ESTIMATES OF CHRISTOFFEL FUNCTION ON MULTIVARIATE DOMAINS

by

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Abstract

We consider Christoffel functions defined on some compact multivariate domain $D \subset \mathbb{R}^d$ with non-empty interior, $d \geq 2$. Christoffel function can be defined as

$$\lambda_n(\boldsymbol{x}, D, w) = \left(\sum_{k=1}^{n} p_k^2(\boldsymbol{x}) \right)^{-1}, \ \boldsymbol{x} \in D,$$

where w is a non-negative integrable weight function, $\{p_k\}_{k=1}^n$ is an orthonormal basis of $\mathcal{P}_{n,d}$ with respect to the inner product $\langle f, g \rangle = \int_D f(\boldsymbol{y}) g(\boldsymbol{y}) w(\boldsymbol{y}) d\boldsymbol{y}$, $\mathcal{P}_{n,d}$ is the space of all real algebraic polynomials of total degree less or equal to n in d variables. Alternatively, the Christoffel functions can be defined using an extremal problem as

$$\lambda_n(\boldsymbol{x}, D, w) = \min_{P \in \mathcal{P}_{n,d}, P(\boldsymbol{x})=1} \int_D P(\mathbf{t})^2 w(\mathbf{t}) d(\mathbf{t}).$$

Christoffel functions have applications in different problems of approximation theory, harmonic analysis, numerical analysis, statistical physics, random matrix theory, spectral theory and other areas of mathematics. Understanding the structure of orthonormal bases of $\mathcal{P}_{n,d}$ on a general multivariate domain, which can be used to compute $\lambda_n(\boldsymbol{x}, D, w)$, can be a very challenging task. Known results are mostly limited only to very specific domains.

In this thesis, we obtain some new estimates for Christoffel functions on multivariate domains from several general classes. In most situations, we managed to compute the Christoffel function up to a constant factor avoiding explicit computation of the corresponding orthonormal system of polynomials. One of the key tools we apply is the technique of comparison between different domains.

Our first results is a lower bound on Christoffel function on planar convex domains in terms of an adaptation of the parallel section function of the domain. For a certain class of planar convex domains, this allows us to compute the pointwise behavior of Christoffel functions using previously known upper bound. In particular, we obtain new estimates for Christoffel functions on l_α-balls B_α in $\mathbb{R}^2$, $1 < \alpha < 2$.

Our second result is the computation, up to a constant factor, of the Christoffel functions on planar domains with boundary consisting of finitely many C^2 curves such that each corner point of the boundary has interior angle strictly between 0 and π.

Finally, we found, up to a constant factor, the Christoffel functions on any simple polytope in $\mathbb{R}^d$, $d \geq 4$, and on any convex polytope in $\mathbb{R}^3$.

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Chapter 1

Introduction

1.1 Short survey on multivariate Christoffel functions

In the one-dimensional case, let w be a measurable positive finite weight on the real line such that

$$\int_{\mathbb{R}} t^n w(t)dt < \infty, \ n = 0, 1, 2, \ldots .$$

Then, the Christoffel function associated with w is given by

$$\lambda_n(x, D, w) = \left(\sum_{k=1}^{N} p_k^2(w, x) \right)^{-1}, \ n \geq 1, \tag{1.1.1}$$

where $\{p_k\}$, $k = 1, \ldots, N$, is a system of orthonormal polynomials w.r.t. to the inner product $\langle f, g \rangle = \int_D f(y)g(y)w(y)dy$. For the uniform weight $w = 1$, we set $\lambda_n(\boldsymbol{x}, D) = \lambda_n(x, D, w)$. Note that in (1.1.1), we have $N = n + 1$.

Let us begin by showing a connection of Christoffel functions to a better known problem of approximate integration. The Gauss-Christoffel quadrature formula is given by

$$\int_{\mathbb{R}} f(t)w(t)dt = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f), \tag{1.1.2}$$

where τ_k, $k = 1, \ldots, n$ are different nodes and A_k are the weights. It is a quadrature formula of the maximal algebraic degree of exactness $2n - 1$. In other words, under proper assumptions on the weight, one can choose the nodes τ_k and the weights A_k so that $R_n(f) = 0$ for any $f \in \mathcal{P}_{2n-1,1}$, where $\mathcal{P}_{2n-1,1}$ is the space of polynomials in one variable of degree less or equal to $2n-1$, see [36] for details. The first formula of this type was obtained by Carl Friedreich Gauss two centuries ago for the case $w(t) \equiv 1$. The formula for more general weight functions was discovered by Christoffel in 1858, see [6]. The weight coefficients A_k from (1.1.2) can be expressed by the Christoffel function $\lambda_n(\tau_k, \mathbb{R}, w)$ in the following way

$$A_k = \lambda_n(\tau_k, \mathbb{R}, w), \; k = 1, \ldots, n,$$

see [6] for details.

Christoffel functions have their origin in Christoffel's works [6] and [7], see also [36] and [5] for details.

Christoffel function are useful tools in a few areas of analysis and mathematics. The first well known works related to Christoffel functions were studies on quadratures and the moment problem of Chebyshev, Gauss, Jacobi, Markov (A. A. Markov), Posse, and Stieltjes. Also, Akhiezer, Carleman, Hamburger, Krein, and Riesz used them to study the uniqueness of the solution to the moment problem. Freud systematically studied Christoffel functions. His work was influenced by the paper [15] of Erdos and Turan, 1950. For more details see the survey [34] by Nevai.

There is a strong connection between Christoffel functions and orthogonal polynomials (see Nevai, [34] and Simon, [41]). There are applications of Christoffel functions in such fields as statistical physics (see Pastur, [37]), universality in random matrix theory (see Lubinsky, [29]), spectral theory (see Simon, [41] and Breuer, Last and Simon, [4]) and some other fields in mathematics (see e.g. [22] and [23]).

For orthogonal polynomials $\{\boldsymbol{\varphi}_n\}$, $n = 0, 1, 2, \ldots$ with respect to the inner product $\langle f, g\rangle = \int_C f(y)g(y)w(y)dy$ on the unit circle C, the Christoffel function λ_n associated with w can be defined in the similar way

$$\lambda_n(x, [0, 2\pi], w) = \left(\sum_{k=0}^{n-1} |\boldsymbol{\varphi}_k(x)|^2\right)^{-1}, \ n = 1, 2, \ldots .$$

The simplest example of a Christoffel function, associated with $w(t) = \frac{1}{2\pi}$, is

$$\lambda_n(x, [0, 2\pi], w) = \left(\sum_{k=0}^{2n-2} \boldsymbol{\varphi}_k^2(t)\right)^{-1} = \frac{1}{n}, \tag{1.1.3}$$

where $\{\boldsymbol{\varphi}_k\}_{0\le k\le 2n-2} = \{1, \sin t, \cos t, \ldots, \sin(n-1)t, \cos(n-1)t\}$ is the standard trigonometric basis of the set T_n of trigonometric polynomials of degree less or equal $n-1$. So clearly by $(\sin(kt))^2 + (\cos(kt))^2 = 1$ we have $\lambda_n(w, t) = \frac{1}{n}$.

An example where Christoffel functions can be obtained in closed form in case of algebraic polynomials on the segment $[-1, 1]$ can be given for the Chebyshev weight

$$w(t) = (1 - t^2)^{-\frac{1}{2}}, \ |t| < 1 \text{ and } w(t) = 0, \ |t| \ge 1.$$

See [34, p. 10] for details. Namely,

$$\lambda_n(x, [-1,1], w)^{-1} = \pi^{-1}\left(n - \frac{1}{2} + \frac{1}{2}U_{2n-2}(x)\right),$$

where U_n is the Chebyshev polynomial of the second kind

$$U_n(x) = 2^n \prod_{k=1}^{n}\left(x - \cos\left(\frac{k\pi}{n+1}\right)\right).$$

Let us introduce the doubling and Jacobi weights. We call a positive weight w on $[-1,1]$ a doubling weight if, for any interval $I \subset [-1,1]$, we have

$$w(2I) \leq Lw(I)$$

with some positive doubling constant L, where for $I = [a,b]$ we denote by $2I$ the interval with length $2(b-a)$ and center at $(a+b)/2$, and $w(I) = \int_I w(u)du$. Here we interpret $\int_E f(x)w(x)dx$ as $\int_{E\cap[-1,1]} f(x)w(x)dx$.

We define Jacobi weight as a weight $w_{\alpha,\beta}$ such that $w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ with $\alpha, \beta \in J_p$, where

$$J_p = \begin{cases} \left(-\frac{1}{p}, \infty\right), & \text{if } 0 < p < \infty, \\ [0, \infty), & \text{if } p = \infty. \end{cases}$$

We will denote throughout this thesis

$$\rho_n(x) = \frac{1}{n^2} + \frac{1}{n}\sqrt{1-x^2},\ x \in \mathbb{R},\ |x| \leq 1. \tag{1.1.4}$$

Uniform weights or Jacobi weights with proper bounds on powers are examples of doubling weights. The following inequality was proved by Mastroianni and Totik for a doubling weight w and $1 \le p < \infty$, see [35, p. 66]. There is a constant C depending only on the doubling constant L and p such that for all n and x,

$$\frac{1}{C}\int_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du \le \lambda_n(x,[-1,1],w) \le C\int_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du.$$

In case of $w=1$, we are drawing the following corollary. There is a constant C depending only on the doubling constant L such that for all n and $x \in [0,1]$

$$\frac{1}{C}\rho_n(x) \le \lambda_n(x,[-1,1],w) \le C\rho_n(x).$$

A generalization of the results of Mastroianni and Totik for quasismooth curves was obtained by Varga [49].

Christoffel functions in the multivariate case are the main focus of the present thesis. We will associate Christoffel functions with a compact set $D \subset \mathbb{R}^d$ with non-empty interior and a positive weight function $w \in L_1(D)$. We denote by $\boldsymbol{x}$ a vector in $\mathbb{R}^d$ and represent by the same notation the corresponding point in $\mathbb{R}^d$. We denote by $\mathcal{P}_{n,d}$ the space of all real algebraic polynomials of total degree less or equal to n in d variables.

Similarly to the univariate case, the associated Christoffel function is defined as

$$\lambda_n(\boldsymbol{x},D,w) = \left(\sum_{k=1}^{N} \boldsymbol{\varphi}_k(\boldsymbol{x})^2\right)^{-1}, \ \boldsymbol{x} \in D, \tag{1.1.5}$$

where $N = \binom{n+d}{d}$ is the dimension of $\{\boldsymbol{\varphi}_k\}_{k=1}^{N}$ is an orthonormal basis on D of $\mathcal{P}_{n,d}$.

The following well-known fact about Christoffel functions can be used to calculate Christoffel functions through an optimization problem. This can also be considered as an alternative definition of Christoffel functions, e.g see [14].

Proposition 1.1. [14, Theorem 3.1] *Suppose $D \subset \mathbb{R}^d$ is a bounded domain and w is a finite measurable positive weight on D. Then, for $\boldsymbol{x} \in D$, we have*

$$\lambda_n(\boldsymbol{x}, D, w) = \min_{P \in \mathcal{P}_{n,d},\ P(\boldsymbol{x})=1} \int_D P(\mathbf{t})^2 w(\mathbf{t}) d\mathbf{t}. \tag{1.1.6}$$

In particular, $\lambda_n(\boldsymbol{x}, D, w)$ does not depend on the choice of orthonormal basis of on D.

The above proposition implies that the Christoffel function is a monotonic function of domain. The following proposition is a straightforward corollary of this fact.

Proposition 1.2. *Suppose $D_1 \subset D_2 \subset \mathbb{R}^d$ are two bounded domains, and w a finite measurable positive weight on D_2. Then for $\boldsymbol{x} \in D_1$,*

$$\lambda_n(\boldsymbol{x}, D_1, w) \le \lambda_n(\boldsymbol{x}, D_2, w). \tag{1.1.7}$$

Another useful simple tool is an affine change of variable in (1.1.6)

Proposition 1.3. [14, Theorem 3.4] *For any non-degenerate affine transform $\mathscr{T}$, $\mathscr{T}\boldsymbol{x} = \boldsymbol{x}_0 + A\boldsymbol{x}$ with $\det \mathscr{T} := \det A \neq 0$ on $\mathbb{R}^d$ and any bounded domain $D \subset \mathbb{R}^d$, we have*

$$\lambda_n(\mathscr{T}\boldsymbol{x}, \mathscr{T}(D)) = \lambda_n(\boldsymbol{x}, D)|\det \mathscr{T}|, \ \boldsymbol{x} \in D. \tag{1.1.8}$$

Here and through the thesis, we denote by $\lambda_n(\boldsymbol{x}, D)$ Christoffel function with $w = 1$, $\lambda_n(\boldsymbol{x}, D, 1)$. Propositions 1.2 and 1.3 are key tools for getting bounds for Christoffel functions in multivariable cases

As we see from (1.1.5), Christoffel functions depend on the structure of the orthogonal polynomials on D which can be hard to study when D is a rather general domain. In the situation when an orthonormal basis of $\mathcal{P}_n$ is available in closed form, we could compute the Christoffel functions by using the classical definition (1.1.5) see, e.g. Xu [53] and (1.1.3). Our main approach will be to utilize (1.1.7), (1.1.8) and the known behaviour of the Christoffel function for some specific domains such as the unit cube or ball. This approach for general domains was introduced in 2015 by Kroo [24], and then developed by Ditzian and Prymak [14] and Prymak [38].

The notation $B^d := B_2^d$ stands for the d dimensional Euclidean unit ball centered at $0 \in \mathbb{R}^d$. In case of $d = 2$ we use notation B omitting the index. For the Euclidean norm of $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we use the notation $\|\boldsymbol{x}\|_2$,

$$\|\boldsymbol{x}\|_2 := \sqrt{|x_1|^2 + \ldots + |x_d|^2}.$$

Rather precise asymptotics of multivariate Christoffel functions were found only for some special domains such as: cube, unit ball, simplex and certain classes of weights on these domains. For the unit ball B^d and central symmetric positive continuous weights w, it was shown in 1998 by Bos, Della Vecchia and Mastroianni [3] that

$$\lim_{n\to\infty} \lambda_n(\boldsymbol{x}, B^d, w)\binom{n+d}{d} = \frac{\pi^{\frac{d+1}{2}} w(\boldsymbol{x})\sqrt{1-\|\boldsymbol{x}\|_2^2}}{\Gamma\left(\frac{d+1}{2}\right)}. \tag{1.1.9}$$

An extension of the above relation to a certain wider class of weights was also given in [3].

In the thesis, we denote by c, $c(\cdot)$, $\ldots$ the constants which are positive and depend only on parameters indicated in the parentheses (if any) and may be different at different occurrences even if the same notation is used. This is in contrast to $c_1(\cdot)$, $c_2(\cdot), \ldots$ which have the same value at different occurrences for the same arguments. The equivalence "$\approx$" is understood with absolute constants, namely, $A \approx B$ means $c^{-1}A \leq B \leq cA$, where $c > 0$ is an absolute constant.

It is possible to distinguish two types of results estimating Christoffel functions. The first one is to consider *asymptotics* of a Christoffel function when a point $\boldsymbol{x} \in D$ is fixed while $n \to \infty$. A typical result has the form

$$\lim_{n\to\infty} n^{\mu}\lambda_n(\boldsymbol{x}, D) = \Phi(\boldsymbol{x}), \tag{1.1.10}$$

for example, as in (1.1.9) or (1.1.3). The second approach is to find the *behaviour* of λ_n as a function of $\boldsymbol{x}$ for each fixed n, namely, to establish that

$$\lambda_n(\boldsymbol{x}, D) \approx \Psi_n(\boldsymbol{x}, D) \tag{1.1.11}$$

for an explicitly computable function Ψ_n as in (1.1). In particular, this allows to compute the behaviour of quantities like

$$\inf_{\boldsymbol{x}\in D} \lambda_n(\boldsymbol{x}, D),$$

where the infimum over $\boldsymbol{x}$ is taken for a fixed n, and then letting $n \to \infty$. The result of the type

$$\inf_{\boldsymbol{x}\in D} \lambda_n(\boldsymbol{x}, D) \approx n^{-\sigma}$$

is important because of the applications mentioned at the end of Section 1.2. Namely, this behaviour is determining for Nikol'skii type inequalities on D (see [14]) and can be crucial for stability and accuracy of discrete least squares approximation.

Note that generally speaking (1.1.10) does not imply (1.1.11), while (1.1.11) does provide information about Φ in (1.1.10) up to a constant. Let us note that, even under a very natural but general hypothesis that D is a convex body, the limit in (1.1.10) is not known to exist, although it is quite natural to conjecture that the limit does exist and this was shown for some specific domains such as cubes, simplexes and balls.

In terms of two types of Christoffel function estimates considered above, (1.1.9) establishes the asymptotics of Christoffel function on a ball B^d. Partial behaviour of Christoffel function on the ball B^d was obtained by Ditzian and Prymak [14, Theorem 4.1] in 2016. Namely,

$$\lambda_n(\boldsymbol{x}, B^d) \approx c(d) n^{-d-1}, \text{ for } \|\boldsymbol{x}\|_2 = 1, \tag{1.1.12}$$

$$\lambda_n(\boldsymbol{x}, B^d) \text{ is an increasing function of } \|x\|_2, \text{ for } \frac{1}{2} \leq \|\boldsymbol{x}\|_2 \leq 1,$$

$$\lambda_n(\boldsymbol{x}, B^d) \approx c(a,d) n^{-d}, \text{ for } \|\boldsymbol{x}\|_2 < a < 1 \text{ with constant of equivalence depending on } a. \tag{1.1.13}$$

This result was derived using the structure of orthogonal polynomials on the ball, see [44] for the details.

Let us introduce notation that we will use in this thesis. The notation ∂D denotes the boundary of $D \subset \mathbb{R}^d$ with respect to the Euclidean metric. The interior of $D \subset \mathbb{R}^d$ is denoted by $\mathrm{Int}D = \mathrm{Int}D$. Also,

$$\overline{D} = D \setminus (\mathrm{Int}D).$$

The distance between a point $\boldsymbol{x} \in \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ is denoted by $\mathrm{dist}(\boldsymbol{x}, D)$, and is defined as

$$\mathrm{dist}(\boldsymbol{x}, D) := \inf_{\boldsymbol{y} \in D} \|\boldsymbol{x} - \boldsymbol{y}\|_2.$$

We set $\mathrm{dist}(\boldsymbol{x}, \emptyset) := \infty$.

The diameter of a set $D \subset \mathbb{R}^d$ is denoted by $\mathrm{diam}(D)$,

$$\mathrm{diam}(D) := \sup_{\boldsymbol{x}, \boldsymbol{y} \in D} \|\boldsymbol{x} - \boldsymbol{y}\|_2.$$

Let us demonstrate a standard technique to obtain the behaviour of a Christoffel function for an interior point $\boldsymbol{x} \in \mathrm{Int} D$, where, recalling our assumptions and notation, D is a compact set in $\mathbb{R}^d$ and B^d will denote the unit ball in $\mathbb{R}^d$. Set $\delta := \mathrm{dist}(\boldsymbol{x}, \partial D) > 0$, $R := \mathrm{diam}(D)$. Then $\delta B^d + \boldsymbol{x} \subset D \subset R B^d + \boldsymbol{x}$, so by (1.1.12) and Propositions 1.2 and 1.3 we have

$$c(d) \delta^d n^{-d} \leq \lambda_n(\boldsymbol{x}, D) \leq c(d) R^d n^{-d}. \tag{1.1.14}$$

This means that if $\boldsymbol{x}$ is "far from the boundary", or $\delta > \delta_0$, then $\lambda_n(\boldsymbol{x}, D) \approx c(\delta_0, D) n^{-d}$. Investigation of the situation when $\boldsymbol{x}$ is "close to the boundary" is the primary focus of most of the current research and requires more complicated arguments. The above equation (1.1.12) illustrates that the behaviour of Christoffel functions on the boundary of the ball is $n^{-(d+1)}$.

We denote a simplex in $\mathbb{R}^d$ by Δ^d,

$$\Delta^d := \{\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, x_1 + \ldots + x_d \leq 1\}. \tag{1.1.15}$$

We define the norm $\|\boldsymbol{x}\|_\alpha$ of $\boldsymbol{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$ with respect to the l_α metric for $1\le\alpha<\infty$ as

$$\|\boldsymbol{x}\|_\alpha := \sqrt[\alpha]{|x_1|^\alpha+\ldots+|x_d|^\alpha}.$$

The Chebyshev weight function W_0 on Δ^d is defined as

$$W_0 := w_0 x_1^{-\frac{1}{2}}\ldots x_d^{-\frac{1}{2}}\left(1-\|\boldsymbol{x}\|_1\right)^{-\frac{1}{2}},$$

where $w_0 := \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$. Xu in [54] obtained the following asymptotics of Christoffel functions over the interior of the simplex Δ^d, for all $\boldsymbol{x}\in\mathrm{Int}\Delta^d$, with respect to

Theorem 1.4. [54, Theorem 3.3] *If $W=W_0q^2$, where q is a polynomial of d variables, then*

$$\lim_{n\to\infty}\binom{n+d}{n}\lambda_n(\boldsymbol{x},\Delta^d,W)=\frac{W(\boldsymbol{x})}{W_0(\boldsymbol{x})},\ \boldsymbol{x}\in\mathrm{Int}\Delta^d. \tag{1.1.16}$$

Also Xu in this paper obtained a similar result for certain Jacobi type weights, namely for the weight function $W_\alpha = w_\alpha x_1^{\alpha_1-\frac{1}{2}}\ldots x_d^{\alpha_d-\frac{1}{2}}\left(1-\|\boldsymbol{x}\|_1\right)^{\alpha_{d+1}-\frac{1}{2}}$, $\alpha\in 2\mathbb{N}^d$ $\alpha_i\in 2\mathbb{N}$, where $w_\alpha = \frac{\prod_{i=1}^{d+1}\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{d+1}\alpha_i)}$ is a normalized constant such that $\int_{\Delta^d}W_\alpha d\boldsymbol{x}=1$. Namely, according to Corollary 3.4 from [54], one has

$$\lim_{n\to\infty}\binom{n+d}{n}\lambda_n(\boldsymbol{x},\Delta^d,W_\alpha)=\frac{W_\alpha(\boldsymbol{x})}{W_0(\boldsymbol{x})},\ \boldsymbol{x}\in\mathrm{Int}\Delta^d.$$

According to Kroo and Lubinsky, see [28] Theorem 1.4, uniformly for $\boldsymbol{x}\in D$, $\boldsymbol{y}\in B(\boldsymbol{x},\frac{1}{\sqrt{n}})$,

$$\lim_{n\to\infty}\binom{n+d}{d}\lambda_n(\boldsymbol{y},\Delta^d,\nu)=\frac{\nu'}{W_0(\boldsymbol{x})}, \tag{1.1.17}$$

$$W_0(\boldsymbol{x}) := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}\prod_{i=1}^d x_i^{-\frac{1}{2}}\left(1-\sum_{i=1}^d x_i\right)^{-\frac{1}{2}}.$$

We could notice that $(\lambda_n(\boldsymbol{x}, D))^{-1}$ in both asymptotic relations (1.1.9) and (1.1.16) has the same magnitude of order $\binom{n+d}{d} \approx n^d$ as $n \to \infty$. This magnitude increases substantially when $\boldsymbol{x}$ is going closer to the boundary of the domain. On the other hand denoting the distance of $\boldsymbol{x}$ to the boundary of the domain D by $\operatorname{dist}(\boldsymbol{x}, \partial D) = \inf_{\boldsymbol{y} \in \partial D} \|\boldsymbol{x} - \boldsymbol{y}\|_2$ we observe a different behaviour of relations (1.1.9) and (1.1.17). Indeed, in (1.1.17) we see that $\lim_{n\to\infty} \lambda_n(\boldsymbol{x}, D) \approx (\operatorname{dist}(\boldsymbol{x}, \partial D))^{\frac{1}{2}}$, and in (1.1.17) we consider that $\lim_{n\to\infty} \lambda_n(\boldsymbol{x}, D) \approx (\operatorname{dist}(\boldsymbol{x}, \partial D))^{\frac{d}{2}}$ as $\boldsymbol{x}$ approaches a vertex of the simplex. So, it seems the different behaviour close to the boundary can be caused by the different geometry (smoothness) of the considered domains.

Once asymptotics of Christoffel functions are established for one weight, it is possible, under some additional conditions, to find the asymptotics for a family of other weights. The asymptotics obtained by Xu in [54] used the structure of the orthonormal basis and did not include the uniform weight. The result for the simplex for the uniform weight was obtained only later through the universality in [28] by Kroo and Lubinsky in 2013. Such results are referred to as universality in the bulk. More precisely, universality in the bulk allows to obtain the asymptotics of Christoffel function for a large family of weights provided the asymptotics is known for one specific weight. A precise statement will be given below in Theorem 1.16. Let us now state the main results of [28].

A compactly supported measure μ on $\mathbb{R}^d$ is said to be regular if

$$\lim_{n\to\infty}\left(\sup_{\deg(P)\le n}\frac{\|P\|^2_{L_\infty(\mathrm{supp}[\mu])}}{\int|P|^2d\mu}\right)^{\frac{1}{n}}=1, \tag{1.1.18}$$

where, as usual, we define support by supp$[\mu]$ and L_∞-norm on $D\subset\mathbb{R}$ of a function $f:D\to\mathbb{R}$ as

$$\|f\|_{L_\infty(D)}=\operatorname*{ess\,sup}_{\boldsymbol{x}\in D}|f(\boldsymbol{x})|.$$

Theorem 1.5. [28, Theorem 1.1] *Let μ, ν be positive measures whose support is a compact set $\mathfrak{K}\in\mathbb{R}^d$ and both are regular. Let $D\subset D_1\subset\mathfrak{K}$, where D is compact and D_1 is open. Assume that ν and that μ are mutually absolutely continuous in D, and the Radon - Nikodym derivative $\frac{d\nu}{d\mu}$ is positive and continuous in D, while uniformly in $\overline{D_1}$,*

$$\lim_{\varepsilon\to0+}\left(\limsup_{n\to\infty}\frac{\lambda_{\lfloor n(1-\varepsilon)\rfloor}(\boldsymbol{x},\mu)}{\lambda_n(\boldsymbol{x},\mu)}\right)=1, \tag{1.1.19}$$

where $\lfloor a\rfloor$ denotes the integer part of real number a and $\lambda_n(\boldsymbol{x},\mu)=\lambda_n(\boldsymbol{x},w)$ with continuous and positive weight $w(\boldsymbol{x})$, so that $d\mu(\boldsymbol{x})=wd\boldsymbol{x}$. Then, uniformly for $\boldsymbol{x}\in D$ and $\boldsymbol{y}\in B(\boldsymbol{x},\frac{1}{\sqrt{n}})$, we have

$$\lim_{n\to\infty}\frac{\lambda_n(\boldsymbol{y},\nu)}{\lambda_n(\boldsymbol{y},\mu)}=\frac{d\nu}{dw}(\boldsymbol{x}). \tag{1.1.20}$$

To prove Theorem 1.5, the authors used the “needle” polynomials constructed in 1992 by Kroo and Swetits in [27].

