Amenability, Weak Amenability and Approximate Amenability of $\ell^1(S)$

by

Filofteia Gheorghe

A thesis submitted to

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in partial fulfillment of the requirements for the degree of

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Abstract

We are doing a survey on amenability, weak amenability and generalized notions of amenability of semigroup algebras $\ell^1(S)$. Based on the characterizations of amenability of a Banach algebra, F. Ghahramani, R.J. Loy and Y. Zhang have introduced approximate amenability and pseudo-amenability for Banach algebras. Since they have been introduced, many results concerning them have been obtained by many researchers. We focus on this topic regarding semigroup algebras. We give some new characterizations for a Banach algebra to be approximately amenable. For a semigroup S with a generating set E, we also give necessary and sufficient conditions so that $\ell^1(S)$ is amenable, weakly amenable or boundedly approximately amenable.

It is known that if the semigroup algebra $\ell^1(S)$ is approximately amenable then S must be a regular amenable semigroup. We prove that the converse is not true by examining the bicyclical semigroup S_1 which is an important semigroup and has been studied by many researchers from various aspects. Precisely, we show that, although S_1 is a regular amenable semigroup, $\ell^1(S_1)$ is not approximately amenable. In the appendix we also give a direct proof to the fact that $\ell^1(S_2)$ is not approximately amenable, where S_2 is the partially bicyclic semigroup defined by $S_2 =< 1, a, b, c \mid ab = ac = 1 >$.

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Contents

Chapter 1

Preliminaries

1.1 ℓ^1- semigroup algebra

Terms and concepts of basic real and functional analysis which we have not defined or discussed can be found in [8] and [49]. In this section we establish some notations and definitions.

Definition 1.1.1. A *complex algebra* is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined, called an *algebra product*, $A \times A \longrightarrow A$; $(a, b) \longrightarrow ab$, that satisfies:

$$a(bc) = (ab)c, (a + b)c = ac + bc, a(b + c) = ab + ac \qquad (a, b, c \in A)$$

$$(a\alpha)b = a(\alpha b) = \alpha(ab) \qquad (\alpha \in \mathbb{C}, a, b \in A).$$

We say that A is *commutative* if $ab = ba$ for all $a, b \in A$. If in addition, A is a Banach space with respect to a norm that satisfies the submultiplicative inequality

$$\| ab \| \leq \| a \| \| b \| \qquad (a, b \in A) \tag{1.1}$$

then A is called a *Banach algebra.* If A contains a unit element e such that $ae = ea = a$ $(a \in A)$ then A is called a *Banach algebra with identity.* If $\| e \| = 1$ then A is a unital Banach algebra.

Suppose that the algebra A does not have a unit. Then we define the *unitization of* A to be $A^{\#} = \mathbb{C} \odot A$. $A^{\#}$ is a unital algebra, with unit $(1, 0)$, for the product

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab) \qquad (\alpha, \beta \in \mathbb{C}, a, b \in A).$$

If A is a Banach algebra, then so is $A^{\#}$ for the norm $\|(\alpha, a)\| = |\alpha| + \|a\|$.

A *subalgebra* of an algebra A is a linear subspace B of A such that $ab \in B$ for all $a, b \in B$.

A *left ideal* in an algebra A is a subalgebra $I \subseteq A$ such that, if $a \in A$ and $b \in I$, then $ab \in I$.

Similarly, we can define a *right ideal* and a *two-sided ideal* for A.

The *radical* of an algebra A is defined to be the intersection of the maximal left ideals of $A^{\#}$; it is denoted by *rad* A. The algebra A is *semisimple* if $rad\{A\} = \{0\}$ (see [9]).

We recall some definitions. For further details see [38].

Definition 1.1.2. A *groupoid* (S, μ) is defined as a non-empty set S on which a binary operation $\mu : S \times S \to S$ is defined. We say that (S, μ) is a *semigroup* if the operation μ is *associative*, that is to say, if, for all x, y and z in S,

$$\mu(\mu(x, y), z)) = \mu(x, \mu(y, z)). \tag{1.2}$$

We shall follow the usual algebraic practice of writing the binary operation as *multiplication.* Thus $\mu(x, y)$ becomes xy, and formula (1.2) takes the simple form

$$(xy)z = x(yz),$$

the familiar associative law of elementary algebra. A non-empty subset T of S is a *subsemigroup* if T is a semigroup for the product in S. For $s \in S$, we set

$$\langle s \rangle = \{s^n : n \in \mathbb{N}\},$$

the semigroup *generated* by s; the subsemigroup of S *generated* by a subset T is denoted by $\langle T \rangle$.

If the semigroup S has the property that, for all x, y in S,

$$xy = yx,$$

we shall say that S is a *commutative* semigroup. (The term *abelian* is also used, by analogy with the group theoretic term). If a semigroup S contains an element 1 with the property that, for all x in S,

$$x1 = 1x,$$

we say that 1 is an *identity element* (or just an *identity*) of S, and that S is a *semigroup with identity* or (more usually) a *monoid.* We now define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

We refer to S^1 as the *monoid obtained from S by adjoining an identity if necessary.*

If a semigroup S with at least two elements contains an element 0 such that, for all x in S,

$$0x = x0 = 0,$$

we say that 0 is a *zero element* (or just a *zero*) of S, and that S is a *semigroup with zero.* By analogy with the case of S^1, we define

$$S^o = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

and refer to S^o as the *semigroup obtained from S by adjoining a zero if necessary.*

If a is an element of a semigroup S without identity then Sa need not contain a. The following notations will be standard:

$$S^1a = Sa \cup \{a\}, aS^1 = aS \cup \{a\}, S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}.$$

Let S be a semigroup. For subsets A and B of S, we set $A \cdot B = \{ st : s \in A, t \in B \}$; we write $S^{[2]}$ for $S \cdot S$.

S is called *simple* if it contains no proper (two-sided) ideal.

Definition 1.1.3. Let S be a semigroup, and consider the Banach space

$$\ell^1(S) = \{f : S \to \mathbb{C} \mid \sum_{s \in S} \mid f(s) \mid < \infty\}.$$

We write δ_s for the characteristic function of $\{ s \}$ for $s \in S$, so $f \in \ell^1(S)$ has the form :

$$f = \sum_{s \in S} \alpha_s \delta_s,$$

where

$$\| f \|_1 = \sum_{s \in S} \mid \alpha_s \mid < \infty.$$

Definition 1.1.4. Let S be a semigroup, and let $f = \sum \alpha_r \delta_r$ and $g = \sum \beta_s \delta_s$ belong to $\ell^1(S)$. Set

$$f \star g = (\sum \alpha_r \delta_r) \star (\sum \beta_s \delta_s) = \sum_{t \in S} (\sum_{rs=t} \alpha_r \beta_s) \delta_t$$

where $\sum_{rs=t} \alpha_r \beta_s = 0$ when there are no elements r and s in S with $rs = t$. Let S be a semigroup, and set $A = (\ell^1(S), \star, \| \cdot \|_1)$. A with the usual pointwise addition, scalar multiplication, the product (convolution) $\star$ and with the norm $\| f \|_1$ is a Banach algebra called the *discrete semigroup algebra of S.*

Moreover if $S = G$ is a group then $\ell^1(S)$ is the discrete group algebra $\ell^1(G)$.

Definition 1.1.5. Let $\ell^\infty(S)$ be the space of all bounded complex-valued functions on S. The dual space of $\ell^1(S)$ is $\ell^\infty(S)$ with the duality

$$\langle f, \lambda \rangle = \sum_{s \in S} f(s)\lambda(s) \qquad (f \in \ell^1(S), \lambda \in \ell^\infty(S)).$$

Given a function f on S *the left (right) translation of* f *by* $x \in S$ is a function on S denoted by $l_x f$ such that $l_x f(s) = f(xs)$ (resp. $r_x f(s) = f(sx)$). A discrete semigroup S is *left amenable* if the space $\ell^\infty(S)$ admits a functional m, called *left invariant mean*, such that $m(\mathbf{1}) = 1 = \| m \|$ and $m(l_x f) = m(f)$, $x \in S$, $f \in \ell^\infty(S)$. Similarly one can defines *right amenable.* If S is both left and right amenable, it is *amenable.*

1.2 More about Banach algebras

Definition 1.2.1. Let A be an algebra. A *left A-module* is a linear space E over $\mathbb{C}$ and a map

$$(a, x) \mapsto a \cdot x : A \times E \to E,$$

such that :

$$a \cdot (\alpha x + \beta y) = \alpha a \cdot x + \beta a \cdot y, (\alpha a + \beta b) \cdot x = \alpha a \cdot x + \beta b \cdot x$$

$$a \cdot (b \cdot x) = ab \cdot x \qquad (\alpha, \beta \in \mathbb{C}, a, b \in A, x, y \in E).$$

A *right A-module* is defined similarly.

An *A-bimodule* is a space E which is both a left A-module and a right A-module and which is such that:

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \qquad (a, b \in A, x \in E).$$

A Banach space E is said to be a *Banach left A-module* if it is a left A-module and there exists $k > 0$ such that $\| ax \| \leq k \| a \| \| x \|$, $a \in A$, $x \in E$.

Banach right A-modules are defined similarly. E is said to be a *Banach A-bimodule* if it is both a Banach left A-module and a Banach right A-module and the module multiplications are related by $a(xb) = (ax)b$, $a, b \in A$, $x \in E$.

For any Banach algebra A, A itself is a Banach A-bimodule with the product of A giving the module multiplications. If E is a Banach left(right) A-module, then E^*, the conjugate space of E, is a Banach right (resp.left) A-module with the natural module multiplications defined by

$$\langle x, fa \rangle = \langle ax, f \rangle \ (\text{ resp.} \langle x, af \rangle = \langle xa, f \rangle), \tag{1.3}$$

for $f \in E^*, a \in A$ and $x \in E$. We call this module *the dual right* (resp. *left*) *module of* E. Here, for $x \in E$ and $f \in E^*$, $\langle x, f \rangle$ denotes the value $f(x)$. If E is a Banach A-bimodule, then multiplications given by Equation (1.3) make E^* into a Banach A-bimodule, called the *dual module of* E.

Since the only bimodules we are concerned with in the following are Banach bimodules we will refer to them simply as A-bimodules.

Suppose that X and Y are Banach spaces. We denote the *projective tensor product* of X and Y by $X \widehat{\otimes} Y$, and denote the elementary tensor of $x \in X$ and $y \in Y$ by $x \otimes y$; we refer to [51] for the details about this kind of product space.

Suppose that A is a Banach algebra. Then $A \widehat{\otimes} A$ is a Banach A-bimodule with the multiplications specified by

$$a \cdot (b \otimes c) = ab \otimes c, (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

We use π to denote the linear map from $A\widehat{\otimes}A$ into A specified by

$$\pi(a\otimes b) = ab \qquad (a, b \in A).$$

A *directed set* is a partially ordered set Λ (admitting Reflexivity, Antisymmetry and Transitivity) such that, given $\lambda_1, \lambda_2 \in \Lambda$, there exist $\lambda \in \Lambda$ with $\lambda \geq \lambda_k$ $(k = 1, 2)$. Let X be a topological space. A *net* in X is a mapping from a directed set Λ into X. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X is said to *converge* to $x \in X$, denoted by $\lim\limits_{\lambda\in\Lambda} x_\lambda = x$, if for every neighborhood U of x, there exist $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Amenability of Banach algebras has been one of the major themes in the homology theory of Banach algebras [34]. The definition of amenability was introduced by B.E. Johnson in 1972. Amenable Banach algebras have since proved themselves to be widely applicable in modern analysis (for example see [10] and [50]).

Definition 1.2.2. Suppose that A is a Banach algebra and let E be a Banach A-bimodule. A *(continuous) derivation* from A into E is a (continuous) linear mapping $D : A \to E$ which satisfies

$$D(ab) = a\cdot D(b) + D(a)\cdot b \qquad (a, b \in A).$$

For any $x \in E$, the mapping $ad_x : A \to E$ given by

$$ad_x(a) = a\cdot x - x\cdot a \qquad (a \in A),$$

is a continuous derivation, called an *inner derivation.* x is called the *implementing element* for ad_x. Denote by $Z^1(A, E)$ the space of all continuous derivations from A into E and by $N^1(A, E)$ the space of all inner derivations from A into E. Then $N^1(A, E)$ is a subspace of

$Z^1(A, E)$. The quotient space

$$H^1(A, E) = \frac{Z^1(A, E)}{N^1(A, E)}$$

is called the first *cohomology group* of A with coefficients in E. For the general theory of the Banach cohomology group $H^n(A, E)$, where $n \in \mathbb{N}$, see [40], [35] and [10].

A Banach algebra A is said to be *contractible* if $H^1(A, E) = 0$ for all Banach A-bimodules E, *amenable* if $H^1(A, E^*) = 0$ for all Banach A-bimodules E, and *weakly amenable* if $H^1(A, A^*) = 0$, where A^* denotes the dual space of A with natural A-bimodule action. Throughout, unless otherwise stated, by a derivation we mean a *continuous* derivation.

