On the Convergence and Analytical Properties of Power Series on non-Archimedean Field Extensions of the Real Numbers

by

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Abstract

In this thesis the analytic properties of power series over a class of non-Archimedean field extensions of the real numbers, a representative of which will be denoted by \( \mathcal{F} \), are investigated. In Chapter 1 we motivate the interest in said fields by recalling work done by K. Shamseddine and M. Berz [6]. We first review some properties of well-ordered subsets of the rational numbers which are used in the construction of such a field \( \mathcal{F} \). Then, we define operations \(+\) and \(\cdot\) which make \( \mathcal{F} \) a field. Then we define an order under which \( \mathcal{F} \) is non-Archimedean with infinitely small and infinitely large elements. We embed the real numbers as a subfield; and the embedding is compatible with the order. Then, in Chapter 2, we define an ultrametric on \( \mathcal{F} \) which induces the same topology as the order on the field. This topology will allow us to define continuity and differentiability of functions on \( \mathcal{F} \) which we shall show are insufficient conditions to ensure intermediate values, extreme values, et cetera. We shall study convergence of sequences and series and then study the analytical properties of power series, showing they have the same smoothness properties as real power series; in particular they satisfy the intermediate value theorem, the extreme value theorem and the mean value theorem on any closed interval within their domain of convergence.
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For my parents Joy and Bob
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Chapter 1

Introduction

1.1 Motivation and Outline

The long standing interest in analysis on the real numbers is easily understood. The arithmetic and algebraic properties of the reals allow us to model physical systems with great accuracy and perhaps because of this the reals satisfy a great deal of our intuition. One particularly intuitive feature of the real numbers is that any two reals are comparable; that is if $a$ and $b$ are real numbers where $|a| > |b| > 0$, there is some whole number $n$ such that $n|b| > |a|$, this is called the Archimedean property. To illustrate how intuitive this property is: imagine trying to measure the height of a stack of papers. Though each sheet is very thin with enough sheets one could make a stack taller than any given ruler. Likewise an ant, even though its steps are very small, can cross a room of any given length in finitely many steps. In a non-Archimedean setting these intuitive results do not hold; there are rooms so vast that ants cannot cross them and sheets so thin that they cannot stack up to challenge some rulers. These non-intuitive results, along with some others, generate some interest in exploring non-Archimedean objects. In this thesis we are concerned with investigating non-Archimedean field extensions of the real numbers which share
some of the arithmetic and algebraic properties that make the real numbers so special
(the existence of $n^{th}$-roots of positive elements, for example, which is not guaranteed in $p$-adic fields[4]).

Previously a great deal of work was done investigating the properties of the non-
Archimedean Levi-Civita field [7], $\mathcal{R}$, and the properties of functions given locally by
a convergent power series on it. These investigations revealed that the disconnected
nature of $\mathcal{R}$ entails that continuous or even infinitely often differentiable functions
do not follow the standard results of calculus on a closed interval (they do not
necessarily have intermediate values, extreme values; they need not satisfy Rolle’s
Theorem, etc.) For a function to achieve sufficient smoothness to guarantee these
properties it must be given locally by a convergent power series.

In this thesis we shall investigate the properties of a class of fields which are
generalizations of the Levi-Civita field, a representative of which we shall call $\mathcal{F}$,
and the properties of analytic functions on them. In Section (1.2) we shall make
some preliminary remarks about the properties of well-ordered subsets of $\mathbb{Q}$ which
will allow us to better discuss the structure of $\mathcal{F}$. In Section (1.3) we shall introduce
the set $\mathcal{F}$ and the operations $+$ and $\cdot$ on it and show that $(\mathcal{F}, +, \cdot)$ is a field by
showing that $\cdot$ is distributive with respect to $+$ and multiplicative inverses of non-
zero elements of $\mathcal{F}$ exist in $\mathcal{F}$ (that $(\mathcal{F}, +)$ and $(\mathcal{F} \setminus \{0\}, \cdot)$ are abelian groups is
trivial with the above mentioned results). We shall also introduce some relations
between elements of $\mathcal{F}$ (including a total order). In Section (2.1) we shall define an
ultrametric on $\mathcal{F}$, and show that it induces the same topology as the order described
in Section (1.3). With the concept of a topology the convergence of sequences and
series can be explored. To motivate our interest in power series in Section (2.3)
we shall provide some examples of continuous and differentiable functions failing
to satisfy the basic theorems of calculus. Because of previous experience in non-
Archimedean analysis, specifically with power series on $\mathcal{R}$ [6] we infer that power
series are the only class of functions that satisfy those theorems without additional conditions on any of the derivatives. In Chapter 3 we will show that power series indeed have the same smoothness properties as real power series.

There are further applications for power series in a non-Archimedean context (Lebesgue-like integration [9], computer differentiation [8], etc.) and in \( R \) functions given locally by power series are the smallest (with a few considerations such as closure under composition and uniform limit, which is crucial in for Lebesgue-like integration) family of basis functions for which these concepts are well defined.

1.2 Properties of Well-Ordered Sets

**Definition 1.1.** (Well-ordered subsets of \( \mathbb{Q} \)). We say that a set \( B \subseteq \mathbb{Q} \) is well-ordered if and only if for every \( B' \subseteq B \) we have that \( B' \) contains a minimum element in the standard order of \( \mathbb{Q} \).

**Lemma 1.1.** If \( A \) is well-ordered then any \( C \) that is a subset of \( A \) is also well-ordered.

*Proof.* Suppose \( A \) is well-ordered, \( C \subset A \) and let \( C' \subset C \). Then \( C' \subset A \) so it must contain a minimum element by definition. Therefore, any subset of \( C \) contains a minimum element and hence \( C \) is well-ordered. \( \square \)

For any two well-ordered subsets \( A \) and \( B \) of \( \mathbb{Q} \), \( A \cap B \) is a subset of \( A \). Hence

**Corollary 1.1.1.** If \( A \) and \( B \) are well-ordered then \( A \cap B \) is well-ordered.

**Lemma 1.2.** (Decreasing sequences in well-ordered sets). If \( (a_n) \) is a decreasing sequence in a well-ordered set \( A \) then there is an \( N \) in \( \mathbb{N} \) such that \( a_n = a_N \) for all \( n \geq N \).

*Proof.* Suppose \( (a_n) \) is a decreasing sequence in a well-ordered set \( A \) and there is no \( N \) in \( \mathbb{N} \) such that \( a_n = a_N \) for \( n \geq N \). Because \( \bigcup_{n \in \mathbb{N}} \{a_n\} \) is a subset of \( A \) it contains
a minimum element, call it $a_L$. Since $a_L$ is in $\cup_{n \in \mathbb{N}} a_n$ there is some $N' \in \mathbb{N}$ such that $a_{N'} = a_L$. Since there is no $N$ such that $a_n = a_N$ for $n \geq N$ and $(a_n)$ is decreasing there must be some $M > N'$ in $\mathbb{N}$ such that $a_M < a_{N'}$, so $a_L$ is not a minimum element of $\cup_{n \in \mathbb{N}} \{a_n\}$. This is a contradiction, therefore for every decreasing sequence $(a_n)$ in $A$ there is an $N$ in $\mathbb{N}$ such that $a_n = a_N$ for $n \geq N$. \hfill \Box

Theorem 1.3. (Well-ordered subsets of $\mathbb{Q}$). For $A, B$ well-ordered subsets of $\mathbb{Q}$ the following hold

- $A \cup B$ is well-ordered.
- $A + B$ is well-ordered.
- For any $r \in A + B$ there are only finitely many pairs $(a, b)$ in $A \times B$ such that $r = a + b$.

Proof. Let $A$ and $B$ be well-ordered subsets of $\mathbb{Q}$.

Suppose $C \subseteq (A \cup B)$, then $C = (C \cap A) \cup (C \cap B)$. As $(C \cap A) \subseteq A$ and $(C \cap B) \subseteq B$ they are both well-ordered by Lemma (1.1) and therefore each contains a minimum element (call them $a$ and $b$ respectively). Then if we denote $\min\{a, b\}$ by $c$ we have that $c \in C$ and for every $x \in (C \cap A)$ we have $c \leq x$ and for every $y \in (C \cap B)$ we have $c \leq y$, and hence $c$ is the minimum element of $C$. Therefore any subset of $(A \cup B)$ contains a minimum element and hence $(A \cup B)$ is well-ordered.

Suppose $C \subseteq A + B$ then for every $c \in A + B$ there is at least one pair $(a, b) \in A \times B$ such that $a + b = c$. For each $c \in C$ choose one such pair and call it $(a_c, b_c)$. Then, using the axiom of choice, define $A_C = \{a \in A \mid \exists c \in C: a = a_c\}$. $A_C$ is well-ordered because it is a subset of $A$. We can similarly define a well-ordered set $B_C = \{b \in B \mid \exists c \in C: b = b_c\}$ by the axiom of choice.
Let \( a_0 = \min \{ A_C \} \), which exists in \( A_C \) as \( A_C \) is well-ordered. There may be many pairs with \( a_c = a_0 \) so let \( B_0 = \{ b \in B_C : \exists c \in C : a_0 + b = c \} \). Because \( B_0 \subseteq B \) it is well-ordered and therefore contains a minimum element; call it \( b_0 \). Now let \( c_0 = a_0 + b_0 \) and \( C_1 = \{ c \in C : c < c_0 \} \) (if \( C_1 \) is empty then \( c_0 \) is a minimum element of \( C \)).

We will construct a chain of \( C_n \) such that eventually a minimum element of some \( C_N \) is a minimum element of \( C \) and is contained in \( C \), so let

- \( A_{C1} = \{ a \in A : \exists c \in C_1 : a = a_c \} \) (exists by axiom of choice)
- \( a_1 = \min \{ A_{C1} \} \) (exists because \( A_{C1} \) is well-ordered)
- \( B_1 = \{ b \in B_C : \exists c \in C_1 : b + a_1 = c \} \) (exists by axiom of choice)
- \( b_1 = \min \{ B_1 \} \) (exists because \( B_1 \) is well-ordered)
- \( c_1 = a_1 + b_1 \)
- \( C_2 = \{ c \in C_1 : c < c_1 \} \)

If \( C_2 \) is empty then \( c_1 \) is a minimum element of \( C_1 \) which means it must be a minimum element of \( C \).

This gives a template that allows us to continue this process generating sequences \((C_n), (a_n), (b_n), \) et cetera. We let

- \( A_{Cn} = \{ a \in A : \exists c \in C_n : a = a_c \} \) (exists by axiom of choice)
- \( a_n = \min \{ A_{Cn} \} \) (exists because \( A_{Cn} \) is well-ordered)
- \( B_n = \{ b \in B_C : \exists c \in C_n : b + a_n = c \} \) (exists by axiom of choice)
- \( b_n = \min \{ B_n \} \) (exists because \( B_n \) is well-ordered)
• \( c_n = a_n + b_n \)

• \( C_{n+1} = \{ c \in C_n \mid c < c_n \} \) (if \( C_{n+1} \) is empty then \( c_n \) is a minimum element of \( C \)).

Either at some finite \( N \) we have that \( C_{N+1} \) is empty and \( c_N \) is a minimum element of \( C \) or we have some infinite sequence \( (C_n) \) which gives an infinite sequence \( (b_n) \) in \( B_C \).

In the case where the process does not terminate and the set \( C_n \) is never empty note the following:

For \( c \in C_{N+1} \) consider:

\[
\begin{align*}
a_N + b_N &> a_c + b_c \\
b_N &> (a_c - a_N) + b_c \\
b_N &> b_c
\end{align*}
\]

and hence for all \( c \) in \( C_{N+1} \) we have \( b_c < b_N \). Therefore \( (b_n) \) is a decreasing sequence in the well-ordered set \( B_C \) which means by Lemma (1.2) there is an \( N \) in \( \mathbb{N} \) such that \( b_n = b_N \) for \( n \geq N \). In other words there is some \( c_N \in C \) such that for all \( c \leq c_N \) we have \( b_c = b_N \). Invoking the axiom of choice again, we can define a set \( A_U = \{ a \in A_C \mid \exists c \in C: a + b_N = c \} \) which is a subset of \( A \) and so is well-ordered. Thus \( A_U \) has a minimum element, call it \( a_U \). It follows that \( a_U + b_N \leq c \) for any \( c \leq c_N \). Thus \( a_U + b_N \) is a minimum element of \( C \).

Therefore every \( C \subset A + B \) must contain a minimum element and hence \( A + B \) is well-ordered.
Finally consider an element \( r \in A + B \). Let \( A_s = \{ a \in A \mid \exists b \in B: a + b = r \} \) and \( B_s = \{ b \in B \mid \exists a \in A: a + b = r \} \). Because \( A_s \subset A \) and \( B_s \subset B \) both are well-ordered, therefore both are bounded below, let \( a_0 = \min\{A_s\} \) and \( b_0 = \min\{B_s\} \). As \( b_0 \) is a minimum element of \( B_s \) we have \( A_s \) is bounded above by \( r - b_0 \), likewise \( B_s \) is bounded above by \( r - a_0 \). Thus there can only be infinitely many pairs \((a, b)\) in \( A \times B \) that sum to \( r \) if there is at least one accumulation point in both \( A_s \) and \( B_s \). Suppose there are such accumulation points and let \( a \) be one such accumulation point of \( A_s \), and let \((a_n)\) be any sequence in \( A_s \) such that \( \lim_{n \to \infty} a_n = a \). Then \((a_n)\) must have a monotone subsequence \((a_{n_k})\). Let \((b_{n_k})\) be a sequence in \( B_s \) where \( b_{n_k} = r - a_{n_k} \), either \((b_{n_k})\) is decreasing or \((a_{n_k})\) is. Without loss of generality say \((b_{n_k})\) is decreasing then, by Lemma (1.2), there is some \( N \in \mathbb{N} \) such that for \( k \geq N \) \( b_{n_k} = b_{n_N} \) and so \( a_{n_k} = a_{n_N} \), or \( a_{n_k} + b_{n_k} \neq r \) which is a contradiction. Then for \( \delta = \frac{1}{2} \min_{1 \leq k \leq N} \{|a - a_k|\} \) we have \( (a - \delta, a + \delta) \cap A_s \) is empty which contradicts that \( a \) is an accumulation point of \( A_s \). So neither \( A_s \) nor \( B_s \) contain an accumulation point and both are bounded above and below hence both are finite. Thus there are only finitely many pairs \((a, b) \in A \times B \) such that \( a + b = r \).

\[
\square
\]

### 1.3 The Field \( \mathcal{F} \)

**Definition 1.2.** (The set \( \mathcal{F} \)). We define

\[
\mathcal{F} = \{ f : \mathbb{Q} \to \mathbb{R} \mid \{ x | f(x) \neq 0 \} \text{ is well-ordered} \}.
\]

The elements of \( \mathcal{F} \) are those functions from \( \mathbb{Q} \) to \( \mathbb{R} \) whose values are non-zero only on well-ordered sets. For an element \( x \) the set where \( x[q] \neq 0 \) is called the support of \( x \), and is denoted by \( \text{supp}(x) \).
We denote elements of $\mathcal{F}$ by $x, y$ etc. and identify their values at $q \in \mathbb{Q}$ with brackets like $x[q]$.

**Definition 1.3.** (Addition and Multiplication on $\mathcal{F}$).

We define addition on $\mathcal{F}$ pointwise: for $x, y$ in $\mathcal{F}

\[ (x + y)[q] = x[q] + y[q]. \] (1.1)

It can be seen that $\text{supp}(x+y)$ is contained in $\text{supp}(x) \cup \text{supp}(y)$, and so by Theorem (1.3) $\text{supp}(x + y)$ is well-ordered. Thus $\mathcal{F}$ is closed under addition.

Multiplication is defined thusly: For $q \in \mathbb{Q}$ we set

\[ (x \cdot y)[q] = \sum_{q_x, q_y \in \mathbb{Q}}^{q_x + q_y = q} x[q_x] \cdot y[q_y] \] (1.2)

The support of $(x \cdot y)$ is contained in $\text{supp}(x) + \text{supp}(y)$; so by Theorem (1.3) $\text{supp}(x \cdot y)$ is well-ordered. This means that the sum in Equation (1.2) must have only finitely many terms by Theorem (1.3) and hence the sum is well defined. This gives that $\mathcal{F}$ is closed under multiplication.

NB: The functions $e_+: \mathbb{Q} \to \mathbb{R}$ and $e.: \mathbb{Q} \to \mathcal{F}$ given by

\[ e_+[q] = 0 \text{ for all } q \text{ and } \]

\[ e.[q] = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \]

are members of $\mathcal{F}$ and for any $x$ in $\mathcal{F}$ we have $x + e_+ = x$ and $e. \cdot x = x$. Moreover
We have shown that \( x \cdot e_+ = e_+ \) for any \( x \) in \( \mathcal{F} \).

**Theorem 1.4.** For \( x, y, z \) in \( \mathcal{F} \), \( z \cdot (x + y) = (z \cdot x) + (z \cdot y) \).

**Proof.** Let \( x, y, z \) in \( \mathcal{F} \), if any are 0 the above result is trivial so without loss of generality assume that none are and so all \( \text{supp}(x), \text{supp}(y) \), and \( \text{supp}(z) \) are non-empty.

Consider that \( q \) is in \( \text{supp}((z \cdot x) + (z \cdot y)) \) if and only if

\[
\sum_{q_z \in \text{supp}(z)} z[q_z] \cdot x[q_z] + \sum_{q_z \in \text{supp}(z)} z[q_z] \cdot y[q_y] \neq 0
\]

and \( ((z \cdot x) + (z \cdot y))[q] \) is the value of that sum. For all \( q_z \) in \( \text{supp}(x) \cup \text{supp}(y) \) and \( q_z \) in \( \text{supp}(z) \) we have \( z[q_z] \cdot y[q_z] + z[q_z] \cdot x[q_z] = z[q_z] \cdot (y[q_z] + x[q_z]) \) and so

\[
\sum_{q_z \in \text{supp}(z)} z[q_z] \cdot x[q_z] + \sum_{q_z \in \text{supp}(z)} z[q_z] \cdot y[q_y] = \sum_{q_z \in \text{supp}(z)} z[q_z] \cdot (y[q_y] + x[q_y]).
\]

Only \( q_z \) in \( \text{supp}(x) \cup \text{supp}(y) \) where \( x[q_z] + y[q_z] \neq 0 \) contribute to the above sum. These are exactly the support points of \( x + y \) and \( (x + y)[q_z] = x[q_z] + y[q_z] \) by definition. Hence

\[
\sum_{q_z \in \text{supp}(z)} z[q_z] \cdot (x[q_z] + y[q_z]) = \sum_{q_z \in \text{supp}(z)} z[q_z] \cdot ((x + y)[q_z])
\]

and \( ((z \cdot x) + (z \cdot y))[q] \) is the value of the sum. The right hand side is the definition of \( (z \cdot (x + y))[q] \) and hence \( \text{supp}(z \cdot (x + y)) = \text{supp}((z \cdot x) + (z \cdot y)) \) and \( ((z \cdot x) + (z \cdot y))[q] = (z \cdot (x + y))[q] \) at all \( q \) in the support. Therefore \( z \cdot (x + y) = (z \cdot x) + (z \cdot y) \). \( \square \)

With these definitions of + and · we intend to show that \((\mathcal{F}, +, \cdot)\) is a field. Given that we have shown that \( \mathcal{F} \) is closed under + and ·, and that we have shown · is
distributive with respect to + on \( F \), we will endeavour to show that non-zero elements in \( F \) have multiplicative inverse elements (it is trivial from here to show \((F, +)\) is an abelian group and that \( \cdot \) is associative and commutative). This will require results related to convergence of sequences of elements of \( F \), which will require the definition of an order. After defining an order \((\geq)\) and (ultra-)metric on \( F \) we shall return to the question of multiplicative inverse elements and show that \((F, +, \cdot, \geq)\) is a totally ordered field.

**Definition 1.4.** \((\lambda, \sim, \approx, =_r)\). For \( x, y \in F \) we define

\[
\lambda(x) = \begin{cases} 
\min\{\text{supp}(x)\} & \text{for } x \neq e_+ \\
\infty & \text{for } x = e_+.
\end{cases}
\]

\[
\cdot \lambda(x) = \begin{cases} 
\text{if } x \neq e_+, y \neq e_+, \text{ and } \lambda(x) = \lambda(y).
\end{cases}
\]

\[
\cdot x \approx y \text{ if } x \neq e_+, y \neq e_+, \lambda(x) = \lambda(y), \text{ and } x[\lambda(x)] = y[\lambda(y)].
\]

\[
\cdot x =_r y \text{ if } x[q] = y[q] \text{ for all } q \leq r.
\]

**Theorem 1.5.** For \( x, y \in F \) we have

- \( \lambda(x + y) \geq \min\{\lambda(x), \lambda(y)\} \).
- For \( x \neq e_+ \lambda(x \cdot y) = \lambda(x) + \lambda(y) \), \((x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)] \).

*Proof.* Since \((x + y)[q] = x[q] + y[q] \text{ and } x[q] = 0 \text{ for } q < \lambda(x) \), \(y[q] = 0 \text{ for } q < \lambda(y) \) we have \((x + y)[q] = 0 \text{ for } q < \min\{\lambda(x), \lambda(y)\} \). Thus \((x + y)[q] \neq 0 \text{ only for } q \geq \min\{\lambda(x), \lambda(y)\} \) so \( \lambda(x + y) \geq \min\{\lambda(x), \lambda(y)\} \).

