Quaternion Polynomial Matrices: Computing Normal Forms

by

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Abstract

The applications of quaternion polynomial matrices appear in many fields like applied mathematics, engineering and statistics. In this thesis, we discuss some well-known normal forms of quaternion polynomial matrices.

In the first chapter, we outline some of the basic mathematical definitions and results relevant to quaternions. In the second chapter, we introduce some properties of polynomial matrices. In the third chapter, we discuss some properties of quaternion polynomial matrices. Firstly, the definitions and algorithms of greatest common right divisors (GCRDs) and least common left multiples (LCLMs) of the quaternion polynomials are given. Secondly, we discuss the algorithms for computing several normal forms including the Hermite form, the Smith form and the Popov form. The Maple codes for constructing examples are presented in the fourth chapter.

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Contents

Notation and Terminology

Here are a list of special symbols to be used in Chapters 1 to 3.

$\mathbb{R}$ Real field

$\mathbb{H}$ Quaternion field

$\Re(x)$ The real part of x

$\Im(x)$ The vector part of x

x^* The conjugate quaternion of x

$|x|$ The norm of x

x^{-1} The inverse of x

$\times$ The cross product

$T_{a,b}$ The linear transformation

$H_0 \oplus Span_R\{x_0\}$ Denoting the 2-dimensional plane in H_0 orthogonal to x_0

$Sim\,(\cdot)$ The relations of similarity

$Con\,(\cdot)$ The relations of congruence

$\langle \cdot, \cdot \rangle$ The inner product of two quaternions

$v_{a,b}^{+}$ Subspace and denoting $Span\,\{x_+, y_+\}$

$v_{a,b}^{-}$ Subspace and denoting $Span\,\{x_-, y_-\}$

σ_i The singular values

Ker Denoting the nuclear space

$Cen(a)$ Denoting the centralizer of a

$[p_{\alpha,\beta}]_{\alpha,\beta=1}^{3}$ 3×3 real orthogonal matrix

ϕ Denoting the an endomorphism or an antiendomorphism map

$\dim_{\mathbb{H}}(M)$ The dimension of a subspace M of $\mathbb{H}^n$

A^* The adjoint matrix of matrix A

$Ran(A)$ The range of quaternion matrix A

$rank(A)$ The rank of quaternion matrix A

$\chi_{n,n}$ The isomorphism of the real algebra $\mathbb{H}^{n\times n}$ into the real subalgebra

$W_*^H(A)$ The numerical range of quaternion field

$W_*^C(A)$ The numerical range of complex field

$W_*^F(A)$ The F-joint numerical range

$\langle u, v\rangle^\phi$ The quaternion-valued ϕ-inner product

$Z^{+\phi}$ The ϕ-orthogonal companion

$Inv\,(\cdot)$ Denoting inverse

$diag$ Denoting diagonal matrix

deg Denoting polynomial times

det Denoting determinant of polynomial matrix

$adj(A)$ Denoting adjoint matrix of A

Introduction

Linear system

We know that the mathematical basis of the complex frequency domain theory of linear system theory is mainly the polynomial matrices and the rational function matrices. Delphine Boucher and Felix Ulmer[3] presented a sufficient and necessary condition for the stability of a polynomial matrix. Dazhong Zheng[49] made a summary and analysis of the complex frequency domain theory for linear systems, the matrix fraction description of polynomial matrix theory and transfer function was defined in detail. O.M.Grasselli and A.Tornambe[14] put forward a way of computing a state-space realization of a linear dynamical system through the polynomial matrix description of the system. As the same time, they gave the algorithm of obtaining its dynamic matrix through unimodular transformations, where it is important intermediate step to find the realization algorithm of a square diagonal polynomial matrix.

In [29], J.L.Ramos has used the block observability to develop a method for finding the block partial fraction expansion of an irreducible right or left matrix fraction description with distinct and/or repeated solvents of the denominator matrix polynomials. In Wei-Dong Zhang and Xiao-Ming Xu[50], the problem of minimal-order stabilization in the case where the plant is minimum phase was studied. A low bound on the order of stabilizers was derived and a set of minimal-order stabilizers were characterized. The low bound is related to the number and location of the plants unstable and lightly damped poles and the number of zeros. How to construct a minimal-order or low-order stabilizer for a general case was also discussed and the algorithm was provided. S.Bingulac and N.F.Al-Muthairi[1] found a one-to-one relation between the matrix fraction description and a pseudo-observable form of a multiple input multiple output system. Hence one form can be given as long as one of the two is given.

Quaternion Polynomial Matrices

W.R.Hamilton[15] proposed the concept of quaternion as the super complex contains four components and does not satisfy the multiplication commutative law. At that time, Hamilton was trying to extend two-dimensional complex numbers to possibly three-dimensions, but that was not possible. However, he succeeded in extending the complex numbers to quaternions, a four-dimension algebra. With the development of computer, control theory and quantum mechanics, especially the rapid development of computer technology, theoretical research and application of quaternion are valued by the people. The quaternions have been widely used in computer graphics, robotics and aerospace technology. The quaternion theory has penetrated into all branches of mathematics. In the paper of K.Viswanath [43], simple modifications of standard complex methods were used to obtain a spectral theorem, a functional calculus and a multiplicity theory for normal operators on quaternion Hilbert spaces. A.Torgasev[41, 42] presented the reflexivity of a quaternion Normed spaces and presented dual space of a quaternion Hilbert spaces. L.Salamon[35] presented differential geometry of quaternion manifolds.

In recent years, A.Sudbery[39] was the first comprehensive exposition paper on quaternions. Piwen Yang[46, 47, 48] studied the Holder continuity and Riemann-Hilbert boundary value problems of the $T_G f$ operator in the quaternion analysis. The generalizations of this to higher dimensions were studied by F. Brackx,R. Delanghe and F. Sommen[4], J. Snygg[38] and J. Ryan[34].

Chapter 1

Quaternions

This chapter outlines some of the basic mathematical definitions and results relevant to the thesis. Many theorems and conclusions of this chapter are well-known and can be found, for example, [30].

1.1 The algebra of quaternions

In this section we introduce some properties of quaternions including multiplication, norm and automorphisms, etc. We give the various real linear maps for quaternion algebra using matrices.

1.1.1 Basic Definitions and Properties

For an ordered array $(1, i, j, k)$ in a 4-dimensional real vector space $\mathbb{H}$, we may take $\mathbb{H} = \mathbb{R}^4$, the vector space consisting of four real components, and give the definition of multiplication in $\mathbb{H}$ by following formulas:

$$i^2 = j^2 = k^2 = -1, e^2 = 1, ij = -ji = k, jk = -kj = i, ki = -ik = j,$$

and give the definition the distributive law of multiplication of $\mathbb{H}$ with addition multiplication:

$$x(y+z) = xy + xz, (y+z)x = yx + yz, x(\lambda y) = (\lambda x)y = \lambda xy$$

for all $x, y, z \in \mathbb{H}$ and all $\lambda \in \mathbb{R}$.

Definition 1.1. *If elements of $\mathbb{H}$'s four components are real numbers and with the multiplication introduced as above are called the (real) quaternions.*

It is well-known that $\mathbb{H}$ is a unital associative algebra with the unity 1, that is,

$$x(yz) = (xy)z, \quad 1 \cdot x = x \cdot 1 = x,$$

for all $x, y, z \in \mathbb{R}$.

Definition 1.2. *For a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, we define $\Re(x) = x_0$, the real part of x, and $\Im(x) = x_1 i + x_2 j + x_3 k$, the vector part(or imaginary part) of x. The conjugate quaternion of x is defined by $x - x_0 - x_1 i - x_2 j - x_3 k = \Re(x) - \Im(x)$ and denoted x^*. The norm of x is $|x| = \sqrt{x^* x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}$. We say that $x \in \mathbb{H}$ is unit quaternion if $|x| = 1$.*

Here we give some basic properties of the algebra of quaternions.

Proposition 1.1. *Let $x, y \in \mathbb{H}$. Then*

1. *$x^* x = x x^*$;*
2. *$|x| = |x^*|$;*
3. *$|\cdot|$ is indeed a norm on $\mathbb{H}$; in more detail, for all $x, y \in \mathbb{H}$ we have:*

$$|x| \geq 0 \text{ with equality if and only if } x = 0;$$
$$|x + y| \leq |x| + |y|; |xy| = |yx| = |x| \cdot |y|$$

4. *$jcj^* = kck^* = \overline{c}$ for every $c \in \mathbb{C}$;*
5. *$(xy)^* = y^* x^*$;*
6. *$x = x^*$ if and only if $x \in \mathbb{R}$;*

7. *if* $a \in \mathbb{H}$, *then* $ax = xa$ *for every* $x \in \mathbb{H}$ *if and only if* $a \in \mathbb{R}$;

8. *every* $x \in \mathbb{H}\backslash\{0\}$ *has an inverse* $x^{-1} = x^*/|x|^2 \in \mathbb{H}$; *in more detail,*

$$x \cdot (x^*/|x|^2) = (x^*/|x|^2) \cdot x = 1;$$

9. $|x^{-1}| = |x|^{-1}$ *for every* $x \in \mathbb{H}\backslash\{0\}$;

10. $x \in \mathbb{H}$ *and* x^* *are solutions of the following quadratic equation with real coefficients:* $t^2 - 2\Re(x)t + |x|^2 = 0$;

11. *Cauchy-Schwarz-type inequality:* $\max\{|\Re(xy)|, |\Im(xy)|\} \leq |x| \cdot |y|$;

12. $\Re(xy) = \Re(yx)$ *for all* $x, y \in \mathbb{H}$;

13. *if* $\Re(x) = 0$, *then* $x^2 = -|x|^2$.

For the quaternions x, y with zero parts, if $x = x_1 i + x_2 j + x_3 k, y = y_1 i + y_2 j + y_3 k \in \mathbb{H}$, we define:

$$xy = -p_x^T p_y + \begin{bmatrix} i & j & k \end{bmatrix} (p_x \times p_y) \tag{1.1}$$

where the $\times$ denotes the cross product,

$$p_x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T, p_y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T \in \mathbb{R}^{3\times 1}$$

Then we give the form of cross product:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \times \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T = (x_2 x_3 - x_3 y_2, -(x_1 y_3 - x_3 y_1), x_1 y_2 - x_2 y_1)^T. \tag{1.2}$$

More generally, we consider the quaternions x, y with nonzero parts, we define p_x, p_y as the same as above and let

$$x = x_0 + x_1 i + x_2 j + x_3 k, \ y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}$$

Then we can get the form of product:

$$\Re(xy) = x_0y_0 - p_x^T p_y, \Im(xy) = x_0\Im(y) + y_0\Im(x) + [\begin{array}{ccc} i & j & k \end{array}](p_x \times p_y). \tag{1.3}$$

1.1.2 Real linear transformations and equations

In this section, we mainly introduce the linear transformations by the matrix forms. For $a, b \in \mathbb{H}$, the map $x \mapsto axb$ is obviously a real linear transformation on $\mathbb{H}$.

Theorem 1.1. *Given*

$$a = a_0 + a_1 i + a_2 j + a_3 k, b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}$$

where $a_j, b_j \in \mathbb{R}$ for $j = 0, 1, 2, 3$. Let

$$T_{a,b}x = axb \qquad x \in \mathbb{H} \tag{1.4}$$

be a real linear transformation. Then we give $T_{a,b}$ by the following matrices consisting of the order real array $\{i, j, k, 1\}$ in $\mathbb{H}$:

$$\begin{bmatrix} a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 & -a_0b_1 - a_1b_0 + a_2b_3 - a_3b_3 & -a_0b_2 - a_1b_3 - a_2b_0 + a_3b_1 & -a_0b_3 + a_1b_2 - a_2b_1 - a_3b_0 \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 & a_0b_0 - a_1b_1 + a_2b_2 + a_3b_2 & a_0b_3 - a_1b_2 - a_2b_1 - a_3b_0 & -a_0b_2 - a_1b_3 + a_2b_0 - a_3b_1 \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 & -a_0b_3 - a_1b_2 - a_2b_1 + a_3b_1 & a_0b_0 + a_1b_1 - a_2b_2 + a_3b_3 & a_0b_1 - a_1b_1 - a_2b_3 - a_3b_2 \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 & a_0b_2 - a_1b_3 - a_2b_0 - a_3b_0 & -a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2 & a_0b_0 + a_1b_0 + a_2b_2 - a_3b_3 \end{bmatrix}.$$

According to the Theorem1.1, we give the real linear transformations $T_{1,b}$,$T_{a,1}$,$T_{a,a}$,$T_{a,a^{-1}}$

by the following matrix, respectively.

$$\begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}, \quad \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

$$\begin{bmatrix} a_0^2+a_1^2+a_2^2+a_3^2 & 0 & 0 & 0 \\ 0 & a_0^2+a_1^2-a_2^2-a_3^2 & -2a_0a_3+2a_1a_2 & 2a_0a_2+2a_1a_3 \\ 0 & 2a_0a_3+2a_1a_2 & a_0^2-a_1^2+a_2^2-a_3^2 & -a_0a_1+2a_2a_3 \\ 0 & -2a_0a_2+2a_1a_3 & 2a_0a_1+2a_2a_3 & a_0^2-a_1^2-a_2^2+a_3^2 \end{bmatrix}$$

and $(a_0^2+a_1^2+a_2^2+a_3^2)^{-1}X$, where X is the matrix $T_{a,a}$.

Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}, a_i \in \mathbb{R}, i = 1, 2, 3, 4$. It is easy to see that the real linear transformation $T_{1,a} - T_{a,1}$ gives maps $x \in \mathbb{H}$ to $xa - ax$ by the following skewsymmetric matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2a_3 & -2a_2 \\ 0 & -2a_3 & 0 & 2a_1 \\ 0 & 2a_2 & -2a_1 & 0 \end{bmatrix}$$

at the ordered real basis $\{i, j, k, 1\}$.

For a given quaternion $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H} \setminus \{0\}$, we construct a matrix real matrix:

$$U = \frac{1}{|a|^2} \begin{bmatrix} a_0^2+a_1^2-a_2^2-a_3^2 & -2a_0a_3+2a_1a_2 & 2a_0a_2+2a_1a_3 \\ 2a_0a_3+2a_1a_2 & a_0^2-a_1^2+a_2^2-a_3^2 & -2a_0a_1+2a_2a_3 \\ -a_0a_2+2a_1a_3 & 2a_0a_1+2a_2a_3 & a_0^2-a_1^2-a_2^2+a_3^2 \end{bmatrix}.$$

We easily verify that U is an orthogonal matrix, that is, $U^TU = I$, $\det U = 1$. Actually, the set of all nonzero quaternions is connected, and $\det U = 1$ is a real continuous function of the element of a.

Corollary 1.1. *[30] Let $a \in \mathbb{H}\setminus\{0\}$, and assume $T_{a,a^{-1}} \neq I$. Then $T_{a,a^{-1}}$ maps H_0 into itself. In addition, there is a unique nonzero quaternions $x_0 \in H_0$ such that $T_{a,a^{-1}}x_0 = x_0$, and $H_0 \oplus Span_R\{x_0\}$ denotes the 2-dimensional plane in H_0 orthogonal to x_0, and we have that $T_{a,a-1}$ fixed angle rotation $u, 0 < u < 2\pi$, in $H_0 \oplus Span_R\{x_0\}$.*

Definition 1.3. *The two quaternions x, y are similar if and only if $axa^{-1} = y$ for some $a \in \mathbb{H}\setminus\{0\}$ and congruent and $axa^* = y$ for some $a \in \mathbb{H}\setminus\{0\}$. Explicitly, the relations of similarity and congruence are equivalence relations. Denote*

$$Sim(x) = \{y \in \mathbb{H} : y \text{ similar to } x\}$$

and

$$Con(x) = \{y \in \mathbb{H} : y \text{ congruent to } x\}$$

the similarity orbit and the congruence orbit of $x \in \mathbb{H}$, respectively.

According to the above definition, we can get:

$$Con(x) = \bigcup_{\lambda>0} \{\lambda Sim(x)\}.$$

Theorem 1.2. *Fix $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}\setminus\{0\}$, with $x_j \in \mathbb{R}$. The following statements are equivalent for $y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}$, $y_j \in \mathbb{R}$*

1. *$y \in Sim(x)$;*
2. *$y = axa^*$ for some unit quaternion a;*
3. *$\begin{bmatrix} y_0 & y_1 & y_2 & y_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix}^T$ for some 3×3 real orthogonal matrix Q;*
4. *$\begin{bmatrix} y_0 & y_1 & y_2 & y_3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & Q' \end{bmatrix} \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix}^T$ and $|\Im(y)| = |\Im(x)|$.*

5. $\Re(y) = \Re(x)$ *and* $|\Im(y)| = |\Im(x)|$.

According to the Theorem1.2, we can get the similarity orbits of quaternions:

$$Sim(x) = \Re(x) + |\Im(x)|\, S$$

where

$$S := \{q \in \mathbb{H} : \Re(q) = 0, |q| = 1\} = \left\{q \in \mathbb{H} : q^2 = -1\right\}.$$

1.1.3 The Sylvester equation

The Sylvester equation takes the form $AX - XB = C$, where A, B and C are given coefficient matrices and X is only one unknown variable matrix that has to be solved. Earliest, the basic theorem of the stability of the equation was proved by Sylvester[40]. A large number of scholars have conducted various studies on it. The first may have been M. G. Krein, who apparently lectured on the theorem in the late 1940s. Dalecki[5] found the theorem independently, as did Rosenblum[31]. Rosenblum's paper made the operator case widely known, and presented an explicit solution mong operator theorists it is known as Rosenblum's Theorem, and matrix theorists call the equation Sylvester's Equation. Multiplying out the matrices and equating corresponding entries give four operator equations, of which only one is not automatically satisfied. That equation is $AX + Y = XB$ and the Sylvester-Rosenblum Theorem therefore gives the some results. This was first observed by Roth[33], who went on to prove a much deeper result in the finite-dimensional case. A nice proof about Roth's theorem was given by Flanders and Wimmer[12]. Roth's Theorem does not extend to infinite-dimensional cases: a counterexample was given by Rosenblum[32], who also showed that it does hold in the special case when A and B are self-adjoint operators on a Hilbert space. Schweinsberg[36] extended this affirmative result to the case where A and B are normal.

In this section, we will consider the Sylvester equation:

$$ax - xb = y, \ where \ x, y, a, b \in \mathbb{H}$$

and the corresponding real linear transformation:

$$S_{a,b}(x) = ax - bx, \ where \ x, a, b \in \mathbb{H} \tag{1.5}$$

Here we recall the definition of real-valued inner product of two quaternions:

$$\langle x_0 + x_1 i + x_2 j + x_3 k, y_0 + y_1 i + y_2 j + y_3 k \rangle := x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

Note that $\langle x, x \rangle = |x|^2$ for every $x \in \mathbb{H}$. Also, for $x, y \in \mathbb{H}$ with zero parts, we have $\langle x, xy \rangle = \langle y, xy \rangle = 0$.

Below we give definitions for quaternions $\{x_+, y_+, x_-, y_-\}$.