A Banach algebra A is *amenable* if for every A-bimodule E every derivation $D : A \longrightarrow E^*$ is inner (i.e. $\exists\, x \in E^*$ such that $D(a) = a \cdot x - x \cdot a \; \forall\, a \in A$).

Trivially, an amenable Banach algebra is weakly amenable; however the class of weakly amenable Banach algebras is considerably larger. See Example 4.1.1 below.

There are many alternative formulations of the notion of amenability, of which we note the following, for further details see [2, 6, 10, 35, 50].

The Banach algebra A is amenable if and only if any, and hence all, of the following hold, where $\pi : A \otimes A \longrightarrow A$ is the natural extension of the product map $a \otimes b \mapsto ab$:

(i) (Johnson [41]) A has a *bounded approximate diagonal*, that is, a bounded net $(m_i) \subset A \otimes A$ such that for each $x \in A$, $m_i \cdot x - x \cdot m_i \longrightarrow 0$, $\pi(m_i) \cdot x \longrightarrow x$;

(ii) (Johnson [41]) A has a *virtual diagonal*, that is, an element $M \in (A \otimes A)^{**}$ such that for each $x \in A$, $x \cdot M = M \cdot x$, $(\pi^{**} M) \cdot x = x$;

(iii) (Gourdeau [24]) any derivation of A into any Banach A-bimodule is the strong limit of a net of inner derivations which have a bounded net of implementing elements.

Definition 1.2.3. Let A be a normed algebra. A *left approximate identity* for A is a net $\{e_\lambda\}_{\lambda\in\Lambda}$ in A such that for all $x \in A$,

$$\lim_{\lambda\in\Lambda} e_\lambda x = x. \tag{1.4}$$

Right approximate identities are similarly defined by replacing $e_\lambda x$ with xe_λ in Equation (1.4). A *two-sided approximate identity* is a net that is both a left and a right approximate identity. If $\{e_\lambda\}$ is norm bounded, then we have a *bounded (left/right) approximate identity.*

B.E. Johnson [40] proved the following general implication:

Theorem 1.2.4. *If a Banach algebra A is amenable then A has a bounded approximate identity.*

Chapter 2

Amenability of $\ell^1(S)$

2.1 Amenability of semigroup algebras

The notion of amenability for Banach algebras is well-known as a general principle. The problem of determining which Banach algebras in certain classes are amenable is often a substantial problem; there are some major theorems. For example, the amenable C^*-algebras, the amenable group algebras, and the amenable measure algebras have been determined in famous theorems.

Let S be a semigroup, and let $\ell^1(S)$ be the corresponding semigroup algebra. We classify the semigroups S for which $\ell^1(S)$ is amenable.

Let us first recall the known theory of the amenability of Banach algebras on locally compact groups G. This result combines two famous theorems of B. E. Johnson [40, 43, 15].

Theorem 2.1.1. *Let G be a locally compact group. Then:*

(i) $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable group;

(ii) $L^1(G)$ is weakly amenable.

In particular, $L^1(G)$ is amenable for each locally compact abelian (LCA) group G. Given a semigroup S the problem of the amenability and weak amenability of $\ell^1(S)$ as a Banach algebra is rather more complicated. We will present some results regarding amenability of $\ell^1(S)$ as a Banach algebra and shall determine exactly when $\ell^1(S)$ is amenable as a Banach algebra. This is due to various authors [16, 12, 32, 31, 44].

We recall some further standard notions from semigroup theory.

Definition 2.1.2. An element $u \in S$ is *idempotent* if $uu = u$ and denote with E the set of idempotents. If S is commutative and satisfies the condition that each u in S is idempotent we call S a *semilattice.*

On E there is a usual order: $e, f \in E, e \leq f$ if $ef = fe = e$. An element $p \in E$ is *minimal* if $q = p$ whenever $q \in E$ with $q \leq p$.

Definition 2.1.3. A semigroup S is called *left cancellative* if for all $r, s, t \in S$, $rs = rt$ implies $s = t$. Similarly, we can define *right cancellative.*

Definition 2.1.4. A semigroup S is a *regular semigroup* if for each $s \in S$ there exists $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. If s^* is unique for each $s \in S$, we say that S is an *inverse semigroup.*

A group G is a regular semigroup with $E = \{e_G\}$, the bicyclic semigroup (defined in 3.3 below) is easily seen to be an elementary inverse semigroup.

Theorem 2.1.5 ([16], Theorem 8). *Let S be an inverse semigroup with E finite. Then $\ell^1(S)$ is amenable if and only if each maximal subgroup of S is amenable.*

Definition 2.1.6. Let S be a semigroup and suppose that there is a semilattice E and disjoint subsemigroups S_s ($s \in E$) of S such that $S = \bigcup_{s \in E} S_s$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ $(\alpha, \beta \in E)$.

Then S is called a *semilattice of the subsemigroups* S_α $(\alpha \in E)$. If E is a finite set, we say S is a *finite semilattice of subsemigroups*.

Definition 2.1.7. (a) A semigroup S is *left reversible* if for all $x, y \in S$, $xS \cap yS \neq \emptyset$.

(b) $H \subseteq S$ is a *left ideal group* if H is a left ideal in S, as well as being a group under the semigroup operation.

(c) The minimum ideal $K(S)$ is called the *kernel* of S in [38, §3.1].

There are several known partial results, which we summarize in the following theorem and which determine exactly when $\ell^1(S)$ is amenable as a Banach algebra.

Theorem 2.1.8. *Let S be a semigroup.*

(i) Suppose that S is abelian and $E = S$. Then $\ell^1(S)$ is weakly amenable [12, Proposition 10.5];

(ii) Suppose that $\ell^1(S)$ is an amenable Banach algebra. Then:

S is an amenable semigroup [16, Lemma 3];

S is (left and right) reversible [25], [52, Lemma 1];

S has only finitely many idempotents and each ideal I in S is regular and, in particular, $I = I^{[2]}$ [17, Theorem 2], S has a minimal idempotent;

$\ell^1(S)$ has an identity and $K(S)$ exists and is an amenable group [12, Corollary 10.6];

$\ell^1(S)$ is a semisimple algebra (this follows from [18, Theorem 5.11]);

S contains exactly one left ideal group S_0, which is also the only right ideal group; furthermore S_0 is amenable [44, Theorem 4.4].

(iii) Suppose that S is unital and left or right cancellative. Then $\ell^1(S)$ is amenable if and only if S is an amenable group [31, Theorem 2.3].

(iv) Suppose that S is abelian. Then $\ell^1(S)$ is amenable if and only if S is a finite semillatice of amenable groups [32, Theorem 2.7].

The force of these results seems to be that $\ell^1(S)$ is amenable if and only if S " is built up from amenable groups ". It can be shown that $\ell^1(S)$ is " left-amenable " if and only if S is a left-amenable semigroup [46]. In these results it is apparent that the condition of amenability imposes strong algebraic constraints on the semigroup.

In fact a characterization is given in [12, Theorem 10.12].

We proceed to describe a kind of semigroup which is the utmost importance in the algebraic theory of semigroups (see [5, §3.1]).

Definition 2.1.9. Let G be a group, I and Λ be arbitrary non-empty sets, and $G^o = G \cup \{0\}$ be a group with zero adjoined (see Definition 1.1.2). A *sandwich matrix* $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries being elements of G^o such that each row and column of P has at least one non-zero entry. The set $S = G \times I \times \Lambda$ with the composition

$$(a,i,j) \circ (b,l,k) = (aP_{jl}b, i, k), \qquad (a,i,j),(b,k,l) \in S$$

is a semigroup that we denote by $\mathcal{M}(G; I, \Lambda; P)$.

Similarly if P is a $\Lambda \times I$ matrix over G^o, then $S = G \times I \times \Lambda \cup \{0\}$ is a semigroup under the following composition operation:

$$(a,i,j) \circ (b,l,k) = \begin{cases} (aP_{jl}b, i, k) & \text{if } P_{jl} \neq 0 \\ 0 & \text{if } P_{jl} = 0 \end{cases}$$

$$(a,i,j) \circ 0 = 0 \circ (a,i,j) = 0 \circ 0 = 0.$$

This semigroup which is denoted by $\mathcal{M}^o(G; I, \Lambda; P)$ also can be described in the following way. An $I \times \Lambda$ matrix A over G^o that has at most one nonzero entry $a = A(i, j)$ is called a *Rees $I \times \Lambda$ matrix over G^o* and is denoted by $(a)_{ij}$. The set of all Rees $I \times \Lambda$ matrices over G^o form a semigroup under the binary operation $A \cdot B = APB$, which is called *the Rees $I \times \Lambda$ matrix semigroup over G^o with the sandwich matrix P* and is isomorphic to $\mathcal{M}^o(G; I, \Lambda; P)$ [37, pp. 61-63].

Definition 2.1.10. A *principal series of ideals* for S is a chain

$$S = I_1 \supsetneq I_2 \supsetneq \ldots \supsetneq I_{m-1} \supsetneq I_m = K(S)$$

where I_1, I_2,...,I_m are ideals in S and there is no ideal of S strictly between I_j and I_{j+1} for each $j \in \mathbb{N}_{m-1}$.

Theorem 2.1.11 ([12], Theorem 10.12). *Let S be a semigroup. Then the Banach algebra $\ell^1(S)$ is amenable if and only if the minimum ideal $K(S)$ exists, $K(S)$ is an amenable group, and S has a principal series $S = I_1 \supsetneq I_2 \supsetneq \ldots \supsetneq I_{m-1} \supsetneq I_m = K(S)$ such that each quotient I_j/I_{j+1} is a regular Rees matrix semigroup of the form $\mathcal{M}^o(G, P, n)$, where $n \in N$, G is an amenable group, and the sandwich matrix P is invertible in $\mathbb{M}_n(\ell^1(G))$.*

Definition 2.1.12. The *Brandt semigroup S over a group G with index set I* is the semigroup consisting of all canonical $I \times I$ matrix units over $G \cup \{0\}$ and a zero matrix Θ. Writing $S = \{(g)_{ij} : g \in G, i, j \in I\} \cup \{\Theta\}$, where $(g)_{ij}$ is the matrix with (k, l)-entry equal to g if $(k, l) = (i, j)$ and 0 if $(k, l) \neq (i, j)$ we get

$$(g)_{ij} \cdot (h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ \Theta & j \neq k \end{cases}$$

Theorem 2.1.13 ([16], Theorem 7). *Let S be the Brandt semigroup over a group G with finite index set I. Then $\ell^1(S)$ is amenable if and only if G is amenable.*

For the situation in which the index set is infinite the result is false.

Theorem 2.1.14 ([16], Theorem 12). *Let S be a Brandt semigroup with an infinite index set over an arbitrary group. Then $\ell^1(S)$ is not amenable.*

2.2 Amenability of $\ell^1(S,\omega)$

Definition 2.2.1. A *weight* (function) ω on a semigroup S is a function from S to the positive reals, satisfying $\omega(st) \leq \omega(s)\omega(t)\ \forall\ t, s \in S$.

Then $\ell^1(S,\omega)$ is the Banach space of functions f from S to $\mathbb{C}$ for which $\sum_{s\in S} |f(s)|\omega(s) < \infty$, this sum being the norm of f, which we denote by $\|f\|_\omega$.

Its dual can be identified with $\ell^\infty(S,\omega^{-1})$, the Banach space of functions $\phi : S \to \mathbb{C}$ for which $\sup_{s\in S}\{\frac{|\phi(s)|}{\omega(s)}\} < \infty$, the norm of ϕ being this supremum.

We define the convolution of two functions $f, g \in \ell^1(S,\omega)$ by $f \star g(s) = \sum_{uv=s} f(u)g(v)$.

With multiplication taken to be convolution, $\ell^1(S,\omega)$ becomes a Banach algebra.

In [31] N. Gronbaek gives a complete description of amenability of $\ell^1(G,\omega)$, where G is an infinite group. $\ell^1(G,\omega)$ is often called a *Beurling algebra.* Put

$$\Omega(g) = \omega(g)\omega(g^{-1}) \qquad (g \in G).$$

Theorem 2.2.2 ([31], Theorem 3.2). *Let G be a discrete group. Then the following are equivalent:*

(i) $\ell^1(G,\omega)$ *is amenable;*

(ii) $\ell^1(G,\Omega)$ *is amenable;*

(iii) there is a bounded net $(\sum_g \mu_g^i \delta_g)_{i\in I}$ *in* $\ell^1(G,\Omega)$ *satisfying:*

(a) $\lim_i(\sum_g \mu_g^i) \to 1;$

(b) $\lim_i \|\delta_h \mu^i - \mu^i\|_\Omega \to 0 \qquad (h \in G).$

(iv) there is a positive left-invariant mean on $\ell^\infty(G,\Omega^{-1})$*;*

(v) G *is amenable and* $\sup\{\,\Omega(g) \mid g \in G\,\} < \infty$.

