ONE-SIDED WEIGHTED APPROXIMATION

by

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A Thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
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Abstract

This thesis is devoted to weighted approximation by polynomials on the real line and deals with what is called one-sided weighted approximation in Approximation Theory. More precisely, we investigate the existence of a sequence of polynomials which lie above (or below) a given function and approximate it arbitrarily well. We present an overview of this subject in the L_p-spaces, starting with the early works of Freud and ending by the latest results in this area. This branch of weighted approximation was developed in the last 30-40 years and now contains a big variety of techniques and approaches. We develop new methods on one-sided approximation in the L_∞-norm and obtain several results. We also discuss a number of open questions and describe possible directions of future research in this area.

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Contents

Chapter 1

Introduction

1.1 Some history of Polynomial Approximation

Back in the 19-th century during the early industrial revolution, physics, chemistry and other natural sciences required a development of a new tool in order to construct new mechanisms, more precise and efficient. Inventions like Watt's idea of parallel motion and designing of electric filter networks were supported by new mathematical observations and methods of approximation (in particular, polynomial approximation). Great contributions were made by the famous Russian mathematicians Chebyshev and Markov.

At the same time, classical mathematics already investigated the existence of a continuous nowhere differentiable function, which generated a lot of interest in the whole mathematical community. Bolzano was the first to construct such a function. Continuous nowhere differentiable functions might seem to some as pathological. This accentuated and intensified the need for analytic approximation of certain classes of functions. As a result, this led to a fundamental theorem of Approximation Theory, which was discovered in 1885 by Weierstrass [32], one of the greatest people in mathematics. Namely, Weierstrass proved that any continuous function on an interval can be approximated by algebraic polynomials as well as desired, hence establishing what we now know as the density of polynomials (in the space of all continuous functions on an interval). Various proofs of the Weierstrass theorem

were presented by Weierstrass, Picard, Fejer, Landau, de la Valle Poussin, Runge, Lebesgue, Mittag-Leffler, Lerch, Volterra. The last of the early proofs was due to Bernstein [4] who is well known by his works on Probability Theory, Constructive function theory, mathematical foundations of genetics, etc. This proof is quite different from the previous proofs, and has had a profound impact in various areas. In [4], Bernstein introduced what we today call Bernstein polynomials.

As any great theorem, the Weierstrass Theorem was significant in its influence on the development of several mathematical areas, one of them being the field of Approximation Theory. It led to many interesting generalizations such as the Stone-Weierstrass theorem, the Muntz theorem, the Bohman-Korovkin theorem, which deal with the density of a given subset in the space of all continuous functions. In particular, this thesis is devoted to a certain extension of the approximation on an interval, namely, here we consider weighted approximation on the whole real line.

Finally, we mention that currently many modern directions in pure and applied mathematics use the methods of approximation. Among them are Numerical analysis, PDE, Learning theory, Geodesy, and more.

1.2 Preliminaries

In this section, we introduce some basic definitions and notations. For more details and references see, for example, [1, 8].

- By $C(S)$ we denote the set of all continuous functions on S. In particular, $C[a,b]$ is the space of all continuous functions f on $[a,b]$, equipped with the norm
$$\|f\|_{C[a,b]} := \max_{x\in[a,b]} |f(x)|.$$
- Let (Ω, μ) be a measure space. By $L_p(\Omega)$ we denote the space of all measurable functions $f : \Omega \to \mathbb{R}$ such that
$$\|f\|_{L_p(\Omega)} := \left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p} < \infty.$$

Everywhere below, we assume that $1 \leq p \leq \infty$ unless specifically stated otherwise.

- Π_n is the set of all algebraic polynomials of degree $\leq n$.

- Let $(X, \|\cdot\|_X)$ be a (quasi) normed space, and $Y \subset X$ be a subspace of X. The number
$$E_Y(f)_X := \inf_{y \in Y} \|f - y\|_X$$
is called the *value of best approximation* of the element $f \in X$ by the subspace Y. In particular, the value of best approximation of $f \in C[a,b]$ by polynomials of degree $\leq n$ is
$$E_n(f)_{C[a,b]} := E_{\Pi_n}(f)_{C[a,b]} = \inf_{P \in \Pi_n} \|f - P\|_{C[a,b]}.$$
An element $y^* \in Y$ is called an *element of best approximation* of $f \in X$ by the subspace Y, if
$$E_Y(f)_X = \|f - y^*\|_X.$$
If X is a Banach space and Y is a finite dimensional subspace of X, then, by Borel's Theorem [5] for any $f \in X$, there exists an element of best approximation of f by the subspace Y.

  Let Z be a subset (usually, a cone) of X (for example, Z can be the set of all monotone polynomials on $[a,b]$). Then
$$E_n(f, Z)_X = \inf_{y \in Z \cap \Pi_n} \|f - y\|_X$$
is the *error of approximation* of f by polynomials *restricted to* Z.

- A function $f \in C(\mathbb{R})$ is said to be of polynomial growth (at ∞) if there exist a polynomial P and a constant $L > 0$ such that $|f(x)| \leq L|P(x)|$, for all $x \in \mathbb{R}$.

1.3 Weighted Approximation on the Real Line

The origins of weighted approximation come from the manuscript [3] by Bernstein, where the problem that is now known as Bernstein's Approximation Problem was posed.

For a continuous $W : \mathbb{R} \to (0,1]$, denote by $C_W := C_W(\mathbb{R})$ *the space of all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{x\to\pm\infty} f(x)W(x) = 0$, equipped with the norm $\|f\|_W := \|f\|_{C_W} := \sup_{x\in\mathbb{R}} |f(x)W(x)|$. We call W a weight function. Bernstein's Approximation Problem is stated as follows:

Problem 1.1 (Bernstein's Approximation Problem). *Characterize all weights W such that, for every $f \in C_W$, there exists a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty} \|f - P_n\|_W = 0?$$

In other words, Bernstein's approximation problem asks for a description of the class of all weight functions W such that polynomials are dense in C_W. This problem was solved independently by Achieser, Mergelyan, and Pollard in 1950's (see, e.g., [20]).

Recall that a subset of the normed space is called fundamental if the closure of its linear span is the whole space. The following theorem was presented by Pollard [28]:

Theorem 1.2 (Solution of Bernstein's Approximation Problem). *Suppose that, for all $n \in \mathbb{N}$, $x^n \in C_W$. Then in order that $\{x^n\}_{n=0}^{\infty}$ be fundamental in C_W it is necessary and sufficient that*

(i) $$\int_{-\infty}^{+\infty} \frac{\log(1/W(t))}{1+t^2} dt = \infty;$$

and

(ii) *there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that, for each x,* $\lim_{n\to\infty} P_n(x)W(x) = 1$ *while* $\sup_{n\geq 1} \|P_n\|_W < \infty$.

Since polynomials are unbounded at infinity, it is natural to require W to decay sufficiently fast at infinity in order to counterweight the growth of every polynomial. The so-called Freud's weights $W_\alpha(x) = e^{-|x|^\alpha}$, which are of particular interest in this thesis, satisfy both conditions (i) and (ii) iff $\alpha \geq 1$, and therefore the linear span of

$x^n, n \in \mathbb{N}$, is dense in C_{W_α} iff $\alpha \geq 1$. More recent results regarding approximation with the exponential weights are discussed in s the survey [22] by Lubinsky. Some important facts on this subject are also mentioned in Chapter 2.

The main goal of this thesis is to investigate a special kind of weighted approximation, the so-called one-sided weighted approximation. One-sided polynomial approximation deals with the question on how well one can approximate a given function by polynomials which lie above (or below) it. Results regarding one-sided approximation on an interval in various functional spaces are presented in [2, 7, 18]. Below, we discuss a classical unweighted (uniform) approximation on an interval $[a, b]$ by algebraic polynomials. The rate of one-sided approximation in that case is equivalent to the rate of the unconstrained approximation. Here is a short proof why this is true. Let $f \in C[a, b]$. For a fixed n, let P_n be a polynomial of best approximation among all polynomials of degree $\leq n$. Weierstrass's theorem implies that $\lim_{n\to\infty} \|f - P_n\|_{C[a,b]} = 0$. For any $n \geq 0$, define polynomials $Q_n := P_n + \|f - P_n\|_{C[a,b]}$. Then, the polynomials Q_n lie above the function f. Indeed, for all $x \in [a, b]$,

$$Q_n(x) - f(x) = P_n(x) + \|f - P_n\|_{C[a,b]} - f(x) \geq \|f - P_n\|_{C[a,b]} - |f(x) - P_n(x)| \geq 0,$$

and the error of the one-sided approximation

$$\|Q_n - f\|_{C[a,b]} \leq \|Q_n - P_n\|_{C[a,b]} + \|P_n - f\|_{C[a,b]} = 2\|f - P_n\|_{C[a,b]} = 2E_n(f)_{C[a,b]}$$

converges to 0 as fast as $E_n(f)_{C[a,b]}$. Hence, one constructs one-sided approximation on an interval by sufficiently lifting P_n.