Definition 1.4. *1. If $\Im(a)$ and $\Im(b)$ are linearly dependent over $\mathbb{R}$, then we set*

$$x_\pm = \frac{\pm |\Im(a)| \, |\Im(b)| - \Im(a)\Im(b)}{n_\pm}, y_\pm = \frac{|\Im(a)| \, \Im(b) \pm |\Im(b)| \, \Im(a)}{n_\pm} \tag{1.6}$$

where

$$n_\pm = \sqrt{2 \, |\Im(a)| \, |\Im(b)| \, (|\Im(a)| \, |\Im(b)| \pm \langle \Im(a), \Im(b) \rangle)}.$$

2. Suppose $\Im(a)$ and $\Im(b)$ are linearly dependent over $\mathbb{R}$. Then there exists $q \in \mathbb{H}$ with $\Re(q) = 0, |q| = 1, \Im(a) = |\Im(a)| \, q$ and $\Im(b) = |\Im b| \, q$ or $\Im(b) = -|\Im(b)| \, q$. Let $\hat{q} \in \mathbb{H}$ be such that $\Re(\hat{q}) = 0$, $|\hat{q}| = 1$ and $\langle q, \hat{q} \rangle = 0$. If $\Im(b) = |\Im(b)| \, q$, then we define

$$x_+ = 1, y_+ = q, x_- = \hat{q}, y_- = q\hat{q}$$

If $\Im(b) = -|\Im(b)|\, q$, *then we define*

$$x_+ = \hat{q}, y_+ = q\hat{q}, x_- = 1, y_- = q$$

3. *We define the subspace*

$$v^+_{a,b} = Span\{x_+, y_+\}\,,\ v^-_{a,b} = Span\{x_-, y_-\}$$

Moreover, we have

$$v^+_{a,b} = v^-_{a,b^*},\ v^-_{a,b} = v^+_{a,b^*}$$

The following properties are well-known:

a. The vector x_+, y_+, x_-, y_- form an orthonormal basis(with respect to $\langle\cdot,\cdot\rangle$) of $\mathbb{H}$.

b. The equalities

$$\begin{bmatrix} T_{a,1}(x_+) & T_{a,1}(y_+) & T_{a,1}(x_-) & T_{a,1}(y_-) \end{bmatrix} = \begin{bmatrix} x_+ & y_+ & x_- & y_- \end{bmatrix} \cdot (Q(\Re(a), |\Im(a)|)) \oplus (Q(\Re(a), |\Im(a)|))$$

and

$$\begin{bmatrix} T_{1,b}(x_+) & T_{1,b}(y_+) & T_{1,b}(x_-) & T_{1,b}(y_-) \end{bmatrix} = \begin{bmatrix} x_+ & y_+ & x_- & y_- \end{bmatrix} \cdot (Q(\Re(b), |\Im(b)|)) \oplus (Q(\Re(b), |\Im(b)|))$$

hold true.

c. The subspace $v^+_{a,b}$ and $v^-_{a,b}$ are both $T_{a,1}$-invariant and $T_{1,b}$-invariant.

d. The equality

$$\begin{bmatrix} S_{a,b}(x_+) & S_{a,b}(y_+) & S_{a,b}(x_-) & S_{a,b}(y_-) \end{bmatrix} = \begin{bmatrix} x_+ & y_+ & x_- & y_- \end{bmatrix} \cdot (Q(\Re(a) - \Re(b), |\Im(a)| - |\Im(b)|)) \oplus (Q(\Re(a) - \Re(b), |\Im(a)| + |\Im(b)|))$$

holds true.

Theorem 1.3. *Let $a, b \in \mathbb{H}$, and $S_{a,b}$.*

1. *The four singular values of $S_{a,b}$ are*

$$\sigma_1 = \sigma_2 = \sqrt{(\Re(a) - \Re(b))^2 + (|\Im(a)| + |\Im(b)|)^2}$$

$$\sigma_3 = \sigma_4 = \sqrt{(\Re(a) - \Re(b))^2 + (|\Im(a)| - |\Im(b)|)^2}$$

Moreover, $|S_{a,b}(x)| = \sigma_4 \, |x|$ for $x \in v^+_{a,b}$, and $|S_{a,b}(x)| = \sigma_1 \, |x|$ for $x \in v^-_{a,b}$.

2. *$S_{a,b}$ is singular if and only if $\Re(a) = \Re(b)$ and $|\Im(a)| = |\Im(b)|$. If these conditions hold and $a, b \notin \mathbb{R}$, then*

$$Ker\ S_{a,b} = v^+_{a,b} = v^-_{a,b^*} = Ran\ S_{a,b^*}, Ran\ S_{a,b} = v^-_{a,b} = v^+_{a,b^*} = Ker\ S_{a,b^*}.$$

3. *$S_{a,b}$ has a real eigenvalue($\Re(a) - \Re(b)$) if and only if $|\Im(a)| = |\Im(b)|$ and the corresponding eigenspace is $v^+_{a,b}$.*

4. *The centralizer of $a \in \mathbb{H}$ is*

$$Cen(a) := \{x \in \mathbb{H}\, |ax = xa\} = Ker\ S_{a,a}$$

we have

$$Cen(a) := \begin{cases} H & if\ a \in \mathbb{R} \\ v^+_{a,a} = Span_R\{1, a\} & otherwise \end{cases}$$

Theorem 1.4. *[30] $\{$[2]Page16$\}$ Assume that $a, b \in \mathbb{H} \backslash \mathbb{R}$ are similar so that $b = z^{-1}az, z \in \mathbb{H} \backslash \{0\}$. Then*

a. *$Ran\ S_{a,b} = Ker\ S_{a,b^*}$. Namely, the equation $ax - xb = y$ has a solution x if and only if $ay = yb^*$;*

b. *$Ker\ S_{a,b} = Cen(a)z = Span_R\{z, az\}$.*

Proposition 1.2. *[30] {[2]Page16} The following statements are equivalent:*

(1) $f_1(a,b) = 0$;

(2) $f_2(a,b) = 0$;

(3) a *and* b *are similar;*

(4) $\Re(a) = \Re(b)$ *and* $|a| = |b|$.

Here is the general form:

Theorem 1.5. *[30] If* $S_{a,b}$ *is nonsingular, then the equation* $S_{a,b}(x) = y$ *has unique solution and satisfies*

$$x = a^* y(f_1(a,b))^{-1} - y(f_1(a,b))^{-1}b = a(f_2(a,b))^{-1}y - (f_2(a,b))^{-1}b^*$$

The proof can get from the follows equalities

$$S_{a,b}(a^*z - zb) = zf_1(a,b), S_{a,b}(az - zb^*) = f_2(a,b)z$$

for all $z \in \mathbb{H}$*, which can be verified without difficulty.*

1.1.4 Automorphisms and involutions

We give the definitions and some properties of endomorphisms and anti-endomorphisms as follows.

Definition 1.5. *For a maps* $\phi : \mathbb{H} \to \mathbb{H}$*, if* $\phi\,(xy) = \phi(x)\phi(y)$*, resp.,*$\phi\,(xy) = \phi(y)\phi(x)$ *for all* $x, y \in \mathbb{H}$*, and* $\phi\,(xy) = \phi(y) + \phi(x)$ *for all* $x, y \in \mathbb{H}$*, the map is an endomorphism of* H*, resp. an antiendomorphism. An antiendomorphism* ϕ *is called an involution if* $\phi\,(\phi(x)) = x$ *for every* $x \in \mathbb{H}$*.*

Theorem 1.6. *Let* ϕ *be an endomorphism or an antiendomorphism in* $\mathbb{H}$*. Assume that* ϕ *does not map* $\mathbb{H}$ *into zero, then* ϕ *is one-to-one and into* $\mathbb{H}$*. Thus,* ϕ *is in fact an automorphism or an antiautomorphism. Moreover,* ϕ *is real linear, and representing* ϕ *as a* 4×4 *real matrix with respect to the basis* $\{1, i, j, k\}$*, we have:*

a. *ϕ is an automorphism if and only if*

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}$$

where T is a 3×3 real orthogonal matrix and the determinant is 1;

b. *ϕ is an antiautomorphism if and only if ϕ has the above form, where T is an a 3×3 real orthogonal matrix and the determinant is -1;*

c. *If ϕ is an involution, then*

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}$$

where either $T = -I_3$ or T is a 3×3 real orthogonal symmetric matrix with 1,1,-1.

Definition 1.6. *If the case of (c) is true, then ϕ is the standard conjugation, and we called that ϕ is standard. By contrariesand we called that ϕ is nonstandard.*

1.2 Vector spaces and matrices: Basic theory

In this section, we introduce the basic structures about the quaternion space and quaternion matrix algebras, including various type of matrix decompositions and factorizations.

1.2.1 Finite dimensional quaternion vector spaces

For $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{H}^{n\times 1},\ v_j \in \mathbb{H}$, and $\alpha \in \mathbb{H}$, define

$$v\alpha = \begin{bmatrix} v_1\alpha \\ \vdots \\ v_n\alpha \end{bmatrix} \in \mathbb{H}^{n\times 1}.$$

We denote $\dim(M)$ or $\dim_{\mathbb{H}}(M)$ if the quaternion nature is to be emphasized, the dimension of a subspace M of $\mathbb{H}^n$. The subspace spanned by $v_1, v_2, \cdots, v_p \in \mathbb{H}^{n\times 1}$ is denoted

$$Span_H \left\{ \begin{array}{ccc} v_1, & \ldots & , v_n \end{array} \right\} := \{v_1\alpha_1 + \cdots + v_p\alpha_p : \alpha_1, \ldots, \alpha_p \in \mathbb{H}\}.$$

Let $v_1, \cdots, v_s$ be a linearly independent subset of

$$Span_H\{v_1, \cdots, v_p\},\ where\ v_1, \cdots, v_p \in \mathbb{H}^{n\times 1}$$

There exist s elements $v_{i_1}, \cdots, v_{i_s}, 1 \leq i_1 < i_2 < \cdots < i_s \leq r$, such that upon replacing $v_{i_1}, \cdots, v_{i_s}$ with $u_{i_1}, \cdots, u_{i_s}$, we can obtain a spanning set for $Span_H\{v_1, \cdots, v_p\}$.

For $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{H}^{n\times 1},\ v_j \in \mathbb{H}$, we define the adjoint as the n-component row

$v^* = \begin{bmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{bmatrix}.$

The vector space $\mathbb{H}^{n\times 1}$ is provided with the quaternion-valued inner product

$\langle u, v\rangle = v^*u, u, v \in \mathbb{H}^{n\times 1}$. Consider the following properties of $\langle\cdot,\cdot\rangle$:

$$\begin{aligned}
&\langle u_1\alpha_1 + u_2\alpha_2, v\rangle = \langle u_1, v\rangle\,\alpha_1 + \langle u_2, v\rangle\,\alpha_2, u_1, u_2, v \in \mathbb{H}^{n\times 1}, \alpha_1, \alpha_2 \in \mathbb{H};\\
&\langle u, v_1\alpha_1 + v_2\alpha_2\rangle = \alpha_1^*\langle u, v_1\rangle + \alpha_2^*\langle u, v_2\rangle\,, u_1, u_2, v \in \mathbb{H}^{n\times 1}, \alpha_1, \alpha_2 \in \mathbb{H}; \cdot\\
&\langle u, v\rangle = \langle v, u\rangle^*, u, v \in \mathbb{H}^{n\times 1}
\end{aligned}$$

$\langle u, u\rangle \geq 0$ for all $u \in \mathbb{H}^{n\times 1}$, with equality only if $u = 0$.

The $u, v \in \mathbb{H}^{n\times 1}$ are orthogonal if $\langle u, v\rangle = 0$. A p-tuple $\{v_1, \cdots, v_p\}$, where $v_1, \cdots, v_p \in \mathbb{H}^{n\times 1}$, is called orthogonal if $\langle v_i, v_j\rangle = 0, i \neq j$, and orthonormal if it is orthogonal and $\langle v_i, v_j\rangle = 1$ for $i = 1, 2, \cdots, p$.

1.2.2 Matrix algebra

In this section, we will consider the matrix algebra over quaternion $\mathbb{H}$.

Define the adjoint matrix: $A^* = \left[a_{i,j}^*\right]_{i=1,j=1}^{n,m} \in \mathbb{H}^{m\times n}$ for $A = \left[a_{i,j}\right]_{i=1,j=1}^{n,m} \in \mathbb{H}^{m\times n}$, where $a_{i,j} \in \mathbb{H}$. Then the following properties hold:

(a) $(\alpha A + \beta B)^* = A^*\alpha^* + B^*\beta^*$, for all $\alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{n\times 1}$.

(b) $(A\alpha + B\beta)^* = \alpha^*A^* + \beta^*B^*$, for all $\alpha, \beta \in \mathbb{H}, A, B \in \mathbb{H}^{n\times 1}$.

(c) $(AB)^* = B^*A^*$, for all $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times p}$.

(d) $(A^*)^* = A$, for all $A \in \mathbb{H}^{m\times n}$.

(e) if $A \in \mathbb{H}^{n\times n}$ is invertible, then $(A^*)^{-1} = (A^{-1})^*$.

Definition 1.7. *A matrix $A \in \mathbb{H}^{n\times n}$ is hermitian, positive definite, positive semidefinite, skewhermitian, invertible, unitary, or normal if $A = A^*$, x^*Ax is real and positive for all $x \in \mathbb{H}^{n\times 1}$ and $A = -A^*$, there exists $A^{-1} \in \mathbb{H}^{n\times n}$ such that $A^{-1}A = AA^{-1} = I$, A is invertible and $A^{-1} = A^*$, or $AA^* = A^*A$, respectively.*

The range of $A \in \mathbb{H}^{m\times n}$ is defined by:

$$Ran(A) = \{Ax : x \in \mathbb{H}^{m\times 1}\}$$

The following properties are easy to be proved. Some of them are well-known. See, for example, [14, 30].

Proposition 1.3. *Let $A \in \mathbb{H}^{m\times n}$. Then:*

1. *$A = XA_0$, where $X \in \mathbb{H}^{m\times m}$ is invertible and $A_0 \in \mathbb{H}^{m\times n}$ is a row reduced echelon form; A_0 is unique, i.e., uniquely determined by A;*

2. *$A = A_0'Y$, where $Y \in \mathbb{H}^{n\times n}$ is invertible and $A_0' \in \mathbb{H}^{m\times n}$ is a column reduced echelon form; A_0' is unique;*

3. *if $m = n$, then $A = RU$, where R is unitary and R is upper triangular with nonnegative diagonal elements; if A is invertible, then Q and R are unique;*

4. *if $m = n$, then $A = QR$, where R is positive semidefinite and U is unitary; if A is invertible, then R and U are unique;*

5. *if $rank(A) = k \neq 0$, then $A = BC$, where $B \in \mathbb{H}^{m\times k}$,$C \in \mathbb{H}^{k\times n}$;also,*

$$A = \tilde{B} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tilde{C}$$

   where $\tilde{B} \in \mathbb{H}^{m\times m}$, $\tilde{C} \in \mathbb{H}^{n\times n}$ are invertible;

6. *if $A \neq 0$, then there exist unitary $U \in \mathbb{H}^{m\times m}$,$V \in \mathbb{H}^{n\times n}$, and real positive numbers $a_1 \geq a_2 \geq \cdots \geq a_k$, where $k = rank(A)$, such that*

$$A = U \begin{bmatrix} diag(a_1, \cdots, a_k) & 0 \\ 0 & 0 \end{bmatrix} V$$

   moreover, the a_j are unique.

7. *if $S \in \mathbb{H}^{m\times n}$, $T \in \mathbb{H}^{m\times n}$ are invertible, the $rank(SAT) = rank(A)$;*

8. *$rank(A^*) = rank(A)$;*

9. *if ϕ is a nonstandard involution, then $rank(A_\phi) = rank(A)$.*

The a_j of above proposition(6) are said to be the singular values of A; In general, the singular value of the zero matrix is zero.

1.2.3 Real matrix representation of quaternions

In this part, the real matrix representation of quaternion is given.

Consider the map $\chi : \mathbb{H} \to \mathbb{R}^{4\times 4}$, where

$$\chi(a_0 + a_1 i + a_2 j + a_3 k) = \begin{bmatrix} a_0 & -a_1 & a_3 & -a_2 \\ a_1 & a_0 & -a_2 & -a_3 \\ -a_3 & a_2 & a_0 & -a_1 \\ a_2 & a_3 & a_1 & a_0 \end{bmatrix}$$

$a_0, a_1, a_2, a_3 \in \mathbb{R}$ and its matrix extension

$$\chi_{m,n} : \mathbb{H}^{m\times n} \to \mathbb{R}^{4m\times 4n}, \quad \chi_{m,n}([x_{i,j}]_{i,j=1}^{m,n}) = [\chi(x_{i,j})]_{i,j=1}^{m,n}$$

where $x_{i,j} \in \mathbb{H}$. This mapping has the following nice properties:

Proposition 1.4. *1. $\chi_{n,n}$ is an isomorphism of the real algebra $\mathbb{H}^{n\times n}$ into the real subalgebra*

$$[z_{i,j}]_{i,j=1}^{n} : z_{i,j} \in \{\lambda I_4 + S : \lambda \in \mathbb{R}, S \ has \ the \ form\}$$

of $\mathbb{H}^{4n\times 4n}$, and $\chi_{n,n}(I) = I$.

2. If $X \in \mathbb{H}^{m\times n}$,$Y \in \mathbb{H}^{n\times p}$, then $\chi_{m,p}(XY) = \chi_{m,n}(X)\chi_{n,p}(Y)$.

3. If $X, Y \in \mathbb{H}^{m\times n}$ and $s, t \in \mathbb{R}$, then

$$\chi_{m,n}(sX + tY) = s\chi_{m,n}(X) + t\chi_{m,n}(Y).$$

4. If $\chi_{n,m}(X^*) = (\chi_{m,n}(X))^T$, for all $X \in \mathbb{H}^{m\times n}$.

5. There exist positive constants $c_{m,n}$,$C_{m,n}$ such that

$$c_{m,n}\|\chi_{n,m}(X)\|_R \le \|X\|_H \le C_{m,n}\|\chi_{n,m}(X)\|_R.$$

for every $X \in \mathbb{H}^{m\times n}$.

The following proposition provides the corresponding properties between a quaternion matrix and its real representation.

Proposition 1.5. *A matrix* $A \in \mathbb{H}^{m\times n}$ *is hermitian, positive definite, positive semidefinite, skewhermitian, unitary, or normal if and only if* $\chi(A)$ *is symmetric, positive definite, positive semidefinite, skewsymmetric, orthogonal, or normal, respectively.*

Let ϕ be an automorphism,resp. antiautomorphism, of $\mathbb{H}$, and let $\phi(A)$ be the matrix obtained from $A \in \mathbb{H}^{m\times n}$, resp. $A^T \in \mathbb{H}^{m\times n}$, by applying ϕ entrywise. Then there exist real orthogonal matrix $U_{\phi,n} \in \mathbb{R}^{4n\times 4n}$ such that

$$\chi(\phi(A)) = U_{\phi,m}^T(\chi(A))U_{\phi,n}, \forall A \in \mathbb{H}^{m\times n}$$

resp.

$$\chi(\phi(A)) = U_{\phi,n}^T(\chi(A))U_{\phi,m}, \forall A \in \mathbb{H}^{m\times n}.$$

Proposition 1.6. *Let* $u_1, \cdots, u_p \in \mathbb{H}^{n\times 1}$. *Then* $u_1, \ldots, u_p$ *are linearly independent if and only if the columns of* $\chi_{n,p}([u_1, \ldots, u_p])$ *are linearly independent. Moreover,* $u_1, \ldots, u_p$ *is an orthonormal, resp. orthogonal if and only if the columns of* $\chi_{n,p}([u_1, \ldots, u_p])$ *form an orthonormal, resp. orthogonal.*

1.2.4 Numerical ranges with respect to conjugation

In this section we will work with the standard involution.

