# GENERALIZATIONS OF AHLFORS LEMMA AND BOUNDARY BEHAVIOR OF ANALYTIC FUNCTIONS

by

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# Abstract

Without any doubts, the most important objects in complex analysis are the analytic functions. In this thesis we will consider and investigate the properties of analytic functions via their behavior near the boundary of the domain on which they are defined.

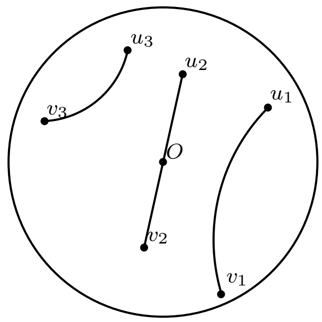
The first part of the thesis (Chapter 1) is an introduction of the hyperbolic metric on the domain and range of an analytic function. Depending on the shape of the domain, the Euclidean metric ceases to be the most natural metric to consider and the hyperbolic metric becomes the best choice, in some sense. For example, every analytic function is a compression with respect to the hyperbolic metric [Schwarz-Pick Lemma].

Further, in Chapter 2 we will look at the the hyperbolic distortion and the hyperbolic derivative, which are the main tools we use to explain the boundary behavior of analytic functions. Classical results state that the hyperbolic derivative is also bounded from above by 1, and we will consider in Section 3.1 the case when it is bounded from below by some positive constant. Boundedness from below implies some nice properties of the function near the boundary. This is described in the first three sections of Chapter 3. For instance, if the function is defined on a nice enough domain, then boundedness from below of the hyperbolic derivative implies that the function has an analytic continuation across the boundary. We extend one of the results given in [9] and state it as Theorem 3.20 and Theorem 3.21.

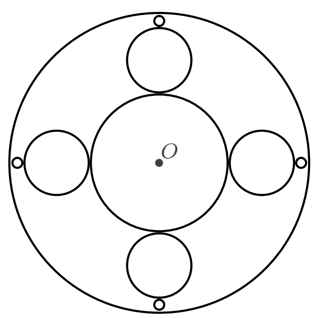
As for the last section, we consider a characterization theorem of Bloch spaces in the case of a simply connected hyperbolic planar domain. Almost all of the functions that we are considering in this thesis belong to the Bloch space, a space

which is deeply connected with the notions of pull-back metric and general distortion. The characterization Theorem 3.27 is given in the same format as a corresponding theorem from [16] for the case of bounded homogeneous domains in $\mathbb{C}^n$, and is a special case of the results proven in [12] for Riemann surfaces. We give a simple proof of this theorem by using the behavior of the hyperbolic derivative and some techniques that were introduced in the previous sections.

# List of figures





# Contents

# Chapter 1

# Preliminaries

## 1.1 Basic facts about the hyperbolic metric

The hyperbolic metric is a specific metric on hyperbolic domains. The simplest hyperbolic domain is the unit disk, and we will start by describing the hyperbolic metric on the disk. In this thesis the hyperbolic metric plays a significant role because it explains some fundamental results of complex analysis from a geometrical point of view. It is also deeply connected with the Bloch-type spaces, and the behavior of self-maps of the unit disk near the boundary. This chapter is mostly based on Beardon and Minda's paper [1] and almost all of the results and their proofs are taken from there.

Through the thesis I will be using the notion of conformal map, by which I mean a map $\phi$, such that $\phi' \neq 0$ in the domain of $\phi$. By a conformal mapping I will mean a univalent analytic function.

First of all, a *metric* on some domain will be just a positive real-valued function that is twice differentiable which can be interpreted as a density of some distance function. In this chapter I will consider a metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$.

**Definition 1.1.** *The hyperbolic metric on* $\mathbb{D}$ *is defined by* $\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$.

Note, that usually a metric is meant to be a symmetric non-negative function $\rho : X \times X \to [0; +\infty]$ on a metric space $X$, that satisfies the triangle inequality. The metric we have defined is just a density of a metric in usual sense. We can define a length of a vector $u$ with initial point $a \in \mathbb{D}$ in a metric $\lambda_{\mathbb{D}}$ by

$$||u||_{\lambda_{\mathbb{D}}} = |u| \cdot \lambda_{\mathbb{D}}(a).$$

Then for any two points $z$ and $w$ in $D$ and any smooth curve $\gamma : [a, b] \to \mathbb{D}$ that joins $z$ and $w$, define the hyperbolic length of $\gamma$ by

$$l_{\mathbb{D}}(\gamma) = \int_{\gamma} \lambda_{\mathbb{D}}(z)|dz| = \int_a^b ||\gamma'||_{\lambda_{\mathbb{D}}} dt.$$

Then we come up with a distance function on $\mathbb{D}$, induced by the hyperbolic metric:

$$d_{\mathbb{D}}(z, w) = \inf_{\gamma} l_{\mathbb{D}}(\gamma),$$

where the infimum is taken over all smooth curves that join $z$ and $w$.

It can be easily verified that $d_{\mathbb{D}}(z, w)$ satisfies all of the requirements for a distance function. It is nonnegative, symmetric and it satisfies the triangle inequality.

Let us consider isometries of both the hyperbolic metric and the hyperbolic distance.

**Definition 1.2.** *An analytic function $\phi : \mathbb{D} \to \mathbb{D}$ is called an isometry of the metric $\lambda_{\mathbb{D}}(z)$ if for all $z$ in $\mathbb{D}$,*

$$\lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| = \lambda_{\mathbb{D}}(z), \tag{1.1}$$

*and $\phi$ is called an isometry of the distance $d_{\mathbb{D}}$ if, for any $z$ and $w$ from $\mathbb{D}$,*

$$d_{\mathbb{D}}(\phi(z), \phi(w)) = d_{\mathbb{D}}(z, w). \tag{1.2}$$

The definition of isometry for the hyperbolic distance is more or less intuitive. However, it is not obvious, why we have such a definition for an isometry of the hyperbolic metric. This can be explained by the fact that if $\phi$ is an isometry of the hyperbolic metric, then for any curve $\gamma$ we have $l_{\mathbb{D}}(\gamma) = l_{\mathbb{D}}(\phi \circ \gamma)$. The next theorem states that two classes of isometries coincide and that they are just the conformal automorphisms of the unit disk. The most general form of a conformal automorphism of the unit disk onto itself is

$$\phi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad \theta \in \mathbb{R}, \quad a \in \mathbb{D}.$$

The set of all conformal automorphisms of $\mathbb{D}$ is denoted by $\mathbf{Aut}(\mathbb{D})$. Note that $\mathbf{Aut}(\mathbb{D})$ is a transitive group on $\mathbb{D}$ under the operation of composition.

**Theorem 1.3.** *For any analytic map $\phi : \mathbb{D} \to \mathbb{D}$ the following are equivalent:*

(*a*) $\phi \in \mathbf{Aut}(\mathbb{D})$.

(*b*) *$\phi$ is an isometry of the metric $\lambda_{\mathbb{D}}$.*

(*c*) *$\phi$ is an isometry of the distance $d_{\mathbb{D}}$.*

To prove that (*b*) implies (*a*), we will need Schwarz's Lemma (see [3], page 130):

**Theorem 1.4** (Schwarz's lemma)**.** *For any analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 0$ there are two possibilities:*

(*a*) *$|f(z)| < |z|$ for every nonzero $z \in \mathbb{D}$, and moreover $|f'(0)| < 1$ or*

(*b*) *$f$ is a rotation, namely there exists a real number $\theta$ such that $f(z) = e^{i\theta} z$.*

We present the proof of Theorem 1.3 as given in [1].

*Proof.* (Theorem 1.3) Let us prove that (*a*) implies (*b*). If $\phi$ is a conformal automorphism of $\mathbb{D}$, then

$$\phi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

for some $a \in \mathbb{D}$ and real $\theta$. Then

$$\phi'(z) = e^{i\theta} \frac{1(1 - \bar{a}z) - (z - a)(-\bar{a})}{(1 - \bar{a}z)^2} = e^{i\theta} \frac{1 - |a|^2}{(1 - \bar{a}z)^2}.$$

Moreover

$$1 - |\phi(z)|^2 = \frac{|1 - \bar{a}z|^2 - |z - a|^2}{|1 - \bar{a}z|^2} = \frac{(1 - \bar{a}z)(1 - a\bar{z}) - (z - a)(\bar{z} - \bar{a})}{|1 - \bar{a}z|^2}$$

$$= \frac{1 + |z|^2|a|^2 - |z|^2 - |a|^2}{|1 - \bar{a}z|^2} = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \bar{a}z|^2}.$$

So, we can write that

$$\lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| = \frac{2|\phi'(z)|}{1 - |\phi(z)|^2} = \frac{2}{1 - |z|^2} = \lambda_{\mathbb{D}}(z),$$

which finishes the proof $(a)$ implies $(b)$.

Now let us prove that $(b)$ implies $(a)$. Suppose that $\phi$ is an isometry of the hyperbolic metric. Let $\psi$ be a conformal automorphism that maps $\phi(0)$ to 0. Then set $\mu = \psi \circ \phi$ with $\mu(0) = \psi(\phi(0)) = 0$. We have that $\mu$ is an isometry of the hyperbolic metric as $\psi$ and $\phi$ are isometries. Then (1.1) implies

$$\lambda_{\mathbb{D}}(\mu(0))|\mu'(0)| = 2|\mu'(0)| = 2 = \lambda_{\mathbb{D}}(0).$$

So, $\mu$ is a self-map of $\mathbb{D}$, it fixes the origin and $|\mu'(0)| = 1$. Applying Schwarz's Lemma we get that $\mu$ is a rotation. That is why $\phi = \psi^{-1} \circ \mu$ is a conformal automorphism.

Proof of $(a) \Rightarrow (c)$. Let $\phi \in \mathbf{Aut}(\mathbb{D})$. Then $\phi$ is an isometry of the hyperbolic metric. So, for any smooth curve $\gamma$ in $\mathbb{D}$,

$$l_{\mathbb{D}}(\phi \circ \gamma) = \int_{\phi\circ\gamma} \lambda_{\mathbb{D}}(w) = \int_{\gamma} \lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| = l_{\mathbb{D}}(\gamma).$$

This implies that for any $z, w \in \mathbb{D}$, we have $d_{\mathbb{D}}(\phi(z), \phi(w)) = d_{\mathbb{D}}(z, w)$ as any curve $\eta$ can be written as $\phi \circ \gamma$.

Proof of $(c) \Rightarrow (a)$. Let $\phi$ be analytic and a $d_{\mathbb{D}}$-isometry. Let $\psi \in \mathbf{Aut}(\mathbb{D})$, mapping $\phi(0)$ to 0. Set $\mu = \psi \circ \phi$. Then $\mu$ is a $d_{\mathbb{D}}$-isometry, and an analytic self-map of $\mathbb{D}$ with $\mu(0) = 0$. That is why

$$d_{\mathbb{D}}(0, \mu(z)) = d_{\mathbb{D}}(\mu(0), \mu(z)) = d_{\mathbb{D}}(0, z).$$

Consequently, $|\mu(z)| = |z|$ . According to Schwarz's Lemma $\mu(z) = e^{i\theta} z$ , i.e. $\mu$ is a conformal automorphism. Thus, $\phi = \psi^{-1} \circ \mu$ is also conformal automorphism of $\mathbb{D}$. $\square$

The next theorem gives an explicit formula for the hyperbolic distance. Before stating that theorem, we need to define the pseudo-hyperbolic distance.

**Definition 1.5.** *The pseudo-hyperbolic distance* $p_{\mathbb{D}}(z, w)$ *is given by the formula*

$$p_{\mathbb{D}}(z, w) = \left| \frac{z - w}{1 - z\bar{w}} \right|.$$

**Theorem 1.6.** *The hyperbolic distance $d_{\mathbb{D}}(z,w)$ in $\mathbb{D}$ is given by*

$$d_{\mathbb{D}}(z,w) = \log \frac{1+p_{\mathbb{D}}(z,w)}{1-p_{\mathbb{D}}(z,w)} = 2\tanh^{-1} p_{\mathbb{D}}(z,w). \tag{1.3}$$

We present the proof as given in [1].

*Proof.* We will start with the case when both $z$ and $w$ are real. Let $\gamma$ be any smooth curve joining $z$ and $w$ in $\mathbb{D}$ and let $\gamma(t) = u(t) + iv(t)$ be the decomposition of $\gamma$ into real and imaginary parts ($0 \le t \le 1$). Then

$$l_{\mathbb{D}}(\gamma) = \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} = \int_0^1 \frac{2\sqrt{((u'(t))^2 + (v'(t))^2)}dt}{1-u(t)^2-v(t)^2} \ge \int_0^1 \frac{2u'(t)dt}{1-u(t)^2}.$$

By evaluating the last integral we get that

$$l_{\mathbb{D}}(\gamma) \ge \log\left(\frac{(1+w)}{(1-w)}\frac{(1-z)}{(1+z)}\right) = \log\left(\frac{\frac{1+w-z-zw}{1-wz}}{\frac{1-w+z-zw}{1-wz}}\right) =$$

$$\log\left(\frac{1+\left(\frac{w-z}{1-wz}\right)}{1-\left(\frac{w-z}{1-wz}\right)}\right).$$

As equality holds for $\gamma(t) = z + t(w-z)$, we see that (1.3) is true for real $z$ and $w$.

Next, let $z$ and $w$ be any two points in $\mathbb{D}$. We know that any rotation is an isometry, so $d_{\mathbb{D}}(0,z) = d_{\mathbb{D}}(0,|z|)$. Now consider $f(z) = (z-w)/(1-z\bar{w})$. Then $f$ is a conformal automorphism and therefore, it is an isometry of the distance $d_{\mathbb{D}}(z,w)$. Thus,

$$d_{\mathbb{D}}(z,w) = d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(f(z),0) = d_{\mathbb{D}}(|f(z)|,0) = d_{\mathbb{D}}(p_{\mathbb{D}}(z,w),0),$$

which finishes the proof as $p_{\mathbb{D}}(z,w)$ is real and we can use formula for $d_{\mathbb{D}}$ with real arguments:

$$d_{\mathbb{D}}(p_{\mathbb{D}}(z,w),0) = \log\frac{1+p_{\mathbb{D}}(p_{\mathbb{D}}(z,w),0)}{1-p_{\mathbb{D}}(p_{\mathbb{D}}(z,w),0)} = \log\frac{1+p_{\mathbb{D}}(z,w)}{1-p_{\mathbb{D}}(z,w)}.$$

□

From the proof of the theorem we can conclude that the hyperbolic geodesic (the curve that has the least hyperbolic length) that connects two points $-1 < x < y < 1$ is an interval that connects $x$ to $y$. As conformal automorphisms of the unit disk map circles into circles and preserve orthogonality, we get the following.

**Definition 1.7.** *The hyperbolic geodesic that connects two points $z$ and $w$ in $\mathbb{D}$ is $\gamma \cap \mathbb{D}$, where $\gamma$ is the circle or a straight line that passes through $z$ and $w$ and is orthogonal to the circle $\partial\mathbb{D}$.*

Here is an example how the hyperbolic geodesic can look like (geodesics that connect $v_1, u_1$, $v_2, u_2$ and $v_3, u_3$):

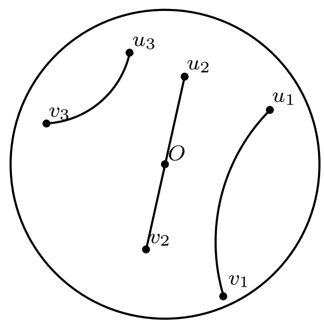



*Figure 1. Hyperbolic geodesic in $\mathbb{D}$*

The unit disk $\mathbb{D}$ with the hyperbolic metric, which is also sometimes referred to as Poincaré metric, is called the Poincaré model of the hyperbolic plane. The name "hyperbolic" comes from the fact that in this type of metrics the sum of interior angles of any triangle will be less than $\pi$ (on the contrary, an elliptic metric is a metric with the sum of interior angles of any triangle being greater then $\pi$). The hyperbolic plane is an example of non-Euclidean geometry, where the Parallel Postulate is removed. That means that there exist plenty of "lines" in the hyperbolic geometry, that are parallel to each other. One can see that if $\gamma$ is a hyperbolic geodesic in $\mathbb{D}$ and $a \in \mathbb{D}$ is a point not on $\gamma$, then there are infinitely many geodesics that pass through $a$ and do not intersect with $\gamma$.

The lines in this plane are the geodesics which are either Euclidean lines or circles, the angle between two hyperbolic lines is just the Euclidean angle between the two tangent lines at the point of intersection. This model satisfies all axioms of the Euclidean geometry, with the exception of the Parallel Postulate.

The hyperbolic distance has a lot of nice properties, however the formula (1.3) for $d_{\mathbb{D}}$ is not simple. Why not using the pseudo-hyperbolic distance $p_{\mathbb{D}}(z, w)$, which has almost the same properties and is much easier to evaluate? The answer is the following: from the next theorem, given in [1], we can see that the hyperbolic distance is additive along the geodesics, but that this is never true for the pseudo-hyperbolic metric.

**Theorem 1.8.** *Let $u, v, w$ be three consecutive points on a hyperbolic geodesic in $\mathbb{D}$, then*

$$d_{\mathbb{D}}(u, w) = d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w).$$

*In contrast for any three distinct points $u, v$ and $w$ in $\mathbb{D}$,*

$$p_{\mathbb{D}}(u, w) < p_{\mathbb{D}}(u, v) + p_{\mathbb{D}}(v, w).$$

*Proof.* Choose three points $u, v, w$ which lie in this order on some geodesic. Then consider an isometry $f$ that maps this geodesic to the real interval $(-1, 1)$ of $\mathbb{D}$, with $f(v) = 0$. Set $x = f(u)$ and $y = f(w)$, so that $-1 < x < 0 < y < 1$. It is sufficient to show that $d_{\mathbb{D}}(x, 0) + d_{\mathbb{D}}(0, y) = d_{\mathbb{D}}(x, y)$. According to Theorem 1.6, we can write

$$d_{\mathbb{D}}(x, y) = \log\left(\frac{1 + \frac{y-x}{1-xy}}{1 - \frac{y-x}{1-xy}}\right) = \log\left(\frac{(1-x)(1+y)}{(1+x)(1-y)}\right) =$$
$$= \log\left(\frac{(1-x)}{(1+x)}\right) + \log\left(\frac{(1+y)}{(1-y)}\right) = d_{\mathbb{D}}(x, 0) + d_{\mathbb{D}}(0, y).$$

In order to show that $p_{\mathbb{D}}$ is a distance function, we will only show that the triangle inequality holds for $p_{\mathbb{D}}(u, v)$. The other properties of a distance are easy to verify. So, for any distinct $u, v$ and $w$,

$$\begin{aligned} p_{\mathbb{D}}(u, w) &= \tanh \frac{1}{2} d_{\mathbb{D}}(u, w) \\ &\leq \tanh \frac{1}{2}(d_{\mathbb{D}}(u, v) + d_{\mathbb{D}}(v, w)) \\ &= \frac{\tanh \frac{1}{2} d_{\mathbb{D}}(u, v) + \tanh \frac{1}{2} d_{\mathbb{D}}(v, w)}{1 + \tanh \frac{1}{2} d_{\mathbb{D}}(u, v) \tanh \frac{1}{2} d_{\mathbb{D}}(v, w)} \\ &< \tanh \frac{1}{2} d_{\mathbb{D}}(u, v) + \tanh \frac{1}{2} d_{\mathbb{D}}(v, w) \\ &= p_{\mathbb{D}}(u, v) + p_{\mathbb{D}}(v, w). \end{aligned}$$

The first inequality follows because $\tanh(x)$ is a monotone increasing function. For the second we used the formula

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

As for the third, one can see that $\tanh(0) = 0$ and the fact that $\tanh(x)$ is a monotone increasing function imply

$$1 + \tanh\frac{1}{2}d_{\mathbb{D}}(u,v)\tanh\frac{1}{2}d_{\mathbb{D}}(v,w) > 1.$$

Moreover, we have shown that there is always a strict inequality in the triangle inequality for $p_{\mathbb{D}}$ for any three distinct points. $\square$

Another unexpected but beautiful result is the following:

**Theorem 1.9.** *The Euclidean topology and the topology derived from the hyperbolic distance are the same. However, the unit disk equipped with the hyperbolic distance is a complete metric space.*

*Proof.* Note that the distance function $d_{\mathbb{D}}$ is not equivalent to the Euclidean distance. To start, let us show that any hyperbolic disk is also an Euclidean disk. First of all, all of the disks $D_{d_{\mathbb{D}},r} = \{z \in \mathbb{D} : d_{\mathbb{D}}(0,z) < r\}$ with a radius $r$ and center 0 are just Euclidean disks with the same center and radius $\tanh(\frac{1}{2}r)$. Really,

$$d_{\mathbb{D}}(0,z) < r \iff 2\tanh^{-1} p_{\mathbb{D}}(0,z) < r \iff |z| < \tanh(\frac{1}{2}r).$$

Now, any hyperbolic disk $D_{d_{\mathbb{D}}}$ with a center at $a$ is mapped onto a hyperbolic disk $D_{d_{\mathbb{D}},r}$ by a hyperbolic isometry $\phi = \frac{z-a}{1-z\bar{a}}$. Then $D_{d_{\mathbb{D}}} = \phi^{-1}(D_{d_{\mathbb{D}},r})$ and, since $\phi^{-1}$ is a conformal automorphism of $\mathbb{D}$, it maps the Euclidean disk $D_{d_{\mathbb{D}},r}$ into the Euclidean disk $D_{d_{\mathbb{D}}}$.

On the other hand, if $D$ is an Euclidean disk, then there is a conformal mapping $\phi$, such that $\phi(D)$ is a Euclidean disk with a radius $R$, that implies $\phi(D) = D_{\mathbb{D},r}$ for $r = 2\tanh^{-1} R$. Since $\phi^{-1}$ is a hyperbolic isometry we get that $D = \phi^{-1}(D_{d_{\mathbb{D}},r})$ is a hyperbolic disk. Since disks are the base for the topology, the hyperbolic topology is the same as the Euclidean one.