Also N. Gronbaek obtained the following generalization of [1, Theorem 2.1].

Corollary 2.2.3 ([31], Corollary 3.3). *If* G *is an abelian group then* $\ell^1(G,\omega)$ *is amenable if and only if*

$$\sup\{\,\omega(g)\omega(g^{-1}) \mid g \in G\,\} < \infty.$$

In Theorem 2.2.2, on the basis of the amenability criterion obtained by A. Ya. Helemskii (see [34]) it was proved that an algebra $\ell^1(G,\omega)$ is amenable if and only if the group G is amenable and the associated weight Ω is bounded above. As a consequence of this assertion R.I. Grigorchuck obtained the following statement.

Theorem 2.2.4 ([27], Theorem 2). *The algebra* $\ell^1(G,\omega)$ *is amenable if and only if the group* G *is amenable and the weight* ω *is equivalent to some multiplicative character* $\chi : G \to \mathbb{R}_+$.

Thus, up to equivalence, the only amenable Beurling algebras are those of the form $\ell^1(G,\chi)$, where G is an amenable group and $\chi : G \to \mathbb{R}_+$ is a multiplicative character.

Similar remarks apply for weighted convolution semigroup algebras. In fact, most of the previous theorems are consequences of the following results due to J. Duncan and A.L.T. Paterson [17]. We first introduce some notation.

$[uu^{-1}] = \{x \in S : xu = u\}$,

$[u^{-1}u] = \{x \in S : ux = u\}$,

$X(u) = uS \bigcap [uu^{-1}]$.

Theorem 2.2.5. *Let S be a semigroup that contains an infinite pairwise disjoint sequence of sets $X(u_n)$. Then $\ell^1(S, \omega)$ is not amenable for any weight function ω.*

Corollary 2.2.6. *Let S be an inverse semigroup with E infinite. Then $\ell^1(S, \omega)$ is not amenable for any weight ω.*

Corollary 2.2.7. *Let S be a left cancellative semigroup with identity with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a group.*

Corollary 2.2.8. *Let S be an abelian semigroup with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a finite semilattice of abelian groups.*

Theorem 2.2.9. *Let S be a semigroup with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a regular semigroup with E finite.*

G.H. Esslamzadeh introduced in [18] the l^1-Munn algebras, defined as follows.

Definition 2.2.10. Let $\mathcal{A}$ be a unital Banach algebra, I and J be arbitary index sets, and P be a $J \times I$ nonzero matrix over $\mathcal{A}$ such that $\| P \|_\infty = \sup\{ \| P_{ji} \| : j \in J, i \in I \} \leq 1$. Let $\mathcal{LM}(\mathcal{A}, P)$ be the vector space of all $I \times J$ matrices A over $\mathcal{A}$ such that

$$\| A \|_1 = \sum_{i \in I, j \in J} \| A_{ij} \| < \infty.$$

Then $\mathcal{LM}(\mathcal{A}, P)$ with the product $A \circ B = APB$, $A, B \in \mathcal{LM}(\mathcal{A}, P)$, and the ℓ^1-norm is a Banach algebra that is called the *ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P* or, briefly, the *ℓ^1-Munn algebra.*

G.H. Esslamzadeh proved the following results for weighted semigroup algebras but for simplicity he considered just the unweighted case.

Lemma 2.2.11 ([18], Lemma 5.2). *If S is a regular semigroup with E finite, then S has a principal series $S = S_1 \supset S_2 \supset S_3 \supset \ldots \supset S_m \supset S_{m+1} = \emptyset$. Moreover for every $k = 1, \ldots, m-1$ there are natural numbers n_k, l_k, a group G_k and a regular $l_k \times n_k$ matrix P_k on G_k^o such that $S_k/S_{k+1} = \mathcal{M}^o(G_k, P_k)$. Also $S_m = \mathcal{M}(G_m, P_m)$ for some $l_m \times n_m$ matrix P_m over a group G_m.*

Theorem 2.2.12 ([18], Theorem 5.9). *With the notations of Lemma 2.2.10 the following conditions are equivalent:*

(i) $\ell^1(S)$ is amenable.

(ii) $\mathcal{LM}(\ell^1(G_k), P_k)$ has an identity and $\ell^1(G_k)$ is amenable, $k = 1, \ldots, m$.

Definition 2.2.13. For a in a semigroup S, $J(a)$ is the principal ideal of $S^1 a S^1$ and J_a is the set of elements $b \in J(a)$ such that $J(b) = J(a)$. The inclusion among the principal ideals induces the following order among the equivalence classes J_a: $J_a \leq J_b$ if $J(a) \subseteq J(b)$ ($J_a < J_b$ if $J(a) \subsetneq J(b)$). Let $I(a)$ denote the ideal $\{\, b \in J(a) : J_b < J_a \,\}$, i.e., $I(a) = J(a) \backslash J_a$. The factors $J(a)/I(a)$, $a \in S$ are called the *principal factors* of S.

Proposition 2.2.14. *For a semigroup S, $\ell^1(S)$ is amenable if and only if S has a principal series $S = S_1 \supset S_2 \supset S_3 \supset \ldots \supset S_m \supset S_{m+1} = \emptyset$ and $\ell^1(T)$ is amenable for every principal factor T of S.*

G.H. Esslamzadeh also gave a generalization of [16, Theorem 8].

Theorem 2.2.15. *Let S be a regular semigroup with a finite number of idempotents. The following conditions are equivalent:*

(i) $\ell^1(S)$ is amenable.

(ii) every maximal subgroup of S is amenable and $\ell^1(T)$ is semisimple for every principal factor T of S.

In particular if $\ell^1(S)$ is amenable, then it is semisimple.

2.3 Amenability of $\ell^1(\beta S)$

In this section we consider the two products $\Box$ and $\Diamond$ on the Stone-Cech compactification βS of S such that $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups. They are the topics of the monograph [36].

Measure algebra $M(G)$: Let G be a locally compact group. The *measure algebra* $M(G)$ is the unital Banach algebra of all finite complex regular Borel measures on G, with the convolution product defined by

$$\langle f, \mu * \nu \rangle = \int_G (\int_G f(gh) d\mu(g)) d\nu(h),\ \mu, \nu \in M(G) \text{ and } f \in C_0(G),$$

where $C_0(G)$ is the space of all continuous functions on G vanishing at infinity.

H. G. Dales, F. Ghahramani and A. Ya. Helemskii proved in [11] that a measure algebra is amenable if and only if G is a discrete and amenable group.

Recently it was proved in [12] that amenability and weak amenability of $\ell^1(S)$ is related to the amenability and weak amenability of $M(\beta S)$.

Proposition 2.3.1 ([12], Lemma 11.6). *Let S be a semigroup such that $M(\beta S, \Box)$ is amenable. Then $\ell^1(S)$ is amenable, S is amenable, S has a finite group ideal, E is finite, and each ideal in S is regular. Further $\ell^1(\beta S, \Box)$ is amenable.*

Theorem 2.3.2 ([12], Theorem 11.9). *Let S be a semigroup such that $\ell^1(\beta S, \Box)$ is an amenable Banach algebra. Then S is finite.*

Definition 2.3.3. A semigroup S is *weakly cancellative* if for any $a, b \in S$, the sets $\{x \in S : xa = b\}$, $\{y \in S : ay = b\}$ are finite.

Proposition 2.3.4 ([12], Proposition 11.13). *Let S be a weakly cancellative semigroup. Suppose that $M(\beta S, \Box)$ is weakly amenable. Then $\ell^1(S)$ is weakly amenable.*

Chapter 3

Weak amenability of $\ell^1(S)$

3.1 Weak amenability of discrete semigroup algebras

The notion of weak amenability for commutative Banach algebras was introduced by W. G. Bade, P. C. Curtis, Jr., and H. G. Dales in [1], and in the general case in [43].

Recall that a Banach algebra A is *weakly amenable* if every derivation $D : A \longrightarrow A^*$ is inner (i.e. $\exists\, x \in A^*$ such that $D(a) = a \cdot x - x \cdot a \;\forall\, a \in A$).
The question whether bounded derivations are necessarily zero has also been considered in [28] and [29]. As the name of concept suggests, weak amenability is derived from the stronger concept of amenability and a principal aim of the paper [1] was to exhibit classes of weakly amenable Banach algebras which are not amenable. It is noted that a commutative Banach algebra is weakly amenable if and only if $H^1(A, E) = 0$ for each symmetric Banach A-module E.

From here on, the term *weakly amenable* will be abbreviated to WA. It is known that $\ell^1(G)$ is weakly amenable for all groups G (in fact $L^1(G)$ is WA for all locally compact groups

G [43]). For the case $\ell^1(G)$, Johnson [42] gives an explicit construction for the implementing element of the inner derivation. Grigorchuk [26, Remark 1.16] provides motivation for the study of cohomology over semigroups.

Throughout S denotes a (discrete) semigroup.

Definition 3.1.1. A *(generalized) inverse* of $s \in S$ is an element $t \in S$ such that $sts = s, tst = t$. If s has an inverse it is called *regular* and if not *singular.* A *completely regular* element is one for which there is $t \in S$, $sts = s$ and $ts = st$ (then tst is an inverse for s). A semigroup is called *(completely) regular* if each of its elements is (completely) regular.

The completely regular elements of a semigroup are those which lie in a subgroup. Completely regular semigroups are those which can be regarded as the disjoint unions of their maximal subgroups.

T.D. Blackmore studied in [4] weak amenability of discrete semigroups algebras where S is completely regular and commutative.

Theorem 3.1.2 ([4], Theorem 3.6 and [33], Corollary 2.8). *If S is completely regular or is a commutative union of groups then $\ell^1(S)$ is WA.*

3.2 Weak amenability of $\ell^1(S,\omega)$ where S is commutative

T.D. Blackmore proved that for S commutative, certain conditions ensure that $\ell^1(S,\omega)$ is not WA for any weight ω. We state his results below [4].

Proposition 3.2.1. *If S is commutative and $\ell^1(S,\omega)$ is WA for some weight ω then $\ell^1(S)$ is WA.*

We denote the set of singular elements by N_S.

Theorem 3.2.2. *Suppose that S is a commutative semigroup and satisfies one of the following:*

i) There is $s \in N_S$ such that for all $t \in N_S \setminus \{s\}$, $s \notin S^1 t$;

ii) S contains one singular element (only);

iii) There exists non-empty $M \subseteq N_S$ such that for all $s, t \in M$, $s \in S^1 t$ and for all $s \in M$ and $u \in N_S \setminus M$, $s \notin S^1 u$;

iv) There exists non-empty, finite subset M of N_S, such that if $s \in M$ and $u \in N_S \setminus M$ then $s \notin S^1 u$;

v) S has a homomorphic image T such that $N_T \neq \emptyset$ is finite.

Then $\ell^1(S, \omega)$ is not WA for any weight ω.

Proposition 3.2.3. *Let S be a commutative semigroup and T a homomorphic image of S. Then $\ell^1(T)$ is WA if $\ell^1(S)$ is WA.*

Proof. The homomorphic image of a commutative WA Banach algebra is WA. □

Proposition 3.2.4. *If S is a commutative, finite semigroup then $\ell^1(S)$ is amenable if (and only if) it is WA.*

Theorem 3.2.5. *Let S be a semigroup, possibly with zero 0. If S satisfies the conditions C1 to C3 given below then $\ell^1(S)$ is WA.*

C1. Whenever $u, v, w, z \in S$ are such that $uv = wz \neq 0$ then there is an $a \in S$ with $v = az$ and $w = ua$.

C2. If $u, v, w, z \in S \setminus \{0\}$ are such that $vu = 0$ and $uv = wz$ then $zw = 0$.

C3. If $u, v \in S \setminus \{0\}$ are such that $uv = vu = 0$ then there are b and c in S with $u = bc$ and $cv = vb = 0$.

These conditions are satisfied if S is a Rees matrix semigroup. Hence:

Corollary 3.2.6. *If S is a Rees matrix semigroup then $\ell^1(S)$ is WA.*

Also in [33] N. Gronbaek characterized the weak amenability of $\ell^1(S,\omega)$ for certain commutative semigroups, in terms of the non-existence of homomorphisms from certain subsets of these semigroups into the semigroup $(\mathbb{C},+)$ that satisfy a boundedness condition depending on ω.

Definition 3.2.7. On a commutative semigroup S we define the preorder $s < t$ by $t \in s+S$. We define the following sets:

$$V(t) = \{\, s \mid s < t \,\}$$

$$V(t)^* = \{\, f \in \mathbb{C}^{V(t)} \mid f(s_1+s_2) = f(s_1)+f(s_2), s_1+s_2 \in V(t) \,\}.$$

Proposition 3.2.8 ([33], Proposition 4.2). *Suppose that there exist $t \in S$ and $f \in V(t)^* \setminus (0)$ such that*

$$\sup\{ \frac{|f(s)|}{\omega(s)\omega(u)} \mid s+u = t \} = \alpha < \infty.$$

Then $\ell^1(S,\omega)$ is not WA.

