theorem. In the next section we introduce this theorem.

3.2.2 A Density Theorem

In order to prove some inequalities regarding the Grunsky operator norm, as in Theorem 3.2.26 ahead, we need a density theorem. This density theorem was obtained in a joint work with Schippers and Staubach [59]. Here we restate this theorem using the notation used here.

Theorem 3.2.15. *Let $\mathfrak{R}$, Σ, Γ_k's, $\Omega_k^{\pm}$'s be the same as Section 3.2. Suppose each boundary curve Γ_k is a quasicircle. For each $k = 1, 2, \ldots, n$, and $\epsilon > 0$ sufficiently small, define*

$$\Omega_{k\epsilon}^+ = \left\{ w \in \Omega_k^+ : g_{\Omega_k^+}(w, p_k) < \epsilon \right\},$$

for some p_k fixed in Ω_k^+. Then the set of the restrictions of functions in $\mathcal{D}(cl(\Sigma) \cup \Omega_{1\epsilon}^+ \cup \cdots \cup \Omega_{n\epsilon}^+)$ to Σ is dense in $\mathcal{D}(\Sigma)$.

Each $\Omega_{k\epsilon}^+$ here is in fact a collar neighbourhood of Γ_k in Ω_k^+ by the definition given in Section 1.4.

3.2.3 Faber and Grunsky Operators

Definition 3.2.16 (Faber Operator)**.** *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be as above. Define the Faber operator corresponding to $\mathbf{f}$ (or Σ), by*

$$\begin{aligned} \mathbf{I_f} : \overline{\mathcal{D}_v(\mathbb{D})}^n &\to \mathcal{D}(\Sigma)_q \\ (\overline{H}_1, \ldots, \overline{H}_n) &\to -K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1, \ldots, \overline{H}_n). \end{aligned}$$

In other words,

$$\mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)(z) = -\sum_{k=1}^{n} [J_q(\Gamma_k)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]_\Sigma(z) \;\; ; \;\; z \in \Sigma.$$

Since the output of each term $[J_q(\Gamma)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]$ in the sum is a harmonic function on Ω_k^-, the sum is a harmonic function on $\Sigma = \cap_{k=1}^n \Omega_k^-$. Since $\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1, \ldots, \overline{H}_n)$ is in $\mathcal{W}'$, the sum is in fact holomorphic on Σ and is zero at the point q by Theorem 3.2.8, so the above definition makes sense.

Remark 3.2.17. *It should be noted that the composition operator $\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}$ used in the definition of the Faber operator is in fact $\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}|_{\overline{\mathcal{D}_v(\mathbb{D})}^n} : \overline{\mathcal{D}_v(\mathbb{D})}^n \to \mathcal{W}'$. However, we still use the same notation for that.*

Corollary 3.2.18. *The Faber operator $\mathbf{I}_{\mathbf{f}}$ is a bounded operator from $\overline{\mathcal{D}_v(\mathbb{D})}^n$ equipped with the norm $\|.\|_{\mathcal{D}_{harm}(\mathbb{D})^n}$ to $\mathcal{D}(\Sigma)_q$ equipped with the norm $\|.\|_{\mathcal{D}_{harm}(\Sigma)}$.*

Here is a corollary to Theorem 3.2.12.

Corollary 3.2.19. *The operator $\widetilde{\mathcal{C}}_{\mathbf{f}}\widetilde{\overline{P}}_{\Omega}\widetilde{\mathfrak{O}}(\Sigma,\Omega)$ is a bounded left inverse for the Faber operator $\mathbf{I}_{\mathbf{f}}$. Therefore, $\mathbf{I}_{\mathbf{f}}$ is a one-to-one operator.*

Similar to the generalization of the Faber operator, we would like to generalize the Grunsky operator corresponding to $\mathbf{f} = (f_1, \ldots, f_n)$ or equivalently to Σ.

Definition 3.2.20 (Grunsky Operator)**.** *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be as above. Assume all the boundary curves of Σ are quasicircles. Define*

$$Gr_{jk}(\mathbf{f}) : \overline{\mathcal{D}(\mathbb{D})} \to \mathcal{D}(\mathbb{D}),$$

by

$$Gr_{jk}(\mathbf{f})\overline{H} = \begin{cases} -P_{\mathbb{D}}\mathcal{C}_{f_j}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}]_{\Omega_j^+}, & \text{if } j \neq k \\ -P_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}], & \text{if } j = k. \end{cases}$$

Then the Grunsky operator corresponding to $\mathbf{f}$ (or Σ) $\mathbf{Gr}_{\mathbf{f}} : \overline{\mathcal{D}_v(\mathbb{D})}^n \to \mathcal{D}(\mathbb{D})^n$ is defined by

$$\mathbf{Gr}_{\mathbf{f}}\left(\overline{H}_1, \ldots, \overline{H}_n\right) = \left(\sum_{k=1}^{n} Gr_{1k}(\mathbf{f})\overline{H}_k, \ldots, \sum_{k=1}^{n} Gr_{nk}(\mathbf{f})\overline{H}_k\right).$$

We emphasize that inside each component we are adding finitely many holomorphic functions on the unit disc in the complex plane, so the component is a holomorphic function on $\mathbb{D}$. Note that each $\mathbf{Gr}_{jk}(\mathbf{f})$ is defined on the bigger domain $\overline{\mathcal{D}(\mathbb{D})}$ whereas $\mathbf{Gr_f}$ is only defined on $\overline{\mathcal{D}_v(\mathbb{D})}^n$ which is not clearly $\left(\overline{\mathcal{D}(\mathbb{D})}\right)^n$.

By Definitions 3.1.9 and 3.1.10 it turns out that $Gr_{jj}(\mathbf{f})$ is simply $\mathbf{Gr}_{f_j}$.

Remark 3.2.21. *For the Riemann sphere, the $\mathfrak{g} = 0$ case, a similar formulation of the Grunsky operator for a family of conformal maps $\mathbf{f} = (f_1, \ldots, f_n)$ with non overlapping images in $\overline{\mathbb{C}}$ was given in [46, Definition 4.6].*

The proof of the boundedness of the operator $\mathbf{Gr_f}$ is not as easy as the one for the operator $\mathbf{I_f}$, and is one of the main results of the thesis. Here we will show that if each boundary curve Γ_k of Ω_k^+ is a quasicircle, then $\mathbf{Gr_f}$ is a bounded operator of the norm less than one. The complete proof of norm is strictly less than one will be done later. To accomplish that we first need a few lemmas.

We did not make use of the holomorphicity of the function h in the previous proof except to prove the existence of the CNT boundary values on Γ. Therefore, Lemma 3.2.11 is also true for harmonic functions on $cl(\Sigma) \cup A$.

Lemma 3.2.22. *Let $\mathfrak{R}$ and Σ be as above and Ω be a simply connected subsurface of $\mathfrak{R}$ bordered by a strip-cutting Jordan curve. Let $f : \mathbb{D} \to \Omega$ be a conformal map onto Ω and $\mathbb{A}$ be an annulus in $\mathbb{D}$ whose outer boundary is the unit circle. Suppose f maps $\mathbb{A}$ conformally to a collar neighbourhood $A = f(\mathbb{A})$ of Γ in Ω. Then for every $h \in \mathcal{D}_{harm}(A)$ one has*

$$\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_f h = \mathcal{C}_f \mathfrak{G}(A, \Omega) h.$$

Proof. First note that the operator $\mathcal{C}_f$ on the left hand side is in fact $\mathcal{C}_{f|_{\mathbb{A}}}$; we, however, use the same notation as $\mathcal{C}_f$ for simplicity. The existence of the CNT limits of h and $\mathcal{C}_f h$ follows from Theorem 1.4.5.

If $h \in \mathcal{D}_{harm}(A)$, then by definition of $\mathfrak{G}(\mathbb{A}, \mathbb{D})$, $\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_f h$ is the unique function in $\mathcal{D}_{harm}(\mathbb{D})$ such that $\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_f h$ and $\mathcal{C}_f h$ have the same non-tangential limits except possibly on a set of logarithmic capacity zero in $\mathbb{S}^1$.

Similarly, $\mathfrak{G}(A, \Omega)h$ is the unique function in $\mathcal{D}_{harm}(\Omega)$ such that $\mathfrak{G}(A, \Omega)h$ and h have the same CNT boundary values on Γ except possibly on a null set in Γ. By Definition 1.4.1 this means that $\mathcal{C}_f\mathfrak{G}(A, \Omega)h$ and $\mathcal{C}_f h$ have the same non-tangential limits on $\mathbb{S}^1$ except possibly on a set of logarithmic capacity zero in $\mathbb{S}^1$.

Therefore, $\mathfrak{G}(\mathbb{A}, \mathbb{D})\mathcal{C}_f h$ and $\mathcal{C}_f\mathfrak{G}(A, \Omega)h$ have the same non-tangential limits on $\mathbb{S}^1$ except possibly on a set of logarithmic capacity zero in $\mathbb{S}^1$, so they are equal on $\mathbb{D}$ by Lemma 1.4.3. □

The Laurent expansion of a holomorphic function h on an annulus $\mathbb{A}$ in the complex plane is well-known. Using the Laurent expansion of $h \in \mathcal{D}(\mathbb{A})$, we obtain two holomorphic functions of finite Dirichlet norm $h_+ \in \mathcal{D}(\mathbb{D})$ and $h_- \in \mathcal{D}(cl(\mathbb{D}^-) \cup \mathbb{A})$ such that $h = h_+ + h_-$ on $\mathbb{A}$. Note that h_- has a holomorphic extension to $\mathbb{C}\backslash\{0\}$. For later use, here we express this theorem on $\mathcal{D}(\mathbb{D})$ in terms of both the $\mathfrak{G}$ and $\mathfrak{O}$ operators.

Lemma 3.2.23 (Laurent Decomposition)**.** *A function $h \in \mathcal{D}(\mathbb{A})$ can be decomposed to $h_+ + h_-$ where h_+ and h_- are as follows,*

$$h_+ = P_{\mathbb{D}}\mathfrak{G}(\mathbb{A}, \mathbb{D})h \in \mathcal{D}(\mathbb{D}),$$

$$h_- = \mathfrak{O}(\mathbb{D}, \mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathfrak{G}(\mathbb{A}, \mathbb{D})h \in \mathcal{D}(\mathbb{D}^-).$$

Proof. Note that $\mathfrak{G}(\mathbb{A},\mathbb{D})z^n = z^n$ for $n \geq 1$, $\mathfrak{G}(\mathbb{A},\mathbb{D})z^{-n} = \bar{z}^n$ for $n \geq 0$, $\mathfrak{O}(\mathbb{D},\mathbb{D}^-)z^n = (\frac{1}{\bar{z}})^n$ for $n \geq 1$, and $\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\bar{z}^n = (\frac{1}{z})^n$ for $n \geq 0$. Then the density of the polynomials in z^n and z^{-n} in the Dirichlet space $\mathcal{D}(\mathbb{A})$ and the boundedness of the $\mathfrak{O}$ and $\mathfrak{G}$ operators completes the proof. $\square$

Lemma 3.2.24. *The transmission operator $\mathfrak{O}(\mathbb{D},\mathbb{D}^-)$ has norm one, so it is an isometry with respect to the Dirichlet semi-norm. That is,*

$$\left\|\mathfrak{O}(\mathbb{D},\mathbb{D}^-)h\right\|_{\mathcal{D}_{harm}(\mathbb{D}^-)} = \|h\|_{\mathcal{D}_{harm}(\mathbb{D})},$$

for every $h \in \mathcal{D}_{harm}(\mathbb{D})$. A similar result is true for $\mathfrak{O}(\mathbb{D}^-,\mathbb{D})$.

Proof. It can be shown that this is true for monomials. Then by the density of these elements in $\mathcal{D}_{harm}(\mathbb{D})$ and the boundedness of the transmission operator we have it in general. $\square$

Recall Remark 1.4.7 for the following theorem.

Lemma 3.2.25. *Let H and G be holomorphic functions on $\mathbb{D}$. Then*

$$\int_{\gamma_r} (\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}(z))'\,\overline{G}(z)\,dz = 0,$$
$$\int_{\gamma_r} \overline{(\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}(z))}\,G'(z)\,dz = 0,$$

where $\gamma_r = \{z \in \mathbb{C} : |z| = r\}$ for $0 < r < 1$.

Proof. The proof is essentially based on the Taylor expansions of $H(z) = \sum_{n=0}^{\infty} a_n z^n$ and $G(z) = \sum_{m=0}^{\infty} b_m z^m$ in the unit disc and Remark 1.4.7. Fix

$0 < r < 1$, then

$$\begin{aligned}
\int_{\gamma_r} (\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}(z))'\,\overline{G}(z)\,dz &= \int_{\gamma_r} (\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\sum_{n=0}^{\infty}\bar{a}_n\bar{z}^n)'\,\overline{(\sum_{m=0}^{\infty}b_m z^m)}\,dz \\
&= \int_{\gamma_r} (\sum_{n=0}^{\infty}\bar{a}_n\,z^{-n})'(\sum_{m=0}^{\infty}\bar{b}_m\,\bar{z}^m)\,dz \\
&= \int_{\gamma_r} (-\sum_{n=0}^{\infty}n\,\bar{a}_n\,z^{-n-1})(\sum_{m=0}^{\infty}\bar{b}_m\,\bar{z}^m)\,dz \\
&= -\int_{\gamma_r} (\sum_{n=1}^{\infty}n\,\bar{a}_n\,z^{-n-1})(\sum_{m=0}^{\infty}\bar{b}_m\,r^{2m}\,z^{-m})\,dz \\
&= -\int_{\gamma_r}\left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}n\,\bar{a}_n\,\bar{b}_m\,r^{2m}\,z^{-m-n-1}\right)dz \\
&= -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}n\,\bar{a}_n\,\bar{b}_m\,r^{2m}\int_{\gamma_r} z^{-m-n-1}dz.
\end{aligned}$$

The only case that one can get a non-zero integral is when $-n-m=0$ which is impossible by non-negativity of these two numbers. The second equality can be proven similarly. □

Here is one of the most important result of the thesis which generalizes similar results explained in Section 2.2 to the many boundary curve case.

