Isomorphisms of Banach Algebras Associated with Locally Compact Groups

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The main theme of this thesis is to study the isometric algebra isomorphisms and the bipositive algebra isomorphisms between various Banach algebras associated with locally compact groups.

Let $LUC(G)$ denote the C^*-algebra of left uniformly continuous functions with the uniform norm and let $C_0(G)^{\perp}$ denote the annihilator of $C_0(G)$ in $LUC(G)^*$. In Chapter 2 of this thesis, among other results, we show that if G is a locally compact group and H is a discrete group then whenever there exists a weak-star continuous isometric isomorphism between $C_0(G)^{\perp}$ and $C_0(H)^{\perp}$, G is isomorphic to H as a topological group. In particular, when H is discrete $C_0(H)^{\perp}$ determines H within the class of locally compact topological groups.

In Chapter 3 of this thesis, we show that if $M(G, \omega_1)$ (the weighted measure algebra on G) is isometrically algebra isomorphic to $M(H, \omega_2)$, then the underlying weighted groups are isomorphic, i.e. there exists an isomorphism of topological groups $\phi : G \to H$ such that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative. Similarly, we show that any weighted locally compact group (G, ω) is completely determined by its Beurling group algebra $L^1(G, \omega)$, $LUC(G, \omega^{-1})^*$ and $L^1(G, \omega)^{**}$, when the two last algebras are equipped with an Arens product. Here, $LUC(G, \omega^{-1})$ is the weighted analogue of $LUC(G)$, for weighted locally compact groups.

In Chapter 4 of this thesis, we show that the order structure combined with the algebra structure of each of the Banach algebras $L^1(G, \omega)$, $M(G, \omega)$, $LUC(G, \omega^{-1})^*$ and $L^1(G, \omega)^{**}$ completely determines the underlying topological group structure together with a constraint on the weight. In particular, we obtain new proofs for a previously known result of Kawada and results of Farhadi as special cases of our results. Finally, we provide an example of a bipositive algebra isomorphism between Beurling measure algebras that

is not an isometry.

We conclude this thesis with a selective list of open problems.

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in an attempt to return for a lifelong gift.

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Chapter 1

Introduction

Let G be a locally compact group with a fixed Haar measure. By the group algebra on G we mean the convolution Banach algebra $L^1(G)$ of Haar-integrable functions on G. A natural question asks to what extent does the algebra structure of $L^1(G)$ reflect the topological group structure of G (see [Rud62, Subsection 4.7.7]). In [Wen51], J. G. Wendel showed that, in general, the algebra structure of a group algebra does not necessarily determine its underlying topological group structure. So, if only the existence of an algebra isomorphism between group algebras is assumed, then the underlying topological groups are isomorphic only if we impose some constraints for instance on the norm of the isomorphism or if we consider some special isomorphism such as a bipositive one.

Wendel [Wen51] proved that if G and H are locally compact topological groups and $T : L^1(G) \to L^1(H)$ is an isometric isomorphism, then there exists an isomorphism $\phi : G \to H$ of the topological groups G and H, a continuous character $\alpha : G \to \mathbb{T}$ and a constant c such that

$$(Tf)\,(g) = c\;\alpha \circ \phi^{-1}(g)\; f \circ \phi^{-1}(g) \quad \big(f \in L^1(G),\; g \in H\big)\,.$$

In particular, the group algebra $L^1(G)$ determines G as a topological group. Wendel's theorem was complemented independently by Johnson in [Joh64b] and Strichartz in [Str65], who proved that a similar result is true if we replace group algebras with measure algebras. In [LM80], Lau and McKennon generalized Johnson's result. In particular, they proved that the dual of the space of left uniformly continuous functions on G, $LUC(G)^*$ with the left Arens multiplication, determines the underlying topological group G. However, there is no description given for isometric isomorphisms on the dual of left uniformly continuous functions in terms of topological group isomorphisms. In [GL88], F. Ghahramani and A.T.M Lau showed that the bidual $L^1(G)^{**}$ of the group algebra with the left Arens product also determines the topological group G.

There are natural canonical embeddings of $L^1(G)$ and $M(G)$ in $LUC(G)^*$ and $L^1(G)^{**}$ equipped with the left Arens multiplication. In [GLL90], Ghahramani, Lau and Losert proved that every isometric isomorphism from $LUC(G)^*$ onto $LUC(H)^*$ sends the canonical copy of $L^1(G)$ onto the canonical copy of $L^1(H)$ and the canonical copy of $M(G)$ onto the canonical copy of $M(H)$. It is also obtained that if T is any isometric isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then the canonical copy of $L^1(G)$ in $L^1(G)^{**}$ is mapped onto the canonical copy of $L^1(H)$ in $L^1(H)^{**}$.

Recently, in the memoir [DLS12], Dales, Lau and Strauss proved that the bidual of the measure algebra, $M(G)^{**}$ with the left Arens multiplication determines the underlying topological group G as well.

Y. Kawada was the first author who studied bipositive algebra isomorphisms. In [Kaw48], Kawada showed that if we have a bipositive algebra isomorphism between group algebras, then the underlying locally compact groups must be isomorphic. H. Farhadi [Far98] proved similar results to Kawada's result for other Banach algebras related to locally compact groups including the biduals of group algebras. He proved that if G and H

are locally compact groups and T is a bipositive algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$ then T is an isometric algebra isomorphism. It then follows from [GL88] that G and H are isomorphic locally compact groups. In the same paper [Far98, Thm. 2.4], Farhadi showed that a similar result holds if we replace the bidual of group algebras with a measure algebra or with dual of the space of left uniformly continuous functions equipped with the left Arens multiplication.

As mentioned above, isometric isomorphisms on the group and measure algebras are very well understood. However, no description is at hand for isometric isomorphisms on the dual of left uniformly continuous functions in terms of topological group isomorphisms. This question is the motivation for the results obtained in Chapter 2. In [GLL90], Ghahramani, Lau and Losert proved that if T is an isometric isomorphism from the dual of left uniformly continuous functions on G, $LUC(G)^*$ onto $LUC(H)^*$ then T maps the canonical copy of the measure algebra $M(G)$ in $LUC(G)^*$ onto the measure algebra $M(H)$. In the same paper, they showed that if we let $C_0(G)^\perp$ denote the annihilator of $C_0(G)$ in $LUC(G)^*$ then

$$LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp,$$

where $\oplus_1$ denotes the ℓ^1-direct sum and $C_0(G)^\perp$ is a weak-star closed ideal in $LUC(G)^*$.

In Chapter 2, we will prove that an isometric isomorphism T of $LUC(G)^*$ onto $LUC(H)^*$ also maps $C_0(G)^\perp$ onto $C_0(H)^\perp$. So, if we have an isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$ then $T = T_1 \oplus_1 T_2$, where $T_1 : M(G) \to M(H)$ and $T_2 : C_0(G)^\perp \to C_0(H)^\perp$ are isometric isomorphisms. As both $LUC(G)^*$ and $M(G)$ determine G, this gives rise to the following difficult and compelling question: does the Banach algebraic structure of $C_0(G)^\perp$ determine the underlying locally compact group G. Another intriguing question is whether or not every isometric isomorphism mapping $C_0(G)^\perp$ onto $C_0(H)^\perp$ extends to an isometric isomorphism mapping $LUC(G)^*$ onto $LUC(H)^*$. We study these questions and provide positive answers when one of the locally compact groups is discrete.

Weighted convolution algebras on the real line were introduced by Beurling and were subsequently defined on arbitrary locally compact groups. Many works have since been devoted to the study of various Beurling algebras on weighted locally compact groups. It is natural to ask to what extent the Banach algebraic structure of each of these Beurling algebras reflects its underlying weighted locally compact group structure. In Chapter 3, we study this question. We will see that the Banach algebraic structure of the Beurling group algebra $L^1(G,\omega)$, the Beurling measure algebra $M(G,\omega)$, the bidual $L^1(G,\omega)^{**}$, and $LUC(G,\omega^{-1})^*$ (where $LUC(G,\omega^{-1})$ is the weighted analogue of $LUC(G)$, the space of left uniformly continuous functions on G, for weighted locally compact groups) all determine the underlying weighted locally compact group (G,ω). We will also provide a complete description of the isometric isomorphisms on Beurling group and measure algebras in terms of topological group isomorphisms and continuous characters.

In Chapter 4, we show that the algebra structure together with the order structure of various Beurling algebras associated with locally compact groups completely determines the structure of the topological groups together with a constraint on the weight. Moreover, we give a complete description of bipositive algebra isomorphisms between Beurling group and measure algebras in terms of topological group isomorphisms. Finally, we provide an example of a bipositive algebra isomorphism between Beurling measure algebras that is not an isometry

We conclude this thesis with a selective list of open problems.

1.1 Preliminaries and Notation

1.1.1 Banach Algebras

This chapter is comprised of a collection of definitions of concepts and notations from theories of Banach algebras and abstract harmonic analysis that we need for our work in the subsequent chapters. It contains no new results.

A **Banach algebra** $\mathcal{A}$ is an associative algebra that at the same time is also a Banach space. The algebra multiplication and the Banach space norm are required to be related via the inequality

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in \mathcal{A}).$$

Throughout this chapter, $\mathcal{A}$ will denote a Banach algebra.

A **left multiplier** on $\mathcal{A}$ is a linear mapping L on $\mathcal{A}$ satisfying

$$L(ab) = L(a)b \quad (a, b \in \mathcal{A}).$$

A **right multiplier** of $\mathcal{A}$ is a linear mapping R on $\mathcal{A}$ satisfying

$$R(ab) = aR(b) \quad (a, b \in \mathcal{A}).$$

Example 1.1.1. Let $\mathcal{A}$ be a Banach algebra. For each $a \in \mathcal{A}$, let L_a and R_a be the linear operators on $\mathcal{A}$ defined by

$$L_a(b) = ab \ \text{ and } \ R_a(b) = ba \quad (b \in \mathcal{A}).$$

Then L_a is a left multiplier on $\mathcal{A}$, R_a is a right multiplier on $\mathcal{A}$.

Definition 1.1.2. Let X be a Banach space and $\mathcal{A}$ be a Banach algebra.

(i) X is called a left Banach $\mathcal{A}-$module if there is a bilinear map

$$(a, x) \mapsto a \cdot x; \ \mathcal{A} \times X \to X,$$

such that for all $a, b \in \mathcal{A}$ and $x \in X$

$$a \cdot (b \cdot x) = ab \cdot x \quad \text{and} \quad \|a \cdot x\| \leq \|a\| \|x\|.$$

A right Banach $\mathcal{A}-$module is defined similarly except that the Banach algebra $\mathcal{A}$ acts on the right.

(ii) X is called a Banach $\mathcal{A}-$bimodule if X is a left and right Banach $\mathcal{A}-$module and the left and right actions are related via the equation

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b,$$

for all $a, b \in \mathcal{A}$ and $x \in X$.

Example 1.1.3. (i) Every Banach algebra $\mathcal{A}$ is a Banach $\mathcal{A}-$bimodule over itself.

(ii) Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$. Then $\mathcal{I}$ is a Banach $\mathcal{A}-$bimodule with respect to the actions $a \cdot x = ax$ and $x \cdot a = xa$ where $a \in \mathcal{A}$ and $x \in \mathcal{I}$.

(iii) Let X be a Banach $\mathcal{A}-$bimodule with dual space X^*. If $f \in X^*$ and $x \in X$ we find it convenient to write $\langle f, x \rangle$ for $f(x)$. Then X^* is a Banach $\mathcal{A}-$bimodule with the module multiplication given by duality as follows

$$\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle,$$
$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle,$$

for every $f \in X^*$, $a \in \mathcal{A}$ and $x \in X$.

Definition 1.1.4. Given a Banach algebra $\mathcal{A}$, as discussed in parts (i) and (iii) of Example 1.1.3, the dual space $\mathcal{A}^*$ can be viewed as a right Banach $\mathcal{A}-$bimodule with the canonical operation

$$\langle f\Box a, b\rangle := \langle f\cdot a, b\rangle = \langle f, ab\rangle,$$

where $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$. (Here we are following the notation in [DL05] and [HNR11]). Let X be a norm closed $\mathcal{A}-$submodule of $\mathcal{A}^*$. Then given $n \in X^*$ and $f \in X$, we may define $n\Box f \in \mathcal{A}^*$ by

$$\langle n\Box f, a\rangle = \langle n, f\Box a\rangle.$$

If $n\Box f \in X$, for all choices of $n \in X^*$ and $f \in X$, then X is called a **left introverted subspace** of $\mathcal{A}^*$. The dual of a left introverted space can be turned into a Banach algebra by defining $m\Box n \in X^*$ through

$$\langle m\Box n, f\rangle = \langle m, n\Box f\rangle \quad (m, n \in X^*, f \in X).$$

In particular, by taking $X = \mathcal{A}^*$ we obtain the **left Arens product** on $\mathcal{A}^{**}$. In addition to $\mathcal{A}^*$, other examples of left introverted subspaces of $\mathcal{A}^*$ include the space of **left uniformly continuous functionals** on $\mathcal{A}$ defined by $LUC(\mathcal{A}) = \overline{\text{lin-span}(\mathcal{A}^*\Box\mathcal{A})}$, the norm closure in $\mathcal{A}^*$ of the linear span of $\mathcal{A}^*\Box\mathcal{A}$, and the space of [weakly] almost periodic functionals on $\mathcal{A}$, $[WAP(\mathcal{A})]$ $AP(\mathcal{A})$, defined as the set of all $f \in \mathcal{A}^*$ such that the linear map $\mathcal{A} \to \mathcal{A}^*,\ a \mapsto a\Box f$, is [weakly] compact. For more information and details see [Sto15, Section 1].

Theorem 1.1.5. *Let $\mathcal{A}$ be a Banach algebra. Then $(\mathcal{A}^{**}, \Box)$ is a Banach algebra containing $\mathcal{A}$ as a closed subalgebra (through the canonical embedding). Moreover, for each $n \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$, the maps $m \mapsto m\Box n$ and $m \mapsto a\Box m$ are weak-star continuous on $(\mathcal{A}^{**}, \Box)$.*

Proof. See [Are51, Page 842]. $\square$

Since $\mathcal{A}$ is weak-star dense in $\mathcal{A}^{**}$, for all $m, n \in \mathcal{A}^{**}$ there are nets (a_α) and (b_α) in $\mathcal{A}$ such that $m = w^* - \lim_\alpha a_\alpha$ and $n = w^* - \lim_\beta b_\beta$. It can be readily seen that

$$m \Box n = w^* - \lim_\alpha \left(w^* - \lim_\beta (a_\alpha b_\beta) \right).$$

Definition 1.1.6. Let $\mathcal{A}$ be a Banach algebra and X be a left (right) Banach $\mathcal{A}$-module.

(i) A **left (right) bounded approximate identity** in $\mathcal{A}$ for X is a bounded net (e_γ) in $\mathcal{A}$ such that

$$\|e_\gamma \cdot x - x\| \to 0 \quad (\|x \cdot e_\gamma - x\| \to 0),$$

for all $x \in X$.

(ii) If X is a Banach $\mathcal{A}$-bimodule, then a **bounded approximate identity** (e_γ) for X is a net in $\mathcal{A}$ which is both a left and right bounded approximate identity for X. When $X = \mathcal{A}$ (as in Example 1.1.3 (i)) we simply call (e_γ) a bounded approximate identity for $\mathcal{A}$.

Recall that a left (right) Banach $\mathcal{A}-$module X is called left (right) **neo-unital** if $\mathcal{A} \cdot X = X$ $(X \cdot \mathcal{A} = X)$.

Theorem 1.1.7. (Cohen's Factorization Theorem) *Let X be a left (right) Banach $\mathcal{A}-$module and suppose that there is a bounded left (right) approximate identity for X in $\mathcal{A}$. Then X is a left (right) neo-unital Banach $\mathcal{A}-$module.*

Proof. See [BD73, 11.10]. □

Let X be a left Banach $\mathcal{A}-$module. It follows from the Cohen factorization theorem that if $\mathcal{A}$ has a left bounded approximate identity, then $\mathcal{A} \cdot X = \overline{\text{lin-span}(\mathcal{A} \cdot X)}$, the norm closure of the linear span of $\mathcal{A} \cdot X$.

A Banach algebra $\mathcal{A}$ is said to be a **dual Banach algebra** if there is a closed $\mathcal{A}-$submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $(\mathcal{A}_*)^* = \mathcal{A}$. For example, ℓ^∞ with pointwise product is a dual Banach algebra with predual ℓ^1.

Proposition 1.1.8. *[Run02, Exercise 4.4.1]) Let $\mathcal{A}$ be a Banach algebra which is a dual space. Then $\mathcal{A}$ is a dual Banach algebra if and only if multiplication in $\mathcal{A}$ is separately weak-star continuous.*

1.1.2 Banach Algebras Related to Locally Compact Groups

A **locally compact group** G is a topological group with a locally compact and Hausdorff topology. A systematic study of analysis on topological groups was developed in 1930's. In 1930 Haar showed that on every second countable locally compact group, there is a measure with properties similar to the Lebesgue measure on the locally compact group $\mathbb{R}$, called the Haar measure. Weil in 1936, reformulated Haar's construction in terms of linear functionals and showed that the second countability assumption can be omitted.

A non-negative Radon measure λ on a locally compact group G, with $\lambda(B) > 0$ for some Borel set B, is called a **left Haar measure** if λ is left translation invariant, that is

$$\lambda(xE) = \lambda(E),$$

for every Borel subset E of G and every $x \in G$.

Theorem 1.1.9. *[HR79, Thm.15.5] Every locally compact group G possesses a unique (up to a positive constant multiple) left Haar measure λ.*

The existence of the Haar measure on every locally compact group G allows us to define various function spaces on G that are similar to those studied in classical harmonic analysis. These function spaces are denoted with similar notations to the function spaces studied in classical harmonic analysis. In what follows, we introduce these spaces (see for example [RS00], [HR70], [HR79] or [Kan09]).

Let $\mathbf{C_b(G)}$ denote the **space of all bounded continuous functions** on G endowed with the uniform norm

$$\|f\|_\infty = \sup_{x\in G}|f(x)| \quad (f \in C_b(G)) .$$

It is easy to see that $C_b(G)$ with the uniform norm and pointwise multiplication forms a Banach algebra.

For $x \in G$, we let δ_x be the point mass concentrated at x. The measure δ_x can also be viewed as a functional on various function spaces on locally compact groups.

A subspace of $C_b(G)$ is $C_c(G)$, the space of **continuous compactly supported** functions on G defined as follows. For a continuous function $f \in C_b(G)$ let the **support of** f be the closure of the set $\{x \in G : \ f(x) \neq 0\}$ in G. Let $\mathbf{C_c(G)}$ denote the subalgebra of $C_b(G)$ containing the elements with compact support. In general, $C_c(G)$ with the uniform norm inherited from $C_b(G)$ is not a closed subalgebra. The closure of $C_c(G)$ in $C_b(G)$ is the Banach subalgebra $\mathbf{C_0(G)}$ of functions that vanish at infinity, where $f \in C_b(G)$ **vanishes at infinity** if

$$\forall \epsilon \ \exists K \subseteq G \text{ where } K \text{ is compact; } \|f|_{G\setminus K}\|_\infty \leq \epsilon.$$

It is easy to see that $C_0(G)$ with the uniform norm and pointwise multiplication forms a Banach subalgebra of $C_b(G)$.

Let $\mathbf{M(G)}$ denote the Banach space of all regular Borel measures with finite total variation, where for $\mu \in M(G)$, the norm of μ is its total variation. It is a standard result that $M(G)$ can be identified with the dual of the Banach space $C_0(G)$. The convolution of two measures in $M(G)$ was first defined in Weil's pioneering monograph [Wei40], where a whole chapter is devoted to the concept of convolution. The convolution product gives a Banach algebra structure to $M(G)$. In what follows, we define the convolution of two measures $\mu, \nu \in M(G)$.

Consider the linear functional

$$I(f) = \int_G \int_G f(xy)\ d\mu(x) d\nu(y)$$

on $C_0(G)$. It is not hard to see that

$$|I(f)| \leq \|f\|_\infty \|\mu\| \|\nu\|.$$

Therefore, I is in fact a bounded linear functional on $C_0(G)$, and we have

$$\|I\| \leq \|\mu\| \|\nu\|. \tag{1.1}$$

It follows from the identification of $M(G)$ with the dual, $C_0(G)^*$, of $C_0(G)$ (see [Con90, III.Thm.5.5]) that there exists a measure $\mu * \nu \in M(G)$, called the **convolution of μ and ν**, such that $I(f) = \mu * \nu(f)$. By equation (1.1) we have that

$$\|\mu * \nu\| \leq \|\mu\| \|\nu\|.$$

Theorem 1.1.10. *[HR79, Thm.19.6] The Banach space $M(G)$ with the convolution product*

$$\mu * \nu(f) = \int_G f(xy) d\mu d\nu, \quad (\mu, \nu \in M(G),\ f \in C_0(G)),$$

is a Banach algebra, called the **measure algebra** *of G.*

We fix a left Haar measure λ and let $L^1(G) = L^1(G, \lambda)$ be the space of (equivalence classes of) all integrable functions with respect to λ, equipped with the convolution multiplication defined $\lambda-$almost everywhere by

$$(f * g)(x) = \int_G f(xy) g(y^{-1})\ dy, \quad \left(f, g \in L^1(G)\right),$$

and the norm

$$\|f\|_1 = \int_G |f(x)| \; dx.$$

Then $L^1(G)$ is a Banach algebra. The mapping $f \mapsto f \; d\lambda$; $L^1(G) \to M(G)$, identifies $L^1(G)$ with a closed two-sided ideal of $M(G)$ (see [HR79, Thm.19.18]).

Let $\mathbf{L}^\infty(\mathbf{G})$ be the space of all bounded, measurable, complex valued functions on G, two functions being regarded as equal if they differ only on a locally null set ($A \subseteq G$ is called locally null if $\lambda(A \cap F) = 0$, for every compact subset F of G). The norm of $f \in L^\infty(G)$ is defined as

$$\|f\|_\infty = \inf \{\alpha \in \mathbb{R} : \alpha \geq 0 \text{ and } \{x \in G : |f(x)| > \alpha\} \text{ is locally null}\}.$$

Then $L^\infty(G)$ can be identified with $L^1(G)^*$, the Banach space dual of $L^1(G)$, through the pairing

$$\langle f, \psi \rangle = \int_G f(x)\psi(x) \; dx, \quad \left(f \in L^\infty(G), \psi \in L^1(G)\right).$$

The concept of uniform continuity can be generalized from the locally compact abelian group $\mathbb{R}$ to any locally compact group G as follows. An element f in $C_b(G)$ is called **left uniformly continuous** if the mapping

$$G \to C_b(G); g \mapsto l_g f$$

is continuous, where $l_g f(x) = f(gx)$. Let $\mathbf{LUC}(\mathbf{G})$ denote the Banach space of all left uniformly continuous functions on G endowed with the uniform norm. It is not hard to see that any continuous compactly supported function on G is also left uniformly continuous. It then follows that $C_0(G)$, the space of continuous functions vanishing at infinity, is also contained in $LUC(G)$. Note that in [HR70] and [HR79], the notation $UCB_r(G)$ is used in the place of the (now) more common notation $LUC(G)$. In the following theorem we are viewing $L^\infty(G) = L^1(G)^*$ as a right dual Banach $L^1(G)$-module, as in Definition 1.1.4.

Theorem 1.1.11. *([HR70, Chapter 32] [HR79, Thm.20.16]) In the $L^1(G)$-module structure of $L^\infty(G)$, we have the factorization $LUC(G) = L^\infty(G)\Box L^1(G)$.*

It follows that $LUC(G) = L^\infty(G)\Box L^1(G)$ is a left introverted subspace of $L^\infty(G) = L^1(G)^*$. As discussed in Section 1.1.1, $LUC(G)^*$ forms a Banach algebra with left Arens product defined by

$$\begin{aligned} \langle m\Box n, f\rangle &= \langle m, n\Box f\rangle \\ \langle n\Box f, \psi_1\rangle &= \langle n, f\Box\psi_1\rangle \\ \langle f\Box\psi_1, \psi_2\rangle &= \langle f, \psi_1 * \psi_2\rangle, \end{aligned} \tag{1.2}$$

where $m, n \in LUC(G)^*$, $f \in LUC(G)$ and $\psi_1, \psi_2 \in L^1(G)$.

It is shown in [Lau78] that the left Arens multiplication defined above on $LUC(G)^*$ is the same as the product on $LUC(G)^*$ defined by

$$\langle mn, f\rangle = \langle m, n_l(f)\rangle \quad \left(m, n \in LUC(G)^* \text{ and } f \in LUC(G)\right), \tag{1.3}$$

where the function $n_l(f)$ in $LUC(G)$ is defined by $n_l(f)(x) = \langle n, l_x f\rangle$ for all $x \in G$. In Chapter 2 of this thesis we will use formula (1.3) for the multiplication on $LUC(G)^*$. When dealing with the weighted version of $LUC(G)^*$ in Chapter 3, we will not have access to this pointwise formula and will therefore use the $\Box$ notation found in (1.2), and used in [DL05] and [HNR11].

The measure algebra $M(G)$ can be naturally isometrically embedded into $LUC(G)^*$ as a Banach subalgebra via

$$\langle \mu, f\rangle = \int_G f\, d\mu \quad \left(\mu \in M(G),\ f \in LUC(G)\right).$$

In fact, we have the ℓ^1-direct sum decomposition

$$LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp,$$

where

$$\mathbf{C_0(G)}^{\perp} := \{m \in LUC(G)^* : \langle m, f\rangle = 0,\ \forall f \in C_0(G)\}.$$

Moreover, $C_0(G)^{\perp}$ is a weak-star closed ideal in $LUC(G)^*$ (see [GLL90, Lemma 1.1]). One can readily check that for each $n \in LUC(G)^*$ and $\mu \in M(G)$ the mappings $m \mapsto mn$ and $n \mapsto \mu n$ are weak-star continuous. In fact, $M(G)$ is the largest subset of elements m in $LUC(G)^*$ for which the left multiplication mapping $n \mapsto mn$ is weak-star continuous on $LUC(G)^*$ (see [Lau86]).

1.1.3 Left Uniform Compactification of a Locally Compact Group

We start this chapter with some definitions from the theory of semigroup compactifications. We then introduce the right topological semigroup compactification, G^{LUC}, as an example of a semigroup compactification with a universal property.

Definition 1.1.12. A **right topological semigroup** is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for each $s \in S$, the mapping $S \to S; t \mapsto ts$ is continuous. The set

$$\mathcal{Z}_t(S) := \{s \in S :\ t \mapsto st \text{ is continuous}\}$$

is the **topological centre** of S.

Definition 1.1.13. A **right topological semigroup compactification** of a locally compact group G is a pair (α, G^{α}) where G^{α} is a compact right topological semigroup, and $\alpha : G \longrightarrow G^{\alpha}$ is a continuous homomorphism with dense range such that $\alpha(G)$ is contained in $\mathcal{Z}_t(G^{\alpha})$.

The Banach space $LUC(G)$ with the usual pointwise operations, the uniform norm and the involution given by pointwise complex conjugation is a commutative unital C^*-algebra.

Therefore, by the remarkable Gelfand representation theorem, $LUC(G) = C(G^{LUC})$, where $\mathbf{G^{LUC}}$ denotes the Gelfand spectrum of $LUC(G)$; that is,

$$G^{LUC} := \{m \in LUC(G)^* \setminus \{0\}; \langle m, fg\rangle = \langle m, f\rangle\langle m, g\rangle \ \forall f, g \in LUC(G)\}$$

with the relative weak-star topology inherited from $LUC(G)^*$. Note that when G is discrete, $C(G^{LUC}) = LUC(G) = \ell^\infty(G) = C(\beta G)$. Therefore, when G is discrete, G^{LUC} is in fact the same as the **Stone-Čech compactification**, βG, of G.

The compact Hausdorff topological space G^{LUC} has a natural multiplication (inherited from the product of $LUC(G)^*$) extending the group multiplication on G. Indeed, it can be readily seen that if m and n belong to G^{LUC}, then the linear functional mn in $LUC(G)^*$ – see equation (1.3) – is also a non-zero multiplicative linear functional and therefore mn belongs to G^{LUC}. Hence, G^{LUC} is a compact right topological semigroup with the Arens multiplication and weak-star topology it inherits from $LUC(G)^*$. Moreover, $\delta : x \mapsto \delta_x$ is a topological embedding of G into $\mathcal{Z}_t(G^{LUC})$ and $\delta(G)$ is dense in G^{LUC}. Therefore, (δ, G^{LUC}) is a right topological semigroup compactification for the locally compact group G. The semigroup compactification (δ, G^{LUC}) is the largest right topological semigroup compactification of G in the sense of Theorem 1.1.14 below.

