TEICHMÜLLER SPACE AND ITS REPRESENTATION WITH THE PERIOD MAPPING

by

Mykhailo Akhtariiev

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University of Manitoba
Winnipeg

UNIVERSITY OF MANITOBA

DEPARTMENT OF DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "*Teichmüller space and its representation with the Period Mapping*" by *Mykhailo Akhtariiev* in partial fulfillment of the requirements for the degree of *Master of Science*.

Dated: ____________________

Supervisor: ____________________

E. Schippers

Readers: ____________________

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Author: *Mykhailo Akhtariiev*

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Abstract

In this thesis we investigate the period mapping of Teichmüller space into the Siegel upper half space.

Chapter 1 of this work consists of the definitions and properties of basic objects we will use in the later chapters. It contains an introduction of the space of all symmetric matrices with positive definite imaginary part, the so-called Siegel upper half space $\mathfrak{H}_g$, and the symplectic group acting on it. The last part of this chapter studies lattices in the complex vector space and the higher-dimensional tori they generate. Later, the Jacobian representation of a Riemann surfaces will use such objects.

The second chapter is an introduction of the object of our study - Riemann surfaces. In this chapter we use the first homology group and holomorphic differentials on Riemann surfaces to define periods and period matrices of a Riemann surface. Later constructions are based on the objects defined in that chapter. Finally we show a representation of Riemann surfaces with their Jacobians - higher dimensional tori associated with every Riemann surface which can be used as their biholomorphic invariants.

In the last chapter we define Riemann, Teichmüller and Torelli moduli spaces of compact Riemann surfaces of genus g and their modular groups and describe the relations between them. Then returning back to period matrices defined in the Chapter 2 we define a period map on the Teichmuller and Torelli moduli spaces into a Siegel upper half space $\mathfrak{H}_g$ and show that this map induces a map on the Riemann moduli space with the image consisting of orbits of the Siegel upper half space under the action of the symplectic group.

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Chapter 0

Introduction and overview

Riemann surfaces are the objects in the category of complex analysis. An important problem is to provide a classification of all Riemann surfaces up to some equivalence relation. In this thesis we will construct a representation of compact Riemann surfaces of genus g with finite-dimensional complex symmetric matrices with positive definite imaginary part.

Chapter 1 consists of the definitions and properties of basic objects we will use in the following chapters. We introduce symplectic vector spaces and symplectomorphisms and we show that all symplectic structures are essentially the same, when considered in appropriate bases. Our attention will mostly be concentrated on matrix representation of such objects and thus, Section 1.1 consists of matrix characteristic and properties of matrices associated with symplectomorphisms. In Section 1.2 we study and show the connection between several different tools in the theory of symplectic spaces: complex structures, Lagrangian subspaces and polarizations. We show that the set of all positive complex structures, the set of all positive polarizations, and the set all Lagrangian subspaces, transverse to a given one in a symplectic vector space, form the same space; and we show that it can be parametrized by the Siegel upper half space $\mathfrak{H}_g$.

In Section 1.4 we provide some facts about multi-dimensional tori, which are the objects we will use in attempts to describe the Riemann space and Teichmuller space of genus g in the Chapter 3. We show that tori with a principal polarization (such tori arise as a representation of Riemann surfaces via Jacobian embedding - more on this later) also have a representation with an element of the Siegel upper half space. This representation varies based on the choice of the basis of the lattice the torus is constructed on and further study reveals that the change of the symplectic basis of the lattice acts on this representation as the symplectic group $Sp_g(\mathbb{Z})$ and we describe this action precisely. This gives us an exact representation of principally polarized tori with elements of $\mathfrak{H}_g/Sp_g(\mathbb{Z})$.

In the Chapter 2 we aim to study the properties of periods and period matrices of a Riemann surface. In order to do so in the early sections (2.1 - 2.4) we give a quick overview of the basic topology theory and the theory of differential forms on Riemann surfaces.

Section 2.5 has a definition of the periods and period matrix of a Riemann surface and we give a proof of the main relations between periods - the Riemann bilinear relations - paying specific attention to its matrix interpretation. It turns out that in matrix form they state precisely that the period matrix belongs to the Siegel upper half space $\mathfrak{H}_g$. This gives us a representation of a Riemann surface with an element of the $\mathfrak{H}_g$ which we will study more later. The construction of period matrices depends on the choice of the basis of the homology group and in Section 2.6 we study the action of basis switching on the period matrix. Once again it appears to be the action of the symplectic group $Sp_g(\mathbb{Z})$ on $\mathfrak{H}_g$ which we describe in detail.

In Section 2.7 we introduce the Jacobian variety of a Riemann surface. We describe two ways to define it: using periods of the Riemann surface and a purely analytical way. The construction of the Jacobian using periods is a g-dimensional torus which can be principally polarized. We show the interpretation of this polarization for both models of the Jacobian. Finally we state the Torelli theorem which says that the representation of Riemann surfaces with their Jacobians embeds the Riemann space into the space of principally polarized tori up to their isomorphisms.

The goal of Chapter 3 is to introduce Riemann and Teichmuller spaces, the relation between them, and find what place does the representation of Riemann surfaces with their period matrices take in between these two.

In Section 3.1.1 we provide the definition for Riemann and Teichmuller moduli spaces, defining the latter in two ways, considering two different types of markings on Riemann surfaces: a system of canonical generators of the fundamental group and orientation preserving diffeomorphisms, and a mapping the Riemann surface onto another one. Even though the markings of Riemann surfaces with orientation preserving diffeomorphisms are more commonly used, we will pay more attention to the markings - system of generators of the fundamental group, because these are more suited for description of the period mappings. Still, we will be using orientation preserving diffeomorphisms definition a frequently and we will construct many parallels between these two ways of looking at Teichmuller space in this chapter.

In Section 3.1.2 we see that the representation of marked Riemann surfaces of genus 1 with their period matrices provides a full description of the Teichmuller space of genus 1 surfaces. More precisely, we show that the Teichmuller space of Riemann surfaces of genus 1 can be modeled by the Siegel upper half plane, by using the period matrix mapping. Motivated by this case, we try to extend this method for higher dimensional case in the following sections. Unfortunately in the case of genus greater than 1, consideration of canonical homology bases is not enough to cover all the markings of the Riemann surfaces with canonical fundamental group bases. This leads to the necessity of the introduction of the Torelli equivalence which is a weaker condition than the Teichmuller equivalence, considering only what is happening with homology bases and not fundamental group bases in general. In Section 3.1.3 we provide the definition of the Torelli equivalence for marked Riemann surfaces considering both cases of markings.

Sections 3.2 and 3.3 are dedicated to studying the actions of Teichmuller and Torelli modular groups on the corresponding spaces. We show that the canonical projections from Teichmuller space onto Torelli space and then onto Riemann space are well defined and are invariant under the action of corresponding modular groups. For instance, this allows one to identify the Riemann space with the orbit set of the Teichmuller space under the action of the Teichmuller modular group.

Finally, in Section 3.4, we consider the mappings from the Teichmuller space and the Torelli space to the Siegel upper half space in attempts to parameterize their elements using the period matrices. It turns out that such a representation for the Torelli space is 1-1 for most of the Riemann surfaces with the exception of hyperelliptic Riemann surfaces, for which this representation is 2-1. The last question which will be answered in this thesis is how does the choice of the marking on the Riemann surface affect the period mapping. As in several cases before, switching the marking acts as a symplectic group $Sp_g(\mathbb{Z})$ on the period matrix and this allows to induce the period map from the Riemann space to the orbit space $\mathfrak{H}_g/Sp_g(\mathbb{Z})$.

Chapter 1

Principally polarized tori and the action of the symplectic group

1.1 Symplectic group

We start with an introduction of the abstract symplectic group consisting of all automorphisms of a vector space preserving its symplectic structure. Although the main object of our interest is a space this group is acting on, starting this way allows us to outline the main properties we will often need later. More information and further study on symplectic geometry can be found in [1] and [9].

Definition 1.1 A pair (V, ω) is called a *symplectic vector space* if V is a vector space and ω is an antisymmetric and nondegenerate bilinear form; that is

$$\omega_x(\cdot) = \omega(x, \cdot) : V \to V \text{ is a non-zero operator for every non-zero } x \in V.$$

Remark. *Equivalently, in finite dimensions ω is nondegenerate if and only if $\omega_1 : V \to V^*$, $\omega_1(x) = \omega_x$ is an isomorphism, where V^* is the dual space to V. It also implies that $\omega_2(y) := \omega(\cdot, y)$ meets the same requirements and vice versa.*

Remark. *The first section contains some general facts about symplectic spaces and although later we will restrict our attention to symplectic vector spaces over $\mathbb{R}$ or over $\mathbb{C}$, the definition and the rest of this section intentionally deals with vector spaces over an arbitrary field $\mathbb{F}$.*

Theorem 1.1.2. *If V is a finite dimensional symplectic vector space (over the field $\mathbb{F}$) then $\dim V = 2g$ and there exists a basis of V in which the matrix, associated with ω, is of form*

$$J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}.$$

Such bases are called symplectic bases of (V, ω).

Proof. This fact can be proven by induction on the dimension of V. The case when $V = 0$ holds with $g = 0$.

Take any nonzero element $v \in V$. By the nondegeneracy of ω there exists a vector $w^0 \in V$ such that $\omega(v, w^0) \neq 0$ and by taking w a multiple of w^0 we can assume that $\omega(v, w) = 1$.

Now take V' to be a subspace of V that is ω-orthogonal to v and w. Every vector u has the unique decomposition $u = \omega(u, w)v - \omega(u, v)w + u'$ where $u' \in V'$ so we have an ω-orthogonal decomposition $V =< v, w > \oplus V'$. The restriction of ω to V' is antisymmetric and nondegenerate, and V' has lower dimension than V, so we can apply the induction hypothesis to V' and $\omega|_{V' \times V'}$.

Finally if $\{u_1, ..., u_{2g-2}\}$ is a symplectic basis of V' then

$$\{v, u_1, ..., u_{g-1}, w, u_g, ..., u_{2g-2}\}$$

is a symplectic basis of V. □

Definition 1.3 A linear transformation $\varphi : V_1 \to V_2$ is called a *symplectomorphism* of two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2) if

$$\omega_2(\varphi(v_1), \varphi(w_1)) = \omega_1(v_1, w_1)$$

for every choice of $v_1, w_1 \in V_1$

Remark. *Symplectomorphisms are necessarily monomorphisms since* $\varphi(v_1) = 0$ *implies that* $\omega(v_1, w_1) = 0$ *for every* w_1 *and thus* $v_1 = 0$. *Hence if* $\dim V_1 = \dim V_2 < \infty$ *every symplectomorphism is an isomorphism. It is easy to check that the inverse is symplectic itself.*

Given two arbitrary symplectic vector spaces V *and* W *of equal finite dimension one can choose symplectic bases* $\{v_1, ..., v_{2g}\}$ *and* $\{w_1, ..., w_{2g}\}$ *and define a symplectomorphism* $\varphi : V \to W$ *by sending* v_i *to* w_i. *This shows that any two finite-dimensional symplectic vector spaces are symplectically isomorphic.*

Definition 1.4 The *symplectic group* $Sp(V)$ of a symplectic vector space V is the collection of all symplectic automorphisms of V.

Remark. *The last remark implies that the term "symplectomorphism"* $V \to V$ *can be used to describe all symplectic automorphisms. In the future I will often use the term "symplectomorphism" meaning symplectic automorphism.*

Proposition 1.1.5. *Let* V *be a symplectic vector space. A* $2g \times 2g$ *matrix* M *represents a symplectomorphism of* V *(in a symplectic basis) if and only if it satisfies*

$$M^T J M = J. \tag{1.1}$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *then equation (1.1) can be written as*

$$A^T C = C^T A,\ B^T D = D^T B,\ A^T D - C^T B = I_g. \tag{1.2}$$

Proof. Fix a symplectic basis of V. Let $[u]$ represent the coordinate vector for $u \in V$ in the symplectic basis. Then

$$\omega(v, w) = [w]^T J[v].$$

Let M be a matrix of a linear transformation $\varphi : V \to V$ in the symplectic basis. Then φ is symplectic if and only if

$$\omega(\varphi(v), \varphi(w)) = \varphi(v, w), \text{ for all } v, w \in V.$$

In the symplectic basis this condition says

$$[\varphi(w)]^T J[\varphi(v)] = [w]^T J[v]$$

or

$$[w]^T M^T JM[v] = [w]^T J[v], \text{ for all } v, w \in V.$$

Since v, w are arbitrary this is equivalent to (1.1). (1.2) is an immediate consequence of (1.1) if one uses the block representation of M. □

Definition 1.6 The matrices that meet condition (1.1) are called *symplectic matrices.*

If we fix a symplectic basis of V with respect to ω we obtain an exact representation of all symplectomoprhisms of (V, ω) by symplectic matrices. Set $Sp_g(\mathbb{F})$ to consist of all symplectic matrices of dimension $2g \times 2g$. It means that any symplectic group $Sp(V)$ can be identified (in many ways, depending on the choice of the symplectic basis) with $Sp_g(\mathbb{F})$ where g is one half of $\dim V$.

Proposition 1.1.7. *If M is a matrix of a symplectic automorphism then so is M^T, and thus (1.1) is equivalent to*

$$AB^T = BA^T,\ CD^T = DC^T,\ AD^T - BC^T = I_g. \tag{1.3}$$

Proof. The equality (1.1) yields $J = M^{-T} JM^{-1}$ which means that M^{-1} is also a matrix of a symplectic automorphism and by taking inverses from both sides we immediately get

$$-J = M(-J)M^T,$$

or

$$(M^T)^T J(M^T) = J$$

and therefore M^T is a matrix of a symplectic automorphism as well. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $M^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$ and equalities (1.2) for this matrix say

$$(A^T)^T B^T = (B^T)^T A^T,\ (C^T)^T D^T = (D^T)^T C^T,\ (A^T)^T D^T - (B^T)^T C^T = I_g,$$

and after simplification we get (1.3). □

Proposition 1.1.8. *All symplectic matrices have unit determinant.*

Proof. Fix a symplectic basis $\{e_1, ..., e_{2n}\}$ of V and consider the dual basis $\{e_1^*, ..., e_{2n}^*\}$ of V^* satisfying

$$e_i^*(e_j) = \delta_{ij}.$$

Then the symplectic form can be written in terms of the exterior product

$$\omega = \sum_{i=1}^{n} e_i^* \wedge e_{i+n}^*.$$

A computation shows that

$$\omega^n = \omega \wedge ... \wedge \omega = (-1)^{\frac{n(n-1)}{2}} e_1^* \wedge ... \wedge e_{2n}^*. \tag{1.4}$$

Take $\tau = e_1^* \wedge ... \wedge e_{2n}^*$. Let $\varphi : V \to V$ be a symplectic automorphism and M be a matrix of this transformation in the symplectic basis.