Theorem 3.2.9 ([33], Theorem 4.7). *Suppose that S is a commutative semigroup and satisfies one of the following:*

(a) S is a cancellative semigroup;

(b) Every element of S is divisible by some $n \in \mathbb{N}$, $n \geq 2$;

(c) If $u \in S\backslash(0)$ then $\langle V(u) \rangle = S$, where $\langle V(u) \rangle$ is the subsemigroup generated by $V(u)$.

Then for every weight $\omega : S \longrightarrow \mathbb{R}_+$, $\ell^1(S,\omega)$ is WA if and only if

$$\{ f \in V(t)^* \mid \sup\{ \frac{|f(s)|}{\omega(s)\omega(u)} \mid s+u=t \} < \infty \} = \{ 0 \}$$

for all $t \in S$.

A consequence of these results is that if G is an abelian group then $\ell^1(G,\omega)$ is WA if and only if there are no non-zero homomorphisms, ϕ, from G into $(\mathbb{C},+)$ for which

$$\sup\{ \frac{|\phi(g)|}{\omega(g)\omega(g^{-1})} : g \in G \} < \infty.$$

See [33, Corollary 4.8].

In [48] it is shown that if G is an abelian group and the weight ω satisfies

$$\lim_{n\to\infty} \frac{\omega(g^n)\omega(g^{-n})}{n} = 0$$

for every $g \in G$, then $\ell^1(G,\omega)$ is WA.

3.3 $H^1(\ell^1(S), \ell^\infty(S))$ and $H^1(\ell^1(S), \ell^1(S))$ for some classes of semigroups

In this section we present the results obtained by S. Bowling and J. Duncan who investigated in [3] two notions of cohomological triviality for Banach algebras: weak amenability and cyclic amenability.

Definition 3.3.1. Recall that S is a *Clifford semigroup* if it is an inverse semigroup with each idempotent central (i.e. $ex = xe$ for every idempotent e and every x in S), or equivalently, if it is a (strong) semilattice of groups (see [37, Chapter IV]). So we can write the

Clifford semigroup as $S = \cup\{ G_e : e \in E \}$ where E is the semilattice of idempotents and each G_e is a group.

B. E. Johnson and J. R. Ringrose proved in [39] that if G is a group then $H^1(\ell^1(G), \ell^1(G)) = \{0\}$. An example given in [3] shows that this conclusion fails for Clifford semigroups, in general.

We recall that a derivation $D : \ell^1(S) \to \ell^\infty(S)$ is *cyclic* if $Ds[t] + Dt[s] = 0$ for all $s, t \in \ell^1(S)$. Every inner derivation is cyclic. We write $H^1_\lambda(\ell^1(S), \ell^\infty(S))$ for the bounded cyclic derivations modulo the inner derivations and we say that $\ell^1(S)$ is *cyclically amenable* if $H^1_\lambda(\ell^1(S), \ell^\infty(S)) = \{0\}$.

Theorem 3.3.2 ([3]). *$H^1(\ell^1(S), \ell^\infty(S)) = \{0\}$ and $H^1_\lambda(\ell^1(S), \ell^\infty(S)) = \{0\}$ for any Clifford semigroup S and for any Rees semigroup S.*

Definition 3.3.3. If A is a non-empty set, let us denote by F_A the set of all non-empty finite words $a_1a_2...a_m$ in the "alphabet" A. A binary operation is defined on F_A by juxtaposition:

$$(a_1a_2...a_m)(b_1b_2...b_m) = a_1a_2...a_mb_1b_2...b_m$$

With respect to this operation F_A is a semigroup, called the *free semigroup* on A. The set A is called the *generating set* of A.

Example 3.3.1. *The bicyclic semigroup* is the semigroup $S_1 = \langle e, p, q \mid pq = e \rangle$,

$$S_1 = \{ q^m p^n \mid m \geq 0, n \geq 0 \}.$$

Then $\ell^1(S_1)$ is not weakly amenable [3]. It is proved that $H^1(\ell^1(S_1), \ell^\infty(S_1)) \cong \ell^\infty(\mathbb{N})$ and $H^1_\lambda(\ell^1(S_1), \ell^\infty(S_1)) = 0$.

Let $FC2$ denote the free commutative semigroup on generators p, q and let $F2$ denote the corresponding free semigroup. It is also proved that

$$H^1(\,\ell^1(FC2), \ell^\infty(FC2)\,), H^1_\lambda(\,\ell^1(FC2), \ell^\infty(FC2)\,) \simeq \ell^\infty(\mathbb{N}).$$

The result clearly extends to any finite generators. In fact, for a semigroup S of a countable number of generators, the $H^1(\ell^1(S), \ell^\infty(S))$ is always isomorphic to $\ell^\infty(\mathbb{N})$. The situation is similar for the non-commutative case but the argument is more difficult.

Theorem 3.3.4. $H^1(\,\ell^1(F2), \ell^\infty(F2)\,) \simeq \ell^\infty(\mathbb{N})$, $H^1_\lambda(\,\ell^1(F2), \ell^\infty(F2)\,) = 0$ *[30].*

Theorem 3.3.5. *i) Let $S = \cup\{\,G_e : e \in E\,\}$ be a Clifford semigroup with identity 1 and suppose that $eG_1 = G_e$ for every $e \in E$. Then $H^1(\,\ell^1(S), \ell^1(S)\,) = 0$.*
ii) Let $S = \cup\{\,G_e : e \in E\,\}$ be a Clifford semigroup with E finite. Then $H^1(\,\ell^1(S), \ell^1(S)\,) = 0$.

S. Bowling and J. Duncan showed that $H^1(\ell^1(S), \ell^1(S)) = 0$ for the bicyclic semigroup, the free commutative semigroup on two generators and the free semigroup on two generators, i.e.

Theorem 3.3.6. $H^1(\,\ell^1(S_1), \ell^1(S_1)\,) = H^1(\,\ell^1(FC2), \ell^1(FC2)\,) = H^1(\,\ell^1(F2), \ell^1(F2)\,) = 0$.

Chapter 4

Approximate amenability of $\ell^1(S)$

4.1 Approximate amenability of a Banach algebra

Based on the characterizations of amenability of a Banach algebra, F. Ghahramani, R.J. Loy and Y. Zhang have introduced several new notions of amenability. In particular, by dropping the requirement that aforementioned nets are bounded, definitions of *approximate* and *pseudo-amenability* were given [20, 22, 21]. The corresponding class of Banach algebras is larger than that of the amenable algebras.

Definition 4.1.1. Let A be a Banach algebra and let E be a Banach A-bimodule. A continuous derivation $D : A \to E$ is *approximately inner* if there is a net ξ_α in E such that

$$D(a) = \lim_{\alpha} ad_{\xi_\alpha}(a) \qquad (a \in A),$$

where the limit is taken with respect to the norm topology on E. When E is a dual module, D is *weak*-approximately inner* if the net converges with respect to the weak*-topology.

Definition 4.1.2. Let A be a Banach algebra.

(i) A is *approximately amenable* if, for each Banach A-bimodule E, every derivation $D : A \longrightarrow E^*$ is approximately inner.

(ii) A is *approximately contractible* if, for each Banach A-bimodule E, every derivation $D : A \longrightarrow E$ is approximately inner.

The qualifier *sequential* prefixed to the above definitions specifies that there is a sequence of inner derivations approximating each given derivation. Similarly, the qualifier *weak** prefixed to the definition of approximate amenability specifies that the convergence is in the *weak** topology over E^*. Moreover, if the implementing net ad_{ξ_α} can be chosen to be bounded in an approximately amenable (contractible) Banach algebra, we call it *boundedly approximately amenable (contractible).*

Of course, each amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras which are not amenable are constructed in [20] and [23].

Definition 4.1.3. Let A be a Banach algebra. A is *weakly approximately amenable* if every derivation $D : A \longrightarrow A^*$ is approximately inner, where A^* denotes the dual space of A with natural A-bimodule action.

Definition 4.1.4. A Banach algebra A is *pseudo-amenable* if there is a net $(u_\alpha) \in A \otimes A$ called an *approximate diagonal* for A, such that $\lim_\alpha (a \cdot u_\alpha - u_\alpha a) = 0$ and $\lim_\alpha \pi(u_\alpha) a = a$ for each $a \in A$.

All the notions of approximate amenability concern with the question of whether every derivation $D : A \longrightarrow E^*$ is approximately inner.

We know that when G is a locally compact group, amenability, approximate amenability, and pseudo-amenability coincide for the group algebra $L^1(G)$ [20, 22].

The Beurling algebra: Recall that a weight ω on a locally compact group G is a continuous function $\omega : G \to (0, \infty)$ satisfying $\omega(xy) \leq \omega(x)\omega(y)$, $(x, y \in G)$. For a locally compact group G and a weight ω on G, $L^1(\omega) = L^1(G, \omega)$ is a Banach algebra under convolution, called the *Beurling algebra* corresponding to ω. The weight ω is *symmetric* if

$$\omega(g) = \omega(g^{-1}) \qquad (g \in G).$$

For any weight ω, its *symmetrization* is the weight defined by

$$\Omega(g) = \omega(g)\omega(g^{-1}) \qquad (g \in G).$$

The following is essentially in [31].

Theorem 4.1.5. *Let G be a locally compact group, ω a weight on G with $\omega(e) = 1$. The following are equivalent:*

(i) $L^1(\omega)$ is amenable;

(ii) $L^1(\Omega)$ is amenable;

(iii) G is amenable and Ω is bounded.

Recently F. Ghahramani, R.J. Loy and Y. Zhang proved in [21] that the Theorem 4.1.5 is still true without condition $\omega(e) = 1$. They also gave a direct proof of (iii) $\Rightarrow$ (i) by constructing a diagonal for $L^1(\omega)$.

We can say more in the special case $G = \mathbb{Z}$. Let ω be a weight on $\mathbb{Z}$ such that $\omega(0) = 1$. Let

$$\ell^1(\omega) = \{\, a = (a(n) : n \in \mathbb{Z}) : \sum_{-\infty}^{\infty} \mid a(n) \mid \omega(n) < \infty \,\}.$$

Then $\ell^1(\omega)$ is a commutative Banach algebra with respect to convolution multiplication

$$(a * b)(n) = \sum_{-\infty}^{\infty} a(n-k)b(k) \qquad (n \in \mathbb{Z}),$$

and the norm

$$\| a \| = \sum_{-\infty}^{\infty} | a(n) | \omega(n) \qquad (a \in \ell^1(\omega)).$$

An algebra of this type is *a Beurling algebra on* $\mathbb{Z}$.

Theorem 4.1.6 ([1], Theorem 2.2). *Let ω be a weight sequence on $\mathbb{Z}$ such that*

$$\frac{\omega(n)\omega(-n)}{n} \longrightarrow 0$$

as $n \to \infty$. Then the Beurling algebra $\ell^1(\omega)$ is weakly amenable.

Theorem 4.1.7 ([1], Theorem 2.3). *Let ω be a weight sequence on $\mathbb{Z}$ such that*

$$\sup\{ \frac{\omega(m+n)}{\omega(m)\omega(n)} (\frac{1+ | n |}{1+ | m+n |}) : m, n \in \mathbb{Z} \}$$

is finite. Then the Beurling algebra $\ell^1(\omega)$ is not weakly amenable.

Theorem 4.1.8 ([1]). *Let* $\omega_\alpha(n) = (1+ | n |)^\alpha \qquad (n \in \mathbb{Z}), \alpha \geq 0.$

(i) If $\alpha = 0$, then $\ell^1(\omega_\alpha)$ is amenable.

(ii) If $\alpha > 0$, then $\ell^1(\omega_\alpha)$ is not amenable.

(iii) If $0 \leq \alpha < 1/2$, then $\ell^1(\omega_\alpha)$ is weakly amenable.

(iv) If $\alpha \geq 1/2$, then $\ell^1(\omega_\alpha)$ is not weakly amenable.

Theorem 4.1.9 ([21], Corollary 8.5). *The Beurling algebras $\ell^1(\mathbb{Z}, \omega)$, $\omega(n) = (1+ | n |)^\alpha$ with $\alpha > 0$, are not sequentially approximately amenable.*

Proposition 4.1.10 ([21], Proposition 8.1). *Suppose the weight ω is bounded away from 0, and that $L^1(G)$ is approximately amenable. Then G is amenable.*

It is conjectured in [21] that $L^1(\omega)$ will fail to be approximately amenable whenever $\Omega \to \infty$ and a weaker result is proved. Suppose that G is a locally compact group, ω a continuous weight on G. Define

$$\hat{\omega}(x) = \lim_{r\to\infty}\inf \frac{\omega(rx)}{\omega(r)} \qquad (x \in G).$$

Theorem 4.1.11 ([21], Theorem 8.4). *Let ω be a weight function on G.*

(1) Suppose that there is a net $(r_\alpha) \subset G$ such that $\lim_\alpha r_\alpha = \infty$ and $(\omega(r_\alpha^{-1})\omega(r_\alpha))$ is bounded. Then $L^1(\omega)$ is boundedly approximately contractible if and only if it is amenable;

(2) Suppose that $\lim_{x\to\infty} \hat{\omega}(x^{-1})\omega(x) = \infty$. Then $L^1(\omega)$ is not boundedly approximately amenable.

