On the Convergence and Analytical Properties of Power Series on non-Archimedean Field Extensions of the Real Numbers

by

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A thesis submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfilment of the requirements of the degree of

MASTER OF SCIENCE

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Abstract

In this thesis the analytic properties of power series over a class of non-Archimedean field extensions of the real numbers, a representative of which will be denoted by $\mathcal{F}$, are investigated. In Chapter 1 we motivate the interest in said fields by recalling work done by K. Shamseddine and M. Berz [6]. We first review some properties of well-ordered subsets of the rational numbers which are used in the construction of such a field $\mathcal{F}$. Then, we define operations $+$ and $\cdot$ which make $\mathcal{F}$ a field. Then we define an order under which $\mathcal{F}$ is non-Archimedean with infinitely small and infinitely large elements. We embed the real numbers as a subfield; and the embedding is compatible with the order. Then, in Chapter 2, we define an ultrametric on $\mathcal{F}$ which induces the same topology as the order on the field. This topology will allow us to define continuity and differentiability of functions on $\mathcal{F}$ which we shall show are insufficient conditions to ensure intermediate values, extreme values, et cetera. We shall study convergence of sequences and series and then study the analytical properties of power series, showing they have the same smoothness properties as real power series; in particular they satisfy the intermediate value theorem, the extreme value theorem and the mean value theorem on any closed interval within their domain of convergence.

Acknowledgements

I would like to thank my advisor Dr. Khodr Shamseddine without whose tireless support and encouragement I could not have finished writing this thesis. He believed in me when I found it hard to, and gave me advice on every word in this thesis. Everything good in this thesis is due to that advice, and all missteps are due to not heeding it well enough.

I would also like to thank my committee members Dr. Gerald Gwinner and Dr. Derek Krepski for providing insight into how to make this work more approachable and relevant.

I would like to thank the University of Manitoba Graduate Fund and the Department of Mathematics for their financial support while I completed this degree. In particular I would like to thank Dr. Steven Kirkland, Dr. Shaun Lui and Dr. Eric Schippers as well as Mrs. Irene Golembioski, Mrs. Sarah Kawecki, and Mrs. Erin Parsons from the Department of Mathematics.

I would like to thank my colleagues Iris Halilović, Nick Mačsai and Chris Rutley for their numerous contributions and consultations as well as their friendship during the writing of this thesis. I would also like to thank all other members of the Organization of Physics Undergraduate Students at the University of Manitoba for their friendship and support.

I would like to thank Dr. Witold Kinsner for helping me to overcome my doubts about being able to finish. His advice set me back on the right path and I will always be grateful for it.

Finally I would like to thank my parents Joy Susan Kovnats and Robert Norman Grafton, to whom I owe everything and who have given me everything. My first, greatest, and most comprehensive teachers everything I know and am is due to them.

For my parents Joy and Bob

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Chapter 1

Introduction

1.1 Motivation and Outline

The long standing interest in analysis on the real numbers is easily understood. The arithmetic and algebraic properties of the reals allow us to model physical systems with great accuracy and perhaps because of this the reals satisfy a great deal of our intuition. One particularly intuitive feature of the real numbers is that any two reals are comparable; that is if a and b are real numbers where $|a| > |b| > 0$ there is some whole number n such that $n|b| > |a|$, this is called the Archimedean property. To illustrate how intuitive this property is: imagine trying to measure the height of a stack of papers. Though each sheet is very thin with enough sheets one could make a stack taller than any given ruler. Likewise an ant, even though its steps are very small, can cross a room of any given length in finitely many steps. In a non-Archimedean setting these intuitive results do not hold; there are rooms so vast that ants cannot cross them and sheets so thin that they cannot stack up to challenge some rulers. These non-intuitive results, along with some others, generate some interest in exploring non-Archimedean objects. In this thesis we are concerned with investigating non-Archimedean field extensions of the real numbers which share some of the arithmetic and algebraic properties that make the real numbers so special (the existence of n^{th}-roots of positive elements, for example, which is not guaranteed in p-adic fields[4]).

Previously a great deal of work was done investigating the properties of the non-Archimedean Levi-Civita field [7], $\mathcal{R}$, and the properties of functions given locally by a convergent power series on it. These investigations revealed that the disconnected nature of $\mathcal{R}$ entails that continuous or even infinitely often differentiable functions do not follow the standard results of calculus on a closed interval (they do not necessarily have intermediate values, extreme values; they need not satisfy Rolle's Theorem, etc.) For a function to achieve sufficient smoothness to guarantee these properties it must be given locally by a convergent power series.

In this thesis we shall investigate the properties of a class of fields which are generalizations of the Levi-Civita field, a representative of which we shall call $\mathcal{F}$, and the properties of analytic functions on them. In Section (1.2) we shall make some preliminary remarks about the properties of well-ordered subsets of $\mathbb{Q}$ which will allow us to better discuss the structure of $\mathcal{F}$. In Section (1.3) we shall introduce the set $\mathcal{F}$ and the operations $+$ and $\cdot$ on it and show that $(\mathcal{F}, +, \cdot)$ is a field by showing that $\cdot$ is distributive with respect to $+$ and multiplicative inverses of non-zero elements of $\mathcal{F}$ exist in $\mathcal{F}$ (that $(\mathcal{F}, +)$ and $(\mathcal{F} \setminus \{0\}, \cdot)$ are abelian groups is trivial with the above mentioned results). We shall also introduce some relations between elements of $\mathcal{F}$ (including a total order). In Section (2.1) we shall define an ultrametric on $\mathcal{F}$, and show that it induces the same topology as the order described in Section (1.3). With the concept of a topology the convergence of sequences and series can be explored. To motivate our interest in power series in Section (2.3) we shall provide some examples of continuous and differentiable functions failing to satisfy the basic theorems of calculus. Because of previous experience in non-Archimedean analysis, specifically with power series on $\mathcal{R}$ [6] we infer that power series are the only class of functions that satisfy those theorems without additional conditions on any of the derivatives. In Chapter 3 we will show that power series indeed have the same smoothness properties as real power series.

There are further applications for power series in a non-Archimedean context (Lebesgue-like integration [9], computer differentiation [8], etc.) and in $\mathcal{R}$ functions given locally by power series are the smallest (with a few considerations such as closure under composition and uniform limit, which is crucial in for Lebesgue-like integration) family of basis functions for which these concepts are well defined.

1.2 Properties of Well-Ordered Sets

Definition 1.1. (Well-ordered subsets of $\mathbb{Q}$). We say that a set $B \subset \mathbb{Q}$ is well-ordered if and only if for every $B' \subseteq B$ we have that B' contains a minimum element in the standard order of $\mathbb{Q}$.

Lemma 1.1. *If A is well-ordered then any C that is a subset of A is also well-ordered.*

Proof. Suppose A is well-ordered, $C \subset A$ and let $C' \subset C$. Then $C' \subset A$ so it must contain a minimum element by definition. Therefore, any subset of C contains a minimum element and hence C is well-ordered. □

For any two well-ordered subsets A and B of $\mathbb{Q}$, $A \cap B$ is a subset of A. Hence

Corollary 1.1.1. *If A and B are well-ordered then $A \cap B$ is well-ordered.*

Lemma 1.2. *(Decreasing sequences in well-ordered sets). If (a_n) is a decreasing sequence in a well-ordered set A then there is an N in $\mathbb{N}$ such that $a_n = a_N$ for all $n \geq N$.*

Proof. Suppose (a_n) is a decreasing sequence in a well-ordered set A and there is no N in $\mathbb{N}$ such that $a_n = a_N$ for $n \geq N$. Because $\cup_{n\in\mathbb{N}}\{a_n\}$ is a subset of A it contains a minimum element, call it a_L. Since a_L is in $\cup_{n\in\mathbb{N}} a_n$ there is some $N' \in \mathbb{N}$ such that $a_{N'} = a_L$. Since there is no N such that $a_n = a_N$ for $n \geq N$ and (a_n) is decreasing there must be some $M > N'$ in $\mathbb{N}$ such that $a_M < a_{N'}$ so a_L is not a minimum element of $\cup_{n\in\mathbb{N}}\{a_n\}$. This is a contradiction, therefore for every decreasing sequence (a_n) in A there is an N in $\mathbb{N}$ such that $a_n = a_N$ for $n \geq N$. $\square$

Theorem 1.3. *(Well-ordered subsets of $\mathbb{Q}$). For A,B well-ordered subsets of $\mathbb{Q}$ the following hold*

- $A \cup B$ *is well-ordered.*
- $A + B$ *is well-ordered.*
- *For any $r \in A + B$ there are only finitely many pairs (a, b) in $A \times B$ such that $r = a + b$.*

Proof. Let A and B be well-ordered subsets of $\mathbb{Q}$.

Suppose $C \subset (A \cup B)$, then $C = (C \cap A) \cup (C \cap B)$. As $(C \cap A) \subset A$ and $(C \cap B) \subset B$ they are both well-ordered by Lemma (1.1) and therefore each contains a minimum element (call them a and b respectively). Then if we denote $min\{a, b\}$ by c we have that $c \in C$ and for every $x \in (C \cap A)$ we have $c \leq x$ and for every $y \in (C \cap B)$ we have $c \leq y$, and hence c is the minimum element of C. Therefore any subset of $(A \cup B)$ contains a minimum element and hence $(A \cup B)$ is well-ordered.

Suppose $C \subset A+B$ then for every $c \in A+B$ there is at least one pair $(a, b) \in A\times B$ such that $a + b = c$. For each $c \in C$ choose one such pair and call it (a_c, b_c). Then, using the axiom of choice, define $A_C = \{a \in A \mid \exists c \in C : a = a_c\}$. A_C is well-ordered because it is a subset of A. We can similarly define a well-ordered set $B_C = \{b \in B \mid \exists c \in C : b = b_c\}$ by the axiom of choice.

Let $a_0 = min\{A_C\}$, which exists in A_C as A_C is well-ordered. There may be many pairs with $a_c = a_0$ so let $B_0 = \{b \in B_C \mid \exists c \in C \colon a_0 + b = c\}$. Because $B_0 \subset B$ it is well-ordered and therefore contains a minimum element; call it b_0. Now let $c_0 = a_0 + b_0$ and $C_1 = \{c \in C \mid c < c_0\}$ (if C_1 is empty then c_0 is a minimum element of C).

We will construct a chain of C_n such that eventually a minimum element of some C_N is a minimum element of C and is contained in C, so let

- $A_{C1} = \{a \in A \mid \exists c \in C_1 \colon a = a_c\}$ (exists by axiom of choice)
- $a_1 = min\{A_{C1}\}$ (exists because A_{C1} is well-ordered)
- $B_1 = \{b \in B_C \mid \exists c \in C_1 \colon b + a_1 = c\}$ (exists by axiom of choice)
- $b_1 = min\{B_1\}$ (exists because B_1 is well-ordered)
- $c_1 = a_1 + b_1$
- $C_2 = \{c \in C_1 \mid c < c_1\}$

If C_2 is empty then c_1 is a minimum element of C_1 which means it must be a minimum element of C.

This gives a template that allows us to continue this process generating sequences (C_n), (a_n), (b_n), et cetera. We let

- $A_{Cn} = \{a \in A \mid \exists c \in C_n \colon a = a_c\}$ (exists by axiom of choice)
- $a_n = min\{A_{Cn}\}$ (exists because A_{Cn} is well-ordered)
- $B_n = \{b \in B_C \mid \exists c \in C_n \colon b + a_n = c\}$ (exists by axiom of choice)
- $b_n = min\{B_n\}$ (exists because B_n is well-ordered)

- $c_n = a_n + b_n$
- $C_{n+1} = \{c \in C_n \mid c < c_n\}$ (if C_{n+1} is empty then c_n is a minimum element of C).

Either at some finite N we have that C_{N+1} is empty and c_N is a minimum element of C or we have some infinite sequence (C_n) which gives an infinite sequence (b_n) in B_C.

In the case where the process does not terminate and the set C_n is never empty note the following:

For $c \in C_{N+1}$ consider:

$$\begin{aligned} a_N + b_N &> a_c + b_c \\ b_N &> (a_c - a_N) + b_c \\ b_N &> b_c \end{aligned}$$

and hence for all c in C_{N+1} we have $b_c < b_N$. Therefore (b_n) is a decreasing sequence in the well-ordered set B_C which means by Lemma (1.2) there is an N in $\mathbb{N}$ such that $b_n = b_N$ for $n \geq N$. In other words there is some $c_N \in C$ such that for all $c \leq c_N$ we have $b_c = b_N$. Invoking the axiom of choice again, we can define a set $A_U = \{a \in A_C \mid \exists c \in C\colon a + b_N = c\}$ which is a subset of A and so is well-ordered. Thus A_U has a minimum element, call it a_U. It follows that $a_U + b_N \leq c$ for any $c \leq c_N$. Thus $a_U + b_N$ is a minimum element of C.

Therefore every $C \subset A + B$ must contain a minimum element and hence $A + B$ is well-ordered.

Finally consider an element $r \in A + B$. Let $A_s = \{a \in A \mid \exists b \in B\colon a + b = r\}$ and $B_s = \{b \in B \mid \exists a \in A\colon a + b = r\}$. Because $A_s \subset A$ and $B_s \subset B$ both are well-ordered, therefore both are bounded below, let $a_0 = \min\{A_s\}$ and $b_0 = \min\{B_s\}$. As b_0 is a minimum element of B_s we have A_s is bounded above by $r - b_0$, likewise B_s is bounded above by $r - a_0$. Thus there can only be infinitely many pairs (a, b) in $A \times B$ that sum to r if there is at least one accumulation point in both A_s and B_s. Suppose there are such accumulation points and let a be one such accumulation point of A_s, and let (a_n) be any sequence in A_s such that $\lim_{n\to\infty} a_n = a$. Then (a_n) must have a monotone subsequence (a_{n_k}). Let (b_{n_k}) be a sequence in B_s where $b_{n_k} = r - a_{n_k}$, either (b_{n_k}) is decreasing or (a_{n_k}) is. Without loss of generality say (b_{n_k}) is decreasing then, by Lemma (1.2), there is some N in $\mathbb{N}$ such that for $k \geq N$ $b_{n_k} = b_{n_N}$ and so $a_{n_k} = a_{n_N}$, or $a_{n_k} + b_{n_k} \neq r$ which is a contradiction. Then for $\delta = \frac{1}{2}\min_{1\leq k\leq N}\{|a - a_k|\}$ we have $(a - \delta, a + \delta) \cap A_s$ is empty which contradicts that a is an accumulation point of A_s. So neither A_s nor B_s contain an accumulation point and both are bounded above and below hence both are finite. Thus there are only finitely many pairs $(a, b) \in A \times B$ such that $a + b = r$.

□

1.3 The Field $\mathcal{F}$

Definition 1.2. (The set $\mathcal{F}$). We define

$$\mathcal{F} = \{f\colon \mathbb{Q} \to \mathbb{R} \mid \{x | f(x) \neq 0\} \text{ is well-ordered}\}.$$

The elements of $\mathcal{F}$ are those functions from $\mathbb{Q}$ to $\mathbb{R}$ whose values are non-zero only on well-ordered sets. For an element x the set where $x[q] \neq 0$ is called the support of x, and is denoted by $supp(x)$.

We denote elements of $\mathcal{F}$ by x, y etc. and identify their values at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Definition 1.3. (Addition and Multiplication on $\mathcal{F}$).

We define addition on $\mathcal{F}$ pointwise: for x, y in $\mathcal{F}$

$$(x+y)[q] = x[q] + y[q]. \tag{1.1}$$

It can be seen that $supp(x+y)$ is contained in $supp(x) \cup supp(y)$, and so by Theorem (1.3) $supp(x+y)$ is well-ordered. Thus $\mathcal{F}$ is closed under addition.

Multiplication is defined thusly: For $q \in \mathbb{Q}$ we set

$$(x \cdot y)[q] = \sum_{\substack{q_x, q_y \in \mathbb{Q} \\ q_x + q_y = q}} x[q_x] \cdot y[q_y] \tag{1.2}$$

The support of $(x \cdot y)$ is contained in $supp(x) + supp(y)$; so by Theorem (1.3) $supp(x \cdot y)$ is well-ordered. This means that the sum in Equation (1.2) must have only finitely many terms by Theorem (1.3) and hence the sum is well defined. This gives that $\mathcal{F}$ is closed under multiplication.

NB: The functions $e_+ : \mathbb{Q} \to \mathbb{R}$ and $e_. : \mathbb{Q} \to \mathbb{F}$ given by

$$e_+[q] = 0 \text{ for all } q \text{ and}$$

$$e_.[q] = \begin{cases} 1 \text{ if } q = 0 \\ 0 \text{ if } q \neq 0 \end{cases}$$

are members of $\mathcal{F}$ and for any x in $\mathcal{F}$ we have $x + e_+ = x$ and $e_. \cdot x = x$. Moreover

$x \cdot e_+ = e_+$ for any x in $\mathcal{F}$.

Theorem 1.4. *For x, y, z in $\mathcal{F}$ $z \cdot (x + y) = (z \cdot x) + (z \cdot y)$.*

Proof. Let x, y, z in $\mathcal{F}$, if any are 0 the above result is trivial so without loss of generality assume that none are and so all $supp(x)$, $supp(y)$, and $supp(z)$ are non-empty.

Consider that q is in $supp((z \cdot x) + (z \cdot y))$ if and only if

$$\sum_{\substack{q_z \in supp(z) \\ q_x \in supp(x) \\ q_z + q_x = q}} z[q_z] \cdot x[q_x] + \sum_{\substack{q_z \in supp(z) \\ q_y \in supp(y) \\ q_z + q_y = q}} z[q_z] \cdot y[q_y] \neq 0$$

and $((z \cdot x) + (z \cdot y))[q]$ is the value of that sum. For all q_α in $supp(x) \cup supp(y)$ and q_z in $supp(z)$ we have $z[q_z] \cdot y[q_\alpha] + z[q_z] \cdot x[q_\alpha] = z[q_z] \cdot (y[q_\alpha] + x[q_\alpha])$ and so

$$\sum_{\substack{q_z \in supp(z) \\ q_x \in supp(x) \\ q_z + q_x = q}} z[q_z] \cdot x[q_x] + \sum_{\substack{q_z \in supp(z) \\ q_y \in supp(y) \\ q_z + q_y = q}} z[q_z] \cdot y[q_y] = \sum_{\substack{q_z \in supp(z) \\ q_\alpha \in supp(x) \cup supp(y) \\ q_z + q_\alpha = q}} z[q_z] \cdot (y[q_\alpha] + x[q_\alpha]).$$

Only q_α in $supp(x) \cup supp(y)$ where $x[q_\alpha] + y[q_\alpha] \neq 0$ contribute to the above sum. These are exactly the support points of $x + y$ and $(x + y)[q_\alpha] = x[q_\alpha] + y[q_\alpha]$ by definition. Hence

$$\sum_{\substack{q_z \in supp(z) \\ q_\alpha \in supp(x) \cup supp(y) \\ q_z + q_\alpha = q}} z[q_z] \cdot (x[q_\alpha] + y[q_\alpha]) = \sum_{\substack{q_z \in supp(z) \\ q_\alpha \in supp(x+y) \\ q_z + q_\alpha = q}} z[q_z] \cdot ((x + y)[q_\alpha])$$

and $((z \cdot x) + (z \cdot y))[q]$ is the value of the sum. The right hand side is the definition of $(z \cdot (x + y))[q]$ and hence $supp(z \cdot (x + y)) = supp((z \cdot x) + (z \cdot y))$ and $((z \cdot x) + (z \cdot y))[q] = (z \cdot (x + y))[q]$ at all q in the support. Therefore $z \cdot (x + y) = (z \cdot x) + (z \cdot y)$. $\square$

With these definitions of $+$ and $\cdot$ we intend to show that $(\mathcal{F}, +, \cdot)$ is a field. Given that we have shown that $\mathcal{F}$ is closed under $+$ and $\cdot$, and that we have shown $\cdot$ is distributive with respect to $+$ on $\mathcal{F}$, we will endeavour to show that non-zero elements in $\mathcal{F}$ have multiplicative inverse elements (it is trivial from here to show $(\mathcal{F}, +)$ is an abelian group and that $\cdot$ is associative and commutative). This will require results related to convergence of sequences of elements of $\mathcal{F}$, which will require the definition of an order. After defining an order ($\geq$) and (ultra-)metric on $\mathcal{F}$ we shall return to the question of multiplicative inverse elements and show that $(\mathcal{F}, +, \cdot, \geq)$ is a totally ordered field.