Theorem 3.2.26. *Let $\mathfrak{R}$, Σ, Γ_k and Ω_k^+, and $f_k, k=1,\ldots,n$ be as in Section 3.2. Assume all the boundary curves are quasicircles. Then the Grunsky operator*

$$\begin{aligned}
\mathbf{Gr_f} : (\overline{\mathcal{D}_v(\mathbb{D})}^n, \|.\|_{\mathcal{D}_{harm}(\mathbb{D})^n}) &\longrightarrow (\mathcal{D}(\mathbb{D})^n, \|.\|_{\mathcal{D}_{harm}(\mathbb{D})^n}) \\
(\overline{H}_1,\ldots,\overline{H}_n) &\longrightarrow \left(\sum_{k=1}^{n} Gr_{1k}(\mathbf{f})\overline{H}_k,\ldots,\sum_{k=1}^{n} Gr_{nk}(\mathbf{f})\overline{H}_k\right),
\end{aligned}$$

is a bounded operator of norm less than or equal to one. Furthermore, one has

$$\left\|\mathbf{I_f}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\Sigma)} = -\left\|\mathbf{Gr_f}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n} + \left\|(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n}. \tag{3.11}$$

Proof. Let us recall the definitions of norms here:

$$\left\|(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n} = \sum_{k=1}^{n}\left\|\overline{H}_k\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})},$$

$$\left\|\mathbf{Gr_f}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n} = \sum_{j=1}^{n}\left\|\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})}.$$

The proof will proceed as follows. We first prove the claim for the set of all $(\overline{H}_1,\dots,\overline{H}_n)$ in $\overline{\mathcal{D}_v(\mathbb{D})}^n$ such that $\mathbf{I_f}(\overline{H}_1,\dots,\overline{H}_n) \in \mathcal{D}(\Sigma)_q$ has a holomorphic extension past the boundary of Σ. More precisely, we assume that $\mathbf{I_f}(\overline{H}_1,\dots,\overline{H}_n)$ carries a holomorphic extension on a collar neighbourhood A_k of the boundary curve Γ_k in Ω_k^+ for each $k = 1,\dots,n$. The proof on this set of functions is rather straightforward. Then by the density theorem, Theorem 3.2.15, and the boundedness of the operators involved in the proof, the claim holds for every element in $\mathcal{D}(\Sigma)_q$. To apply the density theorem properly, we assume $A_k = \Omega^+_{k\epsilon}$ for some sufficiently small $\epsilon > 0$ depending on $(\overline{H}_1,\dots,\overline{H}_n)$.

By the above assumption and the finiteness of the number of the curves, there is a number $0 < R < 1$ such that the image of $\gamma_R = \{z \in \mathbb{C} : |z| = R\}$ under f_k is entirely in A_k for all $k = 1,\dots,n$. In other words, there is an annulus $\mathbb{A} = \{z \in \mathbb{C} : R < |z| < 1\}$ such that

$$f_k|_{\mathbb{A}} : \mathbb{A} \to A_k$$

is a conformal bijection for each $k = 1, \ldots, n$. Note that we may have to shrink A_k's to find such an $\mathbb{A}$ but we keep using the same notation for them.

For fixed Γ_k and $R < r < 1$, the curves $f(\gamma_r)$ are analytic curves in Ω_k^+. These curves approach Γ_k from within Ω_k^+ as r approaches one. If $R < r < 1$, then by the Green's identity and the fact that Γ_k's are negatively oriented with respect to surface Σ we have

$$\begin{aligned}
\left\|\mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\Sigma)} &= \iint_\Sigma \left|\mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)'\right|^2 \\
&= -\frac{1}{2i} \lim_{r \to 1^-} \int_{\cup f_j(\gamma_r)} \mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)'(z)\, \overline{\mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)(z)}\, dz \\
&= -\frac{1}{2i} \lim_{r \to 1^-} \sum_{j=1}^n \int_{f_j(\gamma_r)} \mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)'(z)\, \overline{\mathbf{I_f}(\overline{H}_1, \ldots, \overline{H}_n)(z)}\, dz \\
&= -\frac{1}{2i} \lim_{r \to 1^-} \sum_{j=1}^n \int_{f_j(\gamma_r)} \left(-\sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]_{A_j}(z) \right)' \\
&\quad \times \overline{\left(-\sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]_{A_j}(z) \right)}\, dz \\
&= -\frac{1}{2i} \lim_{r \to 1^-} \sum_{j=1}^n \int_{\gamma_r} \left(-\sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]_{A_j}(f_j(w)) \right)' \\
&\quad \times \overline{\left(-\sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-} \mathcal{C}_{f_k^{-1}} \overline{H}_k]_{A_j}(f_j(w)) \right)}\, dw.
\end{aligned}$$

We have used (Ω_j^+, f_j) as holomorphic charts to set up the integrals on the Riemann surface $\mathfrak{R}$, which defines the variable $z_j = f_j(w)$ in each term. For simplicity we used $z = z_j$. We continue simplifying the above integral by using the composition operator. Here by $\mathcal{C}_{f_j}$ we mean $\mathcal{C}_{f_j|_{\mathbb{A}}}$. We obtained that the above integral is equal to

$$= -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j}(w)\right)' \times \overline{\left(-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j}(w)\right)}\, dw \tag{3.12}$$

Note that the assumption of an existence of the extension of $\mathbf{I_f}$ past the boundary of Σ implies that for fixed $j = 1, 2, \ldots, n$, the sum $\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]$ has a holomorphic extension past the boundary curve Γ_j to A_j. If $k \neq j$, then each term $J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k$ in the sum admits such a holomorphic extension by its definition. So our assumption, in fact, implies that $J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j$ admits a holomorphic extensions to A_j.

We proceed by simplifying the integrand in (3.12). The choice of annulus $\mathbb{A}$ implies that $-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j} \in \mathcal{D}(\mathbb{A})$. Therefore, by applying Lemma 3.2.23, we have the identity

$$\begin{aligned} &-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j} \\ &= \left[\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathfrak{G}(\mathbb{A},\mathbb{D})(-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j})\right] \\ &\quad + \left[P_{\mathbb{D}}\mathfrak{G}(\mathbb{A},\mathbb{D})(-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j})\right], \end{aligned}$$

and by applying Lemma 3.2.22 the above is equal to

$$
\begin{aligned}
= &-\big[\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{G}(A_j,\Omega_j^+)(\sum_{k\neq j}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j})\big] \\
&-\big[\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{G}(A_j,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{A_j}\big] \\
&-\big[P_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{G}(A_j,\Omega_j^+)(\sum_{k\neq j}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j})\big] \\
&-\big[P_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{G}(A_j,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{A_j}\big].
\end{aligned}
$$

Then Lemma 3.2.11 and the definition of $\mathfrak{G}(A_j,\Omega_j^+)$ imply that

$$
\begin{aligned}
= &-\big[\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathcal{C}_{f_j}\sum_{k\neq j}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{\Omega_j^+}\big] \\
&-\big[\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{P}_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{\Omega_j^-}\big] \\
&-\big[P_{\mathbb{D}}\mathcal{C}_{f_j}\sum_{k\neq j}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{\Omega_j^+}\big] \\
&-\big[P_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{\Omega_j^-}\big].
\end{aligned}
$$

By Lemma 3.2.13 and Definition 3.2.20, the above is equal to

$$
\begin{aligned}
= \mathfrak{O}(\mathbb{D},\mathbb{D}^-)\mathcal{C}_{f_j}\big[&-\sum_{k\neq j}(\overline{P}_{\Omega_j^+}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{\Omega_j^+}) \\
&-\overline{P}_{\Omega_j^+}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{\Omega_j^-}\big] \\
&+\sum_{k\neq j}Gr_{jk}(\mathbf{f})\overline{H}_k + Gr_{jj}(\mathbf{f})\overline{H}_j.
\end{aligned}
$$

Applying Equation (3.9) implies that

$$
\begin{aligned}
&= \mathfrak{O}(\mathbb{D}, \mathbb{D}^-)\mathcal{C}_{f_j}\Big[\sum_{k\neq j}(-\overline{P}_{\Omega_j^+}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{\Omega_j^+}) \\
&\quad - \overline{P}_{\Omega_j^+}\mathfrak{O}(\Omega_j^-, \Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{\Omega_j^-}\Big] + \sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k \\
&= \mathfrak{O}(\mathbb{D}, \mathbb{D}^-)\mathcal{C}_{f_j}\left(\overline{P}_{\Omega_j^+}\mathcal{C}_{f_j^{-1}}\overline{H}_j\right) + \sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k;
\end{aligned}
$$

finally, applying Lemma 3.2.13 one more time implies that

$$
= \mathfrak{O}(\mathbb{D}, \mathbb{D}^-)\overline{H}_j + \sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k.
$$

Therefore, we have the following important identity

$$
-\mathcal{C}_{f_j}\sum_{k=1}^{n}[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k]_{A_j} = \mathfrak{O}(\mathbb{D}, \mathbb{D}^-)\overline{H}_j + \sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k. \tag{3.13}
$$

Now by inserting Equation (3.13) in (3.12) and also by the extension property of $\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j$ explained in Remark 1.4.7, we have

$$\begin{aligned}
&\left\|\mathbf{I_f}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\Sigma)} \\
&= -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)+\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)\right)' \\
&\qquad\qquad\times\left(\overline{\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)+\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)}\right)dw \\
&= -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)\right)'\left(\overline{\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)}\right)dw \\
&\quad -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)\right)'\left(\overline{\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)}\right)dw \\
&\quad -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)\right)'\left(\overline{\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)}\right)dw. \\
&\quad -\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k(w)\right)'\left(\overline{\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)}\right)dw.
\end{aligned} \tag{3.14}$$

The first integral in the above identity can be simplified as follows

$$\begin{aligned}
&-\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)\right)'\left(\overline{\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)}\right)dw \\
&= \sum_{j=1}^{n}\left\|\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j\right\|^2_{\mathcal{D}_{harm}(\mathbb{D}^-)} \\
&= \sum_{j=1}^{n}\left\|\overline{H}_j\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})}.
\end{aligned}$$

Here we have used lemma 3.2.24, the Green's identity, and the fact that $\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j$ is in $\mathcal{D}_{harm}(cl(\mathbb{D}^-)\cup\mathbb{A})$ and γ_r's are negatively oriented with respect to $cl(\mathbb{D}^-)\cup\mathbb{A}$.

For the second term in (3.14), we similarly have

$$
\begin{aligned}
&-\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k(w)\right)'\left(\overline{\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k(w)}\right) dw \\
&= -\sum_{j=1}^{n}\iint_{\mathbb{D}}\left|\left(\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k\right)'\right|^2 = -\sum_{j=1}^{n}\left\|\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})}.
\end{aligned}
$$

since γ_r's are positively oriented with respect to the domain of the integration.

Now consider the fourth term in (3.14):

$$
-\frac{1}{2i}\lim_{r\to 1^-}\sum_{j=1}^{n}\int_{\gamma_r}\left(\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k(w)\right)'\left(\overline{\mathfrak{O}(\mathbb{D},\mathbb{D}^-)\overline{H}_j(w)}\right) dw = 0.
$$

by Lemma 3.2.25 and the fact that $G_j := \sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H_k}$ is holomorphic on $\mathbb{D}$. Applying Lemma 3.2.25 one more time shows that the third term in (3.14) is also zero. Therefore, (3.14) becomes

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\left\|\sum_{k=1}^{n} Gr_{jk}(\mathbf{f})\overline{H}_k\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})} + \sum_{j=1}^{n}\left\|\overline{H}_j\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})} \\
&= -\left\|\mathbf{Gr}_{\mathbf{f}}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n} + \left\|(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n}.
\end{aligned}
$$

Finally, by the above equation and (3.12) we have the following

$$
\begin{aligned}
\left\|\mathbf{I}_{\mathbf{f}}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\Sigma)} &= -\left\|\mathbf{Gr}_{\mathbf{f}}(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n} \\
&\quad + \left\|(\overline{H}_1,\dots,\overline{H}_n)\right\|^2_{\mathcal{D}_{harm}(\mathbb{D})^n}.
\end{aligned}
\tag{3.15}
$$

This completes the proof for those $(\overline{H}_1,\dots,\overline{H}_n) \in \overline{\mathcal{D}_v(\mathbb{D})}^n$ for which $\mathbf{I}_{\mathbf{f}}(\overline{H}_1,\dots,\overline{H}_n)$ admits a holomorphic extension past the boundary of Σ.