This approach fails in the case of approximation in the L_p-spaces over a finite interval, for $p < \infty$. It is impossible to get an upper estimate on the rate of the one-sided approximation on $[-1, 1]$ even in terms of $\|f\|_{L_p[-1,1]}$, since for the function f such that $f(x) = 0,\ x \neq 0,\ f(0) = 1$, its L_p-norm is 0, and according to [17], $\inf_{n\in\mathbb{N}\cup\{0\}} \inf_{P\geq f, P\in\Pi_n} \|P - f\|_{L_p[-1,1]} > 0$. Though some kinds of estimates could still be obtained (see, e.g., [17, 18]).

If one tries to generalize the problem of one-sided approximation to the real line, the following question arises:

Question 1.3. *Let $\alpha \geq 1$. Is it true that for any $f \in C_{W_\alpha}$, one can find a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \|f - P_n\|_{W_\alpha} = 0$ and $P_n(x) \geq f(x)$ ($P_n(x) \leq f(x)$), for all $x \in \mathbb{R}$?*

It turns out that the answer to this question is negative. Indeed, consider the function

$$f(x) = \sinh(x/2) = \frac{e^{x/2} - e^{-x/2}}{2},$$

which obviously belongs to C_{W_α}, for all $\alpha \geq 1$. However, there is no polynomial that lies above this function (or below it), since the exponential function grows at infinity more rapidly than any polynomial. That is why Question 1.3 requires a certain correction. In Chapters 3 and 4, following Mastroianni and Szabados [23], we deal with the so-called partially one-sided approximation, requiring that the polynomials lie above a given function only on some specific intervals and not on the whole real line. In Chapter 4, we describe the largest possible (in some sense) such intervals.

Above, we considered only weighted approximation in the space C_W. However, a natural problem is to extend Bernstein's approximation problem to the L_p-spaces, $p < \infty$. The following result is an analogue of Theorem 1.2.

Theorem 1.4 (Pollard [29]). *Let $p \geq 1$, and let the weight W be such that $x^n W(x) \in L_p(\mathbb{R})$, $n \geq 0$. Then for every measurable $f : \mathbb{R} \to \mathbb{R}$ such that $\|fW\|_{L_p(\mathbb{R})} < \infty$, there exist polynomials $\{P_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \|(f - P_n)W\|_{L_p(\mathbb{R})} = 0$ if and only if conditions (i) and (ii) of Theorem 1.2 are satisfied.*

Note that the conditions $\|fW\|_{L_p(\mathbb{R})} < \infty$ and $\lim_{n\to\infty} \|(f - P_n)W\|_{L_p(\mathbb{R})} = 0$ enforce W to satisfy $x^n W(x) \in L_p(\mathbb{R})$, $n \geq 0$. Also, notice that the condition $x^n W(x) \in L_p(\mathbb{R})$, $n \geq 0$ is equivalent to the condition $x^n W(x) \in L_p(\mathbb{R})$, $n \geq k$, $k \in \mathbb{N}$.

By the same reasoning as before, one can not expect "good" one-sided approximation over the whole real line in the L_p-space. However, for special sets of functions, there is a series of positive results on this topic. Methods derived by Nevai [25], Freud [12], Mastroianni and Szabados [23] will be introduced in Chapter 3.

It is worth mentioning that many great contributions to weighted approximation were made by Freud. In particular, Freud established one of the first and the most general results regarding one-sided approximation, which we present next.

Definition 1.5. *For a positive measure σ on the real line, its kth moment σ_k is defined as follows*

$$\sigma_k := \int_{\mathbb{R}} t^k d\sigma(t), \quad k \geq 0.$$

Note that the moments σ_k do not have to be finite.

Definition 1.6. *Let $\{s_k\}$ be a sequence of real numbers. If there is a positive measure σ on the real line such that*

$$s_k = \int_{\mathbb{R}} t^k d\sigma(t), \quad k \geq 0,$$

then we say that the moment problem associated with moments $\{s_k\}$ has a solution. If this solution is unique, the moment problem is said to be **determinate**.

Obviously, in order to make the moment problem solvable, we have to require that all s_k are non-negative, for even k. But even this condition does not guarantee the existence of a solution. For instance, for $s_0 = 1, s_1 = 2, s_2 = 1$, the moment problem is not solvable since, otherwise,

$$s_2 - 2s_1 + s_0 = \int_{\mathbb{R}} \left[t^2 - 2t + 1\right] d\sigma(t) \geq 0,$$

and, on the other hand, $s_2 - 2s_1 + s_0 = -1 < 0$.

Theorem 1.7 (Freud [13]). *Suppose that σ is a positive measure on the real line, with finite moments $\{\sigma_k\}$. Moreover, assume that the moment problem associated with $\{\sigma_k\}$ is determinate. Let $\varepsilon > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function that is Riemann-Stieltjes integrable against $d\sigma$ over every finite interval, and improperly Riemann integrable over the whole real line, and of polynomial growth at ∞. Then there exist polynomials R and S such that*

(1) $S \leq f \leq R$ *on* $\mathbb{R}$,

and

(2) $\displaystyle\int_{\mathbb{R}} (R - S) d\sigma < \varepsilon.$

Remark. If W is a weight function and $d\sigma(t) = W(t)dt$, the previous theorem establishes a result on the one-sided L_1 weighted polynomial approximation.

There is a certain connection between the theory of orthogonal polynomials and the moment problem. So-called Hamburger's and Carleman's conditions are sufficient for the moment problem to be solvable and determinate, respectively (see, e.g. [6]).

Chapter 2

Weighted unconstrained polynomial approximation on the real line

2.1 Rate of approximation

2.1.1 Notations and Definitions

Although Theorem 1.2 establishes the density of algebraic polynomials in the space C_W, another important question is about the rate of such approximation. In order to state results in this direction, we first introduce some basic notations and definitions.

- Classes of Weights:

  (1) We say that $W : \mathbb{R} \to (0, 1]$ is a *general weight* or *general weight function* if W is continuous and satisfies conditions (i), (ii) of Theorem 1.2 as well as

  $$\lim_{x \to \pm\infty} x^n W(x) = 0, \ n \geq 0. \tag{2.1}$$

  (2) We say that $W : \mathbb{R} \to (0, 1]$ is a *Freud-type weight* if $W(x) = \exp(-Q(x))$, where Q is even and continuous on $\mathbb{R}$, Q'' is continuous on $(0, \infty)$, $xQ'(x)$

is positive and increasing in $(0,\infty)$ with limits 0 and ∞ at 0 and ∞, respectively. In addition, for some $A, B > 0$,

$$A \leq \frac{xQ''(x)}{Q'(x)} \leq B, \quad x \in (0,\infty).$$

We denote by $\mathfrak{F}$ the class of all Freud-type weights. Note that any Freud-type weight is a general weight.

(3) *Freud's weights*: $W_\alpha(x) = \exp(-|x|^\alpha)$. Note that $W_\alpha \in \mathfrak{F}$, for all $\alpha > 1$.

- For any weight function W, recall that

$$C_W := C_W(\mathbb{R}) = \{f \in C(\mathbb{R}) \big| \lim_{x\to\pm\infty} f(x)W(x) = 0\},$$

and $\|f\|_W := \|f\|_{C_W} := \sup_{x\in\mathbb{R}} |f(x)W(x)|$.

- Let W be a weight function. For any function f such that $\|fW\|_{L_p(\mathbb{R})} < \infty$ if $p < \infty$ and $f \in C_W$ if $p = \infty$, let $E_n(f,W)_p := \inf_{P\in\Pi_n} \|(f-P)W\|_{L_p(\mathbb{R})}$ denote the value of best weighted approximation by polynomials of degree $\leq n$ in the L_p-norm. For convenience, we denote $E_n(f,W) := E_n(f,W)_\infty$.