Equation (1.4) shows that τ is invariant under $\varphi \otimes ... \otimes \varphi$. Therefore

$$1 = \tau(e_1, ..., e_{2n}) = \tau(\varphi(e_1), ..., \varphi(e_{2n})) = e_1^* \wedge ... \wedge e_{2n}^*(\varphi(e_1), ..., \varphi(e_{2n})) = \det M.$$

□

Equation (1.4) shows that the form τ is independent of the choice of the symplectic basis. This form τ is known as the *volume form* on V.

Remark. *From the proof of the last proposition it immediately follows that a symplectic structure ω on the vector space V induces an orientation τ on V and all symplectic automorphisms are automatically orientation preserving transformations.*

1.2 Complex structures, Lagrangian subspaces and polarizations

The goal of this section is to describe the set of all possible positive complex structures on a real symplectic vector space.

Definition 2.1 A *complex structure* on the vector space V (over $\mathbb{R}$) is a linear automorphism $J \in Aut(V)$ such that $J^2 = -\mathrm{id}_V$.

A complex structure on a symplectic vector space (V, ω) is called *compatible with* ω if it satisfies

$$\omega(Jx, Jy) = \omega(x, y).$$

A compatible complex structure on (V, ω) is called *positive* if

$$\omega(w, Jw) > 0, \text{ for all } w \in V.$$

1.2.1 Lagrangian subspaces

The concept of complex structures is closely related to Lagrangian subspaces and polarizations. In this part we are going to outline some important facts concerning Lagrangian subspaces.

Definition 2.2 Given a symplectic structure ω on the vector space V two vectors v, w are called ω-orthogonal if and only if $\omega(v, w) = 0$.

The ω-orthogonal complement to a subspace $W \subset V$ is the subspace

$$W^{\perp} = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Remark. *ω-orthogonality has similar properties to the regular orthogonality. For example one can show that*

$$\dim W^{\perp} = \dim V - \dim W. \tag{1.5}$$

Definition 2.3 A subspace $L \subset V$ is called *Lagrangian* if and only if $L^{\perp} = L$.

According to the formula (1.5) we immediately obtain that

$$\dim L = \frac{1}{2}\dim V$$

for any Lagrangian subspace.

We now give a brief sketch of the theory of ω-orthogonal decompositions.

Theorem 1.2.4. *For any Lagrangian subspace $L \subset V$ there exists a Lagrangian subspace L' such that*

$$V = L \oplus L'.$$

The last theorem justifies the following definition.

Definition 2.5 Let L and L' be Lagrangian subspaces. L' is called *transverse* to L if and only if

$$V = L \oplus L'.$$

$\mathcal{T}(L)$ denotes the set of all Lagrangian subspaces to L.

Theorem 1.2.6. *Given a decomposition $V = L \oplus L'$ where L and L' are Lagrangian there exists a symplectic basis $\{e_1, ..., e_n, e'_1, ..., e'_n\}$ of V such that $\{e_1, ..., e_n\}$ forms a basis of L and $\{e'_1, ..., e'_n\}$ forms a basis of L'.*

Proof. We sketch the proof which repeats the proof of the Theorem 1.1.2 with small technical changes.

To start with one chooses a pair of vectors $e_1 \in L$ and $e'_1 \in L'$ such that $\omega(e_1, e'_1) = 1$. To do so fix an arbitrary element $e_1 \in L$ and find another vector $\tilde{e} \in V$ such that $\omega(e_1, \tilde{e}) = 1$. This is possible due to the linearity and non-degeneracy of ω. Vector $\tilde{e}$ has a decomposition $\tilde{e} = \tilde{e}^L + \tilde{e}^{L'}$, where $\tilde{e}^L \in L$ and $\tilde{e}^{L'} \in L'$. Set $e'_1 = \tilde{e}^{L'}$. Since L is Lagrangian we have that

$$1 = \omega(e_1, \tilde{e}) = \omega(e_1, \tilde{e}^{L'}) = \omega(e_1, e'_1).$$

Next find a ω-complement space L_1 to $< e_1 >$ in L and a ω-complement space L'_1 to $< e'_1 >$ in L'. Then ω restricted to $V_1 = L_1 \oplus L'_1$ is a symplectic form and L_1 and L'_1 are Lagrangian subspaces of V_1. Therefore we can continue the described procedure applied to V_1. After all the vectors are exhausted we end up with the desired bases for L and L'. $\square$

Definition 2.7 For a decomposition $V = L \oplus L'$, a basis $\{e_1, ..., e_n, e'_1, ..., e'_n\}$ as in Theorem 1.2.6 is called *adapted* to this decomposition.

Every symplectomorphism T takes a Lagrangian subspace to a Lagrangian subspace. This is because $\dim T(L) = \dim L = \frac{1}{2} \dim V$ (and hence $\dim T(L)^{\perp} = \frac{1}{2} \dim V$ by formula (1.5)) and orthogonality of the elements is translated by T:

$$\omega(T(v), T(w)) = \omega(v, w) = 0, \text{ for all vectors } v, w \in V,$$

and so $T(L) \subset (T(L))^{\perp}$. This yields $T(L) = (T(L))^{\perp}$.

Of particular interest are those symplectomorphisms that preserve a fixed Lagrangian subspace.

Theorem 1.2.8. *The set of all symplectomorphisms that preserve L acts transitively on $\mathcal{T}(L)$.*

Moreover

Theorem 1.2.9. *For any two decompositions $V = L \oplus L'$ and $V = L \oplus L''$ there is a symplectomorphism that fixes every vector of L and translates L' to L''.*

Proof. See [1], page 23 for the proof of this and the previous theorem. $\square$

Remark. *A symplectomorphism as in Theorem 1.2.9 in the basis adapted to the decomposition $V = L \oplus L'$ has the form*

$$[T] = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix},$$

where $M^T = M$ by the relations (1.2). It is easily seen that all symmetric matrices parametrize all such symplectomorphisms.

From now on we will operate with both real symplectic vector space and its complexification. To distinguish them denote the real vector space by $V_{\mathbb{R}}$ and define its complexification to be

$$V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}.$$

Given a symplectic vector space $(V_{\mathbb{R}}, \omega_{\mathbb{R}})$, extend the symplectic form to $V_{\mathbb{C}}$ by linearity (over $\mathbb{C}$) so now $V_{\mathbb{C}}$ has a symplectic structure inherited from $V_{\mathbb{R}}$.

Complex conjugation on $V_{\mathbb{C}}$ is defined in the following way

$$\overline{\alpha v} = \overline{\alpha} v, \text{ for } v \in V_{\mathbb{R}},$$

and it is real-linearly extended to the whole $V_{\mathbb{C}}$.

Definition 2.10 A Lagrangian subspace of $V_{\mathbb{C}}$ is called *real* if and only if it is a complexification of a Lagrangian subspace in $V_{\mathbb{R}}$.

A Lagrangian subspace L of $V_{\mathbb{C}}$ is called *positive* if and only if the form

$$L \ni v \to -i\omega(v, \bar{v})$$

is positive definite, meaning

$$-i\omega(v, \bar{v}) > 0, \text{ for all } v \in L.$$

Theorem 1.2.11. *Any real Lagrangian subspace in $V_{\mathbb{C}}$ is transverse to any positive Lagrangian subspace in $V_{\mathbb{C}}$.*

Proof. Let L be a real Lagrangian subspace and let W be a positive Lagrangian subspace of $V_{\mathbb{C}}$. Then L is a complexification of a Lagrangian subspace L' in $V_{\mathbb{R}}$.

Let $u \in L \cap W$. Every vector in L can be written in the form $u = v + i \cdot w$ for some $v, w \in L' \subset V_{\mathbb{R}}$. Then

$$-i\omega_{\mathbb{C}}(u, \bar{u}) = -i\omega_{\mathbb{C}}(v + i \cdot w, v - i \cdot w) = 2\omega_{\mathbb{R}}(v, w) = 0$$

since v, w belong to L' - Lagrangian subspace. Since $u \in W$ and W is positive this equality implies that $u = 0$.

So L and W have zero intersection. This and the fact that $\dim L = \dim W = \frac{1}{2}\dim V_{\mathbb{C}}$ prove that

$$L \oplus W = V_{\mathbb{C}}.$$

□

1.2.2 Polarizations

For arbitrary subspace $W \subset V_{\mathbb{C}}$ let $\overline{W}$ denote the subspace consisting of all elements complex conjugate to elements of W.

$$\overline{W} = \{\bar{w} \mid w \in W\}$$

Definition 2.12 A *positive polarization* of $V_{\mathbb{C}}$ is a decomposition $V_{\mathbb{C}} = W \oplus \overline{W}$ where W is a positive Lagrangian subspace.

Remark. *Any positive Lagrangian subspace $W \subset V_{\mathbb{C}}$ has zero intersection with its conjugate and therefore every positive Lagrangian subspace defines its own polarization.*

Proof. If $w \in W \cup \overline{W}$ then $w, \bar{w} \in W$. w can be written as $w = u + i \cdot v$ for some $u, v \in V_{\mathbb{R}}$. Then $w + \bar{w} = 2u$ and $w - \bar{w} = 2i \cdot v$ belong to W. For these vectors we have

$$-i\omega(2u, \overline{2u}) = -4i\omega(u, u) = 0$$

$$-i\omega(2i \cdot v, \overline{2i \cdot v}) = -4i\omega(v, v) = 0.$$

Positivity of W yields then that $u = v = 0$ and $w = 0$ is the only vector in the intersection.

□

Any positive complex structure $J_{\mathbb{R}}$ on $V_{\mathbb{R}}$ defines a positive polarization

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{+} \oplus \overline{V_{\mathbb{C}}^{+}}$$

of $V_{\mathbb{C}}$ in the following way (we will see later (Theorem 1.2.13) that all positive polarizations can be described in such a way). Take a positive complex structure $J_{\mathbb{R}}$ on $V_{\mathbb{R}}$. Extend the action to $J_{\mathbb{C}}$ on $V_{\mathbb{C}}$ complex linearly. The property $J_{\mathbb{C}}^2 = -\mathrm{id}_{V_{\mathbb{C}}}$ still holds and therefore the only possible eigenvalues of J are $\pm i$. Consider the corresponding eigenspaces:

$$V_{\mathbb{C}}^{+} = \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}} v = iv\}$$

$$V_{\mathbb{C}}^{-} = \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}} v = -iv\}$$

One can show that

$$V_{\mathbb{C}}^{\pm} = \{v \mp iJ_{\mathbb{R}} v \mid v \in V_{\mathbb{R}}\},\ V_{\mathbb{C}}^{-} = \overline{V_{\mathbb{C}}^{+}}$$

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{+} \oplus V_{\mathbb{C}}^{-} = V_{\mathbb{C}}^{+} \oplus \overline{V_{\mathbb{C}}^{+}}$$

and $V_{\mathbb{C}}^{+}$ is a positive Lagrangian subspace. To see this take any $v - iJv, w - iJw \in V_{\mathbb{C}}^{+}$. Then

$$\omega(v - iJv, w - iJw) = \omega(v, w) - i\omega(Jv, w) - i\omega(v, Jw) - \omega(Jv, Jw) = 0$$

because $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jw) = -\omega(J^2 v, Jw) = -\omega(Jv, w)$. Finally

$$\begin{aligned} &- i\omega(v - iJv, \overline{v - iJv}) = -i\omega(v - iJv, v + iJv) = \\ &\qquad = -i\omega(v, v) - i\omega(Jv, Jv) - \omega(Jv, v) + \omega(v, Jv) = 2\omega(v, Jv) > 0 \end{aligned}$$

by positivity of J and antisymmetry of ω.

1.2.3 Connection between positive complex structures, positive polarizations and Largangian subspaces.

Theorem 1.2.13. *The set of all positive complex structures on $V_{\mathbb{R}}$ is in one-to-one correspondence with the set of all positive polarizations of $V_{\mathbb{C}}$.*

Remark. *According to the Theorem 1.2.11, the set of all positive polarizations of $V_{\mathbb{C}}$ can be parameterized by $\mathcal{T}(L)$ for an arbitrary real Lagrangian subspace $L \subset V_{\mathbb{C}}$.*

And finally we get

Theorem 1.2.14. *Let $V_{\mathbb{R}}$ be a symplectic vector space. The set of all positive polarizations $V_{\mathbb{C}} = W \oplus \overline{W}$ is parameterized by the following set of matrices*

$$\mathfrak{H}_g := \left\{ M \in M_g(\mathbb{C}) \;\middle|\; M = M^T,\ \mathrm{Im}\ (M) = \frac{1}{2i}(M - \overline{M}) > 0 \right\}$$

in the following way. Fix an arbitrary decomposition $V_{\mathbb{C}} = L \oplus L'$ into a sum of real and positive Lagrangian subspaces. Then all such W are the images of L' under symplectomorphisms $T : L' \to V_{\mathbb{C}}$ which in any base adapted to the decomposition $V_{\mathbb{C}} = L \oplus L'$ has the form

$$T = \begin{pmatrix} Z \\ I \end{pmatrix},\ Z \in \mathfrak{H}_g$$

Proof. Fix the mentioned decomposition $V_{\mathbb{C}} = L \oplus L'$. Let $\{e_1, ..., e_n, e'_1, ..., e'_n\}$ be an adapted basis for this decomposition.

W, as a positive Lagrangian subspace of $V_{\mathbb{C}}$, can be an arbitrary subspace from $\mathfrak{T}(L)$ (Theorem 1.2.11) and all such subspaces are obtained from L' by a symplectomorphism of the form

$$[T] = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}$$

with $Z^T = Z$ (Theorem 1.2.9 and the Remark following it). Therefore all positive polarizations can be parameterized by symmetric matrices. To understand which symmetric matrices provide us with a positive W take an arbitrary vector $w = \sum_{j=1}^n \alpha_j T(e'_j)$, $(T(e'_j) = \sum_{k=1}^n z_{kj} e_k + e'_j)$. Then by using the fact that $\{e_1, ..., e'_n\}$ is a symplectic basis and by symmetry of Z one can rewrite

$$-i\omega(w, \bar{w}) = -i \sum_{j,k=1}^n (z_{jk} - \overline{z_{jk}})\alpha_j \overline{\alpha_k} = \frac{1}{i} \sum_{j,k=1}^n (z_{jk} - \overline{z_{jk}})\alpha_j \overline{\alpha_k},$$

which proves that positivity of W is equivalent to positive definiteness of $\frac{1}{2i}(Z - \overline{Z}) = \mathrm{Im}\ (Z)$. □

1.3 Siegel upper half space in $\mathbb{C}^g$

In this section we are going to introduce the Siegel upper half space and outline one of the actions of the symplectic group on this space. We will later see that there are many natural ways the symplectic group is acting on the Siegel upper half space which shows how these two spaces are closely connected.

Siegel upper half space is a generalization of upper half space in $\mathbb{C}$:

$$\mathfrak{H} = \{z \in \mathbb{C} \,|\, \text{Im } z > 0\} = \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{2i}(z - \overline{z}) > 0 \right\}$$

Definition 3.1 Siegel upper half space of genus g is the set

$$\mathfrak{H}_g := \left\{ M \in M_g(\mathbb{C}) \;\middle|\; M = M^T,\ \text{Im } (M) = \frac{1}{2i}(M - \overline{M}) > 0 \right\}$$

Remark. *Another convenient model of the Siegel upper half plane is the unit disk.*

The map $z \mapsto i\frac{1+z}{1-z}$ *is a homeomorphism between the unit disk* $\mathbb{S}^1 := \{z|1 - z\overline{z} > 0\}$ *and the upper half plane* $\mathfrak{H}$.

