MAXIMUM NUMBER OF CYCLES IN GRAPHS AND MULTIGRAPHS

by

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Abstract

In this thesis a problem of determining the maximum number of cycles for the following classes of graphs is considered: triangle-free graphs; K_r-free graphs; graphs with m edges; graphs with n vertices and m edges; multigraphs with m edges and multigraphs with n vertices and m edges.

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Chapter 1

Introduction

1.1 Types of problems considered

The main theme of this thesis is determining the maximum number of cycles in a graph with different restrictions. My interest in this topic was initiated by my academic supervisor David Gunderson in January 2014, in the begining of my Ph.D. program. Durocher, Gunderson, Li and Skala [20] were interested in the question of how many cycles can a triangle-free graph have (this question was motivated by the study of path-finding algorithms). The authors of [20] posed the following conjecture.

Conjecture 1.1.1 (Durocher-Gunderson-Li-Skala, 2015 [20]). *For each $n \geq 4$, the balanced complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ contains more cycles than any other n-vertex triangle-free graph.*

Gunderson, Tsaturian and I [10] were able to prove this conjecture for the case

$n \geq 141$ (see Theorems 3.4.1 and 3.4.2 here). We also posed the following conjecture and question.

Conjecture 1.1.2 (Arman-Gunderson-Tsaturian, 2016 [10]). *For any $k > 1$, if an n-vertex C_{2k+1}-free graph has the maximum number of cycles, then $G = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

Question 1.1.3 (Arman-Gunderson-Tsaturian, 2016 [10]). *For $k \geq 4$, what is the maximum number of cycles in a K_k-free graph on n vertices? Could it be that the only cycle-maximal K_k-free graphs are Turán graphs?*

Miklós Simonovits [44] suggested that the Regularity Lemma [46] might be used to answer Question 1.1.3 or prove Conjecture 1.1.2, but at the time we could not overcome certain technical difficulties of such an approach (see Section 5.1 for more details). In January 2014, after a preprint of the paper [10] appeared on arXiv.org, we received an email from Alex Scott [41], in which he informed us that he and a student of his could (by using the Regularity Lemma) prove Conjecture 1.1.2 and answer affirmatively Question 1.1.3 for n large enough. However, their results were not published at the time of the preparation of this thesis.

In February 2016, after a talk in the combinatorics seminar at University of Manitoba, Dr. Karen Gunderson [25] asked us if Sergei Tsaturian and I had considered the question of determining the maximum number of cycles in a graph with a given density, which motivated me and Tsaturian to investigate the following two questions. For any graph G, let $C(G)$ denote the number of cycles in G.

Question 1.1.4. *For $m \geq 3$ let $C(m)$ denote the maximum number of cycles in any graph with m edges. What is $C(m)$ and for which graphs G with m edges does $C(G) = C(m)$?*

Question 1.1.5. *Among all graphs G with a fixed number of vertices and a fixed number of edges what is the maximum of $C(G)$? Which graphs achieve the maximum?*

Tsaturian and I were not able to completely answer either of those two questions, but we did provide [9] useful estimates for the maximum number of cycles in each case (these estimates are also presented in Chapter 4 of this thesis). For instance, we proved that if G is a graph with m edges that has the maximal number of cycles and $C(G)$ is the number of cycles in G, then

$$1.37^m \leq C(G) \leq 1.443^m.$$

Also, Tsaturian and I [9] proved that if G is a graph with the maximum number of cycles among all graphs with n vertices and average degree $d = d(n)$, such that $\lim_{n\to\infty} d(n) = \infty$, then for n large enough,

$$\left(\frac{d}{e}\right)^n (1+o(1))^n \leq C(G) \leq (1+o(1))^n \left(\frac{d}{2}\right)^n.$$

Tsaturian and I could not answer Question 1.1.4 nor Question 1.1.5, and so were motivated to reconsider the main tool used to obtain an upper bounds for the number of cycles (technical Lemma 4.3.1). We came to the conclusion that even if this tool is not very precise in estimating number of cycles in a graph, it is a very precise tool in estimating the number of cycles in a multigraph. We were able to prove (Theorem 4.7.1 here) that for a multigraph G that has the maximum number of cycles among all of the multigraphs with $n \geq 2$ vertices and $m \geq 3$ edges and for $s = \lfloor \frac{m}{n-1} \rfloor$, $\alpha = \frac{m}{n-1} - s$

$$\frac{8}{27} s(s^{1-\alpha}(s+1)^{\alpha})^{n-1} \leq C(G) \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1} \quad , \quad \text{if } \frac{m}{n-1} \geq 3;$$

$$4(\sqrt[3]{3})^{m-4} \leq C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m} \quad , \quad \text{if} \frac{m}{n-1} \geq 3.$$

Also, Tsaturian and I [9] were able to prove (Theorems 4.2.3, 4.2.4, 4.7.2 here) that if G is a graph (or a multigraph) with m edges and the maximum number of cycles, then the maximal degree of G is at most eleven and the minimal degree of G is at least three.

After receiving the email from Alex Scott [41], I was trying to answer Question 1.1.3 affirmatively without using the Regularity Lemma [46]. Lemma 4.3.1 proved to be a useful tool in estimating the number of cycles in a graph, so I used it to approach Question 1.1.3. Recall, that $T(n,r)$ is a Turán graph – complete r-partite graph on n vertices with sizes of partite sets differing by at most one, and $t(n,r)$ is the number of edges in $T(n,r)$. I was able to prove (Theorem 5.3.2 here) that for n large enough, any K_{r+1}-free graph G with n vertices and $m \leq t(n,r) - 2r^4 n \log n$ edges has fewer cycles than $T(n,r)$.

Also, I proved (Theorems 5.2.2, 5.2.6 here and [8]) estimates on the number of cycles in the Turán graph $T(n,r)$, namely that there exists a constant c such that

$$\frac{c}{n^{\frac{2}{3}r^2-\frac{r}{2}+1}}\left(\frac{n(r-1)}{re}\right)^n \leq C(T(n,r)) \leq \left(\frac{e^2n}{r}\right)^{\frac{r}{2}}\left(\frac{n(r-1)}{re}\right)^n.$$

Finally, I want to note that it is possible to estimate the number of cycles in a graph G by using an adjacency matrix of G. If A is a square matrix then the trace of A is defined to be the sum of the diagonal entries of A and is denoted by $\mathrm{tr}(A)$. Perepechko and Voropaev [37] proved that if A is an adjacency matrix of a graph G, and for every subset S of $[n]$, A_S is the submatrix of a matrix A with the set S of rows and columns deleted and for any $3 \leq k \leq n$, c_k is the number of cycles of length k in G, then

$$c_k = \frac{1}{2k}\sum_{i=2}^{k}(-1)^{k-i}\binom{n-i}{n-k}\sum_{|S|=n-i}\mathrm{tr}(A_S^k).$$

Hence, the following formula for the number of cycles in G holds:

$$C(G) = \sum_{k=3}^{n}\frac{1}{2k}\sum_{i=2}^{k}(-1)^{k-i}\binom{n-i}{n-k}\sum_{|S|=n-i}\mathrm{tr}(A_S^k).$$

I believe that this formula might be used to estimate the number of cycles in a K_r-free graph, or in a graph with m edges, but I was not able to use it for a question of maximizing the number of cycles.

1.2 Structure of the thesis

Chapter 2 contains basic notation and Chapters 3, 4, 5 contain original research. The main problem considered in this thesis is the problem of estimating the number of cycles in graphs with different restrictions.

In Chapter 3, the question of determining the maximum number of cycles in a triangle-free graph on n vertices is considered. All of the results in Chapter 3 are present in the paper of Arman, Gunderson, Tsaturian [10]. The main result of Chapter 3, Theorem 3.4.1, establishes the existence of $n_0 \in \mathbb{Z}^+$ such that for any $n \geq n_0$, the only triangle-free graph on n vertices with the largest number of cycles is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

In Chapter 4, the question of determining the maximum number of cycles in a graph with a given number of edges is considered. Most of the results presented in Chapter 4 are also present in the paper of Arman and Tsaturian [9]. Some of other results in Chapter 4 were obtained individually (as noted). The main result of Chapter 4 states that for m large enough,

$$1.37^m \leq C(m) \leq 1.443^m.$$

In Chapter 5, the question of determining the maximum number of cycles in an H-free graph G on n vertices is considered. All of the results in Chapter 5 are original individual research. In Section 5.2, the question of estimating the number of cycles in a Turán graph is considered. The main result of Section 5.2 is Theorem 5.2.6, which states that for n large enough

$$C(T(n,r)) \geq \frac{c}{n^{\frac{2}{3}r^2 - \frac{r}{2} + 1}} \left(\frac{n(r-1)}{re} \right)^n.$$

In Section 5.3 an estimate on the number of cycles in a K_r-free graph is given and the main result of Section 5.3 is Theorem 5.3.2, which states that for n large enough, any K_{r+1}-free graph G with n vertices and $m \leq t(n,r) - 2r^4 n \ln n$ edges,

$$C(G) < C(T(n,r)).$$

Chapter 2

Basic notation and definitions

2.1 Basic Notation

In this thesis notation mostly follows Bollobás' book [13]. For $k \in \mathbb{Z}^+$, let $[k] = \{i \in \mathbb{Z}; 1 \leq i \leq k\}$, and for a set S, denote $[S]^k = \{T \subseteq S : |T| = k\}$. For a set V, let $[V]^2 = \{\{x, y\} : \ x, y \in V, x \neq y\}$ be the collection of all unordered pairs of the elements of V. A graph G is an ordered pair (V, E), where $V \neq \emptyset$ and $E \subseteq [V]^2$. Elements of V are called vertices and elements of E are called edges. An edge $\{x, y\} \in E(G)$ can be denoted by xy. The neighbourhood of a vertex $v \in V(G)$ is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, and the degree of a vertex x is $\deg_G(x) = |N(x)|$. When it is clear what G is, the subscript G can be deleted, writing simply $N(x)$ and $\deg(x)$. If $Y \subset V(G)$, the subgraph of G induced by Y is denoted by $G[Y]$. If $Y \subset V(G)$, the subgraph of G induced by $V(G) \backslash Y$ is denoted by $G \backslash Y$. Denote the average degree of a graph G by $d(G)$, the maximum degree by $\Delta(G)$, and the minimum degree by $\delta(G)$. The complete graph on n vertices is denoted by K_n (where $E = [V]^2$).

A graph $G = (V, E)$ is called bipartite if and only if there is a partition $V = A \cup B$ so that $E \subset \{\{x, y\} : x \in A, y \in B\}$; if $E = \{\{x, y\} : x \in A, y \in B\}$, then G is called the complete bipartite graph on partite sets A and B, denoted by $G = K_{|A|,|B|}$. The balanced complete bipartite graph on n vertices is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ (this graph is also the Turán graph $T(n, 2)$, see below). A cycle on m vertices is denoted by C_m. The complement of a graph G is denoted by $\overline{G}$.

For $n, r \in \mathbb{Z}^+$ define the Turán graph $T(n, r)$ to be the graph with n vertices, such that $V(T(n, r))$ can be partitioned into r sets $V_1, V_2, \ldots, V_r$ in a way such that for any $i \neq j$, $||V_i| - |V_j|| \leq 1$ and so that edge set

$$E(T(n, r)) = \{\{x, y\} : x \in V_i,\ y \in V_j,\ i \neq j\}.$$

Define $t(n, r) = |E(T(n, r))|$ and let $\ell \equiv n \mod r$, then

$$t(n, r) = \frac{n^2}{2}\left(1 - \frac{1}{r}\right) - \frac{\ell(r - \ell)}{2r}.$$

For any graph G let $C(G)$ be the number of undirected cycles in G.

Also, the common notations $o(n), \Omega(n)$ are used: a function $f(n) = o(n)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{n} = 0$; $f(n) = \Omega(n)$ if and only if $\limsup_{n\to\infty} |\frac{f(n)}{n}| > 0$.

2.2 Stirling approximation and relative inequalities

The number e is the base of the natural logarithm. Stirling's approximation formula says that as $n \to \infty$,

$$n! = (1 + o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{2.1}$$

In 1955, Robbins [40] proved the following approximation for factorials, valid for all $n \geq 1$:

$$e^{\frac{1}{12n+1}}\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n < n! < e^{\frac{1}{12n}}\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n.$$

The following consequence of Robbins' approximation is used (valid for all $n \geq 2$):

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < e\cdot\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{2.2}$$

Chapter 3

Counting cycles in triangle-free graphs

3.1 Motivating question

All graphs in this chapter are simple and undirected. All of the results presented in this chapter are taken from the paper of Arman, Gunderson and Tsaturian [10]. The research in this chapter was motivated by a paper of Durocher, Gunderson, Li and Skala [20], where the maximum number of cycles in a triangle-free graph was considered. Durocher, Gunderson, Li and Skala [20] posed the following conjecture:

Conjecture 3.1.1 (Durocher-Gunderson-Li-Skala, 2015 [20]). *For every $n \geq 4$, the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains more cycles than any other n-vertex triangle-free graph.*

The authors of [20] confirmed Conjecture 3.1.1 when $4 \leq n \leq 13$, and made a progress toward this conjecture in general.

Conjecture 3.1.1 holds true for $n \geq 141$ (see Theorems 3.4.1 and 3.4.2 below). Along the way some other results are proved that are of independent interest – *e.g.* estimates for the number of cycles in $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ (Theorem 3.2.4) and estimates on the number of hamiltonian cycles in a triangle-free graphs (Lemma 3.3.3).

Even though Conjecture 3.1.1 arose from a very specific problem in computing (see [15]), it can be considered significant in two areas of graph theory: counting cycles in graphs, and the structure of triangle-free graphs. In recent decades, bounds have been proved for the maximum number of cycles in various classes of graphs. Some of these classes include

- complete graphs [27],
- planar graphs [4, 5, 16],
- outerplaner graphs and series-parallel graphs [19],
- graphs with large maximum degree without a specified odd cycle [11],
- graphs with specified minimum degree [49],
- graphs with a specified cyclomatic number or number of edges [2, 21, 24, 31] (see also [33, Ch4, Ch10]),
- cubic graphs [3, 17],

- graphs with fixed girth [36],
- k-connected graphs [32],
- hamiltonian graphs [38, 42, 49],
- hamiltonian graphs with a fixed number of edges [26],
- 2-factors of the de Bruijn graph [22],
- graphs with a cut-vertex [49],
- complements of trees [29, 39, 51],
- random graphs [47].

In some cases, (*e.g.*, [11, 38]) the associated extremal graphs were found.

By Mantel's theorem [34], among graphs on n vertices, the triangle-free graph with the most number of edges is the balanced complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. Since $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the triangle-free graph on n vertices with the most number of edges Conjecture 3.1.1 might seem reasonable, even though $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains no odd cycles.

3.2 Preliminaries and estimate on the number of cycles in $T(n, 2)$

The following shows that among all bipartite graphs, the balanced one has the most cycles.

Lemma 3.2.1 (Durocher-Gunderson-Li-Skala, 2015 [20]). *For $n \geq 4$, among all bipartite graphs on n vertices, $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ has the greatest number of cycles; that is, $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is the unique cycle-maximal bipartite graph on n vertices.*

So, to settle Conjecture 3.1.1, it is then sufficient to prove that a cycle-maximal triangle-free graph is bipartite. To this end, the following result might be essential:

Theorem 3.2.2 (Andrásfai, 1964 [6]). *Any triangle-free graph G on n vertices with $\delta(G) > 2n/5$ is bipartite.*

See also [7] for an English-language proof of Theorem 3.2.2 and related results. Theorem 3.2.2 is sharp because of C_5 (or a blow-up of C_5).

Lemma 3.2.3 (Durocher-Gunderson-Li-Skala, 2015 [20]). *For $n \geq 4$, the number of cycles in the balanced complete bipartite graph is*

$$C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{\lfloor n/2 \rfloor! \, \lceil n/2 \rceil!}{2k(\lfloor n/2 \rfloor - k)!(\lceil n/2 \rceil - k)!}. \tag{3.1}$$

Two modified Bessel functions (see, *e.g.*, [1]) are used:

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(i!)^2}; \tag{3.2}$$

$$I_1(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k+1} i!(i+1)!}. \tag{3.3}$$

In particular, when $x = 2$ is used in either modified Bessel function, useful approximations are obtained:

$$2.27958 \leq \sum_{i=0}^{\infty} \frac{1}{(i!)^2} = I_0(2) \leq 2.279586; \tag{3.4}$$

$$1.5906 \leq \sum_{i=0}^{\infty} \frac{i}{(i!)^2} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) \leq 1.59064. \tag{3.5}$$

The following form for the number of cycles in $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ gives a way to estimate the right hand side of (3.1) in Lemma 3.2.3:

Theorem 3.2.4 (Arman-Gunderson-Tsaturian, 2016 [10])**.** *For $n \geq 12$,*

$$C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \geq \frac{\lfloor n/2 \rfloor! \, \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even} \\ I_1(2) & \text{if } n \text{ is odd} \end{cases} \tag{3.6}$$

$$\geq \pi \left(\frac{n}{2e}\right)^n \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even} \\ I_1(2) & \text{if } n \text{ is odd,} \end{cases} \tag{3.7}$$

and as $n \to \infty$,

$$C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = (1 + o(1)) \begin{cases} I_0(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is even} \\ I_1(2)\pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is odd.} \end{cases} \tag{3.8}$$

Proof: Using (2.2), the proof that (3.7) follows from (3.6) is elementary (by using Stirling's approximation), and so is omitted.

By Lemma 3.2.3, write

$$C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = \sum_{k=2}^{\lfloor n/2\rfloor} \frac{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}{2k(\lfloor n/2\rfloor - k)!(\lceil n/2\rceil - k)!}$$
$$= \frac{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}{2\,\lfloor n/2\rfloor} \cdot \sum_{k=2}^{\lfloor n/2\rfloor} \frac{\lfloor n/2\rfloor}{k(\lfloor n/2\rfloor - k)!(\lceil n/2\rceil - k)!}. \tag{3.9}$$

Case 1 (n even): Suppose that for $\ell \geq 2$, $n = 2\ell$, and set

$$a_\ell = \sum_{k=2}^{\ell} \frac{\ell}{k((\ell-k)!)^2} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)(i!)^2}.$$

Then equation (3.9) becomes

$$C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = \frac{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}{2\,\lfloor n/2\rfloor} \cdot a_\ell. \tag{3.10}$$

Claim: For $\ell \geq 4$, $a_{\ell+1} < a_\ell$. (This claim is needed later only for $\ell \geq 6$.)

Proof of Claim:

$$\begin{aligned}
a_\ell - a_{\ell+1} &= \sum_{i=0}^{\ell-2} \left(\frac{\ell}{\ell-i} - \frac{\ell+1}{\ell+1-i}\right)\frac{1}{(i!)^2} - \frac{\ell+1}{2((\ell-1)!)^2} \\
&= \sum_{i=0}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)(i!)^2} - \frac{\ell+1}{2((\ell-1)!)^2} \\
&= \sum_{i=2}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)(i!)^2} + \frac{1}{\ell(\ell-1)} - \frac{\ell+1}{2((\ell-1)!)^2} \\
&> \frac{1}{\ell(\ell-1)} - \frac{\ell+1}{2((\ell-1)!)^2} \\
&= \frac{2((\ell-1)!)^2 - (\ell+1)\ell(\ell-1)}{2((\ell-1)!)^2\ell(\ell-1)} \\
&\geq 0 \qquad \text{(for } \ell \geq 4\text{)},
\end{aligned}$$

finishing the proof of the claim.

Since the sequence $\{a_\ell\}$ is non-increasing and bounded below (by 0, for example), $\lim_{\ell\to\infty} a_\ell$ exists. To find this limit, first apply partial fractions:

$$a_\ell = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)(i!)^2} = \sum_{i=0}^{\ell-2} \frac{1}{(i!)^2} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2}.$$

Put $b_\ell = \displaystyle\sum_{i=0}^{\ell-2} \frac{1}{(i!)^2}$ and $c_\ell = \displaystyle\sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2}$. Then

$$\begin{aligned}
c_\ell &= \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)(i!)^2} \\
&= \sum_{i=0}^{3} \frac{i}{(\ell-i)(i!)^2} + \sum_{i=4}^{\ell-2} \frac{i}{(\ell-i)(i!)^2} \\
&\leq \frac{3}{\ell-3} + \sum_{i=4}^{\ell-2} \frac{1}{(i-1)!(\ell-i)i!} \\
&\leq \frac{3}{\ell-3} + \sum_{i=4}^{\ell-2} \frac{1}{i(\ell-i)i!} \\
&\leq \frac{3}{\ell-3} + \sum_{i=4}^{\ell-2} \frac{1}{(2\ell-4)i!} \\
&\leq \frac{3}{\ell-3} + \frac{1}{\ell}\sum_{i=4}^{\ell-2} \frac{1}{i!} & \text{(since } \ell \geq 4) \\
&\leq \frac{3}{\ell-3} + \frac{e}{\ell},
\end{aligned}$$

where the last line is based on $e = \sum_{i=0}^{\infty} \frac{1}{i!}$.

Therefore, $\lim_{\ell\to\infty} c_\ell = 0$, and so

$$\begin{aligned}
\lim_{\ell\to\infty} a_\ell &= \lim_{\ell\to\infty}(b_\ell + c_\ell) \\
&= \lim_{\ell\to\infty} b_\ell \\
&= \sum_{i=0}^{\infty} \frac{1}{(i!)^2} \\
&= I_0(2) && \text{(by (3.4))}.
\end{aligned}$$

Since a_ℓ is non-increasing for $\ell \geq 6$, for $n \geq 12$,

$$C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \geq \frac{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot I_0(2),$$

which proves the even case of (3.6). By (3.4), as $n \to \infty$,

$$C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) = (1+o(1))\frac{\lfloor n/2\rfloor!\,\lceil n/2\rceil!}{2\lfloor n/2\rfloor} \cdot I_0(2),$$

and by Stirling's approximation (2.1), the proof of the even case of (3.8) is complete.