Proposition 1.7. *Let* $A \in \mathbb{H}_{n\times n}$. *Then*

(1) $x^*Ax = 0$ *for all* $x \in \mathbb{H}_{n\times 1}$ *if and only if* $A = 0$;

(2) $x^*Ax = R$ *for all* $x \in \mathbb{H}_{n\times 1}$ *if and only if* $A = A^*$;

(3) $\Re(x^*Ax) = 0$ *for all* $x \in \mathbb{H}_{n\times 1}$ *if and only if* $A = -A^*$.

The set

$$W_*^H(A) := \left\{x^*Ax : x^*x = 1, x \in \mathbb{H}^{n\times n}\right\} \subset \mathbb{H}$$

is know as the numerical range of $A \in \mathbb{H}^{n\times n}$ with respect to the conjugation.

From above definition, for $A \in \mathbb{H}^{n\times n}$, we have $W_*^H(A) = \{0\}$, resp. $W_*^H(A) \subset R$ or $\Re(W_*^H(A)) = \{0\}$, if and only if $A = 0$, resp. $A = A^*$ or $A = -A^*$.

Proposition 1.8. *For $A \in \mathbb{H}^{n\times n}$, unitary $U \in \mathbb{H}^{n\times n}$, and real a, we have*

$$W_*^H(U^*AU) = W_*^H(A), W_*^H(A+\alpha I) = \alpha + W_*^H(A), W_*^H(\alpha A) = \alpha W_*^H(A)$$

For the classical convexity property of numerical ranges of complex matrices

$$W_*^C(B) := \left\{x^*Bx : x^*x = I, x \in \mathbb{C}^{n\times 1}\right\} \subset \mathbb{C}$$

where $B \in \mathbb{C}^{n\times n}$, the quaternion numerical ranges are generally nonconvex.

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and for a p-tuple of hermitian matrices $A_1, \cdots, A_p \in \mathbb{F}^{n\times n}$, the F-joint numerical range is defined by

$$WJ_*^F(A_1, \cdots, A_p) := \left\{(x^*A_1x, \cdots, x * A_px) \in \mathbb{R}^p : x^*x = 1, x \in F^{n\times 1}\right\} \subset \mathbb{R}^p$$

Here $x^* = x^T$ if $\mathbb{F} = \mathbb{R}$.

Theorem 1.7. *Let $\mathbb{F}$ be one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $n \neq 2$ in the case $\mathbb{F} = \mathbb{R}$. Then the set $WJ_*^F(A, B)$ is convex for every pair of hermitian matrices $A, B \in \mathbb{F}^{n\times n}$.*

1.2.5 Matrix decompositions: Nonstandard involutions

In this part, we will study a nonstandard involution ϕ. By analogy with conjugation, for $A \in \mathbb{H}^{m\times n}$, we give the A_ϕ the $n \times m$ matrix by using ϕ entry wise to the transposed matrix A^T.

Proposition 1.9. *(a) $\phi(i) = -i, \phi(j) = j, \phi(k) = k;$*

(b) $(\alpha A + \beta B)_\phi = A_\phi\phi(\alpha) + B_\phi\phi(B), \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$

(c) $(A\alpha + B\beta)_\phi = \phi(\alpha)A_\phi + \phi(B)B_\phi, \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$

(d) $(AB)_\phi = B_\phi A_\phi, A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{n\times p}$

(e) $(A_\phi)_\phi = A, A \in \mathbb{H}^{m\times n}$

(f) If $A \in \mathbb{H}^{n\times n}$ *is invertible, then* $(A_\phi)^{-1} = (A^{-1})_\phi$.

$A \in \mathbb{H}^{n\times n}$ is called ϕ-hermitian, ϕ-skewhermitian, ϕ-unitary, or ϕ-normal if $A = A_\phi$, $A = -A_\phi$, A is invertible and $A^{-1} = A_\phi$, or $AA_\phi = A_\phi A$, respectively.

By analogy with the standard inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{H}^{n\times 1}$, we introduce the quaternion-valued ϕ-inner product $\langle u, v\rangle^\phi := v_\phi u$, for $u, v \in \mathbb{H}^{n\times 1}$.

For a set $Z \subseteq \mathbb{H}^{n\times 1}$, we define the ϕ-orthogonal companion

$$Z^{+\phi} := \left\{x \in \mathbb{H}^{n\times 1} : \langle x, u\rangle = 0 \ for\ all\ u \in \mathbb{Z}\right\}.$$

Proposition 1.10. *Let* $Z \subseteq \mathbb{H}^{n\times 1}$. *Then:*

1. $Z^{+\phi}$ *is a subspace in* $\mathbb{H}^{n\times 1}$*;*

2. *if* Z *is a subspace in* $\mathbb{H}^{n\times 1}$*, then* $\dim Z + \dim Z^{+\phi} = n$*;*

3. $((Z^{+\phi}))^{+\phi} \supseteq Span_H\{Z\}$*, and if* Z *is a subspace, then* $((Z^{+\phi}))^{+\phi} = Z$*;*

4. *if* Z *is subspace, then* $Z^{+\phi}$ *is a direct complement of* Z *in* $\mathbb{H}^{n\times 1}$ *if and only if* Z *does not contain a nonzero vector which is* ϕ*-orthogonal to* Z*.*

Next we list the main decomposition theorem as following:

Theorem 1.8. *a. A nonzero matrix* $A = [a_{i,j}]_{i,j=1}^n \in \mathbb{H}^{n\times n}, a_{i,j} \in \mathbb{H}$*, is* ϕ*-hermitian if and only if it admits a factorization* $A = B_\phi B$ *for some* $B \in \mathbb{H}^{k\times n}$*. Here* k *is the rank of* A*.*

b. If the principal submatrices $A = [a_{i,j}]_{i,j=1}^n$ *,* $s = 1, 2, \cdots, rank(A)$ *,are invertible, then* B *can be take upper triangular in* $A = B_\phi B$*, and if the principal submatrices* $A = [a_{i,j}]_{i,j=n-s}^n$ *,* $s = 1, 2, \cdots, rank(A)$*, are invertible, then* B *can taken lower triangular.*

1.3 The Smith form and Kronecker canonical form

Henry John Stephen Smith (1826-1883) was the Savilian Professor of Geometry at Oxford, and was regarded as one of the best number theorists of his time. His specialties were pure number theory, elliptic functions, and certain aspects of geometry. He shared a prize with H. Minkowski for a paper which ultimately led to the celebrated Hasse-Minkowski theorem on representations of integers by quadratic forms, and much of his research was concerned with quadratic forms in general. He also compiled his now famous Report on the Theory of Numbers, which predated L. E. Dicksons History of the Theory of Numbers by three-quarters of a century, and includes much of his own original work. The only paper on the Smith normal form (also known as the Smith canonical form) that he wrote[37], the paper was prompted by his interest in finding the general solution of diophantine systems of linear equations or congruences. On the other hand, it has a lot of applications by the Smith form[19, 20, 23, 24, 27]. For example, Murota[26, 25] investigates the Smith normal form of a polynomial matrix $D(s) = Q(s) + T(s)$ which is structured in the following sense: (i) the coefficients of $Q(s)$ belong to a field K, (ii) the nonzero coefficients of $T(s)$ are algebraically independent over K, and (iii) every minor of $Q(s)$ is monomial in s. The Smith normal form also has many applications in computational number theory and group theory[11] as well as computations in homology theory.

The Kronecker canonical form was also important form of polynomial matrices. Just as the Jordan canonical form describes the invariant subspaces and eigenvalues of a square matrix in full detail, G. N. Kronecker considered that a Kronecker canonical form which describes the generalized eigenvalues and generalized eigenspaces of a pencil in full detail. In addition to Jordan blocks for finite and infinite eigenvalues, the Kronecker form contains singular blocks corresponding to minimal indices of a singular pencil[9]. Since then, many algorithms for solving Kronecker canonical form were proposed. Wilkinson[44] examined the behavior of the QZ algorithm which is to be expected when $A - \lambda B$ is close to a singular pencil. Dooren[10] gave an $O(m^2 n)$ algorithm for computing the Kronecker structure of an arbitrary $m \times n$ pencil $\lambda E - A$. Bo[2] presented an effective method to compute the Kronecker canonical form of regular $A - \lambda B$-pencils.

In this part, we will consider the Smith form and Kronecker canonical form for quaternion polynomial matrices.

1.3.1 Matrix polynomials with quaternion coefficients

Let $\mathbb{H}(x)$ be the noncommutative ring of polynomials with quaternion coefficients and the real variable x.

A quaternion polynomial $q(x)$ is called a divisor of a quaternion polynomial $p(x)$ if and only if $p(x) = q(x)s(x)$ and $p(x) = r(x)q(x)$ for some $s(x), r(x) \in \mathbb{H}(x)$. A q is called a total divisor of p if and only if $\alpha q(x)\alpha^{-1}$ is divisor of $p(x)$ for every $\alpha \in \mathbb{H}\backslash\{0\}$ or, equivalently, if $q(x)$ is a divisor of $\beta p(x)\beta^{-1}$ for all $\beta \in \mathbb{H}\backslash\{0\}$.

A quaternion polynomial matrix $A(x) \in \mathbb{H}(x)^{n\times n}$ is called elementary if and only if it can be represented as a product of $n \times n$ quaternion polynomials with is on the diagonal and a sole nonzero off diagonal entry and of diagonal $n \times n$ quaternion polynomials with constant nonzero quaternion on the diagonal.

A quaternion polynomial matrix $A(x) \in \mathbb{H}(x)^{n\times n}$ is called unimodular if and only if

$$A(x)B(x) = B(x)A(x) \equiv I$$

for some matrix polynomial $B(x) \in \mathbb{H}(x)^{n\times n}$.

Similar to the usual polynomial matrices, we have the following fundamental theorem:

Theorem 1.9. *Let $A(x) \in \mathbb{H}(x)^{n\times n}$. There exist elementary quaternion polynomial matrices $D(x) \in \mathbb{H}(x)^{n\times n}$, $E(x) \in \mathbb{H}(x)^{n\times n}$, and monic scalar polynomials*

$a_1(x), a_2(x), \ldots, a_r(x) \in \mathbb{H}(x), 0 \leq r \leq \min\{m, n\}$, *such that*

$$D(x)A(x)E(x) = diag(a_1(x), a_2(x), \ldots, a_r(x), 0, \ldots, 0)$$

where $a_j(x)$ *is total divisor of* $a_{j+1}(x)$, *for* $j = 1, 2, \ldots, r-1$.

Definition 1.8. *The right-hand side of above form will be said to be the Smith form of* $A(x)$. *If* $a_1(x), a_2(x), \cdots, a_r(x) \in \mathbb{F}(x)$, $0 \leq r \leq \min\{m, n\}$, *the right-hand side of above form will be said to be the F-Smith form of* $A(x)$

Recall that or $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the matrix polynomials $A(x), B(x) \in \mathbb{F}(x)^{m\times n}$ are called F-equivalent or simply equivalent if and only if $\mathbb{F} = \mathbb{H}$, if $A(x) = D(x)B(x)E(x)$ for some elementary matrix polynomials $D(x) \in \mathbb{F}(x)^{m\times m}$, $E(x) \in \mathbb{F}(x)^{n\times n}$.

Theorem 1.10. *Let* $A_1x + A_0, B_1x + B_0 \in \mathbb{H}(x)^{n\times n}$ *and assume that* A_1 *and* B_1 *are invertible. Then* $A_1x + A_0$ *and* $B_1x + B_0$ *are equivalent if*

$$P(A_1x + A_0)Q = B_1x + B_0$$

for some constant invertible matrices $P, Q \in \mathbb{H}(x)^{n\times n}$.

1.3.2 Nonuniqueness of the Smith form

Due to the non-commutative properties, the Smith form for a given quaternion polynomial matrix is not unique. The two scalar polynomials $a(x), b(x) \in \mathbb{H}(x)^{1\times 1}$ are called H-similar if and only if there exist $\alpha \in \mathbb{H}\backslash\{0\}$ such that $\alpha^{-1}a(x)\alpha = b(x)$ for all real x.

Theorem 1.11. *If* $A(x) \in \mathbb{H}(x)^{m\times n}$ *exist Smith forms*

$$diag(a_1(x), a_2(x), \cdots, a_r(x), 0, \cdots, 0)$$

and

$$diag(b_1(x), b_2(x), \cdots, b_r(x), 0, \cdots, 0)$$

and the $a_j(x)$ *and* $b_j(x)$ *are polynomial with real coefficients, then* $a_j(x) = b_j(x)$ *for* $j = 1, 2, \cdots, r$.

Theorem 1.12. *Two real matrix polynomial are R-equivalent if and only if they are H-equivalent.*

1.3.3 Statement of the Kronecker form

In this part, the definitions and properties of the Kronecker form of a pair of quaternion matrices are given.

The matrix pencils $A_j + xB_j, j = 1, 2$ are said to be strictly equivalent if and only if

$$A_1 = PA_2Q, B_1 = PB_2Q$$

for some invertible quaternion matrices $P \in \mathbb{H}^{m\times m}$ and $Q \in \mathbb{H}^{n\times n}$.

Theorem 1.13. *If every pencil* $A + xB \in \mathbb{H}(x)^{m\times n}$ *is strictly equivalent to a matrix pencil, the block diagonal form as following:*

$$\begin{aligned}
&0_{u\times v} \oplus L_{\varepsilon_1\times(\varepsilon_1+1)} \oplus \cdots \oplus L_{\varepsilon_1\times(\varepsilon_1+1)} \oplus L^T_{\eta_p\times(\eta_p+1)} \oplus L^T_{\eta_q\times(\eta_q+1)} \\
&\oplus (I_{k_1} + xJ_{k_1}(0)) \oplus (I_{k_r} + xJ_{k_r}(0)) \\
&\oplus (xI_{l_1} + J_{l_1}(\alpha_1)) \oplus \cdots \oplus (xI_{l_s} + J_{l_s}(\alpha_s))
\end{aligned}$$

where $\varepsilon_1 \le \cdots \le \varepsilon_p; \eta_1 \le \cdots \le \eta_q;\ k_1 \le \cdots \le k_r$ *are positive integers, and* $\alpha_1, \cdots, \alpha_s \in \mathbb{H}$.

The integers $\varepsilon_1 \le \cdots \le \varepsilon_p$ and $\eta_1 \le \cdots \le \eta_q$ are said to be the left indices and the right indices, respectively of $A + xB$. The integers $k_1 \le \cdots \le k_r$ are said to be the indices, or partial multiplicities, at infinity of $A + xB$. The quaternions $-\alpha_1, \cdots, -\alpha_s$ are said to be the eigenvalues of $A + xB$.

The part

$$0_{u\times v} \oplus L_{\varepsilon_1\times(\varepsilon_1+1)} \oplus \cdots \oplus L_{\varepsilon_p\times(\varepsilon_p+1)} \oplus L^T_{\eta_p\times(\eta_p+1)} \oplus L^T_{\eta_q\times(\eta_q+1)}$$

is termed the singular part, and

$$(I_{k_1} + xJ_{k_1}(0)) \oplus (I_{k_r} + xJ_{k_r}(0)) \oplus (xI_{l_1} + J_{l_1}(\alpha_1)) \oplus \cdots \oplus (xI_{l_s} + J_{l_s}(\alpha_s))$$

is the regular part.

For a fixed eigenvalue α of $A + xB$, let $i_1 < \cdots < i_w$ be all the subspace such that $\alpha_{i_1}, \cdots, \alpha_{i_w}$ are said to be the indices, or partial multiplicities, of the eigenvalue α of $A+xB$. There exist several indices at infinity that are equal to a fixed positive interger; the same comment applies to the indices of fixed eigenvalue $A + xB$, to the right indices of $A + xB$, and to the left indices of $A + xB$.

Chapter 2

Polynomial matrices

The theory of polynomial matriices is the main base of the complex frequency domain theory of linear systems. This chapter introduces some properties which will be used in Chapter 3. Many theorems and results of this chapter are well-known and can be found in, for example, come from the book[49].

2.1 Some properties of polynomial matrices

A polynomial matrix is a matrix and each entry of this matrix is a polynomial. Let $q_{ij}(x) \in \mathbb{R}(x), i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$, the polynomial matrices with element $q_{i,j}(x)$ is:

$$Q(x) = \begin{bmatrix} q_{11}(x) & \cdots & q_{1n}(x) \\ \vdots & & \vdots \\ q_{m1}(x) & \cdots & q_{mn}(x) \end{bmatrix}. \tag{2.1}$$

For a square matrix $Q(x)$, it is called singular if $\det(Q(x)) = 0$. Otherwise, it is nonsingular.

Example 2.1. *Two polynomial matrices are given as following:*

$$Q_1(x) = \begin{bmatrix} x+1 & 1 \\ x^2-1 & x-1 \end{bmatrix},$$

and

$$Q_2(x) = \begin{bmatrix} x+1 & 1 \\ x & x-1 \end{bmatrix}.$$

It is clear that,

$$\det Q_1(x) = (x+1)(x-1) - (x^2-1) = 0$$

$$\det Q_2(x) = (x+1)(x-1) - x = x^2 - x - 1$$

So, the $Q_1(x)$ is singular and $Q_2(x)$ is nonsingular.

The definitions of linear dependence and linear independence are given.

Definition 2.1. *The polynomial vector group $\{q_1(x), q_2(x), \ldots, q_m(x)\}$ is linear dependence if and only if there exists a set of polynomials that are not all zeros $\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_m(x)\}$ satisfied:*

$$\alpha_1(x)q_1(x) + \alpha_2(x)q_2(x) + \cdots + \alpha_m(x)q_m(x) = 0 \tag{2.2}$$

The polynomial vector group $\{q_1(x), q_2(x), \ldots, q_m(x)\}$ is linear independence if and only if there is not a set of incomplete zeros of polynomials

$$\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_m(x)\}$$

that satisfies equation (2.2).

Example 2.2. *A given line 2-d polynomial vector :*

$$q_1(x) = \begin{bmatrix} x+1 & x-2 \end{bmatrix}, \; q_2(x) = \begin{bmatrix} x^2+3x+2 & x^2-4 \end{bmatrix}$$

The polynomial is selected $\alpha_1(x) = x + 2, \alpha_2(x) = -1,$ *so*

$$\begin{aligned}&\alpha_1(x)q_1(x) + \alpha_2(x)q_2(x)\\ &= \begin{bmatrix} x^2+x+2 & x^2-4 \end{bmatrix} - \begin{bmatrix} x^2+3x+2 & x^2-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}\end{aligned}$$

According to the definition, $q_1(x)$ *and* $q_2(x)$ *are linear dependent.*

Rank is the most fundamental property of matrices and the definition of polynomial matrices rank is given below. For the polynomial matrix:

$$Q(x) = \begin{bmatrix} q_{11}(x) & \cdots & q_{1n}(x) \\ \vdots & & \vdots \\ q_{m1}(x) & \cdots & q_{mn}(x) \end{bmatrix} \tag{2.3}$$

Definition 2.2. *The rank of the polynomial matrix* $Q(x)$ *is the* r $(rank(Q(x)) = r)$*. If there is at least one minor order* $r \times r$ *that is not identical to zero, and all minors that are equal to or greater than the order* $r \times r$ *are identical to zero.*

Example 2.3. *A given line* 2×2 *polynomial matrix:*

$$Q(x) = \begin{bmatrix} x+1 & x \\ x^2-1 & x^2-x \end{bmatrix}$$

It is obvious that all the elements of $Q(x)$ *of the* 1×1 *order sub type are not equal to zero, but* $Q(x)$ *of the* 2×2 *order sub type is*

$$Q(x) = \det \begin{bmatrix} x+1 & x \\ x^2-1 & x^2-x \end{bmatrix} \equiv 0.$$

Thus the rank of $Q(x)$ *is 1, that is,* $rank(Q(x)) = 1$.