Remark 1.6. Note that (1.1.19) is satisfied if for some $\beta>0$ and positive continuous function Φ,

$$\lim_{n\to\infty}n^\beta\lambda_n(\boldsymbol{x},\mu)=\Phi(\boldsymbol{x})$$

uniformly in $\overline{D_1}$.

In our situation, we claim the existence of the limit $\lim_{n\to\infty} \lambda_n(\boldsymbol{x}, D)n^2 = F(\boldsymbol{x})$ for $\boldsymbol{x} \in D$, where D is a convex planar body and this convergence is uniform on compact subsets of Int$D$. According to [28], a sufficient condition for the regularity condition (1.1.18) to be satisfied involves a compact set $\mathfrak{K}$ with analytic parametrization. Namely, for any $\boldsymbol{x} \in \mathfrak{K}$, there exists a curve $\gamma(t) \in \mathbb{R}^d$, $t \in [0,1]$, analytic and bounded in an open set $\Omega \subset \mathbb{C}$ that contains $[0,1]$ and $\gamma(0) = \boldsymbol{x}$, $B\left(\gamma(t), \boldsymbol{\varphi}(t)\right) \subset \mathfrak{K}$, $0 < t < 1$. Here $B(\mathbf{a}, r)$ denotes the ball centered at $\mathbf{a}$ with the radius r, Ω and the bound on γ depend only on $\mathfrak{K}$, while $\boldsymbol{\varphi}$ is a positive continuous function tending to 0 as $t \to 0$ that also depends only on $\mathfrak{K}$. In particular, any polygon or convex set with non-empty interior has an analytic parametrization.

The condition (1.1.19) is not known to be met for general domains in $\mathbb{R}^d$. We have the following theorems for ball and simplex.

Theorem 1.7. [28, Theorem 1.3] *Let ν be a regular measure on $\overline{B^d}$ so that $d\nu(x) = w$ and assume that D is a compact subset of the interior of $\overline{B^d}$ such that ν' is positive and continuous in D. Then, uniformly for $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in B(\boldsymbol{x}, \frac{1}{\sqrt{n}})$, we have*

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\boldsymbol{y}, B^d, w) = \frac{\nu'}{W_0^{ball}(\boldsymbol{x})}, \tag{1.1.21}$$

where

$$W_0^{ball}(\boldsymbol{x}) := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \left(1 - \|\boldsymbol{x}\|_2^2\right)^{-\frac{1}{2}}.$$

Theorem 1.8. [28, Theorem 1.4] *Let ν be a regular measure on Δ^d so that $d\nu(x) = w$*

and assume that D is a compact subset of the interior of Δ^d defined in (1.1.15), *such that ν' is positive and continuous in D. Then, uniformly for $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in B(\boldsymbol{x}, \frac{1}{\sqrt{n}})$, we have*

$$\lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\boldsymbol{y}, \Delta^d, w) = \frac{\nu'}{W_0^{simplex}(\boldsymbol{x})}, \tag{1.1.22}$$

where

$$W_0^{simplex}(\boldsymbol{x}) := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \prod_{i=1}^{d} x_i^{-\frac{1}{2}} \left(1 - \sum_{i=1}^{d} x_i\right)^{-\frac{1}{2}}.$$

The rest of this section is dedicated to very recent results on bounds of multivariate Christoffel functions for rather general domains.

In the multivariate case, the orthonormal basis in the definition (1.1.5) of Christoffel functions is hard to compute, hence the use of the variational definition (1.1.6) becomes important. Intuitively, for a fixed point $\boldsymbol{x}$, we want to find a polynomial f of fixed degree satisfying $f(\boldsymbol{x}) = 1$ and having the smallest possible L^2-norm. One standard approach is to use comparison with basic reference domains for which the estimates may be found through the orthonormal basis, as was already illustrated in (1.1.14). A well-known approach to find the lower bounds of Christoffel functions is to use inequalities bounding the norm of the derivative of a polynomial through the norm of that polynomial. A common tool used especially for obtaining upper bounds of Christoffel functions is "needle" or fast decreasing polynomials. The trigonometric "needle" polynomials for the interval $(t_0 - r, t_0 + r) \subset (-\pi, \pi)$ are defined as trigonometric polynomials $0 \le T_n(t) \le 1$, $t \in [-\pi, \pi]$, $T_n(t_0) = 1$ of degree n that satisfies, for some absolute constant $c > 0$,

$$0 \leq T_n(t) \leq e^{-cnr},\ t \in [-\pi, \pi],\ |t - t_0| \geq r,\ n \in \mathbb{N}, \tag{1.1.23}$$

see Abstract of [25], Kroo. For purposes of this thesis, it was more convenient to use the construction from [14] based on Chebyshev polynomials.

The unit ball in $\mathbb{R}^d$, with respect to the l_α metric, is

$$B^d_\alpha := \{\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d :\ \|\boldsymbol{x}\|_\alpha \leq 1\},\ 1 \leq \alpha < \infty. \tag{1.1.24}$$

In the case $d = 2$ we denote $B_\alpha := B^d_\alpha$.

We call a compact set $K \subset \mathbb{R}^d$ a starlike with respect to a point $\mathbf{a} \in \mathrm{Int}K$ if for every $\boldsymbol{x} \in K$ we have that $[\mathbf{a}, \boldsymbol{x}) \subset \mathrm{Int}K$. In 2015, Kroo [24] studied the magnitude of Christoffel functions for general starlike and convex domains. In that paper, Kroo measured the smoothness of the boundary of the domain in terms of the smoothness of its Minkowski functional and combined geometric and analytic methods to obtain a sharp pointwise upper bound which gives the behaviour of Christoffel functions near the boundary of the domain. The Minkowski functional of a compact set with nonempty interior $K \subset \mathbb{R}^d$ centered at the origin, which is starlike with respect to the origin $\mathbf{0} \in \mathrm{Int}K$, is defined as

$$\boldsymbol{\varphi}_K(\boldsymbol{x}) := \inf\{\alpha > 0 :\ \frac{\boldsymbol{x}}{\alpha} \in K\}.$$

In the following theorem, the upper bound for $B^d_\alpha \subset \mathbb{R}^d$ is given.

Theorem 1.9. [24, Theorem 2] *If* $1 < \alpha < 2$ *then there is a constant* $c := c(d, \alpha) > 0$

and integer n_0 *such that for any* $\mathbf{z} = (t, 0, \ldots, 0) \in B^d_\alpha$ *(see* (1.1.24)*) with* $\frac{1}{2} < t < 1$

$$\lambda_n(\boldsymbol{z}, B^d_\alpha, w) \leq c \left(\frac{\ln n}{n}\right)^d w(\mathbf{z})(1 - \boldsymbol{\varphi}_{B^d_\alpha}(\mathbf{z}))^{\gamma(\alpha,d)}, \; n \geq n_0, \tag{1.1.25}$$

where $w(\boldsymbol{x})$ *is continuous and positive in* $\mathrm{Int}K$ *and* $\gamma(\alpha, d) := \frac{1}{2} + \frac{(d-1)(2-\alpha)}{2\alpha}$.

Note that $\boldsymbol{\varphi}_{B^d_\alpha}(\mathbf{z}) = t$.

To obtain this upper bound on Christoffel functions, the author constructed appropriate "needle" polynomials.

The following lower bound on Christoffel functions was obtained by Kroo for domains starlike about the origin. We have to define the class $C^\alpha \subset \mathbb{R}^d$ first.

Definition 1.10. [24, Definition 1] *Let us say that the set* $K \subset \mathbb{R}^d$ *starlike about the origin has Lip* α, $0 < \alpha \leq 2$ *boundary (or is a* C^α *set) if for some* $M > 0$ *depending on* K *the following properties hold:*

1. *When* $0 < \alpha \leq 1$

$$|\varphi_K(\boldsymbol{x}) - \varphi_K(\boldsymbol{x} + \mathbf{h})| \leq M \|\mathbf{h}\|_2^\alpha, \; \boldsymbol{x} \in \mathbb{R}^d, \mathbf{h} \in B^d.$$

2. *When* $1 < \alpha \leq 2$ *the Minkowski functional* $\varphi_K(\boldsymbol{x})$ *is differentiable on* $\mathbb{R}^d \setminus \{\mathbf{0}\}$ *and its gradient* $\nabla\varphi_K(\boldsymbol{x})$ *satisfies with some* $\delta = \delta_K > 0$ *the property*

$$|\nabla\varphi_K(\boldsymbol{x}) - \nabla\varphi_K(\boldsymbol{x} + \mathbf{h})| \leq M \|\mathbf{h}\|_2^{\alpha-1}, \; \boldsymbol{x} \in \partial K, \; \mathbf{h} \in \delta B^d.$$

Theorem 1.11. [24, Theorem 1] *Let* $K \subset \mathbb{R}^d$ *be a* C^α *domain which is starlike about the origin for some* $0 < \alpha \leq 2$. *Consider* w *which is continuous and positive in* $\mathrm{Int}K$.

Then, there exists a constant $c(K,d) > 0$ *such that*

$$\liminf_{n\to\infty} n^d \lambda_n(\boldsymbol{x}, K, w) \geq c(K,d) w(\boldsymbol{x})(1-\boldsymbol{\varphi}_K(\boldsymbol{x}))^{\gamma(\alpha,d)}, \ \boldsymbol{x} \in \mathrm{Int}K, \tag{1.1.26}$$

where $\gamma(\alpha, d) := \frac{1}{2} + \frac{(d-1)(2-\alpha)}{2\alpha}$.

The work [24] also contains domain independent lower bound for $\lambda_n(\boldsymbol{x}, w)$ in terms of the distance from the boundary of any convex body K to $\mathbf{0} \in \mathrm{Int}K$, $r_K := \min_{\boldsymbol{x}\in\partial K} \|\boldsymbol{x}\|_2$. Denote by R_K the radius of the smallest ball centered at the origin containing K, $R_K := \max_{\boldsymbol{x}\in\partial K} \|\boldsymbol{x}\|_2$.

Theorem 1.12. [24, Theorem 4] *Let* $K \subset \mathbb{R}^d$, $\mathbf{0} \in \mathrm{Int}K$ *be any convex body. Assume that* w *is continuous and positive in* $\mathrm{Int}K$. *Then for every* $\boldsymbol{x} \in \mathrm{Int}K$ *satisfying* $\boldsymbol{\varphi}_K(\boldsymbol{x}) \geq 1 - \frac{r_K}{R_k}$

$$\liminf_{n\to\infty} \frac{\lambda_n(\boldsymbol{x}, K, w)}{\binom{n+d}{d}} \geq \frac{\pi^{\frac{d+1}{2}} r_K^d w(\boldsymbol{x})(1-\boldsymbol{\varphi}_K(\boldsymbol{x})^2)^{\frac{d}{2}}}{2^{d+\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right) d^{\frac{d}{2}}}. \tag{1.1.27}$$

To prove (1.1.26) and (1.1.27) the author considered ellipsoids in $\mathbb{R}^d$ inscribed in the domain D and used the monotonic property of Christoffel function (1.1.7).

Prymak in [38] obtained various upper pointwise bounds on $\lambda_n(\boldsymbol{x}, D, w)$ for convex sets $D \subset \mathbb{R}^d$. In particular, the author computed $\lambda_n((1-\delta, 0, \ldots, 0), B_\alpha^d)$, $B_\alpha^d \subset \mathbb{R}^d$, $1 < \alpha < 2$, removing the unnecessary logarithmic factor in (1.1.25). The main result of [38] in the two-dimensional case is the following theorem.

Theorem 1.13. [38, Theorem 1.1] *Suppose a planar convex body* D *is contained in a disc of radius* R, *and for some* $\boldsymbol{x} \in \mathrm{Int}D$ *and unit vector* $\boldsymbol{u} \in \mathbb{R}^2$ *there are* $r > 0$

and $t_0 < 0$ *such that* $rB + \boldsymbol{x} + t_0\boldsymbol{u} \subset D$. *Let* $\delta := \max\{t : \ \boldsymbol{x} + t\boldsymbol{u} \in D\}$ *and* $l_i := \max\{t : \ \boldsymbol{x} + (-1)^i t\boldsymbol{v} \in D\}$, $i = 1, 2$, *where* $\boldsymbol{v}$ *is one of the two unit vectors orthogonal to* $\boldsymbol{u}$. *If* $\delta > \sigma n^{-2}$, $\sigma > 0$, *then*

$$\lambda_n(\boldsymbol{x}, D) \leq c(r, R, \sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}. \tag{1.1.28}$$

The quantities l_i and δ measure the location of $\boldsymbol{x}$ inside the body, see Figure 1.1, and it turns out that these few measurements are sufficient to give a quite precise bound on the Christoffel function. Note that for any $\boldsymbol{x} \in \mathrm{Int} D$ and any unit vector $\boldsymbol{u} \in \mathbb{R}^2$ there always exist $r > 0$ and $t_0 < 0$ satisfying $rB + \boldsymbol{x} + t_0\boldsymbol{u} \subset D$. However, the constant in (1.1.28) depends on r and the magnitudes of l_1 and l_2 depend on the choice of $\boldsymbol{u}$, so such a choice of $\boldsymbol{u}$ and r should be made carefully to result in an effective estimate on $\lambda_n(\boldsymbol{x}, D)$. Prymak in [38] explicitly constructed the corresponding "needle"-like algebraic polynomials of degree n with $P_n(\boldsymbol{x}) = 1$ and

$$\|P_n\|^2_{L_2(D)} \leq c(r, R, \sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}.$$

The following theorem was given in [38] to show that the bound on $\lambda_n(\boldsymbol{x}, D)$ (1.1.28) is sharp in the class of all convex bodies if we are measuring only distances l_1, l_2, δ.

Theorem 1.14. [38, Theorem 1.5] *For any positive* l_1, l_2, δ *with* $10\delta < l_1, l_2 < \frac{1}{10}$ *one can find a planar convex body* D *and a point* $\boldsymbol{x} \in \mathrm{Int} D$ *such that* $B \subset D \subset 4B$ *and with* $\boldsymbol{u} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$ *that* $\delta := \max\{t : \ \boldsymbol{x} + t\boldsymbol{u} \in D\}$ *and* $l_i := \max\{t : \ \boldsymbol{x} + (-1)^i t\boldsymbol{v} \in D\}$,

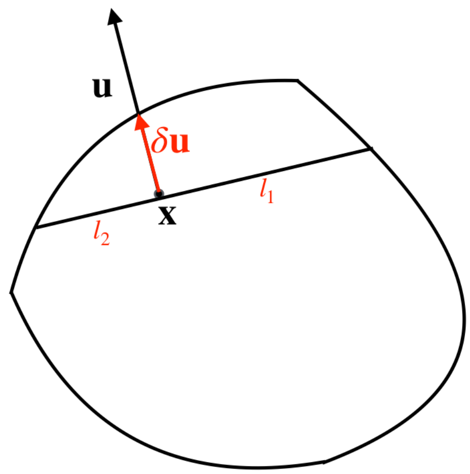

Figure 1.1: Distances l_1 and l_2

$i = 1, 2$, *where* $\boldsymbol{v}$ *is one of the two unit vectors orthogonal to* $\boldsymbol{u}$ *and that for any* n *with* $\delta > \sigma n^{-2}$, $\sigma > 0$, *the following inequality holds:*

$$\lambda_n(\boldsymbol{x}, D) \geq c(\sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}. \tag{1.1.29}$$

In [38], the following result was obtained to show that the behaviour of Christoffel functions does not change for points very close to the boundary (within σn^{-2}).

Proposition 1.15. [38, Proposition 1.4] *If* D *is a starlike body in* $\mathbb{R}^d$, *then for any point* $\boldsymbol{x} \in D$ *we have*

$$\lambda_n(\boldsymbol{x}, D) \approx \lambda_n(\mu \boldsymbol{x}, D), \text{ for all } \mu \in [1 - c(d) n^{-2}, 1],$$

where $c(d) = 2^{-3-\frac{d}{2}}$.

Paper [38] also contains a generalization of Theorems 1.13 and 1.14 to the higher dimensional case.

Let us notice that for lower bounds, we simply restrict the weight given on the body to the ellipsoid and apply Proposition 1.2. For upper bounds, one needs to first extend the weight from the body to some larger domain containing the cube/simplex we will be comparing to. Such an extension which keeps continuity and positivity is well known, see for example [52].

1.2 Overview of the results of the thesis

In this section, we collect the statements of all the results of the thesis and briefly discuss the methods which were used to obtain them. We also discuss some possible generalizations.

In Chapter 2, we estimate the pointwise behaviour of Christoffel functions on planar convex domains in terms of some geometric characteristics of the domain. The main results of this chapter are represented in Section 2.1.

Theorem 2.1. *Suppose $D \subset \mathbb{R}^2$ is a convex compact set with non-empty interior, $\boldsymbol{x} \in \mathrm{Int} D$, $\boldsymbol{u} \in \mathbb{R}^2$ is a unit vector, β and σ are some positive constants. Let $\delta = \max\{q : \boldsymbol{x} + q\boldsymbol{u} \in D\}$ and*

$$l_i(t) := l_i(D, \boldsymbol{x}, t) := \max\{s : \boldsymbol{x} + (\delta - t)\boldsymbol{u} + (-1)^i s\boldsymbol{v} \in D\}, \; i = 1, 2, \; 0 < t < \beta,$$

where $\boldsymbol{v}$ is one of the two unit vectors orthogonal to $\boldsymbol{u}$, such that vectors $(\boldsymbol{u}, 0)$, $(\boldsymbol{v}, 0)$ and $(0, 0, 1)$ form a right-handed system in $\mathbb{R}^3$. If $\sigma n^{-2} < \delta < \beta/2$, then

$$\lambda_n(\boldsymbol{x}, D) \geq c(\beta, \sigma) n^{-2} \sqrt{\delta} \min_{i=1,2} \min_{\delta/2 \leq t \leq \beta} \frac{l_i(t)}{\sqrt{t}}.$$

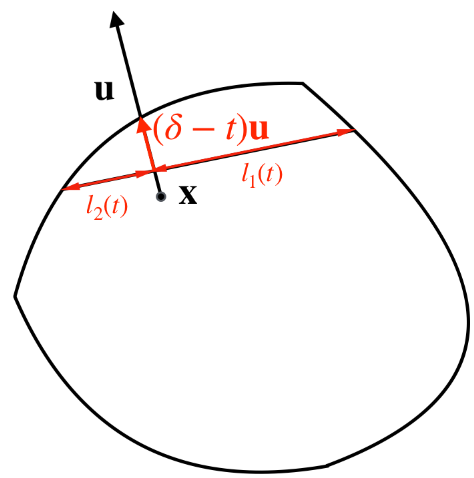

Figure 1.2: Distances $l_1(t)$ and $l_2(t)$

We illustrate definition of $l_i(t)$, $i = 1, 2$, in Figure 1.2.

In Chapter 2 the following class of bodies in $\mathbb{R}^2$ is defined.

Definition 2.2. *We will say that a compact convex body $D \subset \mathbb{R}^d$ belongs to class $\mathfrak{C}$ if there exist positive δ, β, and r satisfying the following conditions.*

- *For any point $\boldsymbol{x} \in \mathrm{Int}D$ there exist $r > 0$ and $s < 0$ such that $rB + \boldsymbol{x} + s\boldsymbol{u} \subset D$, where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) := \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$ and $\boldsymbol{w} \in \partial D$ is such that $\|\boldsymbol{x} - \boldsymbol{w}\|_2 = \min_{\mathbf{b} \in \partial D} \|\boldsymbol{x} - \mathbf{b}\|_2$.*

- *With $l_i(t) = l_i(x, t) := l_i(D, \boldsymbol{x}, \boldsymbol{u}, t)$ defined in Theorem 2.1 we must have*

  $$l_1(t_1, \boldsymbol{x}) \approx c(D) l_2(t_1, \boldsymbol{x}) \text{ and } \frac{l_i(t_1, \boldsymbol{x})}{\sqrt{t_1}} < c(D) \frac{l_i(t_2, \boldsymbol{x})}{\sqrt{t_2}}, \ \delta/2 \le t_1 \le t_2 \le \beta, \ i = 1, 2.$$

Theorem 2.1 in combination with Theorem 1.13 and Proposition 1.15 provides matching lower and upper bounds on Christoffel functions for the class of convex planar domains defined above.

Theorem 2.3. *If the compact convex body $D \subset \mathbb{R}^2$ is from class $\mathfrak{C}$ then for $\delta > \sigma n^{-2}$ we have*

$$\lambda_n(\boldsymbol{x}, D) \approx c(D, \beta, \sigma) n^{-2} l_1(\delta).$$

Let us remark that in this theorem, $l_1(\delta) \approx c(D)(l_1(\delta) + l_2(\delta))$. Hence, the length of the section of D parallel to $\boldsymbol{v}$ through $\boldsymbol{x}$ defined in Theorem 2.1 is responsible for the magnitude of Christoffel function at $\boldsymbol{x}$, provided the main condition of Definition 2.2 is satisfied.

We illustrate how to apply the above result for a specific family of planar convex bodies.

Theorem 2.4. *Let $(x_0, y_0) \in \partial B_\alpha$, $1 < \alpha < 2$ (see (1.1.24)), $0 \le x_0 \le y_0$, $\boldsymbol{u}$ be the outward unit normal at (x_0, y_0). There exists a constant $c_0(\alpha) > 0$ depending only on α such that for*

$$l_i(t) := \max\{s : (x_0, y_0) - t\boldsymbol{u} + (-1)^i s\boldsymbol{v} \in D\},\ i = 1, 2,\ 0 < t < 1,$$

where $\boldsymbol{v}$ is one of the two unit vectors orthogonal to $\boldsymbol{u}$, we have

$$l_i(t) \approx c(\alpha) t^{\frac{1}{2}} (\max\{t, x_0^\alpha\})^{\frac{1}{\alpha} - \frac{1}{2}},\ 0 < t \le c_0(\alpha),\ i = 1, 2.$$

Further, if $\boldsymbol{x} \in \mathrm{Int} B_\alpha$ is such that $\delta := \|\boldsymbol{x} - (x_0, y_0)\|_2 = \min\{\|\boldsymbol{x} - (x, y)\|_2 : (x, y) \in \partial B_\alpha\}$ and $\sigma n^{-2} \le \delta \le 1$, $\sigma > 0$, then

$$\lambda_n(\boldsymbol{x}, B_\alpha) \approx c(\alpha, \sigma) n^{-2} \delta^{\frac{1}{2}} (\max\{\delta, x_0^\alpha\})^{\frac{1}{\alpha} - \frac{1}{2}}.$$

In Theorem 2.4, we obtained the estimate by matching lower and upper bounds on Christoffel functions for the domain B_α. The proof provides techniques showing how the required geometric quantities l_i can be computed (up to a constant factor).

Proposition 1.15 shows that the behaviour of the Christoffel function does not change if we are within a constant factor times n^{-2} of the boundary. Therefore, we formulate all results from Chapter 2 in terms of distance to the boundary under the condition that this distance is at least the order of n^{-2}. Alternatively, one could drop this condition, but then the results would have to include the quantities like $\rho_n^*(x) = \frac{1}{n^2} + \frac{1}{n}\sqrt{x},\ x \in \mathbb{R},\ x \geq 1$ as will be done in Chapters 3 and 4.

In Chapter 3, we consider Christoffel functions on domains in $\mathbb{R}^2$ with boundary consisting of finitely many C^2 curves such that each corner point of the boundary has interior angle strictly between 0 and π. The details can be found in [39].

Our main result in Chapter 3 is the following theorem, see pp. 47-48 for definitions.

Theorem 3.1. *Let $D \subset \mathbb{R}^2$ be a domain with piecewise C^2 boundary with pieces of the boundary Γ_i, $i = 1, \ldots, m$, and corner points $\boldsymbol{v}_j$ with interior angles α_j, $0 < \alpha_j < \pi$, and related corner boundary curves $\Gamma_j^\pm$ as defined above, $j = 1, \ldots, k$. For any point $\boldsymbol{x} \in D$*

$$\lambda_n(\boldsymbol{x}, D) \approx c(D) \min\Big(\min_{1\leq i\leq m} n^{-1}\rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_i)), \min_{1\leq j\leq k} \rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_j^-))\rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_j^+))\Big), \tag{1.2.1}$$

where $c(D) > 0$ is a constant depending only on D,

$$\rho_n^*(x) = \frac{1}{n^2} + \frac{1}{n}\sqrt{x}, \ x \in \mathbb{R}, \ x \geq 1. \tag{1.2.2}$$

When $\boldsymbol{x}$ is not near a corner point (the distances from point $\boldsymbol{x}$ to all corner points are greater that some $\delta > 0$), the arguments of this theorem work without the requirement $0 < \alpha_j < \pi$, see Remark 3.11. Our proof of the above theorem is based on a comparison with ellipses of appropriate size and on an use of the extremal property (1.1.6).

In Chapter 4 we introduce the results in higher dimensional spaces $\mathbb{R}^d$, $d \geq 3$ for Christoffel function over convex polytopes. In Section 4.1 well-known facts about Christoffel functions and polytopes in higher dimensional spaces are presented.