Theorem 1.1.14. *[BJM89, Chapter 4, Thm.4.4] Let G be a locally compact group. Then (δ, G^{LUC}) is the universal right topological semigroup compactification of G. That is, if (α, G^α) is any other right topological semigroup compactification of G, then there exists a continuous semigroup homomorphism θ mapping G^{LUC} onto G^α such that the diagram*

$$\begin{tikzcd} G^{LUC} \arrow[r, "\theta"] & G^{\alpha} \\ G \arrow[u, "\delta"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

The **corona** (outgrowth) of the LUC-compactification of G, $G^{LUC} \setminus G$, is denoted by $\mathbf{G}^*$ and is a closed ideal of the compact semigroup G^{LUC}.

An element $z \in G^*$ is called right cancellable if for $m, n \in G^{LUC}$, $mz = nz$ implies that $m = n$. We have the following:

Theorem 1.1.15. *[FP03, Thm.1] Let G be a locally compact non-compact group. There is a subset of right cancellable elements of G^* that is open and dense in G^*.*

We conclude this section with the celebrated Theorem of Veech. Veech's theorem asserts that any locally compact group acts freely on its $LUC-$compactification. Veech's theorem is very important in topological dynamics as well as in the theory of semigroup compactifications.

Theorem 1.1.16. *([Pym99] or [Ala14])* **(Veech)** *Suppose that G is a locally compact group. Let $g \in G$ with g not equal to e_G. Then for every $m \in G^{LUC}$ we have that $gm \neq m$.*

1.1.4 Beurling Algebras on Locally Compact Groups

Throughout this section, G denotes a locally compact group. A positive function ω on G is called a **weight** if it has the following properties.

(i) $\omega(xy) \leq \omega(x)\omega(y)$ for all x, $y \in G$.

(ii) ω is continuous.

Note that in Chapter 3, we additionally assume that $\omega(e_G) = 1$. We do not need this condition for our results in Chapter 4. By a **weighted locally compact group**, we mean a pair (G, ω) where G is a locally compact group and ω is a weight function on G.

Example 1.1.17. (i) Given $\lambda \in \mathbb{R}^+$, the Freud weight $\omega(x) = exp(-|x|^\lambda)$ defines a weight function on the additive group $\mathbb{R}$.

(ii) If ω_1 and ω_2 are two weights on the locally compact group G, then $\omega_1\omega_2$ is also a weight on G.

(iii) Let ω be a weight on G. Then both $\omega_1(x) := \omega(x)\omega(x^{-1})$ and $\omega_2(x) := \omega(x^{-1})$ are weights on G.

The weighted analogues of the spaces discussed in Section 1.1.2 have been defined and studied by many authors over the years. For example, see [Gha80], [GSZ10], [Grø90], [DL05], [FS07] and [Gha84b]. We conclude this section by providing the basic definitions and theorems about these weighted Banach algebras, the so-called Beurling algebras. The memoir [DL05] contains a wealth of information about Beurling algebras.

Let (G, ω) be a weighted locally compact group. Then

$$L^1(G, \omega) := \{f : G \to \mathbb{C} \cup \{\infty\} \ : \ f\omega \in L^1(G)\}$$

is a linear space that is not necessarily a subset of $L^1(G)$. We equip $L^1(G, \omega)$ with the norm

$$\|f\|_{1,\omega} = \int_G |f(x)|\omega(x) \, dx.$$

Then the mapping $f \mapsto f\omega$ from $L^1(G, \omega)$ onto $L^1(G)$ defines a linear isometry, and therefore $L^1(G, \omega)$ with pointwise operations of addition and scalar multiplication and the norm defined above forms a Banach space isometrically isomorphic to $L^1(G)$. We then define the convolution product on $L^1(G, \omega)$ through

$$f * g(x) = \int_G f(xy)g(y^{-1})dx \quad (x \in G)\,.$$

An easy calculation shows that the convolution product defined as above is well-defined on $L^1(G,\omega)$ and turns $L^1(G,\omega)$ into a Banach algebra. We summarize these facts in the proposition below.

Proposition 1.1.18. *[Kan09, Page 20] Let (G,ω) be a weighted locally compact group. Then $L^1(G,\omega)$ is a Banach space that is isometrically isomorphic to the Banach space $L^1(G)$ via the isometry $f \mapsto f\omega$. Furthermore, $L^1(G,\omega)$ is a Banach algebra with the convolution product.*

The Banach algebra $\mathbf{L^1(G},\omega)$ is called the **Beurling group algebra** on G associated with the weight ω.

Remark 1.1.19. (i) If ω is bounded away from zero, then the Beurling group algebra is a subalgebra of the group algebra, whereas if ω is bounded then the group algebra is a subalgebra of the Beurling group algebra. If G is compact, then $L^1(G) = L^1(G,\omega)$.

(ii) Since $L^1(G,\omega)$ as a Banach space is isometrically isomorphic to $L^1(G)$ via $f \mapsto f\omega$, we can see that the dual of $L^1(G,\omega)$ is

$$\mathbf{L}^\infty(\mathbf{G},\omega^{\mathbf{-1}}) := \{\mathbf{g} : \ \mathbf{g}/\omega \in \mathbf{L}^\infty(\mathbf{G})\}.$$

The duality is given by $f \mapsto \int_G f(x)g(x)\,dx$, where $f \in L^1(G,\omega)$ and $g \in L^\infty(G,\omega^{-1})$.

Lemma 1.1.20. *[Gha84b, Lemma 2.1] The Banach algebra $L^1(G,\omega)$ has a bounded approximate identity.*

Lemma 1.1.21. *[Kan09, Lemma 1.3.5] Let (G,ω) be a weighted locally compact group.*

(i) Every compactly supported function in $L^1(G)$ belongs to $L^1(G,\omega)$.

(ii) $C_c(G)$ is dense in $L^1(G,\omega)$.

Let $\mathbf{C_0(G,}\omega^{\mathbf{-1}})$ denote the linear space of all (continuous) functions f on G such that $f/\omega \in C_0(G)$. Then it can be seen that $C_0(G,\omega^{-1})$ with the norm defined by

$$\|f\|_{\infty,\omega^{-1}} := \|\frac{f}{\omega}\|_{\infty},$$

is a Banach space.

The set of all regular Borel measures on G such that

$$\int_G \omega(s)d|\mu|(s) < \infty$$

forms a Banach space, denoted by $\mathbf{M(G,}\omega)$, with respect to the norm

$$\|\mu\|_\omega = \int_G \omega(s)d|\mu|(s).$$

It can be seen that as a Banach space $M(G,\omega)$ is isometrically isomorphic to $C_0(G,\omega^{-1})^*$ with the duality given by $f \mapsto \int_G f(x)\ d\mu(x)$. The Banach space $M(G,\omega)$ is a Banach algebra if it is equipped with the multiplication

$$\langle \mu * \nu, f\rangle = \int_G \int_G f(st)d\mu(s)\ d\nu(t) \quad \left(f \in C_0(G,\omega^{-1})\right).$$

The Banach space $C_0(G,\omega)$ is isometrically isomorphic with $C_0(G)$ via the isomorphism $f \mapsto \omega f; C_0(G) \to C_0(G,\omega^{-1})$. Its dual map is given by $\mu \mapsto \omega\mu; M(G,\omega) \to M(G)$.

The convolution product defined above is separately weak-star continuous on $M(G,\omega)$. We can identify each $f \in L^1(G,\omega)$, with a measure in $M(G,\omega) = C_0(G,\omega^{-1})^*$ via

$$h \mapsto \int_G h(x)f(x)dx, \quad \left(h \in C_0(G,\omega^{-1})\right),$$

and it can be seen that $L^1(G,\omega)$ is a closed ideal of $M(G,\omega)$ (see [Gha84b, page 72]). The above definitions and identifications can be found in various places, for example [DL05], [Gha84a] or [Gha84b]. We summarize the above discussion in the following proposition.

Proposition 1.1.22. *Let (G, ω) be a weighted locally compact group. Then $M(G, \omega)$ is a Banach space that is isometrically isomorphic to the Banach space $M(G)$ via the weak-star continuous isometric linear isomorphism $\mu \mapsto \omega\mu : M(G, \omega) \to M(G)$. Moreover, the Banach space $C_0(G, 1/\omega)$ is a predual of $M(G, \omega)$, and $M(G, \omega)$ is a unital dual Banach algebra with respect to the convolution product.*

Lemma 1.1.23. *[DL05, Prop.7.2] and [Gha84b, page 72] The Banach algebra $L^1(G, \omega)$ is a closed ideal in $M(G, \omega)$, and $l^1(G, \omega)$ is a closed subalgebra of $M(G, \omega)$. Both $L^1(G, \omega)$ and $l^1(G, \omega)$ are weak-star dense in $M(G, \omega)$.*

We also note the following formula for the product of $f \in L^1(G, \omega)$ and $\mu \in M(G, \omega)$ (see [DL05, page 73]):

$$\mu * f(x) = \int_G f(y^{-1}x) d\mu(y) \quad (x \in G).$$

Let $LUC(G, \omega^{-1})$ denote the linear space of all (continuous) functions f on G such that $f/\omega \in LUC(G)$. Then it can be seen that $LUC(G, \omega^{-1})$ with the norm defined by

$$\|f\|_{\infty,\omega^{-1}} := \|\frac{f}{\omega}\|_\infty,$$

is a Banach space isometrically linear isomorphic to $LUC(G)$ via

$$\Phi : LUC(G, \omega^{-1}) \to LUC(G);\ f \mapsto f\omega^{-1}.$$

Let

$$G_\omega^{LUC} := \Phi^*(G^{LUC}),$$

where, as before, G^{LUC} denotes the Gelfand spectrum of the C^*−algebra $LUC(G)$.

Chapter 2

Isometric Isomorphisms of $C_0(G)^\perp$ in $LUC(G)^*$

Let $LUC(G)$ denote the C^*-algebra of left uniformly continuous functions with the uniform norm and let $C_0(G)^\perp$ denote the annihilator of $C_0(G)$ in $LUC(G)^*$. In this chapter, among other results, we show that if G is a locally compact group and H is a discrete group then whenever there exists a weak-star continuous isometric isomorphism between $C_0(G)^\perp$ and $C_0(H)^\perp$, G is isomorphic to H as a topological group. In particular, when H is discrete $C_0(H)^\perp$ determines H within the class of locally compact topological groups.

As noted already, in this chapter we will use formula 1.3 for the multiplication on $LUC(G)^*$.

2.1 When is G^* an F-space?

Our main goal in this section is to show that if G is a locally compact non-discrete group then neither G^{LUC} nor G^* are F-spaces (see Definition 2.1.1). Therefore G^{LUC} (G^*) is an

F-space if and only if G is discrete. This shows that some interesting facts concerning Stone–Čech compactifications of discrete groups cannot possibly be generalized to LUC-compactifications of general locally compact groups following the same line of proof. We will apply this result in Section 2.2. We conclude this section by showing that the corona of the LUC-compactification of a locally compact group does not contain P-points.

F-spaces were studied in detail by L. Gillman and M. Henriksen in 1956 [GH56] as the class of spaces X for which $C(X)$ is a ring in which every finitely generated ideal is a principal ideal. Several conditions both topological and algebraic were proved equivalent for a space to be an F-space (see [GJ76]). We choose the following characterization as our definition for an F-space.

Definition 2.1.1. A completely regular space X is an **F-space** if for any continuous bounded function f on X there is a continuous bounded function k on X such that $f = k|f|$.

Many of the proofs in the case of discrete groups use the fact that for a discrete space the Stone-Čech compactification and its corona are F-spaces (see for example [PPS00], [Fil], [Zel99], [MS96], [HS94] and [HS12]). This is especially useful due to the following separation lemma. A proof is given in [HS94, Lemma 1.1].

Lemma 2.1.2. *Let X be a compact space. Then X is an F-space if and only if for $\sigma-$compact subsets A and B of X, $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ implies that $\bar{A} \cap \bar{B} = \emptyset$.*

In the proof of Theorem 2.1.4, we make use of absolutely convergent series with alternating partial sums. An example of such a series is given below.

Example 2.1.3. Let $\sum_{n=1}^{\infty} b_n$ be a convergent series with positive terms and let

$$
\begin{aligned}
a_1 &:= b_1 \\
a_2 &:= -b_1 - b_2 \\
a_3 &:= b_2 + b_3 \\
a_4 &:= -b_3 - b_4 \\
&\dots \\
a_{2k} &:= -b_{2k-1} - b_{2k} \\
a_{2k+1} &:= b_{2k} + b_{2k+1} \\
&\dots .
\end{aligned}
$$

Then the series $\sum_{n=1}^{\infty} a_n$ is an example of an absolutely convergent series whose partial sums have alternating sign.

Theorem 2.1.4. *Suppose that X is a Hausdorff locally compact non-discrete topological space with a non-trivial convergent sequence. Then X is not an F-space.*

Proof. Suppose that X contains a non-trivial sequence (x_n) convergent to a (non-isolated) point x_0 in X. Without loss of generality assume that $x_n \neq x_0$ for each $n \in \mathbb{N}$. Since x_0 is not isolated and X is a locally compact space we can inductively construct a nested family $\{K_n\}_{n\geq 2}$ of compact neighbourhoods of x_0 such that $x_1, x_2, ..., x_n \notin K_{n+1}$ and $x_{n+1}, x_{n+2}, ... \in K_{n+1}$, for each n. To see this note that since $x_1 \neq x_0$ there is a pre-compact open set U_0 such that $\{x_2, x_3, ...\} \cup \{x_0\} \in U_0$ and $x_1 \notin U_0$. Since X is a locally compact space for the compact set $\{x_0\}$ and the open set U_0 there is a compact set K_2 such that $\{x_0\} \subseteq K_2^{\circ} \subseteq K_2 \subseteq U_0$, where K_2° denotes the interior of the set K_2. Without loss of generality we can assume that $(x_n)_{n\geq 2} \in K_2$. Suppose that the compact set K_n is given such that $x_1, ..., x_{n-1} \notin K_n$. Since $x_1, ..., x_n \neq x_0$, there is a pre-compact open set U_n such that $\{x_{n+1}, x_{n+2}, ...\} \cup \{x_0\} \subseteq U_n \subseteq K_n$ and $x_1, ..., x_n \notin U_n$. Since X is locally compact there is a compact set K_{n+1} such that $\{x_0\} \subseteq K_{n+1}^{\circ} \subseteq K_{n+1} \subseteq U_n$ and note that $x_1, ..., x_n \notin K_{n+1}$. By discarding some elements of our sequence if necessary we can assume that $x_{n+1}, x_{n+2}, ... \in K_{n+1}$.

Consider an absolutely convergent series $\sum_{n=1}^{\infty} a_n$, with alternating partial sums as in Example 2.1.3. Using Urysohn's lemma for locally compact spaces, for $n \geq 2$ we define the compactly supported function f_n such that $f_n(X \setminus K_n) = 0$ and $f_n(k_{n+1} \cup \{x_n\}) = a_{n-1}$, for $n \geq 2$. By the Weierstrass M-test, the series $\sum_{n \geq 2} f_n$ is uniformly convergent to a continuous function. Let $f := \sum_{n \geq 2} f_n$. Note that $f(X \setminus K_2) = 0$. For any function k where $f = k|f|$, we observe that k is not continuous at x_0. To see this we note that $f(x_n) = \sum_{m=1}^{n} a_n$, $n \geq 2$. Therefore, $f(x_n)f(x_{n+1}) < 0$ as $\sum_{m=1}^{n} a_m \sum_{m=1}^{n+1} a_m < 0$. So the function k alternates on the convergent sequence (x_n) between $+1$ and -1 and it is not continuous at x_0. $\square$

So to show that G^{LUC} (or G^*) is not an F-space, it is enough to show that it contains a non-trivial convergent sequence.

Kuzminov has shown that any compact group is **dyadic**, i.e., a continuous image of a Cantor cube. This implies that every infinite compact group contains a non-trivial convergent sequence. This result can be found for example in J. Van Mill's article in [Pea07, page 190].

Theorem 2.1.5. *Let G be a locally compact non-discrete group. Then G has a non-trivial convergent sequence. In particular, G is not an $F-$space.*

Proof. It is easy to see that if G is a metrizable non-discrete locally compact group then G has a non-trivial convergent sequence. If G is a locally compact non-discrete group then G

has a subgroup H that is sigma-compact, clopen and non-discrete. If H is metrizable then any point in H is a limit point of a non-trivial convergent sequence in H. This sequence is also convergent in G. So suppose that H is not metrizable. By the Kakutani-Kodaira theorem [HR79, Thm. 8.5] there is a compact normal subgroup N of H such that H/N is metrizable, and so by [HR79, 5.38 part (e)], N cannot be metrizable since otherwise H would be metrizable. Because N is not metrizable it cannot be finite. As noted above, since N is compact and non-discrete we can find a non-trivial convergent sequence in N. Any such non-trivial convergent sequence is also convergent in the open subgroup H and therefore in G. □

As stated in the introduction, were G^* an F-space for some non-discrete groups, we would be able to prove stronger versions of our main results in the next section namely, Theorem 2.2.7 and Corollary 2.2.8. However, as the next result shows, G^* is an F-space if and only if G is discrete.

Theorem 2.1.6. *Suppose that G is a locally compact non-discrete group. Then neither G^{LUC} nor G^* is an F-space.*

Proof. Suppose that G is a locally compact non-discrete group. Then G contains a non-trivial sequence, say (x_n), convergent to an element x. Let z be a right cancellable element in G^*. Then the non-trivial sequence $(x_n z)$ converges to xz. So in both cases, neither G^{LUC} nor G^* is an F-space, by Theorem 2.1.4. □

Definition 2.1.7. A point in a topological space is called a P**-point** if every $G_\delta-$set containing the point is a neighbourhood of the point.

Gillman and Henriksen [GJ76] were the first to study P-points. If G is a discrete group then under the continuum hypothesis the set of P-points in G^* forms a dense subset in

G^*. It is a fact that the existence of P-points cannot be proved in ZFC. We refer the reader to [HS12] and the remark in [Fil96, page 385] for an explanation of these statements. From Corollary 2.1.8 (below) we see that under the continuum hypothesis, if G is a locally compact group, then $G^{LUC} \setminus G$ has a P-point if and only if G is discrete. P-points were used in [Fil, Thm. 3] to show that for a discrete group G there are left cancellable elements. In fact these particular P-points in [Fil, Thm. 3] are also right cancellable. It is not known if for the general case of discrete groups there can be left cancellable elements that are not right cancellable (see [HS12, Thm. 8.40]).

Corollary 2.1.8. *Suppose that G is a locally compact non-discrete group. Then G^* does not contain any P-point.*

Proof. First observe that for each point p in G^* there is a non-trivial sequence (x_n) converging to p. To see this, note that from the proof of Lemma 2.1.5 there is a non-trivial sequence (z_n), convergent to a point y in the group. Therefore, the non-trivial sequence (y_n) where $y_n := y^{-1}z_n$ is convergent to e. Now $x_n := y_n p$ is a non-trivial sequence, by Veech's theorem, convergent to p in G^*. Let $\{U_n\}$ be a family of open neighbourhoods of p such that for each n, $x_1, x_2, ..., x_n \notin U_n$. Then $p \in \cap_m U_m$, but for each n, $x_n \notin \cap_m U_m$ so the G_δ-set $\cap_m U_m$ cannot be an open neighbourhood of p. Hence p is not a P-point. $\square$

2.2 Isometric Isomorphisms on $C_0(G)^\perp$

Let G and H be locally compact groups and $\mathbb{T}$ be the circle group. Suppose that $\alpha : G \to \mathbb{T}$ is a continuous character and $\psi : G \to H$ is a continuous isomorphism. Then it is easy to see that

$$j_{\alpha,\psi}(f) := \alpha \cdot f \circ \psi$$

maps $C_0(H)$ into $C_0(G)$ and the dual mapping

$$j^*_{\alpha,\psi} : M(G) \to M(H)$$

is a weak-star continuous isometric isomorphism. It follows from [Joh64b] that every isometric isomorphism $T : M(G) \to M(H)$ is of the form $T = j^*_{\alpha,\psi}$, for some character $\alpha : G \to \mathbb{T}$ and an isomorphism $\psi : G \to H$, and therefore is weak-star continuous. Similarly, if $\alpha : G \to \mathbb{T}$ is a continuous character and $\psi : G \to H$ is a continuous homomorphism, then it is easy to see that

$$j_{\alpha,\psi}(f) := \alpha \cdot f \circ \psi$$

maps $LUC(H)$ into $LUC(G)$ and that the dual map

$$j^*_{\alpha,\psi} : LUC(G)^* \to LUC(H)^*$$

is a homomorphism. When ψ is a topological isomorphism $j^*_{\alpha,\psi} : LUC(G)^* \to LUC(H)^*$ is a weak-star continuous isometric isomorphism. Moreover, every weak-star continuous isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$ also takes this canonical form, but it is not clear if every isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$ is weak-star continuous.

In [GLL90] Ghahramani, Lau and Losert showed that given locally compact groups G and H, every isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$ maps $M(G)$ onto $M(H)$. It can be shown directly that G^{LUC} determines G, within the class of locally compact groups. To see this we note that G^* is an ideal in G^{LUC} and therefore the only invertible elements of G^{LUC} are elements of G. An interesting question is whether $C_0(G)^\perp$ (or G^*) also determines G. In this section we employ Theorem 2.1.6 to show that this is the case when G is discrete (Theorem 2.2.7 and Corollary 2.2.8).

First we show that if G and H are non-compact locally compact groups and if $T : LUC(G)^* \to LUC(H)^*$ is an isometric isomorphism, then $C_0(G)^\perp$ is also mapped onto $C_0(H)^\perp$. We recall the following lemma from [Str65].

Lemma 2.2.1. *Let X be a locally compact space, and let μ and $\nu \in M(X)$. Then μ and ν are mutually singular if and only if $\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\|$.*

Lemma 2.2.1 implies that isometries preserve mutual singularity.

Since $LUC(G) = C(G^{LUC})$, the Banach space of continuous functions on G^{LUC}, we have that $LUC(G)^* = M(G^{LUC})$, the Banach space of all regular Borel measures on G^{LUC}. It can be seen that $C_0(G)^\perp$ is isometrically isomorphic to $M(G^*)$, the Banach space of regular Borel measures on G^*. Therefore, $M(G^{LUC}) = LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp = M(G) \oplus_1 M(G^*)$.

Theorem 2.2.2. *Suppose that G and H are non-compact locally compact groups. If $T : LUC(G)^* \to LUC(H)^*$ is an isometric isomorphism. Then T maps $C_0(G)^\perp$ onto $C_0(H)^\perp$.*

Proof. Letting $m \in C_0(G)^\perp$, we will show that $T(m) \in C_0(H)^\perp$. Suppose that $T(m) = \nu + r$, where $\nu \in M(H)$ and $r \in C_0(H)^\perp$, so that ν and r are mutually singular measures in $M(H^{LUC})$. Since an isometry preserves mutual singularity, if we consider $m = T^{-1}(\nu) + T^{-1}(r)$, then we have that $T^{-1}(\nu)$ and $T^{-1}(r)$ are singular and also we have that $T^{-1}(\nu) \in M(G)$, by [GLL90, Thm. 1.6]. Suppose that $T^{-1}(r) = \nu' + r'$ where $\nu' \in M(G)$ and $r' \in C_0(G)^\perp$. Therefore $m = T^{-1}(\nu) + \nu' + r'$. Now, since $T^{-1}(\nu), \nu' \in M(G)$ we must have $T^{-1}(\nu) + \nu' = 0$ because $M(G) \cap C_0(G)^\perp = 0$. Since ν' is absolutely continuous with respect to $T^{-1}(r)$, it is mutually singular with $T^{-1}(\nu)$, and so we have that $T^{-1}(\nu) = 0$. Hence $T(m) = \nu + r = r \in C_0(G)^\perp$. ∎

Corollary 2.2.3. *Suppose that G and H are locally compact groups. Then $T : LUC(G)^* \to LUC(H)^*$ is an isometric isomorphism if and only if there are isometric isomorphisms $T_1 : M(G) \to M(H)$ and $T_2 : C_0(G)^\perp \to C_0(H)^\perp$ such that $T = T_1 + T_2$, $T_2(\mu m) = T_1(\mu)T_2(m)$ and $T_2(m\mu) = T_2(m)T_1(\mu)$ for all $m \in C_0(G)^\perp$ and $\mu \in M(G)$.*

Note that Corollary 2.2.3 shows that the value of T_1 is connected to that of T_2. It is not clear if every isometric isomorphism $T : C_0(G)^\perp \to C_0(H)^\perp$ can be extended to one on $LUC(G)^*$. When either G or H is abelian and discrete and T is weak-star continuous, we will show that such an extension always exists (Theorem 2.2.11). Proposition 2.2.5 shows that such an extension is always unique. We need the following lemma for the proof of Proposition 2.2.5.

Lemma 2.2.4. *Suppose that G is a locally compact group and let $z \in G^*$ be a right cancellable element in G^{LUC}. The following statements hold:*
(i) z is also right cancellable in $LUC(G)^*$.
(ii) If (n_i) is a bounded net in $LUC(G)^*$ and $n_i z \xrightarrow{w^*} nz$ in $LUC(G)^*$, then $n_i \xrightarrow{w^*} n$ in $LUC(G)^*$.

Proof. (i) Let $z \in G^*$ be right cancellable in G^{LUC}. Then we show that z is also right cancellable in $LUC(G)^*$. To see this, we note that since z is right cancellable, the unital $*$-algebra

$$\{zf, f \in LUC(G)\} \quad \text{where} \quad zf(x) = \langle z, l_x f\rangle$$

separates the points in $LUC(G)$ and thus, by the Stone-Weierstrass theorem, is dense in $LUC(G) = C(G^{LUC})$. So if for some $m, n \in LUC(G)^*$ we have that $mz = nz$, then for all $f \in LUC(G)$ we must have $\langle m, zf\rangle = \langle n, zf\rangle$ and therefore $m = n$.
(ii) Suppose that (n_i) is a net in $LUC(G)^*$ such that (n_i) is bounded in norm by $M > 0$

and $n_i z \xrightarrow{w^*} nz$ in $LUC(G)^*$. We show that $n_i \xrightarrow{w^*} n$ in $LUC(G)^*$. Suppose that $\varepsilon > 0$ and $f \in LUC(G)$ are given. As noted in the proof of part (i), the algebra $\{zg,\ g \in LUC(G)\}$ is norm-dense in $LUC(G)$ so there is $g \in LUC(G)$ such that

$$\|f - zg\| \leq \frac{\varepsilon}{M}.$$

Also $n_i z \xrightarrow{w^*} nz$, so there is i_0 such that for all $i \geq i_0$ we have

$$|n_i z(g) - nz(g)| \leq \varepsilon.$$

Therefore for all $i \geq i_o$,

$$\begin{aligned} |n_i(f) - n(f)| &\leq |n_i(f) - n_i(zg)| + |n_i(zg) - n(zg)| \\ &+ |n(zg) - n(f)| \\ &\leq \varepsilon + \varepsilon + \varepsilon. \quad \square \end{aligned}$$

Proposition 2.2.5. *Suppose that G and H are locally compact groups and $T : C_0(G)^\perp \to C_0(H)^\perp$ is an (algebraic) isomorphism. If there is an isomorphism $\tilde{T} : LUC(G)^* \to LUC(H)^*$ such that $\tilde{T}$ is an extension of T, then $\tilde{T}$ is unique.*

Proof. Suppose that T_1 and T_2 are two such extensions. Let $z \in G^*$ be right cancellable in G^{LUC}. Then, by Lemma 2.2.4, z is also right cancellable in $LUC(G)^*$ and hence, $T(z) = T_1(z) = T_2(z)$ is also right cancellable in $LUC(H)^*$. For each m in $LUC(G)^*$, then we have that

$$T_1(m)T(z) = T(mz) = T_2(m)T(z).$$

Since $T(z)$ is right cancellable we have that $T_1(m) = T_2(m)$, for all m in $LUC(G)^*$. $\square$

The following proposition shows that under certain conditions the extension of an algebraic homomorphism $\phi : G^* \to H^*$ to an algebraic homomorphism $\varphi : G^{LUC} \to H^{LUC}$ is also unique.