Now, let us prove that $\mathbb{D}$ with the hyperbolic distance is a complete metric space. Let $\{z_n\}$ be a Cauchy sequence in $(\mathbb{D}, d_{\mathbb{D}})$. It is a bounded sequence, therefore there is a subsequence $\{z_{n_k}\}$ that converges to some $z \in \overline{\mathbb{D}}$. Suppose, that $z \in \partial\mathbb{D}$. Then, for any $n_0$ and for $k \to \infty$,

$$p_{\mathbb{D}}(z_{n_0}, z_{n_k}) = \frac{|z_{n_0} - z_{n_k}|}{|1 - \overline{z_{n_0}} z_{n_k}|} \to \frac{|z_{n_0} - z|}{|1 - \overline{z_{n_0}} z|} = \frac{|z_{n_0} - z|}{|1 - \overline{z_{n_0}} z| \cdot |\bar{z}|} = \frac{|z_{n_0} - z|}{|\overline{z_{n_0}} - z|} = 1.$$

Therefore $d_{\mathbb{D}}(z_{n_0}, z_{n_k}) = 2\tanh^{-1}(p_{\mathbb{D}}(z_{n_0}, z_{n_k})) \to 2\tanh^{-1}(1) = +\infty$. So we get a contradiction with $\{z_n\}$ being a Cauchy sequence with respect to $d_{\mathbb{D}}$. Therefore $z \in \mathbb{D}$ and we can write

$$d_{\mathbb{D}}(z_n, z) \leq d_{\mathbb{D}}(z_n, z_{n_k}) + d_{\mathbb{D}}(z_{n_k}, z).$$

Taking $k \to \infty$, and $n \to \infty$ we get that $d_{\mathbb{D}}(z_n, z) \to 0$. Therefore, $z_n$ converges to $z$ in $(\mathbb{D}, d_{\mathbb{D}})$.

$\square$

The Euclidean metric is obviously not complete on $\mathbb{D}$. The important property of the hyperbolic metric $d_{\mathbb{D}}$ is that $\lim_{|z|\to 1} d_{\mathbb{D}}(0, z) = +\infty$, which informally means, that $\partial\mathbb{D}$ is infinitely far away from any point $a \in \mathbb{D}$ in the hyperbolic metric. So, in many cases it is more convenient to use the complete metric $d_{\mathbb{D}}$ when considering the domain $\mathbb{D}$.

To end this section, we introduce an illustration of the disks in the hyperbolic metric. The hyperbolic disks with the same radius have different Euclidean size, which depends on the center of the hyperbolic disk. For instance in this Figure 2 all of the hyperbolic disks have the same radius 1.

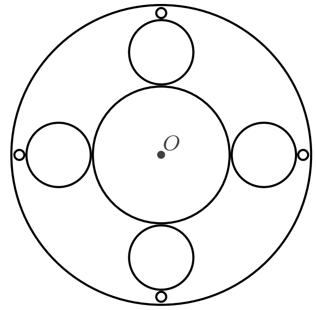



*Figure 2. Hyperbolic disks in $\mathbb{D}$ with radius 1.*

## 1.2 Schwarz-Pick lemma

We stated Schwarz's Lemma in the previous section. One of the conditions in Schwarz's Lemma is that $f(0) = 0$. In terms of the hyperbolic geometry, all points in $\mathbb{D}$ are the same and 0 is not special. That gives the idea that there should be a Schwarz-type result with possibly some other fixed point $a$. Pick in 1915 noticed that actually we can drop the condition that $f$ has a fixed point. Here is Pick's invariant formulation of Schwarz's Lemma.

**Theorem 1.10.** *(The Schwarz-Pick Lemma) Suppose that $f : \mathbb{D} \to \mathbb{D}$ is an analytic function. Then there are two possibilities:*

(a) *$f$ is a hyperbolic contraction, i.e.*

$$d_{\mathbb{D}}(f(z), f(w)) < d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}(f(z))|f'(z)| < \lambda_{\mathbb{D}}(z) \tag{1.4}$$

*for all $z, w \in \mathbb{D}$, or*

(b) *$f$ is a hyperbolic isometry, and*

$$d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(z, w), \quad \lambda_{\mathbb{D}}(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z) \tag{1.5}$$

*for all $z, w \in \mathbb{D}$. That means that $f \in \boldsymbol{Aut}(\mathbb{D})$.*

*Proof.* According to Theorem 1.3, $f$ is an isometry if and only if either of the conditions in (1.5) hold. Now suppose $f$ is analytic, but not a conformal automorphism of $\mathbb{D}$. Then, let us prove inequality (1.4) for two distinct points $z_1$ and $z_2$. Denote $\phi$ to be a conformal automorphism of $\mathbb{D}$ that maps $f(z_1)$ to 0 and $\psi$ to be an automorphism that maps 0 to $z_1$. Then we can apply the Schwarz Lemma to the function $\phi \circ f \circ \psi$ and get $|(\phi \circ f \circ \psi)(z)| < |z|$. Therefore, we get that

$$d_{\mathbb{D}}(0, (\phi \circ f \circ \psi)(z)) = d_{\mathbb{D}}(0, |(\phi \circ f \circ \psi)(z)|) < d_{\mathbb{D}}(0, |z|) = d_{\mathbb{D}}(0, z).$$

Now we can put $z = \psi^{-1}(z_2)$ and using the fact that $\phi$ and $\psi^{-1}$ are isometries for $d_{\mathbb{D}}$ we will get

$$\begin{aligned} d_{\mathbb{D}}(0, (\phi \circ f \circ \psi)(z)) < d_{\mathbb{D}}(0, z) \quad &\Rightarrow \\ d_{\mathbb{D}}(\phi(f(z_1)), \phi(f(\psi(z)))) < d_{\mathbb{D}}(\psi^{-1}(z_1), z) \quad &\Rightarrow \\ d_{\mathbb{D}}(\phi(f(z_1)), \phi(f(z_2))) < d_{\mathbb{D}}(\psi^{-1}(z_1), \psi^{-1}(z_2)) \quad &\Rightarrow \\ d_{\mathbb{D}}(f(z_1), f(z_2)) < d_{\mathbb{D}}(z_1, z_2). \end{aligned}$$

That gives us the first inequality in (1.4). To get the second, let us use the Schwarz Lemma for the function $\phi(f(\psi(z)))$ to get $|\phi(f(\psi))'(0)| < 1$. That implies

$$|\phi'(f(\psi(0))) \cdot f'(\psi(0)) \cdot \psi'(0)| < 1,$$

or in other words

$$|\phi'(f(z_1)) \cdot f'(z_1) \cdot \psi'(0)| < 1,$$

Now,

$$|\psi'(0)|\lambda_{\mathbb{D}}(\psi(0)) = \lambda_{\mathbb{D}}(0),$$

$$|\phi'(f(z_1))|\lambda_{\mathbb{D}}(\psi(f(z_1))) = \lambda_{\mathbb{D}}(f(z_1)),$$

since $\psi$ and $\phi$ are isometries of the hyperbolic metric. As $\psi(0) = z_1$ and $\phi(f(z_1)) = 0$, we get

$$|\psi'(0)| = \frac{\lambda_{\mathbb{D}}(0)}{\lambda_{\mathbb{D}}(z_1)},$$

$$|\phi'(f(z_1))| = \frac{\lambda_{\mathbb{D}}(f(z_1))}{\lambda_{\mathbb{D}}(0)}.$$

Therefore,

$$\begin{aligned} 1 &> |\phi'(f(z_1)) \cdot f'(z_1) \cdot \psi'(0)| \\ &= \left| \frac{\lambda_{\mathbb{D}}(f(z_1))}{\lambda_{\mathbb{D}}(0)} \cdot f'(z_1) \cdot \frac{\lambda_{\mathbb{D}}(0)}{\lambda_{\mathbb{D}}(z_1)} \right| \\ &= \left| \frac{\lambda_{\mathbb{D}}(f(z_1))}{\lambda_{\mathbb{D}}(z_1)} \cdot f'(z_1) \right|. \end{aligned}$$

And finally we get

$$|\lambda_{\mathbb{D}}(f(z_1)) \cdot f'(z_1)| < \lambda_{\mathbb{D}}(z_1)$$

for any $z_1 \in \mathbb{D}$, which finishes the proof of the second inequality (1.4).

□

Note that the Schwarz-Pick Lemma says that every analytic self-map $f$ of the unit disk is a contraction with respect to the hyperbolic metric. Or, one can write,

$$d_{\mathbb{D}}(f(z), f(w)) \le d_{\mathbb{D}}(z, w), \ \frac{\lambda_{\mathbb{D}}(f(z))|f'(z)|}{\lambda_{\mathbb{D}}(z)} \le 1. \tag{1.6}$$

Of course, we can get rid of the expression $d_{\mathbb{D}}$ in the Schwarz-Pick Lemma, get a precise "algebraic" inequality and restate the Schwarz-Pick Lemma in the following form:

**Corollary 1.11.** *(The Schwarz-Pick Lemma, "explicit" version) Let $f$ be self-map of $\mathbb{D}$ and $z_1, z_2$ be two distinct points in $\mathbb{D}$. Then*

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \le \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|,$$

*and for any $z \in \mathbb{D}$,*

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

*If the first equality holds at least for one pair of distinct points $z_1$ and $z_2$, or the second holds for some point $z$, then both inequalities hold for any points $z_1, z_2, z$ and $f$ is a conformal automorphism of $\mathbb{D}$.*

*Proof.* The second inequality follows directly from the fact that any analytic map is a contraction with respect to the hyperbolic distance. The first inequality can be restated as

$$p_{\mathbb{D}}(f(z_1), f(z_2)) \leq p_{\mathbb{D}}(z_1, z_2).$$

Now using the fact that $\tanh^{-1}$ is a monotone function and equation (1.3) we will get that the inequality is equivalent to

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \leq d_{\mathbb{D}}(z_1, z_2).$$

The equality part follows directly from the Schwarz-Pick Lemma. □

Another significant fact about the hyperbolic metric is that it is a unique metric, up to a multiplication by a scalar, that makes every analytic self-map a contraction, and every conformal automorphism an isometry.

**Theorem 1.12.** *For any metric $\rho(z)$ on the unit disk $\mathbb{D}$ the following are equivalent:*

*(a)* $\rho(f(z))|f'(z)| \leq \rho(z)$ *for all analytic $f : \mathbb{D} \to \mathbb{D}$ and $z \in \mathbb{D}$.*

*(b) For any conformal map $f : \mathbb{D} \to \mathbb{D}$ and all $z \in \mathbb{D}$ we get equality in (a), namely* $\rho(f(z))|f'(z)| = \rho(z)$.

*(c)* $\rho(z) = c\lambda_{\mathbb{D}}(z)$ *for some $c > 0$.*

Recall that by a conformal map we mean a map $f$ with $f'(z) \neq 0$ on $\mathbb{D}$.

*Proof.* $(a) \Rightarrow (b)$: Suppose $f \in \mathbf{Aut}(\mathbb{D})$. By the assumption, we have that

$$\rho(f(z))|f'(z)| \leq \rho(z).$$

As $f$ is a conformal self-map of $\mathbb{D}$ so is also its inverse function $g = f^{-1}$. For $g$ we get

$$\rho(g(w))|g'(w)| \leq \rho(w).$$

Using $w = f(z)$ and the fact that $g'(f(z)) = \frac{1}{f'(z)}$ we get

$$\rho(z)|\frac{1}{f'(z)}| \leq \rho(f(z)),$$

or in other words

$$\rho(z) \leq |f'(z)|\rho(f(z)).$$

Therefore, $\rho(z) = |f'(z)|\rho(f(z))$.

$(b) \Rightarrow (c)$: We will find the constant $c$ from the equation $\rho(0) = c\lambda_{\mathbb{D}}(0)$. Let $f$ be a conformal automorphism of $\mathbb{D}$, that maps 0 into $a \in \mathbb{D}$. Then according to $(b)$ and the fact that conformal automorphisms are isometries for $\lambda_{\mathbb{D}}$ we get that

$$\rho(a)|f'(0)| = \rho(f(0))|f'(0)| = \rho(0) \quad = c\lambda_{\mathbb{D}}(0) = c\lambda_{\mathbb{D}}(f(0))|f'(0)| = c\lambda_{\mathbb{D}}(a)|f'(0)|.$$

As $|f'(0)|$ is nonzero, we get $\rho(a) = c\lambda_{\mathbb{D}}(a)$ for all $a \in \mathbb{D}$.

$(c) \Rightarrow (a)$: Follows directly from the Schwarz-Pick Lemma. □

## 1.3 Hyperbolic derivatives and distortions on $\mathbb{D}$

The hyperbolic metric is a natural metric to study the properties of the analytic self-maps of the disk. Therefore one should also use derivatives that are compatible with the hyperbolic metric. To do that we will need to define the hyperbolic difference quotient and the complex pseudo-hyperbolic distance, as it is defined in [1].

**Definition 1.13.** *The expression*

$$[z, w] = \frac{z - w}{1 - \bar{w}z}$$

*defines the complex pseudo-hyperbolic distance between any two points $z$ and $w$ in $\mathbb{D}$.*

Note that in Section 1.1, we denoted the pseudo-hyperbolic distance by $p_{\mathbb{D}}(z, w) = |[z, w]|$. Considering the complex pseudo-hyperbolic distance as an analogue of ordinary distance on the real line, one can define the difference quotient for a self-map of $\mathbb{D}$. In this section we will consider some properties of the hyperbolic difference quotient and the hyperbolic derivative as given and described in [1].

**Definition 1.14.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is an analytic map, then for any two distinct points $z, w \in \mathbb{D}$ we define the hyperbolic difference quotient $f^*(z, w)$ by*

$$f^*(z, w) = \frac{[f(z), f(w)]}{[z, w]}.$$

If we combine equation (1.3), the fact that tanh is monotone increasing, and the Schwarz-Pick Lemma, we get the following

$$\begin{aligned} d_{\mathbb{D}}(f(z), f(w)) \le d_{\mathbb{D}}(z, w) \ &\Rightarrow \ \tanh^{-1} p_{\mathbb{D}}(f(z), f(w)) \le \tanh^{-1} p_{\mathbb{D}}(z, w) \\ &\Rightarrow p_{\mathbb{D}}(f(z), f(w)) \le p_{\mathbb{D}}(z, w), \end{aligned}$$

for any $z, w \in \mathbb{D}$. Moreover, if equality holds for one pair $z$ and $w$ of distinct points, then $f$ is a conformal automorphism of $\mathbb{D}$ and then equality hold for all $z$ and $w$. We get that either $f$ is a hyperbolic isometry with $|f^*(z,w)| = 1$ for all $z$ and $w$, or $f$ is not an isometry and $|f^*(z,w)| < 1$ for all $z$ and $w$.

For our further discussion we will need to introduce the notion of finite Blaschke product.

**Definition 1.15.** *A finite Blaschke product $B : \mathbb{D} \to \mathbb{D}$ of degree $k$ is defined by*

$$B(z) = \xi \prod_{i=1}^{k} \frac{z - a_i}{1 - z\bar{a}_i},$$

*where $\xi \in \partial\mathbb{D}$ and $a_i \in \mathbb{D}$.*

*Remark* 1.16. A conformal automorphism of $\mathbb{D}$ is a Blaschke product of degree 1.

The next Theorem gives us a general description of finite Blaschke products.

**Theorem 1.17.** *$B$ is a finite Blaschke product if and only if $B$ is analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ and $|B(z)| = 1$, for each $z \in \partial\mathbb{D}$.*

*Proof.* The "only if" part is just trivial, as we have the following for $|z| = 1$,

$$|B(z)| = \left| \xi \prod_{i=1}^{k} \frac{z - a_i}{1 - z\bar{a}_i} \right| = |\xi| \prod_{i=1}^{k} \left| \frac{z - a_i}{1 - z\bar{a}_i} \right| = 1.$$

Now let us prove the "if" part. If $B$ has no zeroes in $\mathbb{D}$, then by the Minimum Modulus principle, $B$ is a constant, which must be of modulus one. Now suppose that $B$ does have a zero in $\mathbb{D}$. Note that if $B(z)$ has an infinite number of zeroes in $\mathbb{D}$, then there is a sequence of zeroes that converge to a point $w \in \overline{\mathbb{D}}$. If $w \in \mathbb{D}$, then $B(z)$ is not analytic at $w$, if $w \in \partial\mathbb{D}$, then $B(w) = 0$ since $B(z)$ is continuous in $\overline{\mathbb{D}}$.

Therefore $B(z)$ can only have a finite number of zeroes in $\mathbb{D}$, say $a_1, ..., a_k$, which are written with their multiplicities, and the function

$$B_1(z) = B(z) / \left| \prod_{i=1}^{k} \frac{z - a_i}{1 - z\bar{a}_i} \right|$$

is also analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ and it maps $\partial\mathbb{D}$ onto itself. Moreover, $B_1$ has no zeroes in $\mathbb{D}$. Therefore applying the Minimum Modulus Principle to the function $B_1$, we get

$$B_1(z) = \xi,$$

for some $\xi \in \partial\mathbb{D}$ and that ends the proof. $\square$

Also let us consider the automorphism of $\mathbb{D}$

$$\phi(z) = \frac{z - w}{1 - \bar{w}z},$$

and the Blashcke product

$$B(z) = \xi \prod_{i=1}^{k} \frac{z - a_i}{1 - z\overline{a_i}}$$

of degree $k$. Then the function $\phi(B(z))$ is also a Blaschke product, with all zeroes in $\mathbb{D}$. Really, one can get

$$\phi(B(z)) = 0 \Rightarrow \quad B(z) - w = 0 \Rightarrow$$

$$\xi \frac{\prod_{i=1}^{k}(z - a_i) - (w/\xi)\prod_{i=1}^{k}(1 - z\overline{a_i})}{\prod_{i=1}^{k}(1 - z\overline{a_i})} = 0.$$

As the degree of the numerator is $k$, that means that $\phi(B(z))$ has exactly $k$ zeroes, and all of them are in $\mathbb{D}$. Therefore $\phi(B(z))$ is a Blaschke product of degree $k$. Or, in other words, the degree of a Blaschke product is preserved under a composition with a conformal automorphism of $\mathbb{D}$.

Now we turn to the behavior of the hyperbolic difference quotient. Some of the basic properties are given in the following result in [1].

**Theorem 1.18.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic, and $w$ be some point in $\mathbb{D}$.*

*(a) The function $f^*(z, w)$, as a function of $z$, is analytic in $\mathbb{D}$.*

(b) *Whenever $f$ is not a conformal mapping $\mathbb{D}$ onto itself, $|f^*(z,w)| < 1$ for any $z, w \in \mathbb{D}$.*

(c) *The map $z \to f^*(z,w)$ is a conformal automorphism of $\mathbb{D}$ if and only if $f$ is a Blaschke product of degree 2.*

(The proof is taken from [1] with some modifications).

*Proof.* To show part $(a)$, note that $w$ is a removable singularity for $f^*(z,w)$ as a function of $z$ since

$$f^*(z,w) = \left(\frac{f(z)-f(w)}{z-w}\right)\left(\frac{1-\overline{f(w)}f(z)}{1-\bar{w}z}\right)^{-1}.$$

The function $z \to f^*(z,w)$ is obviously analytic everywhere except possibly the point $z = w$. However, when $z \to w$, we get

$$f^*(z,w) \quad \to \quad f'(w)\left(\frac{1-\overline{f(w)}f(w)}{1-\bar{w}w}\right)^{-1} = \frac{|f'(w)|\lambda_{\mathbb{D}}(f(w))}{\lambda_{\mathbb{D}}(w)} < \infty.$$

Therefore we conclude that the point $z = w$ is a removable singularity of $f^*(z,w)$ as a function of $z$.

As noted before, if $f$ is not a conformal automorphism of $\mathbb{D}$, then we have that $|f^*(z,w)| < 1$ and this gives us (b).

Note that that there are conformal automorphisms $\phi$ and $\psi$ of $\mathbb{D}$ (depending on $w$ ) such that

$$f^*(z,w) = \frac{\phi(f(z))}{\psi(z)}.$$

More precisely, we can take

$$\phi(z) = \frac{z-f(w)}{1-\overline{f(w)}z},$$

$$\psi(z) = \frac{z-w}{1-\bar{w}z}.$$

Therefore we can write

$$f(z) = \phi^{-1}(f^*(z,w)\psi(z)).$$

We will use this to show part $(c)$. If $f^*(z,w)$ is a conformal automorphism, then $f^*(z,w)\psi(z)$ is a Blaschke product of degree 2 and $f(z) = \phi^{-1}(f^*(z,w)\psi(z))$ is a Blaschke product and its degree is equal to two since the composition with a conformal automorphism preserves the degree of a Blaschke product.

On the other hand, if $f$ is a Blashcke product of degree two, then $B(z) := \phi(f(z))$ is also a Blaschke product of degree two, and $f^*(z,w) = \frac{B(z)}{\psi(z)}$. Since we know that $f^*(z,w)$ is analytic in $z$ and $B(z)$ is a product of two automorphisms, we get that there is a conformal automorphism $\psi_1$ such that $B(z) = \psi(z)\psi_1(z)$ and so $f^*(z,w) = \psi_1$, which finishes the proof.

□

Using the hyperbolic difference quotient we can generalize the Schwarz Lemma and derive the so-called three-point Schwarz-Pick Lemma, which, as stated in [1], includes all presently known variations and extensions of the Schwarz Lemma.

