DIFFERENT ASPECTS OF EMBEDDING OF NORMED SPACES OF ANALYTIC FUNCTION

by

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Abstract

In the present work we develop a unified way of looking at normed spaces of analytic functions (NSAF's) and their embedding into the Frechet space of analytic functions on a general domain, by requiring only that the embedding map is bounded. This is a succinct definition of NSAF and derive from it a list of interesting properties. For example Proposition 4.4 describes the behavior of point evaluations and Proposition 4.6 part (i) gives a general sufficient condition for a NSAF to be a Banach space, which as far as we know, are new results. Also, Proposition 4.5, parts (ii) and (iii) of Proposition 4.6 and Proposition 4.7 are results, which are slight generalizations of fairly standard results, which show up elsewhere in a more specific setting.

Some of the facts about NSAF's are stated and proven in a more general context. In particular, a significant part of the material is dedicated to the normed space of continuous functions on a metric space. On the other hand, we provide the necessary background on differential geometry and complex analysis, which further determine the peculiarities in the context of spaces of analytic functions.

At the end we illustrate our results on two specific examples of NSAF's, namely the Bergman and the Bloch Spaces over a general domain in $\mathbb{C}^d$. We give a new proof of the reflexivity of the Bergman Space $A^p(G,\mu)$ for the case $p > 1$ and of the Schur property of $A^1(G,\mu)$. We also give new proofs for the equivalences of some of the definitions of the Bloch functions.

Contents

Chapter 1

Introduction

The theory of analytic spaces, in its classical form, was launched in the first half of the previous century as a tool for the Theory of Functions of a Complex Variable to involve methods of Functional Analysis and vice versa. Most of the work was done for some particular well-known spaces over some specific simple domains. However it was clear that the constructions that were studied arise from more general matters.

As an example, consider the Bloch Space, which has many nice geometric properties. The study of this (and many other) spaces began by considering the unit disc as the basic domain for the functions. The definition of the space in that case was very explicit and involved a use of particular weights.

A natural question to ask is what geometric properties lead to such definitions. In order to answer this question, it was necessary to introduce the notion of such spaces on a much more general class of domains. For example, it turned out (see [3]) that the Bloch space over the general domain is the space of all analytic complex valued Lipschitz functions with respect to a particular choice of metric on the domain.

The research in this area uses methods and concepts from Complex and Functional Analysis, as well as Differential Geometry, and allows exploring some deep connections between the corresponding structures. The synthetic nature of the studied objects requires a lot of different points of view. In this thesis, beside presenting a short review on the needed diverse background, we succeed on connecting the different views and presenting the unifying common threads.

Some aspects of the NSAF are not connected to the complex analytic structures, and hence our first chapter is dedicated to various real-analytic issues. In fact most of Chapter 2 describes our construction with only a linear structure, but without the differential one.

Section 2.1 deals with the most general case. The main object there is the space of continuous maps $\mathcal{C}(M, N)$. We describe some of its topological properties and introduce the concept of dilation. The important results of this section are Propositions 2.9 and 2.11. We will eventually use the first of them to show that the dilation can be used for defining NSCF's, while the second one is a specification of the Arzela-Ascoli Theorem for Lipschitz functions.

Sections 2.2 and 2.3 provide some background material and some preparatory results. The first of them introduces the reader to topological vector spaces, and the second contains few lemmas which will be useful later. The main results of these sections are Proposition 2.19 and Lemma 2.24.

Section 2.4 describes some functional analytic properties of $\mathcal{C}(M)$ and several properties of the Normed Spaces of Continuous Functions (NSCF's). The importance of the results in this section is that they provide the apparatus for studying NSAF's. Section 2.5 is dedicated to one of the NSCF's, namely the Lipschitz space.

Chapter 3 serves two purposes: to introduce differential geometry concepts and notations which will arise later and to state and prove Theorem 3.17 and its Corollary 3.19. We conclude this chapter with a short review of complex differential geometry. In particular, Theorem 3.20 is stated.

Chapter 4 is the central part of the thesis. We join the information, obtained by considering the spaces of continuous functions and with differential geometry results from Chapter 3.

Section 4.1 contains several general facts about NSAF's. The main feature here is Montel's Theorem, which provides a special, "unusual" property of the space of

analytic functions $\mathcal{H}(G)$. Since each continuous inclusion in $\mathcal{H}(G)$ is compact, some results from Section 2.4 are combined (compare Propositions 2.28, 2.30 and 4.4). After that we apply the obtained results to some concrete spaces.

Section 4.2. is dedicated to the Bergman space over the general domain with a rather general measure. Based on the previous section, we give a new proof of the reflexivity of the Bergman Space $A^1(G,\mu)$ for the case $p > 1$ and of the Schur property of $A^1(G,\mu)$.

Section 4.3 introduces the Bergman and the Kobayasi-Royden metrics. The first one is constructed through the Bergman Space. This two metrics help us to study the Bloch Space, which we do in Section 4.4. Here we join some information about the Lipschitz Space with the specific properties of the analytic functions. In particular we give new proofs for some of the equalities of the definitions of the Bloch functions.

Some Conventions

Let X be a topological space and let $\{x_n\}_{n=1}^{+\infty}$ be a sequence in X. We will denote the convergence of this sequence to some $x \in X$ by $x_n \to x$ (omitting $n \to \infty$). If X possesses a metric d_X, then $B_X(x,r)$ stands for the ball centered at x and with the radius r.

If X, Y are some sets, $E \subset X$ and $\Phi : X \to Y$, then $\Phi_{|E}$ stands for the restriction of Φ to the set E. $Ran(\Phi)$ stands for the range of Φ.

Note, that the word *functional* will be used *not only* for the linear functionals on the vector spaces. We will use this word for the functions of functions, or for their more abstract analogues.

The word *metric* will be used with two meanings. In Section 2 we use it for the distance function on the metric space. In Sections 3 and 4 we reserve it for the Riemannian and the Finsler metrics.

The last remark concerns the coordinate representation of some objects. The elements of some spaces will be denoted as columns in $\mathbb{R}^d$ and $\mathbb{C}^d$, and the functionals acting on such spaces will be written as rows.

Chapter 2

Real Analytic Aspects

2.1 Space of Continuous Maps. The Concept of Dilation

Let M be a metrizable topological space, let (N, d_N) be a metric space, and let $\mathcal{C}(M, N)$ denote the set of continuous maps between them. The last set can be turned into a topological space by introducing the so called *open-compact topology*. Namely, let us consider a topology with a subbase consisting of all sets of the type

$$\{\Phi \in \mathcal{C}(M, N) \,|\, \Phi(K) \subset U\},$$

where U is an open subset of N, and K is a compact subset of M. It is clear that such topology does not depend on the particular metric within the fixed topology of N. In the next three propositions and in Theorem 2.4 we state some properties of the introduced topology (for the proofs and more details see [18] or [7]):

Proposition 2.1. *For $E \subset M$ and for $\Phi, \Psi \in \mathcal{C}(M, N)$ define*

$$\rho_E(\Phi, \Psi) = \sup_{p \in E} \{d_N(\Phi(p), \Psi(p))\},$$

which is the uniform distance between Φ and Ψ relative to E. Then:

(i) The local base in the point $\Phi \in \mathcal{C}(M, N)$ can be chosen as the class of all sets of the type

$$B_K(\Phi, r) = \{\Psi \in \mathcal{C}(M, N) \,|\, \rho_K(\Phi, \Psi) < r\},$$

where $K \subset M$ is compact and $r > 0$. This means that the open-compact topology is generated by the family of pseudometrics ρ_K.

(ii) Convergence in the given topology is equivalent to the simultaneous convergence in all metrics ρ_K, i.e. to the uniform convergence on all K.

The picture is simplified whenever we can "exhaust" the space M by some sequence of compact sets:

Proposition 2.2. *Let $M = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is compact and $K_n \subset intK_{n+1}$, for each $n \in \mathbb{N}$. Then:*

(i) Convergence in the compact-open topology is equivalent to the convergence in all uniform metrics ρ_{K_n};

(ii) $\mathcal{C}(M, N)$ is metrizable.

Observe that within our assumption each compact set is contained in K_n, for some $n \in \mathbb{N}$. This fact justifies (i).

An example of the metric on $\mathcal{C}(M, N)$ is given by

$$\rho(\Phi, \Psi) := \sum_{n\in\mathbb{N}} \frac{\rho_{K_n}(\Phi, \Psi)}{2^n (1 + \rho_{K_n}(\Phi, \Psi))}.$$

The completeness of $\mathcal{C}(M, N)$ is determined by the completeness of N.

Proposition 2.3. *Let N be complete. Then:*

(i) If a sequence in $\mathcal{C}(M, N)$ is Cauchy with respect to all ρ_K, then this sequence is convergent.

(ii) Within the conditions of Proposition 2.2, the space $\mathcal{C}(M, N)$ is metrizable by a complete metric.

An example of the above construction is the space of continuous complex valued functions on a complex domain, which will be studied later.

Further, we will extensively consider families of continuous maps between metric spaces. We will need some compactness-type conditions for these families. The

metric which we introduced before will not fit these purposes, since it is "artificially" bounded from above by 1. We will say that $\mathfrak{F} \subset \mathcal{C}(M,N)$ is *bounded*, if it is uniformly bounded on all compact sets (meaning that $\mathfrak{F}$ is bounded in ρ_K for all compacts $K \subset M$). It turns out that on locally compact spaces the boundedness and compactness possess some have characterizations. We state a theorem which reveals these characterizations.

Theorem 2.4. *Let M be locally compact and let $\mathfrak{F} \subset \mathcal{C}(M,N)$.*

(i) The family $\mathfrak{F}$ is bounded if and only if it is locally uniformly bounded, i.e. if for each $p \in M$ there exists an open neighborhood U of p such that $\mathfrak{F}$ is bounded in ρ_U.

(ii) The family $\mathfrak{F}$ is precompact if and only if for all $p \in M$ the set $\{\Phi(p) \,|\, \Phi \in \mathfrak{F}\}$ is precompact and for each $\varepsilon > 0$ there exists an open neighborgood $U \ni p$ such that for all $\Phi \in \mathfrak{F}$ and $q \in U$ we have that $d_N(\Phi(p), \Phi(q)) < \varepsilon$.

Note that part (ii) is the famous Arzela-Ascoli Theorem, which we will use in several instances. This theorem has two conditions: the first of them will be referred to as pontwise boundedness, while the second as local equicontinuity.

Note also, that if M is locally compact, and L is another topological space, then composition is a continuous operation from $\mathcal{C}(L,M) \times \mathcal{C}(M,N)$ to $\mathcal{C}(L,N)$ (see [7]).

Now let M be a metric space with a distance function d_M. We will define a quantitative characteristic of the behavior of continuous maps from the metric point of view.

Let $\Phi \in \mathcal{C}(M,N)$. The *dilation* of Φ is defined by

$$\operatorname{dil}(\Phi) = \sup_{p,q \in M,\ p \neq q} \frac{d_N(\Phi(p), \Phi(q))}{d_M(p,q)}.$$

The first trivial consequence of the definition is that if $E \subset M$, then $\operatorname{dil}\left(\Phi_{|E}\right) \leq \operatorname{dil}(\Phi)$.

The map $\Phi \in \mathcal{C}(M,N)$ is called a *Lipschitz* map, if $\operatorname{dil}(\Phi) < +\infty$. The set of all such maps will be denoted by $Lip(M,N)$. The next proposition contains the simplest examples of Lipschitz maps.

Proposition 2.5.

(i) If $M = N$ and Φ is the identity map, then $\operatorname{dil}(\Phi) = 1$.

(ii) Φ is a constant map if and only if $\operatorname{dil}(\Phi) = 0$.

In fact, $\operatorname{dil}(\Phi) = 1$ holds for any isometry. The first part of the following proposition shows that the dilation "respects" taking the composition. The other two parts contain significant particular cases.

Proposition 2.6. *Let $\Psi \in \mathcal{C}(L, M)$, for some metric space (L, d_L). Then:*

(i) $\operatorname{dil}(\Phi \circ \Psi) \leq \operatorname{dil}(\Psi) \operatorname{dil}(\Phi)$*;*

(ii) If Φ is an isometry, then $\operatorname{dil}(\Phi \circ \Psi) = \operatorname{dil}(\Psi)$*;*

(iii) If Ψ is an onto isometry then $\operatorname{dil}(\Phi \circ \Psi) = \operatorname{dil}(\Phi)$.

Proof. The proofs of parts (i) and (ii) are trivial and we will omit them.

(iii): Since Ψ is an isometry, from the part (i) we have that $\operatorname{dil}(\Phi \circ \Psi) \leq \operatorname{dil}(\Phi) \operatorname{dil}(\Psi) = \operatorname{dil}(\Phi)$. On the other hand, since Ψ is an onto isometry, it is invertible, and Ψ^{-1} is also an isometry. Hence

$$\operatorname{dil}(\Phi \circ \Psi) = \operatorname{dil}(\Phi \circ \Psi) \operatorname{dil}\left(\Psi^{-1}\right) \geq \operatorname{dil}\left(\Phi \circ \Psi \circ \Psi^{-1}\right) = \operatorname{dil}(\Phi).$$

□

Note, that the dilation is a functional on the topological space $\mathcal{C}(M, N)$, having values in $[0, +\infty]$. Let us introduce an important class of functionals on the topological spaces.

Let X be a topological space and let $\beta : X \to [-\infty, +\infty]$. The function β is *lower semi-continuous*, if for all $c \geq -\infty$ we have that $\beta^{-1}((c, +\infty])$ is open in X (or, equivalently, $\beta^{-1}([-\infty, c])$ is closed).

Clearly all continuous real-valued function on X are lower semi-continuous, but the latter class is much wider, which is shown in the next proposition.

Proposition 2.7. *Let $\{\beta_\alpha \,|\alpha \in I\}$ be the set of lower semi-continuous functions on X. Then the function defined by $\beta(x) = \sup_{\alpha \in I} \beta_\alpha(x)$ is lower semi-continuous.*

Proof. For all $c \geq -\infty$ we have that $\beta^{-1}((c,+\infty]) = \bigcup_{\alpha \in I} \beta_\alpha^{-1}((c,+\infty])$. The later is a union of the open sets, hence $\beta^{-1}((c,+\infty])$ is open for all $c \geq -\infty$, which ensures the lower semi-continuity of β. □

If β is real-valued and such that both β and $-\beta$ are lower semi-continuous, then they are actually continuous. This is a consequence of the following characterization of lower semi-continuous functions.

Proposition 2.8. *Let X be a topological space and let $\beta : X \to [-\infty,+\infty]$. The function β is lower semi-continuous if and only if $\liminf_{y \to x} \beta(y) \geq \beta(x)$, for all $x \in X$.*

Proof. We remind the reader that

$$\liminf_{y \to x} \beta(y) = \sup \left\{ \inf_{y \in U} \beta(y) \,|U \text{ is an open neighborhood of } x \right\}.$$

Let us prove the sufficiency. For each $c > -\infty$ and each $x \in X$ such that $c < \beta(x)$, there exists an open neighborhood U of x, such that $\inf_{y \in U} \beta(y) > c$. Hence $x \in U \subset \beta^{-1}((c,+\infty])$, and so the latter set is open.

Next we prove the necessity. For each $c < \beta(x)$ the set $V = \beta^{-1}((c,+\infty])$ is an open neighborhood of x, and so $c \leq \inf_{y \in V} \beta(y) \leq \liminf_{y \to x} \beta(y)$. From the arbitrariness of c we conclude that $\liminf_{y \to x} \beta(y) \geq \beta(x)$. □

Now we can prove a property of the dilation which will play an important role in our studies.

Proposition 2.9. *The dilation is a lower semi-continuous functional on $\mathcal{C}(M,N)$.*

Proof. Let $I = (M \times M) \setminus \{(p,p) \,|p \in M\}$. For each $(p,q) \in I$ consider the continuous functional on $\mathcal{C}(M,N)$ defined by

$$\beta_{(p,q)}(\Phi) = \frac{d_N(\Phi(p), \Phi(q))}{d_M(p,q)}.$$

Then, by definition, $\operatorname{dil}(\Phi) = \sup_{(p,q) \in I} \beta_{(p,q)}(\Phi)$, for all $\Phi \in \mathcal{C}(M,N)$. Thus the previous proposition guarantees the lower semi-continuity of the dilation. □

Since the dilation is a global characteristic of the map, it can not be applied to any investigation of the local behavior. For this reason we will introduce the local version of the dilation.

The *(local) dilation* of Φ at $p \in M$ is

$$\mathrm{dil}_p(\Phi) = \lim_{\varepsilon \to 0} \mathrm{dil}\left(\Phi_{|B_M(p,\varepsilon)}\right).$$

Note that since the dilation of a restriction decreases as the domain gets smaller, the considered quantity either converges, or diverges to infinity. In the latter case we will define $\mathrm{dil}_p(\Phi) = +\infty$. As a trivial consequence of the definition, we have that $\mathrm{dil}(\Phi) \geq \sup_{p \in M} \mathrm{dil}_p(\Phi)$.

The local dilation respects taking composition in the similar way to the global dilation. All parts of the next proposition follow from the corresponding parts of Proposition 2.6.

Proposition 2.10. *Let $\Psi \in \mathcal{C}(L, M)$, for some metric space (L, d_L). Then:*

(i) $\mathrm{dil}_q(\Phi \circ \Psi) \leq \mathrm{dil}_q(\Psi)\, \mathrm{dil}_{\Psi(q)}(\Phi)$, *for every $q \in L$.*

(ii) If Φ is an isometry, then $\mathrm{dil}_q(\Phi \circ \Psi) = \mathrm{dil}_q(\Psi)$, *for every $q \in L$.*

(iii) If Ψ is an onto isometry then $\mathrm{dil}_q(\Phi \circ \Psi) = \mathrm{dil}_{\Psi(q)}(\Phi)$, *for every $q \in L$.*

The next proposition is a useful corollary of the Arzela-Ascoli Theorem, formulated in terms of the dilation of continuous maps. Observe that in the case of locally compact metric space the local equicontinuity can be replaced by the equicontinuity on the compact sets. Namely, $\mathfrak{F} \subset \mathcal{C}(M, N)$ is equicontinuous on the compact set $K \subset M$ if for any $\varepsilon > 0$ there exists $\delta > 0$, such that if $p, q \in K$ and $d_M(p, q) < \delta$, then $d_N(\Phi(p), \Phi(q)) < \varepsilon$, for each $\Phi \in \mathfrak{F}$. Clearly, equicontinuity on all compact sets implies local equicontinuity, since each point has a precompact neighborhood.

Proposition 2.11. *Let M be locally compact. Let $\mathfrak{F} \subset \mathcal{C}(M, N)$ be such that for each compact set K there is $C_K > 0$ such that* $\mathrm{dil}\left(\Phi_{|K}\right) \leq C_K$, *for each $\Phi \in \mathfrak{F}$.*

(i) If for each $q \in M$ the set $\{\Phi(q) \,|\, \Phi \in \mathfrak{F}\}$ is precompact in N, then $\mathfrak{F}$ is precompact in $\mathcal{C}(M, N)$.

(ii) Assume that each bounded subset of N is precompact. If there exists $q \in M$ such that the set $\{\Phi(q) \,|\Phi \in \mathfrak{F}\}$ is bounded in N, then $\mathfrak{F}$ is precompact in $\mathcal{C}(M, N)$.

Proof. (i): This is a direct corollary of the Arzela-Ascoli Theorem. The pointwise boundedness holds automatically, while for each compact set K and $\varepsilon > 0$ we can take $\delta = \frac{\varepsilon}{C_K}$ to show the equicontinuity on K.

(ii): In order to reduce this case to part (i) we only have to prove the pointwise boundedness of $\mathfrak{F}$. Let p be an arbitrary point in M. Let $b = d_M(p, q)$ and let $K = \{p, q\}$, which is a compact set. Since there is $C_K > 0$ such that $\operatorname{dil}\left(\Phi_{|K}\right) \leq C_K$ for each $\Phi \in \mathfrak{F}$, we have that $d_N(\Phi(p), \Phi(q)) \leq C_K b$ for each $\Phi \in \mathfrak{F}$. Hence, as $\{\Phi(q) \,|\Phi \in \mathfrak{F}\}$ is bounded, $\{\Phi(p) \,|\Phi \in \mathfrak{F}\}$ is also bounded. Thus, the second set is also precompact, and so the pointwise boundedness of $\mathfrak{F}$ holds. □

The last proposition requires some remarks. First, observe that we can replace the uniform boundedness of the dilations of $\Phi \in \mathfrak{F}$ on each compact set, with the uniform boundedness of the dilations on neighborhoods of all points. Also, note that the condition on N established in part (ii) is satisfied by both $\mathbb{R}$ and $\mathbb{C}$. In general, this condition arises naturally when N is a Riemannian manifold, which will be the central object of the next chapter (see Theorem 3.8). Finally, the dilation happens to be an especially useful tool in the situation, when both M and N are Riemannian Manifolds, since there is a direct connection between the global and local versions of the dilation (as it is shown in Theorem 3.17).