Proposition 2.2.6. *Suppose that G and H are locally compact groups and $\phi : G^* \to H^*$ is an algebraic homomorphism. Suppose that $\varphi_L, \varphi_{L'} : G^{LUC} \to H^{LUC}$ are homomorphic extensions of ϕ such that either*
(i) $\varphi_L(G), \varphi_{L'}(G) \subseteq H$, or
(ii) the interior of $\phi(G^*)$ is non-empty.
Then, $\varphi_L = \varphi_{L'}$.

Proof. Let $x \in G$. Then, for all $p \in G^*$,

$$\varphi_L(x)\phi(p) = \varphi_L(xp) = \phi(xp) = \varphi_{L'}(xp) = \varphi_{L'}(x)\phi(p).$$

In the case of (i), $\varphi_L(x) = \varphi_{L'}(x)$, by Veech's theorem 1.1.16. By Theorem 1.1.15, the set of right cancellable elements is dense in G^*, so in the second case we can choose $p \in G^*$ such that $\phi(p)$ is right cancellable. $\square$

In fact, if $\varphi : G^{LUC} \to H^{LUC}$ is a surjective homomorphism, then φ satisfies condition (i) in Proposition 2.2.6. To see this, we observe that since φ is onto, there is a $q \in G^{LUC}$, such that $e_H = \varphi(q)$. We have that

$$e_H = \varphi(e_G q) = \varphi(e_G)\varphi(q) = \varphi(e_G)e_H = \varphi(e_G).$$

Thus, for each x in G, we have that $e_H = \varphi(x)\varphi(x^{-1})$ and thus $\varphi(x)$ must belong to H.

When $\psi : G \to H$ is a continuous homomorphism and $\alpha : G \to \mathbb{T}$ is the constant character 1, $j^*_{1,\psi} : LUC(G)^* \to LUC(H)^*$ is a weak-star continuous homomorphism and $\tilde{\psi} := j^*_{1,\psi}|_{G^{LUC}}$ is a continuous homomorphism of G^{LUC} into H^{LUC}. We say that the continuous homomorphism $\phi : G^* \to H^*$ is **induced by a continuous homomorphism** $\psi : G \to H$, if $\tilde{\psi}|_{G^*} = \phi$.

Note that if $\phi : G^* \to H^*$ is a continuous homomorphism, it is not necessarily induced by a homomorphism $\psi : G \to H$. A simple example is $\phi : G^* \to H^*$ where $\phi(p) = \iota$ and ι is an idempotent in H^* (see [HS12, Thm. 2.5] for a proof of the existence of idempotents in G^*). To see this suppose that there is a homomorphism $\psi : G \to H$ such that $\tilde{\psi}|_{G^*} = \phi$. Then since

$$\iota = \tilde{\psi}(xp) = \psi(x)\phi(p) = \psi(x)\iota$$

for (any) $p \in G^*$ and $x \in G$, we have that ψ is the trivial homomorphism $x \mapsto e_G$, by Veech's theorem 1.1.16. By uniqueness, $\tilde{\psi}(p) = e_G$, for all $p \in G^{LUC}$, which is not possible as $\tilde{\psi}|_{G^*} = \phi$.

We apply Theorem 2.1.6 and [PPS00, Thm.6.2] to prove the next result.

Theorem 2.2.7. *Let G and H be locally compact groups. Suppose that $\phi : G^* \to H^*$ is a continuous isomorphism and that either G or H is discrete. Then there is a unique topological isomorphism $\psi : G \to H$ such that $\phi = \tilde{\psi}|_{G^*}$. In particular, G and H are isomorphic, as topological groups.*

Proof. Since $\phi : G^* \to H^*$ is a continuous isomorphism, by Theorem 2.1.6 we have that both G and H are discrete groups. The existence of such a continuous surjection $\psi : G \to H$ now follows from [PPS00, Thm.6.2]. We shall also show that ψ is injective. Suppose that for some $x, y \in G$, we have that $\psi(x) = \psi(y)$. Then for each $q \in G^*$, since H^* is an ideal in H^{LUC}, we have that

$$\phi(xq) = \tilde{\psi}(xq) = \psi(x)\phi(q) = \psi(y)\phi(q) = \tilde{\psi}(yq) = \phi(yq).$$

Since ϕ is injective, $xq = yq$ and so, by Veech's theorem 1.1.16, $x = y$. Thus ψ is an isomorphism of discrete groups and therefore G and H must be isomorphic. $\square$

Is [PPS00, Thm.6.2] true when G and H are not assumed to be discrete? We say that $p \in G^*$ is a prime element if $p = xq$ for $x \in G^{LUC}$ and $q \in G^*$ implies that $x \in G$. Although some details such as the existence of prime elements in the proof of [PPS00, Thm. 6.2] remain valid for non-discrete locally compact SIN-groups (see the proof of [FS04, Thm.1.4]), the proof heavily depends on the Lemma 2.1.2 above. By Theorem 2.1.6, Lemma 2.1.2 cannot be employed for non-discrete locally compact groups, so the same proof will not work.

Corollary 2.2.8. *Let G and H be locally compact groups. Suppose that $T : C_0(G)^\perp \to C_0(H)^\perp$ is a weak-star continuous isometric isomorphism and either G or H is discrete. Then the topological groups G and H are isomorphic.*

Proof. First we note that since $C_0(G)^\perp = M(G^*)$, by [Con90, Thm. V.8.4] the set of extreme points of the unit ball of $C_0(G)^\perp$ is

$$\{\alpha\delta_p;\ \alpha \in \mathbb{T},\ p \in G^*\}.$$

In particular the point masses corresponding to the points in G^* are among the extreme points of the unit ball of $C_0(G)^\perp$. Because T is an isometry, it maps the extreme points of the unit ball of $C_0(G)^\perp$ onto the extreme points of the unit ball of $C_0(H)^\perp$. Therefore there exist maps $\phi : G^* \to H^*$ and $\alpha : G^* \to \mathbb{T}$ such that $T(\delta_p) = \alpha(p)\delta_{\phi(p)}$, for all $p \in G^*$. We first show that α is continuous. Suppose that (p_γ) is a net in G^* that is convergent to $p \in G^*$. Since T is weak-star continuous we have that

$$\alpha(p_\gamma)\delta_{\phi(p_\gamma)} = T(\delta_{p_\gamma}) \xrightarrow{w^*} T(\delta_p) = \alpha(p)\delta_{\phi(p)}.$$

Evaluating this equation at 1_{G^*} implies that $\alpha(p_\gamma) \to \alpha(p)$. Hence α is continuous. Similarly, since T is an isomorphism we can show that α is also multiplicative and thus α is in fact a continuous character. Therefore $\phi = \bar{\alpha}T|_{G^*} : G^* \to H^*$ is a continuous isomorphism. Since G^* is compact and H^* is Hausdorff, the continuous isomorphism $\phi : G^* \to H^*$ is also a homeomorphism. Now the result follows from Theorem 2.2.7 . □

When G and H are both non-discrete locally compact groups, it remains open whether $C_0(G)^\perp$ determines G.

Proposition 2.2.9. *Let G be a locally compact group, K a compact group, and suppose that $\alpha : G^* \to K$ is a continuous homomorphism. Let ι be any idempotent in G^* and define $\alpha_l : G^{LUC} \to K$ by $\alpha_l(x) = \alpha(x\iota)$. Then the following statements hold:*
(i) α_l is a continuous extension of α to G^{LUC}.
(ii) If β is any homomorphic extension of α to G^{LUC}, then $\beta = \alpha_l$. (Thus any such homomorphic extension – if it exists – is unique and automatically continuous.)
(iii) If ι commutes with elements of G, then α_l is a homomorphism on G^{LUC}.

Proof. (i) We note that

$$\alpha(\iota) = \alpha(\iota^2) = \alpha(\iota)^2,$$

so $\alpha(\iota) = e_K$, the identity in K. As $x \mapsto x\iota$ is a continuous mapping of G^{LUC} into G^*, and α is continuous on G^*, α_l is continuous on G^{LUC}.
(ii) If β is any such homomorphic extension of α, then for any $x \in G^{LUC}$,

$$\beta(x) = \beta(x)e_K = \beta(x)\beta(\iota) = \beta(x\iota) = \alpha(x\iota) = \alpha_l(x).$$

(iii) For any $x, y \in G$,

$$\alpha_l(xy) = \alpha(xy\iota) = \alpha(xy\iota^2) = \alpha(x\iota y\iota) = \alpha(x\iota)\alpha(y\iota) = \alpha_l(x)\alpha_l(y).$$

Thus, $\alpha_G := \alpha_l|_G : G \to K$ is a continuous homomorphism. (Note that it is not yet clear that α_l is a homomorphism on G^{LUC}, even though its restrictions to G and G^* are homomorphisms.) As observed in the introduction, α_G extends to a continuous homomorphism $\widetilde{\alpha_G} : G^{LUC} \to K$ (note that since K is compact, $K^{LUC} = K$). As both $\widetilde{\alpha_G}$ and α_l are continuous extensions of $\alpha_G = \alpha_l|_G$ to G^{LUC}, we must have $\alpha_l = \widetilde{\alpha_G}$. Hence, α_l is a homomorphism. $\square$

Corollary 2.2.10. *Let G be an abelian locally compact group and K be a compact group. Then every continuous homomorphism $\alpha : G^* \to K$ has a unique continuous homomorphic extension to G^{LUC}.*

Proof. Let ι be any idempotent in G^* and (x_γ) be a net in G convergent to ι. We have that for each $x \in G$

$$x\iota = x(\lim_\gamma x_\gamma) = \lim_\gamma (x x_\gamma) = \lim_\gamma (x_\gamma x) = (\lim_\gamma x_\gamma)x = \iota x.$$

The result now follows from Proposition 2.2.9. $\square$

We say a linear operator $T : C_0(G)^\perp \to C_0(H)^\perp$ is **positive** if for each positive linear functional $m \in M(G^*) = C(G^*)^*$ we have that $T(m)$ is a positive linear functional. Here, as usual, m is positive if $m(f) \geq 0$ whenever $f \geq 0$.

Theorem 2.2.11. *Let G and H be locally compact groups with either G or H discrete and suppose that $T : C_0(G)^\perp \to C_0(H)^\perp$ is a weak-star continuous isometric isomorphism. If either G is abelian, or T is a positive operator, then there exists a unique weak-star continuous isometric isomorphism $\tilde{T} : LUC(G)^* \to LUC(H)^*$ such that $\tilde{T}|_{C_0(G)^\perp} = T$.*

Proof. The uniqueness follows from Proposition 2.2.5. The proof of Corollary 2.2.8 shows that there exists a continuous character $\alpha : G^* \to \mathbb{T}$ and a topological isomorphism $\phi : G^* \to H^*$ such that

$$T(\delta_x) = \alpha(x)\delta_{\phi(x)} \quad (x \in G^*).$$

By Theorem 2.2.7, there exists a topological isomorphism $\psi : G \to H$ such that $\tilde{\psi}|_{G^*} = \phi$. If G is abelian, by Corollary 2.2.10 there exists a unique continuous character $\alpha_G : G \to \mathbb{T}$ such that $\tilde{\alpha_G}|_{G^*} = \alpha$; if T is positive, $\alpha \equiv 1_{G^*}$ and $\alpha = \widetilde{1_G}|_{G^*}$. As noted in the introduction to Section 2.2, $\tilde{T} = j^*_{\alpha_G,\psi}$ is a weak-star continuous isometric isomorphism of $LUC(G)^*$ onto $LUC(H)^*$. For $x \in G^*$, weak-star continuity and density considerations give

$$\tilde{T}(\delta_x) = \alpha(x)\delta_{\phi(x)} = T(\delta_x)$$

and the proof is complete. $\square$

Suppose that $T : C_0(G)^\perp \to C_0(H)^\perp$ is an isometric isomorphism (not necessarily weak-star continuous). Then for each $x \in H$, the mapping

$$L_x : C_0(G)^\perp \to C_0(G)^\perp, \quad \text{where} \quad L_x(m) = T^{-1}(\delta_x T(m))$$

is an invertible isometric left multiplier on $C_0(G)^\perp$ (i.e. $L_x(mn) = L_x(m)n,\ m, n \in C_0(G)^\perp$) with inverse $L_{x^{-1}}$. Similarly, for each $x \in H$, the mapping

$$R_x : C_0(G)^\perp \to C_0(G)^\perp, \quad \text{where} \quad R_x(m) = T^{-1}(T(m)\delta_x)$$

is an invertible isometric right multiplier on $C_0(G)^\perp$. We call $\mathbf{L_x} : \mathbf{C_0}(\mathbf{G})^\perp \to \mathbf{C_0}(\mathbf{G})^\perp$ a **left-point multiplier** and $\mathbf{R_x} : \mathbf{C_0}(\mathbf{G})^\perp \to \mathbf{C_0}(\mathbf{G})^\perp$ a **right-point multiplier associated with** T. The reader is referred to Section 1.1.1 and [Pal94, Sections 1.2.1-1.2.7] for definitions and basic theorems regarding the left/right multipliers.

Suppose that $T : C_0(G)^\perp \to C_0(H)^\perp$ is a weak-star continuous isometric isomorphism. Moreover, suppose that $T = \tilde{T}|_{C_0(G)^\perp}$, where $\tilde{T} = j^*_{\alpha,\psi} : LUC(G)^* \to LUC(H)^*$ for some character α on G and some topological isomorphism $\psi : G \to H$. (By Theorem 2.2.11, this is the case when G is discrete and either G is abelian or T is positive.) Then given $x \in H$, for each $m \in C_0(G)^\perp$

$$\begin{aligned} L_x(m) = T^{-1}(\delta_x T(m)) &= \tilde{T}^{-1}(\delta_x \tilde{T}(m)) \\ &= \tilde{T}^{-1}(\delta_x)\tilde{T}^{-1}(\tilde{T}(m)) \\ &= \bar{\alpha}(\psi^{-1}(x))\delta_{\psi^{-1}(x)} m. \end{aligned}$$

We shall say that a multiplier $L : C_0(G)^\perp \to C_0(G)^\perp$ is ***given by a point-mass*** if there exist $y \in G$ and $\gamma \in \mathbb{T}$ such that $L(m) = \gamma\delta_y m$, for all $m \in C_0(G)^\perp$. The above argument shows that left-point multipliers associated with "canonical form" isomorphisms $T = j^*_{\alpha,\psi}$ (and their inverses) are given by point-masses. We now prove the converse of this statement.

Theorem 2.2.12. *Let G and H be locally compact groups and $T : C_0(G)^\perp \to C_0(H)^\perp$ be a weak-star continuous isometric isomorphism. Suppose that the left-point multipliers associated with T and T^{-1} are given by point-masses. Then T takes the canonical form $T = j^*_{\beta,\gamma}$ for a character β on G and a topological isomorphism $\gamma : G \to H$. In particular, T extends to a topological isomorphism of $LUC(G)^*$ onto $LUC(H)^*$ and G and H are topologically isomorphic.*

Proof. For each $x \in H$, let $\psi(x) \in G$ and $\alpha(x) \in \mathbb{T}$ be such that $L_x(m) = \alpha(x)\delta_{\psi(x)} m$, for all $m \in C_0(G)^\perp$. Suppose that $z \in G^*$ is right cancellable in $LUC(G)^*$. First we show that both α and ψ are multiplicative. To see this, let $x, y \in H$. Then, we have that

$$\begin{aligned} \alpha(xy)\delta_{\psi(xy)}\delta_z = L_{xy}(\delta_z) &= T^{-1}(\delta_{xy}T(\delta_z)) \\ &= T^{-1}[\delta_x T\left(T^{-1}(\delta_y T(\delta_z))\right)] \\ &= \alpha(x)\delta_{\psi(x)}T^{-1}(\delta_y T(\delta_z)) \\ &= \alpha(x)\alpha(y)\delta_{\psi(x)}\delta_{\psi(y)}\delta_z = \alpha(x)\alpha(y)\delta_{\psi(x)\psi(y)}\delta_z, \end{aligned}$$

so $\alpha(xy) = \alpha(x)\alpha(y)$ and $\psi(xy) = \psi(x)\psi(y)$. Now we show that $\psi : H \to G$ is continuous. Suppose that $x_\gamma \to x$ in H. Since T (and therefore T^{-1}) is weak-star continuous, we have that $L_{x_\gamma}(\delta_z) \to L_x(\delta_z)$ and so $\alpha(x_\gamma)\delta_{\psi(x_\gamma)}\delta_z \xrightarrow{w^*} \alpha(x)\delta_{\psi(x)}\delta_z$ in $C_0(G)^\perp$. Evaluating at the constant function 1 in $LUC(G)$, we see that $\alpha(x_\gamma) \to \alpha(x)$, and therefore $\delta_{\psi(x_\gamma)}\delta_z \xrightarrow{w^*} \delta_{\psi(x)}\delta_z$ in $C_0(G)^\perp$. It follows from Lemma 2.2.4 that $\delta_{\psi(x_\gamma)} \xrightarrow{w^*} \delta_{\psi(x)}$. Thus, ψ is continuous. To see that ψ is a bijection, we note that given $y \in G$, $\gamma(y) \in H$ and $\beta(y) \in \mathbb{T}$ are such that

$$\beta(y)\delta_{\gamma(y)}m = T(\delta_y T^{-1}(m)) \quad (m \in C_0(H)^\perp),$$

then the above argument shows that γ and β are also continuous maps. Letting $x \in G$ and $m = T^{-1}(\delta_z) \in C_0(G)^\perp$, we obtain

$$\begin{aligned} \beta(\psi(x))\delta_{\gamma(\psi(x))z} = \beta(\psi(x))\delta_{\gamma(\psi(x))}\delta_z &= T(\delta_{\psi(x)}T^{-1}(\delta_z)) \\ &= T\left(\overline{\alpha(x)}\alpha(x)\delta_{\psi(x)}m\right) \\ &= \overline{\alpha(x)}T(L_x(m)) \\ &= \overline{\alpha(x)}T\left(T^{-1}(\delta_x T(m))\right) \\ &= \overline{\alpha(x)}\delta_x\delta_z = \overline{\alpha(x)}\delta_{xz}. \end{aligned}$$

Hence $\gamma = \psi^{-1}$ (and $\beta \circ \psi = \bar{\alpha}$; equivalently $\beta = \bar{\alpha} \circ \psi^{-1}$). Now we show that T has the desired canonical form. To see this, suppose that $p \in H^{LUC}$. Let (x_i) be a net in H

converging to p. We have that

$$\begin{aligned}\tilde{\alpha}(p)\delta_{\tilde{\psi}(p)}m &= \lim_i \alpha(x_i)\delta_{\psi(x_i)}m \\ &= \lim_i T^{-1}(\delta_{x_i}T(m)) \\ &= T^{-1}(\delta_p T(m)) = T^{-1}(\delta_p)m,\end{aligned}$$

for all $m \in C_0(G)^\perp$. Taking $m = z \in G^*$, a right cancellable element in G^{LUC}, we have that $T^{-1}(\delta_p) = \tilde{\alpha}(p)\delta_{\tilde{\psi}(p)}$. So $T^{-1}|_{G^*} = j^*_{\alpha,\psi}|_{G^*}$ and, since T^{-1} is weak-star continuous, we have that $T^{-1} = j^*_{\alpha,\psi}|_{C_0(G)^\perp}$. It follows that T has the canonical form $T = j^*_{\tilde{\alpha}\circ\psi^{-1},\psi^{-1}}|_{C_0(G)^\perp} = j^*_{\beta,\gamma}|_{C_0(G)^\perp}$, and therefore extends to $LUC(G)^*$. $\square$

It can be shown that if G and H are locally compact groups then any isometric (algebra) isomorphism $T : M(G) \to M(H)$ is weak-star continuous. It is thus natural to ask if every isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$, equivalently $T : M(G^{LUC}) \to M(H^{LUC})$, is also weak-star continuous. In fact, this question was the motivation for the results we obtained in this section. We have shown that any such isometric isomorphism maps $C_0(G)^\perp$ onto $C_0(H)^\perp$, so a related question is if every isometric isomorphism $T : C_0(G)^\perp \to C_0(H)^\perp$ is weak-star continuous. Similar questions can be asked about algebraic isomorphisms $\phi : G^{LUC} \to H^{LUC}$ and $\varphi : G^* \to H^*$.

2.3 Double Centralizer Algebra $\mathcal{D}(C_0(G)^\perp)$

We conclude with a discussion of the multipliers on $C_0(G)^\perp$, which may be useful with regard to the problems described above and which we think is of independent interest. We recall that a double centralizer of a Banach algebra $\mathcal{A}$ is a pair (L, R) of operators on $\mathcal{A}$

such that L is a left multiplier, R is a right multiplier and

$$mL(n) = R(m)n \quad (m, n \in \mathcal{A}).$$

The Banach algebra $\mathcal{D}(\mathcal{A})$ of continuous double centralizers of $\mathcal{A}$ is the set of all double centralizers (L, R) on $\mathcal{A}$ such that L and R are continuous, with the norm given by

$$\|(L, R)\| = \max\{\|L\|, \|R\|\},$$

($\|L\|$ and $\|R\|$ are the operator norms of L and R respectively), with pointwise linear operations, and multiplication

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1) \quad ((L_1, R_1), (L_2, R_2) \in \mathcal{D}(\mathcal{A}))\,.$$

Since $LUC(G)^*$ is unital, a simple observation shows that every left multiplier L on $LUC(G)^*$ is of the form $L(n) = m_0 n$:

$$L(n) = L(\delta_e n) = L(\delta_e)n.$$

So, in this case the left multipliers are easily characterized. The left multiplier L is weak-star continuous if and only if $m_0 \in M(G)$ (see [Lau86, Cor.3]). Also, $L : LUC(G)^* \to LUC(G)^*$ is an invertible isometric left multiplier if and only if $m_0 = \alpha\delta_x$, $\alpha \in \mathbb{T}$, $x \in G$ (see [GLL90, Cor.1.2]). Therefore, if we could show that every left-point multiplier on $C_0(G)^\perp$ associated with a weak-star continuous isometric isomorphism $T : C_0(G)^\perp \to C_0(G)^\perp$ extends to an isometric invertible left multiplier on $LUC(G)^*$, then every left-point multiplier on $C_0(G)^\perp$ must be given by a point-mass; by Theorem 2.2.12 we could then conclude that $C_0(G)^\perp$ determines G. Similarly, every right multiplier R on $LUC(G)^*$ is of the form $R(n) = nm_0$, for some $m_0 \in LUC(G)^*$, and therefore, every right multiplier on $LUC(G)^*$ is weak-star continuous. As shown in Proposition 2.3.1 below, the right-point multipliers R_x associated with $T : C_0(G)^\perp \to C_0(H)^\perp$ (as defined above) are also always weak-star continuous (without assuming that T is weak-star continuous).

When $T : C_0(G)^\perp \to C_0(H)^\perp$ is an isomorphism and $x \in H$, it is readily verified that the pair (L_x, R_x) is a double centralizer of $C_0(G)^\perp$ (i.e. $mL_x(n) = R_x(m)n$ for all $m, n \in C_0(G)^\perp$). We note that if (L, R) is a double centralizer of $C_0(G)^\perp$ such that $L(n) = m_0 n$, for some $m_0 \in LUC(G)^*$, then using a right cancellable element $z \in C_0(G)^\perp$ - see Lemma 2.2.4 - we have that $R(n) = nm_0$:

$$R(n)z = nL(z) = nm_0 z.$$

Proposition 2.3.1. *Suppose that G and H are locally compact groups and $T : C_0(G)^\perp \to C_0(H)^\perp$ is an isometric isomorphism. Then for each $x \in H$, the isometric right multiplier $R_x(m) = T^{-1}(T(m)\delta_x)$ is weak-star continuous.*

Proof. By [ER00, page 335], it is enough to show that R_x is weak-star continuous on the unit ball of $C_0(G)^\perp$. Suppose that $n_i \xrightarrow{w^*} n$ in $C_0(G)^\perp$, with (n_i) bounded in norm by 1. Clearly $(R_x(n_i))$ is also bounded by 1. Suppose that $z \in G^*$, is right cancellable in G^{LUC}. We have that

$$\begin{aligned} R_x(n_i)\delta_z &= n_i L_x(\delta_z) \\ &\xrightarrow{w^*} nL_x(\delta_z) = R_x(n)\delta_z. \end{aligned}$$

By Lemma 2.2.4, $R_x(n_i) \to R_x(n)$ weak-star in $C_0(G)^\perp$, as needed. $\square$

Another related interesting question is whether there is a characterization for the double centralizer algebra on $C_0(G)^\perp$. Unlike, $LUC(G)^*$ and $L^1(G)$, $C_0(G)^\perp$ does not have an identity element or an approximate identity.

Proposition 2.3.2. *The right topological semigroup G^* does not contain a left approximate identity, and hence does not have a left identity.*

Proof. Suppose first that E is a left identity for G^*. Then the mapping $m \mapsto Em;\ G^* \to G^*$ is the identity mapping on G^* and therefore continuous. But this is not possible since by [PP01, Theorem 2], the topological centre of G^* is empty. Hence G^* has no left identity.

Suppose now that (y_i) is a left approximate identity for G^*. Then, since G^* is compact, (y_i) has a subnet convergent to an element y in G^*. But any such y is clearly a left identity for G^*, a contradiction. □

Proposition 2.3.3. *Let G be a locally compact group. Then the annihilator of $C_0(G)^\perp$ in $LUC(G)^*$ is zero.*

Proof. Since the annihilator of $C_0(G)^\perp$ in $LUC(G)^*$ is the intersection of its left and right annihilators, it suffices to show that the left annihilator is zero. Let m belong to the left annihilator of $C_0(G)^\perp$ in $LUC(G)^*$. Let $z \in G^*$ be a right cancellable element in G^{LUC}. Then

$$mz = 0 = 0z,$$

so $m = 0$ by Lemma 2.2.4. □

Noting that $C_0(G)^\perp$ is a closed ideal in $LUC(G)^*$, there is a natural embedding of $LUC(G)^*$ into $\mathcal{D}(C_0(G)^\perp)$, the double centralizer algebra of $C_0(G)^\perp$ (see [Pal94, Prop.1.2.6]). Does $LUC(G)^* = \mathcal{D}(C_0(G)^\perp)$? Is there any characterization of $\mathcal{D}(C_0(G)^\perp)$?

Chapter 3

Isometric Isomorphisms of Beurling Algebras

Many works have been devoted to the study of Beurling algebras (on locally compact groups). A natural question asks to what extent does the algebraic structure of the weighted convolution Banach algebra $L^1(G, \omega)$ reflect its underlying topological group structure. It was observed by J. G. Wendel that in general, the algebraic structure of the group algebra does not determine the underlying topological group. However, if we impose some restriction on the algebraic isomorphism, success may be achieved. In this chapter I will show that the Banach algebraic structure of many Beurling algebras determine their underlying weighted topological group.

3.1 Some Preliminary Results

We devote this section to some preliminary results that are needed for our work in this chapter. Throughout, G denotes a locally compact topological group with a fixed left Haar measure dx. A **weight** on G is a positive continuous function $\omega : G \to \mathbb{R}^+$ such that $\omega(xy) \leq \omega(x)\omega(y)$, for all x, $y \in G$. In this chapter we require for a weight function ω that $\omega(e_G) = 1$, where e_G is the identity element in G. We do not need this restriction on the weight function for our results in Chapter 4. By a **weighted locally compact group**, we mean a pair $(\mathbf{G}, \omega)$ where G is a locally compact group and ω is a weight function on G.

The following lemma reveals a property of the weight ω that is very useful in our work especially when we need a uniform approximation.

Lemma 3.1.1. *For every weight ω on a locally compact group we have that*

$$\left|\frac{\omega(tx) - \omega(x)}{\omega(x)}\right| \xrightarrow{uniformly} 0 \text{ as } t \to e_G;$$

that is, for any given $\epsilon > 0$, there is a neighbourhood U of the identity such that for each $x \in G$ and $t \in U$ we have that $\left|\dfrac{\omega(tx)}{\omega(x)} - 1\right| < \epsilon$.

