The Eigenvalue Problem of the Uzawa Pressure Operator

by

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Abstract

Knowledge about the spectrum of the Uzawa pressure operator is important for solving and performing an error analysis of the Stokes problem. The infimum of the spectrum of the Uzawa pressure operator is significant, for instance, it gives the information about the rate of convergence of numerical methods for the Stokes problem. The spectrum of Uzawa pressure operator is still not known for the case of a square domain. This thesis provides some results related to this problem. It depicts the efforts made for estimating the infimum of the spectrum of the Uzawa pressure operator. In [24], the authors give an upper bound equal to 0.2260 for the infimum of the spectrum of the Uzawa pressure operator, we have improved it to 0.20164. We conclude this thesis by giving a conjecture that the infimum of the spectrum of the Uzawa pressure operator is equal to $\frac{1}{2} - \frac{1}{\pi} = 0.18169011381\ldots$.

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Introduction

Motivation

Partial differential equations (PDEs) are used to model many physical phenomena, for instance, problems of fluid dynamics. The system of non-linear PDEs which formulates the flow of a viscous fluid is known as the Navier-Stokes equations. This system is considered with a set of boundary conditions and initial conditions. The solution of this problem is the pressure and fluid velocity in the given domain. They are named after Claude-Louis Navier and Sir George Gabriel Stokes, for their enormous contribution to the field of fluid dynamics. The former gave a theory of elasticity in 1821. The Navier-Stokes equations for an incompressible, Newtonian fluid over a bounded domain Ω with a Lipschitz boundary, are given as

$$\begin{aligned} \rho(u_t + (u \cdot \nabla)u) &= -\nabla p + \mu \Delta u + f \text{ in } \Omega, \\ \nabla \cdot u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ u(x,0) &= u_0(x) \text{ in } \Omega. \end{aligned}$$

In the equations above, u represents the velocity vector, p is the pressure, and both are functions of space and time. We consider μ and ρ as positive constants, which represent viscosity and density of the fluid, respectively. These equations are formulated by considering fluid as a continuum. They are used to model weather, air flow around a wing and the study of the flow of blood in a body, etc. Although they are very significant, we do not have a proof of the existence and uniqueness of the solution of the Navier-Stokes equations in three dimensions. Consequently, they are studied widely. These equations are simplified according to the nature of fluid flow being studied. One of the main factors which characterize the flow of a fluid is the Reynolds number, which is defined as the ratio of inertial forces to viscous forces.

The Stokes equations are one of the simplified forms of the Navier-Stokes equations. They are a system of linear PDEs, which model a fluid flow with a very low Reynolds number, i.e., $R_e \ll 1$. We refer to the steady state Stokes equations with Dirichlet boundary conditions as the Stokes problem. The study of existence and uniqueness of the solution of the Stokes problem is related to the Uzawa pressure operator, which is denoted by S. This relationship encourages the study of the properties of the Uzawa pressure operator. It is further observed that the infimum of the spectrum of the Uzawa pressure operator is required for performing an error analysis of the Stokes problem. Therefore, in this thesis, we discuss the spectrum of the Uzawa pressure operator.

Literature Review

The problem of finding $\lambda_{\min}(S)$, the infimum of the spectrum of the Uzawa pressure operator, is related to the problem of finding the optimal constants for some inequalities in a particular domain. These inequalities are Korn's inequality, Friedrichs inequality, Babuška-Aziz inequality and Nečas inequality. The Friedrichs inequality was given by Kurt Friedrichs in 1937 [20]. He further generalised the Friedrichs inequality in [21] and proved the Korn's inequality in [22]. The problem of finding the optimal constants of these two inequalities for smooth domains was solved over the years until 1982 by various authors [40, 30, 29, 31]. They found the best constants for domains such as a disk, a sphere, an ellipse and domains related to them. In [29, 31], the relationship of optimal constants with an eigenvalue problem in theory of elasticity was discussed. In [31], the author also found an upper bound for the optimal constant for star-shaped domains.

In 1974, M. Crouzeix provided a convergence analysis of the Uzawa's algorithm, which can be found in [49], for solving the Stokes problem numerically. It was found that the spectral values of the Schur complement operator for the Stokes problem in continuous form are required for the convergence of this method. This operator was named as Uzawa pressure operator, after the Japnese economist Hirofumi Uzawa. In 1992, the authors of [23] discussed the relationship of the optimal constant in the Nečas inequality with the spectrum of the Uzawa pressure operator. They also performed an approximation of this constant in [24]. In [13], M. Crouziex listed several properties and results related to the Uzawa pressure operator. He proved the existence of the continuous spectrum of the Uzawa pressure operator for domains having corners. In [50], W. Velte studied the relationships of the eigenvalue problems with the problem of finding optimal constants in various inequalities. In [1, 38, 39, 9], the authors found some bounds on the inf-sup constant, which is related to $\lambda_{\min}(S)$, by changing the boundary conditions provided with S, thus, creating a modified S. In paper [9], the authors also describe the full spectrum for a ring. The expression for the spectrum of S on an ellipse was given in [34].

In [47], G. Stoyan defined the Horgan-Payne angle and reformulated the upper bound given in [31]. This gives the best lower bound of 0.1464 on $\lambda_{\min}(S)$, which is the same as in [31]. In [52], S. Zsuppán gave the relationship between the Friedrichs constant for the domains that can be mapped on to each other by a conformal mapping. Several other attempts at finding bounds on $\lambda_{\min}(S)$ can be found in some papers, such as [14, 15, 12]. The expression for the essential spectrum of S was given in [11]. The minimum of the essential spectrum gave us the upper bound of 0.1817 on $\lambda_{\min}(S)$, i.e., $0.1464 \leq \lambda_{\min}(S) \leq 0.1817$. In this thesis, these results are summarized and efforts are made for improving these bounds.

Outline of the thesis

In this thesis, we discuss the eigenvalue problem of the Uzawa pressure operator. This thesis consists of six chapters. Chapter 1, summarizes the concepts that are required in understanding this thesis. Some fundamental aspects of the analysis of PDEs and spectral theory of a bounded linear operator are given in this chapter.

Chapter 2, describes the Stokes problem and the existence of its solutions. It defines the Uzawa pressure operator and gives its general properties. This chapter is concluded with the relationship between both of them.

Chapter 3, depicts the general spectral properties of the Uzawa pressure operator. The significant results for the domains such as a disk and a square are given. The proof of continuous spectrum of the Uzawa pressure operator for a polygonal domain is provided in this chapter.

Chapter 4, defines some of the inequalities related to the Uzawa pressure operator. This chapter explains the relationship among the optimal constants and the relationship of these constants with the infimum of the spectrum of the Uzawa pressure operator. We use these relationships to improve the bounds on the $\lambda_{\min}(S)$ by various ways.

Chapter 5, discusses the study of the Stokes problem as a mixed problem and thus, a saddle point problem. We study the discrete form of the Stokes problem by implementing FEM and describe various methods for solving the linear system.

Chapter 6, concludes all of the results mentioned in this thesis. It enlists some of the open problems related to the study of the spectrum of the Uzawa pressure operator.

1

Mathematical Background

1.1 Partial Differential Equations

Partial Differential Equations (PDEs) arise from the problems emerging from various branches of Science and Engineering. In this section, the fundamental results of the analysis of PDEs will be discussed.

The initial point for the analysis of weak solutions of PDEs is the concept of distributions. A discussion on distribution theory can be found in the book [32]. We require Sobolev spaces to understand the implementation of the concepts provided by the Distribution theory. Sobolev spaces and the theorems provoked by them lead us to the formulation of solving problems in a weak sense, removing all the perplexities possessed by it. Thus, we will examine the notions of Sobolev spaces and ultimately discuss some results related to the weak solution of a PDE. In this section, we consider $\Omega \subset \mathbb{R}^N$, where N is the dimension of the space, to be an open and connected set.

The class of infinitely differentiable functions over Ω is denoted by $C^\infty(\Omega)$. We consider the class of infinitely differentiable functions with compact support over Ω, i.e., $C_c^\infty(\Omega)$ functions and term them as “Test functions”. We say that a sequence

$\{\phi_m\}_{m\in\mathbb{N}} \subset C_c^\infty(\Omega)$ converges to $\phi \in C_c^\infty(\Omega)$, i.e., $\phi_m \to \phi$, if there exists a compact set $K \subset \Omega$ such that $\mathrm{supp}(\phi_m)$, $\mathrm{supp}(\phi) \subset K$ and $D^\alpha \phi_m \to D^\alpha \phi$ uniformly in K, for all multi-indices α.

Definition 1.1. A linear functional T on $C_c^\infty(\Omega)$ is said to be a distribution on Ω if whenever $\phi_m \to 0$ in $C_c^\infty(\Omega)$, we have $T(\phi_m) \to 0$.

1.1.1 Some Important Spaces

For defining Sobolev spaces, we need to know the definition of some spaces which are repeatedly used.

Definition 1.2 (**Bounded linear operator**)**.** A linear operator $T : X \to X$ on a normed linear space $(X, \|\cdot\|)$ is said to be bounded if there exists a real number c such that for all $x \in X$,

$$\|Tx\| \le c\|x\|.$$

The norm of a bounded linear operator $T : X \to X$ is defined as,

$$\|T\|_{\mathrm{op}} := \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|\le 1}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X \\ \|x\|=1}} \|Tx\|. \tag{1.1}$$

Thus, by using the definition of boundedness for a bounded linear operator T defined on X, we have the following inequality for $x \in X$

$$\|Tx\| \le \|T\|_{\mathrm{op}}\|x\|.$$

Definition 1.3 (**Dual Space**)**.** Let $(X, \|\cdot\|)$ be a normed linear space. The space consisting of continuous linear functionals $x' : X \to \mathbb{R}$ is called the dual space of X and it is denoted by X'. A linear functional x' is said to be continuous if there exists a constant c such that for $x \in X$,

$$|\langle x', x\rangle_{X',X}| \leq c\|x\|.$$

The norm on the space X' is defined as

$$\|x'\|' := \sup_{\substack{x \in X \\ x \neq 0}} \frac{|\langle x', x\rangle_{X',X}|}{\|x\|} = \sup_{\|x\|=1} |\langle x', x\rangle_{X',X}| = \sup_{\|x\|\leq 1} |\langle x', x\rangle_{X',X}|. \tag{1.2}$$

The dual space X' together with the norm $\|\cdot\|'$ is a Banach space, i.e., a complete normed linear space.

Definition 1.4 (**Adjoint (Dual) Operator**)**.** Let V and W be two normed linear spaces and let $A : V \to W$. The adjoint (dual) operator $A^* : W' \to V'$ is defined for all $v \in V$ and $w' \in W'$ by,

$$\langle A^* w', v\rangle_{V',V} = \langle w', Av\rangle_{W',W}. \tag{1.3}$$

Definition 1.5 (**Reflexive Banach Space**)**.** A Banach space X is called reflexive if the dual space of X' is the space X itself, i.e., $X'' = X$.

Definition 1.6 (**Orthogonal Complement**)**.** Let H be a Hilbert space equipped with the inner product $(\cdot,\cdot)$ and Z be a subspace of H. The set containing all the elements of H that are orthogonal to all the elements of Z is called the orthogonal complement of Z. It is denoted by $Z^\perp$.

$$Z^\perp := \{x \in H \,|\, (x, z) = 0, \ \forall\, z \in Z\}.$$

Definition 1.7 (**Polar Space** [6])**.** Let H be a Hilbert space and Z be a subspace of H. The subset of the dual of H, i.e., H' containing all the functionals that vanish identically on Z is called the polar space of Z. It is denoted by Z^0.

$$Z^0 = \left\{ f \in H' \mid \langle f, z \rangle_{H',H} = 0,\ \forall\, z \in Z \right\}.$$

Definition 1.8 (L^p **Space).** If $1 \leq p < \infty$, the space $L^p(\Omega)$, is a Banach space of measurable functions f defined on Ω such that

$$\|f\|_{0,p} := \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}, \tag{1.4}$$

is finite. For $p = \infty$, we have that $L^\infty(\Omega)$ is a Banach space of measurable functions whose essential supremum is finite. The norm on the space L^∞ is given as $\|\cdot\|_\infty := \operatorname{ess\,sup}_\Omega(\cdot)$.

The space that we usually use is L^2 space. The L^2 inner product is defined as,

$$(u, v)_0 := \int_\Omega uv, \quad \forall\, u, v \in L^2(\Omega). \tag{1.5}$$

We denote the norm on $L^2(\Omega)$ by $\|\cdot\|_0$ and it is defined for $u \in L^2(\Omega)$ as,

$$\|u\|_0 := \left[\int_\Omega |u|^2 \right]^{\frac{1}{2}}. \tag{1.6}$$

Theorem 1.9. $L^2(\Omega)$ is a Hilbert space, i.e., a complete inner product space, with respect to the $L^2(\Omega)$ inner product.

1.1.2 Sobolev Spaces

This section is mainly taken from the book [19]. We begin with the concept of weak derivatives. For this, assume f to be continuously differentiable with derivative f' and by applying integration by parts, we have $\int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi'$. Similarly, $\int_\Omega (D^\alpha f)\phi = (-1)^{|\alpha|} \int_\Omega f(D^\alpha \phi)$ for some multi-index α. This relation eliminates the need of f to be differentiable. The weak derivative, if it exists, is unique almost everywhere in that set.

Definition 1.10. Let $f \in L^1_{loc}(\Omega)$, we say, f has αth weak derivative if there exists a function $f_\alpha \in L^1_{loc}(\Omega)$ such that,

$$\int_\Omega f D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega f_\alpha \phi dx.$$

1. f_α is called the αth weak derivative of f.

2. if $f \in C^k(\Omega)$, then $f_\alpha = D^\alpha f,$ where $|\alpha| \leq k$.

Fix $1 \leq p \leq \infty$ and let m be a positive integer. We define a certain function space, whose members have weak derivatives of various orders lying in various L^p spaces.

Definition 1.11. The Sobolev space $W^{m,p}(\Omega)$ is defined by $W^{m,p}(\Omega) := \{u \in L^p(\Omega) \,|\, D^\alpha u \in L^p(\Omega),\, |\alpha| \leq m\}$.

Hence, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all of the distributional derivatives up to order m are also in $L^p(\Omega)$. The norm on the space $W^{m,p}(\Omega)$ is defined as,

$$||u||_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p \right)^{\frac{1}{p}}.$$

The semi-norm over it consists of L^p-norms of highest derivatives of it. We also denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and its norm is denoted by $\|\cdot\|_m$. The inner product on the space $H^m(\Omega)$ is defined as,

$$(u, v)_m = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v.$$

If the boundary of Ω is smooth enough, precisely if $\partial\Omega$ is a Lipschitz boundary then $H^m(\Omega)$ is the closure of $C^\infty(\overline{\Omega})$ with respect to $\|\cdot\|_m$, for $m \geq 1$. Also, $C_c^\infty(\Omega) \subset W^{m,p}(\Omega)$, except for $p = \infty$. Define $W_0^{m,p}(\Omega)$ as the closed subspace of $W^{m,p}(\Omega)$ which is the the closure of the space $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. A norm on $W_0^{m,p}(\Omega)$ is defined as,

$$\|u\|_m^* := \sqrt{\sum_{|\alpha|=m} \|D^\alpha u\|_0^2}.$$

In this thesis, the space $H_0^1(\Omega) := \{u \in L^2(\Omega) \,|\, Du \in L^2(\Omega), u = 0 \text{ on } \partial\Omega\}$, will be mainly used. It is a Hilbert space with the inner product,

$$(u, v)_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v,$$

and thus, $\|u\|_1^* := \|\nabla u\|_0$. The following theorem gives us a relationship between the norms of the spaces H^m and H_0^m.

Theorem 1.12 (**Poincaré-Friedrichs inequality**)**.** Let Ω be a bounded domain with a Lipschitz boundary and $m \geq 1$, then there exists a constant $c > 0$, such that for every $u \in H_0^m(\Omega)$,

$$\|u\|_m^2 \leq c \sum_{|\alpha|=m} \|D^\alpha u\|_0^2.$$

Now, we discuss the concept of embeddings. Let X and Y be Banach spaces such that $X \subset Y$, then X is said to be continuously embedded in Y, denoted by $X \hookrightarrow Y$, if and only if there exists a constant $c > 0$, such that $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$. This embedding is said to be compact if weak convergence in X implies convergence in Y. Recall that $x_n \in X$ converges weakly to $x \in X$, i.e., $x_n \rightharpoonup x$ if for every $x' \in X'$, $\langle x', x_n \rangle_{X',X} \to \langle x', x \rangle$.

Theorem 1.13 (**Sobolev Embedding theorem** [35])**.** Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary. Let $0 \leq m < l$ be an integer and $1 \leq p, q < \infty$.

1. If $\frac{1}{p} \leq \frac{1}{q} + \frac{l-m}{N}$, then we have $W^{l,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$ and it is compact if $\frac{1}{p} <$

$\frac{1}{q} + \frac{l-m}{N}$.

2. If $\frac{1}{p} < \frac{l-m}{N}$, then we have $W^{l,p}(\Omega) \hookrightarrow C^m(\overline{\Omega})$ and it is compact.

Definition 1.14 (**Negative Sobolev space**)**.** Let Ω be a bounded domain in $\mathbb{R}^N$ and $m \geq 1$. The space $H^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$, i.e., $H^{-m}(\Omega) := \{\rho : H_0^m(\Omega) \to \mathbb{R},\ \rho$ is linear and bounded$\}$. The norm on $H^{-m}(\Omega)$ is given as,

$$\|\rho\|_{-m} := \sup_{\substack{v \neq 0 \\ v \in H_0^m(\Omega)}} \frac{\langle \rho, v \rangle_{H^{-m}(\Omega), H_0^m(\Omega)}}{\|v\|_m^*}.$$

The concepts of distribution theory and Sobolev spaces provide us with a new type of solution, i.e., a weak solution of a PDE, described in the succeeding section.

1.1.3 Weak Solution

A weak solution of a PDE is a type of solution which may not be differentiable at all the points, but it satisfies the given PDE in a weak sense. In the process of finding a weak solution, firstly, we derive a weak formulation of the given PDE. We have to use integration by parts frequently for finding the weak form of a PDE, so we need the following identities. The book [35] describes the first and second identities, and [18] mentions the third one, which is also known as Divergence formula.

Lemma 1.15 (**Green's identities**)**.** Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary and ν be the unit outward normal on $\partial\Omega$.

$$\int_\Omega u v_{x_i} = \int_{\partial\Omega} u v \nu_i - \int_\Omega u_{x_i} v,\ \forall\, u, v \in H^1(\Omega). \tag{1.7}$$

$$\int_\Omega u \Delta v = \int_{\partial\Omega} u(\nu \cdot \nabla v) - \int_\Omega \nabla u \cdot \nabla v,\ \forall\, u \in H^1(\Omega) \text{ and } v \in H^2(\Omega), \tag{1.8}$$

$$\int_\Omega \nabla \cdot u = \int_{\partial\Omega} u \cdot \nu,\ \forall\, u \in (H^1(\Omega))^2. \tag{1.9}$$

Weak Formulation

Let us consider that X be a Banach space and X' be the dual of X. For a given operator $L : X \to X'$, we have to solve the problem $Lx = f$, where $x \in X$ and $f \in X'$. Thus, we need to find $x \in X$ that satisfies the given equation. Consider for all $y \in X$ so that $\langle Lx, y\rangle_{X',X} = \langle f, y\rangle_{X',X} =: f(y)$.

Definition 1.16 (**Weak Solution**)**.** Consider L and $x \in X$ as defined above. The form $\langle Lx, y\rangle_{X',X} = \langle f, y\rangle_{X',X}$, for all $y \in X$, is called the weak formulation of the problem $Lx = f$. Moreover, the $x \in X$ which satisfies the problem $\langle Lx, y\rangle_{X',X} = \langle f, y\rangle_{X',X}$, for all $y \in X$, is called the weak solution of the problem $Lx = f$.

For example, consider for $u \in H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$ the following Poisson equation with Dirichlet boundary condition,

$$-\Delta u = f \text{ in } \Omega. \tag{1.10}$$

A classical or strong solution of the above problem is a function $u \in C^2(\bar{\Omega})$ which satisfies the equation (1.10) pointwise. We will find the weak form of this problem. On multiplying (1.10) by $v \in H_0^1(\Omega)$ and then by integrating over the domain Ω,

$$\int_\Omega (-\Delta u)v = \int_\Omega fv,$$

by using (1.8) and $u, v = 0$ on $\partial\Omega$, as $u, v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv.$$

Define $B(u, v) := \int_\Omega \nabla u \cdot \nabla v$, then this $B : H_0^1(\Omega) \times H_0^1(\Omega)$ is linear in each component, hence, is a bilinear form. Since $f \in H^{-1}(\Omega)$, it is a linear functional on $H_0^1(\Omega)$. The weak form of the given problem is $B(u, v) = f(v)$, for all $v \in H_0^1(\Omega)$. The $u \in H_0^1(\Omega)$ that satisfies (2.10) is a weak solution of given problem. Therefore,

$u \in H_0^1(\Omega)$ is not necessarily a C^2 function.

After defining the concept of weak formulation and a weak solution, we get concerned about the existence and uniqueness of the weak solution. The following results help us in determining the existence and uniqueness of a weak solution.

Theorem 1.17 (**Riesz Representation Theorem** [35]). Let $(H, (\cdot,\cdot))$ be a Hilbert space and f be a bounded linear functional on H. Then there exists a unique $u \in H$ such that $(u, v) = f(v)$, for all $v \in H$ and $\|u\| = \|f\|'$.

Theorem 1.18 (**Lax-Milgram Theorem** [35]). Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear form for which exist constants $c_1, c_2 > 0$ such that for every $u, v \in H$, the following conditions are satisfied,

- B is bounded, i.e., $|B(u, v)| \leq c_1\|u\|\|v\|$,
- B is coercive, i.e., $B(u, u) \geq c_2\|u\|^2$.

Let $f \in H'$, then there is a unique $u \in H$ and a constant $c > 0$ such that $B(u, v) = f(v)$, for all $v \in H$ and $\|u\| \leq c\|f\|'$, where $\|\cdot\|'$ is the norm on H'.

Theorem 1.19 (**Lions** [45]). Let H be a Hilbert space and $\{V, \|\cdot\|\}$ be a normed linear space. Suppose $B : H \times V \to \mathbb{R}$ is bilinear and that $B(\cdot, v)$ is in H' for each $v \in V$. Then the following are equivalent.

1. For each $f \in V'$ there exists a $u \in H$ such that

$$B(u, v) = f(v), \ \forall\, v \in V.$$

2. There exists a constant $c > 0$ such that

$$\inf_{\|v\|=1} \sup_{\|u\|\leq 1} |B(u, v)| \geq c > 0.$$

Theorem 1.20 (**Banach-Nečǎs-Babuška** [18])**.** Let W be a Banach space and V be a reflexive Banach space. Let $a : W \times V \to \mathbb{R}$ be a continuous bilinear form and $f \in V'$. Then the problem of finding $u \in W$ such that $a(u, v) = f(v)$, for all $v \in V$ has a unique solution if and only if there exists an $\alpha > 0$, such that

$$\inf_{w \in W} \sup_{v \in V} \frac{a(w, v)}{\|w\|_W \|v\|_V} \geq \alpha, \tag{1.11}$$

and,

$$\forall\, v \in V,\ (\forall\, w \in W \quad a(w, v) = 0) \Rightarrow (v = 0). \tag{1.12}$$

Moreover, the following a priori estimate holds:

$$\forall\, f \in V', \quad \|u\|_W \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Lemma 1.21 (**Characterisation of surjective operators** [18])**.** Let V, W be real Banach spaces, $A : V \to W$ be a linear operator and A^* be the adjoint of the operator A. The following statements are equivalent

1. $A : V \to W$ is surjective.

2. $A^* : W' \to V'$ is injective and the range of A^* is closed in V'.

3. There exists $\alpha > 0$ such that

$$\forall\, w' \in W',\ \|A^* w'\|_{V'} \geq \alpha \|w'\|_{W'}. \tag{1.13}$$

4. There exists $\alpha > 0$ such that

$$\inf_{w' \in W'} \sup_{v \in V} \frac{\langle A^* w', v \rangle_{V', V}}{\|w'\|_{W'} \|v\|_V} \geq \alpha. \tag{1.14}$$

1.2 Spectral Theory

This section is concerned with describing various results about spectral theory of linear operators. These results exclude the trivial vector space, i.e., $\{0\}$. We only discuss the case of linear spaces of infinite dimension.

1.2.1 Spectral Theory of Linear Operators

We first discuss the spectral theory of linear operators. Let X be a non-trivial normed space and $T : D(T) \to X$ be a linear operator, where $D(T) \subset X$ is the domain of T. We associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $D(T)$. The operator T may or may not be invertible.

Definition 1.22 (**Resolvent of T** [33])**.** If the operator T_λ is invertible, then we call its inverse as the resolvent of T, and it is denoted by $R_\lambda(T)$. It is also a linear operator. It is expressed as $R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$.

The resolvent of a linear operator is very important for determining the behavior of T. A particular class of the number λ is known as regular values.

Definition 1.23 (**Regular values** [33])**.** A regular value λ of T is a complex number such that

1. $R_\lambda(T)$ exists.

2. $R_\lambda(T)$ is bounded.

3. $R_\lambda(T)$ is defined on a set which is dense in X.

Definition 1.24 (**Resolvent set** [33])**.** The set of all regular values of T is known as the resolvent set of T. It is denoted by $\rho(T)$.

We now define the spectrum of T which is the main reason of existence of spectral theory of operators.

Definition 1.25 (**Spectrum** [33])**.** The complement of the resolvent set of T in the complex plane is known as the spectrum of T. It is denoted by $\sigma(T)$. Thus, $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The elements of the spectrum are known as spectral values of T.

There are two ways of classifying the spectrum. The first way gives us the partition of the spectrum into three disjoint sets. The results of the first type of classification of the spectrum have been taken from the book [33].

1. **Point Spectrum**

   The set of complex numbers λ such that $R_\lambda(T)$ does not exist is known as the point spectrum of T. It is denoted by $\sigma_p(T)$. An element of the point spectrum is known as an eigenvalue of T.

2. **Continuous Spectrum**

   The set of complex numbers λ such that $R_\lambda(T)$ exists and is defined on a set which is dense in X but is unbounded is known as the continuous spectrum of T. It is denoted by $\sigma_c(T)$.

3. **Residual Spectrum**

   The set of complex numbers λ such that $R_\lambda(T)$ exists but the domain of $R_\lambda(T)$ is not dense in X is known as the residual spectrum of T. It is denoted by $\sigma_r(T)$. The boundedness of $R_\lambda(T)$ does not matter for this spectrum.

These three kinds of spectrum are disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. The resolvent set and the spectrum of a linear operator are also disjoint; their union is the whole complex space.

In the case of a finite dimensional linear space the continuous spectrum and residual spectrum do not exist, i.e., if X is finite dimensional and $T : X \to X$ is a linear operator then $\sigma_c(T) = \sigma_r(T) = \phi$. Thus, every spectral value of T is an eigenvalue of T. The spectrum of a linear operator on a finite dimensional space is known as a *pure point spectrum*.

The definition of a point spectrum is similar to the definition of an eigenvalue of a matrix. Since for all $\lambda \in \sigma_p(T)$, we have $R_\lambda(T)$ doesn't exist, which means that the operator $(T - \lambda I)$ is not invertible. Then there exists some $x \neq 0$ in X such that $(T - \lambda I)x = 0$, i.e., $Tx = \lambda x$. This vector x is known as an eigenvector of the eigenvalue λ of T. The pair (λ, x) is called an eigenpair of T. In case X is infinite dimensional, an eigenvalue may have infinite multiplicity.

Definition 1.26 (**Approximate Point Spectrum** [27])**.** Let T be a bounded linear operator. The set of $\lambda \in \mathbb{C}$ such that there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in X$ with $\|\phi_n\| = 1$ and $\|(T - \lambda I)\phi_n\| \to 0$ as $n \to \infty$ is called the approximate point spectrum of T. It is denoted by $\sigma_{ap}(T)$. The elements of this set are called approximate eigenvalues of T.

1.2.2 Spectral Theory of Bounded Self-Adjoint Linear Operator

In this section, we discuss the case of bounded self-adjoint linear operators defined on a Hilbert space. This section is directly relevant to us. Firstly, we provide some basic definitions. Consider a complex Hilbert space H equipped with an inner product $(\cdot, \cdot) : H \times H \to \mathbb{R}$ and $T : H \to H$ to be a bounded linear operator. In contrast to (1.3), the operator $T^* : H \to H$ is said to be the adjoint of a bounded linear operator T if for all $x, y \in H$ it satisfies

$$(Tx, y) = (x, T^*y)\,.$$

The operator T^* is unique and $\|T^*\|_{\text{op}} = \|T\|_{\text{op}}$ on H.

Definition 1.27 (**Self-Adjoint Operator**)**.** A bounded linear operator T defined on a H is said to be self-adjoint if $T = T^*$. Then for all $x, y \in H$,

$$(Tx, y) = (x, Ty) \, .$$

We also have a result that says if T is a bounded linear operator on H then it is self-adjoint if and only if (Tx, x) is real for all $x \in H$. The spectrum of a bounded self-adjoint linear operator on a complex Hilbert space is real.

Theorem 1.28. The spectrum of $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where

$$m = \inf_{\substack{x \in X \\ \|x\|=1}} (Tx, x) \, , \tag{1.15}$$

$$M = \sup_{\substack{x \in X \\ \|x\|=1}} (Tx, x) \, . \tag{1.16}$$

The values m and M are spectral values of T.

Definition 1.29 (**Rayleigh quotient** [33])**.** The Rayleigh quotient of a bounded self-adjoint linear operator is defined as

$$q(x) := \frac{(Tx, x)}{(x, x)}.$$

Thus, we have

$$\sigma(T) \subseteq [\inf_{\substack{x \in H \\ x \neq 0}} q(x), \sup_{\substack{x \in H \\ x \neq 0}} q(x)].$$

We also have a result that the spectrum of T, as defined above, is not empty.

Theorem 1.30 (**Residual spectrum** [33])**.** The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ is empty.

We now give the second classification of the spectrum for the particular case of a bounded self-adjoint linear operator $T : H \to H$. This theory can be found in the book [27]. This classification partitions the spectrum into two disjoint sets.

1. **Discrete Spectrum**

   Let T be bounded self-adjoint linear operator on H. A real λ is said to belong to the discrete spectrum if and only if λ is an isolated point in $\sigma_p(T)$ and it has a finite multiplicity. We denote it by $\sigma_d(T)$.

2. **Essential Spectrum**

   Let T be bounded self-adjoint linear operator on H. The essential spectrum of T is defined as the complement in the spectrum of the discrete spectrum of T. It is expressed as $\sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_d(T)$. A point in the essential spectrum corresponds to either of the following:

   (a) a point in continuous spectrum.