2.2 Topological Vector Spaces

Now we proceed by considering the function-analytic aspects of the above construction. Till the end of this section we will use the word *scalar* for either a real or a complex number. Further, we will specify which field is considered in each particular case. Let us remind the reader some general concepts on topological vector spaces. This section is mostly a review; for the omitted proofs and some further considerations see [16] or [6]. However, part (ii) of Proposition 2.19 is absent in the literature,

as far as we know.

Let X and Y be topological vector spaces.

- A set $E \subset X$ is called bounded if for any open $U \ni 0$ there exists $c \geq 0$ such that $bE \subset U$, for any $b \leq c$.
- A linear operator $J : X \to Y$ is called bounded, if it transforms bounded subsets of X into bounded subsets of Y.
- A linear operator $J : X \to Y$ is called (pre)compact, if it transforms bounded subsets of X into precompact subsets of Y.

The next theorem explains the connection between the boundedness and continuity of linear transformations on topological vector spaces.

Proposition 2.12. *Let J be a linear operator from X into Y.*

(i) If J is continuous, then it is bounded.

(ii) If X is metrizable and J is bounded, then it is continuous.

A nonnegatively-valued functional ρ on X is called a *seminorm*, if for all $x, y \in X$ and all scalars b we have that

- $\rho(x) \geq 0$;
- $\rho(bx) = |b| \rho(x)$ (positive homogeneity);
- $\rho(x + y) \leq \rho(x) + \rho(y)$ (subadditivity).

It is easy to see that the kernel of a seminorm is a linear subspace of X. A collection of seminorms $\{\rho_\alpha | \alpha \in I\}$ is said to *separate points* of X if for each $x \in X$ there is $\alpha \in I$, for which $\rho_\alpha(x) > 0$.

Lemma 2.13. *Let $\{\rho_\alpha | \alpha \in I\}$ be a collection of seminorms on X. Define $\rho(x) = \sup_{\alpha \in I} \rho_\alpha(x)$. Then:*

(i) The set $Y = \{x \in X \,|\rho(x) < +\infty\}$ *is a linear subspace of* X *and* ρ *is a seminorm on* Y*.*

(ii) If additionally $\{\rho_\alpha \,|\alpha \in I\}$ *separates points of* X*, then* ρ *is a norm on* Y*.*

Proof. (i): Let $x, y \in Y$. Then $x + y \in Y$ because

$$\rho(x+y) = \sup_{\alpha \in I} \rho_\alpha(x+y) \leq \sup_{\alpha \in I} \rho_\alpha(x) + \sup_{\alpha \in I} \rho_\alpha(y) = \rho(x) + \rho(y) < +\infty.$$

In the similar way, for any scalar $b \neq 0$ we have that $\rho(bx) = |b|\,\rho(x) < +\infty$ and so $bx \in Y$. Hence, Y is linear and ρ is a seminorm.

(ii): The kernel of ρ is the intersection of the kernels of ρ_α, which is trivial, since $\{\rho_\alpha \,|\alpha \in I\}$ separates points of X. It is easy to see that a seminorm with trivial kernel is actually a norm. □

Note, that since a finite sum of seminorms is a seminorm, the same results hold for the sum of seminorms.

A topological space X is called *locally convex*, if there is a collection of continuous seminorms on X which separates points. The family of sets of the form $\{x \in X \,|\rho_\alpha(x) < c\}$, where $\alpha \in I$ and $c > 0$, forms a local subbase at 0. Bounded subsets of locally convex spaces posses a nice characterization.

Proposition 2.14. *Let* X *be a locally convex space with the collection of seminorms, as it was described before. Then a set* $E \subset X$ *is bounded in* X *if and only if* ρ_α *is bounded on* E *for each* $\alpha \in I$*.*

As a corollary, precompact sets in locally convex spaces are bounded.

A sequence $\{x_n\}_{n=1}^{+\infty}$ in a locally convex space X is called Cauchy, if it is Cauchy with respect to ρ_α for each $\alpha \in I$. This definition is consistent with the usual definition of the Cauchy sequence in the following sense:

Proposition 2.15.

(i) Assume that I is countable. Then

$$d_X(x,y) = \sum_{n\in\mathbb{N}} \frac{\rho_{\alpha_n}(x-y)}{2^n\left(1+\rho_{\alpha_n}(x-y)\right)}$$

is a translation-invariant metric on X, such that Cauchy sequences in this metric coincide with Cauchy sequences in X.

(ii) If X is metrizable, then X possesses a countable family of seminorms, which separates points.

Note also, that if I is finite, the sum of ρ_{α_n}, as well as their supremum is actually a norm on X.

One can adopt the notion of completeness for the locally convex spaces. Locally convex space X is said to be complete if each Cauchy sequence in X is convergent. Metrizable complete locally convex space X is called a *Frechet space*.

Note, that if X is Frechet, the metric constructed in the last proposition is complete. Various properties of Banach spaces also hold for the Frechet ones. An example of such property which we will use is the Closed Graph Theorem.

We conclude this section with some important examples.

Example 2.16. Denote the space of continuous scalar-valued functions on a metrizable space M as $\mathcal{C}(M)$. It is a topological vector space with respect to the pointwise operations and the open-compact topology. For each $E \subset M$ consider the functional defined by $\|f\|_\infty^E = \sup_{p\in E} |f(p)|$. Clearly $\left\{\| \ \|_\infty^K | K \text{ is compact in } M\right\}$ is a collection of continuous seminorms, which separates points and from proposition 2.3, $\mathcal{C}(M)$ is complete with respect to this collection. Note that the definition of bounded sets given before Theorem 2.4 agrees with the boundedness in $\mathcal{C}(M)$ as a locally convex space.

Let Y be a topological vector space and let Y^* be its dual, i.e. the vector space of all continuous scalar-valued linear functionals on Y. The next example is dedicated to introducing some new topologies on Y and Y^*.

Example 2.17. The weak topology on Y is the topology generated by the family $\{\rho_l = |l|, l \in Y^*\}$. Let $\{y_n\}_{n=1}^{+\infty}$ be a sequence in Y. It is easy to see that y_n converges weakly to $y \in Y$, if and only if for any continuous linear functional l on Y we have that $l(y_n) \to l(y)$. The weak convergence will be denoted by $y_n \xrightarrow{w} y$.

Example 2.18. Since Y^* consists of continuous functionals, $Y^* \subset \mathcal{C}(Y)$ (as sets). Hence, one can consider the topology on Y^* induced by the topology of $\mathcal{C}(Y)$. Note that since all singletons are compact, this topology is stronger then the so called weak* topology on Y^*, being induced by the family $\left\{\| \ \|_\infty^{\{y\}} | y \in Y\right\}$.

The strong topology on Y^* is the topology generated by the family

$$\left\{\| \ \|_\infty^{E} | E \text{ is bounded in } Y\right\}.$$

This definition agrees with the definition of the strong topology on the dual of a normed space. Note that since all compacts are bounded, the strong topology on the dual space is stronger then the open-compact topology. By default we consider Y^* to be provided with the strong topology.

We conclude this section with some properties of linear operators.

Proposition 2.19. *Let X and Y be topological vector spaces and let J be a linear operator from X into Y.*

(i) If J is continuous, then it is also continuous with respect to weak topologies on X and Y.

(ii) If J is compact, then the adjoint operator $J^* : Y^* \to X^*$ defined by $J^*(l) = l \circ J$ is bounded with respect to the open-compact topology on Y^* and the strong topology on X^*.

Proof. (i): Let $E \subset Y^*$ and let $\varepsilon > 0$. The set $U = \{y \in Y, |l(y)| < \varepsilon, l \in E\}$ is an open neighborhood of $0 \in Y$. Since J is continuous, we get that $D = \{l \circ J | l \in E\} \subset X^*$ and hence $V = \{x \in X, |e(x)| < \varepsilon, e \in D\}$ is an open neighborhood of $0 \in X$. Since $J(V) \subset U$, while E and ε are arbitrary, we conclude that J is continuous with respect to the weak topologies.

(ii): Let E be bounded in Y^* and let D be bounded in X. Then $J(D)$ is a precompact set in X. By the definition of the adjoint operator we have that

$$\begin{aligned}\sup_{l\in J^*(E)} \|l\|_\infty^D &= \sup_{e\in E} \|J^*e\|_\infty^D = \sup_{e\in E}\sup_{x\in D} |J^*e(x)| = \sup_{e\in E}\sup_{x\in D} |e(Jx)| \\ &= \sup_{e\in E}\sup_{y\in J(D)} |e(y)| = \sup_{e\in E}\sup_{y\in \overline{J(D)}} |e(y)| = \sup_{e\in E} \|e\|_\infty^{\overline{J(D)}} < +\infty,\end{aligned}$$

where the last inequality follows from the boundedness of E in the open-compact topology and compactness of $\overline{J(D)}$. Since D is arbitrary, $J^*(E)$ is bounded in X^*, and as E is arbitrary, J^* is a bounded operator with respect to the given topologies.

□

Observe, that in a similar way, we can prove that if J is bounded, so is J^* with respect to the strong topologies on both Y^* and X^*.

2.3 Some Functional Analysis Facts

We will extensively consider inclusions from a normed space into a Frechet one. In order to do this we need some auxiliary results. Throughout this section X is a normed space with the norm denoted by $\| \ \|$. The following proposition contains a well-known criteria for the weak convergence in a normed space and its strengthening for the case of a reflexive space.

Proposition 2.20.

(i) Let $E \subset X^*$ be such that $\overline{span E} = X^*$. Assume that $\{x_n\}_{n=1}^\infty$ is a bounded sequence in X and for all $e \in E$ we have that $e(x_n) \to 0$. Then $x_n \xrightarrow{w} 0$.

(ii) Assume that X is reflexive, and let $E \subset X^*$ be such that $\bigcap_{e\in E} Ker\ e = \{0\}$. Then $\overline{span E} = X^*$.

Proof. (i): Since the sequence is bounded, there exists $C > 0$ such that $\|x_n\| \le C$, for all $n \in \mathbb{N}$. Let $l \in X^*$. For any $\varepsilon > 0$ there exist $e_1, e_2, ..., e_m \in E$ and scalars $b_1, b_2, ..., b_m$, such that $\left\| l - \sum_{k=1}^m b_k e_k \right\| < \frac{\varepsilon}{2C}$. Since for all $k \in \overline{1,m}$ we have

that $e_k(x_n) \to 0$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$ we have that $e_k(x_n) < \frac{\varepsilon}{2m|b_k|}$. Then for each $n \geq N$ we have that

$$\begin{aligned} l(x_n) &\leq \sum_{k=1}^{m} |b_k e_k(x_n)| + \left| \left(l - \sum_{k=1}^{m} b_k e_k \right)(x_n) \right| \\ &< \sum_{k=1}^{m} \frac{\varepsilon |b_k|}{2m|b_k|} + \|x_n\| \left\| l - \sum_{k=1}^{m} b_k e_k \right\| < m \frac{\varepsilon |b_k|}{2m|b_k|} + C \frac{\varepsilon}{2C} = \varepsilon. \end{aligned}$$

From the arbitrariness of ε, we conclude that $l(x_n) \to 0$, while the arbitrariness of l guaranties that $x_n \xrightarrow{w} 0$.

(ii): Assume to the contrary, that $\overline{span E} \neq X^*$. Then from the Hahn-Banach Theorem (see [16]) and from the reflexivity of X, the set

$$E^{\perp} = \{x \in X^{**} \,|x(e) = 0, \forall e \in E\}$$

is nontrivial in $X^{**} = X$, which contradicts the fact that $\bigcap_{e \in E} Ker\ e = \{0\}$. □

Till the end of this section J is a linear operator from X into a locally convex space Y with the topology generated by a family of the seminorms $\{\rho_\alpha \,|\alpha \in I\}$.

Proposition 2.21.

(i) J is bounded if and only if for each $\alpha \in I$ there is $C_\alpha > 0$, such that $\rho_\alpha(Jx) \leq C_\alpha \|x\|$, for each $x \in X$.

(ii) Assume that X is a Banach space and that J is a compact operator. Then if $\{x_n\}_{n=1}^{+\infty}$ is a sequence in X, weakly convergent to $x \in X$, then $Jx_n \to Jx$.

Proof. (i): Since each bounded set of X is contained in some ball, the boundedness of J is equivalent to the boundedness of the image of the unit ball $B_X(0,1)$. By Proposition 2.14 this image is bounded if and only if for each $\alpha \in I$ there is $C_\alpha > 0$, such that $\rho_\alpha(Jx) \leq C_\alpha$, for each $x \in B_X(0,1)$. Clearly the latter is equivalent to $\rho_\alpha(Jx) \leq C_\alpha \|x\|$, for each $x \in X$.

(ii): Since $\{x_n\}_{n=1}^{+\infty}$ is weakly convergent, it is bounded in X (see [16]), and as J is compact, $\{Jx_n\}_{n=1}^{+\infty}$ is precompact in Y. Hence, it has at least one limit point in Y. On the other hand, compactness implies continuity, as well as weak continuity

according to part (i) of Proposition 2.19, and so $Jx_n \xrightarrow{w} Jx$. Hence, if y is a limit point of $\{Jx_n\}_{n=1}^{+\infty}$ in Y, then $y = Jx$. □

Corollary 2.22. *If J is bounded it maps Cauchy sequences in X into Cauchy sequences in Y.*

Mappings from the reflexive spaces have some additional properties.

Proposition 2.23. *Let X be a reflexive space.*

(i) If J is bounded, that an image of a closed ball in X is closed in Y.

(ii) If J maps weakly convergent sequences in X into convergent sequences in Y, then J is compact.

Proof. (i): Since X is reflexive, a closed ball in X is compact in the weak topology (see [16]). Since from Proposition 2.19, the continuity of the linear operator implies the continuity with respect to the weak topologies on both X and Y, the images of the closed balls are weakly compact in Y, and so they are closed.

(ii): We have to prove that images of closed balls in X are precompact in Y. Let B be a closed ball in X and let A be an infinite subset of its image $J(B)$. By the choice of A, the set $J^{-1}(A)$ is an infinite subset of B. As X is reflexive, B is weakly compact, hence there exists a weakly convergent sequence $\{x_n\}_{n=1}^{\infty} \subset J^{-1}(A)$. This implies that $\{Jx_n\}_{n=1}^{\infty}$ is convergent in Y. Thus, each infinite set contains a convergent sequence, and so $J(B)$ is precompact. □

The following lemma will play an important role in our study of normed spaces of continuous functions. In fact, part (ii) is an abstract version of a result from [21]. We remind that J is a linear operator from a normed space X into the locally convex space Y.

Lemma 2.24. *Let Y be a Frechet Space.*

(i) Assume that X is is a Banach space. Let $L \subset Y^*$ be such that $\bigcap_{l \in L} Ker\ l = \{0\}$. If $l \circ J \in X^*$, for all $l \in L$, then J is continuous.

(ii) Assume that J is a bounded operator such that the images of the closed balls of X are closed in Y. Then X is a Banach space.

Proof. (i): Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that for some $y \in Y$ and $x \in X$ we have that $x_n \to x$ and $Jx_n \to y$. Then from the continuity of both l and $l \circ J$, for all $l \in L$, we have that $(l \circ J)\, x_n \to (l \circ J)\, x$ and $l\,(Jx_n) \to l\,(y)$. Hence, for all $l \in L$ we get that $l\,(Jx) = l\,(y)$, and since $\bigcap_{l \in L} Ker\ l = \{0\}$ we conclude that $y = Jx$. Thus by the Closed Graph Theorem, J is continuous.

(ii): Let $\{x_n\}_{n=1}^{\infty} \subset X$ be Cauchy. From Corollary 2.22 and the completeness of Y, there exists $y \in Y$, such that $Jx_n \to y$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, it is contained in some closed ball in X. As its image is closed, there is $x \in X$ such that $y = Jx$. For a fixed $\varepsilon > 0$, there exists N such that for all $m, n \geq N$ we have that $\|x_m - x_n\| \leq \varepsilon$, or equivalently that $x_m \in B := \overline{B_X\,(x_n, \varepsilon)}$. Then, since $x \in \overline{J\,(B)} = J\,(B)$, we get that $\|x - x_n\| \leq \varepsilon$, for each $n \geq N$. From the arbitrariness of ε we conclude that $\lim\limits_{n \to +\infty} \|x - x_n\| = 0$. Hence each Cauchy sequence has a limit in X, and so X is a Banach space. $\square$

2.4 Normed Spaces of Continuous Functions

Let M be "permissible", i.e. a locally compact metric space, exhausted by compact sets in the sense that $M = \bigcup_{n \in \mathbb{N}} K_n$, where K_n is compact and $K_n \subset intK_{n+1}$, for each $n \in \mathbb{N}$. Under this assumption, the countable collection $\left\{\| \ \|_{\infty}^{K_n} \,|n \in \mathbb{N}\right\}$ separates points of $\mathcal{C}\,(M)$, and hence by Proposition 2.15 it is metrizable. Since we also know that $\mathcal{C}\,(M)$ is complete, we conclude that it is a Frechet space. In order to illustrate the connection between the nature of $\mathcal{C}\,(M)$ with its functional analytic properties consider the following result.

Proposition 2.25. *(Dini) Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}\,(M)$ be a sequence of real-valued functions such that for some $f \in \mathcal{C}\,(M)$ the sequence $\{f_n\,(p)\}_{n=1}^{\infty}$ is monotonously convergent to $f\,(p)$, for each $p \in M$. Then f_n converges to f in $\mathcal{C}\,(M)$.*

Proof. Let $K \subset M$ be compact and let $\varepsilon > 0$. For every $p \in K$ there is $N\,(p) \in$

$\mathbb{N}$ such that $\left|f_{N(p)}(p) - f(p)\right| < \varepsilon$. From the continuity of $f_{N(p)}(p)$ there exists an open neighborhood $U(p)$ of p, such that $\left|f_{N(p)}(q) - f(q)\right| < \varepsilon$, for each $q \in U(p)$. Hence, by the assumption on $\{f_n\}_{n=1}^{\infty}$, for any $n \geq N(p)$ we get that $|f_n(q) - f(q)| \leq \left|f_{N(p)}(q) - f(q)\right| < \varepsilon$. From the compactness of K there is a finite cover $\{U(p_1), U(p_1), ...U(p_k)\}$ of K. Then for $n \geq \max\{N(p_1), N(p_1), ...N(p_k)\}$ and $q \in M$ we have that $|f_n(q) - f(q)| < \varepsilon$. From the arbitrariness of K and ε we conclude that $f_n \to f$. $\square$

Let us introduce an extremely important family of continuous linear functionals on $\mathcal{C}(M)$ and on its subsets. *Point evaluation* in $p \in M$ is a scalar-valued functional defined by $\kappa_p(f) = f(p)$. Clearly, $\kappa_p \in (\mathcal{C}(M))^*$, for each $p \in M$. Note that we will keep the same name and notation for any restriction of such functionals on subsets of $\mathcal{C}(M)$. Clearly $\bigcap_{p \in M} Ker\ \kappa_p = \{0\}$. Point evaluations provide an inclusion of M into $(\mathcal{C}(M))^*$, having the following property:

Proposition 2.26. *The correspondence $p \to \kappa_p$ is continuous with respect to the open-compact topology on $(\mathcal{C}(M))^*$ (see example 2.18).*

Proof. Let $p \in M$, $\varepsilon > 0$ and let E be a compact subset of $\mathcal{C}(M)$. The sets of a type $V = \left\{l \in (\mathcal{C}(M))^*, \|l(f) - \kappa_p(f)\|_{\infty}^{E} < \varepsilon\right\}$ form a local base of $\mathcal{C}(M)$ at κ_p. Hence, it is sufficient to prove that there exists an open neighborhood U of p such that for all $q \in U$ we have that $\kappa_q \in V$. The latter is equivalent to $|\kappa_q(f) - \kappa_p(f)| = |f(q) - f(p)| < \varepsilon$, for each $f \in E$. Since E is compact, by the Arzela-Ascoli Theorem the described U exists. $\square$

Let X be a linear subset of $\mathcal{C}(M)$ and let $\| \ \|$ be a norm on X. We will say that X is a *normed space of continuous functions* (NSCF), if the inclusion map from X into $\mathcal{C}(M)$ is continuous. In this section we develop a few tools of working with NSCF's, which we will apply to the spaces of analytic functions. Some of the results of this section are well-known for the analytic case. For example, parts (ii) and (iii) of Proposition 2.30 for the analytic case were taken from [5]. However, note that the definition of NSCF's there is closer to the condition of part (iv) of Proposition 2.28.