There is also a disk model for the Siegel upper half space for arbitrary genus g.

Let $\mathbb{S}^g := \{M \in M_g(\mathbb{C}) \,|\, M^T = M,\ I - M\overline{M} > 0\}$ *denote the "unit disk" in* $\mathbb{C}^g$. *Then the mapping*

$$Z \mapsto i(I + Z)(I - Z)^{-1},\ Z \in \mathbb{S}^g$$

is a well-defined bijection between $\mathbb{S}^g$ *and* $\mathfrak{H}_g$

The symplectic group acts in natural way on the Siegel upper half space:
If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{R})$ and $Z \in \mathfrak{H}_g$ then

$$Z^M = (AZ + B)(CZ + D)^{-1} \tag{1.6}$$

Theorem 1.3.2. *The action (1.6) is a well-defined action of* $Sp_g(\mathbb{R})$ *on* $\mathfrak{H}_g$.

Proof. Define

$$\mathcal{B} := AZ + B,\ \mathcal{D} := CZ + D,$$

$$Y = \begin{pmatrix} Z \\ E \end{pmatrix}$$

$$X = \begin{pmatrix} \mathcal{B} \\ \mathcal{D} \end{pmatrix} = MY$$

Then $Z^M = \mathcal{B}\mathcal{D}^{-1}$ (Existence of $\mathcal{D}^{-1}$ is still to be proven).

$$\left(\frac{1}{2i}(\mathcal{B}^T\overline{\mathcal{D}} - \mathcal{D}^T\overline{\mathcal{B}}) =\right) \frac{1}{2i}X^TJ\overline{X} = \frac{1}{2i}Y^TM^TJ\overline{MY} =$$

$$= \frac{1}{2i}\begin{pmatrix} Z & E \end{pmatrix} M^TJM \begin{pmatrix} \overline{Z} \\ E \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} Z & E \end{pmatrix} J \begin{pmatrix} \overline{Z} \\ E \end{pmatrix} = \frac{1}{2i}(Z - \overline{Z}) > 0$$

Assume $\mathcal{D}v = 0$ then $\overline{\mathcal{D}}\overline{v} = 0$ and $v^T\mathcal{D}^T = 0$ and thus

$$v^T\frac{1}{2i}(\mathcal{B}^T\overline{\mathcal{D}} - \mathcal{D}^T\overline{\mathcal{B}})\overline{v} = 0$$

which contradicts the positivity of $\frac{1}{2i}(\mathcal{B}^T\overline{\mathcal{D}} - \mathcal{D}^T\overline{\mathcal{B}})$ and hence $\mathcal{D}^{-1}$ does exist.

Now let us check the definition of upper half space.

$$(\mathcal{B}\mathcal{D}^{-1})^T = \mathcal{B}\mathcal{D}^{-1} \Leftrightarrow \mathcal{D}^T\mathcal{B} = \mathcal{B}^T\mathcal{D}$$

$$\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B} = \begin{pmatrix} \mathcal{B}^T & \mathcal{D}^T \end{pmatrix} J \begin{pmatrix} \mathcal{B} \\ \mathcal{D} \end{pmatrix} = \begin{pmatrix} Z^T & E \end{pmatrix} M^TJM \begin{pmatrix} Z \\ E \end{pmatrix} =$$

$$= \begin{pmatrix} Z^T & E \end{pmatrix} J \begin{pmatrix} Z \\ E \end{pmatrix} = Z^T - Z = 0$$

and hence Z^M is symmetric.

$$\frac{1}{2i}\left(\mathcal{B}\mathcal{D}^{-1} - \overline{\mathcal{B}\mathcal{D}^{-1}}\right) = \frac{1}{2i}\mathcal{D}^{-T}\left(\mathcal{B}^T\overline{\mathcal{D}} - \mathcal{D}^T\overline{\mathcal{B}}\right)\overline{\mathcal{D}^{-1}} = \frac{1}{2i}\mathcal{D}^{-T}X^TJ\overline{X}\overline{\mathcal{D}^{-1}} =$$

$$= \frac{1}{2i}\mathcal{D}^{-T}Y^TM^TJ\overline{MY}\overline{\mathcal{D}^{-1}} = \frac{1}{2i}\mathcal{D}^{-T}Y^TJ\overline{Y}\overline{\mathcal{D}^{-1}} = \frac{1}{2i}\mathcal{D}^{-T}\left(Z^T - \overline{Z}\right)\overline{\mathcal{D}^{-1}} =$$

$$= \frac{1}{2i}\mathcal{D}^{-T}\left(Z - \overline{Z}\right)\overline{\mathcal{D}^{-1}} > 0$$

since $\frac{1}{2i}\left(Z - \overline{Z}\right) > 0$. This proves that $Z^M \in \mathfrak{H}_g$. $\square$

1.4 Lattices in $\mathbb{C}^g$ and Higher Dimensional Tori

It is well known that a convenient topological model of tori of genus 1 is parallelograms with identified opposite sides. Another way to phrase it is to say that any torus is topologically equivalent to a complex plane modulo the action of a lattice generated by two sides of the parallelogram. We will use a similar model for a surfaces of genus higher than 1 and this section is devoted to study of these objects.

Definition 4.1 Let $\mathbf{e} = \{e_1, e_2, ..., e_{2g}\}$ be a collection of $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^g$. The subgroup

$$L = \mathbb{Z}e_1 + \mathbb{Z}e_2 + ... + \mathbb{Z}e_{2g}$$

of $\mathbb{C}^g$ is called a *lattice.*

Obviously such a collection of vectors forms a basis for $\mathbb{C}^g$ as a $2g$-dimensional vector space over $\mathbb{R}$.

Definition 4.2 A $g \times 2g$ complex matrix $\Pi = \left(e_1^T \; e_2^T \; ... \; e_{2g}^T\right)$, with vector e_i taken as the ith column, is called the *period matrix* for the lattice L and its given basis.

Theorem 1.4.3. *Two bases generate the same lattice iff the change of basis matrix is integer-valued matrix with determinant* ± 1*.*

Proof. The new basis has integer coordinates in the old system and thus the change of basis matrix must consist of integer elements. We also know that it is an invertible matrix and the inverse is integer-valued itself by the same reason. Hence $1 = \det E \cdot \det E^{-1}$ implies that both determinants are ± 1. $\square$

Definition 4.4 The topological space $\mathbb{C}^g/L$ is called a g-dimensional torus.

Remark. *Because the actions of lattices on* $\mathbb{C}^g$ *are discrete and fixed point free, the complex tori inherit the structure of complex manifold from* $\mathbb{C}^g$*.*

Definition 4.5 A positive definite hermitian form H on $\mathbb{C}^g$ is called a *polarization* on the torus $T = \mathbb{C}^g/L$ if $E = \operatorname{Im} H$ is an integer valued antisymmetric form on L.

Linear isomorphisms φ of $\mathbb{C}^g$ that transform one lattice L_1 into another L_2 induce homeomorphisms between corresponding tori $T_1 = \mathbb{C}^g/L_1$ and $T_2 = \mathbb{C}^g/L_2$. Denote the induced mapping by $\varphi_{\#}$.

$$\begin{tikzcd} \mathbb{C}^g \arrow[r, "\varphi"] \arrow[d] & \mathbb{C}^g \arrow[d] \\ \mathbb{C}^g/L_1 \arrow[r, "\varphi_{\#}"] & \mathbb{C}^g/L_2 \end{tikzcd}$$

Isomorphisms of complex tori $\varphi_{\#} : T_1 = \mathbb{C}^g/L_1 \to T_2 = \mathbb{C}^g/L_2$ are those mappings which are induced by linear isomorphisms φ of $\mathbb{C}^g$ such that $\varphi(L_1) = L_2$.

Definition 4.6 Two polarized tori (T_1, H_1) and (T_2, H_2) are *isomorphic* if there exists an isomorphism $\varphi_{\#} : T_1 \to T_2$ such that pullback of H_2 by $\varphi_{\#}$ equals H_1:

$$H_1(x, y) = H_2(\varphi_{\#}(x), \varphi_{\#}(y)).$$

Now let us give some characterization of all Hermitian forms on $\mathbb{C}^g$. Let the torus $T = \mathbb{C}^g/L$ and the basis $\mathbf{e}$ of the lattice be fixed, let Π denote the period matrix of the lattice and $[E]_{\mathbf{e}}$ be a matrix of the form $E = \operatorname{Im} H$ in the basis $\mathbf{e}$.

Proposition 1.4.7. *There is $1-1$ correspondence between Hermitian forms H and real-valued alternating forms E satisfying $E(iu, iv) = E(u, v)$ via relations*

$$H(u, v) = E(iu, v) + iE(u, v),\ E(u, v) = \operatorname{Im} H(u, v) \tag{1.7}$$

Proof. The proof of this statement can be found in [5], page 29. □

In the following proposition we state the conditions on E in matrix form that generate a positive definite Hermitian for H.

Proposition 1.4.8. *1. A real-valued alternating form E defines a Hermitian form H via the relations (1.7) iff it satisties*

$$\Pi[E]_{\mathbf{e}}^{-1}\Pi^T = 0.$$

2. In the same set up E defines a positive definite form H iff it satisties

$$i\Pi[E]_{\mathbf{e}}^{-1}\overline{\Pi}^T > 0$$

Proof. See [5], page 76. □

Definition 4.9 A polarization H is called *principal* if $\det[E]_{\mathbf{e}} = 1$.

Proposition 1.4.10. *If a torus $T = \mathbb{C}^g/L$ admits a principal polarization H then there exists a basis $\mathbf{e}$ of L in which $E = \mathrm{Im}\, H$ has form*

$$[E]_{\mathbf{e}} = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}.$$

Definition 4.11 Such bases are called *symplectic bases* of the lattice.

Remark. *A polarization on $T = \mathbb{C}^g/L$ provides $\mathbb{C}^g$ with a symplectic form $E = \mathrm{Im}\, H$. The last proposition states that some basis of the lattice can be chosen as a symplectic basis of the symplectic vector space $\mathbb{C}^g$ if the polarization is principal. Conversely, Propositions 1.4.7 and 1.4.8 state conditions on a symplectic basis of $\mathbb{C}^g$ with symplectic form E that provide the corresponding torus with a principal polarization.*

Proof. (of the Proposition)

First, observe that

$$\det E = 1 \Leftrightarrow E \in SL(2g, \mathbb{Z}) \Leftrightarrow E^{-1} \in SL(2g, \mathbb{Z}).$$

Choose $e'_1 = e_1$ and $e'_{g+1} = \sum_{i=1}^{2g} (E^{-1})_{i1} e_i$ (later the first coordinate of this vector will be changed to 0). Then

$$E(e'_1, e'_{g+1}) = \sum_{i=1}^{2g} \left(E^{-1}\right)_{i1} E(e_1, e_i) = \sum_{i=1}^{2g} \left(E^{-1}\right)_{i1} E_{1i} = 1.$$

Note: in the initial basis e'_1 has coordinates $I_{\cdot 1}$ and e'_{g+1} has coordinates $(E^{-1})_{\cdot 1}$.

Let $L' =< e'_1, e'_{g+1} >^{\perp} \cap L$ (here $< ... >$ denotes a linear span) where the orthogonal complement is taken in $\mathbb{R}^{2g}$ with respect to E. Our goal is to prove that L' is generated by $2g-2$ vectors which together with e'_1 and e'_{g+1} form a basis of L.

Solve $\sum_{i=1}^{2g} \alpha_i e_i \perp < e'_1, e'_{g+1} >$.

$$\sum_{i=1}^{2g} \alpha_i e_i \perp e'_1 \Leftrightarrow \sum_{i=1}^{2g} \alpha_i E_{1i} = 0$$

The last condition holds for $\alpha = (E^{-1})_{\cdot 2}, ..., (E^{-1})_{\cdot 2g}$ so we can look at the columns (all except the first one) of E^{-1} as coordinates of vectors which are orthogonal to e'_1. So

$$< e'_1 >^{\perp} =< (E^{-1})_{\cdot 2}, ..., (E^{-1})_{\cdot 2g} > .$$

$$\sum_{i=1}^{2g} \alpha_i e_i \perp e'_{n+1} = \sum_{j=1}^{2g} (E^{-1})_{j1} e_j \Leftrightarrow 0 = \sum_{i,j=1}^{2g} \alpha_i (E^{-1})_{j1} E_{ij} = \alpha_1$$

So the only condition for a vector to be perpendicular to e'_{g+1} is $\alpha_1 = 0$. Note that if we change the first coordinate of $(E^{-1})_{\cdot 2}, ..., (E^{-1})_{\cdot 2g}$ to 0 the first orthogonality will still hold. That is because the only term that might change is $\alpha_1 E_{11}$, but $E_{11} = 0$ so α_1 doesn't actually influence first equality. By the same reason we may change the first coordinate of e'_{g+1} to 0.

Denote by $\widetilde{(E^{-1})}_{\cdot i}$, $i = 1, ..., 2g$ vector $(E^{-1})_{\cdot i} - (E^{-1})_{1i} \cdot e_1$. From above:

$$\left\langle e_1, \left(E^{-1}\right)_{\cdot 1} \right\rangle^{\perp} = \left\langle \widetilde{(E^{-1})}_{\cdot 2}, ..., \widetilde{(E^{-1})}_{\cdot 2g} \right\rangle .$$

Let us now prove that there exist $2g-2$ vectors from the lattice L that form a basis of the lattice on the right. Construct an integer $2g \times 2g$ matrix X with unit determinant given information of which is described below (existence of such a matrix is guaranteed by the Lemma 1.4.12 – see below):

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & (E^{-1})_{2\,1} & (E^{-1})_{3\,1} & \dots & (E^{-1})_{2g\,1} \\ 0 & ? & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ? & ? & \dots & ? \end{pmatrix}$$

Obviously, the first row represents e_1 in $e_1, ..., e_{2g}$ coordinates, the second row represents $\widetilde{(E^{-1})}_{\cdot 1}$. Denote the vectors represented by rest of the lines by $v_3, ..., v_{2g}$. Since rows 3..$2g$ are linearly independent with 1st and 2nd we obtain that

$$\langle v_3, ..., v_{2g} \rangle = \left\langle \widetilde{(E^{-1})}_{\cdot 2}, ..., \widetilde{(E^{-1})}_{\cdot 2g} \right\rangle .$$

This matrix has unit determinant so vectors $e'_1, e'_{g+1}, v_3, ..., v_{2g}$ form a claimed basis of L. The matrix of E in this new basis has form

$$\begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

$L' =< v_3, ..., v_{2g} >$ Form E, restricted to L' still has the same properties (integer-valued, antisymmetric form with unit determinant) so we can apply induction here. In the end, after proper permutation of basic vectors, the matrix of E will have the desired form. $\square$

Lemma 1.4.12. *Given any line $(a_1\ a_2\ ...\ a_g)$ of g relatively prime integers one can complete it to $g \times g$ integer matrix with unit determinant:*

$$X = \begin{pmatrix} a_1 & a_2 & \dots & a_g \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix}$$

Proposition 1.4.13. *Let L be a principally polarized lattice in $\mathbb{C}^g$ with a symplectic basis $\{e_1, ..., e_g, e_{g+1}, ..., e_{2g}\}$. If $\{e_1, ..., e_g\}$ is chosen as basis of $\mathbb{C}^g$ then the period matrix of L in these coordinates has form*

$$\Pi = (I_g \; \Gamma),$$

where Γ is a symmetric complex matrix with positive definite imaginary part.