Both \( \text{supp}(x) \) and \( \text{supp}(y) \) are well-ordered, and non-empty as neither are \( e_+ \), so they each contain a minimum element \( \lambda(x) \) and \( \lambda(y) \) respectively. Let \((a, b)\) in \( \text{supp}(x) \times \text{supp}(y) \) be such that \( a + b = \lambda(x) + \lambda(y) \), then \( \lambda(x) - a = b - \lambda(y) \). If \( \lambda(x) - a > 0 \) then \( \lambda(x) > a \) which is a contradiction, if \( \lambda(x) - a < 0 \) then \( b < \lambda(y) \).
which is a contradiction. Therefore only \((\lambda(x), \lambda(y))\) in \(\text{supp}(x) \times \text{supp}(y)\) sums to \(\lambda(x) + \lambda(y)\) and \(\text{supp}(x \cdot y) \subseteq \text{supp}(x) + \text{supp}(y)\). Therefore \(\lambda(x) + \lambda(y) \leq \lambda(x \cdot y)\).

Consider that \(y[\lambda(y)] \cdot x[\lambda(x)]\) is not zero by definition and there is no other \((a, b)\) in \(\text{supp}(x) \times \text{supp}(y)\) such that \(a + b = \lambda(x) + \lambda(y)\) then

\[(x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)] \neq 0\]

by definition. Therefore \(\lambda(x \cdot y) \leq \lambda(x) + \lambda(y)\). Thus \(\lambda(x \cdot y) = \lambda(x) + \lambda(y)\) and

\[(x \cdot y)[\lambda(x) + \lambda(y)] = x[\lambda(x)] \cdot y[\lambda(y)]\].

\(\square\)

**Definition 1.5.** (Ordering in \(F\)). For \(x, y\) in \(F\) we say that \(x \geq y\) if \(x = y\) or \(x \neq y\) and \((x - y)[\lambda(x - y)] > 0\).

This is a total order on \(F\) because either \(x = y\) or \((x - y)[\lambda(x - y)]\) is in \(\mathbb{R}\) and the standard order on \(\mathbb{R}\) is a total order.

**Definition 1.6.** (Absolute value on \(F\)). The function \(|\cdot|: F \to F\) given by

\[|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } 0 > x \end{cases}\]

is an absolute value on \(F\)

NB: For \(x, y\) in \(F\) we have \(|x \cdot y| = |x| \cdot |y|\) and \(|x + y| \leq |x| + |y|\).

**Definition 1.7.** (Embedding \(\mathbb{R}\) in \(F\)). Define \(\Pi: \mathbb{R} \to F\) by

\[\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}.

This function is an isomorphic injection between \(\mathbb{R}\) and \(\Pi(\mathbb{R})\) \((\Pi(x + y) = \Pi(x) + \Pi(y)\) and \(\Pi(x \cdot y) = \Pi(x) \cdot \Pi(y)\) \(x, y \in \mathbb{R}\) and so embeds \(\mathbb{R}\) in \(F\). Notice also that if \(x \geq y\)
in \( \mathbb{R} \) then \( \Pi(x) \geq \Pi(y) \) so \( \Pi \) is an order preserving embedding. When referencing a real number \( x \) as an element of \( \mathcal{F} \) we mean \( \Pi(x) \). In particular this embedding maps the real number 0 to \( e_+ \) and the real number 1 to \( e_- \). From now on we will use 0 and 1 to denote the neutral elements of \( \mathcal{F} \).

**Definition 1.8. (Order of Magnitude in \( \mathcal{F} \))** For \( x, y > 0 \) in \( \mathcal{F} \) we say

\[
x \gg y \text{ if } \lambda(x) < \lambda(y)
\]

This relation compares the so-called orders of magnitude of elements of \( \mathcal{F} \). For any \( n \) in \( \mathbb{N} \) and \( y > 0 \) in \( \mathcal{F} \) we have \( \lambda(ny) = \lambda(n) + \lambda(y) = \lambda(y) \). Then \( x \gg y \) implies \( x > ny \) for all \( n \) in \( \mathbb{N} \). If \( x \gg y \) we say \( x \) is infinitely larger than \( y \) and \( y \) is infinitely smaller than \( x \). If \( x \sim y \) then we say \( x \) is of the same order as \( y \). Those \( x \) with \( \lambda(x) = 0 \) we call finite (for each such \( x \) there is an \( x' \) in \( \mathbb{R} \) such that \( x \approx \Pi(x') \)). Those \( x > 0 \) with \( \lambda(x) < 0 \) we call infinitely large, as \( x \gg 1 \) we have \( x > n \) for all \( n \) in \( \mathbb{N} \). Those \( x > 0 \) with \( \lambda(x) > 0 \) we call infinitely small or infinitesimal, as \( 1 \gg x \) we have \( 1 > nx \) for all \( n \) in \( \mathbb{N} \).

**Definition 1.9. (The number \( d \)).** Consider the function \( d: \mathbb{Q} \to \mathbb{R} \)

\[
d[q] = \begin{cases} 
1 & \text{if } q = 1 \\
0 & \text{if } q \neq 1
\end{cases}
\]

This function is an element of \( \mathcal{F} \) and by the above definition of order we have \( d > 0 \) and \( 1 \gg d \). Hence \( d \) is infinitely small or infinitesimal. The multiplicative inverse of
this element is given by

\[
d^{-1}[q] = \begin{cases} 
  1 & \text{if } q = -1 \\
  0 & \text{if } q \neq -1 
\end{cases}
\]

and it is easy to check that \( d^{-1} \gg 1 \).

Let \( n \in \mathbb{N} \) for \( \xi_n \) in \( \mathcal{F} \) given by

\[
\xi[q] = \begin{cases} 
  1 & \text{if } q = \frac{1}{n} \\
  0 & \text{if } q \neq \frac{1}{n} 
\end{cases}
\]

then \( \underbrace{\xi_n \cdots \xi_n}_{n \text{ times}} = d \) and so \( \xi_n \) is the \( n \text{th} \) root of \( d \).

For any \( t = \frac{n}{m}, t \neq 0 \), in \( \mathbb{Q} \) we have \( d^t = \underbrace{\xi_m \cdots \xi_m}_{n \text{ times}} \) is an element of \( \mathcal{F} \) whose sole support point is \( t \) and whose value at that support point is 1. Then for \( t \neq 0 \) \( d^{1/t} \) is the \( t \text{th} \) root of \( d \) and for \( i > j \) we have \( d^i \gg d^j \).

NB: Since for every \( q \) in \( \mathbb{Q} \) \( d^q[q] = 1 \) we have for \( a_q \) in \( \mathbb{R} \) \( a_q d^q[q] = a_q \). For every \( x \) in \( \mathcal{F} \) we can enumerate \( supp(x) \) and form a sequence, which must be bounded below as \( supp(x) \) is well-ordered, \( (q_n) \) such that if \( i > j \) we have \( q_i > q_j \). We can then define a real sequence \( a_n = x[q_n] \) and any \( x \) in \( \mathcal{F} \) is given by the formal power series \( \sum_{n=1}^{\infty} a_n d^{q_n} \). While this sequence need not converge (such a sequence converges in the order topology only if \( supp(x) \) is left-finite [5]) it can be helpful to treat elements of \( \mathcal{F} \) as formal generalized power series when we do arithmetic with them or define functions on \( \mathcal{F} \). With this understanding of the form of elements of \( \mathcal{F} \) we can move on to describe a topology on \( \mathcal{F} \).
Chapter 2

Topological Structure and Convergence

2.1 Topology

In this chapter we will introduce the concept of an ultrametric space and describe convergence of sequences and series in such a space. The discussion of convergence of sequences and series will equip us to investigate the analytic properties of power series on $\mathcal{F}$.

Definition 2.1. (Ultrametric). Given a set $X$ a function $\Delta: X \times X \to \mathbb{R}$ is an ultrametric on $X$ if for all $x, y, z$ in $X$

- $\Delta(x, y) \geq 0$, $\Delta(x, y) = 0$ if and only if $x = y$.

- $\Delta(x, y) = \Delta(y, x)$.

- $\Delta(x, z) \leq max\{\Delta(x, y), \Delta(y, z)\}$

The pair $(X, \Delta)$ is called an ultrametric space.
NB: An ultrametric is a metric which satisfies the strong triangle inequality (which entails the triangle inequality). Therefore every ultrametric space is also a metric space.

**Theorem 2.1.** (*F* is an ultrametric space). *F* with the function

\[ \Delta(x, y) = \begin{cases} 
  e^{-\lambda(x-y)} & \text{if } x \neq y \\
  0 & \text{if } x = y 
\end{cases} \]

is an ultrametric space.

**Proof.** Let *x*, *y*, *z* in *F*.

For *x* \neq *y* we have \( \lambda(x - y) \) is in \( \mathbb{Q} \), then \( \Delta(x, y) = e^{-\lambda(x-y)} > 0 \). When *x* = *y* we have \( \Delta(x, y) = 0 \) by definition. Thus \( \Delta(x, y) \geq 0 \) and \( \Delta(x, y) = 0 \) if and only if \( x = y \).

As \( x - y = -(y - x) \) we have \( \lambda(x - y) = \lambda(-1) + \lambda(y - x) \) by Theorem (1.5) and since \( \lambda(-1) = 0 \) we have \( \lambda(x - y) = \lambda(y - x) \) which gives \( \Delta(x, y) = \Delta(y, x) \).

Finally since \( x - z = (x - y) + (y - z) \) we have \( \lambda(x - z) \geq \max\{\lambda(x-y), \lambda(y-z)\} \) by Theorem (1.5). Thus we have that \( -\lambda(x - z) \leq \max\{-\lambda(x-y), -\lambda(y-z)\} \) which gives \( \Delta(x, z) \leq \max\{\Delta(x,y), \Delta(y,z)\} \).

**Theorem 2.2.** The topologies induced by the order and the ultrametric are equal.

**Proof.** Let \( x_0 \in F \) and \( \delta > 0 \) in \( F \) then \( I = \{ x \in F \mid |x-x_0| < \delta \} \), or \( (x_0-\delta, x_0+\delta) \), is an open interval in the order topology. We have that \( \Delta(x, x_0) \leq e^{-\lambda(\delta)} \) for any \( x \) in \( I \).

Let \( \epsilon = e^{-(\lambda(\delta)+1)} \), (NB we have \( \epsilon > 0 \) in \( \mathbb{R} \)) and let \( B(x_0, \epsilon) = \{ x \in F \mid \Delta(x, x_0) < \epsilon \} \).

Then if \( x \in B_U(x_0, \epsilon) \) we have \( e^{-\lambda(x-x_0)} < e^{-(\lambda(\delta)+1)} \) thus \( \lambda(x - x_0) > \lambda(\delta) + 1 \) so \( |x - x_0| < \delta \). It follows that \( B_U(x_0, \epsilon) \subset B_O(x_0, \delta) \).
Now let $x_0 \in \mathcal{F}$ and $\delta > 0$ in $\mathbb{R}$ and consider $B(x_0, \delta)$, an open ball in the ultrametric topology. Let $q \in \mathbb{Q}$ be such that $q < \ln(\delta)$ and let $\epsilon = d^{-q}$ (NB $\epsilon > 0$ in $\mathcal{F}$) then for $x$ in $I$ we have that $|x - x_0| < d^{-q}$ which implies $\lambda(x - x_0) \geq -q$ so $-\lambda(x - x_0) \leq \ln(\delta)$ which gives $\Delta(x, x_0) < \delta$. Thus $I \subset B_U(x_0, \delta)$.

So any open ball in the topology induced by the ultrametric contains an open ball in the topology induced by the order and vice versa. Therefore the topologies are equal [3].

Thus $\mathcal{F}$ with the order topology is an ultrametric (therefore metric) space and certain results about limits of sequences and series will hold like any other metric space.

### 2.2 Convergence of Sequences and Series

**Definition 2.2.** (Convergence of sequences). Let $(s_n)$ be a sequence in $\mathcal{F}$. We say that $(s_n)$ is convergent to the limit $s \in \mathcal{F}$ if for every $\epsilon > 0$ in $\mathcal{F}$ there exists $N \in \mathbb{N}$ such that $\Delta(s_n, s) < \epsilon$ for all $n \geq N$.

**Definition 2.3.** (Cauchy Sequence) A sequence $(a_n)$ in $\mathcal{F}$ is said to be Cauchy if for every $\epsilon > 0$ in $\mathcal{F}$ there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n, m > N(\epsilon)$ we have $\Delta(a_n, a_m) < \epsilon$.

NB: Since the order and ultrametric topologies are equal $\Delta(x, y) < \epsilon$ is equivalent to $|x - y| < \epsilon$.

**Corollary 2.2.1.** (Sequences in a metric space)[3] Let $(a_n)$ and $(b_n)$ be sequences in $\mathcal{F}$ converging to $A$ and $B$ in $\mathcal{F}$ respectively, and $c$ a constant in $\mathcal{F}$ then

- The sequence $(ca_n)$ converges to $cA$. 

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The sequence \((a_n + b_n)\) converges to \(A + B\) in \(\mathcal{F}\).

The sequence \((a_n \cdot b_n)\) converges to \(A \cdot B\) in \(\mathcal{F}\).

The limit of a convergent sequence in \(\mathcal{F}\) is unique.

Convergent sequences are Cauchy.

**Definition 2.4.** (Regularity). A sequence \((s_n)\) in \(\mathcal{F}\) is regular if \(\bigcup_{n=0}^\infty \text{supp}(s_n)\) is well-ordered.

**Theorem 2.3.** (Properties of regularity). Let the sequences \((a_n)\) and \((b_n)\) in \(\mathcal{F}\) be regular. We have the sequence of the sums, the sequence of the products, any subsequence, any rearrangement of either sequence, and the combined sequence \(c_{2n} = a_n\) and \(c_{2n+1} = b_n\), are all regular.

Proof. Let \(A = \bigcup_{n=0}^\infty \text{supp}(a_n)\) and \(B = \bigcup_{n=0}^\infty \text{supp}(b_n)\). For any \(c \in \text{supp}(a_n + b_n)\) either \(c \in \text{supp}(a_n)\) or \(c \in \text{supp}(b_n)\) then \(\bigcup_{n=0}^\infty \text{supp}(a_n + b_n) \subset (A \cup B)\) and is therefore well-ordered by Lemma (1.1) and Theorem (1.3), thus \((a_n + b_n)\) is a regular sequence.

For any two elements \(x, y\) in \(\mathcal{F}\) we have \(\text{supp}(x \cdot y) \subset \text{supp}(x) + \text{supp}(y)\) and hence we have \(\text{supp}(a_n \cdot b_n) \subset \text{supp}(a_n) + \text{supp}(b_n)\) for all \(n\). Therefore for \(c \in \bigcup_{n=0}^\infty \text{supp}(a_n \cdot b_n)\) we have \(c \in A + B\). Thus we have \(\bigcup_{n=0}^\infty \text{supp}(a_n \cdot b_n) \subset A + B\) and is therefore well-ordered by Lemma (1.1) and Theorem (1.3), thus \((a_n \cdot b_n)\) is a regular sequence.

For any subsequence or rearrangement of the sequence \((a_n)\), call it \((s_n)\), we have \(\text{supp}(s_n) \subset A\) for all \(n\) and so \(\bigcup_{n=0}^\infty \text{supp}(s_n) \subset A\) and is therefore well-ordered by
Lemma (1.1). So any rearrangement or subsequence of a regular sequence is regular.

The support of the combined sequence is contained in \( A \cup B \) and is therefore well-ordered by Theorem (1.3), thus the combined sequence \( c_{2n} = a_n, c_{2n+1} = b_n \) is regular. \( \square \)

**Lemma 2.4.** Let \((s_n)\) be a sequence converging to \(s\) in \(\mathcal{F}\). Then \(|s_n|\) converges to \(|s|\) in \(\mathcal{F}\).

**Proof.** Let \(\epsilon > 0\) in \(\mathcal{F}\) be given. Then there is some \(N \in \mathbb{N}\) such that \(|s_n - s| < \epsilon\) for all \(n \geq N\). Given that \(||s_n| - |s|| \leq |s_n - s|\) for all \(n\) we then have that \(||s_n| - |s|| < \epsilon\) for \(n \geq N\). Therefore \(|s_n|\) converges to \(|s|\). \(\square\)

NB: The converse of Lemma (2.4) does not hold, consider the example \(a_n = (-1)^n\) where \(|a_n|\) is 1 for all \(n\) and therefore converges to 1 while \(a_n\) does not converge to any element in \(\mathcal{F}\).

**Definition 2.5.** \((Q_r)\) For \(r \in \mathbb{Q}\) let \(Q_r = \{q \in \mathbb{Q} \mid q \leq r\}\)

**Theorem 2.5.** \(\mathcal{F}\) is Cauchy complete with respect to the order topology

**Proof.** Let \((s_n)\) be a Cauchy sequence in \(\mathcal{F}\) and let \(\epsilon > 0\) in \(\mathcal{F}\). Then there is some \(N(\epsilon)\) such that

\[
|s_n - s_m| < \epsilon \quad \text{for all } n, m \geq N(\epsilon).
\]

Then for all \(r \in \mathbb{Q}\) there is some \(N_r \in \mathbb{N}\) such that

\[
|s_n - s_m| < d^{r+1} \quad \text{for all } n, m \geq N_r.
\]

(2.1)

So we have

\[
s_n[q] = s_{N_r}[q] \text{ for all } n \geq N_r \text{ and for all } q \leq r.
\]

(2.2)
By Equation (2.1) we may assume that

\[ N_{r_1} \leq N_{r_2} \text{ if } r_1 < r_2. \]  \hspace{1cm} (2.3)

Define \( s : \mathbb{Q} \to \mathbb{R} \) by \( s[q] = s_{N_r}[q] \). We will show that \( \text{supp}(s) \) is well-ordered and hence \( s \) is in \( \mathcal{F} \). For every \( r \in \mathbb{Q} \) we have

\[ \text{supp}(s) \cap Q_r = \text{supp}(s_{N_r}) \cap Q_r \]  \hspace{1cm} (2.4)

Let \( A \subset \text{supp}(s) \) be non-empty. \( A \) is non-empty so there is at least one \( r \in A \). Consider that \( A \cap Q_r \subset \text{supp}(s) \cap Q_r = \text{supp}(s_{N_r}) \cap Q_r \). Because \( (s_n) \) is a sequence in \( \mathcal{F} \) we have \( \text{supp}(s_{N_r}) \) is well-ordered which implies \( \text{supp}(s_{N_r}) \cap Q_r \) is well-ordered and therefore \( A \cap Q_r \) is well-ordered by Lemma (1.1). Then \( A \cap Q_r \) contains a minimum element, call it \( a \), and this \( a \) must also be a minimum element of \( A \). So any non-empty \( A \subset \text{supp}(s) \) contains a minimum element therefore \( \text{supp}(s) \) is well-ordered and \( s \in \mathcal{F} \). It remains to be shown that \( (s_n) \) converges to \( s \).

Let \( \epsilon > 0 \) be given. Then there is some \( r \in \mathbb{Q} \) such that \( d^r < \epsilon \). Then

\[ s_n[q] = s_{N_r}[q] = s[q] \text{ for all } n \geq N_r \text{ and for all } q \leq r. \]

Hence

\[ |s_n - s| \ll d^r < \epsilon \text{ for all } n \geq N_r. \]

So \( (s_n) \) converges to \( s \) in \( \mathcal{F} \). Therefore if \( (s_n) \) is Cauchy in the sense of Definition (2.3) it is convergent in the sense of Definition (2.2) and so \( \mathcal{F} \) is Cauchy complete in the order topology.

\[ \square \]

**Theorem 2.6.** (Convergence Criterion for sequences). Let \( (s_n) \) be a sequence in \( \mathcal{F} \). Then \( (s_n) \) converges if and only if for all \( r \in \mathbb{Q} \) there exists \( N(r) \in \mathbb{N} \) such that
Proof. Suppose \((s_n)_n\) is a convergent sequence in \(\mathcal{F}\) and let \(r \in \mathbb{Q}\). We then have that 
\((s_n)_n\) is Cauchy by Corollary (2.2.1) so there is some \(N(r) \in \mathbb{N}\) such that \(|s_m - s_l| < d^{r+1}\) for all \(m, l \geq N(r)\). This implies \(s_m \to r s_l\) for \(m, l \geq N(r)\).

Suppose for all \(r \in \mathbb{Q}\) there is some \(N(r) \in \mathbb{N}\) such that \(s_m = r s_l\) for all \(m, l \geq N(r)\). Let \(\epsilon > 0\) in \(\mathcal{F}\) be given; then there is some \(r \in \mathbb{Q}\) such that \(d^r \ll \epsilon\). There is some \(N(r)\) such that \(s_m = r s_l\) for all \(m, l \geq N(r)\) and hence \(|s_m - s_l| < \epsilon\) for \(m, l \geq N(r)\). We therefore have \((s_n)_n\) is Cauchy and hence it is convergent by Theorem (2.5).

\[\]  

**Theorem 2.7.** If a sequence \((s_n)_n\) is convergent then it is regular.

**Proof.** Let \((s_n)_n\) converge to \(s\) in \(\mathcal{F}\) and let \(A\) be a non-empty subset of \(\bigcup_{n=0}^{\infty} \text{supp}(s_n)\). There is at least one \(r \in A\) and by Theorem (2.6) there is some \(N_r\) such that for \(n, m \geq N_r\) we have \(\text{supp}(s_n) \cap Q_r = \text{supp}(s_m) \cap Q_r = \text{supp}(s) \cap Q_r\) for all \(n, m \geq N_r\).