We define a convex polytope in $\mathbb{R}^d$ following [20]. We say that a halfspace is a set (in $\mathbb{R}^d$) of the form

$$\{\boldsymbol{x} : \ \mathbf{a} \cdot \boldsymbol{x} \leq b\}$$

for vector $\mathbf{a}$, and a real number b.

The following theorem for d-dimensional simple polytope P is proved in Section 4.2.

Theorem 4.7. *For any d-dimensional simple polytope* P *we denote facets by F_i and corresponding $(d-1)$-dimensional planes which contain the facets by H_i, $F_i \subset H_i$, $i = 1, \ldots, k$. For any point $\boldsymbol{x} \in$* P *we denote by $\{d_i^*(\boldsymbol{x})\}_{i=1,\ldots,k}$ the non-decreasing* *rearrangement of* $\{d_i(\boldsymbol{x})\}_{i=1,\dots,k}$*, where* $d_i(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, H_i)$*. Then*

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \approx c(\mathrm{P}) \prod_{i=1}^{d} \rho_n^*(d_i^*).$$

Note that $\operatorname{dist}(\boldsymbol{x}, H_i)$ may be different from $\operatorname{dist}(\boldsymbol{x}, F_i)$ as the foot of the perpendicular from $\boldsymbol{x}$ onto H_i may not belong to F_i.

In Section 4.3 a similar statement to Theorem 4.7 for an arbitrary convex polytope in $\mathbb{R}^3$ is obtained.

Theorem 4.13. *For any* 3*-dimensional convex polytope* P *we denote by* $\{d_i^*(\boldsymbol{x})\}_{i=1,\dots,l}$ *the non-decreasing rearrangement of the distances* $\{d_i(\boldsymbol{x})\}_{i=1,\dots,l}$*, where* $\boldsymbol{x} \in \mathrm{P}$*,* $d_i(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, H_i)$*,* H_i *is the plane containing the facet* F_i*,* $i = 1, \dots, l$*. Then*

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \approx c(\mathrm{P}) \rho_n^*(d_1^*(\boldsymbol{x})) \rho_n^*(d_2^*(\boldsymbol{x})) \rho_n^*(d_3^*(\boldsymbol{x})).$$

In Section 4.4 we give an example to demonstrate that certain properties of polytopes we used for $d = 3$ do not hold for arbitrary polytopes in $\mathbb{R}^d$, $d \geq 4$. This means that the method used to prove Theorem 4.13 cannot be generalized directly for general polytopes in $\mathbb{R}^d$, $d \geq 4$.

Let us discuss methods used in the proofs of the theorems from Chapters 2 - 4. In all these chapters we use the introduced comparison techniques, Propositions 1.2 and 1.3.

In Chapter 2, we consider ellipses and balls in $\mathbb{R}^2$ inscribed in the considered planar convex domain D to obtain the lower bounds for Christoffel functions. To get estimates of a parallel section function of B_α, we compare the function describing the upper half of ∂B_α with lines and parabolas passing through a certain point on ∂B_α.

In Chapter 3, we use the comparison technique with appropriate reference domains and use of the extremal property (1.1.6). For the lower bound, we use "grain"-type domains which are intersections of two discs of the same radius. The estimate for such reference domains relies on a Videsnkii-type inequality. For the upper bound, we compare with domains which are the intersections of two annuli and construct the required polynomials P with small L_2 norm and $P(\boldsymbol{x}) = 1$ at a fixed point $\boldsymbol{x}$.

In Chapter 4, we construct affine transforms of paralleletopes of the corresponding dimension inscribed and circumscribed in and about the d-dimensional polytope P to get the equivalence for Christoffel functions. In the case of 3-dimensional polytope P, we work with a cross-section of P which is a polygon.

As we noticed before, once asymptotics of Christoffel functions are established for a weight, the asymptotics can be found for a family of other weights. Using the universality of Christoffel functions obtained by Kroo and Lubinsky in [28] it is possible to find, under some additional conditions, the asymptotics for a family of other weights in our situations. Namely, asymptotics for Christoffel functions over domains considered in Chapters 2 and 4 can be achieved using the universality type results from [28] in combination with Theorems 1.7 and 1.8 and the introduced comparison technique. This is possible because the methods of Chapters 2 and 4 rely on comparisons with balls and cubes only. The result of Chapter 3 is more involved and cannot be directly generalized like this, to other weights.

For w a weight function on $(-1,1)$, a generalized Christoffel function (see [31] and [33]) can be defined by

$$\Lambda_n(x,w,q) = \min_{P\in\mathcal{P}_n^1} \int_{-1}^{1} \left|\frac{P(t)}{P(x)}\right|^q w(t)dt,$$

where $n = 1,2,\ldots, q \in (0,\infty)$, $|x| \le 1$. When $q = 2$ we get the usual Christoffel function. It is natural to expect that the result on the usual Christoffel function can be carried over to the generalized Christoffel function. In particular, one can generalize the key tools that we used in the thesis, namely, the comparison technique and the affine transform techniques, see Propositions 1.2 and 1.3. This would lead to an extension of the results of Chapter 2 (which would also require to extend the results of [38]) and Chapter 4. Material of Chapter 3 would require some additional arguments apart from the two key techniques mentioned above. So overall, treating the generalized Christoffel function would introduce numerous additional technicalities which contribute little towards the main problem of understanding the behaviour of Christoffel function in terms of geometry of the domain. Therefore, we opted not to consider the generalized Christoffel function in the thesis.

Finally, let us briefly mention two examples of possible applications of the results of the thesis.

The rate of growth of $\inf_{\boldsymbol{x}\in D}(\lambda_n(\boldsymbol{x},D))^{-1}$ as $n\to\infty$ was shown in [14] to be determining for Nikol'skii type inequalities on D. Namely, when $\lambda_n(\boldsymbol{x},D) \approx n^\sigma$ with

$\boldsymbol{x} \in D$, $D \subset \mathbb{R}^d$ and $0 < p \leq q \leq \infty$, one has

$$\|P\|_{L_q(D)} \leq cn^{\sigma\left(\frac{1}{p}-\frac{1}{q}\right)}\|P\|_{L_p(D)}.$$

As per [10], $\inf_{\boldsymbol{x}\in D}(\lambda_n(\boldsymbol{x}, D))^{-1}$ estimate to be crucial for stability and accuracy of discrete least squares approximation. In [11] the weighted analog of $\lambda_n(\boldsymbol{x}, D)$ was considered.

Chapter 2

Pointwise behaviour of Christoffel functions on planar convex domains

2.1 Introduction and results

In this section, we prove a general lower bound on Christoffel functions on planar convex domains in terms of a modification of the parallel section function of the domain. For a certain class of planar convex domains, in combination with a recent general upper bound obtained in [38], this allows to compute the pointwise behavior of Christoffel functions for a class of planar convex domains. We illustrate this approach for the domains $\{(x, y) : |x|^\alpha + |y|^\alpha \leq 1\}$, $1 < \alpha < 2$, and compute up to a constant factor the required modification of the parallel section function, and, consequently, Christoffel functions at an arbitrary interior point of the domain.

Our main result is the following theorem.

Theorem 2.1. *Suppose $D \subset \mathbb{R}^2$ is a convex compact set with non-empty interior, $\boldsymbol{x} \in \mathrm{Int}D$, $\boldsymbol{u} \in \mathbb{R}^2$ is a unit vector, β and σ are some positive constants. Let $\delta = \max\{q : \boldsymbol{x} + q\boldsymbol{u} \in D\}$ and*

$$l_i(t) := l_i(D, \boldsymbol{x}, t) := \max\{s : \boldsymbol{x} + (\delta - t)\boldsymbol{u} + (-1)^i s\boldsymbol{v} \in D\}, \; i = 1, 2, \; 0 < t < \beta, \tag{2.1.1}$$

where $\boldsymbol{v}$ is one of the two unit vectors orthogonal to $\boldsymbol{u}$, such that vectors $(\boldsymbol{u}, 0)$, $(\boldsymbol{v}, 0)$ and $(0, 0, 1)$ form a right-handed system in $\mathbb{R}^3$. If $\sigma n^{-2} < \delta < \beta/2$, then

$$\lambda_n(\boldsymbol{x}, D) \geq c(\beta, \sigma) n^{-2} \sqrt{\delta} \min_{i=1,2} \min_{\delta/2 \leq t \leq \beta} \frac{l_i(t)}{\sqrt{t}}. \tag{2.1.2}$$

Remark that for the points very close to the boundary (within σn^{-2}), the problem can be reduced to the case when $\delta > \sigma n^{-2}$ using Proposition 1.14 ([38, Theorem 1.4]).

We can think of $l_i(t)$ as a modified parallel section function of D, and note that $l_1(t) + l_2(t)$ is the total length of the section of D parallel to $\boldsymbol{v}$ through the point $\boldsymbol{x} + (\delta - t)\boldsymbol{u}$. We can interpret the quantity on the right hand side of (2.1.2) as the size of an ellipse that can be inscribed into D so that $\boldsymbol{x}$ is sufficiently inside the ellipse, see Figure 2.1.

For the upper bound, by Theorem 1.13 ([38, Theorem 1.1]), we have

$$\lambda_n(\boldsymbol{x}, D) \leq c(D, \sigma) n^{-2} \sqrt{\min\{l_1(\delta) l_2(\delta), \delta\}}. \tag{2.1.3}$$

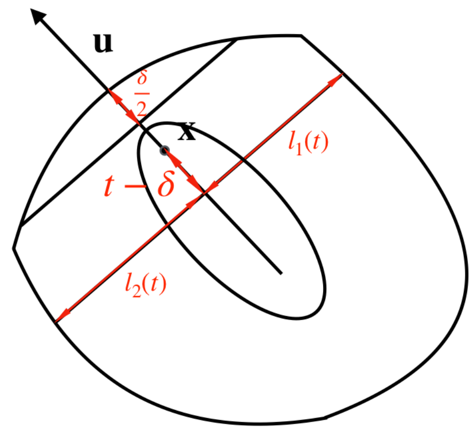

Figure 2.1: Inscribed ellipse

(One can refer to [38] for specific geometric measurements of D that affect the constant $c(D, \sigma)$ and are omitted here for simplicity.)

Let us introduce the following class of bodies in $\mathbb{R}^2$.

Definition 2.2. *We will say that a compact convex body $D \subset \mathbb{R}^d$ belongs to class $\mathfrak{C}$ if there exist positive δ, β, and r satisfying the following conditions.*

- *For any point $\boldsymbol{x} \in \mathrm{Int}D$ and there exists $r > 0$ and $s < 0$ such that $rB + \boldsymbol{x} + s\boldsymbol{u} \subset D$, where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) := \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$ and $\boldsymbol{w} \in \partial D$ is such that $\|\boldsymbol{x} - \boldsymbol{w}\|_2 = \min_{\mathbf{b} \in \partial D} \|\boldsymbol{x} - \mathbf{b}\|_2$.*

- *With $l_i(t) = l_i(x, t) := l_i(D, \boldsymbol{x}, \boldsymbol{u}, t)$ defined in Theorem 2.1 we must have*

$$l_1(t_1, \boldsymbol{x}) \approx c(D) l_2(t_1, \boldsymbol{x}) \text{ and } \frac{l_i(t_1, \boldsymbol{x})}{\sqrt{t_1}} < c(D) \frac{l_i(t_2, \boldsymbol{x})}{\sqrt{t_2}}, \ \delta/2 \leq t_1 \leq t_2 \leq \beta, \ i = 1, 2. \tag{2.1.4}$$

Let us notice that every C^2-smooth body (has C^2-smooth boundary) belongs to the class $\mathfrak{C}$. Our domains of interest are ones that are not C^2-smooth but belongs to the class $\mathfrak{C}$. Example of such domain is B_α, $1 < \alpha < 2$, which is C^1-smooth. In some way, $\mathfrak{C}$ is somewhere between the two classes of C^1-smooth and C^2-smooth domains. It is not hard to see that there are convex bodies that do not belong to $\mathfrak{C}$, for example if the boundary of the body is not smooth, like for a half-disc. We will address the problem of estimating the Christoffel function for such bodies in subsequent sections in Chapter 3.

As $l_1(\delta) \approx c(D)(l_1(\delta) + l_2(\delta))$, we see that the length of the parallel section function of D parallel to $\boldsymbol{v}$ through $\boldsymbol{x}$ is responsible for the magnitude of Christoffel function at $\boldsymbol{x}$, provided (2.1.4) is satisfied. We remark that a natural choice for $\boldsymbol{u}$ would be the direction in which the distance from $\boldsymbol{x}$ to ∂D is attained, as it was defined in Definition 2.2, although other choices are possible depending on specific situation. We believe that the family of convex bodies satisfying (2.1.4) for some choice of $\boldsymbol{u}$ is rather wide. For this family, the combination of Theorem 2.1 and Theorem 1.13 ([38, Theorem 1.1]) provides geometric characterization of the behaviour of Christoffel functions at any point of the domain.

Let us formulate the result for a compact convex body $D \subset \mathbb{R}^2$ in notations of Theorem 2.1 and Definition 2.2.

Theorem 2.3. *If the compact convex body $D \subset \mathbb{R}^2$ is from class $\mathfrak{C}$ then for $\delta > \sigma n^{-2}$*

we have

$$\lambda_n(\boldsymbol{x}, D) \approx c(D, \beta, \sigma) n^{-2} l_1(\delta). \tag{2.1.5}$$

Proof. If $\delta \geq \frac{\beta}{2}$, we apply (1.1.14) with $n = 2$ to obtain (2.1.5).

If $\delta < \frac{\beta}{2}$, by Theorem 2.1 we have

$$\lambda_n(\boldsymbol{x}, D) \geq c(D, \beta, \sigma) n^{-2} \sqrt{\delta} \min_{i=1,2} \min_{\delta/2 \leq t \leq \beta} \frac{l_i(t)}{\sqrt{t}}$$

$$\geq c(D, \beta, \sigma) n^{-2} \sqrt{\delta} \frac{l_1(\frac{\delta}{2})}{\sqrt{\frac{\delta}{2}}} > c(D, \beta, \sigma) n^{-2} l_1(\delta),$$

because $l_1(\frac{\delta}{2}) \approx c(D)(l_1(\frac{\delta}{2}) + l_2(\frac{\delta}{2})) \geq \frac{1}{2}(l_1(\delta) + l_2(\delta))$. On the other hand, by (2.1.3) we have

$$\lambda_n(\boldsymbol{x}, D) \leq c(D, \sigma) n^{-2} \sqrt{\min\{l_1(\delta) l_2(\delta), \delta\}}$$

$$= c(D, \sigma) n^{-2} \sqrt{\min\{l_1^2(\delta), \delta\}} \leq c(D, \sigma) n^{-2} l_1(\delta).$$

Hence, for a body $D \in \mathfrak{C}$, these lower and upper boundaries coincide with $c(D, \beta, \sigma) n^{-2} l_1(\delta)$.

$\square$

Let us note that for the upper estimate (2.1.3), significantly fewer geometric measurements are needed (only δ, $l_1(\delta)$ and $l_2(\delta)$) compared with (2.1.2) which requires the knowledge of $l_i(t)$ for $\delta_2 \leq t \leq \beta$.

We will illustrate our main result for the domains $B_\alpha := \{(x, y) : |x|^\alpha + |y|^\alpha \leq 1\}$, $1 < \alpha < 2$. In particular, we show that these domains belong to the class $\mathfrak{C}$. To this end, for each interior point $\boldsymbol{x}$ within a constant distance from the boundary of the domain, we compute $l_i(D, \boldsymbol{x}, t)$ (see (2.1.1)) explicitly up to a constant factor in terms of t and (x_0, y_0), a nearest point from the boundary to $\boldsymbol{x}$, i.e., (x_0, y_0) is such that $\|\boldsymbol{x} - (x_0, y_0)\|_2 = \min\{\|\boldsymbol{x} - (x, y)\|_2 : (x, y) \in \partial B_\alpha\}$. We note that, generally speaking, to find l_i one needs to solve a non-linear equation. We hope that the techniques developed below to estimate l_i for B_α, which mostly result in equations of degree at most 2, may prove useful for other planar convex bodies.

Theorem 2.4. *Let $(x_0, y_0) \in \partial B_\alpha$, $1 < \alpha < 2$, $0 \le x_0 \le y_0$, $\boldsymbol{u}$ be the outward unit normal at (x_0, y_0). There exists a constant $c_0(\alpha) > 0$ depending only on α such that for*

$$l_i(t) := \max\{s : (x_0, y_0) - t\boldsymbol{u} + (-1)^i s\boldsymbol{v} \in D\}, \ i = 1, 2, \ 0 < t < 1, \tag{2.1.6}$$

where $\boldsymbol{v}$ is one of the two unit vectors orthogonal to $\boldsymbol{u}$, we have

$$l_i(t) \approx c(\alpha) t^{\frac{1}{2}} (\max\{t, x_0^\alpha\})^{\frac{1}{\alpha} - \frac{1}{2}}, \ 0 < t \le c_0(\alpha), \ i = 1, 2. \tag{2.1.7}$$

Further, if $\boldsymbol{x} \in \mathrm{int} B_\alpha$ is such that $\delta := \|\boldsymbol{x} - (x_0, y_0)\|_2 = \min\{\|\boldsymbol{x} - (x, y)\|_2 : (x, y) \in \partial B_\alpha\}$ and $\sigma n^{-2} \le \delta \le 1$, $\sigma > 0$, then

$$\lambda_n(\boldsymbol{x}, B_\alpha) \approx c(\alpha, \sigma) n^{-2} \delta^{\frac{1}{2}} (\max\{\delta, x_0^\alpha\})^{\frac{1}{\alpha} - \frac{1}{2}}. \tag{2.1.8}$$

The behavior of $\lambda_n(\boldsymbol{x}, B_\alpha)$ on $x = 0$ and $y = 0$ which contain the "least smooth" points $(0, \pm 1)$ and $(\pm 1, 0)$ of ∂B_α has been studied in [24] and [38] and was essentially shown to be $n^{-2}\delta^{\frac{1}{\alpha}}$. In contrast, along $x = \pm y$, where the boundary is C^2 smooth, the behavior is $n^{-2}\delta^{\frac{1}{2}}$, see [38, Proposition 3.3]. Theorem 2.4 fills this gap by computing Christoffel functions everywhere inside B_α and specifies how exactly the transition between different smoothness affects the behaviour of Christoffel functions. Also, Theorem 2.4 gives an affirmative answer to [38, Conjecture 3.4], moreover, the theorem provides the right hand side of [38, (3.1)] up to a constant factor.

Due to [38, Proposition 2.4 and (2.3)], we have

$$\lambda_n(\boldsymbol{x}, B^d) \approx n^{-1}\rho_n^*(1 - \|\boldsymbol{x}\|_2), \quad \boldsymbol{x} \in B^d. \tag{2.1.9}$$

For the unit ball B in $\mathbb{R}^2$, $\sigma > 0$ we have

$$\lambda_n((x,y), B) \approx c(\sigma)n^{-2}\sqrt{1 - x^2 - y^2},\ (x,y) \in (1 - \sigma n^{-2})B. \tag{2.1.10}$$

2.2 Proofs

Proof of Theorem 2.1. Denote

$$\Lambda := \sqrt{\frac{\beta}{6}} \min_{i=1,2} \min_{\delta/2 \le t \le \beta} \frac{l_i(t)}{\sqrt{t}}.$$

Consider the ellipse

$$\mathbf{E} := \left\{ \boldsymbol{x} - t\boldsymbol{u} + s\boldsymbol{v} : \left(\frac{\frac{\beta}{3} + \frac{\delta}{2} - t}{\frac{\beta}{3}} \right)^2 + \left(\frac{s}{\Lambda} \right)^2 \le 1 \right\}.$$

If t and s satisfy the inequality from the definition of $\mathbf{E}$, then $\frac{\delta}{2} \le t < \frac{11}{12}\beta$ and

$$\begin{aligned} |s| &\le \Lambda \sqrt{1 - \left(\frac{\frac{\beta}{3} + \frac{\delta}{2} - t}{\frac{\beta}{3}} \right)^2} = \frac{3\Lambda}{\beta} \sqrt{2\frac{\beta}{3}\left(t - \frac{\delta}{2}\right) - \left(t - \frac{\delta}{2}\right)^2} \\ &\le \frac{3\Lambda}{\beta} \sqrt{2\frac{\beta}{3}\left(t - \frac{\delta}{2}\right)} \le \Lambda \sqrt{\frac{6}{\beta}} \sqrt{t} \le \min_{i=1,2} l_i(t), \end{aligned}$$

so $\mathbf{E} \subset D$. Note that for an affine transform T such that $T\mathbf{E} = B$ we have $\det T = \frac{3}{\Lambda\beta}$. Now by (1.1.7), (1.1.8) and (2.1.10),

$$\lambda_n(\boldsymbol{x}, D) \geq \lambda_n(\boldsymbol{x}, \mathbf{E}) = \frac{\Lambda\beta}{3}\lambda_n(T\boldsymbol{x}, B) \approx \Lambda\beta c(\sigma) n^{-2}\sqrt{\delta},$$

implying (2.1.2). □

Proof of Theorem 2.4. In addition to already set notations regarding constants, throughout this proof we emphasize that all the constants below do not depend on x_0 or t. We can assume that $v_x > 0$ in $\boldsymbol{v} = (v_x, v_y)$ from (2.1.6).

First we will show how (2.1.7) implies (2.1.8). Assuming (2.1.7), we can apply Theorem 2.1 with $\beta = c_0(\alpha)$ and obtain the lower bound in (2.1.8) if $\delta \leq c_0(\alpha)/2$. If $\delta > c_0(\alpha)/2$, we note that $\delta B + \boldsymbol{x} \subset B_\alpha$, so by (1.1.7), (1.1.8) and (2.1.10),

$$\lambda_n(\boldsymbol{x}, B_\alpha) \geq \lambda_n(\boldsymbol{x}, \delta B + \boldsymbol{x}) = \delta^2 \lambda_n((0,0), B) \geq c(\alpha) n^{-2},$$

which proves the lower bound in (2.1.8). The upper bound in (2.1.8) readily follows from (2.1.3).

It remains to prove (2.1.7). We remark that one can establish (2.1.7) for a wider range of t, e.g. for $0 < t < \frac{5}{4}$, but this requires some additional technicalities and is not needed for (2.1.8), which was our main goal.

We will select $c_0(\alpha)$ in the end of the proof. Now fix t with $0 < t \leq c_0(\alpha)$ and set $(x_1, y_1) = (x_0, y_0) - t\boldsymbol{u}$. We assume that $x_0 > 0$, the case $x_0 = 0$ will be considered later.

Suppose $y = l(x)$ is the equation of the line $\{(x_0, y_0) - t\boldsymbol{u} + (-1)^i s\boldsymbol{v} : s \in \mathbb{R}\}$. Then

$$l(x) = f(x_0) + f'(x_0)(x - x_0) - t\sqrt{1 + (f'(x_0))^2},$$

where $f(x) = (1 - |x|^\alpha)^{\frac{1}{\alpha}}$ describes the upper half of ∂B_α. We have $x_1 = x_0 + \frac{tf'(x_0)}{\sqrt{1+(f'(x_0))^2}}$. Since $y_0 = f(x_0) \geq x_0$, we obtain $x_0 \leq 2^{-\frac{1}{\alpha}} \leq y_0$. For $0 < x < 1$, we have $f'(x) = -x^{\alpha-1}(1-x^\alpha)^{\frac{1}{\alpha}-1}$, and for $0 < x < 2^{-\frac{1}{\alpha}}$, we get $1 < (1-x^\alpha)^{\frac{1}{\alpha}-1} < \sqrt{2}$. So,

$$x_0^{\alpha-1} < -f'(x_0) < \sqrt{2}x_0^{\alpha-1} \text{ and } 1 < \sqrt{1 + (f'(x_0))^2} < \sqrt{3}. \tag{2.2.1}$$

Now we can verify that $l(\pm 1) > 0 = f(\pm 1)$ provided $t < \frac{2^{-\frac{1}{\alpha}} - 2^{-\frac{1}{2}}}{\sqrt{3}}$, so we will require that $c_0(\alpha) < \frac{2^{-\frac{1}{\alpha}} - 2^{-\frac{1}{2}}}{\sqrt{3}}$. Hence, letting $x_2 < x_3$ be the x-coordinates of the points of intersection of the line $y = l(x)$ with ∂B_α, we obtain that

$$l(x_j) = f(x_j) \text{ and } l_{j-1}(t) = \sqrt{1 + (f'(x_0))^2}(-1)^{j-1}(x_j - x_1), j = 2, 3.$$

Therefore, due to (2.2.1), we need to show

$$|x_j - x_1| \approx c(\alpha)t^{\frac{1}{2}}(\max\{t, x_0^\alpha\})^{\frac{1}{\alpha}-\frac{1}{2}},\ j = 2, 3.$$

We note that (2.2.1) implies

$$0 \leq x_0 - x_1 \leq \sqrt{2}tx_0^{\alpha-1}. \tag{2.2.2}$$

We define tangent parabolas to $y = f(x)$ at $x = x_0$ with varying quadratic term as follows:

$$P(m, x) := f(x_0) + f'(x_0)(x - x_0) + \frac{m}{2}(x - x_0)^2.$$

Note that for $m < 0$ the equation $l(x) = P(m, x)$ has two solutions

$$x = x_0 \pm \sqrt{\frac{2t\sqrt{1 + (f'(x_0))^2}}{-m}}. \tag{2.2.3}$$

Further, for any interval $[a, b] \subset [0, 1]$ containing x_0, we have

$$P(\min\{f''(t) : t \in [a, b]\}, x) \leq f(x) \leq P(\max\{f''(t) : t \in [a, b]\}, x), \;\; x \in [a, b]. \tag{2.2.4}$$

It is straightforward to compute that

$$f'''(x) = -(\alpha - 1)x^{\alpha-3}(1 - x^\alpha)^{\frac{1}{\alpha}-3}((\alpha - 2) + (\alpha + 1)x^\alpha),$$

so $f'''(x) > 0$ for $x \in (0, c_1(\alpha))$, where $c_1(\alpha) := (\frac{2-\alpha}{1+\alpha})^{\frac{1}{\alpha}}$.