Proposition 2.1. *The following properties can be easily verified:*

1. *For any nonzero $m \times n$ polynomial matrix $Q(x)$:*

$$1 \leq rank(Q(x)) \leq \min(m, n)$$

2. *For any nonzero $m \times n$ polynomial matrix $Q(x)$:*

$$Q(x) \quad full \quad rank \Leftrightarrow rank(Q(x)) = \min(m, n)$$

3. *For any m-d polynomial vector $q_1(x), q_2(x), \cdots, q_n(x), n \leq m$:*

$$\begin{aligned} & q_1(x), q_2(x), \cdots, q_n(x) \quad linear \quad dependence \\ & \Leftrightarrow rank[q_1(x), q_2(x), \cdots, q_n(x)] = n \\ & q_1(x), q_2(x), \cdots, q_n(x) \quad linear \quad independence \\ & \Leftrightarrow rank[q_1(x), q_2(x), \cdots, q_n(x)] < n \end{aligned} .$$

4. *For any nonzero $n \times n$ polynomial matrix $Q(x)$:*

$$\begin{aligned} & Q(x) \quad is \quad nonsingular \Leftrightarrow rank(Q(x)) = n \\ & Q(x) \quad is \quad singular \Leftrightarrow \; rank(Q(x)) < n \end{aligned} .$$

5. *For any nonzero $m \times n$ polynomial matrix $Q(x)$, any nonsingular $m \times m$ matrix $P(x)$ and $n \times n$ matrix $R(x)$,*

$$rank(Q(x)) = rankP(x)Q(x) = rankQ(x)R(x).$$

6. *Rank of polynomial product: Suppose any non zero polynomial matrix $Q(x)$ and $R(x)$,*

$$rankQ(x)R(x) \leq \min\left(rank(Q(x)), rank(R(x))\right).$$

The unimodular matrix is a class of important polynomial matrices. The $n \times n$

polynomial matrix $Q(x)$ is unimodular matrix if and only if $\det(Q(x)) = c$, where c is non zero constant.

Example 2.4. *For 2×2 polynomial matrix:*

$$Q(x) = \begin{bmatrix} x+2 & 1 \\ x^2+5 & x-2 \end{bmatrix}$$

It is obvious that $\det Q(x) = (x+2)(x-2) - x^2 + 5 = 1$. Then the $Q(x)$ is a unimodular matrix. Furthermore

Proposition 2.2. *The $n \times n$ polynomial matrix $Q(x)$ is unimodular matrix, if and only if $Q^{-1}(x)$ is unimodular matrix.*

Proof: (1) Necessity. $Q(x)$ is known as an unimodular matrices, which proves that $Q^{-1}(x)$ is a polynomial matrices.

According to the definition of unimodular matrices, because the matrix $Q(x)$ is an unimodular matrices, we have $\det Q(x) = c \neq 0$. So,

$$Q^{-1}(x) = \frac{adjQ(x)}{\det Q(x)} = \frac{1}{c} adjQ(x)$$

where adjoint matrix $adj(Q(x))$ is polynomial matrices. The matrix $Q^{-1}(x)$ is the polynomial matrices.

(2) Sufficiency. $Q^{-1}(x)$ is known as a polynomial matrices, which proves that $Q(x)$ is an unimodular matrices.

Because the matrix $Q^{-1}(x)$ and $Q(x)$ are polynomial matrices, suppose $\det Q(x) = a(x)$, $\det Q^{-1}(x) = b(x)$, where $a(x)$ and $b(x)$ are the polynomial matrices. Because $Q^{-1}(x)Q(x) = I$,

$$\det Q(x) \det Q^{-1}(x) = a(x)b(x) = I \tag{2.4}$$

The equation above is true if and only if $a(x)$ and $b(x)$ is non zero constant. Therefore, $\det Q(x) = a \neq 0$, that is to say $Q(x)$ is an unimodular matrices. Elementary transformation plays an important role in matrices over number fields, and the elementary transformation has three forms. Similarly, the elementary transformation of the polynomial matrices has three forms. This section is a brief discussion of the elementary transformation of the polynomial matrices. The type function, realization and attribute of elementary transformation are mainly introduced.

The first elementary transformation is the two row or two column of the exchange polynomial matrices $Q(x)$. Also, the row exchange is the first row elementary transformation, and the column exchange is the first column elementary transformation.

For the first elementary transformation of $m \times n$ polynomial matrix, it is necessary to introduce the corresponding elementary matrices $m \times m$ E_{1r} and $n \times n$ E_{lc}, and

$$\text{Exchange } Q(x) \text{ two rows } \bar{Q}_{1r}(x) = E_{1r}Q(x)$$
$$\text{Exchange } Q(x) \text{ two columns } \bar{Q}_{1c}(x) = E_{1c}Q(x)$$

The generation of elementary matrices should follow the rules:

$m \times m$ row elementary matrices E_{1r} = If the matrix $Q(x)$ is exchanged with row i and j, the E_{1r} is the constant matrix for the I_m exchange column i and j; $n \times n$ column elementary matrices E_{1c} = If the matrix $Q(x)$ is exchanged with column i and j, the E_{1c} is the constant matrix for the I_n exchange row i and j.

Clearly, the following properties of elementary matrices hold:

Corollory 2.1. *(i) For the elementary matrix E_1, the inverse of E_1 must exists and $(E_1)^{-1} = E_1$.*

(ii) The elementary matrix E_1 is an unimodular matrix.

Example 2.5. *For* 4×3 *polynomial matrix* $Q(x)$*:*

$$Q(x) = \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

The matrix $Q(x)$ *is exchanged with row* 2 *and* 3*, so*

$$E_{1r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore,

$$\bar{Q}_{1r}(x) = E_{1r}Q(x)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2+5 & x+6 & x & x^2+2x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \end{bmatrix}$$

The matrix $Q(x)$ *is exchanged with column* 2 *and* 3*, so*

$$E_{1c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\bar{Q}_{1c}(x) = Q(x)E_{1c}$$

$$= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & x^2+1 & x-5 & x^2+x+3 \\ x^2-x+1 & x^2+3x+2 & x+2 & x-4 \\ x^2+5 & x & x+6 & x^2+2x+3 \end{bmatrix}$$

The function of the second elementary transformation is to multiply the $Q(x)$ of a row or a column with a nonzero constant c, and c multiplied by the row is called second kinds of elementary row transformation, and c multiplied by the column is called second kinds of elementary column transformation.

For the second elementary transformation of $m \times n$ polynomial matrices, it is necessary to introduce corresponding elementary matrices $m \times m$ E_{2r} and $n \times n$ E_{2c}, and

$$\text{The row of } Q(x) \text{ multiplied by the } c \; \bar{Q}_{2r}(x) = E_{2r}Q(x)$$
$$\text{The column of } Q(x) \text{ multiplied by the } c \; \bar{Q}_{2c}(x) = E_{2c}Q(x)$$

The generation of elementary matrices should follow the rules:
$m \times m$ row elementary matrices E_{2r} = If the row i of matrix $Q(x)$ is multiplied by the c, the E_{2r} is the column i of constant matrix I_m is multiplied by the c; $m \times m$ column elementary matrices E_{2c} = If the row i of matrix $Q(x)$ is multiplied by the c, the E_{2c} is the row i of constant matrix I_n is multiplied by the c. In order to keep the inverse, the constant c can not be changed to a non-zero polynomial.

Corollory 2.2. *(i) For the elementary matrix E_2, the inverse of E_2 must exists and $(E_2)^{-1} = E_2$ with the $\frac{1}{c}$ replace to the c.*

(ii) The elementary matrix E_2 is an unimodular matrices.

Example 2.6. *For 4×3 polynomial matrix $Q(x)$:*

$$Q(x) = \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

The matrix $Q(x)$ is multiplied by 2 for row 2, so

$$E_{2r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\underline{C_2 \leftarrow 2C_2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\bar{Q}_{2r}(x) = Q(x)E_{2r}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ 2x^2-2x+2 & 2x+4 & 2x^2+6x+4 & 2x-8 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

The matrix $Q(x)$ is multiplied by 2 for column 2, so

$$E_{2c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\underline{R_2 \leftarrow 2R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\bar{Q}_{2c}(x) = Q(x)E_{2c}$$

$$= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & 2x-10 & x^2+1 & x^2+x+3 \\ x^2-x+1 & 2x+4 & x^2+3x+2 & x-4 \\ x^2+5 & 2x+12 & x & x^2+2x+3 \end{bmatrix}$$

The function of the third elementary transformation is to multiply the $Q(x)$ of a row or a column with a nonzero polynomial $d(x)$ and add it to row or a column, where corresponding row transformation is called the third kinds of elementary row transformation, and the corresponding column transformation is called the third kinds of elementary column transformation.

For the third elementary transformation of $m \times n$ polynomial matrices, it is necessary to introduce corresponding elementary matrices $m \times m$ E_{3r} and $n \times n$ E_{3c}, and

The polynomial matrix $Q(x)$ on row operation $\bar{Q}_{3r}(x) = E_{3r}Q(x)$

The polynomial matrix $Q(x)$ on column operation $\bar{Q}_{3c}(x) = E_{3c}Q(x)$

(c). Generation of elementary matrix E_{3r} and E_{3c}.

The generation of elementary matrices has the following rules:

$m \times m$ row elementary matrix E_{3r} = If the matrix $Q(x)$ row i is multiplied by the $d(x)$ and then added to the row j, the E_{3r} is the $d(x)$ into the matrix $I_m(j,i)$; $n \times n$ row elementary matrix E_{3c} = If the matrix $Q(x)$ column i is multiplied by the $d(x)$ and then added to the column j, the E_{3c} is the $d(x)$ into the matrix $I_m(i,j)$;

Corollary 2.3. *(i) For the elementary matrix E_3, the inverse of E_3 must exists and $(E_3)^{-1} = E_3$ with the $-d(x)$ replace the $d(x)$.*

(ii) The elementary matrix E_3 is an unimodular matrix.

Example 2.7. *For 4×3 polynomial matrix $Q(x)$:*

$$Q(x) = \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

The matrix $Q(x)$ is multiplied by x for row 1 and add row 2, so

$$E_{3r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\underline{C_1 \leftarrow xC_1 + R_2}} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\bar{Q}_{3r}(x) = Q(x)E_{3r}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ 2x^2+1 & x^2-4x+2 & x^3+x^2+4x+2 & x^3+x^2+4x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix}$$

The matrix $Q(x)$ is multiplied by x for column 1 and add column 3 , so

$$E_{3c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\underline{R_1 \leftarrow xR_1 + C_3}} \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$
\begin{aligned}
\bar{Q}_{3c}(x) &= Q(x)E_{3c} \\
&= \begin{bmatrix} x+1 & x-5 & x^2+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^2+3x+2 & x-4 \\ x^2+5 & x+6 & x & x^2+2x+3 \end{bmatrix} \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} x+1 & x-5 & 2x^2+x+1 & x^2+x+3 \\ x^2-x+1 & x+2 & x^3+4x+2 & x-4 \\ x^2+5 & x+6 & x^3+6x & x^2+2x+3 \end{bmatrix}
\end{aligned}
$$

2.2 Canonical forms of polynomial matrices

In 1851, Hermite firstly put forward a canonical form of polynomial matrices in [17] that is called Hermite forms now. Since then, One of the successes of computer algebra over the past three decades has been the development of fast algorithms for computing Hermite canonical forms. Havas[16] considered some algorithms that various different strategies have been proposed for computing the Hermite normal form of integer matrices. Domich[8] described a new class of Hermite normal form solution procedures which perform modulo determinant arithmetic throughout the computation. Micciancio and Warinschi[22] proposed a linear space algorithm for computing the Hermite normal form.

The Popov form also is a important canonical form of polynomial matrices. In particular, Popov form has a wide range of applications for Ore polynomial matrices[13, 7].

2.2.1 The Hermite forms and related results

The Hermite forms is a canonical form of the polynomial matrices. The Hermite forms can be divided into rows and columns Hermitian forms. The function of the canonical form of a polynomial matrix is a prominent feature of the matrix.

Any polynomial matrices can be transformed into the Hermite forms through a series of elementary transformations and unimodular transformations. Here, the row Hermite forms and column Hermite forms are given.

Definition 2.3. *Considering an $m \times n$ polynomial matrix $Q(x)$, $rank(Q(x)) = r \leq \min\{m, n\}$, its row Hermite form is defined as:*

$$H_{Hr}(x) = \begin{bmatrix} 0 & \cdots & 0 & a_{1,k_1}(x) & \cdots & a_{1,k_2}(x) & \cdots & a_{1,k_3}(x) & \cdots & a_{1,k_r}(x) & \cdots \\ \vdots & & \vdots & & & a_{2,k_2}(x) & \cdots & a_{2,k_3}(x) & \cdots & a_{2,k_r}(x) & \cdots \\ \vdots & & \vdots & & & & & a_{3,k_3}(x) & \cdots & a_{3,k_r}(x) & \cdots \\ \vdots & & \vdots & & & & & & & \vdots & \vdots \\ 0 & \cdots & 0 & & & & & & & a_{r,k_r}(x) & \cdots \\ 0 & \cdots & 0 & & \cdots & \cdots & \cdots & & & & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & & & & \cdots & \cdots & \cdots & & & & 0 \end{bmatrix}$$

i. The front r rows of $H_{Hr}(x)$ are non zero rows, after $(m - r)$ rows are zero rows.

ii. In every non zero rows, located in the leftmost nonzero element a_{i,k_i} is a monic polynomial.

ii. The most left non-zero element of the $H_{Hr}(x)$ is presented as a ladder type.

iv. The number of polynomials of most left non-zero element a_{i,k_i} of the $H_{Hr}(x)$ is greatest, compared with other elements in the same column.

Definition 2.4. *Considering an $m \times n$ polynomial matrix $Q(x)$, $rank(Q(x)) = r \leq \min\{m, n\}$, the column Hermite forms is $H_{Hc}(x) = H_{Hr}^T(x)$.*

Given an $m \times n$ polynomial matrix $Q(x)$, the row Hermite forms $Q_{Hr}(x)$ for Hermitian forms can be achieved by an appropriate $m \times m$ unimodular matrix $V(x)$ left-multiplicated $Q(x)$; the row Hermite forms $Q_{Hr}(x)$ for Hermitian forms can be achieved by an appropriate $n \times n$ unimodular matrix $U(x)$ right-multiplicated $Q(x)$, that is,

$$Q_{Hr}(x) = V(x)Q(x)$$
$$Q_{Hc}(x) = Q(x)U(x)$$

Here, the algorithm of computing row Hermite forms is given:

Algorithm: Computing Hermite Form

For the $m \times n$ polynomial matrix $Q(x)$, $rank(Q(x)) = r$.

- Step 1: For a given $Q(x)$, the first column to the $(k-1)$ column is supposed to be zero column, and the k column is the first non zero column.
- Step 2: Suppose $i = 1$.
- Step 3: Through row transformation of the elementary transformation, Q_{ik_i} is changed into the lowest times of elements in k_1 column, and multiplied by a corresponding constant to make it a monic polynomial. It is called $\tilde{q}_{i,k_i}(x)$.
- Step 4: According to the polynomial division, the other elements in k_1 column are expressed

$$item\ of\ \tilde{q}_{i,k_i}(x) + remainder\ of\ \tilde{q}_{i,k_i}(x)$$

- Step 5: According to the row elementary transformation, for all elements in k_1 column, and minus the factor part. Repeat the process above until the following elements of $Q_{i,k_i}(x)$ are 0.
- Step 6: In the i row, search from the k_i column to the right, and suppose $k_i + 1, k_i + 2, \cdots$ as the same change, and the first different change is called k_{i+1}.

- Step 7: Suppose $i+1=i$.
- Step 8: If $i=r$, go into the next Step. Otherwise, go into Step 3.
- Step 9: According to the elementary transformation, suppose $(m-r)$ row to be zero-row.
- Step 10: The result is Hermite forms.

Example 2.8. *Given the following polynomial matrix: is given:*

$$Q(x) = \begin{bmatrix} x & 0 \\ 0 & x \\ 1 & x+3 \end{bmatrix}$$

where rank$(Q(x)) = 2$*. Here the row Hermite form and the corresponding unimodular transformation matrix* $V(x)$ *are calculated:*

$$Q(x) = \begin{bmatrix} x & 0 \\ 0 & x \\ 1 & x+3 \end{bmatrix} \xrightarrow[E_1]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & x+3 \\ 0 & x \\ x & 0 \end{bmatrix}$$

$$\xrightarrow[E_2]{R_3 \leftarrow R_3 - xR_1} \begin{bmatrix} 1 & x+3 \\ 0 & x \\ 0 & -x(x+3) \end{bmatrix} \xrightarrow[E_3]{R_3 \leftarrow R_3 + (x+3)R_2} \begin{bmatrix} 1 & x+3 \\ 0 & x \\ 0 & 0 \end{bmatrix} = Q_{Hr}(x)$$

$$V(x) = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x+3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & x+3 & -x \end{bmatrix}.$$

Hence

$$Q_{Hr}(x) = V(x)Q(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & x+3 & -x \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \\ 1 & x+3 \end{bmatrix} = \begin{bmatrix} 1 & x+3 \\ 0 & x \\ 0 & 0 \end{bmatrix}.$$

Next we consider some properties of Hermite forms.

