Boundedly Pseudo-Amenable and Boundedly Pseudo-Contractible Banach Algebras

by

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Abstract

Over the past few decades, amenable and contractible Banach algebras have been widely researched, and it has been discovered that these notions are often quite strong. This eventually lead to the development of more generalized notions of amenability and contractibility. Since amenability (resp. contractibility) of a Banach algebra $\mathfrak{A}$ is traditionally defined by the property of all continuous derivations from $\mathfrak{A}$ to duals of Banach $\mathfrak{A}$-bimodules (resp. all Banach $\mathfrak{A}$-bimodules) being inner, one could require only these continuous derivations to be approximately inner in the sense of there existing a net of inner derivations which approximate a derivation, to get the notions of approximate amenability and approximate contractibility. We could then require that these inner derivations be bounded in the operator norm to get the notions of bounded approximate amenability and bounded approximate contractibility. To obtain more generalized notions of amenability and contractibility, we could alternatively look at the fact that for a Banach algebra, amenability and the existence of a bounded approximate diagonal are equivalent as well as contractibility and the existence of a diagonal are equivalent. We then define a Banach algebra to be pseudo-amenable if it possesses a (not necessarily bounded) approximate diagonal, and pseudo-contractible if it possesses a (not necessarily bounded) central approximate diagonal. These generalizations have been studied recently in the literature.

In this thesis we will study the notions of bounded pseudo-amenability and bounded pseudo-contractibility for a Banach algebra, where we require the existence of the approximate diagonals and central approximate diagonals to be multiplier bounded. We will investigate various properties of these types of Banach algebras, including: l^p direct sums, relationships to unitizations, hereditary properties on ideal and quotient subalgebras, connections to other generalized notions of amenability, and projective tensor products of these Banach algebras. We will also provide some examples of boundedly pseudo-amenable and boundedly pseudo-contractible Banach algebras.

Contents

1 Background Material

Recall that a Banach space $(X, \|\cdot\|)$ is a complete normed vector space, i.e., every Cauchy sequence in X converges in X. By equipping a Banach space with a product operation for pairs of vectors, we obtain what is called a Banach algebra. The precise definition is given below.

Definition 1.0.1. A **Banach algebra** over the field $\mathbb{F}$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$) is a Banach space $\mathfrak{A}$ over $\mathbb{F}$ equipped with an additional operation $* : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ called the **multiplication on** $\mathfrak{A}$ or **product on** $\mathfrak{A}$ which satisfies the following properties for all $a, b, c \in \mathfrak{A}$ and for all $\alpha \in \mathbb{F}$:

(1) $a * (b * c) = (a * b) * c$.

(2) $a * (b + c) = (a * b) + (a * c)$.

(3) $(\alpha a) * b = \alpha(a * b) = a * (\alpha b)$.

(4) $\|a * b\| \leq \|a\|\|b\|$.

A Banach algebra is said to be **commutative** if in addition, it satisfies $a * b = b * a$ for all $a, b \in \mathfrak{A}$. *We will often write "ab" in place of "$a * b$".*

Sometimes a Banach algebra $\mathfrak{A}$ has a vector e such that $ea = a$ and $ae = a$ for all $a \in \mathfrak{A}$. Such an element is called an **identity** or **unit** for $\mathfrak{A}$, and a Banach algebra is **unital** if it has an identity e such that $\|e\| = 1$.

When a Banach algebra $\mathfrak{A}$ does not have an identity, it is often useful to append $\mathfrak{A}$ with an identity.

Definition 1.0.2. The **unitization** of a Banach algebra $\mathfrak{A}$ is the Banach algebra

$$\mathfrak{A}^\sharp := \{a + \alpha : a \in \mathfrak{A}, \alpha \in \mathbb{C}\},$$

equipped with the operations of addition, scalar multiplication, and product respectively defined for all $a + \alpha, b + \beta \in \mathfrak{A}^\sharp$ and $\lambda \in \mathbb{C}$ by:

(1) $(a + \alpha) + (b + \beta) := (a + b) + (\alpha + \beta)$.

(2) $\lambda(a + \alpha) := (\lambda a) + (\lambda \alpha)$.

(3) $(a + \alpha)(b + \beta) := (ab + \alpha b + \beta a) + (\alpha\beta)$.

Observe that the unitization of a Banach algebra $\mathfrak{A}$ always has an identity, namely $\mathbf{0}+1$ where $\mathbf{0}$ denotes the zero vector in $\mathfrak{A}$ and $1 \in \mathbb{C}$.

Sometimes a Banach algebra $\mathfrak{A}$ possesses an approximate identity or bounded approximate identity that can play the role of an identity.

Definition 1.0.3. Let $\mathfrak{A}$ be a Banach algebra. A **left approximate identity** for $\mathfrak{A}$ is a net $(e_v) \subset \mathfrak{A}$ with the property that for all $a \in \mathfrak{A}$:

$$\lim_v e_v a = a.$$

If additionally there exists a constant $K > 0$ such that for all v, $\|e_v\| \leq K$ then (e_v) is said to be a **bounded left approximate identity** for $\mathfrak{A}$. Similarly, a **right approximate identity** for $\mathfrak{A}$ is a net $(e_v) \subset \mathfrak{A}$ with the property that for all $a \in \mathfrak{A}$:

$$\lim_v a e_v = a.$$

If additionally there exists a constant $K > 0$ such that for all v, $\|e_v\| \leq K$ then (e_v) is said to be a **bounded right approximate identity**. A net $(e_v) \subset \mathfrak{A}$ that is both a left and right (resp. bounded left and bounded right) approximate identity is simply called an **approximate identity** or **two-sided approximate identity** (resp. **bounded approximate identity** or **two-sided bounded approximate identity**) for $\mathfrak{A}$. Furthermore, if (e_v) is an approximate identity (resp. bounded approximate identity) such that for all $a \in \mathfrak{A}$ and for all v, $ae_v = e_v a$ then (e_v) is called a **central approximate identity** (resp. **central bounded approximate identity**).

It is often useful to determine when two Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$ have the same algebra structure.

Definition 1.0.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. A **continuous algebra homomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ is a continous linear mapping $\varphi : \mathfrak{A} \to \mathfrak{B}$ with the property that for all $a_1, a_2 \in \mathfrak{A}$:

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2).$$

A **continuous algebra epimorphism** is a surjective continuous algebra homomorphism.

The simplest example of a Banach algebra is the following. Let X be a compact topological space and let $C(X)$ be the Banach space of all continuous real or complex-valued functions on X with uniform norm $\|f\| := \sup_{x\in X} |f(x)|$ for all $f \in C(X)$. Then $C(X)$ becomes a unital commutative Banach algebra by equipping $C(X)$ with the pointwise multiplication product defined for all $f, g \in C(X)$ and for all $x \in X$ by:

$$(f * g)(x) := f(x)g(x).$$

Let X be a Banach space and let $\mathcal{B}(X) := \mathcal{B}(X, X)$ be the Banach space of all bounded linear operators from X to X with the operator norm $\|T\|_{\mathrm{op}} := \inf\{M : \|T(x)\| \leq M\|x\| \text{ for all } x \in X\}$. Then $\mathcal{B}(X)$ becomes a unital Banach algebra by equipping $\mathcal{B}(X)$ with operator composition $\circ$ as its product, defined for all $S, T \in \mathcal{B}(X)$ and for all $x \in X$ by:

$$(S \circ T)(x) := S(T(x)).$$

Two other major classical Banach algebras are the group algebra on a locally compact group G, and the discrete group algebra on a group G.

Definition 1.0.5. The **group algebra of a locally compact group** G is the Banach algebra:

$$L^1(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; \int_G |f(x)|\, d\mu(x) < \infty \right\},$$

(where μ is the left-Haar measure for G) with the pointwise addition and pointwise scalar multiplication operations, and the convolution product defined for all $f, g \in L^1(G)$, $\alpha \in \mathbb{C}$, and $x \in G$ by:

$$(f * g)(x) := \int_G f(t)g(t^{-1}x)\, d\mu(t),$$

and with the L_1 norm given by $\|f\|_1 := \displaystyle\int_G |f(x)|\, d\mu(x)$ for all $f \in L^1(G)$.

Another very important class of Banach algebras that we will encounter later on in this thesis are the weighted semigroup algebras.

Definition 1.0.6. A **weight** on a semigroup S is a function $\omega : S \to (0, \infty)$ with the property that for all $x, y \in S$:

$$\omega(xy) \leq \omega(x)\omega(y).$$

Definition 1.0.7. The **discrete semigroup algebra with weight** ω **of a semigroup** S is the Banach algebra:

$$\ell^1(S,\omega) := \left\{ f : S \to \mathbb{C} \;\middle|\; \sum_{x \in S} |f(s)|\omega(s) < \infty \right\},$$

with the pointwise addition and pointwise scalar multiplication operations, and the semigroup convolution product defined for all $f, g \in \ell^1(S,\omega)$, $\alpha \in \mathbb{C}$, and $x \in S$ by:

$$(f * g)(x) := \sum_{x=st} f(s)g(t),$$

with the convention that $(f * g)(x) = 0$ whenever the equation $x = st$ has no solutions. The norm of $\ell^1(S,\omega)$ is given by $\|f\|_1 := \sum_{x \in S} |f(x)|\omega(x)$ for all $f \in \ell^1(S,\omega)$.
The discrete semigroup algebra with weight $\omega = 1$ of a semigroup S is just called the **discrete semigroup algebra of** S and is denoted by $\ell^1(S)$.

Ideals and modules of a Banach algebra will frequently be mentioned later, so we will define them now.

Definition 1.0.8. Let $\mathfrak{A}$ be a Banach algebra and let $J \subseteq \mathfrak{A}$ be a subspace of $\mathfrak{A}$. Then J is said to be a **left ideal** of $\mathfrak{A}$ if $\mathfrak{A}J = J$, and similarly, J is said to be a **right ideal** of $\mathfrak{A}$ if $J\mathfrak{A} = J$. J is said to be a **two-sided ideal** of $\mathfrak{A}$ (or simply an **ideal** of $\mathfrak{A}$) if J is both a left and right ideal of $\mathfrak{A}$. *Here, we define the sets* $\mathfrak{A}J$ *by* $\mathfrak{A}J = \{aj : a \in \mathfrak{A}, j \in J\}$. *A similar definition is given for the set* $J\mathfrak{A}$.

Definition 1.0.9. Let $\mathfrak{A}$ be a Banach algebra. A Banach space X equipped with left and right module multiplications from $\mathfrak{A} \times X$ to X written $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto x \cdot a$, is a **Banach** $\mathfrak{A}$**-bimodule** if the following properties are satisfied:

(1) For each fixed $a \in \mathfrak{A}$, the operators from X to X defined for all $x \in X$ by $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are linear.

(2) For each fixed $x \in X$, the operators from $\mathfrak{A}$ to X defined for all $a \in \mathfrak{A}$ by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are linear.

(3) For all $a, b \in \mathfrak{A}$ and for all $x \in X$; $a \cdot (b \cdot x) = (ab) \cdot x$ and $(x \cdot a) \cdot b = x \cdot (ab)$.

(4) For all $a, b \in \mathfrak{A}$ and for all $x \in X$; $a \cdot (x \cdot b) = (a \cdot x) \cdot b$.

(5) There exists a constant $K > 0$ such that for all $a \in \mathfrak{A}$ and for all $x \in X$; $\|a \cdot x\| \leq K\|a\|\|x\|$ and $\|x \cdot a\| \leq K\|a\|\|x\|$.

For example, a Banach algebra $\mathfrak{A}$, when given its product as the bimodule multiplication, is a Banach $\mathfrak{A}$-bimodule.

Given a Banach $\mathfrak{A}$-bimodule X, the dual space X^* can also naturally be made into a Banach $\mathfrak{A}$-bimodule.

Definition 1.0.10. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. For each $a \in \mathfrak{A}$, $f \in X^*$, and $x \in X$:

$$\langle a \cdot f, x\rangle := \langle f, x \cdot a\rangle \quad \text{and} \quad \langle f \cdot a, x\rangle := \langle f, a \cdot x\rangle.$$

define both a left and right module multiplication on X^*, and X^* equipped with this bimodule multiplication is called the **dual Banach $\mathfrak{A}$-bimodule** corresponding to the $\mathfrak{A}$-bimodule X.

Another very prominent type of Banach $\mathfrak{A}$-bimodule we will look at involves tensor products of Banach algebras. We first define what the tensor product of two Banach algebras is.

Definition 1.0.11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach spaces. The **projective tensor norm on $\mathfrak{A} \otimes \mathfrak{B}$** (where $\mathfrak{A} \otimes \mathfrak{B}$ is the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$) is the norm defined for all $u \in \mathfrak{A} \otimes \mathfrak{B}$ by:

$$\|u\| := \inf\left\{\sum_{i=1}^{n} \|a_i\|\|b_i\| : u = \sum_{i=1}^{n} a_i \otimes b_i\right\}.$$

The **projective tensor product** of $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ and is the completion of $\mathfrak{A} \otimes \mathfrak{B}$ under this norm . If $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ becomes a Banach algebra with the product defined for all elementary tensors $a \otimes b, c \otimes d \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ by:

$$(a \otimes b)(c \otimes d) := (ac) \otimes (bd).$$

If $\mathfrak{A}$ is a Banach algebra, then we can consider the projective tensor product of $\mathfrak{A}$ with itself - $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$. Naturally, $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ becomes a Banach $\mathfrak{A}$-bimodule with the left and right module multiplications specified for all $a \in \mathfrak{A}$ and for all elementary tensors $b \otimes c \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ by:

$$a \cdot (b \otimes c) := (ab) \otimes c \quad \text{and} \quad (b \otimes c) \cdot a := b \otimes (ca).$$

If $\mathfrak{A}$ is a Banach algebra and X is a Banach $\mathfrak{A}$-bimodule, then we can make similar definitions for what it means for X to have an approximate identity (or bounded approximate identity) in $\mathfrak{A}$.

Definition 1.0.12. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-Bimodule. A **left approximate identity in $\mathfrak{A}$ for X** is a net $(e_v) \subset \mathfrak{A}$ with the property that for all $x \in X$:

$$\lim_v e_v \cdot x = x.$$

If additionally there exists a constant $K > 0$ such that for all v, $\|e_v\| \leq K$, then (e_v) is called a **bounded left approximate identity in $\mathfrak{A}$ for X**. Similarly, a **right approximate identity in $\mathfrak{A}$ for X** is a net $(e_v) \subset \mathfrak{A}$ with the property that for all $x \in X$:

$$\lim_v x \cdot e_v = x.$$

If additionally there exists a constant $K > 0$ such that for all v, $\|e_v\| \leq K$, then (e_v) is called a **bounded right approximate identity in $\mathfrak{A}$ for X**. A net $(e_v) \subset \mathfrak{A}$ that is both a left and right (resp. bounded left and bounded right) approximate identity in $\mathfrak{A}$ for X is simply called an **approximate identity in $\mathfrak{A}$ for X** (resp. **bounded approximate identity in $\mathfrak{A}$ for X**).

Definition 1.0.13. Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule X is said to be **neo-unital** if $X = \mathfrak{A} \cdot X \cdot \mathfrak{A}$, that is, for every $x \in X$ there exists $a, b \in \mathfrak{A}$ and $y \in X$ such that $x = a \cdot y \cdot b$.

The notion of a Banach $\mathfrak{A}$-bimodule being neo-unital arises in part from a very famous theorem known as Cohen's factorization theorem. We will only need a weak version of Cohen's factorization theorem in this thesis which we state below, but the stronger version can be found in
[1, Theorem 11.10].

Theorem 1.0.1. (**Cohen's Factorization Theorem**) Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ has a bounded approximate identity in $\mathfrak{A}$ for X then for every $x \in X$ there exists $a, b \in \mathfrak{A}$ and $y \in X$ such that $x = a \cdot y \cdot b$.

Two Banach $\mathfrak{A}$-bimodules may be structurally similar. We define this this situation below.

Definition 1.0.14. Let $\mathfrak{A}$ be a Banach algebra and let X and Y be Banach $\mathfrak{A}$-bimodules. A **continuous $\mathfrak{A}$-bimodule morphism** from X to Y is a continuous linear mapping $\varphi : X \to Y$ with the property that for all $a \in \mathfrak{A}$ and for all $x \in X$:

$$\varphi(a \cdot x) = a \cdot \varphi(x) \quad \text{and} \quad \varphi(x \cdot a) = \varphi(x) \cdot a.$$

Definition 1.0.15. Let $\mathfrak{A}$ be a Banach algebra and let X and Y be Banach $\mathfrak{A}$-bimodules. An **approximate A-bimodule morphism** from X to Y is a net $(\varphi_\alpha) \subset \mathcal{B}(X, Y)$ with the property that for all $a \in \mathfrak{A}$ and for all $x \in X$:

$$\lim_\alpha [a \cdot \varphi_\alpha(x) - \varphi_\alpha(a \cdot x)] = 0 \quad \text{and} \quad \lim_\alpha [\varphi_\alpha(x) \cdot a - \varphi_\alpha(x \cdot a)] = 0.$$

Another very important result that we will use later on is [1, Theorem 42.6] which is stated below.