By Theorem 3.2.15 and the boundedness of the operators we have (3.15) for all elements in $\overline{\mathcal{D}_v(\mathbb{D})}^n$ which completes the proof. □

We are as yet unable to prove that the norm of the Grunsky operator is strictly less than one. To do so, we will first show that the Schiffer operator and as a result the Faber operator, for the many boundary curve case, are isomorphisms when the boundary curves are quasicircles. Then we will prove that in this case the Grunsky operator norm is strictly less than one; see Corollary 3.2.45.

Exactly like the case of one boundary curve, to prove some isomorphisms theorems for Faber operator, we need the Schiffer operator. In the next section we will define the Schiffer operator for the many boundary curve case and prove some important properties of that. The dependence to the regularity of the boundary curves will be discussed.

3.2.4 Schiffer Operator for Surfaces with Many Borders

We define the Schiffer operator for the many boundary curve case based on the Schiffer operators $T(\Omega;\Sigma)$ and $T(\Sigma;\Omega)$ for a compact Riemann surface with one boundary curve.

By the definition of $\Omega_j^{\pm}$, for fixed $j = 1, \dots, n$, the Schiffer operator can be written as

$$\begin{aligned} T(\Omega_j^+;\Omega_j^-) : \overline{A(\Omega_j^+)} &\to A(\Omega_j^-) \\ \overline{\alpha} &\to \frac{1}{\pi i}\iint_{\Omega_j^+,w} \partial_z\partial_w g(w;z,q)\wedge\overline{\alpha}(w), \end{aligned}$$

where $z \in \Omega_j^-$. Therefore the following definition is well-defined.

Definition 3.2.27 (Schiffer Operator for $n > 1$)**.** *Let $\mathfrak{R}$, Γ_k's, $\Omega_k^{\pm}$'s and Σ be as Section 3.2. Define the Schiffer operator by*

$$T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma) : \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)} \to A(\Sigma)$$

where for $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)}$ and $z \in \Sigma$ one has

$$\begin{aligned}[T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)](z) &:= \frac{1}{\pi i} \sum_{k=1}^{n} \iint_{\Omega_k, w} \partial_z \partial_w g(w; z, q) \wedge \overline{\alpha}_k(w) \\ &= \sum_{k=1}^{n} [T(\Omega_k^+; \Omega_k^-)\overline{\alpha}_k]_{\Sigma}(z).\end{aligned}$$

For fixed $j = 1, \ldots, n$, the Schiffer operator is defined by

$$T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+) : \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)} \to A(\Omega_j^+)$$

where for $z \in \Omega_j^+$ on has

$$\begin{aligned}[T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)](z) := &\sum_{\substack{k=1 \\ k \neq j}}^{n} [T(\Omega_k^+; \Omega_k^-)\overline{\alpha}_k]_{\Omega_j^+}(z) \\ &+ [T(\Omega_j^+; \Omega_j^+)\overline{\alpha}_j](z).\end{aligned}$$

Similar to the space $\mathcal{V}_\Omega$ defined in (3.2) let $\mathcal{V}$ be defined by

$$\mathcal{V} = \left\{ (\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)} : \sum_{k=1}^{n} \iint_{\Omega_k^+} \beta \wedge \overline{\alpha}_k = 0 \, ; \; \forall \overline{\beta} \in \overline{A(\mathfrak{R})} \right\}.$$

Before proceeding further we would like to first investigate the relationship between the function space $\mathcal{W}$ and space $\mathcal{V}$ of anti-holomorphic 1-forms.

Lemma 3.2.28. *The operator*

$$\tilde{\bar{\partial}} : \bigoplus_{k=1}^{n} \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W} \to \mathcal{V}$$
$$(\overline{h}_1, \dots, \overline{h}_n) \to (\bar{\partial}\, \overline{h}_1, \dots, \bar{\partial}\, \overline{h}_n),$$

is a surjective operator which preserves the semi-norm.

Proof. If $(\overline{h}_1, \dots, \overline{h}_n) \in \bigoplus_{k=1}^{n} \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W}$, then

$$\tilde{\bar{\partial}}(\overline{h}_1, \dots, \overline{h}_n) = (\bar{\partial}\, \overline{h}_1, \dots, \bar{\partial}\, \overline{h}_n) \in \mathcal{V}$$

by Stokes' theorem, the first identity in (3.16) below, and the fact that each $\bar{\partial}\, \overline{h}_k$ is in $\overline{A(\Omega_k^+)}$. So the operator $\tilde{\bar{\partial}}$ is well-defined.

On the other hand, let $(\overline{\alpha}_1, \dots, \overline{\alpha}_n) \in \mathcal{V}$, since each Ω_k^+ is a simply connected domain of $\mathfrak{R}$ there exists (unique up to constant) $\overline{h}_k \in \overline{\mathcal{D}(\Omega_k^+)}$ such that $\bar{\partial}\, \overline{h}_k = \overline{\alpha}_k$. It should be shown that $(\overline{h}_1, \dots, \overline{h}_n) \in \bigoplus_{k=1}^{n} \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W}$. Let $\alpha \in A(\mathfrak{R})$. Stokes' theorem implies that

$$\lim_{\epsilon \to 0^+} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^{p_k}} \overline{h}_k \alpha = \sum_{k=1}^{n} \iint_{\Omega_k^+} \bar{\partial}\, \overline{h}_k \wedge \alpha = \sum_{k=1}^{n} \iint_{\Omega_k^+} \overline{\alpha}_k \wedge \alpha = 0, \tag{3.16}$$

where the last equality holds because $(\overline{\alpha}_1, \dots, \overline{\alpha}_n) \in \mathcal{V}$. Since α was arbitrary this is true for every $\alpha \in A(\mathfrak{R})$. Therefore, $(\overline{h}_1, \dots, \overline{h}_n) \in \bigoplus_{k=1}^{n} \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W}$ and

$$\tilde{\bar{\partial}}(\overline{h}_1, \dots, \overline{h}_n) = (\bar{\partial}\, \overline{h}_1, \dots, \bar{\partial}\, \overline{h}_n) = (\overline{\alpha}_1, \dots, \overline{\alpha}_n).$$

The operator $\tilde{\bar{\partial}}$ preserves the norm since the operator $\bar{\partial}$ is norm preserving. □

The above lemma then implies that

Corollary 3.2.29. *The restriction $\widetilde{\overline{\partial}}$ operator to $\mathcal{W}'$ is an isometric isomorphism onto $\mathcal{V}$.*

Proof. By Lemma 3.2.4 and Lemma 3.1.7 for the one boundary curve case. □

We need an integral operator with the Bergman kernel. Recall the $S(\Omega)$ operator defined in Subsection 3.1.1.

Definition 3.2.30 (Schiffer Operator for $n > 1$). *The Schiffer (comparison) operator is defined by*

$$S(\Omega_1^+, \ldots, \Omega_n^+) : \bigoplus_{k=1}^{n} A(\Omega_k^+) \to A(\mathfrak{R})$$
$$(\alpha_1, \ldots, \alpha_n) \to \sum_{k=1}^{n} \iint_{\Omega_k^+, w} K_{\mathfrak{R}}(z, w) \wedge \alpha_k(w),$$

for $z \in \mathfrak{R}$.

Although the Schiffer comparison operator has the same kernel as the Bergman operator, the domain of the integration is different than the Bergman ones. Schiffer considered this in a chapter in the book by Courant [12]. Similar to the T operator, the S operator can be reduced to a finite sum of operators corresponding to each Ω_k^+. That is,

$$S(\Omega_1^+, \ldots, \Omega_n^+)(\alpha_1, \ldots, \alpha_n) = \sum_{k=1}^{n} S(\Omega_k^+)\alpha_k.$$

The output of the integral operator S is defined everywhere on the Riemann surface $\mathfrak{R}$ even on the boundary curve Γ.

We also need a conjugate of S operator.

$$\overline{S}(\Omega_1^+, \dots, \Omega_n^+) : \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)} \to \overline{A(\mathfrak{R})}$$
$$(\overline{\alpha}_1, \dots, \overline{\alpha}_n) \to \overline{S(\Omega_1^+, \dots, \Omega_n^+)(\alpha_1, \dots, \alpha_n)},$$

or equivalently

$$\overline{S}(\Omega_1^+, \dots, \Omega_n^+)(\overline{\alpha}_1, \dots, \overline{\alpha}_n) = \sum_{k=1}^{n} \overline{S(\Omega_k^+)\alpha_k}.$$

Using our notation, we recall the definition

$$\overline{T}(\Omega_j^+; \Omega_j^-) : A(\Omega_j^+) \to \overline{A(\Omega_j^+)}$$
$$\alpha \to \overline{T(\Omega_j^+; \Omega_j^-)\overline{\alpha}},$$

for every $\alpha \in A(\Omega_j^+)$.

For brevity we may simply say T or S operators to refer to the above operators when the number of boundary curves is clear from the context.

Thanks to [62, Theorem 3.11] the adjoint operators of both T and S can be found in the case of $n > 1$ boundary curves. The following theorem [62, Theorem 3.11] shows one of the applications of this operator. It seems that this result was not investigated by Schiffer; see Schippers and Staubach [62] for more details.

Theorem 3.2.31 (Schippers and Staubach)**.** *Let $\mathfrak{R}$ be a compact Riemann surface. Let Γ be a strip-cutting Jordan curve with measure zero and assume that Γ is separating $\mathfrak{R}$ into two connected components $\Omega^\pm$. Then the adjoint of $T(\Omega^+; \Omega^-)$ operator is $\overline{T}(\Omega^-; \Omega^+)$.*

We frequently need to restrict holomorphic 1-forms from a surface to its subsurfaces or extend them on the other way (if it exists). For that purpose, we define two operators. Let $\mathcal{B}$ and $\mathcal{A}$ be two subsurfaces of a Riemann surface $\mathfrak{R}$ such that $\mathcal{B} \subset \mathcal{A}$. Let $K_{\mathcal{A}}$ be the Bergman kernel of $\mathcal{A}$.

$$\begin{aligned} S(\mathcal{B};\mathcal{A}) : A(\mathcal{B}) &\to A(\mathcal{A}) \\ \alpha &\to \iint_{\mathcal{B}} K_{\mathcal{A}}(z,w) \wedge_w \alpha(w). \end{aligned}$$

for $z \in \mathcal{A}$. We may simply write $S(\mathcal{B})$ when $\mathcal{A} = \mathfrak{R}$ which is the same as the definition given in Subsection 3.1.1 for $S(\mathcal{B})$. Another operator is the restriction operator

$$\begin{aligned} Res(\mathcal{A},\mathcal{B}) : A(\mathcal{A}) &\to A(\mathcal{B}) \\ \alpha &\to \alpha|_{\mathcal{B}}. \end{aligned}$$

similarly $Res(\mathcal{B})$ stands for the restriction from $\mathfrak{R}$ to $\mathcal{B}$. It is clear from the definition and the reproducing property of the Bergman kernel, see Section 1.8, that $S(\mathcal{A};\mathcal{A})$ is identity operator $I_{A(\mathcal{A})}$. Trivially we have $Res(\mathcal{A},\mathcal{A}) = I_{A(\mathcal{A})}$. We are ready to talk about the adjoint of the $S(\mathcal{B};\mathcal{A})$ operator. This result will be used to find the adjoint of T.

Lemma 3.2.32. $S(\mathcal{B};\mathcal{A})^{\star} = Res(\mathcal{A},\mathcal{B})$.

Proof. Let $\alpha \in A(\mathcal{B})$ and $\beta \in A(\mathcal{A})$. By part (5), Theorem 1.8.1 and the reproducing property of the Bergman kernel, one has

$$\begin{aligned}
< S(\mathcal{B};\mathcal{A})\alpha, \beta >_{A_{harm}(\mathcal{A})} &= \iint_{\mathcal{A},z} [S(\mathcal{B};\mathcal{A})\alpha](z)\,\overline{\beta(z)}\,dA_z \\
&= \iint_{\mathcal{A},z} \left(\iint_{\mathcal{B},w} (2i)K_{\mathcal{A}}(z,w)\,\alpha(w)\,dA_w \right) \overline{\beta(z)}\,dA_z \\
&= \iint_{\mathcal{B},w} \left(\iint_{\mathcal{A},z} (2i)K_{\mathcal{A}}(z,w)\,\overline{\beta(z)}\,dA_z \right) \alpha(w)\,dA_w \\
&= \iint_{\mathcal{B},w} \left(\overline{-\iint_{\mathcal{A},z} (2i)\overline{K_{\mathcal{A}}(z,w)}\,\beta(z)\,dA_z} \right) \alpha(w)\,dA_w \\
&= \iint_{\mathcal{B},w} \left(\overline{\iint_{\mathcal{A},z} (2i)K_{\mathcal{A}}(w,z)\,\beta(z)\,dA_z} \right) \alpha(w)\,dA_w \\
&= \iint_{\mathcal{B},w} \overline{\beta(w)}\,\alpha(w)\,dA_w =< \alpha, Res(\mathcal{A},\mathcal{B})\beta >_{A_{harm}(\mathcal{B})} .
\end{aligned}$$

To justify the change in the order of the integration note that the Bergman kernel is holomorphic and bounded on $\mathfrak{R}$; therefore, Fubini's theorem is applicable. See [59] for a complete proof. □

Remark 3.2.33. *The above result for general domains in the complex plane was first shown by E. Schippers in an unpublished note.*

Theorem 3.2.34. *For the Schiffer operator we have*

1. *The adjoint of $T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)$ is*

$$\begin{aligned}
T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)^\star : A(\Sigma) &\to \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)} \\
\alpha &\to \left(\overline{T}(\Omega_1^-;\Omega_1^+)S(\Sigma;\Omega_1^-)\alpha, \dots, \overline{T}(\Omega_n^-;\Omega_n^+)S(\Sigma;\Omega_n^-)\alpha \right).
\end{aligned}$$

2. *For $j = 1, \ldots, n$, the adjoint of $T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)$ is*

$$T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)^\star : A(\Omega_j^+) \to \bigoplus_{k=1}^{n} \overline{A(\Omega_k^+)}$$

$$\alpha \to \left(\overline{T}(\Omega_1^-; \Omega_1^+) S(\Omega_j^+, \Omega_1^-)\alpha, \ldots, \overline{T}(\Omega_n^-; \Omega_n^+) S(\Omega_j^+; \Omega_n^-)\alpha\right).$$

Proof. For the case $n = 1$, we have $\Omega_1^- = \Sigma$. Therefore, $S(\Sigma; \Omega_1^-) = I_{A(\Omega_1^-)}$ and $T(\Omega_1^+; \Sigma)^\star = T(\Omega_1^+; \Omega_1^-)^\star = \overline{T}(\Omega_1^-; \Omega_1^+)$ which is Theorem 3.2.31. We therefore continue by assuming that $n > 1$.