Proposition 2.3. *Suppose $Q(x)$ is $n \times n$ nonsingular polynomial matrices, $\bar{D}(x) = D(x)R(x)$, where $R(x)$ is any $n \times n$ unimodular matrix, then polynomial matrices $D(x)$ and $\bar{D}(x)$ have the same column Hermite forms.*

Proof: Suppose D_{Hc} and $\bar{D}_{Hc}$ are the column Hermite forms of polynomial matrices $D(x)$ and $\bar{D}(x)$. Then $n \times n$ unimodular matrices $U(x)$ and $\bar{U}(x)$ exist and satisfy:

$$D_{Hc}(x) = D(x)U(x), \bar{D}_{Hc}(x) = \bar{D}(x)\bar{U}(x) \tag{2.5}$$

Based on it, according to the equation $\bar{D}(x) = D(x)R(x)$,

$$\bar{D}_{Hc}(x) = D(x)R(x)\bar{U}(x) = D_{Hc}(x)U^{-1}(x)U(x)\bar{U}(x). \tag{2.6}$$

Suppose $W(x) = U^{-1}(x)R(x)\bar{U}(x)$. Then the $W(x)$ is an unimodular matrices. Hence

$$\bar{D}_{Hc}(x) = D_{Hc}(x)W(x) \tag{2.7}$$

According to the equation(2.7), $W^{-1}(x) = D_{Hc}^{-1}(x)\bar{D}_{Hc}(x)$. Because $D_{Hc}(x)$ and $\bar{D}_{Hc}(x)$ are lower triangular matrix, the $W(x)$ is a lower triangular matrix:

$$W(x) = \begin{bmatrix} c_1 & & & \\ w_{21}(x) & c_2 & & \\ \vdots & & \ddots & \\ w_{n1}(x) & \cdots & w_{n,n-1} & c_n \end{bmatrix} \tag{2.8}$$

$\bar{D}_{Hc}(x) = D_{Hc}(x)W(x)$ has been proved. If $\bar{D}_{Hc}(x) = D_{Hc}(x)$ is to be proved,

$W(x)$ must be proved as a unit matrix . According to the $\bar{D}_{Hc}(x) = D_{Hc}W(x)$,

$$\begin{bmatrix} \bar{d}_{11}(x) & & & \\ \bar{d}_{21}(x) & \bar{d}_{22}(x) & & \\ \vdots & & \ddots & \\ \bar{d}_{n1}(x) & \cdots & \cdots & \bar{d}_{nn}(x) \end{bmatrix}$$

$$= \begin{bmatrix} d_{11}(x) & & & \\ d_{21}(x) & d_{22}(x) & & \\ \vdots & & \ddots & \\ d_{n1}(x) & \cdots & \cdots & d_{nn}(x) \end{bmatrix} \begin{bmatrix} c_1 & & & \\ w_{21}(x) & c_2 & & \\ \vdots & & \ddots & \\ w_{n1}(x) & \cdots & w_{n,n-1}(x) & c_n \end{bmatrix}$$

So,

$$\bar{d}_{ii}(x) = d_{ii}(x)c_i, i = 1, 2, \cdots, n \tag{2.9}$$

and the $\bar{d}_{i,i}(x)$ and $d_{i,i}(x)$ are monic polynomial. Thus the following equation should be proved:

$$c_i = 1, i = 1, 2, \ldots, n. \tag{2.10}$$

So, $w_{ik}(x) = 0, i \neq k$ should be proved. According the equation(2.11) and column Hermite forms, by

$$\begin{cases} \bar{d}_{21}(x) = w_{21}(x)d_{22}(x) + d_{21}(x) \\ \deg \bar{d}_{21}(x) < \deg \bar{d}_{22}(x) = \deg d_{22}(x) \end{cases} \tag{2.11}$$

we can get $w_{21} = 0$. By

$$\begin{cases} \bar{d}_{31}(x) = w_{31}(s)d_{33}(x) + d_{31}(x) \\ \deg \bar{d}_{31}(x) < \deg \bar{d}_{33}(x) = \deg d_{33}(x) \end{cases} \tag{2.12}$$

we can get $w_{31} = 0$. Thus

$$w_{ik}(x) = 0,\ i = 2, \cdots, n,\ k = 1, \ldots, i-1.$$

Therefore $W(x) = I$, that is, $\bar{D}_H(x) = D_H(x)$.

Similarly, we have the following column case:

Proposition 2.4. *Suppose $A(x)$ is a $n \times n$ nonsingular polynomial matrix, $\bar{A}(x) = T(x)A(x)$, $T(x)$ is any $n \times n$ unimodular matrix. Then polynomial matrices $A(x)$ and $\bar{A}(x)$ have the same column Hermite forms.*

2.2.2 The Smith forms and related results

Smith forms is an important canonical form of polynomial matrices. Any polynomial matrices can be transformed into the Smith forms through elementary row operations. In this section, the Smith forms are briefly introduced.

For the $q \times p$ polynomial matrix $Q(x)$, we have $rank(Q(x)) = r, 0 \leq r \leq \min(q, p)$. There exist a unimodular matrix pair $U(x), V(x)$ such that

$$U(x)Q(x)V(x) = \begin{bmatrix} \lambda_1(x) & & & 0 \\ & \ddots & & \vdots \\ & & \lambda_r(x) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.13}$$

where $\{\lambda_i(x), i = 1, 2, \cdots, r\}$ is the nonzero polynomials and satisfy:

$$\lambda_i(x) \,|\lambda_{i+1}(x)\, , i = 1, 2, \cdots, r-1$$

We list one algorithm to compute the Smith forms:

Algorithm: Computing the Smith forms:

For $q \times p$ polynomial matrix $Q(x)$, $rank(Q(x)) = r, 0 \leq r \leq min(q, p)$, the purpose is to construct the Smith forms $\bigwedge(x)$.

- Step 1: If $Q(x) \equiv 0$, the Smith forms $\bigwedge(x) = 0$, go into Step 17. Otherwise, go into the next step.

- Step 2: If $Q(x)$ is equation (2.13), go into Step 12. Otherwise, suppose i=1, go into the next step.

- Step 3: For the non diagonalization of parts, change the lowest time element to position (i, i), suppose it $q_{ii}(x)$.

- Step 4: For the non diagonalization of parts, division by $q_{ii}(x)$,

$$\begin{cases} q_{ij}(x) = q_{ii}(x)p_{ij}(x) + f_{ij}(x) \\ q_{ki}(x) = q_{ii}(x)p_{ki}(x) + f_{ki}(x) \end{cases}$$

  where $j = i+1, \cdots, p, k = i+1, \cdots, q$.

- Step 5: If $f_{ij}(x) = 0$ and $f_{ki}(x) = 0$, go into Step 8. Otherwise, go into the next step.

- Step 6: Find the lowest time element at non zero residue, and call it as $f_{ai}(x)$ and row a - row i $\times p_{ai}(x)$.

- Step 7: If $f_{ij}(x) = 0$ and $f_{ki}(x) = 0$, go into the next step. Otherwise, go into Step 3.

- Step 8:

$$\begin{cases} row \quad k - row \quad i \times \tilde{p}_{ki}(x), k = i+1, \cdots, q \\ column \quad j - column \quad i \times \tilde{p}_{ij}(x), j = i+1, \cdots, p \end{cases}$$

  where $\tilde{p}_{ki}$ and $\tilde{p}_{ij}$ are the ith column and the ith row corresponding elementary, and we can get following polynomial matrix:

$$\begin{bmatrix} \lambda_1^*(x) & & & \\ & \ddots & & \\ & & \lambda_i^*(x) & \\ & & & Q_{i+1}(x) \end{bmatrix}$$

- Step 9: Suppose $i + 1 = i$.

- Step 10: If $i \neq r+1$, go into Step 3. If $i = r+1$, go into the next step.

- Step 11: According to the row and column elementary for polynomial matrix $Q_{r+1}(x)$,

$$\begin{bmatrix} \lambda_1^*(x) & & & & \\ & \lambda_2^*(x) & & & 0 \\ & & \ddots & & \\ & & & \lambda_r^*(x) & \\ & 0 & & & 0 \end{bmatrix} \tag{2.14}$$

- Step 12: If $\{\lambda_i^*(x), i = 1, 2, \cdots, r\}$ are monic polynomial, go into Step 14. Otherwise, go into the next step.

- Step 13: For non monic polynomial, divise the row by the coefficient of first element.

- Step 14: If $\{\lambda_i^*(x), i = 1, 2, \cdots, r\}$ satisfies divisible properties:

$$\lambda_i^*(x)i \left| \lambda_{i+1}^*(x) \right., i = 1, 2, \cdots, r-1$$

- Step 15: Take the elementary transformation for equation(2.14), and make it satisfied with divisible properties, go into Step 17.

- Step 16: Suppose $\lambda_i(x) = \lambda_i^*(x), i = 1, 2, \cdots, r$.

- Step 17: Stop.

Example 2.9. *For the* 2×3 *polynomial matrix* $Q(x)$*, rank*$(Q(x)) = 2$*, now we* *can transformed into Smith shape by elementary row and column transformation.*

$$Q(x) = \begin{bmatrix} x^2+1 & x & 1 \\ 2 & x+1 & x+2 \end{bmatrix}$$

$$\xrightarrow{C_1 \leftrightarrow C_3} \begin{bmatrix} 1 & x & x^2+1 \\ x+2 & x+1 & 2 \end{bmatrix}$$

$$\xrightarrow{C_2 \leftarrow C_2 + (-x)C_1} \begin{bmatrix} 1 & 0 & x^2+1 \\ x+2 & -(x+2)(x+1) & 2 \end{bmatrix}$$

$$\xrightarrow{C_2 \leftarrow C_3 + (-x^2-1)C_1} \begin{bmatrix} 1 & 0 & 0 \\ x+2 & -(x+2)(x+1) & 2-(x^2+1)(x+2) \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 + (-x-2)R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(x+2)(x+1) & -x(x+1)^2 \end{bmatrix}$$

$$\xrightarrow{C_3 \leftarrow C_3 + (-x)C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(x+2)(x+1) & x^2+x \end{bmatrix}$$

$$\xrightarrow{C_3 \leftarrow C_3 + C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(x+2)(x+1) & -2x-2 \end{bmatrix}$$

$$\xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2x-2 & -(x+2)(x+1) \end{bmatrix}$$

$$\xrightarrow{C_3 \leftarrow C_3 - \frac{1}{2}(x+2)C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2x-2 & 0 \end{bmatrix}$$

$$\xrightarrow{C_2 \leftarrow -\frac{1}{2}C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x+1 & 0 \end{bmatrix}$$

The Smith forms have very nice properties. Here we start from invariant polynomials.

Proposition 2.5. *For the $q \times p$ polynomial matrix $Q(x)$, $rank(x) = r, 0 \leq r \leq min(q, p)$, the $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ of the Smith forms are invariant polynomials, that is, let*

$$\Delta_i(x) = \gcd\{i \times i \quad sub \quad type \quad polynomial \quad of \quad Q(x)\}, i = 1, 2, \cdots, r.$$

Then

$$\begin{aligned} &\Delta_0(x) = 1 \\ &\lambda_1(x) = \tfrac{\Delta_1(x)}{\Delta_0(x)} \\ &\lambda_2(x) = \tfrac{\Delta_2(x)}{\Delta_1(x)} \\ &\vdots \\ &\lambda_r(x) = \tfrac{\Delta_r(x)}{\Delta_{r-1}(x)} \end{aligned} .$$

Proposition 2.6. *For the $q \times p$ polynomial matrix $Q(x)$, its Smith form is unique. But the unimodular transform matrices $\{U(x), V(x)\}$ of its Smith form is not unique.*

The $q \times p$ polynomial matrix $Q_1(x)$ and $Q_2(x)$ are called Smith equivalent if only and if $Q_1(x)$ and $Q_2(x)$ have the same Smith forms, denoted by $Q_1(x) \sim Q_2(x)$.

And Smith equivalence has the following characteristics:

Reflexive character: $Q_1(x) \sim Q_2(x) \Leftrightarrow Q_2(x) \sim Q_1(x)$.

Reflexivity: $Q_1(x) \sim Q_1(x)$.

Transitive: $Q_1(x) \sim Q_2(x), Q_2(x) \sim Q_3(x) \Rightarrow Q_1(x) \sim Q_3(x)$.

Theorem 2.1. *The $q \times p$ polynomial matrix $Q_1(x)$ and $Q_2(x)$ are Smith equivalence if and only if $q \times q$ and $p \times p$ unimodular matrices $P(x)$ and $Q(x)$ exist and satisfy:*

$$Q_2(x) = P(x)Q_1(x)T(x).$$

Proof: According to the definition of Smith equivalence, we can get:

$$\Lambda(x) = U_1(x)Q_1(x)V_1(x) = U_2(x)Q_2(x)V_2(x) \tag{2.15}$$

where the $U_i(x)$ and $V_i(x)$ $i = 1, 2$ are unimodular matrices. Multiple equation(2.15) left by $U^{-1}(x)$ and right by $V^{-1}(x)$, we can get:

$$Q_2(x) = U_2^{-1}(x)U_1(x)Q_1(x)V_1(x)V_2^{-1}(x).$$

Set

$$P(x) = U_2^{-1}(x)U_1(x), T(x) = V_1(x)V_2^{-1}(x).$$

It is obvious that $P(x)$ and $T(x)$ are unimodular matrices.

2.2.3 The Popov forms and related results

The Popov forms is also called the polynomial-echelon form, which is a canonical form of polynomial matrices.

Definition 2.5. *The $p \times p$ polynomial matrix*

$$D_E(x) = \begin{bmatrix} d_{11}(x) & \cdots & d_{1p}(x) \\ \vdots & & \vdots \\ d_{p1}(x) & \cdots & d_{pp}(x) \end{bmatrix}$$

is a Popov form if

1. *$D_E(x)$ is column-reduce, and $k_{c1} \leq k_{c2} \leq \cdots \leq k_{cp}$.*

2. *For jth column $j = 1, 2, \cdots, p$, the principal index $m_j \in [1, 2, \cdots, p]$ exists, and the principal element $d_{m_j^j}(x)$ satisfies the following condition:*
   a. $\deg \left[d_{m_j^j}(x) \right] = k_{cj}$.
   b. $d_{m_j^j}(x)$ *is a monic polynomial.*

c. $\deg\left[d_{ij}(x)\right] < k_{cj}, \forall i > m_j$.

d. For ith column and jth column , if $i < j$ and $k_{ci} = k_{cj}$, so $m_i < m_j$.

e. $\deg\left[d_{m_j^q}(x)\right] < k_{cj}, \forall q \neq j$.

Example 2.10. *For 3×3 polynomial matrix*

$$D(x) = \begin{bmatrix} 5x+1 & \boxed{x^2+3x+2} & 4x+6 \\ 3x+4 & 2x+1 & \boxed{x^3+x^2+2} \\ \boxed{x+7} & 3 & 5 \end{bmatrix}$$

The following judgement $D(x)$ is Popov form:

1. *According to the $k_{c1} = 1, k_{c2} = 2, k_{c3} = 3$ and $\deg\det(D(x)) = 6$, and it satisfies the condition 1.*

2. *The principal elements are*

$$d_{m_1 1} = x+7, d_{m_2 2} = x^2+3x+2, d_{m_3 3} = x^3+x^2+2$$

   Thus the principal indexes are

$$m_1 = 3, m_2 = 1, m_3 = 2$$

   a. $\deg(d_{31}(x)) = 1 = k_{c1}, \deg(d_{12}(x)) = 2 = k_{c2}, \deg(d_{23}(x)) = 3 = k_{c3}$.

   b. $d_{31}(x), d_{12}(x), d_{23}(x)$ are monic polynomials.

   c. $\deg\left[d_{ij}(x)\right] < k_{cj}, \forall i > m_j$.

   d. $k_{c1} = 1 \neq k_{c2} = 2 \neq k_{c3} = 3$, $k_{ci} \neq k_{cj}, i < j$ does not exist.

   e. It satisfies with the condition that $\deg\left[d_{m_j q}(s)\right] < k_{cj}, \forall q \neq j$.

Hence the polynomial matrix $D(x)$ is Popov form.

Theoretically, the Popov form $D_E(x)$ of any polynomial $D(x)$ can be achieved though right multiplicative appropriate dimension unimodular matrices.

Theorem 2.2. *For $p \times p$ polynomial matrix $\bar{D}(x)$, there exist the Popov forms $D_E(x)$ and unimodular matrix $U(x)$ such that $\bar{D}(x)U(x) = D_E(x)$.*

Here is one algorithm for computing Popov forms:

Algorithm: Computing Popov forms

For a given polynomial matrix $D(x)$,

- Step 1: Judge the column-reduce and the number of non reducing column. If it is column-reduce and the number of non reducing column, go into Step 3. Otherwise, go into Step 2.

- Step 2: Translate $\bar{D}(x)$ into column-reduce and the number of non reducing column. Introduce the $p \times p$ unimodular matrix $V(x)$, make $D(x) = \bar{D}(x)V(x)$ a column-reduce and the number of non reducing column.

- Step 3: Calculate the number of column $D(x)$, $k_{ci}, i = 1, 2, \cdots, p$, $L = \max\{k_{c1}, k_{c2}, \cdots, k_{cp}\}$,

$$\beta(x) = \begin{bmatrix} D(x) & \cdots & x^L D(x) & -I_p & \cdots & -x^L I_p \end{bmatrix}$$

  find the first correlation column at $\begin{bmatrix} -I_p & \cdots & -x^L I_p \end{bmatrix}$, let

$$b_{\beta_1}(x), b_{\beta_2}(x), \cdots, b_{\beta_p}(x).$$

- Step 4: Introduce linear combination equation with constant coefficients p first correlation column:

$$\begin{array}{l} a_{11}b_1(x) + a_{21}b_2(x) + \cdots + a_{2(L+1)p,1}b_{2(L+1)p,1}(x) = 0 \\ \cdots \\ a_{1p}b_1(x) + a_{2p}b_2(x) + \cdots + a_{2(L+1)p,p}b_{2(L+1)p,p}(x) = 0 \end{array}.$$

- Step 5: According to the equation,

$$A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1p} \\ \vdots & & \vdots \\ \alpha_{2(L+1)p,1} & \cdots & \alpha_{2(L+1)p,p} \end{bmatrix}$$

  block it by introducing the $p \times p$ matrix:

$$A = \begin{bmatrix} \bar{U}_0 \\ \vdots \\ \bar{U}_L \\ E_0 \\ \vdots \\ E_L \end{bmatrix}, \bar{U}_i, E_i \in \Re^{p \times p}.$$

- Step 6:

$$\begin{aligned} \bar{U}(x) &= \bar{U}_L x^L + \cdots + \bar{U}_1 x + \bar{U}_0 \\ E(x) &= E_L x^L + \cdots + E_1 x + E_0 \end{aligned}.$$

- Step 7: If $E(x)$ is Popov forms, suppose $E(x) = D_E(x)$ and $\tilde{U}(x) = \bar{U}(x)$, go into Step 9. If $E(x)$ is quasi Popov forms, go into Step 8.

- Step 8: According to characteristic of Popov forms, we can get $D_E(x)$ and $\bar{U}(x)$.

- Step 9: If $\bar{D}(x)$ is the column-reduce and the number of non reducing column, then $U(x) = \tilde{U}(x)$. Otherwise, $U(x) = V(x)\tilde{U}(x)$.

- Step 10: Stop.

2.2.4 The Kronecker forms and related results

Matrix pencil is a special kind of polynomial matrices. The matrix E and A are $m \times n$ real regular matrices, $x \in \mathbb{R}$. Then matrix pencil is defined as

$\triangleq (xE - A)$.

The $m \times n$ matrix pencil $(xE - A)$ is regular if and only if it satisfies:

$$m = n \quad \textit{and} \quad \det(xE - A) \neq 0$$

If the matrix pencil is singular if and only if it is not regular.

For any $m \times n$ matrix pencil $(xE - A)$, Kronecker forms can be get by means of some suitable non singular transformation matrices $U_{m\times m}$ and $V_{n\times n}$.
The format of Kronecker forms is as following:

$$K(x) = U(xE - A)V = \begin{bmatrix} L_{u_1} & & & & & & & \\ & \ddots & & & & & & \\ & & L_{u_a} & & & & & \\ & & & \tilde{L}_{v_1} & & & & \\ & & & & \ddots & & & \\ & & & & & \tilde{L}_{v_\beta} & & \\ & & & & & & xJ - I & \\ & & & & & & & xI - F \end{bmatrix}$$

where the set $\left\{F, J, \{L_{u_i}\}, \left\{\tilde{L}_{v_j}\right\}\right\}$ is unique, and

(i) F is in Jordan form, for example,

$$F = \begin{bmatrix} a & 1 & & & & \\ & a & 1 & & & \\ & & a & 1 & & \\ & & & a & & \\ & & & & b & 1 \\ & & & & & b \end{bmatrix}.$$

(ii) J is in zero eigenvalue Jordan form, for example,

$$J = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

(iii) The form of L_{u_i} is $u_i \times (u_i + 1)$ matrix as following:

$$L_{u_i} = \begin{bmatrix} x & -1 & & & \\ & x & -1 & & \\ & & \ddots & \ddots & \\ & & & x & -1 \end{bmatrix}.$$

(iv) The form of $\tilde{L_{v_j}}$ is $(v_j + 1) \times v_j$ matrix as following:

$$\tilde{L_{v_j}} = \begin{bmatrix} x & & & \\ -1 & x & & \\ & -1 & \ddots & \\ & & \ddots & x \\ & & & -1 \end{bmatrix}.$$

Proposition 2.7. *(Right singularity)*

Kronecker forms $K(x)$, $\{L_{u_i}, i = 1, 2, \ldots, \alpha\}$ corresponds to the right singularity of the reflection matrix pencil $(xE - A)$, and $\{u_1, u_2, \ldots, u_\alpha\}$ is called the right Kronecker index.