Theorem 1.0.2. Let X, Y, and Z be normed spaces. Then for every bilinear mapping $\phi : X \times Y \to Z$ there exists a unique linear mapping $\sigma : X \otimes Y \to Z$ such that for all $x \in X$ and for all $y \in Y$:

$$\sigma(x \otimes y) = \phi(x, y).$$

If $\mathfrak{A}$ is a Banach algebra, then the product on $\mathfrak{A}$ is a bilinear mapping, and thus, there exists a unique linear mapping on $\mathfrak{A} \otimes \mathfrak{A}$ such that $a \otimes b \mapsto ab$. The continuous extension of this linear map to $\mathfrak{A} \widehat{\otimes} \mathfrak{A}$ will be important later.

Definition 1.0.16. Let $\mathfrak{A}$ be a Banach algebra. The corresponding **product map** is the continuous map $\pi : \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathfrak{A}$ specified for all elementary tensors $a \otimes b \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$ by:

$$\pi(a \otimes b) := ab.$$

1.1 Amenability, Contractibility, and Weak Amenability

In 1972, Barry Johnson in [15] first introduced the concept of an amenable Banach algebra and paved the pathway for the study of these types of Banach algebras. In this section, we will survey the notions of ***amenable***, ***contractible***, and ***weakly amenable*** Banach algebras. We will state some of his paramount work, alongside major results developed after his original publication.

Definition 1.1.1. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. A **continuous derivation from $\mathfrak{A}$ to X** is a continuous linear operator $D : \mathfrak{A} \to X$ with the property that for all $a, b \in \mathfrak{A}$:

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

For each $x \in X$, it is easy to verify that the operator $\mathrm{ad}_x : \mathfrak{A} \to X$ defined by:

$$\mathrm{ad}_x(a) := a \cdot x - x \cdot a$$

for all $a \in \mathfrak{A}$ is a continuous derivation. We call it the **continuous inner derivation at** x.

It is natural to wonder exactly when a continuous derivation is inner. This leads to the following notions.

Definition 1.1.2. A Banach algebra $\mathfrak{A}$ is said to be **amenable** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation from $\mathfrak{A}$ to the corresponding dual Banach $\mathfrak{A}$-bimodule X^* is an inner derivation.

Definition 1.1.3. A Banach algebra $\mathfrak{A}$ is said to be **contractible** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation from $\mathfrak{A}$ to X is an inner derivation.

Definition 1.1.4. A Banach algebra $\mathfrak{A}$ is said to be **weakly amenable** if every continuous derivation from $\mathfrak{A}$ to $\mathfrak{A}^*$ is an inner derivation.

From the definitions above, we see that every contractible Banach algebra is an amenable Banach algebra and every amenable Banach algebra is weakly amenable.

The following properties of amenable and contractible Banach algebras were proven in Johnson's work in [15].

Theorem 1.1.1. If $\mathfrak{A}$ is an amenable Banach algebra then $\mathfrak{A}$ has a bounded approximate identity.

Theorem 1.1.2. If $\mathfrak{A}$ is a contractible Banach algebra then $\mathfrak{A}$ has an identity.

There are other useful characterizations of amenability in terms of nets in the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ and through the existence of a special type of element in the second dual $(\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$.

Definition 1.1.5. Let $\mathfrak{A}$ be a Banach algebra. A **diagonal** for $\mathfrak{A}$ is an element $m \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that:

$$a \cdot m = m \cdot a \quad \text{and} \quad \pi(m)a = a,$$

for all $a \in \mathfrak{A}$. An **approximate diagonal** for $\mathfrak{A}$ is a net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that:

$$\lim_\alpha [a \cdot m_\alpha - m_\alpha \cdot a] = 0 \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)a = a,$$

for all $a \in \mathfrak{A}$. A **multiplier bounded approximate diagonal** is an approximate diagonal $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ with the property that there exists a $K > 0$ such that $\|a \cdot m_\alpha - m_\alpha \cdot a\| \leq K\|a\|$ and $\|\pi(m_\alpha)a\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all α. A **bounded approximate diagonal** is an approximate diagonal $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ with the property that there exists a $K > 0$ such that $\|m_\alpha\| \leq K$ for all α.

Note that every diagonal gives rise to a bounded approximate diagonal and every bounded approximate diagonal is a multiplier bounded approximate diagonal.

Recall that if $\mathfrak{A}$ is a Banach algebra then the corresponding product map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is a continuous $\mathfrak{A}$-bimodule homomorphism, i.e., for all $a \in \mathfrak{A}$ and for all $m \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ we have that $\pi(a \cdot m) = a\pi(m)$ and $\pi(m \cdot a) = \pi(m)a$.

If $m \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a diagonal for $\mathfrak{A}$, then $a\pi(m) = \pi(a \cdot m) = \pi(m \cdot a) = \pi(m)a = a$ for all $a \in \mathfrak{A}$ and so $\pi(m)$ is an identity for $\mathfrak{A}$.

In a similar fashion, if $(m_v) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is an approximate diagonal for $\mathfrak{A}$ then $(\pi(m_\alpha))$ is an approximate identity for $\mathfrak{A}$.

Definition 1.1.6. A **virtual diagonal** for a Banach algebra $\mathfrak{A}$ is an element $M \in (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ with the property that:

$$a \cdot M = M \cdot a \quad \text{and} \quad (\pi^{**}(M)) \cdot a = a,$$

for all $a \in \mathfrak{A}$.

The following result given in [1, Lemma 43.8 and Theorem 43.9] gives us the following equivalence.

Theorem 1.1.3. Let $\mathfrak{A}$ be a Banach algebra. The following statements are equivalent:

(1) $\mathfrak{A}$ is amenable.

(2) $\mathfrak{A}$ has a bounded approximate diagonal.

(3) $\mathfrak{A}$ has a virtual diagonal.

For contractibility, we have a similar result, noted in [19, Chapter 4].

Theorem 1.1.4. Let $\mathfrak{A}$ be a Banach algebra. The following statements are equivalent:

(1) $\mathfrak{A}$ is contractible.

(2) $\mathfrak{A}$ has a diagonal.

If $\mathfrak{A}$ and $\mathfrak{B}$ are both Banach algebras and if $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a continuous algebra homomorphism then the following result from [15, Proposition 5.3] shows us when the codomain $\mathfrak{B}$ inherits amenability from $\mathfrak{A}$.

Theorem 1.1.5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is continuous algebra homomorphism with $\text{Range}(\varphi)$ dense in $\mathfrak{B}$ and $\mathfrak{A}$ is amenable, then $\mathfrak{B}$ is amenable.

Let $\mathfrak{A}$ be a Banach algebra and $J \subseteq \mathfrak{A}$ be an ideal of $\mathfrak{A}$. The result from [15, Proposition 5.1] tells us exactly when J inherits amenability from $\mathfrak{A}$, and as a consequence of the previous theorem, $\mathfrak{A}/J$ always inherits amenability from $\mathfrak{A}$.

Theorem 1.1.6. Let $\mathfrak{A}$ be an amenable Banach algebra and let $J \subseteq \mathfrak{A}$ be a closed ideal of $\mathfrak{A}$. Then:

(1) J is amenable if and only if J has a bounded approximate identity.

(2) $\mathfrak{A}/J$ is amenable.

The notion of amenability was first defined in terms of groups. A locally compact group G is said to be amenable if there exists a (left) invariant mean on G, that is, a positive linear functional μ on $L^\infty(G)$ such that $\mu(\mathbf{1}) = 1$ and:

$$\mu(T_h(m)) = \mu(m),$$

for all $m \in L^\infty(G)$ and for all $h \in G$, where $T_h : L^\infty(G) \to L^\infty(G)$ is the left translation operator defined for all $m \in L^\infty(G)$ and for all $g \in G$ by $[T_h(m)](g) = m(h^{-1}g)$. Every finite group is amenable, and as proven in [1, Proposition 43.5], so is every abelian group. Also, every compact group is amenable. In [15], Johnson proved the following famous theorem.

Theorem 1.1.7. (Johnson) Let G be a locally compact group. Then the group algebra $L^1(G)$ is amenable if and only if G is an amenable group.

Over the years, significant effort has been put towards classifying other Banach algebras, and an account of results classifying the amenability and weak amenability of Banach algebras can be found in [4]. For example, it is known that $(L^1(G))^{**}$ is amenable if and only if G is a finite group, and the measurable algebra $M(G)$ is amenable if and only if G is a discrete amenable group.

Let E be a compact Hausdorff topological space and let $C(E)$ denote the space of all continuous real-valued or complex-valued functions on E. As proven in [1, Proposition 43.12], $C(E)$ is always an amenable Banach algebra. It is also known that a C^*-algebra is amenable if and only if it is nuclear.

For weak amenability, again, let G be a locally compact group. Then $L^1(G)$ is always weakly amenable, and moreover, $M(G)$ is weakly amenable if and only if G is a discrete group.

Furthermore, it was proven in [7, Theorem 4.1] that for $1 \leq p < \infty$, the sequence algebra ℓ^p is weakly amenable.

For contractibility, we have similar results. It is well known that $L^1(G)$ is contractible if and only if G is a finite group. It is also known that a C^*-algebra is contractible if and only if it is finite-dimensional, and that a commutative contractible Banach algebra is always finite-dimensional, as noted in [19, Corollary 4.1.3]. It is even conjectured that every contractible Banach algebra is finite-dimensional.

1.2 Approximate Amenability, Approximate Contractibility, and Weak* Approximate Amenability

A strictly weaker notion of amenability and contractibility called ***approximate amenability*** and ***approximate contractibility*** were introduced in [8]. These terms are defined below.

Definition 1.2.1. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. A continuous derivation $D : \mathfrak{A} \to X$ is said to be **approximately inner** if there exists a net $(x_v) \subset X$ such that:

$$D(a) = \lim_v \mathrm{ad}_{x_v}(a) = \lim_v [a \cdot x_v - x_v \cdot a],$$

for all $a \in \mathfrak{A}$. A continuous derivation $D : \mathfrak{A} \to X^*$ is said to be **weak* approximately inner** if there exists a net $(f_v) \subset X^*$ such that:

$$D(a) = \mathrm{wk}^* \lim_v \mathrm{ad}_{f_v}(a) = \mathrm{wk}^* \lim_v [a \cdot f_v - f_v \cdot a],$$

for all $a \in \mathfrak{A}$.

Definition 1.2.2. A Banach algebra $\mathfrak{A}$ is said to be **approximately amenable** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D : \mathfrak{A} \to X^*$ is approximately inner.

Definition 1.2.3. A Banach algebra $\mathfrak{A}$ is said to be **approximately contractible** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D : \mathfrak{A} \to X$ is approximately inner.

Definition 1.2.4. A Banach algebra $\mathfrak{A}$ is said to be **weak* approximately amenable** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D : \mathfrak{A} \to X^*$ is weak* approximately inner.

Fascinatingly enough, it was proven in [10, Theorem 2.1] that the three notions above are in fact equivalent.

Theorem 1.2.1. Let $\mathfrak{A}$ be a Banach algebra. The following statements are equivalent:

(1) $\mathfrak{A}$ is approximately amenable.

(2) $\mathfrak{A}$ is approximately contractible.

(3) $\mathfrak{A}$ is weak* approximately amenable.

Recall from the previous section that an amenable Banach algebra always has a bounded approximate identity. From [8, Lemma 2.2] we at least have the following.

Theorem 1.2.2. If $\mathfrak{A}$ is approximately amenable then $\mathfrak{A}$ has both a left approximate identity and a right approximate identity.

Note that the existence of a bounded left approximate identity and a bounded right approximate identity always guarantees the existence of a bounded (two-sided) approximate identity [1, Proposition 11.6]. However, the existence of an unbounded left approximate identity and an unbounded right approximate identity does not guarantee the existence of a two-sided approximate identity.

Recall from the previous section that a Banach algebra is amenable if and only if it has a bounded approximate diagonal. The following result combines [8, Theorem 2.1] and [8, Proposition 2.6] to give alternative ways for characterizing approximate amenability in terms of nets in $(\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp)^{**}$ and $\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp$.

Theorem 1.2.3. Let $\mathfrak{A}$ be a Banach algebra. The following statements are equivalent:

(1) $\mathfrak{A}$ is approximately amenable.

(2) There exists a net $(M_v) \subset (\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp)^{**}$ such that for all $a \in \mathfrak{A}^\sharp$:

$$\lim_v [a \cdot M_v - M_v \cdot a] = 0 \quad \text{and} \quad \lim_v \pi^{**}(M_v) = e.$$

(3) There exists a net $(M'_v) \subset (\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp)^{**}$ such that for all $a \in \mathfrak{A}^\sharp$:

$$\lim_v [a \cdot M'_v - M'_v \cdot a] = 0 \quad \text{and for all } v \quad \pi^{**}(M_v) = e.$$

(4) There exists a net $(M_v) \subset \mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp$ such that for all $a \in \mathfrak{A}^\sharp$:

$$\lim_v [a \cdot M_v - M_v \cdot a] = 0 \quad \text{and} \quad \lim_v \pi(M_v) = e.$$

(5) There exists a net $(M'_v) \subset \mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp$ such that for all $a \in \mathfrak{A}^\sharp$:

$$\lim_v [a \cdot M'_v - M'_v \cdot a] = 0, \quad \text{and for all } v \quad \pi(M'_v) = e.$$

For images of Banach algebras under algebra homomorphisms, [8, Proposition 2.2] ensures us the following.

Theorem 1.2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a continuous algebra epimorphism and $\mathfrak{A}$ is approximately amenable, then $\mathfrak{B}$ is approximately amenable.

For two-sided ideal of $\mathfrak{A}$, from [10, Corollary 2.1] and as a consequence of the previous theorem, we have the following partial results.

Theorem 1.2.5. Let $\mathfrak{A}$ be a Banach algebra and let $J \subseteq \mathfrak{A}$ be a closed ideal of $\mathfrak{A}$. Then:
(1) If J is amenable and $\mathfrak{A}/J$ is approximately amenable then $\mathfrak{A}$ is approximately amenable.
(2) If $\mathfrak{A}$ is approximately amenable then $\mathfrak{A}/J$ is approximately amenable.

Many Banach algebras have already been determined as approximately amenable. For example, [8, Theorem 3.2] shows that for a locally compact group G, $L^1(G)$ is approximately amenable if and only if G is an amenable group. Furthermore, $M(G)$ is approximately amenable if and only G is a discrete amenable group, and $(L^1(G))^{**}$ is approximately amenable if and only if G is a finite group.

For $1 \leq p < \infty$, the Banach sequence algebras ℓ^p are not approximately amenable as proven in [7, Theorem 4.1], and moreover, if S is any infinite set, then $\ell^p(S)$ is also not approximately amenable.

1.3 Bounded Approximate Amenability and Bounded Approximate Contractibility

The notions of ***bounded approximate amenability*** and ***bounded approximate contractibility*** were investigated in [10] and [3]. These two notions are stronger than approximate amenability and approximate contractibility respectively

Definition 1.3.1. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. A continuous derivation $D : \mathfrak{A} \to X$ is said to be **boundedly approximately inner** if there exists a constant $K > 0$ and a net $(x_v) \subset X$ such that:

$$D(a) = \lim_v \mathrm{ad}_{x_v}(a) = \lim_v [a \cdot x_v - x_v \cdot a],$$

for all $a \in \mathfrak{A}$, and, $\|\mathrm{ad}_{x_v}(a)\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all v.

Definition 1.3.2. A Banach algebra $\mathfrak{A}$ is said to be **boundedly approximately amenable** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D : \mathfrak{A} \to X^*$ is boundedly approximately inner.

Definition 1.3.3. A Banach algebra $\mathfrak{A}$ is said to be **boundedly approximately contractible** if for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D : \mathfrak{A} \to X$ is boundedly approximately inner.

In general, the notions of bounded approximate amenability and bounded approximate contractibility are distinct. In [14] an approximately amenable Banach algebra that is not boundedly approximately amenable was constructed, and so we have the following.

Theorem 1.3.1. There exists an approximately amenable Banach algebra that is not boundedly approximately amenable.

In general, a boundedly approximately amenable Banach algebra need not have a bounded approximate identity as the example constructed in [11] shows. However, a boundedly approximately amenable Banach algebra will always have a bounded approximate identity if it has both a multiplier bounded left approximate identity and a multiplier bounded right approximate identity (i.e., if there exists nets $(e_v), (f_w) \subset \mathfrak{A}$ and constants $K, J > 0$ such that (e_v) is a left approximate identity for $\mathfrak{A}$, (f_w) is a right approximate identity for $\mathfrak{A}$, and such that for all $a \in \mathfrak{A}$, for all v, and for all w, $\|e_v a\| \leq K\|a\|$ and $\|af_w\| \leq J\|a\|$), as shown in [3, Theorem 3.3].