So
\[
(\bigcup_{n=0}^{\infty} \text{supp}(s_n)) \cap Q_r = [\bigcup_{n=0}^{N_r-1} (\text{supp}(s_n) \cap Q_r)] \cup \{\text{supp}(s) \cap Q_r\}
\]

For all \(n\) we have \(\text{supp}(s_n) \cap Q_r\) is well-ordered by Lemma (1.1) and \(\text{supp}(s)\) is well-ordered because \(s\) is in \(\mathcal{F}\) so \((\bigcup_{n=0}^{\infty} \text{supp}(s_n)) \cap Q_r\) is well-ordered by Lemma (1.1) thus \(A \cap Q_r\) is well-ordered by Corollary (1.1.1). Let \(a\) be a minimum element of \(A \cap Q_r\), which must exist as \(A \cap Q_r\) is well-ordered. Then \(a\) must also be a minimum element of \(A\) and if it is in \(A \cap Q_r\) then it must be in \(A\). Therefore \(A \subset \bigcup_{n=0}^{\infty} \text{supp}(s_n)\) contains a minimum element; thus \(\bigcup_{n=0}^{\infty} \text{supp}(s_n)\) is well-ordered and hence \((s_n)_n\) is regular.

\[\]

**Theorem 2.8.** Let \((s_n)_n\) be a sequence in \(\mathcal{F}\). Then \((s_n)_n\) is Cauchy if and only if 
\((s_{n+1} - s_n)_n\) is a null sequence.

**Proof.** Let \((s_n)_n\) be a Cauchy sequence in \(\mathcal{F}\), and let \(\epsilon > 0\) in \(\mathcal{F}\) be given. Then there exists \(N \in \mathbb{N}\) such that \(|s_m - s_l| < \epsilon\) for all \(m, l \geq N\). In particular, \(|s_{m+1} - s_m| < \epsilon\)
for all \( m \geq N \). Hence, \( \lim_{n \to \infty} (s_{n+1} - s_n) = 0 \).

Now assume that \((s_n - s_{n+1})\) is a null sequence. Let \( \epsilon > 0 \) in \( F \), then there is some \( N \in \mathbb{N} \) such that for \( n \geq N \) we have \( |s_{n+1} - s_n| < d\epsilon \). Let \( k > l \geq N \) be given. We then have that

\[
|s_k - s_l| = |s_k - s_{k-1} + s_{k-1} - s_{k-2} + \cdots + s_{l+1} - s_l|
\leq |s_k - s_{k-1}| + |s_{k-1} - s_{k-2}| + \cdots + |s_{l+1} - s_l|
< (k - l)d\epsilon < \epsilon
\]

We know \((k - l)d\epsilon < \epsilon\) for any \( k, l \in \mathbb{N} \) where \( k > l \) because \( d \ll 1 \) so \((k - l)d < 1\) and hence \((k - l)d\epsilon < \epsilon\).

\[\square\]

**Corollary 2.8.1.** Let \((s_n)\) be a sequence in \( F \). Then, \((s_n)\) converges if and only if \((s_{n+1} - s_n)\) is a null sequence with respect to the order topology.

**Proof.** This is a direct consequence of the Cauchy completeness of \( F \) and Theorem (2.8) \[\square\]

**Theorem 2.9.** (Fixed Point Theorem) Let \( q_M \) in \( \mathbb{Q} \) be given and \( M \subset F \) be given by:
\[ M = \{ x \in F \mid \lambda(x) \geq q_M \}, \]
and let \( f: M \to F \) be such that \( f(M) \subset M \) and there is some \( 0 < q_f \ll 1 \) in \( F \) such that for any \( x, y \in M \) we have \( |f(x) - f(y)| \leq q_f |x - y| \), then \( f \) admits a unique fixed point \( x^* \) in \( M \).

**Proof.** Let \( x_0 \) in \( M \) be fixed and define a sequence \((x_n) \subset M \) where \( x_n = f(x_{n-1}) \) for \( n \geq 1 \). Let \( \epsilon > 0 \) in \( F \) be given, then for some \( N(\epsilon) \in \mathbb{N} \) we have \( q_f^{N(\epsilon)} \cdot d^{BM} \ll d\epsilon \).
For $n > N(\epsilon)$ we have

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})|$$

$$\leq q_f |x_{n-1} - x_{n-2}|$$

$$\vdots$$

$$\leq q_f^{n-1} |f(x_0) - x_0|$$

$$\leq q_f^{N(\epsilon)} |f(x_0) - x_0|$$

$$\leq q_f^{N(\epsilon)} d^{RM} < d\epsilon$$

and hence $(x_n - x_{n-1})$ is a null sequence and therefore must converge to some $x^*$ in $\mathcal{F}$ by Corollary (2.8.1).

Since $x^* = \lim_{n \to \infty} x_n$ and $x_n = f(x_{n-1})$, it follows that $x^* = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1})$. Then

$$|f(x^*) - x^*| = |f(x^*) - \lim_{n \to \infty} f(x_{n-1})|$$

$$= | \lim_{n \to \infty} f(x^*) - \lim_{n \to \infty} f(x_{n-1})|$$

$$= | \lim_{n \to \infty} [f(x^*) - f(x_{n-1})]|$$

$$= \lim_{n \to \infty} |f(x^*) - f(x_n)|$$

$$\leq \lim_{n \to \infty} q_f |x^* - x_{n-1}|$$

$$= q_f |x^* - \lim_{n \to \infty} x_{n-1}|$$

$$= q_f |x^* - x^*| = 0.$$}

Thus $f(x^*) - x^* = 0$ and so $x^*$ is a fixed point of $f$.

Let $\lambda(x) = Q$ and $d^{RM} \gg \epsilon > 0$ in $\mathcal{F}$ be given. Because $\lim_{n \to \infty} x_n = x^*$ there is some
$N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$ we have $|x^* - x_n| < \epsilon \ll d^M$ hence $\lambda(x^* - x_n) > q_M$ for $n \geq N(\epsilon)$. Since $(x_n)$ is a sequence in $M$ we have $\lambda(x_{N(\epsilon)}) \geq q_M$ and hence for $n \geq N(\epsilon)$

$$\lambda(x^*) = \lambda(x^* - x_n + x_{N(\epsilon)}) \geq \min\{\lambda(x^* - x_n), \lambda(x_{N(\epsilon)})\} \geq q_M$$

therefore $x^*$ is in $M$.

Let $y^*$ be a fixed point of $f$, then

$$0 \leq |x^* - y^*| = |f(x^*) - f(y^*)| \leq q_f |x^* - y^*|$$

since $q_f < 1$ we have $|x^* - y^*| = 0$ and hence $y^* = x^*$. Thus $f$ has a unique fixed point $x^*$ in $M$.

\[\square\]

**Theorem 2.10.** (Existence of multiplicative inverses of non-zero elements) For non-zero $x$ in $\mathcal{F}$ there is some $\xi$ in $\mathcal{F}$ such that $x \cdot \xi = 1$.

**Proof.** Let $x$ in $\mathcal{F}$ be non-zero and let $a = x[\lambda(x)]$. Then $x = a \cdot d^{\lambda(x)}(1 + y)$ for some $y$ such that $\lambda(y) > 0$. Furthermore $a \cdot d^{\lambda(x)}$ has a well defined inverse: $a^{-1} \cdot d^{-\lambda(x)}$. Thus non-zero elements $x = a \cdot d^{\lambda(x)}(1 + y)$ have multiplicative inverses if and only if elements of the form $1 + y$ where $\lambda(y) > 0$ have multiplicative inverses. So without loss of generality we may assume $x = 1 + y$ where $\lambda(y) > 0$ and look for $\xi$ such that $(1 + y) \cdot \xi = 1$.

Consider the equation

$$\xi \cdot (1 + y) = 1$$

where $\lambda(y) > 0$. Since $\lambda(1 + y) = 0$ and $\lambda(1) = 0$ it follows, if such a $\xi$ exists, that $\lambda(\xi) = 0$ by Theorem (1.5). Moreover, since $x[0] = 1[0] = 1$, we must have $\xi[0] = 1$. 

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Thus $\xi$ must be of the form $1 + z$ where $\lambda(z) > 0$.

We have $(1 + y) \cdot (1 + z) = 1$ if and only if $z = -y(1 + z)$. Let $M = \{ z \in \mathcal{F} : \lambda(z) > 0 \}$ and $f : M \to \mathcal{F}$ be given by $f(z) = -y(1 + z)$. We will show that $f$ has a fixed point $z^*$ in $M$ which entails that $\lambda(z^*) > 0$, $z^* = -y(1 + z^*)$, and $(1 + y) \cdot (1 + z^*) = 1$.

By Theorem (2.9) to show $f$ has a fixed point in $M$ we need to show that $f(M) \subset M$ and that there is some $0 < q_f \ll 1$ such that for any $z_1, z_2$ in $M$ we have $|f(z_1) - f(z_2)| \leq q_f|z_1 - z_2|$.

For $z \in M$ we have $f(z) = -y(1 + z)$ and $\lambda(y) > 0$. By Theorem (1.5) $\lambda(f(z)) = \lambda(-1) + \lambda(y) + \lambda(1 + z)$ and since $\lambda(-1) = \lambda(1 + z) = 0$ and $\lambda(y) > 0$ we have $\lambda(f(z)) > 0$ which gives $f(M) \subset M$.

Let $q_f = |2y|$, since $\lambda(y) > 0$ we have $0 < q_f \ll 1$. For any $z_1, z_2$ in $M$ we have $|f(z_1) - f(z_2)| = |y(1 + z_1) - y(1 + z_2)| = |y||z_1 - z_2| < |2y||z_1 - z_2| = q_f|z_1 - z_2|$.

Thus by Theorem (2.9) there is a fixed point $z^*$ for $f(z)$ in $M$ such that $z^* = -y(1 + z^*)$ or $(1 + z^*)(1 + y) = 1$. Therefore elements of $\mathcal{F}$ of the form $1 + y$ where $\lambda(y) > 0$ have multiplicative inverses and hence non-zero elements of $\mathcal{F}$ have multiplicative inverses.

With the above result we can complete the proof that $(\mathcal{F}, +, \cdot, \geq)$ is a totally ordered field.

Therefore by [1] we have that $(\mathcal{F}, +, \cdot, \geq)$ is a totally ordered real closed field, because $\mathbb{Q}$ is a divisible group and $\mathbb{R}$ is Archimedean, and hence by [2] $\mathcal{F} + i\mathcal{F}$ is algebraically closed and satisfies the fundamental theorem of algebra.
Now we consider the convergence of infinite sums of elements of $\mathcal{F}$, with an eye to eventually study power series.

**Definition 2.6.** (Convergence of series) For a sequence $(a_n)$ in $\mathcal{F}$ the series $\sum_{n=0}^{\infty} a_n$ converges to $s$ in $\mathcal{F}$ if the sequence of partial sums $s_n = \sum_{j=0}^{n} a_j$ converges to $s$. That is for every $\epsilon > 0$ in $\mathcal{F}$ there is an $N(\epsilon) \in \mathbb{N}$ such that for $n \geq N(\epsilon)$ we have $|s_n - s| < \epsilon$.

**Corollary 2.10.1.** If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two series converging to $A$ and $B$ respectively, then the series $\sum_{n=0}^{\infty} (a_n + b_n)$ converges to $A + B$.

**Proof.** $(\mathcal{F}, \Delta)$ is a metric space. \qed

**Corollary 2.10.2.** The infinite series $\sum_{n=0}^{\infty} a_n$ converges in $\mathcal{F}$ if and only if the sequence $(a_n)$ is a null sequence in $\mathcal{F}$.

**Proof.** This follows from Corollary (2.8.1) and Definition (2.6). \qed

NB: The formal power series described at the end of Section (1.3) can converge if and only if the enumeration of the support points of $x$ has $\lim_{n \to \infty} q_n = \infty$. If $\text{supp}(x)$ is a well-ordered set with an accumulation point at some finite $q$ then there is no enumeration where this is possible. As we noted, the said power series converges only in the case where the set of support points is left-finite.

**Corollary 2.10.3.** The series $\sum_{n=0}^{\infty} a_n$ converges if and only if it converges absolutely, that is if and only if $\sum_{n=0}^{\infty} |a_n|$ converges.

**Proof.** This is a direct consequence of Corollary (2.10.2) and the fact that $\lim_{n \to \infty} a_n = 0$ if and only if $\lim_{n \to \infty} |a_n| = 0$. \qed

**Corollary 2.10.4.** If a sequence $(a_n)$ has the property $\lambda(a_n) > R$ for all $n$, and $\sum_{n=0}^{\infty} a_n = s$ then $\lambda(s) > R$. 

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Proof. Let \( s_n = \sum_{i=0}^{n} a_i \) and let \( \epsilon > 0 \) in \( \mathcal{F} \) be given with \( \lambda(\epsilon) = R + 1 \). By the definition of convergence there is some \( N(\epsilon) \) in \( \mathbb{N} \) such that for \( n \geq N(\epsilon) \) we have 
\[ |s - s_n| < \epsilon. \]
Since 
\[ |s - s_n| = |\sum_{i=n+1}^{\infty} a_i| \]
we have \( \lambda(\sum_{i=n+1}^{\infty} a_i) \geq R + 1 \). This implies
\[
\lambda(s) = \lambda \left( \sum_{n=0}^{\infty} a_n + \sum_{n=N(\epsilon)+1}^{\infty} a_n \right) \geq \min_{0 \leq n \leq N(\epsilon)} \{ \lambda(a_n), R + 1 \}.
\]
Since \( \lambda(a_n) > R \) for all \( n \) and \( R + 1 > R \) we must therefore have \( \lambda(s) > R \). \( \square \)

Corollary 2.10.5. If the double sums \( \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \) and \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \) converge to \( A_1 \) and \( A_2 \) respectively and for every \( \epsilon > 0 \) there is some \( N(\epsilon) \) in \( \mathbb{N} \) such that for \( i, j \geq N(\epsilon) \) we have \( |a_{ij}| < \epsilon \) then \( A_1 = A_2 \).

Proof. Note that since both \( \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \) and \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \) are well defined it must be that \( \sum_{i=1}^{\infty} a_{ij} \) converges (to some \( a_j \)) for all \( j \) and \( \sum_{j=1}^{\infty} a_{ij} \) converges (to some \( b_i \)) for all \( i \).

Define \( x_n = \sum_{j=1}^{n} a_j \), \( y_n = \sum_{i=1}^{n} b_i \), and \( s_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \). We will show that 
\[ \lim_{n \to \infty} |x_n - y_n| = 0 \]
which gives \( A_1 = A_2 \).

Let \( \epsilon > 0 \) in \( \mathcal{F} \) be given. Then there is some \( N(\epsilon) \) in \( \mathbb{N} \) such that \( \lambda(a_{ij}) > 3 + \lambda(\epsilon) \) for all \( i, j \geq N(\epsilon) \). Because \( \sum_{j=N(\epsilon)+1}^{\infty} a_{ij} \) converges for all \( i \) and \( \lambda(a_{ij}) \geq 3 + \lambda(\epsilon) \) for \( i, j \geq N(\epsilon) \) we have \( \lambda(\sum_{j=N(\epsilon)+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon) \) for all \( i \geq N(\epsilon) \) by Corollary (2.10.4). For each \( i < N(\epsilon) \) there is some \( N(i, \epsilon) \) in \( \mathbb{N} \) such that \( \lambda(\sum_{j=N(i, \epsilon)+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon) \) because \( \sum_{j=1}^{\infty} a_{ij} \) converges for all \( i \). Let \( N_x = \max\{N(i, \epsilon), N(\epsilon) \mid i < N(\epsilon)\} \) then for \( n \geq N_x \) we have \( \lambda(\sum_{j=n+1}^{\infty} a_{ij}) > 2 + \lambda(\epsilon) \) for all \( i \). Thus for \( n \geq N_x \) we have that 
\[ |x_n - s_n| = |\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} a_{ij}| < d\epsilon. \]

We can similarly find an \( N_y \) such that for \( n \geq N_y \) we have \( |y_n - s_n| < d\epsilon \). Let \( N_0 = \max\{N_x, N_y\} \) then for \( n \geq N_0 \) we have 
\[ |x_n - y_n| \leq |x_n - s_n| + |y_n - s_n| < 2d\epsilon < \epsilon. \]
Thus \( \lim_{n \to \infty} |x_n - y_n| = 0 \) so \( A_1 = A_2 \).

\[ \square \]

**Theorem 2.11.** [5] Consider two convergent series \( \sum_{n=0}^{\infty} a_n = A \) and \( \sum_{n=0}^{\infty} b_n = B \). The series \( \sum_{n=0}^{\infty} c_n \) where \( c_n = \sum_{j=0}^{n} a_j b_{n-j} \), converges to \( A \cdot B \) in \( \mathcal{F} \).

**Proof.** First we show \( \sum_{n=0}^{\infty} c_n \) converges in \( \mathcal{F} \). By Corollary (2.8.1) we need only show that \( (c_n) \) is a null sequence. Because both \( (a_n) \) and \( (b_n) \) are null there is some \( R \) in \( \mathcal{F} \) such that \( |a_n| < R \) and \( |b_n| < R \) for all \( n \). Let \( \epsilon > 0 \) in \( \mathcal{F} \) be given. There is some \( M \) in \( \mathbb{N} \) such that \( |a_m| < \epsilon/R \) and \( |b_m| < \epsilon/R \) for all \( m \geq M \). Let \( N = 2M \) then for \( n \geq N \) we have

\[
|c_n| = |a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0| \\
\leq |a_0b_n| + |a_1b_{n-1}| + \cdots + |a_{n-1}b_1| + |a_nb_0| \\
= |a_0||b_n| + |a_1||b_{n-1}| + \cdots + |a_{n-1}||b_1| + |a_n||b_0| \\
< \frac{\epsilon}{R} + \frac{\epsilon}{R} + \cdots + \frac{\epsilon}{R} + \frac{\epsilon}{R} \\
= (n+1)\epsilon < \epsilon.
\]

So for all \( \epsilon > 0 \) in \( \mathcal{F} \) we can find \( N \in \mathbb{N} \) such that \( |c_n| < \epsilon \) for all \( n \geq N \). Hence \( \lim_{n \to \infty} c_n = 0 \) and thus \( \sum_{n=0}^{\infty} c_n \) converges in \( \mathcal{F} \).

Let \( \sum_{n=0}^{\infty} c_n = C, G = \max\{\sum_{n=0}^{\infty} |a_n|, \sum_{n=0}^{\infty} |b_n|, \sum_{n=0}^{\infty} |c_n|, d^{-1}\} \) and let \( \epsilon > 0 \) in \( \mathcal{F} \) be given. As \( \sum_{n=0}^{\infty} a_n = A, \sum_{n=0}^{\infty} b_n = B, \) and \( \sum_{n=0}^{\infty} c_n = C \) then there are \( N(A, \epsilon), N(B, \epsilon), \) and \( N(C, \epsilon) \) in \( \mathbb{N} \) such that for \( n \geq N(A, \epsilon) \) we have \( |\sum_{i=n}^{\infty} a_i| < G^{-1}d^2\epsilon, n \geq N(B, \epsilon) \) we have \( |\sum_{i=n}^{\infty} b_i| < G^{-1}d^2\epsilon, \) and \( n \geq N(C, \epsilon) \) we have \( |\sum_{i=n}^{\infty} c_i| < G^{-1}d^2\epsilon \) by Corollaries (2.10.4) and (2.10.2). Let \( N_0 = \max\{N(A, \epsilon), N(B, \epsilon), N(C, \epsilon)\} \) then for \( n \geq N_0 \) we have \( |\sum_{i=n}^{\infty} a_i| < G^{-1}d^2\epsilon, \) \( |\sum_{i=n}^{\infty} b_i - B| < G^{-1}d^2\epsilon, \) and \( |\sum_{i=n}^{\infty} c_i| < G^{-1}d^2\epsilon. \) For finite \( n \) we have \( \sum_{i=0}^{n} c_i = (\sum_{i=0}^{n} a_i) \cdot (\sum_{i=0}^{n} b_i). \) Therefore for \( n \geq N_0 \) we have
\[ |C - A \cdot B| = \left| \left( \sum_{i=0}^{n-1} c_i + \sum_{i=n}^{\infty} c_i \right) - \left( \sum_{i=0}^{n-1} a_i + \sum_{i=n}^{\infty} a_i \right) \cdot \left( \sum_{i=0}^{n-1} b_i + \sum_{i=n}^{\infty} b_i \right) \right| \]

\[ \leq \left| \sum_{i=0}^{n-1} c_i \right| - \left| \sum_{i=0}^{n-1} a_i \right| \cdot \left| \sum_{i=n}^{\infty} b_i \right| + \left| \sum_{i=n}^{\infty} c_i \right| \]

\[ + \left| \sum_{i=0}^{n-1} a_i \right| \cdot \left| \sum_{i=n}^{\infty} b_i \right| + \left| \sum_{i=0}^{\infty} a_i \right| \cdot \left| \sum_{i=n}^{\infty} b_i \right| \]

\[ < G^{-1}d^2\epsilon + GG^{-1}d^2\epsilon + GG^{-1}d^2\epsilon + GG^{-1}d^2\epsilon \]

\[ = (G^{-1} + 3)d^2\epsilon < \epsilon \]

Therefore for all \( \epsilon > 0 \) we have \( |C - A \cdot B| < \epsilon \) and so \( C = A \cdot B \) and \( \sum_{n=0}^{\infty} c_n = A \cdot B. \)

With these basic results on the convergence of sequences we shall now define power series and prove a few useful results to do with their convergence.