Proof. First we note that by the submultiplicativity of the weight ω, we have that for any $x, t \in G$

$$\frac{1}{\omega(t^{-1})} \leq \frac{\omega(tx)}{\omega(x)} \leq \omega(t).$$

Now, since $\omega(e_G) = 1$ and ω and $\dfrac{1}{\omega}$ are continuous at e_G, the result follows. $\square$

By Lemma 1.1.23 and Example 1.1.3 parts (ii) and (iii), $M(G, \omega)$ acts by convolution on the ideal $L^1(G, \omega)$ on the left. Dualizing this action gives us a right action of $M(G, \omega)$

on $L^\infty(G,\omega^{-1})$. Explicitly, we have the module action

$$\langle g\cdot\mu,\phi\rangle = \langle g,\mu*\phi\rangle,$$

where $\mu\in M(G,\omega)$ and $g\in L^\infty(G,\omega^{-1})$ and $\phi\in L^1(G,\omega)$. It is not hard to see that for each $g\in L^\infty(G,\omega^{-1})$ and $x\in G$

$$g\cdot\delta_x = l_x g,$$

where $l_x g(y)=g(xy)$, for all $y\in G$.

We proceed by giving a weighted analogue of Lemma 1.1.11 part (a). For this we need the following lemma.

Lemma 3.1.2. *Let (G,ω) be a locally compact weighted group. Then*

$$LUC(G,\omega^{-1}) = \{f\in L^\infty(G,\omega^{-1}); t\mapsto f\cdot\delta_t \text{ is norm continuous at } e_G\}.$$

Proof. See [DL05, Prop.7.15]. □

The following remark is used several times in our work throughout this Chapter.

Remark 3.1.3. Let $\omega\mu\in M(G)$ be defined by $d(\omega\mu)(x)=\omega(x)d\mu(x)$. Then the mapping $\mu\mapsto\omega\mu$ is a weak-star continuous isometric linear isomorphism from $M(G,\omega)$ onto $M(G)$ (Proposition 1.1.22).

Let (μ_α) be a net in $M(G,\omega)$. We say that (μ_α) converges to some μ in $M(G,\omega)$ in the strong operator topology, SO-topology, if $\|\mu_\alpha*\psi-\mu*\psi\|\to 0$, for all $\psi\in L^1(G,\omega)$.

Lemma 3.1.4. *The map $x\mapsto \dfrac{1}{\omega(x)}\delta_x$ from G into $M(G,\omega)$ is strong operator continuous.*

Proof. See [Gha84a, Lemma 1]. □

Lemma 3.1.5. *Let (G,ω) be a weighted locally compact group. For each $\psi \in L^1(G,\omega)$ the map $x \mapsto \psi * \delta_x : \ G \to (L^1(G,\omega), \|\cdot\|_{1,\omega})$ is continuous.*

Proof. Let $\psi \in L^1(G,\omega)$, (x_i) a net in G such that $x_i \to x$ in G. Let U be a fixed compact neighbourhood of x and assume without loss of generality that each x_i belongs to U.

We first consider the case when $\psi \in C_c(G)$. Let K denote the support of ψ and let $M := \sup\{\omega(x) : x \in UK\}$. Then,

$$\begin{aligned}
\|\psi * \delta_{x_i} - \psi * \delta_x\|_{1,\omega} &= \int_G \left|\psi(tx_i^{-1})\Delta(x_i^{-1}) - \psi(tx^{-1})\Delta(x^{-1})\right| \ \omega(t)\ dt \\
&\leq M \int_{UK} \left|\psi(tx_i^{-1})\Delta(x_i^{-1}) - \psi(tx^{-1})\Delta(x^{-1})\right| \ dt \\
&= M\|\psi * \delta_{x_i} - \psi * \delta_x\|_1 \xrightarrow{x_i \to x} 0,
\end{aligned}$$

because $C_c(G) \subseteq L^1(G,\omega) \cap L^1(G)$.

Suppose now that ψ is an arbitrary element of $L^1(G,\omega)$ and $\epsilon > 0$ is given. Let $L := \sup\{\omega(x) : \ x \in U\}$ and using Lemma 1.1.21, choose $\psi_0 \in C_c(G)$ such that $\|\psi - \psi_0\|_{1,\omega} \leq \frac{\epsilon}{3L}$. From the above case, we can take i_0 such that

$$\|\psi_0 * \delta_{x_i} - \psi_0 * \delta_x\|_{1,\omega} \leq \frac{\epsilon}{3},$$

whenever $i \geq i_0$. For $i \geq i_0$, we now have

$$\begin{aligned}
\|\psi * \delta_{x_i} - \psi * \delta_x\|_{1,\omega} &\leq \|\,(\psi - \psi_0) * \delta_{x_i}\|_{1,\omega} + \|\psi_0 * \delta_{x_i} - \psi_0 * \delta_x\|_{1,\omega} + \|\,(\psi_0 - \psi) * \delta_x\|_{1,\omega} \\
&\leq \|\psi - \psi_0\|_{1,\omega}\|\delta_{x_i}\|_\omega + \|\psi_0 * \delta_{x_i} - \psi_0 * \delta_x\|_{1,\omega} + \|\psi_0 - \psi\|_{1,\omega}\|\delta_x\|_\omega \\
&\leq \omega(x_i)\|\psi - \psi_0\|_{1,\omega} + \frac{\epsilon}{3} + \omega(x)\|\psi - \psi_0\|_{1,\omega} \leq \epsilon,
\end{aligned}$$

as needed.

∎

Lemma 3.1.6. *Let (G,ω) be a weighted locally compact group.*

(a) For every $f \in L^\infty(G,\omega^{-1})$ and $\psi \in L^1(G,\omega)$ we have

$$f\Box\psi(x) = \int_G f(yx)\psi(y)dy,$$

locally almost everywhere.

(b) $L^\infty(G,\omega^{-1})\Box L^1(G,\omega) = LUC(G,\omega^{-1})$.

Proof. (a) As noted in [DL05, page 77], part (a) is a straightforward calculation.

(b) We first show that for every $f \in L^\infty(G,\omega^{-1})$ and $\psi \in L^1(G,\omega)$, $f\Box\psi$ belongs to $LUC(G,\omega^{-1})$. Suppose that $t_i \to e_G$ in G. Then

$$\begin{aligned}\| (f\Box\psi)\cdot\delta_{t_i} - (f\Box\psi)\cdot\delta_e\|_{\infty,\omega} &= \|f\Box\left(\psi*\delta_{t_i} - \psi*\delta_{e_G}\right)\|_{\infty,\omega}\\ &\leq \|f\|_{\infty,\omega}\|\psi*\delta_{t_i} - \psi*\delta_{e_G}\|_{1,\omega} \to 0,\end{aligned}$$

by Lemma 3.1.5. By Lemma 3.1.2, $f\Box\psi \in LUC(G,\omega^{-1})$.

Now, we show that $LUC(G,\omega^{-1}) \subseteq L^\infty(G,\omega^{-1})\Box L^1(G,\omega)$. Let (U_i) be a shrinking neighbourhood base of the identity element of G, e_G, where all U_i's are contained in a compact neighbourhood U of e_G. Let $f_i := \dfrac{\chi_{U_i}}{\lambda(U_i)}$, where λ denotes the Haar measure on G. It is easy to see that each f_i is an element of $L^1(G,\omega)$. We now show that for each $f \in LUC(G,\omega^{-1})$, $f\Box f_i \to f$ in $\|.\|_{\infty,\omega}-$norm and so the result follows from Cohen's factorization theorem. To see this, let $\epsilon > 0$ be given. By Lemma 3.1.1, and since $f/\omega \in LUC(G)$, there exists i_0 such that for each $i \geq i_0$ we have that

$$\left|\frac{\omega(yx)-\omega(x)}{\omega(x)}\right| \leq \epsilon \quad \text{and} \quad \left|\frac{f(yx)}{\omega(yx)} - \frac{f(x)}{\omega(x)}\right| \leq \epsilon,$$

for all $y \in U_i$ and $x \in G$. Now we note that for all $i \geq i_0$ and $x \in G \setminus L$, where L is a

$\lambda-$locally null set, we have

$$\begin{aligned}
\left|\frac{f\Box f_i(x)-f(x)}{\omega(x)}\right| &= \frac{1}{\omega(x)}\left|\int_{U_i}\frac{f(yx)}{\lambda(U_i)}\,dy - f(x)\right| \\
&= \frac{1}{\lambda(U_i)}\left|\int_{U_i}\left(\frac{f(yx)}{\omega(yx)}\frac{\omega(yx)}{\omega(x)} - \frac{f(x)}{\omega(x)}\right)dy\right| \\
&\le \frac{1}{\lambda(U_i)}\left[\int_{U_i}\left|\frac{f(yx)}{\omega(yx)}\left(\frac{\omega(yx)}{\omega(x)}-1\right)dy\right| + \int_{U_i}\left|\frac{f(yx)}{\omega(yx)} - \frac{f(x)}{\omega(x)}\right|dy\right] \\
&\le (1+\|f\|_{\omega,\infty})\epsilon;
\end{aligned}$$

here, we have used part (a). As $G\setminus L$ is necessarily dense in G, and $\dfrac{1}{\omega}(f\Box f_i - f)$ is continuous on G, the above inequality holds for all $x\in G$. Therefore, $f\Box f_i \xrightarrow{\|.\|_{\infty,\omega}} f$. $\Box$

It follows from Lemma 3.1.6 that $LUC(G,\omega^{-1})$ is a left introverted subspace of $L^\infty(G,\omega^{-1}) = L^1(G,\omega)^*$. Indeed, given $n\in LUC(G,\omega^{-1})^*$ and $f\in L^\infty(G,\omega^{-1})$, then $n\Box f\in L^\infty(G,\omega^{-1}) = L^1(G,\omega)^*$ where

$$\langle n\Box f,\psi\rangle = \langle n, f\Box\psi\rangle \quad (\psi\in L^1(G,\omega)).$$

Therefore, given $g = f\Box\psi \in LUC(G,\omega^{-1})$ – where $f\in L^\infty(G,\omega^{-1})$ and $\psi\in L^1(G,\omega)$ – $n\Box g = (n\Box f)\Box\psi \in L^\infty(G,\omega^{-1})\Box L^1(G,\omega) = LUC(G,\omega^{-1})$. As noted in Chapter 1, $LUC(G,\omega^{-1})^*$ is thus a Banach algebra with left Arens product defined by

$$\langle m\Box n, f\rangle = \langle m, n\Box f\rangle,$$

where $m,n\in LUC(G,\omega^{-1})^*$, $f\in LUC(G,\omega^{-1})$.

Note that the Banach algebras $LUC(G)^*$ and $LUC(G,\omega^{-1})^*$ are isomorphic if and only if ω is multiplicative. We can embed $M(G,\omega)$ isometrically as a Banach algebra into $LUC(G,\omega^{-1})^*$ via the natural embedding $\mu\mapsto\int_G f d\mu$, $f\in LUC(G,\omega^{-1})$ and $\mu\in M(G,\omega)$.

Lemma 3.1.7. *Let (G,ω) be a weighted locally compact group. Then for each $\mu \in M(G,\omega)$, the mapping $m \mapsto \mu\Box m; LUC(G,\omega^{-1})^* \to LUC(G,\omega^{-1})^*$ is weak-star continuous.*

Proof. It follows from the factorization $LUC(G,\omega^{-1}) = L^\infty(G,\omega^{-1})\Box L^1(G,\omega)$ that $LUC(G,\omega^{-1})$ is a Banach $M(G,\omega)-$submodule of $L^\infty(G,\omega^{-1})$. For $\mu \in M(G,\omega)$, we denote the adjoint of $f \mapsto f\cdot\mu; LUC(G,\omega^{-1}) \to LUC(G,\omega^{-1})$ by $m \mapsto \mu\cdot m; LUC(G,\omega^{-1})^* \to LUC(G,\omega^{-1})^*$. So, to establish weak-star continuity of $m \mapsto \mu\Box m$, it suffices to show that $\mu\Box m = \mu \cdot m$, for all $m \in LUC(G,\omega^{-1})^*$. To this end, first let $\mu = \psi \in L^1(G,\omega)$. We observe that, if $f \in LUC(G,\omega^{-1})$, then $f\Box\psi = f \cdot \psi$, so

$$\begin{aligned}
\langle \psi\Box m, f\rangle &= \langle \psi, m\Box f\rangle \\
&= \langle m\Box f, \psi\rangle \\
&= \langle m, f\Box\psi\rangle \\
&= \langle m, f\cdot\psi\rangle \\
&= \langle \psi\cdot m, f\rangle.
\end{aligned}$$

Hence, $\psi\Box m = \psi \cdot m$. Now let $\mu \in M(G,\omega)$, $m \in LUC(G,\omega^{-1})^*$, and $f \in LUC(G,\omega^{-1})$. Then $f = g\Box\psi = g\cdot\psi$, for some $g \in L^\infty(G,\omega^{-1})$ and $\psi \in L^1(G,\omega)$. Hence,

$$\begin{aligned}
\langle \mu\Box m, f\rangle &= \langle \mu\Box m, g\cdot\psi\rangle \\
&= \langle \psi\cdot(\mu\Box m), g\rangle \\
&= \langle \psi\Box(\mu\Box m), g\rangle \\
&= \langle (\psi * \mu)\Box m, g\rangle \\
&= \langle (\psi * \mu)\cdot m, g\rangle \\
&= \langle \psi\cdot(\mu\cdot m), g\rangle \\
&= \langle \mu\cdot m, g\cdot\psi\rangle = \langle \mu\cdot m, f\rangle.
\end{aligned}$$

Hence, $\mu\Box m = \mu\cdot m$. $\square$

Lemmas 3.1.8 and 3.1.9 below are adaptation of Lemmas 1.1.2 and 1.1.3 of [Gre65] to Beurling algebras. When (G,ω) is a weighted locally compact group and A is a subset of G, we let $\mathcal{E}_A := \{\frac{\delta_x}{\omega(x)} : \ x \in A\}$. Below we let $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$.

Lemma 3.1.8. *Suppose that Q is a compact set in a weighted locally compact group (G,ω). Then the strong operator closure and the weak-star closure of the convex hulls of $\mathbb{T}\mathcal{E}_Q := \{\gamma\frac{\delta_x}{\omega(x)} : \gamma \in \mathbb{T}, \ x \in G\}$ coincide, and they are equal to*

$$\{\mu \in M(G,\omega) : \ \|\mu\|_\omega \leq 1, \ s(\mu) \subset Q\}.$$

On these sets the weak-star and strong operator topologies are the same.

Proof. Let $\mathrm{co}[\mathbb{T}\mathcal{E}_Q]$ denote the convex hull of $\mathbb{T}\mathcal{E}_Q$. We first show that $\mathbb{T}\mathcal{E}_Q$ is SO-compact. To see this, let $\left(\lambda_\alpha \frac{\delta_{x_\alpha}}{\omega(x_\alpha)}\right)$ be a net in $\mathbb{T}\mathcal{E}_Q$. Then (λ_α) and (x_α) are nets in the compact sets $\mathbb{T}$ and Q, respectively. So there exist subnets (λ_{α_i}) and (x_{α_i}) and $\lambda \in \mathbb{T}$, $x \in Q$ such that

$$\lambda_{\alpha_i} \to \lambda \text{ in } \mathbb{T} \quad \text{ and } \quad x_{\alpha_i} \to x \text{ in } Q.$$

By Lemma 3.1.4, $\frac{\delta_{x_{\alpha_i}}}{\omega(x_{\alpha_{x_i}})} \xrightarrow{SO} \frac{\delta_x}{\omega(x)}$, from which $\lambda_{\alpha_i}\frac{\delta_{x_{\alpha_i}}}{\omega(x_{\alpha_i})} \xrightarrow{SO} \lambda\frac{\delta_x}{\omega(x)}$ follows. Indeed, given $\psi \in L^1(G,\omega)$

$$\begin{aligned}\|\lambda_{\alpha_i}\frac{\delta_{x_{\alpha_i}}}{\omega(x_{\gamma_i}} * \psi - \lambda\frac{\delta_x}{\omega(x)} * \psi\|_{1,\omega} \leq |\lambda_{\alpha_i}| \, \| \frac{\delta_{x_{\alpha_i}}}{\omega(x_{\alpha_i})} * \psi - \frac{\delta_x}{\omega(x)} * \psi\|_{1,\omega} \\ + |\lambda_{\alpha_i} - \lambda| \, \| \frac{\delta_x}{\omega(x)} * \psi\|_{1,\omega} \to 0.\end{aligned}$$

This shows that $\mathbb{T}\mathcal{E}_Q$ is SO-compact, and it follows from [DS58, Exercise VI.9.3] that $\overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_Q]$, the SO-closed convex hull of $\mathbb{T}\mathcal{E}_Q$, is also SO-compact. By Proposition A.0.4, $\overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_Q]$ is also weak-star compact and therefore weak-star closed, so $\overline{\mathrm{co}}^{w^*}[\mathbb{T}\mathcal{E}_Q] \subseteq \overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_Q]$. But by Proposition A.0.4, weak-star topology is contained in the strong operator topology, so $\overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_Q] \subseteq \overline{\mathrm{co}}^{w^*}[\mathbb{T}\mathcal{E}_Q]$. Hence

$$\overline{\mathrm{co}}^{w^*}[\mathbb{T}\mathcal{E}_Q] = \overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_Q];$$

moreover, the strong operator and weak-star topologies agree on this set, also by Proposition A.0.4. By [Gre65, Lemma 1.1.2],

$$\overline{\mathrm{co}}^{w^*}[\mathbb{T}\mathcal{E}'_Q] = \{\nu \in M(G) : \|\nu\| \leq 1 \text{ and } \mathrm{Supp}(\nu) \subseteq Q\}, \tag{3.1}$$

where $\mathcal{E}'_Q = \{\delta_x : \ x \in G\}$. By Remark 3.1.3, the map

$$M(G) \to M(G, \omega) : \nu \to \frac{1}{\omega}\nu$$

is a weak-star homeomorphic isometric isomorphism. As $\mathrm{Supp}(\nu) = \mathrm{Supp}(\frac{1}{\omega}\nu)$, applying this map to both sides of equation (3.1) yields

$$\overline{\mathrm{co}}^{w^*}[\mathbb{T}\mathcal{E}_Q] = \{\mu \in M(G, \omega) : \ \|\mu\|_\omega \leq 1, \text{ and } \ \mathrm{Supp}(\mu) \subseteq Q\},$$

as needed. □

The following lemma is an analogue of [Gre65, Lemma 1.1.3], for Beurling algebras.

Lemma 3.1.9. *Let (G, ω) be a weighted locally compact group. Then $\overline{co}^{SO}[\mathbb{T}\mathcal{E}_G]$ is the unit ball of $M(G, \omega)$.*

Proof. Let $\mu \in M(G, \omega)$ with $\|\mu\|_\omega \leq 1$. Then for $n = 1, 2, \ldots,$ we can choose μ_n in $M(G, \omega)$ with compact support K_n such that $\|\mu_n\|_\omega \leq 1$ and $\|\mu_n - \mu\|_\omega \to 0$. By Lemma 3.1.8, for each $n \in \mathbb{N}$,

$$\mu_n \in \overline{\mathrm{co}}^{\mathrm{SO}}[\mathbb{T}\mathcal{E}_{K_n}] \subseteq \overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_G].$$

Clearly, the SO-topology is weaker than the $\|\cdot\|_\omega$-topology, so the latter set is $\|\cdot\|_\omega$-closed. Hence, $\mu \in \overline{\mathrm{co}}^{SO}[\mathbb{T}\mathcal{E}_G]$. □

3.2 Isometric Isomorphisms of Beurling Measure Algebras

Let (G, ω) be a weighted locally compact group. A simple observation is that $M(G, \omega)$ always contains an algebraic copy of G, namely

$$\{\delta_g : g \in G\}.$$

Let τ denote the given topology on the locally compact group G. Then by complete regularity of $C_0(G)$, the topology on G agrees with the relative w^*-topology induced by $M(G)$ and so we have that

$$g_\alpha \to g \ \ in\ \tau \quad \Longleftrightarrow \quad f(g_\alpha) \to f(g) \quad \forall\ f \in C_0(G).$$

Let τ' denote the weak-star topology on G as a subset of $M(G, \omega)$; thus

$$g_\alpha \to g \quad in\ \tau' \quad \Longleftrightarrow \quad f(g_\alpha) \to f(g) \quad \forall\ f \in C_0(G, \omega^{-1}).$$

Lemma 3.2.1. *Let (G, τ) be a locally compact group, ω a weight on G. Then the topologies τ and τ' are the same.*

Proof. Suppose that $g_\alpha \to g$ in τ and let $f \in C_0(G, \omega^{-1})$. Then, f/ω is continuous on G, so $f = \dfrac{f}{\omega} \cdot \omega$ is also continuous on G, hence $f(g_\alpha) \to f(g)$. Similarly, we can show that if $g_\alpha \to g$ in τ' then $g_\alpha \to g$ in τ. □

Lemma 3.2.2. *Let (G, ω) be a weighted locally compact group. Then the set of extreme points of the unit ball of the measure algebra $M(G, \omega)$ is*

$$\left\{\gamma \frac{\delta_g}{\omega(g)} : \ g \in G, \ \gamma \in \mathbb{T}\right\},$$

where $\mathbb{T}$ is the circle group.

Proof. As noted in the introduction, the map $\mu \mapsto \omega\mu$ is an isometric linear isomorphism of $M(G,\omega)$ onto $M(G)$. The inverse map, $\nu \mapsto \dfrac{1}{\omega}\nu$, maps $\{\gamma\delta_g : \gamma \in \mathbb{T},\ g \in G\}$, the set of extreme points of the unit ball of $M(G)$ (see [Con90, Thm.V.8.4]) to $\{\gamma\dfrac{\delta_g}{\omega(g)} :\ g \in G,\ \gamma \in \mathbb{T}\}$, the set of extreme points of the unit ball of $M(G,\omega)$. □

Definition 3.2.3. We say that the weighted locally compact groups (G,ω_1) and (H,ω_2) are isomorphic if there is a topological group isomorphism $\phi : G \to H$ such that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative on G; we call such map ϕ an isomorphism of weighted locally compact groups.

Let $\phi : G \to H$ be an isomorphism of the weighted locally compact groups (G,ω_1) and (H,ω_2) and let $\gamma : G \to \mathbb{T}$ be a continuous character on G. Define the mapping

$$j_{\gamma,\phi} : C_0(H,\omega_2^{-1}) \to C_0(G,\omega_1^{-1}) \ \text{ where } \ j_{\gamma,\phi}(f) = \gamma\frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi.$$

Then it is not difficult to see that $j_{\gamma,\phi}$ is an isometric isomorphism mapping $C_0(H,\omega_2^{-1})$ onto $C_0(G,\omega_1^{-1})$. Hence, the dual mapping $T_{\gamma,\phi} := j^*_{\gamma,\phi} : M(G,\omega_1) \to M(H,\omega_2)$ is also an isometric isomorphism. We observe that $T_{\gamma,\phi}$ also preserves the convolution product. To see this, first note that

$$\langle j^*_{\gamma,\phi}(\delta_x), f\rangle = \langle \delta_x, j_{\gamma,\phi}(f)\rangle = \langle \gamma(x)\frac{\omega_1(x)}{\omega_2 \circ \phi(x)}\delta_{\phi(x)}, f\rangle \quad (f \in C_0(H,\omega_2^{-1})).$$

Since γ, ϕ and $\dfrac{\omega_1}{\omega_2 \circ \phi}$ are multiplicative, it can be readily seen that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is also multiplicative on $M(G,\omega_1)$, note that the linear span of point masses is weak-star dense in $M(G,\omega)$, the convolution product is separately weak-star continuous and $T_{\gamma,\phi} = j^*_{\gamma,\phi}$ is weak-star continuous.

Theorem 3.2.4. *Let (G,ω_1) and (H,ω_2) be weighted locally compact groups. Then the Banach algebras $M(G,\omega_1)$ and $M(H,\omega_2)$ are isometrically isomorphic if and only if the* *weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic. Moreover, if T is any isometric algebra isomorphism from $M(G, \omega_1)$ onto $M(H, \omega_2)$, then there exists a continuous character $\gamma : G \to \mathbb{T}$ and an isomorphism $\phi : G \to H$ of the weighted locally compact groups (G, ω_1) and (H, ω_2) such that for each $g \in G$, we have*

$$\frac{T(\delta_g)}{\omega_1(g)} = \frac{\gamma(g)}{\omega_2(\phi(g))}\delta_{\phi(g)}.$$

Proof. In light of the preceding argument, if the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic, then the weighted measure algebras $M(G, \omega_1)$ and $M(H, \omega_2)$ are isomorphic. We now establish the converse. Suppose that $T : M(G, \omega_1) \to M(H, \omega_2)$ is an isometric algebra isomorphism. Since $T : M(G, \omega_1) \to M(H, \omega_2)$ is an isometric linear isomorphism, it preserves the extreme points of the unit ball. So by Lemma 3.2.2, for each $g \in G$, there exists $\phi(g) \in H$ and $\gamma(g) \in \mathbb{T}$ such that

$$T\left(\frac{\delta_g}{\omega_1(g)}\right) = \gamma(g)\frac{\delta_{\phi(g)}}{\omega_2(\phi(g))}. \tag{3.2}$$

In what follows, we first show that $\phi : G \to H$ and $\gamma : G \to \mathbb{T}$ defined via the equation (3.2) are multiplicative, and that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative.

Since T is multiplicative, we have $T(\delta_{gh}) = T(\delta_g * \delta_h) = T(\delta_g) * T(\delta_h)$, and so

$$\frac{\omega_1(gh)\gamma(gh)}{\omega_2(\phi(gh))}\delta_{\phi(gh)} = \frac{\omega_1(g)\omega_1(h)\gamma(g)\gamma(h)}{\omega_2(\phi(g))\omega_2(\phi(h))}\ \delta_{\phi(g)\phi(h)}. \tag{3.3}$$

Since ω_2 is continuous, $C_c(H)$, the set of continuous compactly supported functions on H, is contained in $C_0(H, \omega_2)$. Taking $f \in C_c(H)$ such that $f(\phi(gh)) = f(\phi(g)\phi(h)) = 1$ gives

$$\frac{\omega_1(gh)\gamma(gh)}{\omega_2(\phi(gh))} = \frac{\omega_1(g)\omega_1(h)\gamma(g)\gamma(h)}{\omega_2(\phi(g))\omega_2(\phi(h))}. \tag{3.4}$$

Since $|\gamma| \equiv 1$, by taking the absolute value of both sides of the above equality, we get

$$\frac{\omega_1(gh)}{\omega_2(\phi(gh))} = \left(\frac{\omega_1(g)}{\omega_2(\phi(g))}\right)\left(\frac{\omega_1(h)}{\omega_2(\phi(h))}\right).$$

Therefore from equation (3.4), we have that

$$\gamma(gh) = \gamma(g)\gamma(h).$$

Equation (3.3) now implies that

$$\phi(gh) = \phi(g)\phi(h).$$

We now show that both $\phi : G \to H$ and $\gamma : G \to \mathbb{T}$ are continuous. Let $x_\alpha \to x$ in G. Then

$$\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \frac{1}{\omega_1(x)}\delta_x * \psi, \quad \left(\psi \in L^1(G, \omega_1)\right),$$

by Lemma 3.1.4. Since the net $\left(T(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha})\right)$ is bounded in $M(H, \omega_2)$, it has a subnet $\left(T(\frac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}})\right)$ converging weak-star to some $\mu \in M(H, \omega_2)$. For $\psi \in L^1(G, \omega_1)$, we have

$$T\left(\frac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}}\right) * T(\psi) \xrightarrow{w^*} \mu * T(\psi),$$

or equivalently

$$T\left(\frac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}} * \psi\right) \xrightarrow{w^*} \mu * T(\psi).$$

But,

$$T\left(\frac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}} * \psi\right) \xrightarrow{\|\cdot\|_{\omega_2}} T\left(\frac{1}{\omega(x)}\delta_x * \psi\right).$$

Hence

$$T\left(\frac{1}{\omega_1(x)}\delta_x * \psi\right) = \mu * T(\psi),$$

for all $\psi \in L^1(G, \omega_1)$. Thus,

$$\frac{1}{\omega_1(x)}\delta_x * \psi = T^{-1}(\mu) * \psi \quad \left(\psi \in L^1(G, \omega_1)\right) \cdot \tag{3.5}$$

Let (ψ_i) be an approximate identity as in [Gha84b, Lemma 2.1]. Then by equation (3.5), we have that

$$\frac{1}{\omega_1(x)}\delta_x * \psi_i = T^{-1}(\mu) * \psi_i. \tag{3.6}$$

Since by [Gha84b, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the weak-star topology of $M(G, \omega_1)$, taking the weak-star limit in (3.13) gives $\dfrac{1}{\omega_1(x)}\delta_x = T^{-1}(\mu)$, or equivalently $\mu = T(\frac{1}{\omega_1(x)}\delta_x)$. This argument also shows that every subnet of $\left(T(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha})\right)$ has a subnet that is weak-star convergent to $T(\dfrac{1}{\omega_1(x)}\delta_x)$ and hence

$$T\left(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha}\right) \xrightarrow{w^*} T\left(\frac{1}{\omega_1(x)}\delta_x\right).$$

That is,

$$\gamma(x_\alpha)\frac{1}{\omega_2(\phi(x_\alpha))}\delta_{\phi(x_\alpha)} \xrightarrow{w^*} \gamma(x)\frac{1}{\omega_2(\phi(x))}\delta_{\phi(x)}. \tag{3.7}$$

If $\phi(x_\alpha) \not\to \phi(x)$ then there is a compact neighbourhood K of $\phi(x)$ such that for each α there is $\alpha' \geq \alpha$ such that $\phi(x_{\alpha'}) \notin K$. Choosing $h \in C_c(H) \subseteq C_0(H, \omega_2^{-1})$ such that $h(\phi(x)) = 1$ and $supp(h) \subseteq K$ and then evaluating (3.7) at h, we obtain a contradiction. Hence $\phi(x_\alpha) \to \phi(x)$, which gives continuity of ϕ. Assuming without loss of generality that $(\phi(x_\alpha))$ is contained in a compact neighbourhood of $\phi(x)$, let $h \in C_c(H) \subseteq C_0(H, \omega_2^{-1})$ be such that for each α, $h(\phi(x_\alpha)) = 1$ and also $h(\phi(x)) = 1$. Then by evaluating (3.7) at h, we get that

$$\frac{\gamma(x_\alpha)}{\omega_2(\phi(x_\alpha))} \to \frac{\gamma(x)}{\omega_2(\phi(x))}.$$

Since $\omega_2 \circ \phi$ is continuous, $\omega_2\left(\phi(x_\alpha)\right) \to \omega_2(\phi(x))$. Therefore, $\gamma(x_\alpha) \to \gamma(x)$, and so γ is continuous.