Proof. Follows from the Proposition 1.4.7 and the Proposition 1.4.8. □

Now we have a representation of a principally polarized torus with an element Γ of Siegel upper half space. Which representations define the same torus?

The representation depends on the choice of the basis of the lattice and the choice of the basis of $\mathbb{C}^g$. Given a lattice with a principal polarization and its symplectic basis with adjusted basis of $\mathbb{C}^g$ (coordinate system), the period matrix has a form $(I \; \Gamma)$. Apply any transformation N of the basis of the lattice which results in a new symplectic basis. N is an integer matrix with determinant ± 1. The period matrix changes to

$$(I \; \Gamma) \mapsto (I \; \Gamma)\, N$$

The matrix of E transforms from J to

$$N^T J N$$

So the new basis is symplectic iff $N^T J N = J$ which means that the matrix N must be symplectic. The set of all such matrices coincides with the restriction of the symplectic group $Sp_g(\mathbb{R})$ to integer matrices and forms a group itself. Denote it with $Sp_g(\mathbb{Z})$. Write the matrix N as

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

So the new period matrix is $(A + \Gamma C, B + \Gamma D)$

Then it is followed by an adjusting change of basis $\mathcal{M}$ in $\mathbb{C}^g$. That is $\mathcal{M} = (A + \Gamma C)^{-1}$. The resulting period matrix is

$$(I \; \Gamma)\, N \mapsto \mathcal{M}\, (I \; \Gamma)\, N = \big(I \;\, (A + \Gamma D)^{-1}(A + \Gamma C)\big)$$

Obviously any integer symplectic matrix defines a symplectic transformation of the basis of the lattice and so we obtain a representation of principally polarized tori with elements of

$$\mathfrak{H}_g\,/Sp_g(\mathbb{Z})\ ,$$

where $Sp_g(\mathbb{Z})$ acts on $\mathfrak{H}_g$ by

$$\Gamma^N = (A+\Gamma D)^{-1}(A+\Gamma C),\ \text{for}\ N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The next theorem summarizes what was said.

Theorem 1.4.14. *Two elements $\Gamma_1, \Gamma_2 \in \mathfrak{H}_g$ represent the same principally polarized torus if and only if there exists a matrix $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{Z})$ such that*

$$\Gamma_2 = \Gamma_1^N = (A+\Gamma_1 D)^{-1}(A+\Gamma_1 C).$$

The set of all principally polarized tori can be identified with the orbit space $\mathfrak{H}_g/Sp_g(\mathbb{Z})$.

Chapter 2

Riemann Surfaces

In this chapter we will provide the basic information about compact Riemann surfaces. Of the most importance are periods of the Riemann surface and its Jacobian. [2] has a very good introduction to the theory of Riemann surfaces, [8] has a very fast but detailed overview, and [3] provides an algebraic approach to the important facts about compact Rieamann surfaces.

2.1 Definition of a Riemann Surface

Definition 1.1 A *Riemann surface* is a 1-dimensional complex manifold with complex structure; that is a triple $(R, \{(U_\alpha, \mu_\alpha)\})$ that satisfies

1. R is a topological space,
2. $\{U_\alpha\}$ is an open covering of R,
3. $\mu_\alpha : U_\alpha \to V_\alpha$ are homeomorphisms onto open sets in the complex plane such that

   $$\mu_\alpha \circ \mu_\beta^{-1} : \mu_\beta(U_\alpha \cap U_\beta) \to V_\alpha$$

   is a holomorphic function whenever the domain is nonempty.

The functions μ_α are called *(local) coordinates* or *charts* on U_α.

Definition 1.2 A mapping $f : R \to S$ between two surfaces R and S with complex structures $\{(U_z, \mu_z)\}$ and $\{(V_z, \nu_z)\}$ respectively is called a *holomorphic* mapping between Riemann surfaces if and only if for any pair of charts (U_α, μ_α) and (V_z, ν_z)

$$\nu_\alpha \circ f \circ \mu_\beta^{-1} : \mu_\beta(U_\beta \cap f^{-1}(V_\alpha)) \to \nu_\alpha(V_\alpha)$$

is holomorphic whenever the domain is nonempty.

Definition 1.3 A holomorphic mapping $f : R \to \mathbb{C}$ is called a *holomorphic function.*

A holomorphic function $f : R \to \mathbb{C} \cup \{\infty\}$ which is not identically ∞ is called a *meromorphic function.*

Let $\mathcal{H}(R)$ denote the algebra of all holomorphic functions and $\mathcal{K}(R)$ denote the algebra of all meromorphic functions.

Proposition 2.1.4. *Let f be a meromorphic function on a Riemann surface R. Then for any point $P \in R$ there exists a coordinate $z : U \to \mathbb{C}$ around P such that $f(z) = z^n$ for unique number n.*

Definition 1.5 The number n is called a *multiplicity* of f at point P.

Remark. *In the rest of the thesis we will deal only with compact Riemann surfaces.*

2.2 Topology

In this section we are going to introduce several basic topology notions which, will be widely used later to describe structures on Riemann surfaces.

2.2.1 Fundamental Group

Let X be a topological space. By a curve we mean a continuous map $\alpha : I \to X$ where I denotes a closed interval $[0; 1]$.

Definition 2.1 Two curves α_0 and α_1 with the same initial points $\alpha_0(0) = \alpha_1(0) = a$ and terminal points $\alpha_0(1) = \alpha_1(1) = b$ are called *homotopic* if there exists a continuous map $F : [0; 1] \times [0; 1] \to X$ such that

$$F(0, \cdot) = \alpha_0(\cdot)$$

$$F(1, \cdot) = \alpha_1(\cdot)$$

$$F(t, 0) = a;\ F(t, 1) = b.$$

The introduced relation is an equivalence relation on the set of all curves.

Define an operation $\cdot$ on two curves α and β with $\alpha(1) = \beta(0)$ by "gluing" them:

$$\gamma = \alpha \cdot \beta$$

$$\gamma(t) = \begin{cases} \alpha(2t), t \leq 1/2 \\ \beta(2t-1), t > 1/2 \end{cases}$$

This operation induces well-defined operation on the homotopy classes of curves. If we now restrict all curves to be closed curves with initial and end points at p then the operation $\cdot$ is defined for every pair of such curves.

Definition 2.2 The fundamental group $\pi_1(X, p)$ is a group consisting of homotopy classes of all closed curves with the base point p and the operation $\cdot$ as above.

Remark. *The identity element is the homotopy class of the constant curve $\alpha(t) = p$.*

The inverse for a given homotopy class $[\alpha]$ is the homotopy class $[\alpha^{-1}]$ where α^{-1} denotes the curve that follows α in the opposite direction.

Remark. *Any two fundamental groups $\pi_1(X, p_1)$ and $\pi_1(X, p_2)$ are isomorphic for a path-connected space so often the base point is not even considered.*

2.2.2 Simplicial homology group

Any Riemann surface has a structure of a simplicial complex, that is, it can be covered by "deformed" triangles such that every edge is contained in exactly two triangles, and for any two triangles their intersection is either empty or it is a common edge or a common vertex or these two triangles coincide.

Triangles are called 2-simplices of this manifold, edges are 1-simplices and vertices are 0-simplices. Denote the collection of all l-simplices by K_l. Any (oriented) edge that joins points P and Q can be written as $< P, Q >$ and any (oriented) triangle with vertices P, Q, R will be written as $< P, Q, R >$.

Consider the abelian group C_l generated by K_l subject to the following relations:

1. No relations on K_0. That is C_0 is a free abelian group on the set of all vertices in given triangulation.

2. For any two points P and Q joined by an edge in the triangulation set

$$< P, Q >= - < Q, P > .$$

3. For any three vertices P, Q, R of one triangle consider the following triangles to be bounded by the relations

$$\begin{aligned} &< P, Q, R >=< Q, R, P >=< R, P, Q >= \\ &\qquad = - < P, R, Q >= - < Q, P, R >= - < R, Q, P > . \end{aligned}$$

Now we want to construct the group generated by all closed curves in which the curve would be represented by the trivial element if it can be filled by a 2-dimensional region.

Define the homomorphisms $\delta_l : C_l \to C_{l-1}$ by setting

$$\delta_1(< P, Q >) =< Q > - < P >$$

$$\delta_2(< P, Q, R >) =< P, Q > + < Q, R > + < R, P >,$$

and extending these actions to the whole groups.

Closed curves now can be represented by the elements $c_1 \in C_1$ which satisfy $\delta_1(c_1) = 0$ and closed curves which can be filled by two-dimensional regions are represented by the elements $c_1' \in C_1$ which belong to the image of δ_2.

Definition 2.3 The *first homology group* is $H_1(R, \mathbb{Z}) = Z_1/B_1$ where $Z_1 = Im(\delta_2)$ and $B_1 = Ker(\delta_1)$.

This definition is consistent provided that

Proposition 2.2.4. *Homomorphisms δ_1 and δ_2 satisfy*

$$\delta_1 \circ \delta_2 = 0.$$

The first homology group is closely related to the fundamental group of the manifold. In fact it is the abelianized version of the fundamental group. For any group G denote its abelianization by $G^{ab} = G/[G; G]$.

Proposition 2.2.5. $H_1(R, \mathbb{Z}) = \pi_1(R, p)/[\pi_1(R, p), \pi_1(R, p)] =: \pi_1(R, p)^{ab}$.

2.2.3 Normal form of the compact orientable 2-manifold

Compact manifolds possess finite triangulations. Using this triangulation one can find the topological representation of the manifold as a polygon in the following way.

Start with an arbitrary triangle Δ_0 in the triangulation. Call it edges a b and c. By the definition of the triangulation there exists a homeomorphism of this triangle onto a triangle on the complex plane.

For every edge (say a) there exists exactly one another triangle Δ_1 in the triangulation which shares this edge with Δ_0. Find a homeomorphism of Δ_1 onto the same plane such that the image of the edge a coincides with the image of a via the first homeomorphism and the image of the second triangle doesn't intersect the image of the first triangle besides this edge.

Continue this process until all triangles in the triangulation are exhausted. We end up with a $n + 2$-gon where n is the number of triangles in the triangulation.

Every edge x in the triangulation was either the edge where two triangles on the plane were "glued" or ended up on the perimeter of the polygon in two copies.

Now let us give every edge in the triangulation a direction and a label. Start walking around the perimeter of the polygon. If the image of the edge a has the same direction as the direction of our movement we assign the symbol a to this edge and a^{-1} in the opposite case. We end up with a sequence of the form $a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} ... a_{i_{n+2}}^{\pm 1}$ where every a occurs twice. We call this expression a *symbol* of this polynomial.

Now it is possible to apply normalization to the polygon; that is finding a different representation of the surface by rearranging the given polygon until it obtain a very specific form. All vertices of the characterizing polygon will correspond to the same vertex of the triangulation and the symbol of this polygon is either $a_1 a_1^{-1}$ or $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ for some g.

Remark. *The number g in the symbol of the normal polygon is a topological invariant of compact orientable 2-manifolds and is called the genus of this surface. All surfaces with the normal polygon whose symbol is $a_1 a_1^{-1}$ are homeomorphic to the sphere $\mathbb{S}^2$ and their genus is defined to be 0.*

This characterizing polygon is a convenient tool for the computation of the fundamental group and the homology group.

Theorem 2.2.6. *Let X be a compact orientable 2-manifold of genus $g > 1$; P be its normal polygon and let $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ be its symbol. Denote by $\tilde{a}_i$ and $\tilde{b}_i$ the closed curves on the X that correspond to the corresponding sides of the normal polygon. Note that all of them are closed curves with the common base point. Then the fundamental group is generated by their homotopy classes subject to the only relation $[\widetilde{a_1}][\widetilde{b_1}][\widetilde{a_1}]^{-1}[\widetilde{b_1}]^{-1}[\widetilde{a_g}][\widetilde{b_g}][\widetilde{a_g}]^{-1}[\widetilde{b_g}]^{-1} = 1$.*

$$\pi_1(X, p) = \left\{ [\widetilde{a_1}], [\widetilde{b_1}], ..., [\widetilde{a_g}], [\widetilde{b_g}] \;\middle|\; [\widetilde{a_1}][\widetilde{b_1}][\widetilde{a_1}]^{-1}[\widetilde{b_1}]^{-1}[\widetilde{a_g}][\widetilde{b_g}][\widetilde{a_g}]^{-1}[\widetilde{b_g}]^{-1} = 1 \right\}$$

Since the first homology group is the abelianized fundamental group, the characterizing relation turns into tautology and therefore the first homology group is a free abelian group on $2g$ generators.

Theorem 2.2.7. $H_1(X,\mathbb{Z}) \simeq < \alpha_1, \beta_1, ..., \alpha_g, \beta_g >$

Remark. *Curves $\widetilde{a_i}, \widetilde{b_i}$ that generate the first homology group constructed as above satisfy the "intersection" property. Here we say that two curves a and b intersect if they have a common point, one curve divides a chart around this point in two connected components and the other one passes through both parts. The "intersection" property says that curves a_i and a_j don't intersect in this sense and any two curves a_i and b_j intersect in exactly one point.*

This explanation is intuitive and not very precise. Later we will give the exact formulation of the intersection property and call such bases to be the canonical bases of the homology group.

2.3 Differential forms

In this section we review definitions and basic facts about complex differential forms. For the purposes of this paper we consider the complexification of the real tangent space of the Riemann surface.

Let a function f be a complex valued function on an open subset of the complex plane. Then the operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ and the differentials dz, $d\bar{z}$ and the differential form $dz \wedge d\bar{z}$ are defined as follows:

$$f_z = \frac{1}{2}(f_x - if_y)$$

$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$$

$$dz = dx + idy$$

$$d\bar{z} = dx - idy$$

$$dz \wedge d\bar{z} = -2i\; dx \wedge dy$$

The formulas above illustrate the relations between two bases $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ and $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right\}$ of the complexifixed tangent space at some point of the Riemann surface and their dual bases $\{dx, dy\}$, $\{dz, d\bar{z}\}$ of the dual space to this tangent space and between generators $dz \wedge\ d\bar{z}$ and $dx \wedge dy$ of the exterior square of this dual space.

Assuming the reader is familiar with definitions of the differential forms in the real setting, now we can define complex differential forms as follows.

Definition 3.1 A complex 0-form on a Riemann surface R is a complex-valued function f on R.