**Theorem 1.19** (Three-point Schwarz-Pick Lemma.)**.** *Suppose that $z, w, v \in \mathbb{D}$ are some not necessary distinct points and $f : \mathbb{D} \to \mathbb{D}$ is analytic, but not an automorphism of $\mathbb{D}$. Then*

$$d_{\mathbb{D}}(f^*(z,v), f^*(w,v)) \le d_{\mathbb{D}}(z,w).$$

*Moreover, if equality holds for some $z, w$ and $v$ then equality holds for any choice of $z, w$ with fixed $v$ and $f$ is a Blashke product of degree two.*

*Proof.* Assuming that $f$ is analytic in $\mathbb{D}$, and is not an automorphism, according to Theorem 1.18(b) we have that $|f^*(z,v)| < 1$ and $|f^*(v,w)| < 1$, which shows that the expression

$$d_{\mathbb{D}}(f^*(z,v), f^*(w,v))$$

is defined. Applying the Schwarz-Pick Lemma to the map $f^*(z,w)$, we get the inequality from the three-point Schwarz-Pick Lemma. Moreover, the Schwarz-Pick Lemma also tells us that equality holds if and only if $f^*(z,w)$ is a conformal automorphism of $\mathbb{D}$ which only happens when $f$ is a Blaschke product of degree two (Theorem 1.18(c) ). □

As was claimed above, this theorem is an improvement of the Schwarz Lemma. From the Schwarz Lemma we get that $\frac{f(z)}{z}$ lies in $\mathbb{D}$ and that $|f'(0)| < 1$. On the other hand if we take $w = 0$ and $v \to 0$ in the Three-point Schwarz-Pick Lemma, we get a stronger result which says that $\frac{f(z)}{z}$ lies in the hyperbolic disk with center $f'(0)$ and radius $d_{\mathbb{D}}(0, z)$.

Now, using the hyperbolic difference quotient, we will define the hyperbolic derivative as it is done in [1].

**Definition 1.20.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic. The hyperbolic derivative $f^h(w)$ of $f$ at $w$ in $\mathbb{D}$ is*

$$f^h(w) = \lim_{z \to w} f^*(z, w) = \lim_{z \to w} \frac{[f(z), f(w)]}{[z, w]} = \frac{(1 - |w|^2) f'(w)}{1 - |f(w)|^2} = \frac{\lambda_{\mathbb{D}}(f(w)) f'(w)}{\lambda_{\mathbb{D}}(w)}.$$

*The hyperbolic distortion of $f$ at $w$ is*

$$|f^h(w)| = \lim_{z \to w} \frac{d_{\mathbb{D}}(f(z), f(w))}{d_{\mathbb{D}}(z, w)} = \frac{(1 - |w|^2)|f'(w)|}{1 - |f(w)|^2} = \frac{\lambda_{\mathbb{D}}(f(w))|f'(w)|}{\lambda_{\mathbb{D}}(w)}.$$

Note that, by the Schwarz-Pick Lemma, we have that for $f : \mathbb{D} \to \mathbb{D}$ analytic either $|f^h(w)| < 1$ for every $w \in \mathbb{D}$, or $|f^h(w)| = 1$ for all $w \in \mathbb{D}$ and $f$ is an automorphism of $\mathbb{D}$.

We will need the notion of the hyperbolic derivative later on, when we are considering the connection between the hyperbolic derivative of $f$ and a possible analytic extension of $f$ across the unit circle.

## 1.4 Angular derivatives

For some of the further results and discussions, we will need to define the angular limit at a point on the boundary. For a function $f : \mathbb{D} \to \mathbb{C}$ and a point $\xi \in \partial\mathbb{D}$ we can consider the unrestricted limit

$$\lim_{z \to \xi} f(z),$$

where $z$ approaches $\xi$ without any restrictions in $\mathbb{D}$. In some cases, it is convenient to separate different types of approaches, namely a tangential approach and an angular approach.

**Definition 1.21.** *For a point $w \in \partial\mathbb{D}$ and $\alpha > 1$, a non-tangential approach domain at $w$ with angle $\alpha$ is defined by*

$$\Gamma(w, \alpha) = \{z \in \mathbb{D} : |z - w| < \alpha(1 - |z|)\}.$$

The term "non-tangential" refers to the fact that near the point $w$, $\Gamma(w, \alpha)$ is a sector that is bounded by two straight lines in $\mathbb{D}$ meeting at $w$ and symmetric about the radius to $w$.

**Definition 1.22.** *An analytic function $f : \mathbb{D} \to \mathbb{C}$ has a non-tangential (angular) limit $L$ at $w$ if the limit of $f(z)$ exists in each non-tangential domain $\Gamma(w, \alpha)$. In this case we will write*

$$L = \angle \lim_{z \to w} f(z).$$

The main fact about non-tangential limits is stated by the following theorem (for the proof see [15]):

**Theorem 1.23** (Julia-Wolff-Caratheodory Lemma)**.** *Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, $\xi \in \partial\mathbb{D}$. Then the following statements are equivalent:*

*(a)*

$$\liminf_{z \to \xi} \frac{1 - |f(z)|}{1 - |z|} =: d(\xi) < \infty,$$

*where the limit is taken as $z$ approaches $\xi$ without any restrictions in $\mathbb{D}$.*

*(b)*

$$\angle \lim_{z \to \xi} \frac{f(z) - \eta}{z - \xi} =: \angle f'(\xi)$$

*exists for some $\eta \in \partial\mathbb{D}$. The value $\angle f'(\xi)$ is called the angular derivative of $f$ at the point $\xi$.*

*(c)* $\angle \lim_{z \to \xi} f'(z)$ *exists, and* $\angle \lim_{z \to \xi} f(z) = \eta \in \partial\mathbb{D}$.

*Moreover,*

*(i)* $d(\xi) > 0$ *in (i).*

*(ii) The boundary points $\eta$ in (ii) and (iii) are the same.*

*(iii)* $\lim_{z \to \xi} f'(z) = \angle f'(\xi) = \bar{\xi}\eta d(\xi)$.

# Chapter 2

# Classical Ahlfors Lemma

## 2.1 The hyperbolic metric on simply connected domains

In this section we will consider the hyperbolic metric on different simply connected domains, restating the basic properties of this metric. Recall that a domain is an open, connected subset of $\mathbb{C}$.

We first define the notion of conformal equivalence of two different domains.

**Definition 2.1.** *Two domains $\Omega_1$ and $\Omega_2$ are conformally equivalent if there is a conformal mapping $\phi$ of $\Omega_1$ onto $\Omega_2$.*

Also, we will need to use the main result about simply connected domains - the Riemann Mapping Theorem (see [3] page 160).

**Theorem 2.2.** *(The Riemann Mapping Theorem) Let $\Omega$ be a proper simply connected subdomain of $\mathbb{C}$ and $a$ be any point in $\Omega$. Then there is a unique conformal mapping $f$ from $\Omega$ onto $\mathbb{D}$, such that $f(a) = 0$ and $f'(a)$ is real and positive. Therefore, any simply connected domain in $\mathbb{C}$ is either $\mathbb{C}$ itself, or it is conformally equivalent to the unit disk $\mathbb{D}$.*

Now, using the Riemann Mapping Theorem we can transfer the hyperbolic metric from the unit disk to any simply connected proper subdomain $\Omega$ of $\mathbb{C}$. Actually, using the conformal mapping $\phi : \Omega \to \mathbb{D}$ we will "pull" the hyperbolic metric from the unit disk back onto the domain $\Omega$.

**Definition 2.3.** *For any semimetric $\rho(w)$ on the domain $\Omega_2$ and analytic function $f : \Omega_1 \to \Omega_2$, the pull-back of $\rho(w)$ by $f$ at point $z \in \Omega_1$ is provided by the formula:*

$$f^*(\rho)(z) = \rho(f(z))|f'(z)|.$$

The pull-back metric $f^*(\rho)(z)$ is also a semimetric on $\Omega_1$ whenever $f$ is not a constant function.

The next theorem says that all of the pull-backs by the conformal mappings $\phi^*(\lambda_{\mathbb{D}})$ are the same and they denote the hyperbolic metric on $\Omega$. The following theorem can be also considered as a theorem of existence and uniqueness of the hyperbolic metric on simply connected domains.

**Theorem 2.4.** *There is a unique hyperbolic metric $\lambda_\Omega$ defined on a simply connected domain $\Omega$, properly contained in $\mathbb{C}$, such that for any conformal mapping $\phi : \Omega \to \mathbb{D}$ we get*

$$\lambda_\Omega(z) = \lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| = \phi^*(\lambda_{\mathbb{D}})(z).$$

*Proof.* First of all, let $\phi : \Omega \to \mathbb{D}$ be any conformal mapping provided by the Riemann Mapping Theorem. Then, define the hyperbolic metric $\lambda_\Omega(z) = \lambda_{\mathbb{D}}(\phi(z))|\phi'(z)|$. Let us show that the definition of $\lambda_\Omega$ is valid, namely that it is independent of the choice of $\phi$ and that it is determined only by $\Omega$. Any conformal mapping from $\Omega$ onto $\mathbb{D}$ can be written as $\mu \circ \phi$, for some $\mu \in \mathbf{Aut}(\mathbb{D})$. Since every element of $\mathbf{Aut}(\mathbb{D})$ is an isometry for $\lambda_{\mathbb{D}}$, we get:

$$\lambda_{\mathbb{D}}(w) = \lambda_{\mathbb{D}}(\mu(w))|\mu'(w)|.$$

Now, for a conformal mapping $\psi = \mu \circ \phi$ that maps $\Omega$ onto $\mathbb{D}$, we get:

$$\lambda_{\mathbb{D}}(\psi(z)) \cdot |\psi'(z)| = \lambda_{\mathbb{D}}(\mu(\phi(z))) \cdot |\mu'(\phi(z))| \cdot |\phi'(z)| = \lambda_{\mathbb{D}}(\phi(z)) \cdot |\phi'(z)|.$$

The last equations show that the value $\lambda_\Omega(z)$ is independent of the choice of the conformal mapping $\phi$ from $\Omega$ onto $\mathbb{D}$. $\square$

Now, we want to define the hyperbolic distance on $\Omega$. We can do that in two different ways. On one hand, from the Riemann Mapping Theorem, we get a conformal mapping $\phi$ from $\Omega$ onto $\mathbb{D}$. Using this map, we define the hyperbolic distance on $\Omega$ by $d_\Omega(z,w) = d_\mathbb{D}(\phi(z),\phi(w))$. One can easily verify that $d_\Omega$ satisfies all the properties of a distance function and it does not depend on the map $\phi$. Really, any other conformal automorphism $\psi : \Omega \to \mathbb{D}$ can be written as $\psi = \mu \circ \phi$ for some $\mu \in \mathbf{Aut}(\mathbb{D})$. Then

$$d_\mathbb{D}(\psi(z),\psi(w)) = d_\mathbb{D}(\mu\circ\phi(z),\mu\circ\phi(w)) = d_\mathbb{D}(\phi(z),\phi(w)),$$

since $\mu$ is a hyperbolic isometry. Therefore the definition of $d_\Omega(z,w)$ is correct.

On the other hand, we can do the same procedure as we did when defining the hyperbolic distance on $\mathbb{D}$. The hyperbolic length of a curve $\gamma$ in $\Omega$ is given by

$$l_\Omega(\gamma) = \int_\gamma \lambda_\Omega(z).$$

Then the distance function between two points $z$ and $w$ in $\Omega$ will be

$$d_\Omega(z,w) = \inf l_\Omega(\gamma),$$

where $\gamma$ is any piecewise smooth curve that connects $z$ and $w$ in $\Omega$. It is easy to see that, in both cases, we get the same distance function. And, same as for the hyperbolic distance on $\mathbb{D}$, the hyperbolic distance $d_\Omega$ is complete on $\Omega$. Namely, let $\{z_n\}$ be a Cauchy sequence in a metric space $(\Omega, d_\Omega)$. Let $\phi$ be a conformal mapping of $\mathbb{D}$ onto $\Omega$. Then $\{\phi^{-1}(z_n)\}$ is a Cauchy sequence in $(\mathbb{D}, d_\mathbb{D})$, since

$$d_\Omega(z_n,z_k) = d_\Omega(\phi(\phi^{-1}(z_n)),\phi(\phi^{-1}(z_k))) = d_\Omega(\phi^{-1}(z_n),\phi^{-1}(z_k)).$$

Therefore, $\{\phi^{-1}(z_n)\}$ converges to some $w \in \mathbb{D}$ and $\{z_n\}$ converges to $\phi(w) \in \Omega$.

The most important consequence of the definition is that the Poincaré's model of the hyperbolic plane $\mathbb{D}$ transfers to any simply connected domain of the plane with its own hyperbolic metric. For example, the set of all automorphisms of $\Omega$ is just the set of isometries of $\Omega$. Also, one can adopt the Schwarz-Pick lemma for the case of simply connected domains by following [1].

Before doing that let us expand the definition of a hyperbolic isometry.

**Definition 2.5.** *A map $f : \Omega_1 \to \Omega_2$ is said to be a hyperbolic isometry between two simply connected regions $\Omega_1$ and $\Omega_2$ if for any $z, w \in \Omega_1$*

$$\lambda_{\Omega_1}(z) = \lambda_{\Omega_2}(f(z)) \cdot |f'(z)|,$$

$$d_{\Omega_1}(z, w) = d_{\Omega_2}(f(z), f(w))$$

Then the following statement holds.

**Theorem 2.6.** *(Conformal Invariance) Any conformal mapping $\phi$ between $\Omega_1$ and $\Omega_2$ - two simply connected proper subdomains of $\mathbb{C}$, is a hyperbolic isometry between $\Omega_1$ and $\Omega_2$.*

*Proof.* Let $\psi$ be a conformal mapping from $\Omega_2$ onto $\mathbb{D}$. Then $\mu = \psi \circ \phi$ is a conformal mapping from $\Omega_1$ onto $\mathbb{D}$. But then $\lambda_{\Omega_1}$ and $\lambda_{\Omega_2}$ are just pull-backs of $\lambda_{\mathbb{D}}$ by the maps $\mu$ and $\psi$. Therefore,

$$\lambda_{\Omega_1}(z) = \lambda_{\mathbb{D}}(\mu(z)) \cdot |\mu'(z)| = \lambda_{\mathbb{D}}((\psi \circ \phi)(z)) \cdot |\psi'(\phi(z))| \cdot |\phi'(z)| =$$

$$= \lambda_{\Omega_2}(\phi(z)) \cdot |\phi'(z)|.$$

Moreover,

$$d_{\Omega_1}(z, w) = d_{\mathbb{D}}(\mu(z), \mu(w)) = d_{\mathbb{D}}(\psi \circ \phi(z), \psi \circ \phi(w)) = d_{\Omega_2}(\phi(z), \phi(w)).$$

Hence, $\phi$ is a hyperbolic isometry between $\Omega_1$ and $\Omega_2$. □

**Theorem 2.7.** *(Schwarz-Pick Lemma for Simply Connected Domains) For any two proper simply connected subdomains $\Omega_1$ and $\Omega_2$ of $\mathbb{C}$ and any analytic map $f : \Omega_1 \to \Omega_2$ the following holds:*

$$d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w),$$

$$\lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z),$$

*for all $z$ and $w$ in $\Omega_1$. If equality holds in any of inequalities at least for one choice of $z$ and $w$, it holds for any choice of $z$ and $w$, and moreover $f$ is a conformal mapping of $\Omega_1$ onto $\Omega_2$.*

*Proof.* This theorem just says that any analytic map from $\Omega_1$ to $\Omega_2$ is either a hyperbolic contraction, ar a hyperbolic isometry. If $f$ is a conformal mapping, then the statement is just a consequence of the Conformal Invariance Theorem .

Now suppose that $f$ is not a conformal mapping from $\Omega_1$ onto $\Omega_2$. Let $\phi$ be a conformal mapping from $\mathbb{D}$ onto $\Omega_1$ and $\psi$ be conformal mapping of $\Omega_2$ onto $\mathbb{D}$. Then for any point $w$ in $\mathbb{D}$ we get

$$\lambda_{\mathbb{D}}(w) = \lambda_{\Omega_1}(\phi(z))|\phi'(z)|,$$

and for any point $u$ in $\Omega_2$,

$$\lambda_{\Omega_2}(u) = \lambda_{\mathbb{D}}(\psi(z))|\psi'(z)|.$$

Since the map $\psi \circ f \circ \phi$ is a self-map of $\mathbb{D}$ and it is not a conformal automorphism, we can use the Schwarz-Pick Lemma for this function and get:

$$\lambda_{\mathbb{D}}(\psi(f(\phi(w)))) \cdot |\psi'(f(\phi(w)))| \cdot |f'(\phi(w))| \cdot |\phi'(w)| < \lambda_{\mathbb{D}}(w).$$

The left side of this inequality can be rewritten as

$$\lambda_{\mathbb{D}}(\psi(f(\phi(w)))) \cdot |\psi'(f(\phi(w)))| \cdot |f'(\phi(w))| \cdot |\phi'(w)| = \lambda_{\Omega_2}(f(\phi(w))) \cdot |f'(\phi(w))| \cdot |\phi'(w)|.$$

The right side of the inequality is just equal to $\lambda_{\Omega_1}(\phi(w)) \cdot |\phi'(w)|$. So we obtain that

$$\lambda_{\Omega_2}(f(\phi(w))) \cdot |f'(\phi(w))| \cdot |\phi'(w)| < \lambda_{\Omega_1}(\phi(w)) \cdot |\phi'(w)|.$$

Put $z = \phi(w)$ and cancel $|\phi'(w)|$ on both sides to get

$$\lambda_{\Omega_2}(f(z))|f'(z)| < \lambda_{\Omega_1}(z).$$

This gives us the second inequality. The first is obtained by integrating the strict inequality for the hyperbolic metrics. $\square$

Now, we are ready to expand the definition of the hyperbolic derivative as it was done in [1]. With the analogue to the definition from Section 1.3 we get:

**Definition 2.8.** *For any analytic $f : \Omega_1 \to \Omega_2$ let us define the hyperbolic derivative of $f$ by*

$$f^h(z) = \frac{\lambda_{\Omega_2}(f(z))f'(z)}{\lambda_{\Omega_1}(z)}$$

*at any point $z \in \Omega_1$. The hyperbolic distortion is defined by*

$$|f^h| = \frac{\lambda_{\Omega_2}(f(z))|f'(z)|}{\lambda_{\Omega_1}(z)}.$$

Then the Schwarz-Pick Lemma can be restated in terms of hyperbolic distortion as follows: For any analytic function $f : \Omega_1 \to \Omega_2$ either the distortion of $f$ is strictly less then 1, or if the distortion is equal to 1 at a single point $z \in \Omega_1$, then $f$ is a conformal mapping from $\Omega_1$ onto $\Omega_2$. That means that distortion is bounded from above in $\Omega_1$. The main results of the next chapter give some answer on what happens with $f$ near the boundary of $\Omega_1$ if the distortion is bounded from below.

Let us note that $\mathbb{C}$ and the extended plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, have another analogue to the hyperbolic metric in the sense that the metric is invariant under the group of conformal automorphisms. For more details see [1], page 27-28. In the case of $\mathbb{C}$, the Euclidean metric is the one that is invariant under the group of conformal automorphisms. As for $\mathbb{C}_\infty$, the spherical metric $\lambda_{\mathbb{C}_\infty}(z) = \frac{2}{1+|z|^2}(\lambda_{\mathbb{C}_\infty}(\infty) = 0)$ is preserved under conformal automorphisms of $\mathbb{C}_\infty$.

Let us consider two examples of the hyperbolic metric on some proper simply connected domains (both examples appear in [1]).

*Example* 2.9. (Disk) Consider $D_R = \{z : |z - z_0| < R\}$ and $\phi(z) = (z - z_0)/R$ , a conformal mapping of $D_R$ onto $\mathbb{D}$. Then we get

$$\lambda_{D_R}(z) = \frac{2R}{R^2 - |z - z_0|^2}.$$

*Example* 2.10. (Half-plane) Let $\mathbb{H}$ be the upper half-plane $\{x + iy : y > 0\}$. Then $\psi(\mathbb{H}) = \mathbb{D}$, with $\psi(z) = (z - i)/(z + i)$. Then

$$\psi'(z) = \frac{2i}{(z + i)^2}$$

and we get the hyperbolic metric at a point $z = x + \mathrm{i}y$ in $\mathbb{H}$ by the formula

$$\begin{aligned}\lambda_{\mathbb{H}}(z) = \lambda_{\mathbb{D}}(\psi(z)) \cdot |\psi'(z)| &= \frac{2}{1 - \left|\frac{z-i}{z+i}\right|^2} \cdot \left|\frac{2i}{(z+i)^2}\right| \\ &= \frac{4}{\left(1 - \left|\frac{z-i}{z+i}\right|^2\right) \cdot |z+i|^2} = \frac{4}{|z+i|^2 - |z-i|^2} \\ &= \frac{4}{(z+i)\cdot(\overline{z}-i) - (z-i)\cdot(\overline{z}+i)} = \frac{4}{2i(\overline{z}-z)} \\ &= \frac{4}{2i(x-iy-x-iy)} = \frac{4}{2i(-2iy)} = \frac{1}{y}.\end{aligned}$$

So $\mathbb{H}$ has a hyperbolic metric

$$\lambda_{\mathbb{H}}(z) = \frac{1}{y} = \frac{1}{\mathrm{Im} z}$$

Moreover, if $\mathcal{H}$ is any open half-plane, then

$$\lambda_{\mathcal{H}}(z) = \frac{1}{d(z, \partial\mathcal{H})},$$

where $d(z, \partial\mathcal{H})$ denotes the Euclidean distance from $z$ to $\partial\mathcal{H}$.

## 2.2 Curvature, pull-back metrics and classical Ahlfors Lemma

One of the most important properties of a metric is its curvature. In this section, we will consider the notion of curvature, and discover some of its properties.

Recall that curvature is one of the main characteristics of a curve. Having a surface $S$, one can define a Gaussian curvature at a point $a \in S$ as a product of principal curvatures at $a$. It is surprising but true that the Gaussian curvature of a metric given on a complex domain has a nice and simple formula.

First of all, one can define curvature for metrics with a continuous density. A *semimetric* on a domain $\Omega$ in $\mathbb{C}$ is a continuous function $\rho : \Omega \to [0; \infty)$ such that $\{z : \rho(z) = 0\}$ is a discrete set in $\Omega$. We can define the notion of a curvature of a semimetric $\rho$ at points where $\rho$ is positive and twice continuously differentiable.