To see that $\phi : G \to H$ is a homeomorphism, consider the isometric isomorphism $T^{-1} : M(H, \omega_2) \to M(G, \omega_1)$. By what we proved above, there is a continuous mapping $\beta : H \to G$ and a character $\gamma' : H \to \mathbb{T}$ such that

$$T^{-1}\left(\frac{\delta_{\phi(g)}}{\omega_2(\phi(g))}\right) = \frac{\gamma'(\phi(g))}{\omega_1(\beta(\phi(g)))}\delta_{\beta(\phi(g))}.$$

Hence,

$$\frac{\delta_g}{\omega_1(g)} = T^{-1}\left(T(\frac{\delta_g}{\omega_1(g)})\right) = \gamma(g)T^{-1}\left(\frac{\delta_{\phi(g)}}{\omega_2(\phi(g))}\right) = \gamma(g)\frac{\gamma'(\phi(g))}{\omega_1\left(\beta(\phi(g))\right)}\delta_{\beta(\phi(g))}.$$

It follows that $\beta(\phi(g)) = g$, and therefore ϕ is a homeomorphism. $\square$

Next we show that every isometric isomorphism $T : M(G, \omega_1) \to M(H, \omega_2)$ is weak-star continuous. Some of the ideas in the proof are based on the proofs of Lemmas 1.4 and 1.5 of [GLL90].

Theorem 3.2.5. *Let $T : M(G, \omega_1) \to M(H, \omega_2)$ be an isometric isomorphism. Then there exists an isomorphism of weighted locally compact groups $\phi : G \to H$ and a continuous character $\gamma : G \to \mathbb{T}$ such that $T = j^*_{\gamma,\phi}$. In particular, T is weak-star continuous.*

Proof. By Theorem 3.2.4 there exists an isomorphism of weighted locally compact groups $\phi : G \to H$ and a continuous character $\gamma : G \to \mathbb{T}$ such that

$$T(\delta_x) = \gamma(x)\frac{\omega_1(x)}{\omega_2 \circ \phi(x)}\delta_{\phi(x)} = j^*_{\gamma,\phi}(\delta_x), \tag{3.8}$$

for each $x \in G$. Letting $\mu \in M(G, \omega_1)$ be such that $\|\mu\|_{\omega_1} = 1$, it suffices to show that $T(\mu) = j^*_{\gamma,\phi}(\mu)$.

Taking μ in $M(G, \omega_1)$ with norm 1, by Lemma 3.1.9, we can find a net (μ_β) in the convex hull of the set $\{\gamma\frac{\delta_x}{\omega(x)} : \ x \in G, \gamma \in \mathbb{T}\}$ such that

$$\lim \|\mu_\beta * \psi - \mu * \psi\|_{\omega_1} = 0, \quad (\psi \in L^1(G, \omega)). \tag{3.9}$$

By equation (3.8),

$$T(\mu_\beta) = j^*_{\gamma,\phi}(\mu_\beta) \tag{3.10}$$

for each β.

We claim that $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$ in $M(H, \omega_2)$. To see this, let ν be a weak-star limit point of $(T(\mu_\beta))$ in $M(H, \omega_2)$ and let $\left(\mu_{\beta(i)}\right)$ be a subnet of (μ_β) such that $T(\mu_{\beta(i)}) \xrightarrow{w^*} \nu$. Observe that it suffices now to show that $\nu = T(\mu)$. To simplify notation, we can assume that $T(\mu_\beta) \xrightarrow{w^*} \nu$. Let $\psi \in L^1(G, \omega_1)$ be fixed. Then

$$\|T(\mu_\beta) * T(\psi) - T(\mu) * T(\psi)\|_{\omega_2} \to 0, \tag{3.11}$$

by equation (3.9). In particular, (3.11) implies that $T(\mu_\beta) * T(\psi) \xrightarrow{w^*} T(\mu) * T(\psi)$ in the weak-star topology of $M(H, \omega_2)$. Now since $T(\mu_\beta) \xrightarrow{w^*} \nu$, using separate weak-star continuity of the convolution product we have that $T(\mu_\beta) * T(\psi) \xrightarrow{w^*} \nu * T(\psi)$ in the weak-star topology of $M(H, \omega_2)$. Thus

$$T\left(\mu * \psi\right) = T(\mu) * T(\psi) = \nu * T(\psi),$$

and so

$$\mu * \psi = T^{-1}(\nu) * \psi, \quad \left(\psi \in L^1(G, \omega_1)\right). \tag{3.12}$$

Let (ψ_i) be a bounded approximate identity as in [Gha84b, Lemma 2.1]. Then by equation (3.12), we have that

$$\mu * \psi_i = T^{-1}(\nu) * \psi_i. \tag{3.13}$$

Since by [Gha84b, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the weak-star topology of $M(G, \omega_1)$, by taking the weak-star limit in (3.13) we have that $\mu = T^{-1}(\nu)$ and therefore $T(\mu) = \nu$. This gives the claim.

Finally, because $j^*_{\gamma,\phi}$ is weak-star continuous and $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$, equation (3.10) yields

$$T(\mu) = w^* - \lim T(\mu_\beta) = w^* - \lim j^*_{\gamma,\phi}(\mu_\beta) = j^*_{\gamma,\phi}(\mu). \qquad \square$$

3.3 Isometric Isomorphisms of Beurling Group Algebras

In this section, among other results, we shall show that every isometric isomorphism $T : L^1(G,\omega_1) \to L^1(H,\omega_2)$ extends to an isometric isomorphism $\tilde{T} : M(G,\omega_1) \to M(H,\omega_2)$. Therefore, if there exists an isometric isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$, then the weighted locally compact groups are isomorphic. To define the extension $\tilde{T} : M(G,\omega_1) \to M(H,\omega_2)$, we shall consider the left multiplier algebra of $L^1(G,\omega_1)$.

We recall that a linear operator L on the Banach algebra $\mathcal{A}$ is a left multiplier if

$$L(ab) = L(a)b \quad (a, b \in \mathcal{A}).$$

By [Joh64a], if the Banach algebra $\mathcal{A}$ has a bounded approximate identity then every left multiplier on $\mathcal{A}$ is continuous.

The space of all continuous left multipliers of a Banach algebra $\mathcal{A}$ equipped with composition of operators as product and with the operator norm is a Banach algebra, called the left multiplier algebra of $\mathcal{A}$. The reader is referred to [Pal94, Sections 1.2.1-1.2.7] for definitions and basic theorems regarding the left multipliers. We denote the Banach algebra of left multipliers of $L^1(G,\omega)$ by $\mathcal{M}(L^1(G,\omega))$. Since $L^1(G,\omega)$ is an ideal in $M(G,\omega)$, each $\mu \in M(G,\omega)$ defines a left multiplier $L_\mu : \psi \mapsto \mu * \psi$ on $L^1(G,\omega)$.

Lemma 3.3.1. *Let (G, ω) be a weighted locally compact group. Suppose that L is a left multiplier on $L^1(G, \omega)$. Then, there exists $\mu \in M(G, \omega)$ such that $L(\psi) = \mu * \psi$, $\psi \in L^1(G, \omega)$.*

Proof. See [Gha84b, Lemma 2.3]. □

The following result is also observed in [DL05, Thm.7.14] when the weight ω satisfies $\omega(x) \geq 1$ for all $x \in G$. We now establish the result for any given weight with a self contained proof.

Theorem 3.3.2. *Let (G, ω) be a weighted locally compact group. The left multiplier algebra of $L^1(G, \omega)$ is isometrically isomorphic to $M(G, \omega)$.*

Proof. Let $\theta : M(G, \omega) \to \mathcal{M}(L^1(G, \omega))$, $\theta(\mu) := L_\mu$. Then by Lemma 3.3.1, θ is a contractive surjective homomorphism.

Letting $\mu \in M(G, \omega)$, we complete the proof by showing that $\|L_\mu\| \geq \|\mu\|$. Let (U_i) be a shrinking neighbourhood system of the identity all contained in a compact neighbourhood of the identity element. Then the proof of [Gha84b, Lemma 2.3] shows that $\psi_i := \dfrac{\chi_{U_i}}{\lambda(U_i)}$ forms a bounded approximate identity for $L^1(G, \omega)$, where λ denotes the Haar measure on G. Then $L_\mu(\psi_i) \xrightarrow{w^*} \mu$ by [Gha84b, Lemma 2.2]. Given $\epsilon > 0$, let $h \in C_0(G, \omega^{-1})$ be such that $\|h\|_{\infty,\omega} = 1$ and $|\mu(h)| \geq \|\mu\|_\omega - \epsilon$. Then by the weak-star convergence, we have that $L_\mu(\psi_i)(h) \to \mu(h)$ and therefore,

$$\lim_i |L_\mu(\psi_i)(h)| = |\mu(h)| \geq \|\mu\| - \epsilon.$$

Now

$$|L_\mu(\psi_i)(h)| \leq \|L_\mu(\psi_i)\|_{1,\omega} \leq \|L_\mu\| \|\psi_i\|_{1,\omega} \leq \|L_\mu\| \frac{\|\chi_{U_i}\|_{1,\omega}}{\lambda(U_i)},$$

and continuity of ω at e gives $\lim_i \frac{\|\chi_{U_i}\|_{1,\omega}}{\lambda(U_i)} = \omega(e) = 1$. Therefore, $\lim_i |L_\mu(\psi_i)(h)| \leq \|L_\mu\|$. Thus, for any $\epsilon > 0$,

$$\|\mu\| - \epsilon \leq |\mu(h)| = \lim_i |L_\mu(\psi_i)(h)| \leq \|L_\mu\|.$$

Since we have already observed that $\|L_\mu\| \leq \|\mu\|$, we have that $\|L_\mu\| = \|\mu\|$. □

Proposition 3.3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then, for every Banach algebra isometric isomorphism T from $\mathcal{A}$ onto $\mathcal{B}$, there is a Banach algebra isometric isomorphism $\tilde{T}$ defined by*

$$\tilde{T}(L) = T \circ L \circ T^{-1},$$

mapping the left multiplier algebra $\mathcal{M}(\mathcal{A})$ onto $\mathcal{M}(\mathcal{B})$. Moreover, $\tilde{T}(L_a) = L_{T(a)}$ for all $a \in \mathcal{A}$.

Proof. Suppose that $T : \mathcal{A} \to \mathcal{B}$ is an isometric isomorphism. We shall first show that $\tilde{T} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ defined by $\tilde{T}(L) = T \circ L \circ T^{-1}$ is also an isometric isomorphism. A simple calculation shows that if L is a left multiplier on $\mathcal{A}$ then $T \circ L \circ T^{-1}$ is a left multiplier on $\mathcal{B}$, so the mapping $\tilde{T} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is well-defined. It is also readily checked that $\tilde{T}$ is a linear algebra isomorphism. Finally, note that since T is an isometry,

$$\|T \circ L \circ T^{-1}\| = \sup_{\|b\| \leq 1} \|T\left(L\left(T^{-1}(b)\right)\right)\| = \sup_{\|b\| \leq 1} \|L\left(T^{-1}(b)\right)\| = \|L\|,$$

so $\tilde{T}$ is also an isometry. □

Theorem 3.3.4. *Let (G, ω_1) and (H, ω_2) be weighted locally compact groups. Suppose that $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is an isometric isomorphism. Then the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic. Conversely, if the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic, then $L^1(G, \omega_1)$ is isometrically isomorphic to $L^1(H, \omega_2)$.*

Proof. The first statement follows from Proposition 3.3.3 and Theorems 3.2.4 and 3.3.2. For the converse, let $\phi : G \to H$ be a topological isomorphism such that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative on G. Then the isometric isomorphisms

$$j_\phi : C_0(H, \omega_2^{-1}) \to C_0(G, \omega_1^{-1}); f \mapsto \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi,$$

$$J_\phi : C_0(H) \to C_0(G); f \mapsto f \circ \phi,$$

$$\theta_G : C_0(G) \to C_0(G, \omega_1^{-1}); f \mapsto \omega_1 f,$$

and

$$\theta_H : C_0(H) \to C_0(H, \omega_2^{-1}); f \mapsto \omega_2 f,$$

satisfy

$$j_\phi \circ \theta_H = \theta_G \circ J_\phi.$$

It is well-known that J_ϕ^* maps $L^1(G)$ onto $L^1(H)$ – e.g. see [Sto11, Prop. 5.1 (iv)] for a general result– and we have already noted that $\theta_G^* : M(G, \omega_1) \to M(G); \mu \mapsto \omega_1 \mu$ maps $L^1(G, \omega_1)$ onto $L^1(G)$. Hence

$$\begin{aligned} j_\phi^* \left(L^1(G, \omega_1)\right) &= \left(\theta_H^{-1}\right)^* \circ J_\phi^* \circ \theta_G^*(L^1(G, \omega_1)) \\ &= L^1(H, \omega_2). \end{aligned}$$

Thus j_ϕ^* maps $L^1(G, \omega_1)$ onto $L^1(H, \omega_2)$ which, by the remarks preceding Theorem 3.2.4, is an isometric isomorphism of Banach algebras. $\square$

Corollary 3.3.5. *Let (G, ω_1) and (H, ω_2) be weighted locally compact groups and suppose that $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is an isometric isomorphism. Then there exists a continuous character $\gamma : G \to \mathbb{T}$ and an isomorphism of weighted locally compact groups $\phi : G \to H$ such that for all $\psi \in L^1(G, \omega_1)$*

$$T(\psi) = j_{\gamma,\phi}^*(\psi) = c\gamma \circ \phi^{-1} \left(\frac{\omega_1 \circ \phi^{-1}}{\omega_2}\right) \cdot \psi \circ \phi^{-1}$$

where c is a measure adjustment constant.

Proof. Let $T : L^1(G,\omega_1) \to L^1(H,\omega_2)$ be an isometric isomorphism, $\tilde{T} : M(G,\omega_1) \to M(H,\omega_2)$ the isometrically isomorphic extension that exists by Theorem 3.3.2 and Proposition 3.3.3. By Theorem 3.2.5, $\tilde{T} = j^*_{\gamma,\phi}$ where $\phi : G \to H$ is an isomorphism of weighted locally compact groups and $\gamma : G \to \mathbb{T}$ is a continuous character on G. As $\psi \mapsto \int_G \psi(\phi(x))dx$ defines a Haar integral on $L^1(H)$, there exists $c > 0$ such that the equation

$$\int_G \psi(\phi(x))dx = c\int_H \psi(y)dy \quad \left(\psi \in L^1(H)\right)$$

holds. Therefore, for $\psi \in L^1(G,\omega_1)$ and $f \in C_0(H,\omega_2^{-1})$ we have

$$\begin{aligned}
\langle \tilde{T}(\psi), f\rangle = \langle j^*_{\gamma,\phi}(\psi), f\rangle &= \int_G \gamma(x)\left(\frac{\omega_1(x)}{\omega_2 \circ \phi(x)}\right) f\circ\phi(x)\psi(x)dx \\
&= \int_H c\gamma\circ\phi^{-1}(y)\left(\frac{\omega_1\circ\phi^{-1}(y)}{\omega_2(y)}\right) f(y)\psi\circ\phi^{-1}(y)dy \\
&= \langle c\gamma\circ\phi^{-1}\left(\frac{\omega_1\circ\phi^{-1}}{\omega_2}\right)\psi\circ\phi^{-1}, f\rangle.
\end{aligned}$$

Hence

$$T(\psi) = \tilde{T}(\psi) = c\gamma\circ\phi^{-1}\left(\frac{\omega_1\circ\phi^{-1}}{\omega_2}\right)\psi\circ\phi^{-1}.$$

□

3.4 Isometric Isomorphisms of the Dual of Weighted $LUC-$functions

In this section, we will first show that if (G,ω_1) and (H,ω_2) are locally compact groups and $T : LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$ is an isometric isomorphism, then T maps $M(G,\omega_1)$ onto $M(H,\omega_2)$, thereby concluding that $LUC(G,\omega^{-1})^*$ determines the weighted locally compact group (G,ω), by Theorem 3.2.4. We will then use this result to prove that the Banach algebraic structure of $L^1(G,\omega)^{**}$, the bidual of the Beurling group algebra, also determines the weighted locally compact group (G, ω).

Let $\phi : G \to H$ be a continuous isomorphism of the weighted locally compact groups (G, ω_1) and (H, ω_2) and let $\gamma : G \to \mathbb{T}$ be a continuous character on G. Define the mapping

$$J_{\gamma,\phi} : LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1}) \quad \text{where} \quad J_{\gamma,\phi}(f) = \gamma \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi.$$

Then it is not difficult to see that $J_{\gamma,\phi}$ is an isometric linear isomorphism mapping $LUC(H, \omega_2^{-1})$ onto $LUC(G, \omega_1^{-1})$. Hence, the dual mapping $T_{\gamma,\phi} := J^*_{\gamma,\phi} : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$ is also an isometric linear isomorphism such that

$$T_{\gamma,\phi}(\delta_x) = \gamma(x) \frac{\omega_1(x)}{\omega_2 \circ \phi(x)} \delta_{\phi(x)}, \; x \in G. \tag{3.14}$$

We observe that $T_{\gamma,\phi}$ also preserves the product. To see this, first note that since γ, ϕ and $\dfrac{\omega_1}{\omega_2 \circ \phi}$ are multiplicative, it can be readily seen from the equation (3.14) that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is multiplicative on $LUC(G, \omega_1^{-1})^*$, note that the linear span of point masses is weak-star dense in $LUC(G, \omega_1^{-1})^*$; for each $n \in LUC(G, \omega_1^{-1})^*$, $m \mapsto m \Box n$ is weak-star continuous on $LUC(G, \omega_1^{-1})^*$; for each $\mu \in M(G, \omega_1)$, $n \mapsto \mu \Box n$ is weak-star continuous; and from equation (3.14), $T_{\gamma,\phi}$ maps the linear span of point masses in $LUC(G, \omega_1^{-1})^*$ into $M(H, \omega_2)$. So, if $m, n \in LUC(G, \omega_1^{-1})^*$ we can find nets (μ_i) and (ν_j) in the linear span of the point masses such that $w^* - \lim \mu_i = m$ and $w^* - \lim \nu_j = n$, and obtain

$$\begin{aligned} T_{\gamma,\phi}(m \Box n) &= T_{\gamma,\phi}\left(w^* - \lim_i \left(w^* - \lim_j (\mu_i \Box \nu_j) \right) \right) \\ &= w^* - \lim_i \left(w^* - \lim_j T_{\gamma,\phi} (\mu_i \Box \nu_j) \right) \\ &= w^* - \lim_i \left(w^* - \lim_j \left(T_{\gamma,\phi}(\mu_i) \Box T_{\gamma,\phi}(\nu_j) \right) \right) = T_{\gamma,\phi}(m) \Box T_{\gamma,\phi}(n), \end{aligned}$$

as needed. In particular, this shows that $LUC(G,\omega_1^{-1})^*$ and $LUC(H,\omega_2^{-1})^*$ are isometrically isomorphic Banach algebras whenever the weighted locally compact groups (G,ω_1) and (H,ω_2) are isomorphic.

Let

$$C_0(G,\omega^{-1})^\perp := \{m \in LUC(G,\omega^{-1})^*;\ m(f) = 0,\ \forall f \in C_0(G,\omega^{-1})\}.$$

The next lemma is a Beurling algebra version of [GLL90, Lemma 1.1].

Lemma 3.4.1. *Suppose that (G,ω) is a weighted locally compact group. Then the ℓ^1-direct sum decomposition*

$$LUC(G,\omega^{-1})^* = M(G,\omega) \oplus_1 C_0(G,\omega^{-1})^\perp$$

holds, and $C_0(G,\omega^{-1})^\perp$ is an ideal in $LUC(G,\omega^{-1})^*$.

Proof. The map $\Phi : LUC(G,\omega^{-1}) \to LUC(G) : f \mapsto \omega^{-1} f$ is an isometric linear isomorphism mapping $C_0(G,\omega^{-1})$ onto $C_0(G)$. As $\Phi^* : \mu \mapsto \omega^{-1}\mu$ maps the copy of $M(G)$ in $LUC(G)^*$ isometrically onto the copy of $M(G,\omega)$ in $LUC(G,\omega^{-1})^*$ and $LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$ by [GLL90, Lemma 1.1], we obtain $LUC(G,\omega^{-1})^* = M(G,\omega) \oplus_1 C_0(G,\omega^{-1})^\perp$.

To show that $C_0(G,\omega^{-1})^\perp$ is an ideal in $LUC(G,\omega^{-1})^*$, we first show that for $\psi \in L^1(G,\omega)$, $\mu \in M(G,\omega) \subseteq LUC(G,\omega^{-1})^*$ and $h \in C_0(G,\omega^{-1})$,

$$h\Box\psi \in C_0(G,\omega^{-1}) \quad \text{and} \quad \mu\Box h \in C_0(G,\omega^{-1}). \tag{3.15}$$

We first note that because multiplication is separately weak-star continuous in $M(G,\omega)$, i.e., $M(G,\omega)$ is a dual Banach algebra, $\mu \cdot h, h \cdot \mu \in C_0(G,\omega^{-1}) = (M(G,\omega), w^*)^*$, where

$$\langle \mu \cdot h, \nu\rangle_{M^*-M} = \langle \nu * \mu, h\rangle_{M-C_0}, \text{ and } \langle h \cdot \mu, \nu\rangle_{M^*-M} = \langle \mu * \nu, h\rangle_{M-C_0}.$$

For $\phi \in L^1(G,\omega)$,

$$\begin{aligned}
\langle h\Box\psi, \phi\rangle_{L^\infty - L^1} &= \langle h, \psi * \phi\rangle_{L^\infty - L^1} \\
&= \langle \psi * \phi, h\rangle_{M - C_0} \\
&= \langle h \cdot \psi, \phi\rangle_{M^* - M} \\
&= \langle \phi, h \cdot \psi\rangle_{M - C_0} \\
&= \langle h \cdot \psi, \phi\rangle_{L^\infty - L^1}
\end{aligned}$$

so $h\Box\psi = h \cdot \psi$ in $L^\infty(G,\omega^{-1})$. By continuity, $h\Box\psi = h \cdot \psi \in C_0(G,\omega^{-1})$. Also from the above argument, $h\Box\phi = h \cdot \phi \in C_0(G,\omega^{-1})$, so

$$\begin{aligned}
\langle \mu\Box h, \phi\rangle_{L^\infty - L^1} &= \langle \mu, h\Box\phi\rangle_{LUC^* - LUC} \\
&= \langle \mu, h \cdot \phi\rangle_{M - C_0} \\
&= \langle \mu \cdot h, \phi\rangle_{M^* - M} \\
&= \langle \phi, \mu \cdot h\rangle_{M - C_0} \\
&= \langle \mu \cdot h, \phi\rangle_{L^\infty - L^1}.
\end{aligned}$$

Hence, $\mu\Box h = \mu \cdot h \in C_0(G,\omega^{-1})$.

Now suppose that $m \in LUC(G,\omega^{-1})^*$, $n \in C_0(G,\omega^{-1})^\perp$, $h \in C_0(G,\omega^{-1})$. Then for $\psi \in L^1(G,\omega)$, equation (3.15) gives $\langle n\Box h, \psi\rangle = \langle n, h\Box\psi\rangle = 0$ and hence $\langle m\Box n, h\rangle = \langle m, n\Box h\rangle = 0$. Thus, $C_0(G,\omega^{-1})^\perp$ is a left ideal in $LUC(G,\omega^{-1})^*$. Writing m as $m = \mu + m_1$ where $\mu \in M(G,\omega)$ and $m_1 \in C_0(G,\omega^{-1})^\perp$, $n\Box m = n\Box\mu + n\Box m_1$, with $n\Box m_1 \in C_0(G,\omega^{-1})^\perp$ from the above. Also, equation (3.15) gives $\langle n\Box\mu, h\rangle = \langle n, \mu\Box h\rangle = 0$, so $n\Box\mu \in C_0(G,\omega^{-1})^\perp$ as well. Thus, $C_0(G,\omega^{-1})^\perp$ is also a right ideal in $LUC(G,\omega^{-1})^*$. $\Box$

Let (m_α) be a net in $LUC(G,\omega^{-1})^*$. We say that (m_α) converges strictly to some m in $LUC(G,\omega^{-1})^*$ if $\|m_\alpha\Box\phi - m\Box\phi\|_\omega \to 0$, for each $\phi \in L^1(G,\omega)$.