Theorem 1.3.2. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly approximately amenable and has both a multiplier bounded left approximate identity and a multiplier bounded right approximate identity, then $\mathfrak{A}$ has a bounded approximate identity.

Consequentially from the previous result and from the definition of bounded approximate contractibility, [3, Corollary 3.4] shows the following.

Theorem 1.3.3. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly approximately contractible then $\mathfrak{A}$ has a bounded approximate identity.

In the previous section we mentioned that approximate amenability and approximate contractibility are equivalent notions (alongside weak* approximate amenability). Unfortunately, it was recently discovered in [11, Corollary 3.4] that this does not hold true for the bounded variants of approximate amenability and approximate contractibility. This was proven using the previous theorem by constructing a Banach algebra that is boundedly approximately amenable but does not have a bounded approximate identity.

Theorem 1.3.4. There exists a boundedly approximately amenable Banach algebra that is not boundedly approximately contractible.

The particular example of a boundedly approximately amenable Banach algebra lacking a bounded approximate identity is actually quite interesting. It is constructed by first considering the Banach algebra $(K(\ell^1), \|\cdot\|_{\text{op}})$ of all compact linear operators from the sequence algebra ℓ^1 to ℓ^1. For each $N \in \mathbb{N}$, $K(\ell^1)$ can be renormed with a norm $\|\cdot\|^{(N)}$ equivalent to the operator norm on $K(\ell^1)$ such that the algebra $(K(\ell^1), \|\cdot\|^{(N)})$ has a bounded left approximate identity constant of 1, and a bounded right approximate identity constant of $N+1$, i.e., the infimum of all bounds of all bounded left approximate identities for $(K(\ell^1), \|\cdot\|^{(N)})$ is no less than 1, and the infimum of all bounds of all bounded right approximate identities for $(K(\ell^1), \|\cdot\|^{(N)})$ is no less than $N+1$. Then the c_0-direct sum:

$$c_0 - \bigoplus_{N=1}^{\infty}(K(\ell^1), \|\cdot\|^{(N)}),$$

has a bounded left approximate identity, but cannot have a bounded right approximate identity, for if such a bounded right approximate identity $(e_v) \subseteq c_0 - \bigoplus_{N=1}^{\infty}(K(\ell^1), \|\cdot\|^{(N)})$ existed with $K > 0$ such that $\|e_v\| \leq K$, then for any $N \in \mathbb{N}$ with $N > K-1$, the projection of (e_v) onto $(K(\ell^1), \|\cdot\|^{(N)})$ is a bounded right approximate identity for $(K(\ell^1), \|\cdot\|^{(N)})$ bounded by K, however, any bounded right approximate identity for $(K(\ell^1), \|\cdot\|^{(N)})$ must be bounded by a constant no smaller than $N+1$, but $N+1 > K$ which is a contradiction.

Hence, $c_0 - \bigoplus_{N=1}^{\infty}(K(\ell^1), \|\cdot\|^{(N)})$ has no bounded right approximate identity and thus no bounded approximate identity. Since every boundedly approximately contractible Banach algebra has a bounded approximate identity, we conclude that $c_0 - \bigoplus_{N=1}^{\infty}(K(\ell^1), \|\cdot\|^{(N)})$ is not boundedly approximately contractible. It is then proven that $c_0 - \bigoplus_{N=1}^{\infty}(K(\ell^1), \|\cdot\|^{(N)})$ is indeed boundedly approximately amenable.

Lastly, from [3, Theorem 5.4] we also have the following characterization of bounded approximate amenability in terms of nets in $(\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp)^{**}$.

Theorem 1.3.5. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly approximately amenable then there exists a net $(M_v) \subset (\mathfrak{A}^\sharp \widehat{\otimes} \mathfrak{A}^\sharp)^{**}$ and a constant $K > 0$ such that:

$$\lim_v [a \cdot M_v - M_v \cdot a] = 0 \quad \text{and} \quad \lim_v \pi^{**}(M_v) = e,$$

for all $a \in \mathfrak{A}^\sharp$ and $\|a \cdot M_v - M_v \cdot a\| \leq K\|a\|$ for all $a \in \mathfrak{A}^\sharp$ and for all v. Conversely, if the latter property holds and if the net $(\pi^{**}(M_v)) \subset \mathfrak{A}^{**}$ is bounded, then $\mathfrak{A}$ is boundedly approximately-amenable

1.4 Pseudo-Amenability and Pseudo-Contractibility

Recall that a Banach algebra is amenable if and only if it has a bounded approximate diagonal, i.e., a bounded net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that for every $a \in \mathfrak{A}$, $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi(m_\alpha)a \to a$. By removing the requirement that the net (m_α) be bounded, we then can define the notions of ***pseudo-amenability*** and ***pseudo-contractibility***.

Definition 1.4.1. A Banach algebra $\mathfrak{A}$ is said to be **pseudo-amenable** if it has a (not necessarily bounded) approximate diagonal, that is, if there exists a net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that:

$$\lim_\alpha [a \cdot m_\alpha - m_\alpha \cdot a] = 0 \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)a = a,$$

for all $a \in \mathfrak{A}$.

Definition 1.4.2. A Banach algebra $\mathfrak{A}$ is said to be **pseudo-contractible** if it has a (not necessarily bounded) central approximate diagonal, that is, if there exists a net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that:

$$a \cdot m_\alpha = m_\alpha \cdot a \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)a = a,$$

for all $a \in \mathfrak{A}$.

We note that since $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is a continuous $\mathfrak{A}$-bimodule homomorphism, we also have that if (m_α) is an approximate diagonal (or central approximate diagonal) for $\mathfrak{A}$ then $\lim_\alpha a\pi(m_\alpha) = a$ for all $a \in \mathfrak{A}$ too.

The following partial result from [12, Theorem 3.2] tells us that whenever a Banach algebra $\mathfrak{A}$ has a bounded approximate identity, then the approximate amenability of $\mathfrak{A}$ implies the pseudo-amenability of $\mathfrak{A}$.

Theorem 1.4.1. Let $\mathfrak{A}$ be a Banach algebra. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in the following:

(1) $\mathfrak{A}$ is approximately amenable and has a bounded approximate identity.

(2) $\mathfrak{A}$ has an approximate diagonal $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ for which $(\pi(m_\alpha))$ is a bounded net.

(3) $\mathfrak{A}$ is pseudo-amenable and has a bounded approximate identity.

If $\mathfrak{A}$ is a Banach algebra with a central approximate identity (not necessarily bounded), then the approximate amenability of $\mathfrak{A}$ still implies the pseudo-amenability of $\mathfrak{A}$ as proven in [12, Proposition 3.3].

Theorem 1.4.2. Let $\mathfrak{A}$ be a Banach algebra with a central approximate identity. If $\mathfrak{A}$ is approximately amenable then $\mathfrak{A}$ is pseudo-amenable.

For a Banach algebra $\mathfrak{A}$, it is still an open question as to whether the pseudo-amenability implies approximate amenability under the assumptions that $\mathfrak{A}$ has a bounded approximate identity or a central approximate identity. Originally [12, Theorem 3.2] claimed the equivalence of approximate amenability with pseudo-amenability under the assumption of the existence of a bounded approximate identity, however, there is a small gap in that direction of the proof.

That said, there are still known (and relatively simple) examples of Banach algebras that are pseudo-amenable but not approximately amenable. For $1 \leq p < \infty$ let:

$$\ell^p = \left\{ (a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^p < \infty \right\},$$

with norm $\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$, and let $\ell^\infty = \{(a_n) : a_n \in \mathbb{C}, (a_n) \text{ bounded}\}$ with norm $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$, and equip all of these spaces with pointwise multiplication to make them Banach algebras. Then, for $1 \leq p < \infty$, ℓ^p is pseudo-amenable (and more precisely, *boundedly pseudo-contractible* which we will define and show later), but in [7, Theorem 4.1] it was proven that these algebras are not approximately amenable.

We now briefly state a relationship that pseudo-contractibility has with the notion of contractibility define earlier. As the following result from [12, Theorem 2.4] shows, if a Banach algebra has an identity then these two notions are in fact equivalent.

Theorem 1.4.3. Let $\mathfrak{A}$ be a Banach algebra. The following statements are equivalent:

(1) $\mathfrak{A}$ is pseudo-contractible and has an identity.

(2) $\mathfrak{A}^\sharp$ is pseudo-contractible.

(3) $\mathfrak{A}$ is contractible.

There are also nice characterizations of pseudo-contractibility and pseudo-amenability in terms of a concept called approximate biprojectivity. There are a few different definitions of approximate biprojectivity in the literature. The two most common ones are given below.

Definition 1.4.3. A Banach algebra $\mathfrak{A}$ is said to be **GZ-approximately biprojective** if there exists a net $(T_\alpha) \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ of continuous $\mathfrak{A}$-bimodule homomorphisms such that:

$$\lim_\alpha \pi(T_\alpha(a)) = a,$$

for all $a \in \mathfrak{A}$.

An alternative equally reasonable definition introduced in [17] is the following.

Definition 1.4.4. A Banach algebra $\mathfrak{A}$ is said to be **P-approximately biprojective** if there exists a net $(T_\alpha) \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ which is an approximate $\mathfrak{A}$-bimodule homomorphism, and such that:

$$\lim_\alpha \pi(T_\alpha(a)) = a,$$

for all $a \in \mathfrak{A}$.

When a Banach algebra $\mathfrak{A}$ has a central approximate identity, from [12, Proposition 3.8] and [17, Corollary 3.6] pseudo-contractibility and pseudo-amenability respectively become equivalent to the notions of GZ-approximate biprojectivity and P-approximate biprojectivity.

Theorem 1.4.4. Let $\mathfrak{A}$ be a Banach algebra. Then the following statements are equivalent:

(1) $\mathfrak{A}$ is pseudo-contractible.

(2) $\mathfrak{A}$ is GZ-approximately biprojective and has a central approximate identity.

Theorem 1.4.5. Let $\mathfrak{A}$ be a Banach algebra. Then the following statements are equivalent:

(1) $\mathfrak{A}$ is pseudo-amenable.

(2) $\mathfrak{A}$ is P-approximately biprojective and has a central approximate identity.

With regards to approximate identities, observe that if $\mathfrak{A}$ is pseudo-amenable (resp. pseudo-contractible) and if $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is an approximate diagonal (resp. central approximate diagonal) for $\mathfrak{A}$ then $(\pi(m_\alpha)) \subset \mathfrak{A}$ is an approximate identity (resp. central approximate identity) for $\mathfrak{A}$.

As per the previous sections, it is always nice to know when pseudo-amenability and pseudo-contractibility can be inherited. The following result from [12, Proposition 2.2] shows us that continuous algebra epimorphisms preserve both pseudo-amenability and pseudo-contractibility.

Theorem 1.4.6. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is continuous algebra epimorphism and $\mathfrak{A}$ is pseudo-amenable (resp. pseudo-contractible) then $\mathfrak{B}$ is pseudo-amenable (resp. pseudo-contractible).

Regarding ideals, we have the following result from [12, Proposition 2.6].

Theorem 1.4.7. Let $\mathfrak{A}$ be a Banach algebra and let $J \subseteq \mathfrak{A}$ be a closed ideal of $\mathfrak{A}$. If $\mathfrak{A}$ is pseudo-amenable and J has an approximate identity (t_i) with the additional property that the operators $L_i, R_i : \mathfrak{A} \to J$ defined for each i and for all $a \in \mathfrak{A}$ by $L_i(a) = t_i a$ and $R_i(a) = at_i$ are uniformly bounded, then J is pseudo-amenable.

As a consequence of the previous two theorem, from [12, Corollary 2.7] we see that if $\mathfrak{A}$ is pseudo-amenable then J is pseudo-amenable whenever it has a bounded approximate identity. Furthermore, since the quotient map $q : \mathfrak{A} \to \mathfrak{A}/J$ is a continuous algebra epimorphism, we see that $\mathfrak{A}/J$ always inherits pseudo-amenability from $\mathfrak{A}$. These results are summarized below.

Theorem 1.4.8. Let $\mathfrak{A}$ be a Banach algebra and let $J \subseteq \mathfrak{A}$ be a closed ideal of $\mathfrak{A}$.
(1) If $\mathfrak{A}$ is pseudo-amenable and J has a bounded approximate identity then J is pseudo-amenable.
(2) If $\mathfrak{A}$ is pseudo-amenable then $\mathfrak{A}/J$ is pseudo-amenable.

In section 4 of [12] it is shown that for a locally compact group G, $L^1(G)$ is pseudo-amenable if and only if G is an amenable group and that $(L^1(G))^{**}$ is pseudo-amenable if and only if G is a finite group. It is also shown that $M(G)$ is pseudo-amenable if and only if G is a discrete amenable group.

On the other hand, it was also proven in [12] that the Segal algebra on a compact group is always pseudo-contractible.

1.5 Bounded Pseudo-Amenability and Bounded Pseudo-Contractibility

In this section we define two new notions of amenability and contractibility called ***bounded pseudo-amenability*** and ***bounded pseudo-contractibility***. These two notions are quite natural, but have not yet been studied extensively. The main results of this thesis in Chapter 2 will investigate these two notions and how they relate to other generalized notions of amenability and contractibility.

Definition 1.5.1. A Banach algebra $\mathfrak{A}$ is said to be **boundedly pseudo-amenable** if it has a multiplier bounded approximate diagonal, that is, if there exists a constant $K > 0$ and a net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\|a \cdot m_\alpha - m_\alpha \cdot a\| \leq K\|a\|$ and $\|\pi(m_\alpha)a\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all α, and:

$$\lim_\alpha [a \cdot m_\alpha - m_\alpha \cdot a] = 0 \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)a = a,$$

for all $a \in \mathfrak{A}$.

Definition 1.5.2. A Banach algebra $\mathfrak{A}$ is said to be **boundedly pseudo-amenable** if it has a multiplier bounded central approximate diagonal, that is, if there exists a constant $K > 0$ and a net $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\|\pi(m_\alpha)a\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all α, and:

$$a \cdot m_\alpha = m_\alpha \cdot a \quad \text{and} \quad \lim_\alpha \pi(m_\alpha)a = a,$$

for all $a \in \mathfrak{A}$.

If $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) with $(m_\alpha) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ as a multiplier bounded approximate diagonal (resp. multiplier bounded central approximate diagonal) then $(\pi(m_\alpha)) \subset \mathfrak{A}$ is a multiplier bounded approximate identity (resp. multiplier bounded central approximate identity) for $\mathfrak{A}$.

The following result was obtained in [3, Proposition 2.2].

Theorem 1.5.1. Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable if and only if $\mathfrak{A}$ is boundedly approximately contractible.

And [3, Proposition 3.2] gives a criterion for when the unitization $\mathfrak{A}^\sharp$ can inherit bounded pseudo-amenability from $\mathfrak{A}$.

Theorem 1.5.2. Let $\mathfrak{A}$ be a Banach algebra with a central bounded approximate identity. If $\mathfrak{A}$ is boundedly pseudo-amenable then $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable.

In the next chapter, we will develop new results on bounded pseudo-amenability and bounded pseudo-contractibility.

2 New Results on the Notions of Bounded Pseudo-Amenability and Bounded Pseudo-Contractibility

In this chapter we investigate bounded pseudo-amenability and bounded pseudo-contractibility in detail, and present the main results of this thesis.

In **Section 2.1** we study the l^p direct sums of collections of boundedly pseudo-amenable and boundedly pseudo-contractible Banach algebras.

In **Section 2.2** we prove that if $\mathfrak{A}$ has a (not necessarily central) bounded approximate identity then the bounded pseudo-amenability of $\mathfrak{A}^\sharp$ implies the bounded pseudo-amenability of $\mathfrak{A}$. We also show that $\mathfrak{A}^\sharp$ is boundedly pseudo-contractible if and only if $\mathfrak{A}$ is boundedly pseudo-contractible and possesses and identity.

In **Section 2.3** we study the inheritability properties of bounded pseudo-amenability and bounded pseudo-contractibility.

In **Section 2.4** we find relationships of bounded pseudo-amenability (and bounded pseudo-contractibility) with other generalized notions of amenability.

In **Section 2.5** we investigate the bounded pseudo-amenability and bounded pseudo-contractibility of tensor products of Banach algebra.

2.1 l^p Direct Sums of Boundedly Pseudo-Amenable and Boundedly Pseudo-Contractible Banach Algebras

In [12, Proposition 2.1] it was shown that if $\{\mathfrak{A}_i : i \in I\}$ is a collection of pseudo-amenable (resp. pseudo-contractible) Banach algebras indexed by some indexing set I, then for $1 \leq p < \infty$, the l^p direct sum algebras $\bigoplus_{i \in I}^{p} \mathfrak{A}_i$ are also pseudo-amenable (resp. pseudo-contractible). In this section, we derive a similar result for collections of boundedly pseudo-amenable and boundedly pseudo-contractible Banach algebras.