Now we show that

$$\text{if } t \leq c_2(\alpha)x_0^\alpha \text{ and } x_0 \leq c_1(\alpha)/2, \text{ then } |x_j - x_1| \approx c(\alpha)t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}}, \;\; j = 2, 3, \tag{2.2.5}$$

where $c_2(\alpha)$ will be selected later. By (2.2.4),

$$P(f''(x_0/2), x) \leq f(x) \leq P(f''(2x_0), x), \;\; x \in [x_0/2, 2x_0]. \tag{2.2.6}$$

Let $z_1 < z_2$ and $z_3 < z_4$ be the solutions of the quadratic equations $l(x) = P(f''(x_0/2), x)$ and $l(x) = P(f''(2x_0), x)$, respectively. Since $-f''(2^{\pm1}x_0) \approx c(\alpha)x_0^{\alpha-2}$, by (2.2.3) we see that $c_3(\alpha)t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}} \leq |z_j - x_0| \leq c_4(\alpha)t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}}$, $j = 1, 2, 3, 4$, for some positive constants $c_3(\alpha)$ and $c_4(\alpha)$ independent of the forthcoming choice of $c_2(\alpha)$. As $t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}} \leq \sqrt{c_2(\alpha)}x_0$, if we impose that $c_2(\alpha) < (2c_4(\alpha))^{-2}$, then $x_0/2 < z_3$ and $z_4 < 2x_0$. Now (2.2.6) implies that $z_3 < x_2 < z_1$ and $z_2 < x_3 < z_4$, so

$c_3(\alpha)t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}} \le |x_j - x_0| \le c_4(\alpha)t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}}$, $j = 2, 3$. If $c_2(\alpha) < (c_3(\alpha))^2/8$, then since $tx_0^{\alpha-1} \le \sqrt{c_2(\alpha)}t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}}$ we can use (2.2.2) to see that $0 \le x_0 - x_1 \le \frac{c_3(\alpha)}{2}t^{\frac{1}{2}}x_0^{1-\frac{\alpha}{2}}$ and conclude that (2.2.5) holds provided $c_2(\alpha)$ is sufficiently small. Namely, we choose arbitrary $c_2(\alpha) > 0$ satisfying $c_2(\alpha) < \min\{(2c_4(\alpha))^{-2}, (c_3(\alpha))^2/8\}$.

Next we claim that

$$\text{if } x_0 \in [c_1(\alpha)/2, 2^{-\frac{1}{\alpha}}], \text{ then } |x_j - x_1| \approx c(\alpha)t^{\frac{1}{2}}, \quad j = 2, 3. \tag{2.2.7}$$

The proof is similar to that of (2.2.5) with certain differences as we will now outline. The interval $[x_0/2, 2x_0]$ is replaced with $[c_1(\alpha)/3, 1/2]$ and then we use that $c_5(\alpha) \le -f''(x) \le c_6(\alpha)$ for $x \in [c_1(\alpha)/3, 1/2]$ and some positive constants $c_5(\alpha)$ and $c_6(\alpha)$, so that

$$P(-c_6(\alpha), x) \le f(x) \le P(-c_5(\alpha), x), \quad x \in [c_1(\alpha)/3, 1/2].$$

Further, instead of requiring that $c_2(\alpha)$ is sufficiently small as was done for (2.2.5), we will require $c_0(\alpha)$ (and, hence, t) not to exceed a specific constant depending on α only, chosen to ensure that the analogs of z_3 and z_4 belong to $[c_1(\alpha)/3, 1/2]$, and that $x_1 - x_0$ does not exceed a sufficiently small constant times $t^{\frac{1}{2}}$. We omit the details.

The proofs of the remaining estimates are different from the proofs of (2.2.5) and (2.2.7) as we will mostly compare f with lines rather than with parabolas. Define $\tilde{x} > 0$ to be the point where $f(\tilde{x}) = l(x_0)$. It is straightforward that

$$\frac{x^\alpha}{\alpha} \le 1 - f(x) \le 2^{1-\frac{1}{\alpha}}\frac{x^\alpha}{\alpha}, \quad 0 < x < 2^{-\frac{1}{\alpha}}. \tag{2.2.8}$$

This implies $1 - f(x_0) \approx c(\alpha)x_0^\alpha$. By (2.2.1), we have $f(x_0) - l(x_0) \approx t$. So, if $t \geq c_2(\alpha)x_0^\alpha$, then $1 - l(x_0) \approx c(\alpha)t$ and due to (2.2.8) we obtain the following:

$$\text{if } t \geq c_2(\alpha)x_0^\alpha, \text{ then } \tilde{x} \approx c(\alpha)t^{\frac{1}{\alpha}}. \tag{2.2.9}$$

Next we establish that

$$\text{if } t \geq c_2(\alpha)x_0^\alpha \text{ and } x_0 \leq c_1(\alpha)/2, \text{ then } x_3 - x_1 \approx c(\alpha)t^{\frac{1}{\alpha}}. \tag{2.2.10}$$

Let f^{-1} be the inverse of f on $[0,1]$. Clearly, f^{-1} is concave. Therefore, $f^{-1}(y) - f^{-1}(y+h)$ is decreasing in y for fixed $h > 0$. Applying this with $h = f(x_0) - l(x_0)$, we see that $\tilde{x} - x_0 = f^{-1}(l(x_0)) - f^{-1}(f(x_0)) > f^{-1}(1-h) - f^{-1}(1) \approx c(\alpha)h^{\frac{1}{\alpha}} \approx c(\alpha)t^{\frac{1}{\alpha}}$. Since $x_3 > \tilde{x}$, by (2.2.2) we have $x_3 - x_1 > \tilde{x} - x_0$, which yields the lower bound on $x_3 - x_1$ in (2.2.10).

To estimate x_3 from above, we consider $L(x)$, the tangent line to f at $\tilde{x}$, which, by concavity of f, satisfies $f(x) \leq L(x)$, $x \in [0,1]$, and has the slope smaller than the slope of l. Therefore, letting $\bar{x}$ be the point of intersection of l and L, we have the bound $x_3 < \bar{x}$ and compute that

$$\bar{x} = \frac{f(\tilde{x}) - f(x_0) + t\sqrt{1 + (f'(x_0))^2} + x_0 f'(x_0) - \tilde{x}f'(\tilde{x})}{f'(x_0) - f'(\tilde{x})}. \tag{2.2.11}$$

Due to (2.2.9) and $\tilde{x} - x_0 \approx c(\alpha)t^{\frac{1}{\alpha}}$, we estimate $f'(x_0) - f'(\tilde{x}) = (x_0 - \tilde{x})f''(\xi) \geq c(\alpha)t^{\frac{1}{\alpha}}\tilde{x}^{\alpha-2} \geq c(\alpha)t^{1-\frac{1}{\alpha}}$, where $\xi \in (x_0, \tilde{x})$. Using (2.2.9) and $x_0 \leq c(\alpha)t^{\frac{1}{\alpha}}$, it is rather straightforward to show that the numerator of (2.2.11) does not exceed $c(\alpha)t$ leading to $\bar{x} \leq c(\alpha)t^{\frac{1}{\alpha}}$. Due to $x_0 \leq c(\alpha)t^{\frac{1}{\alpha}}$ and (2.2.2), we have $-x_1 < x_0 - x_1 \leq$

$c(\alpha)t^{2-\frac{1}{\alpha}} \leq c(\alpha)t^{\frac{1}{\alpha}}$, so, in summary, $x_3 - x_1 \leq \bar{x} - x_1 \leq c(\alpha)t^{\frac{1}{\alpha}}$, which is the upper bound on $x_3 - x_1$ in (2.2.10).

Now we prove that

$$\text{if } t \geq c_2(\alpha)x_0^{\alpha} \text{ and } x_0 \leq c_1(\alpha)/2, \text{ then } x_1 - x_2 \approx c(\alpha)t^{\frac{1}{\alpha}}. \tag{2.2.12}$$

Since f is even and l is decreasing, we have $x_2 > -\tilde{x}$ (recall that $\tilde{x} > 0$ is such that $f(\tilde{x}) = l(x_0) < l(x_1)$), so taking (2.2.2) and (2.2.9) into account, we establish the upper bound on $x_1 - x_2$ in (2.2.12) as follows:

$$x_1 - x_2 \leq x_1 + \tilde{x} \leq x_0 + \tilde{x} \leq c(\alpha)t^{\frac{1}{\alpha}}.$$

Since f is concave, we have $\{x : l(x) \leq f(x)\} = [x_2, x_3]$. Therefore, to prove the lower bound on $x_1 - x_2$ in (2.2.12), it is enough to show that there exists sufficiently small $c_7(\alpha) > 0$ such that $l(x_1 - c_7(\alpha)t^{\frac{1}{\alpha}}) < f(x_1 - c_7(\alpha)t^{\frac{1}{\alpha}})$, which would imply $x_1 - x_2 \geq c_7(\alpha)t^{\frac{1}{\alpha}}$. If $c_7(\alpha)$ satisfies $\sqrt{2}c_7(\alpha)c_2(\alpha)^{\frac{1}{\alpha}-1} < \frac{1}{\sqrt{3}} - \frac{1}{2}$, then by (2.2.1) we get

$$\begin{aligned} l(x_1 - c_7(\alpha)t^{\frac{1}{\alpha}}) &= f'(x_0)(-c_7(\alpha)t^{\frac{1}{\alpha}}) + f(x_0) - \frac{t}{\sqrt{1 + (f'(x_0))^2}} \\ &< \sqrt{2}c_7(\alpha)x_0^{\alpha-1}t^{\frac{1}{\alpha}} + f(x_0) - \frac{t}{\sqrt{3}} \\ &< \sqrt{2}c_7(\alpha)c_2(\alpha)^{\frac{1-\alpha}{\alpha}}t + f(x_0) - \frac{t}{\sqrt{3}} < f(x_0) - \frac{t}{2}. \end{aligned}$$

Next, if $x_1 - c_7(\alpha)t^{\frac{1}{\alpha}} \geq 0$, then $f(x_0) - \frac{t}{2} < f(x_0) < f(x_1) < f(x_1 - c_7(\alpha)t^{\frac{1}{\alpha}})$ as f is decreasing on $[0, 1]$. Before proceeding, we note that by (2.2.2) and $t \geq c_2(\alpha)x_0^{\alpha}$ we have $-x_1 \leq x_0 - x_1 \leq \sqrt{2}c_2(\alpha)^{\frac{1}{\alpha}-1}t^{2-\frac{1}{\alpha}} \leq c_7(\alpha)t^{\frac{1}{\alpha}} < \frac{1}{2}$ if we assume that

$c_0(\alpha) \le (c_7(\alpha)c_2(\alpha)^{1-\frac{1}{\alpha}}2^{-\frac{1}{2}})^{\frac{\alpha}{2(\alpha-1)}}$ and $c_0(\alpha) \le (2c_7(\alpha))^{-\alpha}$. So, if $x_1 - c_7(\alpha)t^{\frac{1}{\alpha}} < 0$, then by monotonicity of f on $[-1,0]$, we see that $f(x_1 - c_7(\alpha)t^{\frac{1}{\alpha}}) > f(-2c_7(\alpha)t^{\frac{1}{\alpha}}) = f(2c_7(\alpha)t^{\frac{1}{\alpha}})$. Now we conclude as follows:

$$f(x_0) - \frac{t}{2} \le 1 - \frac{t}{2} < 1 - \frac{2^{1-\frac{1}{\alpha}}2^{\alpha}c_7(\alpha)^{\alpha}}{\alpha}t < f(2c_7(\alpha)t^{\frac{1}{\alpha}}),$$

where $\frac{2^{1+\alpha-\frac{1}{\alpha}}c_7(\alpha)^{\alpha}}{\alpha} < \frac{1}{2}$ was used for the second step, and (2.2.8) was used in the last step under the assumption that $c_0 \le 2^{-1-\alpha}c_7(\alpha)^{-\alpha}$. We can choose $c_7(\alpha) > 0$ arbitrarily to satisfy $c_7(\alpha) < \min\{c_2(\alpha)^{\frac{1}{\alpha}-1}(\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{2}}), \alpha^{\frac{1}{\alpha}}2^{\frac{1}{\alpha^2}-\frac{2}{\alpha}-1}\}$. Now (2.2.12) is established.

If $x_0 = 0$ and $c_0(\alpha) \le 1 - 2^{-\frac{1}{\alpha}}$, we can invoke (2.2.8) to immediately obtain that $l_i(t) = f^{-1}(1-t) \approx c(\alpha)t^{\frac{1}{\alpha}}$, $i = 1, 2$.

Now we choose $c_0(\alpha) > 0$ so that all the previously stated requirements (which were estimates from above on $c_0(\alpha)$) are fulfilled. The proof of (2.1.7) is complete as a combination of (2.2.5), (2.2.7), (2.2.10), (2.2.12) if $x_0 > 0$ and the argument of the previous paragraph if $x_0 = 0$. $\square$

Chapter 3

Christoffel functions on planar domains with piecewise smooth boundaries

In this chapter, we compute up to a constant factor Christoffel functions on planar domains with boundaries consisting of finitely many C^2 curves such that each corner point of the boundary has interior angle strictly between 0 and π. The formula uses the distances from the point of interest to the curves or certain parts of the curves defining the boundary of the domain.

Through this Chapter we will consider the piecewise C^2 curves with the following parametrization.

A C^2 curve is a parametric curve given by a parametrization $\boldsymbol{\varphi}$ which is a C^2

mapping from $[0,1]$ to $\mathbb{R}^2$ satisfying $|\boldsymbol{\varphi}'(s)| \neq 0$ everywhere and $\boldsymbol{\varphi}(s) \neq \boldsymbol{\varphi}(t)$ when $0 \leq s < t < 1$. In particular, we allow closed curves when $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}(1)$. By ∂D we denote the boundary of a domain $D \subset \mathbb{R}^2$. We call $D \subset \mathbb{R}^2$ a domain with piecewise C^2 boundary if $\partial D = \cup_{i=1}^{m} \Gamma_i$, where each Γ_i is a C^2 curve. A point $\boldsymbol{v}$ is a corner point of ∂D if $v = \boldsymbol{\varphi}(s) = \boldsymbol{\psi}(t)$ where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are parametrizations of Γ_i and Γ_ℓ with $\boldsymbol{\varphi}'(s) \neq \boldsymbol{\psi}'(t)$. Note that this can happen even when $i = \ell$, $\boldsymbol{\varphi} = \boldsymbol{\psi}$ say for $t = 0$ and $s = 1$. However, we do require that if Γ_i with parametrization $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^2$ is closed and $\boldsymbol{\varphi}(0)$ is not a corner point (i.e. $\boldsymbol{\varphi}'(0) = \boldsymbol{\varphi}'(1)$), then necessarily $\boldsymbol{\varphi}''(0) = \boldsymbol{\varphi}''(1)$, so that Γ_i is C^2 everywhere.

Let $\{\boldsymbol{v}_j\}_{j=1}^{k}$ be the set of all corner points of ∂D. For every j, we define the interior angle α_j of D at $\boldsymbol{v}_j$, and two related "corner" boundary curves $\Gamma_j^{\pm}$. Let $\boldsymbol{\varphi} : [-\varepsilon, \varepsilon] \to \mathbb{R}^2$, $\varepsilon > 0$, be the natural parametrization of ∂D in a neighborhood of $\boldsymbol{v}_j$ such that $\boldsymbol{\varphi}(0) = \boldsymbol{v}_j$, D remains on the left when ∂D is traversed as the parameter increases and $\Gamma_j^- := \boldsymbol{\varphi}|_{[-\varepsilon,0]}$ and $\Gamma_j^+ := \boldsymbol{\varphi}|_{[0,\varepsilon]}$ are C^2 curves with no common points except for $\boldsymbol{v}_j$ (it might be that this corner point is formed at the endpoints of a single Γ_i). These conditions can be achieved by considering proper orientation and taking sufficiently small $\varepsilon > 0$. Then the unit vectors $\mathbf{t}_j^+ := \lim_{s\to 0^+} \boldsymbol{\varphi}'(s)$ and $\mathbf{t}_j^- := \lim_{s\to 0^-} \boldsymbol{\varphi}'(s)$ are tangent to ∂D at $\boldsymbol{v}_j$. We define $\alpha_j \in [0, 2\pi)$ as the angle required to turn $\mathbf{t}_j^+$ counterclockwise to get $-\mathbf{t}_j^-$, see the illustration below.

If D is a simple polygon, this definition coincides with the standard definition of the interior angle. We remark that, in the above, we allow piecewise C^2 domains to

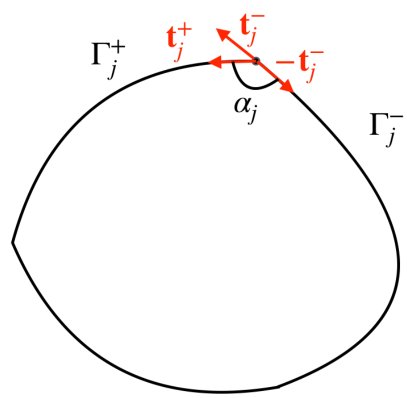

Figure 3.1: Definition of interior angle

have several connected components and to have holes.

We set $\operatorname{dist}(\boldsymbol{x}, \Gamma) := \inf_{\boldsymbol{y}\in\Gamma} \|\boldsymbol{x} - \boldsymbol{y}\|_2$ to be the distance from a point $\boldsymbol{x}$ to a set (or curve) Γ, where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$, and $\operatorname{dist}(\boldsymbol{x}, \emptyset) := \infty$. Recall that we define $\rho_n^*(t) := n^{-2} + n^{-1}\sqrt{t}$, $t \geq 0$.

The central result of this Chapter is the following theorem.

Theorem 3.1. *Let $D \subset \mathbb{R}^2$ be a domain with piecewise C^2 boundary with pieces of the boundary Γ_i, $i = 1, \dots, m$, and corner points $\boldsymbol{v}_j$ with interior angles α_j, $0 < \alpha_j < \pi$, and related corner boundary curves $\Gamma_j^{\pm}$ as defined above, $j = 1, \dots, k$. For any point $\boldsymbol{x} \in D$*

$$\lambda_n(\boldsymbol{x}, D) \approx c(D) \min\Big(\min_{1\leq i\leq m} n^{-1}\rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_i)), \min_{1\leq j\leq k} \rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_j^-))\rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_j^+))\Big), \tag{3.0.1}$$

where $c(D) > 0$ is a constant depending only on D.

We use a comparison with appropriate reference domains and the extremal property (1.1.6) to prove Theorem 3.1. For the lower bound, we use "grain"-type domains which are the intersections of two discs of the same radius. These estimates are established in Section 3.2 using Videnskii-type inequality (3.1.3). To obtain the upper bound, we compare with domains which are intersections of two annuli and explicitly construct in Section 3.3 the required polynomials with small L_2 norm and $f(\boldsymbol{x}) = 1$ at a fixed point $\boldsymbol{x}$.

Our goal was to obtain a description of Christoffel functions which contributes n and $\boldsymbol{x} \in D$. The major part of the proof is focused on the specific situation when $\boldsymbol{x}$ is close to one of the corner points $\boldsymbol{v}_j$.

The methods of this chapter allow to handle non-convex domains but do not apply to angles bigger than π or cusps in the boundary of the domain, which are very interesting directions for future research.

3.1 Preliminaries

We start with the collection of the required preliminaries and introduction of some necessary notation.

Recall that, by $B := \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 \le 1\}$, we denote the unit disc in $\mathbb{R}^2$,

$$\lambda_n(\boldsymbol{x}, B) \approx n^{-1}\rho_n^*(1 - \|\boldsymbol{x}\|_2), \quad \boldsymbol{x} \in B. \tag{3.1.1}$$

We will use without reference the following properties of ρ_n^* which are straightforward to verify: $\rho_n^*(t) \le t + n^{-2}$, $\rho_n^*(t) \le \rho_n^*(t') \le \sqrt{\frac{t'}{t}}\rho_n^*(t)$, and $\rho_n^*(t+n^{-2}) \approx \rho_n^*(t)$, valid for any $0 \le t \le t'$.

A local linear extension of a C^2 curve Γ having a parametrization $\boldsymbol{\varphi}$ is a curve Γ^* with C^1 parametrization $\boldsymbol{\varphi}^*$ satisfying $\boldsymbol{\varphi}^*(t) = \boldsymbol{\varphi}(t)$ for $t \in [0,1]$, $\boldsymbol{\varphi}'(t) = (\boldsymbol{\varphi}^*)'(0)$ for $t < 0$, and $(\boldsymbol{\varphi}^*)'(t) = \boldsymbol{\varphi}'(1)$ for $t > 1$. We choose the domain of $\boldsymbol{\varphi}^*$ to be $[-\epsilon, 1+\epsilon]$ for a sufficiently small $\epsilon > 0$ so that $\boldsymbol{\varphi}^*$ is injective possibly with the exception of $\boldsymbol{\varphi}^*(0) = \boldsymbol{\varphi}^*(1)$ in case $\boldsymbol{\varphi}$ was closed. In other words, Γ^* is obtained by extending Γ beyond the beginning point $\boldsymbol{\varphi}(0)$ and the end point $\boldsymbol{\varphi}(1)$ by straight line segments of strictly positive lengths belonging to the lines tangent to Γ at these two points, respectively.

Suppose $\Gamma \subset \mathbb{R}^2$ and $\boldsymbol{y} \in \Gamma$ are such that Γ is a C^2 curve in a neighborhood of $\boldsymbol{y}$. Let $\boldsymbol{u}$ be a unit vector normal to Γ at $\boldsymbol{y}$, and in the case $\Gamma = \partial D$ for a domain D, we choose $\boldsymbol{u}$ to be pointing outwards from D. For any $r > 0$, we denote by

$$B_+(r, \boldsymbol{y}, \Gamma) := rB + \boldsymbol{y} + r\boldsymbol{u} \quad \text{and} \quad B_-(r, \boldsymbol{y}, \Gamma) := rB + \boldsymbol{y} - r\boldsymbol{u} \tag{3.1.2}$$

the two discs of radius r tangent to Γ at $\boldsymbol{y}$.

We require that closed C^2 curves without corner points possess a *rolling disc property*. Namely, if Γ is such a curve, then there exists $r = r(\Gamma) > 0$ such that for any $0 < r' \le r$ and any point $\boldsymbol{y} \in \Gamma$ we have $B_\pm(r', \boldsymbol{y}, \Gamma) \cap \Gamma = \{\boldsymbol{y}\}$. While in a neighborhood of $\boldsymbol{y}$ such a statement follows from standard differential geometry (curvature is separated from zero), the stated above global version follows from a generalization of Blaschke's rolling theorem [51, Theorem 1 (iii) and (v)]. In the following proof of the main theorem we will extend the rolling disc property to certain non-closed C^2 curves.

We will need the following analog of Bernstein and Markov inequalities due to Videnskii [50] (also can be found in [2, 4.1 E.19, p. 242]): for a trigonometric polynomial T_n of degree $\le n$ and $\theta \in (-\beta, \beta)$, $\beta \in (0, \pi)$ one has

$$|T_n'(\theta)| \le \min\left\{ \frac{n\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\beta}{2} - \sin^2\frac{\theta}{2}}}, (1+o(1))2n^2\cot\frac{\beta}{2} \right\} \|T_n\|_{L_\infty([-\beta,\beta])},$$

where $o(1) = 0$ for every $n > \frac{1}{2}\sqrt{3\tan^2(\beta/2)+1}$, so $(1+o(1))2\cot\frac{\beta}{2} \le c(\beta)$ for all n. (In fact, the Markov-type inequality $|T_n'(\theta)| \le c(\beta)n^2\|T_n\|_{L_\infty([-\beta,\beta])}$ was obtained much earlier by Jackson [21, p. 889]; the Videnskii inequality has sharp constant for large n.) The above implies

$$|T_n'(\theta)| \le \tilde{c}(\beta)\frac{\|T_n\|_{L_\infty([-\beta,\beta])}}{\rho_n^*(\beta - |\theta|)}, \tag{3.1.3}$$

where $\tilde{c}(\beta) > 0$ can be assumed to be a decreasing function of $\beta \in (0, \pi)$.

We will use the notation $\operatorname{meas}_d(\cdot)$ to denote the d-dimensional Lebesgue measure. Recall the notation $\operatorname{dist}(X, Y) := \inf_{\boldsymbol{x}\in X, \boldsymbol{y}\in Y} \|\boldsymbol{x} - \boldsymbol{y}\|_2$ for two subsets $X, Y \subset \mathbb{R}^2$.

3.2 Lower bound for specific domains

We will prove in this section the following lemma establishing an appropriate lower bound for "grain"-type domains, which are the intersection of two discs of the same radius.

Lemma 3.2. *Let* $0 < h < 2$, $D_1 := B$, $D_2 := B + (0,h)$, $D := D_1 \cap D_2$, *for* $\boldsymbol{x} \in D$ $d_i^*(\boldsymbol{x}) := 1 - |\boldsymbol{x} - (i-1)(0,h)|$ *is the distance from* $\boldsymbol{x}$ *to* ∂D_i, $i = 1, 2$. *Then for any* $\boldsymbol{x} \in D$

$$\lambda_n(\boldsymbol{x}, D) \geq c(h)\rho_n^*(d_1^*(\boldsymbol{x}))\rho_n^*(d_2^*(\boldsymbol{x})).$$

Let us prepare for the proof of Lemma .

A convex body in $\mathbb{R}^2$ is any convex compact subset of $\mathbb{R}^2$ with non-empty interior. For a convex body in $D \subset \mathbb{R}^2$, $\boldsymbol{x} \in D$ and $\mu > 0$, denote by $D_{\mu,\boldsymbol{x}} := \boldsymbol{x} + \mu(D - \boldsymbol{x})$ the homothety of D with the ratio μ and the center $\boldsymbol{x}$.

Lemma 3.3. *Let* D *be a convex body in* $\mathbb{R}^2$, $\boldsymbol{x}$ *be an interior point of* D *and* $0 < \mu < 1$. *Then for any* $\boldsymbol{y} \in D_{1-\mu,\boldsymbol{x}}$ *we have* $D_{\mu,\boldsymbol{x}} - \boldsymbol{x} + \boldsymbol{y} \subset D$.