   (b) a limit point of a sequence of eigenvalues with finite multiplicity.

   (c) an eigenvalue of infinite multiplicity.

The essential spectrum of a self-adjoint operator T is closed in $\mathbb{R}$. There is an important result related to the approximate point spectrum of a self-adjoint operator.

Theorem 1.31. Let T be a self-adjoint operator, then $\lambda \in \sigma(T)$ if and only if $\lambda \in \sigma_{ap}(T)$.

There is another criterion of finding the essential spectrum of a bounded self-adjoint linear operator on H which was given by Hermann Weyl in the year 1910.

Definition 1.32 (Weyl's characterization of the essential spectrum [27]). A $\lambda \in \mathbb{R}$ belongs to the essential spectrum, $\sigma_{\text{ess}}(A)$ of a self-adjoint bounded linear operator T defined on H, if and only if there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in H$ with $\|\phi_n\| = 1$, ϕ_n tends weakly to 0 and $\|(T - \lambda I)\phi_n\| \to 0$ as $n \to \infty$. Such a sequence ϕ_n is called singular Weyl's sequence.

A point λ such that there exists an associated Weyl's sequence is said to belong to Weyl's spectrum. It is denoted by $W(T)$. It is a subset of the essential spectrum of T. This characterization is very helpful if we know that there is no residual spectrum and we are aware that a particular λ is not an eigenvalue, then it gives a sufficient condition for λ to be in the continuous spectrum.

Definition 1.33 (**Positive Operator** [33])**.** A bounded self-adjoint linear operator $T : H \to H$ is said to be positive if and only if $(Tx, x) \geq 0$ for all $x \in H$, i.e., T is coercive.

Theorem 1.34. A bounded self-adjoint linear operator T on a complex Hilbert space H is positive if and only if its spectrum consists of nonnegative real values only.

A very important type of an operator is the compact operator. A bounded linear operator $T : X \to Y$ is said to be compact if and only if for every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$, we have the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ is precompact in Y, i.e., there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that the sequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ converges in Y.

Theorem 1.35 (**Compact Operators** [33])**.** Let $T : X \to X$ be a compact linear operator on a Banach Space X. Then every spectral value $\lambda \neq 0$ of T (if it exists) is an eigenvalue of T.

The above theorem gives us the result that for a compact operator the continuous spectrum and residual spectrum do not exist.

2

Stokes Problem and Uzawa Pressure Operator

The Stokes problem is one of the problems occurring in fluid dynamics. The type of flow for which Reynold's number is low, say, $R_e \ll 1$, i.e., the fluid velocity is very small, or the viscosity is very large, or an extremely small length scale is considered, is called the Stokes flow. It is also known as a *creeping flow*. It was named after Sir George Gabriel Stokes. The equations of motion of the Stokes flow are called the Stokes equations. We consider the Stokes equations in the steady state along with Dirichlet boundary conditions, so we term them as the Stokes problem. This extremely slow flow is evident in many cases such as swimming of a microorganism, flow of lava, flow of polymers, etc. In this chapter, we will analyze the existence of the solution of Stokes problem and its dependency on the spectrum of the Uzawa pressure operator.

2.1 Stokes problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary. The velocity field, denoted by u, is a vector quantity. We consider $u \in V := (H_0^1(\Omega))^2 = H_0^1(\Omega) \times H_0^1(\Omega)$.

The pressure, denoted by p, is a scalar quantity. We consider $p \in L_0^2(\Omega)$, where

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \right\}.$$

The norm of p is given as,

$$\|p\|_0 = \left[\int_\Omega |p|^2 \right]^{\frac{1}{2}}, \tag{2.1}$$

and the inner product on the space $L_0^2(\Omega)$ is defined to be the same as that for $L^2(\Omega)$, which is given by equation (1.5). Before we define the norm on the space V, we define the following two bilinear forms. For $u, v \in V$ and $q \in L_0^2(\Omega)$,

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i, \tag{2.2}$$

$$b(q, v) := -\int_\Omega q(\nabla \cdot v). \tag{2.3}$$

The bilinear form $a(u, v)$, for all $u = (u_1, u_2), v = (v_1, v_2) \in V$, is an inner product on the space V. The norm on the space V is denoted by $\|\cdot\|$ and it is defined for all $u = (u_1, u_2) \in V$ as,

$$\|u\| := \sqrt{a(u, u)} = \left[\int_\Omega |\nabla u_1|^2 + \int_\Omega |\nabla u_2|^2 \right]^{\frac{1}{2}}. \tag{2.4}$$

Definition 2.1 (**Inverse Laplacian**). The inverse laplacian is denoted by $\Delta^{-1} : (H^{-1}(\Omega))^2 \to V$. Let $u \in (H^{-1}(\Omega))^2$, we say $\Delta^{-1} u = v \in V$ if

$$\Delta v = u \text{ in } \Omega,$$
$$v = 0 \text{ on } \partial\Omega.$$

Here, Δ is the vector laplacian as $v \in V$ is a vector having two components.

The operator defined above will be used frequently in this text. Now, we define

V' as the dual of the space V, thus, $V' := (H^{-1}(\Omega))^2$. The norm on V' is denoted as $\|\cdot\|'$. For any $f \in V'$, the norm is defined as

$$\|f\|' := \sup_{\substack{v \in V \\ v \neq 0}} \frac{\langle f, v \rangle_{V',V}}{\|v\|}, \tag{2.5}$$

or equivalently as given in the book [35] as,

$$\|f\|'^2 = \int_\Omega \nabla((-\Delta)^{-1} f) \cdot \nabla((-\Delta)^{-1} f). \tag{2.6}$$

Definition 2.2 (**Stokes problem** [35])**.** On a bounded domain Ω with a Lipschitz boundary and for $(u, p) \in V \times L_0^2(\Omega)$, $f \in V'$ the Stokes problem is given as,

$$-\Delta u + \nabla p = f \text{ in } \Omega, \tag{2.7}$$

$$\nabla \cdot u = 0 \text{ in } \Omega, \tag{2.8}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2.9}$$

The different function spaces for velocity and pressure increase the difficulty of this problem. Moreover, it is a system of three PDEs; the first two arise from (2.7) and the last one from (2.8). It is not coercive as well so that Lax-Milgram Lemma can't be applied. Non-coerciveness is proved easily from the weak form of the Stokes problem. This problem has a unique solution $(u, p) \in V \times L_0^2(\Omega)$. We now move on to discuss the existence and uniqueness of the solution of the Stokes problem, for which we have to find its weak formulation.

Weak formulation

We now try to find the weak form of the Stokes problem. The momentum equations are given by (2.7),

$$-\Delta u + \nabla p = f \text{ in } \Omega.$$

Let $v \in V$, so we have

$$\int_\Omega (-\Delta u) \cdot v + \int_\Omega (\nabla p) \cdot v = \int_\Omega f \cdot v,$$
$$\sum_{i=1}^{2} \int_\Omega (-\Delta u_i) v_i + \sum_{i=1}^{2} \int_\Omega p_{x_i} v_i = \int_\Omega f \cdot v.$$

By using integration by parts (1.8) and the fact that $v \in V$, i.e., $v_i = 0$ on $\partial\Omega$,

$$\sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i - \sum_{i=1}^{2} \int_\Omega p (v_i)_{x_i} = \int_\Omega f \cdot v.$$

So,

$$\sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i - \int_\Omega p \nabla \cdot v = f(v).$$

By using (2.2) and (2.3), the weak form of momentum equations, as

$$a(u, v) + b(p, v) = f(v). \tag{2.10}$$

By the conservation of mass, i.e., (2.8),

$$\nabla \cdot u = 0 \text{ in } \Omega.$$

For $q \in L_0^2(\Omega)$,

$$\int_\Omega q(\nabla \cdot u) = 0.$$

On using (2.3), the weak form of conservation of mass

$$b(q, u) = 0. \tag{2.11}$$

Thus, from equations (2.10) and (2.11), the weak formulation of the Stokes problem for all $v \in V$ and $q \in L_0^2(\Omega)$ is

$$\begin{aligned} a(u,v) + b(p,v) &= f(v), \\ b(q,u) &= 0. \end{aligned} \tag{2.12}$$

The solution $(u,p) \in V \times L_0^2(\Omega)$ of the weak formulation is known as the weak solution of the Stokes problem.

Proposition 2.3. 1. The bilinear form $a : V \times V \to \mathbb{R}$ defined by (2.2) is continuous, i.e., bounded.

2. The bilinear form $b : L_0^2(\Omega) \times V \to \mathbb{R}$ defined by (2.3) is continuous, i.e., bounded.

Proof. 1. Consider the bilinear function $a : V \times V \to \mathbb{R}$, defined by (2.2). For $u, v \in V$,

$$a(u,v) = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i = \int_\Omega \nabla u \cdot \nabla v \leq \|\nabla u\|_0 \|\nabla v\|_0 = \|u\|\|v\|.$$

Hence, $a(u,v) \leq \|u\|\|v\|$, i.e., a is bounded or continuous.

2. Consider the bilinear function $b : L_0^2(\Omega) \times V \to \mathbb{R}$, defined by (2.3). For $q \in L_0^2(\Omega)$ and $v \in V$,

$$|b(q,v)| = \left| \int_\Omega q(\nabla \cdot v) \right| \leq \|q\|_0 \|\nabla \cdot v\|_0.$$

We try to estimate $\|\nabla \cdot v\|_0$ as follows.

$$\begin{aligned} \|\nabla \cdot v\|_0^2 = \int_\Omega |\nabla \cdot v|^2 = \int_\Omega ((v_1)_{x_1} + (v_2)_{x_2})^2 &= \int_\Omega (v_1)_{x_1}^2 + (v_2)_{x_2}^2 + 2(v_1)_{x_1}(v_2)_{x_2} \\ \leq 2 \int_\Omega \left[(v_1)_{x_1}^2 + (v_2)_{x_2}^2 \right] &\leq 2 \sum_{i=1}^{2} \int_\Omega |\nabla v_i|^2. \end{aligned} \tag{2.13}$$

So, $\|\nabla \cdot v\|_0 \leq \sqrt{2}\|v\|$ and

$$|b(q, v)| \leq \sqrt{2}\|q\|_0\|v\|,$$

which proves that b is bounded or continuous.

$\square$

2.1.1 Function spaces for the Stokes problem

Definition 2.4 ($\mathbb{H}^1_{0,\text{div}}(\Omega)$ **space**)**.** We define the space $\mathbb{H}^1_{0,\text{div}}(\Omega)$, as the space consisting of the divergence free vectors of the space V.

$$\mathbb{H}^1_{0,\text{div}}(\Omega) := \{v \in V \,|\, \nabla \cdot v = 0\}. \tag{2.14}$$

Let $v \in V$ and we know that the divergence operator, i.e., $\nabla\cdot$ maps the space V to $L^2(\Omega)$, i.e., $\nabla \cdot v \in L^2(\Omega)$. On calculating the average of $\nabla \cdot v$ by using the formula (1.9),

$$\int_\Omega \nabla \cdot v = \int_{\partial\Omega} \nu \cdot v = 0.$$

As $v \in V$, thus, $v = 0$ on $\partial\Omega$. Therefore, for all $v \in V$, the image $\nabla \cdot v \in L^2_0(\Omega)$. Hence, the range space of the divergence operator is the space $L^2_0(\Omega)$.

Lemma 2.5. Let Ω be a bounded open set with a Lipschitz boundary. The operator $\nabla\cdot : V \to L^2_0(\Omega)$ is surjective.

The proof of the above Lemma can be found in Chapter 1 of the book [25].

Corollary 2.6. Under the hypothesis of Lemma 2.5, there exists $\beta > 0$ such that

$$\inf_{q \in L^2_0(\Omega)\setminus 0} \sup_{v \in V \setminus 0} \frac{b(q, v)}{\|v\|\|q\|_0} \geq \beta.$$

Proof. Since $\nabla\cdot : V \to L_0^2(\Omega)$ is a surjective operator, therefore, by using the equivalence (1) $\Leftrightarrow$ (4) of Lemma 1.21 and ∇ is the adjoint of $-\nabla\cdot$, therefore, we get that there exists a positive constant $\beta > 0$ such that the following condition is satisfied.

$$\inf_{q\in L_0^2(\Omega)\setminus 0} \sup_{v\in V\setminus 0} \frac{(\nabla q, v)_0}{\|v\|\|q\|_0} \geq \beta.$$

$\square$

Decomposition of the space V

The velocity component of the solution of the Stokes problem lies in the Hilbert space V. This space can be decomposed in two ways, the latter one being an extension for the former one. The decomposition of the space V is important for understanding the properties of the problem.

First decomposition of V

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary. We defined in the previous section $V := (H_0^1(\Omega))^2$. Let V_1 denote the space defined by the equation (2.14), i.e., $V_1 = \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$. Thus, V_1 is a closed subspace of the space V. Consider the space orthogonal to V_1 defined by using Definition 1.6 as follows,

$$V_1^\perp := \{v \in V \mid a(v, w) = 0,\ \forall\, w \in V_1\}. \tag{2.15}$$

The space V can be written as $V = V_1 \oplus V_1^\perp$. This gives us the first decomposition of the space V. We can find an expression for the space $V_1^\perp$. The following lemma is taken from the book [25].

Lemma 2.7. If $f \in V'$ satisfies $\langle f, v\rangle_{V',V} = 0$ for all $v \in V_1$ then there exists $p \in L^2(\Omega)$ such that $f = \nabla p$. When Ω is connected p is unique up to an additive constant.

Proof. The operator $-\nabla$ maps $L^2(\Omega)$ to V'. It is the dual operator of the divergence operator $\nabla\cdot : V \to L^2(\Omega)$. The range of the gradient, i.e., $\mathcal{R}(\nabla)$ is a closed subspace of V', so that by using *Banach Closed Range theorem* [51],

$$\mathcal{R}(\nabla) = (\mathrm{Ker}(\mathrm{div}))^0 = V_1^0,$$

where V_1^0 represents the polar space of V_1. By using Definition 1.7,

$$V_1^0 := \left\{ f \in V' \mid \langle v, w \rangle_{V',V} = 0,\ \forall\, w \in V_1 \right\}. \tag{2.16}$$

Thus, if $f \in V_1^0$, then there exists $p \in L^2(\Omega)$ such that $f = \nabla p$. $\square$

Corollary 2.8 (**The space** $V_1^\perp$ [25])**.** The space $V_1^\perp$ can be expressed as,

$$V_1^\perp = \left\{ (-\Delta)^{-1} \nabla q,\ \forall\, q \in L^2(\Omega) \right\}. \tag{2.17}$$

Proof. First, we prove that $\{(-\Delta)^{-1}\nabla q,\ \forall\, q \in L^2(\Omega)\} \subset V_1^\perp$. Consider for some $q \in L^2(\Omega)$ and $v \in V_1$,

$$a((-\Delta)^{-1}\nabla q, v) = \int_\Omega \nabla(-\Delta)^{-1}\nabla q \cdot \nabla v = \int_\Omega -\Delta(-\Delta)^{-1}\nabla q \cdot v = \int_\Omega \nabla q \cdot v = \int_\Omega -q \nabla\cdot v = 0.$$

Hence, $\{(-\Delta)^{-1}\nabla q,\ \forall\, q \in L^2(\Omega)\} \subset V_1^\perp$. We prove the converse of this relation, i.e., $V_1^\perp \subset \{(-\Delta)^{-1}\nabla q,\ \forall\, q \in L^2(\Omega)\}$. Let $u \in V_1^\perp$, then $\nabla u = [\nabla u_1, \nabla u_2]^T \in (L^2(\Omega))^2 \subset V'$. So, we define $f \in V'$, such that $\langle f, v \rangle_{V',V} = f(v) = a(u, v)$ for all $v \in V$. Take $w \in V_1$, then $f(w) = a(u, w) = 0$ as $u \in V_1^\perp$. Hence, by using Lemma 2.7, there exists a $p \in L^2(\Omega)$ such that for all $v \in V$, we have $f(v) = \langle \nabla p, v \rangle_{V',V}$, i.e.,

$$\langle \nabla p, v \rangle_{V',V} = a(u, v) = \int_\Omega \nabla u \cdot \nabla v = \langle -\Delta u, v \rangle_{V',V}.$$

Thus, $u = (-\Delta)^{-1}\nabla p$ for some $p \in L^2(\Omega)$. Hence, we get the required result. $\square$

Corollary 2.9 (**Isomorphisms** [25])**.** Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$. Then

1. the operator ∇ is an isomorphism of $L_0^2(\Omega)$ onto V_1^0.

2. the operator $\nabla\cdot$ is an isomorphism of $V_1^\perp$ onto $L_0^2(\Omega)$.

Proof. 1. We know that ∇ maps the space $L_0^2(\Omega)$ to the space V_1^0. By using Lemma 2.7, this operator is a bijection on these Banach spaces. Hence, it's an isomorphism.

2. Since $\nabla\cdot$ is the dual operator of $-\nabla$, it is an isomorphism from $(V_1^0)'$ onto $L_0^2(\Omega)$. We need to prove that $(V_1^\perp)'$ can be identified with the space V_1^0. Let $w \in (V_1^\perp)'$ and we restrict $w \to \hat{w}$, so that $\hat{w}$ operates on all elements of V. That is, by setting for all $v \in V$ and Pv as the orthogonal projection of v on $V_1^\perp$,

$$\langle \hat{w}, v \rangle = \langle w, Pv \rangle.$$

Let $v_1 \in V_1$ then $Pv_1 = 0$, hence

$$\langle \hat{w}, v_1 \rangle = \langle w, Pv_1 \rangle = 0.$$

This implies, $\hat{w} \in V_1^0$ and the mapping $w \to \hat{w}$ is an isomorphism from $(V_1^\perp)'$ onto V_1^0. Thus, we can identify $(V_0^1)'$ with $V_1^\perp$. Hence, we get the result.

$\square$

Second Decomposition of V

First of all, we define the scalar-curl of a vector function $f = (f_1, f_2) \in \mathbb{R}^2$ as follows,

$$\mathbf{curl}\, f := \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}. \tag{2.18}$$

Furthermore, curl of a scalar function ϕ is defined as follows,

$$\nabla^{\perp}\phi := \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right). \tag{2.19}$$

Consider the space $V = (H_0^1(\Omega))^2$ equipped with the norm $\|\cdot\|$, defined for all $v = (v_1, v_2) \in V$ as $\|v\| = \sqrt{a(v,v)}$. By using equation (2.4),

$$\|v\| = \left[\int_\Omega |\nabla v_1|^2 + \int_\Omega |\nabla v_2|^2\right]^{\frac{1}{2}} = \left[\int_\Omega (v_1)_{x_1}^2 + (v_1)_{x_2}^2 + (v_2)_{x_1}^2 + (v_2)_{x_2}^2\right]^{\frac{1}{2}}. \tag{2.20}$$

Now, we define the inner product on the space V in a different way. Let $u, v \in V$, then the inner product $(\cdot,\cdot)_{1,\Omega} : V \times V \to \mathbb{R}$ can be written

$$(u,v)_{1,\Omega} := (\nabla \cdot u, \nabla \cdot v)_0 + (\mathbf{curl}\, u, \mathbf{curl}\, v)_0, \tag{2.21}$$

where $(\cdot,\cdot)_0$ is the $L^2(\Omega)$ inner product. The inner product (2.21) defines a norm on the space V as for any $v \in V$,

$$\|v\| = \left[\|\nabla \cdot v\|_0^2 + \|\mathbf{curl}\, v\|_0^2\right]^{\frac{1}{2}}. \tag{2.22}$$

We claim that the norm on the space V defined by (2.20) is same as the one defined by (2.22). It can be easily proved as follows. Consider the norm on the space V defined by (2.22), so that for any $v \in V$,

$$\begin{aligned}
\|v\| &= \left[\|\nabla \cdot v\|_0^2 + \|\mathbf{curl}\, v\|_0^2\right]^{\frac{1}{2}} \\
&= \left[\int_\Omega |\nabla \cdot v|^2 + \int_\Omega |\mathbf{curl}\, v|^2\right]^{\frac{1}{2}} \\
&= \left[\int_\Omega (v_1)_{x_1}^2 + (v_2)_{x_2}^2 + 2(v_1)_{x_1}(v_2)_{x_2} + \int_\Omega (v_2)_{x_1}^2 + (v_1)_{x_2}^2 - 2(v_2)_{x_1}(v_1)_{x_2}\right]^{\frac{1}{2}} \\
&= \left[\int_\Omega (v_1)_{x_1}^2 + (v_1)_{x_2}^2 + (v_2)_{x_1}^2 + (v_2)_{x_2}^2\right]^{\frac{1}{2}},
\end{aligned}$$

which is equal to the norm on V defined by (2.20). Hence, the two norms are the same. On the basis of norm on V defined by (2.22), we divide the space V into three disjoint subspaces. The first subspace is defined as the **divergence-free space**, $V_1 := \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$. The second subspace is defined as the **curl-free space**, $V_2 := \{v \in V \mid \mathbf{curl}\, v = 0\}$. Note that, this subspace is orthogonal to the subspace V_1, as for any $v_1 \in V_1$ and $v_2 \in V_2$,

$$(v_1, v_2)_{1,\Omega} = (\nabla \cdot v_1, \nabla \cdot v_2)_0 + (\mathbf{curl}\, v_1, \mathbf{curl}\, v_2)_0 = (0, \nabla \cdot v_2)_0 + (\mathbf{curl}\, v_1, 0)_0 = 0.$$

The third subspace of this decomposition is the **orthogonal space** W, which is the space orthogonal to both of the subspaces V_1 and V_2, i.e.,

$$W := \{w \in V \mid (v, w)_{1,\Omega} = 0,\ \forall\, v \in V_1 \cup V_2\}. \tag{2.23}$$

Since the subspaces defined as above are orthogonal,

$$V = V_1 \oplus V_2 \oplus W. \tag{2.24}$$

This decomposition of the space V is called the **Crouzeix-Velte decomposition**, as it was given by M. Crouzeix in [13] and W. Velte in [50] in the year 1996. It is useful in proving some properties of the Uzawa pressure operator, which we will define in the next section.

2.2 Uzawa Pressure Operator

An operator that arises during the study of the Stokes problem is the Uzawa pressure operator. The spectrum of the Uzawa pressure operator plays a major role in the process of finding the solution of the Stokes problem. In this section, we discuss some properties of the Uzawa pressure operator and its relationship with the Stokes problem. We first define the Uzawa pressure operator.

Definition 2.10. The Uzawa pressure operator is denoted by S and it is defined on the space $L_0^2(\Omega)$, i.e., $S : L_0^2(\Omega) \to L_0^2(\Omega)$ such that $S := \nabla \cdot \Delta^{-1}\nabla$.

Let us apply S on $p \in L_0^2(\Omega)$ to see how S maps the space $L_0^2(\Omega)$ to itself. Since $Sp = \nabla\cdot\Delta^{-1}\nabla p$, therefore firstly the gradient operator acts on p, so $\nabla p \in (H^{-1}(\Omega))^2$. In the next step Δ^{-1} maps $(H^{-1}(\Omega))^2$ to V and finally divergence $\nabla\cdot$ maps V to the space $L_0^2(\Omega)$. Hence, $S : L_0^2(\Omega) \to L_0^2(\Omega)$. It is a very special operator which changes its behavior according to the shape of the domain. Such interesting properties of this operator will be discussed later in this text. Firstly, we will review some fundamental properties of this operator, described in the book [35].

Proposition 2.11. The Uzawa pressure operator $S : L_0^2(\Omega) \to L_0^2(\Omega)$ has the following properties.

a) It is self-adjoint with respect to $L^2(\Omega)$ inner product.

b) It is bounded.

c) It is coercive.

d) It is a bijective operator.

Proof. a) Let $p, q \in L_0^2(\Omega)$. Define $\hat{p} := \Delta^{-1}\nabla p$ and $\hat{q} := \Delta^{-1}\nabla q$. Consider,

$$
\begin{aligned}
(q, Sp)_0 = \int_\Omega qSp = \int_\Omega q\nabla \cdot (\Delta^{-1}\nabla p) \\
&= -\int_\Omega \nabla q \cdot \Delta^{-1}\nabla p \\
&= -\int_\Omega \Delta\hat{q} \cdot \hat{p} \\
&= -\sum_{i=1}^{2} \int_\Omega (\Delta\hat{q}_i)\hat{p}_i \\
&= \sum_{i=1}^{2} \int_\Omega \nabla\hat{q}_i \cdot \nabla\hat{p}_i = \int_\Omega pSq = (Sq, p)_0.
\end{aligned}
$$

Thus, S is self adjoint.

b) Since, we have

$$(q, Sp)_0 = \int_\Omega qSp = \sum_{i=1}^{2} \int_\Omega \nabla \hat{q}_i \cdot \nabla \hat{p}_i.$$

Thus,

$$\left| \int_\Omega qSp \right| \leq \sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0 \|\nabla \hat{p}_i\|_0 \leq \sqrt{\sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0^2} \sqrt{\sum_{i=1}^{2} \|\nabla \hat{p}_i\|_0^2}. \tag{2.25}$$

Now, we can prove that

$$\sqrt{\sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0^2} \leq \sqrt{2} \|q\|_0,$$

as follows

$$\begin{aligned} \sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0^2 &= \sum_{i=1}^{2} \int_\Omega \nabla \hat{q}_i \cdot \nabla \hat{q}_i = -\sum_{i=1}^{2} \int_\Omega \hat{q}_i \Delta \hat{q}_i \\ &= -\int_\Omega \hat{q} \cdot \Delta \hat{q} = -\int_\Omega \hat{q} \cdot \nabla q = \int_\Omega q \nabla \cdot \hat{q} \\ &\leq \|q\|_0 \|\nabla \cdot \hat{q}\|_0. \end{aligned}$$

By using (2.13),

$$\|\nabla \cdot \hat{q}\|_0 \leq \sqrt{2} \sqrt{\sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0^2},$$

which implies,

$$\sqrt{\sum_{i=1}^{2} \|\nabla \hat{q}_i\|_0^2} \leq \sqrt{2} \|q\|_0.$$

By using the above result in (2.25),

$$(q, Sp)_0 \leq \left| \int_\Omega qSp \right| \leq 2 \|q\|_0 \|p\|_0.$$

Hence, S is bounded.

c) Consider,

$$\begin{aligned}(p, Sp)_0 = \int_\Omega pSp &= \sum_{i=1}^{2} \int_\Omega \nabla \hat{p}_i \cdot \nabla \hat{p}_i \\ &= \sum_{i=1}^{2} \|\nabla \hat{p}_i\|_0^2 \\ &= \sum_{i=1}^{2} \|\hat{p}_i\|_1^{*2} = \|\hat{p}\|_1^{*2}.\end{aligned}$$

Since $\nabla p = \Delta \hat{p}$, hence, $\|\nabla p\|_{-1} = \|\Delta \hat{p}\|_{-1}$. Since

$$\|\Delta \hat{p}\|_{-1} \leq \|\Delta\|_{-1} \|\hat{p}\|_1^* = \|\hat{p}\|_1^*,$$

so,

$$\|\nabla p\|_{-1}^2 \leq \|\hat{p}\|_1^{*2}.$$

Using the above and Nečas inequality (4.4),

$$\begin{aligned}(p, Sp)_0 = \int_\Omega pSp &\geq \|\nabla p\|_{-1}^2 \\ &\geq c\|p\|_0^2. \end{aligned} \tag{2.26}$$

Hence, S is coercive.

d) Since S is a bounded linear operator on the Hilbert space $L_0^2(\Omega)$ and it is coercive, therefore it is a bijective operator on the space $L_0^2(\Omega)$. It follows from the result that, coerciveness is a sufficient condition for a bounded linear operator on a Banach space to be a bijection [18].

□

The above properties help us in proving the existence of the unique solution of the Stokes problem.

Theorem 2.12 (**Existence of the unique solution**)**.** The Stokes equations given by (2.7) and (2.8) have a unique solution $(u, p) \in V \times L_0^2(\Omega)$.

Proof. Consider the weak form of the Stokes equations given by (2.10) and (2.11), where $u, v \in V$. The equation (2.8) implies that $u \in \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$. Let's take $v \in \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$, then $b(p, v) = 0$. So the weak form becomes $a(u, v) = f(v)$, for $f \in V'$. From Proposition 2.3, we have that $a : V \times V$ is a bounded bilinear form. Also $a(u, u) = \|u\|^2$ so that a is coercive.
Hence, $a : \mathbb{H}^1_{0,\mathrm{div}}(\Omega) \times \mathbb{H}^1_{0,\mathrm{div}}(\Omega) \to \mathbb{R}$ is a bilinear functional such that it is bounded and coercive and by applying Theorem 1.18, i.e., Lax-Milgram Theorem, for $f \in V'$, there exists a unique $u \in \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$ such that $a(u, v) = f(v)$, for all $v \in \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$. For a unique $u \in \mathbb{H}^1_{0,\mathrm{div}}(\Omega)$ consider (2.7),

$$-\Delta u + \nabla p = f \Rightarrow u = \Delta^{-1}\nabla p - \Delta^{-1} f.$$

For this u, we have $\nabla \cdot u = 0$ in Ω, which gives

$$Sp = \nabla \cdot \Delta^{-1}\nabla p = \nabla \cdot \Delta^{-1} f \text{ in } \Omega. \tag{2.27}$$

Since S is a bijective operator on the space $L_0^2(\Omega)$, the above equation determines p uniquely without any boundary conditions on p. Thus, we get a unique $p \in L_0^2(\Omega)$, for the unique $u \in V$. Hence, there exists a unique solution $(u, p) \in V \times L_0^2(\Omega)$ for the Stokes equation. $\square$

The Corollary 2.6 gives us the following condition.