Definition 2.1.1. Let $\{\mathfrak{A}_i : i \in I\}$ be a collection of Banach algebras. For $1 \leq p < \infty$, the l^p **Direct Sum of** $\{\mathfrak{A}_i : i \in I\}$ is defined to be the Banach algebra:

$$\bigoplus_{i \in I}^{p} \mathfrak{A}_i := \left\{ a \in \prod_{i \in I} \mathfrak{A}_i : \sum_{i \in I} \|a(i)\|^p < \infty \right\},$$

equipped with coordinatewise operations, and with norm:

$$\|a\|_p := \left(\sum_{i \in I} \|a(i)\|^p \right)^{1/p}.$$

If $\{\mathfrak{A}_i : i \in I\}$ is a collection of Banach algebras and if $J \subseteq I$ then let $P_J : \bigoplus_{i \in I}^{p} \mathfrak{A}_i \to \bigoplus_{i \in J}^{p} \mathfrak{A}_i$ be the associated projection, where $P_J(a)(j) := a(j)$ for all $a \in \bigoplus_{i \in I}^{p} \mathfrak{A}_i$ and for all $j \in J$. The algebra $\bigoplus_{i \in J}^{p} \mathfrak{A}_i$ can be identified with the subalgebra of $\bigoplus_{i \in I}^{p} \mathfrak{A}_i$ consisting of all $a \in \bigoplus_{i \in I}^{p} \mathfrak{A}_i$ with the property that $a(i) = 0$ for all $i \notin J$. With this identification, we also have that for all $a \in \bigoplus_{i \in I}^{p} \mathfrak{A}_i$ and for all $J \subset I$ that $\|P_J(a)\|_p \leq \|a\|_p$.

In particular, for each singleton set $\{i\}$, $\mathfrak{A}_i$ can be identified with the subalgebra of $\bigoplus_{i \in I}^{p} \mathfrak{A}_i$ consisting of elements that are zero everywhere except possibly at i.

Theorem 2.1.1. Let $\{\mathfrak{A}_i : i \in I\}$ be a collection of boundedly pseudo-amenable (resp. boundedly pseudo-contractible) Banach algebras. If $1 \leq p < \infty$, and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, and if for each $i \in I$ there exists a multiplier bounded approximate diagonal for $\mathfrak{A}_i$ with multiplier bound

K_i such that $(K_i)_i \in \ell^q(I)$, then $\bigoplus_{i\in I}^p \mathfrak{A}_i$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. We will only prove the case when $\{\mathfrak{A}_i : i \in I\}$ is a collection of boundedly pseudo-amenable Banach algebras. The case when $\{\mathfrak{A}_i : i \in I\}$ is a collection of boundedly pseudo-contractible Banach algebras is proven similarly.

Let $\mathfrak{A} := \bigoplus_{i\in I}^p \mathfrak{A}_i$. For each $i \in I$ let $(u_\alpha^{(i)}) \subset \mathfrak{A}_i \widehat{\otimes} \mathfrak{A}_i$ be a multiplier bounded approximate diagonal for $\mathfrak{A}_i$ with multiplier bound K_i, such that $(K_i)_i \in \ell^q(I)$, and let $K := \|(K_i)_i\|_q$.

It suffices to show that for each $\epsilon > 0$ and for each finite $F \subset \mathfrak{A}$, there exists an element $u = u_{(\epsilon,F)} \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that:

$$\|a \cdot u - u \cdot a\| < \epsilon \quad \text{and} \quad \|\pi(u)a - a\| < \epsilon,$$

for all $a \in F$, and:

$$\|a \cdot u - u \cdot a\| \leq K\|a\|_p \quad \text{and} \quad \|\pi(u)a\|_p \leq K\|a\|_p,$$

for all $a \in \mathfrak{A}$. Then $(u_{\epsilon,F})_{\epsilon,F} \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ will be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound K so that $\mathfrak{A}$ is boundedly pseudo-amenable.

Let $\epsilon > 0$ and let $F \subset \mathfrak{A}$ be a finite set. Then there exists a finite $J \subset I$ such that for all $a \in F$:

$$\|P_J(a) - a\|_p < \frac{\epsilon}{2},$$

where as noted earlier, $P_J(a) \in \bigoplus_{i\in J}^p \mathfrak{A}_i$ and $\bigoplus_{i\in J}^p \mathfrak{A}_i$ is identified as a subalgebra of $\mathfrak{A}$.

Since each $\mathfrak{A}_i$ is pseudo-amenable, we can choose $u_i \in (u_\alpha^{(i)})_\alpha \subset \mathfrak{A}_i\widehat{\otimes}\mathfrak{A}_i$ such that for all $a \in F$:

$$\sum_{i\in J} \|P_i(a) \cdot u_i - u_i \cdot P_i(a)\| < \epsilon \quad \text{and} \quad \sum_{i\in J} \|\pi(u_i)P_i(a) - P_i(a)\|_p < \frac{\epsilon}{2},$$

where $P_i(a) := P_{\{i\}}(a) \in \mathfrak{A}_i$ and $\mathfrak{A}_i$ is identified as a subalgebra of $\mathfrak{A}$; and furthermore, since $\mathfrak{A}_i$ is complemented in $\mathfrak{A}$, we can embed $\mathfrak{A}_i\widehat{\otimes}\mathfrak{A}_i$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ and thus regard each element u_i above as an element in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$. Let $u := u_{\epsilon,F} := \sum_{i\in J} u_i$. Then:

$$\|a\cdot u - u\cdot a\| = \left\|\sum_{i\in J}[P_i(a)\cdot u_i - u_i\cdot P_i(a)]\right\| \leq \sum_{i\in J}\|P_i(a)\cdot u_i - u_i\cdot P_i(a)\| < \epsilon,$$

and:

$$\|\pi(u)a - a\|_p = \left\|\pi\left(\sum_{i\in J} u_i\right)a - a\right\|_p \leq \sum_{i\in J}\|\pi(u_i)P_i(a) - P_i(a)\|_p + \|P_J(a) - a\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $a \in F$. On the other hand, by Holder's inequality we have that:

$$\|a\cdot u - u\cdot a\| \leq \sum_{i\in J}\|P_i(a)\cdot u_i - u_i\cdot P_i(a)\| \leq \sum_{i\in J} K_i\|P_i(a)\| = \sum_{i\in J} K_i\|a(i)\| \leq K\|a\|_p,$$

and:

$$\|\pi(u)a\|_p \leq \sum_{i\in J}\|\pi(u_i)P_i(a)\|_p \leq \sum_{i\in J} K_i\|P_i(a)\| = \sum_{i\in J} K_i\|a(i)\| \leq K\|a\|_p,$$

for all $a \in \mathfrak{A}$ and so the proof is complete. ■

Definition 2.1.2. A collection $\{\mathfrak{A}_i : i \in I\}$ of boundedly pseudo-amenable Banach algebras is said to be **uniformly boundedly pseudo-amenable** if there exists a constant $K > 0$ such that each $\mathfrak{A}_i$ has a multiplier bounded approximate diagonal $(u_\alpha^{(i)})_\alpha \subset \mathfrak{A}_i\widehat{\otimes}\mathfrak{A}_i$ with multiplier bound K, so that for all $i \in I$, for all $a^{(i)} \in \mathfrak{A}_i$, and for all α, we have that $\|a^{(i)}\cdot u_\alpha^{(i)} - u_\alpha^{(i)}\cdot a^{(i)}\| \leq K\|a^{(i)}\|$, $\|\pi(u_\alpha^{(i)})a^{(i)}\| \leq K\|a^{(i)}\|$.

Similarly, a collection $\{\mathfrak{A}_i : i \in I\}$ of boundedly pseudo-contractible Banach algebras is said to be **uniformly boundedly pseudo-contractible** if there exists a constant $K > 0$ such that each $\mathfrak{A}_i$ has a multiplier bound central approximate diagonal $(u_\alpha^{(i)})_\alpha \subset \mathfrak{A}_i\widehat{\otimes}\mathfrak{A}_i$ with multiplier bound K, so that for all $i \in I$, for all $a^{(i)} \in \mathfrak{A}_i$, and for all α we have that $\|\pi(u_\alpha^{(i)})a^{(i)}\| \leq K\|a^{(i)}\|$.

As a quick consequence of Theorem 2.1.1 we have the following.

Corollary 2.1.1. If $\{\mathfrak{A}_i : i \in I\}$ is uniformly boundedly pseudo-amenable (resp. uniformly bounded pseudo-contractible) then the l^1 direct sum $\bigoplus_{i\in I}^{1} \mathfrak{A}_i$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. If $\{\mathfrak{A}_i : i \in I\}$ is uniformly boundedly pseudo-amenable (or uniformly boundedly pseudo-contractible) with a uniform multiplier bound $K > 0$ then for each $i \in I$, set $K_i := K$. Then $(K_i)_i \in \ell^\infty(I)$ and apply the previous theorem. ■

And as a special case of the previous corollary we have the following.

Corollary 2.1.2. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) then $\ell^1(\mathfrak{A})$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

2.2 Relationship Between the Bounded Pseudo-Amenability and Bounded Pseudo-Contractibility of $\mathfrak{A}^\sharp$ and $\mathfrak{A}$

Recall that Proposition 1.5.2 states that if $\mathfrak{A}$ has a central bounded approximate identity and if $\mathfrak{A}$ is boundedly pseudo-amenable then $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable. By modifying Theorem 1.4.1 we can obtain a stronger converse to this result.

Theorem 2.2.1. Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity. Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) in the following:

(1) $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable.

(2) $\mathfrak{A}$ has a multiplier bounded approximate diagonal $(m_\mu) \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $(\pi(m_\mu))$ is bounded.

(3) $\mathfrak{A}$ is boundedly pseudo-amenable.

Proof. Let $(e_\alpha) \subset \mathfrak{A}$ be a bounded approximate identity for $\mathfrak{A}$ and let $N > 0$ be such that $\|e_\alpha\| \le N$ for all α.

(1) $\Rightarrow$ **(3)**: Suppose that $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable. Let $(M_\mu)_\mu \subset \mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{A}^\sharp$ be a multiplier bounded approximate diagonal for $\mathfrak{A}^\sharp$ with multiplier bound $K > 0$, and let $E \in \mathfrak{A}^\sharp$ denote the identity in $\mathfrak{A}^\sharp$.

We may assume that $\pi(M_\mu) = E$ for all μ, otherwise, we can take $(M'_\mu)_\mu$ in place of $(M_\mu)_\mu$, where for each μ:

$$M'_\mu := M_\mu - \pi(M_\mu) \otimes E + E \otimes E.$$

The net $(M'_\mu)_\mu$ is also an approximate diagonal for $\mathfrak{A}^\sharp$ and satisfies $\pi(M'_\mu) = E$ for all μ. It is multiplier bounded with multiplier bound $3K + 2$ since:

$$\|A \cdot M'_\mu - M'_\mu \cdot A\| \le \|A \cdot M_\mu - M_\mu \cdot A\| + \|A\pi(M_\mu)\| + \|\pi(M_\mu)\|\|A\| + 2\|A\| \le (3K + 2)\|A\|,$$

and $\|\pi(M'_\mu)A\| = \|A\|$ for all $A \in \mathfrak{A}^\sharp$ and for all μ.

Since $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{A}^\sharp \cong (\mathfrak{A}\widehat{\otimes}\mathfrak{A}) \oplus (\mathfrak{A}\widehat{\otimes}\{E\}) \oplus (\{E\}\widehat{\otimes}\mathfrak{A}) \oplus (\mathbb{C}\{E\} \otimes \{E\})$, for each μ we can write:

$$M_\mu = u_\mu + F_\mu \otimes E + E \otimes G_\mu + c_\mu E \otimes E,$$

where $u_\mu \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$, $F_\mu, G_\mu \in \mathfrak{A}$, and $c_\mu \in \mathbb{C}$. Since $\pi(M_\mu) = E$ for all μ and since $(M_\mu)_\mu$ is an approximate diagonal for $\mathfrak{A}$, it is easy to see that $c_\mu = 1$ for all μ, and that:

$$a \cdot u_\mu - u_\mu \cdot a - F_\mu \otimes a + a \otimes G_\mu \xrightarrow{\mu} 0 \quad \text{and} \quad aF_\mu \xrightarrow{\mu} -a \quad \text{and} \quad G_\mu a \xrightarrow{\mu} -a,$$

for all $a \in \mathfrak{A}$, and:

$$\pi(u_\mu) + F_\mu + G_\mu = 0,$$

for all μ.

Now, for each μ and for each α let:

$$m_{\mu,\alpha} := u_\mu + F_\mu \otimes e_\alpha + e_\alpha \otimes G_\mu + e_\alpha \otimes e_\alpha.$$

We will show that $(m_{\mu,\alpha})_{(\mu,\alpha)}$ satisfies the multiplier boundedness property and that there exists a subnet of $(m_{\mu,\alpha})_{(\mu,\alpha)}$ that is an approximate diagonal for $\mathfrak{A}$. First, we claim that $(F_\mu)_\mu$ and $(G_\mu)_\mu$ are bounded. Let P_1 be the projection of $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{A}^\sharp$ onto $\mathfrak{A}\widehat{\otimes}\{E\}$ and let P_2 be the projection of $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{A}^\sharp$ onto $\{E\}\widehat{\otimes}\mathfrak{A}$. Then:

$$\begin{aligned}
\|aF_\mu\| \le \|aF_\mu + a\| + \|a\| = \|aF_\mu \otimes E + a \otimes E\| + \|a\| &= \|P_1(a \cdot M_\mu - M_\mu \cdot a)\| + \|a\| \\
&\le \|P_1\|\|a \cdot M_\mu - M_\mu \cdot a\| + \|a\| \\
&\le \|P_1\|(K+1)\|a\|,
\end{aligned}$$

for all $a \in \mathfrak{A}$. So $(F_\mu)_\mu \subset \mathfrak{A}$ is a right multiplier bounded net with multiplier bound $\|P_1\|(K+1)$. Since $(e_\alpha)_\alpha$ is a bounded approximate identity for $\mathfrak{A}$ bounded by N, we have that:

$$\|F_\mu\| = \lim_\alpha \|e_\alpha F_\mu\| \le \|P_1\|(K+1)N,$$

for all μ, and so $(F_\mu)_\mu$ is bounded. A similar argument shows that $(G_\mu)_\mu \subset \mathfrak{A}$ is also bounded.

So, for each μ and for each α we can write:

$$m_{\mu,\alpha} = M_\mu + F_\mu \otimes (e_\alpha - E) + (e_\alpha - E) \otimes G_\mu - E \otimes E + e_\alpha \otimes e_\alpha.$$

Since $(M_\mu)_\mu$ is a multiplier bounded approximate diagonal, and $(F_\mu)_\mu$, $(G_\mu)_\mu$, and $(e_\alpha)_\alpha$ are all bounded nets, we have that there exists a $L > 0$ such that $\|a \cdot m_{\mu,\alpha} - m_{\mu,\alpha} \cdot a\| \leq L\|a\|$ and $\|\pi(m_{\mu,\alpha})a\| \leq L\|a\|$ for all μ, for all α, and for all $a \in \mathfrak{A}$.

Furthermore,

$$\begin{aligned}\|a \cdot m_{\mu,\alpha} - m_{\mu,\alpha} \cdot a\| \leq \|a \cdot u_\mu - u_\mu \cdot a - F_\mu \otimes e_\alpha a + ae_\alpha \otimes G_\mu\| \\ + \|(aF_\mu + ae_\alpha) \otimes e_\alpha\| + \|e_\alpha \otimes (G_\mu a + e_\alpha a)\|\end{aligned}$$

and:

$$\|\pi(m_{\mu,\alpha})a - a\| = \|(\pi(u_\mu) + F_\mu e_\alpha + e_\alpha G_\mu + e_\alpha^2)a - a\|$$

for all $a \in \mathfrak{A}$. So the iterated limits $\lim_\mu \lim_\alpha \|a \cdot m_{\mu,\alpha} - m_{\mu,\alpha} \cdot a\|$ and $\lim_\mu \lim_\alpha \|\pi(m_{\mu,\alpha})a - a\|$ both equal 0 for each $a \in \mathfrak{A}$, and so there exists a subnet $(m_i)_i$ of $(m_{\mu,\alpha})_{(\mu,\alpha)}$ such that $\|a \cdot m_i - m_i \cdot a\| \xrightarrow{i} 0$ and $\|\pi(m_i)a - a\| \xrightarrow{i} 0$ for all $a \in \mathfrak{A}$. So $(m_i)_i \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a multiplier bounded approximate diagonal for $\mathfrak{A}$ and thus $\mathfrak{A}$ is boundedly pseudo-amenable.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (2): Let $\mathfrak{A}$ be boundedly pseudo-amenable and let $(m_\mu)_\mu \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be any multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $K > 0$. Then $(\pi(m_\mu))_\mu \subset \mathfrak{A}$ is a multiplier bounded approximate identity for $\mathfrak{A}$. So:

$$\|\pi(m_\mu)e_\alpha\| \leq K\|e_\alpha\| \leq KN,$$

for all μ and for all α. Thus:

$$\|\pi(m_\mu)\| = \lim_\alpha \|\pi(m_\mu)e_\alpha\| \leq KN,$$

for all μ and so $(\pi(m_\mu))_\mu$ is bounded. ∎

Combining the aforementioned Theorem 1.5.2 with Theorem 2.2.1 and we get the following corollary:

Corollary 2.2.1. Let $\mathfrak{A}$ be a Banach algebra with a central bounded approximate identity. Then $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable if and only if $\mathfrak{A}$ is boundedly pseudo-amenable.