Note that if X is a NSCF, $\kappa_p \in X^*$, since κ_p on X can be considered as a composition of the continuous inclusion J from X into $\mathcal{C}(M)$ and the continuous functional κ_p on $\mathcal{C}(M)$, i.e. $J^*\kappa_p$. Now it is obvious that the weak convergence in X implies the pointwise convergence. Consider the following lemma, which is a reformulation of part (i) of Proposition 2.20.

Lemma 2.27. *Let X be a linear subset of $\mathcal{C}(M)$ with a norm, such that the span of $\{\kappa_p \,|p \in M\}$ is a dense subset of X^*. Then if $\{f_n\}_{n=1}^{+\infty}$ is a bounded sequence in X, which weakly convergences in $\mathcal{C}(M)$ to some $f \in X$, then $f_n \xrightarrow{w} f$ in X.*

The following proposition gives some sufficient conditions for the space to be NSCF. Note that the point evaluations ar now considered to be elements of X^*.

Proposition 2.28. *Let X be a linear subset of $\mathcal{C}(M)$ and let $\| \ \|$ be a norm on X. Then:*

(i) X is a NSCF if and only if for each compact $K \subset M$ there exists $C_K > 0$ such that $\|\kappa_p\| \leq C_K$ for each $p \in K$;

(ii) If the correspondence $p \to \|\kappa_p\|$ is continuous on M, then X is a NSCF;

(iii) If the correspondence $p \to \kappa_p$ is continuous on M, then X is a NSCF;

(iv) If X is a Banach space and the point evaluations are continuous functionals on X, then X is a NSCF.

Proof. (i): From Proposition 2.14, the inclusion from X into $\mathcal{C}(M)$ is continuous if and only if for each compact K there is $C_K > 0$ such that $\|f\|_\infty^K \leq C_K \|f\|$ for all $f \in X$. This is equivalent to $|\kappa_p(f)| = |f(p)| \leq C_K \|f\|$ for all $f \in X$ and $p \in K$. The last is equivalent to $\|\kappa_p\| \leq C_K$ for all $p \in K$.

It is easy to see that if the correspondence $p \to \kappa_p$ is continuous on M, the correspondence $p \to \|\kappa_p\|$ is continuous on M too, and if the latter holds, then for each compact set $K \subset M$ the range of $\|\kappa_p\|$ on K is bounded by some C_K. By (i) the last implies the continuity of the inclusion. This proves both parts (ii) and (iii).

(iv): Apply part (i) of Lemma 2.24 to the Banach space X, Frechet space $\mathcal{C}(M)$, the inclusion from X into $\mathcal{C}(M)$ and the set of point evaluations. $\square$

The following proposition, gives a sufficient condition for NSCF to be a Banach space.

Proposition 2.29. *Let X be a NSCF with the norm $\| \ \|$. Assume that the extension of $\| \ \|$ to $\mathcal{C}(M)$ given by $\|f\| = +\infty$ for all $f \in \mathcal{C}(M) \setminus X$ is a lower semi-continuous functional on $\mathcal{C}(M)$. Then X is a Banach space.*

Proof. Let us investigate what the established condition means. For any $r > 0$ consider the set $B_X(0, r) = \{f \in X, \|f\| \leq r\}$ which is a closed ball in X. By the definition of lower semi-continuity, this set is closed in $\mathcal{C}(M)$. Hence, all closed balls of X are mapped into the closed sets in $\mathcal{C}(M)$ by the inclusion map, which is also continuous, since X is a NSCF. Thus X is a Banach space, by part (ii) of Lemma 2.24. □

Let us consider now the case when the inclusion map is a compact operator. We will prove later that each NSCF, consisting of only analytic functions on some domain in $\mathbb{C}^d$ has this property, and so does the Lipschitz Space. Since these spaces are essentially the central objects of our study, the considered case is of great importance.

Proposition 2.30. *Let X be a NSCF such that the inclusion map from X into $\mathcal{C}(M)$ is compact. Then:*

(i) The correspondence $p \to \kappa_p \in X^*$ is continuous on M;

(ii) If additionally X is a Banach space, then the weak convergence in X implies convergence in $\mathcal{C}(M)$;

(iii) If additionally X is reflexive, then for bounded sequences in X, the weak convergence is equivalent to the convergence in $\mathcal{C}(M)$.

Proof. (i): Let J be the inclusion map from X into $\mathcal{C}(M)$. From Proposition 2.19, J^* is continuous map from $(\mathcal{C}(M))^*$ with the open-compact topology into X^*, while from Proposition 2.26, the correspondence $p \to \kappa_p \in (\mathcal{C}(M))^*$ is continuous with respect to the open-compact topology on $(\mathcal{C}(M))^*$. Hence their composition, being the correspondence $p \to J^*\kappa_p = \kappa_p \in X^*$, is continuous on M.

Part (ii) follows immediately from the part (ii) of Proposition 2.21.

(iii): Assume that $\{f_n\}_{n=1}^{+\infty}$ is a bounded sequence in X, which is convergent in $\mathcal{C}(M)$ to some $f \in \mathcal{C}(M)$. Since the sequence is bounded, it is contained in a closed ball of X. Since X is reflexive and the inclusion J is continuous, from Proposition 2.23, this ball is also closed in $\mathcal{C}(M)$, and so $f \in JX = X$. From the part (ii) of Proposition 2.20, the span of $\{\kappa_p \,|p \in M\}$ is a dense subset of X^*. Thus, from Lemma 2.27 we have that $f_n \xrightarrow{w} f$ in X. The converse follows directly from part (ii). $\square$

Note that in the case when X is reflexive and when the weakly convergent sequences in X are also convergent in $\mathcal{C}(M)$, the inclusion map is compact from part (ii) of Proposition 2.23.

Finally, let us consider operators between the normed spaces of continuous functions. Part (ii) for the analytic case is known from [22].

Proposition 2.31. *Let X and Y be normed spaces of continuous functions on the permissable spaces M and N respectively, and let T be an operator from X into Y, which is continuous with respect to the open-compact topologies on M and N, restricted to X and Y. Then:*

(i) If X and Y are Banach spaces, then T is bounded with respect to the norms on this spaces.

(ii) If X is compactly embedded in $\mathcal{C}(M)$, then T is compact with respect to the strong topologies on X and Y if and only if each bounded in X sequence, convergent in $\mathcal{C}(M)$ is transformed into the one convergent in Y.

Proof. (i): Assume that $\{f_n\}_{n=1}^{+\infty}$ is a sequence in X, such that $f_n \to f$, for some $f \in X$ and that $Tf_n \to g$, for some $g \in Y$. Then both $f_n \to f$ and $Tf_n \to g$ hold in the topologies of $\mathcal{C}(M)$ and $\mathcal{C}(N)$ respectively, and since T is continuous with respect to these topologies, $g = Tf$. Hence, since both X and Y are Banach spaces, T is continuous by the Closed Graph Theorem.

(ii): If T transforms each sequence convergent in $\mathcal{C}(M)$ into the one convergent in Y, T is a continuous map from X with the open-compact topology into Y. Hence,

since X is compactly embedded, the unit ball of X is precompact in $\mathcal{C}(M)$ and so its image is precompact in Y. Thus, T is compact.

Assume that T is compact and let $\{f_n\}_{n=1}^{+\infty} \subset X$ be a bounded sequence in X, that is convergent in $\mathcal{C}(M)$ to some $f \in X$. Since $\{Tf_n\}_{n=1}^{+\infty}$ is precompact in Y, we only have to prove that all limit points of this sequence are equal to Tf. Suppose $Tf_n \to g$ in X for some $g \in X$. Then, since T is continuous with respect to the open-compact topologies, $Tf_n \to Tf$ in $\mathcal{C}(N)$ and hence $g = Tf$. $\square$

It is easy to see that we can replace "convergent" with "convergent to zero" in the formulation of the part (ii) of the preceding proposition. Note also that we actually did not use the compactness of the inclusion in the proof of the necessity in part (ii).

2.5 The Lipschitz Space

We remind that M is a locally compact, compactly exhausted metric space. We will apply some of the developed tools to the space of Lipschitz scalar-valued functions

$$Lip(M) = \{f \in \mathcal{C}(M) \,|\mathrm{dil}(f) < +\infty\}.$$

Recall that the dilation of the scalar-valued function f is equal to $\sup\limits_{p,q\in M,\ p\neq q} \frac{|f(p)-f(q)|}{d_M(p,q)}$. Further, we will introduce an analogue of the Lipschitz Space for the case of analytic functions. For $p \in M$ and $f \in Lip(M)$ define $\|f\|_{Lip}^p = \mathrm{dil}(f) + |f(p)|$. We will explain how to turn $Lip(M)$ into a normed space.

Proposition 2.32. *Let $p, q \in M$. Then:*

(i) The dilation is a seminorm on $Lip(M)$.

(ii) $\| \ \|_{Lip}^p$ is a norm on $Lip(M)$.

(iii) $(d_M(p,q)+1)^{-1} \| \ \|_{Lip}^q \leq \| \ \|_{Lip}^p \leq (d_M(p,q)+1) \| \ \|_{Lip}^q$.

Proof. (i): Let $I = (M \times M) \setminus \{(p,p) \,|p \in M\}$. For each $(p,q) \in I$ a functional defined by $\rho_{(p,q)}(f) = \frac{|f(p)-f(q)|}{d_M(p,q)}$ is a seminorm on $\mathcal{C}(M)$. Then $\mathrm{dil} = \sup\limits_{(p,q)\in I} \rho_{(p,q)}$, and hence by Lemma 2.13, $Lip(M)$ is a linear subset of $\mathcal{C}(M)$, while dil is a seminorm.

(ii): By Proposition 2.5, part (ii), the kernel of the dilation consists of all constant functions. Obviously, the correspondence $f \to |f(p)|$ is a seminorm and the two of them separate points of $Lip(M)$. Hence, by part (ii) Lemma 2.13 and the remark after it, we conclude that $\| \|_{Lip}^{p}$ is a norm.

(iii): For $f \in Lip(M)$ we have that

$$\left| \|f\|_{Lip}^{p} - \|f\|_{Lip}^{q} \right| = |f(p) - f(q)| \leq \operatorname{dil}(f)\, d_M(p,q) \leq \|f\|_{Lip}^{q}\, d_M(p,q).$$

Hence $\| \|_{Lip}^{p} \leq (d_M(p,q)+1) \| \|_{Lip}^{q}$, while the lower estimate is obtained by interchanging p and q. □

The third statement of the previous proposition shows that the topology generated by $\| \|_{Lip}^{p}$ does not depend on the choice of $p \in M$. Due to this reason we will not fix any norm in $Lip(M)$. This logic can be illustrated by the following proposition.

Proposition 2.33. *Let $E \subset Lip(M)$ be such that $\sup_{f \in E} \operatorname{dil}(f) < +\infty$. Then for each $p \in M$ the set $\{f - f(p) \,|\, f \in E\}$ is bounded in $Lip(M)$.*

Proof. For each $f \in E$, define $g = f - f(p)$. Then $\|g\|_{Lip}^{p} = \operatorname{dil}(f) + |g(p)| \leq C + 0$, where $C = \sup_{f \in E} \operatorname{dil}(f) < +\infty$. Hence the set $\{f - f(p) \,|\, f \in E\}$ is bounded with respect to $\| \|_{Lip}^{p}$. □

Note that another possible option for turning $Lip(M)$ into a normed space is to first factorize it modulo the set of all constants. Next we describe how $Lip(M)$ is embedded in $\mathcal{C}(M)$.

Proposition 2.34. *The inclusion from $Lip(M)$ into $\mathcal{C}(M)$ is a compact operator.*

Proof. Let $K \subset M$ be compact and let $C_K = \max\left\{1, \sup_{q \in K} d_M(p,q)\right\}$, where $p \in M$. Note that the existence of the supremum is guaranteed by the compactness of K. Then for $f \in Lip(M)$ we have that

$$\begin{aligned} C_K \|f\|_{Lip}^{p} &\geq \operatorname{dil}(f) \sup_{q \in K} d_M(p,q) + |f(p)| \\ &\geq \sup_{q \in K} d_M(p,q) \frac{|f(p) - f(q)|}{d_M(p,q)} + |f(p)| \\ &\geq \sup_{q \in K} |f(q)| = \|f\|_{\infty}^{K}. \end{aligned}$$

From the arbitrariness of K, we conclude that the inclusion map from $Lip\,(M)$ into $\mathcal{C}\,(M)$ is bounded. The bounded sets in $Lip\,(M)$ are the ones which have bounded dilations and bounded values in p. Hence by part (ii) of Proposition 2.11 they are precompact in $\mathcal{C}\,(M)$. This proves the compactness of the inclusion. □

Corollary 2.35. *$Lip\,(M)$ is a Banach space with respect to the norm $\| \ \|_{Lip}^{p}$.*

Proof. From the preceding proposition the inclusion map from $Lip\,(M)$ into $\mathcal{C}\,(M)$ is bounded. Also, since the dilation is lower semi-continuous, the closed balls in $Lip\,(M)$ are also closed in $\mathcal{C}\,(M)$. Hence by Proposition 2.29, $Lip\,(M)$ is a Banach space. □

One of the nice properties of the Lipschitz space is a specific version of the Proposition 2.30.

Proposition 2.36. *For each $p, q \in M$, $\|\kappa_p - \kappa_q\| = d_M\,(p, q)$.*

Proof. Let $w \in M$. In the case when $p = q$ there is nothing to prove. If $p \neq q$, for each $f \in Lip\,(M) \setminus \{0\}$, we have that $|f\,(p) - f\,(q)| \leq \mathrm{dil}\,(f)\, d_M\,(p, q)$ and hence

$$\frac{|\kappa_p\,(f) - \kappa_q\,(f)|}{\|f\|_{Lip}^{w}} = \frac{|f\,(p) - f\,(q)|}{\mathrm{dil}\,(f) + |f\,(w)|} \leq d_M\,(p, q)\,.$$

On the other hand it is easy to see that the function defined by $f\,(z) = d_M\,(z, p) - d_M\,(w, p)$ has dilation equal to 1, while $f\,(w) = 0$ and $f\,(p) - f\,(q) = d_M\,(q, p)$. □

Let $\Phi \in \mathcal{C}\,(N, M)$, where M and N are permissable spaces. A *Composition operator* C_Φ is a transformation from $\mathcal{C}\,(M)$ into $\mathcal{C}\,(N)$, given by $C_\Phi\,(f)\,(q) = f\,(\Phi\,(q))$. Clearly, this map is a continuous linear operator.

The importance of the composition operators is determined by several reasons including for example the Banach-Stone Theorem (see [6]).

We will focus on restrictions of composition operators on the Lipschitz spaces. From Proposition 2.6 we can conclude that if Φ is an onto isometry such that $\Phi\,(q) = p$, then C_Φ is an isometry with respect to the norms $\| \ \|_{Lip}^{p}$ and $\| \ \|_{Lip}^{q}$. The next proposition contains some further details. In the case of analytic functions this results are known from [1].

Proposition 2.37.

(i) If $\operatorname{dil}(\Phi) < +\infty$*, then* C_Φ *is a bounded linear operator with*

$$1 \leq \|C_\Phi\| \leq \max\{1, d_M(\Phi(q), p) + \operatorname{dil}(\Phi)\}.$$

In particular, if $\Phi(q) = p$*, then* $1 \leq \|C_\Phi\| \leq \max\{1, \operatorname{dil}(\Phi)\}$.

(ii) If $\Phi(q) = p$, $\operatorname{dil}(\Phi) \leq 1$ *and there exists a sequence of onto isometries* $\{\Psi_j\}_{n=1}^{+\infty} \subset \mathcal{C}(M, N)$*, such that* $\Phi \circ \Psi_j$ *converges to identity in* $\mathcal{C}(M, M)$*, then* $\operatorname{dil}(\Phi) = 1$ *and* C_Φ *is an isometry.*

Proof. (i): Consider arbitrary $f \in Lip(M)$. Then:

$$\begin{aligned} |f(\Phi(q))| &\leq |f(p)| + |f(\Phi(q)) - f(p)| \leq |f(p)| + d_M(\Phi(p), q)\operatorname{dil}(f) \\ &\leq \|f\|_{Lip}^q + (d_M(\Phi(q), p) - 1)\operatorname{dil}(f). \end{aligned}$$

Hence we have that

$$\|f \circ \Phi\|_{Lip}^q = |f(\Phi(q))| + \operatorname{dil}(f \circ \Phi) \leq \|f\|_{Lip}^p + (d_M(\Phi(q), p) - 1 + \operatorname{dil}(\Phi))\operatorname{dil}(f).$$

If $d_M(\Phi(q), p) - 1 + \operatorname{dil}(\Phi) \leq 0$, we will have $\|f \circ \Phi\|_{Lip}^q \leq \|f\|_{Lip}^p$. Otherwise,

$$\|f \circ \Phi\|_{Lip}^q \leq (d_M(\Phi(p), q) + \operatorname{dil}(\Phi))\operatorname{dil}(f) \leq (d_M(\Phi(p), q) + \operatorname{dil}(\Phi))\|(f)\|_{Lip}^p.$$

For the proof of the lower estimate consider $f \in Lip(M)$, defined by $f(z) = 1$, for each $z \in M$. Then $(C_\Phi f)(w) = 1$, for each $w \in N$ and $\|f\|_{Lip}^p = \|C_\Phi f\|_{Lip}^q = 1$.

(ii): First, from the lower semi-continuity of the dilation and since $\Phi \circ \Psi_j$ converges to identity in $\mathcal{C}(M, M)$, we have that

$$1 = \operatorname{dil}(Id) \leq \liminf_{j \to \infty} \operatorname{dil}(\Phi \circ \Psi_j) = \operatorname{dil}(\Phi) \leq 1.$$

Hence, $\operatorname{dil}(\Phi) = 1$. Similarly, for each $f \in Lip(M)$ we have that $f \circ \Phi \circ \Psi_j$ converges to f in $\mathcal{C}(M)$ and so

$$\operatorname{dil}(f) \leq \liminf_{j \to \infty} \operatorname{dil}(f \circ \Phi \circ \Psi_j) = \operatorname{dil}(f \circ \Phi) \leq \operatorname{dil}(f)\operatorname{dil}(\Phi) = \operatorname{dil}(f).$$

Thus

$$\|f \circ \Phi\|_{Lip}^q = \operatorname{dil}(f \circ \Phi) + |f(\Phi(q))| = \operatorname{dil}(f) + |f(p)| = \|f\|_{Lip}^p.$$

□

Chapter 3

Differential Structures

This chapter contains a brief introduction to differential geometry. In particular, we consider some metric aspects of the theory of the Riemannian manifolds (two of them were already mentioned in the end of section 2.1).

3.1 Differential Manifolds

This section serves to remind the reader of some concepts of differential geometry and to introduce our notations. Most of the omitted proofs consist of simple computations and can be found in most of the textbooks (for example, [9]). The only exception is part (ii) of Proposition 3.4, which is again standard.

Let M be a topological space. A *chart* on M is a pair (U, φ), where U is an open subset of M, and φ is a homeomorphism between U and some open subset of $\mathbb{R}^d$, for some $d \in \mathbb{N}$.

A *differential structure* (or an *atlas*) of dimension d on M is a collection $\mathfrak{F} = \{(U_\alpha, \varphi_\alpha) \,|\alpha \in I\}$ of charts into $\mathbb{R}^d$, such that $M = \bigcup_{\alpha \in I} U_\alpha$ and such that for each $\alpha, \beta \in I$, the *transition* map $\varphi_\beta \varphi_\alpha^{-1}$ is a *smooth* (infinitely differentiable) map between $\varphi_\alpha (U_\alpha \bigcap U_\beta)$ and $\varphi_\beta (U_\alpha \bigcap U_\beta)$.

Sometimes one can add an extra condition for $\mathfrak{F}$ to be a maximal collection. This is not substantial since each differential structure can be extended to a unique maximal one. We will not distinguish two structures if they have the same maximal

extension.

A d *dimensional smooth manifold* is a pair of a Hausdorff second-countable topological space M and a differential structure of dimension d on M.

Since each point has a locally compact neighborhood, we conclude that the smooth manifolds are locally compact.

Throughout this chapter M will denote a smooth d-dimensional manifold with a fixed differential structure.

Example 3.1. $\mathbb{R}^d$ possesses differential structure consisting of the single chart $\left(\mathbb{R}^d, Id\right)$.