Definition 2.13 (**Inf-Sup condition for Stokes problem in continuous form**)**.** The inf-sup condition for Stokes problem is said to be satisfied if and only if there exists a positive constant $\beta_0(\Omega)$, such that

$$\inf_{q\in L_0^2\backslash 0}\sup_{v\in V\backslash 0}\frac{|b(q,v)|}{\|v\|\|q\|_0}\geq\beta_0(\Omega)>0. \tag{2.28}$$

The unique solution of the Stokes problem satisfies the above inf-sup condition. This condition is a necessary and sufficient condition for the existence of the solution of the Stokes problem. In the section 3 of paper [17], Lemma 3 gives us a result regarding the exchange of inf-sup condition. Let Y and Z be two Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \to Z$ be a bijective bounded linear map. Assume that Y is reflexive. Then

$$\inf_{y\in Y}\sup_{z'\in Z'}\frac{|\langle z',By\rangle_{Z',Z}|}{\|z'\|_{Z'}\|y\|_Y}=\inf_{z'\in Z'}\sup_{y\in Y}\frac{|\langle z',By\rangle_{Z',Z}|}{\|z'\|_{Z'}\|y\|_Y}. \tag{2.29}$$

Since $\nabla : L_0^2(\Omega)\to V_1^0$ is bijective by Corollary 2.9, therefore, by using the Lemma stated above,

$$\inf_{q\in L_0^2(\Omega)}\sup_{v\in V_1^{0\prime}}\frac{b(q,v)}{\|q\|_0\|v\|}=\inf_{v\in V_1^{0\prime}}\sup_{q\in L_0^2(\Omega)}\frac{b(q,v)}{\|q\|_0\|v\|}.$$

We now try to relate this inf-sup condition with the Uzawa pressure operator. The **Rayleigh quotient** of S is given as

$$\frac{(Sp,p)_0}{\|p\|_0^2}.$$

Thus, the infimum of the spectrum of S, i.e., $\lambda_{\min}(S)$ and the supremum of the spectrum of S, i.e., $\lambda_{\max}(S)$ are given as

$$\lambda_{\min}(S)=\inf_{\substack{p\in L_0^2(\Omega)\\ p\neq 0}}\frac{(Sp,p)_0}{\|p\|_0^2}, \tag{2.30}$$

$$\lambda_{\max}(S)=\sup_{\substack{p\in L_0^2(\Omega)\\ p\neq 0}}\frac{(Sp,p)_0}{\|p\|_0^2}. \tag{2.31}$$

We give some important properties of the Uzawa pressure operator, i.e., S in the form of the following theorem. These properties will be very helpful in describing the spectrum of this operator, which will be discussed in the next chapter. The first property was given by Michel Crouzeix [13] and the later two were described in the paper [24].

Theorem 2.14 (Properties). The Uzawa pressure operator $S := \nabla \cdot \Delta^{-1} \nabla$ is defined on the space $L_0^2(\Omega)$ and satisfies the following properties,

1. There exists a real number $0 < \alpha < 1$ such that for all $p \in L_0^2(\Omega)$,

$$\alpha \|p\|_0^2 \leq (Sp, p)_0 \leq \|p\|_0^2. \tag{2.32}$$

2. $\lambda_{\max}(S) \leq 1$.

3. $\|S\|_{\text{op}} \leq 1$.

Proof. 1. Let $p \in L_0^2(\Omega)$ and $u \in V$ such that $\Delta u = \nabla p$ in Ω, then $u = \Delta^{-1} \nabla p$ in Ω. So, $Sp = \nabla \cdot \Delta^{-1} \nabla p = \nabla \cdot u$ in Ω. We consider

$$\|Sp\|_0^2 = \|\nabla \cdot u\|_0^2 \leq \|\nabla \cdot u\|_0^2 + \|\mathbf{curl} u\|_0^2 = \|u\|^2. \tag{2.33}$$

We claim that $\|u\|^2 = (p, \nabla \cdot u)_0$. This can be proved as follows.

From equation (2.4),

$$\begin{aligned} \|u\|^2 = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla u_i = -\sum_{i=1}^{2} \int_\Omega (\Delta u_i) u_i \\ = -\int_\Omega \nabla p \cdot u = \int_\Omega p(\nabla \cdot u) = (p, \nabla \cdot u)_0. \end{aligned} \tag{2.34}$$

So by using the above two results,

$$\|Sp\|_0^2 \leq (p, \nabla \cdot u)_0 = (p, Sp)_0 \leq \|Sp\|_0 \|p\|_0.$$

Therefore, $\|Sp\|_0 \leq \|p\|_0$, which further gives

$$(Sp, p)_0 \leq \|Sp\|_0 \|p\|_0 \leq \|p\|_0^2.$$

By using the above equation and result that S is coercive from equation (2.26), there exists a real number $0 < \alpha < 1$ such that for all $p \in L_0^2(\Omega)$,

$$\alpha \|p\|_0^2 \leq (Sp, p)_0 \leq \|p\|_0^2.$$

2. We know by equation (2.31) that $\lambda_{\max}(S)$ is given as

$$\lambda_{\max}(S) = \sup_{\substack{p \in L_0^2(\Omega) \\ p \neq 0}} \frac{(Sp, p)_0}{\|p\|_0^2}.$$

By using the right hand side of the inequality (2.32),

$$\frac{(Sp, p)_0}{\|p\|_0^2} \leq 1.$$

Therefore, $\lambda_{\max}(S) \leq 1$.

3. The operator norm of S is given as,

$$\|S\|_{\text{op}} = \sup_{\substack{p \in L_0^2(\Omega) \\ p \neq 0}} \frac{\|Sp\|_0}{\|p\|_0}.$$

Since we proved in part 1 of this theorem that $\|Sp\|_0 \leq \|p\|_0$, which implies $\|S\|_{\text{op}} \leq 1$.

□

The Uzawa pressure operator plays a very important role not only in finding the solution of Stokes problem, but also in existence of the solution of Stokes problem. The following theorem describes their relationship.

Theorem 2.15 (**Relationship with the Inf-Sup constant** [9])**.** The inf-sup constant is equal to the square-root of the infimum of the spectrum of the Uzawa pressure operator, i.e.,

$$\beta_0(\Omega) = \sqrt{\lambda_{\min}(S)}. \tag{2.35}$$

Proof. Let $q \in L_0^2(\Omega)$ and $v \in V$, then

$$(Sq, q)_0 = (\nabla \cdot \Delta^{-1} \nabla q, q)_0 = -(\Delta^{-1}\nabla q, \nabla q)_0 = -(\Delta^{-1}\nabla q, \Delta\Delta^{-1}\nabla q)_0.$$

Define $w := \Delta^{-1}\nabla q$, so that $w \in V$ and we get

$$(Sq, q)_0 = (-\Delta w, w)_0 = (\nabla w, \nabla w)_0 = \|w\|^2. \tag{2.36}$$

Since $(-\Delta w, w)_0 = \|w\|^2$,

$$\|w\|^2 = \sup_{v \in V} \frac{(-\Delta w, v)_0^2}{\|v\|^2}.$$

Substituting the above in (2.36),

$$(Sq, q)_0 = \sup_{v \in V} \frac{(-\Delta w, v)_0^2}{\|v\|^2} = \sup_{v \in V} \frac{(-\nabla q, v)_0^2}{\|v\|^2} = \sup_{v \in V} \frac{(q, \nabla \cdot v)_0^2}{\|v\|^2}.$$

Therefore,

$$\lambda_{\min}(S) = \inf_{q \in L_0^2(\Omega)} \frac{(Sq, q)_0}{\|q\|_0^2} = \inf_{q \in L_0^2(\Omega)} \sup_{v \in V} \frac{(q, \nabla \cdot v)_0^2}{\|v\|^2} = \beta_0(\Omega)^2.$$

Hence, $\beta_0(\Omega) = \sqrt{\lambda_{\min}(S)}$. □

The above theorem is a very significant result for the existence of solution of Stokes problem. It mainly depends on $\lambda_{\min}(S)$. This Uzawa pressure operator is different than other operators. It is interesting and valuable to study the whole spectrum of this operator.

3

Spectrum of the Uzawa Pressure Operator

The Uzawa pressure operator, i.e., $S := \nabla \cdot \Delta^{-1}\nabla$ portrays a very interesting behavior. Its properties change discontinuously with a change in the shape of the domain, i.e., they depend largely on the boundary of the domain. The spectrum of the Uzawa pressure operator is entirely known for the case of domains of the shape of a circle, ellipse and ring. Several results describe the behavior of S if the boundary is sufficiently smooth. Many papers have published since 1937, such as by Friedrichs [20] to as recent as 2016 [5], which describe the spectrum of this operator both analytically and numerically. The complete knowledge of the spectrum of the Uzawa pressure operator is still not achieved for the case of domains such as a square where the boundary is not smooth enough. In this chapter, we will describe some known results about the spectrum of the Uzawa pressure operator and some of our efforts toward finding the eigenfunctions. Our main focus is on the disk and the square.

3.1 General Results

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected domain with a Lipschitz boundary. Consider the spaces V and $L_0^2(\Omega)$ as defined earlier. As discussed in Chapter 2, we have the Uzawa pressure operator is a linear, self-adjoint, coercive and bounded operator. We will now use some results from the spectral theory of self-adjoint bounded linear operator, given in Chapter 1, to describe the spectrum of the Uzawa pressure operator. We mainly use the first classification of the spectrum, in which the spectrum is partitioned into three disjoint sets namely, point spectrum, continuous spectrum and residual spectrum.

Theorem 3.1 (**Spectrum**)**.** The spectrum of the Uzawa pressure operator consists of only positive real values less than or equal to 1.

Proof. Since $S := \nabla \cdot \Delta^{-1} \nabla$ is a bounded self-adjoint linear operator and it is coercive, i.e., it is a positive operator. Hence, by using Theorem 1.34, we get that the spectrum of S consists only of positive real values. By using Theorem 2.14 part 2, we have that $\lambda_{\max}(S) \leq 1$. Therefore, the spectrum of the Uzawa pressure operator consists only positive real values less than or equal to 1. $\square$

The continuous spectrum of Uzawa pressure operator is really special, as it depends on the compactness of S. Since S changes its behavior according to the shape of the domain, the continuous spectrum also varies. We will discuss the continuous spectrum in the next sections. Let us now define an eigenvalue of Uzawa pressure operator.

Definition 3.2 (**Eigenvalue**)**.** A real number $\lambda > 0$ is said to be an eigenvalue of S if and only if there exists some $0 \neq p \in L_0^2(\Omega)$, such that $Sp = \lambda p$ in Ω. The function p in such case is called an eigenfunction of S corresponding to the eigenvalue λ.

Theorem 3.3 (**Residual Spectrum**)**.** The residual spectrum of Uzawa pressure operator does not exist.

Proof. Since $S := \nabla\cdot\Delta^{-1}\nabla$ is a bounded self-adjoint operator, Theorem 1.30, implies that the residual spectrum of S does not exist. $\square$

We know that the Uzawa pressure operator is defined on the space $L_0^2(\Omega)$, i.e., $S : L_0^2(\Omega) \to L_0^2(\Omega)$. There is a certain subspace of $L_0^2(\Omega)$ that is really important. We define the subspace N, as given in the paper [13] by M. Crouzeix, as follows,

$$N := \left\{p \in L_0^2(\Omega) \,|\, \exists\, \phi \text{ so that } \Delta\phi = p \text{ in } \Omega \text{ and } \nabla\phi \in V\right\}. \tag{3.1}$$

The subset N is a closed set.

Theorem 3.4. The Uzawa pressure operator S restricted to the subspace N of the space $L_0^2(\Omega)$ is the identity operator, i.e., $S_{|_N} = I$.

Proof. Let $p \in N$ then there exists ϕ such that $p = \Delta\phi$ in Ω and $\nabla\phi \in V$. By applying the operator S on p,

$$Sp = \nabla \cdot \Delta^{-1}\nabla p = \nabla \cdot \Delta^{-1}\nabla\Delta\phi.$$

Since ϕ is smooth enough, we can interchange the roles of ∇ and Δ, so

$$Sp = \nabla \cdot \Delta^{-1}\Delta\nabla\phi = \nabla \cdot \nabla\phi = \Delta\phi = p.$$

Therefore, for any $p \in N$, we get $Sp = p = Ip$ which implies that $S = I$ on the subspace N. We can write this as $S_{|_N} = I$. $\square$

The above theorem holds for domains having a Lipschitz boundary. This theorem gives us the largest eigenvalue of S for all domains.

Theorem 3.5 (**Maximum eigenvalue of S**)**.** Let $\Omega \subset \mathbb{R}^2$ be any bounded, connected set with a Lipschitz boundary. The supremum of the spectrum of the Uzawa pressure operator on Ω is the eigenvalue one. It is an eigenvalue of infinite multiplicity.

Proof. From Theorem 3.4, $S_{|_N} = I$ such that $Sp = p$ for all $p \in N$. By using Definition 3.2, $\lambda = 1$ is an eigenvalue of S. Also, from Theorem 2.14, $\lambda_{\max}(S) \leq 1$, thus, we conclude that $\lambda_{\max}(S) = 1$. Since corresponding to every $p \in N$, which is infinite dimensional, we have $Sp = p$, hence, there are infinitely many eigenfunctions of $\lambda = 1$. Hence, it is an eigenvalue of infinite multiplicity. $\square$

Since the operator S is just the identity operator when it is restricted to the subspace N of the space $L_0^2(\Omega)$, we know everything about it on that subspace. For determining the remaining spectrum of S that lies in the interval $(0, 1)$, we define the space orthogonal to the subspace N. Let M be the subspace of the space $L_0^2(\Omega)$ such that $M = N^\perp$, i.e., M is orthogonal to N.

$$M := \left\{ p \in L_0^2(\Omega) \,|\, (p, q)_0 = 0, \; \forall \, q \in N \right\}. \tag{3.2}$$

The space $L_0^2(\Omega) = M \oplus N$. Now, we look at some properties of the Uzawa pressure operator on the subspace M.

Proposition 3.6. The Uzawa pressure operator S, maps the subspace M to M.

Proof. Let $p \in M$ and $q \in N$ then we have to prove that $Sp \in M$. Since S is self-adjoint,

$$(Sp, q)_0 = (p, Sq)_0 = (p, q)_0 = 0.$$

Therefore, $(Sp, q)_0 = 0$ for all $p \in M$ and $q \in N$, which implies that $Sp \in M$. Hence, $S : M \to M$. $\square$

Consider $p \in M$ and $q \in N$ then there exists ϕ such that $q = \Delta\phi$ and $\nabla\phi \in V$. We know that $(p, q)_0 = 0$, which implies $(p, \Delta\phi)_0 = 0$. For all $\phi \in C_c^\infty(\Omega)$, $(\Delta p, \phi)_0 = 0$.

Since the space $C_c^\infty(\Omega)$ is dense in the space $L^2(\Omega)$, this implies that $\Delta p = 0$ a.e. in Ω. It is remarked in [13] that if Ω is a connected set then the space M is a space containing all the harmonic functions of the space $L_0^2(\Omega)$. Since we consider Ω to be a connected set,

$$M := \left\{ p \in L_0^2(\Omega) \,|\, \Delta p = 0 \right\}. \tag{3.3}$$

Theorem 3.7. The Uzawa pressure operator S, satisfies the following properties.

1. There exist two real numbers $0 < \alpha < \beta < 1$ such that for all $p \in M$,

$$\alpha \|p\|_0 \leq \|Sp\|_0 \leq \beta \|p\|_0. \tag{3.4}$$

2. The spectrum of S, i.e., $\sigma(S) \subseteq [\alpha, \beta] \cup \{1\}$ and $\beta = 1 - \alpha$.

Proof. 1. Let $p \in M \subset L_0^2(\Omega)$, so by using the left hand side of the inequality (2.32), there exists a real number $0 < \alpha < 1$, such that $\alpha \|p\|_0^2 \leq (Sp, p)_0$. By using Cauchy-Schwartz inequality,

$$\alpha \|p\|_0^2 \leq \|Sp\|_0 \|p\|_0,$$

which implies

$$\alpha \|p\|_0 \leq \|Sp\|_0. \tag{3.5}$$

Let β be a real number such that

$$\beta := \max_{\substack{p \in M \\ p \neq 0}} \frac{(Sp, Sp)_0}{(Sp, p)_0}.$$

Define $u \in V$ such that $\Delta u = \nabla p$ in Ω, then $u = \Delta^{-1} \nabla p$ in Ω. So, $Sp = \nabla \cdot \Delta^{-1} \nabla p = \nabla \cdot u$ in Ω. Then by (2.34), $(u, u)_{1,\Omega} = \|u\|^2 = (p, \nabla \cdot u)_0 = (Sp, p)_0$.

Thus, we get

$$\beta = \max_{\substack{p \in M \\ p \neq 0}} \frac{(Sp, Sp)_0}{(Sp, p)_0} = \max_{\substack{u \in V \\ u \neq 0}} \frac{\|\nabla \cdot u\|_0^2}{\|u\|^2}.$$

Now, we claim that $u \in W$ (see (2.23)). Let $w \in V$ and then by using equation (2.2), $(u, w)_{1,\Omega} = a(u, w) = (\nabla u, \nabla w)_0 = (-\Delta u, w)_0 = (-\nabla p, w)_0 = (p, \nabla \cdot w)_0$. If $w \in V_1$ then $(u, w)_{1,\Omega} = 0$, so that $u \in V_1^{\perp}$. If $w \in V_2$ then there exists ϕ such that $w = \nabla \phi$, then $(u, w)_{1,\Omega} = (p, \nabla \cdot \nabla \phi)_0 = (p, \Delta \phi)_0 = 0$. Hence, $u \in W$. For all $u \in W$, we have $\|u\|^2 = \|\nabla \cdot u\|_0^2 + \|\mathbf{curl}\, u\|_0^2$, so

$$\beta = \max_{\substack{u \in W \\ u \neq 0}} \frac{\|\nabla \cdot u\|_0^2}{\|u\|^2} = 1 - \min_{\substack{u \in W \\ u \neq 0}} \frac{\|\mathbf{curl}\, u\|_0^2}{\|u\|^2} < 1. \tag{3.6}$$

Thus, from the definition of $0 < \beta < 1$, we get $\|Sp\|_0 \leq \beta \|p\|_0$. By using this inequality and (3.5), there exist two real numbers $0 < \alpha < \beta < 1$, such that for all $p \in M$,

$$\alpha \|p\|_0 \leq \|Sp\|_0 \leq \beta \|p\|_0.$$

2. As we defined β in the first part of this theorem, we also define α, as

$$\alpha = \min_{\substack{u \in V \\ u \neq 0}} \frac{\|\nabla \cdot u\|_0^2}{\|u\|^2}.$$

Since for $v = (v_1, v_2) \in W$, we can define $w = (-v_2, v_1)$ such that $\nabla \cdot v = \mathbf{curl}\, w$ and $\nabla \cdot w = -\mathbf{curl}\, v$,

$$\alpha = \min_{\substack{u \in V \\ u \neq 0}} \frac{\|\nabla \cdot u\|_0^2}{\|u\|^2} = \min_{\substack{u \in V \\ u \neq 0}} \frac{\|\mathbf{curl}\, u\|_0^2}{\|u\|^2}.$$

Thus, by using (3.6), $\beta = 1 - \alpha$. Since the definition of α and β are equivalent to that for $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$, respectively, for $p \in M$ from equations (2.30) and (2.31), the spectrum of S, i.e., $\sigma(S) \subseteq [\alpha, 1 - \alpha] \cup \{1\}$.

$\square$

We know that $\nabla\cdot : V \to L_0^2(\Omega)$ and it is surjective. Since $V = V_1 \oplus V_2 \oplus W$, hence $L_0^2(\Omega) = \nabla\cdot(V_1 \oplus V_2 \oplus W)$. Since $\nabla\cdot V_1 = 0$, $L_0^2(\Omega) = \nabla\cdot(V_2 \oplus W)$. Consider $v_2 \in V_2$, then $\mathbf{curl}\, v_2 = 0$, so there exists some scalar function ϕ such that $v_2 = \nabla\phi$. Let $q = \nabla\cdot v_2 = \nabla\cdot\nabla\phi = \Delta\phi$, thus $q \in N$. Therefore, $\nabla\cdot V_2 = N$ and it is important to note that $\nabla\cdot : W \to M$ is bijective. It can be proved as follows.

For $w \in W$, $(v_2, w)_{1,\Omega} = 0$, this implies $(\nabla\cdot v_2, \nabla\cdot w)_0 = 0$, i.e., $(q, \nabla\cdot w)_0 = 0$ for all $q \in N$. Hence, $\nabla\cdot w \in M$. Conversely, let $p \in M$ then there exists some $q \in M$ such that $p = Sq = \nabla\cdot u$, as defined in part 1 of the previous theorem. Moreover, we have $u \in W$.

Thus, $\nabla\cdot : W \to M$ is bijective and this implies $\nabla\cdot W = M$.

Let Ω be a bounded, connected domain with Lipschitz boundary. Let $(\lambda, p) \in \mathbb{R}\times L_0^2(\Omega)$ be an eigenpair of S, then they satisfy the equation $Sp = \lambda p$ in Ω, i.e., $\nabla\cdot\Delta^{-1}\nabla p = \lambda p$. Let $v = \Delta^{-1}\nabla p$, then

$$\Delta v = \nabla p \text{ in } \Omega, \tag{3.7}$$

$$\nabla\cdot v = \lambda p \text{ in } \Omega, \tag{3.8}$$

$$v = 0 \text{ on } \partial\Omega. \tag{3.9}$$

The solution of the above problem for finding $\lambda \in \mathbb{R}$ corresponding to $p \in L_0^2(\Omega)$ is equivalent to that of the eigenvalue problem of the Uzawa pressure operator.

Another eigenvalue problem associated with the Stokes equation, for $u \in V$ and $p \in L_0^2(\Omega)$, is given as

$$\begin{aligned} -\Delta u + \nabla p &= \lambda u \\ \nabla\cdot u &= 0. \end{aligned} \tag{3.10}$$

If we take $u \in V_1$, then for all $v \in V_1$ the weak form of the problem

$$\int_\Omega (-\Delta u + \nabla p)v = \int_\Omega \lambda uv,$$

thus,

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega p\nabla \cdot v = \int_\Omega uv.$$

The problem simplifies to $-\Delta u = \lambda u$ or $-u = \lambda\Delta^{-1}u$. In this case, Δ^{-1} is compact and it follows from the general theory of compact operators that the spectrum consists entirely of the discrete spectrum $\{\lambda_n,\, n \geq 1\}$, with $\lambda_n \to \infty$ as $n \to \infty$.

3.2 Disk

This section describes the results related to the disk. As stated earlier, the whole spectrum is known for the case of a disk. In order to solve for an eigenfunction, we reformulate the eigenvalue problem of the Uzawa pressure operator.
As 1 is an eigenvalue of S for a disk as well, we have found an example of the infinite family of eigenfunctions corresponding to the eigenvalue 1.

Proposition 3.8. Let Ω be a unit disk. The largest eigenvalue of S is $\lambda_{\max} = 1$ and it has an infinite multiplicity. Let $k \in \mathbb{Z}^+$, the following represents a family of eigenfunctions corresponding to the eigenvalue 1,

$$\phi_k = (2\pi k)\cos(2\pi kr) + \frac{\sin(2\pi kr)}{r}.$$

Proof. Consider $\Omega = \{(r, \theta) \mid 0 \leq r < 1,\, 0 \leq \theta < 2\pi\}$. We consider the eigenvalue problem for $\lambda = 1$.

$$\Delta v = \nabla p \text{ in } \Omega, \tag{3.11}$$

$$\nabla \cdot v = p \text{ in } \Omega, \tag{3.12}$$

$$v = 0 \text{ on } \partial\Omega. \tag{3.13}$$

We take for $k \in \mathbb{Z}^+$, $v = \sin(2\pi kr)\hat{r}$, as on $\partial\Omega$, i.e., when $r = 1$, $v = \sin(2\pi k)\hat{r} = 0$. On substituting v in (3.12),

$$\begin{aligned} p &= \frac{1}{r}\frac{\partial(r\sin(2\pi kr))}{\partial r} + \frac{1}{r}\frac{\partial(0)}{\partial\theta} \\ &= (2\pi k)\cos(2\pi kr) + \frac{\sin(2\pi kr)}{r}. \end{aligned} \tag{3.14}$$

On substituting (3.14) in the right-hand side of the equation (3.11),

$$\begin{aligned} \nabla p &= \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right]\left((2\pi k)\cos(2\pi kr) + \frac{\sin(2\pi kr)}{r}\right) \\ &= \left[-\left((2\pi k)^2 + \frac{1}{r^2}\right)\sin(2\pi kr) + (2\pi k)\frac{\cos(2\pi kr)}{r}\right]\hat{r}, \end{aligned} \tag{3.15}$$

Let v_r and v_θ be the radial and angular components of the vector v, respectively. The vector laplacian in polar coordinates is given by

$$\Delta v = \left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2}\right]\hat{r} + \left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2}\right]\hat{\theta}. \tag{3.16}$$

On substituting $v = \sin(2\pi kr)\hat{r}$ for $k \in \mathbb{Z}^+$, in the left-hand side of (3.11) using (3.16) with $v_\theta = 0$,

$$\begin{aligned} \Delta v &= \left[\frac{\partial^2 \sin(2\pi kr)}{\partial r^2} + \frac{1}{r}\frac{\partial\sin(2\pi kr)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\sin(2\pi kr)}{\partial\theta^2} - \frac{\sin(2\pi kr)}{r^2}\right]\hat{r} \\ &= \left[-\left((2\pi k)^2 + \frac{1}{r^2}\right)\sin(2\pi kr) + (2\pi k)\frac{\cos(2\pi kr)}{r}\right]\hat{r}. \end{aligned}$$

On comparing the above equation with (3.15), $\Delta v = \nabla p$. Hence, for $k \in \mathbb{Z}^+$,

$$p = \phi_k = (2\pi k)\cos(2\pi kr) + \frac{\sin(2\pi kr)}{r} \in N \subset L_0^2(\Omega),$$

solves the eigenvalue problem of S for $\lambda = 1$. Hence, it gives us a family of eigenfunctions for $\lambda_{\max}(S) = 1$. $\square$

Proposition 3.9. Let Ω be a unit disk, $\lambda = \frac{1}{2}$ is an eigenvalue of S and it has an infinite multiplicity. Let $i \in \mathbb{Z}^+$, the following represents a family of eigenfunction corresponding to the eigenvalue $\frac{1}{2}$,

$$\phi_i = r^i \sin(i\theta),$$

and

$$\phi_i = r^i \cos(i\theta).$$

Proof. Consider $\Omega = \{(r,\theta) \,|\, 0 \leq r < 1,\ 0 \leq \theta < 2\pi\}$ and let $\phi_{ai} = r^a \sin(i\theta)$, where $i \in \mathbb{Z}^+$ and $a, \lambda \in \mathbb{R}^+$. Consider,

$$S\phi_{ai} = \lambda\phi_{ai}, \tag{3.17}$$

i.e., the equation

$$\nabla \cdot (\Delta^{-1}\nabla(r^a \sin(i\theta)) = \lambda r^a \sin(i\theta). \tag{3.18}$$

Let us calculate $\nabla(r^a \sin(i\theta))$. In polar coordinates,

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}.$$

So,

$$\nabla(r^a \sin(i\theta)) = ar^{a-1}\sin(i\theta)\hat{r} + ir^{a-1}\cos(i\theta)\hat{\theta}. \tag{3.19}$$

Substituting (3.19) in (3.18),

$$\nabla \cdot (\Delta^{-1}(ar^{a-1}\sin(i\theta)\hat{r} + ir^{a-1}\cos(i\theta)\hat{\theta})) = \lambda r^a \sin(i\theta). \tag{3.20}$$

Consider $\Delta^{-1}(ar^{a-1}\sin(i\theta)\hat{r} + ir^{a-1}\cos(i\theta)\hat{\theta}) = v$, i.e.,

$$\begin{cases} \Delta v = ar^{a-1}\sin(i\theta)\hat{r} + ir^{a-1}\cos(i\theta)\hat{\theta} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

where Δ is the vector laplacian in polar coordinates. Let us take v as the following

$$v = (r^\alpha - r^\beta)(m\sin(i\theta)\hat{r} + n\cos(i\theta)\hat{\theta}), \tag{3.22}$$

where $m, n \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+ \cup \{0\}$ and $i \in \mathbb{Z}^+$. We have $v_r = m(r^\alpha - r^\beta)\sin(i\theta)$ and $v_\theta = n(r^\alpha - r^\beta)\cos(i\theta)$. From (3.16),

$$\begin{aligned} \Delta v = &\Big[(m\alpha^2 - mi^2 + 2ni - m)r^{\alpha-2} - (m\beta^2 - mi^2 + 2ni - m)r^{\beta-2}\Big]\sin(i\theta)\hat{r} \\ &+ \Big[(n\alpha^2 - ni^2 + 2mi - n)r^{\alpha-2} - (n\beta^2 - ni^2 + 2mi - n)r^{\beta-2}\Big]\cos(i\theta)\hat{\theta}. \end{aligned} \tag{3.23}$$

We will determine the suitable constants α, β, m and n, so that (3.22) is a solution of (3.21). On comparing (3.21) and (3.23), we get the answers in two ways as follows.
Case 1: For the first way, we eliminate the r^β. Since (3.23) and (3.21) must be equal, so from the equality of the radial components,

$$\Big[(m\alpha^2 - mi^2 + 2ni - m)r^{\alpha-2} - (m\beta^2 - mi^2 + 2ni - m)r^{\beta-2}\Big]\sin(i\theta) = ar^{a-1}\sin(i\theta),$$

which implies

$$(m\alpha^2 - mi^2 + 2ni - m)r^{\alpha-\beta} - (m\beta^2 - mi^2 + 2ni - m) = ar^{a+1-\beta}.$$

Now, we take $(m\beta^2 - mi^2 + 2ni - m) = 0$, i.e.,

$$m\beta^2 = mi^2 - 2ni + m. \tag{3.24}$$

Therefore,

$$(m\alpha^2 - mi^2 + 2ni - m)r^{\alpha-\beta} = ar^{a+1-\beta},$$

thus,

$$\alpha = a + 1. \tag{3.25}$$

Therefore, we have that $m\alpha^2 - mi^2 + 2ni - m = a$, which implies the following relation,

$$m\alpha^2 - a = mi^2 - 2ni + m. \tag{3.26}$$

By using equations (3.24) and (3.26)

$$m\beta^2 = m\alpha^2 - a,$$

using (3.25), $m\beta^2 = m(a+1)^2 - a$. Thus,

$$m = \frac{a}{(a+1)^2 - \beta^2}. \tag{3.27}$$

From the equality of the angular components of (3.23) and (3.21),

$$\left[(n\alpha^2 - ni^2 + 2mi - n)r^{\alpha-2} - (n\beta^2 - ni^2 + 2mi - n)r^{\beta-2}\right]\cos(i\theta) = ir^{a-1}\cos(i\theta),$$

and, using (3.25),

$$(n(a+1)^2 - ni^2 + 2mi - n)r^{a-1} - (n\beta^2 - ni^2 + 2mi - n)r^{\beta-2} = ir^{a-1}.$$

Therefore, $(n\beta^2 - ni^2 + 2mi - n) = 0$ and

$$n\beta^2 = ni^2 - 2mi + n, \tag{3.28}$$

hence,

$$(n(a+1)^2 - ni^2 + 2mi - n)r^{a-1} = ir^{a-1},$$

Therefore, we have that $n(a+1)^2 - ni^2 + 2mi - n = i$, which implies the following relation,

$$n(a+1)^2 - i = ni^2 - 2mi + n. \tag{3.29}$$

Using equations (3.28) and (3.29),

$$n = \frac{i}{(a+1)^2 - \beta^2}. \tag{3.30}$$

Substituting (3.27), (3.30) in (3.28), $i\beta^2 = i^3 - 2ai + i$, as $i \neq 0$,

$$\beta^2 = i^2 - 2a + 1. \tag{3.31}$$

Substituting (3.27), (3.30) in (3.24) $a\beta^2 = ai^2 - 2i^2 + a$, as $i \neq 0$, hence,

$$\beta^2 = i^2 - \frac{2i^2}{a} + 1. \tag{3.32}$$

From the equality of (3.31) and (3.32),

$$a = i. \tag{3.33}$$

Using (3.33) in (3.31), $\beta = \sqrt{i^2 - 2i + 1} = \sqrt{(i-1)^2}$, therefore,

$$\beta = (i-1). \tag{3.34}$$

Substituting (3.34) and (3.33) in (3.30),

$$m = \frac{i}{(i+1)^2 - (i-1)^2},$$

hence, we have $m = \frac{1}{4}$. Similarly, $n = \frac{1}{4}$. On substituting the values of α, β, m and n,

$$v = \frac{1}{4}(r^{i+1} - r^{i-1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta}). \tag{3.35}$$

Over $\partial\Omega$, we have $r = 1$, also, when $r = 1$, we get $v = 0$ over $\partial\Omega$. Therefore, we get that $v = \frac{1}{4}(r^{i+1} - r^{i-1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta})$, where $i \in \mathbb{Z}^+$ is a solution of (3.21). Thus,

$$\Delta^{-1}(ar^{a-1}\sin(i\theta)\hat{r} + ir^{a-1}\cos(i\theta)\hat{\theta}) = \frac{1}{4}(r^{i+1} - r^{i-1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta}), \tag{3.36}$$

where $i \in \mathbb{Z}^+$. By using (3.36) in (3.20) and $a = i$,

$$\nabla \cdot \left[\frac{1}{4}(r^{i+1} - r^{i-1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta})\right] = \lambda r^i \sin(i\theta). \tag{3.37}$$

On calculating the left-hand side of the above equation,

$$\frac{1}{2}r^i \sin(i\theta) = \lambda r^i \sin(i\theta).$$

Hence, $\lambda = \frac{1}{2}$. Our results show that $\lambda = \frac{1}{2}$ is an eigenvalue of S of infinite multiplicity, having eigenvectors $\phi_{ii} = r^i \sin(i\theta)$, where $i \in \mathbb{Z}^+$.