Definition 1.4. (λ, $\sim$, $\approx$, $=_r$). For $x, y \in \mathcal{F}$ we define

- $\lambda(x) = \begin{cases} min\{supp(x)\} \text{ for } x \neq e_+ \\ \infty \text{ for } x = e_+. \end{cases}$
- $x \sim y$ if $x \neq e_+$,$y \neq e_+$, and $\lambda(x) = \lambda(y)$.
- $x \approx y$ if $x \neq e_+$,$y \neq e_+$, $\lambda(x) = \lambda(y)$, and $x[\lambda(x)] = y[\lambda(y)]$.
- $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

Theorem 1.5. *For $x, y \in \mathcal{F}$ we have*

- $\lambda(x + y) \geq min\{\lambda(x), \lambda(y)\}$.
- *For $x \cdot y \neq e_+$ $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$, $(x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)]$.*

Proof. Since $(x + y)[q] = x[q] + y[q]$ and $x[q] = 0$ for $q < \lambda(x)$, $y[q] = 0$ for $q < \lambda(y)$ we have $(x + y)[q] = 0$ for $q < min\{\lambda(x), \lambda(y)\}$. Thus $(x + y)[q] \neq 0$ only for $q \geq min\{\lambda(x), \lambda(y)\}$ so $\lambda(x + y) \geq min\{\lambda(x), \lambda(y)\}$.

Both $supp(x)$ and $supp(y)$ are well-ordered, and non-empty as neither are e_+, so they each contain a minimum element $\lambda(x)$ and $\lambda(y)$ respectively. Let (a, b) in $supp(x) \times supp(y)$ be such that $a + b = \lambda(x) + \lambda(y)$, then $\lambda(x) - a = b - \lambda(y)$. If $\lambda(x) - a > 0$ then $\lambda(x) > a$ which is a contradiction, if $\lambda(x) - a < 0$ then $b < \lambda(y)$

which is a contradiction. Therefore only $(\lambda(x), \lambda(y))$ in $supp(x) \times supp(y)$ sums to $\lambda(x) + \lambda(y)$ and $supp(x \cdot y) \subset supp(x) + supp(y)$. Therefore $\lambda(x) + \lambda(y) \leq \lambda(x \cdot y)$. Consider that $y[\lambda(y)] \cdot x[\lambda(x)]$ is not zero by definition and there is no other (a, b) in $supp(x) \times supp(y)$ such that $a + b = \lambda(x) + \lambda(y)$ then

$$(x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)] \neq 0$$

by definition. Therefore $\lambda(x \cdot y) \leq \lambda(x) + \lambda(y)$. Thus $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ and $(x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)]$. □

Definition 1.5. (Ordering in $\mathcal{F}$). For x, y in $\mathcal{F}$ we say that $x \geq y$ if $x = y$ or $x \neq y$ and $(x - y)[\lambda(x - y)] > 0$.

This is a total order on $\mathcal{F}$ because either $x = y$ or $(x - y)[\lambda(x - y)]$ is in $\mathbb{R}$ and the standard order on $\mathbb{R}$ is a total order.

Definition 1.6. (Absolute value on $\mathcal{F}$). The function $|\cdot|\colon \mathcal{F} \to \mathcal{F}$ given by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } 0 > x \end{cases}$$

is an absolute value on $\mathcal{F}$

NB: For x, y in $\mathcal{F}$ we have $|x \cdot y| = |x| \cdot |y|$ and $|x + y| \leq |x| + |y|$.

Definition 1.7. (Embedding $\mathbb{R}$ in $\mathcal{F}$). Define $\Pi\colon \mathbb{R} \to \mathcal{F}$ by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}.$$

This function is an isomorphic injection between $\mathbb{R}$ and $\Pi(\mathbb{R})$ ($\Pi(x+y) = \Pi(x)+\Pi(y)$ and $\Pi(x \cdot y) = \Pi(x) \cdot \Pi(y)$ $x, y \in \mathbb{R}$) and so embeds $\mathbb{R}$ in $\mathcal{F}$. Notice also that if $x \geq y$

in $\mathbb{R}$ then $\Pi(x) \geq \Pi(y)$ so Π is an order preserving embedding. When referencing a real number x as an element of $\mathcal{F}$ we mean $\Pi(x)$. In particular this embedding maps the real number 0 to e_+ and the real number 1 to $e_\cdot$. From now on we will use 0 and 1 to denote the neutral elements of $\mathcal{F}$.

Definition 1.8. (Order of Magnitude in $\mathcal{F}$) For $x, y > 0$ in $\mathcal{F}$ we say

$$x \gg y \text{ if } \lambda(x) < \lambda(y)$$

This relation compares the so-called orders of magnitude of elements of $\mathcal{F}$. For any n in $\mathbb{N}$ and $y > 0$ in $\mathcal{F}$ we have $\lambda(ny) = \lambda(n) + \lambda(y) = \lambda(y)$. Then $x \gg y$ implies $x > ny$ for all n in $\mathbb{N}$. If $x \gg y$ we say x is infinitely larger than y and y is infinitely smaller than x. If $x \sim y$ then we say x is of the same order as y. Those x with $\lambda(x) = 0$ we call finite (for each such x there is an x' in $\mathbb{R}$ such that $x \approx \Pi(x')$). Those $x > 0$ with $\lambda(x) < 0$ we call infinitely large, as $x \gg 1$ we have $x > n$ for all n in $\mathbb{N}$. Those $x > 0$ with $\lambda(x) > 0$ we call infinitely small or infinitesimal, as $1 \gg x$ we have $1 > nx$ for all n in $\mathbb{N}$.

Definition 1.9. (The number d). Consider the function $d\colon \mathbb{Q} \to \mathbb{R}$

$$d[q] = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}.$$

This function is an element of $\mathcal{F}$ and by the above definition of order we have $d > 0$ and $1 \gg d$. Hence d is infinitely small or infinitesimal. The multiplicative inverse of this element is given by

$$d^{-1}[q] = \begin{cases} 1 & \text{if } q = -1 \\ 0 & \text{if } q \neq -1 \end{cases}$$

and it is easy to check that $d^{-1} \gg 1$.

Let n in $\mathbb{N}$ for ξ_n in $\mathcal{F}$ given by

$$\xi[q] = \begin{cases} 1 & \text{if } q = \frac{1}{n} \\ 0 & \text{if } q \neq \frac{1}{n} \end{cases}$$

then $\underbrace{\xi_n \cdots \xi_n}_{n \text{ times.}} = d$ and so ξ_n is the n^{th} root of d.

For any $t = \frac{n}{m}$, $t \neq 0$, in $\mathbb{Q}$ we have $d^t = \underbrace{\xi_m \cdot \ldots \cdot \xi_m}_{n \text{ times}}$ is an element of $\mathcal{F}$ whose sole support point is t and whose value at that support point is 1. Then for $t \neq 0$ $d^{1/t}$ is the t^{th} root of d and for $i > j$ we have $d^j \gg d^i$.

NB: Since for every q in $\mathbb{Q}$ $d^q[q] = 1$ we have for a_q in $\mathbb{R}$ $a_q d^q[q] = a_q$. For every x in $\mathcal{F}$ we can enumerate $supp(x)$ and form a sequence, which must be bounded below as $supp(x)$ is well-ordered, (q_n) such that if $i > j$ we have $q_i > q_j$. We can then define a real sequence $a_n = x[q_n]$ and any x in $\mathcal{F}$ is given by the formal power series $\sum_{n=1}^{\infty} a_n d^{q_n}$. While this sequence need not converge (such a sequence converges in the order topology only if $supp(x)$ is left-finite [5]) it can be helpful to treat elements of $\mathcal{F}$ as formal generalized power series when we do arithmetic with them or define functions on $\mathcal{F}$. With this understanding of the form of elements of $\mathcal{F}$ we can move on to describe a topology on $\mathcal{F}$.

Chapter 2

Topological Structure and Convergence

2.1 Topology

In this chapter we will introduce the concept of an ultrametric space and describe convergence of sequences and series in such a space. The discussion of convergence of sequences and series will equip us to investigate the analytic properties of power series on $\mathcal{F}$.

Definition 2.1. (Ultrametric). Given a set X a function $\Delta\colon X \times X \to \mathbb{R}$ is an ultrametric on X if for all x, y, z in X

- $\Delta(x, y) \geq 0$, $\Delta(x, y) = 0$ if and only if $x = y$.

- $\Delta(x, y) = \Delta(y, x)$.

- $\Delta(x, z) \leq max\{\Delta(x, y), \Delta(y, z)\}$

The pair (X, Δ) is called an ultrametric space.

NB: An ultrametric is a metric which satisfies the strong triangle inequality (which entails the triangle inequality). Therefore every ultrametric space is also a metric space.

Theorem 2.1. *($\mathcal{F}$ is an ultrametric space). $\mathcal{F}$ with the function*

$$\Delta(x,y) = \begin{cases} e^{-\lambda(x-y)} & \textit{if } x \neq y \\ 0 & \textit{if } x = y \end{cases}$$

is an ultrametric space.

Proof. Let x, y, z in $\mathcal{F}$.

For $x \neq y$ we have $\lambda(x-y)$ is in $\mathbb{Q}$, then $\Delta(x,y) = e^{-\lambda(x-y)} > 0$. When $x = y$ we have $\Delta(x,y) = 0$ by definition. Thus $\Delta(x,y) \geq 0$ and $\Delta(x,y) = 0$ if and only if $x = y$.

As $(x-y) = -(y-x)$ we have $\lambda(x-y) = \lambda(-1) + \lambda(y-x)$ by Theorem (1.5) and since $\lambda(-1) = 0$ we have $\lambda(x-y) = \lambda(y-x)$ which gives $\Delta(x,y) = \Delta(y,x)$.

Finally since $x-z = (x-y)+(y-z)$ we have $\lambda(x-z) \geq min\{\lambda(x-y), \lambda(y-z)\}$ by Theorem (1.5). Thus we have that $-\lambda(x-z) \leq max\{-\lambda(x-y), -\lambda(y-z)\}$ which gives $\Delta(x,z) \leq max\{\Delta(x,y), \Delta(y,z)\}$. $\square$

Theorem 2.2. *The topologies induced by the order and the ultrametric are equal.*

Proof. Let $x_0 \in \mathcal{F}$ and $\delta > 0$ in $\mathcal{F}$ then $I = \{x \in \mathcal{F} \mid |x - x_0| < \delta\}$, or $(x_0 - \delta, x_0 + \delta)$, is an open interval in the order topology. We have that $\Delta(x, x_0) \leq e^{-\lambda(\delta)}$ for any x in I. Let $\epsilon = e^{-(\lambda(\delta)+1)}$, (NB we have $\epsilon > 0$ in $\mathbb{R}$) and let $B(x_0, \epsilon) = \{x \in \mathcal{F} \mid \Delta(x, x_0) < \epsilon\}$. Then if $x \in B_U(x_0, \epsilon)$ we have $e^{-\lambda(x-x_0)} < e^{-(\lambda(\delta)+1)}$ thus $\lambda(x - x_0) > \lambda(\delta) + 1$ so $|x - x_0| < \delta$. It follows that $B_U(x_0, \epsilon) \subset B_O(x_0, \delta)$.

Now let $x_0 \in \mathcal{F}$ and $\delta > 0$ in $\mathbb{R}$ and consider $B(x_0, \delta)$, an open ball in the ultrametric topology. Let $q \in \mathbb{Q}$ be such that $q < \ln(\delta)$ and let $\epsilon = d^{-q}$ (NB $\epsilon > 0$ in $\mathcal{F}$) then for x in I we have that $|x - x_0| < d^{-q}$ which implies $\lambda(x - x_0) \geq -q$ so $-\lambda(x - x_0) \leq \ln(\delta)$ which gives $\Delta(x, x_0) < \delta$. Thus $I \subset B_U(x_0, \delta)$.

So any open ball in the topology induced by the ultrametric contains an open ball in the topology induced by the order and vice versa. Therefore the topologies are equal [3]. $\square$

Thus $\mathcal{F}$ with the order topology is an ultrametric (therefore metric) space and certain results about limits of sequences and series will hold like any other metric space.

2.2 Convergence of Sequences and Series

Definition 2.2. (Convergence of sequences). Let (s_n) be a sequence in $\mathcal{F}$. We say that (s_n) is convergent to the limit $s \in \mathcal{F}$ if for every $\epsilon > 0$ in $\mathcal{F}$ there exists $N \in \mathbb{N}$ such that $\Delta(s_n, s) < \epsilon$ for all $n \geq N$.

Definition 2.3. (Cauchy Sequence) A sequence (a_n) in $\mathcal{F}$ is said to be Cauchy if for every $\epsilon > 0$ in $\mathcal{F}$ there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n, m > N(\epsilon)$ we have $\Delta(a_n, a_m) < \epsilon$.

NB: Since the order and ultrametric topologies are equal $\Delta(x, y) < \epsilon$ is equivalent to $|x - y| < \epsilon$.

Corollary 2.2.1. *(Sequences in a metric space)[3] Let (a_n) and (b_n) be sequences in $\mathcal{F}$ converging to A and B in $\mathcal{F}$ respectively, and c a constant in $\mathcal{F}$ then*

- *The sequence (ca_n) converges to cA.*

- *The sequence $(a_n + b_n)$ converges to $A + B$ in $\mathcal{F}$.*

- *The sequence $(a_n \cdot b_n)$ converges to $A \cdot B$ in $\mathcal{F}$.*

- *The limit of a convergent sequence in $\mathcal{F}$ is unique.*

- *Convergent sequences are Cauchy.*

Definition 2.4. (Regularity). A sequence (s_n) in $\mathcal{F}$ is regular if $\cup_{n=0}^{\infty} supp(s_n)$ is well-ordered.

Theorem 2.3. *(Properties of regularity). Let the sequences (a_n) and (b_n) in $\mathcal{F}$ be regular. We have the sequence of the sums, the sequence of the products, any subsequence, any rearrangement of either sequence, and the combined sequence $c_{2n} = a_n$ and $c_{2n+1} = b_n$, are all regular.*

Proof. Let $A = \cup_{n=0}^{\infty} supp(a_n)$ and $B = \cup_{n=0}^{\infty} supp(b_n)$. For any $c \in supp(a_n + b_n)$ either $c \in supp(a_n)$ or $c \in supp(b_n)$ then $\cup_{n=0}^{\infty} supp(a_n + b_n) \subset (A \cup B)$ and is therefore well orderd by Lemma (1.1) and Theorem (1.3), thus $(a_n + b_n)$ is a regular sequence.

For any two elements x, y in $\mathcal{F}$ we have $supp(x \cdot y) \subset supp(x) + supp(y)$ and hence we have $supp(a_n \cdot b_n) \subset supp(a_n) + supp(b_n)$ for all n. Therefore for $c \in \cup_{n=0}^{\infty} supp(a_n \cdot b_n)$ we have $c \in A + B$. Thus we have $\cup_{n=0}^{\infty} supp(a_n \cdot b_n) \subset A + B$ and is therefore well-ordered by Lemma (1.1) and Theorem (1.3), thus $(a_n \cdot b_n)$ is a regular sequence.

For any subsequence or rearrangement of the sequence (a_n), call it (s_n), we have $supp(s_n) \subset A$ for all n and so $\cup_{n=0}^{\infty} supp(s_n) \subset A$ and is therefore well-ordered by

Lemma (1.1). So any rearrangement or subsequence of a regular sequence is regular.

The support of the combined sequence is contained in $A \cup B$ and is therefore well-ordered by Theorem (1.3), thus the combined sequence $c_{2n} = a_n$, $c_{2n+1} = b_n$ is regular. $\square$

Lemma 2.4. *Let (s_n) be a sequence converging to s in $\mathcal{F}$. Then $(|s_n|)$ converges to $|s|$ in $\mathcal{F}$.*

Proof. Let $\epsilon > 0$ in $\mathcal{F}$ be given. Then there is some $N \in \mathbb{N}$ such that $|s_m - s| < \epsilon$ for all $n \geq N$. Given that $||s_n| - |s|| \leq |s_n - s|$ for all n we then have that $||s_n| - |s|| < \epsilon$ for $n \geq N$. Therefore $(|s_n|)$ converges to $|s|$. $\square$

NB: The converse of Lemma (2.4) does not hold, consider the example $a_n = (-1)^n$ where $|a_n|$ is 1 for all n and therefore converges to 1 while a_n does not converge to any element in $\mathcal{F}$.

Definition 2.5. (Q_r) For $r \in \mathbb{Q}$ let $Q_r = \{q \in \mathbb{Q} \mid q \leq r\}$

Theorem 2.5. *$\mathcal{F}$ is Cauchy complete with respect to the order topology*

Proof. Let (s_n) be a Cauchy sequence in $\mathcal{F}$ and let $\epsilon > 0$ in $\mathcal{F}$. Then there is some $N(\epsilon)$ such that

$$|s_n - s_m| < \epsilon \text{ for all } n, m \geq N(\epsilon).$$

Then for all $r \in \mathbb{Q}$ there is some $N_r \in \mathbb{N}$ such that

$$|s_n - s_m| < d^{r+1} \text{ for all } n, m \geq N_r. \tag{2.1}$$

So we have

$$s_n[q] = s_{N_r}[q] \text{ for all } n \geq N_r \text{ and for all } q \leq r. \tag{2.2}$$

By Equation (2.1) we may assume that

$$N_{r_1} \leq N_{r_2} \text{ if } r_1 < r_2. \tag{2.3}$$

Define $s\colon \mathbb{Q} \to \mathbb{R}$ by $s[q] = s_{N_q}[q]$. We will show that $supp(s)$ is well-ordered and hence s is in $\mathcal{F}$. For every $r \in \mathbb{Q}$ we have

$$supp(s) \cap Q_r = supp(s_{N_r}) \cap Q_r \tag{2.4}$$

Let $A \subset supp(s)$ be non-empty. A is non-empty so there is at least one $r \in A$. Consider that $A \cap Q_r \subset supp(s) \cap Q_r = supp(s_{N_r}) \cap Q_r$. Because (s_n) is a sequence in $\mathcal{F}$ we have $supp(s_{N_r})$ is well-ordered which implies $supp(s_{N_r}) \cap Q_r$ is well-ordered and therefore $A \cap Q_r$ is well-ordered by Lemma (1.1). Then $A \cap Q_r$ contains a minimum element, call it a, and this a must also be a minimum element of A. So any non-empty $A \subset supp(s)$ contains a minimum element therefore $supp(s)$ is well-ordered and $s \in \mathcal{F}$. It remains to be shown that (s_n) converges to s.

Let $\epsilon > 0$ be given. Then there is some $r \in \mathbb{Q}$ such that $d^r < \epsilon$. Then

$$s_n[q] = s_{N_r}[q] = s[q] \text{ for all } n \geq N_r \text{ and for all } q \leq r.$$

Hence

$$|s_n - s| \ll d^r < \epsilon \text{ for all } n \geq N_r.$$

So (s_n) converges to s in $\mathcal{F}$. Therefore if (s_n) is Cauchy in the sense of Definition (2.3) it is convergent in the sense of Definition (2.2) and so $\mathcal{F}$ is Cauchy complete in the order topology. $\square$

Theorem 2.6. *(Convergence Criterion for sequences). Let (s_n) be a sequence in $\mathcal{F}$. Then (s_n) converges if and only if for all $r \in \mathbb{Q}$ there exists $N(r) \in \mathbb{N}$ such that*

$s_m =_r s_l$ for all $m, l \geq N(r)$.

Proof. Suppose (s_n) is a convergent sequence in $\mathcal{F}$ and let $r \in \mathbb{Q}$. We then have that (s_n) is Cauchy by Corollary (2.2.1) so there is some $N(r) \in \mathbb{N}$ such that $|s_m - s_l| < d^{r+1}$ for all $m, l \geq N(r)$. This implies $s_m =_r s_l$ for $m, l \geq N(r)$.