For $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \bigoplus_{k=1}^n \overline{A(\Omega_k^+)}$ and for $\alpha \in A(\Sigma)$ we have the following

$$\begin{aligned}
&< T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n), \alpha >_{A_{harm}(\Sigma)} \\
&= \sum_{k=1}^{n} < [T(\Omega_k^+; \Omega_k^-)\, \overline{\alpha}_k]_\Sigma, \alpha >_{A_{harm}(\Sigma)} \\
&= \sum_{k=1}^{n} < Res(\Omega_k^-, \Sigma)[T(\Omega_k^+; \Omega_k^-)\, \overline{\alpha}_k], \alpha >_{A_{harm}(\Sigma)} \\
&= \sum_{k=1}^{n} < \overline{\alpha}_k, [Res(\Omega_k^-, \Sigma) T(\Omega_k^+; \Omega_k^-)]^\star \alpha >_{A_{harm}(\Sigma)} \\
&= \sum_{k=1}^{n} < \overline{\alpha}_k, T(\Omega_k^+; \Omega_k^-)^\star Res(\Omega_k^-, \Sigma)^\star \alpha >_{A_{harm}(\Omega_k^+)} \\
&= \sum_{k=1}^{n} < \overline{\alpha}_k, \overline{T}(\Omega_k^-; \Omega_k^+) S(\Sigma; \Omega_k^-)\alpha >_{A_{harm}(\Omega_k^+)} \\
&=< (\overline{\alpha}_1, \ldots, \overline{\alpha}_n), (\overline{T}(\Omega_1^-; \Omega_1^+) S(\Sigma; \Omega_1^-)\alpha, \ldots, \overline{T}(\Omega_n^-; \Omega_n^+) S(\Sigma; \Omega_n^-)\beta) >_{\bigoplus_{k=1}^n A_{harm}(\Omega_k^+)},
\end{aligned}$$

therefore,

$$T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)^\star \alpha = \left(\overline{T}(\Omega_1^-; \Omega_1^+) S(\Sigma; \Omega_1^-)\alpha, \ldots, \overline{T}(\Omega_n^-; \Omega_n^+) S(\Sigma; \Omega_n^-)\alpha\right),$$

which completes the proof of the first part.

The second part can be proven in the same way as the first part using the identity $S(\Omega_j^+, \Omega_j^+) = I_{A(\Omega_j^+)}$. □

It was shown in [62, Theorem 3.10] that for the one boundary curve case the adjoint of operator $S(\Sigma)$ is the restriction operator $Res(\Sigma)$. We will show that the same relationship is also true in the many boundary curve case if we modify the restriction operator a bit.

Definition 3.2.35 (Restriction Operator)**.** *Define*

$$\begin{aligned} Res(\Omega_1^+, \ldots, \Omega_1^+) : A(\mathfrak{R}) \to \bigoplus_{k=1}^{n} A(\Omega_k^+) \\ \alpha \to \left(Res(\Omega_1^+)\alpha, \ldots, Res(\Omega_n^+)\alpha\right). \end{aligned}$$

The above operator can be clearly extended to $A_{harm}(\mathfrak{R})$.

Theorem 3.2.36. *The adjoint of the* $S(\Omega_1^+, \ldots, \Omega_n^+)$ *operator is* $Res(\Omega_1^+, \ldots, \Omega_1^+)$*.*

Proof. Let $(\alpha_1, \ldots, \alpha_n) \in \bigoplus_{k=1}^{n} A(\Omega_k^+)$ and $\beta \in A(\mathfrak{R})$, then

$$\begin{aligned}
&< S(\Omega_1^+, \ldots, \Omega_n^+)(\alpha_1, \ldots, \alpha_n), \beta >_{A(\mathfrak{R})} \\
&=< \sum_{k=1}^{n} S(\Omega_k^+)\alpha_k, \beta >_{A(\mathfrak{R})} \\
&= \sum_{k=1}^{n} < S(\Omega_k^+)\alpha_k, \beta >_{A(\mathfrak{R})} \\
&= \sum_{k=1}^{n} < \alpha_k, S(\Omega_k^+)^\star \beta >_{A_{harm}(\Omega_k^+)} \\
&= \sum_{k=1}^{n} < \alpha_k, Res(\Omega_k^+)\beta >_{A_{harm}(\Omega_k^+)} \\
&=< (\alpha_1, \ldots, \alpha_n), (Res(\Omega_1^+)\beta, \ldots, Res(\Omega_n^+)\beta) >_{\bigoplus_{k=1}^{n} A_{harm}(\Omega_k^+)} \\
&=< (\alpha_1, \ldots, \alpha_n), Res(\Omega_1^+, \ldots, \Omega_1^+)\beta >_{\bigoplus_{k=1}^{n} A_{harm}(\Omega_k^+)}
\end{aligned}$$

where the first equality is just the definition of the S operator and the third is a result of the adjoint of $S(\Omega_k^+)$ when it is considered as the one boundary curve case. Finally, the next to last equality is just the definition of the inner product on $\bigoplus_{k=1}^n A_{harm}(\Omega_k^+)$. The above identity shows that

$$S(\Omega_1^+, \ldots, \Omega_n^+)^\star = Res(\Omega_1^+, \ldots, \Omega_1^+),$$

which completes the proof. □

We want to know what is the relationship between the derivatives of the Cauchy-type integral operator and the Schiffer operator for the many boundary curve case. See Theorem 3.1.2 for the one boundary curve case. This theorem will be used to show some properties of the Faber and Grunsky operators corresponding to Σ.

Theorem 3.2.37. *If $(h_1, \ldots, h_n) \in \bigoplus_{k=1}^n \mathcal{D}_{harm}(\Omega_k^+)$ and $q \in \mathfrak{R}\backslash\Gamma$, then*

$$\begin{aligned}
\partial[J_q(\Gamma)(h_1, \ldots, h_n)]_\Sigma &= -T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\partial} h_1, \ldots, \overline{\partial} h_n), \\
\partial[J_q(\Gamma)(h_1, \ldots, h_n)]_{\Omega_j^+} &= -T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)(\overline{\partial} h_1, \ldots, \overline{\partial} h_n) + \partial h_j, \\
\overline{\partial}[J_q(\Gamma)(h_1, \ldots, h_n)]_{\mathfrak{R}\backslash\Gamma} &= \overline{S}(\Omega_1^+, \ldots, \Omega_n^+)(\overline{\partial} h_1, \ldots, \overline{\partial} h_n).
\end{aligned}$$

Proof. We will use Theorem 3.1.2 for different choices of Σ_i. Take $\Sigma_1 = \Omega_k^+$ and $\Sigma_2 = \Omega_k^-$, $k = 1, \ldots, n$. To prove the first identity, let $z \in \Sigma = \cap\Omega_k^-$ then by (3.3) one has

$$\begin{aligned}
\partial[J_q(\Gamma)(h_1,\dots,h_n)]_\Sigma(z) &= \partial\sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-}h_k]_\Sigma(z) = \sum_{k=1}^n \partial[J_q(\Gamma_k)_{\Omega_k^-}h_k]_\Sigma(z)\\
&= \sum_{k=1}^n [-T(\Omega_k^+;\Omega_k^-)\overline{\partial}h_k]_\Sigma(z)\\
&= -[T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)(\overline{\partial}h_1,\dots,\overline{\partial}h_n)](z).
\end{aligned}$$

Fixing j and leting $z\in\Omega_j^+$, we have

$$\begin{aligned}
\partial[J_q(\Gamma)(h_1,\dots,h_n)]_{\Omega_j^+}(z) &= \partial\sum_{k=1}^n [J_q(\Gamma_k)h_k]_{\Omega_j^+}(z)\\
&= \sum_{k\neq j} \partial[J_q(\Gamma_k)_{\Omega_k^-}h_k]_{\Omega_j^+}(z) + \partial[J_q(\Gamma_j)_{\Omega_j^+}h_j](z)\\
&= \sum_{k\neq j} [-T(\Omega_k^+;\Omega_k^-)\overline{\partial}h_k]_{\Omega_j^+}(z) + \partial h_j(z)\\
&\quad - [T(\Omega_j^+;\Omega_j^+)\overline{\partial}h_j](z)\\
&= -[T(\Omega_1^+,\dots,\Omega_n^+;\Omega_j^+)(\overline{\partial}h_1,\dots,\overline{\partial}h_n)](z) + \partial h_j(z).
\end{aligned}$$

The last equality can be proven similarly. □

Now we are ready to see one of the reasons to define the $\mathcal{V}$ space.

Lemma 3.2.38. *The restriction of $T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)$ operator to $\mathcal{V}$ maps $\mathcal{V}$ into $A(\Sigma)_e$.*

Proof. Let $(\overline{\alpha}_1,\dots,\overline{\alpha}_n)\in\mathcal{V}$. By Corollary 3.2.29 there exists a unique $(\overline{h}_1,\dots,\overline{h}_n)$ in $\mathcal{W}'$ such that $(\overline{\partial}\,\overline{h}_1,\dots,\overline{\partial}\,\overline{h}_n)=(\overline{\alpha}_1,\dots,\overline{\alpha}_n)$. Furthermore, since $\mathcal{W}'\subset\mathcal{W}$, the function $[J_q(\Gamma)(\overline{h}_1,\dots,\overline{h}_n)]_\Sigma$ is a holomorphic on Σ. Thus, by the first part of Theorem 3.2.37 one has

$$\begin{aligned} T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)(\overline{\alpha}_1,\dots,\overline{\alpha}_n) &= T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)(\overline{\partial}\,\overline{h}_1,\dots,\overline{\partial}\,\overline{h}_n) \\ &= -\partial[J_q(\Gamma)(\overline{h}_1,\dots,\overline{h}_n)]_\Sigma \in A(\Sigma)_e. \end{aligned}$$

That is, $T(\Omega_1^+,\dots,\Omega_n^+;\Sigma)|_{\mathcal{V}}(\mathcal{V}) \subset A(\Sigma)_e$ which completes the proof. □

To prove the injectivity of the T operator we need a way to transfer the exact harmonic 1-forms on Σ to exact harmonic 1-forms on Ω_j^+, for each $j = 1,\dots,n$. For an exact harmonic 1-form α on Σ there exists a function $h \in \mathcal{D}_{harm}(\Sigma)$ such that $dh = \alpha$ up to additive constant. For fixed Ω_j^+, we define $\mathfrak{O}_e(\Sigma,\Omega_j^+)\alpha$ to be the differential of the unique element $\mathfrak{O}(\Sigma,\Omega_j^+)h$. In other words,

$$\mathfrak{O}_e(\Sigma,\Omega_j^+) : A_{harm}(\Sigma)_e \to A_{harm}(\Omega_j^+)_e$$

where $\mathfrak{O}_e(\Sigma,\Omega_j^+)\alpha = d\,\mathfrak{O}(\Sigma,\Omega_j^+)(h)$ for $\alpha = dh$. We can extend the definition to include all Ω_k^+. That is,

$$\begin{aligned} \mathfrak{O}_e(\Sigma,\Omega_1^+,\dots,\Omega_n^+) : A_{harm}(\Sigma)_e &\to \bigoplus_{k=1}^n A_{harm}(\Omega_k^+)_e \\ \alpha &\to \left(\mathfrak{O}_e(\Sigma,\Omega_1^+)\alpha,\dots,\mathfrak{O}_e(\Sigma,\Omega_n^+)\alpha\right). \end{aligned}$$

Theorem 3.2.39. *Let $\mathfrak{R}$, Σ, Γ_k and Ω_k^+, and $f_k, k = 1,\dots,n$ be as in Section 3.2. Assume all the boundary curves are quasicircles. Then*

$$-\tilde{\overline{P}}(\Omega)\mathfrak{O}_e(\Sigma,\Omega_1^+,\dots,\Omega_n^+)|_{A(\Sigma)_e}$$

is a bounded left inverse for $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$*; therefore,* $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$ *is an injective operator from* $\mathcal{V}$ *into* $A(\Sigma)_e$*.*