Example 3.2. Each open set of M is a d-dimensional smooth manifold with induced differential structure. Namely, if V is open in M, then $\mathfrak{E} = \left\{ \left(U_\alpha \bigcap V, \varphi_{\alpha|U_\alpha \bigcap V} \right) | \alpha \in I \right\}$ forms a d-dimensional differential structure on V. For example, an open interval on a line is a 1-dimensional manifold.

Note that the logic of these two examples is applicable in order to construct "first natural" examples of the most of objects in differentiable geometry.

After defining a structure of a manifold, it is natural to define morphisms between the objects possessing this structure.

Let M and N be smooth manifolds. The map $\Phi : M \to N$ is called *smooth* if for each chart (U, φ) on M and each chart (V, ψ) on N, the *transition* map $\psi \circ \Phi \circ \varphi^{-1}$ is smooth on $\varphi^{-1}\left(U \bigcap \Phi^{-1}(V)\right)$.

It is easy to see that this definition does not depend on the choice of the particular differential structures on M and N, from their classes of equivalence. Also note, that smoothness of a map implies its continuity. Finally, if (U, φ) is a chart on M, then φ is a smooth map from U into $\mathbb{R}^d$.

The set of all smooth maps from M to N will be denoted by $\mathcal{C}^\infty(M, N)$, or by $\mathcal{C}^\infty(M)$, in case $N = \mathbb{R}$. Naturally, $\mathcal{C}^\infty(M, N)$ possesses an induced topology as a subset of $C(M, N)$. Observe that if $\Psi \in \mathcal{C}^\infty(L, M)$, for some smooth manifold L, then $\Phi \circ \Psi \in \mathcal{C}^\infty(L, M)$.

An important class of smooth maps is formed by maps defined on open connected subsets of $\mathbb{R}$. Such maps are called *smooth curves* in N. Often we will consider curves

defined on closed intervals. In that case we will assume that there exists an extension of the corresponding map to some smooth curve on an open interval.

Now let $p \in M$ and let f and g be smooth real-valued functions on some open neighborhoods of p. We will say that f and g belong to the same *germ* in p if they coincide on some open set containing p. It is easy to see that germs in p form an algebra, which will be denoted by F_pM (for example, a sum of germs is the germ generated by the sum of any pair of their representatives on the intersection of their domains). Note that for each $\tilde{f} \in F_pM$, the value $\tilde{f}(p)$ is well defined since all of the representatives of $\tilde{f}$ coincide at p.

A functional $v \in (F_pM)^*$ is a *tangent vector* at $p \in M$ if for all $\tilde{f}, \tilde{g} \in F_pM$ the so called "Leibniz equality" holds:

$$v\left(\tilde{f}\tilde{g}\right) = \tilde{f}(p)\, v(\tilde{g}) + \tilde{g}(p)\, v\left(\tilde{f}\right).$$

We will call the set of all tangent vectors at p a *tangent space* at p and denote it by T_pM. It is easy to see that it is a linear subspace of $(F_pM)^*$.

Although we defined tangent vectors as functionals on F_pM, we will also consider them as functionals acting directly on the smooth functions: for $v \in T_pM$ and $f \in \mathcal{C}^\infty(U)$, where U is an open neighborhood of p, define $v(f) = v\left(\tilde{f}\right)$, where $\tilde{f}$ is the germ generated by f.

Let us construct some natural examples of tangent vectors. For $x \in \mathbb{R}^d$ and $k \in \overline{1,d}$ we will define a functional of taking the k-th partial derivative at x by $\partial_k(x)$. Namely, if $f : \mathbb{R}^d \to \mathbb{R}$ is a differentiable function, then $\partial_k(x) f = \frac{\partial f(x)}{\partial x_k}$. It is easy to see that such functional is a tangent vector at x.

Let U be an open subset of M and let $p \in U$. Since the definition of germs is local, each vector T_pM can be also considered as an element of T_pU. This is another natural example of a tangent vector.

A little more sophisticated example comes from the idea that all local objects on

a manifold can be transferred from the space $\mathbb{R}^d$ by composing with an appropriate chart. The next proposition contains a formalization of this concept (for the proof and more details see [9]).

Proposition 3.3. *Let (U, φ) be a chart in $p \in M$. For each $k \in \overline{1,d}$ define a functional $\partial_k(p)$ on F_pM by*

$$\partial_k(p)\left(\tilde{f}\right) = \partial_k(\varphi(p))\left(f \circ \varphi^{-1}\right),$$

where $\tilde{f} \in F_pM$ and $f \in \tilde{f}$. Then:

(i) $\partial_k(p)$ is well defined and is a tangent vector at p, for each $k \in \overline{1,d}$.

(ii) The vectors $\partial_1(p), \partial_2(p), ..., \partial_d(p)$ form a base in T_pM. Hence $\dim T_pM = d$.

The next proposition includes an example justifying the name "tangent vector". Both parts follow directly from the definition.

Proposition 3.4. *For $p \in M$ let $\gamma : (a,b) \to M$ be a smooth curve with $\gamma(s) = p$. Define a functional γ_p^* on F_pM by*

$$\gamma_p^*\left(\tilde{f}\right) = \lim_{t \to 0} \frac{f(\gamma(t+s)) - f(\gamma(s))}{t} = \partial_1(s)(f \circ \gamma) = (f \circ \gamma)'(s),$$

where $\tilde{f} \in F_pM$ and $f \in \tilde{f}$. Then:

(i) γ_p^* is well defined and is a tangent vector at p.

(ii) Each vector in T_pM can be generated by a curve: if $a = (a_1, a_2, ..., a_d)^T \in \mathbb{R}^d$ let $\gamma(t) = \varphi^{-1}(\varphi(p) + ta)$. Then $\gamma_p^* = \sum_{k=1}^{d} a_k \partial_k(p)$.

Since we have a notion of smoothness of maps between manifolds, we should introduce some appropriate notion of their derivatives.

Let M and N be smooth manifolds and let $\Phi : M \to N$ be smooth. For each $p \in M$ the map Φ induces a linear transformation from $F_{\Phi(p)}N$ into F_pM, defined by $C_\Phi^p(\tilde{g}) = \tilde{f}$, where $g \in \tilde{g} \in F_{\Phi(p)}N$, and $\tilde{f}$ is a germ in F_pM generated by $f = g \circ \Phi$.

A *Differential* of Φ at $p \in M$ is a map $d\Phi_p : T_pM \to T_{\Phi(p)}N$, defined as $(d\Phi_p v)(\tilde{g}) = v(C_\Phi^p(\tilde{g}))$, for $v \in T_pM$ and $\tilde{g} \in F_pN$.

Observe that $d\Phi_p = (C_\Phi^p)^*_{|T_pM}$ and hence it is a linear transformation. Also if $\Psi \in \mathcal{C}^\infty(L, M)$, for some smooth manifold L, then $d(\Phi \circ \Psi)_p = d\Phi_{\Psi(p)} d\Psi_p$.

Example 3.5. Both examples of tangent vectors, considered in the previous two propositions were constructed by the transfer of the usual differentiation into the manifold through some (smooth) map. Expressing this using the notation of differential gives us the following equalities.

- $\gamma_p^* = d\gamma_s(\partial_1(s))$, where $\gamma : (a,b) \to M$ is a smooth curve with $\gamma(s) = p$;
- $\partial_k(p) = (d\varphi^{-1})_{\varphi(p)}(\partial_k(\varphi(p)))$, where (U,φ) is a chart in p.

Note, that using the first equality for the smooth curve $\Phi \circ \gamma$ in N, we get that

$$(\Phi \circ \gamma)^*_{\Phi(p)} = d(\Phi \circ \gamma)_s(\partial_1(s)) = d\Phi_{\gamma(s)} d\gamma_s(\partial_1(s)) = d\Phi_p \gamma_p^*.$$

The next proposition demonstrates the connection between the established notion of the differential and differential in its usual meaning (see [9]).

Proposition 3.6. *Let (U,φ) be a chart in p and let (V,ψ) be a chart in $\Phi(p)$. Also, let $\dim N = m$. Then the matrix of the linear operator $d\Phi_p$ with respect to the bases $\{\partial_1(p), \partial_2(p), ..., \partial_d(p)\}$ and $\{\partial_1(\Phi(p)), \partial_2(\Phi(p)), ..., \partial_m(\Phi(p))\}$ is equal to the Jacobian matrix of $\psi \circ \Phi \circ \varphi^{-1}$.*

In particular, in the case, when $M = N = \mathbb{R}$, then differential is a multiplication with a constant: $d\Phi_p(\partial_1(p)) = \Phi'(p)\,\partial_1(\Phi(p))$.

Example 3.7. Let $f \in \mathcal{C}^\infty(M)$ and let $f(p) = s$. Since $T_s\mathbb{R}$ is 1-dimensional and $\partial_1(s)$ forms the base of this space, there exists $\nabla_p f \in (T_p\mathbb{R})^*$, such that $df_p(v) = \nabla_p f(v)\,\partial_1(s)$. If $g \in \mathcal{C}^\infty(\mathbb{R})$ is defined by $g(t) = t$, then

$$\nabla_p f(v) = \nabla_p f(v)\,\partial_1(s)\,g = df_p(v)(g) = v(g \circ f) = v(f).$$

Hence, if (U,φ) is a chart in p, then the coordinate representation of $\nabla_p f$ is equal to $(\partial_1(p) f, \partial_2(p) f, ..., \partial_d(p) f)$, which justifies the notation $\nabla_p f$.

3.2 Metric Structures On Differential Manifolds

We will introduce two similar ways to provide a differential manifold with a metric space structure. Note that from now on we will use the word "distance functions" for

the object which was previously called "metric". The word "metric" is now reserved for the Riemannian and Finsler metrics, which we are defining below. However, these concepts are connected, as it will be described further. The main purpose of this section is to prove Theorem 3.17 and its Corollary 3.19. Theorem 3.17 was used in [3] to characterize Bloch functions in several variables, however there was just a part of a proof. Our goal is to give a complete proof of this theorem. We base this proof on Theorem 3.10 and provide the auxiliary statements 3.11-3.14 and 3.16.

- A *Finsler metric* is a correspondence ρ, which in each point $p \in M$ gives rise to a norm ρ_p acting on T_pM, such that for any chart (U, φ) in p and $k \in \overline{1, d}$ the function $\rho_p(\partial_k(p))$ is smooth on U.

- A *Riemannian metric* is a correspondence g, which in each point $p \in M$ gives rise to a scalar product g_p acting on T_pM, such that for any chart (U, φ) in p and $j, k \in \overline{1, d}$ the function $g_p(\partial_j(p), \partial_k(p))$ is smooth on U.

A *Riemannian (Finsler) manifold* is a connected smooth manifold with a fixed Riemannian (Finsler) metric on it. As we will show soon every Riemannian Manifold is also a Finsler one.

The first natural examples come from the same ideas as in the examples 3.1 and 3.2. We can establish a so-called *Euclidean metric* on $\mathbb{R}^d$, being the Riemannian metric defined by $g_x(\partial_j(x), \partial_k(y)) = \delta_k^j$. Open subsets of Riemannian (Finsler) manifolds posses induced metrics being the restrictions of the initial one.

Let us say few words about the Finsler manifolds, before we begin to study extensively the Riemannian ones.

A Finsler metric induces a length functional on the smooth curves: if $\gamma : [a, b] \to M$ is a smooth curve then

$$L_\rho(\gamma) := \int_a^b \rho_{\gamma(t)}\left(\gamma^*_{\gamma(t)}\right) dt.$$

Note that reparametrization does not affect the length of the curve.

Naturally, this functional induces a distance function on M: for $p, q \in M$ define

$$d_\rho(p,q) = \inf_{p,q \in Ran(\gamma)} L_\rho(\gamma).$$

If there exists a curve with length equal to $d_\rho(p,q)$, we will call this curve a *geodesic* between p and q. If w is a point the geodesic then the corresponding segment of this curve will also be a geodesic between p, w. Let us state the famous Hopf–Rinow Theorem for Finsler manifolds. For the proof in a more general setting see [2].

Theorem 3.8. *If a Finsler manifold is a complete metric space then every two points can be joint by a geodesic, and each boundeded set is precompact.*

Throughout the rest of the section (M, g) will be a Riemannian manifold. It can be also considered as a Finsler one, since each tangent space of M has an Euclidean structure, determined by the Riemanian metric. Namely, for $v \in T_pM$, define $\rho_p^g(v) = \sqrt{g_p(v,v)}$, which is a norm on the tangent space at p. Since ρ_p^g is never equal to 0 and the square root is a smooth function on $(0, +\infty)$, for any chart (U, φ) the function $\rho_p^g(\partial_i(p))$ is smooth on U. In order to simplify the notation define $d_g = d_{\rho^g}$. Observe that the Euclidean metric gives rise to the usual Euclidean distance on $\mathbb{R}^d$.

Lemma 3.9. *Let (U, φ) be a chart.*

(i) For $p \in U$, $b = (a_1, a_2, ..., a_d)^T \in \mathbb{R}^d$ and $b = (b_1, b_2, ..., b_d)^T \in \mathbb{R}^d$ – vectors in $\mathbb{R}^d$ define

$$G(p,a,b) = g_p\left(\sum_{j=1}^d b_j \partial_j(p), \sum_{k=1}^d b_k \partial_k(p)\right).$$

Then $G \in \mathcal{C}\left(U \times \mathbb{R}^d \times \mathbb{R}^d\right)$.

(ii) For $p \in U$ and $a = (a_1, a_2, ..., a_d)^T \in \mathbb{R}^d$ define

$$F(p,a) = \rho_p^g\left(\sum_{k=1}^d a_k \partial_k(p)\right).$$

Then $F \in \mathcal{C}\left(U \times \mathbb{R}^d\right)$.

Proof. (i): For $j,k \in \overline{1,d}$ define $f_{jk}(p) = g_p(\partial_j(p), \partial_k(p))$, which is a smooth function on U. Then

$$G(p,a,b) = g_p\left(\sum_{j=1}^{d} b_j \partial_j(p), \sum_{k=1}^{d} b_k \partial_k(p)\right) = \sum_{j,k=1}^{d} a_j b_k f_{jk}(p).$$

The latter is clearly a sum of functions from $\mathcal{C}\left(U \times \mathbb{R}^d \times \mathbb{R}^d\right)$, which ensures the continuity of G.

(ii): Follows from $F(p,a) = \sqrt{G(p,a,a)}$. □

Note that we did not use here neither symmetry nor positivity of g_p.

The next theorem shows that there is a certain geometric connection between the points in the neighborhood of any point in the manifold and the tangent vectors in this point (its proof can be found in [9]).

Theorem 3.10. *For each $p \in M$ there is an open set $U \ni p$, an open set $V \subset T_pM$, containing the zero vector and a unique smooth homeomorphism $\varphi : U \to V$, with the property that for each $q \in U$ there exists a unique geodesic $\theta_q : [0,1] \to U$, such that*

$$\theta_q(0) = p, \ \theta_q(1) = q, \ \ (\theta_q)_p^* = \varphi(q).$$

Moreover, under this conditions $\rho_p^g(\varphi(q)) = d_g(p,q)$.

The next three lemmas, stated using the notation from the preceding theorem for some fixed $p \in M$, will specify the behavior of the map φ.

Lemma 3.11. *For all $t \in [0,1]$ we have that $\theta_q(t) = \varphi^{-1}(t\varphi(q))$.*

Proof. Let the curve $\theta_q^t : [0,1] \to U$ be defined as $\theta_q^t(s) = \theta_q(ts)$. Then $\left(\theta_q^t\right)_p^* = t(\theta_q)_p^*$, and since this curve is the geodesic between p and $\theta_q(t)$, we have that

$$\varphi(\theta_q(t)) = \left(\theta_q^t\right)_p^* = t(\theta_q)_p^* = t\varphi(q),$$

which proves the lemma. □

Lemma 3.12. *For $v \in T_pM$ and $l \in (T_pM)^*$, we have that $v(l \circ \varphi) = l(v)$.*

Proof. Let $q = \varphi^{-1}(v)$. By Theorem 3.10, $v = \varphi(q) = (\theta_q)^*_p$, and by the previous lemma, $\theta_q(t) = \varphi^{-1}(tv)$. Hence

$$v(l \circ \varphi) = \lim_{t \to 0} \frac{l(\varphi(\varphi^{-1}(tv)))}{t} = \lim_{t \to 0} \frac{l(tv)}{t} = \lim_{t \to 0} l(v) = l(v).$$

□

Lemma 3.13. *Let γ be a smooth curve on M, with $\gamma(0) = p$. Then*

$$\lim_{t \to 0} \frac{\varphi(\gamma(t))}{t} = \gamma^*_p.$$

Proof. Without loss of generality we may assume that the range of γ is contained in U (otherwise restrict the curve to the component of $\gamma^{-1}(U)$ which contains 0). Hence for $l \in (T_pM)^*$ we have that

$$\lim_{t \to 0} l\left(\frac{\varphi(\gamma(t))}{t}\right) = \lim_{t \to 0} \frac{l(\varphi(\gamma(t)))}{t} = \lim_{t \to 0} \frac{(l \circ \varphi)(\gamma(t))}{t} = \gamma^*_p(l \circ \varphi) = l\left(\gamma^*_p\right),$$

where the last equality follows from the previous lemma. Since l was arbitrary, and T_pM is finite-dimensional, the desired equality is proven. □

Now let us return to some metric aspects of the Riemannian manifolds and the maps between them. We start with a natural property of smooth curves:

Proposition 3.14. *Let γ be a smooth curve on M, with $\gamma(0) = p$. Then*

$$\lim_{t \to 0} \frac{d_g(p, \gamma(t))}{t} = \rho^g_p\left(\gamma^*_p\right).$$

Proof. From Theorem 3.10, $\frac{d_g(p,\gamma(t))}{t} = \frac{\rho^g_p(\varphi(\gamma(t)))}{t}$, for each $t \in [0,1]$. So, by the continuity of the norm and the previous lemma,

$$\lim_{t \to 0} \frac{d_g(p, \gamma(t))}{t} = \lim_{t \to 0} \frac{\rho^g_p(\varphi(\gamma(t)))}{t} = \rho^g_p\left(\lim_{t \to 0} \frac{\varphi(\gamma(t))}{t}\right) = \rho^g_p\left(\gamma^*_p\right).$$

□

Our next goal, as promised before, is to study the properties of the dilation of smooth maps between Riemannian manifolds. We first make an observation.

Let (N, h) be another Riemannian manifold and let $\Phi : M \to N$ be smooth. For each $p \in M$ the differential $d\Phi_p$ is a linear operator between the Euclidean spaces T_pM and $T_{\Phi(p)}N$. Hence, we can define its norm in a usual way:

$$\Phi^\sharp(p) = \sup_{v \in T_pM \setminus \{0\}} \frac{\rho^h_{\Phi(p)}(d\Phi_p(v))}{\rho^g_p(v)}.$$

Note, that we are not using here the symbol $\|d\Phi_p\|$ in order to avoid further possible confusion and to simplify notations.

The first obvious consequence of the definition is that if $\Psi \in \mathcal{C}^\infty(L, M)$ for some Riemannian manifold (L, f), then $(\Phi \circ \Psi)^\sharp(p) \leq \Psi^\sharp(p)\, \Phi^\sharp(\Psi(p))$.

Example 3.15. In the setting of Example 3.7, assume that $\mathbb{R}$ is equipped with the Euclidean metric $\| \ \|_{\mathbb{R}}$. Then for each $v \in T_pM$, we have that

$$\|df_p(v)\|_{\mathbb{R}} = \|\nabla_p f(v)\, \partial_1(s)\|_{\mathbb{R}} = |\nabla_p f(v)|,$$

and so

$$f^\sharp(p) = \sup_{v \in T_pM \setminus \{0\}} \frac{\|df_p(v)\|_{\mathbb{R}}}{\rho^g_p(v)} = \sup_{v \in T_pM \setminus \{0\}} \frac{|\nabla_p f(v)|}{\rho^g_p(v)} = \left(\rho^g_p\right)^*(\nabla_p f),$$

where $\left(\rho^g_p\right)^*$ is a dual norm to ρ^g_p on $(T_x\mathbb{R})^*$. In particular, if $M \subset \mathbb{R}$ and $\rho^g_p = \| \ \|_{\mathbb{R}}$, then $f^\sharp(p) = |f'(p)|$.

Now, let us prove that $\Phi^\sharp$ does not behave chaotically.

Lemma 3.16. $\Phi^\sharp$ *is a nonnegative function in* $C(M)$.