We will now shift our attention to the bounded pseudo-contractibility of $\mathfrak{A}^\sharp$. Recall from Theorem 1.4.3 that if $\mathfrak{A}$ is a Banach algebra then $\mathfrak{A}^\sharp$ being pseudo-contractible is equivalent to $\mathfrak{A}$ being pseudo-contractible and possessing an identity, which is equivalent to $\mathfrak{A}$ being contractible. We can easily obtain a similar relationship for the bounded pseudo-amenability of $\mathfrak{A}^\sharp$ as a consequence of this theorem.

Proposition 2.2.1. Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}^\sharp$ is boundedly pseudo-contractible if and only if $\mathfrak{A}$ is boundedly pseudo-contractible and has an identity.

Proof. Suppose that $\mathfrak{A}^\sharp$ is boundedly pseudo-contractible. Then $\mathfrak{A}^\sharp$ is trivially pseudo-contractible, which by Theorem 1.4.3 implies that $\mathfrak{A}$ is contractible. But every contractible Banach algebra is boundedly pseudo-contractible and every contractible Banach algebra also possesses an identity.

Conversely, suppose that $\mathfrak{A}$ is boundedly pseudo-contractible with an identity. Then $\mathfrak{A}$ is pseudo-contractible with an identity. By Theorem 1.4.3 we have that $\mathfrak{A}$ is contractible. So $\mathfrak{A}^\sharp$ is contractible. Thus $\mathfrak{A}^\sharp$ is boundedly pseudo-contractible. ∎

2.3 Ideals of Bounded Pseudo-Amenable and Bounded Pseudo-Contractible Banach Algebras

Recall from Theorem 1.4.6 that if there exists a continuous epimorphism between a pseudo-amenable (resp. pseudo-contractible) Banach algebra $\mathfrak{A}$ and a Banach algebra $\mathfrak{B}$ then $\mathfrak{B}$ is necessarily pseudo-amenable (resp. pseudo-contractible). The same is true for the bounded variations of pseudo-amenability and pseudo-contractibility. The only care we need to take is to verify that the continuous epimorphism transfers the multiplier boundedness of the approximate diagonal from $\mathfrak{A}$ to the approximate diagonal for $\mathfrak{B}$.

Proposition 2.3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $T : \mathfrak{A} \to \mathfrak{B}$ is a continuous algebra epimorphism and if $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) then $\mathfrak{B}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. Again, we will only prove the case when $\mathfrak{A}$ is boundedly pseudo-amenable since the case when $\mathfrak{A}$ is boundedly pseudo-contractible is similar.

Let $\mathfrak{A}$ be boundedly pseudo-amenable and let $(u_\alpha)_\alpha \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $K > 0$. Let $(T \otimes T) : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ be specified for each elementary tensor $a_1 \otimes a_2 \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ by:

$$(T \otimes T)(a_1 \otimes a_2) := T(a_1) \otimes T(a_2).$$

We claim that $((T \otimes T)(u_\alpha))_\alpha \subset \mathfrak{B}$ is an approximate diagonal for $\mathfrak{B}$. For each $b \in \mathfrak{B}$ take any $a \in \mathfrak{A}$ such that $b = T(a)$. Then:

$$\begin{aligned}
\|b \cdot (T \otimes T)(u_\alpha) - (T \otimes T)(u_\alpha) \cdot b\| &= \|T(a) \cdot (T \otimes T)(u_\alpha) - (T \otimes T)(u_\alpha) \cdot T(a)\| \\
&= \|(T \otimes T)(a \cdot u_\alpha - u_\alpha \cdot a)\| \\
&\leq \|T\|^2 \|a \cdot u_\alpha - u_\alpha \cdot a\| \xrightarrow{\alpha} 0,
\end{aligned}$$

and:

$$\begin{aligned}
\|\pi((T\otimes T)(u_\alpha))b-b\| &= \|\pi((T\otimes T)(u_\alpha))T(a)-T(a)\| \\
&= \|T(\pi(u_\alpha)a-a)\| \\
&\leq \|T\|\|\pi(u_\alpha)a-a\| \stackrel{\alpha}{\to} 0,
\end{aligned}$$

for all $b\in\mathfrak{B}$.

Furthermore, we claim that $((T\otimes T)(u_\alpha))_\alpha$ is multiplier bounded. To prove this, we will first show that there exists an $r>0$ such that for each $b\in\mathfrak{B}$, an $a\in\mathfrak{A}$ can be chosen such that $b=T(a)$ and $\|a\|\leq\dfrac{\|b\|}{r}$.

Let $B_{\mathfrak{A}}=\{a\in\mathfrak{A}:\|a\|<1\}$ and let $B_{\mathfrak{B}}=\{b\in\mathfrak{B}:\|b\|<1\}$. Since $T:\mathfrak{A}\to\mathfrak{B}$ is a continuous epimorphism between Banach algebras, T is an open map by the Open Mapping Theorem, and so $T(B_{\mathfrak{A}})$ is an open neighbourhood of $0_{\mathfrak{B}}$. So, there exists an $r>0$ such that $r\overline{B_{\mathfrak{B}}}\subset T(B_{\mathfrak{A}})$ where $r\overline{B_{\mathfrak{B}}}:=\{b\in\mathfrak{B}:\|b\|\leq r\}$. For each $b\in\mathfrak{B}\setminus\{0_{\mathfrak{B}}\}$, $\dfrac{rb}{\|b\|}\in r\overline{B_{\mathfrak{B}}}$, and so, there exists an $a'\in B_{\mathfrak{A}}$ such that $T(a')=\dfrac{rb}{\|b\|}$, or equivalently:

$$T\left(\frac{\|b\|}{r}a'\right)=b.$$

So let $a:=\dfrac{\|b\|}{r}a'$. Then a is such that $b=T(a)$ and:

$$\|a\|=\frac{\|b\|}{r}\|a'\|<\frac{\|b\|}{r}.$$

Thus, for each $b\in\mathfrak{B}$, we can choose $a\in\mathfrak{A}$ as above such that:

$$\|b\cdot(T\otimes T)(u_\alpha)-(T\otimes T)(u_\alpha)\cdot b\|\leq\|T\|^2\|a\cdot u_\alpha-u_\alpha\cdot a\|\leq\|T\|^2K\|a\|<\frac{\|T\|^2K}{r}\|b\|,$$

and:

$$\|\pi(u_\alpha)b\| \leq \|T\|\|\pi(u_\alpha)\| \leq \|T\|K\|a\| < \frac{\|T\|K}{r}\|b\|.$$

Hence $((T \otimes T)(u_\alpha))_\alpha$ is a multiplier bounded approximate diagonal for $\mathfrak{B}$ with multiplier bound $\max\left\{\frac{\|T\|^2K}{r}, \frac{\|T\|K}{r}\right\}$. Hence $\mathfrak{B}$ is boundedly pseudo-amenable. ■

As a consequence of Proposition 2.3.1, we immediately have the following.

Corollary 2.3.1. Let $\mathfrak{A}$ be a Banach algebra and let $J \subseteq \mathfrak{A}$ be a closed ideal of $\mathfrak{A}$. If $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) then $\mathfrak{A}/J$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

By requiring that the ideal J has a central approximate identity, we can obtain a corresponding result to Theorem 1.4.7 for bounded pseudo-amenability.

Proposition 2.3.2. Let $\mathfrak{A}$ be a boundedly pseudo-amenable (resp. boundedly pseudo-contractible) Banach algebra and let J be a closed two-sided ideal of $\mathfrak{A}$. If J has a central approximate identity $(t_i) \subset J$ such that the operators $L_i, R_i : \mathfrak{A} \to J$ defined for each $a \in \mathfrak{A}$ by $L_i(a) = t_i a$ and $R_i(a) = at_i$ are uniformly bounded, then J is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. Let $\mathfrak{A}$ be boundedly pseudo-amenable. The case when $\mathfrak{A}$ is boundedly pseudo-contractible is proven similarly.

Let $(t_i)_i \subset J$ be a multiplier bounded central approximate identity for J and let $M > 0$ be such that $\|at_i\| \leq M\|a\|$ and $\|t_i a\| \leq M\|a\|$ for all $a \in \mathfrak{A}$ and for all i. Let $(m_\alpha)_\alpha \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $N > 0$. We claim that the net $(t_i \cdot m_\alpha \cdot t_i)_{(i,\alpha)} \subset J$ is a multiplier bounded approximate diagonal for J.

First observe that for each $m := \sum_{n=1}^{\infty} a_n \otimes b_n \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ we have that:

$$\|t_i \cdot m \cdot t_i\| = \left\|\sum_{n=1}^{\infty} t_i a_n \otimes b_n t_i\right\| \leq \sum_{n=1}^{\infty} \|t_i a_n\|\|b_n t_i\| \leq M^2 \sum_{n=1}^{\infty} \|a_n\|\|b_n\|,$$

and so $\|t_i \cdot m \cdot t_i\| \leq M^2\|m\|$ for each i. Therefore, by the centrality of $(t_i)_i$ and the aforementioned inequality, we have that:

$$\|j \cdot (t_i \cdot m_\alpha \cdot t_i) - (t_i \cdot m_\alpha \cdot t_i) \cdot j\| = \|t_i \cdot (j \cdot m_\alpha - m_\alpha \cdot j) \cdot t_i\| \leq M^2 \|j \cdot m_\alpha - m_\alpha \cdot j\| \overset{\alpha}{\to} 0,$$

and:

$$\begin{aligned}\|\pi(t_i \cdot m_\alpha \cdot t_i)j - j\| = \|t_i \pi(m_\alpha) t_i - j\| &\leq \|t_i(\pi(m_\alpha) - j)t_i\| + \|t_i j t_i - j\| \\ &\leq M^2 \|\pi(m_\alpha) - j\| + \|t_i j t_i - j\| \overset{i,\alpha}{\to} 0,\end{aligned}$$

for all $j \in J$ and so $(t_i \cdot m_\alpha \cdot t_i)_{(i,\alpha)}$ is an approximate diagonal for J. Moreover, it is multiplier bounded since:

$$\|j \cdot (t_i \cdot m_\alpha \cdot t_i) - (t_i \cdot m_\alpha \cdot t_i) \cdot j\| = \|t_i \cdot (j \cdot m_\alpha - m_\alpha \cdot j) \cdot t_i\| \leq M^2 \|j \cdot m_\alpha - m_\alpha \cdot j\| \leq M^2 N \|j\|,$$

and:

$$\|\pi(t_i \cdot m_\alpha \cdot t_i)j\| = \|t_i \pi(m_\alpha) t_i j\| \leq M \|\pi(m_\alpha) t_i j\| \leq MN \|t_i j\| \leq M^2 N \|j\|,$$

for each $j \in J$, for each i, and for each α. So $(t_i \cdot m_\alpha \cdot t_i)_{(i,\alpha)} \subset J \widehat{\otimes} J$ is a multiplier bounded approximate diagonal for J with multiplier bound $M^2 N$, so J is boundedly pseudo-amenable. $\blacksquare$

We immediately obtain the following corollary.

Corollary 2.3.2. Let $\mathfrak{A}$ be a boundedly pseudo-amenable (resp. boundedly pseudo-contractible) Banach algebra and let J be a closed two-sided ideal of $\mathfrak{A}$. If J has a central bounded approximate identity then J is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

2.4 Relations to Other Generalized Notions of Amenability and Contractibility

We begin this section by investigating some relationships between bounded pseudo-amenability and bounded approximate contractibility. Many of the results below follow from the results in [3] alongside Theorem 2.2.1. We begin by showing that bounded approximate contractibility always implies bounded pseudo-amenability.

Proposition 2.4.1. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly approximately contractible then $\mathfrak{A}$ is boundedly pseudo-amenable.

Proof. Suppose that $\mathfrak{A}$ is boundedly approximately contractible. By Theorem 1.5.1, the bounded approximate contractibility of $\mathfrak{A}$ implies the bounded pseudo-amenability of $\mathfrak{A}^\sharp$. But also by Theorem 1.3.3, the bounded approximate contractibility of $\mathfrak{A}$ implies that $\mathfrak{A}$ has a bounded approximate identity. So by Theorem 2.2.1, since $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable and $\mathfrak{A}$ has a bounded approximate identity, $\mathfrak{A}$ is boundedly pseudo-amenable. ■

There certainly seems to be a very large connection between the notions of bounded approximate contractibility and bounded pseudo-amenability. It is unknown whether these two notions are equivalent even while assuming the existence of a bounded approximate identity, however, these notions are equivalent when a Banach algebra has a central bounded approximate identity.

Proposition 2.4.2. Let $\mathfrak{A}$ be a Banach algebra with a central bounded approximate identity. Then the following statements are equivalent:

(1) $\mathfrak{A}$ is boundedly pseudo-amenable.

(2) $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable.

(3) $\mathfrak{A}$ is boundedly approximately contractible.

(4) $\mathfrak{A}^\sharp$ is boundedly approximately contractible.

Proof. **(1)** $\Rightarrow$ **(2)** is [3, Proposition 3.2]. Note that this is the only part of the proof that requires a *central* bounded approximate identity.

(2) $\Rightarrow$ **(1)** is Theorem 2.2.1.

(2) $\Leftrightarrow$ **(3)** is Theorem 1.5.1.

(3) $\Rightarrow$ **(1)** is Proposition 2.4.1.

(3) $\Leftrightarrow$ **(4)** is [3, Proposition 2.4]. ■

We now turn our attention to notions of approximate biprojectivity and how they relate to notions of amenability and contractibility. Recall that for a Banach algebra $\mathfrak{A}$ with a central approximate identity, the notion of $\mathfrak{A}$ being pseudo-contractible is equivalent to $\mathfrak{A}$ being GZ - approximately biprojective and the notion of $\mathfrak{A}$ being pseudo-amenable is equivalent to $\mathfrak{A}$ being P - approximately biprojective. To obtain similar equivalences for bounded approximate contractibility and bounded approximate amenability we need to slightly modify these two definitions.

Definition 2.4.1. A Banach algebra $\mathfrak{A}$ is **multiplier boundedly GZ-approximately biprojective** if there exists a net $(T_\alpha)_\alpha \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ of continuous $\mathfrak{A}$-bimodule homomorphisms such that $(\pi \circ T_\alpha)_\alpha$ is bounded and with the property that:

$$\lim_{\alpha} \pi(T_\alpha(a)) = a,$$

for all $a \in \mathfrak{A}$.

We can now present a modified version of Theorem 1.4.4

Proposition 2.4.3. Let $\mathfrak{A}$ be a Banach algebra. Then $\mathfrak{A}$ is boundedly pseudo-contractible if and only if $\mathfrak{A}$ is multiplier boundedly GZ-approximately biprojective and has a multiplier bounded central approximate identity.

Proof. Suppose that $\mathfrak{A}$ is boundedly pseudo-contractible. Let $(u_\alpha)_\alpha \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded central approximate diagonal for $\mathfrak{A}$ with multiplier bound $K > 0$.

For each α, let $T_\alpha : \mathfrak{A} \to \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be defined for all $a \in \mathfrak{A}$ by:

$$T_\alpha(a) := a \cdot u_\alpha.$$

Then each T_α is a continuous $\mathfrak{A}$-bimodule homomorphism, and furthermore:

$$\|\pi(T_\alpha(a))\| = \|\pi(a \cdot u_\alpha)\| = \|a\pi(u_\alpha)\| \leq K\|a\|,$$

for all $a \in \mathfrak{A}$ and for all α. So $(\pi \circ T_\alpha)_\alpha$ is bounded. Also, since $(u_\alpha)_\alpha$ is an approximate diagonal for $\mathfrak{A}$ we have that $\lim\limits_\alpha \pi(T_\alpha(a)) = a$ for all $a \in \mathfrak{A}$. So $(T_\alpha)_\alpha \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ is a net of continuous $\mathfrak{A}$-bimodule homomorphisms for which $(\pi \circ T_\alpha)_\alpha$ is bounded, and so $\mathfrak{A}$ is multiplier boundedly GZ-approximately biprojective. Furthermore, $(\pi(u_\alpha))_\alpha$ is a multiplier bounded central approximate identity for $\mathfrak{A}$.