Suppose for all $r \in \mathbb{Q}$ there is some $N(r) \in \mathbb{N}$ such that $s_m =_r s_l$ for all $m, l \geq N(r)$. Let $\epsilon > 0$ in $\mathcal{F}$ be given; then there is some $r \in \mathbb{Q}$ such that $d^r \ll \epsilon$. There is some $N(r)$ such that $s_m =_r s_l$ for all $m, l \geq N(r)$ and hence $|s_m - s_l| < \epsilon$ for $m, l \geq N(r)$. We therefore have (s_n) is Cauchy and hence it is convergent by Theorem (2.5). $\square$

Theorem 2.7. *If a sequence (s_n) is convergent then it is regular.*

Proof. Let (s_n) converge to s in $\mathcal{F}$ and let A be a non-empty subset of $\cup_{n=0}^{\infty} supp(s_n)$. There is at least one $r \in A$ and by Theorem (2.6) there is some N_r such that for $n, m \geq N_r$ we have $supp(s_n) \cap Q_r = supp(s_m) \cap Q_r = supp(s) \cap Q_r$ for all $n, m \geq N_r$. So

$$(\cup_{n=0}^{\infty} supp(s_n)) \cap Q_r = [\cup_{n=0}^{N_r - 1} (supp(s_n) \cap Q_r)] \cup [supp(s) \cap Q_r]$$

For all n we have $supp(s_n) \cap Q_r$ is well-ordered by Lemma (1.1) and $supp(s)$ is well-ordered because s is in $\mathcal{F}$ so $(\cup_{n=0}^{\infty} supp(s_n)) \cap Q_r$ is well-ordered by Lemma (1.1) thus $A \cap Q_r$ is well-ordered by Corollary (1.1.1). Let a be a minimum element of $A \cap Q_r$, which must exist as $A \cap Q_r$ is well-ordered. Then a must also be a minimum element of A and if it is in $A \cap Q_r$ then it must be in A. Therefore $A \subset \cup_{n=0}^{\infty} supp(s_n)$ contains a minimum element; thus $\cup_{n=0}^{\infty} supp(s_n)$ is well-ordered and hence (s_n) is regular. $\square$

Theorem 2.8. *Let (s_n) be a sequence in $\mathcal{F}$. Then (s_n) is Cauchy if and only if $(s_{n+1} - s_n)$ is a null sequence.*

Proof. Let (s_n) be a Cauchy sequence in $\mathcal{F}$, and let $\epsilon > 0$ in $\mathcal{F}$ be given. Then there exists $N \in \mathbb{N}$ such that $|s_m - s_l| < \epsilon$ for all $m, l \geq N$. In particular, $|s_{m+1} - s_m| < \epsilon$

for all $m \geq N$. Hence, $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

Now assume that $(s_n - s_{n+1})$ is a null sequence. Let $\epsilon > 0$ in $\mathcal{F}$, then there is some $N \in \mathbb{N}$ such that for $n \geq N$ we have $|s_{n+1} - s_n| < d\epsilon$. Let $k > l \geq N$ be given. We then have that

$$\begin{aligned}
|s_k - s_l| &= |s_k - s_{k-1} + s_{k-1} - s_{k-2} + \cdots + s_{l+1} - s_l| \\
&\leq |s_k - s_{k-1}| + |s_{k-1} - s_{k-2}| + \cdots + |s_{l+1} - s_l| \\
&< (k-l)d\epsilon < \epsilon
\end{aligned}$$

We know $(k-l)d\epsilon < \epsilon$ for any $k, l \in \mathbb{N}$ where $k > l$ because $d \ll 1$ so $(k-l)d < 1$ and hence $(k-l)d\epsilon < \epsilon$.

□

Corollary 2.8.1. *Let (s_n) be a sequence in $\mathcal{F}$. Then, (s_n) converges if and only if $(s_{n+1} - s_n)$ is a null sequence with respect to the order topology.*

Proof. This is a direct consequence of the Cauchy completeness of $\mathcal{F}$ and Theorem (2.8) □

Theorem 2.9. *(Fixed Point Theorem) Let q_M in $\mathbb{Q}$ be given and $M \subset \mathcal{F}$ be given by: $M = \{x \in \mathcal{F} \mid \lambda(x) \geq q_M\}$, and let $f\colon M \to \mathcal{F}$ be such that $f(M) \subset M$ and there is some $0 < q_f \ll 1$ in $\mathcal{F}$ such that for any $x, y \in M$ we have $|f(x) - f(y)| \leq q_f|x - y|$, then f admits a unique fixed point x^* in M.*

Proof. Let x_0 in M be fixed and define a sequence $(x_n) \subset M$ where $x_n = f(x_{n-1})$ for $n \geq 1$. Let $\epsilon > 0$ in $\mathcal{F}$ be given, then for some $N(\epsilon) \in \mathbb{N}$ we have $q_f^{N(\epsilon)} \cdot d^{q_M} \ll d\epsilon$.

For $n > N(\epsilon)$ we have

$$\begin{aligned}
|x_n - x_{n-1}| &= |f(x_{n-1}) - f(x_{n-2})| \\
&\leq q_f|x_{n-1} - x_{n-2}| \\
&\vdots \\
&\leq q_f^{n-1}|f(x_0) - x_0| \\
&\leq q_f^{N(\epsilon)}|f(x_0) - x_0| \\
&\leq q_f^{N(\epsilon)}d^{q_M} < d\epsilon
\end{aligned}$$

and hence $(x_n - x_{n-1})$ is a null sequence and therefore must converge to some x^* in $\mathcal{F}$ by Corollary (2.8.1).

Since $x^* = \lim_{n\to\infty} x_n$ and $x_n = f(x_{n-1})$, it follows that $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1})$. Then

$$\begin{aligned}
|f(x^*) - x^*| &= |f(x^*) - \lim_{n\to\infty} f(x_{n-1})| \\
&= |\lim_{n\to\infty} f(x^*) - \lim_{n\to\infty} f(x_{n-1})| \\
&= |\lim_{n\to\infty}[f(x^*) - f(x_{n-1})]| \\
&= \lim_{n\to\infty} |f(x^*) - f(x_n)| \\
&\leq \lim_{n\to\infty} q_f|x^* - x_{n-1}| \\
&= q_f|x^* - \lim_{n\to\infty} x_{n-1}| \\
&= q_f|x^* - x^*| = 0.
\end{aligned}$$

Thus $f(x^*) - x^* = 0$ and so x^* is a fixed point of f.

Let $\lambda(x) = Q$ and $d^{q_M} \gg \epsilon > 0$ in $\mathcal{F}$ be given. Because $\lim_{n\to\infty} x_n = x^*$ there is some

$N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$ we have $|x^* - x_n| < \epsilon \ll d^{q_M}$ hence $\lambda(x^* - x_n) > q_M$ for $n \geq N(\epsilon)$. Since (x_n) is a sequence in M we have $\lambda(x_{N(\epsilon)}) \geq q_M$ and hence for $n \geq N(\epsilon)$

$$\lambda(x^*) = \lambda(x^* - x_n + x_{N(\epsilon)}) \geq \min\{\lambda(x^* - x_n), \lambda(x_{N(\epsilon)})\} \geq q_M$$

therefore x^* is in M.

Let y^* be a fixed point of f, then

$$0 \leq |x^* - y^*| = |f(x^*) - f(y^*)| \leq q_f |x^* - y^*|$$

since $q_f < 1$ we have $|x^* - y^*| = 0$ and hence $y^* = x^*$. Thus f has a unique fixed point x^* in M.

$\square$

Theorem 2.10. *(Existence of multiplicative inverses of non-zero elements) For non-zero x in $\mathcal{F}$ there is some ξ in $\mathcal{F}$ such that $x \cdot \xi = 1$.*

Proof. Let x in $\mathcal{F}$ be non-zero and let $a = x[\lambda(x)]$. Then $x = a \cdot d^{\lambda(x)}(1+y)$ for some y such that $\lambda(y) > 0$. Furthermore $a \cdot d^{\lambda(x)}$ has a well defined inverse: $a^{-1} \cdot d^{-\lambda(x)}$. Thus non-zero elements $x = a \cdot d^{\lambda(x)}(1+y)$ have multiplicative inverses if and only if elements of the form $1+y$ where $\lambda(y) > 0$ have multiplicative inverses. So without loss of generality we may assume $x = 1+y$ where $\lambda(y) > 0$ and look for ξ such that $(1+y) \cdot \xi = 1$.

Consider the equation

$$\xi \cdot (1+y) = 1$$

where $\lambda(y) > 0$. Since $\lambda(1+y) = 0$ and $\lambda(1) = 0$ it follows, if such a ξ exists, that $\lambda(\xi) = 0$ by Theorem (1.5). Moreover, since $x[0] = 1[0] = 1$, we must have $\xi[0] = 1$.

Thus ξ must be of the form $1+z$ where $\lambda(z) > 0$.

We have $(1+y)\cdot(1+z) = 1$ if and only if $z = -y(1+z)$. Let $M = \{z \in \mathcal{F} \colon \lambda(z) > 0\}$ and $f\colon M \to \mathcal{F}$ be given by $f(z) = -y(1+z)$. We will show that f has a fixed point z^* in M which entails that $\lambda(z^*) > 0$, $z^* = -y(1+z^*)$, and $(1+y)\cdot(1+z^*) = 1$.

By Theorem (2.9) to show f has a fixed point in M we need to show that $f(M) \subset M$ and that there is some $0 < q_f \ll 1$ such that for any z_1, z_2 in M we have $|f(z_1) - f(z_2)| \leq q_f|z_1 - z_2|$.

For $z \in M$ we have $f(z) = -y(1+z)$ and $\lambda(y) > 0$. By Theorem (1.5) $\lambda(f(z)) = \lambda(-1) + \lambda(y) + \lambda(1+z)$ and since $\lambda(-1) = \lambda(1+z) = 0$ and $\lambda(y) > 0$ we have $\lambda(f(z)) > 0$ which gives $f(M) \subset M$.

Let $q_f = |2y|$, since $\lambda(y) > 0$ we have $0 < q_f \ll 1$. For any z_1, z_2 in M we have $|f(z_1) - f(z_2)| = |y(1+z_1) - y(1+z_2)| = |y||z_1 - z_2| < |2y||z_1 - z_2| = q_f|z_1 - z_2|$.

Thus by Theorem (2.9) there is a fixed point z^* for $f(z)$ in M such that $z^* = -y(1+z^*)$ or $(1+z^*)(1+y) = 1$. Therefore elements of $\mathcal{F}$ of the form $1+y$ where $\lambda(y) > 0$ have multiplicative inverses and hence non-zero elements of $\mathcal{F}$ have multiplicative inverses. $\square$

With the above result we can complete the proof that $(\mathcal{F}, +, \cdot, \geq)$ is a totally ordered field.

Therefore by [1] we have that $(\mathcal{F}, +, \cdot, \geq)$ is a totally ordered real closed field, because $\mathbb{Q}$ is a divisible group and $\mathbb{R}$ is Archimedean, and hence by [2] $\mathcal{F} + i\mathcal{F}$ is algebraically closed and satisfies the fundamental theorem of algebra.

Now we consider the convergence of infinite sums of elements of $\mathcal{F}$, with an eye to eventually study power series.

Definition 2.6. (Convergence of series) For a sequence (a_n) in $\mathcal{F}$ the series $\sum_{n=0}^{\infty} a_n$ converges to s in $\mathcal{F}$ if the sequence of partial sums $s_n = \sum_{j=0}^{n} a_j$ converges to s. That is for every $\epsilon > 0$ in $\mathcal{F}$ there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$ we have $|s_n - s| < \epsilon$.

Corollary 2.10.1. *If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two series converging to A and B respectively, then the series $\sum_{n=0}^{\infty}(a_n + b_n)$ converges to $A + B$.*

Proof. $(\mathcal{F}, \Delta)$ is a metric space. □

Corollary 2.10.2. *The infinite series $\sum_{n=0}^{\infty} a_n$ converges in $\mathcal{F}$ if and only if the sequence (a_n) is a null sequence in $\mathcal{F}$.*

Proof. This follows from Corollary (2.8.1) and Definition (2.6). □

NB: The formal power series described at the end of Section (1.3) can converge if and only if the enumeration of the support points of x has $\lim_{n\to\infty} q_n = \infty$. If $supp(x)$ is a well-ordered set with an accumulation point at some finite q then there is no enumeration where this is possible. As we noted, the said power series converges only in the case where the set of support points is left-finite.

Corollary 2.10.3. *The series $\sum_{n=0}^{\infty} a_n$ converges if and only if it converges absolutely, that is if and only if $\sum_{n=0}^{\infty} |a_n|$ converges.*

Proof. This is a direct consequence of Corollary (2.10.2) and the fact that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$. □

Corollary 2.10.4. *If a sequence (a_n) has the property $\lambda(a_n) > R$ for all n, and $\sum_{n=0}^{\infty} a_n = s$ then $\lambda(s) > R$.*

Proof. Let $s_n = \sum_{i=0}^{n} a_i$ and let $\epsilon > 0$ in $\mathcal{F}$ be given with $\lambda(\epsilon) = R + 1$. By the definition of convergence there is some $N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$ we have $|s - s_n| < \epsilon$. Since $|s - s_n| = |\sum_{i=n+1}^{\infty} a_i|$ we have $\lambda(\sum_{i=N(\epsilon)+1}^{\infty} a_i) \geq R + 1$. This implies

$$\lambda(s) = \lambda\left(\sum_{n=0}^{N(\epsilon)} a_n + \sum_{n=N(\epsilon)+1}^{\infty} a_n\right) \geq \min_{0 \leq n \leq N(\epsilon)} \{\lambda(a_n), R+1\}.$$

Since $\lambda(a_n) > R$ for all n and $R + 1 > R$ we must therefore have $\lambda(s) > R$. □

Corollary 2.10.5. *If the double sums $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converge to A_1 and A_2 respectively and for every $\epsilon > 0$ there is some $N(\epsilon)$ in $\mathbb{N}$ such that for $i, j \geq N(\epsilon)$ we have $|a_{ij}| < \epsilon$ then $A_1 = A_2$.*

Proof. Note that since both $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ are well defined it must be that $\sum_{i=1}^{\infty} a_{ij}$ converges (to some a_j) for all j and $\sum_{j=1}^{\infty} a_{ij}$ converges (to some b_i) for all i.

Define $x_n = \sum_{j=1}^{n} a_j$, $y_n = \sum_{i=1}^{n} b_i$, and $s_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$. We will show that $\lim_{n \to \infty} |x_n - y_n| = 0$ which gives $A_1 = A_2$.

Let $\epsilon > 0$ in $\mathcal{F}$ be given. Then there is some $N(\epsilon)$ in $\mathbb{N}$ such that $\lambda(a_{ij}) > 3 + \lambda(\epsilon)$ for all $i, j \geq N(\epsilon)$. Because $\sum_{j=N(\epsilon)+1}^{\infty} a_{ij}$ converges for all i and $\lambda(a_{ij}) \geq 3 + \lambda(\epsilon)$ for $i, j \geq N(\epsilon)$ we have $\lambda(\sum_{j=N(\epsilon)+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon)$ for all $i \geq N(\epsilon)$ by Corollary (2.10.4). For each $i < N(\epsilon)$ there is some $N(i, \epsilon)$ in $\mathbb{N}$ such that $\lambda(\sum_{j=N(i,\epsilon)+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon)$ because $\sum_{j=1}^{\infty} a_{ij}$ converges for all i. Let $N_x = \max\{N(i, \epsilon), N(\epsilon) \mid i < N(\epsilon)\}$ then for $n \geq N_x$ we have $\lambda(\sum_{j=n+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon)$ for all i. Thus for $n \geq N_x$ we have that $|x_n - s_n| = |\sum_{i=1}^{n} \sum_{j=n}^{\infty} a_{ij}| < d\epsilon$.

We can similarly find an N_y such that for $n \geq N_y$ we have $|y_n - s_n| < d\epsilon$. Let $N_0 = \max\{N_x, N_y\}$ then for $n \geq N_0$ we have $|x_n - y_n| \leq |x_n - s_n| + |y_n - s_n| < 2d\epsilon < \epsilon$.

Thus $\lim_{n\to\infty} |x_n - y_n| = 0$ so $A_1 = A_2$. $\square$

Theorem 2.11. *[5] Consider two convergent series $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. The series $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, converges to $A \cdot B$ in $\mathcal{F}$.*

Proof. First we show $\sum_{n=0}^{\infty} c_n$ converges in $\mathcal{F}$. By Corollary (2.8.1) we need only show that (c_n) is a null sequence. Because both (a_n) and (b_n) are null there is some R in $\mathcal{F}$ such that $|a_n| < R$ and $|b_n| < R$ for all n. Let $\epsilon > 0$ in $\mathcal{F}$ be given. There is some M in $\mathbb{N}$ such that $|a_m| < d\epsilon/R$ and $|b_m| < d\epsilon/R$ for all $m \geq M$. Let $N = 2M$ then for $n \geq N$ we have

$$\begin{aligned}
|c_n| &= |a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0| \\
&\leq |a_0 b_n| + |a_1 b_{n-1}| + \cdots + |a_{n-1} b_1| + |a_n b_0| \\
&= |a_0||b_n| + |a_1||b_{n-1}| + \cdots + |a_{n-1}||b_1| + |a_n||b_0| \\
&< R\frac{d\epsilon}{R} + R\frac{d\epsilon}{R} + \cdots + \frac{d\epsilon}{R}R + \frac{d\epsilon}{R}R \\
&= (n+1)d\epsilon < \epsilon.
\end{aligned}$$

So for all $\epsilon > 0$ in $\mathcal{F}$ we can find $N \in \mathbb{N}$ such that $|c_n| < \epsilon$ for all $n \geq N$. Hence $\lim_{n\to\infty} c_n = 0$ and thus $\sum_{n=0}^{\infty} c_n$ converges in $\mathcal{F}$.

Let $\sum_{n=0}^{\infty} c_n = C$, $G = \max\{\sum_{n=0}^{\infty} |a_n|, \sum_{n=0}^{\infty} |b_n|, \sum_{n=0}^{\infty} |c_n|, d^{-1}\}$ and let $\epsilon > 0$ in $\mathcal{F}$ be given. As $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and $\sum_{n=0}^{\infty} c_n = C$ then there are $N(A,\epsilon)$, $N(B,\epsilon)$, and $N(C,\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(A,\epsilon)$ we have $|\sum_{i=n}^{\infty} a_i| < G^{-1}d^2\epsilon$, $n \geq N(B,\epsilon)$ we have $|\sum_{i=n}^{\infty} b_i| < G^{-1}d^2\epsilon$, and $n \geq N(C,\epsilon)$ we have $|\sum_{i=n}^{\infty} c_i| < G^{-1}d^2\epsilon$ by Corollaries (2.10.4) and (2.10.2). Let $N_0 = \max\{N(A,\epsilon), N(B,\epsilon), N(C,\epsilon)\}$ then for $n \geq N_0$ we have $|\sum_{i=n}^{\infty} a_i| < G^{-1}d^2\epsilon$, $|\sum_{i=n}^{\infty} b_i - B| < G^{-1}d^2\epsilon$, and $|\sum_{i=n}^{\infty} c_i| < G^{-1}d^2\epsilon$. For finite n we have $\sum_{i=0}^{n} c_i = (\sum_{i=0}^{n} a_i) \cdot (\sum_{i=0}^{n} b_i)$. Therefore for $n \geq N_0$ we have

$$
\begin{aligned}
|C - A \cdot B| &= \left| \left(\sum_{i=0}^{n-1} c_i + \sum_{i=n}^{\infty} c_i \right) - \left(\sum_{i=0}^{n-1} a_i + \sum_{i=n}^{\infty} a_i \right) \cdot \left(\sum_{i=0}^{n-1} b_i + \sum_{i=n}^{\infty} b_i \right) \right| \\
&\leq \left| \left(\sum_{i=0}^{n-1} c_i \right) - \left(\sum_{i=0}^{n-1} a_i \right) \cdot \left(\sum_{i=0}^{n-1} b_i \right) \right| + \left| \left(\sum_{i=n}^{\infty} c_i \right) \right| \\
&+ \left| \left(\sum_{i=0}^{n-1} a_i \right) \cdot \left(\sum_{i=n}^{\infty} b_i \right) \right| + \left| \left(\sum_{i=n}^{\infty} a_i \right) \cdot \left(\sum_{i=0}^{n-1} b_i \right) \right| + \left| \left(\sum_{i=n}^{\infty} a_i \right) \cdot \left(\sum_{i=n}^{\infty} b_i \right) \right| \\
&< G^{-1} d^2 \epsilon + G G^{-1} d^2 \epsilon + G G^{-1} d^2 \epsilon + G G^{-1} d^2 \epsilon \\
&= (G^{-1} + 3) d^2 \epsilon < \epsilon
\end{aligned}
$$

Therefore for all $\epsilon > 0$ we have $|C - A \cdot B| < \epsilon$ and so $C = A \cdot B$ and $\sum_{n=0}^{\infty} c_n = A \cdot B$.

□

With these basic results on the convergence of sequences we shall now define power series and prove a few useful results to do with their convergence.