Proof. It is enough to show that

$$\mu(D - \boldsymbol{x}) + (1-\mu)(D - \boldsymbol{x}) + \boldsymbol{x} \subset D,$$

which is immediate since D is convex and so $\mu D + (1-\mu)D = D$. $\square$

The following corollary deals with the L_∞ norm of the “needle” polynomial realizing $\lambda_n(\boldsymbol{x}, D)$ (see (1.1.6)).

Corollary 3.4. *Let* D *be a convex body in* $\mathbb{R}^2$, $\boldsymbol{x}$ *be an interior point of* D, $0 < \mu < 1$, *and* $P \in \mathcal{P}_n$ *be a polynomial satisfying* $P(\boldsymbol{x}) = 1$ *and* $\|P\|_{L_2(D)}^2 = \lambda_n(\boldsymbol{x}, D)$. *Then*

$$\|P\|_{L_\infty(D_{1-\mu,\boldsymbol{x}})} \leq \mu^{-1}.$$

Proof. Let $M := \|P\|_{L_\infty(D_{1-\mu,\boldsymbol{x}})} \geq 1$ be attained at a point $\boldsymbol{y} \in D_{1-\mu,\boldsymbol{x}}$. Then

$$\lambda_n(\boldsymbol{y}, D) = \min_{Q \in \mathcal{P}_{n,2}, |Q(\boldsymbol{y})|=1} \|Q\|^2_{L_2(D)} \leq \frac{1}{M^2}\|P\|^2_{L_2(D)} = \frac{1}{M^2}\lambda_n(\boldsymbol{x}, D),$$

so by (1.1.7), Lemma 3.3 and (1.1.8), we conclude that

$$M^2 \leq \frac{\lambda_n(\boldsymbol{x}, D)}{\lambda_n(\boldsymbol{y}, D)} \leq \frac{\lambda_n(\boldsymbol{x}, D)}{\lambda_n(\boldsymbol{y}, D_{\mu,\boldsymbol{x}} - \boldsymbol{x} + \boldsymbol{y})} = \mu^{-2}.$$

□

Remark 3.5. Lemma 3.3 and Corollary 3.4 are valid in $\mathbb{R}^d$ with μ^{-1} replaced by $\mu^{-d/2}$ in the conclusion of Corollary 3.4. The only change in the proof is μ^{-2} replaced by μ^{-d} in the end.

The restriction of an algebraic polynomial to a circular arc is a trigonometric polynomial. This is immediate using polar coordinates to parameterize the arc. With this in mind, the next lemma employs the Videnskii inequality in our settings.

Lemma 3.6. *Let $A \subset \mathbb{R}^2$ be an arc of length l of a circle of radius r, $l < 2\pi r$, $\boldsymbol{x} \in A$, $\eta < l$, $r < \eta^{-1}$, $\eta > 0$, d be the distance from $\boldsymbol{x}$ to the 2-point set consisting of the endpoints of A, $f \in \mathcal{P}_n$, $f(\boldsymbol{x}) = 1$. Then*

$$f(\boldsymbol{y}) \geq \frac{1}{2} \quad \textit{whenever} \quad \boldsymbol{y} \in A \quad \textit{and} \quad |\boldsymbol{x} - \boldsymbol{y}| \leq c(\eta)\|f\|^{-1}_{L_\infty(A)}\rho_n^*(d). \tag{3.2.1}$$

Proof. We can parameterize A so that

$$A = \boldsymbol{\varphi}([-\beta, \beta]), \quad \text{where} \quad \boldsymbol{\varphi}(t) = \boldsymbol{o} + r(\cos(t - t_0), \sin(t - t_0)),$$

$\boldsymbol{o} \in \mathbb{R}^2$ and $\beta = \frac{l}{2r}$. Note that $T_n(t) := f(\boldsymbol{\varphi}(t))$ is a trigonometric polynomial of degree $\leq n$. We can assume that $\boldsymbol{x}$ is closer to $\boldsymbol{\varphi}(\beta)$ than to $\boldsymbol{\varphi}(-\beta)$, then $\boldsymbol{x} = \boldsymbol{\varphi}(\beta - d')$, where $d' = 2\arcsin(\frac{d}{2r}) \approx \frac{d}{r}$ and $\beta - d' \geq 0$. With $\boldsymbol{y} = \boldsymbol{\varphi}(t)$, we will show that

$$f(\boldsymbol{y}) = T_n(t) \geq \tfrac{1}{2} \quad \text{whenever} \quad t \in [-\beta, \beta] \quad \text{and} \quad |\beta - d' - t| \leq \gamma \|f\|^{-1}_{L_\infty(A)} \rho^*_n(d') \tag{3.2.2}$$

for a small enough $\gamma = \gamma(\eta) < \frac{1}{2}$. Assuming t is as in (3.2.2), so by $\gamma < \frac{1}{2}$ we have $|\beta - d' - t| \leq \frac{d'}{2} + \frac{1}{2}n^{-2}$ implying $\beta - \frac{3}{2}d' - \frac{1}{2}n^{-2} \leq t \leq \beta - \frac{1}{2}d' + \frac{1}{2}n^{-2}$. For some θ between $\beta - d'$ and t, using (3.1.3) we have

$$\begin{aligned}
|1 - T_n(t)| = |T_n(\beta - d') - T_n(t)| = |T_n'(\theta)||\beta - d' - t| \\
&\leq \tilde{c}(\beta) \frac{\|T_n\|_{L_\infty([-\beta,\beta])}}{\rho^*_n(\beta - |\theta|)} |\beta - d' - t| \\
&\leq c\tilde{c}(\beta) \frac{\|T_n\|_{L_\infty([-\beta,\beta])}}{\rho^*_n(d')} |\beta - d' - t| \\
&\leq c\tilde{c}(\beta) \frac{\|f\|_{L_\infty(A)}}{\rho^*_n(d')} \gamma \|f\|^{-1}_{L_\infty(A)} \rho^*_n(d') \leq \tfrac{1}{2},
\end{aligned}$$

provided $\gamma = \gamma(\eta)$ is sufficiently small (we have $\tilde{c}(\beta) \geq \tilde{c}(\frac{\eta^2}{2})$). Finally, (3.2.2) implies (3.2.1) because $|\beta - d' - t|r \approx \|\boldsymbol{x} - \boldsymbol{y}\|_2 = |\boldsymbol{\varphi}(\beta - d') - \boldsymbol{\varphi}(t)|$ and $\rho^*_n(d) \leq c(\eta)\rho^*_n(d')$. □

Now we are ready for the proof of the main result of this section.

Proof of Lemma 3.2. Without loss of generality, assume $d^*_1(\boldsymbol{x}) \leq d^*_2(\boldsymbol{x})$. We will consider all possible three cases depending on the values of $d^*_1(\boldsymbol{x})$ and $d^*_2(\boldsymbol{x})$ in relation to a parameter $\delta = \delta(h) > 0$ which will be selected later.

Case 1: $d_1^*(\boldsymbol{x}) \geq \frac{\delta}{2}$. Then $\widetilde{B} := \frac{\delta}{2}B + \boldsymbol{x}$ satisfies $\widetilde{B} \subset D$, so by (1.1.7), (1.1.8) and (3.1.1)

$$\lambda_n(\boldsymbol{x}, D) \geq \lambda_n(\boldsymbol{x}, B) = \tfrac{\delta^2}{4}\lambda_n((0,0), B) \approx c(\delta)n^{-2} \geq c(\delta)\rho_n^*(d_1^*(\boldsymbol{x}))\rho_n^*(d_2^*(\boldsymbol{x})).$$

Case 2: $d_1^*(\boldsymbol{x}) < \frac{\delta}{2}$ and $d_2^*(\boldsymbol{x}) \geq \delta$. Let $\boldsymbol{y} \in \partial B$ be the point closest to $\boldsymbol{x}$, i.e. $\boldsymbol{x} = (1 - d_1^*(\boldsymbol{x}))\boldsymbol{y}$. Now we consider the disc $\widetilde{B} := \frac{\delta}{2}B + (1 - \frac{\delta}{2})\boldsymbol{y}$, clearly $\widetilde{B} \subset B$. Note that due to $d_1^*(\boldsymbol{x}) < \frac{\delta}{2}$, the point $\boldsymbol{x}$ belongs to the line segment joining the center $(1 - \frac{\delta}{2})\boldsymbol{y}$ of $\widetilde{B}$ and $\boldsymbol{y}$, in particular, $\boldsymbol{x} \in \widetilde{B}$. Therefore, for any $\boldsymbol{z} \in \widetilde{B}$

$$\|\boldsymbol{z} - (0,h)\|_2 \leq \|\boldsymbol{z} - \boldsymbol{x}\|_2 + \|\boldsymbol{x} - (0,h)\|_2 \leq \delta + 1 - d_2^*(\boldsymbol{x}) \leq 1,$$

implying $\widetilde{B} \subset B + (0,h)$ and, consequently, $\widetilde{B} \subset D$. By (1.1.7), (1.1.8) and (3.1.1)

$$\lambda_n(\boldsymbol{x}, D) \geq \lambda_n(\boldsymbol{x}, B) = \tfrac{\delta^2}{4}\lambda_n((1 - \tfrac{2d_1^*(\boldsymbol{x})}{\delta})\boldsymbol{y}, B) \approx$$

$$\approx c(\delta)n^{-1}\rho_n^*(d_1^*(\boldsymbol{x})) \geq c(\delta)\rho_n^*(d_1^*(\boldsymbol{x}))\rho_n^*(d_2^*(\boldsymbol{x})).$$

Case 3: $d_2^*(\boldsymbol{x}) < \delta$. Let us make the required choice of $\delta = \delta(h) > 0$ at this time. First, we will impose that

$$h^2 + (1 - \delta)^2 > 1. \tag{3.2.3}$$

Next, the set $\partial D_1 \cap \partial D_2$ consists of two points, one of which is $\boldsymbol{u}_+ := \left(\sqrt{1 - (\frac{h}{2})^2}, \frac{h}{2}\right)$. Observe that the set

$$X(\delta) := \{\boldsymbol{y} = (y_1, y_2) \in D : y_1 \geq 0,\ \mathrm{dist}(\boldsymbol{y}, \partial D_1) \leq \delta,\ \mathrm{dist}(\boldsymbol{y}, \partial D_2) \leq \delta\}$$

satisfies $\boldsymbol{u}_+ \in X(\delta)$ and $\lim\limits_{\delta\to 0+} \operatorname{diam}(X(\delta)) = 0$. So, we can choose $\delta = \delta(h) > 0$ so that

$$\operatorname{dist}((0,h), X(\delta)) > c_1(h) \tag{3.2.4}$$

and

$$y_1 \geq c_1(h) \quad \text{and} \quad y_2 \geq c_1(h) \quad \text{for any} \quad \boldsymbol{y} = (y_1, y_2) \in X(\delta). \tag{3.2.5}$$

We use Corollary 3.4 with $\mu = \frac{1}{2}$. For the dilated polynomial $\widetilde{P}(\cdot) := P(2(\cdot - \boldsymbol{x}) + \boldsymbol{x})$ we have

$$\lambda_n(\boldsymbol{x}, D) = \tfrac{1}{4}\|\widetilde{P}\|^2_{L_2(D_{2,\boldsymbol{x}})} \geq \tfrac{1}{4}\|\widetilde{P}\|^2_{L_2(D)}, \quad \widetilde{P}(\boldsymbol{x}) = 1 \quad \text{and} \quad \|\widetilde{P}\|_{L_\infty(D)} \leq 2, \tag{3.2.6}$$

so to complete the proof it is sufficient to find a set $D' \subset D$ with

$$\operatorname{meas}_2(D') \geq c(h)\rho_n^*(d_1^*(\boldsymbol{x}))\rho_n^*(d_2^*(\boldsymbol{x})) \quad \text{and} \quad \widetilde{P}(\boldsymbol{y}) \geq \tfrac{1}{4} \quad \text{for every} \quad \boldsymbol{y} \in D'. \tag{3.2.7}$$

For any point $\boldsymbol{y} \in D$, it will be convenient to denote by

$$A_i(\boldsymbol{y}) := D \cap \partial(\|\boldsymbol{y} - (i-1)(0,h)\|_2 B + (i-1)(0,h))$$

the largest arc of the circle concentric with ∂D_i, passing through $\boldsymbol{y}$ and located inside D, $i = 1, 2$. Since $\boldsymbol{x} \in X(\delta)$, by (3.2.4), the length of $A_2(\boldsymbol{x})$ is at least $c(h)$ (the radius is clearly ≤ 1), so by Lemma 3.6, there is a choice of $\gamma_1 = \gamma_1(h)$ such that $\widetilde{P}(\boldsymbol{y}) \geq \frac{1}{2}$ for any $\boldsymbol{y} \in A_3$, where

$$A_3 := \{\boldsymbol{y} \in A_2(\boldsymbol{x}) : \|\boldsymbol{x} - \boldsymbol{y}\|_2 \leq \gamma_1 \rho_n^*(d_1^*(\boldsymbol{x}))\}.$$

Additionally, we can assume that $\gamma_1 \cdot (1+\delta) < \frac{c_1(h)}{2}$ so that by $\boldsymbol{x} \in X(\delta)$, $\rho_n^*(\delta) \leq n^{-2} + \delta$ and (3.2.5), we have

$$y_1 \geq \tfrac{1}{2} c_1(h) \quad \text{and} \quad y_2 \geq \tfrac{1}{2} c_1(h) \quad \text{for any} \quad \boldsymbol{y} = (y_1, y_2) \in A_3, \tag{3.2.8}$$

in particular,

$$\inf(\{\|\boldsymbol{y}\|_2 : \boldsymbol{y} \in A_3\}) \geq \tfrac{1}{\sqrt{2}} c_1(h). \tag{3.2.9}$$

We will show that

$$\mathrm{meas}_1(\{\|\boldsymbol{y}\|_2 : \boldsymbol{y} \in A_3\}) \geq c(h)\rho_n^*(d_1^*(\boldsymbol{x})). \tag{3.2.10}$$

Because the circle containing A_3 has its largest first coordinate at the point $(1 - d_2^*(\boldsymbol{x}), h)$, which, by (3.2.3), is outside of D_1, A_3 is the graph of an increasing function (of the first coordinate). Therefore, we can denote the endpoints of A_3 as (z_1, z_2) and $(z_1 + v_1, z_2 + v_2)$, where $v_1 > 0$ and $v_2 > 0$. By the definition of A_3, $|(v_1, v_2)| \approx c(h)\rho_n^*(d_1^*(\boldsymbol{x}))$. Now (3.2.10) follows from (3.2.8) and $A_3 \subset B$ as follows:

$$\begin{aligned} \mathrm{meas}_1(\{\|\boldsymbol{y}\|_2 : \boldsymbol{y} \in A_3\}) &\geq |(z_1 + v_1, z_2 + v_2)| - |(z_1, z_2)| \\ &= \frac{2z_1 v_1 + 2z_2 v_2 + v_1^2 + v_2^2}{|(z_1 + v_1, z_2 + v_2)| + |(z_1, z_2)|} \\ &\geq z_1 v_1 + z_2 v_2 \geq \tfrac{c_1(h)}{2}(v_1 + v_2) \geq c(h)\rho_n^*(d_1^*(\boldsymbol{x})). \end{aligned}$$

By (3.2.9), the length of $A_1(\boldsymbol{y})$ is at least $c(h)$ for any $\boldsymbol{y} \in A_3$ while the radius is clearly ≤ 1, so by Corollary 3.4, we can choose γ_2 so that $\widetilde{P}(\boldsymbol{z}) \geq \frac{1}{4}$ for any $\boldsymbol{z} \in A_4(\boldsymbol{y})$, where

$$A_4(\boldsymbol{y}) := \{\boldsymbol{z} \in A_1(\boldsymbol{y}) : \|\boldsymbol{y} - \boldsymbol{z}\|_2 \leq \gamma_2 \rho_n^*(d_2^*(\boldsymbol{x}))\}. \tag{3.2.11}$$

Remark that this choice of γ_2 is independent of the choice of $\boldsymbol{y} \in A_3$ as it only depends on $c_1(h)$ in (3.2.9).

By construction, $\widetilde{P}(\boldsymbol{z}) \geq \frac{1}{4}$ for any $\boldsymbol{z} \in D'$, where $D' := \bigcup_{\boldsymbol{y} \in A_3} A_4(\boldsymbol{y})$, so it remains to justify the first inequality in (3.2.7). We can establish this using polar coordinates (r, θ). For any point in D' we have $r \geq c(h)$ by (3.2.9). For each fixed r, the measure of θ in D', which is the length of some arc $A_4(\boldsymbol{y})$, $\boldsymbol{y} \in A_3$, is at least $c(h)\rho_n^*(d_2^*(\boldsymbol{x}))$ by (3.2.9) and (3.2.11). The lower bound for the measure of r in D' is provided by (3.2.10). The proof is completed. $\square$

3.3 Lemmas for upper bounds

We begin with a basic univariate construction.

Lemma 3.7. *For any $t \in [0,1]$ and any positive integer n there exists $Q \in \mathcal{P}_{n/2,1}$ such that $Q(1-t) = 1$ and*

$$|Q(s)| \leq c \frac{\rho_n^*(t)}{\rho_n^*(t) + |1-t-s|}, \quad s \in [-1,1]. \tag{3.3.1}$$

Proof. This is a partial case of [14, Lemma 6.1]. $\square$

Using rotation, we get a radial polynomial satisfying required conditions on an annulus.

Lemma 3.8. *For any $0 < r_1 < r_2$ let $D := \{\boldsymbol{x} \in \mathbb{R}^2 : r_1 \leq \|\boldsymbol{x}\|_2 \leq r_2\}$. Then, for* *any* $\gamma \in [r_1, r_2]$ *there exists* $P \in \mathcal{P}_n$ *such that*

$$P(\boldsymbol{y}) = 1 \quad \textit{whenever} \quad \|\boldsymbol{y}\|_2 = \gamma \tag{3.3.2}$$

and

$$|P(\boldsymbol{y})| \le c(r_1, r_2) \frac{\rho_n^*(\gamma - r_1)}{\rho_n^*(\gamma - r_1) + |\gamma - \|\boldsymbol{y}\|_2|}, \quad \boldsymbol{y} \in D. \tag{3.3.3}$$

Proof. Set $t := (\gamma^2 - r_1^2)/(r_2^2 - r_1^2) \in [0, 1]$. Let Q be the polynomial provided by Lemma 3.7. We define $P(\boldsymbol{x}) = Q((r_2^2 - \|\boldsymbol{x}\|_2^2)/(r_2^2 - r_1^2))$, clearly P satisfies (3.3.2). We get (3.3.3) from (3.3.1) because $\rho_n^*(t) \approx c(r_1, r_2)\rho_n^*(\gamma - r_1)$. $\square$

The radial structure from the previous lemma will carry too large overall L_2 norm, which can be rectified by multiplication by a "good" univariate polynomial.

Lemma 3.9. *For any* $0 < r_1 < r_2$ *let* $D := \{\boldsymbol{x} \in \mathbb{R}^2 : r_1 \le \|\boldsymbol{x}\|_2 \le r_2\}$. *Then*

$$\lambda_n(\boldsymbol{x}, D) \le c(r_1, r_2) n^{-1} \rho_n^*(\|\boldsymbol{x}\|_2 - r_1) \quad \textit{for any} \quad \boldsymbol{x} \in D.$$

Proof. Due to (1.1.8), it is enough to show for any fixed $\gamma \in [r_1, r_2]$ that

$$\lambda_n((\gamma, 0), D) \le c(r_1, r_2) n^{-1} \rho_n^*(\gamma - r_1),$$

or, due to (1.1.6), that there exists $\widetilde{P} \in \mathcal{P}_n$ satisfying

$$\widetilde{P}(\gamma, 0) = 1 \quad \text{and} \quad \|\widetilde{P}\|_{L_2(D)}^2 \le c(r_1, r_2) n^{-1} \rho_n^*(\gamma - r_1).$$

Let $P \in \mathcal{P}_{n/2}$ be the polynomial from Lemma 3.8 satisfying (3.3.2) and (3.3.3). Let Q be the polynomial from Lemma 3.7 for $t = 1$. Then we define $\widetilde{P}(x_1, x_2) :=$

$P(x_1,x_2)Q(\frac{x_2}{r_2})$. Clearly $\widetilde{P}(\gamma,0)=1$ and we need to show that

$$\int_D (P(x_1,x_2))^2 (Q(\tfrac{x_2}{r_2}))^2 \, dx_1 dx_2 \le c(r_1,r_2) n^{-1} \rho_n^*(\lambda - r_1).$$

By (3.3.1) and (3.3.3), using polar coordinates, it is sufficient to prove that

$$\int_{r_1}^{r_2} \left(\frac{\rho_n^*(\gamma - r_1)}{\rho_n^*(\gamma - r_1) + |\gamma - r|} \right)^2 \int_0^{2\pi} \left(\frac{\frac{1}{n}}{\frac{1}{n} + |\frac{r \sin\theta}{r_2}|} \right)^2 d\theta \, r \, dr \le c(r_1,r_2) n^{-1} \rho_n^*(\gamma - r_1).$$

Observe that $r \approx c(r_1,r_2)$ for any $r \in [r_1,r_2]$.

With $I := \{\theta \in [0,2\pi] : |\sin\theta| \le \frac{1}{n}\}$ we have

$$\int_0^{2\pi} \left(\frac{\frac{1}{n}}{\frac{1}{n} + |\frac{r \sin\theta}{r_2}|} \right)^2 d\theta \le c(r_1,r_2) \left[\int_I d\theta + \int_{[0,2\pi]\setminus I} \left(\frac{\frac{1}{n}}{|\sin\theta|} \right)^2 d\theta \right] \le c(r_1,r_2) n^{-1},$$

for any $r \in [r_1,r_2]$. So, it remains to show that

$$\int_{r_1}^{r_2} \left(\frac{\rho_n^*(\gamma - r_1)}{\rho_n^*(\gamma - r_1) + |\gamma - r|} \right)^2 dr \le c(r_1,r_2) \rho_n^*(\gamma - r_1),$$

which can be done using the same idea as for the integral w.r.t. θ, namely, splitting the integral as $\int_J \cdots + \int_{[r_1,r_2]\setminus J} \ldots$, where $J = \{r \in [r_1,r_2] : |\gamma - r| \le \rho_n^*(\gamma - r_1)\}$. The details are similar to ones in (3.3.6) from the proof of Lemma 3.10. □

The final lemma provides a polynomial satisfying required conditions on the intersection of two annuli obtained by multiplication of two polynomials provided by Lemma 3.8.