**Definition 2.7.** (Power Series) A series of the form

\[ \sum_{n=0}^{\infty} a_n(x - x_0)^n \]

where \((a_n)\) is a sequence in \( F \), \( x_0 \) is a fixed point in \( F \), and \( x \) varies in \( F \), is called a power series.

For \( a < b \) in \( F \) a function \( f: [a, b] \rightarrow F \) is said to be given by a power series on \([a, b]\) if there is a sequence \((a_n)\) in \( F \) and an \( x_0 \) in \([a, b]\) such that

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n. \]

Power series are a special kind of series and, as we shall see in Chapter 3, functions
given by a power series have the smoothness properties of real power series; in particular they satisfy the intermediate value theorem, the extreme value theorem and the mean value theorem on any closed interval within their domain of convergence.

**Lemma 2.12.** Let $M$ be a well-ordered subset of $\mathbb{Q}$. Define

$$M_{\Sigma, N} = \frac{M + \cdots + M}{N \text{ times}}$$

and

$$M_{\Sigma} = \bigcup_{n=1}^{\infty} M_{\Sigma, n}$$

Then $M_{\Sigma}$ is well-ordered if and only if $\min\{M\} \geq 0$

**Proof.** ($\Leftarrow$): Let $m_0 = \min\{M\} \geq 0$. If $m_0 = 0$ then $M_{\Sigma} = (M\setminus\{0\})_{\Sigma} \cup \{0\}$ so without loss of generality we may assume $m_0 > 0$. Let $m_0 > 0$ and let $A \subset M_{\Sigma}$ be non-empty. Then there is some $r$ in $A$. As $\mathbb{Q}$ is Archimedean there is some $N_r$ in $\mathbb{N}$ such that $N_r \cdot m_0 > r$. Then

$$A \cap Q_r \subseteq \left( \bigcup_{n=1}^{N_r} M_{\Sigma, n} \right) \cap Q_r$$

By Theorem (1.3) $\bigcup_{n=1}^{N_r} M_{\Sigma, n}$ is well-ordered as it is the finite union of finite sums of well-ordered sets. Then by Theorem (1.3) we have $(\bigcup_{n=1}^{N_r} M_{\Sigma, n}) \cap Q_r$ is well-ordered so $A \cap Q_r$ contains a minimum element, call it $a$. This minimum element must also be a minimum element of $A$ and contained in $A$ therefore $M_{\Sigma}$ is well-ordered.

($\Rightarrow$): Suppose $m_0 < 0$ then, as $n \cdot m_0$ is in $M_{\Sigma}$ for all $n$ in $\mathbb{N}$ it follows that $M_{\Sigma}$ is not bounded below and hence $M_{\Sigma}$ is not well-ordered. \qed

**Corollary 2.12.1.** The sequence $(x^n)$ is regular if and only if $\lambda(x) \geq 0$. Let $(a_n)$ be a sequence in $\mathcal{F}$. Then the sequences $(a_n x^n)$ and $(\sum_{j=0}^{n} a_j x^j)$ are regular if $(a_n)$ is regular and $\lambda(x) \geq 0$. 

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Proof. Note that $\cup_{n=1}^{\infty} supp(x^n) = M$ where $M = supp(x)$ which is well-ordered. So by Lemma (2.12) we have that $\cup_{n=1}^{\infty} supp(x^n)$ is well-ordered if and only if $\min\{M\} \geq 0$, or $\lambda(x) \geq 0$. The second part is a consequence of Theorem (2.3) where the sequence of products of terms of two regular sequences is regular.

Theorem 2.13. (Convergence Criterion) Let $(a_n)$ be a sequence in $\mathcal{F}$. Define

$$\lambda_0 = -\liminf_{n \to \infty} \left( \frac{\lambda(a_n)}{n} \right) = \limsup_{n \to \infty} \left( \frac{-\lambda(a_n)}{n} \right)$$

in $\mathbb{R}$. Let $x_0$ in $\mathcal{F}$ be fixed and $x$ in $\mathcal{F}$ be given. Then $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges if $\lambda(x-x_0) > \lambda_0$ and diverges if either $\lambda(x-x_0) < \lambda_0$ or $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n) > n\lambda_0$ for infinitely many $n$.

Proof. Suppose $\lambda(x-x_0) > \lambda_0$, then there is some $t \in \mathbb{Q}$ such that $t > 0$ and $\lambda(x-x_0) - t > \lambda_0$. Then there is some $N \in \mathbb{N}$ such that $\lambda(x-x_0) - t > -\frac{\lambda(a_n)}{n}$ for all $n \geq N$. So for $n \geq N$ we have $n\lambda(x-x_0) + \lambda(a_n) > nt$. Since $\lambda(a_n(x-x_0)^n) = \lambda(a_n) + n\lambda(x-x_0)$ we have for $n \geq N$ $\lambda(a_n(x-x_0)^n) > nt$ where $t > 0$. Thus $\lim_{n \to \infty} \lambda(a_n(x-x_0)^n) = \infty$ which gives $\lim_{n \to \infty} a_n(x-x_0)^n = 0$. Hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is convergent by Corollary (2.8.1).

Suppose $\lambda(x-x_0) < \lambda_0$. Then for all $N \in \mathbb{N}$ there is some $n > N$ such that $\lambda(x-x_0) < -\frac{\lambda(a_n)}{n}$ and hence $n\lambda(x-x_0) + \lambda(a_n) < 0$. This implies $\lim_{n \to \infty} \lambda(a_n(x-x_0)^n)$ may not exist and certainly isn’t infinity. Thus $(a_n(x-x_0)^n)$ is not a null sequence and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is divergent by Corollary (2.8.1).

Suppose $\lambda(x_x_0) = \lambda_0$ and $-\lambda(a_n) > n\lambda_0$ for infinitely many $n$. Then $-\lambda(a_n) > n\lambda(x-x_0)$ for infinitely many $n$. Therefore for any $N \in \mathbb{N}$ we can find an $n > N$ such that $\lambda(a_n) + n\lambda(x-x_0) < 0$. This entails that $(a_n(x-x_0)^n)$ is not a null sequence and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is divergent by Corollary (2.8.1).
2.3 Calculus on $\mathcal{F}$

In this section we will define continuity and differentiability as in $\mathbb{R}$. We shall then motivate the interest in power series by showing that functions on $\mathcal{F}$ which are continuous or differentiable, due to the disconnectedness of our field, do not have the analytic properties of continuous or differentiable functions of real variables.

**Definition 2.8.** (Continuity at a point). Let $D \subset \mathcal{F}$ and let $f: D \to \mathcal{F}$. Then we say that $f$ is continuous at $x_0$ in $D$ if for every $\epsilon > 0$ in $\mathcal{F}$ there is some $\delta > 0$ in $\mathcal{F}$ such that for $x \in D$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$.

**Definition 2.9.** (Continuity on a set). Let $D \subset \mathcal{F}$ and $f: D \to \mathcal{F}$, then $f$ is continuous on $D$ if $f$ is continuous at $x_0$ for all $x_0$ in $D$.

**Lemma 2.14.** Let $D \subset \mathcal{F}$, a function $f: D \to \mathcal{F}$ is continuous at $x_0$ in $D$ if and only if for every sequence $(x_n)$ in $D$ which converges to $x_0$ we have that $(f(x_n))$ converges to $f(x_0)$.

**Proof.** Let $f: D \to \mathcal{F}$.

$(\Rightarrow)$: Suppose $f$ is continuous at $x_0$ in $D$ and suppose $(x_n)$ in $D$ converges to $x_0$. Let $\epsilon > 0$ in $\mathcal{F}$ be given, since $f$ is continuous at $x_0$ there is some $\delta > 0$ such that for $x$ in $D$ with $|x_0 - x| < \delta$ we have $|f(x_0) - f(x)| < \epsilon$. The convergence of $(x_n)$ gives us an $N_\delta$ in $\mathbb{N}$ such that for $n \geq N_\delta$ we have $|x_0 - x_n| < \delta$; and hence for $n \geq N_\delta$ we have $|f(x_0) - f(x_n)| < \epsilon$. Thus $(f(x_n))$ converges to $f(x_0)$.

$(\Leftarrow)$: Suppose $(f(x_n))$ converges to $f(x_0)$ for every $(x_n)$ in $D$ which converges to $x_0$. Let $\epsilon > 0$ in $\mathcal{F}$ be given and consider $A = f^{-1}(B_O(f(x_0), \epsilon))$. Suppose for all $\delta > 0$ we have $[B_O(x_0, \delta) \setminus A] \cap D$ is non-empty. We then have that $[B_O(x_0, d^n) \setminus A] \cap D$ is non-empty for all $n \in \mathbb{N}$. For each $n$ select an element of $[B_O(x_0, d^n) \setminus A] \cap D$ and call it $x_n$. Then we have that $|x_0 - x_n| < d^n$ for all $n$ so $(x_n)$ converges to $x_0$, however
\[ |f(x_0) - f(x_n)| \geq \epsilon \] for all \( n \) because \( x_n \notin A \) and hence \( f(x_n) \notin B_O(f(x_0), \epsilon) \). This is a contradiction as we assumed the convergence of \((x_n)\) to \( x_0 \) gives the convergence of \((f(x_n))\) to \( f(x_0) \). So then there is some \( \delta > 0 \) such that \((B_O(x_0, \delta) \cap D) \subseteq (A \cap D)\) therefore \( f \) is continuous at \( x_0 \).

\[ \square \]

**Theorem 2.15.** Let \( D \subset \mathcal{F} \); let \( \alpha \) in \( \mathcal{F} \), and let \( f: D \to \mathcal{F} \), and \( g: D \to \mathcal{F} \) be continuous at \( x_0 \) in \( D \). Then \( f + \alpha g \) and \( f \cdot g \) are continuous at \( x_0 \).

**Proof.** Let \((x_n)\) be a sequence in \( D \) that converges to \( x_0 \). Then we know by Lemma (2.14) that \((f(x_n))\) and \((g(x_n))\) converge to \( f(x_0) \) and \( g(x_0) \) respectively. We also have that for \( n \geq 1 \)
\[
(f + \alpha g)(x_n) = f(x_n) + \alpha g(x_n)
\]
so by Corollary (2.2.1) we have \((f + \alpha g)(x_n)\) converges to \( f(x_0) + \alpha g(x_0) = (f + \alpha g)(x_0) \). So \( f + \alpha g \) is continuous at \( x_0 \) by Lemma (2.14).

We also have for \( n \geq 1 \)
\[
f(x_n) \cdot g(x_n) = (f(x_n) - f(x_0)) \cdot (g(x_n) - g(x_0))
\]
\[
- g(x_0) \cdot (f(x_0) - f(x_n))
\]
\[
- f(x_0) \cdot (g(x_0) - g(x_n))
\]
\[
+ f(x_0) \cdot g(x_0)
\]
Thus \((f \cdot g)(x_n) = f(x_n) \cdot g(x_n)\) converges to \( f(x_0) \cdot g(x_0)\), as both \( f(x_n) - f(x_0) \) and \( g(x_n) - g(x_0) \) converge to \( 0 \), so \( f \cdot g \) is continuous at \( x_0 \).

\[ \square \]

**Corollary 2.15.1.** Let \( D \subset \mathcal{F} \), \( f, g: D \to \mathcal{F} \) be continuous on \( D \), and \( \alpha \) in \( \mathcal{F} \) be given. Then \( f + \alpha g \) and \( f \cdot g \) are continuous on \( D \).
**Theorem 2.16.** Let $D_f, D_g \subset \mathcal{F}$. Let $f : D_f \to \mathcal{F}$ and $g : D_g \to \mathcal{F}$ be such that $f(D_f) \subset D_g$, $f$ is continuous at $x_0$ in $D_f$, and $g$ is continuous at $f(x_0)$ in $D_g$. Then $(g \circ f) : D_f \to \mathcal{F}$ where $(g \circ f)(x) = g(f(x))$ is continuous at $x_0$.

**Proof.** Suppose $(x_n)$ is a sequence in $D_f$ which converges to $x_0$. Then $(f(x_n))$ is a sequence in $D_g$ which converges to $f(x_0)$ by the continuity of $f$ at $x_0$. Given $(f(x_n))$ is a sequence in $D_g$ such that $(f(x_n))$ converges to $f(x_0)$ we have that $(g(f(x_n)))$ is a sequence in $\mathcal{F}$ which must converge to $g(f(x_0))$ by the continuity of $g$ at $f(x_0)$. Thus for any $(x_n)$ in $D_f$ converging to $x_0$ we have that $(g(f(x_n))) = ((g \circ f)(x_n))$ converges to $g(f(x_0)) = (g \circ f)(x_0)$. Therefore $g \circ f$ is continuous at $x_0$ by Lemma(2.14). □

**Corollary 2.16.1.** Let $D_f, D_g \subset \mathcal{F}$. Let $f : D_f \to \mathcal{F}$ and $g : D_g \to \mathcal{F}$ be such that $f(D_f) \subset D_g$, $f$ is continuous on $D_f$, and $g$ is continuous on $D_g$. Then $g \circ f$ is continuous on $D_f$.

**Definition 2.10.** (Differentiability at a point). Let $D \subset \mathcal{F}$ be open. A function $f : D \to \mathcal{F}$ is differentiable at a point $x_0$ in $D$ if there is some $G$ in $\mathcal{F}$ such that for every $\epsilon > 0$ in $\mathcal{F}$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - G \right| < \epsilon;$$

if this is the case we call $G$ the derivative of $f$ at $x_0$ and write $G = f'(x_0)$.

**Definition 2.11.** (Differentiability on a set). Let $D \subset \mathcal{F}$ be open. A function $f : D \to \mathcal{F}$ is differentiable on $D$ if it is differentiable at $x_0$ for all $x_0$ in $D$.

Again, like in $\mathbb{R}$, the following results about differentiation hold:

**Corollary 2.16.2.** (Differentiation rules)[3] For $f, g$, differentiable functions on sets $D_f$ and $D_g$ respectively

- $(cf)'(x) = cf'(x)$
- $(f + g)'(x) = f'(x) + g'(x)$
\[
\begin{align*}
(f \cdot g)'(x) &= f'(x)g(x) + g'(x)f(x) \\
\left(\frac{f}{g}\right)'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \text{ where both are well defined.} \\
(f \circ g)'(x) &= (f' \circ g)(x)g'(x)
\end{align*}
\]

With the above definitions of continuity and differentiability we shall be able to provide examples of how continuity or differentiability of a function on \(F\) are not sufficient to guarantee any of the calculus theorems hold.

Example 1: Let \(f_1 : [0, 1] \to F\) be given by

\[
f_1(x) = \begin{cases} 
  d^{-1} & \text{if } 0 \leq x < d \\
  d^{-1/\lambda(x)} & \text{if } d \leq x \ll 1 \\
  1 & \text{if } x \sim 1 
\end{cases}
\]

We claim \(f_1\) is continuous on \([0, 1]\) but \(f_1\) is not bounded on \([0, 1]\).

Proof. Let \(\epsilon > 0\) in \(F\) be given and let \(x \in [0, 1]\). For \(x = 0\) let \(\delta = \frac{1}{2}d\), then \(f_1(x)\) is constant on \([0, \delta]\) so \(f_1(x)\) is continuous at \(x = 0\). For \(x \neq 0\) and \(x \neq d\) there is \(q \in \mathbb{Q}\) such that \(q > \lambda(x)\), let \(\delta = d^q\). For all \(y \in (x - \delta, x + \delta) \cap [0, 1]\) we have \(\lambda(y) = \lambda(x)\) and thus \(f_1(x)\) is constant, and therefore continuous, on \((x - \delta, x + \delta) \cap [0, 1]\). For \(x = d\) let \(\delta = d^2\). For \(x \in (d, d + d^2)\) \(\lambda(x) = \lambda(d) = 1\) and so \(f_1(x) = d^{-1}\), for \(x \in (d - d^2, d)\) we have \(x < d\) and so \(f_1(x) = d^{-1}\) and \(f_1(d) = d^{-1}\). Therefore \(f_1(x)\) is constant (and therefore continuous) on \((d - d^2, d + d^2)\), specifically \(f_1(x)\) is continuous at \(x = d\). Therefore \(f_1(x)\) is continuous at all \(x \in [0, 1]\) and hence it is continuous on \([0, 1]\).

Recall for every \(q \in (0, 1] \cap \mathbb{Q}\) there is some \(d \leq x_q \ll 1\) such that \(\lambda(x_q) = q\).

Let \(M > 0\) in \(F\) be given with \(\lambda(M) = q_M\). Then there is some \(q \in (0, 1] \cap \mathbb{Q}\) such that \(-1/q < q_M\). Then \(f_1(x_q) = d^{-1/q}\) and so \(\lambda(f_1(x_q)) < \lambda(M)\) which entails that \(f_1(x_q) \gg M\). Therefore \(f_1\) is not bounded on \([0, 1]\). \(\square\)
Example 2: Let $f_2(x) : [-1, 1] \to \mathcal{F}$ be given by

$$f_2(x) = x - x[0].$$

We claim this function is continuous on $[-1, 1]$ and bounded, however it does not attain a maximum or minimum value on $[-1, 1]$.

**Proof.** Let $x \neq 0$ in $[-1, 1]$ be given. Then $x[0]$ is constant on $(x - d^{\lambda(x)+1}, x + d^{\lambda(x)+1})$ and so $f_2(x)$ is linear, and therefore continuous by taking $\delta = \frac{1}{2} \epsilon$, on $(x - d^{\lambda(x)+1}, x + d^{\lambda(x)+1})$ and hence at $x$. On $(-d, d)$ the function is given by $x$ and is therefore continuous at 0. Therefore $f_2(x)$ is continuous on $[-1, 1]$.

For all $x$ in $[-1, 1]$ we have $\lambda(x) \geq 0$ which entails that $\lambda(x - x[0]) > 0$ for all $x$ in $[-1, 1]$. Therefore $f_2([-1, 1])$ is bounded above by any positive real number and below by any negative real number.

Suppose $f_2(x)$ attains some maximum at $x = x_M$ on $[-1, 1]$. Note $f_2(x) = 0$ at any real number and $f_2(d) = d$, so $x_M$ cannot be real. Then $(x_M - d^{\lambda(x_M)+1}, x_M + d^{\lambda(x_M)+1}) \subset [-1, 1]$ and $f_2(x)$ is linear on $(x_M - d^{\lambda(x_M)+1}, x_M + d^{\lambda(x_M)+1})$. Therefore for $\delta < d^{\lambda(x_M)+1}$ we have $f_2(x_M + \delta) > f_2(x_M)$ which contradicts that $x_M$ is a maximum, therefore $f_2(x)$ does not attain a maximum on $[-1, 1]$. The exact same argument works for why $f_2(x)$ does not attain a minimum value. \hfill \Box

Example 3: Let $f_3 : [0, 1] \to \mathcal{F}$ be given by

$$f_3(x) = \begin{cases} 
1 & \text{if } x \sim 1 \\
0 & \text{if } x \ll 1
\end{cases}$$

We claim this function is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $f'(x) = 0$ everywhere, but is non-constant and does not attain intermediate values between $f(0)$ and $f(1)$.
Proof. For each \( x \) in \([0, 1]\) either \( x \sim 1 \) or \( x \ll 1 \). If \( x \ll 1 \) then \( x + h \ll 1 \) for \( h \) in \((-d, d)\) and so \( f_3 \) is constant, and therefore continuous and differentiable, on \((x - d, x + d) \cap [0, 1] \) with derivative \( f'_3 = 0 \). If \( x \sim 1 \) then \( x + h \sim 1 \) for \( h \) in \((-d, d)\) and \( f_3 \) is constant on \((x - d, x + d) \cap [0, 1] \) and again continuous and differentiable with derivative \( f'_3 = 0 \). Therefore \( f_3 \) has derivative 0 everywhere on \((0, 1)\).

Trivially \( f_3(x) \) is non-constant \((0 \neq 1)\) and does not attain any of the intermediate values between \( f(0) \) and \( f(1) \), for instance there is no \( x \) such that \( f_3(x) = \frac{1}{2} \). \( \square \)
Chapter 3

Power Series

In the previous section it was shown that continuity or differentiability are not strong enough properties to guarantee the basic results of calculus. In this chapter we will show that power series satisfy the intermediate value theorem, the extreme value theorem, and the mean value theorem, among other properties.

3.1 Convergence

Proposition 3.1. Let $[a, b] \subset \mathcal{F}$ and $f : [a, b] \to \mathcal{F}$ be given by a convergent power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$ 

If we define a new sequence $(A_n)$ given by

$$A_n = c a_n$$
where $c$ is some constant in $\mathcal{F}$ then

$$F(x) = \sum_{n=0}^{\infty} A_n(x - x_0)^n$$

is a convergent power series on $[a, b]$ and $F(x) = cf(x)$ for all $x$ in $[a, b]$.

Proof. The series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges on $[a, b]$ and so by Corollary (2.10.2)

$$\lim_{n \to \infty} a_n(x - x_0)^n = 0$$

on $[a, b]$. Then by Corollary (2.2.1)

$$\lim_{n \to \infty} A_n(x - x_0)^n = \lim_{n \to \infty} c a_n(x - x_0)^n = c \lim_{n \to \infty} a_n(x - x_0)^n = 0$$

on $[a, b]$. Hence the series

$$\sum_{n=0}^{\infty} A_n(x - x_0)^n$$

converges on $[a, b]$ by Corollary (2.10.2).