Proof. In Theorem 3.2.38 it was shown that $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}(\mathcal{V}) \subset A_e(\Sigma)$. Now we show the injectivity of the T operator when it is restricted to $\mathcal{V}$. For $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \mathcal{V}$, by Lemma 3.2.28, there exists a $\overline{h}_k \in \overline{\mathcal{D}(\Omega_k^+)}$ such that $\overline{\partial}\, \overline{h}_k = \overline{\alpha}_k$ for $k = 1, \ldots, n$, and $(\overline{h}_1, \ldots, \overline{h}_n) \in \bigoplus_{k=1}^n \overline{\mathcal{D}(\Omega_k^+)} \cap \mathcal{W}$. By (3.7) and Theorem 3.2.10, for each $\overline{h}_k$ we have

$$\overline{h}_k(z) = \sum_{k \neq j} [J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_{\Omega_j^+}(z) + [J_q(\Gamma_j)_{\Omega_j^+} \overline{h}_j](z) - \mathfrak{O}(\Sigma, \Omega_j^+) \sum_{k=1}^n [J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_\Sigma(z),$$

$\forall z \in \Omega_j^+$. By taking the differential $d = \partial + \overline{\partial}$ of both sides, and applying Theorem 3.2.8, we have

$$\begin{aligned} d\overline{h}_j(z) = \overline{\partial}\, \overline{h}_j(z) = &\sum_{k \neq j} [\partial J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k + \overline{\partial} J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_{\Omega_j^+}(z) \\ &+ [\partial J_q(\Gamma_j)_{\Omega_j^+} \overline{h}_j + \overline{\partial} J_q(\Gamma_j)_{\Omega_j^+} \overline{h}_j](z) \\ &- \mathfrak{O}_e(\Sigma, \Omega_j^+) \sum_{k=1}^n [\partial J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k + \overline{\partial} J_q(\Gamma_k)_{\Omega_k^-} \overline{h}_k]_\Sigma(z). \end{aligned}$$

In other words,

$$\begin{aligned} \overline{\alpha}_j(z) = &\sum_{k \neq j} [T(\Omega_k^+; \Omega_k^-)\overline{\alpha}_k]_{\Omega_j^+}(z) + \partial \overline{h}_j(z) + [T(\Omega_j^+; \Omega_j^+)\overline{\alpha}_j](z) \\ &- \mathfrak{O}_e(\Sigma, \Omega_j^+) \sum_{k=1}^n [T(\Omega_k^+; \Omega_k^-)\overline{\alpha}_k]_\Sigma(z) \\ =& [T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)](z) \\ &- \mathfrak{O}_e(\Sigma, \Omega_j^+)[T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)](z). \end{aligned}$$

Therefore, we derive the following important equation on Ω_j^+:

$$\overline{\alpha}_j = T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) - \mathfrak{O}_e(\Sigma, \Omega_j^+)T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n).$$

We now apply the projection operator $\overline{P}(\Omega_j^+)$ to both sides of the above equation:

$$\begin{aligned}\overline{\alpha}_j &= \overline{P}(\Omega_j^+)\overline{\alpha}_j = \overline{P}(\Omega_j^+)T(\Omega_1^+, \ldots, \Omega_n^+; \Omega_j^+)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \\ &\quad - \overline{P}(\Omega_j^+)\mathfrak{O}_e(\Sigma, \Omega_j^+)T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \\ &= -\overline{P}(\Omega_j^+)\mathfrak{O}_e(\Sigma, \Omega_j^+)T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n),\end{aligned}$$

where the last equality follows from the fact that the output of the Schiffer operator is a holomorphic 1-form. The above identity implies that

$$-\tilde{\overline{P}}(\Omega)\mathfrak{O}_e(\Sigma, \Omega_1^+, \ldots, \Omega_n^+)|_{A(\Sigma)_e},$$

is a left inverse for the Schiffer operator $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)$ on $\mathcal{V}$. The Schiffer operator $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)$ restricted on $\mathcal{V}$ is therefore a one-to-one operator.

□

In order to prove that $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$ is surjective, we need to show that the Cauchy-type integral operator $J_q(\Gamma)$, like the case of one boundary curve, is independent of the side that the limiting curves of the integration are taken from. The following important relation shows this relation precisely.

We use the notation $J_q(\Gamma_k, \Omega_k^+)$ to indicate that our curves of integration are taken within Ω_k^+. The following can be considered as to be a generalization of [62, Theorem 4.10] from the one boundary curve to $n > 1$ boundary curves.

Lemma 3.2.40. *Let $\mathfrak{R}$, Σ, Ω_k^+'s and Γ_k's be the same as in Theorem 3.2.39. Then for h in $\mathcal{D}(\Sigma)$ we have*

$$h = \sum_{k=1}^{n} J_q(\Gamma_k, \Omega_k^+)_\Sigma[\mathfrak{O}(\Sigma, \Omega_k^+)h].$$

Proof. Since the function h is holomorphic on Σ, by the residue theorem, it is equal to the sum of the limiting integrals from within Σ along the boundary curves Γ_k. That is

$$h(z) = -\lim_{\epsilon \to 0^+} \frac{1}{\pi i} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^k} \partial_w g(w; z, q)\, h(w)\, dw$$

where $z \in \Sigma$ and by Γ_ϵ^k we mean the level curve of the Green's function $g_{\Omega_k^-}(\,\cdot\, , p_k)$ for some choice of $p_k \in \Omega_k^-$ fixed. We consider the negative orientation with respect to Σ.

For each fixed curve Γ_k, choose a collar neighbourhood A_k in Σ which is contained in Σ. Since the integrand is holomorphic on Σ one can replace each integral in the above sum by the integral over a fixed analytic curve Γ_k' in A_k. That defines an operator $J_q'(\Gamma_k, A_k)$. By [62, Theorem 4.9], for every k we have

$$J_q'(\Gamma_k, A_k)_\Sigma(h|_{A_k}) = J_q(\Gamma_k, \Omega_k^-)_\Sigma[\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k})].$$

Since $\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k}) \in \mathcal{D}_{harm}(\Omega_k^-)$ we can apply [62, Theorem 4.10] separately for each curve Γ_k to obtain the following:

$$J_q(\Gamma_k, \Omega_k^-)_\Sigma[\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k})] = J_q(\Gamma_k, \Omega_k^+)_\Sigma \mathfrak{O}(\Omega_k^-, \Omega_k^+)[\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k})].$$

Finally, by taking a sum over all the terms and by Definition 3.2.4 we have

$$\begin{aligned} h &= \sum_{k=1}^{n} [J'_q(\Gamma_k, A_k)_\Sigma (h|_{A_k})] \\ &= \sum_{k=1}^{n} J_q(\Gamma_k, \Omega_k^-)_\Sigma [\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k})] \\ &= \sum_{k=1}^{n} J_q(\Gamma_k, \Omega_k^+)_\Sigma \mathfrak{O}(\Omega_k^-, \Omega_k^+)[\mathfrak{G}(A_k, \Omega_k^-)(h|_{A_k})] \\ &= \sum_{k=1}^{n} J_q(\Gamma_k, \Omega_k^+)_\Sigma [\mathfrak{O}(\Sigma, \Omega_k^+)h], \end{aligned}$$

which completes the proof. □

Theorem 3.2.41. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as Theorem 3.2.39. If $q \in \Sigma$, then the operator $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$ is an onto operator from $\mathcal{V}$ to $A(\Sigma)_e$.*

Proof. Theorem 3.2.39 shows that $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$ maps $\mathcal{V}$ into $A(\Sigma)_e$. Now we show that every element in $A(\Sigma)_e$ is in the image of T.

If $\beta \in A(\Sigma)_e$, then there exists a unique $h \in \mathcal{D}(\Sigma)_q$ such that $\partial_z h = \beta$. Let $h_k \in \mathcal{D}_{harm}(\Omega_k^+)$ such that $\mathfrak{O}(\Sigma, \Omega_k^+)h = h_k$. That is, h and h_k have the same CNT boundary values on Γ_k except possibly on a null set in Γ_k. Clearly, $dh_k = \partial_z h_k + \overline{\partial}_z h_k \in A(\Omega_k^+)_e \bigoplus \overline{A(\Omega_k^+)_e}$. Lemma 3.2.40 and Stokes' theorem now imply that

$$\begin{aligned}
\beta = \partial_z h &= \partial_z \sum_{k=1}^{n} J_q(\Gamma_k, \Omega_k^+)_\Sigma [\mathfrak{O}(\Sigma, \Omega_k^+) h] \\
&= \partial_z \left(-\frac{1}{\pi i} \lim_{\epsilon \to 0^+} \sum_{k=1}^{n} \int_{\Gamma_\epsilon^{p_k}} \partial_w g(w; z, q) h_k(w) \right) \\
&= \partial_z \left(-\frac{1}{\pi i} \sum_{k=1}^{n} \iint_{\Omega_k^+, w} \partial_w g(w; z, q) \wedge \overline{\partial}_w h_k(w) \right) \\
&= -\frac{1}{\pi i} \sum_{k=1}^{n} \iint_{\Omega_k^+, w} \partial_z \partial_w g(w; z, q) \wedge \overline{\partial}_w h_k(w) \\
&= T(\Omega_1^+, \dots, \Omega_n^+; \Sigma)(\overline{\partial} h_1, \dots, \overline{\partial} h_n).
\end{aligned}$$

The change in the order of the differentiation and integration is legitimate since the integrand is non-singular. We need to prove that $(\overline{\partial} h_1, \dots, \overline{\partial} h_n)$ is in $\mathcal{V}$ which shows that β will be in the image of T operator restricted to $\mathcal{V}$. Suppose $\overline{\alpha} \in \overline{A(\mathfrak{R})}$. We need to show that $\sum_{k=1}^{n} \iint_{\Omega_k^+, w} \alpha \wedge \overline{\partial} h_k = 0$.

$$\begin{aligned}
\sum_{k=1}^{n} \iint_{\Omega_k^+, w} \alpha \wedge \overline{\partial} h_k &= \sum_{k=1}^{n} \iint_{\Omega_k^+, w} \alpha(w) \wedge \overline{\partial}_w h_k(w) \\
&= \sum_{k=1}^{n} \iint_{\Omega_k^+, w} \left(\iint_{\mathfrak{R}, z} K_{\mathfrak{R}}(w, z) \wedge_z \alpha(z) \right) \wedge_w \overline{\partial}_w h_k(w) \\
&= \sum_{k=1}^{n} \iint_{\mathfrak{R}, z} \alpha(z) \wedge_z \left(\iint_{\Omega_k^+, w} \overline{K_{\mathfrak{R}}(z, w)} \wedge_w \overline{\partial}_w h_k(w) \right) \\
&= \sum_{k=1}^{n} \iint_{\mathfrak{R}, z} \alpha(z) \wedge_z \left(\iint_{\Omega_k^+, w} \overline{\partial}_z \partial_w g(w; z, q) \wedge_w \overline{\partial}_w h_k(w) \right) \\
&= \iint_{\mathfrak{R}, z} \alpha(z) \wedge_z \left(\sum_{k=1}^{n} \iint_{\Omega_k^+, w} \overline{\partial}_z \partial_w g(w; z, q) \wedge_w \overline{\partial}_w h_k(w) \right).
\end{aligned}$$

The last equation is zero if

$$\sum_{k=1}^{n} \iint_{\Omega_k^+} \overline{\partial}_z \partial_w g(w; z, q) \wedge_w \overline{\partial} h_k = 0.$$

We now show that this equality is valid. By applying Lemma 3.2.40 one more time, we have

$$h(z) = -\frac{1}{\pi i} \sum_{k=1}^{n} \iint_{\Omega_k^+, w} \partial_w g(w; z, q) \wedge \overline{\partial}_w h_k(w).$$

On the other hand, $\overline{\partial}_z h(z) = 0$ by holomorphicity of h; so

$$\sum_{k=1}^{n} \iint_{\Omega_k^+} \overline{\partial}_z \partial_w g(w; z, q) \wedge_w \overline{\partial} h_k = 0,$$

as was claimed. Therefore, every exact L^2 holomorphic 1-form on Σ is in the image of $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$.

These two steps complete the proof. □

Theorems 3.2.41 and 3.2.39 have the following important consequences.

Corollary 3.2.42. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as Theorem 3.2.39. The restriction of the Schiffer operator $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)$ to $\mathcal{V}$ is a bounded isomorphism from $\mathcal{V}$ onto $A(\Sigma)_e$.*

We now show that how Theorem 3.2.41 implies that the Faber operator $\mathbf{I}_f$ is onto.

Theorem 3.2.43. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as in Theorem 3.2.39. Then the Faber operator $\mathbf{I_f} : \overline{\mathcal{D}_v(\mathbb{D})}^n \to \mathcal{D}(\Sigma)_q$ is onto.*

Proof. Let $h \in \mathcal{D}(\Sigma)_q$. By the surjectivity of the $T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)|_{\mathcal{V}}$ operator there exists $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$ in $\mathcal{V}$ such that

$$T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) = dh = \partial h \in A(\Sigma)_e.$$

By Corollary 3.2.29 and Theorem 3.2.37 there exists a unique $(\overline{h}_1, \ldots, \overline{h}_n) \in \mathcal{W}'$ such that

$$\partial^{-1} T(\Omega_1^+, \ldots, \Omega_n^+; \Sigma)(\overline{\alpha}_1, \ldots, \overline{\alpha}_n) = -[J_q(\Gamma)(\overline{h}_1, \ldots, \overline{h}_n)],$$

or equivalently $h = K(-\overline{h}_1, \ldots, -\overline{h}_n)$. So for an arbitrary $h \in \mathcal{D}(\Sigma)_q$ we have shown that there exists $(-\overline{h}_1, \ldots, -\overline{h}_n) \in \mathcal{W}'$ such that

$$h = K(-\overline{h}_1, \ldots, -\overline{h}_n).$$

This proves the operator K is a surjective operator.