Case 2: In this case, we try to eliminate r^α, then

$$\left[(m\alpha^2 - mi^2 + 2ni - m)r^{\alpha-2} - (m\beta^2 - mi^2 + 2ni - m)r^{\beta-2}\right]\sin(i\theta) = ar^{a-1}\sin(i\theta),$$

and, so

$$(m\alpha^2 - mi^2 + 2ni - m) - (m\beta^2 - mi^2 + 2ni - m)r^{\beta-\alpha} = ar^{a+1-\alpha}.$$

Continuing in the same way as Case 1, $\alpha = i-1$, $\beta = i+1$, $m = -\frac{1}{4}$, $n = -\frac{1}{4}$. On substituting the values of α, β, m and n,

$$v = -\frac{1}{4}(r^{i-1} - r^{i+1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta}). \tag{3.38}$$

Over $\partial\Omega$, we have $r = 1$, also, when $r = 1$, we get $v = 0$ over $\partial\Omega$. Therefore, (3.38) satisfies (3.21). On rearranging (3.38),

$$v = \frac{1}{4}(r^{i+1} - r^{i-1})(\sin(i\theta)\hat{r} + \cos(i\theta)\hat{\theta}). \tag{3.39}$$

Comparing (3.35) and (3.39), we conclude that we obtain the same expression for v in both of the cases.

Hence, we have that $\lambda = \frac{1}{2}$ is an eigenvalue of S with infinite multiplicity, having eigenvectors $\phi_{ii} = r^i \sin(i\theta)$. On performing similar calculations for $\phi_{ai} = r^a \cos(i\theta)$ for a, $i \in \mathbb{N}$, we have the result that $\phi_{ii} = r^i \cos(i\theta)$ are also eigenvectors corresponding to the eigenvalue $\frac{1}{2}$. $\square$

Paper [13], remarks that the number $\frac{1}{2}$ is not just an eigenvalue, but the Uzawa pressure operator behaves like $\frac{1}{2}I$ in the space orthogonal to N.

Theorem 3.10. Let Ω be a unit disk. The Uzawa pressure operator S restricted to the subspace $M \subset L_0^2(\Omega)$ is one half of the identity operator on M, i.e., $S_{|_M} = \frac{1}{2}I$. The smallest eigenvalue of S on a disk is $\frac{1}{2}$ and it has infinite multiplicity.

M. Crouzeix also proved that Uzawa pressure operator is compact for domains having boundary of type C^3. In order to prove this result, Crouzeix recalls the following result in the paper [13].

Lemma 3.11. Assume that p is harmonic, belongs to $L_0^2(\Omega)$ and that $\psi \in C^m(\bar{\Omega}) \cap H_0^m(\Omega)$. Then $\psi \partial^j p \in H_0^{m-|j|}(\Omega)$ for all partial derivatives $\partial^j p$ of order $|j| \leq m$.

M. Crouzeix gives the following lemma in [13] that he subsequently uses to prove the compactness of the Uzawa pressure operator on the subspace M.

Lemma 3.12. Assume that the boundary of Ω is of class C^3. There exists a function $a \in (C^2(\bar{\Omega}) \cap H_0^1(\Omega))^2$ such that $\mathbf{curl}\, a = 0$ in Ω and $\nabla \cdot a - \frac{1}{2} \in C^1(\bar{\Omega}) \cap H_0^1(\Omega)$.

Proof. Firstly, we define ϕ such that

$$\phi := \frac{1}{8}\left[-(x_1^2 + x_2^2) + 2\gamma(x)\,(\delta(x))^2\right], \tag{3.40}$$

where for all $x \in \Omega$, $\delta(x) := \operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |y - x|$ is the distance function and γ is a smooth cut-off function defined on Ω such that $\gamma(x) = 1$ if $x \in \partial\Omega$ and $\gamma(x) = 0$ if $x \in \sum$. Here, $\sum$ denotes the ridge of Ω. Then

$$\begin{aligned}
\nabla\phi &= -\frac{1}{4}(x_1, x_2) + \frac{1}{4}\nabla(\gamma(x)\,(\delta(x))^2) \\
&= -\frac{x}{4} + \frac{1}{2}\delta(x)\gamma(x)\nabla\delta(x) + \frac{1}{4}(\delta(x))^2\nabla\gamma(x).
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\Delta\phi = \nabla \cdot \nabla\phi \\
&\quad = -\frac{1}{2} + \frac{1}{2}\gamma(x)|\nabla\delta(x)|^2 + \delta(x)\nabla\gamma(x)\cdot\nabla\delta(x) + \frac{1}{2}\delta(x)\gamma(x)\Delta\delta(x) + \frac{1}{4}(\delta(x))^2\Delta\gamma(x).
\end{aligned}$$

For $x \in \partial\Omega$, $|\nabla\delta(x)| = 1$, hence $\Delta\phi(x) = 0$. Define $a := \nabla\phi + \frac{x}{4}$, then $\mathbf{curl}\, a = 0$ in Ω. Since $\delta \in (C^\infty(\bar{\Omega}) \setminus \sum)$ and $\gamma = 0$ in $\sum$, thus $a \in (C^2(\bar{\Omega}) \cap H_0^1(\Omega))^2$. Note that, $a = 0$ on $\partial\Omega$ because $\delta = 0$ on $\partial\Omega$. For $x \in \partial\Omega$, $\nabla \cdot a = \Delta\phi + \frac{1}{2} = \frac{1}{2}$. Thus, $\nabla \cdot a - \frac{1}{2} \in C^1(\bar{\Omega}) \cap H_0^1(\Omega)$. $\square$

We will now give the result on compactness of S for smooth domains.

Theorem 3.13 (**Compactness** [13]). Assume that the boundary of Ω is of class C^3. Then the operator $S - \frac{1}{2}I$ maps M into $H^1(\Omega)$. Furthermore, $S - \frac{1}{2}I : M \to M$ is compact.

Proof. Consider $p \in M$ and the space W defined by equation (2.23), then $p \in \nabla \cdot W$ such that there exists a $w \in W$ so that $p = \nabla \cdot w$. Let $v \in V_2$, then $\mathbf{curl}\, v = 0$ in Ω. Since, $w \in W$ and $v \in V_2$, we get $(w, v)_{1,\Omega} = 0$. By using equation (2.21),

$$(\nabla \cdot w, \nabla \cdot v)_0 + (\mathbf{curl}\, w, \mathbf{curl}\, v)_0 = 0,$$

which further gives $(\nabla \cdot w, \nabla \cdot v)_0 = 0$, i.e., $(p, \nabla \cdot v)_0 = 0$. Thus, $(\nabla p, v)_0 = 0$. Since, the curl component of the inner product was already zero, then the vanishing of the divergence component implies there exists $q \in L_0^2(\Omega)$ such that $\nabla^\perp q = \nabla p$, so that $(\nabla p, v)_0 = (\nabla^\perp q, v)_0 = (q, \mathbf{curl}\, v)_0 = 0$. Also $-\Delta q = \mathbf{curl}\, \nabla^\perp q = \mathbf{curl}\, \nabla p = 0$. Let a be defined as in Lemma 3.12 and $u \in V$ be defined as $\Delta u = \nabla p$. Now, we define

$$h = pa + q(a_2, -a_1) - u \in V. \tag{3.41}$$

By using Lemma 3.11, $h \in H_0^1(\Omega)$. Consider,

$$\Delta h_1 = 2(p_{x_1}(a_1)_{x_1} + p_{x_2}(a_1)_{x_2}) + p\Delta a_1 + q\Delta a_2 + (q_{x_1} a_{2_{x_1}} + q_{x_2} a_{2_{x_2}}) - \Delta u_1,$$
$$\Delta h_2 = 2(p_{x_1}(a_2)_{x_1} + p_{x_2}(a_2)_{x_2}) + p\Delta a_2 - q\Delta a_1 - (q_{x_1} a_{1_{x_1}} + q_{x_2} a_{1_{x_2}}) - \Delta u_2.$$

By using $\nabla^\perp q = \nabla p$, $\mathbf{curl}\, a = 0$, Lemma 3.11, and the fact that $\Delta a_j \in C(\bar{\Omega}) \cap L^2(\Omega)$,

$$\Delta h = p\Delta a + q(\Delta a_2, -\Delta a_1) + 2\left(\nabla \cdot a - \frac{1}{2}\right)\nabla p \in L^2(\Omega).$$

Thus, $h \in (H^2(\Omega) \cap H_0^1(\Omega))^2$. Since $\nabla \cdot u = Sp$, use (3.41) to get

$$Sp - \frac{1}{2}p = \nabla \cdot u - \frac{1}{2}p = \left(\nabla \cdot a - \frac{1}{2}\right) p - \nabla \cdot h = -\nabla \cdot h \in H^1(\Omega).$$

Thus, $S - \frac{1}{2}I$ maps M into $H^1(\Omega)$. By using Theorem 1.13, i.e., Sobolev embedding theorem H^1 is compactly embedded in the space $L^2(\Omega)$. Since $M \subset L^2(\Omega)$ and $S : M \to M$, therefore, $S - \frac{1}{2}I : M \to M$ is compact. $\square$

Note that $S : N \to N$ is not compact as $S_{|_N} = I$. Also S has the eigenvalue 1 of infinite multiplicity, thus it lies in the essential spectrum of S. This again implies non-compactness of S on N as the essential spectrum of a compact operator does not exist. We have that the Uzawa pressure operator on the subspace M is compact for the domains having boundary of type C^3. By using Theorem 1.35, we get that the continuous spectrum of the Uzawa pressure operator for the domains having boundary of the type C^3 does not exist. For the case of domains having Lipschitz boundary, we cannot say anything about the continuous spectrum yet. In the next section, we will discuss the simple case of a domain having Lipschitz boundary, i.e., we consider the case of a square domain and study the properties of the spectrum.

3.3 Square

We have seen until now that the Uzawa pressure operator is a well behaved operator if the boundary of domain is smooth, i.e., of type C^3. However, as soon as the boundary loses regularity, certain complications arise. The main being the change in compactness. In this section, we will summarize the results regarding the spectrum of the Uzawa pressure operator for the square domain.

Consider Ω to be a square domain. The maximum eigenvalue of the Uzawa pressure operator is 1 and it has infinite multiplicity. The restriction of S to the subspace $N \subset L_0^2(\Omega)$, defined by equation (3.1) is the identity operator, i.e., $S_{|_N} = I$. The minimum eigenvalue of the Uzawa pressure operator for a square domain is still not known. The elements of the subspace N are eigenfunctions of the eigenvalue 1.

Example 3.3.1:

Let $\Omega := \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ and consider $p = (12x^2-1)\left(y^2 - \frac{1}{4}\right)^2 + (12y^2-1)\left(x^2 - \frac{1}{4}\right)^2$, which has zero average on Ω and is square integrable. Hence, $p \in L_0^2(\Omega)$. We now prove that $Sp = p$. Toward that end,

$$\begin{aligned}\nabla \cdot \Delta^{-1} \nabla p = \nabla \cdot \Delta^{-1} \Bigg[& \left(24x\left(y^2 - \frac{1}{4}\right)^2 + 4x(12y^2-1)\left(x^2 - \frac{1}{4}\right)\right)\hat{\imath} \\ & + \left(4y\left(y^2 - \frac{1}{4}\right)\left(12x^2 - 1\right) + 24y\left(x^2 - \frac{1}{4}\right)^2\right)\hat{\jmath}\Bigg].\end{aligned}$$

Let $\Delta^{-1}\nabla p = v$, so

$$\begin{aligned}\Delta v = & \left(24x\left(y^2 - \frac{1}{4}\right)^2 + 4x(12y^2-1)\left(x^2 - \frac{1}{4}\right)\right)\hat{\imath} \\ & + \left(4y\left(y^2 - \frac{1}{4}\right)\left(12x^2 - 1\right) + 24y\left(x^2 - \frac{1}{4}\right)^2\right)\hat{\jmath} \text{ in } \Omega,\end{aligned} \tag{3.42}$$

$$v = 0 \text{ on } \partial\Omega. \tag{3.43}$$

Let $v = (v_1, v_2)$, then by using the definition of the vector laplacian for Cartesian coordinates, i.e., $\Delta v = (\Delta v_1, \Delta v_2)$, we get the set of two problems given as

$$\begin{aligned}\Delta v_1 &= 24x\left(y^2 - \frac{1}{4}\right)^2 + 4x(12y^2-1)\left(x^2 - \frac{1}{4}\right) \text{ in } \Omega, \\ v_1 &= 0 \text{ on } \partial\Omega,\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\Delta v_2 &= 4y\left(y^2 - \frac{1}{4}\right)\left(12x^2 - 1\right) + 24y\left(x^2 - \frac{1}{4}\right)^2 \text{ in } \Omega, \\ v_2 &= 0 \text{ on } \partial\Omega.\end{aligned} \tag{3.45}$$

It can be easily seen that $v_1 = 4x\left(x^2 - \frac{1}{4}\right)\left(y^2 - \frac{1}{4}\right)^2$ and $v_2 = 4y\left(y^2 - \frac{1}{4}\right)\left(x^2 - \frac{1}{4}\right)^2$ are the solution of the equations (3.44) and (3.45), respectively. Now, the final step is of calculating the divergence of the vector v.

$$\nabla \cdot v = (v_1)_x + (v_2)_y = \left(12x^2 - 1\right)\left(y^2 - \frac{1}{4}\right)^2 + \left(12y^2 - 1\right)\left(x^2 - \frac{1}{4}\right)^2 = p.$$

Thus, $p = \nabla \cdot v = Sp$.

The main problem of a square domain is that M. Crouzeix proved in [13] that the Uzawa pressure operator is not compact in case of the domains having corners.

Theorem 3.14. If the boundary of Ω is piece-wise smooth with corners, then the operator $S - \frac{1}{2}I$ is not compact from M into M.

Proof. Let us consider Ω such that $\partial\Omega$ has one corner at origin and is smooth everywhere else. Suppose the corner C is such that $C := \{z = x + iy = re^{\pm i\theta},\ 0 \leq r \leq r_0,\ \theta \neq \frac{\pi}{2}\}$ and it is symmetric about the axis $y = 0$. Consider the analytic function $f(z) = z^{\frac{1}{n}-1} = q_n + ip_n$, so that p_n and q_n are conjugate harmonic functions. Define $u_n \in V$ such that $\Delta u_n = \nabla p_n$, i.e., $u_n = \Delta^{-1}\nabla p_n$ and define $h_n := (h_{n_1}, h_{n_2})$ such that

$$h_{n_1} = u_{n_1} - \frac{x}{2}p_n + \frac{\cos(\theta)\sin((\frac{1}{n}-1)\theta)}{2\sin(\frac{\theta}{n})}\Im(z^{\frac{1}{n}}), \tag{3.46}$$

$$h_{n_2} = u_{n_2} - \frac{x}{2}q_n + \frac{\cos(\theta)\cos((\frac{1}{n}-1)\theta)}{2\cos(\frac{\theta}{n})}\Re(z^{\frac{1}{n}}). \tag{3.47}$$

This h_n is such that $\Delta h_n = 0$ in Ω and $h_n = 0$ on C and it is uniformly smooth in $\partial\Omega \setminus C$. The author uses the results given in [44] to get that, there exists a constant $c > 0$ independent of n, such that $\|h_n\| \leq c$ and $h_n \in H^{3-2\theta/\pi-\epsilon}$, for all $\epsilon > 0$. We find the divergence of h_n by first differentiating (3.46) and (3.47) partially with respect to x and y,

$$(h_{n_1})_x = (u_{n_1})_x - \frac{1}{2}p_n - \frac{x}{2}(p_n)_x + \frac{\cos(\theta)\sin((\frac{1}{n}-1)\theta)}{2\sin(\frac{\theta}{n})}(\Im(z^{\frac{1}{n}}))_x,$$
$$(h_{n_2})_y = (u_{n_2})_y - \frac{x}{2}(q_n)_y + \frac{\cos(\theta)\cos((\frac{1}{n}-1)\theta)}{2\cos(\frac{\theta}{n})}(\Re(z^{\frac{1}{n}}))_y.$$

On adding the above equations and applying Cauchy-Riemann equations

$$\nabla\cdot h_n = \nabla\cdot u_n - \frac{1}{2}p_n - \frac{x}{2}(p_n)_x + \frac{x}{2}(p_n)_x + \left[\frac{\cos(\theta)\sin((\frac{1}{n}-1)\theta)}{2\sin(\frac{\theta}{n})} - \frac{\cos(\theta)\cos((\frac{1}{n}-1)\theta)}{2\cos(\frac{\theta}{n})}\right](\Im(z^{\frac{1}{n}}))_x.$$

On simplifying the above equation,

$$\nabla\cdot h_n = \nabla\cdot u_n - \frac{1}{2}p_n - \frac{1}{2n}\frac{\sin(2\theta)}{\sin(\frac{2\theta}{n})}\Im(z^{\frac{1}{n}-1}).$$

Since $\nabla\cdot u_n = Sp_n$ and $\Im z^{\frac{1}{n}-1} = p_n$,

$$Sp_n - \frac{1}{2}p_n = \frac{1}{2n}\frac{\sin(2\theta)}{\sin(\frac{2\theta}{n})}p_n + \nabla\cdot h_n.$$

Note that $p_n \notin H^{1/n}$ for all $n > 1$ and $\nabla\cdot h_n \in H^{2-2\theta/\pi-\epsilon}$, this shows that the above expression is no more regular than p_n if $n > \frac{\pi}{2(\pi-\theta)}$. On multiplying the above equation by $\gamma_n := (\|p_n\|_0)^{-1}$ and putting $m_n = \gamma_n p_n$,

$$Sm_n - \frac{1}{2}m_n = \frac{1}{2n}\frac{\sin(2\theta)}{\sin(\frac{2\theta}{n})}m_n + \gamma_n\nabla\cdot h_n. \tag{3.48}$$

Now, $\|m_n\|_0 = \gamma_n\|p_n\|_0 = 1$ and as $n\to\infty$, we have $\gamma_n \to 0$ and $m_n \rightharpoonup 0$ in $L^2(\Omega)$. On passing the limit $n\to\infty$ in (3.48),

$$\left\|Sm_n - \frac{1}{2}m_n\right\|_0 \to \frac{\sin(2\theta)}{4\theta}. \tag{3.49}$$

Since the above limit is not zero we have that $S - \frac{1}{2}I$ is not compact from the subspace M to M. $\square$

Since the operator $S - \frac{1}{2}I$ is not compact, this implies that the operator S is not compact for the domains having corners. Thus, as the boundary of the domain loses its regularity, we lose some properties of the Uzawa pressure operator S. We know for compact operator by Theorem 1.35 that the continuous and residual spectrum do not exist. Once we lose the compactness of our operator, we get the possibility of existence of continuous spectrum. From the last result of the previous theorem, i.e., equation (3.49), we get

$$\left\| Sm_n - \left(\frac{1}{2} + \frac{\sin(2\theta)}{4\theta}\right) m_n \right\|_0 \to 0.$$

Since $\|m_n\|_0 = 1$ and $m_n \rightharpoonup 0$, therefore, by using the Weyl's characterization of the essential spectrum 1.32, we get that $\frac{1}{2} + \frac{\sin(2\theta)}{4\theta}$ lies in the essential spectrum of S. Clearly, from (3.48), this value is not an eigenvalue of S, thus it must be an element of the continuous spectrum. This proves the existence of the continuous spectrum. In the paper [11], the authors gave us the exact interval of the continuous spectrum in the case of a polygonal domain. Let $\Omega \subset \mathbb{R}^2$ be a polygon. It means that it is a bounded domain with Lipschitz boundary $\partial\Omega$ which consists of finite straight segments ending at corners. Each corner c belongs to two adjacent segments. Let Γ_c be a plane sector having an opening angle ω_c. Consider the eigenvalue problem of S of finding $(v, p) \in V \times L_0^2(\Omega)$ given by the equations (3.7) and (3.8). We can reformulate this problem by substituting the value of p obtained from (3.8) in equation (3.7), so that the problem becomes as follows.

Definition 3.15 (Cosserat Problem). The problem of finding $u \in V$, $\lambda \in \mathbb{C}$ such that

$$\lambda \Delta u - \nabla(\nabla \cdot u) = 0 \tag{3.50}$$

is called the Cosserat problem. A number λ is a solution of the Cosserat problem if and only if $\lambda \in \sigma(S)$.

This problem was discussed in detail by Mikhlin [37]. Define the operator L_λ from the space V to V' as $L_\lambda := \lambda\Delta - \nabla(\nabla\cdot)$. The values of $\lambda \in \mathbb{C}$ for which the operator L_λ is not a Fredholm operator constitute the essential spectrum of S. The following lemma from [11], gives us an interval of λ where the operator is a Fredholm operator.

Lemma 3.16. Let Ω be a polygon in the plane. Assume that λ does not belong to $\{0, \frac{1}{2}, 1\}$. Then the Cosserat operator L_λ is Fredholm from V to V' if and only if for each corner $c \in C$ and $\omega = \omega_c$ the characteristic equation

$$(1-2\lambda)\frac{\sin(\lambda\omega)}{\lambda} = \pm\sin(\omega), \tag{3.51}$$

has no solution on the line $\Re(\lambda) = 0$.

The above lemma gives us the following theorem.

Theorem 3.17 (**Essential Spectrum** [11])**.** Let $\Omega \subset \mathbb{R}^2$ be a polygon with corner angles $\omega_c \in (0, 2\pi)$, $c \in C$, which is the set of all corners of Ω. Then the essential spectrum of the Cosserat problem in Ω is given by

$$\sigma_{\text{ess}}(S) = \{1\} \cup \bigcup_{c \in C} \left[\frac{1}{2} - \frac{\sin(\omega_c)}{2\omega_c}, \frac{1}{2} + \frac{\sin(\omega_c)}{2\omega_c}\right]. \tag{3.52}$$

The inf-sup constant satisfies

$$0 < \beta_0(\Omega) \leq \min_{c \in C}\sqrt{\frac{1}{2} - \frac{\sin(\omega_c)}{2\omega_c}}. \tag{3.53}$$

Proof. As stated earlier, the essential spectrum, i.e., $\sigma_{\text{ess}}(S)$ is the set of $\lambda \in \mathbb{C}$ for which the operator L_λ is not Fredholm. From Lemma 3.16, $\sigma_{\text{ess}}(S)$ is the set of λ such that equation (3.51) has solutions λ with $\Re(\lambda) = 0$. Let $\lambda \in \mathbb{C}$ be such that it is a purely imaginary number. Consider the function f defined for $\lambda \in i\mathbb{R}$ by,

$$f(\lambda) := \frac{\sin(\lambda\omega)}{\omega}.$$

Since $\lambda = ia$, where $a \in \mathbb{R}$, we get $\sin(\lambda\omega) = i\sinh(a\omega)$. As $f(ia) = \frac{\sinh(aw)}{aw} \in [1, \infty]$, thus, $f : i\mathbb{R} \to [1, \infty]$. Consider the equation (3.51), on solving it for $\lambda \in i\mathbb{R}$,

$$(1 - 2\lambda) = \pm \frac{\frac{\sin(\omega)}{\omega}}{\frac{\sin(\lambda\omega)}{\omega\lambda}}.$$

Using the range of $f(\lambda)$,

$$-\frac{\sin(\omega)}{\omega} \leq 1 - 2\lambda \leq \frac{\sin(\omega)}{\omega}$$

or the essential spectrum lies in the interval $\left[\frac{1}{2} - \frac{\sin(\omega)}{2\omega}, \frac{1}{2} + \frac{\sin(\omega)}{2\omega}\right]$. Since equation (3.51) holds for every corner $c \in C$ and $\omega = \omega_c$, and we know that 1 is an eigenvalue of infinite multiplicity, we get the required result for the essential spectrum. Thus,

$$0 < \beta_0(\Omega) = \sqrt{\lambda_{\min}(S)} \leq \min_{c \in \mathbb{C}} \sqrt{\frac{1}{2} - \frac{\sin(\omega_c)}{2\omega_c}}.$$

$\square$

For the case of a square, we have four corners each with $\omega = \frac{\pi}{2}$. Thus,

$$\sigma_{\text{ess}}(S) = \left[\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi}\right] \cup \{1\}, \tag{3.54}$$

and we have an upper bound for the inf-sup constant $\beta_0(\Omega)$, as

$$0 < \beta_0(\Omega) \leq \sqrt{\frac{1}{2} - \frac{1}{\pi}}. \tag{3.55}$$

4

Methods for improving bounds on $\lambda_{\min}(S)$

We have seen until now that the Stokes problem has a solution if and only if the inf-sup condition is satisfied, i.e., there should exist a positive inf-sup constant $\beta_0(\Omega)$. This constant is further related to $\lambda_{\min}(S)$, the infimum of the spectrum of the Uzawa pressure operator. In order to estimate $\lambda_{\min}(S)$, we list the inequalities that are related to our problem. There are many inequalities that are related to our problem. We discuss the ones which we will use for approximating $\lambda_{\min}(S)$.

4.1 Inequalities

All the inequalities that we will discuss in this section not only hold for two dimensional domains, but they also hold for domains in higher dimensions. Since we are concerned only with the case of a square, we only state the inequalities for bounded domains in $\mathbb{R}^2$. First of all, we give the Friedrichs inequality, which was given by K. Friedrichs in the paper [20] in 1937. He further gave a generalised form of this inequlaity in the paper [21]. Since we deal only with the two dimensional domains so we discuss the former inequality.

Friedrichs Inequality [20]

Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected domain. There exists a number $\Gamma \geq 1$ such that the following inequality holds

$$\int_{\Omega} u^2 dxdy \leq \Gamma \int_{\Omega} v^2 dxdy, \tag{4.1}$$

for every analytic function $w := u + iv$ in Ω which is square integrable, i.e., the integral

$$\int_{\Omega} |w|^2 dxdy$$

is finite and it satisfies

$$\int_{\Omega} u dxdy = 0. \tag{4.2}$$

This inequality is known as Friedrichs inequality. The least positive constant Γ in (4.1) is called the Friedrichs constant for the domain Ω. It is denoted by Γ_{Ω}.

The Friedrichs constant Γ_{Ω} depends only on the shape of the domain, i.e., boundary of the domain. We have the values of Γ_{Ω} for the case of a disk, ellipse and ring. As stated that $\Gamma_{\Omega} \geq 1$ for all domains, the equality is achieved for the case of a disk, i.e., $\Gamma_{\Omega} = 1$ in the case of a disk. The value of the best constant for Friedrichs inequality is still not known for the case of a square. There is another inequality which was given by I. Babŭska and A. K. Aziz in [2] and was stated again in [31] in relation to the Friedrichs inequality. This inequality is given as follows.

Babŭska-Aziz Inequality [2]

Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected domain with a Lipschitz boundary. There exists a positive constant C such that for every $p \in L^2_0(\Omega)$ there exists $u \in V$ such that

$$\nabla \cdot u = p \text{ in } \Omega,$$

and it satisfies

$$\|u\|^2 \leq C\|p\|_0^2. \tag{4.3}$$

This inequality if known as Babuška-Aziz inequality. The least positive constant in (4.3) is called Babŭska-Aziz constant and is denoted by C_Ω.

Nečas Inequality [9]

Let Ω be a bounded open connected domain with a Lipschitz boundary, there exists some positive constant c_1 such that for all $p \in L_0^2(\Omega)$,

$$c_1\|p\|_0 \leq \|\nabla p\|_{-1} \leq \|p\|_0. \tag{4.4}$$

Proof. Since $\nabla : L_0^2(\Omega) \to V_1^0$ is a bounded linear bijective operator, therefore by using Bounded inverse theorem, the inverse of ∇, say $\nabla^{-1} : V_1^0 \to L_0^2(\Omega)$ is a bounded linear operator, i.e., there exists a constant $c > 0$ such that $\|\nabla^{-1}\|_0 \leq c$. Hence, for all $p \in L_0^2(\Omega)$,

$$\|p\|_0 = \|\nabla^{-1}\nabla p\|_0 \leq \|\nabla^{-1}\|_0\|\nabla p\|_{-1} \leq c\|\nabla p\|_{-1}.$$

Hence, there exists some positive constant c_1 such that $c_1\|p\|_0 \leq \|\nabla p\|_{-1}$.

For the other half of the inequality, we rewrite $\|\nabla p\|_{-1}$ as follows.

$$\begin{aligned}
\|\nabla p\|_{-1}^2 &= \int_\Omega \nabla((-\Delta)^{-1}\nabla p) \cdot \nabla((-\Delta)^{-1}\nabla p) \\
&= -\int_\Omega ((-\Delta)^{-1}\nabla p) \cdot (\nabla \cdot \nabla((-\Delta)^{-1}\nabla p)) \\
&= \int_\Omega ((-\Delta)^{-1}\nabla p) \cdot (-\Delta(-\Delta)^{-1}\nabla p)) \\
&= \int_\Omega ((-\Delta)^{-1}\nabla p) \cdot (\nabla p) \\
&= \int_\Omega (Sp)p = (Sp, p)_0. \qquad (4.5)
\end{aligned}$$

From equation (2.32), we have that $(Sp, p)_0 \leq \|p\|_0$. Hence,

$$\|\nabla p\|_{-1} \leq \|p\|_0.$$

$\square$

The largest number positive c_1 gives us the best constant in the Nečas inequality and is denoted as c_{1_Ω}.