M. Lashkarizadeh Bami and H. Samea studied in [47] the approximate amenability of the discrete semigroup algebras $\ell^1(S)$ for left cancellative semigroups S. It is shown that a left cancellative semigroup S (not necessarily with identity) is left amenable whenever the Banach algebra $\ell^1(S)$ is approximately amenable. The converse is not true. As a consequence it is proved that if S is a finite semigroup and $\ell^1(S)$ is approximately amenable, then S is amenable. Also for finite-dimensional Banach algebras A, approximate amenability and amenability are equivalent. Therefore, for a finite semigroup S, approximate amenability and amenability of $\ell^1(S)$ are equivalent.

Corollary 4.1.12 ([47], Corollary 1.11). *Let S be a Brandt semigroup over an amenable group G with infinite index set. Then $\ell^1(S)$ is approximately amenable but not amenable.*

The Corollary 4.1.12 shows that in general the approximate amenability of a semigroup algebra is not equivalent to its amenability.

In [7] ℓ^1-convolution algebras of totally ordered sets are studied.

Let Λ be a non-empty, totally ordered set, and regard it as a semigroup by defining the product of two elements to be their maximum. The resulting semigroup which is denoted by $\Lambda_\vee$, is a semilattice. For every $t \in \Lambda_\vee$ denote the point mass concentrated at t by e_t. The definition of multiplication in $\ell^1(\Lambda_\vee)$ ensures that $e_s e_t = e_{max(s,t)}$ for all s and t.

Theorem 4.1.13 ([7], Theorem 6.1). *Let $\mathcal{T}$ be any totally ordered set. Then $\ell^1(\mathcal{T}_\vee)$ is boundedly approximately contractible.*

Theorem 4.1.14 ([7], Theorem 6.4). *Let Λ be an uncountable well-ordered set. Then $\ell^1(\Lambda_\vee)$ is not sequentially approximately amenable.*

While sequential approximate amenability implies bounded approximate amenability, the converse is false from the previous two theorems.

Other examples of semigroup algebras of the form $\ell^1(S)$ that are approximately amenable but not amenable are given in [12].

Example 4.1.1. Let S be the semigroup $\mathbb{N}$ with product $mn = min\{m, n\}$ and take $A_\wedge = \ell^1(S)$ with convolution product. Because $A_\wedge$ is abelian and $E(S) = S$ according to Theorem 2.1.8 (i), $A_\wedge$ is weakly amenable but not amenable [17, Theorem 2]. It is sequentially approximately amenable from Theorem 4.1.14. See also [21, Example 4.6].

4.2 Amenability of bicyclic and partially bicyclic semigroups

It is known that if $\ell^1(S)$ is approximately amenable, then S must be a regular amenable semigroup [21]. In section 4.4 we will show that the converse is not true by examining the bicyclical semigroup S_1. We prove that $\ell^1(S_1)$ is not approximately amenable.

In this section, we reveal the class of partially bicyclic semigroups. We have already defined the *bicyclic semigroup* in Section 3.3. It is the semigroup generated by a unit e and two more elements p and q subject to the relation $pq = e$. We denote it by

$$S_1 = \langle\, e, p, q \mid pq = e \,\rangle.$$

Many of its properties can be found in [5, §2.7].

The semigroup generated by a unit e and three more elements a, b and c subject to the relations $ab = ac = e$ is denoted by

$$S_2 = \langle\, e, a, b, c \mid ab = e, ac = e \,\rangle;$$

and the semigroup generated by a unit e and four elements a, b, c, d subject to the relations $ac = bd = e$ is denoted by

$$S_{1,1} = \langle\, e, a, b, c, d \mid ac = e, bd = e \,\rangle.$$

S_2 and $S_{1,1}$ will be called *partially bicyclic semigroups.*

J. Duncan and I. Namioka showed in [16] that S_1 is an amenable semigroup by studying the maximal group homomorphic image of S_1. In [45] A.T.-M. Lau and Y. Zhang showed the same result directly by constructing a left and right invariant mean on $\ell^\infty(S_1)$ and also proved that the partially bicyclic semigroups S_2 and $S_{1,1}$ are not left amenable and S_2 is right amenable.

Because of the recent interest in approximate amenability, we shall give attention to this case.

Theorem 4.2.1 ([21], Theorem 9.2). *Let S be a semigroup such that $\ell^1(S)$ is approximately amenable. Then:*

i) S is regular;

ii) S is amenable.

Since the partially bicyclic semigroups S_2 and $S_{1,1}$ are not amenable, the Banach algebras $\ell^1(S_2)$ and $\ell^1(S_{1,1})$ are not approximately amenable according to the previous theorem. In this chapter we will also give a direct proof of the fact that $\ell^1(S_2)$ is not approximately amenable.

4.3 A characterization of approximate amenability of a Banach algebra

The following theorem contains a list of conditions relating approximate amenability, approximate contractibility, pseudo-amenability, and the existence of certain diagonal-type nets.

Theorem 4.3.1. *(1) For a Banach algebra A the following statements are equivalent:*

(i) A is approximately amenable;

(ii) A is approximately contractible;

(iii) A is weak-approximately amenable;*

(iv) the unitization $A^{\#}$ of A is pseudo-amenable;

(v) the unitization $A^{\#}$ of A is approximately amenable;

(vi) there are nets (m_λ) in $A\widehat{\otimes}A$ and $(f_\lambda), (g_\lambda)$ in A such that for each $a \in A$,

(a) $a \cdot m_\lambda - m_\lambda \cdot a + f_\lambda \otimes a - a \otimes g_\lambda \to 0$,

(b) $af_\lambda \to a$, $g_\lambda a \to a$, *and*

(c) $\pi(m_\lambda) - f_\lambda - g_\lambda \to 0$.

(2) If A has a bounded approximate identity, then A is approximately amenable if and only if A is pseudo-amenable.

In part (1), the equivalence of statements (i), (ii) and (iii) is [21, Theorem 2.1], the equivalence of statements (i) and (v) is [20, Proposition 2.4], while the equivalence of conditions (ii), (iv), and (vi) is [20, Proposition 2.6].

Part (2) of Theorem 4.3.1 is [22, Proposition 3.2].

Also there is another characterization for approximate amenability of a Banach algebra:

Proposition 4.3.2 ([13], Proposition 2.1). *Let A be a Banach algebra. Then A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exist $F \in A \otimes A$ and $u, v \in A$ such that $\pi(F) = u + v$ and, for each $a \in S$:*

(i) $\| a \cdot F - F \cdot a + u \otimes a - a \otimes v \| < \varepsilon$;

(ii) $\| a - au \| < \varepsilon$ *and* $\| a - va \| < \varepsilon$.

We give a characterization for a the Banach algebras A to be approximately amenable.

Proposition 4.3.3. *A Banach algebra A is approximately amenable if and only if the mapping $D : A \longrightarrow A^{\#}\widehat{\otimes}A^{\#}$ defined by $D(a) = a\otimes e - e\otimes a$ $(a \in A)$ is approximately inner as a derivation into* $\ker(\pi)$ *where $A^{\#}$ is the unitization of A and e is the identity of $A^{\#}$.*

Proof.

One implication is straightforward since $\ker(\pi)$ is an A-bimodule, therefore D is approximately inner.

Conversely, suppose that the derivation $D : A \to \ker(\pi)$ is approximately inner.

Then $\exists\ \xi_\alpha \in \ker(\pi)$ such that $D(a) = \lim_\alpha(a\cdot\xi_\alpha - \xi_\alpha\cdot a)$. Hence

$\lim_\alpha a\cdot(e\otimes e - \xi_\alpha) - (e\otimes e - \xi_\alpha)\cdot a = 0$. Let $u_\alpha = e\otimes e - \xi_\alpha$.

We have that $\pi(u_\alpha) = e$. We write $u_\alpha = \sum_{i=1}^{\infty} a_\alpha^i \otimes b_\alpha^i$ where $\sum_{i=1}^{\infty} \|a_\alpha^i\|\|b_\alpha^i\| < \infty$.

Let us consider any derivation $\Delta : A^{\#} \to X$ and the mapping $\psi : A^{\#}\widehat{\otimes}A^{\#} \to X$ where X is a unital $A^{\#}$-bimodule and

$$\psi(a\otimes b) = a\Delta(b).$$

We prove that Δ is approximately inner. For:

$$\begin{aligned} a\cdot\psi(u_\alpha) &= a\psi(\sum_{i=1}^{\infty} a_\alpha^i\otimes b_\alpha^i) = a\sum_{i=1}^{\infty} a_\alpha^i\Delta(b_\alpha^i) \\ &= \sum_{i=1}^{\infty} aa_\alpha^i\Delta(b_\alpha^i) = \sum_{i=1}^{\infty}\psi(aa_\alpha^i\otimes b_\alpha^i) \\ &= \psi(a\cdot u_\alpha) = \psi(a\cdot u_\alpha - u_\alpha\cdot a) + \psi(u_\alpha\cdot a). \end{aligned}$$

$$\begin{aligned} \psi(u_\alpha\cdot a) &= \psi(\sum_{i=1}^{\infty} a_\alpha^i\otimes b_\alpha^i a) = \sum_{i=1}^{\infty} a_\alpha^i\Delta(b_\alpha^i a) \\ &= \sum_{i=1}^{\infty}(a_\alpha^i b_\alpha^i\Delta(a) + a_\alpha^i\Delta(b_\alpha^i)\cdot a) = \pi(u_\alpha)\Delta(a) + \psi(u_\alpha)\cdot a \\ &= \Delta(a) + \psi(u_\alpha)\cdot a. \end{aligned}$$

Combining these identities, we obtain:

$$a \cdot \psi(u_\alpha) = \psi(a \cdot u_\alpha - u_\alpha \cdot a) + \Delta(a) + \psi(u_\alpha) \cdot a$$

and

$$\Delta(a) = \lim_\alpha (a \cdot \psi(u_\alpha) - \psi(u_\alpha) \cdot a).$$

Thus Δ is approximately inner, therefore $A^{\#}$ is approximately contractible and so A is approximately amenable using the previous theorem. □

We can also give a shorter proof using Theorem 4.3.1. If D is approximately inner then $\exists\ \xi_\alpha \subset \ker(\pi)$ such that $D(a) = \lim_{\alpha \to \infty} (a \cdot \xi_\alpha - \xi_\alpha \cdot a)$, $a \in A$. Then

$$\lim_{\alpha \to \infty} a(e \otimes e - \xi_\alpha) - (e \otimes e - \xi_\alpha)a = 0.$$

Take $u_\alpha := e \otimes e - \xi_\alpha$. Then $\lim_\alpha (a \cdot u_\alpha - u_\alpha \cdot a) = 0$ and $\pi(u_\alpha) = e$. Therefore $A^{\#}$ is pseudo-amenable and by Theorem 4.3.1 the Banach algebra A is approximately amenable.

Proposition 4.3.4. *Let S be a semigroup with generating set E.*

Then $\ell^1(S)$ is amenable(respectively weakly amenable) if and only if for every continuous derivation $D : \ell^1(S) \to X^*$ (respectively $D : \ell^1(S) \to \ell^\infty(S)$) there exists ξ in X^* (respectively $\ell^\infty(S)$) such that $D(\delta_s) = \delta_s \cdot \xi - \xi \cdot \delta_s$ for every $s \in E$.

Proof.

"$\Rightarrow$" is trivial.

"$\Leftarrow$": $S = \langle E \rangle$. For each $s \in S$ let $l(s)$=min$\{n$: s is a product of n elements of $E\}$. We prove $D(\delta_s) = \delta_s \cdot \xi - \xi \cdot \delta_s$ for all $s \in S$ by induction on n.

Suppose that $\forall\ s \in S$ with $l(s) \leqslant n$, $D(\delta_s) = \delta_s \cdot \xi - \xi \cdot \delta_s$.