On the other hand, since $\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}$ is an isomorphism (with respect to the Dirichlet semi-norm) this completes the proof. $\square$

Corollary 3.2.44. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as Theorem 3.2.39. Then the Faber operator $\mathbf{I_f} : \overline{\mathcal{D}_v(\mathbb{D})}^n \to \mathcal{D}(\Sigma)_q$ (corresponding to Σ) is a bounded isomorphism.*

It should be noted that, like the case of one boundary curve, the Faber operator is an isomorphism when all the boundary curves are quasicircles (in $\mathfrak{R}$). See Section 2.1 for similar examples of the Faber operator which are isomorphisms for quasicircles.

Corollary 3.2.45. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as in Theorem 3.2.39. Then the norm of the Grunsky operator (corresponding to Σ) is strictly less than one.*

Proof. By Corollary 3.2.44 there exists a constant $0 < c < 1$ such that

$$c \left\| \left(\overline{H}_1, \ldots, \overline{H}_n\right) \right\|_{\mathcal{D}_{harm}(\mathbb{D})^n} \leq \left\| \mathbf{I_f} \left(\overline{H}_1, \ldots, \overline{H}_n\right) \right\|_{\mathcal{D}_{harm}(\Sigma)} .$$

Now (3.15) and the above inequality complete the proof. □

The characterization of $\mathcal{D}(\Sigma)$; that is the first problem presented in introduction, will be discussed in the next section for surface with many borders.

3.2.5 Graph of the Grunsky Operator for Many Borders

We need to define some new operators to show that as in the one boundary curve case, in the many boundary curve case the pull back of the functions in $\mathcal{D}(\Sigma)_q$ under the conformal maps $\mathbf{f} = (f_1, \ldots, f_n)$ is the graph of the Grunsky operator $\mathbf{Gr_f}$.

Theorem 3.2.46. *Let $\mathfrak{R}$, Σ, Ω_k^+ and $\Gamma_k, k = 1, \ldots, n$, be the same as in Theorem 3.2.39. Then*

$$\widetilde{\mathcal{C}}_{\mathbf{f}} \widetilde{\mathfrak{H}}(\Sigma, \Omega) \mathcal{D}(\Sigma)_q = graph(\mathbf{Gr_f}).$$

In other words, the set of pull back of the transmission of functions in $\mathcal{D}(\Sigma)_q$ under $\mathbf{f} = (f_1, \ldots, f_n)$ is the graph of the Grunsky operator $\mathbf{Gr_f}$.

Proof. By Corollary 3.2.44, we know that $\mathbf{I_f}$ is an isomorphism from $\overline{\mathcal{D}_v(\mathbb{D})}^n$ onto $\mathcal{D}(\Sigma)_q$. The proof proceeds in two steps:

Step one: we calculate the composition of $\widetilde{\mathfrak{O}}(\Sigma,\Omega)$ with $\mathbf{I_f}\left(\overline{H}_1,\ldots,\overline{H}_n\right)$.

$$\begin{aligned}
&\widetilde{\mathfrak{O}}(\Sigma,\Omega)\mathbf{I_f}(\overline{H}_1,\ldots,\overline{H}_n)=\widetilde{\mathfrak{O}}(\Sigma,\Omega)(-\sum_{k=1}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_\Sigma)\\
&=-\left(\mathfrak{O}(\Sigma,\Omega_j^+)\sum_{k=1}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_\Sigma\right)_{j=1}^{n}\\
&=-\left(\sum_{\substack{k=1\\k\neq j}}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_{\Omega_j^+}+\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]\right)_{j=1}^{n},
\end{aligned}$$

where $\mathfrak{O}(\Sigma,\Omega_j^+)\sum_{k=1}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_\Sigma$ in the second line is the j-th component of n-tuple $\widetilde{\mathfrak{O}}(\Sigma,\Omega)\mathbf{I_f}(\overline{H}_1,\ldots,\overline{H}_n)$.

Step two: the identity above can be justified as follows. Note that for fixed $j=1,\ldots,n$, if $k\neq j$, the function $J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k$ is harmonic on $\Omega_j^+\subset\Omega_k^-$. Therefore, by Definition 3.2.4 the transmission of this term to Ω_j^+ is itself. For this j, the function $J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j$ is defined and is harmonic on Ω_j^-; this implies that

$$\begin{aligned}
\mathfrak{O}(\Sigma,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_\Sigma&=\mathfrak{O}(\Omega_j^-,\Omega_j^+)\mathfrak{G}(B_j,\Omega_j^-)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]_{B_j}\\
&=\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]
\end{aligned}$$

where B_j is defined in the same way as Definition 3.2.4, for each $j=1,\ldots,n$.

Now let $(\overline{H}_1, \ldots, \overline{H}_n)$ be in $\overline{\mathcal{D}_v(\mathbb{D})}^n$. Then

$$\begin{aligned}
&\left[\widetilde{P}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{O}}(\Sigma,\Omega)\mathbf{I}_{\mathbf{f}}\right](\overline{H}_1,\ldots,\overline{H}_n) = \widetilde{P}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{O}}(\Sigma,\Omega)\left(-\sum_{k=1}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_\Sigma\right)\\
&= -\widetilde{P}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\left(\sum_{\substack{k=1\\k\neq j}}^{n}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_{\Omega_j^+} + \mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]\right)_{j=1}^{n}\\
&= -\widetilde{P}_{\mathbb{D}}\left(\sum_{\substack{k=1\\k\neq j}}^{n}\mathcal{C}_{f_j}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_{\Omega_j^+} + \mathcal{C}_{f_j}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]\right)_{j=1}^{n}\\
&= \left(\sum_{\substack{k=1\\k\neq j}}^{n}(-P_{\mathbb{D}}\,\mathcal{C}_{f_j}\left[J_q(\Gamma_k)_{\Omega_k^-}\mathcal{C}_{f_k^{-1}}\overline{H}_k\right]_{\Omega_j^+}) - P_{\mathbb{D}}\mathcal{C}_{f_j}\mathfrak{O}(\Omega_j^-,\Omega_j^+)[J_q(\Gamma_j)_{\Omega_j^-}\mathcal{C}_{f_j^{-1}}\overline{H}_j]\right)_{j=1}^{n}\\
&= \left(\sum_{\substack{k=1\\k\neq j}}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k + Gr_{jj}(\mathbf{f})\overline{H}_j\right)_{j=1}^{n} = \left(\sum_{k=1}^{n}Gr_{jk}(\mathbf{f})\overline{H}_k\right)_{j=1}^{n}\\
&= \left(\sum_{k=1}^{n}Gr_{1k}(\mathbf{f})\overline{H}_k,\ldots,\sum_{k=1}^{n}Gr_{nk}(\mathbf{f})\overline{H}_k\right) = \mathbf{Gr}_{\mathbf{f}}(\overline{H}_1,\ldots,\overline{H}_n).
\end{aligned}$$

Therefore, $\widetilde{P}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{O}}(\Sigma,\Omega)\,\mathbf{I}_{\mathbf{f}} = \mathbf{Gr}_{\mathbf{f}}$ as bounded operators.

On the other hand, by invoking Lemma 3.2.13 to each map f_j, we have

$$\begin{aligned}
\big[\widetilde{\overline{P}}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\,\mathbf{I}_{\mathbf{f}}\big](\overline{H}_1,\ldots,\overline{H}_n) &= \widetilde{\overline{P}}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\left(-K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1,\ldots,\overline{H}_n)\right)\\
&= \widetilde{\overline{P}}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\left(-\mathfrak{D}(\Sigma,\Omega_j^+)K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1,\ldots,\overline{H}_n)\right)_{j=1}^n\\
&= \widetilde{\overline{P}}_{\mathbb{D}}\left(-\mathcal{C}_{f_j}\mathfrak{D}(\Sigma,\Omega_j^+)K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1,\ldots,\overline{H}_n)\right)_{j=1}^n\\
&= \left(-\overline{P}_{\mathbb{D}}\,\mathcal{C}_{f_j}\mathfrak{D}(\Sigma,\Omega_j^+)K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1,\ldots,\overline{H}_n)\right)_{j=1}^n\\
&= \left(-\mathcal{C}_{f_j}\,\overline{P}_{\Omega_j^+}\,\mathfrak{D}(\Sigma,\Omega_j^+)K\widetilde{\mathcal{C}}_{\mathbf{f}^{-1}}(\overline{H}_1,\ldots,\overline{H}_n)\right)_{j=1}^n\\
&= \left(\mathcal{C}_{f_j}(\mathcal{C}_{f_j^{-1}}\overline{H}_j)\right)_{j=1}^n = (\overline{H}_1,\ldots,\overline{H}_n)\\
&= I_{\overline{\mathcal{D}_v(\mathbb{D})}^n}(\overline{H}_1,\ldots,\overline{H}_n).
\end{aligned}$$

where next to the last equality is coming from Equation (3.10). Therefore, $I_{\overline{\mathcal{D}_v(\mathbb{D})}^n} = \widetilde{\overline{P}}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\,\mathbf{I}_{\mathbf{f}}$. Finally, by the above two equalities we have the following

$$\begin{aligned}
\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\mathbf{I}_{\mathbf{f}}(\overline{H}_1,\ldots,\overline{H}_n) &= \big[(\widetilde{P}_{\mathbb{D}}+\widetilde{\overline{P}}_{\mathbb{D}})\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\mathbf{I}_{\mathbf{f}}\big](\overline{H}_1,\ldots,\overline{H}_n)\\
&= \big[\widetilde{P}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\mathbf{I}_{\mathbf{f}}\big](\overline{H}_1,\ldots,\overline{H}_n)\\
&\quad+\big[\widetilde{\overline{P}}_{\mathbb{D}}\,\widetilde{\mathcal{C}}_{\mathbf{f}}\,\widetilde{\mathfrak{D}}(\Sigma,\Omega)\mathbf{I}_{\mathbf{f}}\big](\overline{H}_1,\ldots,\overline{H}_n)\\
&= \mathbf{Gr}_{\mathbf{f}}(\overline{H}_1,\ldots,\overline{H}_n) + I_{\overline{\mathcal{D}_v(\mathbb{D})}^n}\left(\overline{H}_1,\ldots,\overline{H}_n\right)
\end{aligned}$$

which is in $graph(\mathbf{Gr}_{\mathbf{f}})$. This completes the proof. □

Remark 3.2.47. *For the Riemann sphere case, a similar result was proven by Radnell, Schippers and Staubach. See [46, Theorem 4.1] for the $n = 1$ boundary curve case and [46, Theorem 4.10] for the $n > 1$ boundary curve case.*

As was briefly explained in Subsection 2.2.1, the fact that the Grunsky operator norm is less than one has many consequences in Teichmüller spaces.

In the next chapter, we will explain how in our case, namely open Riemann surfaces with n boundary curves homeomorphic to $\mathbb{S}^1$, we may have similar results.

Chapter 4

A Period Map On the Teichmüller Space of Bordered Surfaces

In the previous chapters, we have seen the definitions and some important properties of the Faber and Grunsky operators on bordered Riemann surfaces. In Chapter 3, we saw that how these operators can be used to characterize the pull-back of $\mathcal{D}(\Sigma)$ under $\mathbf{f}$. As was mentioned in the introduction, our second main problem is to define a period map on the Teichmüller space of a bordered Riemann surface which is analogous to the classical period map.

In this chapter, we will recall the definition of the period map on the universal Teichmüller space given by the Grunsky operator. It will be explained why these maps can be considered to be a generalization of the classical period map defined for compact Riemann surfaces. We will also review the literature concerning the holomorphicity of this period map on Teichmüller spaces in the case $\mathfrak{g} = 0$ and $n \geq 1$. A definition of a period map, may also be called the Grunsky map, for bordered Riemann surfaces of arbitrary genus will be provided here.

This chapter will conclude by a conjecture due to Radnell, Schippers, and Staubach regarding the holomorphicity of the Grunsky map in the case $\mathfrak{g} > 0$ and $n \geq 1$. If this conjecture is true, then the Grunsky map for bordered surfaces would share one more analogy with the classical period map; so it could be considered to be a generalization of the classical period map to bordered Riemann surfaces (for $\mathfrak{g}, n$ as above).

4.1 A Short History of the Problem

Here we briefly recall the relations discovered between the classical period map defined for compact Riemann surfaces and some period maps defined for bordered surfaces. Then in the rest of chapter we will provide precise definitions and theorems in this regard. We start with the universal Teichmüller space $T(1)$ defined in Section 1.9. The universal Teichmüller space is the Teichmüller space for $\mathbb{D}^-$. Clearly, $\mathbb{D}^-$ can be modelled as the Riemann sphere from which the closure of the unit disc removed. Therefore, the existent proofs concerning the holomorphicity of the period map on the universal Teichmüller space and its equality with the Grunsky map, can be interpreted as the case in which Σ is bordered surface with $\mathfrak{g} = 0$ and $n = 1$; see [47].