It follows that

$$cf(x) = c \lim_{N \to \infty} \sum_{n=0}^{N} a_n(x - x_0) = \lim_{N \to \infty} c \sum_{n=0}^{N} a_n(x - x_0)^n$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} c a_n(x - x_0)^n = \lim_{N \to \infty} \sum_{n=0}^{N} A_n(x - x_0)^n$$

$$= \sum_{n=0}^{\infty} A_n(x - x_0)^n = F(x)$$

on $[a, b]$.

Proposition 3.2. Let $[a, b] \subset \mathcal{F}$, $x_0$ in $[a, b]$, and $f : [a, b] \to \mathcal{F}$ be a convergent
power series on \([a, b]\) given by

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n. \]

If we define a new sequence \((A_n)\) and \(\xi_0\) given by

\[ A_n = a_n d^{n\lambda(b-a)} \quad \text{and} \quad \xi_0 = d^{-\lambda(b-a)} x_0 \]

then \(F: [d^{-\lambda(b-a)} a, d^{-\lambda(b-a)} b] \to F\) given by

\[ F(\xi) = \sum_{n=0}^{\infty} A_n (\xi - \xi_0)^n \]

is a convergent power series on \([d^{-\lambda(b-a)} a, d^{-\lambda(b-a)} b]\); and for \(x\) in \([a, b]\) \(f(x) = F(d^{-\lambda(b-a)} x)\).

Moreover \((A_n)\) is a regular sequence.

**Proof.** Let \(n\) in \(\mathbb{N}\) and \(x\) in \([a, b]\). We then have

\[
\begin{align*}
    a_n(x - x_0)^n &= a_n ((d^{\lambda(b-a)}) (d^{-\lambda(b-a)}) (x - x_0))^n \\
    &= a_n (d^{\lambda(b-a)})^n (d^{-\lambda(b-a)} x - d^{-\lambda(b-a)} x_0)^n \\
    &= A_n (\xi - \xi_0)^n
\end{align*}
\]

where \(\xi = d^{-\lambda(b-a)} x\) and is therefore in \([d^{-\lambda(b-a)} a, d^{-\lambda(b-a)} b]\).

Since \(\sum_{n=0}^{\infty} a_n(x - x_0)^n\) converges for all \(x\) in \([a, b]\) we have \(\sum_{n=0}^{\infty} A_n (\xi - \xi_0)^n\) converges for all \(\xi\) in \([d^{-\lambda(b-a)} a, d^{-\lambda(b-a)} b]\) and thus converges at some finite \((\xi - \xi_0)\) so, by Corollary (2.10.2), the sequence \((A_n)\) must be null and it is therefore regular by Theorem (2.7). Thus \(F(\xi)\) is given by a convergent power series on \([d^{-\lambda(b-a)} a, d^{-\lambda(b-a)} b]\) and for \(x\) in \([a, b]\) \(f(x) = F(\xi) = F(d^{-\lambda(b-a)} x)\).

**Lemma 3.1.** (Uniform boundedness of power series) For a convergent power series
Given that \((a_n)\) is a null sequence so is \((d^{-1}|a_n|)\) hence there must be some \(M\) in \(\mathcal{F}\) such that \(\sum_{n=0}^{\infty} |a_n|d^{-1} < M\). Since 0 \(\leq \lambda((x-x_0)^n)\) for all \(x\) in \([a,b]\) and \(a_n \leq |a_n|\) for all \(n\) we have \(M > \sum_{n=0}^{\infty} |a_n|d^{-1} > |\sum_{n=0}^{\infty} a_n(x-x_0)^n|\) and hence \(|f(x)| < M\) on \([a,b]\).

Indeed because \(|a_n|\) is a null sequence there is a \(N_F\) in \(\mathbb{N}\) such that for \(n \geq N_F\) we have \(\lambda(|a_n|) > 0\) and so \(i(f) = \min\{\lambda(|a_n|) \mid n \leq N_F\}\) exists and \(i(f) = \min\{\lambda(a_n)\}\).

Due to the scaling of the argument of \(f\) we have \(\lambda(f(x)) \geq i(f)\) on \([a,b]\) and since \(\sum_{n=0}^{N_F} a_n(x-x_0)^n\) is a polynomial with at least one coefficient \(a_i\) with \(\lambda(a_i) = i(f)\) we have \(\lambda(\sum_{n=0}^{N_F} a_n(x-x_0)^n) = \lambda(f(x)) = i(f)\) for at least one finite \(x_c\) in \([a,b]\).

Therefore \(i(f) = \min\{\lambda(f(x)) \mid x \in [a,b]\}\).

\[\square\]

**Definition 3.1.** (Index of a power series \(i(f)\)) For \([a,b] \subset \mathcal{F}, \ x_0 \in [a,b]\), and \(f: [a,b] \to \mathcal{F}\) a convergent power series given by

\[f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n\]

on \([a,b]\) we define the index of \(f\)

\[i(f) = \min\{\lambda(f(x)) \mid x \in [a,b]\}\]
Corollary 3.1.1. Let \([a, b] \subset \mathcal{F}\) where \(\lambda(b-a) = 0\), \(x_0\) in \([a, b]\), and \(f : [a, b] \rightarrow \mathcal{F}\) be a convergent power series on \([a, b]\) given by

\[
f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n
\]

then if \(\lambda(b-a) \leq 0\) we have

\[
\lambda(a_n) \geq i(f)
\]

Proof. Let \(x_c\) in \([a, b]\) be such that \(\lambda(x_c - x_0) = 0\), then \(\lambda(a_n(x_c - x_0)^n) = \lambda(a_n)\) for all \(n\). Since \(\sum_{n=0}^{\infty} a_n(x_c - x_0)^n\) converges to \(f(x_c)\) we have, by Corollary (2.10.2), \(\lim_{n \to \infty} a_n(x_c - x_0)^n = 0\); so \(\lim_{n \to \infty} a_n = 0\) hence \((a_n)\) is regular by Theorem (2.7). Thus \(m = \min\{\cup_{n \in \mathbb{N}}\text{supp}(a_n)\}\) is well defined. Trivially \(\lambda(a_n) \geq m\); and, as in Lemma (3.1), one can define a polynomial whose coefficients have \(\lambda(a_n) = m\) which must attain a value \(A\) at some \(x_A\) in \([a, b]\) where \(\lambda(A) = m\). Then, as in Lemma (3.1) \(i(f) = m\) and hence \(\lambda(a_n) \geq i(f)\).

\[
S_{k,j;N} = \left\{(n_1, \ldots, n_k) \in \{1, \ldots, N\}^k \mid \sum_{i=1}^{k} n_i = j\right\}
\]

and

\[
S_{k,j} = \left\{(n_1, \ldots, n_k) \in \mathbb{N}^k \mid \sum_{i=1}^{k} n_i = j\right\}
\]

Lemma 3.2. Let \([a, b] \subset \mathcal{F}\) be given, \(x_0\) in \([a, b]\), and \(f : [a, b] \rightarrow \mathcal{F}\) be given by a
convergent power series

\[ f(x) = \sum_{n=1}^{\infty} a_n (x - x_0)^n \]
on \([a, b]\). Then for a fixed \(k \geq 2\) in \(\mathbb{N}\) the function \((f(x))^k\) is also given by a convergent power series on \([a, b]\), that is there is a sequence \((c_n)\) such that

\[ (f(x))^k = \sum_{j=k}^{\infty} c_j (x - x_0)^j \]

for \(x\) in \([a, b]\).

Proof. Since by Theorem (3.2) one can scale the argument of a convergent power series to be finite we shall, without loss of generality, assume \([a, b]\) is an interval of finite length with \(\lambda((x - x_0)) \geq 0\) on \([a, b]\). Thus \((a_n)\) is a null sequence as \(\lim_{n \to \infty} a_n (x - x_0)^n = 0\) for finite \((x - x_0)\) by Corollary (2.10.2). The sequence \((a_n)\) is also regular by Theorem (2.7).

Let \(k \geq 2\) be fixed and consider for some finite \(N\) the product

\[ \left( \sum_{n=1}^{N} a_n (x - x_0)^n \right) \cdots \left( \sum_{n=1}^{N} a_n (x - x_0)^n \right). \]

This product has finitely many terms and can be rearranged and written as

\[ \sum_{j=k}^{Nk} \alpha_{k,j} (x - x_0)^j. \]

where the sequence \((\alpha_{k,j})\) is defined as

\[ \alpha_{k,j} = \sum_{(n_1, \ldots, n_k) \in S_{k,j,N}} \prod_{i=1}^{k} a_{n_i}. \quad (3.1) \]
Likewise \( \left( \sum_{n=1}^{N+1} a_n (x - x_0)^n \right)^k \) can be rewritten as

\[
\sum_{j=k}^{kN+k} \beta_{k, j}(x - x_0)^j
\]

where

\[
\beta_{k, j} = \sum_{(n_1, \ldots, n_k) \in S_{k, j, N+1}} \prod_{i=1}^{k} a_{n_i}.
\]

Note however that, for \( j < N + k \), we have \( S_{k, j, N} = S_{k, j, N+1} \). Thus for \( j < N + k \) we have \( \alpha_{k, j} = \beta_{k, j} \) which gives

\[
\sum_{j=k}^{N+k-1} \alpha_{k, j}(x - x_0)^j = \sum_{j=k}^{N+k-1} \beta_{k, j}(x - x_0)^j.
\]

Thus for any \((k, j)\) pair where \( j \geq k \) there is some finite \( N(k, j) = (j - k) + 1 \) such that for \( m \geq N(k, j) \) we have

\[
\left( \sum_{n=1}^{N(k, j)} a_n (x - x_0)^n \right)^k \quad \text{and} \quad \left( \sum_{n=1}^{m} a_n (x - x_0)^n \right)^k
\]

contribute the same \((x - x_0)^j\) term for all \( j \leq N(k, j) \).

Then define for each \( j \)

\[
c_j = \sum_{(n_1, \ldots, n_k) \in S_{k, j, N(k, j)}} \prod_{i=1}^{k} a_{n_i} = \sum_{(n_1, \ldots, n_k) \in S_{k, j}} \prod_{i=1}^{k} a_{n_i}.
\]
We intend to show that
\[
\sum_{j=k}^{\infty} c_j (x - x_0)^j = \left( \sum_{n=1}^{\infty} a_n (x - x_0)^n \right)^k
\]
on \([a, b]\).

Since \((a_n)\) is a null sequence it must be that there is some \(N_F\) such that for \(n > N_F\) we have \(\lambda(a_n) > 0\). Let

\[ M_0 = \max\{d^{-1}|a_n| \mid n \leq N_F \} \text{ and } M = \max\{M_0, 2\}. \]

Let \(1 > \epsilon > 0\) in \(\mathcal{F}\) be given. Since \((a_n)\) is a null sequence there must be some \(N_1(\epsilon)\) such that for \(n > N_1(\epsilon)\)

\[ \lambda(a_n) > 3 + \lambda(\epsilon) - k \lambda(M) \quad (3.2) \]

and some \(N_2(\epsilon)\) such that for \(n > N_2(\epsilon)\)

\[ \lambda \left( \sum_{n=1}^{\infty} |a_n| \right) > 2 + \lambda(\epsilon) - k \lambda(M). \quad (3.3) \]
By Equation (3.3) we have for $l \geq N_2(\epsilon)$

$$
\left| \left( \sum_{n=1}^{\infty} a_n (x - x_0)^n \right)^k - \left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^k \right|
= \left| \left( \sum_{n=1}^{l} a_n (x - x_0)^n + \sum_{n=l+1}^{\infty} a_n (x - x_0)^n \right)^k - \left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^k \right|
= \sum_{j=0}^{k} \binom{k}{j} \left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^{k-j} \left( \sum_{n=l+1}^{\infty} a_n (x - x_0)^n \right)^j
\leq \sum_{j=1}^{k} \binom{k}{j} \left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^{k-j} \left( \sum_{n=l+1}^{\infty} a_n (x - x_0)^n \right)^j
\leq \sum_{j=1}^{k} \binom{k}{j} \left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^{k-j} \left( \sum_{n=l+1}^{\infty} a_n (x - x_0)^n \right)^j
\leq \sum_{j=1}^{k} (M)^{k-j} (d^2 \epsilon M^{-k})^j
\leq \sum_{j=1}^{k} M^{-1} d^2 \epsilon
= k M^{-1} d^2 \epsilon < d \epsilon.
$$

For any finite $l$ one can write

$$
\left( \sum_{n=1}^{l} a_n (x - x_0)^n \right)^k = \sum_{j=k}^{kl} \alpha_{k,j} (x - x_0)^j
$$

where $\alpha_{k,j}$ is the same as in Equation (3.1). Each $\alpha_{k,j}$ is a sum of products of $k$ terms. In each product there must be at least one term whose index is greater than
or equal to \( \lfloor \frac{j}{k} \rfloor \). Since each term of the product is bounded above by \( M \) by Equation (3.2) we have if \( l > kN_1(\epsilon) \)

\[
|\alpha_{k,l}| = \left| \sum_{(n_1, \ldots, n_k) \in S_{k,l,N}} \prod_{i=1}^{k} a_{n_i} \right|
\leq Na_{n_1}M^{k-1}
< Nd^3\epsilon M^{-k}M^{k-1}
= Nd^3\epsilon M^{-1}
< d^2\epsilon
\]

where \( N \) is the cardinality of \( S_{k,j,l} \) (therefore finite) and \( n_I \) is one of the members of each list in \( S_{k,j,l} \) that has \( \lfloor \frac{n_I}{k} \rfloor > N_1(\epsilon) \).

Now let \( J = \max\{kN_1(\epsilon), N_2(\epsilon), k\} \) and \( x \in [a, b] \). For \( l > J \) we have

\[
\left| \sum_{j=k}^{l} c_j(x-x_0)^j - (f(x))^k \right|
= \left| \sum_{j=k}^{l} c_j(x-x_0)^j - (f(x))^k + \left( \sum_{n=1}^{l} a_n(x-x_0)^n \right)^k - \left( \sum_{n=1}^{\infty} a_n(x-x_0)^n \right)^k \right|
\leq \left| \sum_{j=k}^{l} c_j(x-x_0)^j - \left( \sum_{n=1}^{l} a_n(x-x_0)^n \right)^k \right| + \left| \left( \sum_{n=1}^{\infty} a_n(x-x_0)^n \right)^k - \left( \sum_{n=1}^{l} a_n(x-x_0)^n \right)^k \right|
\leq \left| \sum_{j=k}^{kl} \alpha_{k,j}(x-x_0)^j \right| + d\epsilon
\leq (kl - l - 1)d^2\epsilon + d\epsilon
< d\epsilon + d\epsilon < \epsilon.
\]

Therefore \( \lim_{J \to \infty} \sum_{j=k}^{J} c_j(x-x_0)^j = (f(x))^k \) for \( x \in [a, b] \), which implies \( (c_j) \) converges to 0 by Corollary (2.10.2). Thus \( (f(x))^k \) is given by a convergent power series
\[
\sum_{j=k}^{\infty} c_j (x - x_0)^j \text{ on } [a, b].
\]

**Theorem 3.3.** Let \([a, b] \subset \mathcal{F}\) and \([c, e] \subset \mathcal{F}\). Let \(f : [a, b] \to \mathcal{F}\) and \(g : [c, e] \to \mathcal{F}\) be convergent power series given by

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n(x - x_0)^n
\]

\[
g(y) = c_0 + \sum_{k=1}^{\infty} b_k(y - f(x_0))^k
\]

where \(f([a, b]) \subset [c, e]\).

Then there is a sequence \((c_j)\) such that

\[
c_0 + \sum_{j=1}^{\infty} c_j (x - x_0)^j
\]

converges to \((g \circ f)(x)\) for all \(x\) in \([a, b]\).

**Proof.** By Theorem (3.2) and Theorem (3.1) the function \(F : [d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b] \to \mathcal{F}\) given by

\[
F(x) = d^{-\lambda(e-c)} f(d^{\lambda(b-a)}x)
\]

is given by a convergent power series for \(x\) in \([d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]\) with

\[
F([d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]) = d^{-\lambda(e-c)} f([a, b]) \subset [d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e].
\]

Likewise \(G : [d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e] \to \mathcal{F}\) where

\[
G(y) = g(d^{\lambda(e-c)}y)
\]

is given by convergent power series for \(y\) in \([d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e]\) and

\[
G([d^{-\lambda(e-c)}c, d^{-\lambda(e-c)}e]) = g([c, e]).
\]
Thus for $x$ in $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ we have

$$(G \circ F)(x) = G(d^{-\lambda(e-c)}f(d^\lambda(b-a)x))$$

$$= g(f(d^\lambda(b-a)x)) = (g \circ f)(d^\lambda(b-a)x).$$

Therefore $(g \circ f)(x)$ is given by a convergent power series on $[a, b] = d^\lambda(b-a)[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$ if and only if $(G \circ F)(x)$ is on $[d^{-\lambda(b-a)}a, d^{-\lambda(b-a)}b]$; so without loss of generality we may assume that $\lambda(b-a) = \lambda(e-c) = 0$.

Given that we assume $\lambda(b-a) = \lambda(e-c) = 0$ there must be some $x_c$ in $[a, b]$ and $y_c$ in $[c, e]$ such that $(x_c - x_0)$ and $(y_c - f(x_0))$ ($y_c$ not necessarily $f(x_c)$) are finite. Both $\sum_{n=1}^\infty a_n(x_c - x_0)^n$ and $\sum_{k=1}^\infty b_k(y_c - f(x_0))^k$ converge and hence by Corollary (2.10.2) $\lim_{n \to \infty} a_n = \lim_{k \to \infty} b_k = 0$.

For $x$ in $[a, b]$ we have

$$(g \circ f)(x) = c_0 + \sum_{k=1}^\infty b_k(f(x) - f(x_0))^k$$

$$= c_0 + \sum_{k=1}^\infty b_k \left(a_0 + \sum_{n=1}^\infty a_n(x - x_0)^n - f(x_0)\right)^k$$

$$= c_0 + \sum_{k=1}^\infty b_k \left(\sum_{n=1}^\infty a_n(x - x_0)^n\right)^k.$$

Let

$$V_k(x) = b_k \left(\sum_{n=1}^\infty a_n(x - x_0)^n\right)^k$$

which by Lemma (3.2) can be rewritten as

$$V_k(x) = b_k \sum_{j=k}^\infty c_{k,j}(x - x_0)^j.$$
where

\[ c_{k,j} = b_k \sum_{(n_1, \ldots, n_k) \in S_{k,j}} \prod_{i=1}^{k} a_{n_i}. \]

Since \( S_{k,j} \) is empty for \( k > j \) we can define a sequence \((c_j)\) given by, for \( j \geq 1\),

\[ c_j = \sum_{k=1}^{j} c_{k,j}. \quad (3.4) \]

We wish to show that \( \lambda(a_n) \geq 0 \) for \( n \geq 1 \). By Corollary (3.1.1) we know \( \lambda(a_n) \geq i(f) \), so if \( i(f) \geq 0 \) we are done (note that \( \lambda(e - c) = 0 \) does not guarantee this, for instance if \( e = d^{-1} + 1 \) and \( c = d^{-1} - 1 \) then \( \lambda(e - c) = 0 \) but for all \( y \) in \([c, e]\) we have \( \lambda(y) < 0 \)), however if \( i(f) < 0 \) Corollary (3.1.1) is not enough to ensure \( \lambda(a_n) \geq 0 \) for \( n \geq 1 \). Suppose \( i(f) < 0 \) and for some \( \Lambda \subset \mathbb{N} \) we have \( \lambda(a_n) < 0 \) for \( n \) in \( \Lambda \). Because \( \lim_{n \to \infty} a_n = 0 \) we know \( \Lambda \) must be finite and so we can define a polynomial and \( x_A \), as in Lemma (3.1), such that

\[ P(x) = \sum_{n \in \Lambda} a_n (x - x_0)^n \]

\( \lambda(x_A - x_0) = 0 \) and

\[ \lambda(P(x_A)) = \min\{\lambda(a_n) \mid n \in \Lambda\} = I. \]

Because \([a, b]\) is finite in length there must be some \( x_\alpha \), for each \( 0 < \alpha < 1 \) in \( \mathbb{R} \), in \([a, b]\) such that

\[ (x_\alpha - x_0) = \alpha(x_A - x_0). \]

Then

\[ P(x_\alpha)[I] = \sum_{n \in \Lambda} \alpha^n (a_n(x_A - x_0)^n)[I] \]

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which is a real polynomial in $\alpha$ and therefore can only be equal to $P(x_A)[I]$ at finitely many $\alpha$ and can only be equal to zero at finitely many $\alpha$. Let $x_B = x_\alpha$ for some $\alpha$ such that $P(x_\alpha)[I] \neq P(x_A)[I]$ and $P(x_\alpha)[I] \neq 0$. Then

$$\lambda(P(x_A) - P(x_B)) = I < 0$$

and, as in Lemma (3.1),

$$\lambda(f(x_A) - f(x_B)) = \lambda(P(x_A) - P(x_B)) < 0$$

Let $B = \max\{f(x_A), f(x_B)\}$ and $A = \min\{f(x_A), f(x_B)\}$, then $B - A > e - c$ however we need $B$ and $A$ in $[c, e]$ as $f([a, b]) \subset [c, e]$, this is a contradiction. Therefore if $i(f) < 0$ we have $\lambda(a_n) \geq 0$ for $n \geq 1$ and if $i(f) \geq 0$ then $\lambda(a_n) \geq 0$ for $n \geq 1$ as well. Thus for any $\Lambda \subset \mathbb{N}$

$$\lambda \left( \prod_{n \in \Lambda} a_n \right) \geq \lambda(a_j)$$

for any $j$ in $\Lambda$. Corollary (3.1.1) also ensures that

$$M = \max\{|b_k| \mid k \in \mathbb{N}\}$$

is well defined.