Definition 2.7. (Power Series) A series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where (a_n) is a sequence in $\mathcal{F}$, x_0 is a fixed point in $\mathcal{F}$, and x varies in $\mathcal{F}$, is called a power series.

For $a < b$ in $\mathcal{F}$ a function $f \colon [a, b] \to \mathcal{F}$ is said to be given by a power series on $[a, b]$ if there is a sequence (a_n) in $\mathcal{F}$ and an x_0 in $[a, b]$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Power series are a special kind of series and, as we shall see in Chapter 3, functions given by a power series have the smoothness properties of real power series; in particular they satisfy the intermediate value theorem, the extreme value theorem and the mean value theorem on any closed interval within their domain of convergence.

Lemma 2.12. *Let M be a well-ordered subset of $\mathbb{Q}$. Define*

$$M_{\Sigma,N} = \underbrace{M + \cdots + M}_{N\ times}$$

and

$$M_{\Sigma} = \bigcup_{n=1}^{\infty} M_{\Sigma,n}$$

Then M_{Σ} is well-ordered if and only if $min\{M\} \geq 0$

Proof. ($\Leftarrow$): Let $m_0 = min\{M\} \geq 0$. If $m_0 = 0$ then $M_{\Sigma} = (M \backslash \{0\})_{\Sigma} \cup \{0\}$ so without loss of generality we may assume $m_0 > 0$. Let $m_0 > 0$ and let $A \subset M_{\Sigma}$ be non-empty. Then there is some r in A. As $\mathbb{Q}$ is Archimedean there is some N_r in $\mathbb{N}$ such that $N_r \cdot m_0 > r$. Then

$$A \cap Q_r \subseteq \left(\bigcup_{n=1}^{N_r} M_{\Sigma,n} \right) \cap Q_r$$

By Theorem (1.3) $\cup_{n=1}^{N_r} M_{\Sigma,n}$ is well-ordered as it is the finite union of finite sums of well-ordered sets. Then by Theorem (1.3) we have $(\cup_{n=1}^{N_r} M_{\Sigma,n}) \cap Q_r$ is well-ordered so $A \cap Q_r$ contains a minimum element, call it a. This minimum element must also be a minimum element of A and contained in A therefore M_{Σ} is well-ordered.

($\Rightarrow$): Suppose $m_0 < 0$ then, as $n \cdot m_0$ is in M_{Σ} for all n in $\mathbb{N}$ it follows that M_{Σ} is not bounded below and hence M_{Σ} is not well-ordered. □

Corollary 2.12.1. *The sequence (x^n) is regular if and only if $\lambda(x) \geq 0$. Let (a_n) be a sequence in $\mathcal{F}$. Then the sequences $(a_n x^n)$ and $(\sum_{j=0}^{n} a_j x^j)$ are regular if (a_n) is regular and $\lambda(x) \geq 0$.*

Proof. Note that $\cup_{n=1}^{\infty} supp(x^n) = M_\Sigma$ where $M = supp(x)$ which is well-ordered. So by Lemma (2.12) we have that $\cup_{n=1}^{\infty} supp(x^n)$ is well-ordered if and only if $\min\{M\} \geq 0$, or $\lambda(x) \geq 0$. The second part is a consequence of Theorem (2.3) where the sequence of products of terms of two regular sequences is regular. □

Theorem 2.13. *(Convergence Criterion) Let (a_n) be a sequence in $\mathcal{F}$. Define*

$$\lambda_0 = -\liminf_{n\to\infty}\left(\frac{\lambda(a_n)}{n}\right) = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right)$$

in $\mathbb{R}$. Let x_0 in $\mathcal{F}$ be fixed and x in $\mathcal{F}$ be given. Then $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges if $\lambda(x-x_0) > \lambda_0$ and diverges if either $\lambda(x-x_0) < \lambda_0$ or $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n) > n\lambda_0$ for infinitely many n.

Proof. Suppose $\lambda(x-x_0) > \lambda_0$, then there is some $t \in \mathbb{Q}$ such that $t > 0$ and $\lambda(x-x_0) - t > \lambda_0$. Then there is some $N \in \mathbb{N}$ such that $\lambda(x-x_0) - t > \frac{-\lambda(a_n)}{n}$ for all $n \geq N$. So for $n \geq N$ we have $n\lambda(x-x_0) + \lambda(a_n) > nt$. Since $\lambda(a_n(x-x_0)^n) = \lambda(a_n) + n\lambda(x-x_0)$ we have for $n \geq N$ $\lambda(a_n(x-x_0)^n) > nt$ where $t > 0$. Thus $\lim_{n\to\infty} \lambda(a_n(x-x_0)^n) = \infty$ which gives $\lim_{n\to\infty} a_n(x-x_0)^n = 0$. Hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is convergent by Corollary (2.8.1).

Suppose $\lambda(x-x_0) < \lambda_0$. Then for all $N \in \mathbb{N}$ there is some $n > N$ such that $\lambda(x-x_0) < \frac{-\lambda(a_n)}{n}$ and hence $n\lambda(x-x_0)+\lambda(a_n) < 0$. This implies $\lim_{n\to\infty} \lambda(a_n(x-x_0)^n)$ may not exist and certainly isn't infinity. Thus $(a_n(x-x_0)^n)$ is not a null sequence and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is divergent by Corollary (2.8.1).

Suppose $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n) > n\lambda_0$ for infinitely many n. Then $-\lambda(a_n) > n\lambda(x-x_0)$ for infinitely many n. Therefore for any $N \in \mathbb{N}$ we can find an $n > N$ such that $\lambda(a_n) + n\lambda(x-x_0) < 0$. This entails that $(a_n(x-x_0)^n)$ is not a null sequence and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is divergent by Corollary (2.8.1). □

2.3 Calculus on $\mathcal{F}$

In this section we will define continuity and differentiability as in $\mathbb{R}$. We shall then motivate the interest in power series by showing that functions on $\mathcal{F}$ which are continuous or differentiable, due to the disconnectedness of our field, do not have the analytic properties of continuous or differentiable functions of real variables.

Definition 2.8. (Continuity at a point). Let $D \subset \mathcal{F}$ and let $f\colon D \to \mathcal{F}$. Then we say that f is continuous at x_0 in D if for every $\epsilon > 0$ in $\mathcal{F}$ there is some $\delta > 0$ in $\mathcal{F}$ such that for $x \in D$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$.

Definition 2.9. (Continuity on a set). Let $D \subset \mathcal{F}$ and $f\colon D \to \mathcal{F}$, then f is continuous on D if f is continuous at x_0 for all x_0 in D.

Lemma 2.14. *Let $D \subset \mathcal{F}$, a function $f\colon D \to \mathcal{F}$ is continuous at x_0 in D if and only if for every sequence (x_n) in D which converges to x_0 we have that $(f(x_n))$ converges to $f(x_0)$.*

Proof. Let $f\colon D \to \mathcal{F}$.

($\Rightarrow$): Suppose f is continuous at x_0 in D and suppose (x_n) in D converges to x_0. Let $\epsilon > 0$ in $\mathcal{F}$ be given, since f is continuous at x_0 there is some $\delta > 0$ such that for x in D with $|x_0 - x| < \delta$ we have $|f(x_0) - f(x)| < \epsilon$. The convergence of (x_n) gives us an N_δ in $\mathbb{N}$ such that for $n \geq N_\delta$ we have $|x_0 - x_n| < \delta$; and hence for $n \geq N_\delta$ we have $|f(x_0) - f(x_n)| < \epsilon$. Thus $(f(x_n))$ converges to $f(x_0)$.

($\Leftarrow$): Suppose $(f(x_n))$ converges to $f(x_0)$ for every (x_n) in D which converges to x_0. Let $\epsilon > 0$ in $\mathcal{F}$ be given and consider $A = f^{-1}(B_O(f(x_0), \epsilon))$. Suppose for all $\delta > 0$ we have $[B_O(x_0, \delta)\backslash A] \cap D$ is non-empty. We then have that $[B_O(x_0, d^n)\backslash A] \cap D$ is non-empty for all $n \in \mathbb{N}$. For each n select an element of $[B_O(x_0, d^n)\backslash A] \cap D$ and call it x_n. Then we have that $|x_0 - x_n| < d^n$ for all n so (x_n) converges to x_0, however

$|f(x_0) - f(x_n)| \geq \epsilon$ for all n because $x_n \notin A$ and hence $f(x_n) \notin B_O(f(x_0), \epsilon)$. This is a contradiction as we assumed the convergence of (x_n) to x_0 gives the convergence of $(f(x_n))$ to $f(x_0)$. So then there is some $\delta > 0$ such that $(B_O(x_0, \delta) \cap D) \subseteq (A \cap D)$ therefore f is continuous at x_0.

□

Theorem 2.15. *Let $D \subset \mathcal{F}$; let α in $\mathcal{F}$, and let $f \colon D \to \mathcal{F}$, and $g \colon D \to \mathcal{F}$ be continuous at x_0 in D. Then $f + \alpha g$ and $f \cdot g$ are continuous at x_0.*

Proof. Let (x_n) be a sequence in D that converges to x_0. Then we know by Lemma (2.14) that $(f(x_n))$ and $(g(x_n))$ converge to $f(x_0)$ and $g(x_0)$ respectively. We also have that for $n \geq 1$

$$(f + \alpha g)(x_n) = f(x_n) + \alpha g(x_n)$$

so by Corollary (2.2.1) we have $(f + \alpha g)(x_n)$ converges to $f(x_0) + \alpha g(x_0) = (f + \alpha g)(x_0)$. So $f + \alpha g$ is continuous at x_0 by Lemma (2.14).

We also have for $n \geq 1$

$$\begin{aligned} f(x_n) \cdot g(x_n) &= (f(x_n) - f(x_0)) \cdot (g(x_n) - g(x_0)) \\ &- g(x_0) \cdot (f(x_0) - f(x_n)) \\ &- f(x_0) \cdot (g(x_0) - g(x_n)) \\ &+ f(x_0) \cdot g(x_0) \end{aligned}$$

Thus $(f \cdot g)(x_n) = f(x_n) \cdot g(x_n)$ converges to $f(x_0) \cdot g(x_0)$, as both $f(x_n) - f(x_0)$ and $g(x_n) - g(x_0)$ converge to 0, so $f \cdot g$ is continuous at x_0. □

Corollary 2.15.1. *Let $D \subset \mathcal{F}$, $f, g \colon D \to \mathcal{F}$ be continuous on D, and α in $\mathcal{F}$ be given. Then $f + \alpha g$ and $f \cdot g$ are continuous on D.*

Theorem 2.16. *Let $D_f, D_g \subset \mathcal{F}$. Let $f\colon D_f \to \mathcal{F}$ and $g\colon D_g \to \mathcal{F}$ be such that $f(D_f) \subset D_g$, f is continuous at x_0 in D_f, and g is continuous at $f(x_0)$ in D_g. Then $(g \circ f)\colon D_f \to \mathcal{F}$ where $(g \circ f)(x) = g(f(x))$ is continuous at x_0.*

Proof. Suppose (x_n) is a sequence in D_f which converges to x_0. Then $(f(x_n))$ is a sequence in D_g which converges to $f(x_0)$ by the continuity of f at x_0. Given $(f(x_n))$ is a sequence in D_g such that $(f(x_n))$ converges to $f(x_0)$ we have that $(g(f(x_n)))$ is a sequence in $\mathcal{F}$ which must converge to $g(f(x_0))$ by the continuity of g at $f(x_0)$. Thus for any (x_n) in D_f converging to x_0 we have that $(g(f(x_n))) = ((g \circ f)(x_n))$ converges to $g(f(x_0)) = (g \circ f)(x_0)$. Therefore $g \circ f$ is continuous at x_0 by Lemma(2.14). □

Corollary 2.16.1. *Let $D_f, D_g \subset \mathcal{F}$. Let $f\colon D_f \to \mathcal{F}$ and $g\colon D_g \to \mathcal{F}$ be such that $f(D_f) \subset D_g$, f is continuous on D_f, and g is continuous on D_g. Then $g \circ f$ is continuous on D_f.*

Definition 2.10. (Differentiability at a point). Let $D \subset \mathcal{F}$ be open. A function $f\colon D \to \mathcal{F}$ is differentiable at a point x_0 in D if there is some G in $\mathcal{F}$ such that for every $\epsilon > 0$ in $\mathcal{F}$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ we have $\left|\dfrac{f(x) - f(x_0)}{x - x_0} - G\right| < \epsilon$; if this is the case we call G the derivative of f at x_0 and write $G = f'(x_0)$.

Definition 2.11. (Differentiability on a set). Let $D \subset \mathcal{F}$ be open. A function $f\colon D \to \mathcal{F}$ is differentiable on D if it is differentiable at x_0 for all x_0 in D.

Again, like in $\mathbb{R}$, the following results about differentiation hold:

Corollary 2.16.2. *(Differentiation rules)[3] For f,g, differentiable functions on sets D_f and D_g respectively*

- $(cf)'(x) = cf'(x)$
- $(f + g)'(x) = f'(x) + g'(x)$

- $(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x)$

- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ *where both are well defined.*

- $(f \circ g)'(x) = (f' \circ g)(x)g'(x)$

With the above definitions of continuity and differentiability we shall be able to provide examples of how continuity or differentiability of a function on $\mathcal{F}$ are not sufficient to guarantee any of the calculus theorems hold.

Example 1: Let $f_1 \colon [0,1] \to \mathcal{F}$ be given by

$$f_1(x) = \begin{cases} d^{-1} & \text{if } 0 \leq x < d \\ d^{-1/\lambda(x)} & \text{if } d \leq x \ll 1 \\ 1 & \text{if } x \sim 1 \end{cases}$$

We claim f_1 is continuous on $[0,1]$ but f_1 is not bounded on $[0,1]$.

Proof. Let $\epsilon > 0$ in $\mathcal{F}$ be given and let x in $[0,1]$. For $x = 0$ let $\delta = \frac{1}{2}d$, then $f_1(x)$ is constant on $[0, \delta)$ so $f_1(x)$ is continuous at $x = 0$. For $x \neq 0$ and $x \neq d$ there is q in $\mathbb{Q}$ such that $q > \lambda(x)$, let $\delta = d^q$. For all y in $(x - \delta, x + \delta) \cap [0,1]$ we have $\lambda(y) = \lambda(x)$ and thus $f_1(x)$ is constant, and therefore continuous, on $(x - \delta, x + \delta) \cap [0,1]$. For $x = d$ let $\delta = d^2$. For x in $(d, d + d^2)$ $\lambda(x) = \lambda(d) = 1$ and so $f_1(x) = d^{-1}$, for x in $(d - d^2, d)$ we have $x < d$ and so $f_1(x) = d^{-1}$ and $f_1(d) = d^{-1}$. Therefore $f_1(x)$ is constant (and therefore continuous) on $(d - d^2, d + d^2)$, specifically $f_1(x)$ is continuous at $x = d$. Therefore $f_1(x)$ is continuous at all x in $[0,1]$ and hence it is continuous on $[0,1]$.

Recall for every q in $(0,1] \cap \mathbb{Q}$ there is some $d \leq x_q \ll 1$ such that $\lambda(x_q) = q$.

Let $M > 0$ in $\mathcal{F}$ be given with $\lambda(M) = q_M$. Then there is some q in $(0,1] \cap \mathbb{Q}$ such that $-1/q < q_M$. Then $f_1(x_q) = d^{-1/q}$ and so $\lambda(f_1(x_q)) < \lambda(M)$ which entails that $f_1(x_q) \gg M$. Therefore f_1 is not bounded on $[0,1]$. $\square$

Example 2: Let $f_2(x)\colon [-1,1] \to \mathcal{F}$ be given by

$$f_2(x) = x - x[0].$$

We claim this function is continuous on $[-1,1]$ and bounded, however it does not attain a maximum or minimum value on $[-1,1]$.

Proof. Let $x \neq 0$ in $[-1,1]$ be given. Then $x[0]$ is constant on $(x - d^{\lambda(x)+1}, x + d^{\lambda(x)+1})$ and so $f_2(x)$ is linear, and therefore continuous by taking $\delta = \frac{1}{2}\epsilon$, on $(x - d^{\lambda(x)+1}, x + d^{\lambda(x)+1})$ and hence at x. On $(-d, d)$ the function is given by x and is therefore continuous at 0. Therefore $f_2(x)$ is continuous on $[-1,1]$.

For all x in $[-1,1]$ we have $\lambda(x) \geq 0$ which entails that $\lambda(x - x[0]) > 0$ for all x in $[-1,1]$. Therefore $f_2([-1,1])$ is bounded above by any positive real number and below by any negative real number.

Suppose $f_2(x)$ attains some maximum at $x = x_M$ on $[-1,1]$. Note $f_2(x) = 0$ at any real number and $f_2(d) = d$, so x_M cannot be real. Then $(x_M - d^{\lambda(x_M)+1}, x_M + d^{\lambda(x_M)+1}) \subset [-1,1]$ and $f_2(x)$ is linear on $(x_M - d^{\lambda(x_M)+1}, x_M + d^{\lambda(x_M)+1})$. Therefore for $\delta < d^{\lambda(x_M)+1}$ we have $f_2(x_M + \delta) > f_2(x_M)$ which contradicts that x_M is a maximum, therefore $f_2(x)$ does not attain a maximum on $[-1,1]$. The exact same argument works for why $f_2(x)$ does not attain a minimum value. $\square$

Example 3: Let $f_3\colon [0,1] \to \mathcal{F}$ be given by

$$f_3(x) = \begin{cases} 1 & \text{if } x \sim 1 \\ 0 & \text{if } x \ll 1 \end{cases}$$

We claim this function is continuous on $[0,1]$ and differentiable on $(0,1)$ with $f'(x) = 0$ everywhere, but is non-constant and does not attain intermediate values between $f(0)$ and $f(1)$.

Proof. For each x in $[0,1]$ either $x \sim 1$ or $x \ll 1$. If $x \ll 1$ then $x+h \ll 1$ for h in $(-d,d)$ and so f_3 is constant, and therefore continuous and differentiable, on $(x-d,x+d)\cap[0,1]$ with derivative $f_3'=0$. If $x \sim 1$ then $x+h \sim 1$ for h in $(-d,d)$ and f_3 is constant on $(x-d,x+d)\cap[0,1]$ and again continuous and differentiable with derivative $f_3'=0$. Therefore f_3 has derivative 0 everywhere on $(0,1)$.

Trivially $f_3(x)$ is non-constant $(0 \neq 1)$ and does not attain any of the intermediate values between $f(0)$ and $f(1)$, for instance there is no x such that $f_3(x)=\frac{1}{2}$. $\square$

Chapter 3

Power Series

In the previous section it was shown that continuity or differentiability are not strong enough properties to guarantee the basic results of calculus. In this chapter we will show that power series satisfy the intermediate value theorem, the extreme value theorem, and the mean value theorem, among other properties.

3.1 Convergence

Proposition 3.1. *Let $[a, b] \subset \mathcal{F}$ and $f\colon [a, b] \to \mathcal{F}$ be given by a convergent power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If we define a new sequence (A_n) given by

$$A_n = ca_n$$

where c is some constant in $\mathcal{F}$ then

$$F(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n$$

is a convergent power series on $[a, b]$ and $F(x) = cf(x)$ for all x in $[a, b]$.

Proof. The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges on $[a, b]$ and so by Corollary (2.10.2)

$$\lim_{n \to \infty} a_n (x - x_0)^n = 0$$

on $[a, b]$. Then by Corollary (2.2.1)

$$\lim_{n \to \infty} A_n (x - x_0)^n = \lim_{n \to \infty} c a_n (x - x_0)^n = c \lim_{n \to \infty} a_n (x - x_0)^n = 0$$

on $[a, b]$. Hence the series

$$\sum_{n=0}^{\infty} A_n (x - x_0)^n$$

converges on $[a, b]$ by Corollary (2.10.2).

It follows that

$$\begin{aligned} cf(x) &= c \lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0) = \lim_{N \to \infty} c \sum_{n=0}^{N} a_n (x - x_0)^n \\ &= \lim_{N \to \infty} \sum_{n=0}^{N} c a_n (x - x_0)^n = \lim_{N \to \infty} \sum_{n=0}^{N} A_n (x - x_0)^n \\ &= \sum_{n=0}^{\infty} A_n (x - x_0)^n = F(x) \end{aligned}$$

on $[a, b]$. $\square$

Proposition 3.2. *Let $[a, b] \subset \mathcal{F}$, x_0 in $[a, b]$, and $f \colon [a, b] \to \mathcal{F}$ be a convergent* *power series on $[a,b]$ given by*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If we define a new sequence (A_n) and ξ_0 given by

$$A_n = a_n d^{n\lambda(b-a)} \text{ and } \xi_0 = d^{-\lambda(b-a)} x_0$$

then $F\colon [d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b] \to \mathcal{F}$ given by

$$F(\xi) = \sum_{n=0}^{\infty} A_n (\xi - \xi_0)^n$$

is a convergent power series on $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$; and for x in $[a,b]$ $f(x) = F(d^{-\lambda(b-a)}x)$. Moreover (A_n) is a regular sequence.