As was mentioned in Section 1.10, to each compact Riemann surface of finite genus $\mathfrak{g} > 0$ we may assign a matrix which is symmetric and has positive imaginary part and this assignment depends holomorphically on the Riemann surface. Recall that the Teichmüller spaces of compact Riemann surfaces are finite dimensional complex manifolds.

Let $\mathrm{Diff}(\mathbb{S}^1)$ be the set of all orientation-preserving diffeomorphisms from $\mathbb{S}^1$ onto $\mathbb{S}^1$; also let $\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ be the set of all Möbius transformations of $\overline{\mathbb{C}}$ that preserve $\mathbb{S}^1$. It can be shown that $\mathrm{Diff}(\mathbb{S}^1)/\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ is a subspace of $T(1)$. Kirillov and Yur′ev [28], later continued by S. Nag [38], showed that there is a period map, which we denote by Π, on this subspace of the universal Teichmüller space. Nag [38] showed that this map shares many analogies with the classical period maps defined on Teichmüller spaces of compact surfaces. That is, it is symmetric and $I - \Pi\,\overline{\Pi} > 0$, where by "$> 0$" we mean the left hand side operator is positive definite.

Nag and Sullivan [39] extended the period map Π from $\mathrm{Diff}(\mathbb{S}^1)/\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ to the whole space $T(1)$. Additionally, they provided a proof for the holomorphicity of this map. They proved this map is Gateaux holomorphic, and Takhtajan and Teo showed that it was holomorphic [76, Appendix B].

Takhtajan and Teo [76] defined a map called the "universal period mapping" from $T(1)$ into the Banach space of bounded operators on the Hilbert space $\ell^2(\mathbb{C})$. Then they proved that this map is holomorphic and more importantly showed that it coincides with the period map Π that had been introduced by Kirillov, Yur′ev, Nag and Sullivan (called the KYNS period map). Takhtajan and Teo discovered that the KYNS period map is given by the Grunsky map; see Definition 4.3.1 ahead. Similar results with a bit different formulation regarding the holomorphicity of the Grunsky map on $T(1)$ were independently proven by Shen [67]. We will provide a slightly different model of $T(1)$ in the next section to describe Takhtajan and Teo, and Shen's results.

Later on, Radnell, Schippers and Staubach generalized the above period map to the Teichmüller space of bordered surfaces of genus zero with $n \geq 1$

boundary curves (each one homeomorphic to $\mathbb{S}^1$) and demonstrated its connection to the Grunsky map; see D. Radnell, E. Schippers [44], and Radnell, Schippers and Staubach [47] and the references therein. To explain these connections in more detail we need to introduce an intermediate kind of Teichmüller space, called the rigged Teichmüller space in the next section.

4.2 Rigged and Universal Teichmüller Spaces

For a Riemann surface (either compact or bordered) $\mathfrak{R}$, the Teichmüller space $T_{\mathfrak{R}}$ was defined in Subsection 1.9. For the following surfaces, the rigged Teichmüller space can also be defined.

Definition 4.2.1 (Surfaces of Type $(\mathfrak{g}, n)$)**.** *Let $\mathfrak{g} \geq 0, n \geq 1$. A Riemann surface $\mathfrak{R}^B$ is called a bordered surface of type $(\mathfrak{g}, n)$ if*

1. *the border of $\mathfrak{R}^B$ consists of n ordered non-overlapping simple closed curves, each homeomorphic to $\mathbb{S}^1$.*

2. *there is a biholomorphism between $\mathfrak{R}^B$ and $\mathfrak{S}$, where $\mathfrak{S}$ is a compact Riemann surface of genus $\mathfrak{g}$ with n non-overlapping simply connected domains (each one biholomorphic to $\mathbb{D}$ in $\mathbb{C}$) removed.*

A Riemann surface $\mathfrak{R}^P$ is called a punctured surface of type $(\mathfrak{g}, n)$ if it is biholomorphic to a compact Riemann surface of genus $\mathfrak{g}$ with n ordered points removed.

Remark 4.2.2. *If $2\mathfrak{g} - 2 + n > 0$, then the Teichmüller space of $\mathfrak{R}^P$ given in the above definition is of finite dimension as a complex manifold; see Lehto [31].*

A set of conformal maps can be assigned to a punctured Riemann surface. These maps were used to define rigged Teichmüller spaces; see D. Radnell and E. Schippers [43].

Definition 4.2.3 (Rigging)**.** *Let $\mathfrak{R}^P$ be a punctured Riemann surface of type $(\mathfrak{g}, n)$ with punctures $p_1, \ldots, p_n$. Define $\mathcal{O}^{qc}(\mathfrak{R}^P)$ the set of all $\mathbf{f} = (f_1, \ldots, f_n)$ of injective conformal maps $f_k : \mathbb{D} \to \mathfrak{R}^P$ such that for $k, j = 1, \ldots, n$, we have*

- *the map f_k has a quasiconformal extension to an open neighbourhood of $\mathrm{cl}(\mathbb{D})$,*
- *$f_k(0) = p_k$,*
- *$\mathrm{cl}(f_k(\mathbb{D})) \cap \mathrm{cl}(f_j(\mathbb{D})) = \emptyset$ whenever $k \neq j$.*

Each element $\mathbf{f} \in \mathcal{O}^{qc}(\mathfrak{R}^P)$ is called a rigging of $\mathfrak{R}^P$.

We now ready to define the rigged Teichmüller space which was first introduced by D. Radnell [42] in his Ph.D thesis. For brevity, by “QCM" in the following definition we mean “quasiconformal map".

Definition 4.2.4 (Rigged Teichmüller Space)**.** *Let $\mathfrak{R}^P$ be a punctured Riemann surface of type $(\mathfrak{g}, n)$. The rigged Teichmüller space of $\mathfrak{R}^P$ is*

$$\widetilde{T}_{\mathfrak{R}^P} = \left\{(\mathfrak{R}^P, F_1, \mathfrak{S}^p, \mathbf{f}) : F_1 \text{ is a QCM between } \mathfrak{R}^P \text{ and } \mathfrak{S}^p, \mathbf{f} \in \mathcal{O}^{qc}(\mathfrak{R}^P)\right\} / \sim .$$

The relation $\sim$ is defined as follows: $(\mathfrak{R}^P, F_1, \mathfrak{S}^p_1, \mathbf{f}) \sim (\mathfrak{R}^P, F_2, \mathfrak{S}^p_2, \mathbf{g})$ whenever there is a biholomorphism $\tau : \mathfrak{S}^p_1 \to \mathfrak{S}^p_2$, preserving the punctures and their order, such that $F_2^{-1} \circ \tau \circ F_1$ is homotopic to the identity. The homotopy is constant on the punctures and $g_k = \tau \circ f_k$ for $k = 1, \ldots, n$.

It is easy to show that $\sim$ defines an equivalence relation. Similar to $T_{\mathfrak{R}}$, let $[\mathfrak{R}^P, F_1, \mathfrak{S}^p, \mathbf{f}]$ denote the equivalence class of $(\mathfrak{R}^P, F_1, \mathfrak{S}^p, \mathbf{f})$.

Radnell and Schippers [43, Section 5.4] showed that the Teichmüller space $T_{\mathfrak{R}^B}$ of bordered Riemann surface $\mathfrak{R}^B$ of type $(\mathfrak{g}, n)$ covers the rigged Teichmüller space $\widetilde{T}_{\mathfrak{R}^P}$ of $\mathfrak{R}^P$ for $\mathfrak{g} \geq 0, n \geq 1$. This needs a word of explanation. First of all $\mathfrak{R}^B$ is obtained from $\mathfrak{R}^P$ by removing some caps (=image of $\mathbb{D}$

via f_k's in the rigging) and is not arbitrary. Now let $PModI(\mathfrak{R}^B)$ be the modular group consisting of quasiconformal self maps of $\mathfrak{R}^B$ which are identity on the boundary of $\mathfrak{R}^B$, modulo homotopy rel boundary; see Section 1.9 for the definition. These maps fix the ordering of the boundary components. The modular group $PModI(\mathfrak{R}^B)$ acts by biholomorphisms on $T_{\mathfrak{R}^B}$. More precisely, for $[\rho] \in PModI(\mathfrak{R}^B)$ and $[\mathfrak{R}^B, f, \mathfrak{S}] \in T_{\mathfrak{R}^B}$ one has

$$[\rho]^*[\mathfrak{R}^B, f, \mathfrak{S}] = [\mathfrak{R}^B, f \circ \rho^{-1}, \mathfrak{S}].$$

This action is fixed point free and properly discontinuous [43, Lemmas 5.10, 5.11]. Let also $DB(\mathfrak{R}^B)$ be the subgroup of $PModI(\mathfrak{R}^B)$ generated by Dehn twists around the boundary curve of $\mathfrak{R}^B$. This subgroup is isomorphic to $\mathbb{Z}^n$ (n = the number of boundary curves of $\mathfrak{R}^B$) [43, Proposition 2.2]. Let $\mathcal{P} : T_{\mathfrak{R}^B} \to \widetilde{T}_{\mathfrak{R}^P}$ be the covering map mentioned above. It was shown in [43] that $\mathcal{P}(p) = \mathcal{P}(q)$ if and only if there exists a $[\rho] \in DB(\mathfrak{R}^B)$ such that $[\rho]^*p = q$, where $p, q \in T_{\mathfrak{R}^B}$. Therefore, the quotient map from $T_{\mathfrak{R}^B}/DB(\mathfrak{R}^B)$ into the rigged Teichmüller space $\widetilde{T}_{\mathfrak{R}^P}$ is a bijective map. In other words, this covering induces a complex manifold structure on the rigged Teichmüller space of $\mathfrak{R}^B$. We consider the rigged Teichmüller space endowed with this complex structure.

Remark 4.2.5. *Whenever we say the Teichmüller space of a Riemann surface, we simply mean the usual Teichmüller space of the surface in the sense of Definition 1.9.1.*

To describe the results of Takhtajan and Teo [76] and that of Shen [67] on period maps on Teichmüller spaces, we need the following model of the universal Teichmüller space. This definition is modelled on $\mathbb{D}$ by using Beltrami differentials in contrast to the one given in Subsection 1.9 modelled on $\mathbb{D}^-$ by using quasiconformals. Here is a definition.

$$L^\infty(\mathbb{D}) = \left\{ \mu(z)\frac{dz}{d\bar{z}} \,:\, \mu : \mathbb{D} \to \mathbb{C} \text{ is essentially bounded and measurable on } \mathbb{D} \right\}.$$

This is an infinite dimensional complex Banach space endowed with the essential sup norm. Let $L^\infty(\mathbb{D})_1$ be the open unit ball of $L^\infty(\mathbb{D})$.

Remark 4.2.6. *We do not want to add more technicalities to the thesis here by recalling the definition of (m,n)-differentials; it is, however, good to mention that each element in $L^\infty(\mathbb{D})$ is understood as $(-1,1)$-differential form. This is called a **Beltrami differential** in the literature; see e.g. [31, Section IV. 1.4]. We may interchangeably use terms $(-1,1)$-differential form, or Beltrami differential to call elements in $L^\infty(\mathbb{D})$ when there is no risk of confusion.*

For every $\mu\frac{dz}{d\bar{z}} \in L^\infty(\mathbb{D})_1$ one can extend μ to be zero on $\mathbb{D}^-$, known as the Bers' idea. We continue using the same notation for this extended differential. Now by [31, Theorem I. 4.4] there exists a quasiconformal map $f_\mu : \mathbb{C} \to \mathbb{C}$, unique up to a composition with a conformal map, such that one has

$$\overline{\partial} f_\mu = \mu\, \partial f_\mu$$

almost everywhere in $\mathbb{C}$. Note that f_μ is holomorphic on $\mathbb{D}^-$ since μ is zero there. We normalized f_μ by assuming that $f_\mu \in \mathbf{\Sigma}_0$; see Subsection 1.2.1 for the definition. With this notation, we are ready now to give another model of the universal Teichmüller space.

Definition 4.2.7 (Universal Teichmüller Space). *Let $\mu, \nu \in L^\infty(\mathbb{D})_1$. We say $\mu \sim \nu$ if and only if $f_\mu|_{\mathbb{D}^-} = f_\nu|_{\mathbb{D}^-}$. The universal Teichmüller space is then $L^\infty(\mathbb{D})_1/\sim$. We use $[\mu]$ to denote the equivalence class of the Beltrami differential μ in this space.*

Remark 4.2.8. *The proof that the two definitions of the universal Teichmüller space given in the thesis are equivalent exists in the literature; see for example [31, 37]. Thus, we may use $T(1)$ to denote $L^\infty(\mathbb{D})_1/\sim$ as well.*

The universal Teichmüller space has a unique infinite dimensional complex Banach manifold structor such that the projection

$$p : L^\infty(\mathbb{D})_1 \to T(1),$$

is a holomorphic submersion [76]. That is, the differential of p is surjective everywhere.

4.3 Period Maps and Grunsky Maps

Recall the definition of the period map for compact surfaces from Section 1.10. We saw that for a compact Riemann surfaces $\mathfrak{R}$, the period map is a map from the Teichmüller space of $\mathfrak{R}$ into the Siegel upper half space.