Case 2 (n odd): Suppose that for $\ell \geq 6$, $n = 2\ell + 1$. The proof follows the even case, and so is only outlined. Put

$$a_\ell = \sum_{k=2}^{\ell} \frac{\ell}{k(\ell-k)!(\ell+1-k)!} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)i!(i+1)!}.$$

Claim: For $\ell \geq 4$, $a_{\ell+1} < a_\ell$.

Proof of claim: Letting $\ell \geq 4$,

$$\begin{aligned}
a_\ell - a_{\ell+1} &= \sum_{i=0}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)} \cdot \frac{1}{i!(i+1)!} - \frac{\ell+1}{2(\ell-1)!\ell!} \\
&= \sum_{i=2}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)} \cdot \frac{1}{i!(i+1)!} + \frac{1}{2(\ell-1)\ell} - \frac{\ell+1}{2(\ell-1)!\ell!} \\
&> \frac{(\ell-2)!(\ell-1)! - (\ell+1)}{2(\ell-1)!\ell!} \\
&\geq 0,
\end{aligned}$$

finishing the proof of the claim.

Therefore, $\lim_{\ell\to\infty} a_\ell$ exists. To find this limit, write

$$a_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i+1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!}.$$

Letting $b_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i+1)!}$ and $c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!}$, observe that

$$c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!} \leq \frac{3}{\ell-3} + \frac{e}{\ell},$$

and so $\lim_{\ell\to\infty} c_\ell = 0$. Thus,

$$\lim_{\ell\to\infty} a_\ell = \lim_{\ell\to\infty} b_\ell = \sum_{i=0}^{\infty} \frac{1}{i!(i+1)!} = \sum_{i=0}^{\infty} \frac{i+1}{((i+1)!)^2} = \sum_{i=0}^{\infty} \frac{i}{(i!)^2},$$

which, by (3.5), is equal to $I_1(2)$. Then again

$$\begin{aligned}
C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) &\geq \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2 \lfloor n/2 \rfloor} \cdot I_1(2) \\
&= \frac{\ell!(\ell+1)!}{2\ell} \cdot I_1(2) \\
&= \frac{(\ell!)^2}{2\ell}(\ell+1) \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{\ell}{e}\right)^{2\ell} (\ell+1) \cdot I_1(2) && \text{(by (2.1))} \\
&> (1+o(1))\pi \left(\frac{\ell}{e}\right)^{2\ell} (\ell-1) \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{n-1}{2e}\right)^{n-1} \left(\frac{n-1}{2}\right) \cdot I_1(2) \\
&= (1+o(1))\pi e \left(\frac{n-1}{2e}\right)^{n} \cdot I_1(2) \\
&= (1+o(1))\pi e \left(\frac{n-1}{n}\right)^{n} \left(\frac{n}{2e}\right)^{n} \cdot I_1(2) \\
&= (1+o(1))\pi \left(\frac{n}{2e}\right)^{n} \cdot I_1(2),
\end{aligned}$$

and as $n \to \infty$,

$$C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = (1+o(1))\pi \left(\frac{n}{2e}\right)^{n} \cdot I_1(2).$$

This completes the proof for odd n, and so the proof of the lemma. □

Lemma 3.2.5 (Arman-Gunderson-Tsaturian, 2016 [10]). *Let H be a triangle-free graph on 6 vertices with $x, y \in V(H)$. Then there are at most 9 different x–y paths.*

Proof: Consider two cases.

Case 1: H contains no copy of C_5. Then H contains no odd cycle, and so is bipartite. Without loss of generality, add edges to H to make H a complete bipartite graph. There are only four different complete bipartite graphs on six vertices, namely $\overline{K_6}$, $K_{1,5}$, $K_{2,4}$, and $K_{3,3}$. By inspection, in any of these, the maximum number of paths between any two vertices is at most 9 (which is realized for $K_{3,3}$).

Case 2: H contains a copy of C_5. Suppose that $x_1, x_2, x_3, x_4, x_5, x_1$ forms a cycle C, and that x_6 is the remaining vertex. Then x_6 is adjacent to at most two vertices of C. If x_6 is adjacent to fewer than two vertices of C, add an extra edge or two so that x_6 is adjacent to precisely two vertices of C; without loss of generality, suppose that x_6 is adjacent to x_1 and x_3. Then the maximum number of paths between any two vertices is 4 (for example, between x_2 and x_6). $\square$

3.3 Counting cycles through a vertex or an edge

Lemma 3.3.1 (Arman-Gunderson-Tsaturian, 2016 [10]). *There exists $n_0 \in \mathbb{Z}^+$ so that for every even integer $n \geq n_0$, if G is a triangle-free graph on n vertices, and $x_1x_2 \in E(G)$, then the number of cycles containing the edge x_1x_2 is at most $10\pi\frac{n^{n-1}}{(2e)^n}$.*

Proof: Let G be a triangle-free graph on n vertices, and let $x_1x_2 \in E(G)$. For each $k = 4, \ldots, n$, let c_k denote the number of cycles of length k that contain the edge x_1x_2. The goal is to give an upper bound for $\sum_{k=4}^{n} c_k$.

Let $2 \leq i \leq \frac{n-4}{2}$. An upper bound on $c_{2i} + c_{2i+1}$ is first calculated; to do so, count all possible cycles of the form $x_1, x_2, \ldots, x_{2i}$ or $x_1, x_2, \ldots, x_{2i+1}$. For each $j > 1$, there are at most $d_j = |N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}|$ ways to choose an x_{j+1}. Note that

$N(x_j) \cap N(x_{j+1}) = \emptyset$, since otherwise a triangle is formed with x_j and x_{j+1}. Also,

$$|(N(x_j)\backslash\{x_1,\ldots,x_{j-1}\}) \cup (N(x_{j+1})\backslash\{x_1,\ldots,x_j\})| \leq |V(G)\backslash\{x_1,\ldots,x_j\}| = n-j.$$

Therefore,

$$\begin{aligned} d_j + d_{j+1} &\leq |N(x_j)\backslash\{x_1,\ldots,x_{j-1}\}| + |N(x_{j+1})\backslash\{x_1,\ldots,x_j\}| \\ &= |(N(x_j)\backslash\{x_1,\ldots,x_{j-1}\}) \cup (N(x_{j+1})\backslash\{x_1,\ldots,x_j\})| \\ &\leq n-j, \end{aligned}$$

and thus

$$d_j d_{j+1} \leq \left\lfloor \frac{n-j}{2} \right\rfloor \cdot \left\lceil \frac{n-j}{2} \right\rceil. \tag{3.11}$$

Using (3.11), the number of ways to choose vertices $x_3, x_4, \ldots, x_{2i}$ so that x_1, x_2, x_3, x_4, …, x_{2i} form a path is at most

$$\prod_{j=2}^{2i-1} d_j = \prod_{j=1}^{i-1}(d_{2j}d_{2j+1}) \leq \prod_{j=1}^{i-1}\left(\left\lfloor \frac{n-2j}{2} \right\rfloor \cdot \left\lceil \frac{n-2j}{2} \right\rceil\right) = \prod_{j=1}^{i-1}\left(\frac{n-2j}{2}\right)^2. \tag{3.12}$$

If there is an edge $x_{2i}x_1 \in E(G)$, there is one cycle $x_1, x_2, \ldots, x_{2i}$ of length $2i$, and no cycles of the form $x_1, x_2, \ldots, x_{2i+1}$ because otherwise, x_1, x_{2i}, x_{2i+1} form a triangle. So, in total, there is exactly one cycle that contains the path $x_1, x_2, \ldots, x_{2i}$ and has length $2i$ or $2i+1$. If there is no edge $x_{2i}x_1$, there is no cycle $x_1, \ldots, x_{2i}$ and at most $n-2i$ cycles of the form $x_1, \ldots, x_{2i}x_{2i+1}$. In any case, there are at most $n-2i$ cycles of length $2i$ or $2i+1$ containing the path $x_1, \ldots, x_{2i}$.

By these observations and inequality (3.12),

$$c_{2i} + c_{2i+1} \leq (n-2i)\prod_{j=1}^{i-1}\left(\frac{n-2j}{2}\right)^2. \tag{3.13}$$

To evaluate $\sum_{k=4}^{n} c_k$, separate the sum into two parts:

$$\begin{aligned}
\sum_{k=4}^{n-5} c_k &= \sum_{i=2}^{(n-6)/2} (c_{2i} + c_{2i+1}) \\
&\leq \sum_{i=2}^{(n-6)/2} \left((n-2i) \prod_{j=1}^{i-1} \left(\frac{n-2j}{2} \right)^2 \right) && \text{(by (3.13))} \\
&= \sum_{i=2}^{(n-6)/2} (n-2i) \left(\frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-2i}{2}\right)!} \right)^2 \\
&= \left(\left(\frac{n-2}{2} \right)! \right)^2 \sum_{j=3}^{\frac{n-4}{2}} \frac{2j}{(j!)^2} \\
&= \left(\left(\frac{n-2}{2} \right)! \right)^2 \left(\sum_{j=1}^{\frac{n-4}{2}} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2 \cdot 2}{(2!)^2} \right) \\
&\leq \left(\left(\frac{n-2}{2} \right)! \right)^2 (2 \cdot (1.591) - 3) && \text{(by (3.5))} \\
&< 0.19 \left(\left(\frac{n-2}{2} \right)! \right)^2 . && (3.14)
\end{aligned}$$

To count $\sum_{k=n-4}^{n} c_k$, note that by (3.11), there are at most

$$\prod_{i=2}^{n-5} d_i \leq \prod_{j=1}^{\frac{n-6}{2}} \left(\frac{n-2j}{2} \right)^2$$

ways to choose a path $x_1, x_2, \ldots, x_{n-4}$, and by Lemma 3.2.5, there are at most 9 paths that connect x_{n-4} and x_1 in the graph $G \backslash \{x_1, \ldots, x_{n-5}\}$; that is, there are at most 9 ways to complete the path $x_1, x_2, \ldots, x_{n-4}$ to a cycle. Therefore,

$$\sum_{k=n-4}^{n} c_k \leq 9 \prod_{j=1}^{\frac{n-6}{2}} \left(\frac{n-2j}{2} \right)^2 = 9 \cdot \frac{\left(\left(\frac{n-2}{2}\right)! \right)^2}{(2!)^2} = \frac{9}{4} \left(\left(\frac{n-2}{2} \right)! \right)^2 . \tag{3.15}$$

Adding equations (3.14) and (3.15),

$$\sum_{k=4}^{n} c_k \leq 0.19 \left(\left(\frac{n-2}{2} \right)! \right)^2 + \frac{9}{4} \left(\left(\frac{n-2}{2} \right)! \right)^2 = 2.44 \left(\left(\frac{n-2}{2} \right)! \right)^2 . \tag{3.16}$$

By Stirling's approximation, as $n \to \infty$,

$$
\begin{aligned}
2.44\left(\left(\frac{n-2}{2}\right)!\right)^2 &= (1+o(1))2.44\left(\frac{n-2}{2e}\right)^{n-2} \cdot \pi(n-2) \\
&= (1+o(1))2.44\pi\frac{n^{n-1}}{(2e)^n}(2e)^2\left(\frac{n-2}{n}\right)^{n-1} \\
&= (1+o(1))2.44\pi\frac{n^{n-1}}{(2e)^n}4e^2 \cdot \frac{1}{e^2} \\
&= (1+o(1))9.76\pi\frac{n^{n-1}}{(2e)^n} \\
&< 10\pi\frac{n^{n-1}}{(2e)^n} \qquad \text{(for } n \text{ suff. large)}
\end{aligned}
$$

completing the proof of the lemma. □

By a closer inspection, Lemma 3.3.1 holds for the value $n_0 = 97$.

Lemma 3.3.2 (Arman-Gunderson-Tsaturian, 2016 [10])**.** *There exists $n_0 \in \mathbb{Z}^+$ so that for every odd integer $n \geq n_0$, if G is a triangle-free graph on n vertices, and $x_1x_2 \in E(G)$ with $\deg_G(x_2) \leq \frac{2}{5}n$, then the number of cycles containing the edge x_1x_2 is at most $7.81\pi\frac{n^{n-1}}{(2e)^n}$.*

Proof: The proof is similar to that of Lemma 3.3.1. Let G be a triangle-free graph on n vertices, and let $x_1x_2 \in E(G)$, where $\deg(x_2) \leq \frac{2}{5}n$. For each $k = 4, \ldots, n$, let c_k denote the number of cycles of length k that contain the edge x_1x_2.

For $3 \leq i \leq \frac{n-5}{2}$, an upper bound on $c_{2i-1} + c_{2i}$ is first calculated; to do so, count all possible cycles of the form $x_1, x_2, \ldots, x_{2i-1}$ or $x_1, x_2, \ldots, x_{2i}$. As in Lemma 3.3.1, for each $j > 1$, there are at most $d_j = |N(x_j)\backslash\{x_1, \ldots, x_{j-1}\}|$ ways to choose an x_{j+1},

and

$$d_j d_{j+1} \le \left\lfloor \frac{n-j}{2} \right\rfloor \cdot \left\lceil \frac{n-j}{2} \right\rceil. \tag{3.17}$$

Using (3.17) and the fact that $d_2 \le \frac{2}{5}n$, the number of ways to choose vertices $x_3, x_4, \ldots, x_{2i-1}$ so that $x_1, x_2, x_3, x_4, \ldots, x_{2i-1}$ form a path is at most

$$\begin{aligned} \prod_{j=2}^{2i-2} d_j = d_2 \prod_{j=3}^{2i-2} d_j &\le \frac{2}{5}n \prod_{j=1}^{i-2} (d_{2j+1} d_{2j+2}) \\ &\le \frac{2}{5}n \prod_{j=1}^{i-2} \left(\left\lfloor \frac{n-2j-1}{2} \right\rfloor \cdot \left\lceil \frac{n-2j-1}{2} \right\rceil \right) \\ &= \frac{2}{5}n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2. \end{aligned} \tag{3.18}$$

If $x_{2i-1}x_1 \in E(G)$, there is one cycle of length $2i-1$ and no cycles of length $2i$; if there is no such edge, there are no cycles of length $2i-1$ and at most $n-2i+1$ cycles of length $2i$. By these observations and (3.18),

$$c_{2i-1} + c_{2i} \le (n-2i+1) \frac{2}{5}n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2. \tag{3.19}$$

To evaluate $\sum_{k=4}^{n} c_k$, separate the sum into three parts:

$$\sum_{k=4}^{n} c_k = c_4 + \sum_{k=5}^{n-5} c_k + \sum_{k=n-4}^{n} c_k.$$

First,

$$c_4 \le d_2 d_3 < n \cdot n = n^2. \tag{3.20}$$

Next,

$$\begin{aligned}
\sum_{k=5}^{n-5} c_k &= \sum_{i=3}^{\frac{n-5}{2}} (c_{2i-1} + c_{2i}) \\
&\le \sum_{i=3}^{\frac{n-5}{2}} \left[(n-2i+1) \frac{2}{5} n \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2 \right] && \text{(by (3.19))} \\
&= \frac{2}{5} n \sum_{i=3}^{\frac{n-5}{2}} \left[(n-2i+1) \prod_{j=1}^{i-2} \left(\frac{n-2j-1}{2} \right)^2 \right] \\
&= \frac{2}{5} n \sum_{i=3}^{\frac{n-5}{2}} (n-2i+1) \left(\frac{\left(\frac{n-3}{2}\right)!}{\left(\frac{n-2i+1}{2}\right)!} \right)^2 \\
&= \frac{2}{5} n \left(\left(\frac{n-3}{2} \right)! \right)^2 \sum_{j=3}^{\frac{n-5}{2}} \frac{2j}{(j!)^2} \\
&= \frac{2}{5} n \left(\left(\frac{n-3}{2} \right)! \right)^2 \left(\sum_{j=1}^{\frac{n-5}{2}} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2 \cdot 2}{(2!)^2} \right) \\
&< \frac{2}{5} n \left(\left(\frac{n-3}{2} \right)! \right)^2 (3.19 - 3) && \text{(by (3.5))} \\
&= 0.076 n \left(\left(\frac{n-3}{2} \right)! \right)^2 . && (3.21)
\end{aligned}$$

To count $\sum_{k=n-4}^{n} c_k$, note that by (3.20), there are at most

$$\prod_{i=2}^{n-5} d_i = d_2 \cdot \prod_{j=1}^{(n-7)/2} d_{2j+1} d_{2j+2} \le \frac{2}{5} n \prod_{j=1}^{\frac{n-7}{2}} \left(\frac{n-2j-1}{2} \right)^2$$

ways to choose a path $x_1, x_2, \ldots, x_{n-4}$, and by Lemma 3.2.5, there are at most 9 ways to complete to a cycle (by paths that connect x_{n-4} and x_1) in the graph $G \backslash \{x_1, \ldots, x_{n-5}\}$. Therefore,

$$\sum_{k=n-4}^{n} c_k \le 9 \cdot \frac{2}{5} n \prod_{j=1}^{\frac{n-7}{2}} \left(\frac{n-2j-1}{2} \right)^2 = 9 \cdot \frac{2}{5} n \cdot \frac{\left(\left(\frac{n-3}{2}\right)!\right)^2}{(2!)^2} = \frac{9}{10} n \left(\left(\frac{n-3}{2} \right)! \right)^2 . \tag{3.22}$$

Adding (3.20), (3.21), and (3.22), as $n \to \infty$,

$$\begin{aligned}
\sum_{k=4}^{n} c_k &\leq n^2 + 0.076n\left(\left(\frac{n-3}{2}\right)!\right)^2 + \frac{9}{10}n\left(\left(\frac{n-3}{2}\right)!\right)^2 \\
&= n^2 + 0.976n\left(\left(\frac{n-3}{2}\right)!\right)^2 \qquad (3.23)\\
&= n^2 + (1+o(1))0.976n(n-3)\pi\left(\frac{n-3}{2e}\right)^{n-3} \\
&= (1+o(1))0.976\pi n \cdot \frac{n^{n-2}}{(2e)^n}\left(\frac{n-3}{n}\right)^{n-2}(2e)^3 \\
&= (1+o(1))0.976\pi \cdot \frac{n^{n-1}}{(2e)^n}\frac{1}{e^3}(2e)^3 \\
&= (1+o(1))7.808\pi \cdot \frac{n^{n-1}}{(2e)^n} \\
&< 7.81\pi\frac{n^{n-1}}{(2e)^n} \qquad \text{(for } n \text{ suff. large)},
\end{aligned}$$

completing the proof. □

By a closer inspection, Lemma 3.3.2 holds for the value $n_0 = 24729$.

Lemma 3.3.3 (Arman-Gunderson-Tsaturian, 2016 [10]). *Let H be a triangle-free graph on k vertices. Then H has at most $e^2\left(\frac{k}{2e}\right)^k$ hamiltonian cycles.*

Proof: Let x_1 be the first vertex of a hamiltonian cycle. For each $i \geq 1$, there are at most $d_i = |N(x_i)\backslash\{x_1, \ldots, x_i\}|$ ways to choose a vertex x_{i+1}. Note that $N(x_i) \cap N(x_{i+1}) = \emptyset$ because if the intersection contains some vertex v, then v, x_i, and x_{i+1} form a triangle. Also,

$$|N(x_i)\backslash\{x_1, \ldots, x_i\} \cup N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \leq |V(H)\backslash\{x_1, \ldots, x_i\}| = k - i.$$

Therefore,

$$\begin{aligned}
d_i + d_{i+1} &= |N(x_i)\backslash\{x_1, \ldots, x_i\}| + |N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \\
&= |N(x_i)\backslash\{x_1, \ldots, x_i\} \cup N(x_{i+1})\backslash\{x_1, \ldots, x_{i+1}\}| \\
&\leq k - i,
\end{aligned}$$

and thus $d_i d_{i+1} \leq \left\lfloor \frac{k-i}{2} \right\rfloor \cdot \left\lceil \frac{k-i}{2} \right\rceil$.

When k is odd, the number of hamiltonian cycles is at most

$$\begin{aligned}
\prod_{i=1}^{k-1} d_i = \prod_{j=1}^{\frac{k-1}{2}} d_{2j-1} d_{2j} &\leq \prod_{j=1}^{\frac{k-1}{2}} \left\lfloor \frac{k-2j+1}{2} \right\rfloor \cdot \left\lceil \frac{k-2j+1}{2} \right\rceil \\
&= \prod_{j=1}^{\frac{k-1}{2}} \left(\frac{k-2j+1}{2} \right)^2 \\
&= \left(\left(\frac{k-1}{2} \right)! \right)^2,
\end{aligned}$$

and by (2.2), this number is at most

$$\begin{aligned}
\left(\frac{\left(\frac{k-1}{2}\right)^{\frac{k-1}{2}+\frac{1}{2}}}{e^{\frac{k-1}{2}-1}} \right)^2 &= \frac{\left(\frac{k-1}{2}\right)^k}{e^{k-3}} = e^3 \left(\frac{k-1}{k} \right)^k \left(\frac{k}{2e} \right)^k = e^3 \frac{1}{\left(1+\frac{1}{k-1}\right)^k} \left(\frac{k}{2e} \right)^k \\
&\leq e^2 \left(\frac{k}{2e} \right)^k,
\end{aligned}$$

completing the proof for odd k.