All of these inequalities are related to each other. There are more inequalities which are related to our problem, such as Korn's inequality having mainly four cases which are widely discussed in papers such as [22],[40], [30] and [29]. We only describe the inequalities which will give us a better way of estimating $\lambda_{\min}(S)$. The following theorem gives the relationships among the best constants of these inequalities. The first relation was given in [9] and the second relation was proved in [12].

Theorem 4.1 (**Relationship among the inequalities**)**.** The best constants in the inequalities described above are given as follows.

1. $C_\Omega = c_{1_\Omega}^{-2}$.

2. $C_\Omega = \Gamma_\Omega + 1$.

Proof. 1. The inf-sup constant $\beta_0(\Omega)$ is the largest number such that (2.28) is satisfied, i.e.,

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{(q, \nabla \cdot v)_0}{\|v\|} \geq \beta_0(\Omega) \|q\|_0.$$

Consider the left-hand side of the above inequality,

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{(q, \nabla \cdot v)_0}{\|v\|} \geq \frac{(q, \nabla \cdot v)_0}{\|v\|}.$$

By substituting $\nabla \cdot v = q \in L_0^2(\Omega)$ and using (4.3),

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{(q, \nabla \cdot v)_0}{\|v\|} \geq \frac{\|q\|_0^2}{\|v\|} \geq \frac{1}{\sqrt{C_\Omega}} \|q\|_0,$$

where, C_Ω is the best constant in (4.3), hence, $\frac{1}{C_\Omega}$ is the largest constant such that the above inequality is satisfied. This implies the following relation,

$$\beta_0(\Omega) = \frac{1}{\sqrt{C_\Omega}}. \tag{4.6}$$

From equation (4.5) and (2.35),

$$c_{1_\Omega}^2 = \inf_{\substack{p \in L_0^2(\Omega) \\ p \neq 0}} \frac{\|\nabla p\|_{-1}^2}{\|p\|_0^2} = \inf_{\substack{p \in L_0^2(\Omega) \\ p \neq 0}} \frac{(Sp, p)_0}{\|p\|_0^2} = \beta_0(\Omega)^2.$$

The above equation and (4.6) implies,

$$C_\Omega = \frac{1}{c_{1_\Omega}^2}. \tag{4.7}$$

2. This proof was given in [12]. We first prove that $\Gamma_\Omega \leq C_\Omega - 1$, in the way it was proved in [31].

   Let $q \in L_0^2(\Omega)$ such that $\nabla \cdot w = q$, for some $w \in V$. Then by using the inequality (4.3),

$$\|w\|^2 \leq C_\Omega \|q\|_0^2. \tag{4.8}$$

   By using the definition of norm as (2.22), i.e., $\|w\|^2 = \|\nabla \cdot w\|_0^2 + \|\mathbf{curl}\, w\|_0^2$,

$$\|\mathbf{curl}\, w\|_0^2 \leq C_\Omega \|q\|_0^2 - \|\nabla \cdot w\|_0^2 = (C_\Omega - 1)\|q\|_0^2.$$

   Thus, we can say that $\|\mathbf{curl}\, w\|_0 \leq \sqrt{C_\Omega - 1}\|q\|_0$.

   Let s be the conjugate harmonic function to q such that $\nabla q = \nabla^\perp s$. Now, we try to estimate the norm of q.

$$\|q\|_0^2 = (q, \nabla \cdot w)_0 = (-\nabla q, w)_0 = \left(-\nabla^{\perp} s, w\right)_0 = (-s, \mathbf{curl}\, w)_0 .$$

By using the Cauchy-Schwartz inequality,

$$\|q\|_0^2 \leq \|s\|_0 \|\mathbf{curl}\, w\|_0 \leq \sqrt{C_\Omega - 1}\|q\|_0 \|s\|_0.$$

Therefore, for harmonic conjugates q and s, the following inequality holds,

$$\|q\|_0 \leq \sqrt{C_\Omega - 1}\|s\|_0.$$

Comparing the above inequality with (4.1), as Γ_Ω is the smallest positive constant such that the inequality holds,

$$\Gamma_\Omega \leq C_\Omega - 1.$$

Now, we try to prove that $C_\Omega - 1 \leq \Gamma_\Omega$.
Consider a harmonic $p \in L_0^2(\Omega)$ and define $u \in V$ such that $\Delta u = \nabla p$. So, we get the following weak form for all $v \in V$

$$(\nabla u, \nabla v)_0 = (p, \nabla \cdot v)_0 .$$

Now, define $q = \nabla \cdot u$ and $g = \mathbf{curl}\, u$, so that

$$(p, q)_0 = (p, \nabla \cdot u)_0 = (-\nabla p, u)_0 = (-\Delta u, u)_0 = (\nabla u, \nabla u)_0 = \|u\|^2,$$

also, by using the equation (2.22), $(p,q)_0 = \|u\|^2 = \|q\|_0^2 + \|g\|_0^2$. Thus,

$$\|g\|_0^2 = (p,q)_0 - \|q\|_0^2 = (p,q)_0 - (q,q)_0 = (p-q,q)_0. \tag{4.9}$$

By using Cauchy-Schwartz inequality on the right-hand side of the above equation,

$$\|g\|_0^2 \leq \|p-q\|_0\|q\|_0. \tag{4.10}$$

Now, we note that $p-q$ and g are harmonic functions, as

$$\Delta q = \nabla \cdot \Delta u = \Delta p = 0,$$

and

$$-\Delta g = \mathbf{curl}\,\Delta u = \mathbf{curl}\,\nabla p = 0.$$

Moreover, $p-q$ and g are conjugate harmonic functions, as

$$\nabla^\perp g - \nabla q = (-\Delta u_1, -\Delta u_2) = -\Delta u = -\nabla p,$$

i.e., $\nabla^\perp g = -\nabla(p-q)$. Hence, by using Friedrichs inequality,

$$\|p-q\|_0^2 \leq \Gamma_\Omega \|g\|_0^2. \tag{4.11}$$

By using this in (4.10),

$$\|g\|_0^2 \leq \sqrt{\Gamma_\Omega}\|q\|_0\|g\|_0,$$

which gives

$$\|g\|_0^2 \leq \Gamma_\Omega \|q\|_0^2. \tag{4.12}$$

Note that,

$$\begin{aligned}
\|p\|_0^2 &= \|p-q\|_0^2 + 2(p,q)_0 - \|q\|_0^2 \\
&\le \Gamma_\Omega \|g\|_0^2 + \|q\|_0^2 + 2\|g\|_0^2 \quad \text{by using (4.9) and (4.11)} \\
&\le (\Gamma_\Omega + 1)\|g\|_0^2 + \|g\|_0^2 + \|q\|_0^2 \\
&\le (\Gamma_\Omega + 1)\|g\|_0^2 + (\Gamma_\Omega + 1)\|q\|_0^2 \quad \text{by using (4.12)} \\
&= (\Gamma_\Omega + 1)(\|g\|_0^2 + \|q\|_0^2) \\
&= (\Gamma_\Omega + 1)\|u\|^2.
\end{aligned}$$

Thus,

$$\|p\|_0^2 \le (\Gamma_\Omega + 1)\|\nabla p\|_{-1}^2.$$

By using equation (4.7),

$$C_\Omega = \frac{1}{c_{1\Omega}^2} = \sup_{p \in L_0^2(\Omega)\backslash 0} \frac{\|p\|_0^2}{\|\nabla p\|_{-1}^2} \le \Gamma_\Omega + 1.$$

Hence, we have proved that

$$C_\Omega = \Gamma_\Omega + 1. \tag{4.13}$$

□

Recall that we have shown $\lambda_{\max}(S) = 1$. We will now give the relationship of the best constants in each of the inequalities with $\lambda_{\min}(S)$, the infimum of the spectrum of the Uzawa pressure operator. We have already shown that $\lambda_{\min}(S) = \beta_0(\Omega)^2$. The first relationship was indirectly given in [9], rest of them are the implications we get by using the previous theorem.

Theorem 4.2 (**Relationship with** $\lambda_{\min}(S)$)**.** The best constant in the given inequalities are related to the infimum of the spectrum of the Uzawa pressure operator as follows.

1. $\lambda_{\min}(S) = c_{1_\Omega}^2$.

2. $\lambda_{\min}(S) = \frac{1}{C_\Omega}$.

3. $\lambda_{\min}(S) = \frac{1}{\Gamma_\Omega + 1}$.

Proof. 1. From equation (4.5), we have that $\|\nabla p\|_{-1}^2 = (Sp, p)_0$. So,

$$\lambda_{\min}(S) = \inf_{p \in L_0^2(\Omega)} \frac{(Sp, p)_0}{(p, p)_0} = \inf_{p \in L_0^2(\Omega)} \frac{\|\nabla p\|_{-1}^2}{\|p\|_0^2} = c_{1_\Omega}^2. \tag{4.14}$$

2. From the relations (4.7) and (4.14),

$$\lambda_{\min}(S) = c_{1_\Omega}^2 = \frac{1}{C_\Omega}. \tag{4.15}$$

3. From the relations (4.13) and (4.15), we get that

$$\lambda_{\min}(S) = \frac{1}{C_\Omega} = \frac{1}{\Gamma_\Omega + 1}. \tag{4.16}$$

□

The above relations are very important for determining the estimates of the minimum eigenvalue of the Uzawa pressure operator. Since we know from the Theorem 2.15 that $0 < \beta_0(\Omega) = \sqrt{\lambda_{\min}(S)}$. Also, Friedrichs proved in [20] that $\Gamma_\Omega \geq 1$. Thus, by using the relation (4.16), we get that $\lambda_{\min}(S) \leq \frac{1}{2}$. We have from Chapter 4, the result for essential spectrum of Uzawa pressure operator and the upper bound on (3.55), that gives us the best upper bound on $\lambda_{\min}(S)$ so far as,

$$\lambda_{\min}(S) \leq \frac{1}{2} - \frac{1}{\pi} = 0.18169011381. \tag{4.17}$$

We will now state the best lower bound on $\lambda_{\min}(S)$.

4.2 Lower Bound for $\lambda_{\min}(S)$

In this section, we give the result regarding a lower bound for the infimum of the spectrum of the Uzawa pressure operator. As we know from (4.16) that $\lambda_{\min}(S)$ and the Friedrichs constant Γ_Ω share an inverse relationship. Thus, in order to find a lower bound on $\lambda_{\min}(S)$, we should find an upper bound for Γ_Ω. The first attempt for finding an upper bound on Γ_Ω was made by C. O. Horgan and L. E. Payne in the paper [31] published in the year 1983. In section 6 of [31], the upper bound on Γ_Ω for star-shaped domains is derived.

Theorem 4.3 (**Upper bound for star-shaped domains**)**.** Let Ω be a star-shaped domain with Lipschitz boundary. Let n_r represent the radial component of the outward normal vector n on $\partial\Omega$. An upper bound on the Friedrichs constant for Ω is given as,

$$\Gamma_\Omega \leq \max_{\partial\Omega} \left[\frac{1}{n_r} + \sqrt{\frac{1}{n_r^2} - 1} \right]^2 . \tag{4.18}$$

This inequality is also called as **Horgan-Payne Inequality**.

The proof can be found in paper [31]. The name Horgan-Payne was given by G. Stoyan in the year 2001 in paper [47], where he also gave a much simpler formulation for the upper bound on Γ_Ω. He defines the Horgan-Payne angle as follows.

Definition 4.4 (**Horgan-Payne Angle** [47])**.** Let Ω be a star-shaped domain with Lipschitz boundary $\partial\Omega$ and let $x_0 \in \Omega$. For all $x \in \partial\Omega$, consider the outer angle $\alpha(x)$ between a ray emitted from x_0 (hitting $\partial\Omega$ at x) and the outer normal vector to $\partial\Omega$ at x. Then the angle ω, defined as

$$\omega = \frac{\pi}{2} - \max_{x\in\partial\Omega} \alpha(x), \tag{4.19}$$

is called the Horgan-Payne angle.

The upper bound on Γ_Ω was reformulated in the form of the Horgan-Payne angle and it is given in the paper [47] by the following lemma. This upper bound on Γ_Ω is expressed as a lower bound on the inf-sup constant $\beta_0(\Omega)$.

Lemma 4.5. Let Ω be a star-shaped domain with Lipschitz boundary and positive Horgan-Payne angle ω. Then the inf-sup constant $\beta_0(\Omega)$ satisfies the following estimate:

$$\sin\frac{\omega}{2} \leq \beta_0(\Omega). \tag{4.20}$$

Proof. Let $x \in \partial\Omega$ and n_r represents the radial component of the outward normal vector n on $\partial\Omega$. Consider the outer angle $\alpha(x)$ between a ray emitted from x_0 (hitting $\partial\Omega$ at x) and the outer normal vector to $\partial\Omega$ at x. Then at $x \in \partial\Omega$, we have that $n_r = \cos(\alpha(x))$. By substituting this value of n_r in the inequality (4.18),

$$\begin{aligned}\Gamma_\Omega &\leq \max_{x\in\partial\Omega}\left[\frac{1}{\cos(\alpha(x))} + \sqrt{\left(\frac{1}{\cos^2(\alpha(x))} - 1\right)}\right]^2 \\ &= \max_{x\in\partial\Omega}\left[\frac{1+\sin(\alpha(x))}{\cos(\alpha(x))}\right]^2.\end{aligned}$$

It is easily seen that the maximum in the above inequality is achieved at the maximum value of $\alpha(x)$, which is given by equation (4.19) as $\frac{\pi}{2} - \omega$. Thus,

$$\Gamma_\Omega \leq \left[\frac{1+\cos(\omega)}{\sin(\omega)}\right]^2 = \cot^2\frac{\omega}{2}.$$

By using the relation (4.16),

$$\beta_0(\Omega) = \frac{1}{\sqrt{1+\Gamma_\Omega}} \geq \sin\frac{\omega}{2}.$$

□

The above results holds for star-shaped domains having a positive Horgan-Payne angle. The author remarks that the Horgan-Payne angle is positive for domains for which there is a ball from all the points of which all of the $\partial\Omega$ is visible. Polygons are examples of such a domain.

As we are interested in finding a lower bound for the case of a square domain, we need to know the Horgan-Payne angle for a square. Stoyan mentions in his paper [47] that $\omega = \frac{\pi}{4}$ for a square. Thus, by using this value of ω in the relation (4.20),

$$\lambda_{\min}(S) \geq \sin^2 \frac{\pi}{8} = 0.1464466094.$$

Thus,

$$\lambda_{\min}(S) \in \left[\sin^2 \frac{\pi}{8}, \frac{1}{2} - \frac{1}{\pi}\right] = [0.1464466094, 0.18169011381]. \tag{4.21}$$

While trying to estimate the constant $\beta_0(\Omega)$, one would often think about bounding the domain between some other domains to get the estimate. One such result, given as follows, was proved in [12].

Theorem 4.6. Let $\Omega \subset \mathbb{R}^2$ be a domain contained in a ball of radius $\mathcal{R}$, star-shaped with respect to a concentric ball of radius ρ, then

$$\beta_0(\Omega) \geq \frac{\rho}{\sqrt{2}\mathcal{R}} \left(1 + \sqrt{1 - \frac{\rho^2}{\mathcal{R}^2}}\right)^{-\frac{1}{2}}. \tag{4.22}$$

Consider the square $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ containing the circle with radius $\rho = \frac{1}{2}$ centred at origin and contained in the circle with radius $\mathcal{R} = \frac{1}{\sqrt{2}}$ centred at origin. By using (4.22), $\beta_0(\Omega) \geq 0.38268343236$, which implies, $\lambda_{\min}(S) \geq 0.1464466094$. Thus, the bound given by (4.21), are the best known bounds for $\lambda_{\min}(S)$. In the next section, we will try to improve these bounds.

4.3 Efforts for improving the bounds

In this section, we try to use the inequalities mentioned in this chapter for improving the bounds on $\lambda_{\min}(S)$. Since it is directly related to the best constants in those inequalities, therefore we could estimate the best constant for each of them. The boundary of a square is Lipschitz, so it is not smooth enough. Until now, we just have conjectures about these constants.

4.3.1 Friedrichs Inequality

Let $\Omega := \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ and consider the Friedrichs inequality given by the equation (4.1). Let Γ_Ω be the best constant for Ω; then we can express it as,

$$\Gamma_\Omega = \sup_{\nabla u = \nabla^\perp v} \frac{\int_\Omega u^2}{\int_\Omega (v+c)^2},$$

where u and v are square integrable functions, c is a constant and u has zero average over Ω. Let $G(u,v)$ denote the following expression.

$$G(u,v) := \frac{\int_\Omega u^2}{\int_\Omega (v+c)^2}. \tag{4.23}$$

In this section, we will try to maximize $G(u,v)$ over the set of conjugate harmonic functions u and v such that the analytic function $f(z) = u + iv$ is square integrable and u has zero average. We can easily see that $G(u,v)$ will be maximum if its denominator is minimum, so we minimize the function $g(c) := \int_\Omega (v+c)^2$. The critical points of g are,

$$g'(c) = 2\int_\Omega (v+c) = 0,$$

thus,

$$c = -\frac{1}{|\Omega|}\int_\Omega v,$$

where $|\Omega|$ denotes the measure of the set Ω. Thus, the maximization process includes, constructing a harmonic function u having zero average over Ω and then finding the conjugate harmonic function v of u. We then optimize v to have zero average. Finally, we calculate $G(u, v)$, then find its maximum.

In order to find the set of harmonic functions u, with zero average on Ω, we formulate the following problem.

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.24}$$

On solving the above problem by separation of variables, we find the following possibilities for the function u for the constants a and b.

1. $u = x(ay + b)$.

2. $u = (e^{ax} - e^{-ax}) \sin(ay)$.

3. $u = (e^{ax} - e^{-ax}) \cos(ay)$.

Now, we optimize these functions.

1. For $u = x(ay + b)$, we find that the conjugate harmonic function $v = \frac{a}{2}(y^2 - x^2) + by + c$, where c is a constant. In order that v has zero average, we find $c = 0$. Thus, $v = \frac{a}{2}(y^2 - x^2) + by$. On calculating $G(u, v)$ for these u and v,

$$G(u, v) = 1 + \frac{3a^2}{2a^2 + 60b^2}.$$

The maximum of G is 2.5 and it is attained for $b = 0$ and for all $a \in \mathbb{R}$. Thus, we get $\Gamma_\Omega \geq 2.5$.

2. For $u = (e^{ax} - e^{-ax}) \sin(ay)$, we find that the conjugate harmonic function $v = -(e^{ax} + e^{-ax}) \cos(ay) + c$, where c is a constant. In order that v has zero

average, we find the constant c to as,

$$c = \frac{8}{a^2} \sin\frac{a}{2} \sinh\frac{a}{2}.$$

Thus,

$$v = -(e^{ax} + e^{-ax})\cos(ay) + \frac{8}{a^2} \sin\frac{a}{2} \sinh\frac{a}{2}.$$

On calculating $G(u, v)$ for these u and v,

$$G(u, v) = \frac{a^2(\sinh a - a)(a - \sin a)}{a^2(\sinh a + a)(\sin a + a) - 64\sin^2\frac{a}{2}\sinh^2\frac{a}{2}}.$$

The maximum of G is 2.5 and it is attained for many points near $a = 0$. Thus, $\Gamma_\Omega \geq 2.5$.

3. For $u = (e^{ax} - e^{-ax})\cos(ay)$, we find that the conjugate harmonic function $v = (e^{ax} + e^{-ax})\sin(ay) + c$, where c is a constant. In order that v has zero average, we find the constant $c = 0$. Thus, $v = (e^{ax} + e^{-ax})\sin(ay)$. On calculating $G(u, v)$ for these u and v,

$$G(u, v) = \frac{(\sinh a - a)(\sin a + a)}{(\sinh a + a)(a - \sin a)}.$$

The maximum of G is 1.2777 and it is attained for $a = 7.769$. Thus, $\Gamma_\Omega \geq 1.2777$, which is not better than 2.5.

Hence, until now, we have $\Gamma_\Omega \geq 2.5$. Now, we try to manipulate these functions in order to improve the bounds. We take u to be as following, where a, b are constants.

1. $u = e^{ax}\sin(ay)$.

2. $u = (e^{ay} + be^{-ay})\sin(ax)$.

3. $u = b(x^2 - y^2) + a(y - x)$.

Now, we optimize these functions.

1. For $u = e^{ax}\sin(ay)$, we find that the conjugate harmonic function $v = -e^{ax}\cos(ay)+c$, where is a constant. In order that v has zero average, we find the constant c to be

$$c = \frac{4}{a^2}\sin\frac{a}{2}\sinh\frac{a}{2}.$$

Thus,

$$v = -e^{ax}\cos(ay) + \frac{4}{a^2}\sin\frac{a}{2}\sinh\frac{a}{2}.$$

On calculating $G(u, v)$ for these u and v,

$$G(u, v) = \frac{a^2(a - \sin a)\sinh a}{a^2(\sin a + a)\sinh a - 32\sin^2\frac{a}{2}\sinh^2\frac{a}{2}}.$$

The maximum of G is 1.975 and it is attained for $a = 3.675$. Thus, $\Gamma_\Omega \geq 1.975$, which is not better than the bound 2.5.

2. For $u = (e^{ay} + be^{-ay})\sin(ax)$, we find that the conjugate harmonic function $v = (e^{ay} - be^{-ay})\cos(ax) + c$, where c is a constant. In order that v has zero average, we find the constant c to as,

$$c = \frac{4(b-1)}{a^2}\sin\frac{a}{2}\sinh\frac{a}{2}.$$

Thus,

$$v = (e^{ay} - be^{-ay})\cos(ax) + \frac{4(b-1)}{a^2}\sin\frac{a}{2}\sinh\frac{a}{2}.$$

On calculating $G(u, v)$ for these u and v,

$$G(u, v) = \frac{a^2(a - \sin a)(2ab + (b^2+1)\sinh a)}{a^2(\sin a + a)((b^2+1)\sinh a - 2ab) - 32(b-1)^2\sin^2\frac{a}{2}\sinh^2\frac{a}{2}}.$$

The maximum of G is 2.5 and it is attained for points near $a = 0$. Thus,

$\Gamma_\Omega \geq 2.5$.

3. For $u = b(x^2 - y^2) + a(y - x)$, we find that the conjugate harmonic function $v = 2bxy + a(y-x) + c$, where c is a constant. In order that v has zero average, we find the constant $c = 0$. Thus, $v = 2bxy + a(y - x)$. On calculating $G(u, v)$ for these u and v,
$$G(u, v) = \frac{12b^2 + 18a^2}{20b^2 + 60a^2}.$$
The maximum of G is less than 1. So, this is not a good choice.

Until now, the only best lower bound that we have on Γ_Ω is of 2.5. Now, on looking at the Section 2.2 of paper [46] by G. Stoyan, we observe that he mentions the following data for $z = x + iy$.

- If $u = \Im(z^2)$ and $v = \Re(z^2)$, we get $\Gamma_\Omega \geq 2.5$.
- If $u = \Im(z^6)$ and $v = \Re(z^6)$, we get $\Gamma_\Omega \geq 2.5822$.

He also mentions that if $u = \Im(z^{10})$ and $v = \Re(z^{10})$, we get a number even bigger than the one we got for z^6. We calculated that number with the help of MATLAB and found the value that $\Gamma_\Omega \geq 2.7625$ for z^{10}. Hence, we conclude that by constructing some new analytic function and taking their real and imaginary parts as v and u respectively, we can find some better bounds. Let $f(z) = v + iu$ be an analytic function, such that $\int_\Omega u = 0$. Therefore, we consider the following cases.

1. Let $f(z) = z^{2(2n-1)}$, where n is a positive integer. We performed iterations on n in MATLAB and found the following values.

   (a) For $n = 1002$, we have $G(u, v) = 2.9980$.

   (b) For $n = 4302$, we have $G(u, v) = 2.9985$.

   (c) For $n = 10002$, we have $G(u, v) = 2.9998$.

   (d) For $n = 16002$, we have $G(u, v) = 2.9999$.

Thus, it appears to approaching 3, which is better than the previous bound of 2.5.

2. Let $f(z) = \cos^n(z)$, where n is a positive integer. We performed iterations on n in MATLAB and found that the value of $G(u, v)$ was at most 1. Therefore, it is not a good choice of $f(z)$.

3. Let $f(z) = \sin^n(z)$, where n is a positive integer. We performed iterations on n in MATLAB and found that the value of $G(u, v)$ was at most 2.6653 for $n = 38$. Therefore, it is not a good choice of $f(z)$.

4. Let $f(z) = (z^2 + az)^n$, where a is a constant, here we optimize a in the function $f(z)$ so that $G(u, v)$ is maximized. It was mostly giving us maximum for $a = 0$, thus, case is reduced to that of $f(z) = z^n$.

5. Let $f(z) = (z^3 + z^2 + cz)^n$, where c is a constant, we optimize c in the function $f(z)$ so that $G(u, v)$ is maximized. For $n = 10$ and $c = 0$, it gave us $G(u, v) = 2.9184$.

We conclude that the maximum we have until now is $\Gamma \geq 3$.

4.3.2 Changing the boundary conditions

For square or rectangular domains, some authors change the boundary condition of Δ^{-1}, which has Dirichlet boundary condition while being involved in the Uzawa pressure operator. The modification of the boundary conditions is made intelligently so that the new resultant operator has an easy to determine spectrum. The minimum eigenvalues of new operators give a bound on $\lambda_{\min}(S)$, the infimum of the spectrum of Uzawa pressure operator. This technique was first used by the authors E. V. Chizhonkov and P. P. Aristov in [1], in which they formulated the following eigenvalue problem of finding $\lambda \in \mathbb{R}$ and $p \in L_0^2(\Omega)$ on the unit square.

$$
\begin{aligned}
-\Delta u + \nabla p = 0 \text{ in } \Omega, \\
\nabla \cdot u = \lambda p \text{ in } \Omega,
\end{aligned}
\tag{4.25}
$$

with the boundary conditions

$$
u_1 = 0, \quad u_2 = 0 \text{ on } y = 0 \text{ and } y = 1, \tag{4.26}
$$

and

$$
u_1 = 0, \quad \frac{\partial u_2}{\partial x} = 0 \text{ on } x = 0 \text{ and } x = 1. \tag{4.27}
$$

The operator corresponding to these boundary conditions is denoted by $A_m := \nabla \cdot \Delta_m^{-1}\nabla$. They solved this eigenvalue problem by using the method of separation of variables and found the set of eigenvalues as follows

$$
\Lambda = \{1\} \cup \left\{ \frac{1}{2}\left(1 \pm \frac{t}{\sinh t}\right),\, t = \frac{m\pi}{L},\, m \in \mathbb{N} \right\}. \tag{4.28}
$$

The author M. A. Ol'shanskiĭ in paper [38] introduces another type of eigenvalue problem by doing a variation in the boundary as follows,

$$
\begin{aligned}
-\Delta u + \nabla p = 0 \text{ in } \Omega, \\
\nabla \cdot u = \lambda p \text{ in } \Omega,
\end{aligned}
$$

with the boundary conditions

$$
u \cdot \nu = 0, \quad \frac{\partial (u \cdot \tau)}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{4.29}
$$

where ν and τ are normal and tangent vectors to $\partial\Omega$. The operator corresponding to the above boundary conditions is denoted by $A_p := \nabla \cdot \Delta_p^{-1}\nabla$, which was proved to be equal to the identity operator. In paper [39], the authors gave bounds on the minimum eigenvalue of S w.r.t to the minimum eigenvalue of the two operators A_m and A_p.

Lemma 4.7 (**Bounds** [39])**.** Let $\Omega = \{(x, y) : 0 < x < L_1, 0 < y < L_2\}$ and $l = \max(L_2/L_1, L_1/L_2)$, then the following estimates hold.

$$\frac{1}{60l^2}\lambda_{\min}(A_p) \leq \lambda_{\min}(S) \leq \lambda_{\min}(A_m) \leq \frac{\pi^2}{12l^2}.$$

These bounds, however, do not give us anything better. These are important for mentioning in the text because they provide us with another method for improving the bounds on $\lambda_{\min}(S)$. A similar method is described in papers such as [14] and [15]. One other way to find bounds is to map a domain conformally onto other domain. The Friedrich constants of such domains are related to each other in the form of the following theorem. This relationship was given in paper [52].

Theorem 4.8. Let Ω be a simply connected domain with piecewise smooth boundary. Let η denote the bijective conformal mapping of the domain Ω onto Ω_1 such that $\eta(z_0) = w_0$, where z_0, $w_0 \in \mathbb{C}$. Set $L = \frac{\sup_\Omega |\eta'|}{\inf_\Omega |\eta'|}$. If $0 < L < \infty$, then

$$\frac{1}{L^2}\Gamma_\Omega \leq \Gamma_{\Omega_1} \leq L^2\Gamma_\Omega, \tag{4.30}$$

where Γ_Ω and Γ_{Ω_1} denote the Friedrichs constants on the domain Ω and Ω_1, respectively.

Proof. Let u, v be conjugate harmonic functions on Ω_1 such that they satisfy the condition of Friedrichs inequality and thus,

$$\int_{\Omega_1} u^2 dA \leq \Gamma_{\Omega_1} \int_{\Omega_1} v^2 dA.$$

Let η be a conformal mapping of the domain Ω onto Ω_1. Set $U = u \circ \eta$ and $V = v \circ \eta$.

Since u has zero average

$$\int_\Omega U(z)|\eta'(z)|^2 dA(z) = 0.$$

Hence, there is a point $z_* \in \Omega$ such that $U(z_*) = 0$. This is a condition for the another formulation of the Friedrichs inequality, for which the best constant is greater than or equal to the Friedrichs constant (See section 2 of paper [52]).

$$\Gamma_{\Omega_1} \geq \frac{\int_{\Omega_1} u^2}{\int_{\Omega_1} v^2} = \frac{\int_\Omega U(z)^2|\eta'(z)|^2 dA(z)}{\int_\Omega V(z)^2|\eta'(z)|^2 dA(z)} \geq \frac{\inf_\Omega |\eta'(z)|^2}{\sup_\Omega |\eta'(z)|^2} \cdot \frac{\int_\Omega U^2 dA}{\int_\Omega V^2 dA} = \frac{1}{L^2}\frac{\int_\Omega U^2 dA}{\int_\Omega V^2 dA}.$$

Thus,

$$\int_\Omega U^2 dA \leq L^2 \Gamma_{\Omega_1} \int_\Omega V^2 dA,$$

which is another formulation of Friedrichs inequality on Ω, thus

$$\Gamma_\Omega \leq L^2 \Gamma_{\Omega_1}.$$

Similarly, we can prove the right-hand side of the inequality. □

Corollary 4.9. [52] Let Ω, Ω_1 and L be the same as in Lemma 4.8,

$$\frac{1}{L}\beta_0(\Omega) \leq \beta_0(\Omega_1) \leq L\beta_0(\Omega). \tag{4.31}$$

Proof. From equations (4.16) and (2.35), and the right hand side of the relation (4.30),

$$\beta_0(\Omega_1)^2 = \frac{1}{1+\Gamma_{\Omega_1}} \geq \frac{1}{1+L^2\Gamma_\Omega} \geq \frac{1}{L^2}\frac{1}{1+\Gamma_\Omega} = \frac{1}{L^2}\beta_0(\Omega)^2.$$

Similarly, we can prove the right hand side of (4.31). □

The following example was given in [52] as an application of Theorem 4.8.