Let $\epsilon > 0$ in $\mathcal{F}$ be given. Since $(a_n)$ is a null sequence there is an $N(\epsilon)$ in $\mathbb{N}$ such that for $n \geq N(\epsilon)$

$$\lambda(a_n) > 2 + \lambda(\epsilon) - \lambda(M).$$
There is also a $K(\epsilon)$ in $\mathbb{N}$ such that for $k > K(\epsilon)$

$$\lambda(b_k) > 2 + \lambda(\epsilon).$$

Let $J = \max\{K^2(\epsilon), K(\epsilon) \cdot N(\epsilon)\}$ then for $j > J$ we have

$$c_j = \sum_{k=1}^{j} c_{k,j}$$

$$= \sum_{k=1}^{K(\epsilon)} c_{k,j} + \sum_{k=K(\epsilon)+1}^{j} c_{k,j}.$$ 

Because $j > K(\epsilon) \cdot N(\epsilon)$ we have for $k < K(\epsilon)$ that $\lfloor \frac{j}{k} \rfloor > N(\epsilon)$. Each $(n_1, \ldots, n_k)$, in any $S_{k,j}$ with $j > J$ and $k < K(\epsilon)$, has at least one component $N$ where $N \geq \lfloor \frac{j}{k} \rfloor$. Hence we have when $j > J$ and $k < K(\epsilon)$ for said $N \lambda(a_N) > 2 + \lambda(\epsilon) - \lambda(M)$. This ensures, for $k < K(\epsilon)$ and $j > J$,

$$\lambda(c_{k,j}) = \lambda(b_k) + \lambda\left(\sum_{(n_1, \ldots, n_k) \in S_{k,j}} \prod_{i=1}^{k} a_{n_i}\right)$$

$$> \lambda(M) - 1 + \lambda(a_N)$$

$$> \lambda(M) - 1 + 2 + \lambda(\epsilon) - \lambda(M)$$

$$> \lambda(\epsilon) + 1$$

by Equation (3.5) and the fact that $|S_{k,j}|$ is finite for any finite $k, j$ pair.

Equation (3.5) and the finiteness of $|S_{k,j}|$ also gives us $\lambda(c_{k,j}) \geq \lambda(b_k) - 1$ for any $k, j$ pair and therefore for those $c_{k,j}$ with $k > K(\epsilon)$ and $j > J$.
Thus for \( j > J \)

\[
|c_j| = \left| \sum_{k=1}^{j} c_{k,j} \right| = \left| \sum_{k=1}^{K(\epsilon)} c_{k,j} + \sum_{k=K(\epsilon)+1}^{j} c_{k,j} \right| \leq \left| \sum_{k=1}^{K(\epsilon)} c_{k,j} \right| + \left| \sum_{k=K(\epsilon)+1}^{j} c_{k,j} \right| < \sum_{k=1}^{K(\epsilon)} d\epsilon + \sum_{k=K(\epsilon)+1}^{j} d\epsilon = K(\epsilon)d\epsilon + j - K(\epsilon)d\epsilon = jd\epsilon < \epsilon
\]

Therefore \( \lim_{j \to \infty} c_j = 0 \) and \( \sum_{j=1}^{\infty} c_j(x - x_0)^j \) converges for \((x - x_0)\) which are at most finite, by Corollary (2.10.2), and so \( \sum_{j=1}^{\infty} c_j(x - x_0)^j \) converges on \([a, b]\).

If we once again let \( \epsilon > 0 \), then due to the fact that \((b_k)\) is a null sequence there is some \( K(\epsilon) \) such that for \( k > K(\epsilon) \)

\[
\lambda(b_k) > \lambda(\epsilon) + 1.
\]

Thus if \( k, j > K(\epsilon) \) we have

\[
\lambda(c_{k,j}) = \lambda(b_k) + \lambda \left( \sum_{(n_1, \ldots, n_k) \in S_{k,j}} \prod_{i=1}^{k} a_{n_i} \right) \geq \lambda(b_k) > \lambda(\epsilon) + 1.
\]
Then by Corollary (2.10.5) we have for \( x \) in \([a; b]\)

\[
(g \circ f)(x) = g(f(x_0)) + \sum_{k=1}^{\infty} V_k(x)
\]

\[
= g(f(x_0)) + \sum_{k=1}^{\infty} c_{k,j}(x - x_0)^j
\]

\[
= g(f(x_0)) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{k,j}(x - x_0)^j
\]

\[
= g(f(x_0)) + \sum_{j=1}^{\infty} c_j(x - x_0)^j.
\]

So \((c_j)\), where \(c_0 = g(f(x_0))\) and \(c_n\) is as in Equation (3.4), is a sequence such that \((g \circ f)(x) = c_0 + \sum_{j=1}^{\infty} c_j(x - x_0)^j\) on \([a, b]\). Therefore \((g \circ f)(x)\) is given by a convergent power series on \([a, b]\).

\[\square\]

### 3.2 Analytical Properties

#### Theorem 3.4. (Derivative of power series). If \( f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \) is a convergent power series on \([a, b]\) where \(x_0\) is in \([a, b]\) then for \(x_c\) in \((a, b)\) \( f \) is differentiable at \(x_c\) and

\[
f'(x_c) = \sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1}
\]

where the power series converges on \((a, b)\).

**Proof.** Let \(x_c\) in \((a, b)\). Let \(\epsilon > 0\) in \(F\) be given. Then, since \(\sum_{n=1}^{\infty} a_n(x_c - x_0)^n\) converges, there is an \(N(\epsilon)\) in \(N\) such that for \(n > N(\epsilon)\)

\[
|a_n(x_c - x_0)^n| < d\epsilon|x_c - x_0|.
\]
Hence for \( n > N(\epsilon) \)

\[
|na_n(x_c - x_0)^{n-1}| = \frac{n}{|x_c - x_0|} |a_n(x_c - x_0)^n| < \frac{n}{|x_c - x_0|} d\epsilon |x_c - x_0| < \epsilon.
\]

Therefore \( \sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1} \) converges by Corollary (2.10.2). Let \( G = \sum_{n=1}^{\infty} na_n(x_c - x_0)^{n-1} \) and consider

\[
\left| \frac{\sum_{n=0}^{\infty} a_n(x - x_0)^n - \sum_{n=0}^{\infty} a_n(x_c - x_0)^n}{x - x_c} - G \right| = \left| \frac{\sum_{n=1}^{\infty} a_n((x - x_0)^n - (x_c - x_0)^n)}{x - x_c} - G \right|
\]

\[
= \left| \sum_{n=1}^{\infty} a_n(x - x_c) \frac{\sum_{i=0}^{n-1}(x_c - x_0)^i(x - x_0)^{n-(i+1)}}{x - x_c} - G \right|
\]

\[
= \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x_c - x_0)^i(x - x_0)^{n-(i+1)} - (x_c - x_0)^{n-1} \right|
\]

\[
= \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x_c - x_0)^i (x - x_0)^{n-(i+1)} - (x_c - x_0)^{n-1} \right|
\]

\[
= \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x_c - x_0)^i ((x - x_0)^{n-(i+1)} - (x_c - x_0)^{n-1}) \right|
\]

\[
= \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-2} (x_c - x_0)^i (x - x_c) \sum_{j=0}^{n-(i+2)} (x - x_0)^j (x - x_0)^{n-(i+2+j)} \right|
\]

\[
= |x - x_c| \sum_{n=0}^{\infty} a_n \sum_{i=0}^{n-2} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)}.
\]
Note that
\[
\lim_{x \to x_c} \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)}
\]
\[
= \sum_{n=1}^{\infty} a_n \left( \sum_{i=0}^{n-1} (n - (i + 1)) \right) (x_c - x_0)^{n-2}
\]
and that
\[
\lambda \left( \sum_{i=0}^{n-1} (n - (i + 1)) \right) = 0
\]
for all \( n \).

It follows that
\[
\lambda \left( a_n \left( \sum_{i=0}^{n-1} n - (i + 1) \right) (x_c - x_0)^{(n-2)} \right) = \lambda(a_n(x_c - x_0)^n) - 2\lambda((x_c - x_0)).
\]
Thus
\[
\lim_{n \to \infty} a_n \left( \sum_{i=0}^{n-1} (n - (i + 1)) \right) (x_c - x_0)^{(n-2)} = 0
\]
because
\[
\lim_{n \to \infty} n a_n (x_c - x_0)^{n-1} = 0
\]
and hence
\[
\lim_{x \to x_c} \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)} = L
\]
exists.
Therefore

\[
\lim_{x \to x_c} \left| \frac{f(x) - f(x_c)}{x - x_c} - G \right| = \lim_{x \to x_c} |x - x_c| \left| \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)} \right|
\]

\[
= \lim_{x \to x_c} |x - x_c| \lim_{x \to x_c} \sum_{n=1}^{\infty} a_n \sum_{i=0}^{n-1} (x - x_0)^i \sum_{j=0}^{n-(i+2)} (x_c - x_0)^j (x - x_0)^{n-(i+2+j)}
\]

\[
= 0 \cdot L
\]

\[
= 0
\]

and hence \( f \) is differentiable at \( x_c \) with

\[
f'(x_c) = G = \sum_{n=1}^{\infty} na_n (x_c - x_0)^{n-1}.
\]

Since \( x_c \) was an arbitrary point in \((a, b)\), it follows that \( f(x) \) is differentiable on \((a, b)\) with \( f'(x) = \sum_{n=1}^{\infty} na_n (x - x_0)^{n-1} \) for every \( x \) in \((a, b)\). \(\square\)

**Corollary 3.4.1.** If a function \( f(x) \) is given by a convergent power series on \([a, b]\) it is infinitely often differentiable on \((a, b)\). Moreover \( f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k} \) for \( k \) in \( \mathbb{N} \).

**Proof.** This follows from the fact that if \( f(x) \) is given by a power series on \((a, b)\) then \( f(x) \) is differentiable on \((a, b)\) and \( f'(x) \) is again given by a power series on \((a, b)\) where the terms of the power series representing the derivative are given by the derivatives of the terms of the initial power series. \(\square\)

**Corollary 3.4.2.** *(Re-expansion of Power Series)* If \( f: [a, b] \to \mathcal{F} \) is given by a
convergent power series

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \]

on \([a, b]\) where \(x_0\) is in \([a, b]\) then the power series can be re-expanded around any point in \([a, b]\). Hence \(f(x)\) can be rewritten as

\[ f(x) = \sum_{n=0}^{\infty} b_n(x - x_c)^n \]

where \(x_c\) is any fixed point in \([a, b]\) and the power series converges for any \(x\) in \([a, b]\).

Proof. Without loss of generality by Theorem (3.2) we may assume that \(\lambda(b - a) = 0\) and so for \(x\) in \([a, b]\) \(\lambda(x - x_0) \geq 0\). We may therefore assume, by Corollary (2.10.2), that \(\lim_{n \to \infty} a_n = 0\). Let \(x_c\) be in \([a, b]\). For any fixed \(N\) we have

\[
a_N(x - x_0)^N = a_n((x - x_c) + (x_c - x_0))^N = a_N \sum_{j=0}^{N} \binom{N}{j}(x_c - x_0)^{N-j}(x - x_c)^j = \sum_{j=0}^{\infty} \alpha_{j,N}(x - x_c)^j
\]

where

\[
\alpha_{j,N} = \begin{cases} 
\binom{N}{j}a_N(x_c - x_0)^{N-j} & \text{for } j \leq N \\
0 & \text{for } j > N
\end{cases}
\]

Thus \(f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{j,n}(x - x_c)^j\). Let \(\epsilon > 0\) in \(F\) be given. Since \((a_n)\) is a null sequence there must be some \(N(\epsilon)\) in \(N\) such that for \(n > N(\epsilon)\) we have \(\lambda(a_n) > \lambda(\epsilon) + 1\). Because \(\lambda(\binom{j}{n}) = 0\), for any \(j, n\) pair where \(\binom{j}{n}\) is well defined, and \(\lambda(x_c - x_0) \geq 0\) we have \(\lambda(\binom{j}{n})(x_c - x_0)^{n-j}a_n \geq \lambda(a_n)\) for any \(j, n\) pair where \(j \leq n\). Then if \(j, n > N(\epsilon)\) we have \(\lambda(\alpha_{j,n}) > \lambda(\epsilon) + 1\) so by Corollary (2.10.5) we have, for
\[ f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} c_j (x - x_c)^j = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} c_j (x - x_c)^j = \sum_{j=0}^{\infty} c_j (x - x_c)^j \]

where \( c_j = \sum_{n=0}^{\infty} \binom{j}{n} a_n (x_c - x_0)^n \).

Therefore for any \( x_c \) in \([a, b]\) there is a sequence \((c_j)\) such that if \( f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) on \([a, b]\) then we also have \( f(x) = \sum_{j=0}^{\infty} c_j (x - x_c)^j \) on \([a, b]\).

\[ \text{Corollary 3.4.3.} \text{ Let } [a, b] \subset \mathcal{F} \text{ and } f : [a, b] \rightarrow \mathcal{F} \text{ be given by a convergent power series; let } x_c \text{ in } (a, b) \text{ and } \delta \text{ such that } x_c + \delta \in [a, b]. \text{ Then we have} \]
\[ f(x_c + \delta) = f(x_c) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_c)}{n!} \delta^n. \]

\[ \text{Proof.} \text{ By Corollary (3.4.2) there is a sequence } (a_n) \text{ such that } f(x) \text{ is given by } \sum_{n=0}^{\infty} a_n (x - x_c)^n \text{ on } [a, b]. \text{ Then} \]
\[ f(x_c + \delta) = \sum_{n=0}^{\infty} a_n ((x_c + \delta) - x_c)^n = \sum_{n=0}^{\infty} a_n \delta^n \]

for \( \delta \) such that \( x_c + \delta \in [a, b] \). By Theorem (3.4), and Corollary (3.4.1), we know that \( f(x) \) is infinitely often differentiable on \((a, b)\) and \( f^{(M)}(x) = \sum_{n=M}^{\infty} \frac{n!}{(n-M)!} a_n (x - x_c)^{n-M} \).

Therefore
\[ f^{(M)}(x_c) = \sum_{n=M}^{\infty} \frac{n!}{(n-M)!} a_n (x_c - x_c)^{n-M} = M! a_M \]

and hence \( a_n = \frac{f^{(n)}(x_c)}{n!} \). \( \square \)

\[ \text{Definition 3.3.} \text{ (Quasi-Multiplicity) Recall that } \mathcal{F} + i\mathcal{F} \text{ is algebraically closed like} \]
C. Let $Q(x)$ be a polynomial over $\mathcal{F} + i\mathcal{F}$ of degree $n$, let $\xi_1, \ldots, \xi_n$ be its $n$ roots in $\mathcal{F} + i\mathcal{F}$, let $j \in \{1, \ldots, n\}$ and let $l \leq n$ be given in $\mathbb{N}$. Then we say that $\xi_j$ has quasi-multiplicity $l$ as a root of $Q(x)$ if, for some $j_1 < j_2 < \ldots < j_{l-1}$ in $\{1, \ldots, n\}\{j\}$ we have that

$$\xi_j \approx \xi_k \text{ if and only if } k \in \{j, j_1, j_2, \ldots, j_{l-1}\}$$

**Theorem 3.5.** (Intermediate Value Theorem) Let $a < b$ in $\mathcal{F}$ be given and let $g: [a, b] \to \mathcal{F}$ be given by a convergent power series. Then $g$ assumes on $[a, b]$ every intermediate value between $g(a)$ and $g(b)$.

**Proof.** Define $G: [-1, 1] \to \mathcal{F}$ by

$$G(x) = g\left(\frac{(b - a)}{2}x + \frac{a + b}{2}\right).$$

By Theorem (3.3) $G(x)$ is again given by a convergent power series on $[-1, 1]$. Let $i(G)$ be the index of $G(x)$, which exists by (3.1), and so $d^{-i(G)}G(x)$, by Theorem (3.1), is given by a power series with zero index on $[-1, 1]$.

Since scaling does not affect intermediate values, $g(x)$ assumes all intermediate values on $[a, b]$ if and only if $d^{-i(G)}G(x)$ assumes all intermediate values on $[-1, 1]$ because the map

$$h(x) = \left(\frac{b - a}{2}\right)x + \left(\frac{a + b}{2}\right)$$

is a bijection from $[-1, 1]$ onto $[a, b]$.

If $d^{-i(G)}G(-1) = d^{-i(G)}G(1)$ we are done as there are no intermediate values, so assume $d^{-i(G)}G(-1) \neq d^{-i(G)}G(1)$. Without loss of generality let $G(1) > G(-1)$, if the opposite is true one can compose with $-x$ and the resulting function will remain
a power series by Theorem (3.3).

Let $S$ be between $d^{-i(G)}G(-1)$ and $d^{-i(G)}G(1)$ and define $f(x) = d^{-i(G)}G(x) - S$, then since $d^{-i(G)}G(-1) \neq d^{-i(G)}G(1)$ we have that one must be bigger than the other so $\frac{f(-1)}{f(1)} < 0$. Therefore $g$ assumes the value $S$ on $[a, b]$ if and only if $f$ assumes the value $0$ in $[-1, 1]$.

So without loss of generality let we may assume $g = f : [-1, 1] \to \mathcal{F}$, where $i(f) = 0$ and $f(-1) < S = 0 < f(1)$.

Due to Corollary (3.4.2) we may expand $f(x)$ around $x = 0$ so there is a sequence $(a_n)$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

on $[-1, 1]$. Given $f(1) = \sum_{n=0}^{\infty} a_n$ we know that sum converges so there is some $m_0$ in $\mathbb{N}$ such that, if $A_0 = \min\{|f(1)|, |f(-1)|\}$, then $\lambda(a_n) > \lambda(A_0) + 1$ for $n > m_0$.

Let

$$P_0(x) = a_0 + \sum_{n=1}^{m_0} a_n x^n.$$  

$$R_0(x) = \sum_{n>m_0} a_n x^n.$$  

Then we have

$$P_0(x) = f(x) - R_0(x),$$
which implies

\[ P_0(-1) = f(-1) - R_0(-1) \approx f(-1). \]

\[ P_0(1) = f(1) - R_0(1) \approx f(1). \]

Therefore \( P_0(x) \) has a sign change on \([-1, 1]\) and hence \( P_0(x) = a_{m_0} \prod_{i=1}^{q} (x - r_i)^{n_i} \) has at least one root of odd quasi-multiplicity on \([-1, 1]\). If \( P_0(x) \) has no such root then \((x - r_i)^{n_i}\) must have the same sign for all \( x \) in \([-1, 1]\) and therefore \( P_0(x) \) has no sign change on \([-1, 1]\), this is a contradiction therefore \( P_0(x) \) must have such a root in \([-1, 1]\).

We consider two cases:

*Case I*: All roots \( x_i \) of \( P_0(x) \) with odd quasi-multiplicity in \([-1, 1]\) have the property \( \lambda(x_i) > 0 \).

Since there are only finitely many roots of \( P_0(x) \) on \([-1, 1]\) there are only finitely many roots of odd quasi-multiplicity with \( \lambda(x_i) > 0 \) thus we can order them \( \{x_1, \ldots, x_m\} \) from least to greatest.

We evaluate \( f(x_i) \) for all \( 1 \leq i \leq m \). If \( f(x_i) = 0 \) for any \( i \) we are done, we may therefore assume \( f(x_i) \neq 0 \) for all \( 1 \leq i \leq m \). We consider two subcases:

*Sub-case I.a*: There is some \( k, 1 \leq k < m \), such that

\[ \frac{f(x_k)}{f(x_{k+1})} < 0 \]

In this case there is some sign change in \( f(x) \) between \( x_k \) and \( x_{k+1} \). We expand \( f(x) \) around each. Consider
\[
f(x_k + x) = f(x_k) + \sum_{n=1}^{\infty} a_n(x_k)x^n
\]
\[
f(x_{k+1} - x) = f(x_{k+1}) + \sum_{n=1}^{\infty} a_n(x_{k+1})(-x)^n
\]

Where \(a_n(x) = \frac{i^{(n)}(x)}{n!}\). Since \(i(f) = 0\) by construction there must be some lowest \(M \in \mathbb{N}\) such that \(a_M(x_k) \sim 1\) (if there were no such \(M\) then \(f(x_k + x)\) would not be finite for any value of \(x\)) and since \(\lambda(x_{k+1} - x_k) \geq \min\{\lambda(x_{k+1}), \lambda(x_k)\} > 0\) it must be that \(a_M(x_{k+1}) \sim 1\). Let \(P_{1,\alpha}(x) = \sum_{n=1}^{M} a_n(x_k)x^n\) and let \(\delta_1' = (d^{\frac{2}{3}}A_0)^{\frac{1}{3}}\).