Lemma 3.10. *Let* $\zeta > 0$, $\zeta \le h \le 1-\zeta$, $\boldsymbol{o}_i := (-1)^i(0, r_1 h)$, $0 < r_1 < r_2$, $D_i := \{\boldsymbol{x} \in \mathbb{R}^2 : r_1 \le \|\boldsymbol{x} - \boldsymbol{o}_i\|_2 \le r_2\}$, $i = 1,2$, $D := D_1 \cap D_2$. *Suppose* $\boldsymbol{x} \in D$ *is such that* $\|\boldsymbol{x} - \boldsymbol{o}_i\|_2 \le (1 + \frac{\zeta}{2}) r_1$, $i = 1,2$. *Then*

$$\lambda_n(\boldsymbol{x}, D) \le c(\zeta, r_1, r_2) \rho_n^*(\|\boldsymbol{x} - \boldsymbol{o}_1\|_2 - r_1) \rho_n^*(\|\boldsymbol{x} - \boldsymbol{o}_2\|_2 - r_1).$$

Proof. Due to (1.1.8), we will assume that $r_1 = 1$ and $r = \frac{r_2}{r_1} = r_2 > 1$. For a fixed $\boldsymbol{x} \in D$, denote $t_i := \|\boldsymbol{x} - \boldsymbol{o}_i\|_2 - 1$. According to Lemma 3.8, there is a polynomial $P_i \in \mathcal{P}_{n/2}$ such that

$$P_i(\boldsymbol{y}) = 1 \quad \text{whenever} \quad \|\boldsymbol{y} - \boldsymbol{o}_i\|_2 = t_i + 1 \tag{3.3.4}$$

and

$$|P_i(\boldsymbol{y})| \le c(r) \frac{\rho_n^*(t_i)}{\rho_n^*(t_i) + |t_i + 1 - \|\boldsymbol{y} - \boldsymbol{o}_i\|_2|}. \tag{3.3.5}$$

We will demonstrate that $P(\boldsymbol{y}) := P_1(\boldsymbol{y})P_2(\boldsymbol{y})$ is a required polynomial (see (1.1.6)), namely, it satisfies $P(\boldsymbol{x}) = 1$ and $\|P\|^2_{L_2(D)} \le c(\zeta, r)\rho_n^*(t_1)\rho_n^*(t_2)$. The equality $P(\boldsymbol{x}) = 1$ follows from (3.3.4). Due to (3.3.5), we need to show that

$$\int_D \frac{\rho_n^*(t_1)}{(\rho_n^*(t_1) + |t_1 + 1 - \|\boldsymbol{y} - \boldsymbol{o}_1\|_2|)^2} \cdot \frac{\rho_n^*(t_2)}{(\rho_n^*(t_2) + |t_2 + 1 - \|\boldsymbol{y} - \boldsymbol{o}_2\|_2|)^2} \, d\boldsymbol{y}$$
$$=: \int_D f_1(\boldsymbol{y}) \cdot f_2(\boldsymbol{y}) \, d\boldsymbol{y} \le c(\zeta, r).$$

The integrand is symmetric about the second coordinate axis (containing both $\boldsymbol{o}_i$), therefore it is sufficient to estimate the integral over $D_+ := \{\boldsymbol{y} = (y_1, y_2) \in D : y_1 \ge 0\}$. We will be split into three parts. First, let $D^* := \{\boldsymbol{y} = (y_1, y_2) \in D : y_1 \ge \zeta'\}$, with $\zeta' = \min\{\frac{\sqrt{1-\zeta^2}}{2}, \zeta\}$. Consider the mapping $\Phi : D^* \to [1, r]^2$ defined by $\Phi(y_1, y_2) := (\|\boldsymbol{y} - \boldsymbol{o}_1\|_2, \|\boldsymbol{y} - \boldsymbol{o}_2\|_2)$. It is clear that Φ is injective on D^*. We also observe that Φ is continuously differentiable on D^* and its Jacobian determinant J_Φ satisfies

$$J_\Phi(y_1, y_2) = \frac{2y_1 h}{\|\boldsymbol{y} - \boldsymbol{o}_1\|_2 \|\boldsymbol{y} - \boldsymbol{o}_2\|_2} \ge \frac{2\zeta'\zeta}{r^2} =: c_2(\zeta, r)$$

on D^*. Therefore, with $\boldsymbol{u} = (u_1, u_2)$ and $\tilde{f}_i(u_i) = \dfrac{\rho_n^*(t_i)}{(\rho_n^*(t_i) + |t_i + 1 - u_i|)^2}$, we have

$$\begin{aligned}\int_{D^*} f_1(\boldsymbol{y}) \cdot f_2(\boldsymbol{y})\, d\boldsymbol{y} &= \int_{\Phi(D^*)} \tilde{f}_1(u_1) \cdot \tilde{f}_2(u_2) \frac{1}{J_\Phi(\Phi^{-1}(u_1, u_2))}\, d\boldsymbol{u} \\ &\le \frac{1}{c_2(\zeta, r)} \int_1^r \tilde{f}_1(u_1)\, du_1 \cdot \int_1^r \tilde{f}_2(u_2)\, du_2.\end{aligned}$$

With $I_i := \{u_i \in [1, r] : |t_i + 1 - u_i| \le \rho_n^*(t_i)\}$, we have

$$\begin{aligned}\int_1^r \tilde{f}_i(u_i)\, du_i &= \int_{I_i} \tilde{f}_i(u_i)\, du_i + \int_{[1,r]\setminus I_i} \tilde{f}_i(u_i)\, du_i \\ &\le \int_{I_i} \frac{1}{\rho_n^*(t_i)}\, du_i + 2\int_{\rho_n^*(t_i)}^{\infty} \frac{\rho_n^*(t_i)}{(2s)^2}\, ds \\ &\le 2 + \frac{1}{2},\end{aligned} \tag{3.3.6}$$

so $\int_{D^*} f_1(\boldsymbol{y}) \cdot f_2(\boldsymbol{y})\, d\boldsymbol{y} \le c(\zeta, r)$. It remains to prove that $\int_{D_+ \setminus D^*} f_1(\boldsymbol{y}) \cdot f_2(\boldsymbol{y})\, d\boldsymbol{y} \le c(\zeta, r)$. We notice that $J := [\sqrt{1 - \zeta'^2} + h, r + h]$ we have $D_+ \setminus D^* \subset [0, \zeta'] \times ((-J) \cup J)$. With fixed $y_1 \in [0, \zeta']$, firstly

$$\int_J f_1(y_1, y_2) \cdot f_2(y_1, y_2)\, dy_2 \le c(\zeta, r). \tag{3.3.7}$$

For $y_2 \in J$ we have

$$\begin{aligned}\|\boldsymbol{y} - \boldsymbol{o}_1\|_2 &\ge y_2 + h \ge \sqrt{1 - \zeta'^2} + 2h \ge \sqrt{1 - \zeta^2} + 2\zeta \\ &> \sqrt{(1 - \zeta)^2} + 2\zeta = 1 + \zeta \ge \|\boldsymbol{x} - \boldsymbol{o}_1\|_2 + \tfrac{\zeta}{2} = t_1 + 1 + \tfrac{\zeta}{2},\end{aligned}$$

so

$$f_1(y_1, y_2) \le \frac{\rho_n^*(t_1)}{(\|\boldsymbol{y} - \boldsymbol{o}_1\|_2 - t_1 - 1)^2} \le \frac{\rho_n^*(t_1)}{(\frac{\zeta}{2})^2} \le c(\zeta). \tag{3.3.8}$$

Further, for $y_2 \in J$ one has

$$\sqrt{1 - \zeta'^2} \le y_2 - h \le \|\boldsymbol{y} - \boldsymbol{o}_2\|_2 \le y_1 + (y_2 - h) \le \zeta' + r$$

implying

$$\frac{\partial}{\partial y_2}\|\boldsymbol{y}-\boldsymbol{o}_2\|_2 = \frac{y_2-h}{\|\boldsymbol{y}-\boldsymbol{o}_2\|_2} \geq \frac{\sqrt{1-\zeta'^2}}{r+\zeta'} =: c_3(\zeta, r)$$

and

$$\frac{\partial}{\partial y_2}\|\boldsymbol{y}-\boldsymbol{o}_2\|_2 \leq \frac{r}{\sqrt{1-\zeta'^2}} =: c_4(\zeta, r),$$

so

$$|\|(y_1,y_2)-\boldsymbol{o}_2\|_2 - \|(y_1,y_2')-\boldsymbol{o}_2\|_2| \approx c(\zeta,r)|y_2-y_2'| \quad \text{for any} \quad y_2, y_2' \in J. \quad (3.3.9)$$

Now denote $J_2 := \{y_2 \in J : |t_2+1-\|\boldsymbol{y}-\boldsymbol{o}_2\|_2| \leq \rho_n^*(t_2)\}$, by (3.3.9) we have $\text{meas}_1(J_2) \approx c(\zeta,r)\rho_n^*(t_2)$, and so

$$\int_{J_2} f_2(y_1,y_2)\,dy_2 \leq \int_{J_2} \frac{1}{\rho_n^*(t_2)}\,dy_2 \leq c(\zeta,r).$$

Let $y_2' \in J$ be such that $\|(y_1,y_2')-\boldsymbol{o}_2\|_2 = t_2+1$. Using (3.3.9), we have with sufficiently small $c_5(\zeta,r)$ that

$$\begin{aligned}\int_{J\setminus J_2} f_2(y_1,y_2)\,dy_2 &\leq \frac{1}{4}\int_{J\setminus J_2} \frac{\rho_n^*(t_2)}{(t_2+1-\|\boldsymbol{y}-\boldsymbol{o}_2\|_2)^2}\,dy_2 \leq c(\zeta,r)\int_{J\setminus J_2}\frac{\rho_n^*(t_2)}{(y_2'-y_2)^2}\,dy_2\\ &\leq c(\zeta,r)\int_{\{y_2\in J:|y_2'-y_2|\geq c_5(\zeta,r)\rho_n^*(t_2)\}} \frac{\rho_n^*(t_2)}{(y_2'-y_2)^2}\,dy_2\\ &\leq c(\zeta,r)\int_{\rho_n^*(t_2)}^{\infty}\frac{\rho_n^*(t_2)}{s^2}\,ds = c(\zeta,r).\end{aligned}$$

Finally, $\int_J f_2(y_1,y_2)\,dy_2 \leq c(\zeta,r)$, and taking (3.3.8) into account, we get (3.3.7). Note that (3.3.7) is valid if J is replaced with $-J$ by following essentially the same proof, namely, one simply interchanges $i=1$ and $i=2$ observing that $\|(y_1,y_2)-\boldsymbol{o}_i\|_2 = \|(y_1,-y_2)-\boldsymbol{o}_{3-i}\|_2$. So,

$$\int_{D_+\setminus D^*} f_1(\boldsymbol{y})\cdot f_2(\boldsymbol{y})\,d\boldsymbol{y} \leq \int_{[0,\zeta']\times((-J)\cup J)} f_1(\boldsymbol{y})\cdot f_2(\boldsymbol{y})\,d\boldsymbol{y} \leq c(\zeta,r),$$

and the proof is complete. $\square$

3.4 Proof of the main result

Proof of Theorem 3.1. Denote $d_i(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \Gamma_i)$, $i = 1, \dots, m$, and $d_j^{\pm}(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \Gamma_j^{\pm})$, $j = 1, \dots, k$. Notice that for every j and choice of $\pm$, there exists i such that $\Gamma_j^{\pm} \subset \Gamma_i$ and then $d_j^{\pm}(\boldsymbol{x}) \geq d_i(\boldsymbol{x})$.

We consider three cases depending on a parameter $\delta = \delta(D) > 0$ which will be selected later.

Case 1: If $d_i(\boldsymbol{x}) \geq \frac{\delta}{4}$ for all $i = 1, \dots, m$. Then $\frac{\delta}{4}B + \boldsymbol{x} \subset D$, so by (1.1.7), (1.1.8) and (3.1.1) one has

$$\lambda_n(\boldsymbol{x}, D) \geq \lambda_n(\boldsymbol{x}, \tfrac{\delta}{4}B + \boldsymbol{x}) = \tfrac{\delta^2}{16}\lambda_n((0,0), B) \approx c(\delta)n^{-2}.$$

Let $R := \operatorname{diam}(D)$, then $D \subset RB + \boldsymbol{x}$ and by (1.1.7), (1.1.8) and (3.1.1) one has

$$\lambda_n(\boldsymbol{x}, D) \leq \lambda_n(\boldsymbol{x}, RB + \boldsymbol{x}) = R^2\lambda_n((0,0), B) \approx c(R)n^{-2}.$$

We complete Case 1 by observing that

$$\min\Big(\min_{1\leq i\leq m} n^{-1}\rho_n^*(d_i(\boldsymbol{x})), \min_{1\leq j\leq k} \rho_n^*(d_j^-(\boldsymbol{x}))\rho_n^*(d_j^+(\boldsymbol{x}))\Big) \approx c(\delta, R)n^{-2}$$

due to $d_i, d_j^-, d_j^+ \in [\frac{\delta}{4}, R]$ for any i, j.

Case 2: If $\min_{1\leq i\leq m} d_i(\boldsymbol{x}) = d_{i_0}(\boldsymbol{x}) < \frac{\delta}{4}$ for some i_0 and $\operatorname{dist}(\boldsymbol{x}, \{v_j\}_{j=1}^k) \geq \delta$. We denote by Γ_i' the curve obtained by removing open $\frac{\delta}{4}$-neighborhoods of every corner point from Γ_i. Then Γ_i' is either a closed C^2 curve without corner points (if nothing was removed from Γ_i), in which case we set $\Gamma_i'' := \Gamma_i'$, or Γ_i' can be extended to a closed C^2 curve Γ_i'' without corner points (we can use the connected component of ∂D containing Γ_i' and modify ∂D to be C^2 smooth in arbitrarily small neighborhood of each corner point). Let $r(\Gamma_i'')$ be a radius fulfilling the rolling disc property for Γ_i''. With

$$r_0 := \min\big(\min_{1\le i\le m} r(\Gamma_i''), \tfrac{1}{3}\min_{1\le i_1<i_2\le m} \operatorname{dist}(\Gamma_{i_1}', \Gamma_{i_2}')\big)$$

and $\Gamma' := \cup_{i=1}^m \Gamma_i'$ we have the following extended rolling disc property:

$$B_\pm(r, \boldsymbol{y}, \Gamma') \cap \Gamma' = \{\boldsymbol{y}\} \quad \text{for any} \quad 0 < r \le r_0 \quad \text{and any} \quad \boldsymbol{y} \in \Gamma'. \tag{3.4.1}$$

Once δ was selected, we impose

$$\frac{\delta}{4} \le r_0. \tag{3.4.2}$$

Let $\boldsymbol{y} \in \Gamma_{i_0}$ be a point such that $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \operatorname{dist}(\boldsymbol{x}, \Gamma_{i_0})$, by $\operatorname{dist}(\boldsymbol{x}, \{\boldsymbol{v}_j\}_{j=1}^k) \ge \delta$ clearly $\boldsymbol{y} \in \Gamma'$ and $\boldsymbol{x} - \boldsymbol{y}$ is orthogonal to Γ_{i_0} at $\boldsymbol{y}$. Because of (3.4.1) with $r = \frac{\delta}{4}$, but we claim that we have even stronger $B_\pm(\frac{\delta}{4}, \boldsymbol{y}, \partial D) \cap \partial D = \{\boldsymbol{y}\}$. Indeed, if $\boldsymbol{z} \in \partial D \setminus \Gamma'$, then $\operatorname{dist}(\boldsymbol{z}, \{\boldsymbol{v}_j\}_{j=1}^k) < \frac{\delta}{4}$ and $|\boldsymbol{y} - \boldsymbol{z}| \ge |\boldsymbol{x} - \boldsymbol{z}| - |\boldsymbol{x} - \boldsymbol{y}| > \frac{3\delta}{4} - \frac{\delta}{4} = \operatorname{diam}(B_\pm(\frac{\delta}{4}, \boldsymbol{y}, \partial D))$, as required.

It is clear that $B_\pm(\frac{\delta}{4}, \boldsymbol{y}, \partial D) \cap \partial D = \{\boldsymbol{y}\}$ implies $B_-(\frac{\delta}{4}, \boldsymbol{y}, \partial D) \subset D$. Let $\boldsymbol{u}$ be the outward unit normal vector to ∂D at $\boldsymbol{y}$. By (1.1.7), (1.1.8) and (3.1.1)

$$\begin{aligned}\lambda_n(\boldsymbol{x}, D) \ge \lambda_n(\boldsymbol{x}, B_-(\tfrac{\delta}{4}, \boldsymbol{y}, \partial D)) &= \tfrac{\delta^2}{16}\lambda_n((1 - \tfrac{d_{i_0}(\boldsymbol{x})}{\delta/4})\boldsymbol{u}, B)\\ &\approx c(\delta) n^{-1} \rho_n^*(d_{i_0}(\boldsymbol{x})).\end{aligned}$$

For the other direction, we have that $B_+(\frac{\delta}{4}, \boldsymbol{y}, \partial D) \cap \mathrm{int}(D) = \emptyset$. Further, (recall that $R = \mathrm{diam}(D)$) $D \subset 2RB + \boldsymbol{y} + \frac{\delta}{4}\boldsymbol{u}$, so for

$$D := \{\boldsymbol{z} \in \mathbb{R}^2 : \tfrac{\delta}{3} \leq \|\boldsymbol{y} + \tfrac{\delta}{3}\boldsymbol{u} - \boldsymbol{z}\|_2 \leq 2R\}$$

we have $D \subset D$ and by (1.1.7) and Lemma 3.9

$$\lambda_n(\boldsymbol{x}, D) \leq \lambda_n(\boldsymbol{x}, D) \leq c(\delta, R) n^{-1} \rho_n^*(d_{i_0}(\boldsymbol{x})).$$

To complete (3.0.1) we need to justify that for any j

$$\rho_n^*(d_j^-(\boldsymbol{x}))\rho_n^*(d_j^+(\boldsymbol{x})) \geq c(D) n^{-1} \rho_n^*(d_{i_0}(\boldsymbol{x})). \tag{3.4.3}$$

Indeed, since $\Gamma_j^+ \cap \Gamma_j^- = \{v_j\}$ there exists $\delta' > 0$ depending on D (and δ) such that $\max\{d_j^+(\boldsymbol{z}), d_j^-(\boldsymbol{z})\} < \delta'$ implies $\mathrm{dist}(\boldsymbol{z}, \{\boldsymbol{v}_j\}_{j=1}^k) < \frac{\delta}{4}$. Therefore $\max\{d_j^+(\boldsymbol{x}), d_j^-(\boldsymbol{x})\} \geq \delta'$, as well as $\min\{d_j^+(\boldsymbol{x}), d_j^-(\boldsymbol{x})\} \geq \mathrm{dist}(\boldsymbol{x}, \partial D) = d_{i_0}(\boldsymbol{x})$, so (3.4.3) justified.

Case 3: If $\mathrm{dist}(\boldsymbol{x}, \{\boldsymbol{v}_j\}_{j=1}^k) < \delta$.

For every j and each choice of $\pm$ we let $\Gamma_j^{\pm *}$ be some local linear extension of $\Gamma_j^{\pm}$ satisfying $(\Gamma_j^{\pm *} \setminus \Gamma_j^{\pm}) \cap D = \emptyset$. Such local linear extensions exist since all interior angles are less than π. Such a choice of local linear extensions depends only on D. Every $\Gamma_j^{\pm *}$ can be extended to a closed C^2 curve, so by the rolling disc property,

$$B_{\pm}(r, \boldsymbol{y}, \Gamma_j^{\pm *}) \cap \Gamma_j^{\pm *} = \{\boldsymbol{y}\} \quad \text{for any} \quad 0 < r \leq \tilde{r} \quad \text{and any} \quad \boldsymbol{y} \in \Gamma_j^{\pm *}, \tag{3.4.4}$$

where $\tilde{r}$ is the smallest radius fulfilling the rolling disc property for all (finitely many) curves $\Gamma_j^{\pm *}$. Similarly to Case 2, we will obtain

$$\gamma\delta \leq \tilde{r} \tag{3.4.5}$$

with some $\gamma = \gamma(D) \geq 2$ which will be selected later.

We will need the following properties (i)–(iv) of ∂D and $\Gamma_j^{\pm *}$ required from the choice of δ, see [39].

Property (i). Set $\delta_1 = \delta_1(D)$ to be the smallest length of the line segments $\Gamma_j^{\pm *} \backslash \Gamma_j^{\pm}$ and of the curves $\Gamma_j^{\pm}$ (over all j and choices of $\pm$). Let $\boldsymbol{y}_{\pm}(\boldsymbol{x})$ be a point from $\Gamma_j^{\pm *}$ such that $\operatorname{dist}(\boldsymbol{x}, \Gamma_j^{\pm *}) = \|\boldsymbol{x} - \boldsymbol{y}_{\pm}(\boldsymbol{x})\|_2$. If $\delta \leq \delta_1$, the above implies that for $\boldsymbol{x}$ in δ-neighborhood of $\boldsymbol{v}_j$ the point $\boldsymbol{y}_{\pm}(\boldsymbol{x})$ is not an endpoint of $\Gamma_j^{\pm *}$ so that $\boldsymbol{x} - \boldsymbol{y}_{\pm}(\boldsymbol{x})$ is orthogonal to the unit tangent vector of $\Gamma_j^{\pm}$ at $\boldsymbol{y}_{\pm}(\boldsymbol{x})$. Moreover, $\|\boldsymbol{y}_{\pm}(\boldsymbol{x}) - \boldsymbol{v}_j\|_2 \leq \delta$.

Property (ii). There exists $\delta_2 = \delta_2(D)$ such that for any $\delta \leq \delta_2$ the $(\gamma+1)\delta$-neighborhood U of $\boldsymbol{v}_j$ satisfies $U \cap \Gamma_{j'}^{\pm *} = \emptyset$ for $j' \neq j$ (possible by $(\Gamma_{j'}^{\pm *} \setminus \Gamma_{j'}^{\pm}) \cap D = \emptyset$) and $U \cap \partial D \subset \Gamma_j^{+} \cup \Gamma_j^{-}$.

For properties (iii) and (iv) we assume that $\delta \leq \min\{\delta_1, \delta_2\}$ and $\boldsymbol{x}$ is in δ-neighborhood of $\boldsymbol{v}_j$ for an arbitrary j.

Property (iii). Let $\boldsymbol{u}_{\pm}(\boldsymbol{y})$ be the unit normal vector to $\Gamma_j^{\pm *}$ at $\boldsymbol{y} \in \Gamma_j^{\pm *}$ chosen in continuous manner so that $\boldsymbol{u}_{\pm}(\boldsymbol{y})$ points outward of D when $\boldsymbol{y} \in \partial D$. Since D is a C^2 domain, the angle between $\boldsymbol{u}_{\pm}(\boldsymbol{y})$ and $\boldsymbol{u}_{\pm}(\boldsymbol{y}')$ does not exceed $c(D)|\boldsymbol{y} - \boldsymbol{y}'|$, for any $\boldsymbol{y}, \boldsymbol{y}' \in \Gamma_j^{\pm *}$. Combining this with property (i), we can ensure that the angle between $\boldsymbol{u}_{\pm}(\boldsymbol{y}_{\pm}(\boldsymbol{x}))$ and $\boldsymbol{u}_{\pm}(\boldsymbol{v}_j)$ is less than $\frac{\varepsilon}{3}$ whenever $\delta \leq \delta_3 = \delta_3(D)$, where $\varepsilon = \min_j\{\alpha_j, \pi - \alpha_j\}$ (recall that α_j is the interior angle of D at $\boldsymbol{v}_j$). Consequently, the angle between $\boldsymbol{u}_{+}(\boldsymbol{y}_{+}(\boldsymbol{x}))$ and $\boldsymbol{u}_{-}(\boldsymbol{y}_{-}(\boldsymbol{x}))$ is at least $\frac{\varepsilon}{3}$ and at most $\pi - \frac{\varepsilon}{3}$.

Property (iv). There exists $\delta_4 = \delta_4(D)$ such that for any $\delta \le \delta_4$

$$\mathrm{int}(B_+(\gamma\delta, \boldsymbol{y}_\pm(\boldsymbol{x}), \Gamma_j^{\pm *})) \cap \Gamma_j^{\mp} = \emptyset$$

which is rather obvious by $0 < \alpha_i < \pi$. (We make the same choice of either top or bottom sign in each $\pm$ or $\mp$.)

We consider $\delta := \min\{\delta_1, \dots, \delta_4, 4r_0, \frac{\tilde{r}}{\gamma}\}$, with $\gamma = \gamma(\varepsilon)$ will be selected later and then (3.4.2), (3.4.5) and properties (i)–(iv) are satisfied.

Now, we are ready for the proof. We let $d_\pm^*(\boldsymbol{x}) := \mathrm{dist}(\boldsymbol{x}, \Gamma_j^{\pm *})$ where j is such that $\mathrm{dist}(\boldsymbol{x}, \{\boldsymbol{v}_j\}_{j=1}^k) = \|\boldsymbol{x} - \boldsymbol{v}_j\|_2$. We have $\boldsymbol{y}_\pm(\boldsymbol{x}) \in \Gamma_j^{\pm *}$ is not an endpoint of $\Gamma_j^{\pm *}$ by property (i). Let $D_\pm := B_-(2\delta, \boldsymbol{y}_\pm(\boldsymbol{x}), \Gamma_j^{\pm *})$. By (3.4.4), (3.4.5), $\gamma \ge 2$ and property (ii), $D_\pm \cap \Gamma_j^{\pm *} = \emptyset$ and so $D := D_+ \cap D_-$ (which contains $\boldsymbol{x}$) satisfies $D \subset D$. Observe that D is the intersection of two discs of the same radius 2δ. We will apply Lemma 3.2 to an affine image of D. Let $\boldsymbol{o}_\pm$ be the center of $D_\pm$. We observe that $\boldsymbol{x}$ belongs to each line segment joining $\boldsymbol{o}_\pm$ and $\boldsymbol{y}_\pm(\boldsymbol{x})$, $l_\pm := \|\boldsymbol{o}_\pm - \boldsymbol{x}\|_2 = 2\delta - d_\pm^*(\boldsymbol{x}) \in [\delta, 2\delta]$. Since the property (iii), these two line segments intersect (at $\boldsymbol{x}$) at an angle θ which is between $\frac{\varepsilon}{3}$ and $\pi - \frac{\varepsilon}{3}$. Let $\theta_\pm$ be the angle opposite to $l_\pm$ in the triangle with the vertices at $\boldsymbol{x}$, $\boldsymbol{o}_+$ and $\boldsymbol{o}_-$. Without loss of generality, we can assume $d_+^*(\boldsymbol{x}) \ge d_-^*(\boldsymbol{x})$. Then $\theta_- \ge \theta_+$ and $\theta_- + \theta_+ = \pi - \theta$. We have

$$\tilde{h} := \|\boldsymbol{o}_- - \boldsymbol{o}_+\|_2 = \frac{\sin\theta}{\sin\theta_-} l_- \ge \frac{\sin\frac{\varepsilon}{3}}{1}\delta.$$

If $\theta_- \ge \frac{\pi}{2}$, then $\tilde{h} \le l_- \le 2\delta$. Otherwise,

$$\tilde{h} = \frac{\sin\theta}{\sin\theta_-} l_- \le \frac{\sin\theta}{\sin\frac{\pi-\theta}{2}} 2\delta = 2(\sin\tfrac{\theta}{2})2\delta \le (\cos\tfrac{\varepsilon}{6})4\delta.$$

There is an affine transform T which is a composition of appropriate rotation, translation and homothety with ratio $(2\delta)^{-1}$ such that Lemma 3.2 is applicable to $T(D)$. We obtain the required lower bound on $\lambda_n(\boldsymbol{x}, D)$ in the standard manner using (1.1.7) and (1.1.8):

$$\begin{aligned}\lambda_n(\boldsymbol{x}, D) &\geq \lambda_n(\boldsymbol{x}, D) = 4\delta^2 \lambda_n(T(\boldsymbol{x}), T(D)) \\ &\geq c(D)\rho_n^*(d_-^*(\boldsymbol{x}))\rho_n^*(d_+^*(\boldsymbol{x})) \geq c(D)\rho_n^*(d_-(\boldsymbol{x}))\rho_n^*(d_+(\boldsymbol{x})),\end{aligned}$$

where we used that $\sin(\frac{2\varepsilon}{3})d_\pm(\boldsymbol{x}) \leq d_\pm^*(\boldsymbol{x})$. Indeed, if $d_\pm(\boldsymbol{x}) \neq d_\pm^*(\boldsymbol{x})$ for some choice of $\pm$, then by property (i) $d_\pm(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{v}_j\|_2$ and the needed inequality follows from properties (i) and (iii).