Proof. Let n in $\mathbb{N}$ and x in $[a,b]$. We then have

$$\begin{aligned} a_n(x - x_0)^n &= a_n((d^{\lambda(b-a)})(d^{-\lambda(b-a)})(x - x_0))^n \\ &= a_n(d^{\lambda(b-a)})^n (d^{-\lambda(b-a)}x - d^{-\lambda(b-a)}x_0)^n \\ &= A_n(\xi - \xi_0)^n \end{aligned}$$

where $\xi = d^{-\lambda(b-a)}x$ and is therefore in $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$.

Since $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges for all x in $[a,b]$ we have $\sum_{n=0}^{\infty} A_n(\xi - \xi_0)^n$ converges for all ξ in $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ and thus converges at some finite $(\xi - \xi_0)$ so, by Corollary (2.10.2), the sequence (A_n) must be null and it is therefore regular by Theorem (2.7). Thus $F(\xi)$ is given by a convergent power series on $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ and for x in $[a,b]$ $f(x) = F(\xi) = F(d^{-\lambda(b-a)}x)$. $\square$

Lemma 3.1. *(Uniform boundedness of power series) For a convergent power series*

$f(x)$ on $[a,b]$ given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

where x_0 is in $[a,b]$ we have $f(x)$ is uniformly bounded on $[a,b]$.

Proof. By Proposition (3.2) we can scale the argument of $f(x)$ so without loss of generality we shall assume that $\lambda(b-a)=0$. Then $\lambda(a_n(x-x_0)) \geq \lambda(a_n)$ on $[a,b]$. Let x_c be in $[a,b]$ such that $\lambda(x_c-x_0)=0$, then $\lambda(a_n(x_c-x_0)^n) = \lambda(a_n)$ so we have $\lim_{n\to\infty} a_n(x_c-x_0)^n = 0$ by Corollary (2.10.2) and therefore $\lim_{n\to\infty} a_n = 0$.

Given that (a_n) is a null sequence so is $(d^{-1}|a_n|)$ hence there must be some M in $\mathcal{F}$ such that $\sum_{n=0}^{\infty} |a_n|d^{-1} < M$. Since $0 \leq \lambda((x-x_0)^n)$ for all x in $[a,b]$ and $a_n \leq |a_n|$ for all n we have $M > \sum_{n=0}^{\infty} |a_n|d^{-1} > |\sum_{n=0}^{\infty} a_n(x-x_0)^n|$ and hence $|f(x)| < M$ on $[a,b]$.

Indeed because $(|a_n|)$ is a null sequence there is a N_F in $\mathbb{N}$ such that for $n \geq N_F$ we have $\lambda(|a_n|) > 0$ and so $i(f) = \min\{\lambda(|a_n|) \mid n \leq N_F\}$ exists and $i(f) = \min\{\lambda(a_n)\}$. Due to the scaling of the argument of f we have $\lambda(f(x)) \geq i(f)$ on $[a,b]$ and since $\sum_{n=0}^{N_F} a_n(x-x_0)^n$ is a polynomial with at least one coefficient a_i with $\lambda(a_i) = i(f)$ we have $\lambda(\sum_{n=0}^{N_F} a_n(x-x_0)^n) = \lambda(f(x)) = i(f)$ for at least one finite x_c in $[a,b]$. Therefore $i(f) = \min\{\lambda(f(x) \mid x \in [a,b]\}$.

□

Definition 3.1. (Index of a power series $i(f)$) For $[a,b] \subset \mathcal{F}$, x_0 in $[a,b]$, and $f\colon [a,b] \to \mathcal{F}$ a convergent power series given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

on $[a,b]$ we define the index of f

$$i(f) = \min\{\lambda(f(x)) \mid x \in [a,b]\}$$

.

Corollary 3.1.1. *Let* $[a, b] \subset \mathcal{F}$ *where* $\lambda(b-a) = 0$*,* x_0 *in* $[a, b]$*, and* $f\colon [a, b] \to \mathcal{F}$ *be a convergent power series on* $[a, b]$ *given by*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

then if $\lambda(b-a) \leq 0$ *we have*

$$\lambda(a_n) \geq i(f)$$

.

Proof. Let x_c in $[a, b]$ be such that $\lambda(x_c - x_0) = 0$, then $\lambda(a_n(x_c - x_0)^n) = \lambda(a_n)$ for all n. Since $\sum_{n=0}^{\infty} a_n(x_c - x_0)^n$ converges to $f(x_c)$ we have, by Corollary (2.10.2), $\lim_{n\to\infty} a_n(x_c - x_0)^n = 0$; so $\lim_{n\to\infty} a_n = 0$ hence (a_n) is regular by Theorem (2.7). Thus $m = \min\{\cup_{n\in\mathbb{N}} supp(a_n)\}$ is well defined. Trivially $\lambda(a_n) \geq m$; and, as in Lemma (3.1), one can define a polynomial whose coefficients have $\lambda(a_n) = m$ which must attain a value A at some x_A in $[a, b]$ where $\lambda(A) = m$. Then, as in Lemma (3.1) $i(f) = m$ and hence $\lambda(a_n) \geq i(f)$. □

Definition 3.2. For j, k, N in $\mathbb{N}$ we define

$$S_{k,j,N} = \left\{ (n_1, \ldots, n_k) \in \{1, \ldots, N\}^k \mid \sum_{i=1}^{k} n_i = j \right\}$$

and

$$S_{k,j} = \left\{ (n_1, \ldots, n_k) \in \mathbb{N}^k \mid \sum_{i=1}^{k} n_i = j \right\}$$

Lemma 3.2. *Let* $[a, b] \subset \mathcal{F}$ *be given,* x_0 *in* $[a, b]$*, and* $f\colon [a, b] \to \mathcal{F}$ *be given by a* *convergent power series*

$$f(x) = \sum_{n=1}^{\infty} a_n (x - x_0)^n$$

on $[a, b]$*. Then for a fixed* $k \geq 2$ *in* $\mathbb{N}$ *the function* $(f(x))^k$ *is also given by a convergent power series on* $[a, b]$*, that is there is a sequence* (c_n) *such that*

$$(f(x))^k = \sum_{j=k}^{\infty} c_j (x - x_0)^j$$

for x *in* $[a, b]$*.*

Proof. Since by Theorem (3.2) one can scale the argument of a convergent power series to be finite we shall, without loss of generality, assume $[a, b]$ is an inverval of finite length with $\lambda((x - x_0)) \geq 0$ on $[a, b]$. Thus (a_n) is a null sequence as $\lim_{n \to \infty} a_n (x - x_0)^n = 0$ for finite $(x - x_0)$ by Corollary (2.10.2). The sequence (a_n) is also regular by Theorem (2.7).

Let $k \geq 2$ be fixed and consider for some finite N the product

$$\underbrace{\left(\sum_{n=1}^{N} a_n (x - x_0)^n \right) \cdots \left(\sum_{n=1}^{N} a_n (x - x_0)^n \right)}_{k \text{ times}}.$$

This product has finitely many terms and can be rearranged and written as

$$\sum_{j=k}^{Nk} \alpha_{k,j} (x - x_0)^j.$$

where the sequence $(\alpha_{k,j})$ is defined as

$$\alpha_{k,j} = \sum_{(n_1, \ldots, n_k) \in S_{k,j,N}} \prod_{i=1}^{k} a_{n_i}. \tag{3.1}$$

Likewise $\left(\sum_{n=1}^{N+1} a_n(x-x_0)^n\right)^k$ can be rewritten as

$$\sum_{j=k}^{kN+k} \beta_{k,j}(x-x_0)^j$$

where

$$\beta_{k,j} = \sum_{(n_1,\ldots,n_k)\in S_{k,j,N+1}} \prod_{i=1}^{k} a_{n_i}.$$

Note however that, for $j < N+k$, we have $S_{k,j,N} = S_{k,j,N+1}$. Thus for $j < N+k$ we have $\alpha_{k,j} = \beta_{k,j}$ which gives

$$\sum_{j=k}^{N+k-1} \alpha_{k,j}(x-x_0)^j = \sum_{j=k}^{N+k-1} \beta_{k,j}(x-x_0)^j.$$

Thus for any (k,j) pair where $j \geq k$ there is some finite $N(k,j) = (j-k)+1$ such that for $m \geq N(k,j)$ we have

$$\left(\sum_{n=1}^{N(k,j)} a_n(x-x_0)^n\right)^k \text{ and } \left(\sum_{n=1}^{m} a_n(x-x_0)^n\right)^k$$

contribute the same $(x-x_0)^j$ term for all $j \leq N(k,j)$.

Then define for each j

$$\begin{aligned} c_j &= \sum_{(n_1,\ldots,n_k)\in S_{k,j,N(k,j)}} \prod_{i=1}^{k} a_{n_i} \\ &= \sum_{(n_1,\ldots,n_k)\in S_{k,j}} \prod_{i=1}^{k} a_{n_i}. \end{aligned}$$

We intend to show that

$$\sum_{j=k}^{\infty} c_j (x - x_0)^j = \left(\sum_{n=1}^{\infty} a_n (x - x_0)^n \right)^k$$

on $[a, b]$.

Since (a_n) is a null sequence it must be that there is some N_F such that for $n > N_F$ we have $\lambda(a_n) > 0$. Let

$$M_0 = \max\{d^{-1}|a_n| \mid n \leq N_F\} \text{ and } M = \max\{M_0, 2\}.$$

Let $1 > \epsilon > 0$ in $\mathcal{F}$ be given. Since (a_n) is a null sequence there must be some $N_1(\epsilon)$ such that for $n > N_1(\epsilon)$

$$\lambda(a_n) > 3 + \lambda(\epsilon) - k\lambda(M) \tag{3.2}$$

and some $N_2(\epsilon)$ such that for $n > N_2(\epsilon)$

$$\lambda\left(\sum_{N_2(\epsilon)+1}^{\infty} |a_n| \right) > 2 + \lambda(\epsilon) - k\lambda(M). \tag{3.3}$$

By Equation (3.3) we have for $l \geq N_2(\epsilon)$

$$
\begin{aligned}
&\left| \left(\sum_{n=1}^{\infty} a_n (x-x_0)^n \right)^k - \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^k \right| \\
&= \left| \left(\sum_{n=1}^{l} a_n (x-x_0)^n + \sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right)^k - \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^k \right| \\
&= \left| \sum_{j=0}^{k} \binom{k}{j} \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^{k-j} \left(\sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right)^j - \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^k \right| \\
&= \left| \sum_{j=1}^{k} \binom{k}{j} \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^{k-j} \left(\sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right)^j \right| \\
&\leq \sum_{j=1}^{k} \left| \binom{k}{j} \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^{k-j} \left(\sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right)^j \right| \\
&\leq \sum_{j=1}^{k} \left| \binom{k}{j} \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^{k-j} \right| \left| \left(\sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right)^j \right| \\
&\leq \sum_{j=1}^{k} \left| \binom{k}{j} \left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^{k-j} \right| \left(\left| \sum_{n=l+1}^{\infty} a_n (x-x_0)^n \right| \right)^j \\
&\leq \sum_{j=1}^{k} (M)^{k-j} (d^2 \epsilon M^{-k})^j \\
&\leq \sum_{j=1}^{k} M^{-1} d^2 \epsilon \\
&= k M^{-1} d^2 \epsilon < d\epsilon.
\end{aligned}
$$

For any finite l one can write

$$\left(\sum_{n=1}^{l} a_n (x-x_0)^n \right)^k = \sum_{j=k}^{kl} \alpha_{k,j} (x-x_0)^j$$

where $\alpha_{k,j}$ is the same as in Equation (3.1). Each $\alpha_{k,j}$ is a sum of products of k terms. In each product there must be at least one term whose index is greater than or equal to $\lfloor \frac{j}{k} \rfloor$. Since each term of the product is bounded above by M by Equation (3.2) we have if $l > kN_1(\epsilon)$

$$
\begin{aligned}
|\alpha_{k,l}| &= \left| \sum_{(n_1,\ldots,n_k)\in S_{k,l,N}} \prod_{i=1}^{k} a_{n_i} \right| \\
&\leq N a_{n_I} M^{k-1} \\
&< N d^3 \epsilon M^{-k} M^{k-1} \\
&= N d^3 \epsilon M^{-1} \\
&< d^2 \epsilon
\end{aligned}
$$

where N is the cardinality of $S_{k,j,l}$ (therefore finite) and n_I is one of the members of each list in $S_{k,j,l}$ that has $\lfloor \frac{n_I}{k} \rfloor > N_1(\epsilon)$.

Now let $J = \max\{kN_1(\epsilon), N_2(\epsilon), k\}$ and x in $[a, b]$. For $l > J$ we have

$$
\begin{aligned}
&\left| \sum_{j=k}^{l} c_j (x - x_0)^j - (f(x))^k \right| \\
&= \left| \sum_{j=k}^{l} c_j (x - x_0)^j - (f(x))^k + \left(\sum_{n=1}^{l} a_n (x - x_0)^n \right)^k - \left(\sum_{n=1}^{l} a_n (x - x_0)^n \right)^k \right| \\
&\leq \left| \sum_{j=k}^{l} c_j (x - x_0)^j - \left(\sum_{n=1}^{l} a_n (x - x_0)^n \right)^k \right| + \left| \left(\sum_{n=1}^{\infty} a_n (x - x_0)^n \right)^k - \left(\sum_{n=1}^{l} a_n (x - x_0)^n \right)^k \right| \\
&\leq \left| \sum_{j=l+1}^{kl} \alpha_{k,j} (x - x_0)^j \right| + d\epsilon \\
&\leq (kl - l - 1) d^2 \epsilon + d\epsilon \\
&< d\epsilon + d\epsilon < \epsilon.
\end{aligned}
$$

Therefore $\lim\limits_{J\to\infty} \sum_{j=k}^{J} c_j (x - x_0)^j = (f(x))^k$ for x in $[a, b]$, which implies (c_j) converges to 0 by Corollary (2.10.2). Thus $(f(x))^k$ is given by a convergent power series

$\sum_{j=k}^{\infty} c_j(x-x_0)^j$ on $[a,b]$. $\square$

Theorem 3.3. *Let $[a,b] \subset \mathcal{F}$ and $[c,e] \subset \mathcal{F}$. Let $f\colon [a,b] \to \mathcal{F}$ and $g\colon [c,e] \to \mathcal{F}$ be convergent power series given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n(x-x_0)^n$$
$$g(y) = c_0 + \sum_{k=1}^{\infty} b_k(y-f(x_0))^k$$

where $f([a,b]) \subset [c,e]$.

Then there is a sequence (c_j) such that

$$c_0 + \sum_{j=1}^{\infty} c_j(x-x_0)^j$$

converges to $(g \circ f)(x)$ for all x in $[a,b]$.

Proof. By Theorem (3.2) and Theorem (3.1) the function $F\colon [d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b] \to \mathcal{F}$ given by

$$F(x) = d^{-\lambda(e-c)} f(d^{\lambda(b-a)}x)$$

is given by a convergent power series for x in $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ with

$$F([d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]) = d^{-\lambda(e-c)} f([a,b]) \subset [d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e].$$

Likewise $G\colon [d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e] \to \mathcal{F}$ where

$$G(y) = g(d^{\lambda(e-c)}y)$$

is given by convergent power series for y in $[d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e]$ and

$$G([d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e]) = g([c,e]).$$

Thus for x in $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ we have

$$\begin{aligned}(G \circ F)(x) &= G(d^{-\lambda(e-c)}f(d^{\lambda(b-a)}x)) \\ &= g(f(d^{\lambda(b-a)}x)) = (g \circ f)(d^{\lambda(b-a)}x).\end{aligned}$$

Therefore $(g \circ f)(x)$ is given by a convergent power series on $[a,b] = d^{\lambda(b-a)}[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ if and only if $(G \circ F)(x)$ is on $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$; so without loss of generality we may assume that $\lambda(b-a) = \lambda(e-c) = 0$.

Given that we assume $\lambda(b-a) = \lambda(e-c) = 0$ there must be some x_c in $[a,b]$ and y_c in $[c,e]$ such that $(x_c - x_0)$ and $(y_c - f(x_0))$ (y_c not necessarily $f(x_c)$) are finite. Both $\sum_{n=1}^{\infty} a_n(x_c - x_0)^n$ and $\sum_{k=1}^{\infty} b_k(y_c - f(x_0))^k$ converge and hence by Corollary (2.10.2) $\lim_{n\to\infty} a_n = \lim_{k\to\infty} b_k = 0$.

For x in $[a,b]$ we have

$$\begin{aligned}(g \circ f)(x) &= c_0 + \sum_{k=1}^{\infty} b_k(f(x) - f(x_0))^k \\ &= c_0 + \sum_{k=1}^{\infty} b_k \left(a_0 + \sum_{n=1}^{\infty} a_n(x-x_0)^n - f(x_0)\right)^k \\ &= c_0 + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n(x-x_0)^n\right)^k.\end{aligned}$$

Let

$$V_k(x) = b_k \left(\sum_{n=1}^{\infty} a_n(x-x_0)^n\right)^k$$

which by Lemma (3.2) can be rewritten as

$$V_k(x) = b_k \sum_{j=k}^{\infty} c_{k,j}(x-x_0)^j$$

where

$$c_{k,j} = b_k \sum_{(n_1,\ldots,n_k)\in S_{k,j}} \prod_{i=1}^{k} a_{n_i}.$$

Since $S_{k,j}$ is empty for $k > j$ we can define a sequence (c_j) given by, for $j \geq 1$,

$$c_j = \sum_{k=1}^{j} c_{k,j}. \tag{3.4}$$

We wish to show that $\lambda(a_n) \geq 0$ for $n \geq 1$. By Corollary (3.1.1) we know $\lambda(a_n) \geq i(f)$, so if $i(f) \geq 0$ we are done (note that $\lambda(e-c) = 0$ does not guarantee this, for instance if $e = d^{-1}+1$ and $c = d^{-1}-1$ then $\lambda(e-c) = 0$ but for all y in $[c,e]$ we have $\lambda(y) < 0$), however if $i(f) < 0$ Corollary (3.1.1) is not enough to ensure $\lambda(a_n) \geq 0$ for $n \geq 1$. Suppose $i(f) < 0$ and for some $\Lambda \subset \mathbb{N}$ we have $\lambda(a_n) < 0$ for n in Λ. Because $\lim\limits_{n\to\infty} a_n = 0$ we know Λ must be finite and so we can define a polynomial and x_A, as in Lemma (3.1), such that

$$P(x) = \sum_{n\in\Lambda} a_n(x-x_0)^n$$

$\lambda(x_A - x_0) = 0$ and

$$\lambda(P(x_A)) = \min\{\lambda(a_n) \mid n \in \Lambda\} = I.$$

Because $[a,b]$ is finite in length there must be some x_α, for each $0 < \alpha < 1$ in $\mathbb{R}$, in $[a,b]$ such that

$$(x_\alpha - x_0) = \alpha(x_A - x_0).$$

Then

$$P(x_\alpha)[I] = \sum_{n\in\Lambda} \alpha^n (a_n(x_A - x_0)^n)[I]$$

which is a real polynomial in α and therefore can only be equal to $P(x_A)[I]$ at finitely many α and can only be equal to zero at finitely many α. Let $x_B = x_\alpha$ for some α such that $P(x_\alpha)[I] \neq P(x_A)[I]$ and $P(x_\alpha)[I] \neq 0$. Then

$$\lambda(P(x_A) - P(x_B)) = I < 0$$

and, as in Lemma (3.1),

$$\lambda(f(x_A) - f(x_B)) = \lambda(P(x_A) - P(x_B)) < 0$$

. Let $B = \max\{f(x_A), f(x_B)\}$ and $A = \min\{f(x_A), f(x_B)\}$, then $B - A > e - c$ however we need B and A in $[c, e]$ as $f([a, b]) \subset [c, e]$, this is a contradiction. Therefore if $i(f) < 0$ we have $\lambda(a_n) \geq 0$ for $n \geq 1$ and if $i(f) \geq 0$ then $\lambda(a_n) \geq 0$ for $n \geq 1$ as well. Thus for any $\Lambda \subset \mathbb{N}$

$$\lambda\left(\prod_{n\in\Lambda} a_n\right) \geq \lambda(a_j) \tag{3.5}$$

for any j in Λ. Corollary (3.1.1) also ensures that

$$M = \max\{|b_k| \mid k \in \mathbb{N}\}$$

is well defined.