Proof. Let (U, φ) be a chart and let S be a sphere in $\mathbb{R}^d$. For $p \in U$ and $a = (a_1, a_2, ..., a_d)^T \in S$ consider the function

$$F(p, a) = \frac{\rho^h_{\Phi(p)}\left(d\Phi_p\left(\sum_{k=1}^d a_k \partial_k(p)\right)\right)}{\rho^g_p\left(\sum_{k=1}^d a_k \partial_k(p)\right)}.$$

From Lemma 3.9 and the remark after it, both the numerator and the denominator are continuous on $U \times S$. Also, since the vectors $\partial_1(p), \partial_2(p), ..., \partial_d(p)$ form a base of T_pM, we conclude that the denominator is positive and hence $F \in C(U \times S)$.

Now let $\{q_n\}_{n=1}^{+\infty}$ be a sequence in U, convergent to p, such that $\lim\limits_{n\to+\infty}\Phi^\sharp(q_n) = \limsup\limits_{q\to p}\Phi^\sharp(q)$. Since S is a compact set, for each n there is $a^n \in S$ such that $\Phi^\sharp(q_n) = F(q_n, a^n)$. Again, from the compactness of S, without loss of generality we may assume that $a^n \to a \in S$. Then from the continuity of F, we have that

$$\limsup_{q\to p}\Phi^\sharp(q) = \lim_{n\to+\infty}\Phi^\sharp(q_n) = \lim_{n\to+\infty}F(q_n, a^n) = F(p, a) \leq \Phi^\sharp(p).$$

On the other hand, since the vectors $\partial_1(p), \partial_2(p), ..., \partial_d(p)$ form a base of T_pM, we get that $\Phi^\sharp(p) = \sup\limits_{a\in S}F(p, a)$. Since $F(\cdot, a)$ is continuous on U, for any $a \in S$, from Proposition 2.7 we get that $\Phi^\sharp$ is lower semi-continuous on U, and hence

$$\liminf_{q\to p}\Phi^\sharp(q) \geq \Phi^\sharp(p) \geq \limsup_{q\to p}\Phi^\sharp(q).$$

As p was arbitrary, we conclude that $\Phi^\sharp$ is continuous on U and since we can cover M by charts, $\Phi^\sharp \in C(M)$. □

Let us now describe the connection between $\Phi^\sharp$ and the behavior of Φ from the metric point of view. Recall that

$$\operatorname{dil}(\Phi) = \sup_{p,q\in M,\ p\neq q}\frac{d_h(\Phi(p), \Phi(q))}{d_g(p, q)} \quad \text{and} \quad \operatorname{dil}_p(\Phi) = \lim_{\varepsilon\to 0}\operatorname{dil}\left(\Phi_{|B_g(p,\varepsilon)}\right).$$

We also remind that $\operatorname{dil}(\Phi) \geq \sup\limits_{p\in M}\operatorname{dil}_p$. One of the important features of the distance on the manifold is the following.

Theorem 3.17.

(i) For all $p \in M$ we have that $\operatorname{dil}_p(\Phi) = \Phi^\sharp(p)$.

(ii) $\operatorname{dil}(\Phi) = \sup\limits_{p\in M}\operatorname{dil}_p(\Phi) = \left\|\Phi^\sharp\right\|_\infty^M$.

(iii) Φ is Lipschitz if and only if $\sup\limits_{p\in M}\Phi^\sharp(p) < +\infty$.

Proof. First let us prove that $\operatorname{dil}(\Phi) \leq \left\|\Phi^\sharp\right\|_\infty^M$. The case which requires a proof is the case when $\left\|\Phi^\sharp\right\|_\infty^M < +\infty$. Consider points $p \neq q$ and a smooth curve $\gamma : [a, b] \to M$

with $\gamma(a) = p$ and $\gamma(b) = q$. Then

$$\begin{aligned} d_h(\Phi(p), \Phi(q)) &\leq L_h(\Phi \circ \gamma) = \int_a^b \rho^h_{\Phi(\gamma(t))}\left((\Phi \circ \gamma)^*_{\Phi(\gamma(t))}\right) dt \\ &= \int_a^b \rho^h_{\Phi(\gamma(t))}\left(d\Phi_{\gamma(t)}\gamma^*_{\gamma(t)}\right) dt \leq \int_a^b \Phi^\sharp(\gamma(t))\, \rho^g_{\gamma(t)}\left(\gamma^*_{\gamma(t)}\right) dt \\ &\leq \int_a^b \left(\sup_{w \in M} \Phi^\sharp(w)\right) \rho^g_{\gamma(t)}\left(\gamma^*_{\gamma(t)}\right) dt = \left\|\Phi^\sharp\right\|^M_\infty \int_a^b \rho^g_{\gamma(t)}\left(\gamma^*_{\gamma(t)}\right) dt \\ &= \left\|\Phi^\sharp\right\|^M_\infty L_g(\gamma). \end{aligned}$$

Hence, by the definition of the distance and the arbitrariness of γ, we conclude that $d_h(\Phi(p), \Phi(q)) \leq \left\|\Phi^\sharp\right\|^M_\infty d_g(p, q)$. Thus, $\operatorname{dil}(\Phi) \leq \left\|\Phi^\sharp\right\|^M_\infty$.

Now fix $p \in M$. As a consequence of what we have just proven, for all neighborhoods U of p, we have that $\operatorname{dil}_p(\Phi) \leq \left\|\Phi^\sharp\right\|^U_\infty$. Thus, considering the continuity of $\Phi^\sharp$, we conclude that $\operatorname{dil}_p(\Phi) \leq \Phi^\sharp(p)$.

To prove the opposite inequality, fix an arbitrary neighborhood U of p. Let $\gamma : (a, b) \to U$ be a smooth curve, with $\gamma(0) = p$. Then by Proposition 3.14,

$$\begin{aligned} \operatorname{dil}_{|U} \Phi &\geq \lim_{t \to 0} \frac{d_h(\Phi(p), \Phi(\gamma(t)))}{d_g(p, \gamma(t))} \\ &= \lim_{t \to 0} \frac{d_h(\Phi(p), \Phi(\gamma(t)))}{t} \frac{t}{d_g(p, \gamma(t))} = \frac{\rho^h_{\Phi(p)}\left(d\Phi_p\left(\gamma^*_p\right)\right)}{\rho^g_p\left(\gamma^*_p\right)}. \end{aligned}$$

Hence, since U and γ were arbitrary,

$$\operatorname{dil}_p(\Phi) \geq \sup_\gamma \frac{\rho^h_{\Phi(p)}\left(d\Phi_p\left(\gamma^*_p\right)\right)}{\rho^g_p\left(\gamma^*_p\right)} = \Phi^\sharp(p).$$

Combining the obtained inequalities, we have proven part (i). Part (ii) holds since $\sup_{p \in M} \Phi^\sharp(p) = \sup_{p \in M} \operatorname{dil}_p(\Phi) \leq \operatorname{dil}(\Phi) \leq \left\|\Phi^\sharp\right\|^M_\infty$. Part (iii) is an immediate corollary of (ii) and (i). □

Combining this theorem with the part (iii) of Proposition 2.10, we obtain the following:

Corollary 3.18. *If for some Riemannian manifold (L, f), Ψ is a smooth isometry from L onto M, then for each $p \in L$ we have that $(\Phi \circ \Psi)^{\sharp}(p) = \Phi^{\sharp}(\Psi(p))$.*

Now we can reformulate Proposition 2.11, considering the properties of Riemannian manifolds and the smooth maps between them.

Corollary 3.19. *Let $\mathfrak{F} \subset \mathcal{C}^{\infty}(M, N)$ be such that $\left\{\Phi^{\sharp} | \Phi \in \mathfrak{F}\right\}$ is a bounded set in $\mathcal{C}(M)$.*

(i) If for all $p \in M$ the set $\{\Phi(p) | \Phi \in \mathfrak{F}\}$ is precompact in N, then $\mathfrak{F}$ is also precompact in $\mathcal{C}(M, N)$.

(ii) If N is a complete metric space and there exists $p \in M$ such that $\{\Phi(p) | \Phi \in \mathfrak{F}\}$ is bounded in N, then $\mathfrak{F}$ is precompact in $\mathcal{C}(M, N)$.

Proof. Let $p \in M$. Since manifolds are locally compact metric spaces, p has a precompact neighborhood U. Since $\left\{\Phi^{\sharp} | \Phi \in \mathfrak{F}\right\}$ is a bounded set in $\mathcal{C}(M)$, there exists $C > 0$, such that $C \geq \sup\limits_{\Phi \in \mathfrak{F}} \left\|\Phi^{\sharp}\right\|_{\infty}^{U} = \sup\limits_{\Phi \in \mathfrak{F}} \operatorname{dil}\left(\Phi_{|U}\right)$. Hence, the dilations of the members of $\mathfrak{F}$ are locally uniformly bounded, and hence $\mathfrak{F}$ satisfies the main condition of Proposition 2.11. The condition of part (i) duplicates the condition of part (i) of that proposition, while from Theorem 3.8, the metric space N from part (ii) satisfies the condition of part (ii) of Proposition 2.11. □

3.3 Complex Manifolds

In this section we will adapt the concepts of differential geometry to the complex case. First, consider the following definitions, which are analogous to the appropriate definitions in the real case. Let M be a topological space.

- An *analytic structure* (or an *atlas*) of dimension d on M is a collection $\mathfrak{F} = \{(U_{\alpha}, \varphi_{\alpha}) | \alpha \in I\}$ of charts into $\mathbb{C}^d$, such that $M = \bigcup_{\alpha \in I} U_{\alpha}$ and such that for each $\alpha, \beta \in I$, the *transition* map $\varphi_{\beta}\varphi_{\alpha}^{-1}$ is an analytic map between $\varphi_{\alpha}(U_{\alpha} \bigcap U_{\beta})$ and $\varphi_{\beta}(U_{\alpha} \bigcap U_{\beta})$.

- A d *dimensional analytic manifold* is a pair of a Hausdorff second-countable topological space M and a differential structure of dimension d on M.

- Let M and N be analytic manifolds. The map $\Phi : M \to N$ is called *analytic* if for each chart (U, φ) on M and each chart (V, ψ) on N, the *transition* map $\psi \circ \Phi \circ \varphi^{-1}$ is analytic on $\varphi^{-1}\left(U \bigcap \Phi^{-1}(V)\right)$.

The set of analytic maps between M and N will be denoted by $\mathcal{H}(M, N)$.

We will use the construction of the tangent space in the analytic case which is the same as in the smooth case. However, note that now the vector spaces are complex and the tangent vectors are complex-valued functionals. The definition of the differential of the analytic map is identical to the one in the smooth case. If $U \subset \mathbb{C}$ is open in $\mathbb{C}$ and $h \in \mathcal{H}(U, M)$, h is called a *complex curve* in M. In the same way as a smooth curve, h generates a tangent vector h_z^*, where $z \in Ran(h)$.

Finally, it is clear that all analytic manifolds or maps are also smooth. However the dimensions of the manifolds and the tangent spaces will be doubled.

A *Complex Riemannian metric* is a correspondence H, which in each point $p \in M$ gives rise to a scalar product H_p acting on T_pM, such that for any chart (U, φ) in p and $j, k \in \overline{1, d}$ the function $H_p\left(\partial_j(p), \partial_k(p)\right)$ is analytic on U.

Observe that from any Complex Riemannian metric we can construct a Riemanian one, using the formula $g_p(u, v) = \mathrm{Re} H_p(u, v)$, where $p \in M$ and $u, v \in T_pM$. Since $g_p(u, u) = H_p(u, u)$, we can transfer the "metric" theorems from the smooth case to the analytic one directly.

The first natural example of an analytic manifold is $\mathbb{C}^d$ with the identity chart. In the same way as in the real case, the tangent space at any point $z \in \mathbb{C}^d$ consists of linear combinations of the vectors $\partial_k(z)$, which is the functional of taking the k-th partial (complex) derivative at z, for $k \in \overline{1, d}$. The Euclidean metric on $\mathbb{C}^d$ is a Complex Riemannian metric, defined by $H_z\left(\partial_j(z), \partial_k(z)\right) = \delta_k^j$.

As usually, open subsets of the analytic manifolds inherit all structures of the initial ones.

Note, that we will sometimes use results, obtained for the real case, when working with complex manifolds, if they are *exact* analogues.

We will state the following theorem, revealing some important features of the analytic manifolds, which are absent within the smooth ones.

Theorem 3.20. *Let M and N be analytic manifolds with some Complex Riemannian metrics. Then:*

(i) $\mathcal{H}(M,N)$ is a closed subspace in $\mathcal{C}(M,N)$ and $\Phi \to \Phi^{\sharp}$ is a continuous correspondence between $\mathcal{H}(M,N)$ and $\mathcal{C}(M)$.

(ii) $\mathfrak{F} \subset H(M,N)$ is precompact if and only if $\left\{\Phi^{\sharp} \,|\Phi \in \mathfrak{F}\right\}$ is bounded in $\mathcal{C}(M)$ and for each $p \in M$ we have that $\{\Phi(p)\,|\Phi \in \mathfrak{F}\}$ is precompact in N.

(iii) If N is a complete metric space then $\mathfrak{F}$ is precompact if and only if $\left\{\Phi^{\sharp} \,|\Phi \in \mathfrak{F}\right\}$ is bounded in $\mathcal{C}(M)$ and there exists $p \in M$ such that $\{\Phi(p)\,|\Phi \in \mathfrak{F}\}$ is bounded in N.

The proof of part (i) is based on the Weierstrass theorems, which will be stated later (4.2). The technics of the proof are similar to the proof of Lemma 3.16. The sufficiency of the parts (ii) and (iii) follows directly from part (i) of Corollary 3.19. To proof the necessity of part (ii), observe that from part (i) and the precompactness of $\mathfrak{F}$, the set $\left\{\Phi^{\sharp} \,|\Phi \in \mathfrak{F}\right\}$ has to be precompact in $\mathcal{C}(M)$, while the set $\{\Phi(p)\,|\Phi \in \mathfrak{F}\}$ has to be precompact in N, for for each $p \in M$. The necessity of part (iii) immediately follows from part (ii).

Chapter 4

Normed Spaces of Analytic Functions

4.1 Some General Facts

A *domain* G is an open connected subset of $\mathbb{C}^d$. Let $\mathcal{C}(G)$ denote the set of all continuous complex-valued functions on G. We will consider $\mathcal{C}(G)$ as a topological vector space with operations defined pointwise and with the open-compact topology. Some properties of this topology were mentioned in the Chapter 2. Let us recall that it is generated by the family of seminorms defined by

$$\|h\|_K = \sup_{z \in K} |h(z)| = \left\| h_{|K} \right\|_{\mathcal{C}(K)},$$

where $K \subset G$ is compact and $h \in \mathcal{C}(G)$. Note that G can be exhausted by by the compact sets. For example, $K_n := \left\{ z \in G \,\middle|\, |z| \leq n, dist\,(z, \partial G) \geq \frac{1}{n} \right\}$ forms an exhaustive sequence. Hence we can apply results from Section 2.4 to $\mathcal{C}(G)$. Also, G is a complex manifold, so we can use some results from the Chapter 3.

Now, let us move to $\mathcal{H}(G)$, which is the subset of $\mathcal{C}(G)$ consisting of all analytic functions on G. Description of basic properties of analytic functions can be found in [18]. We would like to mention only one of them (see [17]):

Proposition 4.1. *Let $f \in \mathcal{H}(G)$, $z \in G$ and let $r \in (0, dist(z, \partial G)]$. Then*

$$f(z) = \frac{1}{\lambda(B_{\mathbb{C}^d}(z,r))} \int_{B_{\mathbb{C}^d}(z,r)} f d\nu,$$

where ν is the usual Lebesgue measure on G.

Note that G is completely contained into a single identity chart, and hence, for example, for $f \in \mathcal{H}(G)$ and $z \in G$ the coordinate representation of $\nabla_z f \in (T_z G)^*$ in the basis of $\partial_1(z), \partial_2(z), ..., \partial_d(z)$ is $(\partial_1(z) f, \partial_2(z) f, ..., \partial_d(z) f)$. In particular, this means that $\|\nabla_z f\|^*_{\mathbb{C}^d} = \left(\sum_{k=1}^{d} |\partial_k(z) f|^2\right)^{\frac{1}{2}}$.

The space $\mathcal{H}(G)$ has some significant properties, which are crucial for our study. We start with two classical Weierstrass Theorems (for the proofs see [18]).

Theorem 4.2.

(i) $\mathcal{H}(G)$ is a closed subspace of $\mathcal{C}(G)$;

(ii) For each $k \in \overline{1,d}$ the correspondence $f \to \partial_k f$ is a bounded operator on $\mathcal{H}(G)$.

The Weierstrass Theorems have some important consequence. From the part (i) we immediately conclude that $\mathcal{H}(G)$ is a Frechet space. Part (ii) and the discussion above justify that the correspondence $f \to \|\nabla_{\cdot} f\|^*_{\mathbb{C}^d}$ is continuous. The next theorem reveals the speciality of $\mathcal{H}(G)$ from the functional analytic point of view.

Theorem 4.3. *(Montel) $\mathfrak{F} \subset \mathcal{H}(G)$ is precompact if and only if it is bounded.*

Proof. We only need to prove sufficiency. Any $f \in \mathcal{H}(G)$ can be considered as a map between the Riemannian manifolds G equipped with the usual Euclidean metrics $\| \ \|_{\mathbb{C}^d}$ on each tangent space and $\mathbb{C}$. Then, from Example 3.15, for $z \in G$ we have that $f^\sharp(z) = \|\nabla_z f\|^*_{\mathbb{C}^d}$, and hence the correspondence $f \to f^\sharp$ is continuous. Now we can conclude that, as $\mathfrak{F}$ is bounded in $\mathcal{H}(G)$, for any point $z \in G$, the set $\{f(z) \,|\, f \in \mathfrak{F}\}$ is bounded in $\mathbb{C}$, while the set $\left\{f^\sharp \,|\, f \in \mathfrak{F}\right\}$ is bounded in $\mathcal{C}(G)$. Finally, the last two conditions imply the compactness of $\mathfrak{F}$ by Corollary 3.19. $\square$

Recall that the point evaluation in $z \in G$ is a bounded linear functional defined by $\kappa_z(f) = f(z)$.

Let X be a linear subset of $\mathcal{H}(G)$ and let $\| \ \|$ be a norm on X. X is called a *normed space of analytic functions* (NSAF), if the inclusion map from X into $\mathcal{H}(G)$ is continuous. In the following four propositions we will adapt some properties of the NSAF's to NSAF's, considering Montel's Theorem. Note that most of the proofs will just refer to the corresponding result of Section 2.4.

Proposition 4.4. *Let X be a linear subset of $\mathcal{H}(G)$ and let $\| \ \|$ be a norm on X. Then the following conditions are equivalent:*

(i) X is a NSAF;

(ii) The convergence in X implies convergence in $\mathcal{H}(G)$;

(iii) The bounded sets in X are precompact in $\mathcal{H}(G)$;

(iv) For each compact set $K \subset G$ there exists $C_K > 0$ such that $\|\kappa_z\| \leq C_K$ for each $z \in K$;

(v) The correspondence $z \to \|\kappa_z\|$ is continuous on G;

(vi) The correspondence $z \to \kappa_z$ is continuous on G.

Proof. Observe that the implications (vi)⇒(v)⇒(iv)⇒(i) were proven in Proposition 2.28, while (i)⇔(ii) is obvious.

Condition (iii) means that the inclusion map from X into $\mathcal{H}(G)$ is compact. Hence (iii)⇒(vi) was proven in the part (i) of Proposition 2.30.

Finally, by the Montel's Theorem, boundedness and precompactness coincide in $\mathcal{H}(G)$, and so boundedness and compactness coincide for the linear operators into $\mathcal{H}(G)$. Thus if X is a NSAF, the inclusion map is automatically compact, which justifies (i)⇒(iii). □

We can add two extra conditions for Banach spaces. Note, that the next proposition is partially taken from [5]; in this book they consider Banach Spaces of Analytic Functions, which are defined through the condition (i).

Proposition 4.5. *Let X be a linear subset of $\mathcal{H}(G)$ and let $\| \ \|$ be a norm on X, such that X is a Banach space with respect to this norm. Then the following conditions are equivalent:*

(i) For each $z \in G$ we have that $\kappa_z \in X^*$;

(ii) X is a NSAF;

(iii) Weak convergence in X implies convergence in $\mathcal{H}(G)$.

Proof. (iii)⇒(ii)⇒(i) is obvious, (i)⇒(ii) follows from the part (iv) of Proposition 2.28, while (ii)⇒(iii) follows from the part (ii) of Proposition 2.30. □

The next proposition contains some remaining miscellaneous facts about NSAF's. Parts (ii) and (iii) are taken from [5].

Proposition 4.6.