We claim there must be some \(j_\alpha \in \mathbb{N}\) such that \(\lambda(P_{1,\alpha}(j_\alpha \delta_1')) \leq \frac{1}{2} + \lambda(A_0)\). We observe that \(a_M(x_k)(j\delta_1')^M \sim d^{\frac{2}{3}}A_0\) for all \(j\) in \(\mathbb{N}\). For \(n < M\) we have

\[
a_n(x_k)(j\delta_1')^n \left[\lambda(A_0) + \frac{1}{2}\right] = j^n(a_n(x_k)(\delta_1')^n) \left[\lambda(A_0) + \frac{1}{2}\right].
\]

Gather those \(n\) such that \(a_n(x_k)(j\delta_1')^n[\lambda(A_0) + \frac{1}{2}] \neq 0\) in a set, say \(\Lambda\), then

\[
P_{1,\alpha}(j\delta_1') \left[\lambda(A_0) + \frac{1}{2}\right] = \sum_{n \in \Lambda} a_n(x_k)(\delta_1')^n \left[\lambda(A_0) + \frac{1}{2}\right] j^n
\]

which is a real polynomial of degree \(|\Lambda| < M\) in \(j\) and so cannot be zero at more than \(|\Lambda|\) points by the fundamental theorem of algebra. Thus there must be some \(j_\alpha \in \mathbb{N}\) such that for \(\delta_{1,\alpha} = j_\alpha \delta_1'\) we have \(P_{1,\alpha}(\delta_{1,\alpha})[\lambda(A_0) + \frac{1}{2}] \neq 0\) and hence \(\lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}\).

Define \(P_{1,\beta}(x) = \sum_{n=1}^{M} a_n(x_{k+1})(-x)^n\), and for the exact same reasons as above,
there must be some $j_{\beta}$ such that if $\delta_{1,\beta} = j_{\beta}\delta_1'$ then $\lambda(P_{1,\beta}(\delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$.

Since $\lambda(a_n(x_i)) \geq 0$ and both $\lambda(\delta_{1,\alpha}^n) > \lambda(A_0) + \frac{1}{2}$ and $\lambda(\delta_{1,\beta}^n) > \lambda(A_0) + \frac{1}{2}$ for $n > M$ we have, by Corollary (2.10.4), that

$$\lambda \left( \sum_{n=M+1}^{\infty} a_n(x_k)\delta_{1,\alpha}^n \right) > \lambda(A_0) + \frac{1}{2} \quad \text{and}$$

$$\lambda \left( \sum_{n=M+1}^{\infty} a_n(x_{k+1})\delta_{1,\beta}^n \right) > \lambda(A_0) + \frac{1}{2}.$$  

We then have

$$\lambda(f(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$

$$\lambda(f(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

which implies

$$\lambda(P_{0}(x_k + \delta_{1,\alpha}) + R_0(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$

$$\lambda(P_{0}(x_{k+1} - \delta_{1,\beta}) + R_0(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

which gives

$$\lambda(P_{0}(x_k + \delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}$$

$$\lambda(P_{0}(x_{k+1} - \delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2}$$

as $|R_0(x)| \ll d^\frac{3}{2} A_0$ on $[-1, 1]$.  

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Then we show that at least one of the following must be true:

(i) \[ \lambda(x_{k+1} - x_k) \geq \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta}). \]

(ii) \[ \frac{f(x_k + \delta_{1,\alpha})}{f(x_k)} < 0. \]

(iii) \[ \frac{f(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} < 0. \]

Suppose \( \lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha}) \) and

\[
\frac{f(x_k + \delta_{1,\alpha})}{f(x_k)} > 0 \quad \text{and} \quad \frac{f(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} > 0.
\]

Then \( \frac{P_0(x_k + \delta_{1,\alpha})}{f(x_k)} > 0 \) and \( \frac{P_0(x_{k+1} - \delta_{1,\beta})}{f(x_{k+1})} > 0. \)

It follows that

\[
\frac{P_0(x_k + \delta_{1,\alpha})}{P_0(x_{k+1} - \delta_{1,\beta})} \frac{f(x_{k+1})}{f(x_k)} > 0
\]

and hence

\[
\frac{P_0(x_k + \delta_{1,\alpha})}{P_0(x_{k+1} - \delta_{1,\beta})} < 0
\]

as

\[
\frac{f(x_{k+1})}{f(x_k)} < 0.
\]

This implies a sign change in \( P_0(x) \) on \((x_k + \delta_{1,\alpha}, x_{k+1} - \delta_{1,\beta})\) which is a subset of \([x_k, x_{k+1}]\) as \( \lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta}). \) There must therefore be some \( c \in (x_k + \delta_{1,\alpha}, x_{k+1} - \delta_{1,\beta}) \) such that \( P_0(c) = 0, \) but then \( x_k \) and \( x_{k+1} \) are not consec-
positive roots of $P_0(x)$ which is a contradiction.

Thus if $\lambda(x_{k+1} - x_k) < \lambda(\delta_{1,\alpha})$ either

(i) $\frac{f(x_{k+1}+\delta_{1,\alpha})}{f(x_k)} < 0$; in which case we let $X_0 = x_k$, and $\delta_1 = \delta_{1,\alpha}$ or

(ii) $\frac{f(x_{k+1}+\delta_{1,\beta})}{f(x_{k+1})} < 0$; in which case we let $X_0 = x_{k+1}$, and $\delta_1 = \delta_{1,\beta}$.

In either case $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

If $\lambda(x_{k+1} - x_k) \geq \lambda(\delta_{1,\alpha}) = \lambda(\delta_{1,\beta})$ let $X_0 = x_k$ and $\delta_1 = x_{k+1} - x_k$. Then $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

Therefore in Sub-case I.a, where all roots $x_i$ of $P_0(x)$ have $\lambda(x_i) > 0$ and for some $1 \leq k < m$ we have $\frac{f(x_k)}{f(x_{k+1})} < 0$, we can define an $X_0$ and $\delta_1$ such that $P_0(X_0) = 0$, $\lambda(\delta_1) = \frac{\lambda(A_0)}{M} + \frac{1}{2M}$ and $f(X_0 + x)$ has a sign change on $(\delta_1, \delta_1)$.

Sub-case I.b: For all roots of $P_0(x)$, which we call $\{x_1, \ldots, x_m\}$, we have $\frac{f(x_i)}{f(x_j)} > 0$. In this sub case, as in the previous one, we expand $f(x)$ around a root of $P_0(x)$, which we call $X_0$, to find a $\delta_1$ such that $\lambda(\delta_1) \geq \lambda(A_0) + \frac{1}{2}$ and $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$.

If $f(x_i)$ has the same sign for all $1 \leq i \leq m$ then either $f(x_1) > 0$ or $f(x_m) < 0$. If $f(x_1) > 0$, then since we had $f(-1) < 0$ there is a sign change in $f(x)$ between $-1$ and $x_1$, and we let $X_0 = x_1$. If $f(x_m) < 0$, then since we had $f(1) > 0$ there is a sign change in $f(x)$ between $x_m$ and $1$, and we let $X_0 = x_m$.

Suppose $f(x_1) > 0$ and we let $X_0 = x_1$; there must be some $M \in \mathbb{N}$ such that $f^{(M)}(x_1)$ is the first finite derivative of $f$ at $x_1$. There must also, as in Sub-case
I.a, be some \( P_{1,\beta}(x) = \sum_{n=1}^{M} a_n(x_0)(-x)^n \) and \( \delta_1 \) (where \( \delta_1 \sim (d^{\frac{1}{2}}A_0)^{\frac{1}{2}} \)) such that \( \lambda(P_{1,\beta}(\delta_1)) \leq \lambda(A_0) + \frac{1}{2} \). As before we can be sure that there is some sign change in \( f(x_1 + x) \) on \( (x_1 - \delta_1, x_1) \) or there is some root of \( P_0(x) \) on \( (-1, x_1 - \delta_1) \), which is a contradiction as we took \( x_1 \) to be the smallest root of \( P_0(x) \).

Likewise if \( f(x_m) < 0 \) and we let \( X_0 = x_m \) there must be some \( M \in \mathbb{N} \) such that \( f^{(M)}(x_m) \) is the first finite derivative of \( f \) at \( x_m \). As in Sub case I.a there must be a \( P_{1,\alpha}(x) = \sum_{n=1}^{M} a_n(x_m)x^n \) and \( \delta_1 \sim (d^{\frac{1}{2}}A_0)^{\frac{1}{2}} \) such that \( f(x_m + x) \) has a sign change on \( (x_m, x_m + \delta_1) \) or there is a root of \( P_0(x) \) on \( (x_m + \delta_1, 1) \) which is a contradiction as we took \( x_m \) to be the largest root of \( P_0(x) \).

Therefore in Sub-case I.b where all roots \( r_i \) of \( P_0(x) \) have the properties \( \lambda(r_i) > 0 \) and \( \frac{f(r_i)}{f(x_i)} > 0 \), we can define an \( X_0 \) and \( \delta_1 \) such that \( \lambda(f(X_0)) \geq \lambda(A_0) + \frac{1}{2} \), \( \lambda(\delta_1) = \frac{\lambda(A_0)}{M} + \frac{1}{2M} \), and \( f(X_0 + x) \) has a sign change on \( (-\delta_1, \delta_1) \).

Case II: At least one root of \( P_0(x) \) with odd quasi-multiplicity is finite.

In this case there is a finite root, say \( X \), with odd quasi-multiplicity \( m \) such that there is some finite \( \Delta_0 \) such that

\[
\frac{P_0(X + \Delta)}{P_0(X - \Delta)} < 0
\]

for all finite \( \Delta < \Delta_0 \).

As previously there are finitely many \((m - 1)\) roots of \( P_0(x) \) infinitely close to \( X \), which can be arranged from least to greatest as \( \{X_1, \ldots, X_m\} \) including \( X \). If \( f(X_i) = 0 \) for any \( 1 \leq i \leq m \) we are done so assume \( f(X_i) \neq 0 \) for all \( i \). As in Case
I we consider two sub cases

Sub-case II.a: There is some \(1 \leq k < m\) such that \(\frac{f(X_k)}{f(X_{k+1})} < 0\). Just as in Sub case I.a there must be

- Some \(M \in \mathbb{N}\) such that \(f^{(M)}(X_k) \sim f^{(M)}(X_{k+1})\) and \(f^{(M)}(x_i)\) is the first finite derivative of \(f\) at \(X_i\) (both \(X_k\) and \(X_{k+1}\)).

- A polynomial \(P_{1,\alpha}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_k)}{n!} x^n\).

- A polynomial \(P_{1,\beta}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_{k+1})}{n!} (-x)^n\).

- \(\delta_{1,\alpha} > 0\) such that \(\delta_{1,\alpha} \sim (d^{\frac{3}{2}} A_0)^{\frac{1}{M}}\) and \(\lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}\).

- \(\delta_{1,\beta} > 0\) such that \(\delta_{1,\beta} \sim (d^{\frac{3}{2}} A_0)^{\frac{1}{M}}\) and \(\lambda(P_{1,\beta}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2}\).

Just as in Sub-case I.a, one of the following must be true:

(i) \(\lambda(X_{k+1} - X_k) \geq \lambda(\delta_{1,\alpha})\); in which case we let \(X_0 = X_k\), and \(\delta_1 = X_{k+1} - X_k\).

(ii) \(\lambda(X_{k+1} - X_k) < \lambda(\delta_{1,\alpha}), \ \frac{f(X_k + \delta_{1,\alpha})}{f(X_k)} < 0\); in which case we let \(X_0 = X_k\), and, \(\delta_1 = \delta_{1,\alpha}\).

(iii) \(\lambda(X_{k+1} - X_k) < \lambda(\delta_{1,\alpha}), \ \frac{f(X_{k+1} - \delta_{1,\beta})}{f(X_{k+1})} < 0\); in which case we let \(X_0 = X_{k+1} - \delta_{1,\beta}\), \(\delta_1 = \delta_{1,\beta}\).
Sub-case II.b: \( \frac{f(X_i)}{f(X_j)} > 0 \) for all \( i, j \). Then for some \( \Delta < \Delta_0 \) one of the following must be true

(i) \( \frac{f(X_1 - \Delta)}{f(X_1)} < 0 \) or

(ii) \( \frac{f(X_m + \Delta)}{f(X_m)} < 0 \)

as \( f(X_1) \) and \( f(X_m) \) must have the same sign but \( f(X_1 - \Delta) \) and \( f(X_m + \Delta) \) must have opposite signs as \( X \) is a root with odd quasi-multiplicity.

As in Sub-case I.b, (i) being true implies

a) There is some \( M \in \mathbb{N} \) such that \( \frac{f^{(M)}(X_1)}{M!} \) is the first finite coefficient of expansion of \( f(x) \) about \( X_1 \).

b) There is some polynomial \( P_{1,\beta}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_1)}{n!}(-x)^n \).

c) There is a \( \delta_{1,\beta} > 0 \) such that \( \delta_{1,\beta} \sim (d_1^2 A_0)^{\frac{1}{\delta_1}} \) and \( \lambda(P_{1,\beta}(\delta_{1,\beta})) \leq \lambda(A_0) + \frac{1}{2} \).

d) \( f(x) \) has a sign change on \( (X_1 - \delta_{1,\beta}, X_1) \).

In this case let \( X_0 = X_1 \) and \( \delta_1 = \min\{\delta_{1,\beta}, 1 + X_1\} \).

As in Sub-case I.b, (ii) being true implies

a) There is some \( M \in \mathbb{N} \) such that \( \frac{f^{(M)}(X_m)}{M!} \) is the first finite coefficient of expansion of \( f(x) \) about \( X_m \).

b) There is a polynomial \( P_{1,\alpha}(x) = \sum_{n=1}^{M} \frac{f^{(n)}(X_m)}{n!}x^n \).

c) There is a \( \delta_{1,\alpha} > 0 \) such that \( \delta_{1,\alpha} \sim (d_1^2 A_0)^{\frac{1}{\delta_1}} \) and \( \lambda(P_{1,\alpha}(\delta_{1,\alpha})) \leq \lambda(A_0) + \frac{1}{2} \).

d) \( f(x) \) has a sign change on \( (X_m, X_m + \delta_{1,\alpha}) \).
In this case let $X_0 = X_m$ and $\delta_1 = \min\{\delta_1 \alpha, 1 - X_m\}$.

Then in all sub-cases we have that $\frac{f(X_0 + \delta_1)}{f(X_0)} < 0$ and $\lambda(f(X_0 + \delta_1)) \leq \lambda(A_0) + \frac{1}{2}$.

Re-expand $f(x)$ around $X_0$

$$f(X_0 + x) = f(X_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n \quad (3.6)$$

Since there must be some finite $x$ such that $X_0 + x$ is in $[-1, 1]$, then by Corollary (3.4.3) there must be some finite $x$ at which the sum in Equation (3.6) converges. Therefore there exists $m_1$ in $\mathbb{N}$ such that for $X_0 + x$ in $[-1, 1]$ we have

$$\lambda \left( \sum_{n=m_1+1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n \right) > \lambda(f(X_0)) + 1$$

Define

$$P_1(x) = f(X_0) + \sum_{n=1}^{m_1} \frac{f^{(n)}(X_0)}{n!} x^n$$

$$R_1(x) = \sum_{n=m_1+1}^{\infty} \frac{f^{(n)}(X_0)}{n!} x^n$$

We’ve already established that $f(X_0 + x)$ has a sign change on $(-\delta_1, \delta_1)$ and that $\lambda(f(X_0 + \delta_1)) < \lambda(f(X_0)) < \lambda(R_1(x))$ (or $\lambda(f(X_0 - \delta_1)) < \lambda(f(X_0)) < \lambda(R_1(x))$) and hence $P_1(x)$ has a sign change on $(-\delta_1, \delta_1)$.

We therefore have that $P_1(x)$ has a root of odd quasi-multiplicity on $(-\delta_1, \delta_1)$, as we established polynomial sign changes correspond to roots of odd quasi-multiplicity,
so as in Case I of the previous expansion there must be some $x_1 \in (-\delta_1, \delta_1)$ and
$\delta_2 > 0$ such that $\delta_2 \sim (d^{\frac{1}{2}} f(X_0))^{\frac{1}{M}}$ and $P_1(x_1) = 0$, $\lambda(P_1(x_1 + \delta_2)) \leq \lambda(f(X_0)) + \frac{1}{2}$,
and $\frac{f(X_0 + x_1 + \delta_2)}{f(X_0 + x_1)} < 0$.

NB: $\lambda(\delta_2) = \frac{\lambda(f(X_0))}{M} + \frac{1}{2M} \geq \frac{\lambda(A_0)}{M} + \frac{1}{2M} + \frac{1}{2M}$.

By induction, we obtain sequences $(x_n)$ and $(\delta_n)$ such that

- $x_n \in (-\delta_{n-1}, \delta_{n-1})$

- $f(X_0 + x_1 + \ldots + x_{n-1} + x) = P_n(x) + R_n(x)$

- $\lambda(R_n(x)) > \lambda(d^{\frac{1}{2}} f(X_0 + x_1 + \ldots + x_{n-1}))$ for $X_0 + x_1 + \ldots + x_{n-1}$ in $[-1, 1]$

- $\frac{f(X_0 + x_1 + \ldots + x_{n-1} + \delta_n)}{f(X_0 + x_1 + \ldots + x_{n-1} - \delta_n)} < 0$

- $\lambda(\delta_n) \geq \frac{\lambda(A_0)}{M} + \frac{n}{2M}$

It follows that $\lim_{n \to \infty} \lambda(\delta_n) = \infty$, and hence

$$\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} x_n = 0$$

Then there is some $X$ such that $X = X_0 + \sum_{i=1}^{\infty} x_i$. Since $X_0 + x_1 \in (-1, 1)$
we have $X \in (-1, 1)$ because $\lambda(x_1) < \lambda(\sum_{i=2}^{\infty} x_i)$ so $X \approx X_0 + x_1$ and hence
$X - X_0 \approx x_1$.

We also have
\[
\lim_{n \to \infty} f(X_0 + \ldots + x_n) = \lim_{n \to \infty} R_n(x_n) = 0
\]

Therefore

\[
0 = \lim_{n \to \infty} f(X_0 + \ldots + x_n) = f(\lim_{n \to \infty} (X_0 + \ldots + x_n)) = f(X).
\]

So there is some \(X \in (-1, 1)\) such that \(f(X) = 0\).

\[\square\]

**Lemma 3.6.** If \(P(x)\) is a polynomial of degree \(n \geq 2\) over \(\mathbb{R}\), such that \(P(r) = 0\), and \(P^{(i)}(r) = 0\) for \(0 \leq i \leq n - 1\), then \(P(x) = \alpha(x - r)^n\) for some \(\alpha\) in \(\mathbb{R}\).

**Proof.** We will prove this statement via induction: let \(n \geq 2\) and let \(S_n\) denote the statement “If \(P(x)\) is a polynomial over \(\mathbb{R}\) with \(P(r) = 0\) and \(P^{(i)}(r) = 0\) for \(0 \leq i \leq n - 1\) then \(P(x) = \alpha(x - r)^n\) for some \(\alpha\) in \(\mathbb{R}\).”

In the base case \(n = 2\) \(P(r) = P'(r) = 0\). As \(P(r) = 0\) then \(P(x) = \alpha(x - r)(x - r_2)\) so \(P'(x) = \alpha[(x - r) + (x - r_2)] = \alpha[(2x - (r + r_2))]\). \(P'(r) = 0\) so \(2r = r + r_2\) then \(r = r_2\) and hence \(P(x) = 2(x - r)^2\).

Fix \(k \geq 2\) and suppose \(S_k\) is true. Further, suppose \(P(x)\) is a polynomial of degree \(k + 1\), \(P(r) = 0\), and \(P^{(i)}(r) = 0\) for \(0 \leq i \leq k\). Then \(G(x) = P'(x)\) is a polynomial of degree \(k\) such that \(G(r) = 0\) and \(G^{(i)}(r) = 0\) for \(0 \leq i \leq k - 1\). Therefore, by \(S_k\), there is a \(\beta\) such that \(P'(x) = G(x) = \beta(x - r)^k\). Because \(P(r) = 0\) we have \(P(x) = \alpha(x - r)H(x)\) where \(H(x)\) is a polynomial of degree \(k\). Then \(P'(x) = \alpha[H(x) + (x - r)H'(x)] = \beta(x - r)^k\). Since polynomials over \(\mathbb{R}\) have unique factorizations there is a constant such that \(H(x) = c(x - r)^k\) and hence there is an \(\alpha\) such that \(P(x) = \alpha(x - r)^{k+1}\) and so \(S_{k+1}\) holds.
Therefore by the principle of mathematical induction $S_n$ holds for every $n \geq 2$.

**Corollary 3.6.1.** A convergent power series $f$ on $[a, b]$ has at most finitely many zeros.

**Proof.** As in the proof of the intermediate value theorem, without loss of generality we may assume the domain and range of $f$ are scaled, so that $[a, b] = [-1, 1]$, and $i(f) = 0$. We can once again express $f(x)$ as a sum of a polynomial and a remainder term, $P_0(x)$ and $R_0(x)$, and for every $y$ such that $f(y) = 0$ there is a root $x$ of $P_0$ such that $\lambda(x - y) > 0$. At points $x'$ such that $\lambda(x' - x_k) \leq 0$ for all $k$, where $x_k$ are the roots of $P_0$, we would have $P_0(x')[i(f)] \neq 0$ so $f(x')$ could not be zero. There must be finitely many zeroes of $P_0$ by the fundamental theorem of algebra ($\mathcal{F} + i\mathcal{F}$ is algebraically closed).

Consider that near any zero $x_k$ of $P_0$ we can re-expand $f$ around $x_k$ and there is some $M_k \in \mathbb{N}$ such that $a_{M_k}(x_k) = \frac{f^{(M_k)}(x_k)}{M_k!}$ is the first finite coefficient of the expansion of $f$ about $x_k$. Then

$$f(x_k + \delta) = f(x_k) + \sum_{n=1}^{M_k} a_n(x_k)\delta^n + \sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n$$

where $a_n(x_k) = \frac{f^{(n)}(x_k)}{n!}$.