Proof: For $u_i \times (u_i + 1)$ matrix $L_{u_i}, i = 1, 2, \ldots, \alpha$, we can get:

$$\begin{bmatrix} x & -1 & & & \\ & x & -1 & & \\ & & \ddots & \ddots & \\ & & & x & -1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{u_i} \end{bmatrix} = 0,$$

where $\begin{bmatrix} 1 & x & \cdots & x^{u_i} \end{bmatrix}^T$ is minimum number of polynomial vectors for above equation. Then $n \times 1$ polynomial vector $f_i(x)$ are constructed as follows:

$$f_i(x) = V \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ x \\ \vdots \\ x^{u_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \textit{group} \quad \text{i}, \quad i = 1, 2, \ldots, \alpha. \\ \\ \\ \\ \end{matrix}$$

According to the $K(x) = U(xE - A)V$, we can get:

$$(xE - A)f_i(x) = U^{-1}K(x)V^{-1}(V \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ x \\ \vdots \\ x^{u_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix}) = 0, \quad i = 1, 2, \ldots, \alpha.$$

is minimum number of polynomial vectors for equation above. This indicates that the right singularity of $(sE - A)$ can be reflected by $\{L_{u_i}, i = 1, 2, \cdots, \alpha\}$.

Similarly, we have left singularity. Kronecker forms $K(s)$, $\{\tilde{L}_{v_j}, j = 1, 2, \ldots, \beta\}$ corresponds to the right singularity of the reflection matrix pencil $(sE - A)$, and $\{v_1, v_2, \ldots, v_\alpha\}$ is called the left Kronecker index.

Chapter 3

Quaternion Polynomial Matrices

In this chapter, we consider some normal forms for quaternion polynomial matrices. The definitions and algorithms of greatest common right divisors (GCRDs) and least common left multiples (LCLMs) for quaternion polynomials are explored in section 1. Using these one sided GCRDs and LCLMs, we define a special row/column transformation in section 2. From section 3 to 5 we discuss some algorithms for computing canonical forms of quaternion polynomial matrices including the Hermite form, the Smith form and the Popov form.

3.1 GCRDs and LCLMs

In this section, the definitions of greatest common right divisor(GCRD) and least common left multiple(LCLM) at the quaternion polynomial. Some results can be found in, for example, [6].

Recall that let f, g be quaternion polynomials. The greatest common right divisor s, written as $GCRD(f, g)$, is defines as:

(a) $f = f_1 s$ and $g = g_1 s$ for some quaternion polynomial f_1 and g_1.

(b) If t is a common right divisor of f and g, then t is a right divisor of s.

Let f, g be quaternion polynomials. The least common left multiple s, written as $LCLM(f, g)$, is defined as

(i) there exist k_1 and l_1 satisfying $k_1 g = l_1 f$.

(ii) $\text{GCLD}(k_1, l_1) = 1$.

We first introduce Euclidean algorithm for one side case.

Lemma 3.1. *Let $F(x)$ be a quaternion polynomial and let $\alpha \in \mathbb{H}$. Then there exists a $Q \in \mathbb{H}[x]$ such that $F(x) = (x - \alpha) * Q(x) + F(\alpha)$.*

Lemma 3.2. *Let F and G be nonzero quaternion polynomials. Then there exist $A, B \in \mathbb{H}[x]$ such that $GCLD(F, G) = F * A + G * B$.*

Next we introduce two matrix operations for quaternion polynomial matrices. One is to transform the column of the matrix by GCRDs and another is to transform the column of the matrix by LCLMs. Specific examples are given as following:

Example 3.1. *Let's talk about 2×2 quaternion polynomial matrices. Take*

$$A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}.$$

We can compute $g_1 = GCRD(a_{11}(x), a_{21}(x))$, and then find quaternion polynomials t_1 and s_1 such that:

$$s_1 a_{11}(x) + t_1 a_{21}(x) = g_1.$$

Furthermore, we can compute quaternion polynomials k_1 and l_1 such that

$$k_1 a_{11}(x) = l_1 a_{21}(x) = LCLM\left(a_{11}(x), a_{21}(x)\right).$$

Then we set

$$E_1 = \begin{bmatrix} s_1 & t_1 \\ k_1 & -l_1 \end{bmatrix}.$$

Therefore

$$E_1A = \begin{bmatrix} s_1 & t_1 \\ k_1 & -l_1 \end{bmatrix} \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} = \begin{bmatrix} g_1 & s_1a_{12}(x) + t_1a_{22}(x) \\ 0 & k_1a_{12}(x) - l_1a_{22}(x) \end{bmatrix}.$$

In such way, we could convert the elements of the first column of the matrix A to all $0s$.

Similarly, we can construct a column operation. Set

$$A = \begin{bmatrix} b_{11}(x) & b_{12}\,(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}.$$

Compute $g_2 = GCRD(b_{11}(x), b_{12}(x))$, *and calculate quaternion polynomials* t_2 *and* s_2 *such that:*

$$b_{11}(x)s_2 + b_{12}(x)t_2 = g_2.$$

Moreover, find quaternion polynomials k_2 *and* l_2 *such that*

$$b_{11}(x)k_2 = b_{12}(x)l_2 = LCLM\,(b_{11}(x), b_{12}(x))\,.$$

Next, let

$$F_1 = \begin{bmatrix} s_2 & k_2 \\ t_2 & -l_2 \end{bmatrix}.$$

Then

$$AF_1 = \begin{bmatrix} s_2 & k_2 \\ t_2 & -l_2 \end{bmatrix} \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}$$

.

$$= \begin{bmatrix} g_2 & 0 \\ b_21(x)s_2 + b_22(x)t_2 & b_21(x)k_2 - b_22(x)l_2 \end{bmatrix}$$

That is, we convert the elements of the first row of the matrix A to 0.

Next we give algorithms for computing GCRDs and LCLMs.

Theorem 3.1. *(GCRD) Let F, G be nonzero quaternion polynomials. Then the following list of instructions returns their greatest common right divisor in a finite number of steps:*

Input: $F, G \in \mathbb{H}[x] \setminus \{0\}$

Output: $GCRD(F, G)$

Initialization: $a := F, b := G$

while $b \neq 0$ ***Do***

$t := b$

$b := mod_l(a, b)$

$a := t$

Return a.

Theorem 3.2. *(LCLM) Let F, G be nonzero quaternion polynomials. Then the following list of instructions returns their greatest common right divisor in a finite number of steps:*

Input: $F, G \in \mathbb{H}[x] \setminus \{0\}$

Output: $LCLM(F, G)$

Initialization: $a := F, b := G$

while $b \neq 0$ ***Do***

$t := b$

$b := mod_l(a, b)$

$a := t$

Return $a * b * c$.

Example 3.2. *Consider $F(x) = x * (x - i) * (x - j) = x^3 - (i + j)x^2 + kx$ and $G(x) = x * (x - k) = x^2 - kx$. In order to find $mod_l(F, G)$ we need to perform right division of F by G. This gives:*

$$F = G * (x - i - j + k) + (k + j - i - 1)x.$$

Since the first remainder

$$R_1 = (k + j - i - 1)x$$

is non-zero, we need to perform yet another division. We use the remainder to divide G and have that

$$G = (k + j - i - 1)^{-1} * R_1 * (x - k),$$

which end the iteration since the new remainder is zero. Keeping the common divisor monic, we have

$$\begin{aligned} x &= GCRD(F, G) = (k + j - i - 1)^{-1} * R_1 \\ &= (k + j - i - 1)^{-1} * F - (k + j - i - 1)^{-1} * G * (x - i - j + k) \end{aligned}.$$

Then

$$F = x * (x^2 - (i + j)x + k)$$

$$G = x * (x - k).$$

Hence

$$LCLM(F, G) = x * (x^2 - (i + j)x + k) * (x - k).$$

Example 3.3. *Consider quaternion polynomial matrix*

$$\mathrm{A} = \begin{bmatrix} x^2 - kx & ix^2 - jx \\ x^3 - (i + j)\, x^2 + kx & kx^2 - ix \end{bmatrix}$$

Next, we will change the matrix element $A(2, 2)$ into 0 by applying elementary transformations. Compute

$$GCRD(x^2 - kx, x^3 - (i + j)\, x^2 + kx) = x,$$

$$LCLM(x^2 - kx, x^3 - (i+j)\,x^2 + kx) = x * (x^2 - (i+j)x + k) * (x - k),$$

and

$$LCLM\,(A\,(1,1)\,, A\,(2,1)) = S_1\left(x^2 - kx\right) = S_2\left(x^3 - (i+j)\,x^2 + kx\right)$$

where $S_1 = (x^2 - (i+j)\,x + k))\,, S_2 = (x - k)\,.$ *Then*

$$A \to S_2R_2 - S_1R_1 = \begin{bmatrix} x^2 - kx & ix^2 - jx \\ 0 & -ix^4 + (j-1)\,x^3 + (2-k-2j)\,x^2 + (j-i)\,x \end{bmatrix}.$$

3.2 Elementary Transformations

In quaternion polynomial matrices, we will use the following four elementary row and column operations:

1. Interchange of any two rows (or columns);

2. Addition to any row (or column) of a quaternion polynomial left (or right) multiple of any other row (or column);

3. Scaling any row (or column)by any nonzero quaternion.

4. Convert an element to 0 by GCRDs and LCLMs.

Example 3.4. *The quaternion polynomial matrix $Q(x)$:*

$$Q(x) = \begin{bmatrix} 1 + ix - jx^2 & 1 - jx \\ 1 + ix & 2 \end{bmatrix}.$$

1. *Interchange the first row and the second row.*

$$Q_1(x) = \begin{bmatrix} 1 + ix & 2 \\ 1 + ix - jx^2 & 1 - jx \end{bmatrix}.$$

2. Add to the first row of i left multiple of the second row.

$$Q_2(x) = \begin{bmatrix} 1 + i + (i-1)x - jx^2 & 1 + 2i - jx \\ 1 + ix & 2 \end{bmatrix}.$$

3. The second row to right multiply j.

$$Q_3(x) = \begin{bmatrix} 1 + ix - jx^2 & 1 - jx \\ j + ijx & 2j \end{bmatrix} = \begin{bmatrix} 1 + ix - jx^2 & 1 - jx \\ j + kx & 2j \end{bmatrix}.$$

Consider a quaternion polynomial $Q(x)$, the $d(x)$ is called the right (or left) total divisor of $Q(x)$ if it exists a quaternion polynomial $l(x)$ (or $r(x)$) satisfied

$$Q(x) = l(x)d(x) \quad \text{or} \quad Q(x) = d(x)r(x).$$

Given two quaternion polynomial matrices $Q(x)$ and $P(x)$, the $r(x)$ is called an *mod* (or remainder) of $(Q(x), P(x))$ if it exists a $q(x)$ satisfied

$$Q(x) = q(x)P(x) + r(x)$$

where the $deg(r(x)) < deg(P(x))$.

3.3 Hermite forms

The Hermite forms is a canonical form of the quaternion polynomial matrices. The Hermite forms can be divided into rows and columns Hermitian forms. The function of the canonical form of a quaternion polynomial matrix is a prominent feature of the matrix.

Any quaternion polynomial matrices can be transformed into the Hermite forms through a series of four elementary transformations and unimodular transformations we discussed above. Here, the row Hermite forms and column Hermite forms are given.

Note that the ranks of quaternion polynomial matrices can not be defined by using determinants or minors. The rank of a quaternion polynomial matrix A is defined as the maximum number of columns of A which are right linearly independent.

Definition 3.1. *Considering an $m \times n$ quaternion polynomial matrix $Q(x)$, $rank(Q(x)) = r \leq \min\{m, n\}$, the row Hermite form:*

$$H_{Hr}(x) = \begin{bmatrix} 0 & \cdots & 0 & a_{1,k_1}(x) & \cdots & a_{1,k_2}(x) & \cdots & a_{1,k_3}(x) & \cdots & a_{1,k_r}(x) & \cdots \\ \vdots & & \vdots & & & a_{2,k_2}(x) & \cdots & a_{2,k_3}(x) & \cdots & a_{2,k_r}(x) & \cdots \\ \vdots & & \vdots & & & & & a_{3,k_3}(x) & \cdots & a_{3,k_r}(x) & \cdots \\ 0 & & 0 & & & & & & & \vdots & \vdots \\ 0 & \cdots & 0 & & & & & & & a_{r,k_r}(x) & \cdots \\ 0 & \cdots & 0 & & \cdots & & \cdots & & \cdots & & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & & & & \cdots & & \cdots & & \cdots & & 0 \end{bmatrix}.$$

i. The front r rows of $H_{Hr}(x)$ are non zero rows, and $(m - r)$ rows after are zero rows.

ii. In every non zero rows, located in the leftmost nonzero element a_{i,k_i} is a monic quaternion polynomial.

iii. The most left non-zero element of the $H_{Hr}(x)$ is presented as a ladder type.

iv. The number of quaternion polynomials of most left non-zero element a_{i,k_i} of the $H_{Hr}(x)$ is the greatest, compared with other elements at the same column.

Consider $m \times n$ quaternion polynomial matrix $Q(x)$, $rank(Q(x)) = r \leq \min\{m, n\}$, the column Hermite forms is $H_{Hc}(x) = H_{Hr}^T(x)$.

Given an $m \times n$ quaternion polynomial matrix $Q(x)$, the row Hermite forms $Q_{Hr}(x)$ for Hermitian forms can be achieved by an appropriate $m \times m$ unimodular matrix $V(x)$ left-multiplicated $Q(x)$; the row Hermite forms $Q_{Hr}(x)$ for Hermitian forms can be achieved by an appropriate $n \times n$ unimodular matrix $U(x)$ right-multiplicated $Q(x)$, that is,

$$Q_{Hr}(x) = V(x)Q(x)$$
$$Q_{Hc}(x) = Q(x)U(x)$$

Here one of algorithms for computing row Hermite forms is provided.

Algorithm: Computing Hermite Forms

Input: A Quaternion polynomial matrix $Q(x)$, where $Q(x) \in \mathbb{H}_{m\times n}[\mathrm{x}]$, $rank(Q(x)) = r$.

Output: The Hermitian form of $Q(x)$.

Step 1: For a given $Q(x)$, the first column to the $(k-1)$ column is supposed to be zero column, and the first k column is the first non zero column.

Step 2: $i = 1$.

Step 3: **while** $i \neq r$ **do**

$Q_{ik_i} \leftrightarrow$ the lowest times in k_1th column, call it $\tilde{q}_{i,k_i}(x)$;

At this step, the algorithms of LCLM and GCRD are used to compute the remainder of $\tilde{q}_{i,k_i}(x)$.

$$item\ of\ \tilde{q}_{i,k_i}(x) + remainder\ of\ \tilde{q}_{i,k_i}(x)$$

$Q(:, k_i) \rightarrow 0$;

$i \rightarrow i + 1$;

Step 4: **end while**

Step 5: $(m - r)$ row $\rightarrow$ zero-row.

Example 3.5. *Let*

$$Q(x) = \begin{bmatrix} 1 + ix + jx^2 & 0 \\ 0 & 1 + ix + jx^2 \\ 1 & 4 + ix + jx^2 \end{bmatrix}.$$

It is easy to check that rank$(Q(x)) = 2$. Next the row Hermite form and the corresponding unimodular transformation matrix $V(x)$ are calculated:

$$Q(x) = \begin{bmatrix} 1 + ix + jx^2 & 0 \\ 0 & 1 + ix + jx^2 \\ 1 & 4 + ix + jx^2 \end{bmatrix}$$

$$\xrightarrow[E_1]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 + ix + jx^2 \\ 0 & 1 + ix + jx^2 \\ 1 + ix + jx^2 & 0 \end{bmatrix}$$

$$\xrightarrow[E_2]{R_3 \leftarrow R_3 - (1+ix+jx^2)R_1} \begin{bmatrix} 1 & 4 + ix + jx^2 \\ 0 & 1 + ix + jx^2 \\ 0 & -(1 + ix + jx^2)(4 + ix + jx^2) \end{bmatrix}$$

$$\xrightarrow[E_3]{R_3 \leftarrow R_3 + (4+ix+jx^2)R_2} \begin{bmatrix} 1 & 4 + ix + jx^2 \\ 0 & 1 + ix + jx^2 \\ 0 & 0 \end{bmatrix} = Q_{Hr}(x)$$

$$V(x) = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 + ix + jx^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 + ix + jx^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 4 + ix + jx^2 & -(1 + ix + jx^2) \end{bmatrix}$$

Then

$$Q_{Hr}(x) = V(x)Q(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 4+ix+jx^2 & -(1+ix+jx^2) \end{bmatrix}$$

$$\begin{bmatrix} 1+ix+jx^2 & 0 \\ 0 & 1+ix+jx^2 \\ 1 & 4+ix+jx^2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 4+ix+jx^2 \\ 0 & 1+ix+jx^2 \\ 0 & 0 \end{bmatrix}$$

Next we discuss some properties of Hermite forms.