(i) Let X be a NSAF with the norm $\| \ \|$. Assume that the extension of $\| \ \|$ to $\mathcal{H}(G)$ given by $\|f\| = +\infty$ for all $f \in \mathcal{H}(G) \setminus X$ is a lower semi-continuous functional on $\mathcal{H}(G)$. Then X is a Banach space.

(ii) Let X be a linear subset of $\mathcal{H}(G)$ having a norm, such that the span of $\{\kappa_z \,|z \in G\}$ is a dense subset of X^*. Then, if $\{f_n\}_{n=1}^{+\infty}$ is bounded sequence in X, which weakly converges in $\mathcal{H}(G)$ to some $f \in X$, we have that $f_n \xrightarrow{w} f$ in X.

(iii) Let X be a linear subset of $\mathcal{H}(G)$ and let $\| \ \|$ be a norm on X. If X is reflexive and the point evaluations are continuous functionals on X, then for bounded sequences in X, weak convergence is equivalent to the convergence in $\mathcal{H}(G)$.

Proof. The part (i) is proven in the same way as Proposition 2.29 by using the part (ii) of Lemma 2.24. Part (ii) is stated in the same way as Lemma 2.27, without changes.

(iii): The previous proposition implies, that under our assumptions X is NSAF, and hence it is compactly embedded in $\mathcal{H}(G)$. Thus from part (iii) of Proposition 2.30, the weak convergence is equivalent to the convergence in $\mathcal{H}(G)$. □

Part (i) of the preceding proposition can be viewed in a following way. In order to define a NSAF, we often start with a condition, which can be expressed through a functional. Namely, we consider the set of all analytic functions, where the established functional has finite values, and then this functional plays the role of a norm

on the obtained set. In fact we have shown that if our functional possesses a natural property of lower semi-continuity, the constructed space is always Banach.

This construction was inspired by [21], where it was proven that the Bloch space is Banach via a procedure, implicitly similar to ours.

Finally let $\widehat{G}$ be another domain. Consider the following properties of operators between NSAF's. Note, that part (ii) is known from [22].

Proposition 4.7. *Let X and Y be normed spaces of analytic functions on G and $\widehat{G}$ respectively, and let T be an operator from X into Y, continuous with respect to the open-compact topologies, restricted to X and Y. Then:*

(i) If X and Y are Banach spaces, then T is bounded with respect to the norms on this spaces.

(ii) T is compact with respect to the strong topologies on X and Y if and only if each bounded sequence in X, which is convergent in $\mathcal{H}(G)$, is transformed into the sequence convergent in Y.

Proof. The part (i) is stated in the same way as Proposition 2.31, without changes. For deriving part (ii) from the corresponding part of that proposition, observe that the NSAF's are always compactly embedded in $\mathcal{H}(G)$ and hence in $\mathcal{C}(G)$. □

As in Proposition 2.31, we can replace "convergent" with "convergent to zero" in the formulation of the part (ii) of the preceding proposition.

4.2 The Bergman Space

The classical Bergman Space was introduced by Stefan Bergman in the first half of the 20th century for the Lebesgue measure on the unit disk in the (complex) plane. We apply some of the tools developed in the previous section to construct the Bergman Space over a general domain and with a rather general measure on it.

Let μ be a Borel measure on G, which is finite on the compact subsets of G and let $p \geq 1$. Define a functional $\| \ \|_p$ on the set of Borel complex-valued functions on

G by $\|f\|_p = \left(\int_G |f_n|^p \, d\mu\right)^{\frac{1}{p}}$, being the usual $L^p(G,\mu)$-norm. The next property of this functional will be useful for us.

Lemma 4.8. *$\| \ \|_p$ is lower semi-continuous on $\mathcal{H}(G)$.*

Proof. Let $I = \{K \subset G \,|\, K \text{ is compact}\}$ and for each $K \in I$ consider a functional on $\mathcal{H}(G)$ defined by $\beta_K(f) = \left(\int_K |f_n|^p \, d\mu\right)^{\frac{1}{p}}$. It is easy to see that β_K is a continuous seminorm on $\mathcal{H}(G)$. Since for each positive measurable function g we have that

$$\int_G g d\mu = \sup\left\{\int_K g d\mu \,|\, K \text{ is compact in } G\right\},$$

we conclude that $\|f\|_p = \sup\limits_{K \in I} \beta_K(f)$, and so by Proposition 2.7, $\| \ \|_p$ is lower semi-continuous. □

The *Bergman Space* $A^p(G,\mu)$ is the subset of $\mathcal{H}(G)$ consisting of the $L^p(G,\mu)$ functions, i.e. of the functions having finite value of $\| \ \|_p$. I.e.

$$A^p(G,\mu) = \left\{f \in \mathcal{H}(G), \|f\|_p < +\infty\right\}.$$

It is clear that $A^p(G,\mu)$ is a linear set and, since different analytic functions belong to different classes of equivalence in $L^p(G,\mu)$, the norm of the last space naturally induces the norm $\| \ \|_p$ on $A^p(G,\mu)$. In the case when $\mu = \nu$, the usual Lebesgue measure, we will denote the Bergman Space with $A^p(G)$.

In order to expect some natural properties from the Bergman Space, we have to put some conditions on the measure μ. In this section we will assume that ν is absolutely continuous with respect to μ, and $C_\mu < +\infty$, where $C_\mu = \left\|\frac{d\nu}{d\mu}\right\|_{\frac{p}{p-1}}$, in case when $p > 1$, and $C_\mu = \operatorname{ess\,sup} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, if $p = 1$. The following lemma helps us to describe some basic properties of the Bergman Space, within this assumption.

Lemma 4.9. *For each $f \in A^p(G,\mu)$ and each $z \in G$ we have that*

$$|f(z)| \leq \frac{C_\mu d! \ \pi^{-d}}{dist\,(z, \partial G)^{2d}} \|f\|_p.$$

Proof. Define $r = dist\,(z, \partial G)$. Then from Proposition 4.1, we have that

$$f(z) = \frac{1}{\nu(B(z,r))} \int_{B(z,r)} f d\nu.$$

Since $\nu\left(B\left(z,r\right)\right)=\frac{r^{2d}\pi^{d}}{d!}$, we obtain the following inequalities:

$$\begin{aligned}\left|f\left(z\right)\right| &\leq \frac{d!\ \pi^{-d}}{r^{2d}}\int_{G}\left|f\right|d\nu=\frac{d!\ \pi^{-d}}{r^{2d}}\int_{G}\left|f\right|\frac{d\nu}{d\mu}d\mu\\ &\leq \frac{d!\ \pi^{-d}}{r^{2d}}\left(\int_{G}\left|f\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{G}\left(\frac{d\nu}{d\mu}\right)^{q}d\mu\right)^{\frac{1}{q}}=\frac{C_{\mu}d!\ \pi^{-d}}{r^{2d}}\left\|f\right\|_{p},\end{aligned}$$

where $q=\frac{p}{p-1}$ and the third step follows from the Holder's inequality. □

Theorem 4.10.

(i) $A^{p}\left(G,\mu\right)$ is a Banach space of analytic functions. Moreover, it is a closed subspace of $L^{p}\left(G,\mu\right)$;

(ii) For each $z\in G$ the norm of κ_{z}, as a linear functional on $A^{p}\left(G,\mu\right)$, does not exceed $\frac{C_{\mu}d!\ \pi^{-d}}{dist(z,\partial G)^{2d}}$.

Proof. Part (ii) follows directly from the previous lemma.

(i): Let $K\subset G$ be compact. Define $R=dist\left(K,\partial G\right)$. Then for each $z\in K$ we have that $dist\left(z,\partial G\right)\geq R$ and hence the norms of the point evaluations in the points of K do not exceed $\frac{C_{\mu}d!\ \pi^{-d}}{R^{2d}}$. From the arbitrariness of K, the Bergman Space satisfies the condition (iv) of Proposition 4.4, thus it is a NSAF. Since in Lemma 4.8 we have proven that $\|\ \|_{p}$ is lower semi-continuous on the whole $\mathcal{H}\left(G\right)$, by the part (i) of Proposition 4.6 we obtain that $A^{p}\left(G,\mu\right)$ is a Banach space. As a consequence, it must be a closed subspace of $L^{p}\left(G,\mu\right)$. □

Corollary 4.11.

(i) In case when $p>1$, for each sequence bounded in $A^{p}\left(G,\mu\right)$ weak convergence is equivalent to the convergence in $\mathcal{H}\left(G\right)$.

(ii) In case when $p=1$, weak convergence in $A^{p}\left(G,\mu\right)$ coincides with strong convergence (and hence $A^{1}\left(G,\mu\right)$ has the Schur Property).

Proof. (i): As we have just proven, the point evaluations are continuous functionals on $A^{p}\left(G,\mu\right)$. For $p>1$ the space $L^{p}\left(G,\mu\right)$ is reflexive, and so is its closed subspace $A^{p}\left(G,\mu\right)$. Hence, the desired statement follows from part (iii) of Proposition 4.6.

(ii): Assume that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to 0 weakly. In order to show strong convergence, it is sufficient to prove that f_n also converges in measure on each subset of finite measure (see [6]). Let $E \subset G$ be such set and let $\varepsilon, \delta > 0$. Since G is compactly exhausted, there exists a compact set K, such that $\mu(E \backslash K) < \varepsilon$. As the weak convergence in $A^1(G, \mu)$ implies the convergence on K, there exists $N \in \mathbb{N}$, such that for each $n > N$ we have that $\|f_n\|_{\infty}^{K} < \delta$ and hence $\mu(\{z \in E \,||f_n(z)| > \delta\}) \leq \mu(E \backslash K) < \varepsilon$. The arbitrariness of ε ensures that $\mu(\{z \in E \,||f_n(z)| > \delta\}) \to 0$, while the arbitrariness of δ provides the convergence of f_n in measure on E. $\square$

Note that part (ii) of the preceding proposition was known for the classical case. However, the proof was given through the isomorphism with l^1.

Now we will explore further the case when $p = 2$. The norm of $A^2(G, \mu)$ is induced be the scalar product of $L^2(G, \mu)$ and hence $A^2(G, \mu)$ is a Hilbert space. The next proposition describes the so called *reproducing* properties of the Bergman Space.

Proposition 4.12.

(i) There is a unique function $K \in \mathcal{C}(G \times G)$, analytic in the first variable and anti-analytic in the second, such that for $z \in G$ and $f \in A^2(G, \mu)$ we have that

$$f(z) = \int_G f(w) K(z, w) \, d\mu(w).$$

(ii) $K(z, z) = \|\kappa_z\|_2^2 = \|K(\cdot, z)\|_2^2 = \left\|\overline{K(z, \cdot)}\right\|_2^2$, and so $K(z, z) > 0$, for each $z \in G$.

(iii) For each $w \in G$ the function $K(\cdot, w)$ is an element of $A^2(G, \mu)$ which maximizes the value of the functional $f \to \frac{f(w)}{\|f\|_2}$.

(iv) If $\{e_n\}_{n=1}^{+\infty}$ is an orthonormal basis of $A^2(G, \mu)$, then

$$K(z, w) = \sum_{n=1}^{+\infty} e_n(z) \overline{e_n(w)} = \overline{K(w, z)},$$

and the series converges in $\mathcal{C}(G \times G)$.

(v) Let P be the orthogonal projection from $L^2(G,\mu)$ onto $A^2(G,\mu)$. Then for $f \in L^2(G,\mu)$ we have that

$$Pf(z) = \int_G f(w)\, K(z,w)\, d\mu(w).$$

Proof. (i): Since for each $z \in G$ the linear functional κ_z is bounded on the Hilbert space $A^2(G,\mu)$, from the Riesz representation theorem, there is a unique $k_z \in A^2(G,\mu)$, such that for all $f \in A^2(G,\mu)$ we have that

$$f(z) = \kappa_z(f) = \langle f, k_z \rangle = \int_G f\overline{k_z} d\mu.$$

Hence, for the required equality we can define $K(z,w) = \overline{k_z(w)}$. As

$$K(z,w) = \overline{k_z(w)} = \overline{\langle k_z, k_w \rangle} = \langle k_w, k_z \rangle = k_w(z) = \overline{K(w,z)},$$

we conclude that $K(z,w)$ is analytic in z and anti-analytic in w.

(ii): Follows from $K(z,z) = \overline{\langle k_z, k_z \rangle} = \|k_z\|_2^2 = \|\kappa_z\|_2^2 > 0$.

(iii): Since $K(\cdot,w) = k_w \in A^2(G,\mu)$, from the Cauchy-Schwarz inequality

$$f(w) = \langle f, K(\cdot,w) \rangle \leq \|f\|_2 \|K(\cdot,w)\|_2,$$

while

$$\frac{K(w,w)}{\|K(\cdot,w)\|_2} = \frac{\|K(\cdot,w)\|_2^2}{\|K(\cdot,w)\|_2} = \|K(\cdot,w)\|_2.$$

(iv) By Parseval's identity (see [16]),

$$\begin{aligned} K(z,w) &= k_w(z) = \langle k_w, k_z \rangle = \sum_{n=1}^{+\infty} \langle k_w, e_n \rangle \overline{\langle k_z, e_n \rangle} \\ &= \sum_{n=1}^{+\infty} \langle e_n, k_z \rangle \overline{\langle e_n, k_w \rangle} = \sum_{n=1}^{+\infty} e_n(z) \overline{e_n(w)}. \end{aligned}$$

Since $\sum_{n=1}^{+\infty} |e_n(z)|^2 = \sum_{n=1}^{+\infty} e_n(z) \overline{e_n(z)}$ converges pointwise to $K(z,z)$, and as all terms are positive, Dini's Theorem implies convergence on all compact subsets of G. Hence $\sum_{n=1}^{+\infty} e_n(z) \overline{e_n(w)}$ converges in $\mathcal{C}(G \times G)$ absolutely, because for each $n \in \mathbb{N}$ we have that $\left| e_n(z) \overline{e_n(w)} \right| \leq |e_n(z)|^2 + |e_n(w)|^2$.

(v): For $f \in L^2(G,\mu)$ and $z \in G$, by the definition of the projection,

$$Pf(z) = \langle Pf, k_z \rangle = \langle f, k_z \rangle = \int_G f(w)\, K(z,w)\, d\mu(w).$$

□

4.3 Invariant Metrics On Bounded Domains

Throughout this section we will assume that G is a bounded domain in $\mathbb{C}^d$. Observe that $\mu = \nu$ satisfies the conditions established in the previous section and hence $A^2(G)$ is a nontrivial Hilbert Space. In this case the function K, constructed in Proposition 4.12 is called the *Bergman Kernel* of the domain G. The Bergman Kernel possesses an invariant property which allows us to construct an invariant metric on the domain. Namely, the *Bergman metric* on G is the Complex Riemannian metric given by the relations

$$H_z\left(\partial_j(z), \partial_k(z)\right) = \frac{1}{2}\frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z_k}}.$$

Note, that the following two propositions are gathered from [11] and [19].

Proposition 4.13. *For each $z \in G$ the bilinear form H_z is Hermitean, non-degenerate and positive definite.*

Proof. The next equalities ensure that H_z is Hermitean:

$$\begin{aligned}\overline{\frac{\partial^2 \log K}{\partial z_j \partial \overline{z_k}}} &= \overline{\frac{\partial}{\partial z_j}\frac{\partial \log K}{\partial \overline{z_k}}} = \frac{\partial}{\partial \overline{z_j}}\left(\overline{\frac{\partial \log K}{\partial \overline{z_k}}}\right) \\ &= \frac{\partial}{\partial \overline{z_j}}\frac{\partial \overline{\log K}}{\partial z_k} = \frac{\partial}{\partial \overline{z_j}}\frac{\partial \log K}{\partial z_k} = \frac{\partial^2 \log K}{\partial \overline{z_j}\partial z_k}.\end{aligned}$$

In order to prove the positivity, fix some $z \in G$. Let l be a complex curve $l(a) = z + au$, where $u = (u_1, u_2, ..., u_d)^T \in \mathbb{C}^d$ and $a \in \mathbb{C}$. By the chain rule for $K \circ l$, we have that

$$\frac{\partial^2 (\log K \circ l)(0)}{\partial a \partial \overline{a}} = \sum_{j,k=1}^{d} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z_k}} u_j \overline{u}_k.$$

Therefore, it has to be proven that $\frac{\partial^2 (\log K \circ l)(0)}{\partial a \partial \overline{a}}$ is positive for all $u \in \mathbb{C}^d \setminus \{0\}$. Direct computation, based on part (iv) of Proposition 4.12, implies that

$$\frac{\partial^2 (\log K \circ l)(0)}{\partial a \partial \overline{a}} = \frac{1}{K^2(z,z)}\left(\sum_{n=1}^{\infty}|f_n|^2 \sum_{n=1}^{\infty}|f_n'|^2 - \sum_{n=1}^{\infty} f_n' \overline{f_n} \sum_{n=1}^{\infty} \overline{f_n'} f_n\right)(0),$$

where $f_n = e_n \circ l$. Consider the sequences $\{f_n(0)\}_{n=1}^{+\infty}, \{f_n(0)\}_{n=1}^{+\infty} \in l^2$. Since $\sum\limits_{n=1}^{\infty} |f_n(0)|^2 = K(z,z) > 0$, we get that the first of these sequences is nontrivial. Hence, by the "classical" Cauchy-Schwarz inequality we have that

$$\sum_{n=1}^{\infty} |f_n(0)|^2 \sum_{n=1}^{\infty} |f_n'(0)|^2 \geq \sum_{n=1}^{\infty} f_n'(0)\overline{f_n(0)} \sum_{n=1}^{\infty} \overline{f_n'(0)} f_n(0),$$

and the equality is possible if and only if there exists $b \in \mathbb{C}$, such that $f_n'(0) = bf_n(0)$, for each $n \in \mathbb{N}$. Suppose that we have the equality. By the Weierstrass Theorem and since $A^2(G)$ is a NSAF, the correspondence $g \to (g \circ l)'(0)$ is continuous on $A^2(G)$. Since $\{e_n\}_{n=1}^{+\infty}$ is an orthonormal basis of this space, we get that $(g \circ l)'(0) = bg(z)$, for any $g \in A^2(G)$. Substituting $g(w) = \langle w - z, u\rangle_{\mathbb{C}^d}$, we obtain $\|u\|_{\mathbb{C}^d}^2 = 0$, which leads to contradiction. $\square$

If Φ is a differentiable map from G into $\mathbb{C}^d$ then $J_\Phi(z)$ stands for the Jacobian of Φ at z, i.e. the determinant of the matrix from the partial derivatives at point z. The same notation is used for the real case. Note that Φ can be also considered as a map from $G \subset \mathbb{R}^{2d}$ into $\mathbb{R}^{2d}$. Although in this case the "real" and the "complex" Jacobians will not coincide, for analytic maps they are connected by the equality $J_\Phi^{\mathbb{R}}(z) = \left|J_\Phi^{\mathbb{C}}(z)\right|^2$.

Proposition 4.14. *Let $\widehat{G}$ be another bounded domain with the Bergman Kernel $\widehat{K}$ and the Bergman metric $\widehat{H}$. Assume that $\Phi \in \mathcal{H}\left(G, \widehat{G}\right)$ is a biholomorphism, then:*

(i) $K(z,w) = \widehat{K}(\Phi(z), \Phi(w)) J_\Phi(z) \overline{J_\Phi(w)}$.

(ii) For $z \in G$ and u, v in the tangent space T_zG we have that

$$H_z(u,v) = \widehat{H}_{\Phi(z)}(d\Phi(u), d\Phi(v)),$$

and hence Φ is an isometry with respect to the Bergman metrics on G and $\widehat{G}$.