Recall also the Grunsky operator Gr_g on $\ell^2(\mathbb{C})$ defined in Chapter 2 by (2.4). It is a bounded operator due to the Grunsky inequalities. The Grunsky operator may be seen as an assignment of a bounded operator Gr_g on $\ell^2(\mathbb{C})$ to each holomorphic map g on $\mathbb{D}^-$ univalent on $|z| > R$ for some $R > 0$. The Grunsky operator, therefore, introduces a map from the universal Teichmüller space into a space of bounded linear operators. Recall that $B(X)$

is the Banach space of all bounded linear operators on X. By "UTS", for brevity, we mean the "universal Teichmüller space".

Definition 4.3.1 (Grunsky Map on UTS)**.** *The Grunsky map on UTS is defined by*

$$\begin{aligned} \mathbf{Gr} : L^{\infty}(\mathbb{D})_1/\sim &\to B(\ell^2(\mathbb{C})) \\ [\mu] &\to Gr_{f_\mu|_{\mathbb{D}^-}}, \end{aligned}$$

where $Gr_{f_\mu|_{\mathbb{D}^-}}$ is as (2.4).

This is clearly a well-defined operator.

Here is a nice theorem, first proved by Takhtajan and Teo [76, Theorem B] and then independently by Shen [67, Theorem 1], concerning the holomorphicity of the Grunsky map in the sense of complex Gateaux derivative between complex Banach manifolds. Although both theorems claim the same thing, the formulation of theorems are not the same within these two papers; they used different models for the universal Teichmüller space. Here we state the theorem using the formulation of Shen [67].

Theorem 4.3.2 (Takhtajan-Teo, Shen)**.** *The Grunsky map* $\mathbf{Gr}$ *is a holomorphic map between the complex Banach spaces $T(1)$ (=$L^{\infty}(\mathbb{D})_1/\sim$) and $B(\ell^2(\mathbb{C}))$.*

Remark 4.3.3. *In the context of this thesis, we may say that what Theorem 4.3.2 claims is the holomorphicity of the Grunsky map on the Teichmüller space of a bordered Riemann surface of genus zero with one boundary curve (homeomorphic to $\mathbb{S}^1$). That is, $\mathfrak{g} = 0, n = 1$ or a surface of type $(0,1)$.*

It should be noted that the above boundary curve is a quasicircle because of the definition of the universal Teichmüller space. In other words,

$f_\mu|_{\mathbb{D}^-}$ has a quasiconformal extension to $\mathbb{C}$, as a result the boundary is the quasicircle $f_\mu(\mathbb{S}^1)$. A question which naturally arises here is: what happens if one increases the number of boundary curves in the genus zero case?

The results of Radnell, Schippers, and Staubach [46, 47, 64] address this question. Their formulation is on bordered Riemann surfaces. We require a few more definitions and theorems to explain their result. We may use $\mathfrak{R}_\mathfrak{g}$ (similarly $\mathfrak{R}^P_\mathfrak{g}$, or $\mathfrak{R}^B_\mathfrak{g}$) to indicate that the genus of $\mathfrak{R}$ is $\mathfrak{g}$.

Let $\mathfrak{R}^P_0 = \overline{\mathbb{C}} \backslash \{p_1, \ldots, p_n\}$ for some distinct ordered set of points $p_1, \ldots, p_n \in \overline{\mathbb{C}}$. Let also

$$\begin{aligned} &B(\bigoplus_{k=1}^{n} A(\mathbb{D}^-), \bigoplus_{k=1}^{n} A(\mathbb{D})) \\ &= \left\{ T : \bigoplus_{k=1}^{n} A(\mathbb{D}^-) \to \bigoplus_{k=1}^{n} A(\mathbb{D}) : T \textit{ is a bounded linear operator} \right\}. \end{aligned}$$

For simplicity, we use $B(n)$ to indicate $B(\bigoplus_{k=1}^{n} A(\mathbb{D}^-), \bigoplus_{k=1}^{n} A(\mathbb{D}))$. This shorter notation was used in [47]. In [47, Section 2.1], the Grunsky operator $\widehat{\mathbf{Gr}}$ is modelled as an element in $B(n)$. They first defined the following period map on the rigged Teichmüller space of $\mathfrak{R}^P_0$. Then they lifted it to the Teichmüller space of a bordered surface that was obtained from $\mathfrak{R}^P_0$ by removing caps. For brevity by "RTS" we mean the "rigged Teichmüller space".

Definition 4.3.4 (Period Map on RTS of $\mathfrak{R}^P_0$). *Let $\mathfrak{R}^P_0$ and $B(n)$ be as above. Define the period map $\widetilde{\Pi}_0$ by*

$$\begin{aligned} \widetilde{\Pi}_0 : \widetilde{T}_{\mathfrak{R}^P_0} &\to T_{\mathfrak{R}^P_0} \times B(n) \\ [\mathfrak{R}^P_0, F_1, \mathfrak{R}^P_1, \mathbf{f}] &\to \left([\mathfrak{R}^P_0, F_1, \mathfrak{R}^P_1], \widehat{\mathbf{Gr}}(\mathbf{f}) \right), \end{aligned}$$

if $n > 3$*, and for* $n = 1, 2, 3$ *by*

$$\widetilde{\Pi}_0 : \widetilde{T}_{\mathfrak{R}_0^P} \to B(n)$$
$$[\mathfrak{R}_0^P, F_1, \mathfrak{R}_1^P, \mathbf{f}] \to \widehat{\mathbf{Gr}}(\mathbf{f}).$$

Then by using the fibration $\mathcal{P} : T_{\mathfrak{R}_0^B} \to \widetilde{T}_{\mathfrak{R}_0^P}$ they defined a period map Π_0 on the Teichmüller space of $\mathfrak{R}_0^B$. That is, $\Pi_0 = \widetilde{\Pi}_0 \circ \mathcal{P}$. They showed that these period maps are holomorphic [47, Theorem 3.9].

Theorem 4.3.5 (Radnell, Schippers, and Staubach)**.** *The period map* $\widetilde{\Pi}_0$ *and* Π_0 *are both holomorphic.*

In the context of this thesis, we could say what the above theorem claims is the holomorphicity of the Grunsky map on the usual and rigged Teichmüller spaces of punctured surfaces of genus zero with one or more than one boundary curves. That is, $\mathfrak{g} = 0, n \geq 1$ or equivalently for punctured surfaces of type $(0, n), n \geq 1$.

4.4 Conjecture

We now are able to define a period map on the Teichmüller space of bordered Riemann surface $\mathfrak{R}^B$ by using the Grunsky operator corresponding to $\Sigma = \mathfrak{R}^B$ defined in Chapter 3. Afterwards, we express the conjecture concerning the holomorphicity of this period map which may also be called the **Grunsky map**, on the Teichmüller space of a punctured compact surface of arbitrary genus. Let $\mathfrak{g} > 0$, $n \geq 1$, and $\mathfrak{R}_{\mathfrak{g}}^P$ be a punctured Riemann surface of type $(\mathfrak{g}, n)$ with punctures $p_1, \ldots, p_n$.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a rigging in the sense of Definition 4.2.3 for this surface. If we remove all domains $\mathrm{cl}(f_k(\mathbb{D}))$ from $\mathfrak{R}_{\mathfrak{g}}^P$, then $\mathfrak{R}_{\mathfrak{g}}^B = \mathfrak{R}_{\mathfrak{g}}^P \setminus \cup_{k=1}^n \mathrm{cl}(f_k(\mathbb{D}))$ is a bordered Riemann surface of the same genus $\mathfrak{g}$ with n boundary curves. To the Riemann surface $\mathfrak{R}_{\mathfrak{g}}^B$ one may assign the Grunsky and Faber operators as they were defined in Chapter 3. As a result, the following period map may be assigned to $\mathfrak{R}_{\mathfrak{g}}^P$. Recall Definition 3.2.20 of the Grunsky operator $\mathbf{Gr_f}$ corresponding to $\mathbf{f} = (f_1, \ldots, f_n)$ and the definition of $\overline{\mathcal{D}_v(\mathbb{D})}^n$ given by Equation (3.4).

Definition 4.4.1 (Period Map on RTS of $\mathfrak{R}_{\mathfrak{g}}^P$). *Let $\mathfrak{R}_{\mathfrak{g}}^P$, $\mathfrak{g} > 0$, and $n \geq 1$ be as above. Let*

$$B(n, \mathcal{V}) := B\left(\overline{\mathcal{D}_v(\mathbb{D})}^n, \mathcal{D}(\mathbb{D})^n\right)$$

be the set of all bounded linear operators from $\overline{\mathcal{D}_v(\mathbb{D})}^n$ into $\mathcal{D}(\mathbb{D})^n$. Define the period map $\widetilde{\Pi}_{\mathfrak{g}}$ by

$$\widetilde{\Pi}_{\mathfrak{g}} : \widetilde{T}_{\mathfrak{R}_{\mathfrak{g}}^P} \to T_{\mathfrak{R}_{\mathfrak{g}}^P} \times B(n, \mathcal{V})$$
$$[\mathfrak{R}_{\mathfrak{g}}^P, F_1, \mathfrak{S}_{\mathfrak{g}}^P, \mathbf{f}] \to \left([\mathfrak{R}_{\mathfrak{g}}^P, F_1, \mathfrak{S}_{\mathfrak{g}}^P], \mathbf{Gr_f}\right),$$

where $\mathbf{Gr_f}$ is as Definition 3.2.20.

This map, clearly, generalizes the one given in Definition 4.3.4. Here is the conjecture based on the above theorems and definitions.

Conjecture (Radnell, Schippers, and Staubach): The period maps $\widetilde{\Pi}_{\mathfrak{g}}$, and as a result $\Pi_{\mathfrak{g}} = \widetilde{\Pi}_{\mathfrak{g}} \circ \mathcal{P}$ are holomorphic maps on $\widetilde{T}_{\mathfrak{R}_{\mathfrak{g}}^P}$ and $T_{\mathfrak{R}_{\mathfrak{g}}}$, respectively.

$\widetilde{\Pi}_{\mathfrak{g}}$ and $\Pi_{\mathfrak{g}}$ may also be called the **Grunsky map**. Takhtajan and Teo [76], and Shen [67] modelled the period map on the bounded operators on $\ell^2(\mathbb{C})$

space, whereas the period map of Radnell, Schippers and Staubach [47] was modelled on the bounded operators on $\bigoplus_{k=1}^{n} A(\mathbb{D}^-)$. Despite this, as was mentioned in [47] Radnell, Schippers and Staubach period map is a generalization of the period map defined by Takhtajan and Teo [76]. If we temporary let $\mathfrak{g} = 0$, then $\overline{\mathcal{D}(\mathbb{D})}_v^n$ is simply $\left(\overline{\mathcal{D}(\mathbb{D})}\right)^n$. This is because the $\mathcal{V}$ space is trivial in this case. The anti-holomorphic Dirichlet space $\overline{\mathcal{D}(\mathbb{D})}$ is isomorphic to $A(\mathbb{D}^-)$ as the unit disc $\mathbb{D}$ is simply connected. Therefore, our conjecture generalizes the $\mathfrak{g} = 0$ case and in fact all the previous cases.

4.5 Positive Definiteness of Period Map

Other than the holomorphicity of the period maps (classical and the one above) on Teichmüller spaces, there is another important analogy between the classical period map and the Grunsky map on Teichmüller spaces. As was mentioned in Section 1.10, the classical period map has a positive definite imaginary part. On the other hand, in Chapter 3 we showed that the norm of the Grunsky operator $\mathbf{Gr_f}$ is strictly less than one when all the boundary curves are quasicircles; see Corollary 3.2.45. This corollary can be considered as the generalization of the imaginary part of the classical period map is positive definite. This was pointed out by Takhtajan and Teo [76, Remark 2.3], Radnell, Schippers, and Staubach [47]. Here instead of the Siegel upper half plane we consider an equivalent Siegel disk picture; see C. L. Siegel [72]. In the Siegel disk picture, for a period map Z $(= \pi, \Pi_{\mathfrak{g}}, \dots)$, the condition $I - Z\overline{Z} > 0$ takes the place of the condition that $Im(Z) > 0$. The norm being strictly less than one can be written as $I - ZZ^* > 0$.

4.6 Some Open Problems

Here is a list of problems which now may be posted in continuation of this thesis.

- Prove that the conjecture presented in the previous section is true.

- As was explained in Chapter 2, the Faber operator has an important role in the approximation theory of holomorphic functions both on planar domains and domains on Riemann surfaces. This role has not been investigated in my thesis. To show that the operator $\mathbf{I_f}$ corresponding to Σ can be used to approximate holomorphic functions in $\mathcal{D}(\Sigma)$ seems not to be a far-reaching goal to achieve.

- In the case of $\mathfrak{g} = 1, n \geq 1$ (torus with finitely many non-overlapping simply connected domains removed) one may be able to find explicit formulas for both the Faber and the Grunsky operators via the Weierstrass $\wp$ function. Another advantage of working on the torus is that the space of holomorphic 1-forms has dimension one; that is, $\alpha = dz$ on the plane passes down to the only holomorphic 1-form (up to a multiplicative constant) on the torus. Therefore, we may be able to explicitly talk about the approximation property of Faber operator in this case. As was mentioned in Section 2.3, this was done in a different formulation for one boundary curve using the results of Tietz [77]; see Reimer and Schippers [48].

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