Proposition 3.1. *Suppose $Q(x)$ is a $n \times n$ nonsingular quaternion polynomial matrix, $\bar{D}(x) = D(x)R(x)$, $R(x)$ is any $n \times n$ unimodular matrix. Then quaternion polynomial matrix $D(x)$ and $\bar{D}(x)$ possess the same column Hermite forms.*

Proof: Suppose D_{Hc} and $\bar{D}_{Hc}$ are the column Hermite forms of quaternion polynomial matrices $D(x)$ and $\bar{D}(x)$. Then $n \times n$ unimodular matricesw $U(x)$ and $\bar{U}(x)$ exist:

$$D_{Hc}(x) = D(x)U(x), \bar{D}_{Hc}(x) = \bar{D}(x)\bar{U}(x). \tag{3.1}$$

Based on it, according to the equation $\bar{D}(x) = D(x)R(x)$, we have

$$\bar{D}_{Hc}(x) = D(x)R(x)\bar{U}(x) = D_{Hc}(x)U^{-1}(x)U(x)\bar{U}(x). \tag{3.2}$$

Suppose $W(x) = U^{-1}(x)R(x)\bar{U}(x)$, and the matrix is a unimodular matrix. We obtain that the $W(x)$ is a unimodular matrix. Hence

$$\bar{D}_{Hc}(x) = D_{Hc}(x)W(x). \tag{3.3}$$

According to the equation(3.3), $W^{-1}(x) = D_{Hc}^{-1}(x)\bar{D}_{Hc}(x)$. Because $D_{Hc}(x)$ and $\bar{D}_{Hc}(x)$ are lower triangular matrices, the $W(x)$ is also a lower triangular matrix:

$$W(x) = \begin{bmatrix} c_1 & & & \\ w_{21}(x) & c_2 & & \\ \vdots & & \ddots & \\ w_{n1}(x) & \cdots & w_{n,n-1} & c_n \end{bmatrix} \tag{3.4}$$

$\bar{D}_{Hc}(x) = D_{Hc}(x)W(x)$ has been proved. If $\bar{D}_{Hc}(x) = D_{Hc}(x)$ is to be proved, it must be proved that $W(x)$ is a unit matrix. From $\bar{D}_{Hc}(x) = D_{Hc}W(x)$, we have

$$\begin{bmatrix} \bar{d}_{11}(x) & & & \\ \bar{d}_{21}(x) & \bar{d}_{22}(x) & & \\ \vdots & & \ddots & \\ \bar{d}_{n1}(x) & \cdots & \cdots & \bar{d}_{nn}(x) \end{bmatrix}$$

$$= \begin{bmatrix} d_{11}(x) & & & \\ d_{21}(x) & d_{22}(x) & & \\ \vdots & & \ddots & \\ d_{n1}(x) & \cdots & \cdots & d_{nn}(x) \end{bmatrix} \begin{bmatrix} c_1 & & & \\ w_{21}(x) & c_2 & & \\ \vdots & & \ddots & \\ w_{n1}(x) & \cdots & w_{n,n-1}(x) & c_n \end{bmatrix}$$

Then

$$\bar{d}_{ii}(x) = d_{ii}(x)c_i, i = 1, 2, \ldots, n \tag{3.5}$$

and the $\bar{d}_{i,i}(x)$ and $d_{i,i}(x)$ are quaternion monic polynomials, and thus the following equation should be proved:

$$c_i = 1, i = 1, 2, \ldots, n. \tag{3.6}$$

So, $w_{ik}(x) = 0, i \neq k$ should be proved. According to the equation(3.4) and column Hermite forms, by

$$\begin{cases} \bar{d}_{21}(x) = w_{21}(x)d_{22}(x) + d_{21}(x) \\ \deg \bar{d}_{21}(x) < \deg \bar{d}_{22}(x) = \deg d_{22}(x) \end{cases} \tag{3.7}$$

we can get $w_{21} = 0$. From

$$\begin{cases} \bar{d}_{31}(x) = w_{31}(x)d_{33}(x) + d_{31}(x) \\ \deg \bar{d}_{31}(x) < \deg \bar{d}_{33}(x) = \deg d_{33}(x) \end{cases} \tag{3.8}$$

we get $w_{31} = 0$. Hence

$$w_{ik}(x) = 0, \ i = 2, \cdots, n, \ k = 1, \ldots, i-1.$$

Therefore, $W(x) = I$, that is, $\bar{D}_H(x) = D_H(x)$.

There is a similar result for column Hermite form. Suppose $A(x)$ is $n \times n$ nonsingular quaternion polynomial matrix, $\bar{A}(x) = T(x)A(x)$, $T(x)$ is any $n \times n$ unimodular matrices. Then quaternion polynomial matrices $A(x)$ and $\bar{A}(x)$ possess the same column Hermite forms.

3.4 Smith forms

Smith forms is an important canonical form of quaternion polynomial matrices similar to Jordan forms for matrices over number fields. Any quaternion polynomials can be transformed into Smith forms by elementary row operations.

For a given $p \times q$ quaternion polynomial matrices $Q(x)$ with $\text{rank}(Q(x)) = r$. The following diagonal quaternion polynomial matrix can be achieved by using four elementary transformations:

$$U(x)Q(x)V(x) = \Lambda(x) = \begin{bmatrix} \lambda_1(x) & & & 0 \\ & \ddots & & \vdots \\ & & \lambda_r(x) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.9}$$

where $\{\lambda_i(x), i = 1, 2, \ldots, r\}$ are the nonzero quaternion polynomials and $\lambda_i(x)$ is total divisor of $\lambda_{i+1}(x)$.

Algorithm: Computing Smith Forms

Input: Quaternion polynomial matrix $Q(x)$, where $Q(x) \in \mathbb{H}_{q\times p}[x]$, $rank(Q(x)) = r, \quad 0 \neq r \neq min\{q,p\}$.

Output: Smith form $\Lambda(x)$ of $Q(x)$.

If $Q(x) == 0$, $\Lambda(x) = 0$

else **If** $Q(x) \neq$ equation (3.9)

$i \rightarrow 1$

while $i \neq r$ **do**

find the lowest times element and $Q(i,i) \rightarrow it$

At this step, the algorithms of LCLM and GCRD are used to compute the least common multiple of any two element of $Q(x)$.

$Q(i, i+1 : r) \rightarrow 0;$

$Q(i+1 : r, i) \rightarrow 0;$

else

If $\{\lambda_i(x), i = 1, 2, \cdots, r\} \neq$ monic

$\lambda_i(x) \rightarrow$ monic;

end If

end If

end If

Example 3.6. *For a given* 2×3 *quaternion polynomial matrix* $Q(x)$ *with* $rank(Q(x)) = 2$ *as follows. Now we show that it can be transformed into Smith forms by four elementary row and column transformations.*

$$Q(x) = \begin{bmatrix} -x^2 - x^4 + 1 & ix + kx^2 & 1 \\ 2 & ix + kx^2 + 1 & ix + kx^2 + 2 \end{bmatrix}$$

$$\xrightarrow[E_1]{C_1 \leftrightarrow C_3} \begin{bmatrix} 1 & ix + kx^2 & -x^2 - x^4 + 1 \\ ix + kx^2 + 2 & ix + kx^2 + 1 & 2 \end{bmatrix}$$

$$\xrightarrow[E_2]{C_2 \rightarrow C_2 - (ix + kx^2)C_1} \begin{bmatrix} 1 & 0 & -x^2 - x^4 + 1 \\ ix + kx^2 + 2 & -(ix + kx^2 + 2)(ix + kx^2 + 1) & 2 \end{bmatrix}$$

$$\xrightarrow[E_3]{C_3 \rightarrow C_3 + (x^2 + x^4 - 1)C_1} \begin{bmatrix} 1 & 0 & 0 \\ ix + kx^2 + 2 & \begin{matrix} -(ix + kx^2 + 2) \\ (ix + kx^2 + 1) \end{matrix} & \begin{matrix} 2 - (-x^2 - x^4 + 1) \\ (ix + kx^2 + 2) \end{matrix} \end{bmatrix}$$

$$\xrightarrow[E_4]{simplify} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(ix + kx^2 + 2)(ix + kx^2 + 1) & -ix + kx^2(ix + kx^2 + 1)^2 \end{bmatrix}$$

$$\xrightarrow[E_5]{R_2 \rightarrow R_2 - (ix + kx^2 + 2)R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(ix + kx^2 + 2)(ix + kx^2 + 1) & -x^2 - x^4 + ix + kx^2 \end{bmatrix}$$

$$\xrightarrow[E_6]{C_3 \rightarrow C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(ix + kx^2 + 2)(ix + kx^2 + 1) & -2(ix + kx^2) - 2 \end{bmatrix}$$

$$\xrightarrow[E_7]{C_3 \leftrightarrow C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2(ix + kx^2) - 2 & -(ix + kx^2 + 2)(ix + kx^2 + 1) \end{bmatrix}$$

$$\xrightarrow[E_8]{C_3 \rightarrow C_3 - 0.5(ix + kx^2 + 2)C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2(ix + kx^2) - 2 & 0 \end{bmatrix}.$$

$$\xrightarrow[E_9]{R_2 \rightarrow -0.5R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & ix + kx^2 + 1 & 0 \end{bmatrix}$$

Let $f(x), g(x) \in \mathbb{H}[x]$. $f(x)$ and $g(x)$ are called "similar" if there exist $p(x), q(x) \in \mathbb{H}[x]$ such that

$$f(x)p(x) = q(x)g(x).$$

Clearly, two polynomials over number fields are always similar. But in quaternion polynomial case, it is not always true.

Proposition 3.2. *For a given $q \times p$ quaternion polynomial matrix $Q(x)$, the Smith form is unique up to the similarity of entries on the main diagnoal.*

Proposition 3.3. *The $q \times p$ quaternion polynomial matrix $Q_1(x)$ and $Q_2(x)$ are Smith equivalent if only and if $Q_1(x)$ and $Q_2(x)$ have the "same Smith forms" up to the similarity of entries on the main diagonal, denoted by*

$$Q_1(x) \sim Q_2(x). \tag{3.10}$$

Note that the Smith equivalence has the following characteristics:
Reflexive character: $Q_1(x) \sim Q_2(x) \Leftrightarrow Q_2(x) \sim Q_1(x)$.
Reflexivity: $Q_1(x) \sim Q_1(x)$.
Transitive: $Q_1(x) \sim Q_2(x), Q_2(x) \sim Q_3(x) \Rightarrow Q_1(x) \sim Q_3(x)$.

Moreover, we have

Theorem 3.3. *Two $q \times p$ quaternion polynomial matrix $Q_1(x)$ and $Q_2(x)$ are Smith equivalent if and only if there exist two $q \times q$ and $p \times p$ unimodular matrices $P(x)$ and $Q(x)$ such that*

$$Q_2(x) = P(x)Q_1(x)T(x).$$

Proof: By the definition of Smith equivalence,

$$U_1(x)Q_1(x)V_1(x) = U_2(x)Q_2(x)V_2(x) \tag{3.11}$$

for some unimoduar matices $U_i(x)$ and $V_i(x)$ $i = 1, 2$. Multiply left by $U^{-1}(x)$ and right by $V^{-1}(x)$, we can get

$$Q_2(x) = U_2^{-1}(x)U_1(x)Q_1(x)V_1(x)V_2^{-1}(x).$$

Set

$$P(x) = U_2^{-1}(x)U_1(x), T(x) = V_1(x)V_2^{-1}(x).$$

It is obvious that $P(x)$ and $T(x)$ are unimodular matrices.

3.5 Popov forms

In this section, we discuss the Popov forms for quaternion polynomial matrices.

Definition 3.2. *The $q \times p$ quaternion polynomial matrix*

$$D_E(x) = \begin{bmatrix} d_{11}(x) & \cdots & d_{1p}(x) \\ \vdots & & \vdots \\ d_{q1}(x) & \cdots & d_{qp}(x) \end{bmatrix}$$

is in Popov form if

(i) $D_E(x)$ is column-reduce, and $k_{c1} \leq k_{c2} \leq \cdots \leq k_{cp}$.

(ii) For jth column $j = 1, 2, \cdots, p$, principal index $m_j \in [1, 2, \cdots, p]$ exists, and the principal element $d_{m_j^j}(x)$ satisfies the following conditions:

a. $\deg \left[d_{m_j^j}(x)\right] = k_{cj}$.

b. $d_{m_j^j}(x)$ is a monic quaternion polynomial.

c. $\deg \left[d_{ij}(x)\right] < k_{cj}, \forall i > m_j$.

d. For ith column and jth column , if $i < j$ and $k_{ci} = k_{cj}$, so $m_i < m_j$.

e. $\deg \left[d_{m_j^q}(x)\right] < k_{cj}, \forall q \neq j$.

Example 3.7. *Given a 3×3 quaternion polynomial matrix*

$$D(x) = \begin{bmatrix} 5ix+5jx^2+1 & \boxed{2+3ix+(3j-1)x^2-x^4} & 4ix+4jx^2+6 \\ 3ix+3jx^2+4 & 2ix+xjx^2+1 & \boxed{2-x^2-ix^3-(i+j+1)x^4-jx^6} \\ \boxed{ix+jx^2+7} & 3 & 5 \end{bmatrix}$$

The following judgement to show that $D(x)$ is not in Popov form:

(i) According to the $k_{c1}=2, k_{c2}=4, k_{c3}=6$, it satisfies the condition (i).

(ii) The principal elements are

$$d_{m_1 1} = ix+jx^2+7, d_{m_2 2} = 2+3ix+(3j-1)x^2-x^4, d_{m_3 3} = 2-x^2-ix^3-(i+j+1)x^4-jx^6.$$

Then the principal indexes are

$$m_1 = 3, m_2 = 1, m_3 = 2$$

a. $\deg(d_{31}(x)) = 2 = k_{c1}, \deg(d_{12}(x)) = 4 = k_{c2}, \deg(d_{23}(x)) = 6 = k_{c3}$.

b. $d_{31}(x), d_{12}(x), d_{23}(x)$ are monic quaternion polynomials.

c. $\deg[d_{ij}(x)] < k_{cj}, \forall i > m_j$.

d. $k_{c1} = 2 \neq k_{c2} = 4 \neq k_{c3} = 6$, $k_{ci} \neq k_{cj}, i < j$ do not exist.

e. It satisfies that $\deg\left[d_{m_j q}(x)\right] < k_{cj}, \forall q \neq j$.

Hence the quaternion polynomial $D(x)$ is not in Popov form.

For a given $p \times p$ quaternion polynomial matrix $\bar{D}(x)$, we can use four elementary transforms to find the Popov form $D_E(x)$ and unimodular matrix $U(x)$ such that $\bar{D}(x)U(x) = D_E(x)$.

Algorithm: Computing Popov Forms

Input: A Quaternion polynomial matrix $\bar{D}(x)$, where where $\bar{D}(x) \in \mathbb{H}_{p\times q}[x]$

Output: Popov forms $D_E(x)$ and unimodular matrix $U(x)$

Step 1: **If** $\bar{D}(x) \rightarrow$ column-reduced

Calculate column number of $D(x)$ $k_{ci}, i = 1, 2, \cdots, p$

$L = max \quad k_{c1}, k_{c2}, \cdots, k_{cp}$, constitute:

$$\underset{p\times 2(L+1)p}{\beta(x)} = \begin{bmatrix} D(x) & \cdots & x^L D(x) & -I_p & \cdots & -x^L I_p \end{bmatrix}$$

At $\begin{bmatrix} -I_p & \cdots & -x^L I_p \end{bmatrix}$, find p related columns, suppose they are $b_{\beta_1}(x), b_{\beta_2}(x), \cdots, b_{\beta_p}(x)$.

else

$\bar{D}(x) \rightarrow$ column-reduced

$D(x) = \bar{D}(x)V(x)$

end If

Step 2:

$$a_{11}b_1(x) + a_{21}b_2(x) + \cdots + a_{2(L+1)p,1}b_{2(L+1)p}(x) = 0$$
$$\cdots$$
$$a_{1p}b_1(x) + a_{2p}b_2(x) + \cdots + a_{2(L+1)p,q}b_{2(L+1)p}(x) = 0$$

Step 3:

$$\mathrm{A} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{2(L+1)p,1} & \cdots & a_{2(L+1)p,q} \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} \bar{U}_0 \\ \vdots \\ \bar{U}_L \\ \cdots \\ E_0 \\ \vdots \\ E_L \end{bmatrix}, \bar{U}_i, E_i \in \Re^{p\times q}$$

Step 4:

$$\bar{U}(x) = \bar{U}_L x^L + \cdots + U_1 x + \bar{U}_0$$

$$E(x) = E_L x^L + \cdots + E_1 x + E_0$$

Step 5: **If** $E(x) \rightarrow$ Popov forms

$E(x) \rightarrow D_E(x)$

$\bar{U}(x) \rightarrow U(x)$

If $D(x)$ is column-reduced

$U(x) \rightarrow \tilde{U}(x)$

else

$U(x) \rightarrow V(x)\tilde{U}(x)$

end If

else

At this step, the algorithms of LCLM and GCRD are used to compute the ladder type of $U(x)$.

$A \rightarrow$ a ladder type

end If

Example 3.8. *For* 2×2 *polynomial matrix* $D(x)$

$$D(x) = \begin{bmatrix} -3x & x + 2j \\ -x + j & j \end{bmatrix}$$

(i) Judge column-reduced and column number.

$$k_{c1} = 1, k_{c2} = 1, L = \max\{k_{c1}, k_{c2}\} = 1$$

(ii) Give two related columns

$$\beta(x) = \begin{bmatrix} D(x) & xD(x) & -I & -xI \end{bmatrix}$$

$$= \begin{bmatrix} -3x & x+2j & -3x^2 & x^2+2jx & -1 & 0 & -x & 0 \\ -x+j & j & -x^2+jx & jx & 0 & -1 & 0 & -x \end{bmatrix}$$

and find two related columns

$$b_7(x) = -\begin{bmatrix} x+2j \\ j \end{bmatrix} - 2j\begin{bmatrix} -1 \\ 0 \end{bmatrix} - j\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= -b_2(x) - 2jb_5(x) - jb_6(x)$$

$$b_8(x) = \begin{bmatrix} -3x \\ -x+j \end{bmatrix} + j\begin{bmatrix} 0 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -x \\ 0 \end{bmatrix}$$

$$= b_1(x) + jb_6(x) - 3b_7(x)$$

(iii) Calculate A. Firstly, get linear combination equation with constant coefficients

$$b_2(x) + 2jb_5(x) + jb_6(x) + b_7(x) = 0$$

$$-b_1(x) - jb_6(x) + 3b_7(x) + b_8(x) = 0$$

That is,

$$\beta(x)A = [b_1(x), b_2(x), b_3(x), b_4(x), b_5(x), b_6(x), b_7(x), b_8(x)] \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2j & 0 \\ j & -j \\ 1 & 3 \\ 0 & 1 \end{bmatrix} = 0$$

so,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2j & 0 \\ j & -j \\ 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{U}_0 \\ \bar{U}_1 \\ E_0 \\ E_1 \end{bmatrix}$$

(iv) Get $E(x)$.

$$E(x) = E_1 x + E_0 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2j & 0 \\ j & -j \end{bmatrix} = \begin{bmatrix} x + 2j & 3x \\ j & x - j \end{bmatrix}$$

(v) Get Popov forms.

$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 2j & j & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -j & 3 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 2j & j & 1 & 0 \\ -1 & -3 & 0 & 0 & -6j & -4j & 0 & 1 \end{bmatrix} = \left[\tilde{U}_0^T, \tilde{U}_1^T, \tilde{E}_0^T, \tilde{E}_1^T\right]$$

$$\begin{aligned} D_E(x) = \tilde{E}_1 x + \tilde{E}_0 &= \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 2j & -6j \\ j & -4j \end{bmatrix} \\ &= \begin{bmatrix} x + 2j & -6j \\ j & x - 4j \end{bmatrix} \end{aligned}$$

$$U(x) = \tilde{U}_1 x + \tilde{U}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -3 \end{bmatrix}$$

3.6 Applications and future works

In this section, we outline some possible applications and future works.

- System Stability: the normal forms of quaternion polynomial matrices can be used for studying the stability of linear dynamical systems with quaternionic coefficients. Pereira and Rocha[28] gave the definitions of the polynomial determinant Pdet for quaternionic polynomial matrices and extended to some results on system stability.

  For quaternionic state-space system:

  $$\dot{x} = Ax \tag{3.12}$$

  where $A \in \mathbb{H}^{n \times n}$. The solutions are

  $$x(t) = e^{(At)} x_0 \tag{3.13}$$

where the exponential is defined as usual. Consider now a quaternionic system described by a higher-order matrix differential equation

$$R\left(\frac{d}{dt}\right) w = 0 \tag{3.14}$$

where $R(x) = R_m x^m + \cdots + R_1 x + R_0 \in \mathbb{H}^{n\times n}[x]$. This enabled a characterization of the stability of linear systems with quaternionic coefficients, $R\left(\frac{d}{dt}\right) w = 0$ in terms of the zeros of $Pdet(R(s))$, that generalizes the results obtained for the real and complex cases.