Let $t \in S$, $l(t) = n+1$. We can write $t = sx$ where $l(s) \leqslant n$, $x \in E$. Then

$$\begin{aligned} D(\delta_t) = D(\delta_s\delta_x) &= D(\delta_s)\cdot\delta_x + \delta_s \cdot D(\delta_x) \\ &= (\delta_s\cdot\xi - \xi\cdot\delta_s)\cdot\delta_x + \delta_s\cdot(\delta_x\cdot\xi - \xi\cdot\delta_x) \\ &= \delta_s\cdot\delta_x\cdot\xi - \xi\cdot\delta_s\cdot\delta_x = \delta_t\cdot\xi - \xi\cdot\delta t. \end{aligned}$$

By linearity and continuity of D we then have $D(f) = f\cdot\xi - \xi\cdot f\ \forall\ f \in \ell^1(S)$, which means that $\ell^1(S)$ is amenable. □

Proposition 4.3.5. *Let S be a semigroup with generating set E. Then $\ell^1(S)$ is boundedly approximately amenable if and only if for every continuous derivation $D : \ell^1(S) \longrightarrow X^*$ there is a net (ξ_i) in X^* such that ad_{ξ_i} is bounded and $D(\delta_s) = \lim_i(\delta_s\cdot\xi_i - \xi_i\cdot\delta_s)$ for every $s \in E$.*

Proof.

"$\Rightarrow$" is clear, so we only need to prove "$\Leftarrow$". Let $D_i = ad_{\xi_i}$. From hypothesis $\| D_i \|$ is bounded by a constant $M > 0$. For $f \in \ell^1(S)$ and $\varepsilon > 0$, there exists a finite sum $\sum_{n=1}^m c_n\delta_{s_n}$ such that

$$\| f - \sum_{n=1}^m c_n\delta_{s_n} \| < \frac{\varepsilon}{1+\|D\|+M}.$$

$$D_i(\sum_{n=1}^m c_n\delta_{s_n}) \to D(\sum_{n=1}^m c_n\delta_{s_n}).$$

Hence there is i_0 such that for all $i \geq i_0$,

$$\| D_i(\sum_{n=1}^m c_n\delta_{s_n}) - D(\sum_{n=1}^m c_n\delta_{s_n}) \| < \frac{\varepsilon}{1+\|D\|+M}$$

So for $i \geq i_0$,

$$\begin{aligned}
\| D_i(f) - D(f) \| &= \| D_i(\sum_{n=1}^{m} c_n\delta_{s_n}) + D_i(f - \sum_{n=1}^{m} c_n\delta_{s_n}) - D(f) \| \\
&\leq \| D_i(\sum_{n=1}^{m} c_n\delta_{s_n}) - D(f) \| + \| D_i \| \| f - \sum_{n=1}^{m} c_n\delta_{s_n} \| \\
&\leq \| D_i(\sum_{n=1}^{m} c_n\delta_{s_n}) - D(\sum_{n=1}^{m} c_n\delta_{s_n}) \| + \| D(\sum_{n=1}^{m} c_n\delta_{s_n}) - D(f) \| \\
&+ M\frac{\varepsilon}{1+\|D\|+M} \\
&\leq \frac{\varepsilon}{1+\|D\|+M} + \| D \| \frac{\varepsilon}{1+\|D\|+M} + M\frac{\varepsilon}{1+\|D\|+M} \\
&= \varepsilon.
\end{aligned}$$

Therefore

$$\lim_i D_i(f) = D(f) \qquad (f \in \ell^1(S)).$$

□

4.4 Approximate amenability of $\ell^1(S_1)$, where S_1 is the bicyclic semigroup

We consider the problem of approximate amenability for $l^1(S)$ where S is the bicyclic semigroup. We have the following result:

Theorem 4.4.1. *The Banach algebra $\ell^1(S_1)$ is not approximately amenable.*

Proof. In the proof we will use S to denote S_1. Let $\pi : \ell^1(S \times S) \to \ell^1(S)$ be the product mapping. Consider the derivation $D : \ell^1(S) \to \ker(\pi)$ defined by

$$D(f) = f \otimes \delta_e - \delta_e \otimes f \qquad (f \in \ell^1(S)).$$

If $\ell^1(S)$ is approximately amenable, then by Proposition 4.3.3 there exists a net $(\xi_i) \subset \ker(\pi)$ such that $D(f) = \lim_i (f \cdot \xi_i - \xi_i \cdot f)$, $f \in \ell^1(S)$. Let $\xi_i = \sum_{m,n} c^i_{m,n} \delta_{m,n}$, where $c^i_{m,n}$ satisfy $\sum_{mn=t} c^i_{m,n} = 0$ for each $t \in S$, and $\sum_{m,n} \mid c^i_{m,n} \mid < \infty$.
Then for every $u \in S$, we have :

$$\delta_{u,e} - \delta_{e,u} = \lim_i \sum_{m,n} c^i_{m,n}(\delta_{um,n} - \delta_{m,nu}) = \lim_i \sum_{m,n} \Big(\sum_{uk=m} c^i_{k,n} - \sum_{ku=n} c^i_{m,k} \Big) \delta_{m,n}.$$

The convergence is in the norm topology of $\ell^1(S \times S)$.

If $u \in S, u \neq e$, the above implies

$$\left\{ \begin{array}{r} \lim_i \sum_{uk=u} c^i_{k,e} - \sum_{ku=e} c^i_{u,k} = 1 \\ \lim_i \sum_{uk=e} c^i_{k,u} - \sum_{ku=u} c^i_{e,k} = -1 \\ \lim_i \sum_{(m,n)\neq(u,e),(e,u)} \left| \sum_{uk=m} c^i_{k,n} - \sum_{ku=n} c^i_{m,k} \right| = 0 \end{array} \right\} \qquad (*)$$

Taking $u = p$ we have :

$$\lim_i c^i_{e,e} + c^i_{qp,e} = 1 \tag{4.1}$$

$$\lim_i c^i_{q,p} - c^i_{e,e} = -1 \tag{4.2}$$

Taking $u = q$ we have :

$$\lim_i c^i_{e,e} - c^i_{q,p} = 1 \tag{4.3}$$

$$\lim_i c^i_{e,e} + c^i_{e,qp} = 1 \tag{4.4}$$

We prove that $\lim_i c^i_{e,e} = 1$. In the relation (*) take $u = q$ and let

$$\Gamma = \{ (m,n) \in S \times S : m = q^{r+1}, n = p^r, r \geq 1, r \in \mathbb{N} \}.$$

Then

$$\Gamma \subseteq \{ (m,n) \in S \times S : (m,n) \neq (q,e), (e,q) \}.$$

So

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{qk=m} c^i_{k,n} - \sum_{kq=n} c^i_{m,k} \mid = 0.$$

If $qk = q^{r+1}$ then $k = q^r$, and if $kq = p^r$ then $k = p^{r+1}$. Hence

$$\lim_i \sum_{r=1}^{\infty} \mid c^i_{q^r,p^r} - c^i_{q^{r+1},p^{r+1}} \mid = 0,$$

and therefore

$$\lim_i \sum_{r=1}^{\infty} (c^i_{q^r,p^r} - c^i_{q^{r+1},p^{r+1}}) = 0.$$

But $\sum_{r=1}^{\infty}(c^i_{q^r,p^r} - c_{q^{r+1},p^{r+1}}) = c^i_{q,p}$ (since the series $\sum_{m,n} \mid c^i_{m,n} \mid$ converges which implies $\lim_{r\to\infty} c^i_{q^{r+1},p^{r+1}} = 0$).

Therefore, $\lim_i c^i_{q,p} = 0$. So, using relation (4.2) we have that

$$\lim_i c^i_{e,e} = 1.$$

In the relation (*) take $u = q$ and let

$$\Gamma = \{ (m,n) \in S \times S : m = q^{l+1}p^r, n = q^r p^l : r \geq 1, l \geq 1, r, l \in \mathbb{N} \}.$$

Then

$$\Gamma \subseteq \{ (s,t) \in S \times S : (s,t) \neq (p,e), (e,p) \}$$

and

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{qk=m} c^i_{k,n} - \sum_{kq=n} c^i_{m,k} \mid = 0.$$

If $qk = q^{l+1}p^r$ then $k = q^l p^r$, and if $kp = q^r p^l$ then $k = q^r p^{l+1}$. Hence

$$\lim_i \sum_{l\geq 1} \sum_{r\geq 1} \mid c^i_{q^l p^r, q^r p^l} - c^i_{q^{l+1}p^r, q^r p^{l+1}} \mid = 0,$$

and so

$$0 = \lim_i \sum_{l\geq 1} \sum_{r\geq 1} (c^i_{q^l p^r, q^r p^l} - c^i_{q^{l+1} p^r, q^r p^{l+1}}) = \lim_i \sum_{r\geq 1} c^i_{qp^r, q^r p}.$$

Therefore,

$$\lim_i \sum_{r\geq 1} c^i_{qp^r, q^r p} = 0. \tag{4.5}$$

In the relation (*) take $u = qp$ and let

$$\Gamma = \{ (m,n) \in S \times S : m = qp^{r+1}, n = q^r, r \geq 1, r \in \mathbb{N} \}.$$

Then

$$\Gamma \subseteq \{ (s,t) \in S \times S, (s,t) \neq (qp, e), (e, qp) \}.$$

If $qpk = qp^{r+1}$ then $k = p^r$ or $k = qp^{r+1}, r \geq 1$. Also $kqp = q^r$

does not have any solutions k for $r \geq 1$. Therefore,

$$\lim_i \sum_{r\geq 1} \mid c^i_{p^r, q^r} + c^i_{qp^{r+1}, q^r} \mid = 0.$$

But since $\sum_{mn=e} c^i_{m,n} = 0$, $c^i_{e,e} + \sum_{r\geq 1} c^i_{p^r, q^r} = 0$ and so $\lim_i \sum_{r\geq 1} c^i_{p^r, q^r} = -1$. So

$$\lim_i \sum_{r\geq 1} c^i_{qp^{r+1}, q^r} = 1. \tag{4.6}$$

In the relation (*) take $u = qp$ and let

$$\Gamma = \{ (m,n) \in S \times S : m = p^r, n = q^{r+1} p, r \geq 1, r \in \mathbb{N} \}.$$

Then

$$\Gamma \subseteq \{ (s,t) \in S \times S, (s,t) \neq (qp, e), (e, qp) \}.$$

The equation $qpk = p^r, r \geq 1$ does not have any solutions. If $kqp = q^{r+1}p$ then

$k = q^r$ or $k = q^{r+1}p$. Hence

$$\lim_i \sum_{r\geq 1} \mid c^i_{p^r, q^r} + c^i_{p^r, q^{r+1}p} \mid = 0.$$

So arguing as before, we have

$$\lim_i \sum_{r\geq 1} c^i_{p^r,q^{r+1}p} = 1. \tag{4.7}$$

Also we have that $\sum\limits_{mn=qp} c^i_{m,n} = 0$, so we get

$$c^i_{qp,e} + c^i_{e,qp} + c^i_{q,p} + \sum_{r\geq 1} c^i_{qp^{r+1},q^r} + \sum_{r\geq 1} c^i_{p^r,q^{r+1}p} + \sum_{r\geq 1} c^i_{qp^r,q^rp} = 0.$$

We have also that $\lim\limits_i c^i_{qp,e} = \lim\limits_i c^i_{e,qp} = \lim\limits_i c^i_{q,p} = 0$ by (4.1), (4.2) and (4.4).

So using relations (4.6) and (4.7),

$$\lim_i \sum_{r\geq 1} c^i_{qp^r,q^rp} = -2$$

which is a contradiction with relation (4.5).

Therefore, $\ell^1(S)$ is not approximately amenable.

□

Since we couldn't answer to the problem if $\ell^1(S_1)$ is or not approximately weakly amenable we wander this fact.

Appendix

We know from Theorem 4.2.1 that $\ell^1(S_2)$ is not approximately amenable since S_2 is not amenable, but the proof is not strightforward. In this Appendix we will give a direct proof to this result.

Let S_2 be the partially bicyclic semigroup. $S_2 = \langle\, e, p, v_0, v_1 \mid pv_0 = pv_1 = e \,\rangle$ and consider the free semigroup of three generators $F = \langle\, e, v_0, v_1 \,\rangle$. An element of S_2 has the form xp^r, $x \in F$, $r \in \mathbb{N}$. Consider the set $A_p = \{v_0, v_1\}$. We make the convention that every element to the power zero is the identity e.

Denote $l(x)$ the length of the element $x \in F$ (i.e. example $x = v_1^2 v_0^3 v_1$ has the length 6 and we write $l(x) = 6$, $l(e) = 0$).

The following three claims are true for partially bicyclic semigroup S_2.

Claim 1 $xp^r \neq e$, $\forall\, r \geq 1$, $x \in F$.

Proof. Suppose by contradiction that $xp^r = e$. Multiplying to the right with v^r we get $x = v^r\ \forall\ v \in A_p$. This implies that $v_0^r = v_1^r$. If $r = 1$ we have $v_0 = v_1$ which is a contradiction.