Example 4.3.1:

Consider the map $\tilde{z} = \eta(z) = e^z$ and the rectangular domain Ω defined as,

$$\Omega := \{z \in \mathbb{C} : \log(r) \leq \Re z \leq \log(R),\, 0 \leq \Im z \leq \theta\},$$

where $0 < r < R$ and $0 \leq \theta < 2\pi$. The conformal map η maps Ω onto Ω_1 defined as,

$$\Omega_1 := \{\tilde{z} \in \mathbb{C} : r \leq \tilde{z} \leq R,\, 0 \leq \arg \tilde{z} \leq \theta\}.$$

Since $\eta'(z) = e^z$, therefore $|\eta'(z)|^2 = e^{2\Re z}$. By using the definition of the set Ω,

$$r^2 \leq |\eta'(z)|^2 \leq R^2,\ \forall\, z \in \Omega.$$

Thus, $L = \frac{R}{r}$ and by using (4.8) and (4.31),

$$\Gamma_{\Omega_1} \leq \Gamma_\Omega \frac{R^2}{r^2},$$

and $\beta_0(\Omega_1) \geq \beta_0(\Omega)\frac{r}{R}$.

4.3.3 Nečas Inequality

In this section, we try to approximate the best constant c_{1_Ω} in the Nečas inequality, which is given by the equation (4.4). This attempt was made by the authors Maurice Gaultier and Mikel Lezaun [24]. Firstly, they try to approximate the constant by constructing a sequence.

Proposition 4.10. Let for some $k \in \mathbb{N}$, α_k be the smallest eigenvalue of the matrix A_k of the quadratic form defined on $\mathbb{R}^k$ by the following formula

$$Q_k(\xi) = \sum_{i,j=1}^{k} \lambda_i \lambda_j (S\phi_i, \phi_j)_0, \tag{4.32}$$

where ϕ_i are the elements of the set $E := \{\phi_j \,|\, 1 \leq j < \infty\}$, which is an orthonormal basis of $L^2_0(\Omega)$ and $\xi = (\lambda_1, \ldots, \lambda_k)$. Then the sequence $\alpha_k \to c^2_{1_\Omega}$, when $k \to \infty$.

Proof. We know that the space $L^2_0(\Omega)$ is separable. For any positive integer k, P_k be the subspace of the space $L^2_0(\Omega)$ such that $P_k := \text{span}\{\phi_k \,|\, 1 \leq j \leq k\}$. Therefore, we have $P_k \subset P_{k+1}$, for all $k \in \mathbb{N}$.

Now, we define the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$, as follows

$$\alpha_k := \inf_{\substack{q\in P_k \\ p\neq 0}} \frac{\|\nabla q\|^2_{-1}}{\|q\|^2_0} = \inf_{\substack{q\in P_k \\ p\neq 0}} \frac{(Sq, q)_0}{\|q\|^2_0}. \tag{4.33}$$

The sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ is a monotonically decreasing sequence, as $P_k \subset P_{k+1}$, we have $\alpha_k \geq \alpha_{k+1}$ for all $k \in \mathbb{N}$. It is bounded below by $\lambda_{\min}(S)$. Hence, the sequence is convergent. Let it converge to the number α, then as we have $\alpha_k \geq c^2_{1_\Omega}$ since, $P_k \subset L^2_0(\Omega)$, for all $k \in \mathbb{N}$

$$\alpha \geq c^2_{1_\Omega}. \tag{4.34}$$

We need to prove the other side of the inequality now. Let u be an element of $L^2_0(\Omega)$. We can construct a sequence, $\{u_k\}_{k\in\mathbb{N}}$, of $u_k \in P_k$ for all $k \in \mathbb{N}$ such that $u_k \to u$ in $L^2_0(\Omega)$ as $k \to \infty$. As we know that $\nabla : L^2_0(\Omega) \to (H^{-1}(\Omega))^2$ and it is a continuous mapping, so we get $\nabla u_k \to \nabla u$ in $(H^{-1}(\Omega))^2$. From the definition of α_k, i.e., the equation (4.34) and by using the left hand side of Nečas inequality, we get $\|\nabla u_k\|^2_{-1} \geq \alpha_k \|u_k\|^2_0$, by passing on the limit $k \to \infty$,

$$\|\nabla u\|^2_{-1} \geq \alpha \|u\|^2_0.$$

Thus,

$$\alpha \leq c^2_{1_\Omega}. \tag{4.35}$$

Hence, from the equations (4.34) and (4.35), we get that $\alpha = c^2_{1_\Omega}$. Now, in order to prove the proposition, we just need to specify α_k and the matrix A_k. Since E is an orthonormal basis of $L_0^2(\Omega)$, so each $u \in P_k$ can be written uniquely as the linear combination of the basis functions of the space P_k, i.e.,

$$u = \sum_{j=1}^{k} \lambda_j \phi_j.$$

Let $G : P_k \to \mathbb{R}^k$, such that $\xi = Gu = (\lambda_1, \ldots, \lambda_k)$. Now, the space P_k is equipped with the norm induced by the norm of the space $L_0^2(\Omega)$ and $\mathbb{R}^k$ is equipped with the Euclidean norm. Therefore, G is an isometric isomorphism from P_k to the space $\mathbb{R}^k$. We can write,

$$\xi \to Q_k(\xi) = (S(G^{-1}\xi), G^{-1}\xi)_0 = \sum_{i,j=1}^{k} \lambda_i \lambda_j (S\phi_i, \phi_j)_0.$$

Moreover, by using (4.33),

$$(Su, u)_0 = (S(G^{-1}\xi), G^{-1}\xi)_0 \geq \alpha_k \|\xi\|_0^2.$$

So,

$$\alpha_k = \inf_{\substack{\xi \in \mathbb{R}^k \\ \xi \neq 0}} \frac{Q_k(\xi)}{\|\xi\|_0^2},$$

which converges to $c_{1\Omega}^2 = \lambda_{\min}(S)$. $\square$

In Section 6 of the paper [24] give us an implementation of the above proposition in a rectangle.

Implementation of Proposition 4.10 to a Rectangular Domain [24]

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < L, 0 < y < l\}$. The formulation involves computing the entries of the matrix A_k given by the quadratic form $Q_k(\xi)$ defined by (4.32).

The authors of [24] define, for $m, p \in \mathbb{N}$, the following notations

$$\omega_m = \frac{m\pi}{L}, \tag{4.36}$$

$$\delta_p = \frac{p\pi}{l}, \tag{4.37}$$

$$\alpha_{m,p} = \omega_m^2 + \delta_p^2, \tag{4.38}$$

and the constants,

$$c_{0,p}^2 = c_{m,0}^2 = \frac{1}{2}c_{m,p}^2 = \frac{2}{Ll}. \tag{4.39}$$

The set E of orthonormal basis of $L_0^2(\Omega)$ is given as follows

$$E := \left\{\phi_{m,p}(x,y) = c_{m,p}\cos(\omega_m x)\cos(\delta_p y) \,|\, (x,y) \in \Omega,\, m,p \in \mathbb{N},\, m+p \geq 1\right\}. \tag{4.40}$$

It is important to note that the $(m(k+1)+p, j(k+1)+q)$ element of the matrix A_k is the given as $(S\phi_{m,p}, \phi_{j,q})_0$ for $m, p, n, j \in \mathbb{N}$, $0 \leq m, n, p, j \leq k$, $m+p \geq 1$ and $j+q \geq 1$. The matrix is of order $(k+1)^2 - 1$. The authors further denote for some $r, s \in \mathbb{N}$, the function a as follows

$$a(r,0) = a(0,s) = 1 \text{ and } a(r,s) = \sqrt{2}. \tag{4.41}$$

We now evaluate the term $(S\phi_{m,p}, \phi_{j,q})_0$ for some $m, j, p, q \in \mathbb{N}$, such that $m+j \geq 1$ and $p+q \geq 1$.

$$\begin{aligned}(S\phi_{m,p}, \phi_{j,q})_0 &= \int_\Omega S(c_{m,p}\cos(\omega_m x)\cos(\delta_p y))(c_{j,q}\cos(\omega_j x)\cos(\delta_q y)) \\ &= -c_{m,p}c_{j,q}\int_\Omega \Delta^{-1}\nabla(\cos(\omega_m x)\cos(\delta_p y)) \cdot \nabla(\cos(\omega_j x)\cos(\delta_q y)).\end{aligned} \tag{4.42}$$

The longer process is the calculation of $\Delta^{-1}\nabla(\cos(\omega_m x)\cos(\delta_p y))$. Let v be such that

$$\begin{aligned} \Delta v &= \nabla(\cos(\omega_m x)\cos(\delta_p y)) \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.43}$$

Since $v = (v_1, v_2)$ is a vector we get further the following two problems,

$$\begin{aligned} \Delta v_1 &= -\omega_m \sin(\omega_m x)\cos(\delta_p y) \text{ in } \Omega, \\ v_1 &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \Delta v_2 &= -\delta_p \cos(\omega_m x)\sin(\delta_p y) \text{ in } \Omega, \\ v_2 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.45}$$

We solve the both of the problems by using separation of variables. Let us consider the problem (4.44) first. Define $v_1 := X(x)Y(y)$ and set

$$\frac{X''}{X} = -\lambda^2 = -\frac{Y''}{Y},$$

with the boundary conditions, $X(0) = 0$ and $X(L) = 0$ for X and for y, we will later use the conditions $v_1(x, 0) = v_1(x, l) = 0$.

On solving (4.44) with respect to the boundary conditions, we get for some constant A and some $n \in \mathbb{N}$,

$$X_n(x) = A \sin \frac{n\pi x}{L}. \tag{4.46}$$

Let $v_1(x, y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y)$. On substituting v_1 in (4.44), we get

$$\sum_{n=1}^{\infty} \left[Y_n''(y) - \left(\frac{n\pi}{L}\right)^2 Y_n(y) \right] \sin \frac{n\pi x}{L} = -\omega_m \sin(\omega_m x)\cos(\delta_p y).$$

By using the Fourier sine series formula, we get $m = n$ and

$$Y_m''(y) - \omega_m^2 Y_n(y) = -\omega_m \cos(\delta_p y).$$

On solving the above ODE without any boundary conditions, we get, for some constants A_1, B_1, the solution

$$Y_m(y) = A_1 \exp(\omega_m y) + B_1 \exp(-\omega_m y) + \frac{\omega_m}{\delta_p^2 + \omega_m^2} \cos(\delta_p y). \tag{4.47}$$

Thus,

$$v_1(x,y) = \sum_{m=1}^{\infty} \left[A_1 \exp(\omega_m y) + B_1 \exp(-\omega_m y) + \frac{\omega_m}{\alpha_{m,p}^2} \cos(\delta_p y) \right] \sin(\omega_m x). \tag{4.48}$$

By applying the boundary conditions $v_1(x,0) = v_1(x,l) = 0$ and solving the resultant two equations for A_1 and B_1,

$$A_1 = -\frac{\omega_m}{\alpha_{m,p}} \cdot \frac{[(-1)^p - \exp(-\omega_m l)]}{[\exp(\omega_m l) - \exp(-\omega_m l)]}, \tag{4.49}$$

$$B_1 = \frac{\omega_m}{\alpha_{m,p}} \cdot \frac{[(-1)^p - \exp(\omega_m l)]}{[\exp(\omega_m l) - \exp(-\omega_m l)]}. \tag{4.50}$$

The above constants along with (4.48) give us the complete solution of the problem (4.44). We similarly solve the problem (4.45), to get the solution

$$v_2(x,y) = \sum_{p=1}^{\infty} \left[A_2 \exp(\delta_p y) + B_2 \exp(-\delta_p y) + \frac{\delta_p}{\alpha_{m,p}^2} \cos(\omega_m x) \right] \sin(\delta_p y), \tag{4.51}$$

where the constants A_2 and B_2 are given as

$$A_2 = -\frac{\delta_p}{\alpha_{m,p}} \cdot \frac{[(-1)^m - \exp(-\delta_p L)]}{[\exp(\delta_p L) - \exp(-\delta_p L)]}, \tag{4.52}$$

$$B_2 = \frac{\delta_p}{\alpha_{m,p}} \cdot \frac{[(-1)^m - \exp(\delta_p L)]}{[\exp(\delta_p L) - \exp(-\delta_p L)]}. \tag{4.53}$$

By using the solution of these problems in the equation (4.42) and on simplifying we get

$$(S\phi_{m,p}, \phi_{j,q})_0 = -c_{m,p}c_{j,q}\int_\Omega (v_1, v_2)\cdot(-\omega_j \sin(\omega_j x)\cos(\delta_q y), -\delta_q \cos(\omega_j x)\sin(\delta_q y)).$$

We can also drop the summations on v_1, v_2 as we take a particular case of m and p in each element. Now, this integral takes four different values according the values of $m, p, j, q \in \mathbb{N}$, such that $1 \le m, p, j, q \le k$, $m + p \ge 1$ and $j + q \ge 1$. Thus, the four different types of entries for the matrix A_k are given as follows

1. For $m \neq j$ and $p \neq q$, we get $(S\phi_{m,p}, \phi_{j,q})_0 = 0$.

2. For $m = j$ and $p \neq q$, we get

$$(S\phi_{m,p}, \phi_{m,q})_0 = -\frac{\sqrt{2}a(p,q)\omega_m^3(1+(-1)^{p+q})B_{m,p}}{l\alpha_{m,p}\alpha_{m,q}}.$$

3. For $m \neq j$ and $p = q$, we get

$$(S\phi_{m,p}, \phi_{j,p})_0 = -\frac{\sqrt{2}a(m,j)\delta_p^3(1+(-1)^{m+j})\Lambda_{m,p}}{L\alpha_{m,p}\alpha_{j,p}}.$$

4. For $m = j$ and $p = q$, we get

$$(S\phi_{m,p}, \phi_{j,p})_0 = 1 - a(m,p)^2\left[\frac{2\omega_m^3 B_{m,p}}{l\alpha_{m,p}^2} + \frac{2\delta_p^3\Lambda_{m,p}}{L\alpha_{m,p}^2}\right].$$

In the above expressions, $B_{m,p}$ for all $m \ge 1$ and $p \ge 0$ is defined as,

$$\begin{aligned} B_{0,p} &= 0, \\ B_{m,p} &= \frac{\exp(\omega_m l) + \exp(-\omega_m l) - 2(-1)^p}{\exp(\omega_m l) - \exp(-\omega_m l)}. \end{aligned} \tag{4.54}$$

Also, $\Lambda_{m,p}$ for all $m \geq 0$ and $p \geq 1$ is defined as,

$$\begin{aligned} \Lambda_{m,0} &= 0, \\ \Lambda_{m,p} &= \frac{\exp(\delta_p L) + \exp(-\delta_p L) - 2(-1)^m}{\exp(\delta_p L) - \exp(-\delta_p L)}. \end{aligned} \tag{4.55}$$

Since we have the expressions for the entries of the matrix A_k, we design a program in MATLAB which evaluates the minimum eigenvalue, denoted by $\lambda^k_{\min}$ of the matrix A_k for a unit square domain. Since the matrix A_k is a sparse square matrix of order $(k+1)^2 - 1$, we present the spy graphs of A_{50} of order 2600 given by Figure 4.1 and of A_{100} of order 10200 provided by Figure 4.2.

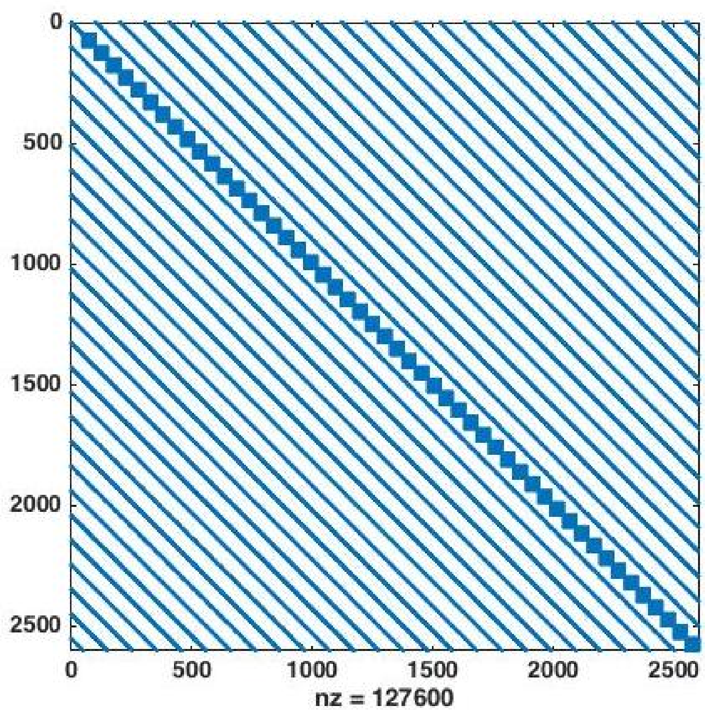

Figure 4.1: Spy graph of A_{50}.

The structure of spy graphs implies that the matrix A_k contains non-zero super, sub and main diagonals, followed by more non-zero diagonals. Also, A_{50} has 1.89% and A_{100} has 0.97% non-zero entries. By plotting the Greshgorin discs of A_k and then finding the least point in the plot; we tried to estimate $\lambda_{min}(S)$. We get that

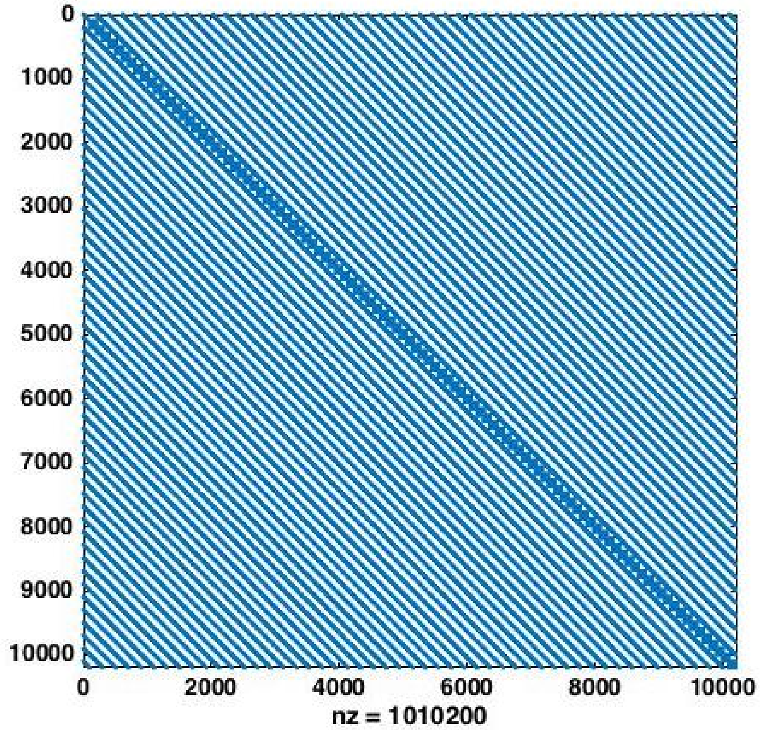

Figure 4.2: Spy graph of A_{100}.

for $k = 2, 3, 4$, the estimate is 0.2344, 0.1075, 0.1670. Moreover, for $k = 50$ it is -0.1075; thus it does not give us anything.

As stated in Proposition 4.10, $\lim_{k\to\infty} \lambda_{\min}^k = \lambda_{\min}(S)$. We observe that $\lambda_{\min}^k$ decreases with an increase in k. The lowest that we have found until now is 0.20164 for $k = 225$, for which A_k is a square matrix of order 51075. It is better than 0.2260 as stated in [24].

In order to estimate $\lambda_{\min}(S)$, we designed five models which relate $\lambda_{\min}^k$ to the corresponding value of k, i.e., $\lambda_{\min}^k = f(k)$, where f is some real valued function. The models contain the parameters a, b, c, d, e and f. We collect data for different values of k for performing a least square fit of the models.

Table 4.1 mentions the data over which we perform least square fit of the models for estimating $\lambda_{\min}^k$. Table 4.2 describes the result corresponding to each model. Since $\lim_{k\to\infty} \lambda_{\min}^k = a$ for each model, thus $a = \lambda_{\min}(S)$. The residual measures

k	$\lambda_{\min}^k$
2	0.30308
4	0.26527
8	0.24167
16	0.22668
32	0.21675
64	0.20984
81	0.20786
100	0.20643
128	0.20481
150	0.20385
175	0.20295
200	0.20230
225	0.20164

Table 4.1: Data for least square fit.

the accuracy of the model, the least value of residual represents the best model. There are significant improvements in norm of the residual for the first four models, however, there is an insignificant improvement for the Model 5. Hence, we do not try any model for more than six parameters. The norm of residual will further decrease by increasing the parameters, but there will not be a good improvement.

We provide the figures in which we plot the data points $(k, \lambda_{\min}^k)$ and the model used for performing the least square fit at the same points. It is evident from the figures that our models present good approximations for the data. We find that $\lambda_{\min}(S)$ is 0.19116, which is not a good value because we require some data points for higher values of k.

Sr. No.	Model ($\lambda_{\min}^k =$)	$\lambda_{\min}(S) = a$	Norm of residual
1	$a + bk^{-0.5}$	0.19048	0.0062698
2	$a + bk^{-0.5} + ck^{-1}$	0.19402	0.00094577
3	$a + bk^{-0.5} + ck^{-1} + dk^{-1.5}$	0.19304	0.00050181
4	$a + bk^{-0.5} + ck^{-1} + dk^{-1.5} + ek^{-2}$	0.19179	0.0001482
5	$a + bk^{-0.5} + ck^{-1} + dk^{-1.5} + ek^{-2} + fk^{-2.5}$	0.19116	0.00010168

Table 4.2: Least square fit results.

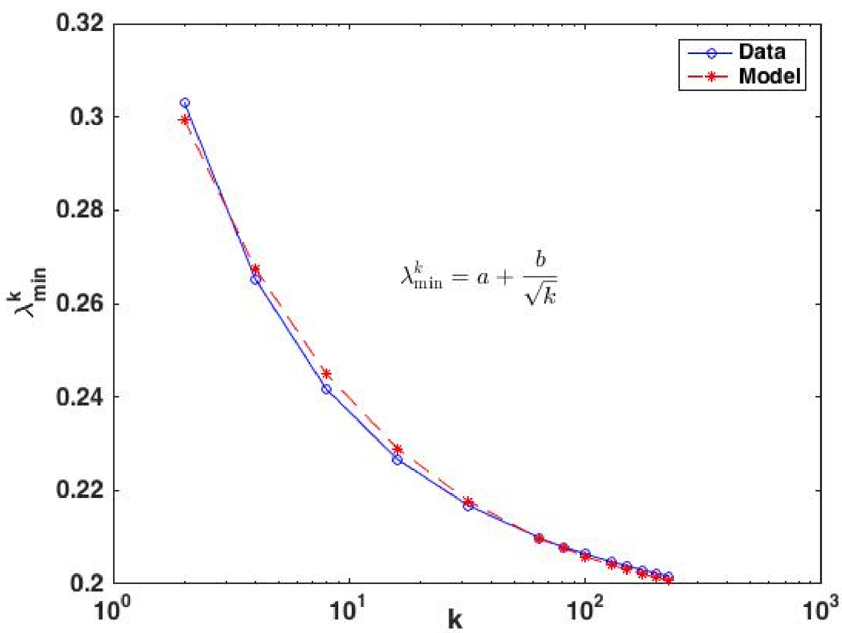

Figure 4.3: Model 1.

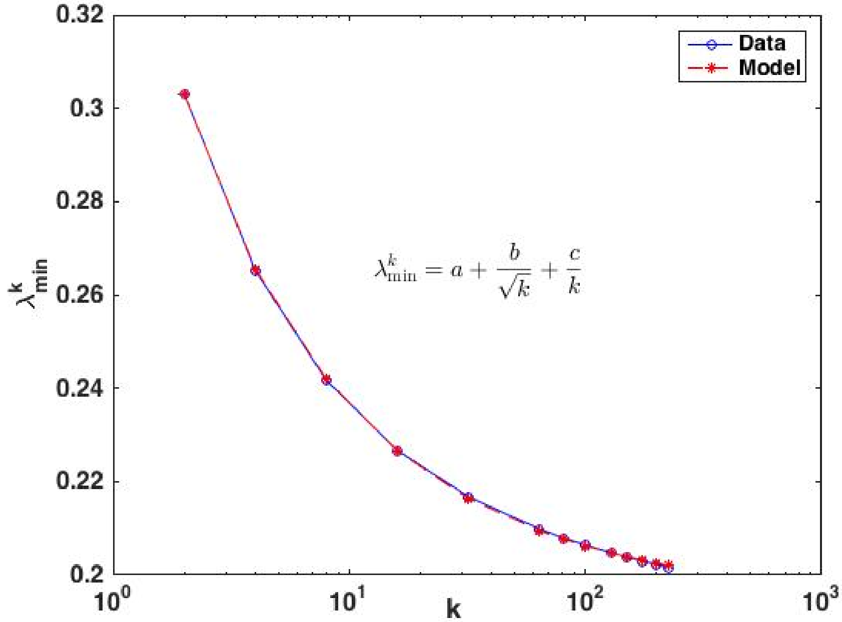

Figure 4.4: Model 2.

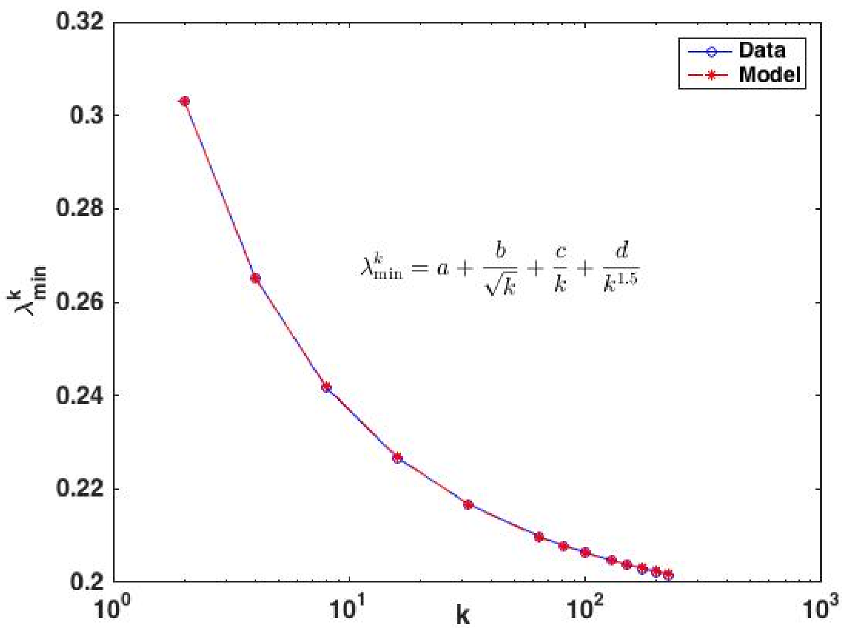

Figure 4.5: Model 3.

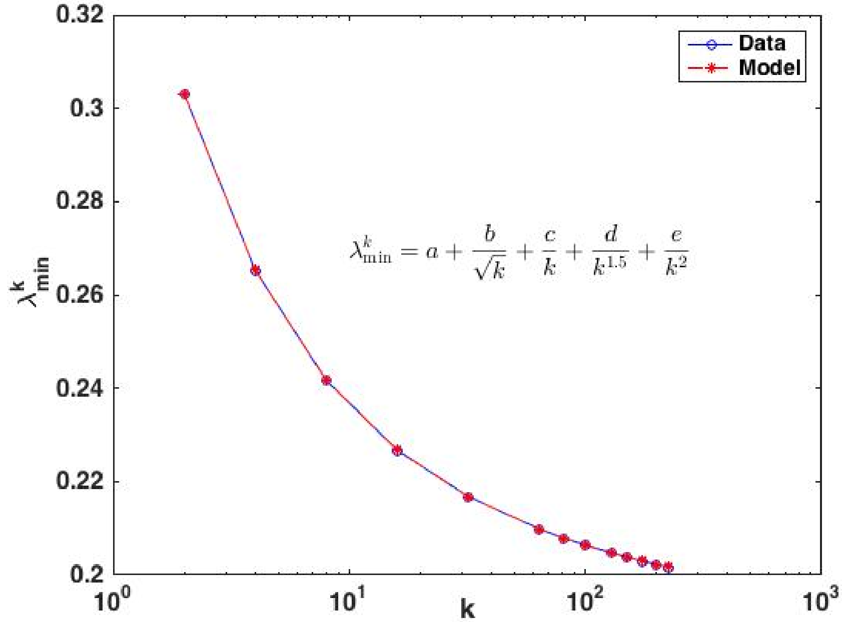

Figure 4.6: Model 4.

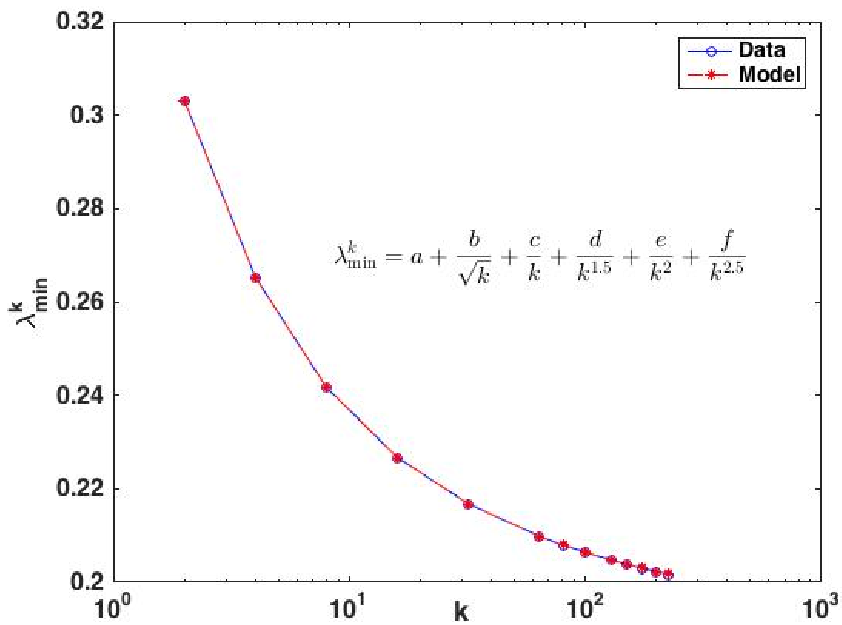

Figure 4.7: Model 5.