Lemma 3.4.2. *Suppose that* (G,ω_1) *and* (H,ω_2) *are weighted locally compact groups and* $T : LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$ *is an isometric isomorphism. Let* (m_α) *be a net in* $M(G,\omega_1)$ *converging strictly to* $m \in M(G,\omega_1)$ *with* $\|m_\alpha\|_{\omega_1} = \|m\|_{\omega_1} = 1$*. Then* $T(m_\alpha)$ *converges to* $T(m)$ *in the weak-star topology of* $LUC(H,\omega_2^{-1})^*$*.*

Proof. Our argument is very similar to the proof of [GLL90, Theorem 1.4], but is included for the convenience of the reader. Let n be a weak-star cluster point of the bounded net $(T(m_\alpha))$. By passing to a subnet, if necessary, we may assume that $T(m_\alpha)$ converges to n in the weak-star topology of $LUC(H,\omega_2^{-1})^*$. Let $\phi \in L^1(G,\omega)$. Since (m_α) converges strictly to m

$$\|m_\alpha \Box \phi - m \Box \phi\|_{1,\omega_1} \to 0,$$

and therefore

$$\|T(m_\alpha)\Box T(\phi) - T(m)\Box T(\phi)\|_{\omega_2} \to 0.$$

Hence, for every $f \in LUC(H,\omega_2^{-1})$,

$$\begin{aligned}\langle T(m)\Box T(\phi), f\rangle &= \lim_\alpha \langle T(m_\alpha)\Box T(\phi), f\rangle \\ &= \lim_\alpha \langle T(m_\alpha), T(\phi)\Box f\rangle \\ &= \langle n, T(\phi)\Box f\rangle = \langle n\Box T(\phi), f\rangle\cdot\end{aligned}$$

Thus,

$$T(m)\Box T(\phi) = n\Box T(\phi),$$

and so

$$m\Box\phi = T^{-1}(n)\Box\phi.$$

By Lemma 3.4.1, there exists $m_1 \in M(G,\omega_1)$ and $m_2 \in C_0(G,\omega_1^{-1})^\perp$ such that $T^{-1}(n) = m_1 + m_2$ giving

$$m\Box\phi = T^{-1}(n)\Box\phi = m_1\Box\phi + m_2\Box\phi.$$

As $m\Box\phi = m*\phi$, $m_1\Box\phi = m_1*\phi \in M(G,\omega_1)$ and $m_2\Box\phi$ lies in the ideal $C_0(G,\omega_1^{-1})^\perp$, we must have

$$m_2\Box\phi = 0 \quad \text{and} \quad m*\phi = m_1*\phi. \tag{3.16}$$

As $\phi \in L^1(G,\omega_1)$ was chosen arbitrarily, if (ψ_i) is a bounded approximate identity as in [Gha84b, Lemma 2.1], then by equation (3.16), we have

$$m*\psi_i = m_1*\psi_i.$$

Since by [Gha84b, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the weak-star topology of $M(G,\omega_1)$, taking the weak-star limit gives $m = m_1$. Since

$$1 + \|m_2\| = \|m\| + \|m_2\| = \|m_1\| + \|m_2\| = \|T^{-1}(n)\| = \|n\| \leq 1,$$

we must have $\|m_2\| = 0$, and therefore $m_2 = 0$. So, $T^{-1}(n) = m$ or $n = T(m)$. $\square$

Lemma 3.4.3. *Let (G,ω) be a weighted locally compact group. Then $G_\omega^{LUC} = G_\omega \dot\cup G_\omega^*$, where $G_\omega = \{\omega(x)^{-1}\delta_x : x \in G\} = G_\omega^{LUC} \cap M(G,\omega)$ and $G_\omega^* = G_\omega^{LUC} \setminus G_\omega = G_\omega^{LUC} \cap C_0(G,\omega^{-1})^\perp$. Moreover, $\{\gamma p : \gamma \in \mathbb{T}, p \in G_\omega^{LUC}\}$ is the set of extreme points of the unit ball of $LUC(G,\omega^{-1})^*$.*

Proof. Since $LUC(G)$ is a commutative unital C^*-algebra, the Gelfand representation theorem for commutative unital C^*-algebras implies that $LUC(G) = C(G^{LUC})$, where G^{LUC} denotes the Gelfand spectrum of $LUC(G)$. It follows from [Con90, Thm.V.8.4] that the set of extreme points of the unit ball of $LUC(G)^* = M(G^{LUC})$ is

$$\{\gamma\delta_x : \ \gamma \in \mathbb{T}, \ x \in G^{LUC}\}.$$

As noted in the introduction

$$\Phi : LUC(G,\omega^{-1}) \to LUC(G) : f \mapsto \omega^{-1} f$$

is an isometric linear isomorphism of Banach spaces. Therefore

$$\Phi^* : LUC(G)^* \to LUC(G, \omega^{-1})^*$$

is also an isometric linear isomorphism and so Φ^* maps the extreme points of the unit ball of $LUC(G)^*$ onto the extreme points of the unit ball of $LUC(G, \omega^{-1})^*$. Now, by the definition of G_ω^{LUC}, we have that $\Phi^*(G^{LUC}) = G_\omega^{LUC}$. Thus, the set of the extreme points of the unit ball of $LUC(G, \omega^{-1})^*$ is

$$\{\gamma p : \gamma \in \mathbb{T},\ p \in G_\omega^{LUC}\}.$$

Moreover, as noted in the proof of Lemma 3.4.1,

$$\Phi^*(M(G)) = M(G, \omega) \ \text{ and } \ \Phi^*(C_0(G)^\perp) = C_0(G, \omega^{-1})^\perp.$$

Hence,

$$\begin{aligned} G_\omega^{LUC} &= \Phi^*(G^{LUC}) = \Phi^*(G) \dot{\cup} \Phi^*(G^{LUC} \setminus G) \\ &= \left\{ \frac{\delta_x}{\omega(x)} : x \in G \right\} \dot{\cup} \left(G_\omega^{LUC} \setminus \Phi^*(G) \right) \\ &= G_\omega \dot{\cup} \left(G_\omega^{LUC} \setminus G_\omega \right), \end{aligned}$$

$$G_\omega = \Phi^*(G) = \Phi^* \left(G^{LUC} \cap M(G) \right) = G_\omega^{LUC} \cap M(G, \omega)$$

and

$$G_\omega^{LUC} \setminus G_\omega = \Phi^* \left(G^{LUC} \setminus G \right) = \Phi^* \left(G^{LUC} \cap C_0(G)^\perp \right) = G_\omega^{LUC} \cap C_0(G, \omega^{-1})^\perp. \quad \square$$

Before proceeding to the next theorem, note that δ_{e_G} is the identity for $LUC(G, \omega^{-1})^*$ where e_G is the identity in G. To see this, first note that

$$\delta_{e_G} \Box \frac{\delta_x}{\omega(x)} = \frac{\delta_x}{\omega(x)} \ \text{ and } \ \frac{\delta_x}{\omega(x)} \Box \delta_{e_G} = \frac{\delta(x)}{\omega(x)},$$

for each x in G. The linear span of $\{\frac{\delta_x}{\omega(x)} : x \in G\}$ is weak-star dense in $LUC(G, \omega^{-1})^*$ so, by the weak-star continuity of the product by δ_{e_G} on both left and right (see Lemma 3.1.6), we obtain $\delta_{e_G} \Box m = m$ and $m \Box \delta_{e_G} = m$, for each m in $LUC(G, \omega^{-1})^*$. Thus δ_{e_G} is the identity.

Theorem 3.4.4. *Suppose that (G, ω_1) and (H, ω_2) are weighted locally compact groups and $T : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$ is an isometric isomorphism. Then T maps $M(G, \omega_1)$ onto $M(H, \omega_2)$ and hence (G, ω_1) and (H, ω_2) are isomorphic weighted locally compact groups.*

Proof. As T is an isometric isomorphism, it maps an extreme point of the unit ball of $LUC(G, \omega_1^{-1})^*$ onto an extreme point of the unit ball of $LUC(H, \omega_2^{-1})^*$. Hence, by Lemma 3.4.3, for each $x \in G$ there exists $\gamma(x) \in \mathbb{T}$ and $p_x \in H_{\omega_2}^{LUC}$ such that

$$T\left(\frac{\delta_x}{\omega_1(x)}\right) = \gamma(x) p_x.$$

Moreover, as a surjective algebra homomorphism, T maps δ_{e_G}, the identity of $LUC(G, \omega_1^{-1})^*$ to the identity, δ_{e_H}, of $LUC(H, \omega_2^{-1})^*$. Therefore, for $x \in G$,

$$\begin{aligned}\delta_{e_H} = T(\delta_{e_G}) &= T(\delta_x) \Box T(\delta_{x^{-1}}) \\ &= \omega_1(x)\omega_1(x^{-1})\gamma(x)\gamma(x^{-1}) p_x \Box p_{x^{-1}}.\end{aligned}$$

If p_x belongs to $H_{\omega_2}^* = H_{\omega_2}^{luc} \cap C_0(H, \omega_2^{-1})^\perp$, then because $C_0(H, \omega_2^{-1})^\perp$ is an ideal in $LUC(H, \omega_2^{-1})^*$, we would have $\delta_{e_H} \in C_0(H, \omega_2^{-1})^\perp$ as well. This is not possible since $\delta_{e_H} \in M(H, \omega_2)$ and $M(H, \omega_2) \cap C_0(H, \omega_2^{-1})^\perp = \{0\}$. Hence, p_x belongs to $H_{\omega_2} = \{\omega_2(y)^{-1}\delta_y : y \in H\}$. Therefore there exists $\phi(x)$ in H such that $p_x = \omega_2(\phi(x))^{-1}\delta_{\phi(x)}$. Thus we have mappings $\gamma : G \to \mathbb{T}$ and $\phi : G \to H$ such that

$$T(\delta_x) = \gamma(x) \frac{\omega_1(x)}{\omega_2(\phi(x))} \delta_{\phi(x)} \quad (x \in G).$$

Moreover, the arguments found in the proof of Theorem 3.2.4 show that γ, $\dfrac{\omega_1}{\omega_2 \circ \phi}$ and ϕ are multiplicative on G. To see that γ and ϕ are continuous, let $x_\alpha \to x$ in G. Then by Lemmas 3.1.4 and 3.4.2,

$$T\left(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha}\right) \xrightarrow{w^*} T\left(\frac{1}{\omega_1(x)}\delta_x\right)$$

in $LUC(H, \omega_2^{-1})^*$. It follows that

$$\gamma(x_\alpha)\frac{1}{\omega_2(\phi(x_\alpha))}\delta_{\phi(x_\alpha)} \xrightarrow{w^*} \gamma(x)\frac{1}{\omega_2(\phi(x))}\delta_{\phi(x)}$$

in $M(H, \omega_2^{-1})$, which is exactly equation (3.7) in the proof of Theorem 3.2.4. The proof of Theorem 3.2.4 hence shows that γ and ϕ are continuous.

Hence,

$$J_{\gamma,\phi} : LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1}) : f \mapsto \gamma\frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi$$

is a well-defined isometric isomorphism and $J^*_{\gamma,\phi} : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$ satisfies

$$J^*_{\gamma,\phi}(\delta_x) = T(\delta_x) \quad (x \in G). \tag{3.17}$$

Moreover, for any $\mu \in M(G, \omega_1)$, $J^*_{\gamma,\phi}(\mu) \in M(H, \omega_2)$. To see this note that by Lemma 3.4.1, $J^*_{\gamma,\phi}(\mu) = \nu + m$ where $\nu = J^*_{\gamma,\phi}(\mu)|_{C_0(H,\omega_2^{-1})}$ and $m \in C_0(H, \omega_2^{-1})^\perp$. Letting $j^*_{\gamma,\phi}$: $M(G, \omega_1) \to M(H, \omega_2)$ be the isometric isomorphism defined in Section 2, it is clear $\nu = J^*_{\gamma,\phi}(\mu)\,|_{C_0(H,\omega_2^{-1})} = j^*_{\gamma,\phi}(\mu)$, so

$$\|\mu\| = \|J^*_{\gamma,\phi}(\mu)\| = \|j^*_{\gamma,\phi}(\mu)\| + \|m\| = \|\mu\| + \|m\|.$$

Hence, $m = 0$, as needed.

Let $\mu \in M(G, \omega_1)$ with $\|\mu\| = 1$. As with the proof of Theorem 3.2.5, we can choose a net (μ_β) in the convex hull of $\{\gamma\frac{\delta_x}{\omega_1(x)},\ \gamma \in \mathbb{T}\ , x \in G\}$, such that $\mu_\beta \to \mu$ in $M(G, \omega_1)$

strictly. Therefore $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$ in $LUC(H, \omega_2^{-1})$, by Lemma 3.4.2. But equation (3.17) and weak-star continuity of $J_{\gamma,\phi}^*$ give

$$T(\mu_\beta) = J_{\gamma,\phi}^*(\mu_\beta) \xrightarrow{w^*} J_{\gamma,\phi}^*(\mu)$$

in $LUC(H, \omega_2^{-1})^*$, so $T(\mu) = J_{\gamma,\phi}^*(\mu)$ belongs to $M(H, \omega_2)$, as needed. $\square$

3.5 Isometric Isomorphisms on the Biduals of Weighted Group Algebras

We conclude this chapter by showing that if $T : L^1(G, \omega_1)^{**} \to L^1(H, \omega_2)^{**}$ is an isometric isomorphism, then the Banach algebras $LUC(G, \omega_1^{-1})^*$ and $LUC(H, \omega_2^{-1})^*$ are isometrically isomorphic. Therefore, by Theorem 3.4.4, the bidual of the weighted group algebra $L^1(G, \omega)^{**}$ determines the weighted topological group (G, ω). We make use of the following lemma in the proof of Proposition 3.5.2.

Lemma 3.5.1. *Let (G, ω) be a weighted locally compact group. Then $L^1(G, \omega)^{**}$ has a right identity of norm one.*

Proof. First we show that $L^1(G, \omega)$ has a bounded approximate identity whose terms are of norm one. To see this let U be a compact neighbourhood of the identity element e_G of G and let $(U_i)_{i \in I}$ be a neighbourhood system of e_G directed downwards by inclusion such that $U_i \subseteq U$, for each $i \in I$. Then the proof of [Gha84b, Lemma 2.3] shows that the net $(f_i)_{i \in I}$ where $f_i := \dfrac{\chi_{U_i}}{\lambda(U_i)}$, for each $i \in I$ is a bounded approximate identity of $L^1(G, \omega)$. Let $\psi_i := \frac{1}{\omega} f_i$, for each $i \in I$. Then $\|\psi_i\|_{1,\omega} = 1$, for each $i \in I$. Routine calculations using continuity of ω at e_G show that

$$\|\psi_i - f_i\|_{1,\omega} \to 0.$$

Hence, for any $\phi \in L^1(G,\omega)$,

$$\begin{aligned}\|\psi_i * \phi - \phi\|_{1,\omega} &\leq \|\psi_i * \phi - f_i * \phi\|_{1,\omega} + \|f_i * \phi - \phi\|_{1,\omega} \\ &\leq \|\psi_i - f_i\|_{1,\omega}\|\phi\|_{1,\omega} + \|f_i * \phi - \phi\|_{1,\omega} \to 0.\end{aligned}$$

Therefore, (ψ_i) is a bounded approximate identity for $L^1(G,\omega)$, with each term having $\|\psi_i\|_{1,\omega} = 1$.

Let E denote a weak-star cluster point of the canonical image of the bounded approximate identity (ψ_i) of $L^1(G,\omega)$ in $L^1(G,\omega)^{**}$. Then by [BD73, Thm.7], E is a right identity for $L^1(G,\omega)^{**}$. Since $\|\psi_i\|_{1,\omega} = 1$ for each $i \in I$, it follows that $\|E\| \leq 1$. Moreover,

$$\|E\| = \|E \Box E\| \leq \|E\|\|E\|,$$

so $\|E\| \geq 1$ as well. □

Proposition 3.5.2. *Suppose that (G,ω_1) and (H,ω_2) are weighted locally compact groups, and suppose that $T : L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is an isometric isomorphism. Then $LUC(G,\omega_1^{-1})^*$ is isometrically isomorphic to $LUC(H,\omega_2^{-1})^*$, and hence (G,ω_1) and (H,ω_2) are isomorphic weighted locally compact groups.*

Proof. Replacing $L^1(G)^{**}$, $L^1(H)^{**}$, $LUC(G)^*$ and $LUC(H)^*$ with their weighted counterparts, the proof of the main theorem of [GL88] works in exactly the same way. Specifically, if E is a given right identity of $L^1(G,\omega_1)^{**}$ with $\|E\| = 1$, corresponding to a bounded approximate identity whose terms are of norm one, then $E \Box L^1(G,\omega_1)^{**}$ is isometrically isomorphic to $LUC(G,\omega_1^{-1})^*$, and therefore $L^1(G,\omega_1)^{**}$ contains a copy of $LUC(G,\omega_1^{-1})^*$. Now if $T : L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is an isometric isomorphism, T maps $E \Box L^1(G,\omega_1)^{**} = LUC(G,\omega_1^{-1})^*$ onto $T(E) \Box L^1(H,\omega_2)^{**} = LUC(H,\omega_2^{-1})^*$. Therefore, $LUC(G,\omega_1^{-1})^*$ is isometrically isomorphic to $LUC(H,\omega_2^{-1})^*$. Hence, by Theorem 3.4.4, the weighted locally compact groups (G,ω_1) and (H,ω_2) are isomorphic. □

Chapter 4

Bipositive Algebra Isomorphisms on Beurling Algebras

The results in this chapter are joint work with professor F. Gharamani.

In [Kaw48], Y. Kawada showed that if we have a bipositive algebra isomorphism between group algebras, then the underlying locally compact groups must be isomorphic. H. Farhadi proved similar results to Kawada's result for other Banach algebras related to locally compact groups by first showing that any such bipositive algebra isomorphism must in fact be an isometric isomorphism.

In Section 4.2, we provide an example of a bipositive algebra isomorphism between Beurling measure algebras that is not an isometry. The same construction is carried out in the following sections of this chapter to provide bipositive algebra isomorphisms between other types of Beurling algebras (e.g., Beurling group algebras) that are not isometries. This example shows that the same approach as the one in Farhadi's paper [Far98], cannot be used to settle similar problems when the weight is non-trivial.

In this chapter we show that the algebra structure together with the order structure of various Beurling algebras associated with locally compact groups completely determines the structure of the topological groups together with a constraint on the weights. Moreover, we give a complete description of bipositive algebra isomorphisms between Beurling group algebras and between Beurling measure algebras in terms of topological group isomorphisms.

4.1 Some Preliminary Results

We recall that a weight function on G is a positive continuous function $\omega : G \to \mathbb{R}^+$ such that $\omega(xy) \leq \omega(x)\omega(y)$, for all x, $y \in G$. We note that in this chapter we **do not** assume the condition $\omega(e_G) = 1$ on the weight function.

Definition 4.1.1. (i) The function $f \in C_0(G, \omega^{-1})$ is called positive if for every $x \in G$, $f(x) \geq 0$.

(ii) The measure $\mu \in M(G, \omega)$ is called positive if for every positive $f \in C_0(G, \omega^{-1})$, $\langle \mu, f \rangle \geq 0$.

(iii) An element $\psi \in L^1(G, \omega)$ is called positive if $\psi(x) \geq 0$, λ a.e. $x \in G$.

(iv) The function $f \in LUC(G, \omega^{-1})$ is called positive if $f(x) \geq 0$, for all $x \in G$.

(v) The functional $m \in LUC(G, \omega^{-1})^*$ is positive if $\langle m, f \rangle \geq 0$, for every positive function f in $LUC(G, \omega^{-1})$.

(vi) Let $\mathcal{A}$ and $\mathcal{B}$ be ordered vector spaces (see [AB06, page 16]). Then an operator $T : \mathcal{A} \to \mathcal{B}$ is called positive if for each positive element $a \in \mathcal{A}$, $T(a) \geq 0$ in $\mathcal{B}$. The operator T is called bipositive if T is a bijection and both T and T^{-1} are positive operators.

(vii) For a space S of functions, S^+ denotes the subset of all positive elements in S.

We conclude this section with some lemmas that are needed for our work in the subsequent sections.

Lemma 4.1.2. *Let (G,ω) be a weighted locally compact group. Then we have*

$$\big(LUC(G,\omega^{-1})^*\big)^+ = M(G,\omega)^+ \oplus \big(C_0(G,\omega^{-1})^{\perp}\big)^+.$$

Proof. Let $p \in \big(LUC(G,\omega^{-1})^*\big)^+$ and let $\mu \in M(G,\omega)$ be the measure obtained by restricting p to $C_0(G,\omega^{-1})$. Then μ can be regarded as an element of $LUC(G,\omega^{-1})^*$ through the pairing

$$\langle \mu, f\rangle = \int_G f(t)\ d\mu(t) \quad \big(f \in LUC(G,\omega^{-1})\big).$$

Obviously, μ is positive. Let $m := p - \mu \in C_0(G,\omega^{-1})^{\perp}$. We prove that $m \geq 0$. To this end, we take $g \in \big(LUC(G,\omega^{-1})\big)^+$. Given $\epsilon > 0$, there exists a compact subset $K \subseteq G$ such that

$$\int_{G\setminus K} \omega(t)\ d|\mu|(t) < \epsilon.$$

Let $f \in C_c(G)$ such that $0 \leq f \leq 1$ and $f \equiv 1$ on K. We set $g_1 := fg$; pointwise product. Then $g_1 \in C_c(G) \subseteq C_0(G,\omega^{-1})$, and $g - g_1 \in LUC(G,\omega^{-1})^+$. We have

$$\begin{aligned}
\langle m, g\rangle &= \langle m, g-g_1\rangle - \langle m, g_1\rangle \\
&= \langle p, g-g_1\rangle - \langle \mu, g-g_1\rangle \\
&\geq -\langle \mu, g-g_1\rangle \\
&\geq -\|g-g_1\|_{\infty,\omega^{-1}} \int_{G\setminus K} \omega(t) d|\mu|(t) \\
&\geq -\epsilon\|g-g_1\|_{\infty,\omega^{-1}} \geq -\epsilon\|g\|_{\infty,\omega^{-1}}.
\end{aligned}$$

Hence, $\langle m, g\rangle \geq 0$. $\square$

Lemma 4.1.3. *Let (G,ω) be a weighted locally compact group. Suppose that μ and ν are positive measures in $M(G,\omega)$ and $\mu * \nu = 0$. Then $\mu = 0$ or $\nu = 0$.*

Proof. Suppose that $\|\mu\|_\omega \neq 0$ and $\|\nu\|_\omega \neq 0$. Then there are compact sets K and L such that

$$\int_K \omega(x)\ d\mu(x) > 0 \quad \text{and} \quad \int_L \omega(x)\ d\nu(x) > 0. \tag{4.1}$$

Since on K and L, ω is bounded away from 0, we have

$$\int_K d\mu(x) > 0 \quad \text{and} \quad \int_L d\nu(x) > 0. \tag{4.2}$$

Since $\|\mu * \nu\|_\omega = 0$,

$$\int_G \int_G \omega(xy)\ d\mu(x)d\nu(y) = 0. \tag{4.3}$$

But

$$\begin{aligned}\int_G \int_G \omega(xy)d\mu(x)d\nu(y) &\geq \int_K \int_L \omega(xy)d\mu(x)d\nu(y) \\ &\geq \min\{\omega(t) : t \in KL\} \int_K d\mu(x) \int_L d\nu(y) > 0,\end{aligned}$$

a contradiction to (4.2). ☐

Lemma 4.1.4. *Let (G,ω) be a weighted locally compact group.*

(i) Suppose that $\mu \in (M(G,\omega))^+$ is invertible and μ^{-1} is also positive. Then there exist a positive number γ and an element x in G such that $\mu = \gamma\delta_x$.

(ii) Suppose that $m \in (LUC(G,\omega^{-1})^*)^+$ is such that m is invertible and m^{-1} is also positive. Then there exist a positive number γ and an element x in G such that $m = \gamma\delta_x$.

Proof. (i) First we observe that there exist positive discrete measures μ_d and ν_d and positive continuous measures μ_c and ν_c such that

$$\mu = \mu_d + \mu_c \qquad \text{and } \mu^{-1} = \nu_d + \nu_c.$$

To see this, we note that since $\omega\mu$ belongs to $M(G)$, by [HR79, 19.20 and 19.21], there exist a discrete measure $\omega\mu_d$ and a continuous measure $\omega\mu_c$ in $M(G)$ such that

$$\omega\mu = \omega\mu_d + \omega\mu_c,$$

and

$$\|\omega\mu\| = \|\omega\mu_d\| + \|\omega\mu_c\|. \tag{4.4}$$

Now, since $\omega\mu$ is positive, we have

$$\begin{aligned} \|\omega\mu\| = \langle \omega\mu, 1\rangle &= \langle \omega\mu_d, 1\rangle + \langle \omega\mu_c, 1\rangle \\ &= \mathrm{R}e\left(\langle \omega\mu_d, 1\rangle + \langle \omega\mu_c, 1\rangle\right) \\ &= \mathrm{Re}\langle \omega\mu_d, 1\rangle + \mathrm{Re}\langle \omega\mu_c, 1\rangle \end{aligned} \tag{4.5}$$

Obviously, $|\mathrm{Re}\langle \omega\mu_d, 1\rangle| \leq \|\omega\mu_d\|$ and $|\mathrm{Re}\langle \omega\mu_c, 1\rangle| \leq \|\omega\mu_c\|$. Now if either of these last two inequalities is strict, then from (4.5) we would have

$$\|\omega\mu\| < \|\omega\mu_d\| + \|\omega\mu_c\|,$$

a contradiction. Therefore we must have $\mathrm{Re}\langle \omega\mu_d, 1\rangle = \|\omega\mu_d\|$ and $\mathrm{R}e\langle \omega\mu_c, 1\rangle = \|\omega\mu_c\|$ and therefore, $\omega\mu_d$ and $\omega\mu_c$ are positive measures. It then follows that μ_d and μ_c are positive measures.

Since $\mu * \mu^{-1} = \delta_{e_G}$ we have that

$$(\mu_d + \mu_c) * (\nu_d + \nu_c) = \delta_{e_G},$$

that is

$$\mu_d * \nu_d + \mu_d * \nu_c + \mu_c * \nu_d + \mu_c * \nu_c = \delta_{e_G}.$$

Hence, $\mu_d * \nu_d = \delta_{e_G}$ and $\mu_d * \nu_c + \mu_c * \nu_d + \mu_c * \nu_c = 0$, because $M_d(G,\omega) \cap M_c(G,\omega) = \{0\}$ and $M_c(G,\omega)$ is an ideal in $M(G,\omega)$. Since all the measures above are positive, we have that $\mu_d * \nu_d = \delta_{e_G}$, $\mu_d * \nu_c = 0$, $\mu_c * \nu_d = 0$ and $\mu_c * \nu_c = 0$. Since $\mu_d \neq 0$ and $\nu_d \neq 0$, from Lemma 4.1.3, the last equations yield $\mu_c = \nu_c = 0$. Suppose that

$$\mu_d = \sum_{n=1}^{\infty} a_n \delta_{x_n} \quad \text{ and } \quad \nu_d = \sum_{n=1}^{\infty} b_n \delta_{x'_n}$$

where (x_n) and (x'_n) are sequences of distinct elements of G and a_n and b_n's are positive numbers such that $\sum_n a_n \omega(x_n) < \infty$ and $\sum_n b_n \omega(x'_n) < \infty$. Since $a_n \geq 0$ and $b_n \geq 0$, from $\mu_d * \nu_d = \delta_{e_G}$, it follows that μ_d is concentrated at a single point.

(ii) By Lemma 4.1.2, we have $m = \mu_1 + n_1$ and $m^{-1} = \mu_2 + n_2$, with $\mu_i \geq 0$ and $n_i \geq 0$, for all $i = 1, 2$. From

$$\delta_{e_G} = m \Box m^{-1} = \mu_1 * \mu_2 + \mu_1 \Box n_2 + n_1 \Box \mu_2 + n_1 \Box n_2,$$

it follows that

$$\mu_1 * \mu_2 = \delta_{e_G} \quad \text{ and } \quad \mu_1 \Box n_2 + n_1 \Box \mu_2 + n_1 \Box n_2 = 0,$$

since $M(G,\omega) \cap C_0(G,\omega^{-1})^{\perp} = \{0\}$ and $C_0(G,\omega^{-1})^{\perp}$ is an ideal in $LUC(G,\omega^{-1})^*$. Similarly,

$$\mu_2 * \mu_1 = \delta_{e_G} \tag{4.6}$$

and

$$n_2 \Box \mu_1 + \mu_2 \Box n_1 + n_2 \Box n_1 = 0. \tag{4.7}$$

Hence, by equation (4.6) and part (i), there exist $\gamma > 0$ and an element $x \in G$ such that $\mu_1 = \gamma\delta_x$ and $\mu_2 = \gamma^{-1}\delta_{x^{-1}}$. Since all the terms in equation (4.7) are non-negative we have

$$\mu_1 \Box n_2 = 0 \quad \text{and} \quad n_1 \Box \mu_2 = 0$$

and thus by invertibility of μ_1 and μ_2 we have that $n_1 = n_2 = 0$. $\square$

4.2 Bipositive Isomorphisms of Beurling Measure Algebras

In this section we show that the existence of a bipositive algebra isomorphism between the Beurling measure algebras $M(G, \omega_1)$ and $M(H, \omega_2)$ implies that their underlying locally compact groups must be isomorphic. This extends [Far98, Thm.2.4] to the context of Beurling measure algebras, giving Farhadi's result a new proof.