- Rotation algorithm: Quaternion polynomial matrices have been found many applications in the field of digital signal processing, communications and other areas where they are used to describe the transfer function for multiple-input multiple-output systems. Menanno and Bihanhave[21] introduced a quaternion version of the SBR2 algorithm which is based on eigenvalue decomposition of quaternion para-Hermitian polynomial matrices. They used it to perform polarized source separation in convolutive mixture models.

- j-conjugate product: Wu et al[45] proposed the concept of j-conjugate product over quaternion polynomial matrices. They introduced two new operations, right j -conjugate product and left j-conjugate product, for quaternion polynomial matrices. Their basic properties are derived. As an application, the consimilarity of quaternion matrices is investigated by means of j-conjugate product. It has been revealed that two quaternion matrices A and B are consimilar if and only if $sI - A$ and $sI - B$ right (or, left) conequivalent and this coincidence with the cases of real and complex matrices.

Chapter 4

Maple Codes

In this thesis, we use Maple to produce some examples. Some of quaternion codes are picked from [18]. Here are a part of Maple codes:

```
Qdef := proc (a, b, c, d)
#the definition of quaternion
 a+I*b+c*J+d*K
end proc;
Qreal := proc (a)
#the real part of quaternion
subs(I = 0, J = 0, K = 0, a)
end proc;
Qimag := proc (a)
#the imag part of quaternion
a-Qreal(a)
end proc;
Qnorm := proc (a)
(Qreal(a)^2+coeff(a, I)^2+
coeff(a, J)^2+coeff(a, K)^2)^(1/2)
end proc;
sortcollect := proc (A, a)
```

```
return sort(collect(A, a), a)
end proc;
Qinv := proc (a)
sortcollect(Qreal(a)-Qimag(a), x)
/sortcollect(Qreal(a)^2+
coeff(a, I)^2+coeff(a, J)^2+coeff(a, K)^2, x)
end proc;
M := proc(a,b)
#M is the multiplication operator for
#quaternions and quaternion polynomials.
local a1 := Qreal(a),
b1 := coeff(a, I),
c1 := coeff(a, J),
d1 := coeff(a, K),
a2 := Qreal(b),
b2 := coeff(b, I),
c2 := coeff(b, J),
d2 := coeff(b, K);
return sortcollect (a1*a2 - b1*b2 - c1*c2 - d1*d2
+ (a1*b2 + b1*a2 + c1*d2 - d1*c2)*I
+ (a1*c2 - b1*d2 + c1*a2 + d1*b2)*J
+ (a1*d2 + b1*c2 - c1*b2 + d1*a2)*K, x);
end proc;
QMatrixMultiply := proc(A::Matrix, B::Matrix)
 #procedure for quaternion matrix multiplication.
 local nrowsA := LinearAlgebra:-RowDimension(A),
 ncolsA := LinearAlgebra:-ColumnDimension(A),
 ncolsB := LinearAlgebra:-ColumnDimension(B),
 AB := Matrix(1..nrowsA, 1..ncolsB), i, j;
```

```
for i from 1 to nrowsA do
for j from 1 to ncolsB do
AB(i, j) := simplify(add(M(A(i, k), B(k, j)),
 k = 1 .. ncolsA));
end do;
end do;
end proc
QMultiplyScalar := proc(A::Matrix, B)
#Local procedure to multiply a
#quaternion matrix by a quaternion scalar.
local nrowsA := LinearAlgebra:-RowDimension(A),
ncolsA := LinearAlgebra:-ColumnDimension(A),
 AB := Matrix(1..nrowsA, 1..ncolsA),
 i, j;
 for i from 1 to nrowsA do
 for j from 1 to ncolsA do
 AB(i, j) := simplify(M(A(i, j), B));
 end do;
 end do;
 end proc;
GCRD:=proc(F,G)
a:=F;
b:=G;
while b<>0 do
t:=b;
b:=mod(a,b);
a=:t;
od
return a;
```

```
end proc;
LCLM:=proc(F,G)
a:=F;
b:=G;
while b<>0 do
t:=b;
b:=mod(a,b);
a=:t;   od
return a*b*t;
end proc;
Hermite:=proc(Q)
r=rand(Q);
[m,n]=size(Q);
a:=find(Q);
x:=findx(Q);
y:=findy(Q);
Q:=cort(Q,a,x,y);
i:=1;
while i<>r do
for j form 2 to n do
 q:=findleast(Q(i,:));
 p:=rem(q);
Q(:,j):=0;
i:=i+1;
end do
end do
return row(Q);
end proc
Smith:=proc(Q)
```

```
r:=rand(Q);
[m,n]=size(Q);
a:=find(Q);
x:=findx(Q);
y:=findy(Q);
Q:=cort(Q,a,x,y);
i:=1;
while i<>r do
for j form 2 to m do
b:=LCLM(Q(i,i),Q(i,j));
c:=GCRD(Q(i,i),Q(i,j));
c1:=Qrem(Q(i,i));
c2:=Qrem(Q(i,j));
Q(j,:)=Q(j,:)*c2-Q(i,:)*c1;
end do
Q(i,:):=0;
end do
return Q;
end proc
Popov:=proc(Q)
r:=rank(Q);
Q1=:reduce(Q);
B(1,:):=Q1;
for i form 2 to r do
B(i,:):=*Q1;
end do
B(r+1,:):=-ones(r,r);
for i form r+2 to 2*r do
B(i,:):=-s^((i-1))*ones(r,r);
```

```
end do
Ro:=RowOperation(IdentityMatrix(m),[n,p]);
U:=Ro.U;
V:=Ro.V;
for i from n+1 to m do
Ro:=RowOp(U,n,i);
U:=Ro.U;
V:=Ro.V;
od;
for i to m do
for j to n do
U[i,j]:= expand(U[i,j]);
end do;
end do;
A=V*U;
return A;
end proc
```

```
> Qdef := proc(a, b, c, d)
   a + b·I + c·J + d·K;
   end proc
                    Qdef := proc(a, b, c, d) a + I*b + c*J + d*K end proc      (1)
> a := Qdef(1, 2, 3, 4)
                    a := 1 + 2 I + 3 J + 4 K                                   (2)
> Qreal := proc(a)
  #Calculates the real part of a quaternion.
  subs(I = 0, J = 0, K = 0, a);
   end proc:
>
> Qreal(a)
                    1                                                          (3)
> Qnorm := proc(a)
  #Calculates the 2 norm of a quaternion.
  (Qreal(a)^2 + coeff(a, I)^2 + coeff(a, J)^2 + coeff(a, K)^2)^(1/2);
  end proc:
> Qnorm(a)
                    √30                                                        (4)
>
> b := Qdef(4, 3, 2, 1)
                    b := 4 + 3 I + 2 J + K                                     (5)
>
> Qimag := proc(a)
  #Calculates the imaginary part of a quaternion.
  a - Qreal(a);
   end proc:
> Qimag(b)
                    3 I + 2 J + K                                              (6)
```

```
> a + b
                                5 + 5 I + 5 J + 5 K                                  (7)
> a·b
                       (1 + 2 I + 3 J + 4 K) (4 + 3 I + 2 J + K)                     (8)
> sortcollect := proc (A, a)
  #Returns A as a polynomial in standard form of variable a.
  return sort(collect(A, a), a)
  end proc;
          sortcollect := proc(A, a) return sort(collect(A, a), a) end proc           (9)
> M := proc(a, b)
  #M is the multiplication operator for quaternions and quaternion polynomials.
  local a1 := Qreal(a), b1 := coeff(a, I),
  c1 := coeff(a, J), d1 := coeff(a, K),
  a2 := Qreal(b), b2 := coeff(b, I),
  c2 := coeff(b, J), d2 := coeff(b, K);
   F := a1*a2 - b1*b2 - c1*c2 - d1*d2
   + (a1*b2 + b1*a2 + c1*d2 - d1*c2)*I
   + (a1*c2 - b1*d2 + c1*a2 + d1*b2)*J
   + (a1*d2 + b1*c2 - c1*b2 + d1*a2)*K;
   return sort(collect(F, x), x)
   end proc;
> M(a, b)
                              - 12 + 6 I + 24 J + 12 K                               (10)
> Qinv := proc(a)
  #Calculates the inverse of a quaternion or a quaternion polynomial.
  sortcollect(Qreal(a) - Qimag(a), x)/sortcollect((Qreal(a)^2 + coeff(a, I)^2 + coeff(a, J)^2 + coeff(a, K)^2),
      x)
   end proc:
```

```
> Qinv(a)
```

$$\frac{1}{30} - \frac{1}{15} I - \frac{1}{10} J - \frac{2}{15} K \qquad (11)$$

```
>  QMatrixMultiply := proc(A::Matrix, B::Matrix)
  #Local procedure for quaternion matrix multiplication.
  local nrowsA := LinearAlgebra:-RowDimension(A),
  ncolsA := LinearAlgebra:-ColumnDimension(A),
  ncolsB := LinearAlgebra:-ColumnDimension(B),
  AB := Matrix(1..nrowsA, 1..ncolsB), i, j;
  for i from 1 to nrowsA do
  for j from 1 to ncolsB do
  AB(i, j) := simplify(add(M(A(i, k), B(k, j)), k = 1 .. ncolsA));
  end do;
   end do;
   end proc
> A := [{1 + 2·I + 3·J + 4·K, 2 + 3·I}, {3 + 4·K, 5 + 4·I + 6·K}]
              A := [{2 + 3 I, 1 + 2 I + 3 J + 4 K}, {3 + 4 K, 5 + 4 I + 6 K}]        (12)
> B := [{a, b}, {b, a}]
     B := [{1 + 2 I + 3 J + 4 K, 4 + 3 I + 2 J + K}, {1 + 2 I + 3 J + 4 K, 4 + 3 I + 2 J + K}]   (13)
```

```
> t := x^3 + (I + J)x^2 + Kx
```

$$t := x^3 + (I + J)\ x^2 + Kx \qquad (14)$$

```
> s := x^2 + Kx
```

$$s := x^2 + Kx \qquad (15)$$

```
> t1 := x
```

$$t1 := x \qquad (16)$$

```
> s1 := (x^3 - (I + J)x^2 + Kx)·(x - K)
```

$$s1 := \left(x^3 - (I + J)\ x^2 + Kx\right)\ (x - K) \qquad (17)$$

```
> GCRD(t, s)
>
                                    x                                        (18)
> LCLM(t, s)
              (x^3 - (I + J) x^2 + Kx) (x - K)                               (19)
>
> H1 := [{1 + Ix + Jx^2, 0}, {0, 1 + Ix + Jx^2}, {1, 4 + Ix + Jx^2}]
       H1 := [{0, Jx^2 + Ix + 1}, {0, Jx^2 + Ix + 1}, {1, Jx^2 + Ix + 4}]    (20)
> Hermite(H1)
              [{1, Jx^2 + Ix + 4}, {0, Jx^2 + Ix + 1}, {0}]                  (21)
> S1 := [{-x^2 - x^4 + 1, Ix + Kx^2, 1}, {2, Ix + Kx^2 + 1, Ix + Kx^2 + 2}]
     S1 := [{1, Kx^2 + Ix, -x^4 - x^2 + 1}, {2, Kx^2 + Ix + 1, Kx^2 + Ix + 2}]   (22)
>
> Smith(S1)
                     [{0, 1}, {0, Kx^2 + Ix + 1}]                            (23)
> P1 := [{-3 Ix - 3·Jx^2, 2 + Ix}, {1 - Ix - Jx^2, 1}]
          P1 := [{2 + Ix, -3 Jx^2 - 3 Ix}, {1, -Jx^2 - Ix + 1}]              (24)
>
> Popov(P1)
               [{-6, Jx^2 + Ix + 2}, {1, Jx^2 + Ix - 4}]                     (25)
                        [{-1, 0}, {-3, 1}]                                   (26)
```

Bibliography

[1] S.Bingulac and N.F.Al-Muthairi, *Relation between state space and MFD models for MIMO systems*, IEEE International Conference on Systems **3** (1993), no. 2, 429–434.

[2] K.Bo, *On computing the Kronecker canonical form of regular $A-\lambda B$-pencils*, Lecture Notes in Mathematics **973** (1983), 30–57.

[3] D.Boucher and F.Ulmer, *Stability of polynomial matrices*, IEEE Transactions on Automatic Control **32** (1987), no. 3, 228.

[4] F.Brackx, R.Delanghe and F.Sommen, *Clifford ana lysis operator in the four element analysis*, Pitman, Boston-London-Melburn, 1982.

[5] J.L.Dalecki, *On the asymptotic solution of a vector differential equation*, Doklady Akad Nauk Sssr **92** (1953), 881–884.

[6] A.Damiano, G.Gentili and D.Struppa, *Computations in the ring of quaternionic polynomials*, Journal of Symbolic Computation **45** (2010), 38–45.

[7] P.Davies and H.Cheng, *Computing Popov form of Ore polynomial matrices*, Acm Communications in Computer Algebra (2010), 149–156.

[8] P.D.Domich, R.Kannan and L.E.Trotter, *Hermite normal form computation using moduls determinant arithmetic*, Mathematics of Operations Research **12** (1987), 50–59.

[9] P.V.Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications **27** (1979), 103–140.

[10] T.B.V.Dooren, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications **105** (1988), 9–65.

[11] J.G.Dumas, F.Heckenbach, D. Saunders and V. Welker, *Computing simplicial homology based on efficient Smith normal form algorithms*, Algebra, Geometry, and Software Systems **1** (2003), 177–206.

[12] H.Flanders and H.K.Wimmer, *On the matrix equations $AX - XB = C$ and $AX - YB = C$*, SIAM Journal on Applied Mathematics **32** (1977), 707–710.

[13] M.Giesbrecht, G.Labahn and Y.Zhang, *Computing valuation Popov forms, International Conference on Computational Science* **3516** (2005), 619–626.

[14] O.M.Grasselli and A.Tornambe, *On the computation of a state-space realization of a polynomial matrix description of a system*, Proceedings of the 29th IEEE Conference on Decision and Control, Honolulu, Hawaii, USA **2** (1991), 1050–1052.

[15] W.R.Hamilton, *On quaternions: or a new system of imaginaries in algebra*, Philosophical Magazine **25** (1884), 489–495

[16] G.Havas and B.S.Majewski, *Hermite normal form computation for integer matrices*, Congressus Numerantium (1994), 1–10.

[17] C.Hermite, *Sur lintroduction des variables continues dans la thorie des nombres*, Journal Fr Die Reine Und Angewandte Mathematik **41** (1851), 191C-216.

[18] L.j.Huang, *Quaternion equations and Quaternion polynomial matrices*, The University of Manitoba (2012).

[19] R.Kannan, *Solving systems of linear equations over polynomials*, Theoretical Computer Science **39** (1985), 69–88.

[20] R.Kannan, *Some recent results in combinatorial approaches to dynamica systems*, Theoretical Computer Science **122** (1989), 725–759.

[21] G.M.Menanno and N.L.Bihan, *Quaternion polynomial matrix diagonalization for the separation of polarized convolutive mixture*, Signal Processing **90** (2010), 2219–2231.

[22] D.Micciancio and B.Warinschi, *A linear space algorithm for computing the Hermite normal form*, International Symposium on Symbolic and Algebraic Computation **18** (2001), 231–236.

[23] K.Murota, *On the irreducibility of layered mixed matrices*, Linear and Multilinear Algebra **24** (1989), 273–288.

[24] K.Murota, *On the Smith normal orm o structured polynomial matrices*, SIAM Journal on Matrix Analysis and Applications **12** (1991), 747–765.

[25] K.Murota, *On the Smith normal form of structured polynomial matrices,II*, Society for Industrial and Applied Mathematics **14** (1993), 1103–1111.

[26] K.Murota, *On the Smith normal form of structured polynomial matrices*, SIAM Journal on Matrix Analysis and Applications **12** (2006), 747–765.

[27] K.Murota, M.Iri and M. Nakamura, *Combinatorial canonical form of layered mixed matrices and its application to block-triangularization of Systems of equations*, Society for Industrial and Applied Mathematics **8** (1987), 123–149.

[28] R.Pereira and P.Rocha, *On the determinant of quaternionic polynomial matrices and its application to system stability*, Mathematical Methods in the Applied Sciences **31** (2010), 99–122

[29] J.L.Ramos, *A new method for block partial fraction expansion of matrix fraction descriptions*, IEEE Transactions on Automatic Control **36** (1991), no. 2, 1482–1485.

[30] L.Rodman, *Topics in quaternion linear algebra*, Princeton University Press, 2014.

[31] M.Rosenblum, *On the operator equation $BX-XA=Q$*, Duke Mathematical Journal **23** (1956), 263–269.

[32] M.Rosenblum, *The operator equation $BX-XA=Q$ with self-adjoint A and B*, Proceedings of the American Mathematical Society **20** (1969), 115–120.

[33] W.E.Roth, *The equations $AX-YB=C$ and $AX-XB=C$ in matrices*, Proceedings of the American Mathematical Society **3** (1952), 392–396.

[34] J.Ryan, *Clifford algebras in analysis and related topics*, CRC Press, Boca Raton-New York-London-Tokyo, 1996.

[35] L.Salamon, *Differential geometry of quaternionic manifolds*, Annales Scientifiques De L cole Normale Suprieure **19** (1987), 31–55.

[36] A.Schweinsberg, *The operator equation $AX-XB=C$ with normal A and B*, Pacific Journal of Mathematic **102** (1982), 447–453.

[37] H.J.S.Smith, *On systems of linear indeterminate equations and congruences*, Philosophical Transactions of the Royal Society of London **151** (1861), 293–326.

[38] J.Snygg, *Clifford algebra: A computation tool for physicists*, New York, 1997.

[39] A.Sudbery, *Quaternionic analysis*, Annales Scientifiques De L cole Normale Suprieure **85** (1979), 199–225.

[40] J.Sylvester, *Sur l'equation en matrices $px = xq$*, C.R.Acad. Sci.Paris **99** (1884), 7–116.

[41] A.Torgasev, *On reflexivity of a quaternion normed spaces*, Biochimica Et Biophysica Acta **27** (1997), 57–64.

[42] A.Torgasev, *Dual space of a quaternionic Hilbert spaces*, Facta Univ.Ser.Math.Inform **14** (1999), 71–77.

[43] K.Viswanath, *Normal operations on quaternionic Hilbert spaces*, Transactions of the American Mathematical Society **162** (1971), 337–350

[44] J.H.Wilkinson, *Kronecker's canonical form and the QZ algorithm, Linear Algebra and its Applications* **28** (1979), 85–303.

[45] A.G.Wu, W.Liu, C.Li and G.R.Duan, *On j-conjugate product of quaternion polynomial matrices*, Applied Mathematics and Computation **219** (2013), 11223–11232.

[46] P.Yang, *The regular function and some boundary value problems in the quaternion space*, Journal of Sichuan Normal University **17** (1994), no 6, 1–7.

[47] P.W.Yang, *T operator and two kinds of boundary value problems in the quaternion analysis*, Journal of Mathematics **44** (2001), no 2, 343–350.

[48] P.W.Yang, *Holder continuity and Riemann-Hilbert boundary value problem of $G_t f$ operator in the quaternion analysis*, Journal of Mathematics **46** (2003), no 5, 993–998.

[49] D.Z.Zheng, *Linear Systems Theory*, Tsinghua Press, 2002.

[50] W.D.Zhang and X.M.Xu, *On minimal-order stabilization of minimum phase plants*, Automatica **38** (2002), no. 7, 1243–1246.