**Definition 2.11.** *For any semimetric $\rho(z)$ on a domain $\Omega$, define the Gaussian curvature by*

$$\kappa_\rho(a) = -\frac{\Delta \log(\rho(a))}{\rho^2(a)},$$

*at any point $a \in \Omega$, such that $\rho(a)$ is nonzero and $\rho$ is twice continuously differentiable at the neighborhood of $a$.*

In this definition $\Delta$ is the Laplacian operator, namely

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Further on, we will simply use the word curvature instead of Gaussian curvature. In computing the Laplacian it is convenient to use the fact that

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\frac{\partial^2}{\partial \bar{z} \partial z},$$

where the complex derivatives are given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Note that for $f$ analytic we have that

$$\frac{\partial}{\partial \bar{z}} f(z) = 0,$$
$$\frac{\partial}{\partial \bar{z}} \overline{f(z)} = \overline{f'(z)}$$

This is true because of the Cauchy-Riemann equations for the functions $f$ and $\bar{f}$. Let us introduce the following example, which we will use later on.

*Example* 2.12. Let $f$ be analytic function and $f(z) \neq 0$. Then the function $\log|f(z)|$ is harmonic. That means that

$$\Delta(\log|f|)(z) = 0.$$

That is because

$$\Delta(\log|f|)(z) = \Delta(\frac{1}{2}\log f\bar{f})(z) = \Delta\frac{1}{2}(\log f \cdot \log \bar{f})(z) = \Delta\frac{1}{2}(\log f)(z) + \Delta\frac{1}{2}(\log \bar{f})(z)$$
$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(\log f)(z) + 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}(\log \bar{f})(z) = 0,$$

since $\log f$ is analytic.

Let us also introduce the chain rule for the differentiation in the complex plane. For continuous functions $f$ and $g$, such that $g$ is differentiable at $z$ and $f$ is differentiable at $g(z)$ we have the following:

$$\frac{\partial}{\partial z} f(g(z)) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial z}(z),$$
$$\frac{\partial}{\partial \bar{z}} f(g(z)) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \bar{z}}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial \bar{z}}(z),$$

Here are the first three simple examples of curvature for the Euclidean metric, the hyperbolic metric and the spherical metric.

*Example* 2.13. (Euclidean Plane) The curvature $\kappa_\rho = 0$ for $\mathbb{C}$ taken with the Euclidean metric $\rho(z) = 1$. That is because $\log(\rho(z)) = 0$ for any $z \in \mathbb{C}$.

*Example* 2.14. (Disk) Recall that the hyperbolic metric of $\mathbb{D}$ is given by the formula

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2} = \frac{2}{1-z\bar{z}}.$$

Then we get

$$\Delta \log \lambda_{\mathbb{D}}(z) = \Delta \log \frac{2}{1-z\bar{z}} = -4\frac{\partial^2}{\partial \bar{z}\partial z}\log(1-z\bar{z})$$
$$= 4\frac{\partial}{\partial \bar{z}}\frac{\bar{z}}{1-\bar{z}z} = \frac{4}{(1-\bar{z}z)^2} = \lambda_{\mathbb{D}}^2(z).$$

So, $\kappa_{\lambda_{\mathbb{D}}}(z) = -1$ for all $z \in \mathbb{D}$.

*Example* 2.15. (Spherical Plane) Let $\rho(z) = \frac{2}{1+|z|^2}$ be the spherical metric on $\mathbb{C}$. Then with the same argument as in the previous example we get that $\kappa_\rho(z) = 1$ for any $z \in \mathbb{C}$. Or in other words, the spherical metric has constant curvature 1.

An important fact about curvature is that it is invariant under the pull-back of the metric. We have already used the notion of a pull-back and here let us prove the composition property for the pull-back. Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ be an analytic functions and $\rho$ is a semimetric on $\Omega_3$. Then at any $z \in \Omega_1$

$$(f \circ g)^*(\rho)(z) = g^*(f^*(\rho))(z).$$

That is because

$$(f \circ g)^*(\rho)(z) = \rho(f(g(z)) \cdot |f'(g(z))| \cdot |g'(z)| = f^*(g(z)) \cdot |g'(z)| = g^*(f^*(\rho))(z).$$

*Example* 2.16. Set $\Omega_2 = \mathbb{C}$ with $\rho(w) = 1$. Then the pull-back of $\rho$ will be

$$f^*(\rho)(z) = |f'(z)|.$$

In the last section we will consider the Bloch space

$$\mathcal{B}(\Omega) = \{f : \Omega \to \mathbb{C} \ : \ \sup_{z\in\Omega} \frac{f^*(\rho)(z)}{\lambda_\Omega(z)} < \infty\},$$

or in other words, the space of analytic functions whose pull-back of the Euclidean metric is bounded by the hyperbolic metric.

Let us choose $\Omega_1 = \Omega_2 = \mathbb{D}$ and $\rho = \lambda_\mathbb{D}$. Then the Schwarz-Pick Lemma states that the pull-back metric $f^*(\lambda_\mathbb{D})$ is less or equal to $\lambda_\mathbb{D}$.

Usually we will write just $f^*(\rho)$ to denote the pull-back semimetric on $\Omega_1$. Note that for a simply connected domain $\Omega$, the hyperbolic metric $\lambda_\Omega$ on it is a pull-back of $\lambda_\mathbb{D}$ under the map $f : \Omega \to \mathbb{D}$ from the Riemann mapping theorem. Note also that according to the definition of hyperbolic distortion, the hyperbolic distortion at $z \in \Omega_1$ is just the ratio of the pull-back metric of the function $f$ and the hyperbolic metric at $z$.

Now, we can state the following well-known result, which says that the curvature is invariant under taking a pull-back.

**Theorem 2.17.** *(Invariance of the Gaussian curvature) Let $f$ be an analytic map between two domains $\Omega_1$ and $\Omega_2$, and let $\rho(w)$ be a semimetric on $\Omega_2$. Then for any $z \in \Omega_1$ such that $f'(z) \neq 0$, $\rho(f(z)) > 0$ and $\rho$ is of class $C^2$ at $f(z)$ we have*

$$\kappa_{f^*(\rho)}(z) = \kappa_\rho(f(z)).$$

*Proof.* As defined above, $f^*(\rho)(z) = \rho(f(z))|f'(z)|$ and so for $f'(z) \neq 0$ :

$$\log(\rho(f(z))|f'(z)|) = \log(\rho(f(z))) + \log|f'(z)|$$

Let $\Delta_z$ be the Laplacian operator taken with respect to the $z$ variable. Then we can write according to Remark 2.12:

$$\Delta_z(f^*(\rho)(z)) = \Delta_z(\log\rho(f(z))) + \Delta_z(\log|f'(z)|) = \frac{\partial}{\partial\bar{z}}\left(\frac{\partial}{\partial z}\log\rho(f(z))\right).$$

Using the Chain rule and taking $w = f(z)$, we have that

$$\frac{\partial}{\partial z}\log\rho(f(z)) = \frac{\partial}{\partial w}(\log\rho(w))\frac{\partial w}{\partial z} = \frac{\partial}{\partial w}(\log\rho(w))|f'(z)|.$$

Next,

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\log\rho(f(z))\right) &= \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial w}(\log\rho(w))|f'(z)|\right)\\ &= \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial w}(\log\rho(w))\right)\cdot f'(z),\end{aligned}$$

since $f'(z)$ is analytic and $\frac{\partial}{\partial \bar{z}}f'(z) = 0$.

Furthermore using the Chain rule again, one gets

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial w}(\log\rho(w))\right)\cdot f'(z) &= \frac{\partial}{\partial \bar{w}}\left(\frac{\partial}{\partial w}(\log\rho(w))\right)\cdot\frac{\partial \bar{w}}{\partial \bar{z}}\cdot f'(z)\\ &= \frac{\partial^2}{\partial \bar{w}\partial w}(\log\rho(w))\cdot \overline{f'(z)}\cdot f'(z)\\ &= \frac{\partial^2}{\partial \bar{w}\partial w}(\log\rho(w))\cdot |f'(z)|^2,\end{aligned}$$

since $\frac{\partial \bar{w}}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\overline{f(z)} = \overline{f'(z)}$.

Hence,

$$\Delta_z(\log(\rho(f(z)))) = \Delta_w\log(\rho(w))|f'(z)|^2,$$

and we can write

$$\begin{aligned}\kappa_{f^*(\rho)}(z) &= -\frac{\Delta_z(\log(\rho(f(z)))|f'(z)|)}{\rho^2(f(z))|f'(z)|^2}\\ &= -\frac{(\Delta_w\log\rho)(w)|f'(z)|^2}{\rho^2(w)|f'(z)|^2} = \frac{(\Delta_w\log\rho)(w)}{\rho^2(w)}\\ &= \kappa_\rho(w) = \kappa_\rho(f(z)).\end{aligned}$$

□

Now we are ready to calculate the curvature for the hyperbolic metric on a simply connected proper subdomain in $\mathbb{C}$, , which is yet another interesting and well known result about the simply connected proper subdomains of $\mathbb{C}$.

**Theorem 2.18.** *The curvature of the hyperbolic metric on a simply connected domain $\Omega$ of $\mathbb{C}$ is equal to -1 at any point of $\Omega$.*

*Proof.* Recall that the curvature of the unit disk with the hyperbolic metric is equal to -1, as shown in Example 2.14. Now the general case for a simply connected domain $\Omega$ follows from the previous theorem, considering a conformal mapping $\phi : \Omega \to \mathbb{D}$. Since $\phi^*(\lambda_{\mathbb{D}})(z) = \lambda_\Omega(z)$, for $z \in \Omega$, we get that

$$\kappa_{\lambda_\Omega}(z) = \kappa_{\phi^*(\lambda_{\mathbb{D}})}(w) = \kappa_{\lambda_{\mathbb{D}}}(\phi(z)) = -1.$$

□

An important result of Ahlfors states that if the curvature of a metric is bounded by the curvature of the hyperbolic metric, the metric itself is bounded by the hyperbolic metric.

**Theorem 2.19** (Ahlfors Lemma)**.** *Let $\lambda_\Omega$ is a hyperbolic metric on a simply connected proper subdomain $\Omega$ of $\mathbb{C}$ and let $\rho(z)$ be any $C^2$ semimetric on $\Omega$ such that $\kappa_\rho(z) \le \kappa_{\lambda_\Omega}(z) = -1$, whenever $\rho(z) > 0$. Then $\rho \le \lambda_\Omega$ on $\Omega$.*

*Proof.* First, let us note that the general case follows from the case $\Omega = \mathbb{D}$. Indeed, for general $\Omega$ consider a conformal mapping $\phi : \mathbb{D} \to \Omega$. Then $\sigma(z) = \phi^*(\rho)(z)$ is a $C^2$ semimetric on $\mathbb{D}$ with $\kappa_\sigma(z) \le -1$, for all $z$ such that $\sigma(z) > 0$. Since the curvature is preserved under a "pull-back" by the map $\phi$, we get

$$\sigma(z) = \rho(\phi(z))|\phi'(z)| \le \lambda_{\mathbb{D}}(z) = \lambda_\Omega(\phi(z))|\phi'(z)|,$$

and so for $w = \phi(z)$ we have $\rho(w) \le \lambda_\Omega(w)$. Since $\phi$ is a conformal automorphism, $w$ is an arbitrary point in $\Omega$.

Now, let us prove the theorem for the case when $\Omega = \mathbb{D}$. For some $r > 1$ consider a semimetric on $\mathbb{D}$ given by $\rho_r = \rho(\frac{z}{r})$. Then

$$\kappa_{\rho_r} = -\frac{\Delta \log(\rho(\frac{z}{r}))}{\rho^2(\frac{z}{r})} = -\frac{\Delta \log(\rho)(\frac{z}{r})}{r^2\rho^2(\frac{z}{r})} = \kappa_\rho(\frac{z}{r}) \cdot \frac{1}{r^2}.$$

Then, consider the function

$$f_r(z) = \frac{\rho_r(z)}{\lambda_{\mathbb{D}}(z)},$$

defined on $\mathbb{D}$. This function is continuous on $\mathbb{D}$. Moreover $f_r(z) \geq 0$ and $\lim_{|z|\to 1} f(z) = 0$. That means that $f_r$ attains a maximum in an interior point $z_0 \in \mathbb{D}$. Suppose $\rho_r(z_0) \neq 0$. Then $\log f_r(z)$ has a local maximum at $z_0$, which means all of the second order partial derivatives of the function $\log f_r(z)$ at the point $z_0$ are negative. Therefore,

$$\begin{aligned} 0 \geq \Delta(\log f_r)(z_0) &= \Delta(\log \rho_r)(z_0) - \Delta(\log \lambda_{\mathbb{D}})(z_0) \\ &= -\kappa_{\rho_r}(z_0)\rho_r^2(z_0) + \kappa_{\lambda_{\mathbb{D}}}(z_0)\lambda_{\mathbb{D}}^2(z_0) \\ &= -\frac{1}{r^2}\kappa_\rho(\frac{z_0}{r})\rho_r^2(z_0) + \kappa_{\lambda_{\mathbb{D}}}(z_0)\lambda_{\mathbb{D}}^2(z_0) \\ &\geq \frac{1}{r^2}\rho_r^2(z_0) - \lambda_{\mathbb{D}}^2(z_0). \end{aligned}$$

Hence,

$$f_r(z_0) \leq r^2.$$

Since $z_0$ is a maximum for $f_r$, we get $f_r(z) \leq r^2$ for all $z \in \mathbb{D}$. Note that, for any fixed $z \in \mathbb{D}$ we have that $f_r(z) \to f_1(z) = \frac{\rho(z)}{\lambda_{\mathbb{D}}(z)}$, as $r \to 1$ to get that

$$\frac{\rho(z)}{\lambda_{\mathbb{D}}(z)} \leq 1.$$

□

The Schwarz-Pick Lemma is a simple consequence of the previous theorem. For $f : \Omega_1 \to \Omega_2$ analytic and not a constant, set $\rho(z) = f^*(\lambda_{\Omega_2})(z)$, for $z \in \Omega_1$. Then we have that $\kappa_\rho(z) = \kappa_{\lambda_{\Omega_2}}(w) = -1$ (or $\kappa_\rho(z) = 0$ if $f'(z) = 0$). Therefore $\kappa_\rho(z) \leq -1$ and $\rho(z) \leq \lambda_{\Omega_1}(z)$ on $\Omega_1$, i.e.,

$$\rho(z) = \lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z),$$

for all $z \in \Omega_1$. That means that for the analytic function $f : \Omega_1 \to \Omega_2$ the bounds on the curvature of the pull-back hyperbolic metric imply bounds on the hyperbolic distortion. We will see this again in Chapter 3, when we consider further generalizations of the Ahlfors Lemma.

We have considered the hyperbolic metric on different simply connected subdomains of $\mathbb{D}$. But we never considered $\mathbb{C}$ itself. As a matter of fact, there is no hyperbolic metric on $\mathbb{C}$.

**Theorem 2.20.** *There is no $C^2$ semimetric $\rho(z)$ on $\mathbb{C}$ for which $\kappa_\rho(z) \le -1$ whenever $\rho(z) > 0$.*

*Proof.* We will prove this by contradiction. Let $\rho(z)$ be such a semimetric and let $\rho(z_0) > 0$ for some $z_0$. Then consider the hyperbolic metric

$$\lambda_r(z) = \frac{2r}{r^2 - |z|^2}$$

on the disk $D_r$ with a center 0 and a radius $r > |z_0|$. Then $\kappa_{\lambda_r}(z) = -1 \ge \kappa_\rho(z)$ for any $z \in D_r$ such that $\rho(z) > 0$. Therefore, according to the Ahlfors Lemma

$$\rho(z_0) \le \lambda_r(z_0) = \frac{2r}{r^2 - |z_0|^2}.$$

Again, take limit as $r \to \infty$ on both sides of this inequality. We get $\rho(z_0) \le 0$, which is a contradiction. □

Before ending this chapter, we will show how using some of the properties of the curvature we can also give a very short and nice proof of Liouville's theorem. Recall that entire function is just an analytic function defined on the whole $\mathbb{C}$.

**Theorem 2.21** (Liouville)**.** *There are no bounded entire functions, except for constant functions.*

*Proof.* Let $f$ be a non-constant bounded entire function. Suppose that $|f(z)| < r$ for all $z \in \mathbb{C}$ and let $\lambda_r = \frac{2r}{r^2 - |z|^2}$. Then $f^*(\lambda_r)(z)$ would be a semimetric on $\mathbb{C}$ with curvature at most -1. This contradicts the previous theorem. □

## 2.3 Hyperbolic metric on general hyperbolic domains

Later in the thesis, we will consider the hyperbolic metric, defined not only on simply connected domains of $\mathbb{C}$. Basically, in this section, we will construct the hyperbolic metric on almost arbitrary subdomain of $\mathbb{C}$ by following [1]. In order to define the hyperbolic metric on an arbitrary domain by again using the unit disk, we need a generalization of the Riemann Mapping Theorem. For this reason, we will need to introduce the notion of a topological covering which is an extension of the notion of a conformal mapping. For the general theory of topological covering spaces and more details, we refer to [10].

**Definition 2.22.** *Let $f : \Omega_1 \to \Omega_2$ be an analytic function. $f$ is called an analytic covering if for any point $a \in \Omega_2$ there is a open neighborhood $U$, such that $f^{-1}(U) = \cup_{\alpha \in A} U_\alpha$ - disjoint union of open sets $U_\alpha$, such that the restriction of $f$ on each $U_\alpha$ is a conformal mapping from $U_\alpha$ onto $U$.*

Obviously, any conformal mapping $f : \Omega_1 \to \Omega_2$ is an analytic covering.

*Example* 2.23. For any positive integer $n$, the function $f(z) = z^n : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ is an analytic covering. Let $w$ be any point in $\mathbb{C}\backslash\{0\}$ with $\theta = arg(w)$. Let $U = \mathbb{C}\backslash\{re^{i(\theta+\pi)} : r \geq 0\}$, and $U_i = \{z : \frac{(\theta+\pi)}{n} + (i-1)\frac{(\theta+2\pi)}{n} < arg(z) < \frac{(\theta+\pi)}{n} + i\frac{(\theta+2\pi)}{n}\}$ for any integer $i$ between 1 and $n$. Then the restriction of $f$ on any $U_i$ is a conformal mapping from $U_i$ onto $U$.

**Definition 2.24.** *Any domain $\Omega$ in $\mathbb{C} \cup \infty$ that omits three distinct points in $\mathbb{C} \cup \infty$ (i.e. $\mathbb{C} \cup \{\infty\}\backslash\Omega$ contains at least three points) is called a hyperbolic domain.*

We want to construct the hyperbolic metric on any hyperbolic domain $\Omega$. To do that, we will need a "nice" map from $\Omega$ onto $\mathbb{D}$ to pull-back the hyperbolic metric on $\mathbb{D}$. The existence of such a map is provided by the following theorem. It is just an analogue for the Riemann Mapping Theorem if one replaces the conformal mapping by an analytic covering.

**Theorem 2.25.** *(Planar Uniformization Theorem) For any hyperbolic domain $\Omega$ and any point $a \in \Omega$ there is unique analytic covering $f : \mathbb{D} \to \Omega$ such that $f(0) = a$ and $f'(0) > 0$.*

We refer to [17] for a proof of the Planar Uniformization Theorem. Note that the Riemann Mapping Theorem is just a consequence of the Planar Uniformization Theorem since any analytic covering from $\mathbb{D}$ onto a simply connected domain $\Omega$ is a conformal mapping. We will also need the fact that if $f : \mathbb{D} \to \Omega$ is an analytic covering, then all of the analytic coverings from $\mathbb{D}$ onto $\Omega$ are of the form $f \circ h$, where $h \in \mathbf{Aut}(\mathbb{D})$. Now we are ready to transfer the hyperbolic metric from the disk onto any hyperbolic domain $\Omega$ in the same way as we did it for simply connected regions. The next theorem and its proof are as in [1].

**Theorem 2.26.** *On any hyperbolic domain $\Omega$, there exists a unique hyperbolic real analytic metric $\lambda_\Omega$ with curvature $-1$ such that for any analytic covering $h : \mathbb{D} \to \Omega$, the pull-back metric $h^*(\lambda_\Omega)$ will be the hyperbolic metric on $\mathbb{D}$.*

*Proof.* First, we will construct the hyperbolic metric locally. Let $\Omega$ be a hyperbolic domain and $h$ be an analytic covering of $\mathbb{D}$ onto $\Omega$. For a simply connected neighborhood $U$ of a point $w_0 \in \Omega$, consider the branch of the inverse function $H = h^{-1}$ defined on $U$. Then the hyperbolic metric at a point $w \in U$ is given by

$$\lambda_\Omega(w) = \lambda_{\mathbb{D}}(H(w))|H'(w)|.$$

This metric has curvature $-1$ in $U$. Now, we need to verify that the metric defined locally matches on the intersection of the neighborhoods. Let $U_1$ and $U_2$ be two simply connected neighborhoods of $w_0$ with non-empty intersection. Let $H_1$ and $H_2$ be two conformal branches of $h^{-1}$ defined on $U_1$ and $U_2$ respectively. Then there is $g \in \mathbf{Aut}(\mathbb{D})$, such that $H_2 = g \circ H_1$. Using the fact that $g$ is an isometry of $\lambda_{\mathbb{D}}$ we get:

$$\begin{aligned} H_2^*(\lambda_{\mathbb{D}})(z) = (g \circ H_1)^*(\lambda_{\mathbb{D}})(z) &= |g'(H_1(z))| \cdot |H_1'(z)| \cdot \lambda_{\mathbb{D}}(g(H_1(z))) \\ = |H_1'(z)| \cdot |g'(H_1(z))| \cdot \lambda_{\mathbb{D}}(g(H_1(z))) &= |H_1'(z)| \cdot g^*(\lambda_{\mathbb{D}})(H_1(z)) = H_1^*(g^*(\lambda_{\mathbb{D}}))(z) \\ &= H_1^*(\lambda_{\mathbb{D}})(z). \end{aligned}$$

Therefore, we have proven that the hyperbolic metric on $\Omega$ is defined properly at any point $w_0$ and it is independent on the choice of the branch of $h^{-1}$. Moreover, $h^*(\lambda_\Omega) = \lambda_{\mathbb{D}}$. We need also prove that the hyperbolic metric on $\Omega$ does not depend on the choice of the analytic covering. Let $k : \mathbb{D} \to \Omega$ be another covering from $\mathbb{D}$ onto $\Omega$. According to the remark after the Planar Uniformization Theorem, $k = h \circ g$

for some $g \in \mathbf{Aut}(\mathbb{D})$, and therefore

$$k^*(\lambda_\Omega)(z) = (h \circ g)^*(\lambda_\Omega)(z) = |h'(g(z))| \cdot |g'(z)| \cdot \lambda_\Omega(h(g(z)))$$
$$= |h'(g(z))| \cdot \lambda_\Omega(h(g(z))) \cdot |g'(z)| = |g'(z)| \cdot h^*(\lambda_\Omega)(g(z)) = g^*(h^*(\lambda_\Omega))(z)$$
$$= g^*(\lambda_{\mathbb{D}})(z) = \lambda_{\mathbb{D}}(z)$$

for any $z \in \mathbb{D}$.