Lemma 4.2.1. *Let (G, ω) be a weighted locally compact group. The map $x \mapsto \delta_x : G \to M(G, \omega)$ is strong operator continuous.*

Proof. This can be proved using an argument like the one used to prove Lemma 3.1.5. $\square$

Lemma 4.2.2. *Let (G, ω) be a weighted locally compact group. The right annihilator of $L^1(G, \omega)$ in $M(G, \omega)$ is zero.*

Proof. Let $\mu \in M(G, \omega)$ be a right annihilator of $L^1(G, \omega)$ and let (f_i) denote the bounded approximate identity of $L^1(G, \omega)$ given in the proof of [Gha84b, Lemma 2.1]. Then by [Gha84b, Lemma 2.2], we have that

$$\mu = \delta_e * \mu = w^* - \lim_i (f_i * \mu) = 0.$$ $\square$

Suppose that (G, ω_1) and (H, ω_2) are weighted locally compact groups, $\phi : G \to H$ is a topological group isomorphism from G onto H and $\gamma : G \to ((0, +\infty), \times)$ is a continuous homomorphism. Furthermore, suppose that there are positive numbers M and m such that

$$m \leq \gamma(x) \frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M, \quad (x \in G)\,.$$

We define the mapping

$$j_{\gamma,\phi} : C_0(H, \omega_2^{-1}) \to C_0(G, \omega_1^{-1}) \ \text{ where } \ j_{\gamma,\phi} := \gamma \cdot f \circ \phi.$$

Then it is straightforward to check that $j_{\gamma,\phi}$ is a well-defined bounded bipositive linear isomorphism. Hence, the dual mapping

$$T_{\gamma,\phi} := j^*_{\gamma,\phi} : M(G, \omega_1) \to M(H, \omega_2)$$

is also a bounded bipositive linear isomorphism. We observe that $T_{\gamma,\phi}$ is multiplicative. To see this, first we note that for each $x \in G$ we have

$$T_{\gamma,\phi}(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

Hence, since γ and ϕ are multiplicative, it can be readily seen that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is also multiplicative on $M(G, \omega_1)$, we note that the linear span of point masses is weak-star dense in $M(G, \omega)$, the convolution product is separately weak-star continuous and $T_{\gamma,\phi} = j^*_{\gamma,\phi}$ is weak-star continuous. Clearly, $T_{\gamma,\phi}$ is invertible with ${T_{\gamma,\phi}}^{-1} = T_{\beta,\phi^{-1}}$, where $\beta := \dfrac{1}{\gamma} \circ \phi^{-1}$. Therefore, $T_{\gamma,\phi}$ is a bipositive algebra isomorphism. Finally, we observe that $T_{\gamma,\phi}(L^1(G, \omega_1)) = L^1(H, \omega_2)$. Since $\psi' \mapsto \int_G \psi'(\phi(x))dx$ defines a Haar integral on $L^1(H)$, there exists $c > 0$ such that the equation

$$\int_G \psi'(\phi(x))dx = c \int_H \psi'(y)dy \quad \big(\psi' \in L^1(H)\big)$$

holds. Let $\psi \in L^1(G,\omega_1)$. Then for each $f \in C_0(H,\omega_2)$ we have

$$\begin{aligned}\langle T_{\gamma,\phi}(\psi), f\rangle = \langle j^*_{\gamma,\phi}(\psi), f\rangle &= \langle \psi, \gamma \cdot f \circ \phi\rangle \\ &= \int_G \psi(x)\gamma(x) f\circ\phi(x)dx \\ &= c\int_H \psi\circ\phi^{-1}(y)\gamma\circ\phi^{-1}(y)f(y)dy \\ &= \langle c\gamma\circ\phi^{-1}\cdot\psi\circ\phi^{-1}, f\rangle.\end{aligned}$$

Therefore, $T_{\gamma,\phi}(\psi) = c\gamma\circ\phi^{-1}\cdot\psi\circ\phi^{-1} \in L^1(H,\omega_2)$, and hence

$$T_{\gamma,\phi}(L^1(G,\omega_1)) \subseteq L^1(H,\omega_2).$$

A similar argument using ${T_{\gamma,\phi}}^{-1} = T_{\beta,\phi^{-1}}$ where $\beta := \dfrac{1}{\gamma}\circ\phi^{-1}$, shows that

$$T_{\beta,\phi^{-1}}(L^1(H,\omega_2)) \subseteq L^1(G,\omega_1). \tag{4.8}$$

Now, by applying $T_{\gamma,\phi}$ to each side of (4.8), we have that $L^1(H,\omega_2) \subseteq T_{\gamma,\phi}(L^1(G,\omega_1))$ and therefore $T_{\gamma,\phi}(L^1(G,\omega_1)) = L^1(H,\omega_2)$.

Theorem 4.2.3. *Let (G,ω_1) and (H,ω_2) be weighted locally compact groups. A necessary and sufficient condition for the existence of a bipositive algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$ is that there exist an isomorphism of topological groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and positive constants M and m such that*

$$m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M.$$

Furthermore, if T is such a bipositive isomorphism we have that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)},$$

for some γ and ϕ as above (depending on T) and all $x \in G$.

Proof. In light of the preceding argument, if $\phi : G \to H$ is an isomorphism of the locally compact groups G and H such that for a continuous homomorphism $\gamma : G \to (0, +\infty)$ and positive constants M and m

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M,$$

then, there is a bipositive algebra isomorphism from $M(G, \omega_1)$ onto $M(H, \omega_2)$. For the converse, suppose that $T : M(G, \omega_1) \to M(H, \omega_2)$ is a bipositive algebra isomorphism. By [AB85, Thm.4.3], T and T^{-1} are bounded operators. Suppose that x in G is given. Since δ_x is a positive measure and T is a positive operator, $T(\delta_x)$ is a positive measure. Also since T is an algebra isomorphism and δ_x is an invertible measure, $T(\delta_x)$ is also invertible, with the inverse $T(\delta_{x^{-1}})$. Thus we have that $T(\delta_x)$ is a positive invertible measure with a positive inverse. It now follows from Lemma 4.1.4 that there exists an element $\phi(x) \in H$ and a positive number $\gamma(x)$ such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

Since T is an algebra isomorphism and $\delta_x * \delta_y = \delta_{xy}$ for each $x, y \in G$, we can readily see that both $\gamma : G \to (0, \infty)$ and $\phi : G \to H$ are multiplicative. We shall now show that γ and ϕ are continuous. Suppose that (g_α) is a net in G that tends to e_G, the identity element of G. By Lemma 4.2.1, for every $\psi \in L^1(G, \omega)$,

$$\delta_{g_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \psi.$$

Since T is bounded we that

$$T(\delta_{g_\alpha} * \psi) \xrightarrow{\|\cdot\|_{1,\omega_2}} T(\psi)$$

in $M(H, \omega_2)$. Hence

$$T(\delta_{g_\alpha}) * T(\psi) \xrightarrow{\|\cdot\|_{1,\omega_2}} T(\psi). \tag{4.9}$$

Let U be a precompact neighbourhood of e_G. Without loss of generality we can assume $g_i \in U$, for all i. Then

$$\|T(\delta_{g_\alpha})\| \leq \|T\|\|\delta_{g_\alpha}\| \leq \|T\|\omega_1(g_\alpha) \leq \|T\| \sup\{\omega_1(t); t \in \overline{U}\}.$$

Hence, the net $(T(\delta_{g_\alpha}))$ is bounded in $M(H, \omega_2)$, and so it has a subnet $\Big(T(\delta_{g_{\alpha(i)}})\Big)$ converging weak-star to some $\mu \in M(H, \omega_2)$. Then by Equation (4.9), we have that

$$\mu * T(\psi) = T(\psi).$$

Applying T^{-1} to the two sides of this equation yields

$$T^{-1}(\mu) * \psi = \psi.$$

Hence by Lemma 4.2.2, $T^{-1}(\mu) = \delta_{e_G}$, or equivalently $T(\delta_{e_G}) = \mu$. Hence $\mu = \gamma(e_G)\delta_{\phi(e_G)} = \gamma(e_G)\delta_{e_H}$. This, in particular, shows that $\gamma(g_i) \to \gamma(e_G)$ and $\phi(g_i) \to \phi(e_G)$. Hence, ϕ and γ are continuous.

To prove that ϕ is a bijection, we note that corresponding to T^{-1}, there exist $\beta : H \to (0, +\infty)$ and $\psi : H \to G$ such that

$$T^{-1}(\delta_y) = \beta(y)\delta_{\psi(y)} \quad (y \in H).$$

It follows from the equations $T(T^{-1}(\delta_y)) = \delta_y$ and $T^{-1}(T(\delta_x)) = \delta_x$ that ψ is a bijection, $\psi = \phi^{-1}$ and $\beta(\phi(x)) = \dfrac{1}{\gamma(x)}$, for all $x \in G$. By symmetry, ψ is continuous. Therefore, ϕ is an isomorphism of topological groups from G onto H.

Now, we show that for every $x \in G$ we have

$$\|T^{-1}\|^{-1} \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq \|T\|. \tag{4.10}$$

Since T is a bounded operator, for all $x \in G$, we have that

$$\gamma(x)\omega_2(\phi(x)) = \gamma(x)\|\delta_{\phi(x)}\| = \|T(\delta_x)\| \leq \|T\|\|\delta_x\| = \|T\|\omega_1(x).$$

A similar argument using T^{-1} shows that

$$\omega_1(x) \leq \|T^{-1}\|\gamma(x)\omega_2(\phi(x)).$$

Thus we have established the inequalities in equation (4.10). □

Proposition 4.2.4. *Let (G, ω_1) and (H, ω_2) be locally compact weighted groups. Suppose that $\phi : G \to H$ is an isomorphism of topological groups and $\gamma : G \to (0, +\infty)$ is a continuous homomorphism such that for positive constants M and m we have*

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M.$$

Then $\dfrac{\omega_2(\phi(x))}{\omega_1(x)}$ is bounded below if and only if $\gamma \equiv 1$.

Proof. We show that if $\dfrac{\omega_2(\phi(x))}{\omega_1(x)}$ is bounded below by a constant, say $C > 0$, then $\gamma \equiv 1$. The converse is obvious. First we note that if there is an element $x \in G$ such that $\gamma(x) < 1$ then $\gamma(x^{-1}) > 1$, since $1 = \gamma(e_G) = \gamma(x)\gamma(x^{-1})$. Suppose that there exists $x \in G$ such that $\gamma(x) > 1$. Then, since γ is a homomorphism, using powers of x, we can assume that there is a sequence (y_n) of elements of G such that $\gamma(y_n) > n^3$. Let $\mu := \sum \frac{1}{n^2}\omega_1(y_n)^{-1}\delta_{y_n}$. Then, since $T_{\gamma,\phi}$, as introduced before the statement of Theorem 4.2.3, is positive, for each $n \in \mathbb{N}$,

$$T_{\gamma,\phi}(\mu) \geq \frac{1}{n^2}T_{\gamma,\phi}(\omega_1(y_n)^{-1}\delta_{y_n}) = \frac{1}{n^2}\omega_1(y_n)^{-1}\gamma(y_n)\delta_{\phi(y_n)} \geq n\omega_1(y_n)^{-1}\delta_{\phi(y_n)}.$$

This means that for each $n \in \mathbb{N}$,

$$\|T_{\gamma,\phi}(\mu)\| \geq n\omega_1(y_n)^{-1}\omega_2(\phi(y_n))) \geq nC,$$

a contradiction. □

Theorem 4.2.5. *Let T be a bipositive algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$. Then there exist an isomorphism of locally compact groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and positive constants M and m such that*

$$T = j^*_{\gamma,\phi} \quad \textit{and} \quad m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M.$$

In particular, T maps $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$.

Proof. By Theorem 4.2.3 there exist an isomorphism of locally compact groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and positive constants M and m such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)} = j^*_{\gamma,\phi}(\delta_x) \ \text{ and } \quad m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M, \tag{4.11}$$

for each $x \in G$. The remainder of the proof is identical to that of Theorem 3.2.5. For the convenience of the reader, we provide the details. Letting $\mu \in M(G,\omega_1)$ be such that $\|\mu\|_{\omega_1} = 1$, it suffices to show that $T(\mu) = j^*_{\gamma,\phi}(\mu)$.

Taking μ in $M(G,\omega_1)$ with norm 1, by Lemma 3.1.9, we can find a net (μ_β) in the convex hull of the set $\{\gamma\frac{\delta_x}{\omega(x)} : \ x \in G, \gamma \in \mathbb{T}\}$ such that

$$\lim \|\mu_\beta * \psi - \mu * \psi\|_{\omega_1} = 0 \quad (\psi \in L^1(G,\omega_1)). \tag{4.12}$$

By equation (4.11),

$$T(\mu_\beta) = j^*_{\gamma,\phi}(\mu_\beta) \tag{4.13}$$

for each β. We claim that $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$ in $M(H,\omega_2)$. To see this, let ν be a weak-star limit point of $(T(\mu_\beta))$ in $M(H,\omega_2)$ and let $\left(\mu_{\beta(i)}\right)$ be a subnet of (μ_β) such that $T(\mu_{\beta(i)}) \xrightarrow{w^*} \nu$. Observe that it suffices now to show that $\nu = T(\mu)$. To simplify notation, we can assume that $T(\mu_\beta) \xrightarrow{w^*} \nu$. Let $\psi \in L^1(G,\omega_1)$ be fixed. Then

$$\|T(\mu_\beta) * T(\psi) - T(\mu) * T(\psi)\|_{\omega_2} \to 0,$$

by equation (4.12). Note that because $M(H,\omega_2) = C_0(G,\omega_2^{-1})^*$ is a dual Banach algebra (multiplication in $M(H,\omega_2)$ is separately weak-star continuous), straightforward calculations show that $C_0(H,\omega_2^{-1})$ is a submodule of the dual Banach $M(H,\omega_2)$-module $M(H,\omega_2)^* = C_0(H,\omega_2^{-1})^{**}$. Therefore, given $k \in C_0(H,\omega_2^{-1})$, $T(\psi)\cdot k \in M(H,\omega_2)\cdot C_0(H,\omega_2^{-1}) \subseteq C_0(H,\omega_2^{-1})$, and hence

$$\begin{aligned}\langle T(\mu) * T(\psi), k\rangle &= \lim\langle T(\mu_\beta) * T(\psi), k\rangle \\ &= \lim\langle T(\mu_\beta), T(\psi)\cdot k\rangle \\ &= \langle \nu, T(\psi)\cdot k\rangle \\ &= \langle \nu * T(\psi), k\rangle.\end{aligned}$$

Thus

$$T(\mu * \psi) = T(\mu) * T(\psi) = \nu * T(\psi),$$

and so

$$\mu * \psi = T^{-1}(\nu) * \psi, \quad \left(\psi \in L^1(G,\omega_1)\right). \tag{4.14}$$

Let (ψ_i) be an approximate identity as in [Gha84b, Lemma 2.1]. Then by equation (4.14), we have that

$$\mu * \psi_i = T^{-1}(\nu) * \psi_i. \tag{4.15}$$

Since by [Gha84b, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the weak-star topology of $M(G,\omega_1)$, by taking the weak-star limit in (4.15) we have that $\mu = T^{-1}(\nu)$ and therefore $T(\mu) = \nu$. This proves the claim.

Finally, because $j^*_{\gamma,\phi}$ is weak-star continuous and $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$, equation (4.13) yields

$$T(\mu) = w^* - \lim T(\mu_\beta) = w^* - \lim j^*_{\gamma,\phi}(\mu_\beta) = j^*_{\gamma,\phi}(\mu).$$

□

In the following corollary we give an alternative proof of [Far98, Thm. 2.4].

Corollary 4.2.6. *Suppose that G and H are locally compact groups. If T is a bipositive algebra isomorphism from $M(G)$ onto $M(H)$ then T is an isometry.*

Proof. By Theorem 4.2.3 and 4.2.5 and Proposition 4.2.4, there is an isomorphism of topological groups ϕ from G onto H such that $T = j_\phi^*$. It is readily seen that j_ϕ is an isometry and therefore T is an isometric algebra isomorphism. $\square$

In the Corollary 4.2.7 below we give a characterization of all bipositive algebra isomorphisms from $M(G, \omega_1)$ onto $M(H, \omega_2)$ that are also isometries.

Corollary 4.2.7. *Suppose that T is a bipositive algebra isomorphism from $M(G, \omega_1)$ onto $M(H, \omega_2)$. Then T is an isometric isomorphism if and only if there exists an isomorphism of topological groups ϕ from G onto H such that*

$$T(\delta_x) = \frac{\omega_1(x)}{\omega_2(\phi(x))}\delta_{\phi(x)} \quad \textit{and} \quad \frac{\omega_1(x)}{\omega_2(\phi(x))} \textit{ is multiplicative.}$$

Proof. Suppose that $T : M(G, \omega_1) \to M(H, \omega_2)$ is a bipositive isometric isomorphism. Then by the proof of Theorem 4.2.3, there exist an isomorphism of topological groups ϕ from G onto H and a continuous homomorphism $\gamma : G \to (0, +\infty)$ such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)} \quad \text{and} \quad \|T^{-1}\|^{-1} \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq \|T\|. \tag{4.16}$$

Since T is an isometry, $\|T\| = \|T^{-1}\| = 1$. Equation (4.16) now implies that

$$\gamma = \frac{\omega_1(x)}{\omega_2(\phi(x))} \quad \text{and} \quad T(\delta_x) = \frac{\omega_1(x)}{\omega_2(\phi(x))}\delta_{\phi(x)}. \quad \square$$

The following example shows that a bipositive algebra isomorphism between Beurling measure algebras is not necessarily an isometry.

Example 4.2.8. Let $\mathbb{R}$ denote the additive group of real numbers. Define the subadditive function g on $\mathbb{R}$ by

$$g(x) := \begin{cases} 1 & x \leq -1; \\ |x| & -1 \leq x \leq 1; \\ 1 & 1 \leq x. \end{cases}$$

Consider the weight functions $\omega_1 = 1$ and $\omega_2 := e^g$ on $\mathbb{R}$. Let $\phi_0 : \mathbb{R} \to \mathbb{R}; x \mapsto x$ be the identity isomorphism and $\gamma_0 : \mathbb{R} \to (0, +\infty); x \mapsto 1$ be the trivial homomorphism on $\mathbb{R}$. Using the notation introduced after Lemma 4.2.2, we define the operator

$$T_{\gamma_0,\phi_0} : M(\mathbb{R}) \to M(\mathbb{R}, \omega_2).$$

It can be readily seen that T is a bipositive algebra isomorphism that is not an isometry.

We remark that the above construction can be carried out in the following sections to provide bipositive algebra isomorphisms between other types of Beurling algebras (e.g., Beurling group algebras) that are not isometries.

4.3 Bipositive Isomorphisms of Beurling Group Algebras

In this section, among other results, we shall show that every bipositive algebra isomorphism T from $L^1(G, \omega_1)$ onto $L^1(H, \omega_2)$ extends to a bipositive algebra isomorphism $\tilde{T}$ from $M(G, \omega_1)$ onto $M(H, \omega_2)$. Therefore, by Theorem 4.2.3, if there exists a bipositive algebra isomorphism from $L^1(G, \omega_1)$ onto $L^1(H, \omega_2)$, then the locally compact groups G

and H are isomorphic. This generalizes the result of Kawada [Kaw48] to the context of weighted group algebras, giving it a new proof. We shall use the left multiplier algebra of $L^1(G, \omega_1)$ to define the extension $\tilde{T} : M(G, \omega_1) \to M(H, \omega_2)$.

Lemma 4.3.1. *Let (G, ω) be a weighted locally compact group. Then a left multiplier $L_\mu : \psi \mapsto \mu * \psi$ on $L^1(G, \omega)$ is positive if and only if μ is positive.*

Proof. It is obvious that for a positive measure $\mu \in M(G, \omega)$, the left multiplier $L_\mu(\psi) = \mu * \psi$ is positive. For the converse, suppose that L_μ is positive. Then the proof of [Gha84b, Lemma 2.3] shows that $\mu = w^* - \lim_i L_\mu(f_i)$, where $f_i := \dfrac{\chi_{U_i}}{\lambda(U_i)}$. Since the f_i's are positive elements in $L^1(G, \omega)$, we have that the $L_\mu(f_i)$'s are also positive and therefore $\mu = w^* - \lim_i \ L_\mu(f_i)$ is also positive. □

Let (G, ω) be a weighted locally compact group. Recall that $\mathcal{M}_l(L^1(G, \omega)$ denotes the Banach algebra of all left multipliers on $L^1(G, \omega)$ equipped with the multiplication given by composition of operators and the operator norm. The following theorem shows that there is a bipositive algebra isomorphism from $\mathcal{M}_l(L^1(G, \omega))$ onto $M(G, \omega)$.

Theorem 4.3.2. *Let (G, ω) be a weighted locally compact group. The left multiplier algebra of $L^1(G, \omega)$ is bipositively and algebraically isomorphic to $M(G, \omega)$.*

Proof. Let $\theta : M(G, \omega) \to \mathcal{M}_l \left(L^1 \left(G, \omega\right)\right)$, $\theta(\mu) := L_\mu$. It follows from [Gha84b, Lemma 2.3] and Lemma 4.2.2 that θ is an algebra isomorphism. Lemma 4.3.1, now shows that θ is bipositive. □

Lemma 4.3.3. *Let (G, ω) be a weighted locally compact group. A bounded operator $L : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is a left multiplier if and only if*

$$Lr_g = r_g L,$$

for all $g \in G$, where $r_g\psi(x) = \psi(xg)$.

Proof. The proof is the same as in [Str66, Proposition 1]. $\square$

Before proceeding to the next theorem we remark that if the left multiplier γL_{δ_x} is an algebra isomorphism, for some x in G and a positive number γ, then $x = e_G$, the identity element in G, and $\gamma = 1$.

Theorem 4.3.4. *A necessary and sufficient condition for the existence of a bipositive algebra isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$ is the existence of a topological group isomorphism ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and positive constants M and m such that*

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M \quad (x \in G).$$

Moreover, for every bipositive algebra isomorphism T from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$ there exist ϕ, γ, m and M as above such that

$$T(f) = c(\gamma \circ \phi^{-1}) \cdot f \circ \phi^{-1} \qquad (f \in L^1(G,\omega_1)),$$

where c is the measure adjustment constant given by $\dfrac{\lambda_1(E)}{\lambda_2(\phi(E))}$.

Proof. Suppose that T is a bipositive algebra isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$. Then it is easy to see that for each $\mu \in M(G,\omega)$

$$L_\mu : L^1(H,\omega_2) \to L^1(H,\omega_2);\ \psi \mapsto T(\mu * T^{-1}(\psi))$$

is a multiplier. So, by [Gha84b, Lemma 2.3], there is a measure $\tilde{T}(\mu) \in M(H,\omega_2)$ such that

$$L_\mu(\psi) = \tilde{T}(\mu) * \psi.$$

If $\mu \in M(G,\omega_1)^+$ then by Lemma 4.3.1, L_μ is also a positive multiplier, and therefore $\widetilde{T}(\mu)$ belongs to $M(H,\omega_2)^+$. An easy calculation shows that $\widetilde{T} : \mu \mapsto \tilde{T}(\mu)$ from $M(G,\omega_1)$ onto $M(H,\omega_2)$ is a bipositive algebra isomorphism extending T. Therefore, by Theorem 4.2.3, there exist an isomorphism of topological groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and positive numbers M and m such that

$$m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M \quad (x \in G).$$

For the converse, suppose that there exist a group isomorphism ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$ and constants $M > 0,\ m > 0$ such that

$$m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M.$$

We define $T : L^1(G,\omega_1) \to L^1(H,\omega_2)$ by

$$T\psi(x) = c\gamma(\phi^{-1}(x))\psi(\phi^{-1}(x)) \quad (x \in H),$$

where c is the measure adjustment constant given by $\dfrac{\lambda_1(E)}{\lambda_2(\phi(E))}$. We show that T is a bipositive algebra isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$. First, we show that T is an algebra homomorphism. For $\psi_1,\psi_2 \in L^1(G,\omega_1)$ and $y \in H$ we have

$$\begin{aligned} T\psi_1 * T\psi_2(y) &= \int_H T\psi_1(z)\ T\psi_2(z^{-1}y)\ dz \\ &= c^2\int_H \Big(\gamma\circ\phi^{-1}(z)\gamma\circ\phi^{-1}(z^{-1}y)\Big)\Big(\psi_1\circ\phi^{-1}(z)\psi_2\circ\phi^{-1}(z^{-1}y)\Big)\ dz \end{aligned}$$

Since ϕ^{-1} and $\gamma \circ \phi^{-1}$ are multiplicative,

$$\begin{aligned}
T\psi_1 * T\psi_2(y) &= c^2 \int_H \gamma \circ \phi^{-1}(y) \Big(\psi_1 \circ \phi^{-1}(z)\ \psi_2 \circ \phi^{-1}(z^{-1}y) \Big)\ dz \\
&= c^2 \gamma \circ \phi^{-1}(y) \int_H \psi_1 \circ \phi^{-1}(z)\ \psi_2 \circ \phi^{-1}(z^{-1}y)\ dz \\
&= \left[c\gamma \circ \phi^{-1}(y) \right] \left[c \int_H \psi_1 \circ \phi^{-1}(z)\ \psi_2 \circ \phi^{-1}(z^{-1}y)\ dz \right] \\
&= c\gamma \circ \phi^{-1}(y) \int_G \psi_1(x)\ \psi_2 \left(x^{-1} \phi^{-1}(y) \right)\ dx \\
&= c\gamma \circ \phi^{-1}(y)\ \psi_1 * \psi_2 \left(\phi^{-1}(y) \right) = T(\psi_1 * \psi_2)(y).
\end{aligned}$$

Thus, T is an algebraic homomorphism. We can see that the mapping S defined by

$$L^1(H, \omega_2) \to L^1(G, \omega_1);\ \psi \mapsto c^{-1}\ \overline{\gamma} \cdot \psi \circ \phi$$

from $L^1(H, \omega_2)$ to $L^1(G, \omega_1)$ is an inverse to T. To see this, let $\psi \in L^1(H, \omega_2)$. We have that

$$\begin{aligned}
T(S(\psi))(x) &= T(c^{-1}\ \overline{\gamma} \cdot \psi \circ \phi)(x) \\
&= \Big(\gamma \circ \phi^{-1} \cdot (\overline{\gamma} \cdot \psi \circ \phi) \circ \phi^{-1} \Big)(x) \\
&= \gamma \circ \phi^{-1}(x) \overline{\gamma} \circ \phi^{-1}(x) \psi \circ \phi \circ \phi^{-1}(x) = \psi(x).
\end{aligned}$$

Similarly, we can see that $S \circ T = id_{L^1(G, \omega_1)}$. So, T is an algebra isomorphism. It can be readily seen that T is bipositive. Hence, T is a bipositive algebra isomorphism.

Now suppose that $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is a bipositive algebra isomorphism. By the above argument, to T there corresponds a group isomorphism ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0, +\infty)$ and positive constants M and m such that

$$m \leq \gamma(x) \frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M.$$

Let $U : L^1(G,\omega_1) \to L^1(H,\omega_2)$ be defined by

$$U(\psi) = c\left(\gamma \circ \phi^{-1}\right) \cdot \psi \circ \phi^{-1},$$

where c is the measure adjustment constant given by $\dfrac{\lambda_1(E)}{\lambda_2(\phi(E))}$.It is readily seen that U is a bipositive algebra isomorphism. A simple calculation shows that for every y in H

$$TU^{-1}r_yUT^{-1} = r_y,$$

where $r_yf(x) = f(xy)$. To see this, let $\tilde{T}$ be the extension of T to $M(G,\omega_1)$ as described above. For every ψ in $L^1(H,\omega_2)$ and y in H we have

$$\begin{aligned}
TU^{-1}r_yUT^{-1}(\psi)(x) &= TU^{-1}r_y\Big(c\ \gamma \circ \phi^{-1} \cdot T^{-1}(\psi) \circ \phi^{-1}\Big)(x)\\
&= TU^{-1}\Big(c\ \gamma \circ \phi^{-1} \cdot T^{-1}(\psi) \circ \phi^{-1}\Big)(xy)\\
&= T\Big(c^{-1}\overline{\gamma} \cdot \Big(c\gamma \circ \phi^{-1} \cdot T^{-1}(\psi) \circ \phi^{-1}\Big) \circ \phi\Big)(xy)\\
&= T\left(T^{-1}\psi\right)(xy)\\
&= \psi(xy) = r_y(\psi)(x),
\end{aligned}$$

since

$$\begin{aligned}
\Big(c^{-1}\overline{\gamma} \cdot \Big(c\gamma \circ \phi^{-1} \cdot T^{-1}(\psi) \circ \phi^{-1}\Big) \circ \phi\Big)(z) &= c^{-1}\overline{\gamma}(z)\Big(c\gamma \circ \phi^{-1} \cdot T^{-1}(\psi) \circ \phi^{-1}\Big)(\phi(z))\\
&= \left(c^{-1}\overline{\gamma}(z)\right)\left(c\gamma(z)\right)T^{-1}(\psi)(z) = T^{-1}(\psi)(z).
\end{aligned}$$

Therefore, by Lemma 4.3.3, the algebra isomorphism TU^{-1} is also a left multiplier. We note that the left multiplier TU^{-1} is in fact a bipositive invertible left multiplier. Since TU^{-1} is positive it follows from Lemma 4.3.1 that the measure μ associated with TU^{-1} is also positive. Since TU^{-1} is an invertible left multiplier, μ is also invertible. Since TU^{-1} is bipositive, μ^{-1} is also positive. It follows from Lemma 4.1.4 that there exist a positive constant γ and an element x in H such that $TU^{-1} = \gamma L_{\delta_x}$. Now by the remark mentioned before the statement of this theorem we have that TU^{-1} is the identity operator on $L^1(H, \omega_2)$, and so $T = U$. $\square$

The following corollary gives an alternative proof to [Wen51, Thm. 2].