When k is even, similarly obtain

$$\begin{aligned}
\prod_{i=1}^{k-1} d_i &= \left(\prod_{j=1}^{\frac{k-2}{2}} d_{2j-1}d_{2j}\right)\cdot d_{k-1} \leq \left(\prod_{j=1}^{\frac{k-2}{2}} \left\lfloor \frac{k-2j+1}{2}\right\rfloor \cdot \left\lceil \frac{k-2j+1}{2}\right\rceil\right)\cdot 1 \\
&= \prod_{j=1}^{\frac{k-1}{2}} \left(\frac{k-2j}{2}\right)\left(\frac{k-2j+2}{2}\right) = \frac{k}{2}\left(\left(\frac{k-2}{2}\right)!\right)^2 \leq \frac{k}{2}\left(\frac{\left(\frac{k-2}{2}\right)^{\frac{k-2}{2}+\frac{1}{2}}}{e^{\frac{k-2}{2}}}\right)^2 \\
&= k\frac{(k-2)^{k-1}}{e^{k-4}2^k} = e^4\left(\frac{k-2}{k}\right)^{k-1}\left(\frac{k}{2e}\right)^k = e^4\frac{1}{\left(1+\frac{2}{k-2}\right)^{k-1}}\left(\frac{k}{2e}\right)^k \\
&\leq e^2\left(\frac{k}{2e}\right)^k,
\end{aligned}$$

completing the proof for even k, and hence for the lemma. □

3.4 Main theorems

In Theorem 3.4.1, Conjecture 3.1.1 is proved for sufficiently large n. Then in Theorem 3.4.2, a lower bound on such n is given.

Theorem 3.4.1 (Arman-Gunderson-Tsaturian, 2016 [10])**.** *There exists $n_0 \in \mathbb{Z}^+$ so that for any $n \geq n_0$, the triangle-free graph on n vertices with the largest number of cycles is $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.*

Proof: Let G be a triangle-free graph on n vertices. It is first shown that if G contains a vertex of small degree, then G has far fewer cycles than does $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

Let $x \in V(G)$, and assume that $\deg(x) \leq \frac{2}{5}n$. Cycles in G are counted according to whether or not they contain x.

First, the number of cycles not containing x are counted. Any cycle in $G\backslash x$ is a hamiltonian cycle for some subgraph, and so the number of cycles in G not containing x is loosely bounded above by

$$
\begin{aligned}
&\sum_{Y\subseteq V(G)\backslash x} (\text{number of ham. cycles in } G[Y]) && (3.24)\\
&\quad \le \sum_{k=4}^{n-1} \binom{n-1}{k} e^2 \left(\frac{k}{2e}\right)^k && \text{(by Lemma 3.3.3)}\\
&\quad < e^2 \sum_{k=4}^{n-1} \binom{n-1}{k} \left(\frac{n-1}{2e}\right)^k \\
&\quad < e^2 \left(1+\frac{n-1}{2e}\right)^{n-1} \\
&\quad = e^2 \left(\frac{n+2e-1}{2e}\right)^{n-1} \\
&\quad = e^2 \left(\frac{n}{2e}\right)^{n-1} \left(\frac{n+2e-1}{n}\right)^{n-1} \\
&\quad < e^2 \left(\frac{n}{2e}\right)^{n-1} \left(1+\frac{2e-1}{n}\right)^{n} \\
&\quad \le e^2 \left(\frac{n}{2e}\right)^{n-1} e^{2e-1} \\
&\quad = \frac{2e^{2e+2}}{n} \left(\frac{n}{2e}\right)^{n}. && (3.25)
\end{aligned}
$$

Next, the number of cycles containing x are counted. Each cycle C containing x has exactly two edges (in C) incident with x, and so the number of cycles containing x is

$$\frac{1}{2} \sum_{y\in N(x)} (\text{number of cycles containing } xy). \tag{3.26}$$

By Lemma 3.3.1, for even n, the expression (3.26) is at most

$$\frac{1}{2} \cdot \frac{2}{5} n \cdot 10\pi \frac{n^{n-1}}{(2e)^n} = 2\pi \left(\frac{n}{2e}\right)^n.$$

In this case, for n sufficiently large, the total number of cycles in G is at most

$$2\pi\left(\frac{n}{2e}\right)^n + \frac{2e^{2e+2}}{n}\left(\frac{n}{2e}\right)^n = \left(2\pi + \frac{2e^{2e+2}}{n}\right)\left(\frac{n}{2e}\right)^n \leq 2.01\pi\left(\frac{n}{2e}\right)^n.$$

However, by (3.6), the number of cycles in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is (for n even) at least $2.27958\pi\left(\frac{n}{2e}\right)^n$.

Let n be odd; then by Lemma 3.3.2, the expression (3.26) is at most

$$\frac{1}{2}\cdot\frac{2}{5}n\cdot 7.81\pi\frac{n^{n-1}}{(2e)^n} = 1.562\pi\left(\frac{n}{2e}\right)^n. \tag{3.27}$$

Thus, for odd n sufficiently large, by (3.27) and (3.25) the total number of cycles in G is at most

$$1.562\pi\left(\frac{n}{2e}\right)^n + \frac{2e^{2e+2}}{n}\left(\frac{n}{2e}\right)^n \leq 1.57\pi\left(\frac{n}{2e}\right)^n.$$

By (3.6) in Theorem 3.2.4, the number of cycles in $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ for n odd is at least $1.5906\pi\left(\frac{n}{2e}\right)^n$.

In both the even and odd case, if G contains a vertex of degree at most $\frac{2}{5}n$, then G has far fewer cycles than does $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

So assume that $\delta(G) > \frac{2}{5}n$. Then by Theorem 3.2.2, G is bipartite. By Lemma 3.2.1, the number of cycles in G is maximized by $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. □

Theorem 3.4.2 (Arman-Gunderson-Tsaturian, 2016 [10]). *The statement of Theorem 3.4.1 with $n_0 = 141$ is true.*

Proof: To show that $n_0 = 141$ works, further estimations on $C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil})$ are needed for $n \geq 141$. Both when n is even and when n is odd, (3.10) holds (but the expression for a_ℓ changes). Since each (one for odd, one for even) sequence of a_ℓs are non-increasing for $n \geq 140$, by (3.10),

$$\begin{aligned} C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) &\leq \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot \begin{cases} a_{71} & \text{for } n \text{ even} \\ a_{70} & \text{for } n \text{ odd} \end{cases} \\ &\leq \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot \begin{cases} 2.302786 & \text{for } n \text{ even} \\ 1.60067 & \text{for } n \text{ odd.} \end{cases} \end{aligned} \tag{3.28}$$

(The values of a_{70} and a_{71} were calculated by computer.) With these estimates in hand, now Theorem 3.4.1 is proved with $n_0 = 141$. Let G be a triangle-free graph on $n \geq 141$ vertices. First, it is proved that $C(G) \leq 6 \cdot C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$ for G having odd number of vertices. This result is then used to prove the statement of Theorem 3.4.2 for even number of vertices in G. Finally, Theorem 3.4.2 is verified for odd number of vertices in G.

Without loss of generality, assume that there is a vertex of degree at most $\frac{2}{5}n$ (since otherwise, the theorem is proved by Theorem 3.2.2 and Lemma 3.2.1). In the following calculations, bounds given in (2.2) and Theorem 3.2.4 are used freely.

Case 1: Let $n \geq 141$ be odd. By (3.23) from the proof of Lemma 3.3.2, the number of cycles passing through an edge xy in G is at most $n^2 + 0.976n \left(\left(\frac{n-3}{2} \right)! \right)^2$.

Then the number of cycles in G is bounded by

$$
\begin{aligned}
C(G) &\leq \frac{1}{2} \cdot \frac{2}{5} n \cdot \left[n^2 + 0.976 n \left(\left(\frac{n-3}{2} \right)! \right)^2 \right] + \frac{2e^{2e+2}}{n} \left(\frac{n}{2e} \right)^n \\
&= \frac{\frac{n-1}{2}!\frac{n+1}{2}!}{n-1} \cdot I_1(2) \cdot \left(\frac{\frac{n}{5} \left[n^2 + 0.976 n \left(\left(\frac{n-3}{2} \right)! \right)^2 \right] (n-1)}{\frac{n-1}{2}!\frac{n+1}{2}! \cdot I_1(2)} \right. \\
&\qquad\qquad \left. + I_1(2) \cdot \pi \left(\frac{n}{2e} \right)^n \left(\frac{2e^{2e+2}}{n \cdot I_1(2) \pi} \right) \right) \\
&\leq C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \cdot \left(10^{-10} + \frac{8}{5} \cdot (0.976) \left(\frac{n^2}{n^2 - 1} \right) + \frac{2e^{2e+2}}{n\pi} \right) \cdot \frac{1}{I_1(2)} \\
&\leq C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \cdot 6.
\end{aligned}
$$

Case 2: Let n be even and $n \geq 142$. Then by (3.16), the proof of Theorem 3.4.1, and by the result in Case 1,

$$
\begin{aligned}
C(G) &\leq \frac{1}{2} \cdot \frac{2}{5} n \cdot 2.44 \left(\left(\frac{n-2}{2} \right)! \right)^2 + 6 \cdot C(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}) \\
&= \frac{\frac{n}{5} 2.44 \left(\left(\frac{n-2}{2} \right)! \right)^2}{\frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot I_0(2)} \cdot \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor} \cdot I_0(2) \\
&\qquad\qquad + C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \frac{6 \cdot C(K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil})}{C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})} \\
&\leq C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \cdot \left(\frac{\frac{4}{5} \cdot 2.44}{I_0(2)} + \frac{6 \cdot 1.60067 \cdot \frac{\lfloor \frac{n-1}{2} \rfloor! \lceil \frac{n-1}{2} \rceil!}{2 \lfloor \frac{n-1}{2} \rfloor}}{I_0(2) \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2 \lfloor n/2 \rfloor}} \right) \qquad \text{(by (3.28))} \\
&= C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \left(\frac{\frac{4}{5} \cdot 2.44}{I_0(2)} + \frac{6 \cdot 1.60067}{I_0(2)} \cdot \frac{2}{n} \right) \\
&\leq C(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \quad \text{(for } n \geq 142\text{)}.
\end{aligned}
$$

Returning to the case when n is odd, using equation (3.28) again,

$$\begin{aligned}
C(G) &\leq C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \cdot \left(\frac{10^{-10} + \frac{8}{5}(0.976)\left(\frac{n^2}{n^2-1}\right)}{I_1(2)} + \frac{2.302786 \cdot \frac{\left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil !}{2\left\lfloor \frac{n-1}{2} \right\rfloor}}{I_1(2) \cdot \frac{\lfloor n/2\rfloor ! \lceil n/2\rceil !}{2\lfloor n/2\rfloor}} \right) \\
&\leq C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \cdot \left(\frac{10^{-10} + \frac{8}{5}(0.976)\left(\frac{n^2}{n^2-1}\right)}{I_1(2)} + \frac{2.302786}{I_1(2) \cdot (n+1)} \right) \\
&\leq C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) \cdot 0.9947 \\
&< C(K_{\lfloor n/2\rfloor,\lceil n/2\rceil}).
\end{aligned}$$

This completes the proof of the theorem for $n \geq 141$. □

3.5 Concluding remarks

Another question related to Conjecture 3.1.1 that might be interesting is:

Question 3.5.1 (Arman-Gunderson-Tsaturian, 2016 [10]). *What is the maximum number of cycles in a graph on n vertices with girth at least g?*

The case $g = 3$ is trivial and Theorem 3.4.1 addresses this question for $g = 4$; however, there seems to be little known for $g \geq 5$.

A type of stability result also follows from the techniques given in this chapter. Theorem 3.4.1 shows that among all triangle-free graphs with n vertices and $m = \left\lfloor \frac{n^2}{4} \right\rfloor$ edges, $K_{\lceil \frac{n}{2}\rceil,\lfloor \frac{n}{2}\rfloor}$ has the most number of cycles. Let $\ell = o(n)$, and set $m = \left\lfloor \frac{n^2}{4} \right\rfloor - \ell$. If G has n vertices and m edges, and has the most number of cycles among all triangle-free n-vertex graphs with m edges, then the same argument as in the proof of Theorem 3.4.1 shows that G is bipartite. By the maximality of the number of cycles, one possibly can show that G is a subgraph of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

For $14 \leq n \leq 140$, Conjecture 3.1.1 remains open. With a bit more care, it appears that with the techniques in this chapter, one might be able to prove Conjecture 3.1.1 for the even n to $n \geq 100$ or so, but the techniques used here do not seem to leave much room for the odd n. Skala [45] has suggested that Lemma 3.2.5 might be proved for graphs with slightly more vertices, and such an improvement might yield modest improvements for the bound on n for which Theorem 3.4.1 holds.

Chapter 4

Counting cycles in a graph with given number of edges

4.1 Overview of the results

Counting the number of cycles in a graph is a problem that was studied for different classes of the graphs: graphs with given cyclomatic number, planar graphs, 3-regular and 4-regular graphs, and many others (see the list in Section 3.1). However, only a few general bounds (that are only based on number of vertices and edges, or only number of edges) for a number of cycles in a graph are known. In this chapter bounds on the number of cycles in a graph as a function of number of edges, or vertices and edges, are presented.

As in Chapter 3, let $C(G)$ denote the number of cycles in a graph G. In 1897, Ahrens [2] proved that for a graph G with n vertices, m edges and k components,

$$m - n + k \leq C(G) \leq 2^{m-n+k} - 1. \tag{4.1}$$

The lower bound in (4.1) is tight; for example, it is achieved by any disjoint union of cycles and trees. In 1976, the tightness of the upper bound in (4.1) was shown by Mateti and Deo [35] and the only graphs for which the upper bound is tight are K_3, K_4, $K_{3,3}$ and $K_4 - e$. In 2008, Aldred and Thomassen [4] improved the upper bound in (4.1) by showing that for a connected graph G,

$$C(G) \leq \frac{15}{16} 2^{m-n+1}. \tag{4.2}$$

In 1981, Entringer and Slater [21] considered $C(G)$ for the class of connected graphs with a fixed cyclomatic number $r = m - n + 1$. It follows from the results of [21] that there is a 3-regular connected graph G for which $C(G) > 2^{r-1}$. Shi [42], in 1994, presented for all $r \geq 1$ an example of a hamiltonian graph G with $C(G) = 2^{r-1} + (r-1)(r-2) + 1$.

In 2009, Király [30] investigated $C(G)$ for several classes of graphs: the union and the sum of two trees (the sum of two trees is the multigraph that is formed by the disjoint union of edges of two trees), 3-regular and 4-regular graphs, and graphs with the average degree 4. Király also conjectured that there is a constant c, such that for any graph G that has m edges,

$$C(G) \leq c(1.4)^m.$$

Aldred and Thomassen [4] also studied $C(G)$ for the class of planar graphs. Arman, Gunderson and Tsaturian [10] studied $C(G)$ for the class of triangle-free graphs on n vertices (our findings are also presented in Chapter 3). In 2006, Teunter and van der Poort [48] considered the question of counting the number of hamiltonian cycles in a graph with a given number of vertices and edges by using techniques similar to those used in Section 4.3.

In this chapter, $C(G)$ is investigated for two classes of graphs and multigraphs: those with n vertices and m edges, and those with m edges.

Theorem 4.4.2 below states that if a graph G has n vertices and m edges, then

$$C(G) \leq \begin{cases} \frac{3}{4}\Delta(G)(\frac{m}{n-1})^{n-1}, \text{ if } \frac{m}{n-1} \geq 3, \\ \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m, \text{ if } \frac{m}{n-1} < 3. \end{cases} \tag{4.3}$$

For graphs with sufficiently large number of edges and average degree at least 4.25 the bound in (4.3) is better than in (4.2).

In Section 4.6 (see Theorem 4.6.1) it is shown that for n large enough and $d = d(n)$, such that $\lim_{n\to\infty} d(n) = \infty$, there exists a graph G with n vertices and $m = \frac{d}{2}n$ edges such that

$$C(G) \geq (1 + o(1))^n \left(\frac{d}{e}\right)^n = (1 + o(1))^n \left(\frac{2m}{en}\right)^n.$$

For $m \in \mathbb{Z}^+$, let $C(m)$ be the maximum number of cycles in a graph with m edges. In Corollary 4.4.3 it is shown that for all $m \geq 1$

$$C(m) < 8.25(\sqrt[3]{3})^m \sim 8.25(1.44229..)^m,$$

which, for $m > 4056$, implies

$$C(m) < 1.443^m.$$

In Section 4.2 it is shown that the extremal graphs for $C(m)$ have bounded degrees. Namely, it is shown that if G is a graph with m edges with $C(G) = C(m)$, then $\Delta(G) \leq 11$ (Theorem 4.2.3) and $\delta(G) \geq 3$ (Theorem 4.2.4).

In Section 4.5, for m sufficiently large, a graph G with m edges is constructed, such that

$$C(G) \geq (2 + \sqrt{8})^{\frac{m}{5} - 1} \geq 1.37^m. \tag{4.4}$$

Corollary 4.4.3 and inequality (4.4) imply that for m large enough,

$$1.37^m \leq C(m) \leq 1.443^m. \tag{4.5}$$

In Section 4.7, the problems of maximizing the number of cycles in a multigraph with a given number of edges or with a given number of vertices and edges are considered. It is shown (Theorem 4.7.3) that if G is a multigraph that has the most cycles among all multigraphs with m multi-edges, then

$$\frac{9}{10}(\sqrt[3]{3})^m \leq C(G) \leq 8.25(\sqrt[3]{3})^m.$$

4.2 Maximal and minimal degree of graphs with $C(m)$ cycles

Recall that, for $m \in \mathbb{Z}^+$, $C(m)$ is the maximum number of cycles in a graph with m edges. The main result of this section is Theorem 4.2.3 that states that maximum degree in a graph with m edges that has $C(m)$ cycles is at most eleven.

The proof of Theorem 4.2.3 relies on the following two technical lemmas.

Lemma 4.2.1 (Arman-Tsaturian, 2017$^+$ [9]). *Let $k \geq 6$ be a positive integer. For $i, j \in [k]$ such that $i \neq j$, let $w_{i,j}$ be non-negative real numbers, such that $w_{i,j} = w_{j,i}$, and let $S = \sum_{1 \leq i < j \leq k} w_{i,j}$. Then there exists a 6-element set $D \subseteq [k]$ such that*

$$\sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \geq \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

Proof. The proof relies on an averaging argument. For each $i \in [k]$ set $w_i = \sum_{j \in [k], j \neq i} w_{i,j}$. Note that

$$\sum_{i \in [k]} w_i = 2S.$$

Let X be a collection of all 6-element subsets of $[k]$. For $D \in X$ let

$$\begin{aligned} S(D) &= \sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \\ &= S - \sum_{i \in D} \left(\sum_{j \in [k],\, j \neq i} w_{i,j} \right) + \sum_{i,j \in D,\, i<j} w_{i,j} \\ &= S - \sum_{i \in D} w_i + \sum_{i,j \in D,\, i<j} w_{i,j}. \end{aligned}$$

Let $\overline{S(D)}$ be the average of $S(D)$ over all $D \in X$. Then

$$\begin{aligned}
\overline{S(D)} &= \frac{\sum_{D\in X}\left(S - \sum_{i\in D} w_i + \sum_{i,j\in D, i<j} w_{i,j}\right)}{\binom{k}{6}} \\
&= S - \frac{\sum_{i\in[k]}\sum_{D\in X, i\in D} w_i}{\binom{k}{6}} + \frac{\sum_{1\le i<j\le k}\sum_{D\in X:\, i,j\in D} w_{i,j}}{\binom{k}{6}} \\
&= S - \frac{\sum_{i\in[k]}\binom{k-1}{5} w_i}{\binom{k}{6}} + \frac{\sum_{1\le i<j\le k}\binom{k-2}{4} w_{i,j}}{\binom{k}{6}} \\
&= S - \frac{\binom{k-1}{5}\cdot 2S}{\binom{k}{6}} + \frac{\binom{k-2}{4}\cdot S}{\binom{k}{6}} \\
&= \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.
\end{aligned}$$

Therefore, there exists $D \in X$, such that $S(D) \ge \overline{S(D)}$, i.e.,

$$\sum_{\substack{1\le i<j\le k \\ i\notin D,\ j\notin D}} w_{i,j} \ge \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

$\square$

Lemma 4.2.2 (Arman-Tsaturian, 2017^{+} [9]). *Let $k \ge 2$ be a positive integer. For $i, j \in [k]$ such that $i \ne j$, let $w_{i,j}$ be non-negative real numbers, such that $w_{i,j} = w_{j,i}$, and let $S = \sum_{1\le i<j\le k} w_{i,j}$. Then there exists a partition $A_1 \cup A_2 \cup A_3 \cup A_4 = [k]$, such that*

$$\sum_{1\le l<m\le 4}\ \sum_{\substack{i\in A_\ell \\ j\in A_m}} w_{i,j} \ge \left(\frac{3k^2-4}{4k(k-1)}\right) S.$$

Proof. For all $\ell \in [4]$ let $a_\ell = \left\lfloor \frac{k+l-1}{4} \right\rfloor$ (note that $a_1 + a_2 + a_3 + a_4 = k$). Let X be the collection of all ordered quadruples (A_1, A_2, A_3, A_4), such that $\pi = A_1 \cup A_2 \cup A_3 \cup A_4$ is a partition of $[k]$ and for all $\ell \in [4]$, $|A_\ell| = a_\ell$. Note that

$$|X| = \frac{k!}{a_1!a_2!a_3!a_4!}.$$

For $\mathbf{p} = (A_1, A_2, A_3, A_4) \in X$ define

$$\begin{aligned} S(\mathbf{p}) &= \sum_{1 \le l < m \le 4} \sum_{\substack{i \in A_\ell \\ j \in A_m}} w_{i,j} \\ &= S - \sum_{l \in [4]} \sum_{\substack{i < j \\ i,j \in A_\ell}} w_{i,j}. \end{aligned}$$