Definition 3.2 A complex 1-form on a Riemann surface R is an assignment of an expression on U

$$f_U(z)\ dz_U + g_U(z)\ d\bar{z}_U$$

for every chart (U, μ), such that it is invariant under coordinate changes. That is, if U_α and U_β are two intersecting coordinate systems and $\varphi = \mu_\beta \circ \mu_\alpha^{-1} : \mu_\alpha(U_\alpha \cap U_\beta) \to \mu_\beta(U_\alpha \cap U_\beta)$ is the change of coordinates then the assigned forms satisfy

$$\begin{pmatrix} f_{U_\beta}(z_\beta) \\ g_{U_\beta}(z_\beta) \end{pmatrix} = \begin{pmatrix} \varphi_{z_\alpha} & 0 \\ 0 & \overline{\varphi_{z_\alpha}} \end{pmatrix} \cdot \begin{pmatrix} f_{U_\alpha}(z_\alpha) \\ g_{U_\alpha}(z_\alpha) \end{pmatrix}, \ z_\beta = \varphi(z_\alpha) \tag{2.1}$$

Remark. *The change of coordinates matrix in (2.1) is the simplified regular change of coordinates matrix*

$$\begin{pmatrix} \varphi_z & \varphi_{\bar{z}} \\ \overline{\varphi}_z & \overline{\varphi}_{\bar{z}} \end{pmatrix}$$

using the fact that the change of coordinates φ is holomorphic and hence $\varphi_{\bar{z}} = 0$ and with the equality $\bar{f}_{\bar{z}} = \overline{f_z}$.

Definition 3.3 A complex 2-form on a Riemann surface R is an assignment of an expression

$$f_\alpha\ dz_\alpha \wedge\ d\bar{z}_\alpha$$

to each chart (U_α, μ_α) such that it is invariant under coordinate changes:

$$f_{U_\beta}(z_\beta) = f_{U_\alpha}(z_\alpha) \left|\varphi_{z_\alpha}\right|^2,$$

where $\varphi = \mu_\beta \circ \mu_\alpha^{-1} : \mu_\alpha(U_\alpha \cap U_\beta) \to \mu_\beta(U_\alpha \cap U_\beta)$ is the change of coordinates map and $z_\beta = \varphi(z_\alpha)$.

Usually functions in these forms are at least C^1 in both variables x and y. Such differentials will be referred to as C^1-forms.

Definition 3.4 The *Hodge star* operator $*$ acts on 1-forms as follows:
for a 1-form $\omega = f\ dx + g\ dy = u\ dz + v\ d\bar{z}$ set $*\omega = -g\ dx + f\ dy = -iu\ dz + iv\ d\bar{z}$

Denote by $L^2(R)$ the space of measurable square-integrable 1-forms, that is forms ω such that

$$\int_R \omega \wedge *\overline{\omega} < \infty.$$

With the hermitian form $(\omega_1, \omega_2) = \int_R \omega_1 \wedge *\overline{\omega_2}$, $L^2(R)$ becomes a Hilbert space.

A 0-form is called *harmonic* if $\triangle f = (f_{xx} + f_{yy})\ dx \wedge dy = 0$. A 1-form ω is *harmonic* if locally $\omega = df$ with f harmonic.

Exterior product

The exterior product extends complex linearly to complex forms. The product of a k-form and a l-form is a $k+l$-form provided $k + l \leq 2$ and 0 otherwise satisfying

$$dz \wedge d\bar{z} = -\ d\bar{z} \wedge dz$$

$$dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0$$

Thus

$$(f_1\ dz + g_1\ d\bar{z}) \wedge (f_2\ dz + g_2\ d\bar{z}) = f_1 g_2\ dz \wedge d\bar{z} + f_2 g_1\ d\bar{z} \wedge dz = (f_1 g_2 - f_2 g_1)\ dz \wedge d\bar{z}$$

Operator d

The operator d extends complex-linearly to the complex case as well. Let $\wedge^k$ denotes the space of all k-forms on a Riemann surface R. Now the operator d acts on C^1 complex differential forms and satisfies $\wedge^k \to \wedge^{k+1}$ (k=0,1).

$$df = f_z\ dz + f_{\bar{z}}\ d\bar{z}$$

$$d(f\ dz + g\ d\bar{z}) = df \wedge dz + dg \wedge d\bar{z} = (g_{\bar{z}} - f_z)\ dz \wedge d\bar{z}.$$

It can be shown that this is a well-defined operation on a Riemann surface.

Denote by $\wedge = \wedge^0 \oplus \wedge^1 \oplus \wedge^2$ the space of all differential forms. Then $\wedge$ is an algebra and $d : \wedge \to \wedge$ is a graduation. An important property of this operator is

$$d^2 = 0$$

Definition 3.5 A differential ω is called *closed* if $d\omega = 0$. A 1-form ω is called *exact* if $\omega = df$ for some function f on R.

Remark. *Any locally exact differential is closed, and every closed form is locally exact.*

For a simply connected Riemann surface we have a stronger statement.

Proposition 2.3.6. *If Riemann surface R is simply connected then any closed differential is exact on R.*

Given an arbitrary Riemann surface R and a closed differential ω on it, one may consider a polygon $\mathcal{M}$ assigned to this Riemann surface and differential $\widetilde{\omega}$ on $\mathcal{M}$ which is restriction of ω on $\mathcal{M}$. Note that $\widetilde{\omega}$ is the same on the sides $a_i,\ a_i^{-1}$ and $b_j,\ b_j^{-1}$ where $\mathcal{M}$ has representation $\Pi_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$. The interior $int(\mathcal{M})$ is simply connected and hence $\widetilde{\omega}$ is exact on $int(\mathcal{M})$.

Integration

Let us introduce k-chains ($k = 0, 1, 2$).

Definition 3.7 k-chain is a formal finite sum of k-simplices.

A k-form can be integrated over k-chains.

A 0-chain can be written in the form $\sum_{i=1}^{m} n_i P_i$. If f is a 0-form on R then the integral of f over $\sum_{i=1}^{m} n_i P_i$ is

$$\sum_{i=1}^{m} n_i f(P_i).$$

If c is a piecewise differentiable path on R and $\omega = f(z)\ dz + g(z)\ d\bar{z} = u(x,y)\ dx + v(x,y)\ dy$ then

$$\int_c \omega = \int_0^1 \left(u(x(t),y(t))\frac{dx}{dt} + v(x(t),y(t))\frac{dy}{dt} \right) dt.$$

It can be shown that the integral above is well defined on different charts by applying transformation formulas of differentials and change of variables.

In the similar way one can define integrals of 2-differentials over 2-chains: if $\Omega = f(z)\ dz \wedge d\bar{z} = g(x,y)\ dx \wedge dy$ and D is a 2-chain then

$$\int_D \Omega = \int_D g(x,y)\ dx \wedge dy$$

The boundary of k-chain (k=1,2) is a $(k-1)$-chain. Denote the operator that maps the k-chain to its boundary by ∂.

The important connection between operators d and ∂ is depicted in Stokes' theorem.

Theorem 2.3.8 (Stokes). *Let ω be a k-form (k=0,1,2) and D be a $(k+1)$-chain. Then*

$$\int_{\partial D} \omega = \int_D d\omega$$

We will need the following integral identity.

Proposition 2.3.9. *Let $\{a_1, ..., a_g, b_1, ..., b_g\}$ be a canonical basis of the homology group of compact Riemann surface R. For any closed differentials θ and $\widetilde{\theta}$*

$$\int_R \theta \wedge \widetilde{\theta} = \sum_{i=1}^{g} \left(\int_{a_i} \theta \int_{b_i} \widetilde{\theta} - \int_{b_i} \theta \int_{a_i} \widetilde{\theta} \right)$$

Proof. On $\mathcal{M}$: $\omega = df$, where f can be chosen by $f(z) = \int_{z_0}^{z} \omega$ for some point z_0. For points on the edge of $\mathcal{M}$, z and z' denote corresponding points. Then

$$\begin{aligned}\int_R \theta \wedge \widetilde{\theta} = \int_{\mathcal{M}} \theta \wedge \widetilde{\theta} = \int_{\mathcal{M}} df \wedge \widetilde{\theta} = \int_{\mathcal{M}} d(f\widetilde{\theta}) = \int_{\delta\mathcal{M}} f\widetilde{\theta} = \\ = \sum_{i=1}^{g} \left(\int_{a_i} f\widetilde{\theta} + \int_{b_i} f\widetilde{\theta} + \int_{a_i^{-1}} f\widetilde{\theta} + \int_{b_i^{-1}} f\widetilde{\theta} \right) = \\ = \sum_{i=1}^{g} \left(\int_{a_i} \left(\int_{z_0}^{z} \theta \right) \widetilde{\theta} + \int_{a_i^{-1}} \left(\int_{z_0}^{z} \theta \right) \widetilde{\theta} + \ldots \right) = \\ = \sum_{i=1}^{g} \left(- \int_{a_i} \left(\int_{b_i} \theta \right) \widetilde{\theta} + \int_{b_i} \left(\int_{a_i} \theta \right) \widetilde{\theta} \right).\end{aligned}$$

□

Meromorphic differentials

Definition 3.10 Let R be a Riemann surface with complex structure $\{(U_\alpha, \mu_\alpha)\}$.

A *Meromorphic differential* on R is an assignment of a form $f_\alpha(z)dz$ to each coordinate chart, such that all $f_\alpha : U_\alpha \to \mathbb{C}$ are meromorphic functions and it is invariant under change of coordinates:

$$f_\beta(\mu(z)) \cdot \mu'(z) = f_\alpha(z)$$

if μ is a change of coordinates from $\mu_\alpha(U_\alpha \cap U_\beta)$ to $\mu_\beta(U_\alpha \cap U_\beta)$.

The *order* of meromorphic differential at point P is defined as

$$\mathrm{ord}_P \omega = \mathrm{ord}_0 f$$

where $\omega = f(z)dz$ around P and z is a local coordinate vanishing at P.

In the same notation the *residue* of ω is defined as

$$\mathrm{res}_P \omega = \mathrm{res}_0 f.$$

The following theorems are standard and taken from [2].

Theorem 2.3.11. *1. Let $P \in R$ and z be a local coordinate on R vanishing at P, For every integer $n \geq 1$. there exists a meromorphic differential on R which is holomorphic on $R \setminus \{P\}$ and with singularity $1/z^{n+1}$ at P.*

2. Given two distinct points P_1 and P_2 on R and local coordinates z_j vanishing at $P_j, j = 1, 2$, there exists a meromorphic differential ω, holomorphic on $R \setminus \{P_1, P_2\}$, with singularity $1/z_1$ at P_1 and singularity $-1/z_2$ at P_2.

Proof. The proof can be found in [2], page 51. □

Theorem 2.3.12. *Let $P_1, ..., P_k$ be $k > 1$ distinct points on a Riemann surface R. Let $c_1, ..., c_k$ be complex numbers with $\sum_{j=1}^{k} c_j = 0$. Then there exists a meromorphic abelian differential ω on M, holomorphic on $R \setminus \{P_1, ..., P_k\}$ with*

$$ord_{P_j}\omega = -1, \; res_{P_j}\omega = c_j$$

Proof. See [2], page 52. □

Theorem 2.3.13. *Let R be a compact Riemann surface and ω an abelian differential on M. Then*

$$\sum_{P \in R} res_P \omega = 0$$

Proof. See [2], page 52 for details. □

Holomorphic differentials

Definition 3.14 A meromorphic differential $\omega = f dz$ is called *holomorphic* if the functions f_α in local coordinates are holomorphic functions on U_α. The space of all holomorphic differentials on a Riemann surface R is denoted by $\mathcal{H}(R)$.

Proposition 2.3.15. *Every holomorphic differential is locally exact: $\omega = dF_\alpha$ on U_α.*

Proof. This is a direct consequence of the properties of holomorphic functions. □

Theorem 2.3.16. *Let R be a compact Riemann surface with genus g. Then $\mathcal{H}(R)$ is a g-dimensional complex vector space.*

Proof. See [2], page 62 for further details. $\square$

Let $H_1(R)$ denotes the first homology group of R. Recall that $H_1(R)$ a free group of rank $2g$.

Corollary 2.3.17. *Given a homology basis $\{a_1, ..., a_g, b_1, ..., b_g\}$ of $H_1(M)$, there exists a unique basis $\{\omega_1, ..., \omega_g\}$ of $\mathcal{H}(R)\}$ normalized on the first g elements of homology group:*

$$\int_{a_i} \omega_j = \delta_{ij}$$

Definition 3.18 The unique basis $\{\omega_1, ..., \omega_g\}$ of $\mathcal{H}(R)$ from the corollary is called the adapted basis to a homology basis $\{a_1, ..., a_g, b_1, ..., b_g\}$.

Fix a canonical homology basis $\{a_1, ..., a_g, g_1, ..., b_g\}$. Let ω be any holomorphic differential. Numbers

$$\int_{a_1} \omega, ..., \int_{a_g} \omega, \int_{b_1} \omega, ..., \int_{b_g} \omega$$

are called *periods* of ω.

Proposition 2.3.19. *If $\{a_1, ..., b_g\}$ is a homology basis and $\{\omega_1, ..., \omega_g\}$ is a basis of $\mathcal{H}(R)$ dual to the homology basis then any holomorphic differential ω has representation*

$$\omega = \sum_{i=1}^{g} \mu_i \omega_i,$$

where $\{\mu_i\}$ are first g periods of ω.

2.4 Intersection theory on a Riemann surface

In this section we are giving the sketch of the intersection theory and more details can be found in [2].

Given any simple closed curve c on R one can assign differential forms to it in the following way. Choose an annular neighborhood Ω of c and find a smaller strip Ω_0 that also contain c. Let $\Omega^{\pm}$ be the annular strips Ω is divided into by c. Let us choose Ω^{-} to lie to the left of c if one follows the orientation of c. In the same way $\Omega_0^{\pm}$ are the components of Ω_0 separated by c and Ω_0^{-} is to the left of the curve if following its orientation.

Construct a real-valued function f_c on R with properties

$$f_c(P) = 1,\ P \in \Omega_0^{-}$$

$$f_c(P) = 0,\ P \in R \setminus \Omega^{-}$$

$$f_c \text{ is } C^{\infty} \text{ on } R \setminus c$$

Then a form η_c defined by

$$\eta_c = \begin{cases} df_c, \text{ on } \Omega \setminus c \\ 0 \text{ on } (R \setminus \Omega) \cup c \end{cases}$$

is a closed real-valued C^{∞} differential form with compact support.