- Recall that the Gamma and the Beta functions are defined as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \text{for } \operatorname{Re} z > 0,$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \text{for } \operatorname{Re} x,\ \operatorname{Re} y > 0.$$

2.1.2 Jackson-Favard inequality

Since the general weight function W decays rapidly to 0 (see condition (2.1)), it turns out that the weighted norm of any polynomial outside of the fixed finite interval can be estimated in terms of the weighted norm of this polynomial inside the interval. These observations lead to a powerful tool in weighted approximation called restricted range inequalities. We now introduce Freud's and Mhaskar-Rakhmanov-Saff's numbers, which arise in these inequalities.

Definition 2.1. *Let $W \in \mathfrak{F}$. For $n \geq 1$, let q_n be the positive root of equation*

$$n = q_n Q'(q_n). \tag{2.2}$$

Then q_n is called nth **Freud's number**. *The positive root a_n of the equation*

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt \tag{2.3}$$

is called nth **Mhaskar-Rakhmanov-Saff's number**.

Note that conditions on the function $Q(x)$ guarantee existence and uniqueness of such numbers. For instance, for Freud's numbers, the function $xQ'(x)$ is continuous on $(0, \infty)$, and has limits 0 and ∞ at 0 and ∞, respectively. Therefore, by the Intermediate Value Theorem, for all $n \geq 1$, there exists a solution of $n = xQ'(x)$ while uniqueness is guaranteed by monotonicity of $xQ'(x)$.

For Freud's weights $W_\alpha(x) = \exp(-|x|^\alpha), \alpha \geq 1$, and $n \geq 1$:

$$q_n = (n/\alpha)^{1/\alpha}, \tag{2.4}$$

and

$$a_n = \left(2^{\alpha-2} \frac{\Gamma(\alpha/2)^2}{\Gamma(\alpha)}\right)^{1/\alpha} n^{1/\alpha}. \tag{2.5}$$

Indeed, for $Q(x) = x^\alpha$, $\alpha \geq 1$, $x \geq 0$, equality (2.2) implies

$$n = q_n \alpha q_n^{\alpha-1},$$

and so (2.4) follows. For Mhaskar-Rakhmanov-Saff's numbers, recalling that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(1/2) = \sqrt{\pi},$$

and using equality (2.3) together with the duplication formula $\Gamma(z + 1/2) =$

$2^{1-2z}\sqrt{\pi}\Gamma(2z)$, we obtain

$$\begin{aligned}
n &= \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}dt = \frac{2\alpha a_n^\alpha}{\pi}\int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}}dt \\
&[u = t^2, dt = \frac{du}{2\sqrt{u}}] \\
&= \frac{\alpha a_n^\alpha}{\pi}\int_0^1 u^{(\alpha-1)/2}(1-u)^{-1/2}du = \frac{\alpha a_n^\alpha}{\pi}B\left(\frac{\alpha+1}{2},\frac{1}{2}\right) \\
&= \frac{\alpha a_n^\alpha}{\pi}\frac{\Gamma((\alpha+1)/2)\Gamma(1/2)}{\Gamma(\alpha/2+1)} = \frac{\alpha a_n^\alpha}{\pi}\frac{\Gamma(\alpha+1)\pi}{2^\alpha\Gamma(\alpha/2+1)^2} \\
&= \frac{\alpha a_n^\alpha \alpha\Gamma(\alpha)}{2^\alpha(\alpha/2)^2\Gamma(\alpha/2)^2} = \frac{a_n^\alpha\Gamma(\alpha)}{2^{\alpha-2}\Gamma(\alpha/2)^2},
\end{aligned}$$

which implies (2.5). Observe that a_n and q_n are both of order $n^{1/\alpha}$.

We are now ready to state a theorem which provides an estimate for the rate of weighted (unconstrained) approximation of f in terms of the norm of its derivative. Recall that a function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that whenever a sequence of pairwise disjoint sub-intervals $[x_k, y_k]$ of $[a,b]$, $k = 1,2,\ldots,n$, satisfies $\sum_{k=1}^{n}|y_k - x_k| < \delta$ then $\sum_{k=1}^{n}|f(y_k) - f(x_k)| < \varepsilon$. Note that if f is absolutely continuous on $[a,b]$, then its derivative f' exists almost everywhere on $[a,b]$ and is Lebesgue-integrable over $[a,b]$.

Theorem 2.2 (Jackson-Favard inequality). *Let $\alpha > 1, 1 \le p \le \infty$ and $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous over each finite interval, with $\|f'W_\alpha\|_{L_p(\mathbb{R})} < \infty$. Then*

$$E_n(f, W_\alpha)_p \le C_0\frac{a_n}{n}\|f'W_\alpha\|_{L_p(\mathbb{R})}, \tag{2.6}$$

where C_0 does not depend on f and n.

This theorem was proved by Freud [14], [15] in the case $\alpha \ge 2$, and by Levin and Lubinsky [21] in the case $1 < \alpha < 2$. It turns out that (2.6) is no longer valid in the case $\alpha = 1$, and therefore this case requires a separate consideration (for more details see, for example, [22]).

Remark. Throughout this thesis, by C we denote a positive constant, which does not depend on f and n (and may be different even when they appear in the same line). By C_0, C_1, $C_2, \ldots$ we denote constants which are used more than once.

As a consequence of (2.6), one can derive an estimate for the rate of weighted approximation of f in terms of f'. For any polynomial $P_n \in \Pi_n$,

$$E_n(f, W_\alpha)_p = E_n(f - P_n, W_\alpha)_p \leq C_0 \frac{a_n}{n} \|(f' - P_n')W_\alpha\|_{L_p(\mathbb{R})}.$$

This immediately yields

$$E_n(f, W_\alpha)_p \leq C_0 \frac{a_n}{n} E_{n-1}(f', W_\alpha)_p, \ 1 \leq p \leq \infty.$$

Using iteration, the following corollary followa:

Corollary 2.3. *Let $\alpha > 1, 1 \leq p \leq \infty, r \in \mathbb{N}$. Assume that the $(r-1)$-th derivative of f is locally absolutely continuous and $\|f^{(r)}W_\alpha\|_{L_p(\mathbb{R})} < \infty$. Then*

$$E_n(f, W_\alpha)_p \leq C_0^r \left(\frac{a_n}{n}\right)^r \|f^{(r)}W_\alpha\|_{L_p(\mathbb{R})}.$$

2.1.3 Jackson Theorem

It is well known that there exist differentiable functions that do not have bounded derivatives, as well as continuous functions which are not differentiable at all. For example, the function

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^2}\right), & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0, \end{cases}$$

is differentiable on $\mathbb{R}$, but f' is unbounded in the neighbourhood of $x = 0$. Thus $\|f'\|_{W_\alpha} = \infty$ and therefore we cannot apply the result of Theorem 2.2. How is it possible to establish the rate of approximation for such class of functions?

In the case of approximation on an interval, introduction of moduli of continuity and moduli of smoothness by Lebesgue (1909) and De la Vallee Poussin (1919) resolved this problem. Since then, this tool has been widely used by many mathematicians in different areas and still is in use today. Let

$$\Delta_h^r f(x) := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh), \quad r \in \mathbb{N},$$

be the r**th (forward) difference**. For a function $f \in C[a,b]$, the modulus of smoothness of order r is defined as follows:

$$\omega_r(f;t) := \omega(t;f;[a,b]) := \begin{cases} \sup\limits_{h\in[0,t]} \max\limits_{x\in[a,b-rh]} |\Delta_h^r f(x)|, & t \in [0,(b-a)/r], \\ \omega((b-a)/r;f;[a,b]), & t > (b-a)/r. \end{cases}$$

One can now estimate the rate of approximation of any continuous function in terms of the above moduli. Such estimates are also called Jackson-type estimates (inequalities).

Theorem 2.4 (Classical Jackson). *Let r be a positive integer. Then, for any $f \in C[a,b]$,*

$$E_n(f) \le C(r)\omega_r((b-a)/n, f, [a,b]), \quad n \ge r-1,$$

where the constant $C(r)$ depends only on r.

A similar tool and approach is going to be used in our case. For $r \ge 1$, define the r**th order weighted modulus of smoothness** (with the weight W_α) as follows:

$$\omega_{r,p}(f,W_\alpha,t) := \sup_{0<h\le t} \|(\Delta_h^r f)W_\alpha\|_{L_p[-h^{1/(1-\alpha)},h^{1/(1-\alpha)}]} + \inf_{P\in\Pi_{r-1}} \|(f-P)W_\alpha\|_{L_p(\mathbb{R}\setminus[-t^{1/(1-\alpha)},t^{1/(1-\alpha)}])}.$$

These moduli were first introduced in the early works of Freud, and then were used by Ditzian and Totik [11] and Mhaskar [24]. Note that if f is a polynomial of degree $< r$, then $\omega_{r,p}(f,W_\alpha,t)$ is identically zero. In fact, the weighted moduli can be defined for wider class of weights from $\mathfrak{F}$ as well (see, for example, survey [22]).