Proof. (i): Observe that $|J_\Phi(z)|^2$ is the real Jacobian of Φ. Hence, for each $\widehat{f} \in A^2\left(\widehat{G}\right)$, using the change of variable in the integration, we have that

$$\left\|\widehat{f}\right\|_{A^2(\widehat{G})}^2 = \int_{\widehat{G}} \left|\widehat{f}(w)\right|^2 d\nu(w) = \int_G \left|\widehat{f} \circ \Phi(z)\right|^2 |J_\Phi(z)|^2 d\nu(z) = \left\|J_\Phi\left(\widehat{f} \circ \Phi\right)\right\|_{A^2(G)}^2,$$

thus the correspondence $\widehat{f} \to J_\Phi\left(\widehat{f}\circ\Phi\right)$ is an isometry from $A^2\left(\widehat{G}\right)$ onto $A^2(G)$. It is easy to see that this isometry is onto. Let $\{\widehat{e}_n\}_{n=1}^{+\infty}$ be an orthonormal basis of $A^2\left(\widehat{G}\right)$. By what we have just proven, $\{e_n := J_\Phi(\widehat{e}_n\circ\Phi)\}_{n=1}^{+\infty}$ is an orthonormal basis of $A^2(G)$, and so

$$\begin{aligned} K(z,w) &= \sum_{n=1}^{+\infty} e_n(z)\overline{e_n(w)} = \sum_{n=1}^{+\infty} J_\Phi(z)(\widehat{e}_n\circ\Phi)(z)\overline{J_\Phi(w)(\widehat{e}_n\circ\Phi)(w)} \\ &= J_\Phi(z)\overline{J_\Phi(w)}\sum_{n=1}^{+\infty}\widehat{e}_n(\Phi(z))\overline{\widehat{e}_n(\Phi(w))} \\ &= \widehat{K}(\Phi(z),\Phi(w))J_\Phi(z)\overline{J_\Phi(w)}. \end{aligned}$$

(ii): For $j,k\in\overline{1,d}$ let $g_z^{jk} = \frac{\partial^2\log K(z,z)}{\partial z_j\partial\overline{z_k}}$, for $z\in G$ and let $\widehat{g}_w^{jk} = \frac{\partial^2\log\widehat{K}(w,w)}{\partial w_j\partial\overline{w_k}}$, for $w\in\widehat{G}$. Since $K(z,z) = \widehat{K}(\Phi(z),\Phi(z))J_\Phi(z)\overline{J_\Phi(z)}$, we have that

$$\begin{aligned} g_z^{jk} &= \frac{\partial^2\log K(z,z)}{\partial z_j\partial\overline{z_k}} = \frac{\partial^2\log\left(\widehat{K}(\Phi(z),\Phi(z))J_\Phi(z)\overline{J_\Phi(z)}\right)}{\partial z_j\partial\overline{z_k}} \\ &= \frac{\partial^2\log\widehat{K}(\Phi(z),\Phi(z))}{\partial z_j\partial\overline{z_k}} + \frac{\partial^2\log J_\Phi(z)}{\partial z_j\partial\overline{z_k}} + \frac{\partial^2\log\overline{J_\Phi(z)}}{\partial z_j\partial\overline{z_k}}. \end{aligned}$$

As $J_\Phi(z)$ is holomorphic, $\frac{\partial\log J_\Phi(z)}{\partial\overline{z_k}} = \frac{\partial\log\overline{J_\Phi}(z)}{\partial z_k} = 0$. Hence,

$$g_z^{jk} = \frac{\partial^2\log\widehat{K}(\Phi(z),\Phi(z))}{\partial z_j\partial\overline{z_k}} = \sum_{m,n=1}^{d}\widehat{g}_{\Phi(z)}^{mn}\frac{\partial\phi_m(z)}{\partial z_j}\overline{\frac{\partial\phi_n(z)}{\partial\overline{z_k}}},$$

where $\Phi(z) = (\phi_1(z),\phi_2(z),...,\phi_d(z))^T$. Thus for each $u,v\in T_zG$, we have that

$$\begin{aligned} H_z(u,v) &= \frac{1}{2}\sum_{j,k=1}^{d} g_z^{jk}dz_j(u)\overline{dz_k(v)} \\ &= \frac{1}{2}\sum_{j,k=1}^{d}\sum_{m,n=1}^{d}\widehat{g}_{\Phi(z)}^{mn}\frac{\partial\phi_m(z)}{\partial z_j}\overline{\frac{\partial\phi_n(z)}{\partial\overline{z_k}}}dz_j(u)\overline{dz_k(v)} \\ &= \frac{1}{2}\sum_{m,n=1}^{d}\widehat{g}_{\Phi(z)}^{mn}\sum_{j=1}^{d}\frac{\partial\phi_m(z)}{\partial z_j}dz_j(u)\sum_{k=1}^{d}\overline{\frac{\partial\phi_n(z)}{\partial\overline{z_k}}}\overline{dz_k(v)} \\ &= \frac{1}{2}\sum_{m,n=1}^{d}\widehat{g}_{\Phi(z)}^{mn}dw_m(d\Phi(u))\overline{dw_n(d\Phi(v))} = \widehat{H}_{\Phi(z)}(d\Phi(u),d\Phi(v)). \end{aligned}$$

□

For $z \in G$ and $v \in T_zG$ define $\lambda_z(v) = \sqrt{H_z(v,v)}$ to be the Bergman Finsler metric generated by the Bergman Riemannian metric. We denote by d_λ the Bergman distance function, generated by λ.

Example 4.15. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. We will calculate the Bergman Kernel and the Bergman metric for $\mathbb{D}$ in the way as it is done in [12]. First, recall that

$$Aut(\mathbb{D}) = \{\alpha\varphi_b \,|\alpha \in \mathbb{C}, |\alpha| = 1, b \in \mathbb{D}\}, \text{ where } \varphi_b(a) = \frac{b-a}{1-\bar{b}a}.$$

Note, that φ_b is an involution, which interchanges b and 0. Also, simple calculation show that $\varphi_b'(a) = \frac{1-|b|^2}{(1-\bar{b}a)^2}$.

Let f be a function, constantly equal to 1. Since $\mathbb{D}$ is bounded, $f \in A^2(\mathbb{D})$, and hence, by part (i) of Proposition 4.12, for each $a \in \mathbb{D}$ we have that

$$\frac{1}{\pi} = \frac{1}{\pi} f(a) = \frac{1}{\pi}\int_{\mathbb{D}} f(b)\, K(a,b)\, d\nu(b) = \frac{1}{\pi}\overline{\int_{\mathbb{D}} K(b,a)\, d\nu(b)} = \overline{K(0,a)} = K(a,0),$$

where the last equality follows from Proposition 4.1. Thus $K(a,0) = \frac{1}{\pi}$, and hence, by part (i) of Proposition 4.14, for any $a, b \in \mathbb{D}$ we have that

$$\begin{aligned} K(a,b) &= K(\varphi_b(a), \varphi_b(b))\, \varphi_b'(a)\, \overline{\varphi_b'(b)} = K(\varphi_b(a), 0)\, \frac{1-|b|^2}{(1-\bar{b}a)^2}\, \overline{\frac{1-|b|^2}{(1-\bar{b}b)^2}} \\ &= \frac{1}{\pi}\frac{1-|b|^2}{(1-\bar{b}a)^2}\frac{1-|b|^2}{(1-|b|^2)^2} = \frac{1}{\pi}\frac{1}{(1-\bar{b}a)^2}. \end{aligned}$$

Now we are able to calculate the Bergman metric. Since $\mathbb{D}$ is 1-dimensional, it is determined by the only coefficient

$$H_a(\partial_1(a), \partial_1(a)) = \frac{1}{2}\frac{\partial^2 \log K(a,a)}{\partial z_1 \partial \overline{z_1}} = \frac{1}{(1-|a|^2)^2}.$$

Observe that in particular $H_0(\partial_1(0), \partial_1(0)) = 1$, and so at the center of the disk, the Bergman metric coincides with the Euclidean one.

The Bergman metric is not the only metric having invariant properties.

The *Kobayashi-Royden pseudo-metric* on G is a Finsler pseudo-metric defined by

$$\rho_z(v) = \inf\{\alpha > 0 \,|\exists g \in \mathcal{H}(\mathbb{D}, G), g(0) = z, \alpha g_z^* = v\},$$

for $z \in G$ and $v \in T_zG$. If the set considered in the definition is empty, we define $\rho_z(v) = 0$.

Proposition 4.16. *Let $\widehat{G}$ be another domain with Kobayashi-Royden pseudo-metric $\widehat{\rho}$ and let $\Phi \in \mathcal{H}\left(G, \widehat{G}\right)$. In each $z \in G$ and each $v \in T_z G$ we have the following:*

(i) $\widehat{\rho}_{\Phi(z)}\left(d\Phi_z v\right) \leq \rho_z\left(v\right)$;

(ii) If additionally Φ is a biholomorphism, then $\widehat{\rho}_{\Phi(z)}\left(d\Phi_z v\right) = \rho_z\left(v\right)$.

Proof. (i): Let us prove first that for each $z \in G$ and each $v \in T_z G$ we have that

$$\begin{aligned} &\left\{\alpha > 0 \,|\exists g \in \mathcal{H}\left(\mathbb{D}, G\right), g\left(0\right) = z, \alpha g_z^* = v\right\} \\ &= \left\{r^{-1} \,|r > 0, \exists h \in \mathcal{H}\left(r\mathbb{D}, G\right), h\left(0\right) = z, h_z^* = v\right\}. \end{aligned}$$

Assume that for some $r > 0$ there is $h \in \mathcal{H}\left(r\mathbb{D}, G\right)$ such that $h\left(0\right) = z$ and $h_z^* = v$. Let $\alpha = r^{-1}$ and define $g \in \mathcal{H}\left(\mathbb{D}, G\right)$ by $g\left(b\right) = h\left(rb\right)$. Then $g\left(0\right) = h\left(0\right) = z$ and $\alpha g_z^* = r^{-1} r h_z^* = v$. Similarly, if for some $\alpha > 0$ there is $g \in \mathcal{H}\left(\mathbb{D}, G\right)$ such that $g\left(0\right) = z$ and $\alpha g_z^* = v$, then $r = \alpha^{-1}$ and $h \in \mathcal{H}\left(r\mathbb{D}, G\right)$ defined by $h\left(b\right) = g\left(\alpha b\right)$ will satisfy our purposes.

Now, assume that for some $r > 0$ there is $h \in \mathcal{H}\left(r\mathbb{D}, G\right)$ such that $h\left(0\right) = z$ and $h_z^* = v$. Then $g := \Phi \circ h$ belongs to $\mathcal{H}\left(r\mathbb{D}, \widehat{G}\right)$, $g\left(0\right) = \Phi\left(z\right)$ and $g_{\Phi(z)}^* = d\Phi_z v$. Hence $\rho_{\Phi(z)}\left(d\Phi_z v\right) \leq r^{-1}$. Taking the infinum among all r completes the proof.

(ii): From the part (i) we get that $\rho_z\left(v\right) \geq \widehat{\rho}_{\Phi(z)}\left(d\Phi_z v\right)$ as well as $\widehat{\rho}_{\Phi(z)}\left(d\Phi_z v\right) \geq \rho_{\Phi^{-1}(\Phi(z))}\left(d\Phi_{\Phi(z)}^{-1}\left(d\Phi_z v\right)\right) = \rho_z\left(v\right)$, which proves (ii). □

A domain G is *homogenous* if the group of its biholomorphisms $Aut\left(G\right)$ acts transitively, i.e. for any $z, w \in G$ there exists $\Phi \in Aut\left(G\right)$, such that $\Phi\left(z\right) = w$.

Proposition 4.17. *Let G be a homogenous domain. Then the Kobayashi-Royden pseudo-metric on G is non-degenerate.*

We refer the reader to [10]. The reference includes a definition of the so-called *Kobayashi pseudo-distance* and a theorem stating that on the bounded homogenous domains it is non-degenerate. On the other hand, in [15], it is proven that the Kobayashi pseudo-distance is generated by Kobayashi-Royden pseudo-metric. Since on bounded homogenous domains the pseudo-distance is non-degenerate, so is the corresponding pseudo-metric.

The next two results originate from [21].

Proposition 4.18. *If G is homogenous, there is $C_G > 0$ such that for each $z \in G$ and each $v \in T_zG$ we have that $C_G^{-1}\lambda_z(v) \leq \rho_z(v) \leq C_G\lambda_z(v)$.*

Proof. Fix some $w \in G$. Both ρ_w and λ_w are norms on the finitely-dimensional vector space T_wG. Hence, they are equivalent, i.e. there exists a constant $C_G > 0$ such that for each $v \in T_wG$ we have that $C_G^{-1}\lambda_w(v) \leq \rho_w(v) \leq C_G\lambda_w(v)$.

Now consider any $z \in G$. Since G is homogenous, there exists $\Phi \in Aut(G)$ such that $\Phi(z) = w$. Then, from part (ii) of Proposition 4.14 and part (ii) of Proposition 4.16, for any $v \in T_zG$ we have that $\rho_z(v) = \rho_w(d\Phi_z v)$ and that $\lambda_z(v) = \lambda_w(d\Phi_z v)$. Hence $C_G^{-1}\lambda_z(v) \leq \rho_z(v) \leq C_G\lambda_z(v)$. □

Corollary 4.19. *If G and $\widehat{G}$ are homogenous domains, then each $\Psi \in \mathcal{H}\left(G, \widehat{G}\right)$ is Lipschitz with respect to d_λ and $d_{\widehat{\lambda}}$, with* $\operatorname{dil}(\Psi) \leq C_G C_{\widehat{G}}$.

Proof. Let ρ and $\widehat{\rho}$ be the Royden-Kobayashi metrics on G and $\widehat{G}$ respectively. Then by part (i) of Proposition 4.16 we get that

$$\widehat{\lambda}_{\Psi(z)}(d\Psi w) \leq C_{\widehat{G}}\widehat{\rho}_{\Psi(z)}(d\Psi w) \leq C_{\widehat{G}}\rho_z(w) \leq C_G C_{\widehat{G}}\lambda_z(w),$$

and so

$$\Psi^\sharp(z) = \sup_{v \in T_zG\setminus\{0\}} \frac{\widehat{\lambda}_{\Psi(z)}(d\Psi_z(v))}{\lambda_z(v)} \leq C_G C_{\widehat{G}}.$$

Thus, using part (ii) of Theorem 3.17, we conclude that $\operatorname{dil}(\Psi) = \left\|\Psi^\sharp\right\|_\infty^G \leq C_G C_{\widehat{G}}$. □

Example 4.20. By the Schwarz-Pick Lemma (see [12]), we have that

$$\sup\{g'(0) \,|\, g \in \mathcal{H}(\mathbb{D}, \mathbb{D}), g(0) = a\} = \frac{1}{\left(1 - |a|^2\right)}.$$

Hence, since for $g \in \mathcal{H}(\mathbb{D}, \mathbb{D})$, we have that $g_a^* = g'(0)\,\partial_1(a)$, we conclude that

$$\inf\{\alpha > 0 \,|\, \exists g \in \mathcal{H}(\mathbb{D}, \mathbb{D}), g(0) = a, \alpha g_a^* = \partial_1(a)\} = \left(1 - |a|^2\right).$$

Thus, for the unit disk $\rho = \lambda$ and $C_{\mathbb{D}} = 1$.

4.4 The Bloch Space

The Bloch Space over a general domain in $\mathbb{C}^d$ was first extensively studied by Timoney in [21]. A great part of the results of the present section are reworkings of the results of the mentioned paper. However, our main way of treating the Bloch Space was introduced by Cohen and Colonna in [3].

In this section we will continue to use the notation from the previous one: G is a bounded domain in $\mathbb{C}^d$ and λ is the Bergman Finsler metric of G. In this section we will study the analytic version of the Lipschitz Space, which was considered in Section 2.5. One can expect that adding analytic structure to the Lipschitz space will cause some additional properties. Our goal is to describe those properties.

First, let us consider the metric space (G, d_λ). The choice of the Bergman distance is determined by the fact that it is generated by a Riemannian metric, which agrees with the analytic structure on the domain.

The *Bloch Space* over G is a subset of $Lip\,(G)$, consisting of the analytic functions. In other words, $\mathcal{B}\,(G) = Lip\,(G) \bigcap \mathcal{H}\,(G)$, inheriting the norm defined by

$$\|f\|_{Lip}^{p} = \mathrm{dil}\,(f) + |f\,(p)|\,,$$

for some $p \in G$. The elements of the Bloch space are called *Bloch functions*.

Recall that by Theorem 3.17, for $f \in \mathcal{H}\,(G)$ we have that $\mathrm{dil}\,(f) = \left\|f^{\sharp}\right\|_{\infty}^{G}$, where $f^{\sharp}\,(z) = \lambda_z^*\,(\nabla_z f)$, and λ_z^* is a dual norm to λ_z on $(T_z G)^*$. This fact gives us another way of defining the Bloch Space:

$$\mathcal{B}\,(G) = \left\{ f \in \mathcal{H}\,(G)\,, \left\|f^{\sharp}\right\|_{\infty}^{G} < +\infty \right\}.$$

In order to provide the Bloch space with a norm, we will use the norms of the Lipschitz space. Namely, for $z \in G$ and $f \in \mathcal{B}\,(M)$ define $\|f\|_{Lip}^{z} = \mathrm{dil}\,(f) + |f\,(z)|$. By Proposition 2.32, the topology generated by this norm does not depend on the choice of z, so we will not fix it. We will state two more properties of $\mathcal{B}\,(G)$ induced by the corresponding properties of $Lip\,(G)$.

Proposition 4.21.

(i) The Bloch Space is a Banach Space of analytic functions.

(ii) Let $E \subset \mathcal{B}(G)$ be such that $C = \sup_{f \in E} \mathrm{dil}(f) < +\infty$. Then for each $z \in G$ the set $\{f - f(z) \,|\, f \in E\}$ is bounded in $\mathcal{B}(G)$.

Proof. (i): Observe that $\mathcal{B}(G)$ is a preimage of $\mathcal{H}(G)$ via the inclusion from $Lip(G)$ into $\mathcal{C}(G)$, which is continuous by Proposition 2.34. This implies that the Bloch Space is a NSAF. On the other hand, by Theorem 4.2, $\mathcal{H}(G)$ is closed in $\mathcal{C}(G)$, and hence $\mathcal{B}(G)$ is closed in $Lip(G)$. Thus $\mathcal{B}(G)$ is a Banach space.

Part (ii) is identical to the Proposition 2.33 and hence does not require a proof. □

Till the end of the section we will assume that G is a bounded homogenous domain.

The definition of a Bloch function contains two possible disadvantages. First of all, it is formulated in terms of the Bergman Metric, which is a rather sophisticated object, usually not given explicitly. Another problem is that the dilation is a global characteristic of the function. It turns out that the homogeneity of the domain allows us to come up with a local characterization of Bloch functions in terms of the usual Euclidean metric.

Proposition 4.22. *For each $z \in G$ there exists $C_z > 0$ such that for each $f \in \mathcal{H}(G)$ we have*

$$C_z^{-1}\mathrm{dil}(f) \leq \sup\{\|\nabla_z (f \circ \Phi)\|_{\mathbb{C}^d}^* \,|\Phi \in Aut(G)\} \leq C_z \mathrm{dil}(f).$$

Proof. Since λ_z is a norm on the finitely-dimensional vector space T_zG, there is a constant $C_z > 0$, such that $C_z^{-1}\lambda_z(v) \leq \|v\|_{\mathbb{C}^d} \leq C_z\lambda_z(v)$, for all $v \in T_zG$. Then for any linear functional l acting on T_zG we have that $C_z^{-1}\lambda_z^*(l) \leq \|l\|_{\mathbb{C}^d}^* \leq C_z\lambda_z^*(l)$. Observe that from Propositions 3.15 and 3.18, for $\Phi \in Aut(G)$ we have that

$$\lambda_z^*(\nabla_z (f \circ \Phi)) = (f \circ \Phi)^\sharp(z) = f^\sharp(\Phi(z)).$$

Hence

$$C_z^{-1} f^\sharp(\Phi(z)) \leq \|\nabla_z (f \circ \Phi)\|_{\mathbb{C}^d}^* \leq C_z f^\sharp(\Phi(z)).$$

Finally, since G is homogenous, it can be represented as the set $\{\Phi(z) \,|\Phi \in Aut(G)\}$ which implies that $\operatorname{dil}(f) = \sup\left\{f^{\sharp}(\Phi(z)) \,|\Phi \in Aut(G)\right\}$. Hence, taking the supremum in the previous inequality gives us the statement of the proposition. $\square$

We will say that $f \in \mathcal{H}(G)$ has a *schlicht disk* at $z \in G$ of radius r if there exists $g \in \mathcal{H}(\mathbb{D}, G)$ such that $g(0) = z$, and such that $(f \circ g)(a) = f(z) + ra$. Let $R_f(z)$ stand for the supremum of the radii of the schlicht disks of f at z, and define $R_f = \sup\limits_{z \in G} R_f(z)$. It is easy to see that for $\Phi \in \mathcal{H}\left(\widehat{G}, G\right)$, where $\widehat{G}$ is another domain, $R_{f\circ\Phi}(w) \leq R_f(\Phi(w))$, for any $w \in \widehat{G}$, and so $R_{f\circ\Phi} \leq R_f$.

We will need the following important well-known result (for the proof see [4]).