If $r \geq 2$ in the relation $v_0^r = v_1^r$ multiply to the left with p^{r-1} we get $v_0 = v_1$ which is a contradiction. Therefore, $xp^r \neq e$, $\forall r \geq 1$, $x \in F$. $\square$

Claim 2 Let irreducible $\alpha, \beta \in S_2$, $\alpha\beta = e$. Then $\alpha = p^r$ for some $r \in \mathbb{N}$ and $\beta \in F$.

Proof. $\alpha, \beta \in S_2 \Rightarrow \alpha = xp^r, \beta = yp^t$ for some $r, t \in \mathbb{N}$ and $x, y \in F$

$$\alpha\beta = e \Rightarrow xp^r yp^t = e$$

Suppose $t \geq 1$. Then

$$xp^r yp^t = zp^s$$

for some $z \in F, s \geq 1$. Therefore, $zp^s = e$ which is a contradiction with Claim 1.

Hence

$$t = 0$$

$$\beta = y \in F$$

$$\alpha\beta = xp^r y = e$$

- If $l(y) = r \Rightarrow x = e \Rightarrow \alpha = p^r$.
- If $l(y) < r \Rightarrow xp^r y = xp^{r-l(y)} = e$ which is a contradiction with Claim 1 since $r - l(y) \geq 1$.
- If $l(y) > r$ in the relation $xp^r y = e \Rightarrow z = e$ for some $z \in F$, $l(z) \geq 1$. In the relation $z = e$ we multiply to the left with $p^{l(z)}$ and we get $p^{l(z)} = e$ which is a contradiction with Claim 1.

□

Claim 3 Let $x, y \in F$ such that there exists $v \in A_p$, $xv = yv$. Then $x = y$.

Proof. For suppose that $x \neq y$. If $l(x) \neq l(y)$ then for sure we have contradiction with Claim 1 by multiplying with some $p^{\max\{l(x), l(y)\}+1}$ to the left getting $p^{|l(x)-l(y)|} = e$. Therefore, $l(x) = l(y)$.

Then there exist $0 < n < l(x)$, $n \in \mathbb{N}$ such that multiplying $xv = yv$ with p^n to the left we get $v_0z = v_1z$ for some $z \in F$.

But $v_0S_2 \cap v_1S_2 = \emptyset$. Hence we have a contradiction. $\square$

The following six results are true for $S_2 = \langle\, e, p, v_0, v_1 \mid pv_0 = pv_1 = e\,\rangle$.

$F = \langle\, e, v_0, v_1\,\rangle$, $A_p = \{v_0, v_1\}$, $\forall\ \alpha, \beta \in S_2$, $\alpha\beta = e$ and $\forall\ v \in A_p$.

Lemma 1 $\{\, k \mid v_0pk = v_0p\alpha\,\} \subseteq \{\,\alpha\,\} \cup \{\, vp\alpha \mid v \in A_p\,\}$.

Proof. For, suppose that $v_0pk = v_0p\alpha$ has the solution $k = xp^t$ for some $x \in F$ and $t \in \mathbb{N}$. From Claim 2, $\alpha = p^r$ for some $r \in \mathbb{N}$.

$$v_0pk = v_0p\alpha \Leftrightarrow v_0pxp^t = v_0p^{r+1} \Leftrightarrow pxp^t = p^{r+1} \qquad (1)$$

• If $l(x) = 0$ then $p^{t+1} = p^{r+1} \Rightarrow t = r$ (otherwise multiplying to the right with $v^{min\{r+1,t+1\}}$ we get contradiction with Claim 1). So, $t = r \Rightarrow k = p^r = \alpha \Rightarrow k = \alpha$.

• Suppose $l(x) = 1$. Then $p^t = p^{r+1}$. Therefore by the same argument $t = r + 1$ which implies that

$$k = vp^{r+1} = vp\alpha$$

• Suppose $l(x) \geq 2$. Then in the relation (1), $yp^t = p^{r+1}, l(y) \geq 1, y \in F, y = px$. In the relation $yp^t = p^{r+1}$ multiply to the left with $p^{l(y)}$ and we get $p^t = p^{l(y)+r+1}$. By the same argument $t = l(y) + r + 1$. But from (1) $pxp^t = p^{r+1} \Rightarrow pxp^{l(y)+r+1} = p^{r+1} \Rightarrow pxp^{l(y)} = e$. Therefore $yp^{l(y)} = e$ which is a contradiction by Claim 1 since $l(y) \geq 1$. $\square$

Lemma 2 For each $k \in S_2$, $kvp \neq \beta$ and $vpk \neq \alpha$ $\forall v \in A_p$.

Proof. For suppose, that there exists $k \in S_2$ such that $k = xp^r$ ($x \in F$ and $r \in \mathbb{N}$) then $kvp = \beta \Rightarrow xp^rvp = \beta$. We showed in Claim 2 that $\beta \in F$.

Multiplying to the left with $p^{l(\beta)}$ we get

$$p^{l(\beta)}xp^{r}vp = e \Leftrightarrow zp = e$$

for some $z \in F$. But this is a contradiction with Claim 1.

Therefore, $kvp \neq \beta \ \forall \ v \in A_p$.

For the second equation suppose that $vpk = \alpha \Rightarrow vpk = p^r$, $r \in \mathbb{N}$ since $\alpha = p^r$ by Claim 2. Let consider the solution $k = xp^t$, $x \in F$, $t \in \mathbb{N}$.

$$vpxp^t = p^r \qquad (2)$$

- If $l(x) = 0$ then the relation (2) $\Leftrightarrow vp^{t+1} = p^r$. Multiplying to the left with p we get $p^{t+1} = p^{r+1} \Rightarrow t = r$. So the relation (2) $\Leftrightarrow vpp^t = p^t \Rightarrow vp = e$ which is a contradiction with Claim 1.
- If $l(x) = 1$ then the relation (2) becomes $vp^t = p^r \Rightarrow p^t = p^{r+1} \Rightarrow t = r+1$ (otherwise we have contradiction with Claim 1 since if we multiply to the right with $v^{\min\{t,r+1\}}$ we get $p^s = e$ for some $s \geq 1$). So the relation (2) $\Rightarrow vp^{r+1} = p^r \Rightarrow vp = e$ which is a contradiction with Claim 1.
- If $l(x) \geq 2$ then the relation (2) $\Rightarrow vyp^t = p^r$, $l(y) \geq 1$. Multiplying to the left with $p^{1+l(y)}$ we get that $vyp^t = p^r \Leftrightarrow p^t = p^{r+l(y)+1} \Leftrightarrow t = r+1+l(y)$ (otherwise we have contradiction by the same argument). So, $vyp^t = p^r \Leftrightarrow vyp^{r+1+l(y)} = p^r \Leftrightarrow vyp^{1+l(y)} = e$ which is a contradiction with Claim 1.

Therefore $vpk \neq \alpha \ \forall v \in A_p$. □

Lemma 3 $\{\, k \mid kvp = \beta vp \,\} = \{\, \beta, \beta vp \,\}$

Proof. For suppose, that $k = xp^r$ for some $x \in F$ and $r \in \mathbb{N}$ is a solution of the equation $kvp = \beta vp$.

$$xp^r vp = \beta vp \qquad (3)$$

If we multiply relation (3) with some $v \in A_p$ to the right we get

$$xp^r v = \beta v \qquad (4).$$

• If $r = 0$ then the relation (4) becomes $xv = \beta v \Rightarrow x = \beta$ by Claim 3.

• If $r \geq 1$ then (4) $\Leftrightarrow xp^{r-1} = \beta v$. Multiplying to the right with v^{r-1} we get $x = \beta v^r$ Hence $xp^r = \beta v^r p^r$. So from (3) $\beta v^r p^r vp = \beta vp$. Multiplying with $p^{l(\beta)}$ to the left deduce that $v^r p^r = vp \Leftrightarrow v^{r-1}p^{r-1} = e$. If $r \geq 2$ we have contradiction with Claim 1.

So $r = 1$. Therefore, $k = \beta vp$. We proved that $\{ k \mid kvp = \beta vp \} = \{ \beta, \beta vp \}$. □

Lemma 4 For each $r \in \mathbb{N}^*$, $\{ k \mid kv = \beta p^r \} = \{ \beta p^{r+1} \}$.

Proof. For suppose $k = xp^t$ for some $x \in F$ and $t \in \mathbb{N}$ is a solution of $kv = \beta p^r$,

$$xp^t v = \beta p^r \qquad (5)$$

• If $l(x) = l(\beta)$ then in the relation (5) multiplying to the left with $p^{l(x)}$ we get $p^t v = p^r$. If $t = 0$ then $v = p^r \Leftrightarrow p^{r+1} = e$ (multiply with p to the left). This is a contradiction with Claim 1.

Hence $t \geq 1$. $p^t v = p^r \Leftrightarrow p^{t-1} = p^r \Leftrightarrow t - 1 = r \Leftrightarrow t = r + 1$. Therefore relation (5) becomes

$$xp^{r+1}v = \beta p^r \Rightarrow xp^r = \beta p^r \Rightarrow x = \beta \Rightarrow k = \beta p^t = \beta p^{r+1} \Rightarrow k = \beta p^{r+1}.$$

• If $l(x) > l(\beta)$ then we multiply relation (5) to the left with $p^{l(\beta)}$ and we get

$$yp^t v = p^r \qquad (6)$$

for some $y \in F$, $l(y) \geq 1$.

If $t = 0$ then $yp^t v = p^r \Leftrightarrow yv = p^r \Leftrightarrow yv^{r+1} = e$ (the last equality is obtained multiplying the previous one to the right with v^r). So, $yv^{r+1} = e$ which is a contradiction. (we get $p^{l(y)+r+1} = e$ which is a contradiction with Claim 1).

Hence $t \geq 1$. Therefore, $yp^t v = p^r \Leftrightarrow yp^{t-1} = p^r$. Multiplying with $p^{l(y)}$ to the left we get $p^{t-1} = p^{r+l(y)} \Leftrightarrow t - 1 = r + l(y) \Leftrightarrow t = 1 + r + l(y)$. So the relation (6) becomes $yp^{1+r+l(y)}v = p^r \Leftrightarrow yp^{r+l(y)} = p^r \Leftrightarrow yp^{l(y)} = e$ which is a contradiction with Claim 1 since $l(y) \geq 1$.

- If $l(x) < l(\beta)$ then we multiply relation (5) with $p^{l(x)}$ to the left and we get

$$p^t v = yp^r \qquad (7)$$

for some $y \in F$, $l(y) \geq 1$.

Multiplying the relation (7) with $p^{l(y)}$ to the left we get $p^{l(y)+t}v = p^r \Leftrightarrow p^{t+l(y)-1} = p^r \Leftrightarrow t + l(y) - 1 = r \Rightarrow l(y) = r + 1 - t$. But $l(y) \geq 1$. Therefore

$$r + 1 - t \geq 1$$

In the relation (7) we have that $p^t v = yp^r$. If $t = 0$ from (7) we have that $v = yp^r$. Multiplying with p to the left we get $e = zp^r$ for some $z \in F$. But $r \geq 1$ and so we have a contradiction with Claim 1.

If $t \geq 1$ the relation (7) imply that $p^{t-1} = yp^r$. Multiplying with v^{t-1} to the right we get $e = yp^{r-t+1}$. This is a contradiction with Claim 1 since we proved that $r + 1 - t \geq 1$. $\square$

Lemma 5 i) $\{k \mid pk = p\} \subseteq \{e\} \bigcup \{vp \mid v \in A_p\}$;

ii) $\{k \mid kv = v\} = \{e\} \bigcup \{vp\}$.

Proof. i) Suppose $k = xp^t$, $(x \in F, t \in \mathbb{N})$ is a solution of equation $pk = p$. Then

$$pxp^t = p$$

- If $l(x) = 0$ then $t = 0$ and $k = e$
- If $l(x) = 1$ then $t = 1$ and $k = vp$
- If $l(x) \geq 2$ then $yp^t = p$, $y \in F$, $l(y) \geq 1$. Then $yp^{t-1} = e$ which is a contradiction with Claim 1. (If $t = 0$ we have $y = p \Rightarrow p^{l(y)+1} = e$ which is a contradiction with Claim 1 and if $t = 1$ we have $y = e \Rightarrow p^{l(y)} = e$ which contradict Claim 1.

ii) we want to prove that $\{k \mid kv = v\} = \{e\} \bigcup \{vp\}$.

Suppose $k = xp^t$, $(x \in F, t \in \mathbb{N})$ is a solution of equation $kv = v$. Then

$$xp^t v = v$$

- If $t = 0$ and $l(x) = 0$ then $k = e$.

If $l(x) = 1$ then multiplying with p to the left we get $v = e$ which is a contradiction with Claim 1.

If $l(x) \geq 2$ then multiplying to the left with $p^{l(x)+1}$ we get $p^{l(x)} = e$ which is a contradiction with Claim 1.