In order to describe the behavior of moduli $\omega_{r,p}$, recall their basic properties (see e.g. [9,10]):

Properties of weighted moduli of smoothness on the real line:

Let $\alpha > 1, 1 \le p \le \infty$. Suppose that $f \in C_{W_\alpha}$ if $p = \infty$, and $fW_\alpha \in L_p(\mathbb{R})$ if $p < \infty$. Then

(i) $\omega_{r,p}(f,W_\alpha,t)$ is non-decreasing on $[0,+\infty)$.

(ii) $\lim\limits_{t\to 0+} \omega_{r,p}(f,W_\alpha,t) = 0$.

(iii) $\omega_{r,p}(f, W_\alpha, 2t) \leq C\omega_{r,p}(f, W_\alpha, t)$, $t \geq 0$, where C is a positive constant independent of f and t.

(iv) If $f^{(r-1)}$ is locally absolutely continuous and $f^{(r)}W_\alpha \in L_p(\mathbb{R})$, then $\omega_{r,p}(f, W_\alpha, t) \leq Ct^r \|f^{(r)}W_\alpha\|_{L_p(\mathbb{R})}$, $t \geq 0$, where C is a positive constant independent of f and t.

(v) If $r > 1$, then $\omega_{r,p}(f, W_\alpha, t) \leq C\omega_{r-1,p}(f, W_\alpha, t)$, $t \geq 0$, where C is a positive constant independent of f and t.

(vi) Assume that a Markov-Bernstein inequality holds, *i.e.*,

$$\|PW_\alpha\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n}\|P'W_\alpha\|_{L_p(\mathbb{R})},$$

for all $n \geq 1$ and any polynomial $P \in \Pi_n$. Then the Marchaud-type inequality is true

$$\omega_{r,p}(f, W_\alpha, t) \leq At^r \left[\int_1^B \frac{\omega_{r+1,p}(f, W_\alpha, u)}{u^{r+1}}du + \|fW_\alpha\|_{L_p(\mathbb{R})}\right],$$

for some $A, B > 0$ independent of f and t.

Remark. For Freud's weights W_α and $1 \leq p < \infty$, the Markov-Bernstein inequality holds for $\alpha > 1$. It was proven by Nevai and Lubinsky in 1987. If $p = \infty$, the Markov-Bernstein inequality holds for $\alpha \geq 1$ (see, for example, [24]).

(vii) Suppose that for some $0 < \delta < r$, $\omega_{r,p}(f, W_\alpha, t) = O(t^\delta)$, $t \to 0+$. Let $k := [\delta]$ (integer part of δ). Then $f^{(k)}$ exists a.e. on $\mathbb{R}$, and

$$\omega_{r-k,p}(f^{(k)}, W_\alpha, t) = O(t^{\delta-k}),\ t \to 0+.$$

If $p = \infty$, $f^{(k)}$ is continuous on $\mathbb{R}$.

The following result is an analogue of the classical Jackson theorem (for finite intervals) for weighted polynomial approximation on the real line.

Theorem 2.5 (Jackson-type inequality, see e.g. [22]). *Let $\alpha > 1$ and $r \geq 1$. If $f \in C_{W_\alpha}$, for $p = \infty$, and $fW_\alpha \in L_p(\mathbb{R})$, for $p < \infty$, then,*

$$E_n(f, W_\alpha)_p \leq C\omega_{r,p}(f, W_\alpha, \frac{a_n}{n}), \quad n \geq r - 1$$

where C is a positive constant independent of f and n.

Under conditions of Property (vi), *i.e.*, if the Markov-Bernstein inequality holds, a converse Bernstein-type theorem has the following form (see [9]):

Theorem 2.6. *Let $0 < \delta < r$. Then*

$$\omega_{r,p}(f, W_\alpha, t) = O(t^\delta),\ t \to 0+ \Longleftrightarrow E_n(f, W_\alpha)_p = O(\left(\frac{a_n}{n}\right)^\delta),\ n \to \infty.$$

The previous result provides a constructive characterization of Lipschitz-type classes in terms of the rate of weighted approximation. Once we know that a function is smooth enough, we can guarantee a certain rate of approximation, and conversely, if the convergence of $E_n(f, W_\alpha)_p$ to 0 is sufficiently fast, then it is possible to determine how smooth the function is.

2.2 Restricted Range Inequalities

Restricted range inequalities play an important role in weighted approximation. For instance, one uses this tool in order to prove the Jackson-type inequality. According to Paul Nevai, Freud's discovery of these inequalities is one of the most significant of Freud's contributions to weighted approximation theory.

The following theorem deals with the case $p = \infty$, and will be used in Chapter 4.

Theorem 2.7 (Freud [13]). *Let $\alpha \geq 1$. Then, for any $n \geq 1$ and polynomial $P \in \Pi_n$,*

$$\|PW_\alpha\|_{L_\infty(\mathbb{R})} = \|PW_\alpha\|_{L_\infty[-4q_{2n}, 4q_{2n}]}.$$

Since one of the intermediate steps in the proof of this theorem will be used later, we mention the main ideas of this proof following [22]. By T_n, we denote the classical

Chebyshev polynomial of degree n, *i.e.*, $T_n(x) := \cos(n \arccos x)$, $x \in [-1,1]$ (and uniquely extended to the whole real line).

Proof. We use the fact (see, for example, [24]) that for any polynomial $P \in \Pi_n$,

$$|P_n(x)| \le T_n(|x|)\|P\|_{L_\infty[-1,1]} \le (2|x|)^n \|P\|_{L_\infty[-1,1]}, \quad |x| \ge 1.$$

Scaling from the interval $[-1,1]$ to $[-q_{2n}, q_{2n}]$, we get

$$|P_n(x)| \le \left(\frac{2|x|}{q_{2n}}\right)^n \|P\|_{L_\infty[-q_{2n},q_{2n}]} \le \left(\frac{2|x|}{q_{2n}}\right)^n W_\alpha^{-1}(q_{2n})\|PW_\alpha\|_{L_\infty[-q_{2n},q_{2n}]}, \ |x| \ge q_{2n},$$

where the last inequality follows from the monotonicity of $W_\alpha(x)$ on $(0,\infty)$.

Hence,

$$|P_n(x)W_\alpha(x)| \le 2^n \frac{|x|^n W_\alpha(x)}{q_{2n}^n} W_\alpha^{-1}(q_{2n})\|PW_\alpha\|_{L_\infty[-q_{2n},q_{2n}]}.$$

In addition, if $x \ge 4q_{2n}$, then

$$\log \frac{|x|^n W_\alpha(x)}{q_{2n}^n} W_\alpha^{-1}(q_{2n}) = \int_{q_{2n}}^{x} \frac{n - \alpha u^\alpha}{u} du \le -n \int_{q_{2n}}^{4q_{2n}} \frac{du}{u} = -n \log 4,$$

since αu^α is an increasing function on $(0,\infty)$, and $\alpha q_{2n}^\alpha = 2n$.

Together with the previous inequalities this implies that for $x \ge 4q_{2n}$,

$$|P_n(x)W_\alpha(x)| \le 2^{-n}\|PW_\alpha\|_{L_\infty[-q_{2n},q_{2n}]}. \tag{2.7}$$

For negative values of x, it is enough to consider the polynomial $Q(x) := P(-x)$, and apply (2.7) for Q. Thus,

$$\|PW_\alpha\|_{L_\infty(\mathbb{R})} = \|PW_\alpha\|_{L_\infty[-4q_{2n},4q_{2n}]}. \qquad \square$$

The proof of Theorem 2.7 shows the exponential decay of PW_α outside the interval $[-4q_{2n}, 4q_{2n}]$, depending on the value of n. An analogue of Theorem 2.7 takes place in the L_p-spaces [22]:

Theorem 2.8. *For any $p > 0, \alpha \ge 1$,*

$$\|P_n W_\alpha(x)\|_{L_p(\mathbb{R})} \le (1 + e^{-An})\|PW_\alpha\|_{L_p[-Bq_{2n},q_{2n}]}, \tag{2.8}$$

where $A > 0$ and B is large enough.