Let $\overline{S(\mathbf{p})}$ be the average of $S(\mathbf{p})$ over all possible choices of $\mathbf{p} \in X$.

$$\begin{aligned} \overline{S(\mathbf{p})} &= \frac{\sum_{p \in X} (S - \sum_{l \in [4]} \sum_{i,j \in A_\ell,\, i<j} w_{i,j})}{|X|} \\ &= S - \frac{\sum_{l \in [4]} \sum_{p \in X} \sum_{i,j \in A_\ell,\, i<j} w_{i,j}}{|X|} \\ &= S - \frac{\sum_{l \in [4]} \sum_{1 \le i < j \le k} \sum_{p \in X:\, i,j \in A_\ell} w_{i,j}}{|X|} \end{aligned}$$

Note that for any choice of $\ell \in [4]$ and any choice of i, j, such that $1 \le i < j \le k$, there are exactly

$$\frac{(k-2)!(a_\ell)(a_\ell - 1)}{a_1!a_2!a_3!a_4!}$$

quadruples $\mathbf{p} \in X$, such that $i, j \in A_\ell$. Then,

$$
\begin{aligned}
\overline{S(\mathbf{p})} &= S - (\sum_{l\in[4]} \sum_{1\le i<j\le k} \frac{(k-2)!(a_\ell)(a_\ell-1)}{a_1!a_2!a_3!a_4!} w_{i,j})/|X| \\
&= S - (\sum_{l\in[4]} \frac{(k-2)!(a_\ell)(a_\ell-1)}{a_1!a_2!a_3!a_4!} \cdot S) \cdot \frac{1}{|X|} \\
&= S - (\sum_{l\in[4]} \frac{(k-2)!(a_\ell)(a_\ell-1)}{a_1!a_2!a_3!a_4!}) \cdot S \cdot \frac{a_1!a_2!a_3!a_4!}{k!} \\
&= S - (\sum_{l\in[4]} \frac{\left\lfloor \frac{k+l-1}{4} \right\rfloor (\left\lfloor \frac{k+l-1}{4} \right\rfloor - 1)}{k(k-1)}) \cdot S \\
&= S \left(1 - \frac{1}{k(k-1)} \cdot \begin{cases} \frac{k(k-4)}{4}, & \text{if } k \equiv 0 \bmod 4 \\ \frac{(k-1)(k-3)}{4}, & \text{if } k \equiv \pm 1 \bmod 4 \\ \frac{(k-2)^2}{4}, & \text{if } k \equiv 2 \bmod 4 \end{cases} \right) \\
&\ge S \left(1 - \frac{(k-2)^2}{4k(k-1)} \right).
\end{aligned}
$$

Hence, there exists a $\mathbf{p} = (A_1, A_2, A_3, A_4) \in X$, such that $S(\mathbf{p}) \ge \overline{S(\mathbf{p})}$. Therefore, the partition $A_1 \cup A_2 \cup A_3 \cup A_4$ satisfies the statement of Lemma 4.2.2. □

Theorem 4.2.3 (Arman-Tsaturian, 2017$^+$ [9])**.** *If G is a graph with m edges such that $C(G) = C(m)$, then $\Delta(G) \le 11$.*

Proof. Let m be a positive integer and G be a graph with m edges. To prove Theorem 4.2.3, it is sufficient to prove that if $\Delta(G) \ge 12$, then there is a graph H with m edges and with $C(H) > C(G)$.

Let $\Delta(G) \geq 12$ and u be a vertex of maximal degree in G. Let $N(u) = \{u_1, u_2, \ldots, u_k\}$ be the neighbourhood of u (note that $k \geq 12$). For $i, j \in [k]$, $i \neq j$, define $w_{i,j}$ to be the number of paths from vertex u_i to vertex u_j in the graph $G \backslash u$. Then the number of cycles in G that pass through vertex u is $S = \sum_{1 \leq i < j \leq k} w_{i,j}$. By Lemma 4.2.1, there is a 6-element set $D = \{i_1, i_2, \ldots, i_6\}$, such that

$$\sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \geq \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{4.6}$$

Suppose, upon re-indexing, that $D = \{k-5, k-4, \ldots, k-1, k\}$. Lemma 4.2.2 applied to the collection of real numbers $w_{i,j}$ with $1 \leq i < j \leq k-6$ gives a partition $A_1 \cup A_2 \cup A_3 \cup A_4 = [k-6]$ with

$$\sum_{1 \leq l < m \leq 4} \sum_{\substack{i \in A_\ell \\ j \in A_m}} w_{i,j} \geq \left(\frac{3(k-6)^2 - 4}{4(k-6)(k-7)}\right) \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{4.7}$$

For $i \in [4]$, let $U_i = \{u_j \ : \ j \in A_i\}$. Construct a graph H by deleting u and all of the edges incident to u, adding four new vertices v_1, v_2 , v_3, v_4, then for all $1 \leq i \leq 4$ adding edges from v_i to each of the vertices of U_i, and for all $1 \leq i < j \leq 4$ adding edges $v_i v_j$ (see Figure 4.1). Then $|E(H)| = |E(G)|$.

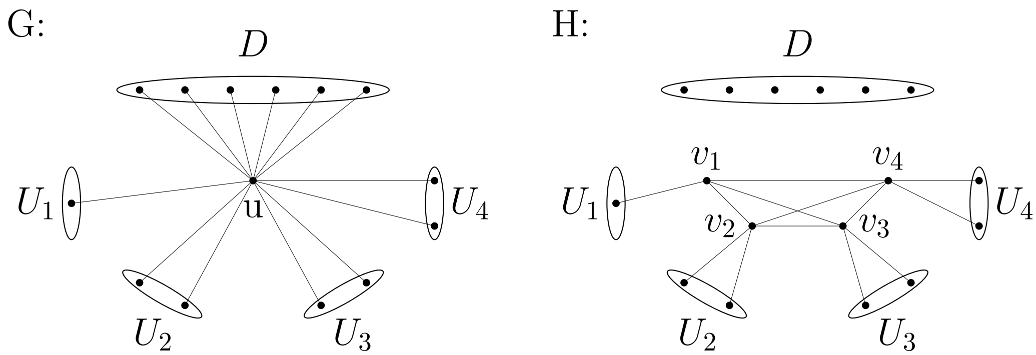

Figure 4.1: Constructing the graph H.

To count the number of cycles in H, note the following:

- Every cycle in G that does not pass through the vertex u is still a cycle in H. There are $C(G) - S$ such cycles.

- Let C be a cycle in G that for some $1 \leq i < j \leq k-6$ contains a path $u_i u u_j$. If for some $\ell \in [4]$ u_i and u_j are in the same class U_ℓ, then C corresponds to the cycle in H that uses the path $u_i v_\ell u_j$ instead of $u_i u u_j$. In the case if for some $1 \leq l < m \leq 4$, $u_i \in U_\ell$ and $u_j \in U_m$, the cycle C corresponds to the cycle that uses the path $u_i v_\ell v_m u_j$ instead of $u_i u u_j$. By (4.6), there are at least

$$\left(1 - \frac{6(2k-7)}{k(k-1)}\right) S$$

cycles in G that use a path $u_i u u_j$ with $u_i, u_j \in N(u) \backslash D$.

- Every cycle in G that for some $i \in A_\ell$ and $j \in A_m$ with $\ell \neq m$ contains a path $u_i u u_j$ gives rise to additional 4 cycles in H(except the one containing $u_i v_\ell v_m u_j$).

For example, if $\ell = 1$, $m = 2$ the four new cycles contain paths $u_iv_1v_3v_2u_j$, $u_iv_1v_4v_2u_j$, $u_iv_1v_3v_4v_2u_j$ and $u_iv_1v_4v_3v_2u_j$ instead of the path u_iuu_j. According to (4.7), there are at least

$$\left(\frac{3(k-6)^2-4}{4(k-6)(k-7)}\right)\left(1-\frac{6(2k-7)}{k(k-1)}\right)S = \left(\frac{3k^2-36k+104}{4k(k-1)}\right)S$$

cycles in G that for some $i \in A_\ell$ and $j \in A_m$ with $\ell \neq m$ pass through a path u_iuu_j.

- There are 7 new cycles in H spanned by the vertices v_1, v_2, v_3, v_4.

By all of the observations above, the number of cycles in H is

$$\begin{aligned} C(H) &\geq C(G) - S + \left(1-\frac{6(2k-7)}{k(k-1)}\right)S + 4\left(\frac{3k^2-48k+104}{4k(k-1)}\right)S + 7 \\ &= C(G) + 7 + S\left(\frac{3(k-4)(k-12)}{k(k-1)}\right) \\ &> C(G). \end{aligned}$$

Therefore, H has more cycles than G. $\square$

By inspection, for $m = 7$ the graphs that have the most cycles are K_4 plus an edge and K_4 with one edge replaced by a path of length two. In the first case the minimum degree is one and in the second case the minimum degree is two.

Tsaturian and I stated the following theorem in [9], but we didn't present the proof in the paper. Here the proof is added for completeness.

Theorem 4.2.4 (Arman-Tsaturian, 2017^+ [9])**.** *If $m > 7$ and $C(G) = C(m)$, then $\delta(G) \geq 3$.*

Proof. Assume the contrary, namely that there exists a graph G with $m > 7$ edges and $C(m)$ cycles, such that $\delta(G) \leq 2$. Let $m > 7$ and let G be a graph having the least number of vertices of degree two among all graphs having $C(m)$ cycles.

First, assume that G is connected (otherwise, identifying a pair of different vertices from one component with a pair of different vertices from another component increases the number of cycles in G.)

Also, assume that G does not contain a vertex of degree one. Indeed, if u is a vertex of degree 1, and $G \backslash \{u\}$ is not a complete graph, then deleting the edge from u and adding it to $G \backslash \{u\}$ increases the number of cycles. If $G \backslash \{u\}$ is a complete graph, and $vu \in E(G)$, then the graph obtained from G by deleting two edges vv_1, vv_2 and adding the edges uv_1, uv_2 has more cycles than G.

Hence, assume that $\delta(G) = 2$ and let u be a vertex of degree 2 in G. Let u_1 and u_2 be the neighbours of u in G. Assume that $u_1u_2 \in E(G)$ (otherwise the graph H that is obtained from a graph G by deleting the edge uu_1 and adding the edge u_1u_2 has the same number of cycles as G and the vertex u has degree 1).

There are four cases to consider:

Case 1: $\deg(u_1) = 2$ or $\deg(u_2) = 2$.

Without loss of generality assume that $\deg(u_1) = 2$. Since G is connected, $\deg(u_2) \geq 3$. If $G \backslash \{u, u_1\}$ is a complete graph, let v_1, v_2 be two vertices of $G \backslash \{u, u_1, u_2\}$. Then the graph H obtained from G by deleting the edges uu_1, u_1u_2 and adding the edges uv_1, uv_2 has at least 2 more cycles than G.

In the case when $G\backslash\{u, u_1\}$ is not a complete graph, let v_1, v_2 be two nonadjacent vertices of $G\backslash\{u, u_1, u_2\}$. Then the graph H obtained from G by deleting the edges uu_1, uu_2, u_1u_2 and adding the edges uv_1, uv_2, v_1v_2 has more cycles than G.

Case 2: $\deg(u_1) = \deg(u_2) = 3$ and there is a vertex $u_3 \neq u$, such that $u_1u_3, u_2u_3 \in E(G)$.

In this case, the only cycles in G that pass through at least one of the vertices u, u_1 or u_2 are cycles uu_1u_2u, $uu_1u_3u_2u$ and $u_1u_2u_3u_1$.

Let v_1v_2 be an edge of $G\backslash\{u, u_1, u_2\}$, then the graph H obtained from G by deleting the edges uu_1, uu_2, u_1u_3, u_2u_3 and adding the edges u_1v_1, u_1v_2, u_2v_1, u_2v_2 has at least four more cycles than G (see Figure 4.2).

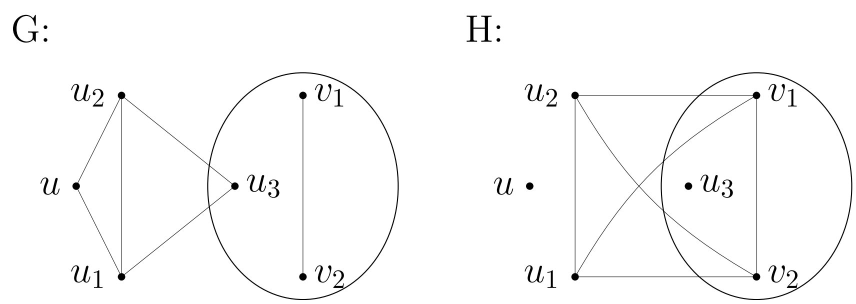

Figure 4.2: Case 2. Constructing the graph H.

Case 3: $\deg(u_1) = \deg(u_2) = 3$ and there exist two distinct vertices u_3 and u_4 (different from u), such that $u_1u_3, u_2u_4 \in E(G)$.

In this case, the only cycle in G that contains the path u_1uu_2, but does not contain the path $u_3u_1uu_2u_4$ is the cycle $u_1uu_2u_1$.

Then the graph H obtained from G by deleting the edges uu_1, uu_2 and adding the edges u_1u_4 and u_2u_3 has more cycles than G (see Figure 4.3).

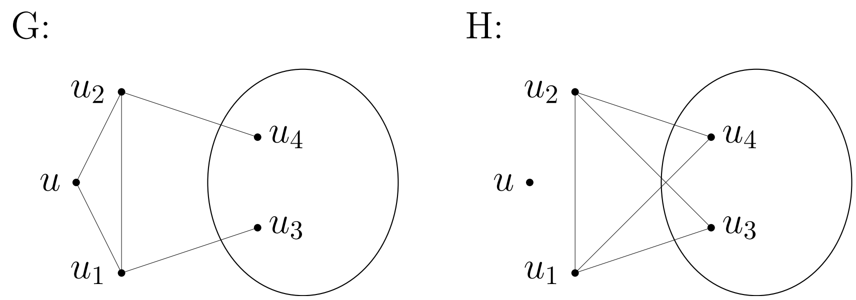

Figure 4.3: Case 3. Constructing the graph H.

Case 4: $\deg(u_1) \geq 4$ or $\deg(u_2) \geq 4$.

Assume that $\deg(u_1) \geq 4$ and let v_1 and v_2 be the other two neighbours of u_1 (different from u_2 and u). Consider the graph H obtained from G by deleting the edge u_1v_2 and adding the edge uv_2. Every cycle in G that did not pass through edge u_1v_2 is still a cycle in H. Cycles in G that contain an edge u_1v_2 and do not contain the vertex u correspond to the cycles in H that use the path u_1uv_2 instead of an edge u_1v_2. Every cycle in G through the path $u_2uu_1v_2$ corresponds to the cycle in H that uses the path u_2uv_2 instead.

Therefore, the number of cycles in H is at least $C(G) = C(m)$.

If there is at least one path P from v_2 to $N(u_1)\backslash\{u, u_2, v_2\}$ that does not use vertices $\{u_1, u_2\}$, then uPu_1u_2u is a new cycle in H; hence H has more cycles than G.

Assume that for all $v \in N(u_1)\backslash\{u, u_2\}$, any path from v to $N(u_1)\backslash\{u, u_2, v\}$ in G omits both vertices u_1 and u_2. Therefore, for any $v \in N(u_1)\backslash\{u, u_2\}$ any cycle that contains the edge u_1v also contains the vertex u_2. By symmetry, for any $v \in N(u_2)\backslash\{u, u_1\}$, any cycle that contains the edge u_2v also contains the vertex u_1. Also, assume that G has no cut vertices, (deletion of which makes G disconnected) otherwise identifying two vertices from different components produces a graph with more cycles.

If there is $v \in N(u_1) \triangle N(u_2)\backslash\{u_1, u_2\}$, assume that $vu_1 \in E(G)$, then $vu_2 \notin G$ and any cycle containing the edge vu_1 is of length at least 4. There is no path from v to $N(u_1)\backslash\{u, u_2, v\}$ that does not use the vertices $\{u_1, u_2\}$, so there is no cycle that contains the vertices u_1 and v, but does not contain the edge u_1v. Then the graph H obtained from G by contracting the edge u_1v and adding an edge anywhere else in a graph has more cycles than G.

If $N(u_1)\triangle N(u_2)\backslash\{u_1, u_2\} = \emptyset$, then the condition that for any $v \in N(u_1)\backslash\{u, u_2\}$, there is no path from v to $N(u_1)$ together with the observation that the deletion of the vertex v does not disconnect the graph G yields that $\deg(v) = 2$. Hence, G is the graph obtained by gluing $k > 3$ triangles by an edge. In this case G has $k + \binom{k}{2}$ cycles. However, the graph H obtained from G by removing the common edge from all triangles and adding it to different place in the graph has $\binom{k}{2} + 2(k-2) + 3$ cycles.

This finishes the last case and the proof of Theorem 4.2.4. $\square$

4.3 Counting Lemma

The main result of this section is Lemma 4.3.1, which is the major tool used for the upper bounds in this chapter and in Chapter 5.

Multigraphs are defined as in Bollobás's book [14]. Let G be a multigrpah, the degree $\deg_G(V)$ of a vertex $v \in V(G)$ is the number of edges incident to v. For two vertices $u, v \in V(G)$, denote by $E(u, v)$ the set of all edges between u and v. For a vertex $v \in V(G)$, denote by $N(v)$ the set of all vertices adjacent to v (by at least one edge). A cycle of length $k \geq 2$ in a multigraph G is an alternating sequence of k distinct vertices and k distinct edges $v_1, e_1, v_2, e_2, \ldots, e_k, v_1$, where for each $i \in [k]$, $v_i \in V(G)$, $e_i \in E(G)$ and any consecutive vertex and edge are incident. As in the case of simple graphs, denote the number of cycles in a multigraph G by $C(G)$. No loop can be a part of a cycle, hence only multigraphs without loops are considered.

Lemma 4.3.1 (Arman-Tsaturian, 2017$^+$ [9])**.** *Let G be a multigraph with n vertices. For any $\ell \in [n]$, and any vertices $v_1, \ldots, v_\ell \in V(G)$, define $F(v_1, \ldots, v_\ell) = N(v_\ell) \backslash \{v_1, \ldots, v_{\ell-1}\}$ and define $f(v_1, \ldots, v_\ell) = \max\{\deg_{G \backslash \{v_2, \ldots, v_{\ell-1}\}}(v_\ell), 1\}$. Denote the number of cycles in G that contain the path $v_1 e_1 v_2 \ldots e_{\ell-1} v_\ell$ by $C(v_1 e_1 v_2 \ldots e_{\ell-1} v_\ell)$ (note that $C(v_1)$ is the number of cycles containing the vertex v_1). For brevity, write $F_\ell = F(v_1, \ldots, v_\ell)$, $f_\ell = f(v_1, \ldots, v_\ell)$, $C_\ell = C(v_1 e_1 \ldots e_{\ell-1} v_\ell)$. For a $k \in [n]$, let*

$v_1e_1v_2e_2\dots v_k$ be a path in G. If $F_k \neq \emptyset$, then

$$C_k \le f_k \cdot \max_{\substack{k+1\le t\le n\\ v_{k+1}\in F_k\\ \vdots\\ v_t\in F_{t-1}}} \{f_{k+1}\cdot f_{k+2}\cdots f_t\}.$$

(the maximum is taken over all paths $v_{k+1}\dots v_t$, such that $v_1\dots v_k e_k v_{k+1}\dots v_t$ extends $v_1\dots v_k$)

Proof. Fix $n \ge 2$. Let G be a multigraph on n vertices. The proof is by mathematical induction on $\ell = n - k$.

Base case. Let $\ell = 1$. Let $v_1e_1\dots v_{n-1}$ be a path in G; C_{n-1} is to be bounded.

The condition $F_{n-1} \neq \emptyset$ means that $F_{n-1} = \{v_n\}$ and it remains to be proved that $C_{n-1} \le f_{n-1}f_n$. Let s be the number of edges between v_{n-1} and v_1. Then $C_{n-1} \le s + (f_{n-1} - s)f_n$. By definition, $f_n \ge 1$; therefore $s + (f_{n-1} - s)f_n \le sf_n + (f_{n-1} - s)f_n = f_{n-1}f_n$, which proves the base case.

Inductive step. Let $i \in [n-1]$. Assume that the statement of the lemma holds for $\ell = i$, and prove it for $\ell = i + 1$; i.e., let $v_1e_1\dots v_{n-i-1}$ be a path in G, and $C_{n-i-1} = C(v_1e_1\dots e_{n-i-2}v_{n-i-1})$ is to be bounded.