Proposition 2.4.1. *Let $\alpha \in L^2(R)$ be closed and C^1. Then for any closed curve c*

$$\int_c \alpha = (\alpha, *\eta_c)$$

For any 2 closed curves c_1 and c_2 define $c_1 \cdot c_2 := (\eta_{c_1}, - * \eta_{c_2}) = \int_R \eta_{c_1} \wedge \eta_{c_2}$ to be the *intersection number* of these curves. It “counts” the number of times c_1 intersects c_2

Proposition 2.4.2. *The intersection number of two curves is well defined and depends only on homology classes of c_1 and c_2 and satisfies the following properties:*

$$a \cdot b \in \mathbb{Z}$$

$$a \cdot b = -b \cdot a$$

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

Definition 4.3 The basis $\{a_1, ..., a_g, b_1, ..., b_g\}$ of $H_1(R)$ is called *canonical* if

$$a_i \cdot b_j = \delta_{i,j}$$

$$a_i \cdot a_j = 0,\ b_i \cdot b_j = 0$$

Denote by $H^1_{DR}(R)$ the *first de Rham cohomology group*; that is the space of all smooth closed differentials on R modulo the space of smooth exact differentials. According to the Hodge decomposition every equivalence class of H^1_{DR} has exactly one harmonic representative. To conclude

Theorem 2.4.4. *For any closed differential form η there exists exactly one harmonic form α in its equivalence class in $H^1_{DR}(R)$. Moreover for any element c of homology group $H_1(R)$:*

$$\int_c \eta = \int_c \alpha$$

So we can assign a harmonic form α_c to any closed curve c. Finally we can construct the holomorphic 1-form ω_c according to:

Proposition 2.4.5. *A form ω is holomorphic if and only if it has a form $\omega = \alpha + i * \alpha$ where α is harmonic.*

2.5 Period matrices and Riemann bilinear relations

Let R be a Riemann surface and $\{a_1, ..., a_g, b_1, ..., b_g\}$ be a canonical base of its homology group $H_1(R)$. Let $\{\omega_1, ..., \omega_g\}$ be a normalized basis of the space of all holomorphic differentials as in Corollary 2.3.17.

Even though the normalization of a basis $\{\omega_1, ..., \omega_g\}$ of the space of all holomorphic differentials for a fixed basis of the homology group $\{a_1, ..., a_g, b_1, ..., b_g\}$ requires the first g periods to satisfy

$$\int_{a_i} \omega_j = \delta_{ij}$$

the last g periods are not independent. The relations between these periods are known as the Riemann bilinear relations. Here we establish these relations in terms of the period matrices.

Definition 5.1 The matrix $\Pi = \Pi(R)$ with entries $\pi_{ij} = \int_{b_i} \omega_j$ is called a *period matrix*

Periods of normalized basis of holomorphic differentials satisfy additional conditions which are called Riemann bilinear relations.

The Proposition 2.3.9 for $\theta = \omega_i$ and $\widetilde{\theta} = \omega_j$ provides

$$\sum_{k=1}^{g} \left(\int_{a_k} \omega_i \int_{b_k} \omega_j - \int_{b_k} \omega_i \int_{a_k} \omega_j \right) = \int_R \omega_i \wedge \omega_j$$

or

$$\int_{b_i} \omega_j - \int_{b_j} \omega_i = 0$$

since $\theta \wedge \widetilde{\theta} = 0$ for any holomorphic differentials.

In terms of periods it means that $\pi_{ij} = \pi_{ji}$ or simply that Π is a symmetric matrix.

Now choose θ to be any holomorphic differential and $\widetilde{\theta} = \overline{\theta}$. $\widetilde{\theta}$ is no longer holomorphic but it is still closed. Let

$$\theta = \begin{pmatrix} \omega_1 & \omega_2 & \ldots & \omega_g \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_g \end{pmatrix} = \omega^T \cdot \mathbf{v}.$$

$\theta \wedge \widetilde{\theta}$ is positive and hence $\frac{1}{-2i} \int_R \theta \wedge \widetilde{\theta} > 0$ ($-2i$ arises as $\frac{\partial(z,\bar{z})}{\partial(x,y)}$). The expression from Proposition 2.3.9 simplifies as follows:

$$\int_R \theta \wedge \widetilde{\theta} = \sum_{i=1}^{g} \left(v_i \overline{\int_{b_i} \theta} - \bar{v}_i \int_{b_i} \theta \right) = \sum_{i,j=1}^{g} \left(v_i \overline{v_j \pi_{ij}} - \overline{v_i} v_j \pi_{ij} \right) =$$
$$\sum_{i,j=1}^{g} \left(v_i \overline{v_j \pi_{ij}} - \overline{v_j} v_i \pi_{ji} \right) = -2i \sum_{i,j=1}^{g} v_i \overline{v_j} \mathrm{Im}\,(\pi_{ij}),$$

and so $\sum_{i,j=1}^{g} v_i \overline{v_j} \mathrm{Im}\,(\pi_{ij})$ is positive for all $\mathbf{v}$. In matrix form it means that Im (Π) is positive definite.

We summarize all these properties with a proposition

Proposition 2.5.2 (Riemann bilinear relations)**.** *Let R be a Riemann surface and let a canonical base $\{a_1, ..., a_g, b_1, ..., b_g\}$ of $H_1(R)$ the period matrix Π be as in the definition above, then*

$$\Pi^T = \Pi$$

$$\mathrm{Im}\,\Pi > 0.$$

This proposition shows that fixing a canonical homology basis gives a representation of Riemann surface by an element of Siegel upper half space. We will return to this representation in the section 3.4.

2.6 Dependence of the period matrix on the base of the first homology group

The construction of the period matrix depends on the choice of the canonical basis of the first homology group $\Sigma = (A_1, ..., A_g, B_1, ..., B_g)^T$ (the order of the elements matters, so from here we consider an ordered basis represented as a $2g$-tuple of elements from $H_1(R)$). In this section we will describe the transformation of the period matrix under a change of the basis of the first homology group.

To emphasize the dependence of the period matrix on the canonical base Σ of $H_1(R)$ from now on we will use notation $\Pi(R, \Sigma)$.

We start with the description of all possible transformations of the canonical bases.

Proposition 2.6.1. *Let Σ and Σ' be two canonical bases of the first homology group $H_1(R)$. Then the change of basis matrix M belongs to the symplectic group $Sp_g(\mathbb{Z})$. That is:*

$$\Sigma' = M\Sigma \textit{ for some } M \in Sp_g(\mathbb{Z})$$

$$\begin{pmatrix} A'_1 \\ \vdots \\ A'_g \\ B'_1 \\ \vdots \\ B'_g \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{M} \begin{pmatrix} A_1 \\ \vdots \\ A_g \\ B_1 \\ \vdots \\ B_g \end{pmatrix}$$

Proof. First of all, any element in $H_1(R)$ can be expressed as integer linear combination of elements of the basis Σ and therefore we immediately get that M is an integer matrix.

Define the product of two vectors consisting of elements from $H_1(R)$ by using intersection number operation:

$$\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \cdot \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} = \begin{pmatrix} C_1 \cdot D_1 & \dots & C_1 \cdot D_n \\ \vdots & \ddots & \vdots \\ C_n \cdot D_1 & \dots & C_n \cdot D_n \end{pmatrix}.$$

Intersection number is an antilinear function (Proposition 2.4.2) and therefore this product is associative with respect to multiplication by any integer transformation from both sides:

$$\left[M \cdot \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \right] \cdot \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} = M \left[\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \cdot \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} \right]$$

$$\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \cdot \left[\begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} M \right] = \left[\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \cdot \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix} \right] M.$$

Now, definition of the canonical homology basis in these terms says that

$$\Sigma \cdot \Sigma^T = -J \text{ and } \Sigma' \cdot \Sigma'^T = -J,$$

where

$$J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}.$$

If $\Sigma' = M\Sigma$ then we have the following relation:

$$-J = \Sigma' \cdot \Sigma'^T = M\Sigma \cdot \Sigma^T M^T = M(-J)M^T,$$

or

$$MJM^T = J.$$

That means that M^T and hence M as well are symplectic matrices. $\square$

Now we describe how the basis transformation $M \in Sp_g(\mathbb{Z})$ affects the period matrix.

Proposition 2.6.2. *Let Σ and Σ' be two canonical bases of the homology group $H_1(R)$. If the transformation matrix*

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is so

$$\Sigma' = M\Sigma$$

then the period matrices $\Pi = \Pi(R, \Sigma)$ and $\Pi' = \Pi(R, \Sigma')$ are related by

$$\Pi' = (\gamma + \delta\Pi)(\alpha + \beta\Pi)^{-1}.$$

Proof. Propositions 2.4.1-2.4.5 allow us to define the pairing of the elements from the $H_1(R)$ and holomorphic differentials on R. For convenience let us denote it by $[c, \alpha] = \int_c \alpha$ (α is holomorphic differential and c is an element of the first homology group). Using this definition we can now define the product of a vector with components from $H_1(R)$ and a vector consisting of holomorphic differentials:

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \cdot \begin{pmatrix} \omega_1 & \dots & \omega_n \end{pmatrix} = \begin{pmatrix} [c_1, \omega_1] & \dots & [c_1, \omega_n] \\ \vdots & \ddots & \vdots \\ [c_n, \omega_1] & \dots & [c_n, \omega_n] \end{pmatrix}.$$

The definition of the adapted basis of the holomorphic differentials to the canonical basis of the first homology group in these terms is the following:

A basis $(\omega_1, ..., \omega_g)$ *of* $\mathcal{H}(R)$ *is adapted to the canonical basis* $(a_1, ..., a_g, b_1, ..., b_g)$ *of* $H_1(R)$ *if and only if*

$$\begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix} \cdot \begin{pmatrix} \omega_1 & \dots & \omega_g \end{pmatrix} = \begin{pmatrix} I_g \\ *_g \end{pmatrix}.$$

By the Corollary 2.3.17 there is the unique way to choose the adapted basis of $\mathcal{H}(R)$ for a fixed canonical basis of the $H_1(R)$. The matrix that appears as the bottom half is exactly the period matrix of R with with respect to the canonical basis Σ of $H_1(R)$, so we can say that the canonical basis of the first homology group and the adapted basis of the space of all holomorphic differentials satisfy

$$\begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix} \cdot \begin{pmatrix} \omega_1 & \dots & \omega_g \end{pmatrix} = \begin{pmatrix} I_g \\ \Pi \end{pmatrix}.$$

Again, due to the linearity of the pairing (over $\mathbb{Z}$ and $\mathbb{C}$), this product of vectors is associative with respect to multiplication by an integer matrix from the left and by a complex matrix from the right.

Now assume that the canonical basis Σ was changed by the transformation $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ into Σ' ($\Sigma' = M\Sigma$). Then the full period matrix changes in the following way:

$$\begin{pmatrix} I_g \\ \Pi \end{pmatrix} = \Sigma\Omega \mapsto \Sigma'\Omega = M\Sigma\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} I_g \\ \Pi \end{pmatrix} = \begin{pmatrix} \alpha + \beta\Pi \\ \gamma + \delta\Pi \end{pmatrix},$$

and to normalize it by changing the basis of $\mathcal{H}(R)$ we need to multiply it by $(\alpha+\beta\Pi)^{-1}$ from the right, i.e. the basis of $\mathcal{H}(R)$ changes in the following way:

$$\Omega \mapsto \Omega \cdot (\alpha+\beta\Pi)^{-1}.$$

The new period matrix (the bottom half of the full period matrix) becomes

$$\Pi' = (\gamma+\delta\Pi)\,(\alpha+\beta\Pi)^{-1}.$$

□

2.7 The Jacobian variety

Let $\{a_1, ..., a_g, b_1, ..., b_g\}$ be a canonical homology basis of the Riemann surface R and $\{\omega_1, ..., \omega_g\}$ be the dual basis of $\mathcal{H}$. The periods of forms ω_i form a matrix $(I_g\ \Pi)$. Denote the columns of this matrix by $e^1, ..., e^g, \pi^1, ..., \pi^g$. According to the Theorem 2.3.16 these vectors are linearly independent over $\mathbb{R}$.

Consider a lattice $L(R)$ constructed on vectors $\{e^1, ..., e^g, \pi^1, ..., \pi^g\}$.

Definition 7.1 The g-dimensional torus $\mathbb{C}^g/L(R)$ is called the *Jacobian variety* of Riemann surface R.

Construct a real-valued antisymmetric form E on $\mathbb{R}^{2g} \simeq \mathbb{C}^g$ with matrix

$$J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$$

corresponding to the basis $\{e^1, ..., e^g, \pi^1, ..., \pi^g\}$ and form H defined by

$$H(u, v) = E(iu, v) + iE(u, v)$$

as in the section 1.4.

The conditions of Propositions 1.4.7 and 1.4.8 are equivalent to the Riemann bilinear relations (Theorem 2.5.2) and hence form H defines a principal polarization on the Jacobian. We conclude this in the following proposition.

Proposition 2.7.2. *The Jacobian of R is a principally polarized g-dimensional torus.*

Jacobian and hermitian form H may be described in different manner without attaching it to the canonical basis of homology group. First of all $\mathbb{C}^g$ may be considered as $(\mathcal{H}(R))^* = Hom_{\mathbb{C}}(\mathcal{H}(R), \mathbb{C})$ and in this notations $L(R)$ is the action of $H_1(R)$ on $\mathcal{H}(\mathcal{R})$:

$$c(\omega) = \int_c \omega$$

Then the Jacobian $J(R) = (\mathcal{H}(R))^* / H_1(R)$.

An antisymmetric form E is the intersection number form on $H_1(R)$ and can be linearly extended to $(\mathcal{H}(\mathcal{R}))^* =< H_1(R) >_{\mathbb{R}}$. Then a hermitian form H on $\mathcal{H}(R)^* \times \mathcal{H}(R)^*$ can be obtained by duality from the form $< \cdot\, ,\ \cdot >$ on $\mathcal{H}(R) \times \mathcal{H}(R)$ where

$$< \omega_1, \omega_2 >= \frac{i}{2} \int_R \omega_1 \wedge \overline{\omega_2}$$

$$H(c_1, c_2) =< \omega_{c_1}, \omega_{c_2} >$$

Let us show the validity of the above formula for the basic vectors of $(\mathcal{H}(R))^*$.

$$\begin{aligned}\text{Im } < \omega_{a_k}, \omega_{b_j} >= \frac{1}{2}\text{Re} \int_R \omega_{a_k} \wedge \overline{\omega_{b_j}} = -\frac{1}{2}\text{Re } (\omega_{a_k}, *\omega_{b_j}) = \\ -\frac{1}{2}\text{Re } (2(\alpha_{a_k}, *\alpha_{b_j}) + 2i(\alpha_{a_k}, \alpha_{b_j})) = (\alpha_{a_k}, - * \alpha_{b_j}) = a_k \cdot b_j\end{aligned}$$

The Jacobian exists together with a natural map

$$\varphi : R \longrightarrow J(R)$$

defined by choosing arbitrary point P_0 in R and setting

$$\varphi(P) = \left(\int_{P_0}^{P} \omega_1, ..., \int_{P_0}^{P} \omega_g \right)^T.$$

In fact

Proposition 2.7.3. *The map φ is a holomorphic embedding of R into its Jacobian.*

Proof. The proof can be found in [2], page 92. □

Moreover, every Riemann surface is uniquely determined by its Jacobian. This is known as the Torelli theorem

Theorem 2.7.4 (Torelli)**.** *Two Riemann surfaces are biholomorphic if and only if their Jacobians are isomorphic as principally polarized tori.*

See [8] and [6] for more information about Jacobians and proofs.