According to the previous theorem, the L_p norm of the weighted polynomials "lives" inside the intervals $[-Bq_{2n}, Bq_{2n}]$, with appropriate B. A natural question arises: what is the best (smallest) B? This problem is deeply discussed in the monograph of Saff and Totik [30] for general weights. The case of exponential weights is treated in [24]. The following result allows us to write a strict inequality in (2.8) (see [24]):

Theorem 2.9. *Let $W \in \mathfrak{F}$, $0 < p < \infty$ and let P be a polynomial of degree $\leq n+2/p$, which is not identically zero. Then*

$$\|PW\|_{L_p(\mathbb{R}\setminus[-a_n,a_n])} < \|PW\|_{L_p[-a_n,a_n]},$$

and

$$\|PW\|_{L_p(\mathbb{R})} < 2^{1/p}\|PW\|_{L_p[-a_n,a_n]},$$

where a_n denotes the nth Mhaskar-Rakhmanov-Saff's number.

Chapter 3

One-Sided L_p Methods

3.1 Freud and Nevai's One-sided L_1 method

One of the methods in proving weighted Jackson-type Theorems is the one-sided L_1 method developed by Freud and Nevai. Usually, the L_1 method allows to obtain estimates on the rate of approximation of special functions such as characteristic functions. After that, using duality and other tools, one can get necessary estimates in the L_p-spaces, $1 < p \leq \infty$. However, using this approach one loses the property of one-sidedness, and therefore, this approach is not useful for our purposes. Hence, we only discuss the L_1 method.

Mainly, this method is based on the theory of orthogonal polynomials and Gaussian quadratures. Some important definitions and theorems on this subject are stated below. A nice survey of this material could be found in Mhaskar's monograph [24].

For a given general weight function W, define orthonormal polynomials $p_0, p_1, \ldots$ by

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + \ldots, \quad \gamma_n > 0,$$

and such that

$$\int_{\mathbb{R}} p_n(x) p_m(x) W^2(x) dx = \delta_{mn}, \tag{3.1}$$

where δ_{mn} is the Kronecker δ-function, *i.e.*, $\delta_{mn} = 0$, if $m \neq n$, and $\delta_{mn} = 1$, if $m = n$.

Remark. We use the weight W^2 and not W in the definition of orthonormal polynomials. This convention simplifies some formulations later on.

Note that each p_n has exactly n simple real zeroes. Indeed, otherwise, p_n has at most $n-1$ distinct odd order zeroes $y_1, \ldots, y_m$, $m \leq n-1$. Then the polynomial $\tilde{p}(x) := \prod_{i=1}^{m}(x - y_i)$ ($\tilde{p}(x) := 1$ if $m = 0$) is of degree less than n and $\tilde{p}(x)p_n(x) \geq 0$, for all $x \in \mathbb{R}$. Hence,

$$\int_{\mathbb{R}} \tilde{p}(x)p_n(x)W^2(x)dx > 0.$$

On the other hand, it is well known that the system of orthonormal polynomials $\{p_0, \ldots, p_{n-1}\}$ is a basis in Π_{n-1} (see, for example, [24]). This together with (3.1) implies that

$$\int_{\mathbb{R}} \tilde{p}(x)p_n(x)W^2(x)dx = 0.$$

This contradiction yields that p_n has exactly n simple zeroes. Denote them by

$$x_{nn} < x_{(n-1)n} < \cdots < x_{2n} < x_{1n}.$$

Then the n**th Christoffel function** for W^2 is defined as follows

$$\lambda_n(W^2, x) = \inf_{P \in \Pi_{n-1}} \frac{\int_{\mathbb{R}} (P(x)W(x))^2 dx}{P^2(x)}.$$

As was already mentioned, the system of orthonormal polynomials $\{p_0, \ldots, p_{n-1}\}$ is a basis for the space Π_{n-1}. Hence, for any $P \in \Pi_{n-1}$, we may write

$$P(x) = \sum_{j=0}^{n-1} c_j p_j(x), \quad x \in \mathbb{R}.$$

Then, using Parseval's identity, we have

$$\int_{\mathbb{R}} P^2(x)W^2(x)dx = \sum_{j=0}^{n-1} c_j^2.$$

Now, Cauchy-Schwarz inequality implies that

$$\frac{\int_{\mathbb{R}} (P(x)W(x))^2 dx}{P^2(x)} = \frac{\int_{\mathbb{R}} (P(x)W(x))^2 dx}{\left(\sum_{j=0}^{n-1} c_j p_j(x)\right)^2} \geq \frac{\sum_{j=0}^{n-1} c_j^2}{\sum_{j=0}^{n-1} p_j^2(x) \sum_{j=0}^{n-1} c_j^2} = \frac{1}{\sum_{j=0}^{n-1} p_j^2(x)}.$$

Moreover, for any fixed $x = x_0$, if $P(x) = k\sum_{j=0}^{n-1} p_j(x_0)p_j(x)$ for some constant $k \neq 0$, then the last inequality becomes equality at $x = x_0$. Thus we proved the following identity for the Christoffel function

$$\lambda_n(W^2, x) = \frac{1}{\sum_{j=0}^{n-1} p_j^2(x)}.$$

Recall now (see e.g. [19]) that the Gaussian quadrature formula

$$\int_{\mathbb{R}} f(x)W^2(x)dx \approx \sum_{j=1}^{n} \lambda_n(W^2, x_{jn})f(x_{jn}),$$

is exact for all polynomials $P \in \Pi_{2n-1}$ ([24, p.10]). As an application of the quadrature formula, one can prove the following estimate known as the Possé inequality (see e.g. [24, Theorem 1.2.3]): for any function f such that its first $2n-1$ derivatives are positive on $(-\infty, x_{kn})$,

$$\sum_{j=k+1}^{n} \lambda_n(W^2, x_{jn})f(x_{jn}) \leq \int_{-\infty}^{x_{kn}} f(x)W^2(x)dx \leq \sum_{j=k}^{n} \lambda_n(W^2, x_{jn})f(x_{jn}).$$

This can be used to estimate the error of one-sided approximation of characteristic functions. For A a subset of $\mathbb{R}$, denote

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following result.

Lemma 3.1 (see Mhaskar [24, Corollary 1.2.6]). *Let W be a general weight function, $n \geq 1$, and $\xi \in (x_{(k+1)n}, x_{kn})$, where $x_{(k+1)n}$ and x_{kn} are the consecutive zeroes of p_n. Then there exist two polynomials R_ξ and r_ξ such that*

$$r_\xi \leq \chi_{(-\infty,\xi]} \leq R_\xi \text{ on } \mathbb{R},$$

and

$$\int_{\mathbb{R}} [R_\xi(x) - r_\xi(x)]W^2(x)dx \leq \lambda_n(W^2, x_{kn}) + \lambda_n(W^2, x_{(k+1)n}).$$

Thus, upper estimates for the Christoffel functions immediately yield bounds for the error of one-sided polynomial approximation. There are several different ways to estimate Christoffel functions (see, for instance, [26]). For our purposes, we only mention one quite simple method introduced by Freud [13].

Lemma 3.2 ([22, Lemma 4.2]). *If $\alpha > 1$, then there exist $C_1, C_2 > 0$ such that, for $n \geq 1$ and $|\xi| \leq C_1 a_n$,*

$$\lambda_n(W_\alpha^2, \xi) \leq C_2 \frac{a_n}{n} W_\alpha^2(\xi).$$

Using results of Lemmas 3.1 and 3.2, we follow Lubinsky [22] and construct one-sided L_1 approximation for a special class of functions (of polynomial growth at infinity).