5

Finite Element Methods for the Stokes Problem

The problem of estimating $\lambda_{\min}(S)$, which is further related to the problem of finding the optimal constants in various inequalities, is significant to the study of the Stokes problem. It is because the inf-sup constant $\beta_0(\Omega) = \lambda_{\min}(S)$ and thus an estimate on $\lambda_{\min}(S)$ provides us with an estimate of $\beta_0(\Omega)$. We require the constant $\beta_0(\Omega)$ for the existence, stability and convergence analysis of the Stokes problem. This importance motivates us to study the Stokes problem in discrete form. We will demonstrate the ***mixed formulation*** of the Stokes problem, which is essential for the study the discrete Stokes problem.

An abstract framework for a mixed problem will be described in this chapter. We will also discuss the concept of a saddle point problem. Finally, we will give an optimal choice for the finite elements for solving the Stokes problem and the methods for solving the linear system. The books such as [25, 6, 18, 7, 8, 10, 48], provide a detailed explanation of the abstract mixed problem and mixed finite elements. A comprehensive study of the discrete Stokes problem is given in [16]. Paper [4] is an excellent reference for solving a saddle point problem numerically.

5.1 The Stokes problem as a Mixed problem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected domain. Consider the Stokes problem for some $(u, p) \in V \times L_0^2(\Omega)$ and $f \in V'$, given by (2.7), (2.8) and (2.9). Consider the space $V_1 := \mathbb{H}^1_{0,\text{div}}(\Omega)$ defined by equation (2.14). A weak formulation of the Stokes problem, for all $v \in V_1$, can be evaluated as follows

$$\begin{aligned}
\int_\Omega (-\Delta u) \cdot v + \int_\Omega \nabla p \cdot v &= \int_\Omega f \cdot v, \\
\int_\Omega \nabla u \cdot \nabla v - \int_\Omega p \nabla \cdot v &= \int_\Omega f \cdot v, \\
\int_\Omega \nabla u \cdot \nabla v &= \int_\Omega f \cdot v.
\end{aligned}$$

Thus, by using equation (2.2) the weak form of the problem becomes, finding $u \in V_1$ such that for all $v \in V_1$, $a(u, v) = f(v)$, where $f \in V'$. Note that the continuity equation is trivially satisfied by the definition of the space V_1. We easily proved the existence of a unique solution of this problem in Theorem 2.12 by using Theorem 1.18. However, this formulation is not easy from a practical point of view because of the following drawbacks.

1. Pressure is not being considered in this formulation.
2. The solution lies in the space V_1, which is not a classical finite element space. We desire to work on spaces such as the space V.

Hence the demerits of the coercive formulation, i.e., Lax-Milgram formulation, provokes the need of using another formulation for the Stokes problem.

5.1.1 Mixed Formulation

Consider the weak form given of the Stokes problem given by equation (2.12) and define a mixed space $X := V \times L_0^2(\Omega)$, equipped with the norm denoted by $\| \cdot \|_X$

and defined as

$$\|(v,q)\|_X^2 := \|v\|^2 + \|q\|_0^2. \tag{5.1}$$

Define a bilinear form $B : X \times X \to \mathbb{R}$ as

$$B((u,p),(v,q)) := a(u,v) + b(p,v) - b(q,u). \tag{5.2}$$

In order to express the right hand side of the first equation in the mixed formulation (2.12), we define a functional $F \in X'$, i.e., $F : X \to \mathbb{R}$ as follows,

$$F((v,q)) = \langle F,(v,q)\rangle_{X',X} := \langle f,v\rangle_{V',V} = f(v). \tag{5.3}$$

Definition 5.1 (**Mixed-Formulation**)**.** The mixed formulation of the Stokes problem is defined as finding $(u,p) \in X$ such that for all $(v,q) \in X$,

$$B((u,p),(v,q)) = \langle F,(v,q)\rangle_{X',X}\,, \tag{5.4}$$

where $F \in X'$.

Note that,

$$B((u,p),(u,p)) = \int_\Omega |\nabla u|^2 = a(u,u) = \|u\|^2.$$

Clearly $B((u,p),(u,p))$ cannot be bounded below by $\|(u,p)\|_X^2$, hence it is not coercive. Thus, we cannot apply Lax-Milgram theorem in order to prove the existence of the unique solution of the above problem. This leads us to use some other framework, i.e., the theory of abstract mixed problems.

5.1.2 Abstract Mixed Problems

Let $(H, \|\cdot\|_H)$ and $(M, \|\cdot\|_M)$ be two Hilbert spaces equipped with the inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_M$ respectively. Let H' and M' be the dual spaces of H and M

respectively. Consider some bilinear forms $\phi : H \times H \to \mathbb{R}$ and $\psi : M \times H \to \mathbb{R}$. Let $\Phi : H \to H'$ and $\Psi : H \to M'$ be operators such that

$$\langle \Phi u, v \rangle_{H',H} = \phi(u,v), \ \forall\, (u,v) \in H \times H, \tag{5.5}$$

and

$$\langle \Psi v, q \rangle_{M',M} = \psi(q,v), \ \forall\, (v,q) \in H \times M. \tag{5.6}$$

Let $\Psi^* : M \to H'$ be the dual (or adjoint) operator of Ψ, then we have

$$\psi(q,v) = \langle \Psi v, q \rangle_{M',M} = \langle \Psi^* q, v \rangle_{H',H}\,, \ \forall\, (v,q) \in H \times M. \tag{5.7}$$

Let H_1 denote the null space of the operator Ψ,

$$H_1 = \operatorname{Ker}\Psi := \{u \in H \,|\, \Psi u = 0, \ \text{i.e.,}\ \psi(q,u) = 0, \ \ \forall\, q \in M\}\,, \tag{5.8}$$

and H_1° represent the polar space of the operator Ψ,

$$H_1^\circ = (\operatorname{Ker}\Psi)^\circ := \left\{ f \in H' \,|\, \langle f, u \rangle_{H',H} = 0, \ \forall\, u \in H_1 \right\}. \tag{5.9}$$

Consider the problem of finding $(u,p) \in H \times M$ such that for all $(v,q) \in H \times M$,

$$\begin{aligned} \phi(u,v) + \psi(p,v) &= \langle f, v \rangle_{H',H}\,, \\ \psi(q,u) &= \langle g, q \rangle_{M',M}\,, \end{aligned} \tag{5.10}$$

where $f \in H'$ and $g \in M'$. Equivalently, we can rewrite the above problem as,

$$\begin{aligned} \Phi u + \Psi^* p &= f, \\ \Psi u &= g. \end{aligned} \tag{5.11}$$

Define a canonical injection $\imath$ of H' into H_1', i.e., it restricts a continuous linear form defined on H to the space H_1. For $f \in H'$, $\imath f \in H_1'$ is defined as,

$$\langle \imath f, v \rangle_{H_1', H_1} = \langle f, v \rangle_{H', H_1}, \ \forall\, v \in H_1. \tag{5.12}$$

The above equation gives us the estimate, $\|\imath f\|_{H_1'} \leq \|f\|_{H'}$. Note that $H_1^\circ = \operatorname{Ker} \imath$. We now give an important result required to prove the well-posedness of mixed formulations, the proof of which is illustrated in [6].

Theorem 5.2. The problem defined by (5.11) has a unique solution if and only if

(i) $\imath \circ \Phi : H_1 \to H_1'$ is an isomorphism.

(ii) $\Psi : H \to M'$ is surjective.

Proof. Let us suppose that a unique solution of (5.11) exists. First, we prove that Ψ is surjective. In (5.11) take $f = 0$ and $g = h$, where $h \in M'$, since it has a unique solution, therefore there exists $u \in H$ such that $\Psi u = h$. Hence $\Psi : H \to M'$ is surjective. We will now prove that $\imath \circ \Phi$ is an isomorphism. To prove $\imath \circ \Phi$ is surjective from H_1 to H_1', let $f \in H_1'$ and by using Hahn-Banach theorem, extend f from H_1 to H. Let $\tilde{f} \in H'$ be the extension of $f \in H_1'$, then $\imath \tilde{f} = f$. Since there exists a unique solution $(u, p) \in H \times M$ of the problem

$$\Phi u + \Psi^* p = \tilde{f},$$

$$\Psi u = 0.$$

Thus for $v \in H_1$,

$$\langle \Phi u, v \rangle + \langle \Psi^* p, v \rangle = \langle \tilde{f}, v \rangle = \langle f, v \rangle,$$

$$\Rightarrow \langle \imath\, \Phi u, v \rangle + \langle p, \Psi v \rangle = \langle f, v \rangle,$$

where in the last step we have used (5.12). Hence for all $f \in H_1'$ there exists $u \in H_1$ such that $\imath\,\Phi u = f$, i.e., $\imath \circ \Phi$ is surjective. In order to prove that $\imath \circ \Phi$ is injective, let $u \in H_1$ be such that $\imath\,\Phi u = 0$. This implies for all $v \in H_1$,

$$\langle \Phi u, v \rangle = 0.$$

Therefore, $\Phi u \in H_1^\circ = (\operatorname{Ker}\Psi)^\circ$. Since Ψ is surjective and $\operatorname{Im}\Psi = M'$, thus by using *Banach Closed Range Theorem*, $(\operatorname{Ker}\Psi)^\circ = \operatorname{Im}\Psi^*$. Thus $\Phi u \in \operatorname{Im}\Psi^*$, therefore, there exists $p \in M$ such that $\Psi^* p = -\Phi u$. Hence $(u,p) \in H_1 \times M$ satisfies,

$$\begin{aligned} \Phi u + \Psi^* p &= 0, \\ \Psi u &= 0. \end{aligned}$$

Since the above problem has a unique solution, this solution is $(0,0)$. Thus, for all $u \in H_1$ such that $\imath\,\Phi u = 0$ implies $u = 0$, i.e., $\imath \circ \Phi$ is injective.
Assume that statements (i) and (ii) are satisfied. We need to prove that the problem (5.11) has a unique solution. Given $g \in M'$ and Ψ surjective from $H \to M'$, define $u_g \in H$ such that $\Psi u_g = g$. Given $f \in H'$ and as $\Phi u_g \in H'$, we get $\imath f - \imath\Phi u_g \in H_1'$. By using the statement (i), there exists $u_0 \in H_1$ such that

$$\imath\,\Phi u_0 = \imath f - \imath\,\Phi u_g.$$

Let $u = u_0 + u_g$, the above equation gives $\imath\,\Phi u = \imath f$, thus by using (5.12)

$$\langle f - \Phi u, v \rangle_{H', H_1} = 0, \ \forall\, v \in H_1.$$

Therefore, $f - \Phi u \in (\operatorname{Ker}\Psi)^\circ = \operatorname{Im}\Psi^*$. Thus, there exists $p \in M$ such that

$$f - \Phi u = \Psi^* p,$$

which gives

$$\Phi u + \Psi^* p = f, \tag{5.13}$$

and note that

$$\Psi u = \Psi u_0 + \Psi u_g = \Psi u_g = g. \tag{5.14}$$

The equations (5.13) and (5.14) prove that (5.11) has a solution. In order to prove the uniqueness of this solution, consider (u, p) such that

$$\Phi u + \Psi^* p = 0,$$
$$\Psi u = 0,$$

which implies $\imath\,\Phi u + \imath\,\Psi^* p = 0$, since $\imath\,\Psi^* p = 0$, we have $\imath\,\Phi u = 0$. By using statement (i), $u = 0$. Thus, $\Psi^* p = 0$. Since Ψ is surjective, therefore Ψ^* is injective. Hence, $\Psi^* p = 0$ gives $p = 0$. This proves that (5.11) has a unique solution. □

Corollary 5.3. The mixed formulation of the Stokes problem given by (5.4) has a unique solution.

Proof. Consider the above theorem for $H = V$, $M = L^2_0(\Omega)$, $\Phi = -\Delta$, $\Psi = -\nabla\cdot$, $\Psi^* = \nabla$, $H_1 = V_1$, $\phi = a$ and $\psi = b$. Since the bilinear form a is coercive we get that the first condition of the above theorem is satisfied. Lemma 2.5 gives us the result that the second condition of the above theorem is satisfied. Hence the problem (2.12), or equivalently (5.4) has a unique solution. □

If $(u, p) \in H \times M$ is a solution of (5.10) or (5.11), then Ψ is surjective. By using Lemma 1.21, we get that (u, p) satisfies the following inf-sup condition.

$$\inf_{q\in M} \sup_{v\in H} \frac{\langle \Psi v, q\rangle_{M',M}}{\|v\|_H \|q\|_M} = \inf_{q\in M} \sup_{v\in H} \frac{\psi(q, v)}{\|v\|_H \|q\|_M}. \tag{5.15}$$

Thus, the second condition in Theorem 5.2 might be changed to state that the above inf-sup condition is satisfied.

5.1.3 Saddle point problem

Define the energy functional of the problem (5.10) denoted by $J : H \to \mathbb{R}$ defined for $v \in H$ as

$$J(v) = \frac{1}{2}\phi(v, v) - \langle f, v\rangle_{H',H} . \tag{5.16}$$

For the mixed problem the term ***Lagrangian*** corresponds to the notion of an energy functional in coercive type problems. Let us recall the following result for the coercive type problems.

Theorem 5.4 (**Energy Minimization**)**.** Let $(H, \|\cdot\|_H)$ be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on $H \times H$. Assume that $a(\cdot,\cdot)$ is continuous, symmetric and coercive. Let $f \in H'$, then the following are equivalent.

1. $u \in H$ is such that $a(u, v) = \langle f, v\rangle$, $\forall\, v \in H$.

2. $u \in H$ minimizes the energy functional $J(v) = \frac{1}{2}a(v, v) - \langle f, v\rangle$ on H, i.e., $J(u) = \min_{v\in H} J(v)$.

Definition 5.5 (**Lagrangian**)**.** The mapping $\mathcal{L} : H \times M \to \mathbb{R}$, defined by

$$\mathcal{L}(v, q) := J(v) + \psi(q, v) - \langle g, q\rangle_{M',M} , \tag{5.17}$$

for some $(v, q) \in H \times M$ and $g \in M'$ is called the Lagrangian for problem (5.10).

Definition 5.6 (**Saddle point** [18])**.** A pair $(u, p) \in H \times M$ is said to be a saddle point of the Lagrangian $\mathcal{L}$ if the following is satisfied

$$\mathcal{L}(u, q) \le \mathcal{L}(u, p) \le \mathcal{L}(v, p),\ \forall\, (v, q) \in H \times M. \tag{5.18}$$

If a mixed problem of the type (5.10) is such that the bilinear form $\phi(\cdot,\cdot)$ is symmetric then the problem is called a *saddle-point* problem. This statement implies that **the Stokes problem is a saddle-point problem**.

Theorem 5.7 ([18])**.** If $\phi(\cdot,\cdot)$ is symmetric and positive, then the following two assertions are equivalent.

1. (u,p) is a saddle point of $\mathcal{L}$.

2. (u,p) is a solution of (5.10).

Proof. Consider for a given $u \in H$, the left-hand side of (5.18)

$$\begin{aligned}
\mathcal{L}(u,q) \leq \mathcal{L}(u,p),\ \forall\, q \in M \\
\Leftrightarrow J(u) + \psi(q,u) - \langle g, q\rangle_{M',M} \leq J(u) + \psi(p,u) - \langle g,p\rangle_{M',M}\,,\ \forall\, q \in M \\
\Leftrightarrow \psi(q-p,u) \leq \langle g, q-p\rangle_{M',M}\,,\ \forall\, q \in M \\
\Leftrightarrow \psi(t\tilde{q},u) \leq \langle g, t\tilde{q}\rangle_{M',M}\,,\ \forall\, \tilde{q} \in M,\, t \in \mathbb{R} \\
\Leftrightarrow \psi(\tilde{q},u) = \langle g, \tilde{q}\rangle_{M',M}\,,\ \forall\, \tilde{q} \in M.
\end{aligned}$$

Thus, the left-hand side of (5.18) gives us the second equation of problem (5.10). Let us consider for a given $p \in M$ the right-hand side of (5.18)

$$\begin{aligned}
\mathcal{L}(u,p) \leq \mathcal{L}(v,p),\ \forall\, v \in H \\
\Leftrightarrow J(u) + \psi(p,u) - \langle g,p\rangle_{M',M} \leq J(v) + \psi(p,v) - \langle g,p\rangle_{M',M}\,,\ \forall\, v \in H \\
\Leftrightarrow J(u) + \psi(p,u) \leq J(v) + \psi(p,v).\ \forall\, v \in H.
\end{aligned}$$

Set $J_1(v) := J(v) + \psi(p,v)$, for all $v \in H$. From the above equation, $J_1(u) \leq J_1(v)$, for all $v \in H$, thus $u \in H$ is a minimizer of J_1, hence it is a solution of the problem of finding $u \in H$ such that $\phi(u,v) + \psi(p,v) = f(v)$ for all $v \in H$, which is the first equation of (5.10). Hence we have proved that (u,p) is a saddle point of $\mathcal{L}$ if and only if it is a solution of the problem (5.10). $\square$

An important property of a saddle point of $\mathcal{L}$ is that if (u,p) is a saddle point of $\mathcal{L}$ then

$$\mathcal{L}(u,p) = \min_{v\in H}\max_{q\in M}\mathcal{L}(v,q) = \max_{q\in M}\min_{v\in H}\mathcal{L}(v,q).$$

The proof of the above property can be found in [18]. Define the set $H_g \subset H$ as,

$$H_g := \{u \in H,\ \Psi u = g \in M',\ \text{ i.e.}, \psi(q,u) = \langle g, q\rangle\,,\ \forall\, q \in M\}\,. \tag{5.19}$$

We will give a theorem which describes the relationship between the mixed problem (5.10) and the minimization of the energy functional J subject to the constraint that the solution belongs to the set V_g.

Theorem 5.8. Assume that the bilinear form $\phi(\cdot,\cdot)$ is symmetric and coercive and that H_g is non-empty. The problem of finding $u \in H$ such that

$$J(u) = \inf_{v\in H_g} J(v), \tag{5.20}$$

has a unique solution. Furthermore, if the bilinear form $\psi(\cdot,\cdot)$ satisfies the inf-sup condition (5.15), then there exists a unique $p \in M$ such that (u,p) is the unique saddle point of $\mathcal{L}$ and the unique solution of the mixed problem (5.10).

Proof. Let $u_g \in H_g$. Since $\Psi v = 0$, for all $v \in H_1$, we can write $H_g = u_g + H_1$. Thus, we can rewrite (5.20), as $\min_{v\in H_1} \tilde{J}(v) = \min_{v\in H_1} J(u_g + v)$. Since the bilinear form ϕ is symmetric and coercive, by using Theorem 1.18, there exists a unique solution $u \in H_1$. As a consequence of Theorem 5.4, $u \in H_1$ is a minimizer for $\tilde{J}$. Thus, (5.20) has a unique solution $u = u_g + u_0$, where $u_0 \in H_1$ is the unique solution of

$$\phi(u_0, v) = \langle f, v\rangle - \phi(u_g, v),\ \forall\, v \in H_1.$$

Given the bilinear form ψ satisfies the inf-sup condition, we get that Ψ is surjective. Therefore, Ψ has a closed range and $H_1^\circ = (\operatorname{Ker}\Psi)^\circ = \operatorname{Im}\Psi^*$. Since $\phi(u,v) = \langle f, v\rangle$ for all $v \in H_1$, we have $f - \Phi u \in H_1^\circ = \operatorname{Im}\Psi^*$. Hence there exists $p \in M$ such that $f - \Phi u = \Psi^* p$, i.e., $\Phi u + \Psi^* p = f$. Since Ψ^* is injective, this $p \in M$ is unique. Also, by definition $\Psi u = g$. Thus, (u,p) is a unique solution of the mixed problem (5.11) and is a saddle point of $\mathcal{L}$. $\square$

We stated earlier that the Stokes problem is a saddle point problem. Thus, from the above theorem the existence of a unique solution of the Stokes problem is proved. Furthermore, the solution is a saddle point of the Lagrangian $\tilde{L} : V \times M \to \mathbb{R}$, defined by

$$\tilde{L}(v,q) = [a(v,v) - \langle f, v\rangle] + b(v,q).$$

The idea is to minimize the energy functional $J_1(v) := a(v,v) - \langle f, v\rangle$ subject to the constraint $\nabla \cdot v = 0$, with q being a Lagrange multiplier.

5.2 Finite Element Method for the Stokes problem

In this section, we will discuss a finite element implementation of the Stokes problem. Firstly, we approximate the Stokes problem by a Galerkin method. Define $V_h \subset V$ and $Q_h \subset L_0^2(\Omega)$, to be the finite dimensional subspaces for velocity and pressure respectively. Thus, we say that $(u_h, p_h) \in V_h \times Q_h$ is a solution of the discrete Stokes problem, if it satisfies the weak form of the Stokes problem

$$a(u_h, v_h) + b(p_h, v_h) = f(v_h),\ \forall\, v_h \in V_h, \tag{5.21}$$

$$b(q_h, v_h) = 0,\ \forall\, q_h \in Q_h. \tag{5.22}$$

We define the operators $A_h : V_h \to V_h'$ and $B_h : V_h \to Q_h'$ by

$$\langle A_h u_h, v_h \rangle = a(u_h, v_h),\ \forall\, (u_h, v_h) \in V_h \times V_h, \tag{5.23}$$

$$\langle B_h v_h, q_h \rangle = b(q_h, v_h),\ \forall\, (v_h, q_h) \in V_h \times Q_h. \tag{5.24}$$

Let $B_h^T : Q_h \to V_h'$ denote the dual (or adjoint) operator of B_h and is thus defined as

$$\langle B_h^T q_h, v_h \rangle = b(q_h, v_h) = \langle B_h v_h, q_h \rangle,\ \forall\, (v_h, q_h) \in V_h \times Q_h. \tag{5.25}$$

Consider the basis functions for V_h to be the set $\{\phi_1, \ldots, \phi_n\}$. We can write for some vector $\xi \in \mathbb{R}^n$,

$$u_h = \sum_{i=1}^{n} \xi_i \phi_i. \tag{5.26}$$

Consider the basis functions for Q_h to be the set $\{\psi_1, \ldots, \psi_m\}$. We can write for some vector $\eta \in \mathbb{R}^m$,

$$p_h = \sum_{i=1}^{m} \eta_i \psi_i. \tag{5.27}$$

Define the norm $\|\cdot\|_* : \mathbb{R}^n \to \mathbb{R}$, for any $\xi \in \mathbb{R}^n$ as

$$\|\xi\|_* := \sup_{w \in \mathbb{R}^n} \frac{(\xi, w)}{\|w\|}, \tag{5.28}$$

where $\|w\| := \|w_h\|$, such that $w_h = \sum_{i=1}^{n} w_i \phi_i \in V_h$. Similarly, define $\|\cdot\|_0 : \mathbb{R}^m \to \mathbb{R}$, for any $\eta \in \mathbb{R}^m$ as,

$$\|\eta\|_0 := \|p_h\|_0, \tag{5.29}$$

where $p_h = \sum_{i=1}^{m} \eta_i \psi_i$.

By using (5.21) and (5.22), we get the discrete Stokes problem as follows,

$$\sum_{i=1}^{n} \xi_i a(\phi_i, \phi_j) + \sum_{i=1}^{m} \eta_i b(\psi_i, \phi_j) = f(\phi_j),\ j = 1, 2, \ldots, n.$$
$$\sum_{i=1}^{n} \xi_i b(\psi_j, \phi_i) = 0,\ j = 1, 2, \ldots, m.$$

In matrix notation, we can state the *discrete Stokes* problem as finding $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\begin{bmatrix} A_h & C_h \\ C_h^T & O \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} f_h \\ O \end{bmatrix}, \tag{5.30}$$

where A_h is symmetric and positive definite $n \times n$ matrix corresponding to the negative laplacian, C_h is $n \times m$ matrix corresponding to the gradient operator, and C_h^T is $m \times n$ matrix corresponding to the negative divergence operator. This scheme is called a **mixed Galerkin approximation** of the Stokes problem.

We know that the solution of the Stokes problem satisfies the inf-sup condition (2.28), this condition also plays a significant role in solving the Stokes problem numerically. The spaces V_h and Q_h are chosen such that they satisfy the *discrete inf-sup condition.*

Definition 5.9 (**Discrete inf-sup condition** [35])**.** The spaces V_h and Q_h are said to satisfy the inf-sup condition in discrete form, if for every $q_h \in Q_h$, there exists a positive constant β independent of h, such that

$$\inf_{q_h \in Q_h \setminus 0} \sup_{v_h \in V_h \setminus 0} \frac{|b(q_h, v_h)|}{\|v_h\| \|q_h\|_0} \geq \beta, \tag{5.31}$$

or equivalently,

$$\sup_{v_h \in V_h \setminus 0} \frac{|b(q_h, v_h)|}{\|v_h\|} \geq \beta \|q_h\|_0,\ \forall q_h \in Q_h. \tag{5.32}$$

Similar to the case of the Stokes problem in continuous form, the discrete inf-sup condition guarantees the existence and uniqueness of solution of the discrete stokes problem. It is in fact the necessary and sufficient condition for the discrete stokes equation to have a unique solution. The proof of existence and uniqueness of the solution of the discrete Stokes problem can be found in [35]. This result is followed by the stability and convergence of the solution of the discrete Stokes problem. A unique solution of any mixed problem has to satisfy an inf-sup condition.

For a coercive problem, i.e., the problem that can be solved by using Lax-Milgram lemma on a space V, well-posedness of the discrete problem is a direct consequence of the well-posedness of the continuous problem. Because any finite dimensional approximation on $V_h \subset V$ can also be solved by the same approach. Generally on the case of mixed problem, the well-posedness of the discrete problem is not so straightforward because of the following two reasons.

- Define a finite dimensional subspace $X_h \subset V_1$ as follows,

$$X_h = \mathrm{Ker} B_h = \{u_h \in V_h \,|\, b(q_h, u_h) = 0, \forall\, q_h \in Q_h\}\,.$$

  The space X_h is not a subspace of V_1, since $u_h \in X_h$ does not imply that $\nabla \cdot u_h = 0$, i.e., $u_h \in V_1$.

- The inf-sup condition on $V \times L_0^2(\Omega)$, given by (2.28), only implies the existence of $\beta > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{v \in V} \frac{b(q_h, v)}{\|v\| \|q_h\|_0} = \beta.$$

  The above is an inf-sup condition on $V \times Q_h$, since $V_h \subset V$, this does not imply an inf-sup condition on $V_h \times Q_h$.

We need to choose the spaces V_h and Q_h suitably. The problem (5.30) is known as the **primal problem**, which gives rise to the following two equations,

$$A_h \xi + C_h \eta = f_h,$$

$$C_h^T \xi = O.$$

On eliminating u from the above problem,

$$(C_h^T A_h^{-1} C_h)\eta = (C_h^T A_h^{-1}) f_h, \tag{5.33}$$

this problem is known as the **dual problem**. In order to solve this system (5.30), we solve the dual problem of obtaining η and then we find ξ by the equation,

$$\xi = A_h^{-1} f_h - C_h \eta.$$

Define $S_h := C_h^T A_h^{-1} C_h$ that occurs in (5.33). This matrix S_h is the Schur complement with respect to η, thus it is the discrete form of the Uzawa pressure operator and hence S_h is called the **Uzawa matrix**. Let B denote the coefficient matrix in (5.30),

$$B := \begin{bmatrix} A_h & C_h \\ C_h^T & O \end{bmatrix}. \tag{5.34}$$

Proposition 5.10. Given that $a(\cdot,\cdot)$ is symmetric and coercive on $V_h \times V_h$ and that the inf-sup condition holds on $V_h \times Q_h$.

1. Matrix A_h is symmetric positive definite.

2. Matrix C_h is injective.

3. Matrix S_h is symmetric positive definite.

4. Matrix B is symmetric, invertible, non-definite. More precisely it has n positive and m negative eigenvalues.

Proof. 1. Since the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive, hence the matrix A_h is symmetric and positive definite.

2. Since inf-sup condition holds on $V_h \times Q_h$ from (5.32)

$$\sup_{v_h \in V_h} \frac{b(q_h, v_h)}{\|v_h\|} \geq \beta_h \|q_h\|_0, \; \forall\, q_h \in Q_h.$$

Let the vectors $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ represent v_h and q_h. The above inequality can be rewritten as,

$$\sup_{\xi \in \mathbb{R}^n} \frac{(C_h^T \xi, \eta)}{\|\xi\|} \geq \beta_h \|\eta\|_0, \; \forall\, \eta \in \mathbb{R}^m,$$

where on the right hand side we have used (5.29). By using the definition of $\|\cdot\|_*$ given by (5.28),

$$\|C_h \eta\|_* \geq \beta_h \|\eta\|_0.$$

Thus, C_h is injective.

3. Since C_h is injective, this means C_h^T has full column rank. As $S_h = C_h^T A_h^{-1} C_h$ and A_h is symmetric positive definite, this implies that S_h is symmetric positive definite.

4. We can write B as,

$$B = \begin{bmatrix} A_h & C_h \\ C_h^T & O \end{bmatrix} = \begin{bmatrix} A_h & O \\ C_h^T & I \end{bmatrix} \begin{bmatrix} A_h^{-1} & O \\ O & -C_h^T A_h^{-1} C_h \end{bmatrix} \begin{bmatrix} A_h & C_h \\ O & I \end{bmatrix} =: P \tilde{B} P^T.$$

Note that the matrix P is non-singular, since $\det P = \det A \neq 0$. Thus B and $\tilde{B}$ represents the same quadratic form in different bases. From the *Sylvester inertia theorem*, we have the result that the number of positive, negative and zero eigenvalues is independent of the basis in which it is written. Since A_h^{-1} has n positive eigenvalues and $-C_h^T A_h^{-1} C_h$ has m negative eigenvalues, we get the desired result.

□

Now we go towards providing the finite element implementation of the Stokes problem. We know that the finite element spaces $V_h \subset V$ and $Q_h \subset L_0^2(\Omega)$ must satisfy inf-sup condition (5.31). In order to prove the proposed finite element spaces satisfy the inf-sup condition, Fortin's lemma is generally used.

Theorem 5.11 (**Fortin's Lemma** [18]). Assume the inf-sup condition holds on $H \times M$, i.e.,

$$\inf_{q \in M} \sup_{v \in H} \frac{b(q,v)}{\|v\|_H \|q\|_M} \geq \beta > 0.$$

Let $H_h \subset H$ and $M_h \subset M$. An inf-sup condition holds on $H_h \times M_h$ with a constant β^* independent of h if and only if there exists a restriction operator $\imath : H \to H_h$ and a constant $c > 0$ independent of h such that

1. $b(\imath\, v - v, q_h) = 0$, $\forall\, q_h \in M_h$.