Let s be the number of edges between v_{n-i-1} and v_1. Then

$$C_{n-i-1} = s + \sum_{\substack{v_{n-i}\in F_{n-i}\\ e_{n-i-1}\in E(v_{n-i},v_{n-i-1})}} C(v_1e_1\dots v_{n-i-1}e_{n-i-1}v_{n-i}).$$

For all possible choices of v_{n-i} and e_{n-i-1}, according to inductive hypothesis,

$$C(v_1e_1\dots e_{n-i-1}v_{n-i}) \leq \begin{cases} f_{n-i}\max_{\substack{n-i+1\leq t\leq n\\ v_{n-i+1}\in F_{n-i}\\ \vdots \\ v_t\in F_{t-1}}}\{f_{n-i+1}\cdots f_t\}, \text{ if } F_{n-i}\neq\emptyset \\ f_{n-i}, \text{ if } F_{n-i}=\emptyset \end{cases}$$

$$\leq \max_{\substack{n-i\leq t\leq n\\ v_{n-i}\in F_{n-i-1}\\ \vdots \\ v_t\in F_{t-1}}}\{f_{n-i}\cdots f_t\}.$$

Therefore,

$$C_{n-i-1} \leq s+(f_{n-i-1}-s)\cdot \max_{\substack{n-i\leq t\leq n\\ v_{n-i}\in F_{n-i-1}\\ \vdots \\ v_t\in F_{t-1}}}\{f_{n-i}\cdots f_t\}$$

$$\leq f_{n-i-1}\cdot \max_{\substack{n-i\leq t\leq n\\ v_{n-i}\in F_{n-i-1}\\ \vdots \\ v_t\in F_{t-1}}}\{f_{n-i}\cdots f_t\}.$$

This proves that the statement of the lemma holds for $\ell = i+1$, and therefore by induction, the statement holds for all $\ell \in [n-1]$. □

4.4 Upper bound for number of cycles in graphs or multigraphs

The main result of this section is Theorem 4.4.2, where an upper bound for the number of cycles in a graph (or a multigraph) with a fixed number of vertices and edges is given.

The proof of Theorem 4.4.2 relies on the following lemma.

Lemma 4.4.1 (Arman-Tsaturian, 2017^+ [9]). *Let G be a multigraph with $n \geq 3$ vertices and m edges, and let v_1 be a vertex in G of degree $\Delta(G)$.*

If $\frac{m}{n-1} \geq 3$, and $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, $\frac{m}{n-1} - s = \alpha$, then there are at most $\frac{\Delta(G)}{2}\left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1}$ cycles in G that contain v_1.

If $\frac{m}{n-1} < 3$, then there are at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$ cycles in G that contain v_1.

Proof. Let G be a multigraph with $n \geq 3$ vertices and m edges, and let v_1 be a vertex of degree $\Delta(G)$.

For any edge $e = v_1v_2$ incident with v_1, by Lemma 4.3.1, the number of cycles that contain e is at most

$$f_2 \cdot \max_{\substack{3 \leq t \leq n \\ v_3 \in F_2 \\ \vdots \\ v_t \in F_{t-1}}} \{f_3 \cdots f_t\} \leq \max_{\substack{2 \leq t \leq n \\ v_2 \in F_1 \\ \vdots \\ v_t \in F_{t-1}}} \{f_2 \cdots f_t\}.$$

Every cycle through v_1 contains two edges incident to v_1; therefore the number of cycles that contain v_1 is at most

$$\frac{\Delta}{2} \cdot \max_{\substack{2 \leq t \leq n \\ v_2 \in F_1 \\ \vdots \\ v_t \in F_{t-1}}} \{f_2 \cdots f_t\}. \tag{4.8}$$

Let $v_2, \ldots v_t$ be a collection of vertices that give the maximum in (4.8) with the smallest possible t. Then $f_t \geq 2$ (otherwise remove all $f_i = 1$ after the last $f_k \geq 2$ to obtain the smaller collection of vertices that gives maximum in (4.8)). Then for all

$2 \leq i \leq t$,

$$f_i = \deg_{G \setminus \{v_2,\ldots,v_{i-1}\}}(v_i).$$

For $2 \leq i \leq t$, all of the edge sets $\{v_i u \in E(G) : u \in V(G) \setminus \{v_2, \ldots, v_i\}\}$ are mutually disjoint, so $f_2 + \cdots + f_t \leq m$. Therefore,

$$\frac{\Delta}{2} f_2 \cdots f_t \leq \frac{\Delta}{2} \cdot \max_{\substack{2 \leq t \leq n \\ x_2 + \ldots + x_t \leq m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdots x_t\}.$$

So the number of cycles in G that contain v_1 is at most

$$\frac{\Delta}{2} \cdot \max_{\substack{2 \leq t \leq n \\ x_2 + \ldots + x_t \leq m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdots x_t\}. \tag{4.9}$$

For a fixed t, the product $x_2 \cdots x_t$ in (4.9) attains its maximum when x_is $(i \geq 2)$ are as equal as possible (for all i, j $|x_i - x_j| \leq 1$), and their sum is equal to m. Let $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, $\frac{m}{n-1} = s + \alpha$.

If $s \geq 3$ (which is equivalent to $\frac{m}{n-1} \geq 3$), let the maximum in (4.9) be achieved for some $t \leq n$ and let $x_2, \ldots, x_t$ be a collection of x_is that gives the maximum in (4.9). If $t < n$, then $s \geq 3$ implies that either for some $i \in [t]$, $x_i \geq 5$, or for two different $i, j \in [t]$, $x_i = x_j = 4$. In the first case, replacing x_i by $x_i - 2$ and setting $x_{t+1} = 2$ gives a collection of x_is with a larger product. In the second case, setting $x_i = x_j = 3$ and $x_{t+1} = 2$ increases the product of x_is. Hence, the maximum in (4.9) is achieved when $t = n$. For all $2 \leq i \leq n$, $x_i = s$ or $x_i = s + 1$. Then the number of cycles in G that pass through v_1 is at most

$$\frac{\Delta}{2} x_2 \cdots x_n = \frac{\Delta}{2} s^{(1-\alpha)(n-1)} (s+1)^{\alpha(n-1)} = \frac{\Delta}{2} (s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

If $s < 3$, let the maximum of (4.9) be achieved for some $2 \leq t \leq n$ and let $x_2, \ldots, x_t$ be the collection of x_is that gives the maximum in (4.9). Recall that for all i, j, $|x_i - x_j| \leq 1$. If for two different $i, j \in [t]$ $x_i = x_j > 3$, then $m > 6 + 3(t-2) = 3t$, and $s < 3$ implies that $t < n$. Replacing x_i by $x_i - 1$, x_j by $x_j - 1$ and setting $x_{t+1} = 2$ increases the product. Therefore, there is at most one i, such that $x_i = 4$. If there is i such that $x_i = 1$, then replacing any x_j $(j \neq i)$ by $x_j + 1$ and deleting x_i increases the product. If for some $i, j, k \in [t]$ $x_i = x_j = x_k = 2$, then replacing x_i by 3, x_j by 3 and deleting x_k increases the product. Therefore, $\{x_2, \ldots, x_t\} \in \{\{3, 3, \ldots, 3, 2, 2\}, \{3, 3, \ldots, 3, 4\}, \{3, 3, \ldots, 3, 2\}, \{3, 3, \ldots, 3\}\}$. Then $x_2 \ldots x_t$ is at most $3^{\frac{m}{3}}$, so the number of cycles that pass through v_1 is at most

$$\frac{\Delta}{2} x_2 \cdots x_t \leq \frac{\Delta}{2} 3^{\frac{m}{3}}.$$

□

Theorem 4.4.2 (Arman-Tsaturian, 2017$^+$ [9])**.** *Let G be a multigraph with $n \geq 2$ vertices and m edges.*

If $\frac{m}{n-1} < 3$, then

$$C(G) < \frac{3}{4} \Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} \geq 3$, and $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, $\alpha = \frac{m}{n-1} - s$, then

$$C(G) < \frac{3}{4} \Delta(G) (s^{1-\alpha}(s+1)^{\alpha})^{n-1} = \frac{3}{4} \Delta(G) ((s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}})^m.$$

Proof. The proof is by mathematical induction on n.

Base case. If $n = 2$, there is only one multigraph on n vertices with m edges, namely two vertices connected by m edges. In this case $s = \frac{m}{n-1} = m$, and G has $\max\{\binom{m}{2}, 0\}$ cycles, which is less than $\frac{3}{4}m(\sqrt[3]{3})^m$ (for the case $m < 3$), and less than $\frac{3}{4}m \cdot m$ (for the case $m \geq 3$).

Inductive step. Let $k \geq 3$ be an integer, and suppose that the statement of the theorem is proved for $n = k - 1$. Let G be a multigraph with k vertices, m edges and let v_1 be a vertex of the maximal degree in G. There are two cases to consider.

Case 1: $\frac{m}{k-1} < 3$.

If $\Delta(G) \leq 2$, then every edge is contained in at most one cycle, and every cycle contains at least two edges, so the number of cycles in G is at most

$$\frac{m}{2} \leq \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\Delta(G) \geq 3$, then the multigraph $G \backslash v_1$ has at most $m-3$ edges, $\Delta(G\backslash v_1) \leq \Delta(G)$ and $\frac{|E(G\backslash v_1)|}{|V(G\backslash v_1)|-1} \leq \frac{m}{k-1} < 3$, therefore, by the inductive assumption, the number of cycles in $G\backslash v_1$ is at most $\frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3}$. By Lemma 4.4.1, the number of cycles that contain v_1 is at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$, therefore the total number of cycles in G is at most

$$\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m + \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3} = \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

Case 2: $\frac{m}{k-1} \geq 3$.

Let $s = \left\lfloor \frac{m}{k-1} \right\rfloor$, $\alpha = \frac{m}{k-1} - \left\lfloor \frac{m}{k-1} \right\rfloor$. Note that $\Delta(G\backslash v_1) \leq \Delta(G)$ and let

$$y = \frac{|E(G\backslash v_1)|}{|V(G\backslash v_1)| - 1} \leq \frac{m}{k-1}.$$

The function

$$f(x) = (\lfloor x \rfloor)^{1-x+\lfloor x \rfloor}(\lfloor x \rfloor + 1)^{x-\lfloor x \rfloor}$$

is non-decreasing on every interval $[a, a+1], a \in \mathbb{Z}_{\geq 0}$ (and hence on $\mathbb{R}^+$); therefore

$$s^{1-\alpha}(s+1)^{\alpha} \geq f(3) = 3. \tag{4.10}$$

If $y \geq 3$, then, by the induction hypothesis,

$$\begin{aligned} |E(G \backslash v_1)| &\leq \frac{3}{4}\Delta(G)\left((\lfloor y \rfloor)^{1-y+\lfloor y \rfloor}(\lfloor y \rfloor + 1)^{y-\lfloor y \rfloor}\right)^{k-2} \\ &\leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}. \end{aligned}$$

If $y < 3$, then $|E(G \backslash v_1)| < 3(k-2)$, and by the induction hypothesis

$$\begin{aligned} |E(G \backslash v_1)| &\leq \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{|E(G \backslash v_1)|} < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{3(k-2)} \\ &= \frac{3}{4}\Delta(G) \cdot 3^{k-2} \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}. \end{aligned}$$

Hence, for any y, $|E(G \backslash v_1)| \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{k-2}$, which together with Lemma 4.4.1 and (4.10) implies that

$$\begin{aligned} C(G) &= \frac{3\Delta(G)}{4}(s^{1-\alpha}(s+1)^{\alpha})^{k-2} + \frac{\Delta(G)}{2}(s^{1-\alpha}(s+1)^{\alpha})^{k-1} \\ &\leq \frac{3\Delta(G)}{4}(s^{1-\alpha}(s+1)^{\alpha})^{k-1}, \end{aligned}$$

which proves the inductive step and hence the theorem. □

A consequence of Theorem 4.4.2 is

Corollary 4.4.3 (Arman-Tsaturian, 2017$^+$ [9])**.** *For any positive integer m*

$$C(m) < 8.25(\sqrt[3]{3})^m.$$

Proof. Let G be a graph with n vertices and m edges, such that $C(G) = C(m)$.

If $\frac{m}{n-1} < 3$, then, by Theorem 4.2.3 and 4.4.2,

$$C(m) = C(G) < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^m \leq 8.25(\sqrt[3]{3})^m.$$

So suppose that $\frac{m}{n-1} \geq 3$. Let $f(s,\alpha) = (s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}}$, then for any $s > 0$, $f(s,\alpha)$ is monotone in α and $\max\limits_{s\in\mathbb{Z}_+,\alpha\in[0,1)} f(s,\alpha) = \max\limits_{s\in\mathbb{Z}_+} s^{\frac{1}{s}} = \sqrt[3]{3}$. This, together with Theorem 4.4.2 and Theorem 4.2.3, implies that for $s = \left\lfloor \frac{m}{n-1} \right\rfloor$ and $\alpha = \frac{m}{n-1} - \left\lfloor \frac{m}{n-1} \right\rfloor$

$$C(m) = C(G) < \frac{3}{4}\Delta(G)((s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}})^m \leq 8.25(\sqrt[3]{3})^m.$$

$\square$

Theorem 4.4.2 is stated in rather technical terms, so the following corollary is intended to be a more readable version of Theorem 4.4.2. This corollary is stated in the paper [9], but the proof does not appear there, so I add the proof here for completeness.

Corollary 4.4.4 (Arman-Tsaturian, 2017$^+$ [9]). *Let G be a graph on n vertices with average degree $d \geq 6$. Then*

$$C(G) < 3\Delta(G)\left(\frac{d}{2}\right)^{n-1}.$$

Proof. Let G be a graph with n vertices, m edges and the average degree $d \geq 6$. Then $\frac{m}{n-1} > \frac{m}{n} = \frac{d}{2} \geq 3$, so Theorem 4.4.2 implies that for $s = \left\lfloor \frac{m}{n-1} \right\rfloor$ and $\alpha = \frac{m}{n-1} - s$

$$C(G) < \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

Here, the weighted AM-GM inequality can be used to prove that

$$s^{1-\alpha}(s+1)^{\alpha} \leq \frac{m}{n-1}. \tag{$\star$}$$

Alternatively, it is easy to verify that for a fixed $s \geq 3$ the function $f(\alpha) = s^{1-\alpha}(s+1)^{\alpha}$ is convex, and so for $\alpha \in [0,1]$, $f(\alpha) \leq (1-\alpha)f(0) + \alpha f(1)$. Therefore

$$s^{1-\alpha}(s+1)^{\alpha} \leq (1-\alpha)s + \alpha(s+1) = \frac{m}{n-1}.$$

Finally,

$$\begin{aligned}
C(G) &\leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1} \\
&\leq \frac{3}{4}\Delta(G)\left(\frac{m}{n-1}\right)^{n-1} && \text{(by } (\star)\text{)} \\
&\leq \frac{3}{4}\Delta(G)\left(\frac{m}{n}\right)^{n-1}\left(\frac{n}{n-1}\right)^{n-1} \\
&\leq 3\Delta(G)\left(\frac{d}{2}\right)^{n-1}.
\end{aligned}$$

(The last inequality is based on the fact that for $n \geq 2$, $\left(1+\frac{1}{n-1}\right)^{n-1}$ is an increasing function and $\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1} = e$.) $\square$

4.5 Example of a graph with $(1.37)^m$ cycles

For $n \geq 1$ let H_n be the graph on $2n+2$ vertices with

$$V(H_n) = \{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots v_{n+1}\} \qquad \text{and}$$

$$E(H_n) = \{u_i v_j : i, j \in [n+1], |i-j| \leq 1\} \cup \{u_i u_{i+1} : i \in [n]\} \cup \{v_i v_{i+1} : i \in [n]\}.$$

For example, see Figure 4.4 for H_{12}.

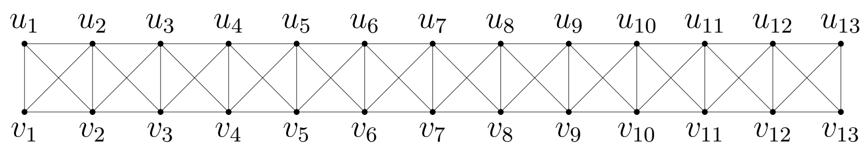

Figure 4.4: The graph H_{12}.

Claim 4.5.1 (Arman-Tsaturian, 2017^+ [9]). *For $n \geq 1$ denote by $P(n)$ the number of paths from the vertex u_1 to the vertex u_{n+1} in H_n. For all $n \geq 2$,*

$$P(n) = 4P(n-1) + 4P(n-2).$$

Note that $P(n)$ is also equal to the number of paths from u_1 to v_{n+1} in H_n.

Proof sketch. The proof of the claim relies on an inductive argument and an observation that each path from u_1 to u_{n+1} in H_n corresponds to exactly one of the following eight types of paths:

- A path from u_1 to u_n in H_{n-1} followed by the path $u_n u_{n+1}$.
- A path from u_1 to u_n in H_{n-1} followed by the path $u_n v_{n+1} u_{n+1}$.
- A path from u_1 to v_n in H_{n-1} followed by the path $v_n u_{n+1}$.
- A path from u_1 to v_n in H_{n-1} followed by the path $v_n v_{n+1} u_{n+1}$.
- A path from u_1 to u_{n-1} in H_{n-2} followed by the path $u_{n-1} u_n v_{n+1} v_n u_{n+1}$.
- A path from u_1 to u_{n-1} in H_{n-2} followed by the path $u_{n-1} v_n v_{n+1} u_n u_{n+1}$.

- A path from u_1 to v_{n-1} in H_{n-2} followed by the path $v_{n-1}u_n v_{n+1} v_n u_{n+1}$.

- A path from u_1 to v_{n-1} in H_{n-2} followed by the path $v_{n-1}v_n v_{n+1} u_n u_{n+1}$.

$\square$

Also, note that $P(0) = 1$, $P(1) = 5$. Solving the recurrence relation $P(n) = 4P(n-1) + 4P(n-2)$ leads to the inequality

$$P(n) \geq (2 + 2\sqrt{2})^n.$$

Define the graph G_n by identifying the vertices u_1 and u_n in H_n (see Figure 4.5 for G_{12}). Then G_n has $2n+1$ vertices, $m = 5n+1$ edges and

$$C(G_n) \geq (2 + 2\sqrt{2})^n.$$

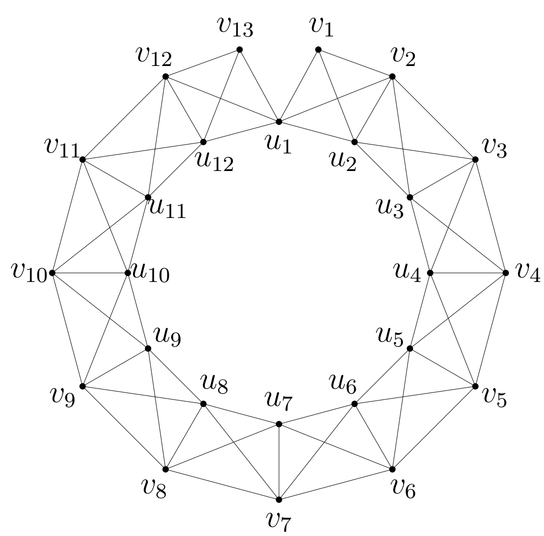

Figure 4.5: G_{12} with 25 vertices and 61 edges.

For an integer m, let the graph G be obtained from $G_{\left\lfloor \frac{m-1}{5} \right\rfloor}$ by adding $(m - 5\left\lfloor \frac{m-1}{5} \right\rfloor - 1)$ edges (anywhere). Then G has m edges and for m large enough (for $m > 16855$)

$$C(G) \geq C(G_{\left\lfloor \frac{m-1}{5} \right\rfloor}) \geq (2+2\sqrt{2})^{\left\lfloor \frac{m-1}{5} \right\rfloor} \geq (2+2\sqrt{2})^{\frac{m}{5}-1} > 1.37^m.$$

4.6 Lower bound for the number of cycles with given number of vertices and edges

In this section the lower bound for the number of cycles in a graph G with n vertices and m edges is established. This result was never published before, but was discovered by myself and Tsaturian independently.