Theorem 3.3 ([22, Theorem 4.3]). *Let $\alpha > 1$, and let f be of polynomial growth at ∞. Moreover, suppose that f' is continuous, and $f'W_\alpha^2 \in L_1(\mathbb{R})$. Then, for any $n \in \mathbb{N}$, there exist polynomials $\bar{\mathfrak{p}}_n$ and $\mathfrak{p}_n$ of degree $\leq n$ such that*

$$\mathfrak{p}_n \leq f \leq \bar{\mathfrak{p}}_n \text{ on } \mathbb{R},$$

and

$$\int_{\mathbb{R}} (\bar{\mathfrak{p}}_n(x) - \mathfrak{p}_n(x)) W_\alpha^2(x) dx \leq C \frac{a_n}{n} \left(\int_{\mathbb{R}} |f'(x)| W_\alpha^2(x) dx + \|f' W_\alpha^2\|_{L_\infty(|x| \geq C a_n)} \right).$$

Proof. We prove this theorem under the assumption that f' is nonnegative on the real line and moreover, $\operatorname{supp}(f') \subset (x_{nn}, x_{1n})$, *i.e.*, $f' = 0$ outside (x_{nn}, x_{1n}), where x_{nn}, x_{1n} are the smallest and the largest zeros of $p_n(W_\alpha^2, x))$, respectively. By the Fundamental Theorem of Calculus

$$\begin{aligned} f(x) =& f(0) + \int_0^x f'(\xi) d\xi \\ =& f(0) + \int_0^\infty (1 - \chi_{(-\infty,\xi)}(x)) f'(\xi) d\xi - \int_{-\infty}^0 \chi_{(-\infty,\xi)}(x) f'(\xi) d\xi. \end{aligned}$$

Now, using polynomials R_ξ and r_ξ from Lemma 3.1, define

$$\bar{\mathfrak{p}}_n(x) = f(0) + \int_0^\infty (1 - r_\xi(x)) f'(\xi) d\xi - \int_{-\infty}^0 r_\xi(x) f'(\xi) d\xi,$$

and

$$\mathfrak{p}_n(x) = f(0) + \int_0^\infty (1 - R_\xi(x))f'(\xi)d\xi - \int_{-\infty}^0 R_\xi(x)f'(\xi)d\xi.$$

Conditions $f' \geq 0$ and $r_\xi \leq \chi_{(-\infty,\xi)} \leq R_\xi$ immediately yield that $\mathfrak{p}_n \leq f \leq \bar{\mathfrak{p}}_n$ on the whole real line. Observing that

$$(\bar{\mathfrak{p}}_n - \mathfrak{p}_n)(x) = \int_{\mathbb{R}} (R_\xi - r_\xi)(x)f'(\xi)d\xi,$$

we conclude

$$\int_{\mathbb{R}} (\bar{\mathfrak{p}}_n(x) - \mathfrak{p}_n(x))W_\alpha^2(x)dx \leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} (R_\xi(x) - r_\xi(x))W_\alpha^2(x)dx\right] f'(\xi)d\xi.$$

Now, Lemmas 3.1 and 3.2 imply that for at least $|\xi| \leq C_1 a_n$ (note that the largest zero x_{1n} is of the same order as a_n),

$$\int_{\mathbb{R}} (R_\xi(x) - r_\xi(x))W^2(x)dx \leq C_2 \frac{a_n}{n} W^2(\xi).$$

Hence,

$$\int_{\mathbb{R}} (\bar{\mathfrak{p}}_n(x) - \mathfrak{p}_n(x))W_\alpha^2(x)dx \leq C_2 \frac{a_n}{n} \int_{-C_1 a_n}^{C_1 a_n} f'(\xi)W_\alpha^2(\xi)d\xi$$
$$+ \|f'W_\alpha^2\|_{L_\infty(|x|\geq C_1 a_n)} \int_{|\xi|>C_1 a_n} \left[W_\alpha^{-2}(\xi) \int_{\mathbb{R}} (R_\xi(x) - r_\xi(x))W_\alpha^2(x)dx\right] d\xi.$$

The first term on the right-hand side has the required form. Using estimates for Christoffel functions, which are given in details in [16], [24, p.83], one can complete the proof of the theorem. □

Remark. Under the conditions of the previous theorem $\|f'W_\alpha^2\|_{L_\infty(|x|\geq Ca_n)}$ does not have to be finite. It is possible to construct a counterexample.

Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation if

$$\sup_{n\geq 1} \sup_{a\leq x_0<\ldots<x_n\leq b} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \infty.$$

In 1974, using the same approach as in the proof of the previous theorem Freud proved the following result.

Theorem 3.4 (Freud [12, p.297]). *Let $W \in \mathfrak{F}$. Assume that for some $a, b > 0$,*

$$1 + a < \frac{Q'(2x)}{Q'(x)} < 1 + b.$$

Let $r \geq 0$ and $f^{(r)}$ be of bounded variation over every finite interval, and of polynomial growth at ∞, satisfying for some $A, B > 0$ and integer m, $|f(x)| \leq A + Bx^{2m}$. Then there exist polynomials $\bar{\mathfrak{p}}_n$ and $\mathfrak{p}_n$ from Π_n such that

$$\mathfrak{p}_n \leq f \leq \bar{\mathfrak{p}}_n \text{ on } \mathbb{R}, \tag{3.2}$$

and

$$\int_{\mathbb{R}} (\bar{\mathfrak{p}}_n(x) - \mathfrak{p}_n(x)) W^2(x) dx \leq C \left(\frac{a_n}{n}\right)^{r+1} \left(\int_{\mathbb{R}} W^2(x) df^{(r)}(x) + A + B\right), \tag{3.3}$$

where the constant C is independent of f, n, A, B.

Remark 1. The weights $W_\alpha = \exp(-|x|^\alpha), \alpha > 1$ satisfy the conditions of Theorem 3.4.

Remark 2. Theorem 3.4 implies an estimate for the rate of unconstrained approximation:

$$E_n(f, W^2)_1 \leq C \left(\frac{a_n}{n}\right)^{r+1} \left(\int_{\mathbb{R}} W^2(x) df^{(r)}(x) + A + B\right).$$

These estimates can be extended to other L_p-spaces, using duality (see e.g. [22]).

3.2 Partially one-sided polynomial approximation on the real line and its applications

In [23], Mastroianni and Szabados proved a similar result to Theorem 3.4 for a wider class of functions. Namely, they constructed one-sided approximants for the functions of exponential growth and presented estimates for the rate of this approximation in the L_p-norm ($1 \leq p \leq \infty$). The main generalization is that not only functions of polynomial growth are allowed. On the other hand, the restriction made in [23] is requiring (3.2) to hold only on some finite intervals instead of the whole real line. This restriction is necessary in order to include functions of exponential growth.

Obtained results can be used in the applications (estimating the ordinary weighted best polynomial approximation, Lagrange interpolation, quadrature errors).

3.2.1 Estimates for certain classes of functions

Let $W \in \mathfrak{F}$ be a Freud-type weight, $W(x) = \exp(-Q(x))$. Recall that Mhaskar-Rakhmanov-Saff numbers a_n are defined by (2.3), and by $x_{1n} > \cdots > x_{nn}$ denote roots of n-th orthonormal polynomial $p_n(W)$ with respect to W^2.

Theorem 3.5 ([23, Theorem 1]). *Let $r \geq 0$ be an integer, and let f be such that $f^{(r)}$ is of bounded variation on each finite subinterval of $\mathbb{R}$. If*

$$n^{(r+1)/3} \int_{|t| \geq x_{1n}/2} W^2 |df^{(r)}(t)| \leq C \int_{\mathbb{R}} W^2(t) |df^{(r)}(t)| < \infty, \quad n \geq 1, \tag{3.4}$$

with some absolute constant $C > 0$, then there exist polynomials $r_n, R_n \in \Pi_{2n-1}$ such that

$$r_n(x) \leq f(x) \leq R_n(x), \quad \text{if } |x| \leq x_{1n}, \quad n \geq 1,$$

and

$$\|(R_n - r_n)W^2\|_{L_p(\mathbb{R})} = O\left(\left(\frac{a_n}{n}\right)^{r+1/p}\right) \int_{\mathbb{R}} W^2(t) |df^{(r)}(t)|, \quad 1 \leq p \leq \infty.$$

Remark 1. Theorem 1 does not establish a convergent to 0 estimate in the case $r = 0$ and $p = \infty$.

Remark 2. Note that condition (3.4) is rather weak, and the class of functions of exponential growth which satisfy it, is quite rich.

However, when the derivative of the function is not of bounded variation, Theorem 3.5 can't be applied. The following theorem resolves this problem.

Theorem 3.6 ([23, Theorem 2]). *Let $r \geq 1$ be an integer, $1 \leq p \leq \infty$, $0 < \mu < 1$, let*

$$b_n := a_n \left(1 + \lambda_r \left(\frac{\log n}{n}\right)^{2/3}\right), \tag{3.5}$$

where $\lambda_r > 0$ is a suitable constant, and let f be such that

$$\exp(\gamma n^{1/3}) \max_{1\le j\le r} \|f^{(j)}W^2\|_{L_p(|x|\ge \mu a_n)} \le \|f^{(r)}W^2\|_{L_p(\mathbb{R})} < \infty, \quad n \ge 1, \tag{3.6}$$

with a suitable constant $\gamma > 0$. Then there exist polynomials $R_n, r_n \in \Pi_{2n-1}$ such that

$$r_n(x) \le f(x) \le R_n(x), \quad \text{if } |x| \le b_n, n \ge 1$$

and

$$\|(R_n - r_n)W^2\|_{L_p(\mathbb{R})} = C_r \left(\frac{a_n}{n}\right)^r \max_{1\le j\le r} \|f^{(j)}W^2\|_{L_p(\mathbb{R})}.$$

In the case $p = 1$, condition (3.6) is more restrictive than (3.4) in general, since it requires an "exponential" estimate in the inequality, and not "polynomial" as (3.4) does. However, (3.6) still allows functions like $f(x) = \exp(2Q(x) - \nu Q(x)^{1/3})$ $(\nu > \gamma)$, with exponential growth at ∞.