2. $\|\imath\, v\|_H \leq C\|v\|_H$.

We restrict our case to triangular finite element method. Let Ω be divided into a finite number of non-overlapping triangular elements K_i. For some triangle K, let h_K denote the longest side of K and define $h := \max_{K \in \mathcal{T}_h} h_K$, where $\mathcal{T}_h$ denote the triangulation of Ω. Define the space $\mathcal{P}_1(K) := \{ax + by + c \,|\, a, b, c \in \mathbb{R},\, (x,y) \in K\}$ and $\mathcal{P}_2(K) := \{a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\, a_i \in \mathbb{R},\, (x,y) \in K\}$. Different combinations of finite element pair were analyzed for their suitability for the discrete Stokes problem. The pair $\mathcal{P}_1 \times \mathcal{P}_1$ has been proved to be unstable as it does not satisfy the inf-sup condition. One of the few pairs that were proved to be stable for the Stokes problem is $\mathcal{P}_2 \times \mathcal{P}_1$ finite element pair, also known as "Taylor-Hood finite elements".

For Taylor-Hood finite element method, we define the subspace $V_h \subset V$ as

$$V_h = \left\{ v \in C(\bar{\Omega}),\, v|_K \in \mathcal{P}_2(K),\, \forall\, K \in \mathcal{T}_h \right\}, \tag{5.35}$$

and the subspace $Q_h \subset L_0^2(\Omega)$ as

$$Q_h = \left\{ q \in C(\bar{\Omega}),\, q|_K \in \mathcal{P}_1(K),\, \forall\, K \in \mathcal{T}_h \right\}. \tag{5.36}$$

Theorem 5.12 (**Second-order convergence of Taylor-Hood finite element scheme** [**35**])**.** Suppose the triangulation is quasi-uniform and no triangle has two edges on $\partial\Omega$. Then the Taylor-Hood finite element scheme is convergent of order 2, i.e., it satisfies

$$\|u - u_h\| + \|p - p_h\|_0 \leq ch^2(\|u\|_3^* + \|p\|_2^*),$$

where $\|\cdot\|_3^*$ and $\|\cdot\|_2^*$ denote the norm on the space $H_0^3(\Omega)$ and $H_0^2(\Omega)$ respectively.

For implementing the Taylor-Hood finite elements, we consider for pressure the linear triangular mesh, i.e., 3 nodes corresponding to the three vertices of the triangle. For velocity, we create mid-points on the edges to get 6 nodes per triangle. Since pressure has zero average, we take η to be zero at one of the nodes, say the first node. A MATLAB implementaion of such higher order finite elements is described in [26]. Difficulty arises if our domain has curved boundaries, in which case we use isoparametric finite elements. The paper [3] provides a good implementation of Taylor-Hood isoparametric finite elements. We have used a unit square domain for every numerical result depicted in this section. We also implemented FEM for elliptical domain. The error in the case of isoparametric FEM is not less than FEM for polygonal domains. We need good information about the boundary nodes to get a significant decrease in the error. Since we only discuss the convergence of numerical methods for solving the linear system, we restrict ourselves to the unit square domain.

After constructing a primal problem of the type (5.30), we can solve this sparse linear system in various ways. A detailed text describing all of these methods is given in [42] and [4]. Paper [41] provides detailed convergence results for methods such as PCG and MINRES for the Stokes problem. We list some of these methods as follows.

- **Direct Method**

  The Stokes problem primal can be solved directly, as we explained earlier by eliminating ξ in (5.30) and then forming the Uzawa matrix. The disadvantage of using a direct method is finding the inverse of the sparse matrix A_h can prove to be costly.

- **MINRES**

  We can use MINRES to solve the problem (5.30); however, it will converge slowly without any preconditioning. We used the preconditioner, say $P1$, given by the matrix

$$P1 = \begin{bmatrix} A_h & O \\ O & S_h \end{bmatrix}.$$

  The preconditioned matrix has only three eigenvalues; thus MINRES converges in three iterations. Another preconditioner, say $P2$, that we can use is given as follows

$$P2 = \begin{bmatrix} \mathrm{diag}(A_h) & O \\ O & I \end{bmatrix}.$$

  Here $\mathrm{diag}(A)$ represents the diagonal matrix containing the main diagonal of A_h. The number of iterations required for the convergence of MINRES reduces slightly on using $P2$ as a preconditioner. Figure 5.1 represents the number of iterations needed for the convergence of MINRES for different discretizations, i.e., various values of h. Figure 5.2 represents the norm of the residual for each iteration of MINRES for a fixed $h = 0.1$. In both of the graphs, the preconditioner $P2$ has been used. The book [43] describes MINRES carefully. This method is available in MATLAB as **minres** function.

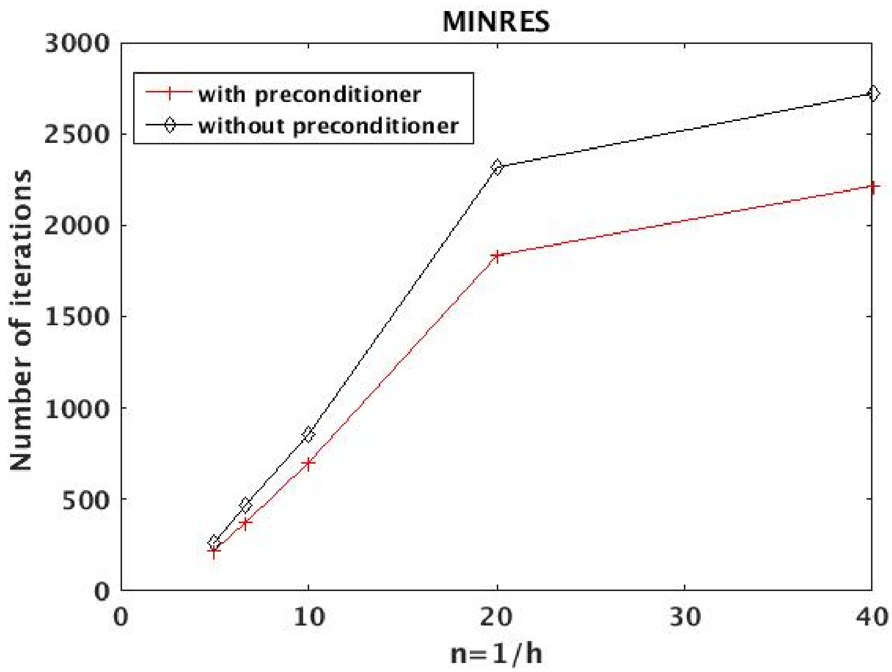

Figure 5.1: Plot of number of iterations required for the convergence of MINRES for various discretizations.

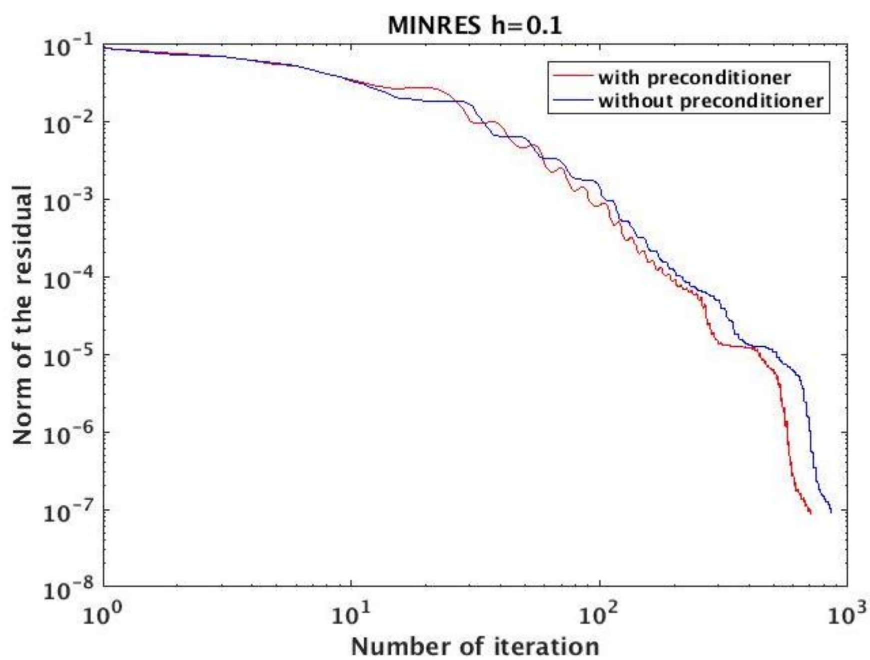

Figure 5.2: Convergence history of MINRES for $h = 0.1$.

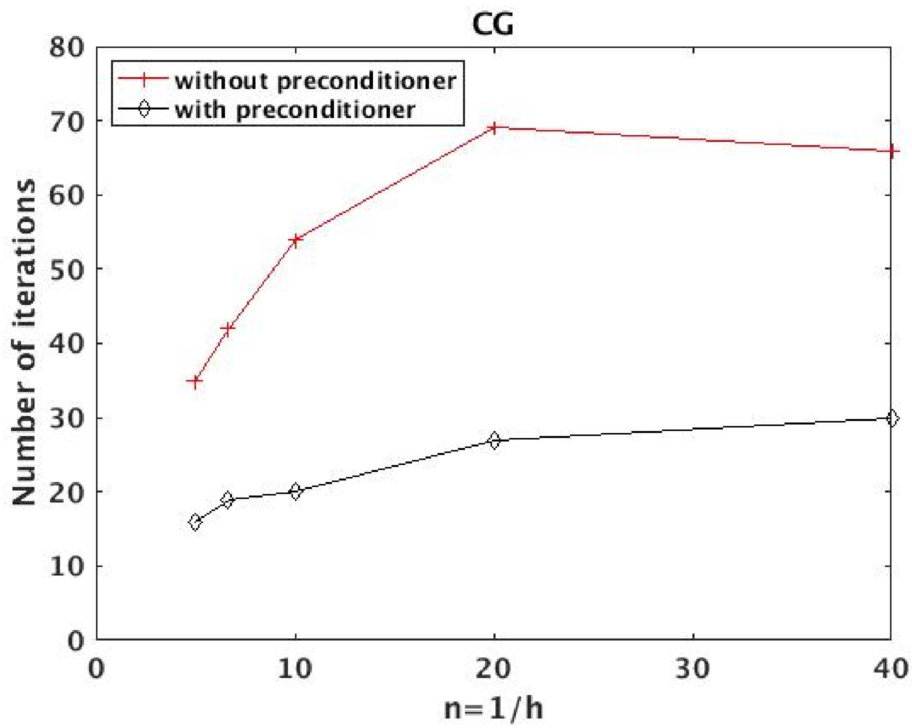

Figure 5.3: Plot of number of iterations required for the convergence of CG for various discretizations.

- **Conjugate Gradient**

  We used MINRES for finding the solution of the primal problem given by (5.30). The dual problem given by (5.33) for finding the vector η can be solved by using Conjugate gradient method. This method is also available in MATLAB as **pcg** function. The preconditioned conjugate method can be employed for solving (5.33) by using the diagonal matrix containing the main diagonal of S_h as a preconditioner. This simple formulation for a preconditioner helped us in getting much better results for convergence, i.e., the number of iterations decreases significantly. This trend is presented in the form of a graph given by Figure 5.3. The convergence history, i.e., the plot of the norm of the residual in each iteration of CG for a fixed $h = 0.1$, is given by Figure 5.4.

- **Uzawa Algorithm**

  Consider the dual problem (5.33), i.e., $S_h\eta = C_h^T A_h^{-1} f_h$. By applying one

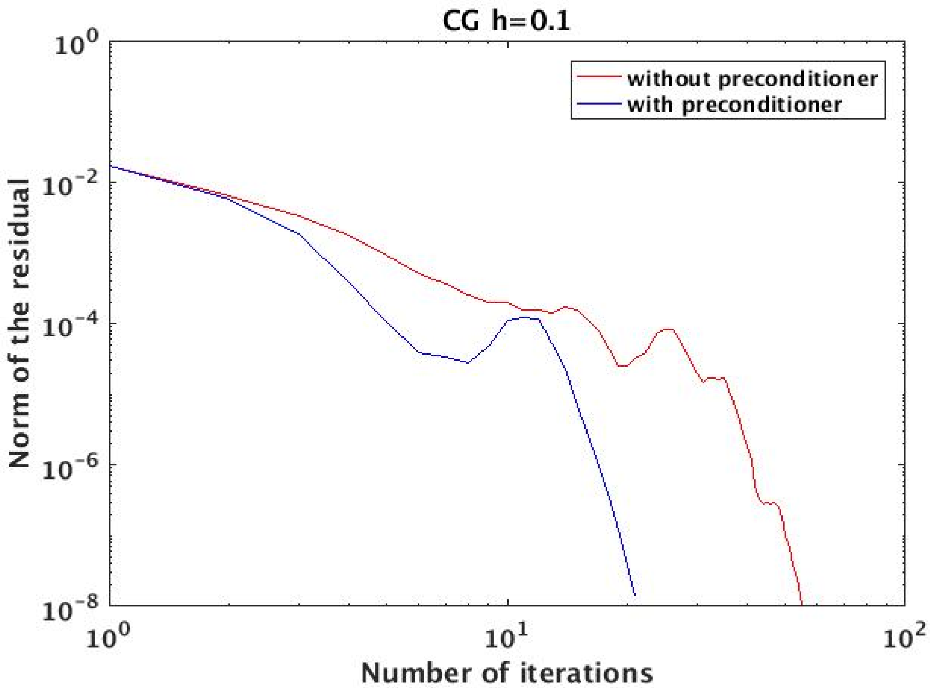

Figure 5.4: Convergence history of CG for $h = 0.1$.

iteration of Richardson method of solving this equation, we obtain

$$\eta^{k+1} = \eta^k + \omega(C_h^T A_h^{-1} f_h - S_h \eta^k).$$

Now we have $C_h^T A_h^{-1} f_h - S_h \eta^k = C_h^T A_h^{-1}(f_h - C_h \eta^k)$. Set $\xi^{k+1} = A_h^{-1}(f_h - C_h \eta^k)$, so we get the iterations of the Uzawa method as,

1. $\xi^{k+1} = A_h^{-1}(f_h - C_h \eta^k)$.
2. $\eta^{k+1} = \eta^k + \omega C_h^T \xi^{k+1}$.

The convergence results of Richardson iteration implies that the Uzawa algorithm converges if and only if $0 < \omega < \frac{2}{\lambda_{\max}(S_h)}$, moreover, the optimal value of ω is given as, $\omega_{\text{opt}} = \frac{2}{\lambda_{\min}(S_h) + \lambda_{\max}(S_h)}$ (see [43]). The use of $\omega = \omega_{\text{opt}}$ reduces the number of iterations required for the convergence of the Uzawa algorithm. Once h is considered as small as 0.05 for a unit square, we get the result that the number of iterations required for the convergence of the Uzawa algorithm increases drastically. The reason of this abrupt change is that as h become smaller $\omega_{\text{opt}} \approx \frac{2}{\lambda_{\max}(S_h)}$ and the convergence occurs for $\omega < \frac{2}{\lambda_{\max}(S_h)}$. In order to avoid this problem and get better results, we take ω to be either $\omega_1 := \frac{1}{\lambda_{\min}(S_h)+\lambda_{\max}(S_h)}$ or $\omega_2 := \frac{1.5}{\lambda_{\min}(S_h)+\lambda_{\max}(S_h)}$.

- **Augmented Lagrangian Uzawa Method**

  The Uzawa method for the Stokes problem was slightly changed in order to give a faster convergence (see [4]). Consider a parameter $r > 0$. By multiplying the second equation of (5.30) by rC_h and adding it to the first equation,

$$\begin{bmatrix} A_h + rC_hC_h^T & C_h \\ C_h^T & O \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} f_h \\ O \end{bmatrix}. \tag{5.37}$$

  The augmented Lagrangian Uzawa method for (5.30) is applying the Uzawa method to the above problem. This method converges if $0 < \omega < 2r$ and $r \gg 1$.

  We compare the convergence of the Uzawa method and the augmented Lagrangian Uzawa method by taking $\omega = \omega_1, \omega_2$ and $r = \omega$. Figure 5.5 represents the graph of number of iterations required for the convergence of the Uzawa and augmented Lagrangian Uzawa algorithm for $\omega = \omega_1$ and Figure 5.6 describes the same for $\omega = \omega_2$. It can be seen that on using ω_2 instead of ω_1 the number of iterations decrease for all h. In fact, if we use ω_{opt} it decreases even further but the trend changes for finer discretizations, in which case they increases abruptly.

  We then present the convergence history of these algorithms for a fixed $h = 0.1$. Figure 5.7 is the loglog plot of norm of the residual in each iteration of the Uzawa and the augmented Lagrangian Uzawa algorithm for $\omega = \omega_1$. For $\omega = \omega_2$ this comparison is shown by Figure 5.8. It can be seen that the aug-

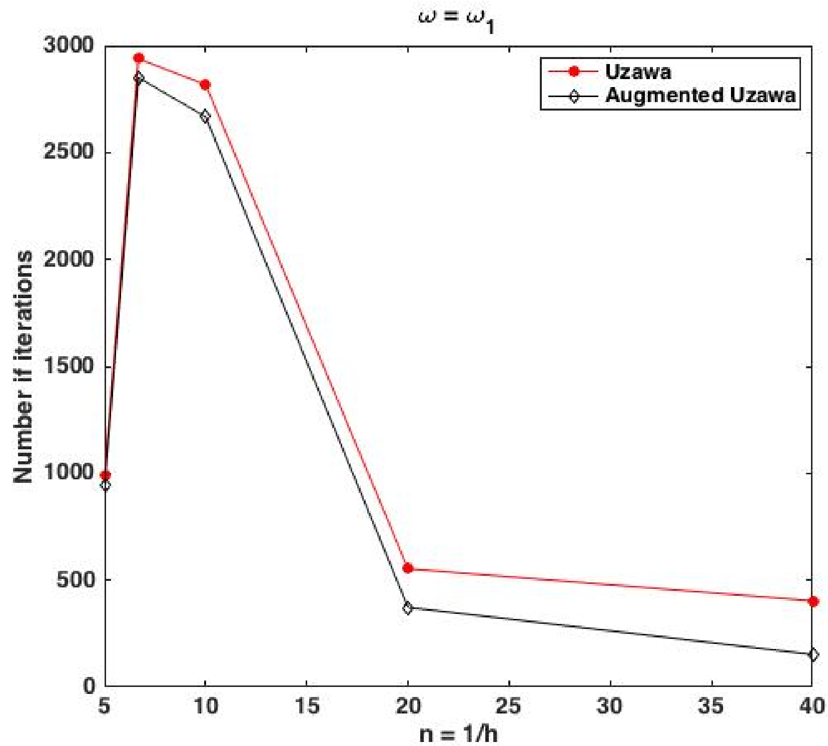

Figure 5.5: Plot of number of iterations required for the convergence of the Uzawa algorithms for various discretizations and $\omega = \omega_1$.

mented Lagrangian Uzawa algorithm possesses smaller residual norm for each iteration. These two graphs are very similar to each other, but on considering $\omega = \omega_{\text{opt}}$, Figure 5.9 gives a different trend. In this case, the Uzawa algorithm with augmentation shows a much better improvement from the Uzawa algorithm.

There are various versions of the Uzawa algorithm, namely, inexact Uzawa algorithm [43], accelerated Uzawa algorithm [28] etc. All of these represents some modifications done in the Uzawa algorithm for getting a faster convergence.

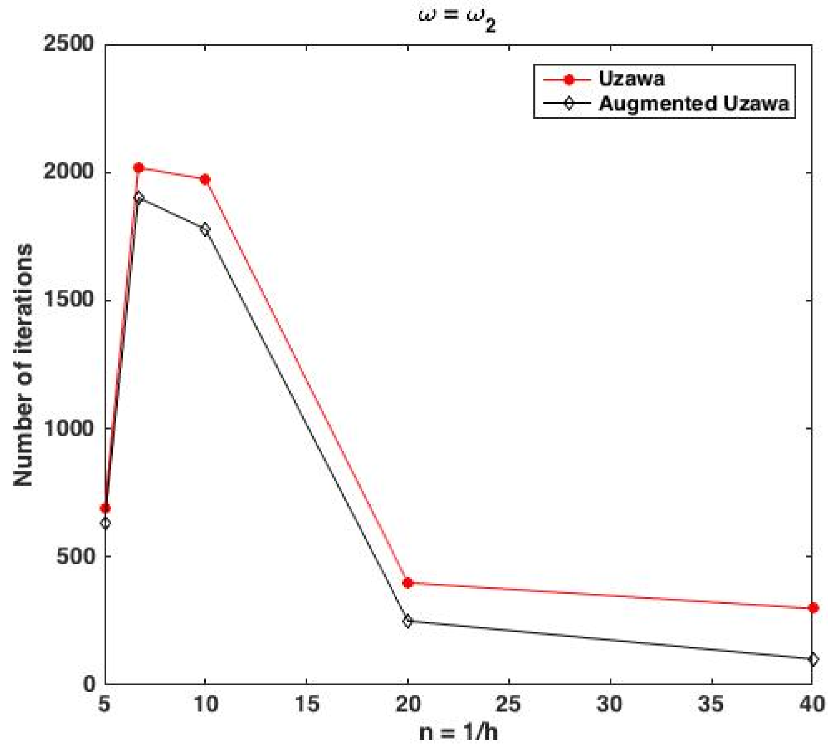

Figure 5.6: Plot of number of iterations required for the convergence of the Uzawa algorithms for various discretizations and $\omega = \omega_2$.

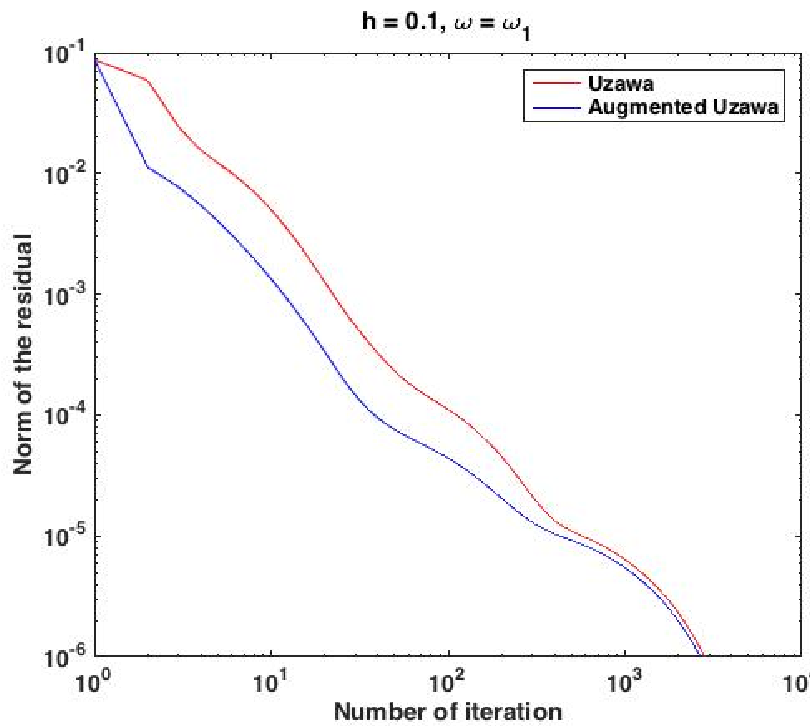

Figure 5.7: Convergence history of the Uzawa algorithms for $h = 0.1$ and $\omega = \omega_1$.

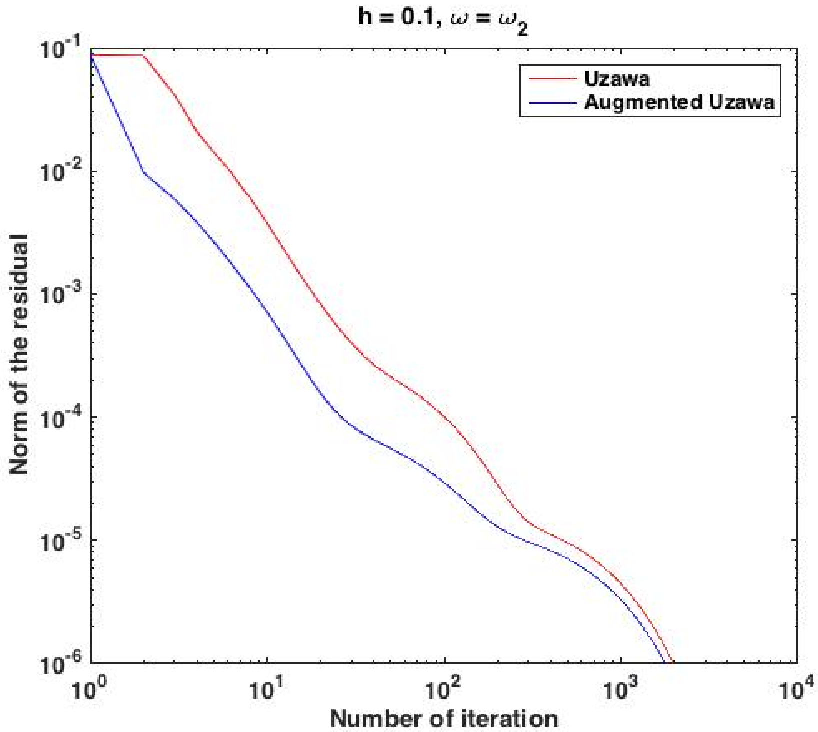

Figure 5.8: Convergence history of the Uzawa algorithms for $h = 0.1$ and $\omega = \omega_2$.

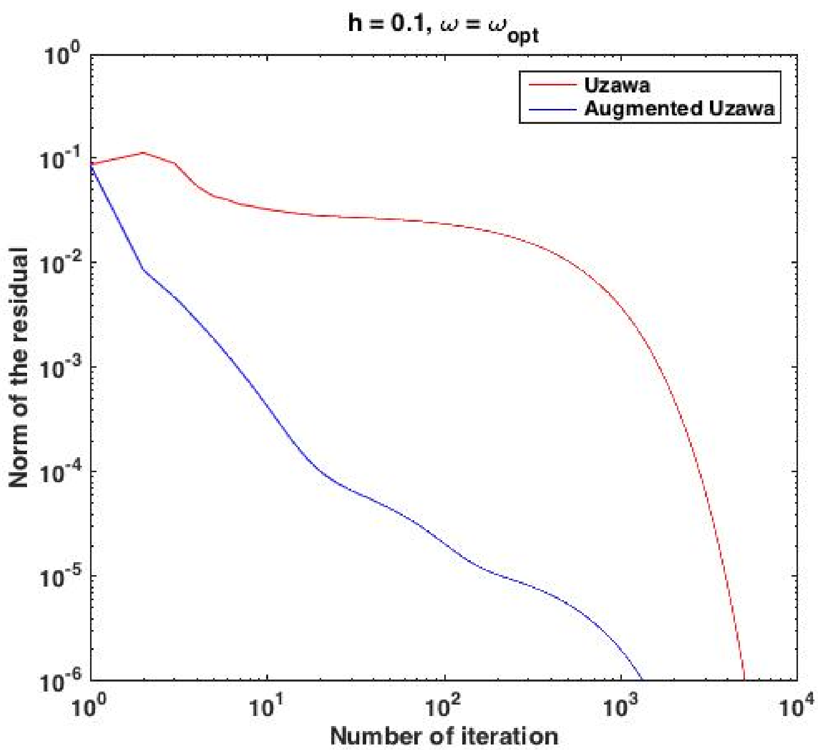

Figure 5.9: Convergence history of the Uzawa algorithms for $h = 0.1$ and $\omega = \omega_{\text{opt}}$.

6

Conclusion

The primary goal of this work is to find the infimum of the spectrum of the Uzawa pressure operator, i.e., $\lambda_{\min}(S)$ for a square domain. The best bounds on $\lambda_{\min}(S)$ implies, $\lambda_{\min}(S) \in \left[\sin^2 \frac{\pi}{8}, \frac{1}{2} - \frac{1}{\pi}\right]$. Since 1983 [31], the lower bound on $\lambda_{\min}(S)$ has not been improved. We need a good estimate on the lower bound to find $\lambda_{\min}(S)$. We tried several ways for improving the bounds, but we could not get an estimate better than the existing one. Some authors such as [11, 5, 36], have tried to find an eigenvalue numerically between this interval, but they could not find any. If there was any eigenvalue in that interval, then it should have been found by using eigenvalue solvers. The upper bound of $\lambda_{\min}(S)$, given above, gives a lower bound on the Friedrichs constant for a square domain as $\Gamma_\Omega \geq 4.5$.

We performed a numerical estimation for the best constant in Nečas inequality, square of which is equal to $\lambda_{\min}(S)$. This estimation was also performed by [24], and they found an upper bound to be 0.2260. We have improved the result to 0.20164. A model for least square fit on the data gives us $\lambda_{\min}(S) = 0.19116$. It is the best result for a numerical estimation of $\lambda_{\min}(S)$.

In [13], M. Crouzeix gave a significant result regarding the spectrum of S in case of two-dimensional domains, which is presented in this thesis in part 2 of Theorem 3.7.

It states that there exists $0 < \alpha < 1$ such that the spectrum of S, $\sigma(S) = [\alpha, 1-\alpha]$. The continuous spectrum of a square domain is given as,

$$\left[\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi}\right],$$

looks exactly of the form $[\alpha, 1-\alpha]$. We conjecture that there is no eigenvalue less than $\frac{1}{2} - \frac{1}{\pi}$. Hence, the infimum of the spectrum of the Uzawa pressure operator is conjectured to be equal to 0.18169011381. Note that, this spectral value lies in the continuous spectrum of S; thus it is never achieved by any $p \in M$. We can try to prove this conjecture for our future work.

There are a lot of open problems for the eigenvalue problem of the Uzawa pressure operator. Some of them are listed below.

1. Improvement in the lower bound of $\lambda_{\min}(S)$ for a square domain.

2. Finding the relationship between the eigenvalue problem of the Uzawa pressure operator, which is given by (3.7), (3.8) and the eigenvalue problem (3.10). The eigenvalues of the latter problem may give an estimate for $\lambda_{\min}(S)$.

3. To give a proof of the conjecture given in this thesis that there does not exist any eigenvalue of S lower than the value 0.18169011381 for a square domain.

4. Finding the harmonic conjugates u and v on a square domain occurring in the Friedrichs inequality corresponding to a lower bound on Friedrichs constant of 4.5.

5. Investigating the change in $\lambda_{\min}(S)$, for a change in size and shape of a particular domain.

A comprehensive study of the discrete Stokes problem can be easily found in the literature. However, there is much scope for improving the methods used for solving the linear system, in particular, the Uzawa algorithm.

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