It is also easy to see that the hyperbolic metric $\lambda_\Omega$ is real analytic, as it is just a product of an absolute value of an analytic function and $\lambda_{\mathbb{D}}$. The curvature of $\lambda_\Omega$ is $-1$ because the curvature of $\lambda_{\mathbb{D}}$ is preserved under taking a pull-back by the map $h$. $\square$

We can also define a hyperbolic distance on the hyperbolic domain $\Omega$ in the same way as we did it for simply connected domains.

$$d_\Omega(z, w) = \inf l_\Omega(\gamma),$$

where the infinum is taken over all piecewise smooth curves that connect $z$ and $w$ in $\Omega$. The analytic coverings are no longer the isometries of the hyperbolic distance. But, we still have that for any $f : \mathbb{D} \to \Omega$ and $z, w \in \mathbb{D}$ that

$$d_\Omega(f(z), f(w)) \leq d_{\mathbb{D}}(z, w).$$

This holds because any curve $\gamma$ that joins $z$ and $w$ will be mapped to the curve $\eta = f(\gamma)$ that joins $f(z)$ and $f(w)$. Equality does not hold, because generally not every curve $\eta$ can be represented as $f(\gamma)$.

However, we can adapt the Conformal Invariance theorem for the analytic coverings and use it to get a very general version of the Schwarz-Pick Lemma, as it is shown in [1].:

**Theorem 2.27.** *(Covering Invariance) The analytic coverings are local isometries of the hyperbolic domains. Namely, for any analytic covering $f : \Omega_1 \to \Omega_2$ of the hyperbolic domains and any $z \in \Omega_1$*

$$f^*(\lambda_{\Omega_2})(z) = \lambda_{\Omega_1}(z).$$

*Proof.* Choose $h : \mathbb{D} \to \Omega_1$ to be any analytic covering provided by the Planar Uniformization Theorem . Then $k = f \circ h : \mathbb{D} \to \Omega_2$ is an analytic covering and therefore

$$\lambda_{\mathbb{D}}(z) = k^*(\lambda_{\Omega_2})(z) = (f \circ h)^*(\lambda_{\Omega_2})(z)$$
$$= h^*(f^*(\lambda_{\Omega_2}))(z).$$

Hence, $f^*(\lambda_{\Omega_2})$ is a metric on $\Omega_1$. Its pull-back by the map $h$ is $\lambda_{\mathbb{D}}$. By Theorem 2.26, we can conclude that $f^*(\lambda_{\Omega_2})$ is the hyperbolic metric on $\Omega_1$. ☐

Another expected property of the hyperbolic metric on the hyperbolic domain is its maximality.

**Theorem 2.28.** *(Schwarz-Pick Lemma - the most general version) Let $f : \Omega_1 \to \Omega_2$ be an analytic map between two hyperbolic domains. Then*

$$f^*(\lambda_{\Omega_2})(z) = \lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z)$$

*for every $z \in \Omega_1$. If equality holds at least for one point in $\Omega_1$, then $f$ is a covering and equality holds for every point in $\Omega_1$.*

*Proof.* Choose $k$ to be an analytic covering from $\mathbb{D}$ onto $\Omega_1$ and $h$ to an analytic covering from $\mathbb{D}$ onto $\Omega_2$. Then, let $F$ be an analytic self-map of $\mathbb{D}$ such that $(f \circ k)(w) = (h \circ F)(w)$ for all $w \in \mathbb{D}$. We get the following

$$k^*(f^*(\lambda_{\Omega_2})) = (f \circ k)^*(\lambda_{\Omega_2}) = (h \circ F)^*(\lambda_{\Omega_2})$$
$$= F^*(h^*(\lambda_{\Omega_2})) = F^*(\lambda_{\mathbb{D}}) \leq \lambda_{\mathbb{D}} = k^*(\lambda_{\Omega_1}).$$

The inequality holds, because of the Schwarz-Pick Lemma for the function $F$. The inequality

$$k^*(f^*(\lambda_{\Omega_2})) \leq k^*(\lambda_{\Omega_1})$$

implies the inequality

$$\lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z),$$

because $k$ is surjective and a local bijection. $f$ is a covering if and only if $F$ is a conformal mapping. This means that if for any point $z \in \Omega_1$ we get

$$f^*(\lambda_{\Omega_2})(z) = \lambda_{\Omega_2}(f(z))|f'(z)| = \lambda_{\Omega_1}(z),$$

then $F^*(\lambda_{\mathbb{D}}) = \lambda_{\mathbb{D}}$ and $F$ is a conformal mapping. This implies that $f$ is also a covering. $\square$

To end this chapter, let us note that the existence of a hyperbolic metric on $\mathbb{C}_{a,b} = \mathbb{C}\backslash\{a, b\}$ will be used later in Section 3.1 in order to get a more general extension of the Ahlfors Lemma. We used extensively the Riemann Mapping Theorem and the Planar Uniformization Theorem to get hyperbolic metrics on different domains. However, sometimes it is really hard to construct the map $f : \mathbb{D} \to \Omega$. For the case $\Omega = \mathbb{C}_{a,b}$ this function $f$ is a modular form - a class of analytic functions that have a huge impact in Number Theory.

# Chapter 3

# Boundary version of Ahlfors Lemma

## 3.1 Boundary Ahlfors lemma for the unit disk

The results in this chapter are mostly based on the paper of Kraus, Roth and Ruscheweyh [9]. We will consider the boundary behavior of an analytic map connected to the hyperbolic derivative and the curvature of the metric induced by this map via the pull-back metric.

One of the main results for the boundary behavior of an analytic function on the unit disk is the following theorem, which was stated and proved in [9] :

**Theorem 3.1.** *For any analytic self-map $f$ of the unit disk and any open arc $\Gamma \subseteq \partial\mathbb{D}$ the following conditions are equivalent:*

*(a) For every $\xi \in \Gamma$,*

$$\lim_{z\to\xi} f^*(\lambda_{\mathbb{D}})(z) = \lim_{z\to\xi} \frac{2|f'(z)|}{1-|f(z)|^2} = +\infty.$$

*(b) For every $\xi \in \Gamma$,*

$$\liminf_{z\to\xi} \frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)} = \liminf_{z\to\xi} (1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2} > 0.$$

*(c) For every* $\xi \in \Gamma$,

$$\liminf_{z\to\xi} \frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)} = 1.$$

*(d) For every* $\xi \in \Gamma$,

$$\lim_{z\to\xi} |f(z)| = 1.$$

*(e)* $f$ *has an analytic extension across the arc* $\Gamma$ *with* $f(\Gamma) \subseteq \partial\mathbb{D}$.

Here and further on, when we say that $f : \Omega \to \Omega$ has an analytic extension across $\Gamma \subseteq \partial\Omega$ we mean that there is an open set $U$, such that $\Gamma \subset U$ and $f$ extends analytically from $\Omega$ to $\Omega \cup U$.

Basically, Theorem 3.1 says that if $\lambda$ is a pull-back of the hyperbolic metric $\lambda_{\mathbb{D}}$ by $f$ and if

$$\lambda(z) = f^*(\lambda_{\mathbb{D}})(z) = \frac{2|f'(z)|}{1 - |f(z)|^2}$$

'blows up' on some arc $\Gamma$, then $\lambda$ grows exactly as fast as $\lambda_{\mathbb{D}}(z)$ as $z \to \xi$ on the arc $\Gamma$. Also, if

$$\frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)}$$

is bounded away from 0 as $z \to \xi$, then $f$ has an analytic extension across the arc $\Gamma$ with $f(\Gamma) \subset \partial\mathbb{D}$.

Recall that the hyperbolic distortion of $f$ at $z$ is

$$|f^h(z)| = \frac{\lambda_{\mathbb{D}}(f(z))|f'(z)|}{\lambda_{\mathbb{D}}(z)} = \frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)}.$$

Thus, Theorem 3.1 connects the behavior of the hyperbolic distortion of $f$ close to $\Gamma$ with the boundary behavior of $f$ on $\Gamma$. Namely, the fact that $|f^h(z)|$ is bounded away from 0 close to $\Gamma$ is equivalent to the fact that $f$ is analytic across $\Gamma$ with $f(\Gamma) \subset \partial\mathbb{D}$.

Note that the implications "*(e)* $\Rightarrow$ *(d)*" and "*(c)* $\Rightarrow$ *(b)* $\Rightarrow$ *(a)*" in Theorem 3.1 are trivial.

The implication "*(d)* $\Rightarrow$ *(e)*" is nothing else, but the classical Schwarz-Carathéodory reflection principle which states the following (see [2] for more details).

**Theorem 3.2** (Reflection principle)**.** *Let* $f : \mathbb{D} \to \mathbb{D}$ *be an analytic function and let* $\Gamma$ *be an open arc on* $\partial\mathbb{D}$*. If the modululus of* $f$ *has a continuous extension to* $\Gamma$ *with* $|f(\Gamma)| = 1$*, then* $f$ *has an analytic continuation across* $\Gamma$*.*

The implication "$(e) \Rightarrow (a)$" is also simple enough since when $|f(\xi)| = 1$,

$$\begin{aligned}\lim_{z\to\xi} \frac{|f'(z)|}{1-|f(z)|^2} &= \lim_{z\to\xi} \frac{|f(\xi)-f(z)|}{(1-|f(z)|^2)|z-\xi|} \\ &\geq \lim_{z\to\xi} \frac{|f(\xi)|-|f(z)|}{(1-|f(z)|)(1+|f(z)|)|z-\xi|} \\ &= \lim_{z\to\xi} \frac{1}{(1+|f(z)|)|z-\xi|} = +\infty\end{aligned}$$

The main parts of the proof of Theorem 3.1 are the implications "$(a) \Rightarrow (c)$" and "$(b) \Rightarrow (d)$". The implication "$(a) \Rightarrow (c)$" can be restated only in terms of metrics. We will not provide the proof of this implication in the thesis. For the proof, we refer to [9]. This result is called the Boundary Ahlfors Lemma.

**Theorem 3.3** (Boundary Ahlfors Lemma)**.** *Suppose $\lambda$ and $\mu$ are two metrics on the unit disk, such that for any $z \in \mathbb{D}$ and some positive constants $C_\mu$ and $c_\lambda$ we have that $\kappa_\lambda \geq -c_\lambda$ and $\kappa_\mu \leq -C_\mu$. Furthermore, let $\Gamma$ be an open arc of $\partial\mathbb{D}$, such that*

$$\lim_{z\to\xi} \lambda(z) = +\infty$$

*for every point $\xi \in \Gamma$. Then*

$$\liminf_{z\to\xi} \frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

*for every $\xi \in \Gamma$.*

We see that the condition that the metric $\lambda_{\mathbb{D}}$ is unbounded at the boundary arc is very important. One can prove the following generalization of the Classical Ahlfors Lemma. This is a version of the Boundary Ahlfors Lemma in the case, when $\Gamma = \partial\mathbb{D}$.

**Theorem 3.4** (Ahlfors Lemma)**.** *Suppose $\lambda$ and $\mu$ are two metrics on the unit disk, such that for any $z \in \mathbb{D}$ and some positive constants $C_\mu$ and $c_\lambda$ we have that $\kappa_\lambda \geq -c_\lambda$ and $\kappa_\mu \leq -C_\mu$. Furthermore, let*

$$\lim_{|z|\to 1} \lambda(z) = +\infty,$$

*then*

$$\frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

*for every $z \in \mathbb{D}$.*

The proof of this version of Ahlfors Lemma is very similar to the proof of the classical Ahlfors Lemma from Section 2.2, and we will not include it here. (See [9], Theorem 2.1)

Note that in Theorem 3.1, it is insufficient to consider restricted limits. For example, the implication $(d) \Rightarrow (e)$ is no longer correct under the weaker assumption

$$\angle \lim_{z \to \xi} |f(z)| = 1$$

for every $\xi \in \Gamma$, where $\angle \lim_{z \to \xi}$ is a limit over any nontangential approach (see Section 1.4). In fact, Heins [5] provides an example of an infinite Blaschke product $B(z)$, such that the zeroes of $B$ have 1 as an accumulation point and $B$ has an angular limit equal to 1 at $z = 1$. Therefore, the angular limit of the modulus of $B$ is equal to 1 on $\partial\mathbb{D}$, but $B$ has no analytic extension to a neighborhood of $\partial\mathbb{D}$, since $\xi = 1$ is a singularity for $B$.

Moreover, we can produce an explicit example of an analytic function $f$ without analytic extension in the neighborhood of 1, but such that $\angle \lim_{z \to \xi} |f(z)| \to 1$, as $z \to \partial\mathbb{D}$. The main tool is the result of Frostman (see [4] for details), which asserts that the Blaschke product

$$f(z) = \prod_{j=1}^{\infty} \frac{z_j}{|z_j|} \cdot \frac{z_j - z}{1 - \overline{z_j} z}$$

has an angular limit of modulus one at a point $\xi \in \partial\mathbb{D}$ if and only if

$$d(\xi) = \sum_{j=1}^{\infty} \frac{1 - |z_j|}{|\xi - z_j|} < \infty.$$

Note that if $z_j \to \xi$, as $j \to \infty$ and $d(\xi) < \infty$, then for any other $\zeta \in \partial\mathbb{D}$ we get $d(\zeta) < d(\xi) < \infty$, because $|\zeta - z_j| > |\xi - z_j|$ starting from some $j$. That means that the angular limit of $|f|$ would exist and would be equal to 1 at any point $\zeta \in \partial\mathbb{D}$.

However, $f$ will not have an analytic continuation in the neighborhood of $\xi$, because $\xi$ is an accumulation point for the zeroes of $f$.

To see this, we construct a sequence $\{z_j\}$ such that $z_j \to \xi = 1$ and $d(1) < \infty$. Suppose that each $z_j \in \partial D(\frac{1}{2}, \frac{1}{2}) = \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$. Then for any $z_j = r_j e^{i\phi_j}$ we have that

$$\frac{1-|z_j|}{|1-z_j|}=\frac{1-\cos(\phi_j)}{\sin(\phi_j)}\to 0,$$

as $\phi_j \to 0$. Then choose $\phi_j$ in such a way that

$$\frac{1-|z_j|}{|1-z_j|}=\frac{1}{2^j},$$

leading to $d(1)=1$. Therefore, as we proved before, for any $\zeta \in \partial\mathbb{D}$,

$$\angle \lim_{z\to\zeta}|f(z)|=1,$$

but $f$ has no analytic extension in the neighborhood of $\xi=1$.

Another application of Theorem 3.1 can be written as follows. In the case when $\Gamma=\partial\mathbb{D}$, we get the following corollary of Theorem 3.1.

**Corollary 3.5.** *Let $f:\mathbb{D}\to\mathbb{D}$ be an analytic function. Then, the following conditions are equivalent:*

*(a)* $\lim_{|z|\to 1}\frac{|f'(z)|}{1-|f(z)|^2}=+\infty.$

*(b) $f$ is a finite Blaschke product.*

Part (a) of the Corollary is part (a) in Theorem 3.1 with $\Gamma=\partial\mathbb{D}$. Hence $(a)$ holds if and only if $|f(z)|=1$ on $\partial\mathbb{D}$ and $f$ has an analytic extension across $\partial\mathbb{D}$. Thus Theorem1.17 implies that $f$ must be a finite Blaschke product.

The corollary is also related to a result of Heins from [6], which was originally proved by using completely different methods.

**Theorem 3.6.** *An analytic function $f:\mathbb{D}\to\mathbb{D}$ is a finite Blaschke product if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2}=1$$

Therefore Theorem 3.1 can be also considered as a localized form of Heins' result.

A special case of Theorem 3.1 is when $f$ is not only analytic in $\mathbb{D}$, but is a rational function. In this case we can state even more: if infinitely many points of the unit circle are mapped on the unit circle, then $f$ must map $\mathbb{D}$ onto $\mathbb{D}$. We consider the following proposition.

**Proposition 3.7.** *Suppose $f$ is a non-constant rational function that maps $\mathbb{D}$ into itself. Let $f(\xi) \in \partial\mathbb{D}$ for infinitely many $\xi \in \partial\mathbb{D}$. Then $f(z)$ must be a finite Blaschke product.*

*Proof.* Suppose that $f(z)$ has zeroes $a_1, a_2, ..., a_k$ inside $\mathbb{D}$. Then the function

$$h(z) = \frac{f(z)}{\prod_{i=1}^{k} \frac{z-a_i}{1-\bar{a}_i z}}$$

also is rational and maps infinitely many point from $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. Therefore, we can assume that $f$ has no zeroes in $\mathbb{D}$. The fact that $f(\xi) \in \partial\mathbb{D}$ for infinitely many $\xi \in \partial\mathbb{D}$ means that

$$f(\xi)\overline{f(\xi)} = 1$$

for infinitely many $\xi \in \partial\mathbb{D}$. But then, the function $f(z)\overline{f(z)}$ can be written as $g(z)\frac{1}{z^l}$ for $z \in \partial\mathbb{D}$ for some rational function $g$ and integer $l$. Indeed, one can write

$$\begin{aligned}
f(\xi)\overline{f(\xi)} &= \frac{\sum_{k=0}^{n} a_k \xi^k}{\sum_{k=0}^{m} b_k \xi^k} \cdot \frac{\sum_{k=0}^{n} \overline{a_k}\overline{\xi}^k}{\sum_{k=0}^{m} \overline{b_k}\overline{\xi}^k} \\
&= \frac{\sum_{k=0}^{n} a_k \xi^k}{\sum_{k=0}^{m} b_k \xi^k} \cdot \frac{\sum_{k=0}^{n} \overline{a_k}(\frac{1}{\xi})^k}{\sum_{k=0}^{m} \overline{b_k}(\frac{1}{\xi})^k} \\
&= \frac{\sum_{k=0}^{n} a_k \xi^k}{\sum_{k=0}^{m} b_k \xi^k} \cdot \frac{\frac{1}{\xi^n}\sum_{k=0}^{n} \overline{a_k}\xi^{n-k}}{\frac{1}{\xi^m}\sum_{k=0}^{m} \overline{b_k}\xi^{m-k}} \\
&= \frac{\sum_{k=0}^{n} a_k \xi^k}{\sum_{k=0}^{m} b_k \xi^k} \cdot \frac{\sum_{k=0}^{n} \overline{a_k}\xi^{n-k}}{\sum_{k=0}^{m} \overline{b_k}\xi^{m-k}} \cdot \xi^{m-n} = g(\xi) \cdot \xi^{m-n}.
\end{aligned}$$

This is true because $\overline{\xi} = \frac{1}{\xi}$ for $\xi \in \partial\mathbb{D}$. Therefore, for $l = n - m$,

$$g(\xi)\frac{1}{\xi^l} = 1$$

and

$$g(\xi) = \xi^l$$

for infinitely many $\xi \in \partial\mathbb{D}$. Two different meromorphic functions can coincide only on a finite set, therefore

$$g(z) = z^l$$

for all $z$ and

$$f(z)\overline{f(z)} = 1$$

for all $z \in \mathbb{D}$. So, $f$ must be a constant according to the Minimum Modulus principle.

$\square$

## 3.2 Proof of (b) implies (d)

Here we will present the proof of Theorem 3.1 that involves the implication "$(b) \Rightarrow (d)$", as given in [9]. The proof is broken into two Lemmas.