For the converse, let $(T_\alpha)_\alpha \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ be a net of continuous $\mathfrak{A}$-bimodule homomorphisms and let $M > 0$ be such that $\|\pi \circ T_\alpha\| \leq M$ for all α. Let $(e_i)_i \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded central approximate identity for $\mathfrak{A}$ and let $K > 0$ be such that $\|ae_i\| = \|e_i a\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all i. We claim that a subnet of $(T_\alpha(e_i))_{(\alpha,i)}$ is a central multiplier bounded approximate diagonal for $\mathfrak{A}$.

Observe that:

$$a \cdot T_\alpha(e_i) = T_\alpha(ae_i) = T_\alpha(e_i a) = T_\alpha(e_i) \cdot a,$$

for all $a \in \mathfrak{A}$, for all α, and for all i. Furthermore, since:

$$\lim_i \lim_\alpha \|\pi(T_\alpha(e_i))a - a\| = \lim_i \|e_i a - a\| = 0,$$

for all $a \in \mathfrak{A}$, there exists a subnet $(T_\alpha(e_i))_{(\alpha,i)}$ that is a central approximate diagonal for $\mathfrak{A}$. It is also necessarily multiplier bounded since:

$$\|\pi(T_\alpha(e_i))a\| = \|\pi(T_\alpha(e_i) \cdot a)\| = \|\pi(T_\alpha(e_i a))\| \leq M\|e_i a\| \leq MK\|a\|,$$

for all $a \in \mathfrak{A}$, for all α, and for all i. So $\mathfrak{A}$ is boundedly pseudo-contractible. ■

We now introduce two additional notions of bounded approximate biprojectivity.

Definition 2.4.2. A Banach algebra $\mathfrak{A}$ is **multiplier boundedly P-approximately biprojective** if there exists a net $(T_\alpha)_\alpha \subset \mathcal{B}(\mathfrak{A}, \mathfrak{A}\widehat{\otimes}\mathfrak{A})$ which is an approximate $\mathfrak{A}$-bimodule homomorphism such that $(\pi \circ T_\alpha)$ is bounded and with the property that:

$$\lim_\alpha \pi(T_\alpha(a)) = a,$$

for all $a \in \mathfrak{A}$.

Whenever $\mathfrak{A}$ is boundedly approximately contractible or boundedly pseudo-amenable we are guaranteed that $\mathfrak{A}$ is multiplier boundedly P-approximately biprojective.

Theorem 2.4.1. Let $\mathfrak{A}$ be a Banach algebra.
(1) If $\mathfrak{A}$ is boundedly approximately contractible then $\mathfrak{A}$ is multiplier boundedly P-approximately biprojective.
(2) If $\mathfrak{A}$ is boundedly pseudo-amenable then $\mathfrak{A}$ is multiplier boundedly P-approximately biprojective.

Proof. **(1)**: Suppose that $\mathfrak{A}$ is boundedly approximately contractible. By [3, Theorem 2.5] there exists nets $(M_\alpha)_\alpha \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$, $(F_\alpha)_\alpha, (G_\alpha)_\alpha \subset \mathfrak{A}$, and a constant $C > 0$ with the following properties:

(a) $\pi(M_\alpha) = F_\alpha + G_\alpha$ for all α, and in particular, $\pi(M_\alpha)a - F_\alpha a = G_\alpha a$ for all α and for all $a \in \mathfrak{A}$.
(b) $aF_\alpha \to a$ for all $a \in \mathfrak{A}$ and $\|aF_\alpha\| \leq C\|a\|$ for all $a \in \mathfrak{A}$ and for all α (that is, (F_α) is a right multiplier bounded right approximate identity for $\mathfrak{A}$).
(c) $G_\alpha a \to a$ for all $a \in \mathfrak{A}$ and $\|G_\alpha a\| \leq C\|a\|$ for all $a \in \mathfrak{A}$ and for all α (that is, (G_α) is a left multiplier bounded left approximate identity for $\mathfrak{A}$).
(d) $a \cdot M_\alpha - M_\alpha \cdot a - a \otimes G_\alpha + F_\alpha \otimes a \to 0$ for all $a \in \mathfrak{A}$ with $\|a \cdot M_\alpha - M_\alpha \cdot a - a \otimes G_\alpha + F_\alpha \otimes a\| \leq C\|a\|$ for all $a \in \mathfrak{A}$ and for all α.

For each α let $T_\alpha : \mathfrak{A} \to \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be be the continuous linear operator defined for all $a \in \mathfrak{A}$ by:

$$T_\alpha(a) := M_\alpha \cdot a - F_\alpha \otimes a.$$

Then, from properties (a) and (c) we have that:

$$\|\pi(T_\alpha(a))\| = \|\pi(M_\alpha \cdot a - F_\alpha \otimes a)\| = \|\pi(M_\alpha)a - F_\alpha a\| = \|G_\alpha a\| \leq C\|a\|,$$

so $(\pi \circ T_\alpha)_\alpha$ is bounded by C. From property (d) we have that:

$$\begin{aligned}\lim_\alpha [a \cdot T_\alpha(b) - T_\alpha(ab)] &= \lim_\alpha [(a \cdot M_\alpha \cdot b - aF_\alpha \otimes b) - (M_\alpha \cdot (ab) - F_\alpha \otimes (ab))] \\ &= -a \cdot \lim_\alpha [b \cdot M_\alpha - M_\alpha \cdot b - b \otimes G_\alpha + F_\alpha \otimes b] \\ &\quad + \lim_\alpha [(ab) \cdot M_\alpha - M_\alpha \cdot (ab) - (ab) \otimes G_\alpha + F_\alpha \otimes (ab)] \\ &= 0,\end{aligned}$$

for all $a, b \in \mathfrak{A}$, and also trivially, $T_\alpha(a) \cdot b - T_\alpha(ab) = 0$ for all $a, b \in \mathfrak{A}$, so $(T_\alpha)_\alpha$ is an approximate $\mathfrak{A}$-bimodule homomorphism. Lastly, from properties (a) and (c) we have that:

$$\lim_\alpha (\pi \circ T_\alpha)(a) = \lim_\alpha [\pi(M_\alpha)a - F_\alpha a] = \lim_\alpha G_\alpha a = a,$$

for all $a \in \mathfrak{A}$. Hence $\mathfrak{A}$ is multiplier boundedly P-approximately biprojective.

(2): Now suppose that $\mathfrak{A}$ is boundedly pseudo-amenable. Let $(m_\alpha)_\alpha \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$, and let $K > 0$ such that $\|\pi(m_\alpha)a\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all α.

For each α let $T_\alpha : \mathfrak{A} \to \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be the continuous linear operator defined for all $a \in \mathfrak{A}$ by:

$$T_\alpha(a) := a \cdot m_\alpha.$$

Then $\|(\pi \circ T_\alpha)(a)\| = \|\pi(a \cdot m_\alpha)\| = \|a \cdot \pi(m_\alpha)\| \leq K\|a\|$ for all $a \in \mathfrak{A}$ and for all α, so $(\pi \circ T_\alpha)_\alpha$ is bounded.

We trivially have $a \cdot T_\alpha(b) - T_\alpha(ab) = 0$ for all $a, b \in \mathfrak{A}$, and since $(m_\alpha)_\alpha$ is an approximate diagonal for $\mathfrak{A}$ we also have that:

$$\lim_\alpha [T_\alpha(a) \cdot b - T_\alpha(ab)] = \lim_\alpha [(a \cdot m_\alpha) \cdot b - (ab) \cdot m_\alpha] = \lim_\alpha a \cdot [m_\alpha \cdot b - b \cdot m_\alpha] = 0,$$

for all $a, b \in \mathfrak{A}$. So $(T_\alpha)_\alpha$ is an approximate $\mathfrak{A}$-bimodule homomorphism. Lastly, we have that:

$$\lim_{\alpha} \pi \circ T_{\alpha}(a) = \lim_{\alpha} \pi(a \cdot m_{\alpha}) = \lim_{\alpha} a \cdot \pi(m_{\alpha}) = a,$$

for all $a \in \mathfrak{A}$. Hence $\mathfrak{A}$ is multiplier boundedly P-approximately biprojective. ■

We now turn our attention to continuous derivations defined on boundedly pseudo-amenable Banach algebras. By modifying [12, Proposition 3.5] we obtain the following result.

Proposition 2.4.4. Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. If $\mathfrak{A}$ is boundedly pseudo-amenable and such that every multiplier bounded approximate identity in $\mathfrak{A}$ for $\mathfrak{A}$ is also a left or right multiplier bounded approximate identity in $\mathfrak{A}$ for X then:

(1) Every continuous derivation $D : \mathfrak{A} \to X$ is boundedly approximately inner.

(2) Every continuous derivation $D : \mathfrak{A} \to X^*$ is boundedly weak* approximately inner.

Proof. Let $\mathfrak{A}$ be boundedly pseudo-amenable and let $(u_{\alpha})_{\alpha} \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $K > 0$. Then $(\pi(u_{\alpha}))_{\alpha} \subset \mathfrak{A}$ is a multiplier bounded approximate identity for $\mathfrak{A}$ with multiplier bound K. Let X be a Banach $\mathfrak{A}$-bimodule.

(1): Assume that $(\pi(u_{\alpha}))_{\alpha}$ is a multiplier bounded right approximate identity in $\mathfrak{A}$ for X. The proof for when $(\pi(u_{\alpha}))_{\alpha}$ is a multiplier bounded left approximate identity in $\mathfrak{A}$ for X is similar. Then, there exists $K^* > 0$ be such that $\|x \cdot \pi(u_{\alpha})\| \leq K^* \|x\|$ for all $x \in X$ and for all α.

Let $D : \mathfrak{A} \to X$ be a continuous derivation and let $T : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to X$ be the continuous operator specified for each elementary tensor $a \otimes b \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ by:

$$T(a \otimes b) := -D(a) \cdot b.$$

It is easy to see that $\|T\| \leq \|D\|$, and:

$$T(a \cdot (b \otimes c)) = T((ab) \otimes c) = -D(ab) \cdot c = -[a \cdot D(b) + D(a) \cdot b] \cdot c = a \cdot T(b \otimes c) - D(a) \cdot (bc),$$

and $T((b \otimes c) \cdot a) = T(b \otimes c) \cdot a$ for all $a, b, c \in \mathfrak{A}$. Therefore:

$$a \cdot T(b \otimes c) - T(b \otimes c) \cdot a = T(a \cdot (b \otimes c) - (b \otimes c) \cdot a) + D(a) \cdot (bc),$$

for all $a, b, c \in \mathfrak{A}$, and consequently:

$$a \cdot T(u) - T(u) \cdot a = T(a \cdot u - u \cdot a) + D(a) \cdot \pi(u),$$

for all $a \in \mathfrak{A}$ and for all $u \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$.

For each α let $\xi_\alpha := T(u_\alpha)$. Then:

$$\lim_\alpha [a \cdot \xi_\alpha - \xi_\alpha \cdot a - D(a) \cdot \pi(u_\alpha)] = 0, \tag{$\star$}$$

for each $a \in \mathfrak{A}$. Since $D(a) \in X$ and since $(\pi(u_\alpha))_\alpha$ is a right approximate identity in $\mathfrak{A}$ for X we have that $\lim\limits_\alpha [D(a) \cdot \pi(u_\alpha)] = D(a)$. Therefore:

$$D(a) = \lim_\alpha [a \cdot \xi_\alpha - \xi_\alpha \cdot a],$$

for all $a \in \mathfrak{A}$ and so D is approximately inner. Moreover:

$$\begin{aligned} \|a \cdot \xi_\alpha - \xi_\alpha \cdot a\| = \|a \cdot T(u_\alpha) - T(u_\alpha) \cdot a\| &= \|T(a \cdot u_\alpha - u_\alpha \cdot a) + D(a)\pi(u_\alpha)\| \\ &\leq \|T\|\|a \cdot u_\alpha - u_\alpha \cdot a\| + \|D(a)\pi(u_\alpha)\| \\ &\leq [K + K^*]\|D\|\|a\|, \end{aligned}$$

for all $a \in \mathfrak{A}$ and for all α. So D is boundedly approximately inner.

(2): Assume that $(\pi(u_\alpha))_\alpha$ is a multiplier bounded left approximate identity in $\mathfrak{A}$ for X (again, the proof for when $(\pi(u_\alpha))_\alpha$ is a multiplier bounded right approximate identity in $\mathfrak{A}$ for X is similar). Then $(\pi(u_\alpha))_\alpha$ is a multiplier bounded right weak* approximate identity in $\mathfrak{A}$ for X^*. By replacing X with X^* in the previous argument, and noting that the convergence in $(\star)$ holds in the weak* topology on X^*, the conclusion follows immediately. ■

2.5 Tensor Products of Boundedly Pseudo-Amenable and Boundedly Pseudo-Contractible Banach Algebras

It is currently unknown whether the projective tensor product of two pseudo-amenable Banach algebras is still pseudo-amenable. The first result below shows that the projective tensor product of a boundedly pseudo-amenable and a boundedly pseudo-contractible Banach algebra is at least pseudo-amenable.

Theorem 2.5.1. If $\mathfrak{A}$ is boundedly pseudo-amenable and $\mathfrak{B}$ is boundedly pseudo-contractible then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is pseudo-amenable.

Proof. It is sufficient to show that for all $\epsilon > 0$ and for all finite $\mathcal{F} \subset \mathfrak{A}\widehat{\otimes}\mathfrak{B}$, there exists a $u := u_{\epsilon,\mathcal{F}} \in (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B})$ such that:

$$\|f \cdot u - u \cdot f\| < \epsilon \quad \text{and} \quad \|\pi(u)f - f\| < \epsilon, \tag{$\star$}$$

for all $f \in \mathcal{F}$. Then $(u_{\epsilon,\mathcal{F}})_{(\epsilon,\mathcal{F})} \subset (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B})$ will be an approximate diagonal for $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$.

Let $\epsilon > 0$ and let $\mathcal{F} \subset \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ be a finite set. Let $(U_i)_i \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $C > 0$ and let $(V_j)_j \subset \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ be a multiplier bounded central approximate diagonal for $\mathfrak{B}$ with multiplier bound $D > 0$.

Let $T : (\mathfrak{A}\widehat{\otimes}\mathfrak{A})\widehat{\otimes}(\mathfrak{B}\widehat{\otimes}\mathfrak{B}) \to (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B})$ be the continuous linear isometry specified for all elementary tensors $a_1 \otimes a_2 \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ and $b_1 \otimes b_2 \in \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ by:

$$T((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) := (a_1 \otimes b_1) \otimes (a_2 \otimes b_2).$$

From the continuity of T, choose $\delta > 0$ be such that if $z \in (\mathfrak{A}\widehat{\otimes}\mathfrak{A})\widehat{\otimes}(\mathfrak{B}\widehat{\otimes}\mathfrak{B})$ and $\|z\| < \delta$ then $\|T(z)\| < \epsilon$.

We will choose $U \in (U_i)_i$ and $V \in (V_j)_j$ appropriately, and set $u := T(U \otimes V)$, so that the inequalities at $(\star)$ are satisfied for all $f \in \mathcal{F}$.

First, for each $f \in \mathcal{F}$, write:

$$f := \sum_{n=1}^{\infty} a_n^{(f)} \otimes b_n^{(f)},$$

where $a_n^{(f)} \in \mathfrak{A}$ and $b_n^{(f)} \in \mathfrak{B}$ with $\|b_n^{(f)}\| = 1$ for each $n \in \mathbb{N}$.

Let $K > 0$ be such that $\sum_{n=1}^{\infty} \|a_n^{(f)}\| = \sum_{n=1}^{\infty} \|a_n^{(f)}\|\|b_n^{(f)}\| < K$ for all $f \in \mathcal{F}$. Then choose $N \in \mathbb{N}$ such that:

$$\sum_{n=N+1}^{\infty} \|a_n^{(f)}\| < \min\left\{\frac{\epsilon}{4D}, \frac{\epsilon}{4CD}\right\}.$$

Since $(V_j)_j$ is a central approximate diagonal for $\mathfrak{B}$, there exists a $V \in (V_j)_j$ such that $b \cdot V = V \cdot b$ for all $b \in \mathfrak{B}$ and:

$$\sum_{n=1}^{N} \|\pi(V)b_n^{(f)} - b_n^{(f)}\| < \frac{\epsilon}{4K},$$

for all $f \in \mathcal{F}$. Then choose $M \in \mathbb{N}$ such that:

$$\sum_{n=M+1}^{\infty} C\|a_n^{(f)}\|\|V\| < \frac{\delta}{2},$$

for all $f \in \mathcal{F}$.