For the upper bound, we will apply Lemma 3.10. we recall and reuse some notations. Let $D_\pm := B_+(\gamma\delta, \boldsymbol{y}_\pm(\boldsymbol{x}), \Gamma_j^{\pm *})$. By (3.4.4) and (3.4.5), $\operatorname{int}(D_\pm) \cap \Gamma_j^{\pm *} = \emptyset$. Moreover, by property (iv) $\operatorname{int}(D_\pm) \cap \Gamma_j^{\mp} = \emptyset$. Using property (ii) into account, we have $\operatorname{int}(D_- \cup D_+) \cap D = \emptyset$. Let $\boldsymbol{o}_\pm$ be the center of $D_\pm$. We have (due to $\delta \leq \operatorname{diam}(D) = R$) $D \subset 2RB + \boldsymbol{o}_\pm$, so we will apply Lemma 3.10 to an appropriate affine image of the set

$$D := ((2RB + \boldsymbol{o}_-) \setminus D_-) \cap ((2RB + \boldsymbol{o}_+) \setminus D_+)$$

containing D and get the proper estimate. Let us first justify the conditions of Lemma 3.10. Arguing similarly to the lower bound, we let $l_\pm := |\boldsymbol{o}_\pm - \boldsymbol{x}| = \gamma\delta + d_\pm^*(\boldsymbol{x}) \in [\gamma\delta, (\gamma+1)\delta]$. Due to property (iii), the lines containing l_i intersect (at $\boldsymbol{x}$) at an angle θ which is between $\frac{\varepsilon}{3}$ and $\pi - \frac{\varepsilon}{3}$. Let $\theta_\pm$ be the angle opposite to $l_\pm$

in the triangle with the vertices at $\boldsymbol{x}$, $\boldsymbol{o}_-$ and $\boldsymbol{o}_+$, then (because of the assumption $d_+^*(\boldsymbol{x}) \geq d_-^*(\boldsymbol{x})$) $\theta_- \leq \theta_+$ and $\theta_- + \theta_+ = \pi - \theta$. We have

$$\tilde{h} := \|\boldsymbol{o}_- - \boldsymbol{o}_+\|_2 = \frac{\sin\theta}{\sin\theta_+} l_+ \geq \frac{\sin\frac{\varepsilon}{3}}{1}\gamma\delta.$$

If $\theta_+ \geq \frac{\pi}{2}$, then $\tilde{h} \leq l_+ \leq (\gamma+1)\delta = \frac{\gamma+1}{2\gamma}\cdot 2\gamma\delta$. Otherwise,

$$\tilde{h} = \frac{\sin\theta}{\sin\theta_+} l_+ \leq \frac{\sin\theta}{\sin\frac{\pi-\theta}{2}}(\gamma+1)\delta = 2(\sin\tfrac{\theta}{2})(\gamma+1)\delta \leq \frac{\gamma+1}{\gamma}(\cos\tfrac{\varepsilon}{6})2\gamma\delta.$$

We need to choose a large enough $\gamma = \gamma(\varepsilon)$ so that for some $\zeta \in (0, \frac{1}{2})$ we have $\frac{\gamma+\nu}{\gamma}(\cos\frac{\varepsilon}{6}) < 1 - \zeta$, $\sin\frac{\varepsilon}{3} \geq \zeta$, $\frac{\gamma+\nu}{2\gamma} < 1 - \zeta$ and $\frac{\gamma+\nu}{\gamma} \leq 1 + \frac{\zeta}{2}$ (since the last condition we have $\|\boldsymbol{x} - \boldsymbol{o}_\pm\|_2 \leq (1 + \frac{\zeta}{2})\gamma\delta$). Now with appropriate T which is a composition of a rotation and a translation and with $r_1 = \gamma\delta$ and $r_2 = 2R$ by Lemma 3.10, (1.1.7) and (1.1.8)

$$\begin{aligned}\lambda_n(\boldsymbol{x}, D) &\leq \lambda_n(\boldsymbol{x}, D) = \lambda_n(T(\boldsymbol{x}), T(D)) \leq c(D)\rho_n^*(d_-^*(\boldsymbol{x}))\rho_n^*(d_+^*(\boldsymbol{x})) \\ &\leq c(D)\rho_n^*(d_-(\boldsymbol{x}))\rho_n^*(d_+(\boldsymbol{x})).\end{aligned}$$

To establish (3.0.1), by property (ii) we observe $\operatorname{dist}(\boldsymbol{x}, \Gamma_{j'}^{\pm}) \geq \delta$, for $j' \neq j$. $\square$

Remark 3.11. The arguments of Case 1 and Case 2 do not use the hypothesis that $0 < \alpha_j < \pi$, so under the conditions of Theorem 3.1, but without this hypothesis, for any $\delta > 0$ and any $\boldsymbol{x} \in D$ such that $\operatorname{dist}(\boldsymbol{x}, \{\boldsymbol{v}_j\}_{j=1}^k) \geq \delta$, we have

$$\lambda_n(\boldsymbol{x}, D) \approx c(\delta, D) \min_{1\leq i\leq m} n^{-1}\rho_n^*(\operatorname{dist}(\boldsymbol{x}, \Gamma_i)).$$

Chapter 4

Christoffel functions in higher dimensional spaces

This chapter contains our new results in higher dimensional spaces for Christoffel functions over convex polytopes.

4.1 Facts and notations

Let us introduce a few results about Christoffel functions in higher dimensional spaces of which we know. We will use the following equivalence for a cube in $\mathbb{R}^d$

$$\lambda_n\left((x_1, \ldots, x_d), [-1,1]^d\right) \approx c(d) \prod_{i=1}^{d} \rho_n(x_i). \tag{4.1.1}$$

Equation (4.1.1) can be obtained applying standard tensor product arguments to the one-dimensional version (1.1).

We will use the following equivalence for the unit cube in $\mathbb{R}^d$

$$\lambda_n\left((x_1,\ldots,x_d),[0,1]^d\right) \approx c(d)\prod_{i=1}^{d}\rho_n^*(x_i),\ \text{if } x_i\in[0,\frac{1}{2}],\ 1\le i\le d, \tag{4.1.2}$$

which follows from (4.1.1). We define the convex hull of a set of points $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_k$ as

$$\{\lambda_1\boldsymbol{x}_1+\lambda_2\boldsymbol{x}_2+\ldots+\lambda_k\boldsymbol{x}_k:\ \lambda_i\ge 0,\ \sum_{i=1}^{k}\lambda_i=1\}.$$

Next, we give some preliminaries about convex polytopes in $\mathbb{R}^d$ following [20].

We say that a halfspace is a set (in $\mathbb{R}^d$) of the form

$$\{\boldsymbol{x}\in\mathbb{R}^d:\ \mathbf{a}\cdot\boldsymbol{x}\le b\}$$

for some vector $\mathbf{a}\in\mathbb{R}^d$, and a real number b.

Definition 4.1. *A polytope* P *in* $\mathbb{R}^d$ *is a bounded intersection of finitely many half-spaces,*

$$\mathrm{P}=\{\boldsymbol{x}\in\mathbb{R}^d:\ \mathbf{a}_i\cdot\boldsymbol{x}\le b_i,\ 1\le i\le k.\}$$

There is an equivalent definition of a polytope in $\mathbb{R}^d$.

Definition 4.2. *A polytope in* $\mathbb{R}^d$ *is the convex hull of a finite set of points in* $\mathbb{R}^d$*.*

Definition 4.3. *Convex hull* $\mathrm{conv}\{\mathbf{a},\mathbf{b},\ldots,\mathbf{c}\}$ *of a set* $\{\mathbf{a},\mathbf{b},\ldots,\mathbf{c}\}$ *of points in an affine space over the reals is the smallest convex set that contains* $\{\mathbf{a},\mathbf{b},\ldots,\mathbf{c}\}$*.*

Let $\mathrm{P} \subset \mathbb{R}^d$ be a convex polytope. We say that a linear equation $\mathbf{a} \cdot \boldsymbol{x} \leq b_0$ is valid for P if it is satisfied for all $\boldsymbol{x} \in \mathrm{P}$.

The affine hull of a finite point set S is defined as the set of all affine combinations of elements of S,

$$\mathrm{aff(S)} := \Big\{ \sum_{\mathrm{i}=1}^{\mathrm{k}} \alpha_{\mathrm{i}} \boldsymbol{x}_{\mathrm{i}} : \ \mathrm{k} > 0, \boldsymbol{x}_{\mathrm{i}} \in \mathrm{S}, \alpha_{\mathrm{i}} \in \mathbb{R}, \sum_{\mathrm{i}=1}^{\mathrm{k}} \alpha_{\mathrm{i}} = 1 \Big\}. \tag{4.1.3}$$

Definition 4.4. *The dimension* $\dim(\mathrm{aff(S)})$ *of affine hull of S is the minimum possible cardinality of a set T such that* $\mathrm{aff(S)} = \mathrm{aff(T)}$.

A set of $n \geq 1$ points is called affinely independent if its affine hull has dimension $n-1$, that is, if every proper subset has a smaller affine hull.

Definition 4.5. *A face of* P *is any set of the form*

$$F = \mathrm{P} \cap \{\boldsymbol{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \boldsymbol{x} = b_0\},$$

where $\mathbf{a} \cdot \boldsymbol{x} \leq b_0$ *is a valid inequality for* P*. The dimension of a face is the dimension of its affine hull:* $\dim(\mathrm{F}) = \dim(\mathrm{aff(F)})$.

The faces of P of dimensions 0 are called vertices, the faces of dimension 1 are called edges, the faces of dimension $\dim(\mathrm{P}) - 1$ are called facets.

We call a d-dimensional polytope in $\mathbb{R}^d$ a simple polytope if each of its vertices is adjacent to exactly d edges, where an edge is a 1-dimensional face. So, in the case of $d = 2$ any convex polygon is a simple polytope.

We need the following properties:

Property (i)[20, Property 2.3 (ii)]. Every intersection of faces of a d-dimensional polytope P is a face of P.

A k-dimensional face of a polytope P is a face of P we call by k-face.

Property (ii)[20, Property 2.16 (iv)]. Every k-face of a simple polytope P is contained in precisely $d-k$ facets of P.

Definition 4.6. *We call a subset S of $\mathbb{R}^d$ a hyperplane if there exist real numbers $a_0, \ldots, a_d \in \mathbb{R}$ so that $|a_0| + \ldots + |a_d| \neq 0$ and*

$$S = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \; a_1 x_1 + \ldots + a_d x_d = a_0\}.$$

Definition 4.7. *The angle $\angle(H, G)$ between two planes H and G in $\mathbb{R}^3$ is the angle between the normals to these planes.*

4.2 Simple polytopes

Let us formulate the central result of this section.

Theorem 4.8. *For any d-dimensional simple polytope* P*, we denote facets by F_i and the corresponding $(d-1)$-dimensional planes which contain the facets by H_i, $F_i \subset H_i$, $i = 1, \ldots, l$. For any point $\boldsymbol{x} \in \mathrm{P}$ we denote by $\{d_i^*(\boldsymbol{x})\}_{i=1,\ldots,l}$ the non-decreasing rearrangement of $\{d_i(\boldsymbol{x})\}_{i=1,\ldots,l}$, where $d_i(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, H_i)$. Then*

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \approx c(\mathrm{P}) \prod_{i=1}^{d} \rho_n^*(d_i^*(\boldsymbol{x})). \tag{4.2.1}$$

To prove this theorem, we will use the following lemma.

Lemma 4.9. *For any k, $k \leq d$, assume that facets F_j, $1 \leq j \leq k$, of a simple polytope* P*, $1 \leq j \leq k$, have non-empty intersection. Let $\mathbf{n}_j$, $1 \leq j \leq k$, be the outward unit normal vector to F_j, $1 \leq j \leq d$. Then the vectors $\mathbf{n}_j$, $1 \leq j \leq k$, are linearly independent.*

Proof. By the property (i) the intersection $\cap_{j=1}^{k} F_j$ is a face of P which (being non-empty) must contain a vertex, call it $\boldsymbol{v}$. Without loss of generality we suppose that the vertex $\boldsymbol{v}$ is the origin. Then, by property (ii) there are exactly d facets containing $\boldsymbol{v}$, and k of them must be those that we already have. The equation of the hyperplane containing a facet F_i, $1 \leq i \leq d$ looks like $\mathbf{n}_j \cdot \boldsymbol{x} = 0$ with outward normal $\mathbf{n}_j$ to F_j, $1 \leq j \leq d$. So, without loss of generality we can assume that F_j, $1 \leq j \leq d$ are the facets containing $\boldsymbol{v}$. Hence, the system of equations $\mathbf{n}_j \cdot \boldsymbol{x} = 0$, $1 \leq j \leq d$ has a unique solution because these d facets meet exactly at the vertex $\boldsymbol{v}$, which is the origin. By standard linear algebra, the system of vectors $\mathbf{n}_j$, $1 \leq j \leq d$, is linearly independent implying linear independence of $\mathbf{n}_j$, $1 \leq j \leq k$. $\square$

Remark 4.10. The statement of Lemma 4.9 holds true for $d = 3$ for any (not necessarily simple) polytope.

Indeed, there are two cases, $k = 2$ and $k = 3$. In the first case, two outward normal vectors to two distinct facets are linearly dependent if and only if they are parallel. Then the intersection of these facets is empty. Hence, if $F_i \cap F_j \neq \emptyset$ then the vectors $\mathbf{n}_i$ and $\mathbf{n}_j$ are linearly independent. For $k = 3$, it follows from this statement and the proof of Lemma 4.9. For any three distinct facets F_1, F_2, F_3 of a polytope

$\mathrm{P} \subset \mathbb{R}^3$ such that $F_1 \cap F_2 \cap F_3 \neq \emptyset$ we have $F_1 \cap F_2 \cap F_3 = \boldsymbol{v}$, where $\boldsymbol{v}$ is a vertex of P. Let us assume that the statement fails. This means that there is a common for all three facets F_i, $1 \leq i \leq 3$ edge e of the polytope, $F_1 \cap F_2 \cap F_3 = e$. Let us consider three corresponding planes H_i, $1 \leq i \leq 3$, so that $e \subset \bigcap_{i=1}^{3} H_i$. We can consider the plane H_0 perpendicular to e. We define then the lines e_i, $1 \leq i \leq 3$ as intersections $e_i = H_i \cap H_0$. Because $e_i \in H_0$, $1 \leq i \leq 3$ they are linearly dependent. Therefore, H_i formed by e and e_i, $1 \leq i \leq 3$ are linearly dependent and one can be written as convex hull of two others which contradicts to Definition of polytope. Let us notice that the statement of this Remark can be obtain due to the fact [20, Theorem 2.7], "Diamond property".

Definition 4.11. *Let us consider* $S = \cup_{2 \leq k \leq d} S_k$*, where* S_k *is the collection of all* k*-element subsets* $\{F_1, \ldots, F_k\}$ *of* $\mathfrak{F}$ *such that* $\cap_{1 \leq j \leq k} F_j \neq \emptyset$ *and* $\mathfrak{F}$ *is the set of all facets of a polytope* P*.*

Let us introduce the following subsets of P for any $\delta > 0$:

$$Y_i(\delta) := \{\boldsymbol{y} \in \mathrm{P} : \operatorname{dist}(\boldsymbol{y}, H_i) \leq \delta\}. \tag{4.2.2}$$

Property (iii). There exists δ_0 which depends on the polytope P so that for any set $\{F_1, \ldots, F_k\}$, $1 \leq k \leq d$, of facets of P we have $Y_1(\delta) \cap Y_2(\delta) \cap \cdots \cap Y_k(\delta) \neq \emptyset$ for any $\delta < \delta_0$ if and only if $F_1 \cap F_2 \cap \cdots \cap F_k \neq \emptyset$.

Indeed, if $Y_1(\delta) \cap Y_2(\delta) \cap \cdots \cap Y_k(\delta) \neq \emptyset$ for any $0 < \delta < \delta_0$, then there exist $n_0 \in \mathbb{N}$ so that $\frac{1}{n} \leq \delta_0$ if $n \geq n_0$. We define $W := \bigcap_{n \geq n_0} \left(Y_1(\frac{1}{n}) \cap \ldots \cap Y_k(\frac{1}{n})\right)$,

$W \subset \mathrm{P}$. Because all sets $Y_j(\frac{1}{n})$ are compact and nonempty, the sequence $\{Y_1(\frac{1}{n}) \cap \ldots \cap Y_k(\frac{1}{n})\}_{n \geq n_0}$ is decreasing, we deduce by Cantor's Intersection Theorem that $W \neq \emptyset$. Then any point $\boldsymbol{y} \in W$ belongs to $Y_j(\frac{1}{n})$, $1 \leq j \leq k$, for any $n \geq n_0$. It means that $\mathrm{dist}(\boldsymbol{y}, H_i) = 0$, hence $F_1 \cap F_2 \cap \cdots \cap F_k \neq \emptyset$.

On the other hand, if $F_1 \cap F_2 \cap \cdots \cap F_k \neq \emptyset$ for any $\boldsymbol{y} \in F_1 \cap F_2 \cap \cdots \cap F_k$ we have $\boldsymbol{y} \in F_j$, $\boldsymbol{y} \in Y_j(\delta)$ for any $1 \leq j \leq k$ and any $\delta > 0$.

To formulate one more lemma, which will be used to prove the main theorem, we have to define the following collection of transforms.

Definition 4.12. *For any set $\{F_1, \ldots, F_k\} = \mathcal{F} \in S_k$ with the corresponding outward unit normal vectors $\{\mathbf{n}_1, \ldots, \mathbf{n}_k\}$, there is a non-singular affine transform $T_{\mathcal{F}}$ that maps the inward unit normal vectors $-\mathbf{n}_j$, $1 \leq j \leq k$ to the first k basic vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k \in \mathbb{R}^d$. Then we define $\mathcal{T}$ as the set of all such transforms $T_{\mathcal{F}}$ for all possible $\mathcal{F} \in S_k$, $2 \leq k \leq d$.*

Notice that the set $\mathcal{T}$ is finite because S_k has at most $\binom{p}{k}$ elements if the polytope P has p facets.

Lemma 4.13. *There exists positive constant $M > 0$ depending on the polytope and independent of $\boldsymbol{x}$ so that for any $2 \leq k \leq d$ and for any $\{F_1, \ldots, F_k\} = \mathcal{F} \in S_k$ and the corresponding $\mathbf{n}_1, \ldots, \mathbf{n}_k$, $T_{\mathcal{F}}$ as in Definition 4.12, we have*

$$M^{-1} d_j(\boldsymbol{x}) \leq \overline{d}_j(\boldsymbol{x}) \leq M d_j(\boldsymbol{x}),\ 1 \leq j \leq k, \tag{4.2.3}$$

where $\overline{d}_j(\boldsymbol{x}) = \mathrm{dist}(T_{\mathcal{F}}(\boldsymbol{x}), T_{\mathcal{F}}(H_j))$, $d_j(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, H_j)$, and H_j is the hyperplane *containing F_j.*

Proof. We have $T_{\mathcal{F}}(\cdot) = A_{\mathcal{F}} \cdot + \mathbf{a}_{\mathcal{F}}$ for a non-singular $d \times d$ matrix $A_{\mathcal{F}}$ and a vector $\mathbf{a}_{\mathcal{F}} \in \mathbb{R}^d$. With

$$M_{\mathcal{F}} := \max\{\|A_{\mathcal{F}}\|_2, \|A_{\mathcal{F}}\|_2^{-1}\},$$

where

$$\|A\|_2 := \sup_{\boldsymbol{y}\in\mathbb{R}^d, \boldsymbol{y}\neq\mathbf{0}} \frac{\|A\boldsymbol{y}\|_2}{\|\boldsymbol{y}\|_2},$$

we clearly have

$$M_{\mathcal{F}}^{-1} d_j(\boldsymbol{x}) \leq \overline{d}_j(\boldsymbol{x}) \leq M_{\mathcal{F}} d_j(\boldsymbol{x}).$$

We conclude by taking $M := \max\{M_{\mathcal{F}},\ \mathcal{F} \in S\}$. □

We are ready to prove Theorem 4.8 using Lemmas 4.9 and 4.13.

Proof of Theorem 4.8. Recall that $\{d_i^*(\boldsymbol{x})\}$ is the non-decreasing rearrangement of $\{d_i(\boldsymbol{x})\}_{i=1,\dots,l}$, where $d_i(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, H_i)$. Without loss of generality, let F_i be the facet so that $d_i^*(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, F_i)$, $1 \leq i \leq l$. For the fixed point $\boldsymbol{x}$, we will use shorthand d_i, d_i^* and $\overline{d}_i$ for simplicity.

For any set of distances d_j^*, $1 \leq j \leq d$ we define the following set of indices J as

$$J := \{j :\ 0 \leq j \leq d-1,\ d_{j+1}^* > \delta_1 \gamma^{j+1}\}, \tag{4.2.4}$$

where constants $\gamma > 1$ and $\delta_1 > 0$ will be chosen later. If $J = \emptyset$ we set $k := d$. Otherwise, we define k, $0 \leq k \leq d-1$ as

$$k := \min\{j :\ j \in J\}. \tag{4.2.5}$$

Then

$$d_j^* \le \delta_1 \gamma^j \text{ for } 1 \le j \le k, \tag{4.2.6}$$

while

$$d_j^* > \delta_1 \gamma^{k+1} \text{ for } j \ge k+1 \text{ if } k < d. \tag{4.2.7}$$

If

$$\delta_1 \gamma^k \le \delta_0 \text{ for } 1 \le j \le k, \tag{4.2.8}$$

then $\boldsymbol{x} \in Y_1(\delta_0) \cap Y_2(\delta_0) \cap \cdots \cap Y_k(\delta_0)$, so by Property (iii), $F_1 \cap F_2 \cap \cdots \cap F_k \neq \emptyset$. We can guarantee (4.2.8) if we impose

$$\delta_1 \gamma^d \le \delta_0, \tag{4.2.9}$$

which will be done at a later time. As $F_1 \cap F_2 \cap \cdots \cap F_k \neq \emptyset$, by Definition 4.12 there is a transform $T_{\mathcal{F}}$ from the collection $\mathcal{T}$ so that

$$T(\mathrm{P}) \subset T(H_1^+) \cap \ldots \cap T(H_k^+), \tag{4.2.10}$$

where $T(H_i^+) = \{\boldsymbol{x} \in \mathbb{R}^d : \ x_i \ge 0\}$, and H_i^+ is the half-space containing P such that $\partial H_i^+ = H_i$, $i = 1, \ldots, k$. Then, by Lemma 4.13 with $d_j = d_j^*$ and (4.2.6) there exist a transform $\widetilde{T}(\cdot) = T(\cdot) - (0, \ldots, 0, -z_{k+1}, \ldots, -z_d)$ so that $T(\boldsymbol{x}) = (z_1, \ldots, z_d)$ and $\widetilde{T}(\boldsymbol{x}) = \big(\overline{d}_1, \ldots, \overline{d}_i, 0, \ldots, 0\big) \in \mathbb{R}^d$. Also, $0 \neq |\det \widetilde{T}| \le c(\mathrm{P})$, so

$$\max\{|\det \widetilde{T}|, |\det \widetilde{T}^{-1}|\} \le c(\mathrm{P}), \tag{4.2.11}$$

since there are finitely many $\widetilde{T} \in \mathcal{T}$.

By the argument from the proof of Lemma 4.13, the diameter of $T\,(\mathrm{P})$ is at most MD, where D is the diameter of P. So, $T\,(\mathrm{P}) \subset \big(MD \cdot B^d + T(\boldsymbol{x})\big)$ and $T\,(\mathrm{P}) \subset [-MD, MD]^d + T(\boldsymbol{x})$. Therefore, by

$$\overline{d}_i \le M d_i^* \le MD,\ 1 \le i \le k$$

and (4.2.10), we have the following inclusion

$$T(\boldsymbol{x}) \in T\,(\mathrm{P}) \subset [0, 2MD]^k \times [-MD, MD]^{d-k}.$$

Due to the above inclusion, (4.1.1), (1.1.7), (1.1.8) and (4.2.11) we obtain the upper bound

$$\lambda_n(\boldsymbol{x}, T\,(\mathrm{P})) \le \lambda_n(\boldsymbol{x}, [0, 2MD]^k \times [-MD, MD]^{d-k}) \approx c(\mathrm{P}) \prod_{i=1}^{k} \rho_n^*(d_i^*). \qquad (4.2.12)$$

To find a lower bound on $\lambda_n(\boldsymbol{x}, T\,(\mathrm{P}))$ we define the following k-dimensional parallelotope $P_k = [0, \alpha]^k$ and $(d-k)$-dimensional parallelotope $Q_k = [-\beta, \beta]^{d-k}$, where positive constants α and β will be selected later. We want to have

$$\widetilde{T}(\boldsymbol{x}) \in \overline{P}_k \times Q_k,\ \text{ and } P_k \times Q_k \subset T\,(\mathrm{P})\,, \qquad (4.2.13)$$

where $\overline{P}_k = [0, \frac{\alpha}{2}]^k$. Since $\widetilde{T}(\boldsymbol{x}) = (\overline{d}_1, \ldots, \overline{d}_k, 0, \ldots, 0)$, to guarantee $T(\boldsymbol{x}) \in \overline{P}_k \times Q_k$, it is enough to impose $\overline{d}_i \le \frac{\alpha}{2}$, $1 \le i \le k$. Let us impose the stronger constraint

$$\overline{d}_i \le \frac{\alpha}{5},\ 1 \le i \le k, \qquad (4.2.14)$$

which will be used in Section 4.3. For this, it is sufficient to have $d_i^* < \frac{\alpha}{5} M^{-1}$, $1 \le i \le k$, or because of (4.2.6) $\delta_1 \gamma^k \le \frac{\alpha}{5} M^{-1}$, so we will choose

$$\alpha := 5 \delta_1 M \gamma^k. \qquad (4.2.15)$$

Let us show $P_k \times Q_k \subset \widetilde{T}\,(\mathrm{P})$, or $\mathrm{Int}\,(P_k \times Q_k) \subset \mathrm{Int}\widetilde{T}\,(\mathrm{P})$ which implies the required inclusion. We have $\mathrm{Int}\ \widetilde{T}\,(\mathrm{P}) \subset \bigcap_{i=1}^{k} \mathrm{Int}\left(\widetilde{T}\left(H_i^+\right)\right)$. As $\mathrm{Int}\left(\widetilde{T}\left(H_i^+\right)\right) = \{x_i > 0\}$ for $1 \leq i \leq k$, we have

$$\mathrm{Int}\,(P_k \times Q_k) = (0,\alpha)^k \times (-\beta,\beta)^{d-k} \subset \bigcap_{i=1}^{k} \mathrm{Int}\left(\widetilde{T}\left(H_i^+\right)\right).$$

Let us fix an arbitrary $i > k$. We have $T(\boldsymbol{x}) \in T(H_i^+)$. For any point $\boldsymbol{y} \in P_k \times Q_k$ we will show that $\boldsymbol{y} \overline{\in} \widetilde{T}(H_i)$ which in combination with $\widetilde{T}(\boldsymbol{x}) \in \widetilde{T}(H_i^+)$ implies the required

$$\mathrm{Int}\,(P_k \times Q_k) \subset \mathrm{Int}\ \widetilde{T}\left(H_i^+\right).$$

So, we need

$$\mathrm{dist}\left(\boldsymbol{y}, \widetilde{T}(H_i)\right) > 0. \tag{4.2.16}$$

To prove the inequality (4.2.16) we will consider two different cases.