Theorem 4.6.1 (Arman, Tsaturian, 2017)**.** *If $d = d(n)$ is such that $\lim_{n\to\infty} d(n) = \infty$, then there exists a graph G with n vertices and average degree d, such that*

$$C(G) \geq \left(\frac{d}{e}\right)^n (1+o(1))^n.$$

If $d > 1$ is fixed, then for $c = \frac{d}{2}$, $\alpha = 1 - \frac{1}{d}$ and n large enough, there exists a graph G with n vertices and average degree d such that

$$C(G) \geq (1+o(1))^n \left(\frac{c^c 2^\alpha}{e^{2\alpha}(1-\alpha)^{(1-\alpha)}(c-\alpha)^{c-\alpha}} \right)^n.$$

Proof. The statement of the theorem follows from an averaging argument for graphs on n vertices and m edges. Let $c = \frac{d}{2} = c(n)$, then $m = cn$. Let $N = \binom{n}{2}$. Let E be the average of the number of cycles in all graphs with n vertices and m edges. The lower bound on E is obtained below. First,

$$E = \sum_{k=3}^{n} \binom{n}{k} \frac{k!}{2k} \binom{N-k}{m-k} / \binom{N}{m} = \sum_{k=3}^{n} \frac{n!}{(n-k)!} \frac{1}{2k} \binom{N-k}{m-k} / \binom{N}{m}.$$

To simplify calculations, for $3 \leq k \leq n$, put

$$a_k = \frac{n!}{(n-k)!} \frac{1}{2k} \binom{N-k}{m-k} / \binom{N}{m}.$$

Let $\alpha = 1 - \frac{1}{2c} + o(1)$, be such that αn is an integer, then

$$\begin{aligned}
E &= \sum_{k=3}^{n} a_k \geq a_{\alpha n} \\
&= \frac{n!}{(n-\alpha n)!} \frac{1}{2\alpha n} \binom{N-\alpha n}{m-\alpha n} / \binom{N}{m} \\
&= \frac{n!}{(n-\alpha n)!} \frac{1}{2\alpha n} \binom{N-\alpha n}{cn-\alpha n} / \binom{N}{cn} \\
&\geq (1+o(1))^n \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{(1-\alpha)n}{e}\right)^{(1-\alpha)n}} \frac{(N-\alpha n)!(cn)!}{((c-\alpha)n)!N!} \\
&\geq (1+o(1))^n \frac{n^{\alpha n}}{e^{\alpha n}\left(1-\alpha\right)^{(1-\alpha)n}} \frac{(\binom{n}{2}-\alpha n)!(cn)!}{((c-\alpha)n)!\binom{n}{2}!} \\
&\geq (1+o(1))^n \frac{n^{\alpha n}}{e^{\alpha n}\left(1-\alpha\right)^{(1-\alpha)n}} \frac{(\binom{n}{2}-\alpha n)^{\binom{n}{2}-\alpha n} e^{\alpha n}}{\binom{n}{2}^{\binom{n}{2}}} \frac{(cn)^{cn}}{e^{\alpha n}((c-\alpha)n)^{(c-\alpha)n}} \\
&= (1+o(1))^n \frac{n^{\alpha n}}{(e^{\alpha}\left(1-\alpha\right)^{(1-\alpha)})^n} \frac{(1-\frac{2\alpha}{n-1})^{\binom{n}{2}-\alpha n}}{\binom{n}{2}^{\alpha n}} n^{\alpha n} \left(\frac{(c)^c}{(c-\alpha)^{(c-\alpha)}}\right)^n \\
&= (1+o(1))^n n^{\alpha n} \left(\frac{2^{\alpha}}{e^{\alpha}\left(1-\alpha\right)^{(1-\alpha)}}\right)^n \frac{(1-\frac{2\alpha}{n-1})^{\left(\frac{n-1}{2\alpha}\right)\left(\alpha n-\frac{2\alpha^2 n}{n-1}\right)}}{n^{2\alpha n}(1-\frac{1}{n})^{\alpha n}} n^{\alpha n} \left(\frac{(c)^c}{(c-\alpha)^{(c-\alpha)}}\right)^n \\
&\geq (1+o(1))^n \left(\frac{2^{\alpha}}{e^{\alpha}\left(1-\alpha\right)^{(1-\alpha)}}\right)^n e^{-\alpha n} \left(\frac{(c)^c}{(c-\alpha)^{(c-\alpha)}}\right)^n \\
&\geq (1+o(1))^n \left(\frac{c^c 2^{\alpha}}{e^{2\alpha}\left(1-\alpha\right)^{(1-\alpha)}(c-\alpha)^{c-\alpha}}\right)^n.
\end{aligned}$$

If $\lim_{n\to\infty} d(n) = \infty$, then $\lim_{n\to\infty} c(n) = \infty$, $\lim_{n\to\infty} \alpha(n) = 1$ and

$$\begin{aligned}
E &\geq (1+o(1))^n \left(\frac{c^\alpha 2^\alpha}{e^{2\alpha}\left(1-\alpha\right)^{(1-\alpha)}} \left(1+\frac{\alpha}{c-\alpha}\right)^{c-\alpha}\right)^n \\
&\geq (1+o(1))^n \left(\frac{c^\alpha 2^\alpha}{e^{2\alpha}\left(1-\alpha\right)^{(1-\alpha)}} e^\alpha \right)^n. \\
&\geq (1+o(1))^n \left(\frac{2c}{e}\right)^n. \\
&= (1+o(1))^n \left(\frac{d}{e}\right)^n.
\end{aligned}$$

If $\lim_{n\to\infty} d(n) \neq \infty$, then

$$E \geq (1+o(1))^n \left(\frac{c^c 2^\alpha}{e^{2\alpha}\left(1-\alpha\right)^{(1-\alpha)}\left(c-\alpha\right)^{c-\alpha}} \right)^n.$$

□

4.7 Maximum number of cycles in multigraphs

The problems of maximizing the number of cycles with a fixed number of edges or a fixed average degree can be also considered for multigraphs.

Theorem 4.7.1 (Arman-Tsaturian, 2017^+ [9])**.** *Let G be a multigraph that has the maximum number of cycles among all the multigraphs with $n \geq 2$ vertices and $m \geq 3$ edges. Let $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, and put $\alpha = \frac{m}{n-1} - s$.*

If $\frac{m}{n-1} \geq 3$, then

$$\frac{8}{27} s (s^{1-\alpha}(s+1)^\alpha)^{n-1} \leq C(G) \leq \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^\alpha)^{n-1}.$$

If $\frac{m}{n-1} \le 3$, *then*

$$4(\sqrt[3]{3})^{m-4} \le C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

Theorem 4.7.1 was stated in the paper [9], but was not proven. Here I present the proof of Theorem 4.7.1.

Proof. The upper bounds in Theorem 4.7.1 follow directly from Theorem 4.4.2.

Define $C_{n,m}$ to be the multigraph obtained from the cycle C_n by replacing each of some $m - \lfloor \frac{m}{n} \rfloor n$ consecutive edges with $\lfloor \frac{m}{n} \rfloor + 1$ multi-edges and the remaining $\lfloor \frac{m}{n} \rfloor n - m + n$ edges with $\lfloor \frac{m}{n} \rfloor$ multi-edges.

Lower bound for case $\frac{m}{n-1} \ge 3$:

Let $\lfloor \frac{m}{n-1} \rfloor = s$, $\frac{m}{n-1} = s + \alpha$. If $\lfloor \frac{m}{n} \rfloor = s$, then $\frac{m}{n} = s + \alpha - \frac{m}{n(n-1)}$ and

$$\begin{aligned}
C(C_{n,m}) &= \left(s^{1-\left(\alpha - \frac{m}{n(n-1)}\right)}(s+1)^{\alpha - \frac{m}{n(n-1)}}\right)^n \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^n s^{\frac{m}{n-1}}(s+1)^{-\frac{m}{n-1}} \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} s^{1-\alpha+\frac{m}{n-1}}(s+1)^{\alpha-\frac{m}{n-1}} \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} s \left(\frac{s}{s+1}\right)^s \\
&\ge \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} \frac{s}{e}.
\end{aligned}$$

(The last inequality holds, since $(\frac{s}{s+1})^s$ is decreasing function and $\lim_{s\to\infty}(\frac{s}{s+1})^s = \frac{1}{e}$.)

If $\lfloor \frac{m}{n-1} \rfloor = s$, $\frac{m}{n-1} = s + \alpha$ and $\lfloor \frac{m}{n} \rfloor = s - 1$, then $\frac{m}{n} = (s-1) + \left(1 + \alpha - \frac{m}{n(n-1)}\right)$

and

$$\begin{aligned}
C(C_{n,m}) &= \left((s-1)^{\frac{m}{n(n-1)}-\alpha} s^{1+\alpha-\frac{m}{n(n-1)}}\right)^n \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}(s+1)^{-\alpha} s^{2\alpha-\frac{m}{n(n-1)}}(s-1)^{\frac{m}{n(n-1)}-\alpha}\right)^n \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^n \left(\left(\frac{s}{s+1}\right)^{\alpha}\left(\frac{s-1}{s}\right)^{\frac{m}{n(n-1)}-\alpha}\right)^n \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} s \left(\frac{s}{s+1}\right)^{\alpha(n-1)} \left(\frac{s-1}{s}\right)^{\frac{m}{n-1}-\alpha n} \\
&= \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} s \left(\frac{s}{s+1}\right)^{\alpha(n-1)} \left(\frac{s-1}{s}\right)^{s-\alpha(n-1)} \\
&\geq \left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1} s \left(\frac{s-1}{s}\right)^{s} \\
&\geq \frac{8s}{27}\left(s^{1-\alpha}(s+1)^{\alpha}\right)^{n-1}.
\end{aligned}$$

(The last inequality holds, since $(\frac{s-1}{s})^s$ is increasing function and $s \geq 3$.)

The lower bound in the case $\frac{m}{n-1} \leq 3$ is achieved by the graph $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ with additional $n - \lfloor \frac{m+1}{3} \rfloor$ isolated vertices. □

To derive an upper bound for the number of cycles in a multigraph with m edges the following theorem (a direct analogue of Theorem 4.2.3) is used. This theorem has not yet been published, but its statement appeared (without proof) in [9].

Theorem 4.7.2 (Arman-Tsaturian)**.** *Let G be a multigraph with m edges such that $C(G) = C(m)$. Then $\Delta(G) \leq 11$.*

Proof. The proof of Theorem 4.7.2 relies on Lemma 4.2.1 and 4.2.2.

Let m be a fixed positive integer and G be a multigraph with m edges. To prove

Theorem 4.2.3 it is sufficient to prove that if $\Delta(G) \geq 12$, then there is a multigraph H with m edges and with $C(H) > C(G)$.

Let $\Delta(G) \geq 12$, and let u be a vertex of maximal degree in G. Let $N'(u) = \{e_1, e_2, \ldots, e_k\}$ be the edge neighbourhood of u (note that $k \geq 12$). Let $N(u) = \{u_1, u_2, \ldots, u_k\}$ be a multiset, such that for any $1 \leq i \leq k$, $e_i = uu_i$. For $i, j \in [k]$, $i \neq j$, if $u_i \neq u_j$ define $w_{i,j}$ to be the number of paths from vertex u_i to vertex u_j in the graph $G \backslash u$, and if $u_i = u_j$ let $w_{i,j} = 0$. Then the number of cycles in G that pass through the vertex u is equal to $S = \sum_{1 \leq i < j \leq k} w_{i,j}$. By Lemma 4.2.1, there is a six element set $D = \{i_1, i_2, \ldots, i_6\}$, such that

$$\sum_{\substack{1 \leq i < j \leq k \\ i \notin D, j \notin D}} w_{i,j} \geq \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

Suppose, upon re-indexing, that $D = \{k-5, k-4, \ldots, k-1, k\}$. Lemma 4.2.2 applied to the collection of real numbers $w_{i,j}$ with $1 \leq i < j \leq k-6$ gives a partition $\pi = A_1 \cup A_2 \cup A_3 \cup A_4$ of $[k-6]$ with

$$\sum_{1 \leq \ell < m \leq 4} \sum_{\substack{i \in A_\ell \\ j \in A_m}} w_{i,j} \geq \left(\frac{3(k-6)^2 - 4}{4(k-6)(k-7)}\right) \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{4.11}$$

For $1 \leq i \leq 4$, let $U_i = \{u_j \ : \ j \in A_i\}$ (U_i is a multiset) and let $E_i = \{uu_j \ : \ j \in A_i\}$. Construct a graph H by deleting all of the edges incident to u; adding four new vertices v_1, v_2 , v_3, v_4, for all $1 \leq i \leq 4$ adding the edges from v_i to each vertex of U_i, and for all $1 \leq i < j \leq 4$ adding the edges $v_i v_j$. Then $|E(H)| = |E(G)|$.

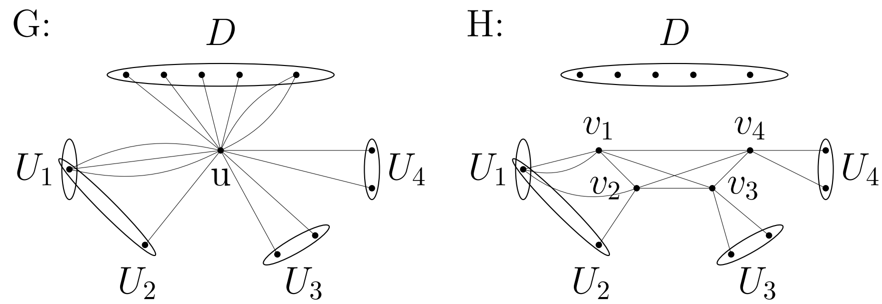

Figure 4.6: Constructing the multigraph H.

To count the number of cycles in H, note the following:

- Every cycle in G that does not pass through the vertex u is still a cycle in H. There are $C(G) - S$ such cycles.

- Let C be a cycle in G that for some $1 \leq i < j \leq k-6$, contains the path $u_i u u_j$. If for some $\ell \in [4]$, u_i and u_j are in the same class U_ℓ, then C corresponds to the cycle in H that uses the path $u_i v_\ell u_j$ instead of $u_i u u_j$. In the case if for some $1 \leq l < m \leq 4$, $u_i \in U_\ell$ and $u_j \in U_m$, the cycle C corresponds to the cycle that uses the path $u_i v_\ell v_m u_j$ instead of $u_i u u_j$. By Lemma 4.2.1, there are at least
$$\left(1 - \frac{6(2k-7)}{k(k-1)}\right) S$$
cycles in G that use a path $u_i u u_j$ with $uu_i, uu_j \in N'(u)$ and $i, j \in [k-6]$.

- Every cycle in G that for some $i \in A_\ell$ and $j \in A_m$ with $\ell \neq m$ pass through the path $u_i u u_j$ give rise to four additional cycles (except the one containing

$u_iv_1v_2u_j$): the ones containing the paths $u_iv_1v_3v_2u_j$, $u_iv_1v_4v_2u_j$, $u_iv_1v_3v_4v_2u_j$ and $u_iv_1v_4v_3v_2u_j$. According to (4.11), there are at least

$$\left(\frac{3(k-6)^2-4}{4(k-6)(k-7)}\right)\left(1-\frac{6(2k-7)}{k(k-1)}\right)S=\left(\frac{3k^2-36k+104}{4k(k-1)}\right)S$$

cycles in G that for some $i \in A_\ell$ and $j \in A_m$ with $\ell \neq m$ pass through a path u_iuu_j.

- There are 7 new cycles in H spanned by the vertices v_1, v_2, v_3, v_4.

By all of the observations above, the number of cycles in H is

$$\begin{aligned} C(H) &\geq C(G) - S + \left(1-\frac{6(2k-7)}{k(k-1)}\right)S + \left(\frac{3k^2-48k+104}{k(k-1)}\right)S + 7 \\ &= C(G) + 7 + S\left(\frac{3k^2-36k+104}{k(k-1)} - \frac{12k-42}{k(k-1)}\right) \\ &= C(G) + 7 + S\left(\frac{3k^2-48k+144}{k(k-1)}\right) \\ &= C(G) + 7 + S\left(\frac{3(k-4)(k-12)}{k(k-1)}\right) \\ &> C(G). \end{aligned}$$

Therefore, H has more cycles than G. $\square$

Theorem 4.7.3 (Arman-Tsaturian, 2017$^+$ [9]). *Let G be a multigraph with $m \geq 3$ edges that has the maximum number of cycles among all the multigraphs with m edges. Then*

$$\frac{9}{10}(\sqrt[3]{3})^m < 4(\sqrt[3]{3})^{m-4} \leq C(G) \leq 8.25(\sqrt[3]{3})^m$$

Proof. The upper bound in Theorem 4.7.3 is obtained by repeating the argument of Corollary 4.4.3 and using Theorem 4.7.2. The example that implies the lower bound is the same as for the second case of Theorem 4.7.1, namely $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ (see proof of Theorem 4.7.1 for definition). $\square$

Theorems 4.7.1 and 4.7.3 answer both questions of determining, up to a constant factor, the number of cycles for multigraphs with a given number of edges and with a given number of vertices and edges. I would like to conclude this section with saying that I believe that for $m \geq 9$ the graph $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ has the most cycles among all multigraphs with m edges, since it is the graph, for which the lower bound in Theorem 4.7.3 is sharp.

4.8 Concluding remarks

Theorem 4.4.2 gives an upper bound for the number of cycles in a graph G with n vertices and m edges. For a graph G with n vertices and the average degree $d \geq 6$, Corollary 4.4.4 implies

$$C(G) \leq 3\Delta(G) \left(\frac{d}{2}\right)^{n-1}.$$

For $d = \Omega(\ln n)$, let G be a random graph $G(n,p)$ with $p = \frac{d}{n-1}$. Glebov and Krivelevich [23] proved that asymptotically almost surely the number of cycles in $G(n,p)$ is at least $\left(\frac{d}{e}\right)^n (1+o(1))^n$. Also, Theorem 4.6.1 implies that if G is a graph with the maximum number of cycles among all graphs with n vertices and average degree $d = d(n)$ such that $\lim_{n\to\infty} d(n) = \infty$, then for n large enough

$$\left(\frac{d}{e}\right)^n (1+o(1))^n \le C(G) \le (1+o(1))^n \left(\frac{d}{2}\right)^n. \qquad (\star\star)$$

Inequality $(\star\star)$ and the fact that $C(K_n) \approx \frac{c}{\sqrt{n}}\left(\frac{n}{e}\right)^n$ for some constant c (see [10] for details) motivates the following conjecture.

Conjecture 4.8.1 (Arman-Tsaturian, 2017$^+$ [9])**.** *For any $\alpha \in (0,1]$ and integer n large enough, any graph G on n vertices with average degree $d = \alpha n$ satisfies*

$$C(G) \le (1+o(1))^n \left(\frac{d}{e}\right)^n.$$

As mentioned in the introduction, Theorem 4.4.3 and the result of Section 4.5 imply that $1.37^m \le C(m) \le 1.443^m$.

Király [30] proved that if G is a 4-regular graph, then there are constants c, ϵ, such that $C(G) \le cn^2(2-\epsilon)^n$; he also conjectured that $C(m) < 1.4^m$. The upper bound in Corollary 4.4.3 is $8.25(\sqrt[3]{3})^m$, which inspires the following conjecture (directly contradicting Király's conjecture).

Conjecture 4.8.2 (Arman-Tsaturian, 2017$^+$ [9])**.** *For sufficiently large m, there exists a graph G with m edges and at least $(1+o(1))^m(\sqrt[3]{3})^m$ cycles.*

Chapter 5

Counting cycles in K_r-free graphs

5.1 Motivation

The authors of [10] posted a list of new conjectures.

Conjecture 1.1.2 (Arman-Gunderson-Tsaturian, 2016 [10]). *For any $k > 1$, if an n-vertex graph C_{2k+1}-free graph has the maximum number of cycles, then $G = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

Question 1.1.3 (Arman-Gunderson-Tsaturian, 2016 [10]). *For $k \geq 4$ what is the maximum number of cycles in a K_k-free graph on n vertices? Could it be that the cycle-maximal K_k-free graphs are indeed Turán graphs?*

Shortly after submitting the paper [10], we received an email from Alex Scott [41] in which he informed that he and a student of his have proved Conjecture 1.1.2 and answered Question 1.1.3 affirmatively for n large enough, by using the Regularity

Lemma [46]. Regularity Lemma implies that for any $\epsilon > 0$, graph H and integer k there is an integer N, such that for any H-free graph G with $n \geq k$ vertices there is integer ℓ, $k \leq \ell \leq N$, such that G can be made ℓ-partite by deleting at most ϵn^2 edges. Regularity Lemma potentially allows to reduce the problem of counting cycles in H-free graphs to the problem of counting cycles in ℓ-partite graphs. Recall that a graph H is called edge-critical if there exists an edge, deletion of which reduces the chromatic number of H. Alex Scott states that if H is an edge-critical graph with chromatic number k, then there exists integer n_0 so that for all integers $n \geq n_0$ and any H-free graph G with n vertices

$$C(G) \leq C(T(n, k-1)).$$

Result of Scott was not published at the time of the preparation of this thesis, however it inspired me to investigate Question 1.1.3 further. The first step toward such an investigation is an estimate on the number of cycles in a Turán graph, which is done in Section 5.2, namely in Theorems 5.2.2 and 5.2.6. A stability result (Theorem 5.3.2) for the number of cycles in K_r-free graphs is given in Section 5.3.

All of the theorems in Chapter 5 are the result of my original research. In Section 5.2 the question of estimating the number of cycles in a Turán graph is considered. The main result of Section 5.2 (Theorem 5.2.6) shows that for any positive integer r there exists a constant $c = c(r)$, such that for n large enough,

$$C(T(n,r)) \geq \frac{c}{n^{\frac{2}{3}r^2 - \frac{r}{2} + 1}} \left(\frac{n(r-1)}{re} \right)^n.$$

In Section 5.3 an estimate on the number of cycles in a K_r-free graph is given and the main result of Section 5.3 (Theorem 5.3.2) shows that for any positive integer r there is positive integer n_0, such that any K_{r+1}-free graph G with $n \geq n_0$ vertices and $m \leq t(n,r) - 2r^4 n \ln n$ edges has fewer cycles than $T(n,r)$.

5.2 Estimate on the number of cycles in a Turán graph

The main result of this section is Theorem 5.2.6, which is later used in Section 5.3.

Theorem 5.2.2 provide useful upper bounds for the number of cycles in a Turán graph. The following lemma is used in the proof of Theorem 5.2.2.

Lemma 5.2.1. *Let $s(n)$ be the number of sequences of length n of symbols from the alphabet $\{1, 2, \ldots, r\}$, such that no two consecutive symbols are the same (n-th and 1-st symbols are also consecutive). Then*

$$s(n) = (r-1)^n + (r-1)(-1)^n.$$

This Lemma is one of the classical result about chromatic polynomials of a cycle, for reference see Birkhoff and Lewis paper [12] (thanks to Dr. Bill Kocay for pointing that this Lemma is a result about chromatic polynomials). Lemma 5.2.3 and Lemma 5.2.5 can also be rewritten in terms of colourings of a cycle.

Proof of Lemma. The sequence $s(n)$ satisfies the following recurrence:

$$s(2) = r(r-1), \quad s(3) = r(r-1)(r-2),$$

$$s(n+2) = (r-2)s(n+1) + (r-1)s(n) \qquad \text{for } n \geq 2.$$

Solving for the roots of the characteristic polynomial and using the initial conditions yields

$$s(n) = (r-1)^n + (r-1)(-1)^n.$$

□

Estimates in Theorem 5.2.2 show that the result of Theorem 5.2.6 is not far away from being sharp.

Theorem 5.2.2 (Arman, 2017$^+$ [8])**.** *For an integer $r \geq 3$ and an integer n large enough*

$$C(T(n,r)) \leq \left(\frac{e^2 n}{r}\right)^{\frac{r}{2}} \left(\frac{n(r-1)}{re}\right)^n.$$

Proof. Let $V(T(n,r)) = V_1 \cup V_2 \cup \dots \cup V_r$, such that $|V_1| \geq |V_2| \geq \dots \geq |V_r|$ and for any $i \neq j$, $|(|V_i| - |V_j|)| \leq 1$. For any $i \in [r]$ let $V_i = \{v_1^i, v_2^i, \dots, v_{|V_i|}^i\}$. Finally, let $t = n - \left\lfloor \frac{n}{r} \right\rfloor \cdot r$.