Since $(U_i)_i$ is an approximate diagonal for $\mathfrak{A}$, there exists a $U \in (U_i)_i$ such that:

$$\sum_{n=1}^{M} \|a_n^{(f)} \cdot U - U \cdot a_n^{(f)}\|\|V\| < \frac{\delta}{2} \quad \text{and} \quad \sum_{n=1}^{N} \|\pi(U)a_n^{(f)} - a_n^{(f)}\| < \frac{\epsilon}{4D},$$

for all $f \in \mathcal{F}$.

Write $U = \sum_{s=1}^{\infty} A_s \otimes A_s'$ and write $V = \sum_{t=1}^{\infty} B_t \otimes B_t'$ where $A_s, A_s' \in \mathfrak{A}$ and $B_t, B_t' \in \mathfrak{B}$ for all $s, t \in \mathbb{N}$. Let:

$$u := T(U \otimes V) = \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}(A_s \otimes B_t) \otimes (A_s' \otimes B_t') \in (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B}).$$

Observe that:

$$(a_n^{(f)} \otimes b_n^{(f)}) \cdot u = \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}(a_n^{(f)}A_s \otimes b_n^{(f)}B_t) \otimes (A_s' \otimes B_t') = T(a_n^{(f)} \cdot U \otimes b_n^{(f)} \cdot V) = T(a_n^{(f)} \cdot U \otimes V \cdot b_n^{(f)}),$$

and:

$$u \cdot (a_n^{(f)} \otimes b_n^{(f)}) = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} (A_s \otimes B_t) \otimes (A'_s a_n^{(f)} \otimes B'_t b_n^{(f)}) = T(U \cdot a_n^{(f)} \otimes V \cdot b_n^{(f)}),$$

for all $f \in \mathcal{F}$ and for all $n \in \mathbb{N}$. Therefore:

$$\begin{aligned} f \cdot u - u \cdot f = \left[\sum_{n=1}^{\infty} a_n^{(f)} \otimes b_n^{(f)}\right] \cdot u - u \cdot \left[\sum_{n=1}^{\infty} a_n^{(f)} \otimes b_n^{(f)}\right] &= \sum_{n=1}^{\infty} [(a_n^{(f)} \otimes b_n^{(f)}) \cdot u - u \cdot (a_n^{(f)} \otimes b_n^{(f)})] \\ &= \sum_{n=1}^{\infty} T((a_n^{(f)} \cdot U - U \cdot a_n^{(f)}) \otimes V \cdot b_n^{(f)}), \end{aligned}$$

for all $f \in \mathcal{F}$.

We are now ready to show that $\|f \cdot u - u \cdot f\| < \epsilon$ and $\|\pi(u)f - f\| < \epsilon$ for all $f \in \mathcal{F}$. First, we have that:

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} (a_n^{(f)} \cdot U - U \cdot a_n^{(f)}) \otimes V \cdot b_n^{(f)} \right\| &\leq \sum_{n=1}^{M} \|a_n^{(f)} \cdot U - U \cdot a_n^{(f)}\| \|V\| \|b_n^{(f)}\| \\ &\quad + \sum_{n=M+1}^{\infty} \|a_n^{(f)} \cdot U - U \cdot a_n^{(f)}\| \|V\| \|b_n^{(f)}\| \\ &\leq \sum_{n=1}^{M} \|a_n^{(f)} \cdot U - U \cdot a_n^{(f)}\| \|V\| + \sum_{n=M+1}^{\infty} C \|a_n^{(f)}\| \|V\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &< \delta, \end{aligned}$$

for all $f \in \mathcal{F}$, and so by the continuity of T we have that $\|f \cdot u - u \cdot f\| < \epsilon$ for all $f \in \mathcal{F}$.

On the other hand:

$$
\begin{aligned}
\|\pi(u)f - f\| &= \left\| [\pi(U)\otimes\pi(V)]\sum_{n=1}^{\infty} a_n^{(f)}\otimes b_n^{(f)} - \sum_{n=1}^{\infty} a_n^{(f)}\otimes b_n^{(f)} \right\| \\
&= \left\| \sum_{n=1}^{\infty}[\pi(U)a_n^{(f)}]\otimes[\pi(V)b_n^{(f)}] - \sum_{n=1}^{\infty} a_n^{(f)}\otimes b_n^{(f)} \right\| \\
&\leq \left\| \sum_{n=1}^{\infty}[\pi(U)a_n^{(f)} - a_n^{(f)}]\otimes[\pi(V)b_n^{(f)}] \right\| + \left\| \sum_{n=1}^{\infty} a_n^{(f)}\otimes[\pi(V)b_n^{(f)} - b_n^{(f)}] \right\| \\
&\leq \sum_{n=1}^{\infty} \|\pi(U)a_n^{(f)} - a_n^{(f)}\|\|\pi(V)\| + \sum_{n=1}^{\infty} \|a_n^{(f)}\|\|\pi(V)b_n^{(f)} - b_n^{(f)}\| \\
&\leq \sum_{n=1}^{N} \|\pi(U)a_n^{(f)} - a_n^{(f)}\|\|\pi(V)\| + \sum_{n=N+1}^{\infty} \|\pi(U)a_n^{(f)} - a_n^{(f)}\|\|\pi(V)\| \\
&\quad + \sum_{n=1}^{N} \|a_n^{(f)}\|\|\pi(V)b_n^{(f)} - b_n^{(f)}\| + \sum_{n=N+1}^{\infty} \|a_n^{(f)}\|\|\pi(V)b_n^{(f)} - b_n^{(f)}\| \\
&\leq D\sum_{n=1}^{N} \|\pi(U)a_n^{(f)} - a_n^{(f)}\| + CD\sum_{n=N+1}^{\infty} \|a_n^{(f)}\| \\
&\quad + \sum_{n=1}^{N} \|a_n^{(f)}\|\|\pi(V)b_n^{(f)} - b_n^{(f)}\| + D\sum_{n=N+1}^{\infty} \|a_n^{(f)}\| \\
&< D\frac{\epsilon}{4D} + CD\frac{\epsilon}{4CD} + K\cdot\frac{\epsilon}{4K} + D\frac{\epsilon}{4D} = \epsilon,
\end{aligned}
$$

for all $f \in \mathcal{F}$.

Hence $(u_{\epsilon,\mathcal{F}})_{(\epsilon,\mathcal{F})} \subset (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B})$ is an approximate diagonal for $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$, and so $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is pseudo-amenable. ■

By strengthening the assumption so that $\mathfrak{B}$ is contracitble in the previous theorem, we can at least guarantee the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is *boundedly* pseudo-amenable.

Theorem 2.5.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}$ is boundedly pseudo-amenable and $\mathfrak{B}$ is contractible, then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable.

Proof. As per the previous proof, let $(U_i)_i \subset \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ be a multiplier bounded approximate diagonal for $\mathfrak{A}$ with multiplier bound $C > 0$. Since $\mathfrak{B}$ is contractible, it has a diagonal. Let $V \in \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ be such that:

$$b \cdot V = V \cdot b \quad \text{and} \quad \pi(V)b = b = b\pi(V),$$

for all $b \in \mathfrak{B}$. Take any directed set J and set $V_j := V$ for all $j \in J$ so that $(V)_j := (V_j)_j$ trivially becomes a multiplier bounded central approximate diagonal for $\mathfrak{B}$.

Construct the approximate diagonal $(u_{\epsilon,\mathcal{F}}) \subset (\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}(\mathfrak{A}\widehat{\otimes}\mathfrak{B})$ as in the previous proof from $(U_i)_i$ and $(V)_j$. All that remains to show is that this approximate diagonal is multiplier bounded.

Observe that:

$$\|\pi(U_i)a\| \leq \|\pi(U_i)a - a\| + \|a\| \leq (C+1)\|a\|,$$

for each i and for each $a \in \mathfrak{A}$.

For each $f \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ write $f = \sum_{n=1}^{\infty} a_n \otimes b_n$. Then:

$$\begin{aligned}
\|f \cdot u_{\epsilon,\mathcal{F}} - u_{\epsilon,\mathcal{F}} \cdot f\| &= \left\| \sum_{n=1}^{\infty} T((a_n \cdot U_{\epsilon,\mathcal{F}} - U_{\epsilon,\mathcal{F}} \cdot a_n) \otimes (V \cdot b_n)) \right\| \\
&\leq \sum_{n=1}^{\infty} \|T((a_n \cdot U_{\epsilon,\mathcal{F}} - U_{\epsilon,\mathcal{F}} \cdot a_n) \otimes (V \cdot b_n))\| \\
&\leq \sum_{n=1}^{\infty} \|a_n \cdot U_{\epsilon,\mathcal{F}} - U_{\epsilon,\mathcal{F}} \cdot a_n\|\|V \cdot b_n\| \leq C\|V\| \sum_{n=1}^{\infty} \|a_n\|\|b_n\|,
\end{aligned}$$

for all $f \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$, and so, $\|f \cdot u_{\epsilon,\mathcal{F}} - u_{\epsilon,\mathcal{F}} \cdot f\| \leq C\|V\|\|f\|$ for all $f \in A\widehat{\otimes}\mathfrak{B}$. We also have that:

$$\begin{aligned}
\|\pi(u_{\epsilon,\mathcal{F}})f\| = \left\|\pi(u_{\epsilon,\mathcal{F}}) \sum_{n=1}^{\infty} a_n \otimes b_n\right\| = \left\|\pi(U_{\epsilon,\mathcal{F}}) \otimes \pi(V) \sum_{n=1}^{\infty} a_n \otimes b_n\right\| &= \left\|\sum_{n=1}^{\infty} \pi(U_{\epsilon,\mathcal{F}})a_n \otimes \pi(V)b_n\right\| \\
&\leq \sum_{n=1}^{\infty} \|\pi(U_{\epsilon,\mathcal{F}})a_n\|\|b_n\| \\
&\leq (C+1) \sum_{n=1}^{\infty} \|a_n\|\|b_n\|,
\end{aligned}$$

for all $f \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$, and so, $\|\pi(u_{\epsilon,\mathcal{F}})f\| \leq (C+1)\|f\|$ for all $f \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$.

Hence $(u_{\epsilon,\mathcal{F}})_{(\epsilon,\mathcal{F})}$ is a multiplier bounded approximate diagonal for $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ with multiplier bound $\max\{C\|V\|, C+1\}$, and so $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable. ■

As a consequence of the previous theorems and well known results, we have the following result regarding the pseudo-amenability of tensor products of various types of Banach algebras.

Corollary 2.5.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. Then:

(1) If $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable and $\mathfrak{B}$ is boundedly pseudo-contractible then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is pseudo-amenable.

(2) If $\mathfrak{A}$ is boundedly approximately contractible and $\mathfrak{B}$ is boundedly pseudo-contractible then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is pseudo-amenable.

(3) If $\mathfrak{A}^\sharp$ is boundedly pseudo-amenable and $\mathfrak{B}$ is contractible then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable.

(4) If $\mathfrak{A}$ is boundedly approximately contractible and $\mathfrak{B}$ is contractible then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable.

We will now look at how bounded pseudo-amenability and bounded pseudo-contractibility of $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ affect $\mathfrak{A}$ and $\mathfrak{B}$. We begin by assuming that one of $\mathfrak{A}$ or $\mathfrak{B}$ has an identity, while the other has at least a central bounded approximate identity.

Proposition 2.5.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable and one of $\mathfrak{A}$ or $\mathfrak{B}$ has an identity, while the other has a central bounded approximate identity, then both $\mathfrak{A}$ and $\mathfrak{B}$ are boundedly pseudo-amenable.

Proof. Without loss of generality, suppose that $\mathfrak{A}$ has an identity $e \in \mathfrak{A}$ and $\mathfrak{B}$ has a central bounded approximate identity $(f_\beta)_\beta \subset \mathfrak{B}$. Then $(e \otimes f_\beta)_\beta \subset \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is a central bounded approximate identity for $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$.

Since $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable and has a central bounded approximate identity, we have by Theorem 1.5.1 that $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly approximately contractible. By [9, Theorem 4.4], we have that both $\mathfrak{A}$ and $\mathfrak{B}$ are boundedly approximately contractible. But since $\mathfrak{A}$ and $\mathfrak{B}$ both have central bounded approximate identities, by Theorem 2.2.1, the bounded approximate contractibility of $\mathfrak{A}$ and $\mathfrak{B}$ implies the boundedly pseudo-amenability of $\mathfrak{A}$ and $\mathfrak{B}$. ■

As a consequence of the previous proposition, we get the following:

Corollary 2.5.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable and both $\mathfrak{A}$ and $\mathfrak{B}$ have an identity, then $\mathfrak{A}$ and $\mathfrak{B}$ are both boundedly pseudo-amenable.

Definition 2.5.1. Let $\mathfrak{A}$ be a Banach algebra. A **character** or **nonzero multiplicative linear functional** on $\mathfrak{A}$ is a linear functional $\chi : \mathfrak{A} \to \mathbb{C}$ that is not identically the zero functional on $\mathfrak{A}$ and such that $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathfrak{A}$.

If $\mathfrak{A}$ is any Banach algebra then the unitization $\mathfrak{A}^\sharp$ always has a character. In fact, $\chi : \mathfrak{A}^\sharp \to \mathbb{C}$ defined for all $a + \lambda \in \mathfrak{A}^\sharp$ by:

$$\chi(A) = \chi(a + \lambda) := \lambda,$$

is a multiplicative linear functional on $\mathfrak{A}^\sharp$.

Proposition 2.5.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{B}^\sharp$ is boundedly pseudo-contractible then $\mathfrak{A}$ and $\mathfrak{B}$ are both boundedly pseudo-contractible.

Proof. Suppose that $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{B}^\sharp$ is boundedly pseudo-contractible. Let $\chi : \mathfrak{A}^\sharp \to \mathbb{C}$ be a character on $\mathfrak{A}^\sharp$. Let $\Phi : \mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{B}^\sharp \to \mathfrak{B}^\sharp$ be specified for all elementary tensors $a \otimes b \in \mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{B}^\sharp$ by:

$$\Phi(a \otimes b) := \chi(a)b.$$

Then it is readily checked that Φ is a continuous algebra epimorphism.

Since $\mathfrak{A}^\sharp\widehat{\otimes}\mathfrak{B}^\sharp$ is boundedly pseudo-amenable, we have by the Theorem 2.3.1 that $\mathfrak{B}^\sharp$ is boundedly pseudo-contractible. A similar argument shows that $\mathfrak{A}^\sharp$ is also boundedly pseudo-contractible.

But by Theorem 2.2.1, bounded pseudo-contractibility of $\mathfrak{A}^\sharp$ and $\mathfrak{B}^\sharp$ respectively imply the bounded pseudo-contractibility of $\mathfrak{A}$ and $\mathfrak{B}$. ■

We now remove the assumption of having an identity or an approximate identity on $\mathfrak{A}$ or $\mathfrak{B}$ and just assume that at least one of $\mathfrak{A}$ or $\mathfrak{B}$ is a commutative Banach algebra. We will use the well known fact that every unital commutative Banach algebra has a character.

Proposition 2.5.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) and $\mathfrak{A}$ (resp. $\mathfrak{B}$) is unital and commutative, then $\mathfrak{B}$ (resp. $\mathfrak{A}$) is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. If $\mathfrak{A}$ is unital and commutative, let $\chi : \mathfrak{A} \to \mathbb{C}$ be a character on $\mathfrak{A}$ and let $\Phi : \mathfrak{A}\widehat{\otimes}\mathfrak{B} \to \mathfrak{B}$ be specified for all elementary tensors $a \otimes b \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ by:

$$\Phi(a \otimes b) := \chi(a)b.$$

Then Φ is a continuous algebra epimorphism of $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ onto $\mathfrak{B}$ and since $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-amenable, by Theorem 2.3.1, so is $\mathfrak{B}$. ■

In [22, Theorem 3.7] it was proven that if $\mathfrak{A}$ and $\mathfrak{B}$ are both boundedly pseudo-contractible then so it $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$. As a consequence of the previous proposition, we have that the converse is true when $\mathfrak{A}$ and $\mathfrak{B}$ are also commutative.

Corollary 2.5.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $\mathfrak{A}$ and $\mathfrak{B}$ are unital and commutative, then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is boundedly pseudo-contractible if and only if both $\mathfrak{A}$ and $\mathfrak{B}$ are boundedly pseudo-contractible

We now turn our attention to how the bounded pseudo-amenability or bounded pseudo-contractibility of $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ affect that of $\mathfrak{A}$. We begin with the following result.