Let $\epsilon > 0$ in $\mathcal{F}$ be given. Since (a_n) is a null sequence there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$

$$\lambda(a_n) > 2 + \lambda(\epsilon) - \lambda(M).$$

There is also a $K(\epsilon)$ in $\mathbb{N}$ such that for $k > K(\epsilon)$

$$\lambda(b_k) > 2 + \lambda(\epsilon).$$

Let $J = \max\{K^2(\epsilon), K(\epsilon) \cdot N(\epsilon)\}$ then for $j > J$ we have

$$\begin{aligned} c_j &= \sum_{k=1}^{j} c_{k,j} \\ &= \sum_{k=1}^{K(\epsilon)} c_{k,j} + \sum_{k=K(\epsilon)+1}^{j} c_{k,j}. \end{aligned}$$

Because $j > K(\epsilon) \cdot N(\epsilon)$ we have for $k < K(\epsilon)$ that $\lfloor \frac{j}{k} \rfloor > N(\epsilon)$. Each $(n_1, \ldots, n_k)$, in any $S_{k,j}$ with $j > J$ and $k < K(\epsilon)$, has at least one component N where $N \geq \lfloor \frac{j}{k} \rfloor$. Hence we have when $j > J$ and $k < K(\epsilon)$ for said N $\lambda(a_N) > 2 + \lambda(\epsilon) - \lambda(M)$. This ensures, for $k < K(\epsilon)$ and $j > J$,

$$\begin{aligned} \lambda(c_{k,j}) &= \lambda(b_k) + \lambda \left(\sum_{(n_1,\ldots,n_k) \in S_{k,j}} \prod_{i=1}^{k} a_{n_i} \right) \\ &> \lambda(M) - 1 + \lambda(a_N) \\ &> \lambda(M) - 1 + 2 + \lambda(\epsilon) - \lambda(M) \\ &> \lambda(\epsilon) + 1 \end{aligned}$$

by Equation (3.5) and the fact that $|S_{k,j}|$ is finite for any finite k, j pair.

Equation (3.5) and the finitness of $|S_{k,j}|$ also gives us $\lambda(c_{k,j}) \geq \lambda(b_k) - 1$ for any k, j pair and therefore for those $c_{k,j}$ with $k > K(\epsilon)$ and $j > J$.

Thus for $j > J$

$$
\begin{aligned}
|c_j| &= \left| \sum_{k=1}^{j} c_{k,j} \right| \\
&= \left| \sum_{k=1}^{K(\epsilon)} c_{k,j} + \sum_{k=K(\epsilon)+1}^{j} c_{k,j} \right| \\
&\leq \left| \sum_{k=1}^{K(\epsilon)} c_{k,j} \right| + \left| \sum_{k=K(\epsilon)+1}^{j} c_{k,j} \right| \qquad < \sum_{k=1}^{K(\epsilon)} d\epsilon + \sum_{k=K(\epsilon)+1}^{j} d\epsilon \\
&= K(\epsilon)d\epsilon + (j - K(\epsilon))d\epsilon = jd\epsilon < \epsilon
\end{aligned}
$$

Therefore $\lim\limits_{j\to\infty} c_j = 0$ and $\sum_{j=1}^{\infty} c_j(x - x_0)^j$ converges for $(x - x_0)$ which are at most finite, by Corollary (2.10.2), and so $\sum_{j=1}^{\infty} c_j(x - x_0)^j$ converges on $[a, b]$.

If we once again let $\epsilon > 0$, then due to the fact that (b_k) is a null sequence there is some $K(\epsilon)$ such that for $k > K(\epsilon)$

$$\lambda(b_k) > \lambda(\epsilon) + 1.$$

Thus if $k, j > K(\epsilon)$ we have

$$
\begin{aligned}
\lambda(c_{k,j}) &= \lambda(b_k) + \lambda\left(\sum_{(n_1,\ldots,n_k)\in S_{k,j}} \prod_{i=1}^{k} a_{n_i} \right) \\
&\geq \lambda(b_k) \\
&> \lambda(\epsilon) + 1.
\end{aligned}
$$

Then by Corollary (2.10.5) we have for x in $[a,b]$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x_0)) + \sum_{k=1}^{\infty} V_k(x) \\
&= g(f(x_0)) + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_{k,j}(x - x_0)^j \\
&= g(f(x_0)) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{k,j}(x - x_0)^j \\
&= g(f(x_0)) + \sum_{j=1}^{\infty} c_j(x - x_0)^j.
\end{aligned}
$$

So (c_j), where $c_0 = g(f(x_0))$ and c_n is as in Equation (3.4), is a sequence such that $(g \circ f)(x) = c_0 + \sum_{j=1}^{\infty} c_j(x - x_0)^j$ on $[a,b]$. Therefore $(g \circ f)(x)$ is given by a convergent power series on $[a,b]$.

□

3.2 Analytical Properties

Theorem 3.4. *(Derivative of power series). If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is a convergent power series on $[a,b]$ where x_0 is in $[a,b]$ then for x_c in (a,b) f is differentiable at x_c and*

$$
f'(x_c) = \sum_{n=1}^{\infty} n a_n (x_c - x_0)^{n-1}
$$

where the power series converges on (a,b).

Proof. Let x_c in (a,b). Let $\epsilon > 0$ in $\mathcal{F}$ be given. Then, since $\sum_{n=1}^{\infty} a_n(x_c - x_0)^n$ converges, there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n > N(\epsilon)$

$$
|a_n(x_c - x_0)^n| < d\epsilon |x_c - x_0|.
$$

Hence for $n > N(\epsilon)$

$$\begin{aligned}|na_n(x_c - x_0)^{n-1}| &= \frac{n}{|x_c - x_0|}|a_n(x_c - x_0)^n| \\ &< \frac{n}{|x_c - x_0|} d\epsilon |x_c - x_0| \\ &< \epsilon.\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1}$ converges by Corollary (2.10.2). Let $G = \sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1}$ and consider

$$\begin{aligned}
& \left| \frac{\sum_{n=0}^{\infty} a_n(x - x_0)^n - \sum_{n=0}^{\infty} a_n(x_c - x_0)^n}{x - x_c} - G \right| \\
= & \left| \frac{\sum_{n=1}^{\infty} a_n((x - x_0)^n - (x_c - x_0)^n)}{x - x_c} - G \right| \\
= & \left| \frac{\sum_{n=1}^{\infty} a_n(x - x_c) \sum_{i=0}^{n-1}(x_c - x_0)^i (x - x_0)^{n-(i+1)}}{x - x_c} - G \right| \\
= & \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x_c - x_0)^i (x - x_0)^{n-(i+1)} - \sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1} \right| \\
= & \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} \left((x_c - x_0)^i (x - x_0)^{n-(i+1)} - (x_c - x_0)^{n-1} \right) \right| \\
= & \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x_c - x_0)^i \left((x - x_0)^{n-(i+1)} - (x_c - x_0)^{n-(i+1)} \right) \right| \\
= & \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-2} (x_c - x_0)^i (x - x_c) \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)} \right| \\
= & |x - x_c| \left| \sum_{n=0}^{\infty} a_n \sum_{i=0}^{n-2} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)} \right|.
\end{aligned}$$

Note that

$$\begin{aligned}\lim_{x\to x_c}\sum_{n=1}^{\infty}a_n\sum_{i=0}^{n-1}(x-x_0)^i\sum_{j=0}^{n-(i+2)}(x_c-x_0)^j(x-x_0)^{n-(i+2+j)}\\ =\sum_{n=1}^{\infty}a_n\left(\sum_{i=0}^{n-1}(n-(i+1))\right)(x_c-x_0)^{n-2}\end{aligned}$$

and that

$$\lambda\left(\sum_{i=0}^{n-1}(n-(i+1))\right)=0$$

for all n.

It follows that

$$\lambda\left(a_n\left(\sum_{i=0}^{n-1}n-(i+1)\right)(x_c-x_0)^{(n-2)}\right)=\lambda(a_n(x_c-x_0)^n)-2\lambda((x_c-x_0)).$$

Thus

$$\lim_{n\to\infty}a_n\left(\sum_{i=0}^{n-1}(n-(i+1))\right)(x_c-x_0)^{(n-2)}=0$$

because

$$\lim_{n\to\infty}na_n(x_c-x_0)^{n-1}=0$$

and hence

$$\lim_{x\to x_c}\sum_{n=1}^{\infty}a_n\sum_{i=0}^{n-1}(x-x_0)^i\sum_{j=0}^{n-(i+2)}(x_c-x_0)^j(x-x_0)^{n-(i+2+j)}=L$$

exists.

Therefore

$$
\begin{aligned}
&\lim_{x\to x_c}\left|\frac{f(x)-f(x_c)}{x-x_c}-G\right| \\
&=\lim_{x\to x_c}|x-x_c|\left|\sum_{n=1}^{\infty}a_n\sum_{i=0}^{n-1}(x-x_0)^i\sum_{j=0}^{n-(i+2)}(x_c-x_0)^j(x-x_0)^{n-(i+2+j)}\right| \\
&=\lim_{x\to x_c}|x-x_c|\lim_{x\to x_c}\left|\sum_{n=1}^{\infty}a_n\sum_{i=0}^{n-1}(x-x_0)^i\sum_{j=0}^{n-(i+2)}(x_c-x_0)^j(x-x_0)^{n-(i+2+j)}\right| \\
&=0\cdot L \\
&=0
\end{aligned}
$$

and hence f is differentiable at x_c with

$$f'(x_c)=G=\sum_{n=1}^{\infty}na_n(x_c-x_0)^{n-1}.$$

Since x_c was an arbitrary point in (a,b), it follows that $f(x)$ is differentiable on (a,b) with $f'(x)=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}$ for every x in (a,b). $\square$

Corollary 3.4.1. *If a function $f(x)$ is given by a convergent power series on $[a,b]$ it is infinitely often differentiable on (a,b). Moreover $f^{(k)}(x)=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n(x-x_0)^{n-k}$ for k in $\mathbb{N}$.*

Proof. This follows from the fact that if $f(x)$ is given by a power series on (a,b) then $f(x)$ is differentiable on (a,b) and $f'(x)$ is again given by a power series on (a,b) where the terms of the power series representing the derivative are given by the derivatives of the terms of the initial power series. $\square$

Corollary 3.4.2. *(Re-expansion of Power Series) If $f\colon[a,b]\to\mathcal{F}$ is given by a* *convergent power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on $[a, b]$ *where* x_0 *is in* $[a, b]$ *then the power series can be re-expanded around any point in [a,b]. Hence* $f(x)$ *can be rewritten as*

$$f(x) = \sum_{n=0}^{\infty} b_n (x - x_c)^n$$

where x_c *is any fixed point in* $[a, b]$ *and the power series converges for any* x *in* $[a, b]$.

Proof. Without loss of generality by Theorem (3.2) we may assume that $\lambda(b - a) = 0$ and so for x in $[a, b]$ $\lambda(x - x_0) \geq 0$. We may therefore assume, by Corollary (2.10.2), that $\lim_{n\to\infty} a_n = 0$. Let x_c be in $[a, b]$. For any fixed N we have

$$\begin{aligned} a_N (x - x_0)^N &= a_n((x - x_c) + (x_c - x_0))^N \\ &= a_N \sum_{j=0}^{N} \binom{N}{j} (x_c - x_0)^{N-j} (x - x_c)^j \\ &= \sum_{j=0}^{\infty} \alpha_{j,N} (x - x_c)^j \end{aligned}$$

where

$$\alpha_{j,N} = \begin{cases} \binom{N}{j} a_N (x_c - x_0)^{N-j} & \text{for } j \leq N \\ 0 & \text{for } j > N \end{cases}.$$

Thus $f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{j,n} (x - x_c)^j$. Let $\epsilon > 0$ in $\mathcal{F}$ be given. Since (a_n) is a null sequence there must be some $N(\epsilon)$ in $\mathbb{N}$ such that for $n > N(\epsilon)$ we have $\lambda(a_n) > \lambda(\epsilon) + 1$. Because $\lambda(\binom{j}{n}) = 0$, for any j, n pair where $\binom{j}{n}$ is well defined, and $\lambda(x_c - x_0) \geq 0$ we have $\lambda(\binom{j}{n}(x_c - x_0)^{n-j} a_n) \geq \lambda(a_n)$ for any j, n pair where $j \leq n$. Then if $j, n > N(\epsilon)$ we have $\lambda(\alpha_{j,n}) > \lambda(\epsilon) + 1$ so by Corollary (2.10.5) we have, for

x in $[a, b]$,

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{j,n}(x - x_c)^j \\
&= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{j,n}(x - x_c)^j \\
&= \sum_{j=0}^{\infty} c_j (x - x_c)^j
\end{aligned}
$$

where $c_j = \sum_{n=0}^{\infty} \binom{j}{n} a_n (x_c - x_0)^n$.

Therefore for any x_c in $[a, b]$ there is a sequence (c_j) such that if $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ on $[a, b]$ then we also have $f(x) = \sum_{j=0}^{\infty} c_j (x - x_c)^j$ on $[a, b]$.

□

Corollary 3.4.3. *Let $[a, b] \subset \mathcal{F}$ and $f \colon [a, b] \to \mathcal{F}$ be given by a convergent power series; let x_c in (a, b) and δ such that $x_c + \delta \in [a, b]$. Then we have*

$$
f(x_c + \delta) = f(x_c) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_c)}{n!} \delta^n.
$$

Proof. By Corollary (3.4.2) there is a sequence (a_n) such that $f(x)$ is given by $\sum_{n=0}^{\infty} a_n (x - x_c)^n$ on $[a, b]$. Then

$$
f(x_c + \delta) = \sum_{n=0}^{\infty} a_n ((x_c + \delta) - x_c)^n = \sum_{n=0}^{\infty} a_n \delta^n
$$

for δ such that $x_c + \delta \in [a, b]$. By Theorem (3.4), and Corollary (3.4.1), we know that $f(x)$ is infinitely often differentiable on (a, b) and $f^{(M)}(x) = \sum_{n=M}^{\infty} \frac{n!}{(n-M)!} a_n (x - x_c)^{n-M}$. Therefore

$$
f^{(M)}(x_c) = \sum_{n=M}^{\infty} \frac{n!}{(n-M)!} a_n (x_c - x_c)^{n-M} = M! a_M
$$

and hence $a_n = \frac{f^{(n)}(x_c)}{n!}$. □

Definition 3.3. (Quasi-Multiplicity) Recall that $\mathcal{F} + i\mathcal{F}$ is algebraically closed like

$\mathbb{C}$. Let $Q(x)$ be a polynomial over $\mathcal{F}+i\mathcal{F}$ of degree n, let $\xi_1, \ldots, \xi_n$ be its n roots in $\mathcal{F}+i\mathcal{F}$, let $j \in \{1, \ldots, n\}$ and let $l \leq n$ be given in $\mathbb{N}$. Then we say that ξ_j has quasi-multiplicity l as a root of $Q(x)$ if, for some $j_1 < j_2 < \ldots < j_{l-1}$ in $\{1, \ldots, n\} \backslash \{j\}$ we have that

$$\xi_j \approx \xi_k \text{ if and only if } k \in \{j, j_1, j_2, \ldots, j_{l-1}\}$$

Theorem 3.5. *(Intermediate Value Theorem) Let $a < b$ in $\mathcal{F}$ be given and let $g\colon [a,b] \to \mathcal{F}$ be given by a convergent power series. Then g assumes on $[a,b]$ every intermediate value between $g(a)$ and $g(b)$.*

Proof. Define $G\colon [-1,1] \to \mathcal{F}$ by

$$G(x) = g\left(\frac{(b-a)}{2}x + \frac{a+b}{2}\right).$$

By Theorem (3.3) $G(x)$ is again given by a convergent power series on $[-1,1]$. Let $i(G)$ be the index of $G(x)$, which exists by (3.1), and so $d^{-i(G)}G(x)$, by Theorem (3.1), is given by a power series with zero index on $[-1,1]$.

Since scaling does not affect intermediate values, $g(x)$ assumes all intermediate values on $[a,b]$ if and only if $d^{-i(G)}G(x)$ assumes all intermediate values on $[-1,1]$ because the map

$$h(x) = \left(\frac{b-a}{2}\right)x + \left(\frac{a+b}{2}\right)$$

is a bijection from $[-1,1]$ onto $[a,b]$.

If $d^{-i(G)}G(-1) = d^{-i(G)}G(1)$ we are done as there are no intermediate values, so assume $d^{-i(G)}G(-1) \neq d^{-i(G)}G(1)$. Without loss of generality let $G(1) > G(-1)$, if the opposite is true one can compose with $-x$ and the resulting function will remain a power series by Theorem (3.3).

Let S be between $d^{-i(G)}G(-1)$ and $d^{-i(G)}G(1)$ and define $f(x) = d^{-i(G)}G(x) - S$, then since $d^{-i(G)}G(-1) \neq d^{-i(G)}G(1)$ we have that one must be bigger than the other so $\frac{f(-1)}{f(1)} < 0$. Therefore g ssumes the value S on $[a, b]$ if and only if f assumes the value 0 in $[-1, 1]$.

So without loss of generality let we may assume $g = f\colon [-1, 1] \to \mathcal{F}$, where $i(f) = 0$ and $f(-1) < S = 0 < f(1)$.

Due to Corollary (3.4.2) we may expand $f(x)$ around $x = 0$ so there is a sequence (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

on $[-1, 1]$. Given $f(1) = \sum_{n=0}^{\infty} a_n$ we know that sum converges so there is some m_0 in $\mathbb{N}$ such that, if $A_0 = \min\{|f(1)|, |f(-1)|\}$, then $\lambda(a_n) > \lambda(A_0) + 1$ for $n > m_0$.

Let

$$P_0(x) = a_0 + \sum_{n=1}^{m_0} a_n x^n.$$
$$R_0(x) = \sum_{n>m_0} a_n x^n.$$

Then we have

$$P_0(x) = f(x) - R_0(x),$$

which implies

$$\begin{aligned} P_0(-1) &= f(-1) - R_0(-1) \approx f(-1). \\ P_0(1) &= f(1) - R_0(1) \approx f(1). \end{aligned}$$

Therefore $P_0(x)$ has a sign change on $[-1,1]$ and hence $P_0(x) = a_{m_0} \prod_{i=1}^{q}(x-r_i)^{n_i}$ has at least one root of odd quasi-multiplicity on $[-1,1]$. If $P_0(x)$ has no such root then $(x-r_i)^{n_i}$ must have the same sign for all x in $[-1,1]$ and therefore $P_0(x)$ has no sign change on $[-1,1]$, this is a contradiction therefore $P_0(x)$ must have such a root in $[-1,1]$.

We consider two cases:

Case I : All roots x_i of $P_0(x)$ with odd quasi-multiplicity in $[-1,1]$ have the property $\lambda(x_i) > 0$.

Since there are only finitely many roots of $P_0(x)$ on $[-1,1]$ there are only finitely many roots of odd quasi-multiplicity with $\lambda(x_i) > 0$ thus we can order them $\{x_1,\ldots,x_m\}$ from least to greatest.

We evaluate $f(x_i)$ for all $1 \le i \le m$. If $f(x_i) = 0$ for any i we are done, we may therefore assume $f(x_i) \neq 0$ for all $1 \le i \le m$. We consider two subcases:

Sub-case I.a: There is some k ,$1 \le k < m$, such that

$$\frac{f(x_k)}{f(x_{k+1})} < 0$$

In this case there is some sign change in $f(x)$ between x_k and x_{k+1}. We expand $f(x)$ around each. Consider

$$f(x_k + x) = f(x_k) + \sum_{n=1}^{\infty} a_n(x_k)x^n$$
$$f(x_{k+1} - x) = f(x_{k+1}) + \sum_{n=1}^{\infty} a_n(x_{k+1})(-x)^n$$

Where $a_n(x_i) = \frac{f^{(n)}(x_i)}{n!}$. Since $i(f) = 0$ by construction there must be some lowest $M \in \mathbb{N}$ such that $a_M(x_k) \sim 1$ (if there were no such M then $f(x_k + x)$ would not be finite for any value of x) and since $\lambda(x_{k+1} - x_k) \geq \min\{\lambda(x_{k+1}), \lambda(x_k)\} > 0$ it must be that $a_M(x_{k+1}) \sim 1$. Let $P_{1,\alpha}(x) = \sum_{n=1}^{M} a_n(x_k)x^n$ and let $\delta_1' = (d^{\frac{1}{2}}A_0)^{\frac{1}{M}}$.