3.2.2 Applications

In [23], Mastroianni and Szabados also present some applications of obtained results. Below, the weight function $W \in \mathfrak{F}$.

As was mentioned before, the following quadrature formula

$$\int_{\mathbb{R}} f(x)W^2(x)dx \approx \sum_{k=1}^{n} \lambda_{kn} f(x_k),$$

becomes an equality whenever f is a polynomial of degree $\le 2n-1$ and x_k are zeroes of $p_n(W^2)$. The coefficients

$$\lambda_{kn} := \int_{\mathbb{R}} l_{kn}(x)W^2(x)dx,$$

where

$$l_{kn}(x) := \frac{p_n(W^2, x)}{p_n'(W^2, x_k)(x - x_k)},$$

are called Cotes numbers of the quadrature procedure with respect to weight W^2. More detailed treatment on this topic can be found in [24, pp.11–13].

The following corollary provides an estimate of the error of such quadrature formula

$$e_n(f) := e_n(f, W^2) = \left| \int_{\mathbb{R}} f(x) W^2(x) dx - \sum_{k=1}^{n} \lambda_{kn} f(x_k) \right|.$$

Corollary 3.7 (Quadrature formula). *If* $\|f^{(r)} W^2\|_{L_1(\mathbb{R})} < \infty$ *and* (3.6) *holds with* $p = 1$, *then*

$$e_n(f) \leq C \left(\frac{a_n}{n}\right)^r \max_{1 \leq j \leq r} \|f^{(j)} W^2\|_{L_1(\mathbb{R})}.$$

Another applications describes the rate of Lagrange interpolation in the L_2-metric.

Definition 3.8. *A Lagrange polynomial*

$$L(z; f) := L(z; f; z_1, z_2, \ldots, z_m)$$

that interpolates a function f at points $z_1, z_2, \ldots, z_m$ (interpolation nodes) is defined as an algebraic polynomial of at most mth order that takes the same values at these points as the function f, i.e.,

$$L(z_i; f) = f(z_i), \quad i = 0, \ldots, m.$$

Let $L_n(f, W)$ be the Lagrange interpolation polynomial with nodes at zeroes of $p_n(W)$. Then the following estimate is valid:

Corollary 3.9 (L_2-Lagrange interpolation). *If* $\|f^{(r)}\|_{L_2(\mathbb{R})} < \infty, r \geq 1$, *and* (3.6) *holds with* $p = 2$ *and* $0 \leq j \leq r$, *then*

$$\|(f - L_n(f, W))W\|_{L_2(\mathbb{R})} = O\left(\left(\frac{a_n}{n}\right)^r\right) \max_{1 \leq j \leq r} \|f^{(j)} W\|_{L_2(\mathbb{R})}.$$

Chapter 4

One-Sided L_∞ Weighted Approximation

As we have discussed after Question 1.3, direct generalization of the problem of one-sided approximation on an interval to the problem of one-sided approximation on the whole real line becomes ill-posed. Therefore, we repose the problem in the following way.

Problem 4.1. Problem definition:
Does there exist a system of intervals $\{[-d_n, d_n], n \geq 1\}$ *such that*

Condition A. $d_n \to \infty, n \to \infty$;

Condition B. *For each function* $f \in C_{W_\alpha}, \alpha \geq 1$, *there exists a sequence of polynomials* $\{P_n \in \Pi_n, n \geq 1\}$ *such that* $\lim_{n\to\infty} \|f - P_n\|_{W_\alpha} = 0$, *and* $P_n(x) \geq f(x)$ $(P_n(x) \leq f(x))$, *for all* $x \in [-d_n, d_n]$?

Further, we present the largest possible in some sense sequence of $\{d_n\}$, so that Problem 4.1 has a positive answer. Recall that $\alpha \geq 1$ is a parameter in the weight $W_\alpha = \exp(-|x|^\alpha)$.

Lemma 4.2. *If the sequence* $\{d_n\}_{n=1}^\infty$ *is such that Condition B is satisfied, then* $d_n < C_3 n^\alpha$, *for all* $n \geq 1$, *and for some constant* $C_3 > 0$.

Proof. Recall that $q_n := (n/\alpha)^{1/\alpha}, n \geq 1$, is Freud's number (see (2.4)). We will prove that $d_n < 4q_{2n}$.

Suppose to the contrary that, for all functions $f \in C_{W_\alpha}$ and all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and a polynomial P_n of degree $\leq n$ such that $\|f - P_n\|_{W_\alpha} < \varepsilon$, and $P_n(x) \geq f(x)$, for all $x \in [-4q_{2n}, 4q_{2n}]$.

Consider the function

$$f^*(x) := \left(\frac{1}{2}\right)^{\alpha(|x|/4)^\alpha/4} (W_\alpha(x))^{-1}, \quad x \in \mathbb{R},$$

which is clearly from C_{W_α}. Then, there exist $n \in \mathbb{N}$ and a polynomial $P_n^* \in \Pi_n$ such that $\|f^* - P_n^*\|_{W_\alpha} < 1/4$ and $P_n^*(x) \geq f^*(x)$, for all $x \in [-4q_{2n}, 4q_{2n}]$. In particular,

$$P_n^*(4q_{2n})W_\alpha(4q_{2n}) \geq f^*(4q_{2n})W_\alpha(4q_{2n}) = \frac{1}{2^{n/2}}. \tag{4.1}$$

Now, using restricted range inequality (2.7), we get

$$P_n^*(4q_{2n})W_\alpha(4q_{2n}) \leq \frac{1}{2^n}\|P_n^* W_\alpha\|_{L_\infty[-q_{2n},q_{2n}]}. \tag{4.2}$$

Note that

$$\|P_n^* W_\alpha\|_{L_\infty[-q_{2n},q_{2n}]} \leq \|f^* - P_n^*\|_{W_\alpha} + \|f^*\|_{W_\alpha} = 1/4 + 1 = 5/4, \tag{4.3}$$

since $\max_{x\in\mathbb{R}} f^*(x)W_\alpha(x) = f^*(0)W_\alpha(0) = 1$.

Inequalities (4.1),(4.2) and (4.3) imply

$$\frac{1}{2^{n/2}} \leq \frac{5}{2^{n+2}},$$

which is not true for any $n \geq 1$. This contradiction completes the proof. □

Note that one can use the function $f = -f^*$ in order to obtain the same result for approximation from below.

We now prove an auxiliary lemma, which will be used in the proofs of the positive results.

Lemma 4.3. *For all $\alpha > 1$, there exists a positive constant $C_4 = C_4(\alpha)$ and a sequence of polynomials $Q_n \in \Pi_n, n \geq 1$ such that*

(1) $Q_n(x) \geq (W_\alpha(x))^{-1}$, *for* $x \in [-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}]$,

and

(2) $\sup_{n\geq 1} \|Q_n\|_{W_\alpha} < \infty$.