**Lemma 3.8.** *Suppose that $f$ is a self-map of $\mathbb{D}$ that fixes 0, $\lambda_{\mathbb{D}}$ is the hyperbolic metric on $\mathbb{D}$ and $\Gamma$ is an open subarc of $\partial\mathbb{D}$, such that, for any $\xi \in \Gamma$,*

$$\liminf_{z\to\xi} \frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)} > 0.$$

*Then, the function*

$$g(z) = \frac{zf'(z)}{f(z)}$$

*has a meromorphic extension to a neighborhood of $\Gamma$, such that $Im(g(\xi)) = 0$ or $g(\xi) = \infty$ for any $\xi \in \Gamma$.*

*Proof.* We can apply the Schwarz Lemma to the function $f$ and get $|f(z)| \leq |z|$. Therefore,

$$\frac{1-|f(z)|^2}{1-|z|^2} \geq 1$$

for any $z \in \mathbb{D}$. Hence, for any $\xi \in \Gamma$,

$$\liminf_{z\to\xi} |f'(z)| = \liminf_{z\to\xi} (1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2} \cdot \frac{1-|f(z)|^2}{1-|z|^2}$$

and therefore,

$$\liminf_{z\to\xi} |f'(z)| \geq \liminf_{z\to\xi} (1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2} > 0.$$

Fix $\xi \in \Gamma$. Then there is $c > 0$ and a neighborhood $U$ of $\xi$, such that $|f'(z)| > c$ for any $z \in U \cap \mathbb{D}$. Denote $\Gamma_1 = U \cap \partial\mathbb{D}$ and $U(\xi) = U \cap \mathbb{D}$. Then the function

$$h(z) = \frac{1}{g(z)} = \frac{f(z)}{zf'(z)}$$

is analytic in $U(\xi)$. We want to show that $h$ has an analytic extension in the neighborhood of $\Gamma_1$. Suppose that $\xi_1 \in \Gamma_1$. Then there are two possibilities:

$$\liminf_{z \to \xi_1} \frac{1-|f(z)|}{1-|z|} = d(\xi) = \infty,$$

and then

$$\liminf_{z \to \xi_1} |f'(z)| = \liminf_{z \to \xi_1} (1-|z|^2) \frac{|f'(z)|}{1-|f(z)|^2} \frac{1-|f(z)|}{1-|z|} \frac{1+|f(z)|}{1+|z|} = \infty.$$

Hence, $\lim_{z \to \xi_1} h(z) = 0$. On other hand, if

$$\liminf_{z \to \xi_1} \frac{1-|f(z)|}{1-|z|} = d(\xi) < \infty,$$

then we can apply the Julia-Wolff-Carathéodory Theorem stated in Section 1.4. We get

$$\angle \lim_{z \to \xi_1} f(z) = \angle f(\xi_1) \in \partial \mathbb{D},$$
$$\angle \lim_{z \to \xi_1} f'(z) = \angle f'(\xi_1) \in \mathbb{C},$$

and $\angle f'(\xi_1) = \bar{\xi_1} \angle f(\xi_1) d(\xi)$, or in other words

$$\angle \frac{f'(\xi_1)\xi_1}{f(\xi_1)} = d(\xi) \geq 1,$$

The inequality $d(\xi) \geq 1$ holds, according to the Schwarz Lemma. Then, define $\angle h(\xi) = \frac{1}{d_\xi} \in (0,1)$.

In both cases, the analytic function $h(z)$ has an angular limit at $\xi_1 \in \Gamma_1$ and this limit is real. According to the classical Schwarz-Carathéodory reflection principle (see [7], page 87), we may conclude that $h$ has an analytic extension to the neighborhood of $\Gamma_1$ which is real on $\Gamma_1$. Therefore, $h$ has an analytic extension for some neighborhood of $\Gamma$, and $g(z)$ has a meromorphic extension to the neighborhood of $\Gamma$ that is real (or equal to $\infty$) on $\Gamma$.

$\square$

Now we are ready to prove the implication "$(b) \Rightarrow (d)$" in Theorem 3.1.

**Lemma 3.9.** *Let $f$ be an analytic self-map of $\mathbb{D}$ and let $\Gamma$ be an open subarc of $\partial\mathbb{D}$. If we have that*

$$\liminf_{z\to\xi} \frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)} = \liminf_{z\to\xi}(1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2} > 0$$

*for every $\xi \in \Gamma$, then $f$ has an analytic extension across $\Gamma$ and*

$$\lim_{z\to\xi} |f(z)| = 1$$

*for every $\xi \in \Gamma$.*

*Proof.* First, the expression

$$\frac{f^*(\lambda_{\mathbb{D}})}{\lambda_{\mathbb{D}}(z)}$$

does not change if we consider $f\circ\phi$ instead of $f$, where $\phi$ is a conformal automorphism of $\mathbb{D}$. Therefore, without loss of generality we may assume that $f(0) = 0$. According to Lemma 3.8,

$$g(z) = \frac{zf'(z)}{f(z)}$$

has a meromorphic extension to a neighborhood of $\Gamma$, which is real on $\Gamma$.

The main part of the proof is the fact that $g(z)$ is not only meromorphic, but even analytic in some neighborhood of $\Gamma$.

If $g$ has an analytic extension in the neighborhood of $\Gamma$, then the function $f$ is the solution of the complex ODE

$$y' = (g(z)/z)y$$

and it also has an analytic extension to a neighborhood of $\Gamma$. (Namely, $f(z) = exp(\int_{z_0}^{z} \frac{g(t)}{t}dt)$.) Hence, for any $\xi \in \Gamma$ the limits

$$\lim_{z\to\xi} |f(z)| = |f(\xi)|$$

and

$$\lim_{z\to\xi} |f'(z)| = |f'(\xi)|$$

exist. If $|f(\xi)| < 1$, then the expression

$$\liminf_{z\to\xi}(1-|z|^2)\frac{|f'(z)|}{1-|f(z)|^2} = (1-|\xi|^2)\frac{|f'(\xi)|}{1-|f(\xi)|^2} = 0,$$

which contradicts the assumption of Lemma 3.9. Since $f$ maps $\mathbb{D}$ into $\mathbb{D}$ we get that $|f(\xi)| = 1$ for any point $\xi \in \Gamma$.

Now, let us prove that $g(z)$ is analytic in some neighborhood of $\Gamma$. Assume to the contrary that $g$ has a pole of order $N \geq 1$ at some point $\xi \in \Gamma$. Note, that if we compose $f$ with a rotation then the expression $\frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)}$ does not change as far as we have the condition $f(0) = 0$. Hence, we can assume that $\xi = -1$ and write

$$g(z) = \frac{h(z)}{(1+z)^N},$$

where $h$ is some function that is analytic at $z = -1$ and, moreover $h(-1) \neq 0$. There is some neighborhood $U$ of $-1$, that contains no other pole of $g$. Set $\Gamma_1 = U \cap \Gamma$. We have that $g$ is real on $\Gamma_1$ and therefore

$$\overline{g(\frac{1}{\bar{\mu}})} = g(\mu),$$

for $\mu \in \Gamma_1 \backslash \{\xi\}$ and so

$$\overline{h(\frac{1}{\bar{\mu}})} = \frac{h(\mu)}{\mu^N}.$$

Now, since $h$ is analytic at $-1$, let $\mu \to -1$ to get

$$\overline{h(-1)} = h(-1)(-1)^N.$$

Multiplying the last equality by $(-1)^N h(-1)$ we get that $|h(-1)|^2(-1)^N = h^2(-1)$. Therefore, $h(-1)^2$ is real and

$$h(-1) \in \begin{cases} \mathbb{R}\backslash\{0\}, & \text{if } N \text{ is even} \\ i\mathbb{R}\backslash\{0\}, & \text{if } N \text{ is odd.} \end{cases}$$

We expect to get the formula for $f(z)$ by solving the ODE

$$g(z) = \frac{zf'(z)}{f(z)}.$$

Consider a Laurent expansion of $\frac{g(z)}{z}$ near the point $z = -1$

$$\frac{g(z)}{z} = \sum_{j=-N}^{+\infty} \frac{c_{j+N+1}}{(1+z)^{-j}}.$$

Note that then $c_1 = -h(-1)$. Then, we get that in $\mathbb{D}_* := \{z \in \mathbb{D} : \mathrm{Re} z < 0\}$

$$f(z) = \exp\left(\int_{z_0}^{z} \frac{g(u)}{u}\right) =$$

$$= \exp\left(\frac{-h(-1)}{(1-N)(1+z)^{N-1}} + \frac{c_2}{(2-N)(1+z)^{N-2}} + \dots + c_N \log(1+z) + \dots\right),$$

where $z_0$ is some point in $\mathbb{D}_*$ near $-1$, log is the principal branch of the logarithm.

To get a contradiction with a fact that $g$ has a pole at $z = -1$, we will approach $-1$ in different ways. To do that, it is necessary to consider three cases:

($N$ *is even*) Then $h(-1) \in \mathbb{R}\backslash\{0\}$, and for any $x$ in the interval $(-1, 0)$ we have that

$$|f(x)| =$$

$$\exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}} + \frac{\mathrm{Re}(c_2)}{(2-N)(1+x)^{N-2}} + \dots + \mathrm{Re}(c_N) \log(1+x) + \dots\right).$$

Suppose that $h(-1) > 0$, then

$$|f(x)| = \exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}}\right)$$

$$\cdot \exp\left(a_1(1+x) + a_2(1+x)^2 + \dots + a_{n-1}(1+x)^{n-1}\log(1+x) + a_n(1+x)^n + \dots\right)$$

for some real coefficients $\{a_i\}$. Therefore,

$$\lim_{x \to -1} |f(x)| = \lim_{x \to -1} \exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}}\right) \cdot 1 = +\infty.$$

But, $f$ is a self-map of $\mathbb{D}$, therefore $\liminf_{x \to -1} |f(x)| \le 1$. A contradiction follows.

Suppose that $h(-1) < 0$, then by the same argument as in the previous case

$$\lim_{x \to -1} |f(x)| = \lim_{x \to -1} \exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}}\right) \cdot 1 = 0.$$

Then

$$\lim_{x \to -1} |f'(x)| = \lim_{x \to -1} |\frac{g(x)}{x}| \cdot |f(x)| = \lim_{x \to -1} |\frac{h(x)}{x(1+x)^N}| \cdot |\exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}}\right)| = 0,$$

since

$$\lim_{x \to -\infty} \frac{e^{-x}}{x^{-n}} = 0$$

for every negative $n$. Then, we get a contradiction with assumption of Lemma 3.9.

($N \geq 3$ *is odd*) In this case $h(-1) \in i\mathbb{R}\backslash\{0\}$. Then approach $-1$ along a ray. Choose $\eta = e^{\frac{i\pi}{2(N-1)}}$ and let $\zeta_r = -1 + r\eta$ for $r \in (0,1)$. Then

$$|f(\zeta_r)|$$

$$= \exp\left(\frac{ih(-1)}{(1-N)r^{N-1}} + \mathrm{Re}\left(\frac{(c_2)}{(2-N)(r\eta)^{N-2}}\right) + \cdots + \mathrm{Re}(c_N \log(r\eta)) + \ldots\right),$$

With the same argument as in the previous case $-ih(-1) > 0$ leads to $\lim_{r\to 1} |f(z)| = \infty$, and for $-ih(-1) < 0$ we have $\lim_{r\to 1} |f(z)| = 0$, and $\lim_{r\to 1} |f'(z)| = 0$. In any case we get a contradiction.

($N = 1$) In this case $h(-1) \in i\mathbb{R}\backslash\{0\}$. Set $\gamma = -ih(-1) \in \mathbb{R}\backslash\{0\}$. We will approach $-1$ along a suitable arc of the circle $\partial D = \{z : |z + \frac{1}{2}| = \frac{1}{2}\}$. In this case the function has the following form

$$f(z) = \exp(-i\gamma \log(1+z)) \exp(\tilde{h}(z)),$$

where $\tilde{h}$ is an analytic function in some neighborhood of $z = -1$ and $\arg(z)$ is an argument function of $z$ with a range $(-\pi, \pi)$.

Then,

$$|f(z)| = \exp(\gamma \arg(1+z)) \exp(\mathrm{Re}\tilde{h}(z)).$$

Suppose that $\gamma < 0$ and set $z_\phi = -\frac{1}{2} + \frac{1}{2}e^{i\phi}$ with $\phi \in (-\pi; \frac{-\pi}{2})$. This is an approach along the lower arc of the circle $\partial D$. Then $\gamma \arg(1 + z_\phi) > 0$, and therefore $\exp(\gamma \arg(1 + z_\phi)) > 1$. Then, since $|f(z)| < 1$ for all $z \in \mathbb{D}$, we get that $\exp(\mathrm{Re}\tilde{h}(z_\phi)) < 1$ and $\mathrm{Re}\tilde{h}(z_\phi) < 0$. Letting $\phi \to -\pi$ we get $\mathrm{Re}\tilde{h}(-1) \leq 0$.

If $\gamma > 0$, then set $z_\phi = -\frac{1}{2} + \frac{1}{2}e^{i\phi}$ with $\phi \in (\frac{\pi}{2}; \pi)$. This is an approach along the upper arc of the circle $\partial D$. Then again $\gamma \arg(1+z_\phi) > 0$, and therefore $\mathrm{Re}\tilde{h}(-1) \leq 0$. This means that $\mathrm{Re}\tilde{h}(-1) \leq 0$ does not depend on the sign of $\gamma$. Now, again consider two cases.

If $\gamma > 0$, then $z_\phi = -\frac{1}{2} + \frac{1}{2}e^{i\phi}$ with $\phi \in (-\pi; -\frac{\pi}{2})$ and

$$\lim_{z\to -\pi^+} |f(z_\phi)| \leq \exp\left(-\gamma\frac{\pi}{2}\right) < 1.$$

Therefore, we get

$$\begin{aligned}
&\liminf_{\phi\to-\pi}(1-|z_\phi|^2)\frac{f'(z_\phi)}{1-|f(z_\phi)|^2}\\
&=\liminf_{\phi\to-\pi}(1-|z_\phi|^2)\cdot\frac{|f(z_\phi)|\cdot|-i\gamma/(1+z_\phi)+\tilde{h}'(z_\phi)/(1+z_\phi)^2|}{1-|f(z_\phi)|^2}\\
&=\liminf_{\phi\to-\pi}\left(\frac{1-|z_\phi|^2}{|1+z_\phi|^2}\right)\cdot\left(\frac{|f(z_\phi)|\cdot|-i\gamma(1+z_\phi)+(1+z_\phi)^2\tilde{h}'(z_\phi)|}{1-|f(z_\phi)|^2}\right)=0.
\end{aligned}$$

This is because the first multiplier is equal to 1, and the second tends to 0. Really, we get that the triangle with vertices at $-1$, $0$ and $z_\phi$ is a right triangle. Then $|1+z_\phi|^2+|z_\phi|^2=1$ according to the Pythagorean theorem.

If $\gamma<0$, then set $z_\phi=-\frac{1}{2}+\frac{1}{2}e^{i\phi}$ with $\phi\in(\frac{\pi}{2};\pi)$. Then

$$\lim_{z\to\pi^-}|f(z_\phi)|\le exp\left(\gamma\frac{\pi}{2}\right)<1$$

and we can get as before that

$$\liminf_{\phi\to-\pi}(1-|z_\phi|^2)\frac{f'(z_\phi)}{1-|f(z_\phi)|^2}=0.$$

Both cases contradict assumption of the Lemma.

□

Basically, this Lemma also provides a proof of the implication "$(b)\Rightarrow(e)$".

We finish this section with two examples from [9] which demonstrate that Theorem 3.1 does not hold in general if the arc $\Gamma$ is replaced by a single point.

*Example* 3.10. The analytic function $f:\mathbb{D}\to\mathbb{D}$ given by

$$f(z)=\frac{\sqrt{1-z}}{\sqrt{1-z}+\sqrt{1+z}}$$

has no analytic extension to a neighborhood of $z=1$. But the condition (a) of Theorem 3.1 is satisfied at $z=1$. However,

$$\lim_{z\to1}|f'(z)|=+\infty \text{ and } \lim_{z\to1}|f(z)|=0.$$

Hence, the implication "$(a)\Rightarrow(e)$" and "$(a)\Rightarrow(d)$" both fail if $\Gamma$ is a single point.

*Example* 3.11. The function $f : \mathbb{D} \to \mathbb{D}$ given by

$$f(z) = T^{-1}(\sqrt{T(z)}),$$

where $T(z) = (1+z)/(1-z)$, satisfies condition (d) of Theorem 3.1 at $z = 1$, although it has no analytic extension to any neighborhood of $z = 1$. Moreover, $f$ satisfies condition (a) of Theorem 3.1. It does not satisfy condition (b), because for the points

$$z_\phi = \frac{1}{2} + \frac{1}{2}e^{i\phi},$$

we get

$$\lim_{\phi \to 0}(1 - |z_\phi|^2)\frac{|f'(z_\phi)|}{1 - |f(z_\phi)|^2} = 0.$$

In particular,

$$\liminf_{z \to 1}(1 - |z|^2)\frac{|f'(z)|}{1 - |f(z)|^2} = 0.$$

Therefore the implications "$(d) \Rightarrow (b)$" , "$(a) \Rightarrow (b)$" and "$(d) \Rightarrow (e)$" of Theorem 3.1 are no longer true if $\Gamma$ is replaced by a single point.

## 3.3 Further extensions of Ahlfors Lemma

We have considered a few versions of the Ahlfors Lemma so far. The classical Ahlfors Lemma was considered is Section 2.2 as Theorem2.19. The Boundary Version of Ahlfors Lemma was given in Section 3.1 as Theorem3.3 as part of the proof of Theorem 3.1. Here we will show the connections between the generalized distortion of self maps of some general domains $\Omega$, and the behavior of $f$ near the boundary $\partial\Omega$. More precisely, we will see the connection between the boundedness of the curvature of some metrics on $\Omega$ from above and below, the boundness from above of the general distortion of analytic functions $f : \Omega \to \Omega$ near the boundary of $\Omega$ and the boundary behavior of these analytic maps.

Also, we will introduce some new results that were not stated in [9]. Those are contained in Theorem 3.20 and Theorem 3.21, which could be considered as a version of Ahlfors Lemma for the case of domains with "not so good" boundaries.

We first extend the Ahlfors Lemma to domains in $\mathbb{C}$ with "nice" boundaries. Let us begin with a definition of a smooth boundary subset of a domain in a complex plane.

Recall that by Jordan curve we mean a continuous injection from $\partial\mathbb{D}$ into $\mathbb{C}$. We say that the domain $\Omega$ is Jordan if its boundary is a Jordan curve.

**Definition 3.12.** *A Jordan domain $\Omega$ is said to be smooth if and only if there is a conformal mapping $\phi$ from $\mathbb{D}$ onto $\Omega$, such that $|\phi'(z)|$ extends continuously to a nonzero function on $\partial\mathbb{D}$.*

An example of a smooth Jordan domain is a domain bounded by a Dini-smooth curve (see [13]). Note that a smooth Jordan domain is not just a Jordan domain with a smooth boundary. For a counterexample we refer to [13], page 46. Also note that according to the Carathéodory extension principle, $\phi$ extends to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. The following definition is a localized form of the previous one.

**Definition 3.13.** *A subset $\Gamma$ of the boundary of a proper domain $\Omega$ is called smooth if for any point $\xi \in \Gamma$ there is an open set $U \subseteq \mathbb{C}$ and a smooth Jordan domain $\Phi \subseteq \Omega$ such that $\xi \in \partial\Phi \cap U \subseteq \Gamma$.*

Now we have the following extension of the Boundary Ahlfors Lemma for a smooth Jordan subset $\Gamma$, as stated and proven in [9].

**Theorem 3.14** (Boundary Ahlfors Lemma for Jordan domains)**.** *Let $\Omega$ be a domain with a smooth boundary set $\Gamma$. Suppose that $\lambda$ and $\mu$ are two metrics on $\Omega$, such that for some positive constants $c_\lambda$ and $C_\mu$ we have that $\kappa_\lambda(z) \geq -c_\lambda$ and $\kappa_\lambda(z) \leq -C_\lambda$ for any $z \in \Omega$. If*

$$\lim_{z\to\xi} \lambda(z) = +\infty,$$

*for every $\xi \in \Gamma$, then*

$$\liminf_{z\to\xi} \frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

*and*

$$\lim_{z\to\xi} d_\lambda(z_0, z) = +\infty$$

*for every $\xi \in \Gamma$ and every $z_0 \in \Omega$.*

*Proof.* In order to prove the first statement, consider a point $\xi \in \Gamma$ and a smooth Jordan domain $\Phi \subseteq \Omega$, such that $\xi \in \Phi \cap U \subseteq \Gamma$, for some open neighborhood $U$ of $\xi$. Note that $\partial\Phi \cap \Gamma \subset \bar{U}$. Let $\phi$ be a conformal mapping from $\mathbb{D}$ onto $\Phi$ that has a continuous extension to a homeomorphism from $\overline{\mathbb{D}}$ onto $\bar{\Phi}$ with $\phi' \neq 0$ on $\partial\mathbb{D}$. Then, there is an open arc $\Gamma_0 \subseteq \partial\mathbb{D}$, such that $\phi(\Gamma_0) \subseteq \Gamma$ and $\phi(\xi_0) = \xi$ for some $\xi_0 \in \Gamma_0$. We want to apply the Boundary Ahlfors Lemma for $\Gamma_0$ and for the pull-back metrics $\phi^*(\lambda)(z)$ and $\phi^*(\mu)(z)$. Since the curvature is preserved under the pull-back, we get that $\kappa_{\phi^*(\lambda)} = \kappa(\lambda) \geq -c_\lambda$ and $\kappa_{\phi^*(\mu)} = \kappa(\mu) \leq -C_\mu$. Since $|\phi'| \neq 0$ on $\partial\mathbb{D}$, it obtains its minimum in $\partial\mathbb{D}$ and $|\phi'(z)| \geq c$ for some $c > 0$ and therefore

$$\lim_{z\to\xi_1} \phi^*\lambda(z) = \lim_{z\to\xi_1} \lambda(\phi(z))|\phi'(z)| \geq c \cdot \lim_{z\to\xi_1} \lambda(\phi(z)) = +\infty$$

for any $\xi_1 \in \Gamma_0$. So we can apply the Boundary Ahlfors Lemma (Theorem 3.3) and get that for $\xi_0 \in \Gamma$

$$\liminf_{z\to\xi_0} \frac{\phi^*(\lambda)(z)}{\phi^*(\mu)(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}.$$

Using the fact that $\phi^*(\lambda)(z) = \lambda(\phi(z))|\phi'(z)|$ and $\phi^*(\mu)(z) = \lambda(\phi(z))|\mu'(z)|$, and substitution $w = \phi^{-1}(z)$ with $\phi^{-1}(\xi_0) = \xi$ we get that

$$\liminf_{w\to\xi} \frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for any $\xi \in \Gamma$.

Now let us prove that any boundary point $\xi \in \Gamma$ is infinitely far from any other point $z_0 \in \Omega$. Let $\Phi$ be the same smooth Jordan domain as in the proof of the first part. Let $\Gamma_1$ be an open arc containing $\xi$, such that $\Gamma_1 \subseteq \partial\Phi \cap \Gamma$. Now consider $\mu = \lambda_\Phi$ the hyperbolic metric on $\Phi$ with a fixed curvature $-1$. From the first part of the proof we get some neighborhood $U$ of a point $\xi$ that does not contain $z_0$ and such that $\lambda(z) \geq c\mu(z)$ for some constant $c$. Therefore

$$d_\lambda(z, z_0) \geq c \cdot \min_{w\in\partial U} d_\mu(z, w).$$

As far as $\mu$ is complete near $\xi$, the expression $d_\mu(z, w) \to \infty$ as $z \to \xi$. Hence $d_\mu(z, z_0) \to \infty$ as $z \to \xi$.