Chapter 3

Representation of Teichmüller space and Teichmüller modular group

This chapter aims to describe several ways to represent Riemann surfaces with respect to different equivalence relations. The resulting spaces form so called moduli spaces. Then we will introduce the transformations of the moduli spaces which form the modular groups, and finally we will describe the relations between different moduli spaces and groups. We use information from [4] and [7] in this chapter. See them for further details.

3.1 Riemann, Teichmüller, and Torelli spaces

3.1.1 Riemann and Teichmüller moduli spaces of compact Riemann surfaces

By a canonical basis of $\pi_1(R, p)$ in this section we mean a set of equivalence classes of closed curves that generate the fundamental group which also form a canonical basis of the first homology group $H_1(R)$.

The first very natural relation to consider would be the relation of biholomorphic equivalence.

Definition 1.1 The *Riemann moduli space* $\mathcal{M}(g)$ consists of all equivalence classes of Riemann surfaces of genus g modulo the biholomorphic equivalence.

$$\mathcal{M}(g) = \{T \mid T \text{ is a Riemann surface of genus } g\}/\sim,$$

where

$$T \sim S \text{ if and only if } \exists f : T \to S \text{ - biholomorphism.}$$

The Teichmüller space of compact Riemann surface can be defined in two ways. For the rest of this chapter R denotes a compact Riemann surface of genus g.

The first way to define the Teichmüller space T_g is by considering *markings* of the Riemann surface consisting of the canonical bases of the fundamental group.

Definition 1.2 The *marking* of R is an ordered system $\Sigma_p = \{a_1, ..., a_g, b_1, ..., b_g\}$ of canonical generators of $\pi_1(R, p)$ for some point $p \in R$.

Definition 1.3 Two markings $\Sigma_p = \{a_i, b_i\}_{i=1}^g$ and $\Sigma_{p'} = \{a_i', b_i'\}_{i=1}^g$ of a Riemann surface R are called *equivalent* if there exists a curve c joining p and p' which induces an isomorphism between $\pi_1(R, p)$ and $\pi_1(R, p')$ such that a_i' and b_i' are the images of a_i and b_i respectively i.e.

$$a_i' \approx c^{-1} a_i c,$$

$$b_i' \approx c^{-1} b_i c,$$

where $\approx$ stands for the homotopy relation.

Denote this relation by the same symbol: $\Sigma \approx \Sigma'$.

Definition 1.4 The Teichmuller space T_g is defined as

$$T_g = \left\{ (R, \Sigma_p) \;\middle|\; \begin{array}{c} R \text{ is a Riemann surface of genus } g; \\ \Sigma_p \text{ is a marking on } R \end{array} \right\} / \sim,$$

where $(R, \Sigma_p) \sim (R', \Sigma_{p'})$ if and only if there exists a biholomorphic mapping $f : R \to R'$ such that $f(\Sigma_p) \approx \Sigma_{p'}$.

The equivalence class of a marked Riemann surface (R, Σ_p) is denoted by $[R, \Sigma_p]$.

To define the Teichmuller space $T(R)$ consider the images of R under orientation preserving diffeomorphisms.

Definition 1.5 Fix a Riemann surface R. The Teichmuller space $T(R)$ is

$$T(R) = \left\{ (R, S, f) \;\middle|\; \begin{array}{c} S \text{ is a Riemann surface} \\ f : R \to S \text{ is an orientation preserving diffeomorphism} \end{array} \right\} / \sim,$$

where $(R, S, f) \sim (R, S', f')$ if and only if $f' \circ f^{-1} : S \to S'$ is homotopic to a biholomorphic mapping $h : S \to S'$.

The equivalence class of (R, S, f) in $T(R)$ is denoted by $[R, S, f]$.

Remark. *Obviously the genus of a surface doesn't change under a diffeomorphism therefore all Riemann surfaces S that appear in different markings of R are of genus g as well.*

Remark. *Unfortunately an orientation preserving diffeomorphism $f : R \to S$ we used in the last definion of the Teichmüller space $T(R)$ is also called a marking of R in various sources. Every time we use this term in the future refers to markings of the fundamental group only.*

The two Teichmüller spaces T_g and $T(R)$ turnes out to be equivalent. To state this fix R and a marking $\Sigma = \{a_1, ..., a_g, b_1, ..., b_g\}$ on it. Each element $[R, S, f]$ of $T(R)$ now is followed with a marked Riemann surface $[S, f(\Sigma)]$ where $f(\Sigma) = \{f(a_1), ..., f(a_g), f(b_1), ..., f(b_g)\}$ is a marking on S. Denote this mapping by $\Phi_\Sigma : T(R) \to T_g$.

Theorem 3.1.6. *Let R be a compact Riemann surface of genus g. The mapping $\Phi_\Sigma : T(R) \to T_g$ such that $[R, S, f] \mapsto [S, f(\Sigma)]$ is bijective.*

Proof. See [4], page 14 for the outline. □

In particular this statement implies that any two Riemann surfaces with fixed markings of the fundamental groups can be mapped one to another via a diffeomorphism which sends one marking into another.

The second definition of Teichmuller space depends on the base surface R and another meaning of the last theorem says that all these spaces are isomorphic. Moreover there is a natural isomorphism between any two $T(R)$ and $T(S)$.

3.1.2 Example: the case of simple tori

We will consider the so-called period mapping which we will investigate in the Section 3.4:

$$\phi : T_g \to \mathfrak{H}_g$$
$$(R, \Sigma) \mapsto \omega = \Pi(R, \Sigma) = (\pi_{ij})_{i,j=1}^{g}, \ \pi_{ij} = \int_{b_i} \omega_j$$

($\Pi(R, \Sigma)$ is a period matrix of R with respect to the basis of homology group Σ; $\{\omega_1, ..., \omega_g\}$ is the basis of the space of all holomorphic differentials adapted to the canonical basis Σ of the first homology group).

The goal of this section is to describe the Teichmüller space of Riemann surfaces of genus 1. As a matter of fact the Teichmüller space of genus 1 is isomorphic to the Siegel upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \text{Im }(z) > 0\}$. Later will see that the Teichm̈uller space and the Siegel upper half space can be related in a similar way in a higher dimensional case, but they are no longer isomorphic and the relation is much more complicated.

According to the Riemann bilinear relations (Theorem 2.5.2) the reduced period matrix of any marked Riemann surface R of genus 1 consists of only one complex number ω which satisfies the only relation $\text{Im }(\omega) > 0$ i.e. $\omega \in \mathfrak{H}$. The rest of this chapter is dedicated to showing that the period mapping ϕ induces an isomorphism $T_1 \to \mathfrak{H}$.

Note that in the case $g = 1$ the Riemann surface R is biholomorphically isomorphic to its Jacobian $J(R, \Sigma) = \mathbb{C}/ < 1, \ \omega >$ via the mapping described in the Theorem 2.7.3. The only thing needed to show this and which can be constructively verified is that the embedding φ in this theorem is onto $J(R)$, therefore it is invertible and the inverse must also be holomorphic. Thus all the Jacobians (for $g = 1$) possess the structure of a Riemann surface themselves.

To justify surjectivity one may consider specific Jacobians which can be represented by the parallelograms with intervals $[0, 1]$ and $[0, \omega]$ as its sides and a complex structure inherited from $\mathbb{C}$. Choose a marking Σ to consist of two closed curves along these sides. Then dz (inherited from dz in $\mathbb{C}$) forms the associated basis of the holomorphic differentials and the period with respect to the second component of Σ is exactly ω.

In this way we get all possible values for the period of a Riemann surface of genus g. Thus the mapping $\phi : \{(R, \Sigma)\} \to \mathfrak{H}$, $(R, \Sigma) \mapsto \Pi(R, \Sigma)$ is onto.

The well-definiteness and injectivity follow from the following theorem.

Theorem 3.1.7. *Two marked Riemann surfaces* (R, Σ) *and* (R', Σ') *are Teichmüller-equivalent if and only if* $\omega = \omega'$ *where* $\omega = \Pi(R, \Sigma)$ *and* $\omega' = \Pi(R', \Sigma')$.

Proof. The biholomorphism between R and $J(R)$ maps the chosen basis of homotopy to the image of two intervals $[0; 1]$ and $[0; \omega]$ in $\mathbb{C}$. Hence if two marked Riemann surfaces share the same value $\omega = \Pi(R, \Sigma) = \Pi(R', \Sigma')$ then the combination $\phi \circ \phi'^{-1}$ is an equivalence between (R, Σ) and (R', Σ').

Conversely suppose there is a biholomorphism between R and R' which also maps Σ to Σ'. It induces a biholomoprism between their Jacobians $f : J(R) \to J(R')$ that can be lifted by the monodromy theorem to the holomorphic mapping $\tilde{f} : \mathbb{C} \to \mathbb{C}$ and we can assume that $\tilde{f}(0) = 0$. The only holomorphic mappings of $\mathbb{C}$ onto itself are linear mappings and taking to account the last remark we can write it as $\tilde{f}(z) = \alpha z$.

The markings are mapped one into another so $\tilde{f}(1) = 1$ and $\tilde{f}(\omega) = \omega'$ which implies that $\omega = \omega'$. □

This ends the goal of this chapter and it can be summarized as

Theorem 3.1.8. *The mapping*

$$\begin{aligned} \phi : T_1 &\to \mathfrak{H} \\ [R, \Sigma] &\mapsto \Pi(R, \Sigma) \end{aligned}$$

is a bijective representation of the Teichmüller space of genus 1 *with the Siegel upper half plane.*

The example of tori of genus 1 shows that the Tiechmüller space is a manifold itself and has a complex structure on it. This result was generalized for arbitrary genus case and is an important result about Teichmüller spaces.

3.1.3 Torelli space

Given a marked Riemann surface $[R, \Sigma]$ we immediately obtain a period matrix (I, Π) (or simply Π).

In this section we will describe the representation of Teichmuller space with period matrices. The period matrix depends on homology classes of curves so any two markings with the same images in $H_1(R, \mathbb{Z})$ define the same period matrix. This immediately shows that this representation is not faithful.

The last remark leads to the following equivalence on Teichmuller space.

Definition 1.9 Two triples (R, S_1, f_1) and (R, S_2, f_2) are *Torelli equivalent* if there exists a conformal mapping $f : S_1 \to S_2$ which induces the same action on $H_1(R)$ as $f_2 f_1^{-1}$.

In terms of the marked Riemann surfaces of genus g this equivalence can be formulated as follows.

Definition 1.10 Two marked Riemann surfaces (R, Σ) and (R', Σ') are *Torelli equivalent* if there exists a conformal mapping $f : R \to R'$ which sends homology images of Σ into Σ' component-wise.

The set of equivalence classes forms the *Torelli space* $U(R)$ (or $U(g)$).

The equivalence of these definitions is stated in the next proposition. Recall that according to the Theorem 3.1.6 for a fixed marking Σ on a Riemann surface R all other marked Riemann surfaces are of the form $(S, f(\Sigma))$ for some orientation preserving diffeomorphism $f : R \to S$.

Proposition 3.1.11. *Fix a marked Riemann surface* (R, Σ). *Then two triples* (R, S_1, f_1) *and* (R, S_2, f_2) *are Torelli equivalent if and only if two marked Riemann surfaces* $(S_1, f_1(\Sigma))$ *and* $(S_2, f_2(\Sigma))$ *are Torelli equivalent.*

Proof. The statement follows directly from Theorem 3.1.6: the mapping $f : S_1 \to S_2$ which induces the same action on the homology groups for (R, S_1, f_1) and (R, S_2, f_2) equivalence, and a mapping which transforms the marking $f_1(\Sigma)$ on $(S_1, f_1(\Sigma))$ to the marking on $(S_2, f_2(\Sigma))$ component-wisely are actually the same. $\square$

3.2 Modular groups

Definition 2.1 For a Riemann surface R

1. Let $Diff^+(R)$ denote the group of all orientation preserving diffeomorphisms $\rho : R \to R$ under composition.

$$Diff^+(R) = (\{f \mid f : R \to R - \text{orientation preserving diffeomorphism}\}, \circ)\,.$$

2. Let $Diff_0^+(R)$ denote the subgroup of $Diff^+(R)$ consisting of all orientation preserving diffeomorphisms which are homotopic to the identity.

$$Diff_0^+(R) = \{f \in Diff^+(R) \mid f \text{ is homotopic to } id_R : R \to R\}$$

Proposition 3.2.2. $Diff_0^+(R)$ *is a normal subgroup of* $Diff^+(R)$.

Proof. For $g \in Diff_0^+(R)$ denote by $g_t : R \to R,\ t \in [0;1]$ a homotopy with $g_1 = g$ and $g_0 = id_R$.

For any two elements $g, h \in Diff_0^+(R)$ the homotopy $g_t \circ h_t : R \to R,\ t \in [0;1]$ is a homotopy between $g_1 \circ h_1 = g \circ h$ and $g_0 \circ h_0 = id_R$, and g_t^{-1} is a homotopy for g^{-1} so it is indeed a subgroup.

For any $g \in Diff_0^+(R)$ and $f \in Diff^+(R)$ the composition $f^{-1} \circ g_t \circ f : R \to R,\ t \in [0;1]$ is a homotopy between $f^{-1} \circ g_1 \circ f = f^{-1} \circ g \circ f$ and $f^{-1} \circ g_0 \circ f =$

$f^{-1} \circ \mathrm{id}_R \circ f = \mathrm{id}_R$ so $f^{-1} \circ g \circ f \in Diff_0^+(R)$ and thus $Diff_0^+(R)$ is a normal subgroup of $Diff^+(R)$. $\square$

The group $Diff^+(R)$ acts on the Teichmüller space in the following way. For $g \in Diff^+(R)$ and $[R, S, f] \in T(R)$

$$g[R, S, f] = [R, S, f \circ g^{-1}] \tag{3.1}$$

Proposition 3.2.3. *The action (3.1) is a well-defined group action on $T(R)$.*

Proof. The proof is trivial. $\square$

Proposition 3.2.4. *The action of the subgroup $Diff_0^+(R) \lhd Diff^+(R)$ on $T(R)$ is trivial.*

Proof. All that need to be checked is that for any $g \in Diff_0^+(R)$ and $[T, S, f] \in T(R)$

$$(T, S, f) \sim (T, S, f \circ g^{-1}),$$

and that is true since $Diff_0^+(R)$ is normal and so $f \circ (f \circ g^{-1})^{-1} = f \circ g \circ f^{-1} \in Diff_0^+(R)$. Therefore it is homotopic to the identity which is a biholomorphic map on S. $\square$

Definition 2.5 The *(Teichmüller) modular group* $Mod(R)$ is

$$Mod(R) = Diff^+(R)/Diff_0^+(R).$$

According to the Proposition 3.2.4 the $Diff^+(R)$ induces the action of the modular group $Mod(R)$ on $T(R)$ by the formula (3.1) for any representative of the element in $Mod(R)$ by an element from $Diff^+(R)$.