Lemma 4.23. *(Bloch's Theorem): There exists a constant $B > 0$ such that for each $h \in \mathcal{H}(\mathbb{D})$ there is a subdisk $D \subset \mathbb{D}$ such that h is one-to-one on D and $h(D)$ contains a disk of radius $B\,|h'(0)|$.*

In other words the Bloch Theorem says that $R_h(0) \geq B\,|h'(0)|$, for each $h \in \mathcal{H}(\mathbb{D})$. This fact was proven by Andre Bloch in 1925. The name of the Bloch Space was given because of the connection of the characterization of the space and the theorem. Namely, as it will be shown in the next proposition, the Bloch Theorem provides another seminorm on $\mathcal{B}(\mathbb{D})$, equivalent to the dilation.

We remind the reader that $C_G > 0$ is a constant such that for each $z \in G$ and each $v \in T_zG$ we have that $C_G^{-1}\lambda_z(v) \leq \rho_z(v) \leq C_G\lambda_z(v)$.

Proposition 4.24. *For $f \in \mathcal{H}(G)$, $z \in G$ and $a \in \mathbb{D}$ we have that:*

(i) $C_G^{-1} f^{\sharp}(z) \leq \sup\left\{(f \circ g)^{\sharp}(a) \,|g \in \mathcal{H}(\mathbb{D}, G), g(a) = z\right\} \leq C_G f^{\sharp}(z)$;

(ii) $R_f(z) \leq \sup\left\{(f \circ g)^{\sharp}(a) \,|g \in \mathcal{H}(\mathbb{D}, G), g(a) = z\right\} \leq B^{-1} R_f(z)$.

Proof. Since from Proposition 3.18, for any $g \in \mathcal{H}(\mathbb{D}, G)$ we have that

$$(f \circ g)^{\sharp}(a) = \left(f \circ g \circ \varphi_a^{-1}\right)^{\sharp}(0),$$

where φ_a as in Example 4.15, it is sufficient to prove both (i) and (ii) for the case $a = 0$. Note, that from examples 3.15 and 4.15, we have that $(f \circ g)^{\sharp}(0) = \left|(f \circ g)'(0)\right|$.

(i): Let us first prove the upper estimate. From Corollary 4.19, for any $g \in \mathcal{H}(\mathbb{D}, G)$ we have that $\operatorname{dil}(g) \leq C_G$, and hence if $g(0) = z$, we have that

$$(f \circ g)^{\sharp}(0) \leq f^{\sharp}(z)\, g^{\sharp}(0) \leq C_G f^{\sharp}(z).$$

Now we will prove the lower estimate. Since $T_z G$ is a finitely-dimensional vector space, there exists $v \in T_z G$, such that $\lambda_z(v) = 1$ and $|\nabla_z f(v)| = f^{\sharp}(z)$. From Proposition 4.18, we get that $\rho_z(v) \leq C_G$, and hence for any $\varepsilon > 0$ there is $g \in \mathcal{H}(\mathbb{D}, G)$ such that $g(0) = z$ and $\alpha g_z^* = v$, where $\alpha \leq C_G + \varepsilon$. Then

$$\begin{aligned}(C_G + \varepsilon)(f \circ g)^{\sharp}(0) &\geq \alpha (f \circ g)^{\sharp}(0) = \alpha \left|(f \circ g)'(0)\right| = \alpha \left|g_z^*(f)\right| \\ &= \alpha \left|\nabla_z f(g_z^*)\right| = \left|\nabla_z f(\alpha g_z^*)\right| = \left|\nabla_z f(v)\right| = f^{\sharp}(z).\end{aligned}$$

As ε is arbitrary we conclude that

$$C_G \sup\left\{(f \circ g)^{\sharp}(0) \,|g \in \mathcal{H}(\mathbb{D}, G), g(0) = z\right\} \geq f^{\sharp}(z).$$

(ii): Let f have a schlicht disk of radius r at z. Then there exists $g \in \mathcal{H}(\mathbb{D}, G)$, such that $g(0) = z$ and such that $(f \circ g)(b) = f(g(0)) + rb$. But then $r = (f \circ g)^{\sharp}(0)$, and so $R_f(z) \leq \sup\left\{(f \circ g)^{\sharp}(0) \,|g \in \mathcal{H}(\mathbb{D}, G), g(0) = z\right\}$.

From Lemma 4.23, for each $g \in \mathcal{H}(\mathbb{D}, G)$ such that $g(0) = z$, we have that

$$B(f \circ g)^{\sharp}(0) = B\left|(f \circ g)'(0)\right| \leq R_{f \circ g}(0) \leq R_f(z),$$

and hence $B \sup\left\{(f \circ g)^{\sharp}(a) \,|g \in \mathcal{H}(\mathbb{D}, G), g(0) = z\right\} \leq R_f(z)$. □

Taking the supremum over $z \in G$, in the proposition gives the following.

Corollary 4.25. *For $f \in \mathcal{H}(G)$, and $a \in \mathbb{D}$ we have that:*

(i) $C_G^{-1} \operatorname{dil}(f) \leq \sup\left\{(f \circ g)^{\sharp}(a) \,|g \in \mathcal{H}(\mathbb{D}, G)\right\} \leq C_G \operatorname{dil}(f)$;

(ii) $R_f \leq \sup\left\{(f \circ g)^{\sharp}(a) \,|g \in \mathcal{H}(\mathbb{D}, G)\right\} \leq B^{-1} R_f$.

Corollary 4.26. *If $f \in \mathcal{H}(G)$ is bounded, then $f \in \mathcal{B}(G)$.*

Proof. The radii of the schlicht disks cannot exceed the half of the diameter of the range of f, which does not exceed $\|f\|_{\infty}^{G}$. Hence, the radii of the schlicht disks of f are bounded from above, which from Corollary 4.25 implies $f \in \mathcal{B}(G)$. □

Lemma 4.27. *Let $\widehat{G}$ be another bounded homogenous domain and let f be a Bloch function on G. Then for each $w \in \widehat{G}$ the set $\left\{ f \circ \Phi - f(\Phi(w)) \middle| \Phi \in \mathcal{H}\left(\widehat{G}, G\right) \right\}$ is bounded in $\mathcal{B}\left(\widehat{G}\right)$ and precompact in $\mathcal{H}\left(\widehat{G}\right)$.*

Proof. By Corollary 4.19, for each $\Phi \in \mathcal{H}\left(\widehat{G}, G\right)$, we have that

$$\operatorname{dil}(f \circ \Phi) \leq \operatorname{dil}(\Phi) \operatorname{dil}(f) \leq C_G C_{\widehat{G}} \operatorname{dil}(f).$$

Hence, by part (ii) of Proposition 4.21, the set $\left\{ f \circ \Phi - f(\Phi(q)) \middle| \Phi \in \mathcal{H}\left(\widehat{G}, G\right) \right\}$ is bounded in $\mathcal{B}\left(\widehat{G}\right)$, and since the Bloch space is NSAF, this set is precompact in $\mathcal{H}\left(\widehat{G}\right)$. □

Now we gather some of the preceding results and add few more in order to list the variety of the ways of looking at the Bloch Space.

Theorem 4.28. *(Timoney) For $f \in \mathcal{H}(G)$ the following conditions are equivalent:*

(i) $f \in \mathcal{B}(G)$;

(ii) f is a uniformly continuous map between G and $\mathbb{C}$ endowed with the Bergman and Euclidean distance respectively;

(iii) The set $\{f \circ \Phi - f(\Phi(z)) | \Phi \in Aut(G)\}$ is precompact in $\mathcal{H}(G)$ for each $z \in G$;

(iv) The set $\{f \circ \Phi - f(\Phi(z)) | \Phi \in Aut(G)\}$ is precompact in $\mathcal{H}(G)$ for some $z \in G$;

(v) $\sup\{\|\nabla_z (f \circ \Phi)\|^*_{\mathbb{C}^d} | \Phi \in Aut(G)\} < +\infty$ *for each $z \in G$;*

(vi) $\sup\{\|\nabla_z (f \circ \Phi)\|^*_{\mathbb{C}^d} | \Phi \in Aut(G)\} < +\infty$ *for some $z \in G$;*

(vii) The set $\{f \circ g - f(g(a)) | g \in \mathcal{H}(\mathbb{D}, G)\}$ is precompact in $\mathcal{H}(\mathbb{D})$ for each $a \in \mathbb{D}$;

(viii) The set $\{f \circ g - f(g(a)) | g \in \mathcal{H}(\mathbb{D}, G)\}$ is precompact in $\mathcal{H}(\mathbb{D})$ for some $a \in \mathbb{D}$;

(ix) $\sup\{\operatorname{dil}(f \circ g) | g \in \mathcal{H}(\mathbb{D}, G)\} < +\infty$;

(x) $\sup\left\{(f \circ g)^\sharp(a) | g \in \mathcal{H}(\mathbb{D}, G)\right\} < +\infty$ *for some $a \in \mathbb{D}$;*

(xi) The radii of the schlicht disks of f are bounded from above.

Proof. First, let us list what is in fact already known. Since by the definition Bloch functions are Lipschitz with respect to the Bergman and Euclidean distance, (i)⇒(ii) is obvious. (vi)⇒(i)⇒(v) is a direct consequence of Proposition 4.22, while (i)⇒(ix)⇒(vii) was proven in Lemma 4.27. (i)⇔(x)⇔(xi) is stated in the Corollary 4.25. Finally, (iii)⇒(iv), (v)⇒(vi), (vii)⇒(viii) and (ix)⇒(x) are trivial.

(ii)⇒(iii): Fix some $z \in G$. From Montel's Theorem and part (i) of Theorem 2.4, it is sufficient to prove the local uniform boundness. Since f is uniformly continuous, there is $\delta > 0$ such that for all $p, q \in G$ with $d_\lambda(p,q) < \delta$ we have that $|f(z) - f(w)| < 1$.

Let $p \in G$ and let $w \in B_\lambda(p,\delta)$. As G is connected, there is a path, which joins z with p. Choose $p = z_0, z_1, z_2, ..., z_n = z$ on the path, such that $d_\lambda(z_{k-1}, z_k) < \delta$ for each $k \in \overline{1,n}$. Also let $z_{n+1} = w$. Since automorphisms preserve the distance, for each $\Phi \in Aut(G)$ we have that $d_\lambda(\Phi(z_{k-1}), \Phi(z_k)) < \delta$ for $k \in \overline{1,n+1}$. Hence

$$\begin{aligned} |(f \circ \Phi)(w) - f(\Phi(z))| &= \left| \sum_{k=1}^{n+1} (f(\Phi(z_k)) - f(\Phi(z_{k-1}))) \right| \\ &\leq \sum_{k=1}^{n+1} |f(\Phi(z_k)) - f(\Phi(z_{k-1}))| < n+1, \end{aligned}$$

and so the set $\{f \circ \Phi - f(\Phi(z)) \,|\Phi \in Aut(G)\}$ is bounded on $B_\lambda(p,\delta)$. The arbitrariness of p ensures the local uniform boundedness.

(iv)⇒(vi): Let the set $\{f \circ \Phi - f(\Phi(z)) \,|\Phi \in Aut(G)\}$ be precompact in $\mathcal{H}(G)$ for some $z \in G$. Since the correspondence $g \to \|\nabla_z g\|^*_{\mathbb{C}^d}$ is continuous, we immediately get that $\{\|\nabla_z (f \circ \Phi)\|^*_{\mathbb{C}^d} \,|\Phi \in Aut(G)\}$ is a bounded set. By the same argument, (iii)⇒(v).

(viii)⇒(vii): Since $\{f \circ g - f(g(a)) \,|g \in \mathcal{H}(\mathbb{D}, G)\}$ is precompact in $\mathcal{H}(\mathbb{D})$, for any $b \in \mathbb{D}$ the set $\{f(g(b)) - f(g(a)) \,|g \in \mathcal{H}(\mathbb{D}, G)\}$ is bounded. Since for any $g \in \mathcal{H}(\mathbb{D}, G)$ we have that

$$f \circ g - f(g(b)) = f \circ g - f(g(a)) - (f(g(b)) - f(g(a))),$$

the set $\{f \circ g - f(g(b)) \,|g \in \mathcal{H}(\mathbb{D}, G)\}$ is bounded in $\mathcal{H}(\mathbb{D})$, and hence precompact.

□

We will dedicate the end of the chapter to some information about the composition operators. Let $\widehat{G}$ be another bounded homogenous domain and let $\Phi \in \mathcal{H}\left(\widehat{G}, G\right)$. We remind that a composition operator C_Φ is a linear transformation from $\mathcal{H}(G)$ into $\mathcal{H}\left(\widehat{G}\right)$, given by $C_\Phi(f)(z) = f(\Phi(z))$. Since the composition of holomorphic maps is holomorphic, C_Φ is continuous.

We will focus on the restriction of composition operators on the Bloch Space. The information about them will help us to characterize Bloch functions via compositions. From Corollary 4.19, $dil\Phi \leq C_G C_{\widehat{G}}$, and so $C_\Phi(\mathcal{B}(G)) \subset \mathcal{B}\left(\widehat{G}\right)$. Let $z \in G$ and $w \in \widehat{G}$. Since the Bloch Space is a subset of the Lipschitz Space, we can reformulate Proposition 2.37, for $\| \|_{Lip}^{z}$ and $\| \|_{Lip}^{w}$. In this form this fact was proven by Allen and Colonna in [1].

Proposition 4.29.

(i) C_Φ is a bounded linear operator with

$$1 \leq \|C_\Phi\| \leq \max\left\{1, d_\lambda(\Phi(w), z) + C_G C_{\widehat{G}}\right\}.$$

In particular, if $\Phi(w) = z$, then $1 \leq \|C_\Phi\| \leq \max\left\{1, C_G C_{\widehat{G}}\right\}$.

(ii) If $\Phi(w) = z$, $\operatorname{dil}(\Phi) \leq 1$ and there exists a sequence of onto isometries $\{\Psi_j\}_{n=1}^{+\infty} \subset \mathcal{H}\left(G, \widehat{G}\right)$, such that $\Phi \circ \Psi_j$ converges to identity in $\mathcal{H}(G, G)$, then $\operatorname{dil}\Phi = 1$ and C_Φ is an isometry.

We will add a sufficient condition for the composition operator to be compact. This is a result of Shi and Luo ([20]).

Theorem 4.30. *If for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Phi^\sharp(p) < \varepsilon$ whenever $dist(\Phi(p), \partial G) < \delta$, then C_Φ is compact.*

Proof. From Proposition 4.7, we have to prove that if $\{f_n\}_{n=1}^{+\infty} \subset \mathcal{B}(G)$ is an arbitrary bounded sequence in $\mathcal{B}(G)$, which converges to 0 in $\mathcal{H}(G)$, then $f_n \circ \Phi \to 0$ in $\mathcal{B}\left(\widehat{G}\right)$. Since $|f_n(\Phi(z))| \to 0$, it is sufficient to prove that $\operatorname{dil}(f_n \circ \Phi) \to 0$.

Define $M = \sup\limits_{n \in \mathbb{N}} \operatorname{dil}(f_n)$. Fix any $\varepsilon > 0$. There is $\delta > 0$ such that in case when $dist(\Phi(z), \partial G) < \delta$ we have that $\Phi^\sharp(z) < \frac{\varepsilon}{M}$, and so

$$(f_n \circ \Phi)^\sharp(z) \leq \Phi^\sharp(z) f_n^\sharp(\Phi(z)) < \frac{\varepsilon}{M} M = \varepsilon.$$

Let $K = \{z \in G \,|dist\,(\Phi\,(z)\,, \partial G) \geq \delta\}$, which is a compact set. Since $f_n \circ \Phi \to 0$ in $\mathcal{H}\,(G)$, from Theorem 3.20, $(f_n \circ \Phi)^\sharp \to 0$ uniformly on compact sets. Thus, there exists $N \in \mathbb{N}$ such that $(f_n \circ \Phi)^\sharp\,(z) < \varepsilon$, whenever $n > N$ and $z \in K$. Hence,

$$\begin{aligned} \mathrm{dil}\,(f_n \circ \Phi) &= \left\|(f_n \circ \Phi)^\sharp\right\|_\infty^G \\ &= \max\left\{\left\|(f_n \circ \Phi)^\sharp\right\|_\infty^K, \left\|(f_n \circ \Phi)^\sharp\right\|_\infty^{G\setminus K}\right\} \leq \max\{\varepsilon, \varepsilon\} = \varepsilon, \end{aligned}$$

for all $n > N$. The arbitrariness of ε provides that $\mathrm{dil}\,(f_n \circ \Phi) \to 0$. □

Composition operators are closely related to the notion of Mobius Invariance. We will not define this notion precisely, since there is a lot of different similar ways to look at it. Roughly speaking, a NSAF is considered to be Mobius invariant if it possesses some invariant property with respect to the group of composition operators with symbols from $Aut\,(G)$. The following proposition states that the Bloch Space is the maximal Mobius invariant space, in one of the possible definition of the Mobius Invariance.

Proposition 4.31. *Let $(X, \|\ \|)$ be a NSAF on G, such that for each $\Phi \in Aut\,(G)$, the operator C_Φ is bounded on X and also $C_X = \sup\{\|C_\Phi\|\,|\Phi \in Aut\,(G)\} < \infty$. Then $X \subset \mathcal{B}\,(G)$ and the inclusion from X into $\mathcal{B}\,(G)$ is bounded.*

Proof. Let $z \in G$. Since the inclusion from X into $\mathcal{H}\,(G)$ is continuous, as well as the correspondence $g \to \|\nabla_z g\|_{\mathbb{C}^d}^*$, there exists $C_1 > 0$, such that $\|\nabla_z g\|_{\mathbb{C}^d}^* \leq C_1\,\|g\|$, for any $g \in X$. Then, considering Proposition 4.22, for any $f \in X$ we have that

$$\begin{aligned} C_z^{-1}\mathrm{dil}\,(f) &\leq \sup\{\|\nabla_z\,(f \circ \Phi)\|_{\mathbb{C}^d}^*\,|\Phi \in Aut\,(G)\} \leq C_1 \sup\{\|f \circ \Phi\|\,|\Phi \in Aut\,(G)\} \\ &= C_1 \sup\{\|C_\Phi f\|\,|\Phi \in Aut\,(G)\} \leq C_1 C_X\,\|f\|\,. \end{aligned}$$

Hence $\mathrm{dil}\,(f) \leq C_z C_1 C_X\,\|f\|$. Similarly, there exists $C_2 > 0$, such that $|f\,(z)| \leq C_2\,\|f\|$, and so $\|f\|_{Lip}^z \leq (C_z C_1 C_X + C_2)\,\|f\|$. Thus the inclusion from X into $\mathcal{B}\,(G)$ is bounded. □

The preceding proposition is a technical corollary of Timoney's Theorem, and is close to the Rubel's and Timoney's result ([14]) for the classical Bloch Space. However, surprisingly, it has not appeared in [21] for the general case.

Chapter 5

Questions

We conclude the thesis with a list of questions. The generality and preciseness of formulations varies among the list. Some of the questions may have (partial) answers, but we are not aware of any reference to support that.

Questions on NSCF's (Section 2.4):

- Is it true that X is reflexive whenever $span\,\{\kappa_p\,|p \in M\} = X^*$?
- Is it true that if X is a Banach NSCF, then $\|\ \|$ is lower semi-continuous with respect to the open-compact topology on X? (I.e. does the converse of Proposition 2.29 hold).
- Is the correspondence $\kappa_p \to p$ continuous in any natural topology? (I.e. find some analogue of Proposition 2.26 and part (i) of 2.30 for the inverse map).
- What can be said about the metric $d_X\,(p,q) = \|\kappa_p - \kappa_q\|_{X^*}$?

Questions on the Lipschitz Space (Section 2.5):

- Compare $Lip\,(M)$ with the space of bounded Lipschitz functions.
- Find a criteria for the compactness of a set in $Lip\,(M)$.

- Find a criteria for weak convergence.
- Find a criteria for the compactness of a operator on $Lip\,(M)$.
- Is it true that if the composition operator on $Lip\,(M)$ is an isometry, then it is generated by an isometry?
- Is it true that if the composition operator on $Lip\,(M)$ is bounded, then it is generated by a Lipschitz map?

Questions on NSAF's (Sections 4.1 and 4.2):

- The same questions as for NSCF's are valid for the NSAF's.
- Find all borel measures μ on G such that $A^p\,(G,\mu)$ is a NSAF for some (all) $p \geq 1$.

Questions on the Bloch Space (Sections 4.4):

- Find a criteria for compactness and weak convergence in $\mathcal{B}\,(G)$.
- Is it true that in $\mathcal{B}\,(G)$, for any $z, w \in G$ we have that $d_\lambda\,(z,w) = \|\kappa_z - \kappa_w\|$? (I.e. does the analogue of Proposition 2.36 hold for the Bloch Space).
- Is the condition in Theorem 4.30 also necessary for compactness of C_Φ?

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