□

The next Corollary from [9] shows the characterization of locally complete metrics for smooth boundary sets in terms of the boundary behavior of their density functions.

**Corollary 3.15.** *Let $\Omega \subset \mathbb{C}$ be a domain, let $\Gamma$ be a smooth subset of $\partial\Omega$ and let $\lambda(z)$ be a metric on $\Omega$ with $\kappa_\lambda \geq -c_\lambda$ for some positive constant $c_\lambda$. Then the following are equivalent:*

*(a)* $\lambda(z)$ *is locally complete near* $\Gamma$,

*(b)* $\lim_{z\to\xi} \lambda(z) = +\infty$ *for every* $\xi \in \Gamma$.

Now, combining Ahlfors Lemma and Theorem 3.1, we can prove the following generalization of the statements "$(a) \Leftrightarrow (b) \Leftrightarrow (e)$" for the unit disk $\mathbb{D}$. This can be considered as a type of Ahlfors Lemma that involves an analytic map $f$ which is a self-map of $\mathbb{D}$.

Define here the *generalized derivative* of $f : (\mathbb{D}, \mu) \to (\mathbb{D}, \lambda)$ at $z$ as

$$f^{\lambda,\mu}(z) = \frac{\lambda(f(z))f'(z)}{\mu(z)},$$

and the *general distortion* of $f$ at $z$ as

$$|f^{\lambda,\mu}(z)| = \frac{\lambda(f(z))|f'(z)|}{\mu(z)}.$$

Then the next theorem (see [9]) states that the boundedness of the general distortion from below near the boundary set $\Gamma$ is equivalent to $f$ having an analytic extension across the boundary arc $\Gamma$.

**Theorem 3.16.** *Suppose that $f$ is an analytic self-map of $\mathbb{D}$ and $\mu$ is a metric on $\mathbb{D}$ with $\kappa_\mu \leq -C_\mu$. If $\Gamma$ is an open arc in $\partial\mathbb{D}$, such that*

$$\lim_{z\to\xi} \mu(z) = +\infty$$

*for any $\xi \in \Gamma$, then the following conditions are equivalent.*

*(a)* *For some semimetric $\lambda(z)$ such that $-c_\lambda \leq \kappa_\lambda \leq -C_\lambda$, we get that for every $\xi \in \Gamma$*

$$\lim_{z\to\xi} f^*(\lambda)(z) = +\infty.$$

*(b) For some semimetric $\lambda(z)$ such that $-c_\lambda \leq \kappa_\lambda \leq -C_\lambda$, we get that for every $\xi \in \Gamma$*

$$\liminf_{z\to\xi} \frac{f^*(\lambda)(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}},$$

*(e) The function $f$ extends analytically across the boundary arc $\Gamma$ with $f(\Gamma) \subset \partial\mathbb{D}$.*

*Proof.* $(a) \Rightarrow (b)$ Take the same $\lambda$ in (b) as given in (a). Then this is just a simple consequence of the Boundary Ahlfors Lemma, i.e. Theorem 3.3 stated in Section 3.1. Moreover this is true even if we only assume $\kappa_\lambda \geq -c_\lambda$.

$(b) \Rightarrow (e)$ Applying Theorem 3.4 for metrics $\lambda_1$ and $\mu_1$, where $\lambda_1(z) = \lambda_{\mathbb{D}}(z)$ and $\mu_1(z)$ is $\lambda(z)$ and using the fact that $\kappa_{\lambda_{\mathbb{D}}} = -1$, we get

$$\lambda_{\mathbb{D}}(z) \geq \sqrt{\frac{C_\lambda}{1}}\lambda(z)$$

near every point $\xi \in \Gamma$.

Then $f^*(\lambda)(z) \geq \sqrt{\frac{C_\mu}{c_\lambda}}\mu(z)$ near every point $\xi$. So, the fact that $\mu(z) \to +\infty$ as $z \to \xi \in \Gamma$ implies

$$\lim_{z\to\xi} \lambda_{\mathbb{D}}(f(z))|f'(z)| \geq \sqrt{\frac{C_\lambda}{1}} \lim_{z\to\xi} \lambda(f(z))|f'(z)| \geq$$

$$\sqrt{\frac{C_\lambda \cdot C_\mu}{c_\lambda}} \lim_{z\to\xi} \mu(z) = +\infty,$$

and the statement $(e)$ follows from Theorem 3.1, since

$$\lambda_{\mathbb{D}}(f(z))|f'(z)| = \frac{|f'(z)|}{1 - |f(z)|^2}.$$

$(e) \Rightarrow (a)$ We get the result by choosing $\lambda(z) = \lambda_{\mathbb{D}}(z)$ and applying part $(e) \to (a)$ from the Theorem 3.1. $\square$

Thus, Theorem 3.16 says that for an arc $\Gamma$ and a metric $\mu$ with curvature bounded from above by a negative constant, the existence of a metric $\lambda$ such that the general distortion of $f$ with respect to $\lambda$ and $\mu$ is bounded from below near $\Gamma$ is equivalent to the fact that $f$ has an analytic extension across $\Gamma$, and it is equivalent to the fact that the pullback metric $f^*(\lambda)$ tends to $\infty$ on $\Gamma$.

One can ask the following:

**Question 1:** Under the assumptions of Theorem 3.16, is the generalized distortion bounded from above? Or, in other words, does there exist $c > 0$ such that

$$\frac{\lambda(f(z))|f'(z)|}{\mu(z)} \leq c$$

for all $z \in \mathbb{D}$?

In the case when $\mu = \lambda = \lambda_\Omega$ - is the hyperbolic metric on $\Omega$, we can choose $c = 1$ since by the Schwarz-Pick Lemma

$$\frac{\lambda(f(z))|f'(z)|}{\mu(z)} = \frac{\lambda_\Omega(f(z))|f'(z)|}{\lambda_\Omega(z)} \leq 1.$$

However we don't know the answer to this question in the general case.

Another natural extension of the Ahlfors Boundary Lemma is an extension for more general domains. The following result was proven in [9] in a general version for Riemann surfaces. The original statement was called Ahlfors Lemma on a Riemann surface. However we give a prove, modifying it for domains in $\mathbb{C}$.

**Theorem 3.17.** *Let $\Omega$ be a simply connected domain in $\mathbb{C}$ with analytic boundary $\partial\Omega$. Let $\Gamma$ be an open arc of $\partial\Omega$ and let $\lambda(z)$ be a complete metric with curvature bounded below and above by negative constants $-c_\lambda$ and $-C_\lambda$, respectively. If $f$ is an analytic self-map of $\Omega$, then the following conditions are equivalent.*

*(a) $f$ has an analytic extension across $\Gamma$ with $f(\Gamma) \subset \partial\Omega$.*

*(b) For every $\xi \in \Gamma$*

$$\lim_{z \to \xi} \lambda(f(z))|f'(z)| = +\infty.$$

*(c) For a semimetric $\mu$ on $\Omega$ with $\kappa_\mu \leq -C_\mu$ and such that for every $\xi \in \Gamma$*

$$\lim_{z \to \xi} \mu(z) = +\infty,$$

*we get the following*

$$\liminf_{z \to \xi} \frac{f^*(\lambda)(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

*for every $\xi$ in $\Gamma$.*

To prove this Theorem we will need to state few additional Theorems which can be considered as a generalization of the Carathéeodory extension principle.

**Theorem 3.18.** *For any simply connected bounded domain $\Omega$ with analytical boundary, there is a conformal mapping $\phi$ from $\mathbb{D}$ onto $\Omega$, such that $\phi$ extends analytically to a conformal mapping defined in the neighborhood of $\overline{\mathbb{D}}$ with $\phi(\partial\mathbb{D}) = \partial\Omega$.*

The proof is rather simple application of the Reflection principle and can be found in [13].

The next Theorem shows a crucial connection between the boundary behavior of an analytic function and its extension outside the region on which it is defined.

**Theorem 3.19.** *Suppose that $\Omega$ is a simply connected domain with analytical boundary equipped with a complete metric $\lambda$, such that $-C_\lambda \geq \kappa_\lambda \geq -c_\lambda$ for some positive constants $c_\lambda$ and $C_\lambda$. Then, for an open subarc $\Gamma$ of $\partial\mathbb{D}$ and an analytic map from $\mathbb{D}$ onto $\Omega$*

$$\lim_{z \to \xi} f^*(\lambda)(z) = +\infty$$

*for every $\xi \in \Gamma$ if and only if $f$ extends analytically in the neighborhood of $\Gamma$ with $f(\Gamma) \subseteq \partial\Omega$.*

*Proof.* Suppose that $\phi$ is a conformal mapping from $\mathbb{D}$ onto $\Omega$ provided by Theorem 3.18. Then, $\phi$ has a conformal extension in the neighborhood of $\overline{\mathbb{D}}$. Consider a pull-back metric on $\mathbb{D}$ that is given for any $z \in \mathbb{D}$ by the formula

$$\mu(z) = \phi^*(\lambda)(z).$$

Since the curvature is preserved under taking the pull-back, we have $-C_\lambda \geq \kappa_\mu \geq -c_\lambda$. Moreover, $\mu(z)$ is a complete metric because $\phi$ is conformal in the neighborhood of $\overline{\mathbb{D}}$. Therefore, according to Corollary 3.15, we get

$$\lim_{|z| \to 1} \mu(z) = +\infty.$$

Also define a self-map $g$ of $\mathbb{D}$ by $g = \phi^{-1} \circ f$. Then, we have the following connection between the pull-back of the metrics $\lambda$ and $\mu$ by the maps $f$ and $g$ :

$$f^*(\lambda)(z) = (\phi \circ g)^*(\lambda)(z) = g^*(\phi^*(\lambda))(z) = g^*(\mu)(z).$$

Now suppose that $f$ has an analytic extension with $f(\Gamma) \subseteq \partial\Omega$. Therefore, $g$ also has an analytic extension in the neighborhood of $\Gamma$, because it is a composition of two analytic functions that are analytic in the neighborhood of $\Gamma$.

Moreover, $g' \neq 0$ on $\Gamma$. To prove that, let us consider the conformal mapping $\psi$ of $\mathbb{D}$ onto itself, such that $\psi(g(0)) = 0$. Then define an analytic self map of $\mathbb{D}$ by $h : \mathbb{D} \to \mathbb{D} := \psi(g(z))$. Note that $h(0) = 0$, and $h(\Gamma) \subset \partial\mathbb{D}$. Suppose that $g'(\xi_0) = 0$ for some $\xi_0 \in \Gamma$. Then $h'(\xi_0) = 0$. That means

$$\lim_{z \to \xi_0} \frac{h(z) - h(\xi_0)}{z - \xi_0} = 0.$$

Choose $z_0 = r\xi_0 \in \mathbb{D}$ for some $1 > r > 0$, such that $\left|\frac{h(z_0)-h(\xi_0)}{z_0-\xi_0}\right| < 1$ or in other words $|h(z_0) - h(\xi_0)| < |z_0 - \xi_0|$. On the other hand, one can get

$$|h(z_0) - h(\xi_0)| \geq |h(\xi_0)| - |h(z_0)| = 1 - |h(z_0)| \geq 1 - |z_0| = |z_0 - \xi_0|.$$

The last inequality holds because of Schwarz Lemma. Therefore we got a contradiction and with $g'(\xi_0) = 0$.

Note that for $\xi \in \Gamma$ when $z \to \xi$ we have that $f(z) \to f(\xi) \in \partial\Omega$, and consequently $g(z) = \phi^{-1}(f(z)) \to \phi^{-1}(f(\xi)) \in \partial\mathbb{D}$. Therefore

$$\mu(g(z)) \to \infty,$$

when $z \to \xi$. Finally we get that

$$f^*(\lambda)(z) = g^*(\mu)(z) \to \infty,$$

as $z \to \xi$.

Now, suppose that $g^*(\mu)(z) = f^*(\lambda)(z) \to \infty$ for every $\xi \in \Gamma$. Then $c_\lambda \leq \kappa_\mu \leq C_\lambda$ and we can apply the Boundary Ahlfors Lemma for the metrics $\lambda_{\mathbb{D}}$ and $\mu$. For any $\xi \in \Gamma$ we get for some constant $c > 0$ that $c\mu \leq \lambda_{\mathbb{D}}$ in some neighborhood of $\xi$. Therefore,

$$\lim_{z \to \xi} \lambda_{\mathbb{D}}(g(z))|g'(z)| \geq c \cdot \lim_{z \to \xi} \lambda_{\mathbb{D}}(g(z))|g'(z)| = \infty.$$

Hence, according to Theorem 3.18, we get that $g$ extends analytically to the neighborhood of $\Gamma$ with $g(\Gamma) \subseteq \partial\mathbb{D}$. Then, $f = \phi \circ g$ also has an analytic extension in the neighborhood of $\Gamma$ with $f(\Gamma) \subseteq \partial\Omega$.

$\square$

Now, we are ready to prove Theorem 3.17 and get the most general version of the Ahlfors Lemma.

*Proof of Theorem 3.17.* Let $\phi$ be a conformal map from $\mathbb{D}$ onto $\Omega$ provided by Theorem 3.18. Then, let $\Gamma_1 \subseteq \partial\mathbb{D}$ be such that $\phi(\Gamma_1) = \Gamma$. Again, instead of working with $f$, we introduce two functions $g = f \circ \phi : \mathbb{D} \to \Omega$ and $h = \phi^{-1} \circ f \circ \phi : \mathbb{D} \to \mathbb{D}$. Also, for $\xi \in \Gamma$ let $\xi_1$ be such that $\phi(\xi_1) = \xi$. Let us prove that "$(a) \Rightarrow (b)$". $f$ has an analytic extension across $\Gamma$ with $f(\Gamma) \subseteq \partial\Omega$ implies that $g$ has an analytic extension across $\Gamma_1$ with $g(\Gamma_1) \subseteq \partial\Omega$. Therefore for $\xi_1 \in \Gamma_1$

$$\lim_{z \to \xi_1} g^*(\lambda)(z) = \infty,$$

according to Theorem 3.19. Now, suppose that $\{u_n\} \subset \Omega$ converges to $\xi$. Then the sequence $\{z_n = \phi^{-1}(u_n)\}$ converges to $\xi_1$. As far as $\phi'$ obtains a maximum on $\overline{\mathbb{D}}$, there is $c > 0$, such that $|\phi(z)| < c$ for every $z \in \overline{\mathbb{D}}$. Hence,

$$f^*(\lambda)(u_n) = \lambda(f(u_n))|f'(u_n)| = \lambda(f(\phi(z_n)))|f'(\phi(z_n))|$$
$$\geq \frac{1}{c} \cdot \lambda(f(\phi(z_n))) \cdot |f'(\phi(z_n))| \cdot |\phi'(z_n)| = \frac{1}{c}\lambda(g(z_n))|g'(z_n)| = \frac{1}{c} g^*(\lambda)(z_n),$$

and so

$$\lim_{u \to \xi} f^*(\lambda)(u) \geq \frac{1}{c} \lim_{z \to \xi_1} g^*(\lambda)(z) = +\infty.$$

Let us prove "$(b) \Rightarrow (c)$". Let $\mu(z)$ be a semimetric on $\Omega$, such that $\kappa_\mu \leq -C_\mu$. Pulling back $\mu(z)$ with the conformal map $\phi$, we get

$$\phi^*(\mu)(u) = \mu(\phi(u))|\phi'(u)|,$$

which defines a semimetric on $\mathbb{D}$ with curvature $\kappa_{\phi^*(\mu)} \leq -C_\mu$. Then, according to the Boundary Ahlfors Lemma, we get that for $\xi \in \Gamma$

$$\liminf_{u \to \xi} \frac{\lambda(f(u))|f'(z)|}{\mu(u)} = \liminf_{z \to \xi_1} \frac{\lambda(g(z)) \cdot |f'(\phi(z))| \cdot |\phi'(z)|}{\mu(\phi(z)) \cdot |\phi'(z)|}$$
$$= \liminf_{z \to \xi_1} \frac{g^*(\lambda)(z)}{\phi^*(\mu)(z)} \geq \sqrt{\frac{C_\lambda}{c_\lambda}}.$$

Finally, here is the proof of $(c) \Rightarrow (a)$. Define on $\mathbb{D}$ the metric $\phi^*(\lambda)(z) = \lambda(\phi(z))|\phi'(z)|$, with $-C_\lambda \geq \kappa_{\phi^*(\lambda)} \geq -c_\lambda$. Note that

$$\phi^*(\lambda)(h(z))|h'(z)| = \lambda(\phi(h(z)) \cdot |(\phi'(h(z))| \cdot |h'(z)| = \lambda(g(z))|g'(z)|$$

for $z \in \mathbb{D}$. Hence, by the assumption for $\xi \in \Gamma$,

$$\begin{aligned}\sqrt{\frac{C_\mu}{c_\lambda}} &\leq \liminf_{u\to\xi} \frac{\lambda(f(u))|f'(u)|}{\mu(u)} \\ &= \liminf_{z\to\xi_1} \frac{\lambda(f(\phi(z))) \cdot |f'(\phi(z))| \cdot |\phi'(z)|}{\mu(\phi(z)) \cdot |\phi'(z)|} \\ &= \liminf_{z\to\xi_1} \frac{\lambda(g(z))|g'(z)|}{\phi^*(\mu)(z)} = \liminf_{z\to\xi_1} \frac{\phi^*(\lambda)(h(z))|h'(z)|}{\phi^*(\mu(z))}.\end{aligned}$$

Note that $\phi^*(\mu)(z) \to \infty$, as $z \to \xi_1$, because $|\phi'|$ is bounded and $\lim_{u\to\xi} \mu(u) = \infty$. Therefore $\phi^*(\lambda)(h(z))|h'(z)| \to \infty$, as $z \to \xi_1$. Then, we can use the Boundary Ahlfors Lemma for the metrics $\lambda_{\mathbb{D}}$ and $\phi^*(\lambda)$ and get

$$\liminf_{z\to\xi_1} \frac{\lambda_{\mathbb{D}}(z)}{\phi^*(\lambda)(z)} \geq \sqrt{C_\lambda},$$

and therefore

$$\lim_{z\to\xi_1} \lambda_{\mathbb{D}}(h(z))|h'(z)| \geq \sqrt{C_\lambda} \lim_{z\to\xi_1} \phi^*(\lambda)(h(z))|h'(z)| = +\infty$$

for every $\xi_1 \in \Gamma_1$. By Theorem 3.1, $h$ has an analytical extension in the neighborhood of $\Gamma_1$ with $h(\Gamma_1) \subseteq \partial\mathbb{D}$, therefore $f$ also has an analytical extension in the neighborhood of $\Gamma$ with $f(\Gamma) \subseteq \partial\Omega$. $\square$

It is natural to consider the following question: knowing that the pull-back metric $\lambda(f(z))|f'(z)| \to \infty$ on some arc $\Gamma$ of the boundary of the domain $\Omega$, how much can we say about the behavior of the $f$ near the boundary?

If $\Omega$ has an analytic boundary, then Theorem 3.17 says that we can extend $f$ analytically across the arc $\Gamma$. If we take $\Omega$ to be a domain which is "a little bit worse", say a smooth Jordan domain, and $\lambda(z)$ to be the hyperbolic metric on the domain $\Omega$, then we get the following result:

**Theorem 3.20.** *Let $\Omega$ be a smooth Jordan domain and $\Gamma$ be an open arc of $\partial\Omega$. Let $\lambda_\Omega(z)$ be the hyperbolic metric on $\Omega$ and let $f : \Omega \to \Omega$ be an analytic function. If*

$$\lim_{z\to\xi} \lambda_\Omega(f(z))|f'(z)| = +\infty.$$

*for every $\xi \in \Gamma$, then $f(z)$ and $f'(z)$ have a continuous extension on $\Gamma$, and $f(\Gamma) \subseteq \partial\Omega$.*

*Proof.* As $\Omega$ is a smooth Jordan domain, there is a conformal mapping $\phi$ of $\mathbb{D}$ onto $\Omega$, such that it can be extended to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$ with $\phi'(z) \neq 0$ for $z \in \partial\mathbb{D}$. Then we can write

$$\lambda_\Omega(w) = \lambda_\mathbb{D}(\phi^{-1}(w)) \left| \frac{1}{|\phi'(\phi^{-1}(w))|} \right|.$$

That is because $\lambda_\Omega$ can be considered as a pull-back metric of $\lambda_\mathbb{D}$ under the map $\phi^{-1}$. Then, we can write, for $\phi(z) = w$,

$$\begin{aligned}\lambda_\Omega(f(w))|f'(w)| &= \lambda_\mathbb{D}(\phi^{-1}(f(w))) \left| \frac{1}{|\phi'(\phi^{-1}(f(w)))|} \right| \cdot |f'(w)| \\ &= \lambda_\mathbb{D}(\phi^{-1}(f(\phi(z)))) \left| \frac{1}{|\phi'(\phi^{-1}(f(\phi(z))))|} \right| \cdot |f'(\phi(z))|.\end{aligned}$$

Here, define the function $g : \mathbb{D} \to \mathbb{D}$ by

$$g = \phi^{-1} \circ f \circ \phi.$$

Since $\phi'(z) \neq 0$ on $\partial\mathbb{D}$, and $|\phi'|$ is continuous on $\overline{\mathbb{D}}$, $\phi'$ obtains its minimum on $\mathbb{D}$ and the minimum is equal to $c > 0$, since $\phi$ is a conformal mapping on $\mathbb{D}$. Therefore, $|\phi'(z)| \geq c > 0$ for any $z \in \overline{\mathbb{D}}$. Let $\phi^{-1}(\Gamma) = \Gamma_1 \subset \partial\mathbb{D}$, set $\xi_1 = \phi^{-1}(\xi)$, for any $\xi \in \Gamma$. Then, for any $\xi_1 \in \Gamma_1$ we can write

$$\begin{aligned}\lim_{z\to\xi_1} \lambda_\mathbb{D}(g(z))|g'(z)| &= \lim_{z\to\xi_1} \lambda_\mathbb{D}((\phi^{-1} \circ f \circ \phi)(z)) \left| \frac{1}{|\phi'(\phi^{-1}(f(\phi(z))))|} \right| \cdot |f'(\phi(z))| \cdot |\phi'(z)| \\ &\geq c \cdot \lim_{z\to\xi_1} \lambda_\mathbb{D}((\phi^{-1} \circ f \circ \phi)(z)) \left| \frac{1}{|\phi'(\phi^{-1}(f(\phi(z))))|} \right| \cdot |f'(\phi(z))| \\ &= c \cdot \lim_{w\to\xi} \lambda_\Omega f(w)|f'(w)| = +\infty.\end{aligned}$$

Therefore, we can apply Theorem 3.16 for the map $g$ and the arc $\Gamma_1$. We get that $g$ has an analytic extension across $\Gamma_1$. This implies that the map $\phi^{-1} \circ f \circ \phi$ has an analytic continuation across $\Gamma_1$ and so, that $f$ and $f'$ have continuous extensions to $\Gamma$. Moreover, as $g(\Gamma_1) \subseteq \partial\mathbb{D}$, we get that $f(\Gamma) \subseteq \partial\Omega$. This finishes the proof. $\square$

Note that, in Theorem 3.20, it is necessary to consider $\Omega$ as a smooth Jordan domain. It is not sufficient to consider domains which are bounded by a smooth Jordan curve. (A counterexample can be found in [13], page 46.)