As $Diff_0^+(R)$ acts trivially on $T(G)$, to define the Torelli modular group we need to identify the class of mappings that act trivially on the $U(R)$. Let K be the subgroup of the $Diff^+(R)$ whose induced action on the homology group is trivial.

$$K = \{f \in Diff^+(R) \mid f^* : H_1(R) \to H_1(R), f^* = \mathrm{id}_{H_1(R)}\}$$

Proposition 3.2.6. *K is a normal subgroup of $Diff^+(R)$.*

Proof. For $f, g \in K$ we have that $(f \circ g)^* = f^* \circ g^* = \mathrm{id}$, and $(g^{-1})^* = (g^*)^{-1} = \mathrm{id}$ so K is a subgroup of $Diff^+(R)$.

If $g \in K$ and $f \in Diff^+(R)$ then $(f^{-1} \circ g \circ f)^* = (f^{-1})^* \circ g^* \circ f^* = (f^{-1})^* \circ \mathrm{id} \circ f^* = \mathrm{id}$ and therefore K is normal. □

Now we can give the definition for the Torelli modular group.

Definition 2.7 The *Torelli modular group* is

$$Tor(R) = Diff^+(R)/K = \{f \mid f \in Diff^+(R)\}/\sim,$$

where $f \sim g$ if and only if they induce the same transformation of the first homology group $H_1(R)$.

Since $Diff_0^+(R) \lhd K$ as well it is possible to describe the Torelli modular group as a quotient group of the Teichmuller modular group.

Proposition 3.2.8.

$$Tor(R) \simeq Mod(R) \big/ \big(K/Diff_0^+(R)\big) \ .$$

Proof.

$$\begin{aligned} Tor(R) = Diff^+(R)/K &\simeq \big(Diff^+(R)/Diff_0^+(R)\big) \big/ \big(K/Diff_0^+(R)\big) \simeq \\ &\simeq Mod(R) \big/ \big(K/Diff_0^+(R)\big) \end{aligned}$$

since $Diff_0^+(R) \lhd K$. □

We now define the Torelli group which should not be confused with the Torelli modular group.

Definition 2.9 The *Torelli group* is

$$Tg(R) = K/Diff_0^+(R).$$

In other words $Tg(R)$ consists of all diffeomorphisms that acts identically on the homology group up to the homotopy relation.

Now we can restate the last proposition

Proposition 3.2.10.

$$Tor(R) \simeq Mod(R)/Tg(R).$$

Remark. *Since the Torelli group is a subgroup of the modular group one may consider it acting on the Teichmuller space as well. In the next section we will see that it connects the Teichmuller space and the Torelli space.*

The definition of the Torelli modular space allows us to define its action on the Torelli moduli space by formula (3.1).

Proposition 3.2.11. *The group action of $Tor(R)$ on $U(R)$ by the formula (3.1) is well-defined.*

Proof. Choose any two orientation preserving diffeomorphisms g_1, g_2 that represent the same class in the $Tor(R)$. Then $g_1^{-1} \circ g_2$ acts trivially on the homology group of S_2. We have that $g_1(R, S, f) = (R, S, f \circ g_1^{-1}) \sim_{Torelli} (R, S, f \circ g_2^{-1}) = g_2(R, S, f)$ if and only if $(f \circ g_1^{-1}) \circ (f \circ g_2^{-1})^{-1}$ has the same act on the homology group as some holomorphic map.

$$(f \circ g_1^{-1}) \circ (f \circ g_2^{-1})^{-1} = f \circ g_1^{-1} \circ g_2 \circ f^{-1} = f \circ (g_1^{-1} \circ g_2) \circ f^{-1}.$$

Since $g_1^{-1} \circ g_2$ acts trivially on $H_1(S_2)$ the last map also acts trivially on the homology. Therefore g_1 and g_2 act identically on the $U(R)$.

$\square$

3.3 Projections

In this section we will see that the construction of modular groups is adapted to the projections of the Teichmuller and Torelli spaces onto the Riemann space. But first we need to define these projections.

To distinguish elements from the Teichmüller space $T(R)$ and the Torelli space $U(R)$ we will use the notation $[R, S, f]_{T(R)}$ and $[R, S, f]_{U(R)}$ to denote the Teichmüller equivalence classes and the Torelli equivalence classes of a triple (R, S, f).

Definition 3.1 Let R be a compact Riemann surface of genus g. Define

$$\mathbf{P}_{TM} : T(R) \to \mathcal{M}(g)$$
$$[R, S, f]_{T(R)} \mapsto [S]$$

$$\mathbf{P}_{TU} : T(R) \to U(R)$$
$$[R, S, f]_{T(R)} \mapsto [R, S, f]_{U(R)}$$

$$\mathbf{P}_{UM} : U(R) \to \mathcal{M}(g)$$
$$[R, S, f]_{U(R)} \mapsto [S]$$

Proposition 3.3.2. *The mappings $pr_{T(R)\to R(g)}$, $pr_{T(R)\to U(R)}$ and $pr_{U(R)\to R(g)}$ are well-defined.*

Proof.

1. $pr_{T(R)\to R(g)}$. Let (R, S_1, f_1) and (R, S_2, f_2) be Teichmüller equivalent. It means $\exists h : S_1 \to S_2$ biholomorphic, such that $f_2 \circ f_1^{-1}$ is homotopic to h. In particular S_1 and S_2 must be Riemann equivalent.

2. $pr_{U(R)\to R(g)}$. As above, the definition of the Torelli equivalence requires S_1 and S_2 be biholomorphic equivalent.

3. $pr_{T(R)\to U(R)}$. If (R, S_1, f_1) and (R, S_2, f_2) are Teichmüller equivalent then $f_2 \circ f_1^{-1}$ is homotopic to some holomorphic $h : S_1 \to S_2$. Then $f_2 \circ f_1^{-1}$ induces the same transformation $H_1(S_1) \to H_1(S_2)$ as h.

□

Proposition 3.3.3.

1. *$pr_{T(R)\to R(g)}$ is invariant under the modular group $Mod(R)$.*

2. *$pr_{T(R)\to U(R)}$ is invariant under the Torelli group $Tg(R)$.*

3. *$pr_{U(R)\to R(g)}$ is invariant under the Torelli modular group $Tor(R)$.*

Proof. 1. Let an orientation preserving diffemorphism g represent an element $\tilde{g} \in Mod(R)$. Then

$$\tilde{g}[R, S, f] = [R, S, f \circ g^{-1}]$$

and thus $pr([R, S, f]) = pr(\tilde{g}[R, S, f]) = [S]$.

2. Choose g to represent an element in $Tg(R)$. Then

$$pr_{T(R) \to U(R)}(g[R, S, f]_{T(R)}) = [R, S, f \circ g^{-1}]_{U(R)} = [R, S, f]_{U(R)}$$

since $f \circ (f \circ g^{-1})^{-1} = f \circ g \circ f^{-1}$ acts trivially on the homology group.

3. Let g represents an element of the $Tor(R)$. Then

$$pr_{U(R) \to R(g)}(g[R, S, f]) = [S]$$

is independent of g.

□

Remark. *This means that we are able to consider these projections acting on the orbit space of the corresponding sets:*

$$pr_{T(R) \to R(g)} : T(R)/Mod(R) \to R(g)$$

$$pr_{T(R) \to U(R)} : T(G)/Tg(R) \to U(R)$$

$$pr_{U(R) \to R(g)} : U(g)/Tor(R) \to R(g).$$

Finally we can identify the Riemann moduli space with the orbit space of the Teichmuller space under the action of the Teichmuller modular group via the map $pr_{T(R) \to R(g)}$.

Theorem 3.3.4. *Let R be a compact Riemann surface of genus g. The Riemann moduli space $R(g)$ is in bijective correspondence $pr_{T(R) \to R(g)}$ with the set of all orbits of the $T(R)$ under the action of the Teichmuller modular group $Mod(R)$.*

Proof. The map is obviously onto.

To prove injectivity choose two preimages $[R, S_1, f_1]$ and $[R, S_2, f_2]$ of some class $[S_1] = [S_2]$ in the Riemann moduli space. The goal is to show that $[R, S_1, f_1]$ and $[R, S_2, f_2]$ belong to the same orbit under the $Mod(R)$.

Let $\sigma : S_1 \to S_2$ be a biholomorphism. Construct the diffeomorphism on R by setting it to be

$$\rho = (f_2^{-1} \circ \sigma \circ f_1)^{-1}.$$

Now we show that $[R, S_1, f_1]$ is the image of $[R, S_2, f_2]$ under the action of ρ:

$$\rho[R, S_2, f_2] = [R, S_2, f_2 \circ \rho^{-1}] = [R, S_2, \sigma \circ f_1].$$

Consider $[R, S_1, f_1]$ and $[R, S_2, \rho \circ f_1]$. The composition $f_1 \circ (\sigma \circ f_1)^{-1} = \sigma^{-1}$ is biholomorphic which shows that two mentioned elements of the Teichmuller space are actually equal. □

3.4 Period mapping

As we have seen earlier the representation of the Teichmuller space of genus 1 with the upper half plane using periods of the surface led to a Siegel upper half plane model of this space. In this section we apply the same method to the surfaces of genus higher than 1.

Recall that for a Riemann surface R of genus g with fixed canonical homology basis $\Sigma = \{a_1, ..., a_g, b_1, ..., b_g\}$ the adapted basis $\{\omega_1, ..., \omega_g\}$ of the space of all holomorphic differentials on R is uniquely chosen in such a way that first g periods of them satisfy:

$$\int_{a_i} \omega_j = \delta_{ij}$$

The remaining g periods of these differentials form so called period matrix

$$\Pi = \Pi(R, \Sigma) = (\pi_{ij})_{i,j=1}^{g},\ \pi_{ij} = \int_{b_i} \omega_j.$$

The period matrix was proven to belong to the Siegel upper half space

$$\mathfrak{H}_g := \left\{ M \in M_g(\mathbb{C}) \,\middle|\, M = M^T,\ \text{Im}\,(M) = \frac{1}{2i}(M - \overline{M}) > 0 \right\} 0$$

(for precise definitions, constructions and more details see Ch. 2.5).

This allows us to provide the following definitions.

Definition 4.1 For the Teichmuller space of genus g the mapping

$$\Pi : T(g) \to \mathfrak{H}_g,\ [R, \Sigma] \mapsto \Pi(R, \Sigma)$$

is called the *period mapping.*

Remark. *In the definition of the Teichmuller space marking* Σ *refers to a selection of a canonical basis of the fundamental group. Even though it is not exactly what is used for a period matrix, a canonical basis of the fundamental group forms a basis of the homology group and can be used for a construction of the period matrix.*

Since the period matrix cares more about a basis of the homology group and not a basis of the fundamental group the Torelli space is more reasonable space to consider as its domain. In order to see this we need the following statement.

Proposition 3.4.2. *The period mapping is invariant under the action of the Torelli group* $Tg(R)$*.*

Proof. By the definition 3.2 of the Torelli group, its elements act identically on the first homology group. Therefore the action of the Torelli group does not change the canonical basis of the homology group and hence does not change the period matrix. $\square$

Definition 4.3 The mapping $\#\Pi : U(R) \to \mathfrak{H}_g$ induced from $\Pi : T(R) \to \mathfrak{H}_g$ is also called the *period matrix.*

Now we have a commutative diagram

$$\begin{tikzcd}
T(R) \arrow[d] \arrow[dr, "\Pi"] & \\
U(R) \arrow[r, "\#\Pi"] \arrow[d] & \mathfrak{H}_g \\
R(g) &
\end{tikzcd}$$

Denote the image of Σ in the homology group by $\widetilde{\Sigma}$.

It turns out that even the representation of the Torelli space with the period matrices is not faithful itself (for $g > 1$).

Another possible homology marking that induces the same period matrix is the "opposite" one:

$$\Pi(R, \widetilde{\Sigma}) = \Pi(R, -\widetilde{\Sigma})$$

To understand which $(R, \widetilde{\Sigma})$ are Torelli equivalent to its "opposite" $(R, -\widetilde{\Sigma})$ we must introduce hyperelliptic Riemann surfaces.

Definition 4.4 For $g \geq 2$ a Riemann surface R is called *hyperelliptic* if it admits a conformal involution with precisely $2g + 2$ fixed points.

The following lemma clarifies the question.

Lemma 3.4.5. *Fix a canonical homology basis $\widetilde{\Sigma}$ on R (for $g \geq 2$). There exists a conformal involutive self-map which maps $\widetilde{\Sigma}$ to its opposite if and only if R is hyperelliptic.*

Now I can describe the projection $U(R) \to \mathfrak{H}_g$. Let $H(R)$ denote those points in $U(R)$ which represent hyperelliptic Riemann surfaces.

Proposition 3.4.6. *The map $\#\Pi : U(R) \to \mathfrak{H}_g$ is $1 - 1$ on $H(R)$ and $2 - 1$ on $U(R) \setminus H(R)$.*

Now we describe how does the corresponding representation of the modular group work.

Proposition 3.4.7. *Elements of the modular group $Mod(R)$ induce symplectic transformations of the first homology group $H_1(R)$.*

Proof. The intersection number of any two elements of the first homology group is invariant under orientation preserving diffeomorphisms and therefore the elements $g \in Mod(R)$ induce automorphisms of the $H_1(R)$ and send canonical bases to the canonical bases. According to the Proposition 2.6.1 this homomorphism is a symplectic transformation. $\square$

Definition 4.8 Let

$$\varphi : Mod(R) \to Aut(H_1(R)) = Sp(H_1(R))$$

$$g \mapsto g*$$

denotes the corresponding mapping.

Proposition 3.4.9. *The kernel of the mapping φ is the Torelli group.*

$$Ker(\varphi) = Tg(R).$$

Proof. This is a paraphrasing of the definition of the Torelli group. $\varphi(g)$ acts trivially on $H_1(R)$ if and only if g is in $Tg(R)$. □

Therefore we get that

Proposition 3.4.10. *The sequence*

$$1 \longrightarrow Tg(R) \longrightarrow Mod(R) \xrightarrow{\varphi} Sp(H_1(R)) \longrightarrow 1$$

is exact.

Proposition 3.4.11. *Let $g \in Mod(T)$ be acting on the element $[R, \Sigma]$ of the Teichmuller space T_g or the Torelli space $U(g)$. Then $\varphi(g) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is symplectic and transforms the period matrix Π of (R, Σ) into the period matrix Π' of $(R, g(\Sigma))$ by*

$$\Pi' = (\gamma + \delta\Pi)(\alpha + \beta\Pi)^{-1}. \tag{3.2}$$

Proof. This is an immediate consequence of the Proposition 2.6.2 and the Proposition 3.4.7. □

Proposition 3.4.12. *Let R be a compact Riemann surface of genus g. Then the period mapping Π (or $\#\Pi$) together with φ induce the well-defined map*

$$\tilde{\Pi} : R(g) \to \mathfrak{H}_g / Sp_g(\mathbb{Z}),$$

where $\mathfrak{H}_g / Sp_g(\mathbb{Z})$ is the orbit space of $\mathfrak{H}_g$ under the action (3.2) of the symplectic group.

Proof. Proposition 3.3.4 implies that $R(g) \simeq T(R)/Mod(R)$. Clearly the orbit $Mod(R)g \mapsto \varphi(Mod(R))\Pi(g) \subset Sp_g(\mathbb{Z})\Pi(g) \in \mathfrak{H}/Sp_g(\mathbb{Z})$. $\square$

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