Corollary 4.3.5. *Suppose that G and H are locally compact groups. If T is a bipositive algebra isomorphism from $L^1(G)$ onto $L^1(H)$, then T is an isometry.*

Proof. By Theorem 4.2.3 and 4.3.4, there is an isomorphism of topological groups ϕ from G onto H such that for each ψ in $L^1(G)$ we have that $T = c\psi \circ \phi^{-1}$, where c is the measure adjustment constant introduced above. It is now readily seen that T is an isometry and therefore an isometric algebra isomorphism. $\square$

4.4 Bipositive Isomorphisms of the Duals of Weighted $LUC-$functions

Suppose that (G, ω_1) and (H, ω_2) are weighted locally compact topological groups. Let ϕ from G onto H be an isomorphism of locally compact groups, $\gamma : G \to (0, +\infty)$ be a continuous homomorphism and M and m be positive constants such that

$$m \leq \gamma(x) \frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M \quad (x \in G).$$

Proceeding just as in Section 3.4, we define the mapping

$$J_{\gamma,\phi} : LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1}) \quad \text{where} \quad J_{\gamma,\phi}(f) = \gamma \cdot f \circ \phi.$$

Then it is not difficult to see that $J_{\gamma,\phi}$ is a bipositive (bounded) linear isomorphism mapping $LUC(H, \omega_2^{-1})$ onto $LUC(G, \omega_1^{-1})$. We note that $J_{\gamma,\phi}$ maps $C_0(H, \omega_2)$ onto $C_0(G, \omega_1)$. Now, the dual mapping

$$T_{\gamma,\phi} := J_{\gamma,\phi}^* : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$$

is also a bipositive (bounded) linear isomorphism such that

$$T_{\gamma,\phi}(\delta_x) = \gamma(x)\delta_{\phi(x)} \quad (x \in G). \tag{4.17}$$

We observe that $T_{\gamma,\phi}$ is also multiplicative. To see this, first note that since γ and ϕ are multiplicative, it can be readily seen from the equation (4.17) above, that $T_{\gamma,\phi}$ is multiplicative on the linear span of point masses. Now, to see that $T_{\gamma,\phi}$ is multiplicative on $LUC(G, \omega_1^{-1})^*$, we note that the linear span of point masses is weak-star dense in $LUC(G, \omega_1^{-1})^*$; for each $n \in LUC(G, \omega_1^{-1})^*; m \mapsto m\Box n$ is weak-star continuous on $LUC(G, \omega_1^{-1})^*$; for each $\mu \in M(G, \omega_1)$, $n \mapsto \mu\Box n$ is weak-star continuous; and from equation (4.17), $T_{\gamma,\phi}$ maps the linear span of point masses in $LUC(G, \omega_1^{-1})^*$ into $M(H, \omega_2)$. So, if $m, n \in LUC(G, \omega_1^{-1})^*$ we can find nets (μ_i) and (ν_j) in the linear span of the point masses such that $w^* - \lim \mu_i = m$ and $w^* - \lim \nu_j = n$, and obtain

$$\begin{aligned}
T_{\gamma,\phi}(m\Box n) &= T_{\gamma,\phi}\left(w^* - \lim_i \left(w^* - \lim_j (\mu_i\Box\nu_j)\right)\right) \\
&= w^* - \lim_i \left(w^* - \lim_j T_{\gamma,\phi}(\mu_i\Box\nu_j)\right) \\
&= w^* - \lim_i \left(w^* - \lim_j \left(T_{\gamma,\phi}(\mu_i)\Box T_{\gamma,\phi}(\nu_j)\right)\right) = T_{\gamma,\phi}(m)\Box T_{\gamma,\phi}(n),
\end{aligned}$$

as required. In particular, this shows that $LUC(G, \omega_1^{-1})^*$ and $LUC(H, \omega_2^{-1})^*$ are bipositively algebraically isomorphic Banach algebras.

Theorem 4.4.1. *The Banach algebras $LUC(G, \omega_1^{-1})^*$ and $LUC(H, \omega_2^{-1})^*$ are bipositively algebraically isomorphic if and only if there exists a topological group isomorphism ϕ from*

G onto H, a continuous homomorphism $\gamma : G \to \big(0, +\infty\big)$, and positive constants m and M such that

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M \quad \big(x \in G\big).$$

Proof. Let $T : LUC(G, \omega_1)^* \to LUC(H, \omega_2)^*$ be a bipositive algebra isomorphism. By [AB85, Thm.4.3], T and T^{-1} are bounded operators. Given $x \in G$, since T is a positive operator $T(\delta_x)$ is a positive measure. Since T is an algebra isomorphisms $T(\delta_x)$ is also invertible with a positive inverse $T(\delta_{x^{-1}})$. Therefore by Lemma 4.1.4, there exist $\phi(x) \in H$ and a positive number $\gamma(x)$ such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

Since T is an algebra isomorphism, we have that both γ and ϕ are multiplicative.

We shall now show that γ and ϕ are continuous. Suppose that $\big(x_\alpha\big)$ is a net in G such that $x_\alpha \to x$ in G. Then by Lemma 4.2.1, for every ψ in $L^1(G, \omega_1)$ we have that

$$\delta_{x_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \delta_x * \psi.$$

Since T is a bounded algebra isomorphism

$$T\left(\delta_{x_\alpha}\right) \Box T\left(\psi\right) \xrightarrow{\|\cdot\|_{1,\omega_2}} T\left(\delta_x\right) \Box T\left(\psi\right).$$

Without loss of generality, we can assume that x_α are contained in a compact neighbourhood U of x. Since

$$\|T(\delta_{x_\alpha})\| \leq \|T\| \sup\{\omega_1(t) : t \in U\},$$

$\big(T(\delta_{x_\alpha})\big)$ is a bounded net in $M(H, \omega_2)$. Thus, there is a subnet $\big(T(\delta_{x_{\alpha_i}})\big)$ and an element m in $LUC(H, \omega_2^{-1})^*$ such that $T(\delta_{x_{\alpha_i}}) \to m$ in the weak-star topology of $LUC(H, \omega_2^{-1})^*$. Therefore,

$$T\left(\delta_{x_{\alpha_i}}\right) \Box T\left(\psi\right) \xrightarrow{w^*} m \Box T\left(\psi\right).$$

Hence, for every ψ in $L^1(G,\omega_1)$

$$\delta_x \Box \psi = T^{-1}(m) \Box \psi.$$

Therefore, $T(\delta_x) = m$, since $LUC(G,\omega_1^{-1}) = L^1(G,\omega_1)\Box LUC(G,\omega_1^{-1})$. A similar discussion then shows that every subnet of $\big(T(\delta_{x_\alpha})\big)$ has a subnet convergent to $T(\delta_x)$. Hence,

$$\gamma(x_\alpha)\delta_{\phi(x_\alpha)} = T(\delta_{x_\alpha}) \xrightarrow{w^*} T(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

An argument similar to that in the proof of Theorem 4.2.3 shows that γ and ϕ are continuous. By considering T^{-1} we can show that ϕ is surjective with a continuous inverse. $\Box$

4.5 Bipositive Isomorphisms between the Biduals of Beurling Algebras

We conclude this chapter by showing that the order structure combined with algebra structure of the bidual of the weighted group algebra $L^1(G,\omega)^{**}$ determines the locally compact group G. This result generalizes Farhadi [Far98, Thm.2.2] to the context of Beurling algebras. We remark that the same ideas as in the proof of Farhadi [Far98, Thm.2.2] cannot be followed to provide a proof for Theorem 4.5.1 below. This is mainly because when $\omega \not\equiv 1$ the function ω is not a multiplicative linear functional on $L^1(G,\omega)$.

Theorem 4.5.1. *The Banach algebras $L^1(G,\omega_1)^{**}$ and $L^1(H,\omega_2)^{**}$ are bipositively and algebraically isomorphic if and only if there exist an isomorphism of topological groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0,+\infty)$, and positive numbers m and M such that*

$$m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M \quad \big(x \in G\big).$$

Proof. First we suppose that there exists a bipositive algebra isomorphism T from $L^1(G,\omega_1)^{**}$ onto $L^1(H,\omega_2)^{**}$. Let (f_i) be a bounded approximate identity of $L^1(G,\omega_1)$ with $f_i \geq 0$, for all i (see [Gha84b, Lemma 2.1]), and let E be a weak-star cluster point of (f_i). Then, $E \geq 0$. By [BD73, Prop.III.28.7], E is a right identity of $L^1(G,\omega_1)^{**}$. Clearly, E is positive. Hence $T(E)$ is also a positive right identity of $L^1(H,\omega_2)^{**}$. Now we argue as in [GL88]. The maps

$$\tau_E : EL^1(G,\omega_1)^{**} \longrightarrow LUC(G,\omega_1^{-1})^*;\ En \mapsto n|_{LUC(G,\omega_1^{-1})^*}$$

and

$$\tau_{T(E)} : T(E)L^1(H,\omega_2)^{**} \longrightarrow LUC(H,\omega_2^{-1})^*;\ T(E)m \mapsto m|_{LUC(H,\omega_2^{-1})^*}$$

establish bipositive algebra isomorphisms between each domain and target algebra. Hence $\tau_E^{-1} \circ T \circ \tau_{T(E)}$ is a bipositive algebra isomorphism from $LUC(G,\omega_1^{-1})^*$ onto $LUC(H,\omega_2^{-1})^*$. It now follows from Theorem 4.4.1 that there exist a topological group isomorphism ϕ from G onto H, a group isomorphism $\gamma : G \to (0,+\infty)$ and positive numbers m and M such that

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M \quad \big(x \in G\big).$$

For the converse, suppose that there exist a topological group isomorphism ϕ from G onto H, a group isomorphism $\gamma : G \to (0,+\infty)$ and positive numbers m and M such that

$$m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M \quad \big(x \in G\big).$$

Then by Theorem 4.3.4, there exists a (bounded) bipositive isomorphism $T : L^1(G,\omega_1) \to L^1(H,\omega_2)$. Then $T^{**} : L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is a bipositive algebra isomorphism. □

Chapter 5

Open Questions

We conclude this thesis with a selective list of open problems.

1. By [Joh64b], every isometric isomorphism $T : M(G) \to M(H)$ is weak-star continuous, so is natural to ask if every isometric isomorphism $T : LUC(G)^* \to LUC(H)^*$ – equivalently, $T : M(G^{LUC}) \to M(H^{LUC})$ – is weak-star continuous. An affirmative answer to this question would provide us with a description of T in terms of a topological isomorphism between G and H, and a continuous character on G.

2. Related to the first question, we have the following interesting problem. Is every isometric isomorphism $T : C_0(G)^\perp \to C_0(H)^\perp$ weak-star continuous. Similarly, we can ask if every algebraic isomorphism $\phi : G^* \to H^*$ must be continuous. Are these two questions equivalent? That is, if $T : C_0(G)^\perp \to C_0(H)^\perp$ is an isometric isomorphism and $T|_{G^*}$ is continuous, is T weak-star continuous?

3. In Chapter 2, we showed that if $T : LUC(G)^* \to LUC(H)^*$ is an isometric isomorphism then $T = T_1 \oplus_1 T_2$, where $T_1 = T|_{M(G)} : M(G) \to M(H)$ and $T_2 = T|_{C_0(G)^\perp}$:

$C_0(G)^\perp \to C_0(H)^\perp$ are isometric isomorphisms. Moreover, every isometric isomorphism $T : M(G) \to M(H)$ extends to a weak-star continuous isometric isomorphism of $LUC(G)^*$ onto $LUC(H)^*$. This gives rise to the following enthralling question. Does every (weak-star continuous) isometric isomorphism $T : C_0(G)^\perp \to C_0(H)^\perp$ extend to an isometric isomorphism $\widetilde{T} : LUC(G)^* \to LUC(H)^*$?

4. When G is discrete, we proved that the Banach algebra structure of $C_0(G)^\perp$ determines G within the class of locally compact groups. Is this also true when G is not discrete? Similarly we can ask if the existence of a continuous isomorphism $\phi : G^* \to H^*$ implies that the underlying locally compact topological groups G and H must be isomorphic.

5. Is every algebraic isomorphism $\phi : G^{LUC} \to H^{LUC}$ automatically continuous? The answer to this question is not even known for the case of the discrete group $\mathbb{Z}$ of integers. This is a very difficult problem. As mentioned before, when G is a discrete group the LUC-compactification of G, G^{LUC}, is the same as its Stone-Čech compactification. Therefore, this question is of interest to people working not only in analysis but also in combinatorics and mathematical logic.

6. Is $LUC(G)^*$ the normed double centralizer algebra for $C_0(G)^\perp$? If the answer is negative, then is there a nice description for the normed double centralizer algebra for $C_0(G)^\perp$? In view of [Zadb], studying this question will open the door to a description of the weak-star continuous isometric isomorphism $T : C_0(G)^\perp \to C_0(H)^\perp$ in terms of a topological isomorphism between locally compact groups G and H.

7. Suppose that G and H are locally compact groups. If $T : LUC(G)^* \to LUC(H)^*$ is a bipositive algebra isomorphism, is there a description of T in terms of a topological isomorphism between G and H?

Appendix A

In an earlier draft of [Zada], we made significant use of Lemma A.0.3. As we feel that Lemma A.0.3 is of independent interest, we include its proof in this appendix.

As in Example 1.1.3 part (ii), $M(G,\omega)$ acts by convolution on the ideal $L^1(G,\omega)$ on the right. Dualizing this action as in Example 1.1.3 part (iii), gives us a left action of $M(G,\omega)$ on $L^\infty(G,\omega^{-1})$. Explicitly, we have the following module action

$$\langle \mu \cdot g, \phi \rangle = \langle g, \phi * \mu \rangle,$$

where $\mu \in M(G,\omega)$ and $g \in L^\infty(G,\omega^{-1})$ and $\phi \in L^1(G,\omega)$. It is not hard to see that for each $g \in L^\infty(G,\omega^{-1})$ and $x \in G$

$$\delta_x \cdot g = r_x g,$$

where $r_x g(y) = g(yx)$, for all $y \in G$.

In part (a) of Lemma A.0.3, we will give a point-wise description of the action $\mu \cdot g$ defined above. For this purpose we need the following lemma.

Lemma A.0.2. *Suppose that $g \in LUC(G,\omega^{-1})$ and $\mu \in M(G,\omega)$. Then the function F defined by*

$$F(x) := \int_G g(xy)d\mu(y)$$

belongs to $C_b(G,\omega^{-1})$.

Proof. First we note that $\dfrac{F}{\omega}$ is bounded since

$$\Big|\frac{F(x)}{\omega(x)}\Big| = \Big|\int_G \frac{g(xy)}{\omega(x)}d\mu(y)\Big| \leq \int_G \Big|\frac{g(xy)}{\omega(xy)}\Big|\Big|\frac{\omega(xy)}{\omega(x)}\Big|d|\mu|(y)$$
$$\leq \|g\|_{\infty,\omega^{-1}} \int_G \omega(y)d|\mu|(y) \leq \|g\|_{\infty,\omega^{-1}}\|\mu\|_\omega.$$

Now we show that $\dfrac{F}{\omega}$ is continuous at any given point $x \in G$. Let $\epsilon > 0$ be given. By the fact that $g \in LUC(G,\omega^{-1})$ and the above lemma, there is a symmetric neighbourhood U of the identity such that if $xz^{-1}, u \in U$ we have that

$$\left|\frac{g(xy)}{\omega(xy)} - \frac{g(zy)}{\omega(zy)}\right| \leq \epsilon \quad \text{and} \quad \left|\frac{\omega(uy)}{\omega(y)} - 1\right| \leq \epsilon \quad \forall y \in G. \tag{A.1}$$

Next, we shall show that for x and z chosen above

$$\left|\frac{\omega(xy)}{\omega(x)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)}\right| \leq 3\epsilon \quad (y \in G).$$

(Here we are assuming, without loss of generality that $\epsilon < \frac{1}{3}$.) To see this, first note that since $xz^{-1} \in U$, there is $u_0 \in U$ such that $x = u_0 z$. So, we have that

$$\Big|\frac{\omega(xy)}{\omega(x)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)}\Big| = \Big|\frac{\omega(u_0zy)}{\omega(u_0z)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)}\Big|.$$

Now, by the equation (A.1) we have that

$$1-\epsilon \leq \frac{\omega(u_0zy)}{\omega(zy)} \leq 1+\epsilon \quad \text{and} \quad 1-\epsilon \leq \frac{\omega(u_0z)}{\omega(z)} \leq 1+\epsilon.$$

It follows that

$$\frac{\omega(u_0zy)}{\omega(u_0z)\omega(y)} \leq \frac{(1+\epsilon)\omega(zy)}{\omega(u_0z)\omega(y)} \leq \frac{(1+\epsilon)\omega(zy)}{(1-\epsilon)\omega(z)\omega(y)}. \tag{A.2}$$

Similarly we have that

$$\frac{\omega(u_0zy)}{\omega(u_0z)\omega(y)} \geq \frac{(1-\epsilon)\omega(zy)}{\omega(u_0z)\omega(y)} \geq \frac{(1-\epsilon)\omega(zy)}{(1+\epsilon)\omega(z)\omega(y)}. \tag{A.3}$$

So, by the equations (A.2) and (A.3)

$$\frac{(1-\epsilon)\omega(zy)}{(1+\epsilon)\omega(z)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)} \leq \frac{\omega(u_0zy)}{\omega(u_0z)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)} \leq \frac{(1+\epsilon)\omega(zy)}{(1-\epsilon)\omega(z)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)}.$$

But

$$\Big(\frac{1-\epsilon}{1+\epsilon} - 1\Big)\frac{\omega(zy)}{\omega(z)\omega(y)} \geq (-3\epsilon)\frac{\omega(zy)}{\omega(z)\omega(y)},$$

and

$$\Big(\frac{1+\epsilon}{1-\epsilon} - 1\Big)\frac{\omega(zy)}{\omega(z)\omega(y)} \leq (3\epsilon)\frac{\omega(zy)}{\omega(z)\omega(y)},$$

so

$$\Big|\frac{\omega(u_0zy)}{\omega(u_0z)\omega(y)} - \frac{\omega(zy)}{\omega(z)\omega(y)}\Big| \leq (3\epsilon)\frac{\omega(zy)}{\omega(z)\omega(y)} \leq 3\epsilon,$$

as we claimed. Now,

$$\begin{aligned}
\left|\frac{F(x)}{\omega(x)} - \frac{F(z)}{\omega(z)}\right| &= \left|\int_G \left(\frac{g(xy)}{\omega(x)} - \frac{g(zy)}{\omega(z)}\right) d\mu(y)\right| \\
&\leq \int_G \Big|\frac{g(xy)}{\omega(xy)}\frac{\omega(xy)}{\omega(x)} - \frac{g(zy)}{\omega(zy)}\frac{\omega(zy)}{\omega(z)}\Big| d|\mu|(y) \\
&\leq \|g\|_{\infty,\omega^{-1}} \int_G \Big|\frac{\omega(xy)}{\omega(x)} - \frac{\omega(zy)}{\omega(z)}\Big| d|\mu|(y) + \int_G \Big|\frac{g(xy)}{\omega(xy)}\frac{\omega(zy)}{\omega(z)} - \frac{g(zy)}{\omega(zy)}\frac{\omega(zy)}{\omega(z)}\Big| d|\mu|(y)
\end{aligned}$$

$$= \|g\|_{\infty,\omega^{-1}} \int_G \frac{1}{\omega(y)} \Big| \frac{\omega(xy)}{\omega(x)} - \frac{\omega(zy)}{\omega(z)} \Big| \omega(y) d|\mu|(y)$$

$$+ \int_G \left| \frac{g(xy)}{\omega(xy)} - \frac{g(zy)}{\omega(zy)} \right| \frac{\omega(zy)}{\omega(y)\omega(z)} \omega(y) d|\mu|(y)$$

$$\leq 3\epsilon \ \|g\|_{\infty,\omega^{-1}} \|\mu\|_\omega + \epsilon \|\mu\|_\omega,$$

using (A.1) and Lemma 3.1.1. Therefore, $\dfrac{F}{\omega}$ is continuous. □

Various parts of the following lemma are also obtained in [DL05, Prop.7.17], in the special case when $\omega \geq 1$. We will give a self-contained proof that works for any given weight ω.

Lemma A.0.3. *Let (G, ω) be a weighted locally compact group. Suppose that $\mu \in M(G, \omega)$ and $g \in LUC(G, \omega^{-1})$. Then*

(a) $\mu \cdot g$ is equivalent in $L^\infty(G, \omega^{-1})$ to a continuous function given by $F(x) = \int_G g(xy) d\mu(y)$, for each $x \in G$.

(b) If $g \in C_0(G, \omega^{-1})$ then $\mu \cdot g$ belongs to $C_0(G, \omega^{-1})$, i.e., $C_0(G, \omega^{-1})$ is a left Banach $M(G, \omega)$-module.

(c) $L^1(G, \omega) \cdot C_0(G, \omega^{-1}) = C_0(G, \omega^{-1})$.

(d) If $\phi \in L^1(G, \omega)$ and $f \in C_0(G, \omega^{-1})$ then we have $\langle \mu, \phi \cdot f \rangle = \langle \mu * \phi, f \rangle$.

Proof. (a) Let $g \in LUC(G, \omega^{-1})$ and $\mu \in M(G, \omega)$. For each $x \in G$, we define

$$F(x) := \int_G g(xy) d\mu(y).$$

By Lemma A.0.2, F belongs to $C_b(G,\omega^{-1})$. For each $\phi \in L^1(G,\omega)$

$$\begin{aligned}\langle F,\phi\rangle &= \int_G\int_G \phi(t)g(ts)d\mu(s)dt\\ &= \int_G\int_G \phi(ts^{-1})\Delta(s^{-1})g(t)d\mu(s)dt\\ &= \langle g,\phi*\mu\rangle = \langle \mu\cdot g,\phi\rangle.\end{aligned}$$

Thus, $F=\mu\cdot g$ in $L^\infty(G,\omega^{-1})$.

(b) First suppose that $g\in C_c(G,\omega^{-1})$ and that $\mu\in M(G,\omega)$ has compact support. Then, by part (a) $\mu\cdot g$ is continuous and since it has a compact support it belongs to $C_c(G)$. Since $C_c(G,\omega^{-1})$ is dense in $C_0(G,\omega^{-1})$ and the measures with compact support are dense in $M(G,\omega)$, it follows that $\mu\cdot g$ belongs to $C_0(G,\omega^{-1})$. Therefore, $(C_0(G,\omega^{-1}),\cdot)$ is a left Banach $M(G,\omega)$-module. (We remark that it is not difficult to obtain part (b) without part (a). To see this, note that $M(G,\omega)$ is a dual Banach algebra, so $C_0(G,\omega^{-1})$ is a submodule of $M(G,\omega)^*$ and one can show that the two module operations agree. However, we do seem to need part (a) to obtain part (c).)

(c) By part (b), it suffices to show that $C_0(G,\omega^{-1})\subseteq L^1(G,\omega)\cdot C_0(G,\omega^{-1})$. To this end, given $h\in C_c(G)=C_c(G,\omega^{-1})$ we will show that h belongs to the norm closure of $L^1(G,\omega)\cdot C_0(G,\omega^{-1})$ in $C_0(G,\omega^{-1})$. The result will then follow from the density of $C_c(G,\omega^{-1})$ in $C_0(G,\omega^{-1})$ and the Cohen factorization theorem.

Let U be a fixed compact neighbourhood of the identity e in G and let $(U_i)_i$ be a base for the neighbourhood system at identity e, with each U_i contained in U. Observe that for each i, $f_i := \dfrac{\chi_{U_i}}{\lambda(U_i)} \in L^1(G,\omega)$, where λ denotes the Haar measure. We will complete the proof by showing that $\|f_i\cdot h-h\|_{\infty,\omega^{-1}}\to 0$. Let $\epsilon>0$. Observe that by part (a), the support, $\mathrm{Supp}\,(f_i\cdot h-h)\subseteq KU^{-1}$ where $K:=\mathrm{Supp}(h)$. Let $M:=\sup\{\omega(x)^{-1}: x\in KU^{-1}\}$. As $h\in C_c(G)$ is right uniformly continuous on G, we can choose i_0 such that $\|r_yh-h\|_\infty\le \dfrac{\epsilon}{M}$, whenever $i \geq i_0$ and $y \in U_i$. Then for $i \geq i_0$ and any $x \in G$

$$\begin{aligned}
\left|\frac{(f_i \cdot h - h)(x)}{\omega(x)}\right| &\leq M\Big|(f_i \cdot h - h)(x)\Big| \\
&= M\left|\int_G h(xy) f_i(y) dy - h(x)\right| \\
&= M\left|\frac{1}{\lambda(U_i)} \int_{U_i} (h(xy) - h(x))\, dy\right| \\
&\leq \frac{M}{\lambda(U_i)} \int_{U_i} \Big|(r_y h - h)(x)\Big| dy \\
&\leq \frac{M}{\lambda(U_i)} \int_{U_i} \|r_y h - h\|_\infty dy \\
&\leq \frac{M}{\lambda(U_i)} \int_{U_i} \frac{\epsilon}{M} dy = \epsilon.
\end{aligned}$$

(d) Let (ψ_i) be a bounded approximate identity for $L^1(G, \omega)$ as in [Gha84b, Lemma 2.1]. As $M(G, \omega)$ is a dual Banach algebra, $f \cdot \mu$ – as used below – is in $C_0(G, \omega^{-1})$ and therefore

$$\begin{aligned}
\langle \mu * \phi, f \rangle &= \langle \phi, f \cdot \mu \rangle \\
&= \lim_i \langle \psi_i * \phi, f \cdot \mu \rangle \\
&= \lim_i \langle \mu * \psi_i, \phi \cdot f \rangle \\
&= \langle \mu, \phi \cdot f \rangle,
\end{aligned}$$

since by [Gha84b, Lemma 2.2], $\psi_i \to \delta_{e_G}$ weak-star and the convolution product is separately weak-star continuous. $\square$

A consequence of the above lemma is the following result, which was used in Chapter 3.

Proposition A.0.4. *The weak-star topology on $M(G,\omega)$ is contained in the strong operator topology. Hence, the strong operator topology and the weak-star topology agree on strong operator compact subsets of $M(G,\omega)$.*

Proof. It suffices to show that the identity map on $M(G,\omega)$ is SO-w* continuous. To see this let (μ_i) be a net in $M(G,\omega)$ such that $\mu_i \to \mu$ in the strong operator topology and let $g \in C_0(G,\omega^{-1})$. Then $g = \phi \cdot f$ for some $\phi \in L^1(G,\omega)$ and $f \in C_0(G,\omega^{-1})$, so

$$\begin{aligned} |\langle \mu_i - \mu, g\rangle| &= |\langle \mu_i - \mu, \phi \cdot f\rangle| \\ &= |\langle (\mu_i - \mu) * \phi, f\rangle| \\ &\le \|\mu_i * \phi - \mu * \phi\|_{1,\omega} \|f\|_{\infty,\omega^{-1}} \to 0, \end{aligned}$$

where we have used parts (c) and (d) of Lemma A.0.3. Observe that if L is an SO-compact subset of $M(G,\omega)$, because $id_L : L \to L$ is continuous and the weak-star topology is Hausdorff, id_L is also a closed map, and therefore a SO-w* homeomorphism. $\square$

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