We claim there must be some $j_\alpha \in \mathbb{N}$ such that $\lambda(P_{1,\alpha}(j_\alpha\delta_1')) \leq \frac{1}{2} + \lambda(A_0)$. We observe that $a_M(x_k)(j\delta_1')^M \sim d^{\frac{1}{2}}A_0$ for all j in $\mathbb{N}$. For $n < M$ we have

$$a_n(x_k)(j\delta_1')^n \left[\lambda(A_0) + \frac{1}{2}\right] = j^n(a_n(x_k)(\delta_1')^n) \left[\lambda(A_0) + \frac{1}{2}\right].$$

Gather those n such that $a_n(x_k)(j\delta_1')^n[\lambda(A_0) + \frac{1}{2}] \neq 0$ in a set, say Λ, then

$$P_{1,\alpha}(j\delta_1') \left[\lambda(A_0) + \frac{1}{2}\right] = \sum_{n\in\Lambda} a_n(x_k)(\delta_1')^n \left[\lambda(A_0) + \frac{1}{2}\right] j^n$$

which is a real polynomial of degree $|\Lambda| < M$ in j and so cannot be zero at more than $|\Lambda|$ points by the fundamental theorem of algebra. Thus there must be some $j_\alpha \in \mathbb{N}$ such that for $\delta_{1,\alpha} = j_\alpha\delta_1'$ we have $P_{1,\alpha}(\delta_{1,\alpha})[\lambda(A_0) + \frac{1}{2}] \neq 0$ and hence $\lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$.

Define $P_{1,\beta}(x) = \sum_{n=1}^{M} a_n(x_{k+1})(-x)^n$, and for the exact same reasons as above, there must be some j_β such that if $\delta_{1,\beta} = j_\beta \delta_1'$ then $\lambda(P_{1,\beta}(\delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$.

Since $\lambda(a_n(x_i)) \geq 0$ and both $\lambda(\delta_{1,\alpha}^n) > \lambda(A_0) + \frac{1}{2}$ and $\lambda(\delta_{1,\beta}^n) > \lambda(A_0) + \frac{1}{2}$ for $n > M$ we have, by Corollary (2.10.4), that

$$\lambda\left(\sum_{n=M+1}^{\infty} a_n(x_k)\delta_{1,\alpha}^n\right) > \lambda(A_0) + \frac{1}{2} \text{ and}$$
$$\lambda\left(\sum_{n=M+1}^{\infty} a_n(x_{k+1})\delta_{1,\beta}^n\right) > \lambda(A_0) + \frac{1}{2}.$$

We then have

$$\lambda(f(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$
$$\lambda(f(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

which implies

$$\lambda(P_0(x_k + \delta_{1,\alpha}) + R_0(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$
$$\lambda(P_0(x_{k+1} - \delta_{1,\beta}) + R_0(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

which gives

$$\lambda(P_0(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$
$$\lambda(P_0(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

as $|R_0(x)| \ll d^{\frac{1}{2}} A_0$ on $[-1, 1]$.

Then we show that at least one of the following must be true:

(i) $\lambda(x_{k+1} - x_k) \geq \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta})$.

(ii) $\dfrac{f(x_k + \delta_{1,\alpha})}{f(x_k)} < 0$.

(iii) $\dfrac{f(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} < 0$.

Suppose $\lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha})$ and

$$\frac{f(x_k + \delta_{1,\alpha})}{f(x_k)} > 0$$
$$\frac{f(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} > 0.$$

Then

$$\frac{P_0(x_k + \delta_{1,\alpha})}{f(x_k)} > 0 \text{ and } \frac{P_0(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} > 0.$$

It follows that

$$\frac{P_0(x_k + \delta_{1,\alpha})}{P_0(x_{k+1} - \delta_{1,\beta})} \frac{f(x_{k+1})}{f(x_k)} > 0$$

and hence

$$\frac{P_0(x_k + \delta_{1,\alpha})}{P_0(x_{k+1} - \delta_{1,\beta})} < 0$$

as

$$\frac{f(x_{k+1})}{f(x_k)} < 0$$

.

This implies a sign change in $P_0(x)$ on $(x_k + \delta_{1,\alpha}, x_{k+1} - \delta_{1,\beta})$ which is a subset of $[x_k, x_{k+1}]$ as $\lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta})$. There must therefore be some $c \in (x_k + \delta_{1,\alpha}, x_{k+1} - \delta_{1,\beta})$ such that $P_0(c) = 0$, but then x_k and x_{k+1} are not consecutive roots of $P_0(x)$ which is a contradiction.

Thus if $\lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha})$ either

(i) $\frac{f(x_k+\delta_{1,\alpha})}{f(x_k)} < 0$; in which case we let $X_0 = x_k$, and $\delta_1 = \delta_{1,\alpha}$ or

(ii) $\frac{f(x_{k+1}-\delta_{1,\beta})}{f(x_{k+1})} < 0$; in which case we let $X_0 = x_{k+1}$, and $\delta_1 = \delta_{1,\beta}$.

In either case $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

If $\lambda(x_{k+1} - x_k) \geq \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta})$ let $X_0 = x_k$ and $\delta_1 = x_{k+1} - x_k$. Then $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

Therefore in *Sub-case I.a*, where all roots x_i of $P_0(x)$ have $\lambda(x_i) > 0$ and for some $1 \leq k < m$ we have $\frac{f(x_k)}{f(x_{k+1})} < 0$, we can define an X_0 and δ_1 such that $P_0(X_0) = 0$, $\lambda(\delta_1) = \frac{\lambda(A_0)}{M} + \frac{1}{2M}$ and $f(X_0 + x)$ has a sign change on (δ_1, δ_1)

Sub-case I.b: For all roots of $P_0(x)$, which we call $\{x_1, \ldots, x_m\}$, we have $\frac{f(x_i)}{f(x_j)} > 0$. In this sub case, as in the previous one, we expand $f(x)$ around a root of $P_0(x)$, which we call X_0, to find a δ_1 such that $\lambda(\delta_1) \geq \lambda(A_0) + \frac{1}{2}$ and $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

If $f(x_i)$ has the same sign for all $1 \leq i \leq m$ then either $f(x_1) > 0$ or $f(x_m) < 0$. If $f(x_1) > 0$, then since we had $f(-1) < 0$ there is a sign change in $f(x)$ between -1 and x_1, and we let $X_0 = x_1$. If $f(x_m) < 0$, then since we had $f(1) > 0$ there is a sign change in $f(x)$ between x_m and 1, and we let $X_0 = x_m$.

Suppose $f(x_1) > 0$ and we let $X_0 = x_1$; there must be some $M \in \mathbb{N}$ such that $f^{(M)}(x_1)$ is the first finite derivative of f at x_1. There must also, as in *Sub-case*

I.a, be some $P_{1,\beta}(x) = \sum_{n=1}^{M} a_n(X_0)(-x)^n$ and δ_1 (where $\delta_1 \sim (d^{\frac{1}{2}}A_0)^{\frac{1}{M}}$) such that $\lambda(P_{1,\beta}(\delta_1)) \leq \lambda(A_0) + \frac{1}{2}$. As before we can be sure that there is some sign change in $f(x_1 + x)$ on $(x_1 - \delta_1, x_1)$ or there is some root of $P_0(x)$ on $(-1, x_1 - \delta_1)$, which is a contradiction as we took x_1 to be the smallest root of $P_0(x)$.

Likewise if $f(x_m) < 0$ and we let $X_0 = x_m$ there must be some $M \in \mathbb{N}$ such that $f^{(M)}(x_m)$ is the first finite derivative of f at x_m. As in *Sub case I.a* there must be a $P_{1,\alpha}(x) = \sum_{n=1}^{M} a_n(x_m)x^n$ and $\delta_1 \sim (d^{\frac{1}{2}}A_0)^{\frac{1}{M}}$ such that $f(x_m + x)$ has a sign change on $(x_m, x_m + \delta_1)$ or there is a root of $P_0(x)$ on $(x_m + \delta_1, 1)$ which is a contradiction as we took x_m to be the largest root of $P_0(x)$.

Therefore in *Sub-case I.b* where all roots x_i of $P_0(x)$ have the properties $\lambda(x_i) > 0$ and $\frac{f(x_i)}{f(x_j)} > 0$, we can define an X_0 and δ_1 such that $\lambda(f(X_0)) \geq \lambda(A_0) + \frac{1}{2}$, $\lambda(\delta_1) = \frac{\lambda(A_0)}{M} + \frac{1}{2M}$, and $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

Case II : At least one root of $P_0(x)$ with odd quasi-multiplicity is finite.

In this case there is a finite root, say X, with odd quasi-multiplicity m such that there is some finite Δ_0 such that

$$\frac{P_0(X + \Delta)}{P_0(X - \Delta)} < 0$$

for all finite $\Delta < \Delta_0$.

As previously there are finitely many $(m - 1)$ roots of $P_0(x)$ infinitely close to X, which can be arranged from least to greatest as $\{X_1, \ldots, X_m\}$ including X. If $f(X_i) = 0$ for any $1 \leq i \leq m$ we are done so assume $f(X_i) \neq 0$ for all i. As in *Case*

I we consider two sub cases

Sub-case II.a: There is some $1 \leq k < m$ such that $\frac{f(X_k)}{f(X_{k+1})} < 0$. Just as in *Sub case I.a* there must be

- Some $M \in \mathbb{N}$ such that $f^{(M)}(X_k) \sim f^{(M)}(X_{k+1})$ and $f^{(M)}(x_i)$ is the first finite derivative of f at X_i (both X_k and X_{k+1}).

- A polynomial $P_{1,\alpha}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_k)}{n!} x^n$.

- A polynomial $P_{1,\beta}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_{k+1})}{n!} (-x)^n$.

- $\delta_{1,\alpha} > 0$ such that $\delta_{1,\alpha} \sim (d^{\frac{1}{2}} A_0)^{\frac{1}{M}}$ and $\lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$.

- $\delta_{1,\beta} > 0$ such that $\delta_{1,\beta} \sim (d^{\frac{1}{2}} A_0)^{\frac{1}{M}}$ and $\lambda(P_{1,\beta}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$.

Just as in *Sub-case I.a*, one of the following must be true:

(i) $\lambda(X_{k+1} - X_k) \geq \lambda(\delta_{1,\alpha})$; in which case we let $X_0 = X_k$, and $\delta_1 = X_{k+1} - X_k$.

(ii) $\lambda(X_{k+1} - X_k) < \lambda(\delta_{1,\alpha})$, $\frac{f(X_k+\delta_{1,\alpha})}{f(X_k)} < 0$; in which case we let $X_0 = X_k$, and, $\delta_1 = \delta_{1,\alpha}$.

(iii) $\lambda(X_{k+1} - X_k) < \lambda(\delta_{1,\alpha})$, $\frac{f(X_{k+1}-\delta_{1,\beta})}{f(X_{k+1})} < 0$; in which case we let $X_0 = X_{k+1} - \delta_{1,\beta}$, $\delta_1 = \delta_{1,\beta}$.

Sub-case II.b: $\frac{f(X_i)}{f(X_j)} > 0$ for all i, j. Then for some $\Delta < \Delta_0$ one of the following must be true

(i) $\dfrac{f(X_1 - \Delta)}{f(X_1)} < 0$ or

(ii) $\dfrac{f(X_m + \Delta)}{f(X_m)} < 0$

as $f(X_1)$ and $f(X_m)$ must have the same sign but $f(X_1 - \Delta)$ and $f(X_m + \Delta)$ must have opposite signs as X is a root with odd quasi-multiplicity.

As in *Sub-case I.b*, (i) being true implies

a) There is some $M \in \mathbb{N}$ such that $\dfrac{f^{(M)}(X_1)}{M!}$ is the first finite coefficient of expansion of $f(x)$ about X_1.

b) There is some polynomial $P_{1,\beta}(x) = \sum_{n=1}^{M} \dfrac{f^{(n)}(X_1)}{n!}(-x)^n$.

c) There is a $\delta_{1,\beta} > 0$ such that $\delta_{1,\beta} \sim (d^{\frac{1}{2}} A_0)^{\frac{1}{M}}$ and $\lambda(P_{1,\beta}(\delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$.

d) $f(x)$ has a sign change on $(X_1 - \delta_{1,\beta}, X_1)$.

In this case let $X_0 = X_1$ and $\delta_1 = \min\{\delta_{1,\beta}, 1 + X_1\}$.

As in *Sub-case I.b*, (ii) being true implies

a) There is some $M \in \mathbb{N}$ such that $\dfrac{f^{(M)}(X_m)}{M!}$ is the first finite coefficient of expansion of $f(x)$ about X_m.

b) There is a polynomial $P_{1,\alpha}(x) = \sum_{n=1}^{M} \dfrac{f^{(n)}(X_m)}{n!} x^n$.

c) There is a $\delta_{1,\alpha} > 0$ such that $\delta_{1,\alpha} \sim (d^{\frac{1}{2}} A_0)^{\frac{1}{M}}$ and $\lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$.

d) $f(x)$ has a sign change on $(X_m, X_m + \delta_{1\alpha})$.

In this case let $X_0 = X_m$ and $\delta_1 = \min\{\delta_{1\alpha}, 1 - X_m\}$.

Then in all sub-cases we have that $\dfrac{f(X_0 + \delta_1)}{f(X_0)} < 0$ and $\lambda(f(X_0+\delta_1)) \leq \lambda(A_0)+\frac{1}{2}$.

Re-expand $f(x)$ around X_0

$$f(X_0 + x) = f(X_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n \tag{3.6}$$

Since there must be some finite x such that X_0+x is in $[-1, 1]$, then by Corollary (3.4.3) there must be some finite x at which the sum in Equation (3.6) converges. Therefore there exists m_1 in $\mathbb{N}$ such that for $X_0 + x$ in $[-1, 1]$ we have

$$\lambda\left(\sum_{n=m_1+1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n\right) > \lambda(f(X_0)) + 1$$

Define

$$\begin{aligned} P_1(x) &= f(X_0) + \sum_{n=1}^{m_1} \frac{f^{(n)}(X_0)}{n!} x^n \\ R_1(x) &= \sum_{n=m_1+1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n \end{aligned}$$

We've already established that $f(X_0+x)$ has a sign change on $(-\delta_1, \delta_1)$ and that $\lambda(f(X_0 + \delta_1)) < \lambda(f(X_0)) < \lambda(R_1(x))$ (or $\lambda(f(X_0 - \delta_1)) < \lambda(f(X_0)) < \lambda(R_1(x))$) and hence $P_1(x)$ has a sign change on $(-\delta_1, \delta_1)$.

We therefore have that $P_1(x)$ has a root of odd quasi-multiplicity on $(-\delta_1, \delta_1)$, as we established polynomial sign changes correspond to roots of odd quasi-multiplicity, so as in *Case I* of the previous expansion there must be some $x_1 \in (-\delta_1, \delta_1)$ and $\delta_2 > 0$ such that $\delta_2 \sim (d^{\frac{1}{2}} f(X_0))^{\frac{1}{M}}$ and $P_1(x_1) = 0$, $\lambda(P_1(x_1 + \delta_2)) \leq \lambda(f(X_0)) + \frac{1}{2}$, and $\frac{f(X_0+x_1+\delta_2)}{f(X_0+x_1)} < 0$.

NB: $\lambda(\delta_2) = \frac{\lambda(f(X_0))}{M} + \frac{1}{2M} \geq \frac{\lambda(A_0)}{M} + \frac{1}{2M} + \frac{1}{2M}$.

By induction, we obtain sequences (x_n) and (δ_n) such that

- $x_n \in (-\delta_{n-1}, \delta_{n-1})$

- $f(X_0 + x_1 + \ldots + x_{n-1} + x) = P_n(x) + R_n(x)$

- $\lambda(R_n(x)) > \lambda(d^{\frac{1}{2}} f(X_0 + x_1 + \ldots + x_{n-1}))$ for $X_0 + x_1 + \ldots + x_{n-1}$ in $[-1, 1]$

- $\dfrac{f(X_0 + x_1 + \ldots + x_{n-1} + \delta_n)}{f(X_0 + x_1 + \ldots + x_{n-1} - \delta_n)} < 0$

- $\lambda(\delta_n) \geq \dfrac{\lambda(A_0)}{M} + \dfrac{n}{2M}$

It follows that $\lim\limits_{n\to\infty} \lambda(\delta_n) = \infty$, and hence

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} x_n = 0$$

Then there is some X such that $X = X_0 + \sum_{i=1}^{\infty} x_i$. Since $X_0 + x_1 \in (-1, 1)$ we have $X \in (-1, 1)$ because $\lambda(x_1) < \lambda(\sum_{i=2}^{\infty} x_i)$ so $X \approx X_0 + x_1$ and hence $X - X_0 \approx x_1$.

We also have

$$\lim_{n\to\infty} f(X_0 + \ldots + x_n) = \lim_{n\to\infty} R_n(x_n) = 0$$

Therefore

$$0 = \lim_{n\to\infty} f(X_0 + \ldots + x_n) = f(\lim_{n\to\infty}(X_0 + \ldots + x_n)) = f(X).$$

So there is some $X \in (-1, 1)$ such that $f(X) = 0$.

$\square$

Lemma 3.6. *If $P(x)$ is a polynomial of degree $n \geq 2$ over $\mathbb{R}$, such that $P(r) = 0$, and $P^{(i)}(r) = 0$ for $0 \leq i \leq n-1$, then $P(x) = \alpha(x-r)^n$ for some α in $\mathbb{R}$.*

Proof. We will prove this statement via induction: let $n \geq 2$ and let S_n denote the statement "If $P(x)$ is a polynomial over $\mathbb{R}$ with $P(r) = 0$ and $P^{(i)}(r) = 0$ for $0 \leq i \leq n-1$ then $P(x) = \alpha(x-r)^n$ for some α in $\mathbb{R}$."

In the base case $n = 2$ $P(r) = P'(r) = 0$. As $P(r) = 0$ then $P(x) = \alpha(x-r)(x-r_2)$ so $P'(x) = \alpha[(x-r)+(x-r_2)] = \alpha[(2x-(r+r_2))]$. $P'(r) = 0$ so $2r = r + r_2$ then $r = r_2$ and hence $P(x) = 2(x-r)^2$.

Fix $k \geq 2$ and suppose S_k is true. Further, suppose $P(x)$ is a polynomial of degree $k+1$, $P(r) = 0$, and $P^{(i)}(r) = 0$ for $0 \leq i \leq k$. Then $G(x) = P'(x)$ is a polynomial of degree k such that $G(r) = 0$ and $G^{(i)}(r) = 0$ for $0 \leq i \leq k-1$. Therefore, by S_k, there is a β such that $P'(x) = G(x) = \beta(x-r)^k$. Because $P(r) = 0$ we have $P(x) = \alpha(x-r)H(x)$ where $H(x)$ is a polynomial of degree k. Then $P'(x) = \alpha[H(x) + (x-r)H'(x)] = \beta(x-r)^k$. Since polynomials over $\mathbb{R}$ have unique factorizations there is a constant such that $H(x) = c(x-r)^k$ and hence there is an α such that $P(x) = \alpha(x-r)^{k+1}$ and so S_{k+1} holds.

Therefore by the principle of mathematical induction S_n holds for every $n \geq 2$. $\square$

Corollary 3.6.1. *A convergent power series f on $[a, b]$ has at most finitely many zeros.*

Proof. As in the proof of the intermediate value theorem, without loss of generality we may assume the domain and range of f are scaled, so that $[a, b] = [-1, 1]$, and $i(f) = 0$. We can once again express $f(x)$ as a sum of a polynomial and a remainder term, $P_0(x)$ and $R_0(x)$, and for every y such that $f(y) = 0$ there is a root x of P_0 such that $\lambda(x - y) > 0$. At points x' such that $\lambda(x' - x_k) \leq 0$ for all k, where x_k are the roots of P_0, we would have $P_0(x')[i(f)] \neq 0$ so $f(x')$ could not be zero. There must be finitely many zeroes of P_0 by the fundamental theorem of algebra ($\mathcal{F} + i\mathcal{F}$ is algebraically closed).