Another extension of Ahlfors Lemma for analytic functions can be obtained as a combination of Theorem 3.20 and of the Boundary Ahlfors Lemma for Jordan domains.

**Theorem 3.21.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $\Gamma$ be a smooth subset of $\partial\Omega$. Further, let $\lambda_\Omega(z)$ be the hyperbolic metric on $\Omega$ and let $\mu(z)$ be a semimetric on $\Omega$ with $\kappa_\mu \leq -C_\mu$ for some positive constant $C_\mu$. Let $f : \Omega \to \Omega$ be analytic. If*

$$\lim_{z\to\xi} \lambda_\Omega(f(z))|f'(z)| = +\infty$$

*for every $\xi \in \Gamma$, then*

$$\liminf_{z\to\xi} \frac{\lambda_\Omega(f(z))|f'(z)|}{\mu(z)} \geq \sqrt{C_\mu}$$

*for every $\xi \in \Gamma$.*

*Moreover, $f(z)$ and $f'(z)$ have continuous extension on $\Gamma$, and $f(\Gamma) \subseteq \partial\Omega$.*

*Proof.* According to the Boundary Ahlfors Lemma for Jordan domains applied to $\mu(z)$ and $f^*(\lambda_\Omega)(z)$, we have $\kappa_\mu \leq -C_\mu$,

$$\kappa_{f^*(\lambda_\Omega)} = \kappa_{\lambda_\Omega} = -1$$

and

$$\lim_{z\to\xi} \lambda_\Omega(f(z))|f'(z)| = \lim_{z\to\xi} f^*(\lambda_\Omega)(z) = +\infty.$$

Therefore,

$$\liminf_{z\to\xi} \frac{f^*(\lambda_\Omega)(z)}{\mu(z)} = \liminf_{z\to\xi} \frac{\lambda_\Omega(f(z))|f'(z)|}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_{\lambda_\Omega}}} = \sqrt{C_\mu}.$$

The second statement follows directly from Theorem 3.20. $\square$

Hence this Theorem states that that if $f^*(\lambda_\Omega)$ tends to $\infty$ on some boundary set $\Gamma$, then the general distortion is bounded away from 0 near $\Gamma$ and, moreover, $f$ and $f'$ have continuous extension on $\Omega \cup \Gamma$ with $f(\Gamma) \subseteq \partial\Omega$.

To finish this section we will state one more open question (at least we do not see an easy solution):

**Question 2.** In Theorem 3.21, can we consider a general metric $\lambda$ with some curvature conditions, instead of the hyperbolic metric $\lambda_\Omega$ and still get the same type of result?

## 3.4 Bloch-type spaces on unbounded simply connected domains

In this section we will consider the Bloch spaces and their extensions on different domains. In 1980 (see [16]), Timoney considered several equivalent characterizations of Bloch functions. In the same paper, he extended the notion of Bloch space to bounded homogeneous domains in $\mathbb{C}^n$. Note that the generalized Bloch spaces and similar type of characterizations of Bloch functions were considered by many other authors beside Timoney. See for example, [11], [8], [12].

Here we will consider the Bloch space on a simply-connected, but possibly unbounded domain of $\mathbb{C}$, and will prove an analogue of Timoney's theorem. First of all, let us define the classical Bloch space.

**Definition 3.22.** *An analytic function $f : \mathbb{D} \to \mathbb{C}$ is called a Bloch function if*

$$||f||_{\mathcal{B}(\mathbb{D})} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty.$$

We will see that the Bloch space is directly related to the ideas of the previous sections. Recall that in Example2.16 we already introduced the Bloch space: Set $\rho(w) = 1$ to be the Euclidean metric on $\mathbb{C}$ and $\lambda_\mathbb{D}$ to be the hyperbolic metric on $\mathbb{D}$. Then the pull-back of $\rho$ is

$$f^*(\rho)(z) = |f'(z)|,$$

for $z \in \mathbb{D}$. Therefore,

$$\mathcal{B}(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ \sup_{z \in \mathbb{D}} \frac{f^*(\rho)(z)}{\lambda_{\mathbb{D}}(z)} < \infty \right\},$$

or in other words, $\mathcal{B}(\mathbb{D})$ is the space of analytic functions whose pull-back of the Euclidean metric is bounded by the hyperbolic metric.

Recall that the curvature of the Euclidean metric $\kappa_\rho(z) = 0$ for all $z \in \mathbb{C}$, and even though $f(\mathbb{D}) = \Omega_2$ might be a simply connected proper subdomain of $\mathbb{C}$, we cannot consider any Schwarz-Pick type results for $f$ in such a case. Namely, the generalized distortion $|f^{\rho,\lambda_{\mathbb{D}}}(z)|$ can be unbounded, and so $f$ might not belong to $\mathcal{B}(\mathbb{D})$ (See example 3.23 below). On the other hand, if $f(\mathbb{D})$ is a bounded subdomain of $\mathbb{C}$, it is not to hard to show that $f$ must belong to $\mathcal{B}(\mathbb{D})$.

In particular, if we consider $f : \mathbb{D} \to \mathbb{D}$, then we can use the Schwarz-Pick Lemma to get that

$$\frac{f^*(\lambda_{\mathbb{D}})(z)}{\lambda_{\mathbb{D}}(z)} \leq 1,$$

since

$$|f'(z)|(1 - |z|^2) \leq 1 - |f(z)|^2 \leq 1.$$

This proves that all of the self-maps of $\mathbb{D}$ are in the Bloch space.

Let us also consider an example of a function which is not in the Bloch space.

*Example* 3.23. Consider

$$f = \frac{1}{1+z}.$$

The function $f$ has a derivative

$$f'(z) = \frac{-1}{(1+z)^2}.$$

Looking only for the real supremum

$$\begin{aligned} \sup_{z \in \mathbb{D},\ \mathrm{Im} z = 0} |f'(z)|(1 - |z|^2) &= \sup_{z \in \mathbb{D},\ \mathrm{Im} z = 0} \left| \frac{1}{(1+z)^2} \right| (1 - z)(1 + z) \\ &= \sup_{z \in \mathbb{D},\ \mathrm{Im} z = 0} \left| \frac{1-z}{1+z} \right| = \infty. \end{aligned}$$

Therefore the function $f$ is not in the Bloch space. Note that $f$ maps $\mathbb{D}$ onto the half plane $\{z : \mathrm{Re}(z) > \frac{1}{2}\}$, i.e. onto an unbounded domain.

As we said before, none of the Ahlfors type lemmas can be applied to the Euclidean metric $\rho$, since $\kappa_\rho = 0$, and so we turn to different geometric characterizations of the functions in the Bloch space. One such example is Timoney's characterization via schlicht disks which in the case of the unit disk has been given in [14] and in the case of Riemann surfaces, including general planar domains, in [11], [12].

To state the theorem of Timoney we will also need the definition of a schlicht disk.

**Definition 3.24.** *Let $f : \Omega \to \mathbb{C}$ be an analytic function, where $\Omega$ is a domain in $\mathbb{C}$. A disk*

$$D = \{w \in \mathbb{C} : |w - w_0| < r\} \qquad (w \in \mathbb{C}, r > 0)$$

*is called a schlicht disk in the range of $f$ if there exists an analytic function $g : \mathbb{D} \to \Omega$ so that $f \circ g$ maps $\mathbb{D}$ bijectively onto $D$.*

Now we are ready to state the theorem as given in [16]:

**Theorem 3.25.** *Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. Then, the following conditions are equivalent:*

(1) *The function $f$ is a Bloch function.*

(2) *The radii of the schlicht disks in the range of $f$ are bounded above.*

(3) *As a function from the metric space $(\mathbb{D}, d_{\mathbb{D}})$ to the metric space ($\mathbb{C}$, Euclidean distance), the function $f$ is uniformly continuous.*

(4) *The family $\{f \circ \phi(z) - f \circ \phi(0) : \phi \in \boldsymbol{Aut}(\mathbb{D})\}$ is a normal family on $\mathbb{D}$.*

(5) *The supremum*

$$\sup\{|(f \circ \phi)'(0)| : \phi \in \boldsymbol{Aut}(\mathbb{D})\}$$

*is finite.*

(6) *The family*

$$\{\Sigma_{j=1}^{n} a_j (f \circ \phi_j)(z) : n \in \mathbb{N}, \phi_j \in \boldsymbol{Aut}(\mathbb{D}), a_j \in \mathbb{C}, \Sigma_{j=1}^{n} |a_j| < 1\}$$

*is a normal family on $\mathbb{D}$. (This family is the absolute convex hull of the orbit of $f$ under $\boldsymbol{Aut}(\mathbb{D})$.).*

Note that for Example 3.23 the radii of the schlicht disks are not bounded above, since $\{z : \mathrm{Re}(z) > \frac{1}{2}\}$ is unbounded region and contains disks of unbounded radii and $\frac{1}{1+z}$ is a univalent function in its domain.

We can restate this theorem for a function $f : \Omega \to \mathbb{C}$ in the case when $\Omega$ is a simply connected domain of $\mathbb{C}$. To do that, we need to define the Bloch space on $\Omega$, which will be denoted by $\mathcal{B}(\Omega)$.

**Definition 3.26.** *An analytic function $f : \Omega \to \mathbb{C}$ is called a Bloch function on the simply-connected domain $\Omega$ if*

$$||f||_{\mathcal{B}(\Omega)} = \sup_{z\in\Omega}\Big\{\frac{|f'(z)|}{\lambda_\Omega(z)}\Big\} < \infty.$$

As it was discussed before in this chapter, using that for $\lambda_{\mathbb{C}}(z) = 1$, where $\lambda_{\mathbb{C}}$ is the Euclidean metric, and for $\lambda_\Omega(z)$ being the hyperbolic metric on $\Omega$ we have that

$$||f||_{\mathcal{B}(\Omega)} = \sup_{z\in\Omega}\Big\{\frac{f^*(\lambda_{\mathbb{C}})(z)}{\lambda_\Omega(z)}\Big\} < \infty.$$

So the Bloch space is the space of functions $f : \Omega \to \mathbb{C}$ with bounded generalized distortion $|f^{\lambda_{\mathbb{C}},\lambda_\Omega}(z)|$.

An interesting problem, regarding the similar fact for $\mathbb{D}$, is the following:

**Question 3.** If $\Omega$ is a Jordan domain, does this imply that any self-map $f$ of $\Omega$ is in the Bloch space $\mathcal{B}(\Omega)$? As far as we know, this is an open problem.

We noted that Theorem 3.25 can be further extended for the Bloch spaces. There have been a number of generalizations of parts of Theorem 3.25, such as the ones in [11], [12] or [16]. We will present a simple proof of one of the generalizations in [12], in the case when $\Omega$ is a simply connected domain in $\mathbb{C}$.

**Theorem 3.27.** *Let $\Omega$ be a simply connected hyperbolic domain and let $f : \Omega \to \mathbb{C}$ be an analytic function. Then the following conditions are equivalent:*

(1) *The function $f$ is a Bloch function on $\Omega$.*

(2) *The radii of the schlicht disks in the range of $f$ are bounded above.*

(3) *As a function from the metric space $(\Omega, d_\Omega)$ to the metric space ($\mathbb{C}$, Euclidean distance), the function $f$ is uniformly continuous.*

(4) *The family $\{f \circ \phi(z) - f \circ \phi(a) : \phi \in \boldsymbol{Aut}(\Omega)\}$ is a normal family on $\Omega$ for any $a \in \Omega$.*

(5) *The supremum*

$$\sup\{|(f \circ \phi)'(a)| : \phi \in \boldsymbol{Aut}(\Omega)\}$$

*is finite for any $a \in \Omega$.*

(6) *The family*

$$\{\Sigma_{j=1}^{n} a_j (f \circ \phi_j)(z) : n \in \mathbb{N}, \phi_j \in \boldsymbol{Aut}(\mathbb{D}), a_j \in \mathbb{C}, \Sigma_{j=1}^{n} |a_j| < 1\}$$

*is a normal family on $\mathbb{D}$. (This family is the absolute convex hull of the orbit of $f$ under $\boldsymbol{Aut}(\mathbb{D})$.).*

*Proof.* The main idea is the following: as a metric spaces $(\Omega, d_\Omega)$ and $(\mathbb{D}, d_\mathbb{D})$ are the same, i.e. they are isometric. The conformal isometry $h : \Omega \to \mathbb{D}$ is provided by the Riemann Mapping theorem. Note that we can also use that for some point $a \in \Omega$ we have that $h(a) = 0$ and $h'(a) > 0$. Furthermore, that isometry between $(\Omega, d_\Omega)$ and $(\mathbb{D}, d_\mathbb{D})$ can be viewed as an isometry between the Bloch spaces on $\Omega$ and $\mathbb{D}$. This can be proved in the following way.

Any analytic function $g : \Omega \to \mathbb{C}$ can be considered as $g = g \circ h^{-1} \circ h$, where $f = g \circ h^{-1}$ is an analytic function from $\mathbb{D}$ to $\mathbb{C}$. On the other hand, having that $f : \mathbb{D} \to \mathbb{C}$ is analytic, construct an analytic function $g : \Omega \to \mathbb{C}$ by the formula $g = f \circ h$. Now using Theorem 2.6 for the map $h$ and the Chain Rule we can write the following for any $z \in \Omega$:

$$\begin{aligned} \frac{|g'(z)|}{\lambda_\Omega(z)} = \frac{|f(h(z))'|}{\lambda_\mathbb{D}(h(z))|h'(z)|} &= \frac{|f'(h(z))||h'(z)|}{\lambda_\mathbb{D}(h(z))|h'(z)|} \\ &= \frac{|f'(h(z))|}{\lambda_\mathbb{D}(h(z))}. \end{aligned}$$

As $z$ varies through all of $\Omega$, $h(z)$ varies through all of $\mathbb{D}$. That is why one can write:

$$||g||_{\mathcal{B}(\Omega)} = \sup\left\{\frac{|g'(z)|}{\lambda_\Omega(z)} :\ z \in \Omega\right\} = \sup\left\{\frac{|f'(h(z))|}{\lambda_{\mathbb{D}}(h(z))} :\ h(z) \in \mathbb{D}\right\}$$

$$= \sup\left\{\frac{|f'(w)|}{\lambda_{\mathbb{D}}(w)} :\ w \in \mathbb{D}\right\} = ||f||_{\mathcal{B}(\mathbb{D})}.$$

This shows that $h$ is an isometry between Bloch spaces on $\mathbb{D}$ and on $\Omega$.

Let us prove that (2) for $g$ is equivalent to (2) from Theorem 3.25 for $f$. We have that $D$ is a schlicht disk for $f$ if there is an analytic $k : \mathbb{D} \to \mathbb{D}$, such that $f \circ k$ is a bijection from $\mathbb{D}$ to $D$. Therefore for $g$ there is a map $s = h^{-1} \circ k$, such that $g \circ s = f \circ h \circ h^{-1} \circ k = f \circ k$ is a bijection from $\mathbb{D}$ onto $D$. And vice versa : if $D$ is a schlicht disk for $g$, then there is $s : \mathbb{D} \to \Omega$, such that $g \circ s$ is 1-1 and onto. Then, for $f$ consider $k = h \circ s$, such that $f \circ k = f \circ h \circ s = g \circ s$ is 1-1 and onto. Therefore, $D$ is a schlicht disk for $f$ if and only if it is also a schlicht disk for $g$, and items (2) from Theorem 3.25 and 3.27 for $f$ and $g$ respectively coincide.

Now let us prove that (3) in Theorem 3.27 for $g = f \circ h$ is equivalent to (3) in Theorem 3.25 for $f$. Condition (3) in theorem 3.27 just means that for every $\epsilon > 0$ there is $\delta > 0$, such that for every $z, w \in \Omega$ we get

$$d_\Omega(z, w) < \delta \Rightarrow |g(z) - g(w)| < \epsilon,$$

or in other words , that for every $\epsilon > 0$ there is $\delta > 0$, such that for any $t = h(z), u = h(w)$ in $\mathbb{D}$

$$d_{\mathbb{D}}(t, u) < \delta \Rightarrow |f(t) - f(u)| < \epsilon,$$

as $h$ is an isometry and $g = f \circ h$.

So we get that (1) and (3) from Theorem 3.27 are equivalent, because (1) and (3) are equivalent in Theorem 3.25.

To prove that (4) and (5) are equivalent to (1) we will need to use that there is a homeomorphism $H$ between $\mathbf{Aut}(\Omega)$ and $\mathbf{Aut}(\mathbb{D})$, given by the formula $H(\phi) = h \circ \phi \circ h^{-1}$.

Now let us prove that (4) in Theorem 3.27 for $g = f \circ h$ , as above, is equivalent to (4) in Theorem 3.25 for $f$. We have

$$\begin{aligned}
&\{g\circ\phi(z)-g\circ\phi(a):\phi\in\mathbf{Aut}(\Omega)\}\\
=&\{f\circ h\circ\phi(z)-f\circ h\circ\phi(a):\phi\in\mathbf{Aut}(\Omega)\}\\
=&\{f\circ H(\phi)\circ h(z)-f\circ H(\phi)\circ h(a):\phi\in\mathbf{Aut}(\Omega)\}\\
=&\{f\circ\psi(z)-f\circ\psi(0):\psi\in\mathbf{Aut}(\mathbb{D})\},
\end{aligned}$$

and the family stays normal after composing each element with a conformal map.

Also, part (5) is equivalent to part (5) from Theorem 3.25 because of the following:

$$\begin{aligned}
\sup\{|(g\circ\phi(a))'|:\phi\in\mathbf{Aut}(\Omega)\} &= \sup\{|(f\circ h\circ\phi(a))'|:\phi\in\mathbf{Aut}(\Omega)\}\\
&= \sup\{|(f\circ H(\phi)\circ h(a))'|:\phi\in\mathbf{Aut}(\Omega)\}\\
&= \sup\{|(f\circ\psi\circ h(a))'|:\psi\in\mathbf{Aut}(\mathbb{D})\}\\
&= \sup\{|(f'(\psi(h(a)))||\psi'(h(a))||h'(a)|:\psi\in\mathbf{Aut}(\mathbb{D})\}\\
&= |h'(a)|\sup\{|(f'(\psi(0))||\psi'(0)|:\psi\in\mathbf{Aut}(\mathbb{D})\}\\
&= |h'(a)|\sup\{|(f(\psi(0))'|:\psi\in\mathbf{Aut}(\mathbb{D})\}.
\end{aligned}$$

The same idea as in the proof of the equivalence of the statements (4) from Theorem 3.25 and 3.27 works for the equivalence of the statements (6). Therefore we have proved that the statements from Theorem 3.27 are the same as for Theorem 3.25, when one considers a composition with a conformal map $h$. That is why Theorem 3.25 implies Theorem 3.27. $\square$

Let us state some consequences of Theorem 3.27 for the case when $\Omega$ is an unbounded domain. For instance, $f\in\mathcal{B}(\Omega)$ only if $\lim_{z\to\infty}|f'(z)|=0$. That is because if $\sup\{\frac{|f'(z)|}{\lambda_\Omega(z)}\}=C<\infty$ and $\lambda_\Omega(z)\to\infty$, as $z\to\infty$, then

$$|f'(z)|\le\frac{C}{\lambda_\Omega(z)}\to 0,$$

as $z\to\infty$.

For example, as a consequence of that we get the following: the Bloch space on some unbounded domains $\Omega$ does not contain any polynomials.

One of the special cases is when $\Omega$ is an open half-plane. Then, according to Example 2.10, the Bloch space $\mathcal{B}(\Omega)$ is just a space of functions, such that $\sup_{z\in\Omega} d(z)|f'(z)| < \infty$, where $d(z)$ is the distance from $z$ to $\partial\Omega$.

There are a number of articles that show the connection between $\lambda_{\mathbb{D}}(z)$ and $d(z)$ for different type of domains. For instance, in 1988, Minda has actually shown that the spaces with $\sup_{z\in\Omega} d(z)|f'(z)| < \infty$ and $\sup_{z\in\Omega} |f'(z)|/\lambda_{\Omega}(z) < \infty$ coincide not only for the half-planes, but for any planar domain (see [11] for details). Results describing the connection between $d(z)$ and $\lambda_{\mathbb{D}}(z)$ in the case of simply connected hyperbolic domains are given in [8]. Minda also considered the characterization of normal families of functions with respect to different metrics in [12].

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