For $\delta$ such that $0 < |\delta| \ll 1$ we have $\lambda(a_{M_k}(x_k)\delta^{M_k}) < \lambda(\sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n)$ and so $f(x_k + \delta) = 0$ only if $\lambda\left(f(x_k) + \sum_{n=0}^{M_k} a_n(x_k)\delta^n\right) > \lambda(a_{M_k}(x_k)\delta^{M_k})$. In that case $f(x_k)[q] + \sum_{n=1}^{M_k} (a_n(x_k)\delta^n)[q] = 0$ for all $q \leq M_k \lambda(\delta)$.

We claim there are only finitely many $\delta$ such that $0 < |\delta| \ll 1$ and $f(x_k + \delta) = 0$. As in the proof of the intermediate value theorem for any such $\delta$ we have $a_{M_k}(x_k + \delta)$ must be the first finite term in $(a_n(x_k + \delta))$ as no coefficient of expansion can change by a finite amount over an infinitely small interval if all coefficients are at most finite.
to start with.

For $0 \leq i < M_k$ where $a_i(x_k) \neq 0$ let $\alpha_{i,k} = \lambda(a_i(x_k))$. Let

$$U_0 = \min_{0 \leq i < M_k} \left\{ \frac{\alpha_{i,k}}{M_k} \right\}$$

then for $\delta$ such that $0 < \lambda(\delta) < U_0$ we have $\lambda(a_{M_k}(x_k)\delta^{M_k}) < \lambda(a_n(x_k)\delta^n)$ for all $n$, hence $f(x_k + \delta) \approx a_{M_k}(x_k)\delta^{M_k} \neq 0$ so $f(x_k + \delta)$ is not equal to zero. We therefore require that $\lambda(\delta) \geq U_0$.

Let $A = \{ q \in \mathbb{Q} \mid \exists i \neq j, 0 \leq i \leq M_k, 0 \leq j \leq M_k, q = \frac{\alpha_{i,k} - \alpha_{j,k}}{j - i} \}$ and let $Q' = \{ q \in A \mid q \geq U_0 \}$. If $\lambda(\delta) = q$ and $q \geq U_0$ and $q$ is not in $A$ then

$$\lambda \left( \sum_{n=0}^{M_k} a_n(x_k)\delta^n \right) = \min_{0 \leq i \leq M_k} \{ \lambda(a_i(x_k)\delta^i) \} \leq M_kq < \lambda \left( \sum_{n=M_k+1}^{\infty} a_n(x_k)\delta^n \right)$$

and hence $f(x_k + \delta) \neq 0$. Therefore if $\lambda(\delta) > 0$ and $f(x_k + \delta) = 0$ we know $\lambda(\delta)$ is in $Q'$, so there are only finitely many possible values for $\lambda(\delta)$ because $A$ is finite.

Let $q_c \in Q'$ and let $\delta = ad^c + \gamma$ where $a$ is in $\mathbb{R} \setminus \{0\}$ and $\lambda(\gamma) > q_c$. Define $\beta_{qc} = \min_{0 \leq i \leq M_k} \{ \alpha_{i,k} + iq_c \}$ and $A = \{ 0 \leq i \leq M_k \mid \alpha_{i,k} + iq_c = \beta_{qc} \}$. Then for $i$ not in $A$ we have $\lambda(a_i(x_k)\delta^i) > \beta_{qc}$ and so $f(x_k + \delta)[\beta_{qc}] = P(a) = \sum_{i \in A} a_i(x_k)[\alpha_{i,k}]a^i$ (for $n > M_k$ we have $\lambda(a_n(x_k)\delta^n) > \lambda(a_{M_k}(x_k)\delta^{M_k})$ so they need not be considered).

This is a real polynomial in $a$ of degree at most $M_k$ and therefore equal to zero at no more than $M_k$ values of $a$. Since $f^{(i)}(x_k + \delta) = \sum_{n=i}^{\infty} \frac{n!}{(n-i)!}a_n(x_k)\delta^{n-i}$ and $\lambda(\frac{n!}{(n-i)!}a_n(x_k)) = \lambda(a_n(x_k))$ wherever $(n-i)!$ is well defined we have

$$f^{(i)}(x_k + \delta)[\beta_{qc} - iq_c] = \left( \sum_{n=i}^{M_k} \frac{n!}{(n-i)!}a_n(x_k)[\beta_{qc} - (n-i)q_c]a^{n-i} \right)$$

$$= \frac{d^i}{da^i}P(a).$$
Either \( P^{(i)}(a) = 0 \) for all \( 0 \leq i \leq M_k - 1 \) and so there is only one possible value for \( a \) by Lemma (3.6) or there is some greatest \( M_{k,1} \leq M_k - 1 \) such that \( P^{(M_{k,1})}(a) \neq 0 \) for each root \( a \) of \( P(a) \). For this \( M_{k,1} \) if \( \lambda(\gamma) > q_c \)

\[
f^{(M_{k,1})}(x_k + ad^{qc} + \gamma)[\beta_{qc} - M_{k,1}\lambda(\gamma)] \]

\[
= \sum_{n=M_{k,1}}^{M_k} a_n(x_k + ad^{qc})[\beta_{qc} - (n - M_{k,1})\lambda(\gamma)](\gamma[\lambda(\gamma)])^{n-M_{k,1}}
\]

\[
\neq 0
\]

as \( a_{M_{k,1}}(x_k + ad^{qc})[\beta_{qc} - M_{k,1}q_c] \neq 0 \) by assumption and \( \lambda(a_j(x_k + ad^{qc})) > \beta_{qc} - nq_c > \beta_{qc} - n\lambda(\gamma) \) for \( M_k \geq j > M_{k,1} \). Moreover \( \lambda(a_{M_{k,1}}(x_k + ad^{qc})\gamma^{M_{k,1}}) < \lambda(a_j(x_k + adq_c)\gamma^j) \) for \( j > M_{k,1} \) and \( \gamma \) with \( \lambda(\gamma) > q_c \). Then \( M_{k,1} \) takes on the roll of \( M_k \) in the next expansion and we consider the \( \gamma \) such that \( \lambda\left(\sum_{n=M_{k,1}}^{\infty} a_n(x_k + ad^{qc})\gamma^n\right) = \lambda\left(\sum_{n=M_{k,1}+1}^{\infty} a_n(x_k + ad^{qc})\gamma^n\right) \).

As in the proof of the intermediate value theorem we can establish an iterative process by which we generate a sequence \( (a_n) \) and \( (q_c,n) \) such that \( \delta = \sum_{n=0}^{\infty} a_n d^{qc,n} \) and \( f(x + \delta) = 0 \). Each \( a_i \) is a root of a polynomial of degree \( M_{k,1} \). Because \( M_{k,i+1} \leq M_{k,i} \) and \( M_{k,i+1} = M_{k,i} \) only if there is only one option for \( a_i \) there is an \( N \) in \( \mathbb{N} \) such that for \( i > N \) there is only one option for \( a_i \) (either the degree of the leading polynomial becomes fixed or the iterative process terminates and \( a_i = 0 \) for all \( i > N \)). Therefore there are only finitely many \( \delta \) such that \( \lambda(\delta) > 0 \) and \( f(x_k + \delta) = 0 \) and since there can only be finitely many roots \( x_k \) of the leading polynomial of \( f \) there are only finitely many roots \( y \) of \( f \) in total.

\( \square \)

**Theorem 3.7.** (Local Extrema)

Let \( [a,b] \subset \mathcal{F} \) be given and \( f : [a,b] \to \mathcal{F} \) be a non-constant convergent power
series given by

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} a_n(x - x_0)^n$$

where $x_0 \in (a,b)$. Let $m \in \mathbb{N}$ be the smallest $m$ such that $f^{(m)}(x_0)$ is not equal to zero. Then $f$ has a local extremum at $x_0$ if and only if $m$ is even. The extremum is a minimum if $f^{(m)}(x_0) > 0$ and a maximum if $f^{(m)}(x_0) < 0$.

Proof. As in Theorems (3.2) and (3.1) as well as Corollary (3.4.3) we may assume without loss of generality that $[a,b] = [-1,1]$, $x_0 = 0$, and $i(f) = 0$. Then $\lambda(x - x_0) \geq 0$ for all $x$ in $[-1,1]$, $\lim_{n \to \infty} a_n = 0$, and $\lambda(a_n) \geq 0$ for all $n$ in $\mathbb{N}$.

Given that $(a_n)$ is a null sequence and $i(f) = 0$ there is some $M$ in $\mathbb{N}$ such that $a_M$ is the first finite term in the sequence $(a_n)$. Then for $x$ such that $\lambda(x - x_0) > 0$ we have

$$\lambda(a_M(x - x_0)^M) < \lambda(a_n(x - x_0)^n)$$

for $n > M$.

Let $\Lambda$ be the set of $m \leq i \leq M$ such that $a_i \neq 0$, let $\alpha_i = \lambda(a_i)$. Then for $i$ in $\Lambda$ and $x = x_0 + d^i$ we have

$$\lambda(a_i(x - x_0)^i) = \alpha_i + iq.$$

For $i$ in $\Lambda$ let $q_i$ be such that

$$\alpha_m + mq_i = \alpha_i + iq_i,$$

and let $q_0 = \min_{i \in \Lambda} \{q_i\}$ then for $q > q_0$ we have $\alpha_i + iq > \alpha_m + mq$ which entails that $\lambda(f(x_0 + d^i) - f(x_0)) = \lambda(a_m d^{mq})$ for such $q$. Then for $\delta > 0$ such that $\lambda(\delta) > q_0$ we have

$$(f(x_0 + \delta) - f(x_0))[\lambda(f(x_0 + \delta) - f(x_0))] = a_m[\lambda(a_m)]$$

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and 
\[(f(x_0 - \delta) - f(x_0))[\lambda(f(x_0 - \delta) - f(x_0))] = a_m[\lambda(a_m)](-1)^m\]
by Corollary (3.4.3). Therefore if \(m\) is even we have that for \(\delta > 0\) such that \(\lambda(\delta) > q_0\)

\[f(x_0 + \delta) - f(x_0) \approx f(x_0 - \delta) - f(x_0) \approx a_m\delta^m\]

which entails \(f(x_0)\) is a local maximum if \(a_m < 0\) and \(f(x_0)\) is a local minimum if \(a_m > 0\). In the case where \(m\) is odd then \(\frac{f(x_0 + \delta)}{f(x_0 - \delta)} < 0\) for \(\delta > 0\) such that \(\lambda(\delta) > q_0\) so \(f(x_0)\) is neither a local maximum nor minimum.

\[\Box\]

**Theorem 3.8. (Extreme Value Theorem)**

Let \([a, b] \subset \mathcal{F}, x_0 \in [a, b], \text{ and } f : [a, b] \to \mathcal{F} \text{ be given by a non-constant convergent power series}

\[f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n\]

then \(f\) assumes a maximum and a minimum value on \([a, b]\).

**Proof.** As in Corollary 3.6.1 without loss of generality we can assume \(i(f) = 0, x_0 = 0, \text{ and } [a, b] = [-1, 1]\).

Since \(\lim_{n \to \infty} a_n = 0\) and \(i(f) = 0\) there is some finite set \(\Lambda\) such that \(\lambda(a_i) = 0\) for \(i \in \Lambda\) and \(\lambda(a_i) > 0\) for \(i \notin \Lambda\). Define \(P(x)\) and \(R(x)\) by

\[P(x) = \sum_{n \in \Lambda} a_n x^n\]
\[R(x) = \sum_{n \notin \Lambda} a_n x^n\]

then \(f(x) = P(x) + R(x)\) and \(f'(x) = P'(x) + R'(x)\). Let \(\{X_1, \ldots, X_K\}\) be the set of points where \(P(x)\) attains its global maximum on \([-1, 1]\), and recall that for all
Let 1 ≤ k ≤ K and expand f(x) around X_k then

\[ f(X_k + x) = f(X_k) + \sum_{n=1}^{\infty} a_n(X_k) x^n \]

\[ f(X_k + x) - f(X_k) = \sum_{n=1}^{\infty} a_n(X_k) x^n \]

where \( a_n(X_k) = \frac{f^{(n)}(X_k)}{n!} \). Given that \( P(X_k) \) is a global and therefore local maximum for \( P(x) \) we have that the smallest \( M_k \) such that \( P(M_k)(X_k) \neq 0 \) is even and \( P(M_k)(X_k) < 0 \). Since \( f^{(M_k)}(X_k) = P^{(M_k)}(X_k) + R^{(M_k)}(X_k) \) we have \( a_{M_k}(X_k) \) is the first finite coefficient of expansion of \( f(x) \) about \( X_k \), \( M_k \) is even, and \( a_{M_k}(X_k) < 0 \).

For 1 ≤ i ≤ M_k such that \( a_i(X_k) \neq 0 \) let

\[ \alpha_i = \lambda(a_i(X_k)) \]
\[ q_i = \frac{\alpha_i}{M_k - i} \]
\[ q'_k = \frac{1}{2} \min_{1 \leq i < M_k} \{ q_i \} \]
\[ \delta_k = d q'_k \]

then for \( q \leq q'_k \) and 1 ≤ i < M_k

\[ q \leq q'_k < q_i \] and hence it follows that

\[ q \leq q'_k < \frac{\alpha_i}{M_k - i} \]. Therefore

\[ (M_k - i)q \leq (M_k - i)q'_k < \alpha_i \] and hence

\[ M_kq - i(q - q'_k) \leq M_kq'_k < \alpha_i + iq'_k. \]
Thus, \( \alpha_i + iq_k > M_kq_k \), so \( a_{M_k}(X_k)x^{M_k} \) dominates \( f(X_k + x) - f(X_k) \) for \( \delta_k < |x| < 1 \) therefore \( f(X_k + x) - f(X_k) < 0 \) for \( \delta_k < |x| < 1 \).

We now consider the local maxima of \( f(x) \) on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1] \). These coincide with the local maxima of \( g(x) = f(X_k + x) - f(X_k) \) which is given by a convergent power series and therefore by Corollary (3.4.1) so is its derivative \( g'(x) \) and so by Corollary (3.6.1) \( g'(x) = 0 \) at only finitely many points on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1] \), say \( \{x_1, \ldots, x_l \} \). Let \( G_k = \max_{1 \leq i \leq l} \{g(x_i)\} \) we claim this must be the global maximum of \( g(x) \) on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1] \). Suppose not, and there is some \( x_{A_1} \) in \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1] \) such that \( g(x_{A}) = A > G_k \). Let \( h(x) = g(x) - A \), \( h(x) \) is given by a convergent power series on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1] \) and therefore can only have finitely many zeros \( \{x_{A_1}, \ldots, x_{A_2} \} \) on that interval. We know \( h(x_{A_1}) = 0 \) and \( h(x_i) < 0 \) for all \( 1 \leq i \leq l \). If we expand \( h(x) \) around \( x_{A_1} \) because \( f'(x_{A_i}) \neq 0 \) for any \( i \) we must be able to find, as in Theorem (3.7), a \( \delta_i > 0 \) small enough such that for \( x \) in \((-\delta, \delta)\) we have \( h'(x_{A_i})x \) dominates \( h(x_{A_i} + x) \).

We now consider two cases:

**Case I:** There is some \( j \) such that \( x_j < x_{A_1} < x_{j+1} < x_{A_2} \). In this case we expand \( h(x) \) around \( x_{A_1} \) and note that \( h'(x_{A_1}) \neq 0 \), therefore as in Theorem (3.7) we can find a \( \delta \) small enough such that \( h'(x_{A_1})x \) dominates \( h(x_{A_1} + x) \) on \((-\delta, \delta)\). Then there must be a point \( p \) in \((-\delta, \delta)\) such that \( h(x_{A_1} + p) > 0 \) and \( x_{A_1} + p \) is in \([x_j, x_{j+1}] \). But \( h(x_j) < 0 \) and \( h(x_{j+1}) < 0 \) then \( h(x_{A_1} + p) > 0 \) implies there is a zero of \( h(x) \) in \([x_j, x_{j+1}] \) by Theorem (3.5) which is a contradiction of our assumption that \( x_j < x_{A_1} < x_{j+1} < x_{A_2} \).

**Case II:** There is some \( j \) such that \( x_j < x_{A_1} < x_{A_2} < x_{j+1} \). In this case we expand have \( h(x_j) < 0 \), \( h(x_{A_1}) = h(x_{A_2}) = 0 \), and \( h(x_{j+1}) < 0 \). As we can find small \( \delta_1 \) and \( \delta_2 \) around \( x_{A_1} \) and \( x_{A_2} \) such that both \( h(x_{A_1} + x) \) and \( h(x_{A_2} + x) \) change
sign on \((-\delta_1, \delta_1)\) and \((-\delta_2, \delta_2)\) respectively the only way we can have \(h(x)\) satisfy the intermediate value theorem is to have \(\frac{h'(x_{A_1})}{h'(x_{A_2})} < 0\). This however implies that \(g'(x) = h'(x)\) has a zero between \(x_{A_1}\) and \(x_{A_2}\) which contradicts our assumption.

Therefore if \(f(X_k + x)\) attains a value greater than the greatest local maximum on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]\) we arrive at a contradiction therefore the greatest local maximum is the global maximum on \([X_k - \delta_k, X_k + \delta_k] \cap [-1, 1]\).

We repeat this process for all \(1 \leq k \leq K\) and take \(F = \max_{1 \leq k \leq K} \{G_k + f(X_k), f(-1), f(1)\}\) which is global maximum of \(F\) on \([-1, 1]\).

This process can be repeated on \(-f(x)\), finding the maximum of \(-f(x)\) gives the minimum of \(f(x)\).

Corollary 3.8.1. For a convergent power series \(f\) on \([a, b]\) there are some \(m\) and \(M\) in \(F\) such that \(f([a, b]) = [m, M]\).

Corollary 3.8.2. (Rolle’s Theorem) For \(f\) a convergent power series on \([a, b]\) if \(f(a) = f(b)\) there is some \(c \in (a, b)\) such that \(f'(c) = 0\).

Proof. If \(f(x)\) is constant on \([a, b]\) then \(f'(x) = 0\) for all \(x\) in \((a, b)\) and we are done. We therefore assume \(f(x)\) is not constant. Let \(m\) and \(M\) be as above. Then \(m < M\).

Since \(f(a) \neq f(b)\) either \(M\) or \(m\) is attained at some \(c\) in \((a, b)\). Therefore \(f\) has a local extremum at \(c\) and so \(f'(c) = 0\) by Theorem (3.7).

Corollary 3.8.3. (Mean Value Theorem) For \(f\) a convergent power series on \([a, b]\) there is a \(c \in (a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Proof. Let \(g: [a, b] \to F\) be given by
\[ g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} (x - a) \]  
(3.7)

The function \( g(x) \) is again a power series. As \( g(a) = g(b) = 0 \) then, by Corollary (3.8.2), there is \( c \in (a, b) \) such that

\[ g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \]  
(3.8)

and hence

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]  
(3.9)

\[ \square \]

**Corollary 3.8.4.** For a convergent power series \( f \) on \([a, b]\) the following are true

(i) If \( f'(x) \neq 0 \) for all \( x \in (a, b) \) then either \( f'(x) > 0 \) on \((a, b)\) and \( f \) is strictly increasing on \([a, b]\), or \( f'(x) < 0 \) on \((a, b)\) and \( f \) is strictly decreasing on \([a, b]\).

(ii) If \( f'(x) = 0 \) for all \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

**Proof.** (i) Assume \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\) then, by Theorem (3.5), if \( f'(x) > 0 \) \( f'(y) \) for some \( x, y \in (a, b) \) then there must be some \( c \in (x, y) \) such that \( f'(c) = 0 \) which is a contradiction so either \( f'(x) > 0 \) for all \( x \in (a, b) \) or \( f'(x) < 0 \) for all \( x \in (a, b) \). Assume \( f'(x) > 0 \) for all \( x \) in \((a, b)\); and let \( z > y \) in \([a, b]\) be given. Then by Corollary (3.8.3) there is some \( c \in (y, z) \subset (a, b) \) such that

\[ f'(c) = \frac{f(z) - f(y)}{z - y} \]  
(3.10)
\( f'(c) > 0 \) by assumption, hence \( f(z) - f(y) > 0 \) and \( f \) is increasing on \([a, b]\). Similarly if we assume \( f'(x) < 0 \) for all \( x \) in \((a, b)\) we have \( f(z) - f(y) < 0 \) for all \( z > y \) in \((a, b)\) and \( f \) is decreasing on \([a, b]\).

\((ii)\) Assume \( f'(x) = 0 \) on \((a, b)\) and let \( y \) in \((a, b)\). Then, by Corollary (3.8.3), there is some \( c \) in \((a, y) \subset (a, b)\) such that

\[
 f'(c) = \frac{f(y) - f(a)}{y - a} \quad (3.11)
\]

Since \( c \in (a, b) \) we must have \( f'(c) = 0 \) and hence \( f(y) = f(a) \). Hence \( f \) is constant on \([a, b]\).

In conclusion, in this thesis we reviewed the algebraic and topological structure of \( F \) a non-Archimedean field extension of the real numbers; and we showed that the topology induced by the order is equal to an ultrametric topology. We showed that there are functions on \( F \) which are continuous and infinitely often differentiable that do not satisfy the familiar theorems of real calculus. We studied the convergence and analytical properties of power series on \( F \) and showed that they do have all the smoothness properties of real power series. In particular, they satisfy the intermediate value theorem, the extreme value theorem, the mean value theorem and power series on \( F \) will have a unique anti-derivative up to a constant within the family of power series. Because the family of power series was shown to be closed under addition, multiplication, composition, and uniform limit this makes power series an appropriate family of functions to develop a theory of integration on \( F \) in future.
Bibliography