Proof. Define the function f as follows

$$f(x) := \begin{cases} (W_\alpha(x))^{-1}, & \text{if } |x| \leq \lambda a_n, \\ (W_\alpha(\lambda a_n))^{-1}, & \text{if } |x| > \lambda a_n, \end{cases}$$

where a_n represents nth Mhaskar-Rakhmanov-Saff's number defined by equality (2.5) ($a_n = Ln^{1/\alpha}, L = L(\alpha)$), and the parameter λ is to be chosen. Obviously, f is absolutely continuous over each finite interval and $\|f'W_\alpha\|_{L_\infty(\mathbb{R})} < \infty$. Hence, one can apply Theorem 2.2 in order to estimate the rate of approximation of f by Π_n:

$$\begin{aligned} E_n(f, W_\alpha) &\leq C_0 \frac{a_n}{n} \|f'W_\alpha\|_{L_\infty \mathbb{R}} = C_0 \frac{a_n}{n} \|\alpha |x|^{\alpha-1}\|_{L_\infty[-\lambda a_n, \lambda a_n]} \\ &= C_0 \frac{a_n}{n} \alpha \lambda^{\alpha-1} (a_n)^{\alpha-1} = \alpha C_0 \lambda^{\alpha-1} \frac{a_n^\alpha}{n} = \alpha C_0 L^\alpha \lambda^{\alpha-1}, \end{aligned}$$

and since $\lambda^{\alpha-1} \to 0, \lambda \to 0$, we can now choose λ so that

$$E_n(f, W_\alpha) = \|f - P_n\|_{W_\alpha} < 1/2, \tag{4.4}$$

for some polynomial P_n of degree $\leq n$. Therefore, for $|x| \leq \lambda a_n$, we have

$$|(f(x) - P_n(x))W_\alpha(x)| < 1/2,$$

which implies

$$|1 - P_n(x)W_\alpha(x)| < 1/2.$$

Hence

$$1/2 < P_n(x)W_\alpha(x) < 3/2,$$

and so for $Q_n := 2P_n$,

$$Q_n > (W_\alpha)^{-1} \text{ on } [-\lambda a_n, \lambda a_n], \text{ and } \|Q_n W_\alpha\|_{L_\infty[-\lambda a_n, \lambda a_n]} < 3.$$

Finally, for $|x| > \lambda a_n$,

$$\begin{aligned}|Q_n(x)W_\alpha(x)| &\le 2(|(P_n(x) - f(x))W_\alpha(x)| + |f(x)W_\alpha(x)|)\\ &< 2(1/2+1) = 3.\end{aligned}$$

□

Note that our approach fails for $\alpha = 1$.

The following lemma together with Lemma 4.2 prove that $d_n \sim n^{1/\alpha}$ are asymptotically the largest possible.

Theorem 4.4. *For all $\alpha > 1$, there exist $C_4 = C_4(\alpha) > 0$ such that the system of intervals $\{[-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}], n \ge 1\}$ satisfies both Conditions A and B of Problem 4.1.*

Proof. Clearly, we only need to prove that Condition B is satisfied for some constant, whereas Condition A always holds for chosen system of intervals. We use the same constant C_4 introduced in Lemma 4.3.

Let $f \in C_{W_\alpha}$ be fixed. There exists a sequence of polynomials $\{P_n \in \Pi_n, n \ge 1\}$ such that $\|f - P_n\|_{W_\alpha} = E_n(f, W_\alpha)$. The polynomial $R_n(x) := P_n(x) + E_n(f, W_\alpha)Q_n(x)$ is of degree $\le n$, where Q_n are polynomials constructed in Lemma 4.3. We will now show that the sequence of polynomials $\{R_n\}$ satisfies Condition B, with $d_n = C_4 n^{1/\alpha}$. Indeed,

$$\|f - R_n\|_{W_\alpha} \le \|f - P_n\|_{W_\alpha} + E_n(f, W_\alpha)\|Q_n\|_{W_\alpha} \le 4E_n(f, W_\alpha), \tag{4.5}$$

which implies that $\|f - R_n\|_{W_\alpha} \to 0, n \to \infty$, since $E_n(f, W_\alpha) \to 0, n \to \infty$, for all $f \in C_{W_\alpha}$.

It remains to show that R_n lies above f on $[-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}]$. Indeed,

$$\begin{aligned}R_n(x) - f(x) &= P_n(x) - f(x) + E_n(f, W_\alpha)Q_n(x)\\ &\ge E_n(f, W_\alpha)Q_n(x) - (W_\alpha(x))^{-1}|(P_n(x) - f(x))W_\alpha(x)|\\ &\ge E_n(f, W_\alpha)(Q_n(x) - (W_\alpha(x))^{-1}) \ge 0,\end{aligned}$$

since $Q_n(x) > (W_\alpha(x))^{-1}$ on $[-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}]$ by Lemma 4.3.

The proof of Theorem 4.4 is now complete. □

Inequality (4.5) implies the following Jackson-type estimate for this kind of approximation.

Corollary 4.5. *Let $\alpha > 1$ and $f \in C_{W_\alpha}$. Then, for every $n \in \mathbb{N}$, there exists a polynomial $R_n \in \Pi_n$ such that*

(1) $\|f - R_n\|_{W_\alpha} \leq C_5 \omega_{r,p}(f, W_\alpha, \frac{a_n}{n})$, *where the constant C_5 is independent of f and n (but depends on α),*

(2) $R_n(x) \geq f(x)$, *for all $x \in [-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}]$.*

Using Lemma 4.4, one can establish a similar result in the weighted L_p norm for a certain class of functions.

Corollary 4.6. *Let $\alpha > 1$, $0 < p < \infty$, and $\varepsilon > 0$. Suppose that f is a continuous function with $f^{1+\varepsilon} \in C_{W_\alpha}$. Then there is a sequence of polynomials $\{P_n \in \Pi_n, n \geq 1\}$ and a constant $C_6 = C_6(\alpha) > 0$ such that*

(1) $\lim_{n\to\infty} \|(f - P_n)W_\alpha\|_{L_p(\mathbb{R})} = 0,$

and

(2) *for all $n \geq 1$,* $\quad P_n(x) \geq f(x), \quad x \in [-C_6 n^{1/\alpha}, C_6 n^{1/\alpha}].$

Proof. Note that the condition $f^{1+\varepsilon} \in C_{W_\alpha}$ implies:

$$\lim_{|y|\to\infty} |f\left((1+\varepsilon)^{1/\alpha} y\right)| e^{-|y|^\alpha} = \lim_{|y|\to\infty} \left(|f\left((1+\varepsilon)^{1/\alpha} y\right)|^{1+\varepsilon} e^{-|(1+\varepsilon)^{1/\alpha} y|^\alpha}\right)^{\frac{1}{1+\varepsilon}} = 0,$$

and therefore by Theorem 4.4 there exists a sequence of polynomials $R_n \in \Pi_n$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \{|f\left((1+\varepsilon)^{1/\alpha} y\right) - R_n(y)| W_\alpha(y)\} = 0, \tag{4.6}$$

and for any $n \in N$,

$$R_n(y) \geq f\left((1+\varepsilon)^{1/\alpha} y\right), \quad y \in [-C_4 n^{1/\alpha}, C_4 n^{1/\alpha}]. \tag{4.7}$$

Define $P_n(x) = R_n\left((1+\varepsilon)^{-1/\alpha}x\right), n \geq 1$, and denoting $x := (1+\varepsilon)^{1/\alpha}y$ and using (4.6), we get condition

$$
\begin{aligned}
&\|(f - P_n)W_\alpha\|_{L_p(\mathbb{R})} \\
&= \left(\int_{\mathbb{R}} \left[|f(x) - R_n\left((1+\varepsilon)^{-1/\alpha}x\right)| \exp(-|x|^\alpha)\right]^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}} (1+\varepsilon)^{1/\alpha}\left[|f\left((1+\varepsilon)^{1/\alpha}y\right) - R_n(y)| \exp\left(-(1+\varepsilon)|y|^\alpha\right)\right]^p dy\right)^{1/p} \\
&\leq (1+\varepsilon)^{1/(\alpha p)}\|f\left((1+\varepsilon)^{1/\alpha}y\right) - R_n(y)\|_{W_\alpha}\left(\int_{\mathbb{R}} \exp(-\varepsilon p|y|^\alpha)dy\right)^{1/p} \to 0, \quad n \to \infty,
\end{aligned}
$$

and so (1) is verified. Inequality (4.7) now implies that for all $n \geq 1$ and $|x| \leq C_4(1+\varepsilon)^{1/\alpha}n^{1/\alpha}$,

$$P_n(x) = R_n\left((1+\varepsilon)^{-1/\alpha}x\right) \geq f\left((1+\varepsilon)^{1/\alpha}(1+\varepsilon)^{-1/\alpha}x\right) = f(x),$$

which verifies part (2) of the corollary. The proof is now complete. □

Chapter 5

Conclusions

In this thesis, we solved the problem dealing with partially one-sided approximation in the space C_{W_α}, $\alpha > 1$. We are only aware of one paper on this subject [23] that contains estimates of such approximation in the weighted L_p norms, but does not establish the density of polynomials in the most general case. It uses technique and theory of orthogonal polynomials. In this thesis, a different approach is presented, which allows to obtain the same rate of partially one-sided approximation as the rate of arbitrary unconstrained approximation.

It is worth mentioning that the case $\alpha = 1$ still requires a separate consideration. Also, a generalization of obtained asymptotical results to the L_p-spaces is a natural problem. These problems are still open and deserve further investigation.

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