1. If $k < d$, we have

   $$\|\boldsymbol{y} - \widetilde{T}(\boldsymbol{x})\|_2 \leq \alpha k + \beta(d-k)$$

   and because of Lemma 4.13 and (4.2.7), we have

   $$\mathrm{dist}\left(\widetilde{T}(\boldsymbol{x}), \widetilde{T}(H_i)\right) \geq M^{-1} d_{k+1}^{*} > aM^{-1}\gamma^{-d+k+1}.$$

   Therefore,

   $$\mathrm{dist}\left(\boldsymbol{y}, \widetilde{T}(H_i)\right) \geq \mathrm{dist}\left(\widetilde{T}(\boldsymbol{x}), \widetilde{T}(H_i)\right) - \|\boldsymbol{y} - \widetilde{T}(\boldsymbol{x})\|_2$$

   $$> \delta_1 M^{-1}\gamma^{k+1} - \alpha k - \beta(d-k) = 0$$

if we set

$$\beta := \frac{1}{d-k}\left(\delta_1 M^{-1}\gamma^{k+1} - \alpha k\right),$$

which will be positive (see (4.2.15)) if $\delta_1 M^{-1}\gamma^{k+1} > 5\delta_1 k M \gamma^k$, which, in turn, requires $\gamma > 5M^2 k$. Then we set

$$\gamma := 6M^2 d. \tag{4.2.17}$$

2. The second case happens when $k = d$, by (4.2.6). In this case we have that $d_j^* \leq \delta_1 \gamma^d$ for all $1 \leq j \leq d$. Due to Proposition (ii) there are exactly d facets of $T(\mathrm{P})$ that intersect at the origin which are $T(F_i)$, $1 \leq i \leq d$. Since any $T(H_i)$ with $i > d$ does not contain the origin (as that would be at least $d+1$ facets meeting at a vertex), $\mathrm{dist}(\mathbf{0}, T(H_i)) \geq M^{-1}\delta$, where $\delta = \delta(\mathrm{P})$ is the smallest distance from a vertex of P to a facet not containing this vertex (this is clearly strictly positive number depending only on P). Now we are ready to choose a so that $[0,\alpha]^d = [0, 5\delta_1\gamma^d M]^d$ does not intersect $T(H_i)$. As $|\boldsymbol{y}| \leq \sqrt{d}\alpha$, we need $\sqrt{d}\alpha < M^{-1}\delta$. Taking into account (4.2.9), (4.2.15) and (4.2.14) we can guarantee these inequalities setting

$$\delta_1 := \min\left\{\frac{\delta}{10M^2\sqrt{d}(6M^2 d)^d}, \ \frac{\delta_0}{(6M^2 d)^d}\right\}. \tag{4.2.18}$$

We established (4.2.16), and, consequently, (4.2.13). This establishes the following lower bound of Christoffel functions using (4.1.1), (1.1.7), and (1.1.8)

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \geq c(\mathrm{P}) \prod_{i=1}^{d} \rho_n^*(d_i^*). \tag{4.2.19}$$

Because the upper bound and lower bounds (4.2.12) and (4.2.19) coincide, we get the needed equivalence

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \approx c(\mathrm{P}) \prod_{i=1}^{k} \rho_n^*(d_i^*).$$

□

4.3 Polytopes in $\mathbb{R}^3$

We will establish the statement similar to the one from Theorem 4.8 for an arbitrary convex polytope P in $\mathbb{R}^3$.

Theorem 4.14. *For any* 3*-dimensional convex polytope* P*, we denote by* $\{d_i^*(\boldsymbol{x})\}_{i=1,\ldots,l}$ *the non-decreasing rearrangement of the distances* $\{d_i(\boldsymbol{x})\}_{i=1,\ldots,l}$*, where* $\boldsymbol{x} \in \mathrm{P}$*,* $d_i(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, H_i)$*,* H_i *is the plane containing the facet* F_i*,* $i = 1, \ldots, l$*. Then*

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \approx c(\mathrm{P}) \rho_n^*(d_1^*(\boldsymbol{x})) \rho_n^*(d_2^*(\boldsymbol{x})) \rho_n^*(d_3^*(\boldsymbol{x})). \tag{4.3.1}$$

We observe that, taking into account Remark 4.10 and following the proof of the upper bound on $\lambda_n(\boldsymbol{x}, \mathrm{P})$ in Theorem 4.8, we readily obtain the upper bound on $\lambda_n(\boldsymbol{x}, \mathrm{P})$ in (4.3.1).

We remark that one cannot proceed in the same way for the lower bound on $\lambda_n(\boldsymbol{x}, \mathrm{P})$ because Property (ii) is, generally speaking, invalid for general polytopes: there may be more that three facets intersecting at a vertex. However, some arguments will remain very similar and ultimately considering an appropriate planar section of the polytope, we will reduce the problem to the 2-dimensional case where any polygon is a simple polytope.

Before moving far, we need the following Lemma.

Lemma 4.15. *With the above assumptions and notations, for any* $\boldsymbol{y} \in \overline{\mathrm{P}}$ *and any line* $l \in \overline{H}$ *satisfying* $l \cap \overline{\mathrm{P}} \neq \emptyset$*, we have*

$$d^{\circ}(\boldsymbol{y}, l) \approx c(\mathrm{P})\mathrm{dist}(\boldsymbol{y}, \mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}})).$$

In other words, the distance from the point $\boldsymbol{y} \in \overline{\mathrm{P}}$ *to the line* l *is equivalent to the distance from* $\boldsymbol{y}$ *to the plane* $\mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}})$*.*

Proof. We consider the right triangle formed by perpendicular of length $d^{\circ}(\boldsymbol{y}, l)$ from the point $\boldsymbol{y}$ to l in the plane $\overline{H}$ and the perpendicular of length $\mathrm{dist}(\boldsymbol{y}, \mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}}))$ to the plane $\mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}}))$. In the defined triangle the angle opposite to $d^{\circ}(\boldsymbol{y}, l)$ equals to the angle between planes $\mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}}))$ and $\overline{H}$, so

$$\frac{d^{\circ}(\boldsymbol{y}, l)}{\mathrm{dist}(\boldsymbol{y}, \mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}}))} = \sin\angle(\mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}})), \overline{\mathrm{H}}) \text{ and } \mathrm{k}(\mathrm{P}) \leq \frac{\mathrm{d}^{\circ}(\boldsymbol{y}, \mathrm{l})}{\mathrm{dist}(\boldsymbol{y}, \mathrm{aff}(\mathrm{l}, \boldsymbol{v}_{\mathrm{j}}))} \leq \sin\alpha,$$

where

$$\alpha := \min_{1 \leq i \leq q} \angle(\mathbf{m}_j, \mathbf{n}_i), \ 1 \leq i \leq q.$$

□

Proof of lower bound on $\lambda_n(\boldsymbol{x}, \mathrm{P})$ *in* (4.3.1). For the fixed point $\boldsymbol{x}$ we will use short-hand d_i and d_i^* for simplicity. Let us consider any vertex $\boldsymbol{v}_j$ of P. Without loss of generality we denote the facets which correspond to the distances $d_i^*,\ i = 1, 2, 3$, by F_1, F_2 and F_3.

We will get a lower bound for Christoffel functions in two separate situations in $\mathbb{R}^3$.

1. We consider the set J and k, $0 \le k \le 2$, defined (with $d = 3$) as in (4.2.4) and (4.2.5), assume $J \neq \emptyset$. Then we have (4.2.6) and (4.2.7) and taking into account Remark 4.10, we can follow the scheme of proof of Theorem 4.8 to obtain the lower bound
$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \ge c(\mathrm{P}) \prod_{i=1}^{k} \rho_n^*(d_i^*) \approx c(\mathrm{P}) \prod_{i=1}^{3} \rho_n^*(d_i^*).$$
The constants γ and δ_1 satisfying (4.2.9) will be chosen later.

2. The only remaining case which cannot be done similarly to the proof of Theorem 4.8 is the case when $J = \emptyset$, in other words $k = d = 3$, so $d_i^* \le \delta_1 \gamma^3$, $1 \le i \le 3$. In this case by (4.2.9), Remark 4.10 and Property (iii) $F_1 \cap F_2 \cap F_3 = \boldsymbol{v}_j$ for some vertex $\boldsymbol{v}_j$, but in contrast to Theorem 4.8, there maybe other facets that also contain $\boldsymbol{v}_j$.

   We define $\mathbf{m}_j$ as an outward vector for polytope P starting at vertex $\boldsymbol{v}_j$ such that is not collinear to any normal vector of the planes H_i, $1 \le i \le l$. Then there exists a constant $k(\mathrm{P}) > 0$ so that $\angle(H_i, H) \ge \sin\angle(H_i, H) \ge k(\mathrm{P})$ for any facet H_i containing vertex $\boldsymbol{v}_j$. Namely, we can take
$$k(\mathrm{P}) := \min_{1 \le j \le m,\ 1 \le i \le l} \left\{ \sqrt{1 - (\mathbf{m}_j \cdot \mathbf{n}_i)^2} \right\}.$$

We denote by H the supporting plane of the vertex $\boldsymbol{v}_j$ having $\mathbf{m}_j$ as its normal vector, $1 \leq j \leq m$. Let $\overline{H}$ be a translation of H such that $\boldsymbol{x} \in \overline{H}$.

We want to ensure that the point $\boldsymbol{x} \in \mathrm{P}$ is "close enough" to the vertex $\boldsymbol{v}_j$. Here we call a point as "close enough" to vertex $\boldsymbol{v}_j$ if the translated plane $\overline{H}$ intersects all the edges of P coming out of $\boldsymbol{v}_j$ and does not intersect any other edges. In fact we will require a slightly stronger condition:

$$\|\boldsymbol{x} - \boldsymbol{v}_j\|_2 \leq \frac{\delta'}{2}, \tag{4.3.2}$$

where

$$\delta' := \min_{1 \leq i \leq m} \min_{1 \leq j \leq m, i \neq j} (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot (-\mathbf{m}_j). \tag{4.3.3}$$

Remark that the polytope $\overline{\mathrm{P}}$ defined as $\overline{\mathrm{P}} = \overline{H} \cap \mathrm{P} \subset \mathbb{R}^2$ is a simple one, namely, it is a convex polygon.

Without loss of generality we can assume that n facets of P containing the vertex $\boldsymbol{v}_j$ are F_i, $1 \leq i \leq q$, $q \geq 3$.

We will guarantee that (4.3.2) holds by imposing

$$\boldsymbol{x} \in I(\delta_2),\ \ I(\delta_2) := \left(Y_1(\delta_2) \cap \ldots \cap Y_n(\delta_2)\right), \tag{4.3.4}$$

where $Y_i(\cdot)$ are defined as in (4.2.2) with $d = 3$ and δ_2 is sufficiently small.

Using similar arguments as in the proof of Proposition (iii) we can obtain

$$\lim_{\delta \to 0^+} \operatorname{diam}\left(I(\delta)\right) = 0.$$

Because of the above statement, there exists $\delta_2 > 0$ such that for any $\delta < \delta_2$ if $\boldsymbol{x} \in I(\delta)$ then we have (4.3.2).

Combining (4.2.9) with the new requirement on δ_1, we will need

$$\delta_1 \gamma^3 \leq \min\{\delta_0, \delta_2\}. \tag{4.3.5}$$

Let us denote the distances from a point $\boldsymbol{y} \in \overline{\mathrm{P}}$ to a line $l \in \overline{H}$ such that $l \cap \overline{\mathrm{P}} \neq \emptyset$ by $d^\circ(\boldsymbol{y}, l)$. The line containing $F_i \cap \overline{H}$ will be denoted by l_i, $1 \leq i \leq q$.

We now turn back to the proof. Let us denote by $d_i^\circ = d_i^\circ(\boldsymbol{x}, l_i)$, $i = 1, 2, 3$ for the fixed $\boldsymbol{x}$. According to Lemma 4.15, we have $d_i^\circ \approx c(\mathrm{P}) d_i^*$, $i = 1, 2, 3$. So, our aim is to obtain the lower bound

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \geq c(\mathrm{P}) \prod_{i=1}^{3} \rho_n^*(d_i^\circ).$$

Without loss of generality we assume that $d_1^\circ \leq d_2^\circ \leq d_3^\circ$. Let us denote by B the diameter of the cross-section $\overline{\mathrm{P}}$. Then we have

$$\frac{d_3^\circ}{B} \geq k_0(\mathrm{P}), \tag{4.3.6}$$

where $k_0(\mathrm{P})$ is a positive constant such that

$$\sin \angle(F_i \cap \overline{H}, F_l \cap \overline{H}) \geq 6k_0(\mathrm{P}), \ 1 \leq i < l \leq q. \tag{4.3.7}$$

Indeed, let us assume the opposite, $\frac{d_3^\circ}{B} < k_0(\mathrm{P})$, then $d_i^\circ < k_0(\mathrm{P})B$ for $1 \leq i \leq 3$. We consider the triangle Δ which is formed by the lines containing the line segments $F_i \cap \overline{H}$, $1 \leq i \leq 3$. We denote the area of Δ by S_Δ and the lengths of the three sides of Δ by a_1, a_2, and a_3.

Because Δ contains $\overline{\mathrm{P}}$, the longest side of Δ, which we denote by a_i, $1 \leq i \leq 3$, has length that is at least $1 = \mathrm{diam}(\overline{\mathrm{P}})$. We denote the second by length side of Δ by a_j, $1 \leq j \leq 3$, $i \neq j$. By triangle inequality we obtain $a_j > \frac{1}{2}$. Then if α is the angle of this triangle between a_i and a_j, then $\sin\alpha \geq 6k_0(\mathrm{P})$ by (4.3.7). Then

$$S_\Delta = \frac{1}{2}(d_1^\circ a_1 + d_2^\circ a_2 + d_3^\circ a_3) < \frac{1}{2}k_0(\mathrm{P})(a_1 + a_2 + a_3) \leq \frac{3}{2}k_0(\mathrm{P})a_i.$$

On the other hand,

$$S_\Delta = \frac{1}{2}ab\sin\alpha > \frac{1}{4}\cdot 6k_0(\mathrm{P})a_i = \frac{3}{2}k_0(\mathrm{P})a_i,$$

contradiction. Hence, we proved (4.3.6).

Because $\frac{d_3^\circ}{B} \geq k_0(\mathrm{P})$, under the condition

$$\delta_1\gamma^3 \leq k_0(\mathrm{P}), \tag{4.3.8}$$

there are three possible cases (a) - (c) described below depending on the distances from $\boldsymbol{x}$ to the sides of $\overline{\mathrm{P}}$. In each of these cases, we use the construction of the inscribed parallelogram provided by the proof of Theorem 4.8 in Section 4.2 applied to $\overline{\mathrm{P}}$. For convenience we will use notation d_i° for $d_i^\circ(\boldsymbol{x}, l_i)$, $1 \leq i \leq 3$.

We note that up to a homothety, there are finitely many polygons $\overline{\mathrm{P}}$ since that for each vertex of P all the corresponding planar sections are similar to each other. Therefore, there will be finitely many affine transforms $\widetilde{T}$, arising from the proof of Theorem 4.8 invoked to $h_B(\overline{\mathrm{P}})$, where h_B is an appropriate homothety with the ratio B^{-1}. So, we have

$$\max\{|\det \widetilde{T}|, |\det \widetilde{T}^{-1}|\} \leq c(\mathrm{P}).$$

We will use one triangle from the parallelogram to construct an appropriate inscribed triangular pyramid (which is a simple polytope) and invoke (1.1.7) and (1.1.8) to get the required bound on $\lambda_n(\boldsymbol{x}, \mathrm{P})$.

(a) If $d_1^\circ \geq \delta_1\gamma$ we follow the construction of the affine image of a cube (square) Q_0 in Section 4.2. Note that in contrast to the case 1 above, here we apply the arguments for the case $d = 2$ from Section 4.2. For described above affine transform $\widetilde{T}$ and the composition $\widetilde{T}_h := \widetilde{T} \circ h_B$ we have $\widetilde{T}_h(F_1) \subset \{x_1 \geq 0\}, \widetilde{T}_h(F_2) \subset \{x_2 \geq 0\}$, $\widetilde{T}_h(\boldsymbol{x}) = (0,0)$, $\widetilde{T}_h\,(\mathrm{P}) \subset \widetilde{T}_h(H_1^+) \cap \widetilde{T}_h(H_2^+)$. Also,

$$\widetilde{T}_h(\boldsymbol{x}) = (0,0) \in Q_0 := [-\beta, \beta]^2 \subset \widetilde{T}_h\left(\overline{\mathrm{P}}\right), \ \beta := \frac{\delta_1\gamma}{2M}.$$

We consider the triangle Δ with the vertices at the points $(0, \beta)$, $(-\beta, -\beta)$, and $(-\beta, \beta)$ and define the triangular pyramid $S := \mathrm{conv}\{\boldsymbol{v}_j, \Delta\}$. The distances from $\widetilde{T}_h(\boldsymbol{x}) = (0,0)$ to all sides of Δ are positive constants depending only on β. Also, if we denote the heights of Δ by h_i, $1 \leq i \leq 3$, we have

$$h_i = c_i\beta,\ 1 \le i \le 3. \tag{4.3.9}$$

Then we consider the pyramid S_0 which is homothetic with ratio $c_0(\mathrm{P})+\frac{\delta'}{2}$ to S, so that $S_0 \subset \mathrm{P}$ for some constant $c_0(\mathrm{P})$. The base Δ_0 of S_0, $\Delta_0 = (c_0(\mathrm{P})+\frac{\delta'}{2})\left(\widetilde{S} - \boldsymbol{v}_j\right)+\boldsymbol{v}_j$, is contained into plane $\underline{H}$, so that $\underline{H}\|H$ (planes $\underline{H}$ and H are parallel) and $\underline{H}$ is on distance $c_0(\mathrm{P})+\frac{\delta'}{2}$ from $\boldsymbol{v}_j$ lying on the same side from $\boldsymbol{v}_j$ as $\boldsymbol{x}$. Because of the choice of $c_0(\mathrm{P})$ so that $S_0 \subset \mathrm{P}$, the triangle Δ_0 is inside polytope P. The distance between $\boldsymbol{x}$ and $\underline{H}$ greater than $c_0(\mathrm{P})$.

Using Propositions 1.2 and 1.3 we obtain

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \ge \lambda_n(\widetilde{T}_h(\boldsymbol{x}), S_0).$$

Denote $S_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_1, x_2, x_3 \ge 0, x_1 + x_2 + x_3 \le 1\}$. There exists an affine transform T_0 so that $T_0(S_0) = S_3$. Also, $|\det T_0| = \frac{V_3}{V_0} \le c(\mathrm{P})$, where V_3 is the volume of S_3, V_0 is the volume of S_0. Indeed, the height h of S perpendicular to $\overline{H}$ is equivalent to $B = 1$, $h \approx 1$. If we denote the height of S_0 perpendicular to $\underline{H}$ by h_0, then $\frac{h_0}{h} = c_0(\mathrm{P}) + \frac{\delta'}{2}$, $h_0 \approx c(\mathrm{P})$. So, $\frac{V_3}{V_0} \le c(\mathrm{P})\frac{S_{\Delta_3}}{S_{\Delta_0}}$, where Δ_3 is the base of S_3. Because of formula of area of a triangle in terms of heights, (4.3.9) and Lemma 4.15, $\frac{S_{\Delta_3}}{S_{\Delta_0}} \approx c(\mathrm{P})$. Therefore, we have

$$\lambda_n(\widetilde{T}_h(\boldsymbol{x}), S_0) = |\det T_0^{-1}|\lambda_n(T_0(\widetilde{T}_h(\boldsymbol{x})), S_3) \ge c(\mathrm{P})\lambda_n(T_0(\widetilde{T}_h(\boldsymbol{x})), S_3).$$

Because of Lemma 4.15, the distances between $T_0(\widetilde{T}_h(\boldsymbol{x}))$ and sides of Δ_3 are equivalent to the corresponding distances d_i°, $1 \le i \le 3$. Using Theorem 4.8 for the triangular pyramid S_3, we have

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \ge c(\mathrm{P})\lambda_n(\widetilde{T}_h(\boldsymbol{x}), S_0) \ge c(\mathrm{P})\lambda_n(T_0(\widetilde{T}_h(\boldsymbol{x})), S_3) \approx c(\mathrm{P}).$$

(b) If $d_1^\circ < \delta_1\gamma$, $d_2^\circ \ge \delta_1\gamma^2$ we consider $P_1 \times Q_1$,

$$P_1 := [0, \alpha], Q_1 := [-\beta, \beta],\ \alpha := 5\delta_1 M\gamma,\ \beta := \frac{\delta_1\gamma^2}{M} - \alpha.$$

The triangle Δ with vertices at the points $(0, -\beta)$, $(0, \beta)$, and $(\alpha, 0)$ is inside the rectangle $P_1 \times Q_1$. Similarly to the previous case, we consider the triangular pyramid S_0 with the base containing in the plane $\underline{H}$, $\underline{H} \| H$ which is on distance $c_0(\mathrm{P}) + \frac{\delta'}{2}$ from $\boldsymbol{v}_j$ lying on the same side from $\boldsymbol{v}_j$ as point $\boldsymbol{x}$, $S_0 \subset \mathrm{P}$. Here again the point $\boldsymbol{x}$ is inside the pyramid S_0 and two shortest distances between $\boldsymbol{x}$ and facets of the pyramid coincide with d_1^* and d_2^*. Therefore,

$$\widetilde{T}_h(\boldsymbol{x}) = (d_1^\circ, 0) \in P_1 \times Q_1 \subset \widetilde{T}_h\left(\overline{\mathrm{P}}\right).$$

Using the results of Theorem 4.8 for the triangular pyramid $T_0(S_0) = S_3$ we obtain

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \ge c(\mathrm{P})\lambda_n(\widetilde{T}_h(\boldsymbol{x}), S_0) \ge c(\mathrm{P})\lambda_n(T_0(\widetilde{T}_h(\boldsymbol{x})), S_3)$$

$$\approx c(\mathrm{P})\rho_n^*(d_1^\circ) \approx c(\mathrm{P})\rho_n^*(d_1^*).$$

(c) If $d_1^\circ < \delta_1\gamma$, $d_2^\circ < \delta_1\gamma^2$, $d_3^\circ \geq \delta_1\gamma^3$ similarly to Section 4.2 we consider the square $P_2 = [0,\alpha]^2$. Then the triangle Δ is defined by the vertices at the points $(\alpha, 0)$, $(0, \alpha)$, and $(0, 0)$. We have

$$\Delta \subset P_2 \subset \widetilde{T}_h\,(\mathrm{P})$$

for some affine transform $\widetilde{T}$ and composition $\widetilde{T}_h := \widetilde{T} \circ h_B$ so that $\widetilde{T}_h(\boldsymbol{x}) = (d_1^\circ, d_2^\circ)$, $\widetilde{T}_h\,(\mathrm{P}) \subset \widetilde{T}_h(H_1^+) \cap \widetilde{T}_h(H_2^+)$. Constructing the pyramid in the same way as in the previous cases, by Theorem 4.8 we have

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \geq c(\mathrm{P})\lambda_n(\widetilde{T}_h(\boldsymbol{x}), S_0) \geq c(\mathrm{P})\lambda_n(T_0(\widetilde{T}_h(\boldsymbol{x})), S_3)$$

$$\approx c(\mathrm{P})\rho_n^*(d_1^\circ)\rho_n^*(d_2^\circ) \approx \rho_n^*(d_1^*)\rho_n^*(d_1^*).$$

Taking into account (4.2.17) with $d = 2$ the constraints (4.3.5) and (4.3.8) we set

$$\gamma := \max\{1, 12M^2\},\ \ \delta_1 = \frac{1}{\gamma^3}\min\{\delta_0, \delta_2, k_0(\mathrm{P})\}.$$

Combining above results we get the following lower bound:

$$\lambda_n(\boldsymbol{x}, \mathrm{P}) \geq c(\mathrm{P})\prod_{i=1}^{3}\rho_n^*(d_i^\circ) \approx c(\mathrm{P})\prod_{i=1}^{3}\rho_n^*(d_i^*). \tag{4.3.10}$$

$\square$

4.4 Difficulty in higher dimensional spaces

It is natural to conjecture that the generalization of Theorem 4.14 for arbitrary polytopes in $\mathbb{R}^d$, $d \geq 4$, holds true. In the following example we will demonstrate that the analog of Remark 4.10 does not hold true for arbitrary polytopes in $\mathbb{R}^d$, $d \geq 4$. This means that the method used in Chapter 4 cannot be generalized directly for general polytopes in $\mathbb{R}^d$, $d \geq 4$.

Example 4.16. Consider the Egyptian pyramid $E = \text{conv}\{(\pm 1, \pm 1, 0), (0, 0, 1)\}$ in $\mathbb{R}^3$ with the vertex $\boldsymbol{v} = (0, 0, 1)$ adjacent to some four facets F_i, $1 \leq i \leq 4$ of E and let $I = [0, 1] \subset \mathbb{R}$. Take the polytope $E_1 := E \times I$, $E_1 \subset \mathbb{R}^4$, which is a polytope in $\mathbb{R}^4$. Then $F_i \times I$, $1 \leq i \leq 4$, are facets of E_1. By construction, the four facets F_i, $1 \leq i \leq 4$, of E have common vertex and the facets $F_i \times I$, $1 \leq i \leq 4$, of E_1 have non-empty intersection $\{\boldsymbol{v}\} \times I$. This is a 1-dimensional face of E_1 which is not a vertex of E_1. So, the statement of Remark 4.10 fails.

The nature of this example can be seen from the so called "diamond property", see [20, Theorem 2.7, p.57].

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