For $3 \leq k \leq n$, let $\mathbf{s} = (s_1, s_2, \dots, s_k)$ be a sequence of symbols from the alphabet $\{1, 2, \dots, r\}$ such that no two consecutive symbols are the same (k-th and 1-st symbols are also consecutive) and for any $i \in [r]$ the symbol i appears at most $|V_i|$ times in $\mathbf{s}$. For any $i \in [k]$ let n_i be the number of times that a symbol s_i appears among the first i symbols of $\mathbf{s}$. For any $i \in [r]$ let π^i be a permutation of $[|V_i|]$. Finally, let

$$C = \left(v^{s_1}_{\pi^{s_1}(n_1)}, v^{s_2}_{\pi^{s_2}(n_2)}, v^{s_3}_{\pi^{s_3}(n_3)}, \dots, v^{s_{k-1}}_{\pi^{s_{k-1}}(n_{k-1})}, v^{s_k}_{\pi^{s_k}(n_k)}, v^{s_1}_{\pi^{s_1}(n_1)}\right)$$

be the cycle of length k that arises from $\mathbf{s}$ and $\pi^1, \pi^2, \dots, \pi^r$. Moreover, every cycle C of the length k arises from at least $2k$ different choices of $\mathbf{s}$, $\pi^1, \pi^2, \dots, \pi^r$ (there are $2k$ ways to choose starting point and a direction on a cycle of length k, each choice corresponds to different collection of $\mathbf{s}$, $\pi^1, \pi^2, \dots, \pi^r$). Therefore, the number of cycles of the length k in $T(n,r)$ is at most the number of ways to choose $\mathbf{s}$, $\pi^1, \pi^2, \dots, \pi^r$ divided by $2k$, so is at most

$$\begin{aligned}
&\frac{1}{2k}\left((r-1)^k + (r-1)\right)|V_1|! \cdot |V_2|! \cdots |V_r|! \\
&= \frac{1}{2k}\left((r-1)^k + (r-1)\right) \cdot \left(\left\lfloor \frac{n}{r} \right\rfloor !\right)^{r-t} \cdot \left(\left\lfloor \frac{n}{r} + 1 \right\rfloor !\right)^{t} \\
&< \frac{1}{2k}\left((r-1)^k + (r-1)\right) \cdot \left(e \left\lfloor \frac{n}{r} \right\rfloor^{\frac{1}{2}} \left(\frac{\left\lfloor \frac{n}{r} \right\rfloor}{e} \right)^{\left\lfloor \frac{n}{r} \right\rfloor} \right)^{r-t} \cdot \left(e \left\lfloor \frac{n}{r} + 1 \right\rfloor^{\frac{1}{2}} \left(\frac{\left\lfloor \frac{n}{r} + 1 \right\rfloor}{e} \right)^{\left\lfloor \frac{n}{r} + 1 \right\rfloor} \right)^{t} \\
&\leq \frac{1}{2k}\left((r-1)^k + (r-1)\right) \cdot \left(e \left(\frac{n}{r}\right)^{\frac{1}{2}} \right)^{r} \left(\frac{n}{er}\right)^{n}.
\end{aligned}$$

Therefore, the number of cycles in $T(n,r)$ is at most

$$\begin{aligned}
\sum_{k=3}^{n} &\frac{1}{2k}\left((r-1)^k + (r-1)\right) \cdot \left(e \left(\frac{n}{r}\right)^{\frac{1}{2}} \right)^{r} \left(\frac{n}{er}\right)^{n} \\
&\leq \sum_{k=3}^{n} \frac{1}{2n}\left((r-1)^n + (r-1)\right) \cdot \left(e \left(\frac{n}{r}\right)^{\frac{1}{2}} \right)^{r} \left(\frac{n}{er}\right)^{n} \\
&< \frac{1}{2}\left((r-1)^n + (r-1)\right) \cdot \left(e \left(\frac{n}{r}\right)^{\frac{1}{2}} \right)^{r} \left(\frac{n}{er}\right)^{n} \\
&< (r-1)^n \cdot \left(e \left(\frac{n}{r}\right)^{\frac{1}{2}} \right)^{r} \left(\frac{n}{er}\right)^{n} \\
&= \left(e^2 \left(\frac{n}{r}\right) \right)^{\frac{r}{2}} \left(\frac{(r-1)n}{er}\right)^{n}.
\end{aligned}$$

$\square$

Let r be a positive integer. The following lemmas are used for the proof of Theorem 5.2.6.

Lemma 5.2.3 (Arman, 2017$^+$ [8]). *Let m and k be integers with $1 \le k \le \frac{m}{2}$. Let $a_{m,k}$ be the number of sequences of length m of symbols from the alphabet $\{1, 2, \ldots, r\}$, such that no two consecutive symbols are the same (m-th and 1-st symbols are also consecutive), and the symbol "1" appears precisely k times. Then*

$$a_{m,k} = \binom{m-k}{k}(r-1)^k(r-2)^{m-2k},$$

and

$$a_{m,0} = (r-2)^m + (-1)^m(r-2).$$

Proof. Let S be the collection of sequences $\mathbf{s} = (s_1, s_2, \ldots s_m)$, such that for all $i \in [m]$ $s_i \in [r]$, $s_i \neq s_{i+1}$ (with $s_{m+1} = s_1$) and there are exactly k indices $i_1, i_2, \ldots, i_k$, such that for all $j \in [k]$, $s_{i_j} = 1$.

Every element of S can be constructed in the following steps. First, choose the positions of "1"'s; namely choose a set $S_1 = \{i_1, i_2, \ldots, i_k : i_1 < i_2 < \cdots < i_k\} \subseteq S$ with the property that for all $j_1, j_2 \in [k]$, $|j_1 - j_2| > 1$ and m and 1 cannot both belong to S_1. There are $\binom{m-k}{k}$ ways to choose such a set S_1. For all $j \in S_1$ set $s_j = 1$.

Now, define all other values of s_j for $j \in [m]/S_1$. For an index $\ell \in [k]$ consider the interval $[s_{i_\ell}+1, s_{i_{\ell+1}}-1]$ with $[s_{i_k}+1, s_{i_1}-1] = [s_{i_k}+1, m] \cup [1, s_{i_1}-1]$. For the value of $s_{i_\ell+1}$ there are $(r-1)$ possibilities to choose from (since $s_{i_\ell+1} \neq 1$); for the value $s_{i_\ell+2}$ there are $(r-2)$ possibilities to choose from ($s_{i_\ell+2} \neq s_{i_\ell+2}$ and $s_{i_\ell+2} \neq 1$); for $s_{i_\ell+3}$ there are $(r-2)$ possibilities ($s_{i_\ell+3} \neq s_{i_\ell+2}$ and $s_{i_\ell+3} \neq 1$); similarly for all $3 < j < i_{\ell+1-i_\ell}$, for s_{i_l+j} there are $(r-2)$ possibilities ($s_{i_l+j} \neq s_{i_l+j-1}$ and $s_{i_l+j} \neq 1$). So, there are $(r-1)(r-2)^{i_{\ell+1}-i_\ell-1}$ ways to choose the values of all of the s_j for $j \in [s_{i_\ell}+1, s_{i_{\ell+1}}-1]$ (here I set $i_{k+1}-i_k = m-i_k+i_1-1$). Hence, there are

$$\prod_{l=1}^{k}(r-1)(r-2)^{i_{\ell+1}-i_l-1} = (r-1)^k(r-2)^{m-2k}$$

ways to choose the values of s_j for $j \in [m]/S_1$. Finally,

$$|S| = \binom{m-k}{k}(r-1)^k(r-2)^{m-2k}.$$

$\square$

Note, that according to the Lemma 5.2.3 and Lemma 5.2.1,

$$\sum_{k=0}^{\frac{m}{2}} a_{m,k} = (r-1)^m + (-1)^m(r-1).$$

Lemma 5.2.4 (Arman, 2017$^+$ [8])**.** *Let r, m be integers, such that $r > 2$ and $m > r^3$. Let $\frac{1}{r^2} > \epsilon > 0$ be such that $m\epsilon$ is an integer and set $c = 1-\epsilon$. Then there exist positive constants $c_1(r), c_2(r)$, such that*

$$\sum_{k=\frac{m}{r}}^{\frac{cm}{2}} a_{cm,k} < c_1(r-1)^{cm}\sqrt{m}e^{-c_2\epsilon^3 m}.$$

Proof. Let

$$S = \sum_{k=\frac{m}{r}}^{\frac{cm}{2}} a_{cm,k}.$$

Claim: For $\frac{cm}{2} \geq k \geq \frac{m}{r}$ sequence $a_{cm,k}$ is decreasing in k.

Proof of claim: Let $k \geq \frac{m}{r}$, then

$$\begin{aligned}\frac{a_{cm,k+1}}{a_{cm,k}} =& \frac{\binom{cm-k-1}{k+1}(r-1)^{k+1}(r-2)^{cm-2k-2}}{\binom{cm-k}{k}(r-1)^k(r-2)^{cm-2k}} \\ =& \frac{r-1}{(k+1)} \Big/ \frac{(cm-k)(r-2)^2}{(cm-2k)(cm-2k-1)} \\ =& \frac{(cm-2k)(cm-2k-1)}{(k+1)(cm-k)} \cdot \frac{(r-1)}{(r-2)^2}.\end{aligned}$$

The function $f(x) = \frac{(cm-2x)(cm-2x-1)}{(x+1)(cm-x)}$ is decreasing on the interval $[\frac{m}{r}, \frac{cm}{2}]$, so

$$\begin{aligned}\frac{a_{cm,k+1}}{a_{cm,k}} \leq& \frac{a_{cm,\frac{m}{r}+1}}{a_{cm,\frac{m}{r}}} \\ =& \frac{(cm-2\frac{m}{r})(cm-2\frac{m}{r}-1)}{(\frac{m}{r}+1)(cm-\frac{m}{r})} \cdot \frac{(r-1)}{(r-2)^2} \\ =& \frac{(c-\frac{2}{r})(c-\frac{2}{r}-\frac{1}{m})}{(\frac{1}{r}+\frac{1}{m})(c-\frac{1}{r})} \cdot \frac{(r-1)}{(r-2)^2} \\ =& \frac{(cr-2)(cr-2-\frac{r}{m})}{(1+\frac{r}{m})(cr-1)} \cdot \frac{(r-1)}{(r-2)^2} \\ =& \frac{(r-2-\epsilon r)(r-2-\epsilon r-\frac{r}{m})}{(1+\frac{r}{m})(r-1-\epsilon r)} \cdot \frac{(r-1)}{(r-2)^2}.\end{aligned}$$

In order to verify that $a_{cm,k}$ is decreasing it is sufficient to show

$$((r-2)-\epsilon r)\left((r-2)-\epsilon r-\frac{r}{m}\right)(r-1) \leq \left(1+\frac{r}{m}\right)((r-1)-\epsilon r)(r-2)^2.$$

After simplifying, the last inequality becomes

$$\begin{aligned}-2\epsilon r(r-1)(r-2) - \frac{r(r-1)(r-2)}{m} + \epsilon^2 r^2(r-1) + \frac{\epsilon r^2(r-1)}{m} \\ \leq -\epsilon r(r-2)^2 + \frac{r}{m}(r-1)(r-2)^2 - \frac{\epsilon r^2}{m}(r-2)^2.\end{aligned}$$

Further this last inequality can be rewritten as

$$\begin{aligned}
&-\frac{r(r-1)(r-2)}{m}+\epsilon^2 r^2(r-1)+\frac{\epsilon r^2(r-1)}{m}\\
\leq&\epsilon r^2(r-2)+\frac{r}{m}(r-1)(r-2)^2-\frac{\epsilon r^2}{m}(r-2)^2.
\end{aligned}$$

The left-hand side of the last inequality (using the assumptions $r > 2$, $\epsilon < r^{-2}$ and $m > r^3$) is at most $0 + \epsilon(r-1) + \epsilon = r\epsilon$ and the right-hand side is at least $\epsilon r^2(r-2) + 0 - \epsilon(r-2) = \epsilon(r+1)(r-1)(r-2)$, which is greater than ϵr. This finishes the proof of the claim.

Hence, the sequence $(a_{cm,k})_{m/r}^{cm/2}$ is decreasing in k and

$$\begin{aligned}
S &= \sum_{k=\frac{m}{r}}^{\frac{cm}{2}} a_{cm,k}\\
&\leq \left(\frac{cm}{2}-\frac{m}{r}+1\right) a_{cm,\frac{m}{r}}\\
&= \left(\frac{cm}{2}-\frac{m}{r}+1\right)\binom{cm-\frac{m}{r}}{\frac{m}{r}}(r-1)^{\frac{m}{r}}(r-2)^{cm-\frac{2m}{r}}\\
&\leq \left(\frac{c}{2}-\frac{1}{r}+\frac{1}{r}\right) m\frac{e\sqrt{cm-\frac{m}{r}}(cm-\frac{m}{r})^{cm-\frac{m}{r}}(r-1)^{\frac{m}{r}}(r-2)^{cm-\frac{2m}{r}}}{\sqrt{2\pi}\sqrt{\frac{m}{r}}(\frac{m}{r})^{\frac{m}{r}}\sqrt{2\pi}\sqrt{cm-\frac{2m}{r}}(cm-\frac{2m}{r})^{cm-\frac{2m}{r}}}\\
&\leq \frac{e\sqrt{2r}}{4\pi}\sqrt{m}\frac{(cm-\frac{m}{r})^{cm-\frac{m}{r}}(r-1)^{\frac{m}{r}}(r-2)^{cm-\frac{2m}{r}}}{(\frac{m}{r})^{\frac{m}{r}}(cm-\frac{2m}{r})^{cm-\frac{2m}{r}}}\\
&= c_1(r)\sqrt{m}\frac{(c-\frac{1}{r})^{(c-\frac{1}{r})m}(r-1)^{\frac{m}{r}}(r-2)^{cm-\frac{2m}{r}}}{(\frac{1}{r})^{\frac{m}{r}}(c-\frac{2}{r})^{(c-\frac{2}{r})m}}\\
&= c_1(r)\sqrt{m}(r-1)^{cm}\frac{(cr-1)^{(c-\frac{1}{r})m}(r-2)^{cm-\frac{2m}{r}}}{(r-1)^{cm-\frac{m}{r}}(cr-2)^{(c-\frac{2}{r})m}}\\
&= c_1(r)\sqrt{m}(r-1)^{cm}\left(\frac{cr-1}{r-1}\right)^{\frac{cr-1}{r}m}\left(\frac{r-2}{cr-2}\right)^{\frac{cr-2}{r}m}.
\end{aligned}$$

To finish the proof, it is sufficient to show that for some positive constant $c_2(r)$,

$$\left(\frac{cr-1}{r-1}\right)^{\frac{cr-1}{r}m}\left(\frac{r-2}{cr-2}\right)^{\frac{cr-2}{r}m} \leq e^{-\epsilon^3 c_2(r)m}. \tag{5.1}$$

Taking logarithms of both sides of the inequality (5.1), using the fact that $c = 1-\epsilon$, and applying the inequality $-x - x^2(\frac{1}{2}+\frac{1}{2(r-1)}) \leq \ln(1-x) \leq -x - \frac{x^2}{2}$ for $x < \frac{1}{r^4}$ yields

$$\begin{aligned}
&\frac{cr-1}{r}m\ln\left(\frac{cr-1}{r-1}\right) - \frac{cr-2}{r}m\ln\left(\frac{cr-2}{r-2}\right)\\
&\quad = \left(1-\epsilon-\frac{1}{r}\right)m\ln\left(1-\frac{\epsilon r}{r-1}\right) - \left(1-\epsilon-\frac{2}{r}\right)m\ln\left(1-\frac{\epsilon r}{r-2}\right)\\
&\quad \leq m\left(1-\epsilon-\frac{1}{r}\right)\left(-\frac{\epsilon r}{r-1}-\frac{\epsilon^2 r^2}{(r-1)^2}\right)\\
&\quad - m\left(1-\epsilon-\frac{2}{r}\right)\left(-\frac{\epsilon r}{r-2}-\frac{\epsilon^2 r^2}{(r-2)^2}-\frac{\epsilon^2 r^2}{2(r-1)(r-2)^2}\right)\\
&\quad = -\epsilon^3 m\left(\frac{r^2}{2(r-1)^2(r-2)^2}\right)\left(2r-3-\frac{1}{r-1}\right)\\
&\quad = -\epsilon^3 c_2(r)m.
\end{aligned}$$

This finishes the proof of inequality (5.1) and the lemma. □

Lemma 5.2.5 (Arman, 2017^+ [8])**.** *Let r, t be integers and let $n = rt$. Let $s(n,t)$ be the number of sequences of the length n of symbols from the alphabet $\{1,2,\ldots,r\}$ such that no two consecutive symbols are the same (n-th and 1-st symbols are also consecutive) and the number of times that every symbol appears is exactly t. Then there is a constant $c = c(r)$, such that for n large enough*

$$s(n,t) \geq c\frac{(r-1)^n}{(n)^{\frac{2}{3}r^2+1}}.$$

Proof. Let m be an integer, $\frac{1}{r^2} > \epsilon \sim \ln m/m^{\frac{1}{3}}$ and $c = 1-\epsilon$ be such that $t = cm-1$.

Consider the set S_1 of the sequences of length cm from the alphabet $\{1, 2, \ldots, r\}$ such that every symbol appears at most $\frac{m}{r}$ times and any two consecutive elements are different. By Lemma 5.2.4, the number of such sequences, when m is large enough, is at least

$$(r-1)^{cm} - (r-1) - rc_1(r-1)^{cm}\sqrt{m}e^{-c_2\epsilon^3 m} = (1-o(1))(r-1)^{cm}.$$

By the pigeonhole principle (PHP), there are $i, j \in [r]$, such that $i \neq j$ and the number of sequences from S_1 that start with i and end at j is at least

$$\frac{1}{r(r-1)}(1-o(1))(r-1)^{cm}.$$

Without loss of generality, assume that $i = 1$ and $j = 2$. Set $S_2 \subset S_1$ to be the set of all sequences from S_1 that start with "1" and end with "2".

If $s \in S_2$, then the number of times that any symbol from the alphabet appears is at least $cm - (r-1)\frac{m}{r} = \left(\frac{1}{r} - \epsilon\right)m$ and at most $\frac{m}{r}$. By the PHP, there is a sequence $n_1, n_2, \ldots, n_r$, such that for each $i \in [r]$, $\left(\frac{1}{r} - \epsilon\right)m \leq n_i \leq \frac{m}{r}$, and $\sum_{i=1}^{r} n_i = cm$, and the number of sequences $\mathbf{s} \in S_2$ such that for each $i \in [r]$ the number of times that symbol "i" appears in $\mathbf{s}$ is n_i is at least

$$\frac{|S_2|}{(\epsilon m+1)^r} \geq (1-o(1))\frac{1}{r(r-1)}\frac{(r-1)^{cm}}{(\epsilon m)^r}.$$

Let $S_3 \subseteq S_2$ be all sequences $\mathbf{s}$ from S_2 such that for each $i \in [r]$ the number of times that symbol "i" appears in $\mathbf{s}$ is n_i.

Let S be a set of sequences of length n from the alphabet $[r]$ such that any two consecutive symbols are different and every symbol appears exactly $t = \frac{n}{r}$ times. Let π be permutation of $[r]$ that takes all elements to its successor and takes r to 1. For an $\mathbf{s} = (s_1, s_2, \ldots, s_{cm}) \in S_3$ define $\pi(\mathbf{s}) = (\pi(s_1), \pi(s_2), \ldots, \pi(s_{cm}))$. Let $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r \in S_3$. For all $i \in [r]$ let $\mathbf{c}_i = \pi^{(i-1)}(\mathbf{s}_i)$, and let $\mathbf{s}'_i$ to be $\mathbf{c}_i$ minus its last element. Let permutations $\mathbf{s} \in S$ be $\mathbf{s} = \mathbf{s}'_1\mathbf{s}'_2 \ldots \mathbf{s}'_r$. Note that every symbol from the alphabet appears exactly $n_1 + n_2 + \cdots + n_r - 1 = cm - 1 = t$ times and every two consecutive elements are different.

The map $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r) \to \mathbf{s}$ is a 1-1 map from S_3^r to S. Hence,

$$\begin{aligned}
|S| =& s(n,t) \\
\geq& |S_3|^r \\
\geq& \left((1 - o(1)) \frac{1}{r(r-1)} \frac{(r-1)^{cm}}{(\epsilon m)^r} \right)^r \\
\geq& (1 - o(1))^r \frac{1}{r^r} \frac{(r-1)^n}{(\epsilon m)^{r^2}} \\
\geq& \left((1 - o(1)) \frac{1}{r(r-1)} \frac{(r-1)^{cm}}{(\epsilon m)^r} \right)^r \\
\geq& c \frac{(r-1)^n}{(n)^{\frac{2}{3}r^2+1}}.
\end{aligned}$$

□

Theorem 5.2.6 (Arman, 2017$^+$ [8])**.** *Let r be a positive integer, then there is constant $c = c(r)$, such that for n large enough*

$$C(T(n,r)) \geq \frac{c}{n^{\frac{2}{3}r^2 - \frac{r}{2} + 1}} \left(\frac{n(r-1)}{re}\right)^n.$$

Proof. The proof follows the lines of the proof given for Theorem 5.2.2.