Consider that near any zero x_k of P_0 we can re-expand f around x_k and there is some $M_k \in \mathbb{N}$ such that $a_{M_k}(x_k) = \frac{f^{(M_k)}(x_k)}{M_k!}$ is the first finite coefficient of the expansion of f about x_k. Then

$$f(x_k + \delta) = f(x_k) + \sum_{n=1}^{M_k} a_n(x_k)\delta^n + \sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n$$

where $a_n(x_k) = \frac{f^{(n)}(x_k)}{n!}$.

For δ such that $0 < |\delta| \ll 1$ we have $\lambda(a_{M_k}(x_k)\delta^{M_k}) < \lambda\left(\sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n\right)$ and so $f(x_k + \delta) = 0$ only if $\lambda\left(f(x_k) + \sum_{n=0}^{M_k} a_n(x_k)\delta^n\right) > \lambda(a_{M_k}(x_k)\delta^{M_k})$. In that case $f(x_k)[q] + \sum_{n=1}^{M_k}(a_n(x_k)\delta^n)[q] = 0$ for all $q \leq M_k\lambda(\delta)$.

We claim there are only finitely many δ such that $0 < |\delta| \ll 1$ and $f(x_k + \delta) = 0$. As in the proof of the intermediate value theorem for any such δ we have $a_{M_k}(x_k + \delta)$ must be the first finite term in $(a_n(x_k + \delta))$ as no coefficient of expansion can change by a finite amount over an infinitely small interval if all coefficients are at most finite to start with.

For $0 \le i < M_k$ where $a_i(x_k) \neq 0$ let $\alpha_{i,k} = \lambda(a_i(x_k))$. Let

$$U_0 = \min_{0 \le i < M_k} \left\{ \frac{\alpha_{i,k}}{M_k} \right\}$$

then for δ such that $0 < \lambda(\delta) < U_0$ we have $\lambda(a_{M_k}(x_k)\delta^{M_k}) < \lambda(a_n(x_k)\delta^n)$ for all n, hence $f(x_k + \delta) \approx a_{M_k}(x_k)\delta^{M_k} \neq 0$ so $f(x_k + \delta)$ is not equal to zero. We therefore require that $\lambda(\delta) \ge U_0$.

Let $A = \{q \in \mathbb{Q} \mid \exists i \neq j, 0 \le i \le M_k, 0 \le j \le M_k, q = \frac{\alpha_{i,k} - \alpha_{j,k}}{j - i}\}$ and let $Q' = \{q \in A \mid q \ge U_0\}$. If $\lambda(\delta) = q$ and $q \ge U_0$ and q is not in A then

$$\lambda\left(\sum_{n=0}^{M_k} a_n(x_k)\delta^n\right) = \min_{0 \le i \le M_k} \{\lambda(a_i(x_k)\delta^i)\} \le M_k q < \lambda\left(\sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n\right)$$

and hence $f(x_k + \delta) \neq 0$. Therefore if $\lambda(\delta) > 0$ and $f(x_k + \delta) = 0$ we know $\lambda(\delta)$ is in Q', so there are only finitely many possible values for $\lambda(\delta)$ because A is finite.

Let $q_c \in Q'$ and let $\delta = ad^{q_c} + \gamma$ where a is in $\mathbb{R} \setminus \{0\}$ and $\lambda(\gamma) > q_c$. Define $\beta_{q_c} = \min_{0 \le i \le M_k} \{\alpha_{i,k} + iq_c\}$ and $\Lambda = \{0 \le i \le M_k \mid \alpha_{i,k} + iq_c = \beta_{q_c}\}$. Then for i not in Λ we have $\lambda(a_i(x_k)\delta^i) > \beta_{q_c}$ and so $f(x_k + \delta)[\beta_{q_c}] = P(a) = \sum_{i \in \Lambda} a_i(x_k)[\alpha_{i,k}]a^i$ (for $n > M_k$ we have $\lambda(a_n(x_k)\delta^n) > \lambda(a_{M_k}(x_k)\delta^{M_k})$ so they need not be considered). This is a real polynomial in a of degree at most M_k and therefore equal to zero at no more than M_k values of a. Since $f^{(i)}(x_k + \delta) = \sum_{n=i}^{\infty} \frac{n!}{(n-i)!} a_n(x_k)\delta^{n-i}$ and $\lambda(\frac{n!}{(n-i)!} a_n(x_k)) = \lambda(a_n(x_k)$ wherever $(n-i)!$ is well defined we have

$$\begin{aligned} f^{(i)}(x_k + \delta)[\beta_{q_c} - iq_c] &= \left(\sum_{n=i}^{M_k} \frac{n!}{(n-i)!} a_n(x_k)[\beta_{q_c} - (n-i)q_c]a^{n-i}\right) \\ &= \frac{d^i}{da^i} P(a). \end{aligned}$$

Either $P^{(i)}(a) = 0$ for all $0 \leq i \leq M_k - 1$ and so there is only one possible value for a by Lemma (3.6) or there is some greatest $M_{k,1} \leq M_k - 1$ such that $P^{(M_{k,1})}(a) \neq 0$ for each root a of $P(a)$. For this $M_{k,1}$ if $\lambda(\gamma) > q_c$

$$
\begin{aligned}
& f^{(M_{k,1})}(x_k + ad^{q_c} + \gamma)[\beta_{q_c} - M_{k,1}\lambda(\gamma)] \\
= & \sum_{n=M_{k,1}}^{M_k} a_n(x_k + ad^{q_c})[\beta_{q_c} - (n - M_{k,1})\lambda(\gamma)](\gamma[\lambda(\gamma)])^{n-M_{k,1}} \\
\neq & 0
\end{aligned}
$$

as $a_{M_{k,1}}(x_k + ad^{q_c})[\beta_{q_c} - M_{k,1}q_c] \neq 0$ by assumption and $\lambda(a_j(x_k + ad^{q_c})) > \beta_{q_c} - nq_c > \beta_{q_c} - n\lambda(\gamma)$ for $M_k \geq j > M_{k,1}$. Moreover $\lambda(a_{M_{k,1}}(x_k + ad^{q_c})\gamma^{M_{k,1}}) < \lambda(a_j(x_k + adq_c)\gamma^j)$ for $j > M_{k,1}$ and γ with $\lambda(\gamma) > q_c$. Then $M_{k,1}$ takes on the roll of M_k in the next expansion and we consider the γ such that $\lambda\left(\sum_{n=0}^{M_{k,1}} a_n(x_k + ad^{q_c})\gamma^n\right) = \lambda\left(\sum_{n=M_{k,1}+1}^{\infty} a_n(x_k + ad^{q_c})\gamma^n\right)$.

As in the proof of the intermediate value theorem we can establish an interative process by which we generate a sequence (a_n) and $(q_{c,n})$ such that $\delta = \sum_{n=0}^{\infty} a_n d^{q_{c,n}}$ and $f(x + \delta) = 0$. Each a_i is a root of a polynomial of degree $M_{k,1}$. Because $M_{k,i+1} \leq M_{k,i}$ and $M_{k,i+1} = M_{k,i}$ only if there is only one option for a_i there is an N in $\mathbb{N}$ such that for $i > N$ there is only one option for a_i (either the degree of the leading polynomial becomes fixed or the iterative process terminates and $a_i = 0$ for all $i > N$). Therefore there are only finitely many δ such that $\lambda(\delta) > 0$ and $f(x_k + \delta) = 0$ and since there can only be finitely many roots x_k of the leading polynomial of f there are only finitely many roots y of f in total.

□

Theorem 3.7. *(Local Extrema)*

Let $[a, b] \subset \mathcal{F}$ be given and $f\colon [a, b] \to \mathcal{F}$ be a non-constant convergent power *series given by*

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} a_n (x - x_0)^n$$

where $x_0 \in (a, b)$. *Let* $m \in \mathbb{N}$ *be the smallest* m *such that* $f^{(m)}(x_0)$ *is not equal to zero. Then* f *has a local extremum at* x_0 *if and only if* m *is even. The extremum is a minimum if* $f^{(m)}(x_0) > 0$ *and a maximum if* $f^{(m)}(x_0) < 0$.

Proof. As in Theorems (3.2) and (3.1) as well as Corollary (3.4.3) we may assume without loss of generality that $[a, b] = [-1, 1]$, $x_0 = 0$, and $i(f) = 0$. Then $\lambda(x - x_0) \geq 0$ for all x in $[-1, 1]$, $\lim\limits_{n\to\infty} a_n = 0$, and $\lambda(a_n) \geq 0$ for all n in $\mathbb{N}$.

Given that (a_n) is a null sequence and $i(f) = 0$ there is some M in $\mathbb{N}$ such that a_M is the first finite term in the sequence (a_n). Then for x such that $\lambda(x - x_0) > 0$ we have

$$\lambda(a_M (x - x_0)^M) < \lambda(a_n (x - x_0)^n)$$

for $n > M$.

Let Λ be the set of $m \leq i \leq M$ such that $a_i \neq 0$, let $\alpha_i = \lambda(a_i)$. Then for i in Λ and $x = x_0 + d^q$ we have

$$\lambda(a_i (x - x_0)^i) = \alpha_i + iq.$$

For i in Λ let q_i be such that

$$\alpha_m + mq_i = \alpha_i + iq_i$$

and let $q_0 = \min\limits_{i\in\Lambda}\{q_i\}$ then for $q > q_0$ we have $\alpha_i + iq > \alpha_m + mq$ which entails that $\lambda(f(x_0 + d^q) - f(x_0)) = \lambda(a_m d^{mq})$ for such q. Then for $\delta > 0$ such that $\lambda(\delta) > q_0$ we have

$$(f(x_0 + \delta) - f(x_0))[\lambda(f(x_0 + \delta) - f(x_0))] = a_m[\lambda(a_m)]$$

and

$$(f(x_0-\delta)-f(x_0))[\lambda(f(x_0-\delta)-f(x_0))=a_m[\lambda(a_m)](-1)^m$$

by Corollary (3.4.3). Therefore if m is even we have that for $\delta>0$ such that $\lambda(\delta)>q_0$

$$f(x_0+\delta)-f(x_0)\approx f(x_0-\delta)-f(x_0)\approx a_m\delta^m$$

which entails $f(x_0)$ is a local maximum if $a_m<0$ and $f(x_0)$ is a local minimum if $a_m>0$. In the case where m is odd then $\frac{f(x_0+\delta)}{f(x_0-\delta)}<0$ for $\delta>0$ such that $\lambda(\delta)>q_0$ so $f(x_0)$ is neither a local maximum nor minimum.

□

Theorem 3.8. *(Extreme Value Theorem)*

Let $[a,b]\subset\mathcal{F}$, x_0 in $[a,b]$, and $f\colon[a,b]\to\mathcal{F}$ be given by a non-constant convergent power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

then f assumes a maximum and a minimum value on $[a,b]$.

Proof. As in Corollary 3.6.1 without loss of generality we can assume $i(f)=0$, $x_0=0$, and $[a,b]=[-1,1]$.

Since $\lim\limits_{n\to\infty}a_n=0$ and $i(f)=0$ there is some finite set Λ such that $\lambda(a_i)=0$ for i in Λ and $\lambda(a_i)>0$ for i not in Λ. Define $P(x)$ and $R(x)$ by

$$P(x)=\sum_{n\in\Lambda}a_nx^n$$
$$R(x)=\sum_{n\notin\Lambda}a_nx^n$$

then $f(x)=P(x)+R(x)$ and $f'(x)=P'(x)+R'(x)$. Let $\{X_1,\dots,X_K\}$ be the set of points where $P(x)$ attains its global maximum on $[-1,1]$, and recall that for all

X such that $\lambda(X - X_i) = 0$ for any $1 \le i \le K$ we have $P(X)[0] < P(X_i)[0]$ and so $f(X)[0] < f(X_i)[0]$. Therefore if $f(x)$ has a maximum value it must be attained at some x in $[-1, 1]$ such that $\lambda(x - X_i) > 0$ for some i.

Let $1 \le k \le K$ and expand $f(x)$ around X_k then

$$f(X_k + x) = f(X_k) + \sum_{n=1}^{\infty} a_n(X_k)x^n$$
$$f(X_k + x) - f(X_k) = \sum_{n=1}^{\infty} a_n(X_k)x^n$$

where $a_n(X_k) = \frac{f^{(n)}(X_k)}{n!}$. Given that $P(X_k)$ is a global and therefore local maximum for $P(x)$ we have that the smallest M_k such that $P^{(M_k)}(X_k) \neq 0$ is even and $P^{(M_k)}(X_k) < 0$. Since $f^{(M_k)}(X_k) = P^{(M_k)}(X_k) + R^{(M_k)}(X_k)$ we have $a_{M_k}(X_k)$ is the first finite coefficient of expansion of $f(x)$ about X_k, M_k is even, and $a_{M_k}(X_k) < 0$. For $1 \le i \le M_k$ such that $a_i(X_k) \neq 0$ let

$$\alpha_i = \lambda(a_i(X_k))$$
$$q_i = \frac{\alpha_i}{M_k - i}$$
$$q'_k = \frac{1}{2} \min_{1 \le i < M_k} \{q_i\}$$
$$\delta_k = d^{q'_k}$$

then for $q \le q'_k$ and $1 \le i < M_k$

$$q \le q'_k < q_i \text{ and hence it follows that}$$
$$q \le q'_k < \frac{\alpha_i}{M_k - i}. \text{ Therefore}$$
$$(M_k - i)q \le (M_k - i)q'_k < \alpha_i \text{ and hence}$$
$$M_k q - i(q - q'_k) \le M_k q'_k < \alpha_i + i q'_k.$$

Thus, $\alpha_i + iq'_k > M_k q'_k$, so $a_{M_k}(X_k)x^{M_k}$ dominates $f(X_k + x) - f(X_k)$ for $\delta_k < |x| < 1$ therefore $f(X_k + x) - f(X_k) < 0$ for $\delta_k < |x| < 1$.

We now consider the local maxima of $f(x)$ on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$. These coincide with the local maxima of $g(x) = f(X_k + x) - f(X_k)$ which is given by a convergent power series and therefore by Corollary (3.4.1) so is its derivative $g'(x)$ and so by Corollary (3.6.1) $g'(x) = 0$ at only finitely many points on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$, say $\{x_1, \ldots, x_l\}$. Let $G_k = \max\limits_{1 \le i \le l}\{g(x_i)\}$ we claim this must be the global maximum of $g(x)$ on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$. Suppose not, and there is some x_{A_1} in $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$ such that $g(x_A) = A > G_k$. Let $h(x) = g(x) - A$, $h(x)$ is given by a convergent power series on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$ and therefore can only have finitely many zeros $\{x_{A_1}, \ldots, x_{A_r}\}$ on that interval. We know $h(x_{A_1}) = 0$ and $h(x_i) < 0$ for all $1 \le i \le l$. If we expand $h(x)$ around x_{A_i} because $f'(x_{A_i}) \neq 0$ for any i we must be able to find, as in Theorem (3.7), a $\delta_i > 0$ small enough such that for x in $(-\delta_i, \delta_i)$ we have $h'(x_{A_i})x$ dominates $h(x_{A_i} + x)$.

We now consider two cases:

Case I: There is some j such that $x_j < x_{A_1} < x_{j+1} < x_{A_2}$. In this case we expand $h(x)$ around x_{A_1} and note that $h'(x_{A_1}) \neq 0$, therefore as in Theorem (3.7) we can find a δ small enough such that $h'(x_{A_1})x$ dominates $h(x_{A_1} + x)$ on $(-\delta, \delta)$. Then there must be a point p in $(-\delta, \delta)$ such that $h(x_{A_1} + p) > 0$ and $x_{A_1} + p$ is in $[x_j, x_{j+1}]$. But $h(x_j) < 0$ and $h(x_{j+1}) < 0$ then $h(x_{A_1} + p) > 0$ implies there is a zero of $h(x)$ in $[x_j, x_{j+1}]$ by Theorem (3.5) which is a contradiction of our assumption that $x_j < x_{A_1} < x_{j+1} < x_{A_2}$.

Case II: There is some j such that $x_j < x_{A_1} < x_{A_2} < x_{j+1}$. In this case we expand have $h(x_j) < 0$, $h(x_{A_1}) = h(x_{A_2}) = 0$, and $h(x_{j+1}) < 0$. As we can find small δ_1 and δ_2 around x_{A_1} and x_{A_2} such that both $h(x_{A_1} + x)$ and $h(x_{A_2} + x)$ change sign on $(-\delta_1, \delta_1)$ and $(-\delta_2, \delta_2)$ respectively the only way we can have $h(x)$ satisfy the intermediate value theorem is to have $\frac{h'(x_{A_1})}{h'(x_{A_2})} < 0$. This however implies that $g'(x) = h'(x)$ has a zero between x_{A_1} and x_{A_2} which contradicts our assumption.

Therefore if $f(X_k + x)$ attains a value greater than the greatest local maximum on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$ we arrive at a contradiction therefore the greatest local maximum is the global maximum on $[X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]$.

We repeat this process for all $1 \leq k \leq K$ and take $F = \max_{1 \leq k \leq K} \{G_k + f(X_k), f(-1), f(1)\}$ which is global maximum of F on $[-1, 1]$.

This process can be repeated on $-f(x)$, finding the maximum of $-f(x)$ gives the minimum of $f(x)$.

□

Corollary 3.8.1. *For a convergent power series f on $[a, b]$ there are some m and M in $\mathcal{F}$ such that $f([a, b]) = [m, M]$.*

Corollary 3.8.2. *(Rolle's Theorem) For f a convergent power series on $[a, b]$ if $f(a) = f(b)$ there is some $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. If $f(x)$ is constant on $[a, b]$ then $f'(x) = 0$ for all x in (a, b) and we are done. We therefore assume $f(x)$ is not constant. Let m and M be as above. Then $m < M$. Since $f(a) \neq f(b)$ either M or m is attained at some c in (a, b). Therefore f has a local extremum at c and so $f'(c) = 0$ by Theorem (3.7). □

Corollary 3.8.3. *(Mean Value Theorem) For f a convergent power series on $[a, b]$ there is a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let $g \colon [a, b] \to \mathcal{F}$ be given by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{3.7}$$

The function $g(x)$ is again a power series. As $g(a) = g(b) = 0$ then, by Corollary (3.8.2), there is $c \in (a, b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \tag{3.8}$$

and hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.9}$$

$\square$

Corollary 3.8.4. *For a convergent power series f on $[a, b]$ the following are true*

(i) If $f'(x) \neq 0$ for all $x \in (a, b)$ then either $f'(x) > 0$ on (a, b) and f is strictly increasing on $[a, b]$, or $f'(x) < 0$ on (a, b) and f is strictly decreasing on $[a, b]$.

(ii) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

Proof. (i) Assume $f'(x) \neq 0$ for all x in (a, b) then, by Theorem (3.5), if $f'(x) > 0 > f'(y)$ for some $x, y \in (a, b)$ then there must be some $c \in (x, y)$ such that $f'(c) = 0$ which is a contradiction so either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Assume $f'(x) > 0$ for all x in (a, b); and let $z > y$ in $[a, b]$ be given. Then by Corollary (3.8.3) there is some $c \in (y, z) \subset (a, b)$ such that

$$f'(c) = \frac{f(z) - f(y)}{z - y} \tag{3.10}$$

$f'(c) > 0$ by assumption, hence $f(z) - f(y) > 0$ and f is increasing on $[a, b]$. Similarly if we assume $f'(x) < 0$ for all x in (a, b) we have $f(z) - f(y) < 0$ for all $z > y$ in (a, b) and f is decreasing on $[a, b]$.

(ii) Assume $f'(x) = 0$ on (a, b) and let y in $(a, b]$. Then, by Corollary (3.8.3), there is some c in $(a, y) \subset (a, b)$ such that

$$f'(c) = \frac{f(y) - f(a)}{y - a} \tag{3.11}$$

Since $c \in (a, b)$ we must have $f'(c) = 0$ and hence $f(y) = f(a)$. Hence f is constant on $[a, b]$ $\square$

In conclusion, in this thesis we reviewed the algebraic and topological structure of $\mathcal{F}$ a non-Archimedean field extension of the real numbers; and we showed that the topology induced by the order is equal to an ultrametric topology. We showed that there are functions on $\mathcal{F}$ which are continuous and infinitely often differentiable that do not satisfy the familiar theorems of real calculus. We studied the convergence and analytical properties of power series on $\mathcal{F}$ and showed that they do have all the smoothness properties of real power series. In particular, they satisfy the intermediate value theorem, the extreme value theorem, the mean value theorem and power series on $\mathcal{F}$ will have a unique anti-derivative up to a constant within the family of power series. Because the family of power series was shown to be closed under addition, multiplication, composition, and uniform limit this makes power series an appropriate family of functions to develop a theory of integration on $\mathcal{F}$ in future.

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