Proposition 2.5.4. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible), $\mathfrak{A}$ is commutative, and $\mathfrak{A}$ has a bounded approximate identity then $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. Trivially regard $\mathfrak{A}$ itself as a Banach $\mathfrak{A}$-bimodule with the action of $\mathfrak{A}$ on $\mathfrak{A}$ being just the multiplication defined on $\mathfrak{A}$. Since $\mathfrak{A}$ has a bounded approximate identity, by Theorem 1.0.1, for all $a \in \mathfrak{A}$ there exists $b, c \in \mathfrak{A}$ such that $bc = a$. In particular, $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is a continuous $\mathfrak{A}$-bimodule epimorphism. Since $\mathfrak{A}$ is commutative, π is also a continuous algebra epimorphism. So by Theorem 2.3.1, $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible). ■

When $\mathfrak{A}$ has a bounded approximate identity, Cohen's Factorization Theorem guarantees that for $a \in \mathfrak{A}$ there exists $b, c, d \in \mathfrak{A}$ such that $a = bcd = b(cd)$, i.e., every $a \in \mathfrak{A}$ can be written as a product of two elements in $\mathfrak{A}$, and such algebras are called *factorable.* When $\mathfrak{A}$ is commutative, $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is guaranteed to be an algebra morphism, and when $\mathfrak{A}$ has a bounded approximate identity, π is guaranteed to be surjective.

The notions of a Banach algebra being factorable is a strictly weaker notion that the existence of a bounded approximate identity. The simplest example of a factorable Banach algebra without a bounded approximate identity is the sequence Banach algebra ℓ^1 over $\mathbb{R}$, which has only a central (unbounded) approximate identity. To see its factorability, let $(x_n) \subset \ell^1$ be a sequence of real numbers. Take any conjugate pair p and q with $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For each $n \in \mathbb{N}$ let:

$$y_n := \begin{cases} x_n^{1/p} & \text{if } x_n \geq 0 \\ -(-x_n)^{1/p} & \text{if } x_n < 0 \end{cases} \quad \text{and} \quad z_n := \begin{cases} x_n^{1/q} & \text{if } x_n \geq 0 \\ (-x_n)^{1/q} & \text{if } x_n < 0 \end{cases}.$$

Then $(y_n) \in \ell^p \subset \ell^1$ and $(z_n) \in \ell^q \subset \ell^1$ with $\|(y_n)\|_p = \|(x_n)\|_1^{1/p}$ and $\|(z_n)\|_q = \|(x_n)\|_1^{1/q}$, and also $(x_n) = (y_n)(z_n)$. So ℓ^1 over $\mathbb{R}$ is factorable, but again, does not have a *bounded* approximate identity.

In [21] a few other notions, all which guarantee the map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is surjective, are introduced, and we use these notions to refine Theorem 2.5.4

Definition 2.5.2. Let $\mathfrak{A}$ be a Banach algebra.

(1) $\mathfrak{A}$ is said to **Factor** if $\mathfrak{A} = \mathfrak{A}^{[2]}$, that is, for every $a \in \mathfrak{A}$ there exists $a_1, b_1 \in \mathfrak{A}$ such that $a = a_1 b_1$.

(2) $\mathfrak{A}$ is said to **Factor Weakly** if $\mathfrak{A} = \mathfrak{A}^2$, that is, for every $a \in \mathfrak{A}$ there exists finitely many $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m \in \mathfrak{A}$ such that $a = \sum_{n=1}^{m} a_n b_n$.

(3) $\mathfrak{A}$ is said to **Factor Projectively** if π is surjective, that is, for every $a \in \mathfrak{A}$ there exists sequences $(a_n), (b_n) \subset \mathfrak{A}$ such that $a = \sum_{n=1}^{\infty} a_n b_n$.

The notion of a Banach algebra being weakly factorable is a strictly weaker notion than being factorable. There exists a separable Banach function algebra $\mathfrak{A}$ that is not factorable (so that there exists an element of $\mathfrak{A}$ that cannot be factored), but such that every $a \in \mathfrak{A}$ can be written in the form $a = a_1 b_1 + a_2 b_2$ with $a_1, a_2, b_1, b_2 \in \mathfrak{A}$. An example can be found in [21, Example 7].

Definition 2.5.3. Let $\mathfrak{A}$ be a Banach algebra.

We say that **Null Sequences in $\mathfrak{A}$ (Left) Factor** if for every sequence $(a_n) \subset \mathfrak{A}$ with $\lim_{n\to\infty} \|a_n\| = 0$ there exists a $u \in \mathfrak{A}$ and a sequence $(b_n) \subset \mathfrak{A}$ with $\lim_{n\to\infty} \|b_n\| = 0$ such that $a_n = b_n u$ for all $n \in \mathbb{N}$.
We say that **Null Sequences in $\mathfrak{A}$ Weakly (Left) Factor** if for every sequence $(a_n) \subset \mathfrak{A}$ with $\lim_{n\to\infty} \|a_n\| = 0$ there exist a $J \in \mathbb{N}$, elements $u_1, u_2, ..., u_J \in \mathfrak{A}$, and sequences $(b_n^{(1)}), (b_n^{(2)}), ..., (b_n^{(J)}) \subset \mathfrak{A}$ with $\lim_{n\to\infty} \|b_n^{(j)}\| = 0$ for each $j \in \{1, 2, ..., J\}$ and such that $a_n = \sum_{j=1}^{J} b_n^{(j)} u^{(j)}$ for all $n \in \mathbb{N}$.
Analogous definitions can be defined for the notion of "Null Sequences in $\mathfrak{A}$ (Right) Factor" and "Null Sequences in $\mathfrak{A}$ Weakly (Right) Factor".

If null sequences in $\mathfrak{A}$ factor, then $\mathfrak{A}$ trivially factors, since every $a \in \mathfrak{A}$ can be associated with the null sequence $(a, 0, 0, ...)$. Similarly, if null sequences in $\mathfrak{A}$ weakly factor, then $\mathfrak{A}$ trivially weakly factors.

The existence of a bounded left or bounded right approximate identity in $\mathfrak{A}$ implies each of the five definitions above. Also, each of the five definitions above imply that $\mathfrak{A}$ projectively factors as proven in [21]. But note that a Banach algebra being projectively factorable is equivalent to the map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ being surjective, and so we obtain the following result.

Proposition 2.5.5. Let $\mathfrak{A}$ be a commutative Banach algebra. If $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) then $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) if $\mathfrak{A}$ has any of the following properties:

(1) $\mathfrak{A}$ has a bounded left approximate identity or a bounded right approximate identity.

(2) Null sequences in $\mathfrak{A}$ left factor or right factor.

(3) Null sequences in $\mathfrak{A}$ weakly left factor or weakly right factor.

(4) $\mathfrak{A}$ is factorable.

(5) $\mathfrak{A}$ is weakly factorable.

(6) $\mathfrak{A}$ is projectively factorable.

Proof. Each of the six properties listed above implies $\mathfrak{A}$ is projectively factorable or equivalently that $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is surjective, in which case π is a continuous algebra epimorphism from the boundedly pseudo-amenable (resp. boundedly pseudo-contractible) $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ onto $\mathfrak{A}$. Then apply Proposition 2.3.1. ■

3 Examples

In this chapter we provide examples of boundedly pseudo-amenable and boundedly pseudo-contractible Banach algebras to show that these notions are distinct.

Consider the semigroup $\mathbb{N}$ equipped with the associative binary operation $\max : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We can then consider the discrete semigroup algebra $\ell^1(\mathbb{N}_{\max})$ (see Section 1.1 for the definition).

Proposition 3.0.1. $\ell^1(\mathbb{N}_{\max})$ is a unital boundedly pseudo-amenable Banach algebra that is not boundedly pseudo-contractible.

Proof. Let $\mathbb{N}$ be the semigroup with product $mn := \max\{m, n\}$ as defined earlier. Then $\ell^1(\mathbb{N}_{\max})$ is a unital sequentially approximately contractible Banach algebra as shown in [10, Example 4.6] and is thus boundedly approximately contractible by the Uniform Boundedness principle. So $(\ell^1(\mathbb{N}_{\max}))^\sharp$ is boundedly pseudo-amenable by Theorem 1.5.1. Since $\ell^1(\mathbb{N}_{\max})$ is unital, it trivially has a central bounded approximate identity. So $\ell^1(\mathbb{N}_{\max})$ is boundedly pseudo-amenable by Theorem 2.2.1.

If $\ell^1(\mathbb{N}_{\max})$ were also boundedly pseudo-contractible, then by Proposition 2.2.1, since $\ell^1(\mathbb{N}_{\max})$ is unital we would have that $(\ell^1(\mathbb{N}_{\max}))^\sharp$ is boundedly pseudo-contractible too, and so trivially, $(\ell^1(\mathbb{N}_{\max}))^\sharp$ is pseudo-contractible. By Theorem 1.4.3, this implies that $\ell^1(\mathbb{N}_{\max})$ is contractible. But this is a contradiction since $\ell^1(\mathbb{N}_{\max})$ is not even amenable as noted in [3, Appendix A.1].

So $\ell^1(\mathbb{N}_{\max})$ is boundedly pseudo-amenable but not boundedly pseudo-contractible. ∎

It is still an open question as to whether or not $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ is boundedly pseudo-amenable or not. We observe the following.

Proposition 3.0.2. $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ is not boundedly pseudo-contractible.

Proof. The algebra $\ell^1(\mathbb{N}_{\max})$ has an identity, namely the function $e : \mathbb{N} \to \mathbb{C}$ defined by:

$$e(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

Thus it trivially has a bounded approximate identity. If $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ were boundedly pseudo-contractible, then by Proposition 2.5.4 we would have that $\ell^1(\mathbb{N}_{\max})$ is necessarily bounded pseudo-contractible, which is a contradiction from Proposition 3.0.1. Thus $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ is not boundedly pseudo-contractible. ■

It should be remarked that the result above can also be obtained with about the same amount of effort, by already known results about pseudo-contractibility and amenability. Note that since $\ell^1(\mathbb{N}_{\max})$ has the identity e, the tensor product $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ has the identity $e \otimes e$. Thus, if $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ were boundedly pseudo-contractible (or event just pseudo-contractible) then Theorem 1.4.3 implies that $\ell^1(\mathbb{N}_{\max})\widehat{\otimes}\ell^1(\mathbb{N}_{\max})$ is contractible and thus amenable. By [9, Theorem 4.9], this implies that $\ell^1(\mathbb{N}_{\max})$ is amenable, which is a contradiction as noted in [3, Appendix A.1].

Now consider the semigroup $\mathbb{N}_{\min}$ whose underlying set is just $\mathbb{N}$ and whose associative binary operation is the minimum function, min. Observe that any function $\omega : \mathbb{N}_{\min} \to [1, \infty)$ is a weight function as clearly $\omega(\min\{m, n\}) = \omega(m) \leq \omega(m)\omega(n)$ for all $m, n \in \mathbb{N}$, $m \leq n$. So $\ell^1(\mathbb{N}_{\min}, \omega)$ is a weighted semigroup algebra for each function $\omega : \mathbb{N}_{\min} \to [1, \infty)$.

The Banach algebra $\ell^1(\mathbb{N}_{\min}, \omega)$ has been studied extensively in [6]. It is commutative, spanned by its idempotents, always has an approximate identity, and has a bounded approximate identity if and only if $\liminf_{n\to\infty} \omega(n) < \infty$. Furthermore, $\ell^1(\mathbb{N}_{\min}, \omega)$ is always weakly amenable and is always pointwise approximately amenable. Under the additional condition that $\liminf_{n\to\infty} \omega(n) < \infty$, $\ell^1(\mathbb{N}_{\min}, \omega)$ is also boundedly approximately contractible, and thus, from [6, Corollary 3.6.2] and Proposition 2.4.2 we have the following.

Proposition 3.0.3. Let $\omega : \mathbb{N}_{\min} \to [1, \infty)$. If $\liminf_{n\to\infty} \omega(n) < \infty$ then $\ell^1(\mathbb{N}_{\min}, \omega)$ is boundedly pseudo-amenable.

Definition 3.0.1. Let $\mathfrak{A}$ be a Banach algebra and let $n \in \mathbb{N}$. The $n \times n$ **matrix algebra of** $\mathfrak{A}$ is the Banach algebra $M_n(\mathfrak{A})$ of all $n \times n$ matrices with entries in $\mathfrak{A}$ equipped with the matrix algebra operations.

In [5, Theorem 2.7] it was proven that a Banach algebra $\mathfrak{A}$ is amenable if and only if $M_n(\mathfrak{A})$ is amenable. In [6, Proposition 1.6.7], it was shown that $\mathfrak{A}$ is approximately amenable if and only if $M_n(\mathfrak{A})$ is approximately amenable.

We can obtain similar partial results for bounded pseudo-amenability and bounded pseudo-contractibility through the identification of $M_n(\mathfrak{A})$ with $M_n\widehat{\otimes}\mathfrak{A}$.

Proposition 3.0.4. Let $\mathfrak{A}$ be a Banach algebra. If $\mathfrak{A}$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible) then $M_n(\mathfrak{A})$ is boundedly pseudo-amenable (resp. boundedly pseudo-contractible).

Proof. First suppose that $\mathfrak{A}$ is boundedly pseudo-amenable. Since M_n is finite-dimensional and semisimple, it is contractible. So by Theorem 2.5.2 we have that $M_n(\mathfrak{A}) \cong M_n \widehat{\otimes} \mathfrak{A}$ is boundedly pseudo-amenable.

Now suppose that $\mathfrak{A}$ is boundedly pseudo-contractible. Since M_n is contractible it is trivially boundedly pseudo-contractible too. But by [22, Theorem 3.7] the tensor product of two boundedly pseudo-contractible Banach algebras is still boundedly pseudo-contractible, so $M_n(\mathfrak{A}) \cong M_n \widehat{\otimes} \mathfrak{A}$ is boundedly pseudo-contractible. ■

Corollary 3.0.1. Let $\mathfrak{A}$ be a Banach algebra with a central bounded approximate identity. Then $\mathfrak{A}$ is boundedly pseudo-amenable if and only if $M_n(\mathfrak{A})$ is boundedly pseudo-amenable.

Proof. If $\mathfrak{A}$ is boundedly pseudo-amenable then $M_n(\mathfrak{A})$ is too by Proposition 3.0.4. Conversely, if $M_n(\mathfrak{A}) \cong M_n \widehat{\otimes} \mathfrak{A}$ is boundedly pseudo-amenable, then since $\mathfrak{A}$ has a central bounded approximate identity and M_n has an identity, we have that this implies the bounded pseudo-amenability of $\mathfrak{A}$ by Proposition 2.5.1. ■

We now turn our attention to the sequence algebras ℓ^p where $1 \leq p < \infty$.

Proposition 3.0.5. For each $1 \leq p < \infty$, the sequence algebra ℓ^p is boundedly pseudo-contractible.

Proof. For each $n \in \mathbb{N}$ let $E_n = \sum_{m=1}^{n} \mathbf{e_m} \otimes \mathbf{e_m}$ where $\mathbf{e_m} \in \ell^p$ is the sequence whose terms are all zero except the m^{th} term. Then $(E_n) \subset \ell^p \widehat{\otimes} \ell^p$. It is easily checked that $\mathbf{a} \cdot E_n = E_n \cdot \mathbf{a}$ for all $\mathbf{a} \in \ell^p$ and for all $n \in \mathbb{N}$. Also, given $\mathbf{a} = (a_1, a_2, ...) \in \ell^p$ and given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\displaystyle\sum_{n=N+1}^{\infty} |a_n|^p < \epsilon^p$. So if $n \geq N$ then:

$$\|\mathbf{a} - \pi(E_n)\mathbf{a}\|_p = \left\| \mathbf{a} - \sum_{m=1}^{n} \mathbf{e_m} a \right\|_p = \left(\sum_{m=n+1}^{\infty} |a_m|^p \right)^{1/p} \leq \left(\sum_{m=N+1}^{\infty} |a_m|^p \right)^{1/p} < \epsilon.$$

So $(E_n) \subset \ell^p \widehat{\otimes} \ell^p$ is a central approximate diagonal for ℓ^p. It is furthermore multiplier bounded since for all $\mathbf{a} = (a_1, a_2, ...) \in \ell^p$ and for all $n \in \mathbb{N}$:

$$\|\pi(E_n)\mathbf{a}\|_p = \left\| \sum_{m=1}^{n} \mathbf{e_m a} \right\|_p = \left(\sum_{m=1}^{n} |a_m|^p \right)^{1/p} \leq \left(\sum_{m=1}^{\infty} |a_m|^p \right)^{1/p} = \|\mathbf{a}\|_p,$$

and hence ℓ^p is boundedly pseudo-contractible. ■

Corollary 3.0.2. For each $1 \leq p < \infty$, the unitization $(\ell^p)^\sharp$ is not boundedly pseudo-amenable.

Proof. Suppose that $(\ell^p)^\sharp$ is boundedly pseudo-amenable. Then $(\ell^p)^\sharp$ is pseudo-amenable. But by [12, Proposition 3.1] this implies that ℓ^p is approximately amenable. But by [7, Theorem 4.1], for $1 \leq p < \infty$, ℓ^p is never approximately amenable. ■

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