Let $m = r\left\lfloor \frac{n}{r} \right\rfloor$. Let $\mathbf{s} = (s_1, s_2, \dots, s_m)$ be a sequence of length m of symbols from the alphabet $\{1, 2, \dots, r\}$, such that no two consecutive symbols are the same (m-th and 1-st symbols are also consecutive) and for any $i \in [r]$ the symbol i appears exactly $\frac{m}{r}$ times in $\mathbf{s}$. For any $i \in [k]$ let n_i be the number of times symbol s_i appears among the first i symbols of $\mathbf{s}$. For any $i \in [r]$ let π^i be a permutation of $[|V_i|]$. Finally, let

$$C = (v^{s_1}_{\pi^{s_1}(n_1)}, v^{s_2}_{\pi^{s_2}(n_2)}, v^{s_3}_{\pi^{s_3}(n_3)}, \dots, v^{s_{k-1}}_{\pi^{s_{k-1}}(n_{k-1})}, v^{s_k}_{\pi^{s_k}(n_k)}, v^{s_1}_{\pi^{s_1}(n_1)})$$

be the cycle of length m arising from $\mathbf{s}, \pi^1, \pi^2, \dots, \pi^r$. The map $(\mathbf{s}, \pi^1, \dots, \pi^r) \to C$ is 1-1. Therefore, by Lemma 5.2.5, the number of cycles in $T(n,r)$ is at least

$$\begin{aligned}
T(n,r) &\geq c_1 \frac{(r-1)^m}{(m)^{\frac{2}{3}r^2+1}} \cdot \prod_{i=1}^{r} |V_i|! \\
&\geq c_2 \frac{(r-1)^n}{(n)^{\frac{2}{3}r^2+1}} \cdot \left(\sqrt{2\pi}\left(\frac{n}{r}\right)^{\frac{1}{2}}\right)^r \left(\frac{n}{er}\right)^n \\
&\geq \frac{c}{(n)^{\frac{2}{3}r^2 - \frac{r}{2} + 1}} \left(\frac{(r-1)n}{er}\right)^n.
\end{aligned}$$

$\square$

5.3 Maximum number of cycles in a K_r-free graph

The main result in this section is Theorem 5.3.2. Lemma 4.3.1 is proved in Section 4.3 and here I restate it for convenience.

Lemma 4.3.1 (Arman-Tsaturian, 2017^+ [9]) *Let G be a multigraph with n vertices. For any $\ell \in [n]$, and any vertices $v_1, \dots, v_\ell \in V(G)$, define $F(v_1, \dots, v_\ell) = N(v_\ell) \backslash \{v_1, \dots, v_{\ell-1}\}$ and define $f(v_1, \dots, v_\ell) = \max\{\deg_{G\backslash\{v_2,\dots,v_{\ell-1}\}}(v_\ell), 1\}$. Denote the number of cycles in G that contain the path $v_1e_1v_2\dots e_{\ell-1}v_\ell$ by $C(v_1e_1v_2\dots e_{\ell-1}v_\ell)$ (note that $C(v_1)$ is the number of cycles containing the vertex v_1). For brevity, write $F_\ell = F(v_1, \dots, v_\ell)$, $f_\ell = f(v_1, \dots, v_\ell)$, $C_\ell = C(v_1e_1\dots e_{\ell-1}v_\ell)$. For a $k \in [n]$, let $v_1e_1v_2e_2\dots v_k$ be a path in G. If $F_k \neq \emptyset$, then*

$$C_k \leq f_k \cdot \max_{\substack{k+1\leq t\leq n \\ v_{k+1}\in F_k \\ \vdots \\ v_t\in F_{t-1}}} \{f_{k+1} \cdot f_{k+2} \cdots f_t\}.$$

(the maximum is taken over all paths $v_{k+1}\dots v_t$, such that $v_1\dots v_k e_k v_{k+1}\dots v_t$ extends $v_1\dots v_k$)

Theorem 4.4.2 is also proved in Section 4.4 and here I restate it for convenience.

Theorem 4.4.2 (Arman-Tsaturian, 2017^+ [9]) *Let G be a multigraph with $n \geq 2$ vertices and m edges.*

If $\frac{m}{n-1} < 3$, then

$$C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} \geq 3$, *and* $\left\lfloor \frac{m}{n-1} \right\rfloor = s$, $\alpha = \frac{m}{n-1} - s$, *then*

$$C(G) < \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1} = \frac{3}{4}\Delta(G)((s^{1-\alpha}(s+1)^{\alpha})^{\frac{1}{s+\alpha}})^{m}.$$

Also, the following lemma is used for the proof of the main result of this section, Theorem 5.3.2.

Lemma 5.3.1 (Arman, 2017^{+} [8])**.** *Let* r *and* n *be positive integers such that* $n \geq 2r+4$. *Then,*

$$\prod_{k=0}^{n}\left(k+2-\left\lceil\frac{k+2}{r}\right\rceil\right) \leq \frac{e^2}{\sqrt{r}}n^2\left(\frac{n(r-1)}{re}\right)^{n}.$$

Proof. Let $m+1 = n - (n \mod r)$, then

$$\begin{aligned}\prod_{k=0}^{m}\left(k+2-\left\lceil\frac{k+2}{r}\right\rceil\right) &= \left(m+1-\frac{m+1}{r}\right)!(r-1)(2r-2)\cdots\left(\frac{m+1}{r}(r-1)\right)\\ &= \left(\frac{(r-1)(m+1)}{r}\right)!(r-1)^{\frac{m+1}{r}}\left(\frac{m+1}{r}\right)!\\ &= S.\end{aligned}$$

Using the upper bound of factorial (inequalities (2.2)),

$$\begin{aligned}&\prod_{k=0}^{n}\left(k+2-\left\lceil\frac{k+2}{r}\right\rceil\right) = S\cdot\prod_{k=m+1}^{n}\left(k+2-\left\lceil\frac{k+2}{r}\right\rceil\right)\\ &\leq S\cdot n^{n-m}\\ &\leq \frac{e^2(m+1)}{\sqrt{r}}\left(\frac{(r-1)(m+1)}{re}\right)^{\frac{(r-1)(m+1)}{r}}(r-1)^{\frac{m+1}{r}}\left(\frac{m+1}{re}\right)^{\frac{m+1}{r}}n^{n-m}\\ &\leq \frac{e^2(n)}{\sqrt{r}}\left(\frac{(r-1)(n)}{re}\right)^{\frac{(r-1)(m+1)}{r}}(r-1)^{\frac{m+1}{r}}\left(\frac{n}{re}\right)^{\frac{m+1}{r}}n^{n-m}\\ &\leq \frac{e^2n^2}{\sqrt{r}}\left(\frac{n(r-1)}{re}\right)^{n}.\end{aligned}$$

□

Theorem 5.3.2 (Arman, 2017^+ [8]). *For any integer $r \geq 2$ there exists a number n_0, such that any K_{r+1}-free graph G with $n \geq n_0$ vertices and $m \leq t(n,r) - 2r^4 n \log n$ edges has fewer cycles than $T(n,r)$.*

Proof. Let G be a K_{r+1}-free graph on n vertices and with m edges. If the graph G has average degree less then $\frac{n(r-1)}{er}$, then Theorem 4.4.2 implies that G has at most

$$3n\left(\frac{n(r-1)}{2er}\right)^n$$

cycles, which is smaller than the number of cycles in a $T(n,r)$. Hence, assume that the graph G has average degree at least $\frac{n(r-1)}{er}$.

Let v_1 be a vertex of G. Let $C(v_1)$ be the number of cycles in G that contain v_1. Then,

$$C(v_1) \leq f_1 \cdot \max_{\substack{2 \leq t \leq n \\ v_2 \in F_1 \\ \vdots \\ v_t \in F_{t-1}}} \{f_2 \cdot f_3 \cdots f_t\}. \tag{5.2}$$

Let the path $v_1 v_2 \ldots v_t$ be the one that gives the maximum in the right hand side of the inequality (5.2). According to the definition of sets F_i and the fact that any subgraph of G is K_{r+1}-free , for any $k \geq 0$,

$$f_{t-k} + f_{t-k+1} + f_{t-k+2} + \cdots + f_t \leq e(G \backslash \{v_2, v_3, v_4, \ldots, v_{t-k-1}]) \leq t(n+k-t+2, r).$$

Set

$$S = \{(f_2, f_3, \ldots, f_t) | \forall\, t-2 \geq k \geq 0 : f_{t-k} + \cdots + f_t \leq \max\{t(n+k-t+2, r), m\}\}.$$

Hence,

$$C_1 \leq f_1 \cdot \max_{\substack{2\leq t\leq n\\ v_2\in F_1\\ \vdots\\ v_t\in F_{t-1}}} \{f_2 \cdot f_3 \cdots f_t\}$$

$$\leq f_1 \cdot \max_{(f_2,\cdots,f_t)\in S}\{f_2 \cdot f_3 \cdots f_t\}. \tag{5.3}$$

Let $(f_2, f_3, \ldots, f_t) \in S$ be the sequence that gives the maximum in (5.3). Note that $t = n$, otherwise splitting one of the f_i into 2 and $f_i - 2$ increases the product.

Let $k_0 \leq n$ be the largest number such that

$$f_{n-k_0} + f_{n-k_0+1} + \cdots + f_n = t(k_0 + 2, r).$$

Also assume that $(f_2, f_3, \ldots, f_n) \in S$ is the sequence that gives the maximum in (5.3) and has the largest possible k_0. Then

$$f_{n-k_0} = t(k_0 + 2, r) - t(k_0 + 1, r).$$

For any ℓ, the difference between the number of edges in $T(\ell + 1, r)$ and $T(\ell, r)$ is equal to the minimal degree of $T(\ell + 1, r)$, so (using $\delta(G)$ for the minimal degree of G)

$$f_{n-k_0} = \delta(T(k_0 + 2, r)) = k_0 + 2 - \left\lceil \frac{k_0 + 2}{r} \right\rceil.$$

With the same arguments,

$$f_{n-k_0-1} + f_{n-k_0} + \cdots + f_n \leq t(k_0 + 3, r) - 1$$

implies

$$f_{n-k_0-1} \leq t(k_0 + 3, r) - 1 - t(k_0 + 2, r),$$

and

$$f_{n-k_0-1} \leq \delta(T(k_0+3,r)) - 1 = k_0 + 3 - \left\lceil \frac{k_0+3}{r} \right\rceil - 1 = k_0 + 2 - \left\lceil \frac{k_0+3}{r} \right\rceil .$$

Note that $f_{n-k_0-1} \leq f_{n-k_0}$. If $f_{n-k_0} - f_{n-k_0-1} \geq 2$, then increasing f_{n-k_0-1} by 1 and decreasing f_{n-k_0} by 1 results in the sequence $(f_2', \ldots, f_n') \in S$ that has a larger product. Hence,

$$0 \leq f_{n-k_0} - f_{n-k_0-1} \leq 1.$$

If there is a $k > k_0 + 1$, such that $f_{n-k} - f_{n-k_0-1} > 1$, then decreasing f_{n-k} by 1 and increasing f_{n-k_0-1} by 1 results in the sequence $(f_2', \ldots, f_n') \in S$ that has a larger product.

If there is a $k > k_0$, such that $f_{n-k} - f_{n-k_0} < -1$, then increasing f_{n-k} by 1 and decreasing f_{n-k_0} by 1 results in the sequence $(f_2', \ldots, f_n') \in S$ that has a larger product.

Finally, if there is $k > k_0+1$ such that $f_{n-k_0}+1 = f_{n-k}$, then $f_{n-k_0-1} = f_{n-k_0}$ and decreasing f_{n-k} by 1 and increasing f_{n-k_0-1} by 1 results in the sequence $(f_2', \ldots, f_n') \in S$ with the same product but larger k_0.

Hence, the sequence $(f_2, \ldots, f_n)$ satisfies the following conditions:

- For all $k \leq k_0$ $f_{n-k} + f_{n-k+1} + \cdots + f_n = t(k+2,r)$. This can be alternatively rewritten as $f_{n-k} = k + 2 - \left\lceil \frac{k+2}{r} \right\rceil$.

- For all $k > k_0$, $-1 \leq f_{n-k} - f_{n-k_0} \leq 0$.

Moreover,

$$\begin{aligned}
m &\geq \sum_{i=2}^{n} f_i \\
&= \sum_{k=0}^{k_0} f_{n-k} + \sum_{k=k_0+1}^{n-2} f_{n-k} \\
&\geq t(k_0+2, r) + (n-2-k_0)(f_{n-k_0} - 1) \\
&\geq (k_0+2)^2 \left(\frac{r-1}{2r}\right) - \frac{r}{2} + (n-2-k_0)\left(k_0 + 1 - \left\lceil \frac{k_0+2}{r} \right\rceil\right).
\end{aligned}$$

Set $k_0 = \delta n$, with $\delta < 1$. Then the last inequality implies

$$\begin{aligned}
\frac{n^2}{2}\left(\frac{r-1}{r}\right) - 2r^4 n \log n \geq m \geq (\delta n + 2)^2 \left(\frac{r-1}{2r}\right) - \frac{r}{2} \\
+ (n-2-\delta n)\left(\delta n + 1 - \left(\frac{\delta n + 2}{r} + 1\right)\right).
\end{aligned}$$

Dividing by $n(\frac{r-1}{2r})$ and simplifying the left-hand side and the right-hand side of the last inequality gives

$$n - 2r^4 \frac{2r}{r-1} \ln n \geq (2\delta - \delta^2)n - (1-\delta)\frac{4}{r-1} + \frac{1}{n}\left(4 - \frac{r^2}{r-1} + \frac{8}{r-1}\right).$$

Provided that n is large enough, the last inequality implies

$$n - 2r^4 \frac{r}{r-1} \ln n \geq (2\delta - \delta^2)n.$$

After setting $\delta = 1 - x$, the last inequality implies

$$\begin{aligned}
& n - 2r^4 \frac{r}{r-1} \log n \geq (1 - x^2)n \\
\Leftrightarrow\ & -2r^4 \frac{r}{r-1} \log n \geq -x^2 n \\
\Leftrightarrow\ & x \geq \sqrt{\frac{2r}{r-1} r^4 \frac{\log n}{n}} \\
\Rightarrow\ & x \geq r^2 \sqrt{\frac{2 \log n}{n}}.
\end{aligned}$$

So $k_0 = \delta n$, with $\delta \leq 1 - r^2\sqrt{\frac{2\log n}{n}}$.

Hence, the following holds:

$$
\begin{aligned}
f_2 \cdots f_n &\leq (f_{n-k_0})^{n-2-k_0} \prod_{k=0}^{k_0} \left(k + 2 - \left\lceil \frac{k+2}{r} \right\rceil\right) \\
&= \left(k_0 + 2 - \left\lceil \frac{k_0+2}{r} \right\rceil\right)^{n-2-k_0} \prod_{k=0}^{k_0} \left(k + 2 - \left\lceil \frac{k+2}{r} \right\rceil\right) \\
&\leq \left(k_0 + 2 - \left\lceil \frac{k_0+2}{r} \right\rceil\right)^{n-2-k_0} c_1(k_0)^2 \left(\frac{k_0(r-1)}{re}\right)^{k_0} \\
&\leq \left(\delta n + 2 - \frac{\delta n + 2}{r}\right)^{n-2-\delta n} c_1(\delta n)^2 \left(\frac{\delta n(r-1)}{re}\right)^{\delta n} \\
&\leq c_1(\delta n)^2 \left(\frac{r-1}{r}\right)^{n-2} (\delta n + 2)^{n-2-\delta n} \left(\frac{\delta n}{e}\right)^{\delta n} \\
&\leq c_1(\delta n)^2 \left(\frac{r-1}{r}\right)^{n-2} (\delta n)^{n-2} \left(1 + \frac{2}{\delta n}\right)^{n-2-\delta n} e^{-\delta n} \\
&\leq c_2 \left(\frac{n(r-1)}{r}\right)^{n} (\delta)^n \left(e^{\frac{2}{\delta n}}\right)^{n-2-\delta n} e^{-\delta n} \\
&\leq c_3 \left(\frac{n(r-1)}{er}\right)^{n} \left(\delta e^{\frac{2}{\delta n} - \delta + 1}\right)^{n}.
\end{aligned}
$$

There are two cases to consider.

Case 1: $\delta \leq \frac{3}{n}$.

Since $\delta n = k_0$ and k_0 is positive integer, there are three possibilities for the value of delta: $\delta = \frac{1}{n}$, $\delta = \frac{2}{n}$ and $\delta = \frac{3}{n}$. In all of this cases

$$
\begin{aligned}
f_2 \cdots f_n &\leq c_3 \left(\frac{n(r-1)}{er}\right)^{n} \left(\delta e^{\frac{2}{\delta n} - \delta + 1}\right)^{n} \\
&\leq c_3 \left(\frac{n(r-1)}{er}\right)^{n} \left(\frac{3e^3}{n}\right)^{n}.
\end{aligned}
$$

Recall that $C(v_1)$ is the number of cycles through a vertex v_1, then

$$C(v_1) \le c_3 \left(\frac{n(r-1)}{er}\right)^n \left(\frac{3e^3}{n}\right)^n \le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\frac{1}{n^{r^4}}\right).$$

Case 2: $\delta \ge \frac{4}{n}$.

Recall that $\delta = 1 - x \le 1 - r^2\sqrt{\frac{2\log n}{n}}$, so $\delta \in (\frac{4}{n}, 1 - r^2\sqrt{\frac{2\log n}{n}})$. For these values of δ the function $f(\delta) = \delta e^{1+\frac{2}{\delta n}-\delta}$ is increasing, so

$$\begin{aligned}
C(v_1) &\le c_3 \left(\frac{n(r-1)}{er}\right)^n \left(\delta e^{\frac{2}{\delta n}-\delta+1}\right)^n \\
&\le c_3 \left(\frac{n(r-1)}{er}\right)^n \left(\left(1 - r^2\sqrt{\frac{2\log n}{n}}\right) e^{r^2\sqrt{\frac{2\log n}{n}}}\right)^n e^{\frac{2}{1-r^2\sqrt{\frac{2\log n}{n}}}} \\
&\le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\left(1 - r^2\sqrt{\frac{2\log n}{n}}\right) e^{r^2\sqrt{\frac{2\log n}{n}}}\right)^n.
\end{aligned}$$

For all positive x, $(1-x)e^x \le 1 - \frac{x^2}{2}$, so

$$\begin{aligned}
C(v_1) &\le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\left(1 - r^2\sqrt{\frac{2\log n}{n}}\right) e^{r^2\sqrt{\frac{2\log n}{n}}}\right)^n \\
&\le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(1 - r^4\frac{\log n}{n}\right)^n \\
&\le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(e^{-r^4\frac{\log n}{n}}\right)^n \\
&\le c_4 \left(\frac{n(r-1)}{er}\right)^n (n)^{-r^4} \\
&\le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\frac{1}{n^{r^4}}\right).
\end{aligned}$$

In both cases

$$C(v_1) \le c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\frac{1}{n^{r^4}}\right).$$

So, by Theorem 5.2.6,

$$
\begin{aligned}
C(G) &\leq \sum_{k=1}^{n} C(v_k) \\
&\leq n c_4 \left(\frac{n(r-1)}{er}\right)^n \left(\frac{1}{n^{r^4}}\right) \\
&\leq c_4 \left(\frac{1}{n^{r^4-1}}\right) \left(\frac{n(r-1)}{er}\right)^n \\
&\leq \frac{c_4}{n} \left(\frac{1}{n^{\frac{2}{3}r^2-\frac{r}{2}+1}}\right) \left(\frac{n(r-1)}{er}\right)^n \leq C(T(n,r)).
\end{aligned}
$$

□

5.4 Concluding remarks

It seems that to improve the result of Theorem 5.3.2, a new version of Lemma 4.3.1 is needed. Indeed, it is easy to verify that if Lemma 4.3.1 gives a tight bound for the number of cycles, then Theorem 5.3.2 provides a tight bound for the number of cycles in a Turán graph, which is possible, but I think is quite unlikely.

It is quite possible that with some effort one can prove a version of Theorem 5.3.2 for H-free graphs, where H is an edge-critical graph:

Theorem 5.4.1 (Arman, 2017$^+$ [8]). *For any graph H with a critical edge and the chromatic number $r+1$ there exists a number n_0, such that any H-free graph G with $n \geq n_0$ vertices and $m \leq t(n,r) - 2r^4 n \ln n$ edges has fewer cycles than $T(n,r)$.*

Note, that a form of this theorem is claimed to be correct by Alex Scott [41].

Chapter 6

Future work and extensions

Dr. Robert Craigen [18] mentioned to me that the question of determining the minimum number of cycles in a graph is also interesting. I have not studied this question thoroughly, but based on my intuition, I conjecture that for a fixed d the minimum number of cycles in a graph G on n vertices and with density d is obtained for a graph, obtained from a tree by identifying each vertex with a vertex of some clique.

It seems that the number of cycles in a Turán graph $T(n, r)$ is not determined precisely for $r \geq 3$. For $r \geq 4$, it is not even known what is the exact number of hamiltonian cycles in $T(n, r)$. The number of hamiltonian cycles in $T(n, 2)$, the number of cycles in $T(n, 2)$ and the asymptotic order of the number of cycles in $T(n, 2)$ is determined by myself, Gunderson and Tsaturian [10]. The formulas (that involve summation) for the number of cycles in $T(n, 3)$ and $T(n, 4)$ are given by Vrba [50].

Also, a recursive formula for the number of hamiltonian cycles in $T(n, r)$ was given by Horák and Továrek [28] in 1979. An interesting and maybe easy problem, in my opinion, is to find the asymptotical order of the number of cycles in $T(n, r)$ for fixed r and n tending to infinity.

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