- If $t = 1$ then $x = v \Rightarrow k = vp$
- If $t \geq 2$ then $xp^{t-1} = v$. Multiplying with p to the left we get yp^{t-1}=e for some $y \in F$ which is a contradiction with Claim 1 since $t - 1 \geq 1$. □

Lemma 6 $\{(m, n) \in S_2 \times S_2 \mid mn = vp\} \subseteq \{(vp\alpha, \beta), (v\alpha, \beta p), (\alpha, \beta vp)\}$

Proof. We want to show that the only decomposition of vp is $vp = vpe = evp = vep$.

For, suppose that

$$vp = xp^t \qquad (8)$$

$x \in F$, $t \in \mathbb{N}$

If $l(x) = 0$ and $t = 0$ we have $vp = e$ which is a contradiction with Claim 1.

If $l(x) = 0$ and $t \geq 1$ then $vp = p^t$. Multiplying with p to the left we have $p = p^{t+1} \Rightarrow p^t = e$ which is a contradiction with Claim 1.

If $l(x) \geq 1$ and $t = 0$ we have $vp = x$. Multiplying with $p^{l(x)}$ to the left we get $p^{l(x)} = e$ which is a contradiction with Claim 1.

If $l(x) \geq 1$ and $t \geq 1$, multiplying relation (8) with v to the right we get

$$v = xp^{t-1} \qquad (9).$$

If $t = 1$ we have $v = x$ and hence $vp = vp$.

If $t \geq 2$ multiply the relation (9) with v^{t-1} to the right and we get $v^t = x$.

In the relation (8) from the beginning we have $vp = v^t p^t$, $t \geq 2$. This imply $v^{t-1}p^{t-1} = e$ which is a contr. with Claim 1 since $t - 1 \geq 1$. □

We summarize the previous six lemmas in the following Proposition:

Proposition Let S_2 be the partially bicyclic semigroup. Denote $A_p = \{v_0, v_1\}$, $|A_p| = 2$. The following six conditions are fulfilled for every $\alpha, \beta \in S_2$, $\alpha\beta = e$, $\forall\, v \in A_p$:

(i) $\{\, k \mid v_0pk = v_0p\alpha \,\} \subseteq \{\, \alpha \,\} \cup \{\, vp\alpha \mid v \in A_p \,\}$;

(ii) For each $k \in S$, $kvp \neq \beta$ and $vpk \neq \alpha$;

(iii) $\{\, k \mid kvp = \beta vp \,\} = \{\, \beta, \beta vp \,\}$;

(iv) For each $r \in \mathbb{N}^*$, $\{\, k \mid kv = \beta p^r \,\} = \{\, \beta p^{r+1} \,\}$;

(v) $\{\, k \mid pk = p \,\} \subseteq \{\, e \,\} \cup \{\, vp \mid v \in A_p \,\}$; $\{\, k \mid kv = v \,\} = \{\, e \,\} \cup \{\, vp \,\}$;

For each $k \in S$, $kp \neq e$ and $vk \neq e$;

(vi) $\{ (m,n) \mid mn = vp \} \subseteq \{ (vp\alpha, \beta), (v\alpha, \beta p), (\alpha, \beta vp) \mid \alpha\beta = e \}$.

then the semigroup algebra $\ell^1(S_2)$ is not approximately amenable.

Based on the proof of non-approximate amenability of $\ell^1(S)$ where S is the bicyclic semigroup we will try to prove similarly that $l^1(S_2)$ is not approximately amenable.

Theorem The Banach algebra $l^1(S_2)$ is not approximately amenable.

Proof.

Note: In the proof when we are using conditions (i)-(vi) are in fact conditions from the above proposition.

Let $\pi : \ell^1(S \times S) \to \ell^1(S)$ be the product mapping, consider the derivation $D : \ell^1(S) \to \ker(\pi)$ defined by

$$D(f) = f \otimes \delta_e - \delta_e \otimes f \qquad (f \in \ell^1(S)).$$

If $\ell^1(S)$ is approximately amenable, then there exists a net $(\xi_i) \subset \ker(\pi)$ such that $D(f) = \lim_i (f \cdot \xi_i - \xi_i \cdot f)$, $f \in \ell^1(S)$

Let $\xi_i = \sum_{m,n} c^i_{m,n}\delta_{m,n}$, where $c^i_{m,n}$ satisfy $\sum_{mn=t} c^i_{m,n} = 0$ for each $t \in S$, and $\sum_{m,n} \mid c^i_{m,n} \mid < \infty$.
Then for every $u \in S$, we have :

$$\delta_{u,e} - \delta_{e,u} = \lim_i \sum_{m,n} c^i_{m,n}(\delta_{um,n} - \delta_{m,nu}) = \lim_i \sum_{m,n} (\sum_{uk=m} c^i_{k,n} - \sum_{ku=n} c^i_{m,k})\delta_{m,n}$$

The convergence is in the norm topology of $\ell^1(S \times S)$

If $u \in S, u \neq e$, the above implies

$$\left\{\begin{array}{r} \lim_i \sum_{uk=u} c^i_{k,e} - \sum_{ku=e} c^i_{u,k} = 1 \\ \lim_i \sum_{uk=e} c^i_{k,u} - \sum_{ku=u} c^i_{e,k} = -1 \\ \lim_i \sum_{(m,n)\neq(u,e),(e,u)} \mid \sum_{uk=m} c^i_{k,n} - \sum_{ku=n} c^i_{m,k} \mid= 0 \end{array}\right\} \quad (*)$$

Take $u = v_0$, we have $\lim_i c^i_{e,e} - c^i_{v_0,p} = 1$. We will prove that $\lim_i c^i_{e,e} = 1$

Take $u = v_0$ and let

$$\Gamma = \{ (m,n) \in S \times S : m = v_0^{r+1}, n = p^r, r \geq 1, r \in \mathbb{N} \}.$$

where $\mathbb{N}$ denotes the set of all positive integers.

Then

$$\Gamma \subseteq \{(m,n) \in S \times S : (m,n) \neq (v_0,e),(e,v_0) \}$$

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{v_0k=m} c^i_{k,n} - \sum_{kv_0=n} c^i_{m,k} \mid= 0$$

The sequence $(v_0,p),(v_0^2,p^2),...,(v_0^r,p^r)$ is infinite since p and v_0 have infinite order:

Suppose, by way of contradiction, that $p^{h+k} = p^h$ for some positive integers h and k. Multiplying on the right by v_0^h, we obtain $p^k = e$. Then $v_0 = ev_0 = p^k v_0 = p^{k-1}e = p^{k-1}$ and $v_0 p = p^k = e$, which is a contradiction with Claim 1.

the equation $v_0 k = v_0^{r+1} \Rightarrow k = v_0^r$ and $kv_0 = p^r \Rightarrow k = p^{r+1}$ by (iv).

Precisely $\lim_i \sum_{r=1}^{\infty} \mid c^i_{v_0^r,p^r} - c^i_{v_0^{r+1},p^{r+1}} \mid= 0$ and this implies that $\lim_i \sum_{r=1}^{\infty} (c^i_{v_0^r,p^r} - c^i_{v_0^{r+1},p^{r+1}}) = 0$.

But $\sum_{r=1}^{\infty} (c_{v_0^r,p^r} - c_{v_0^{r+1},p^{r+1}}) = c^i_{v_0,p}$

Note (the series $\sum_{m,n} \mid c^i_{m,n} \mid$ converges then $\lim_{r\to\infty} c^i_{q^{r+1},p^{r+1}} = 0$), therefore $\lim_i c^i_{v_0,p} = 0$.

So we have that $\lim_i c^i_{e,e} = 1$.

In the relation (*) take $u = v_0 p$, v_0 is fixed and let

$$\Gamma = \{ (m,n) \in S \times S : m = v_0 p\alpha, n = \beta : (\alpha,\beta) \neq (e,e) \}$$

Then

$$\Gamma \subseteq \{(s,t) \in S \times S : (s,t) \neq (v_0 p, e), (e, v_0 p) \}$$

and

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{v_0 pk=m} c^i_{k,n} - \sum_{kv_0 p=n} c^i_{m,k} \mid= 0$$

The function $(\alpha,\beta) \to (m,n)$ is injective. For suppose $v_0 p\alpha = v_0 p\alpha_1$ and $\alpha\beta = \alpha_1\beta = e$. If $\alpha_1 \neq \alpha$ then by (i) $\alpha_1 = vp\alpha$ for some $v \in A_p$. Therefore $vp = vp\alpha\beta = \alpha_1\beta = e$ which is a contradiction. So $\alpha_1 = \alpha$.

the equation $v_0 pk = v_0 p\alpha$ have solutions $k = \alpha$ and $k \in \{vp\alpha \mid v \in A_p\}$ due to (i) and $kv_0 p = \beta$ do not have solutions due to (ii).

Therefore,

$$\lim_i \sum_{\substack{\alpha\beta=e \\ (\alpha,\beta)\neq(e,e)}} (c^i_{\alpha,\beta} + \sum_{v\in A_p} c^i_{vp\alpha,\beta}) = 0 \tag{4.8}$$

In the relation (*) take $u = vp$, $v \in A_p$ is fixed and let

$$\Gamma' = \{ (m,n) \in S \times S : m = \alpha, n = \beta vp : (\alpha,\beta) \neq (e,e) \}$$

Then

$$\Gamma' \subseteq \{(s,t) \in S \times S : (s,t) \neq (vp, e), (e, vp)\}$$

and

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{vpk=m} c^i_{k,n} - \sum_{kvp=n} c^i_{m,k} \mid= 0$$

the equation $vpk = \alpha$ do not have solution due (ii) and $kvp = \beta vp$ have two solutions $k = \beta$ and $k = \beta vp$ due to (iii).

The function $(\alpha, \beta) \to (m, n)$ is injective. For suppose $\beta vp = \beta_1 vp$ and $\alpha\beta = \alpha\beta_1 = e$. If $\beta_1 \neq \beta$ then by (iii) $\beta_1 = \beta vp$. Therefore $vp = \alpha\beta vp = \alpha\beta_1 = e$ which is a contradiction. So $\beta_1 = \beta$.

Therefore,

$$\lim_i \sum_{\substack{\alpha\beta=e \\ (\alpha,\beta)\neq(e,e)}} (c^i_{\alpha,\beta} + c^i_{\alpha,\beta vp}) = 0 \qquad (\forall v \in A_p) \tag{4.9}$$

and we have $|A_p|$ relations.

So, if we are adding relations (4.8) and (4.9) and use the last condition (vi) :

$$\lim_i[(|A_p|+1)(\sum_{\substack{\alpha\beta=e \\ (\alpha,\beta)\neq(e,e)}} c^i_{\alpha,\beta}) + \sum_{v\in A_p}(\sum_{\substack{\alpha\beta=e \\ (\alpha,\beta)\neq(e,e)}} c^i_{vp\alpha,\beta} + \sum_{\substack{\alpha\beta=e \\ (\alpha,\beta)\neq(e,e)}} c^i_{\alpha,\beta vp})] = 0$$

$$\lim_i[(|A_p|+1)(\sum_{mn=e} c^i_{m,n} - c^i_{e,e}) + \sum_{v\in A_p}(\sum_{mn=vp} c^i_{m,n} - \sum_{\alpha\beta=e} c^i_{v\alpha,\beta p} - c^i_{vp,e} - c^i_{e,vp})] = 0$$

$$\lim_i[(|A_p|+1)(\sum_{mn=e} c^i_{m,n} - c^i_{e,e}) + \sum_{v\in A_p}\sum_{mn=vp} c^i_{m,n} - \sum_{v\in A_p}\sum_{\alpha\beta=e} c^i_{v\alpha,\beta p} - \sum_{v\in A_p}(c^i_{vp,e} + c^i_{e,vp})] = 0$$

Taking in the relation (*) $u = v$ we have:

$$\Gamma'' = \{\, (m,n) \in S\times S : m = v^{r+1}\alpha, n = \beta p^r : \alpha\beta = e, r \in \mathbb{N}^*\}$$

Then

$$\Gamma'' \subset \{(m,n) \in S\times S : (m,n) \neq (v,e),(e,v)\}$$

since $v^r p^r \neq e \ \forall\ r \in \mathbb{N}$.

$$\lim_i \sum_{(m,n)\in\Gamma} \mid \sum_{vk=m} c^i_{k,n} - \sum_{kv=n} c^i_{m,k} \mid = 0$$

The equation $vk = v^{r+1}\alpha$ has the solution $k = v^r\alpha$ and the equation $kv = \beta p^r$ has the solution $k = \beta p^{r+1}$ due to (iv). The function $(\alpha, \beta) \to (m, n)$ is also injective and the sequence $(v^r\alpha, \beta p^r)_r$ is infinite.

Therefore,

$$\lim_i \sum_{\alpha\beta=e} (\sum_{r\geq 1}(c^i_{v^r\alpha,\beta p^r} - c^i_{v^{r+1}\alpha,\beta p^{r+1}})) = 0$$

$$\lim_i \sum_{\alpha\beta=e} c^i_{v\alpha,\beta p} = 0$$

Taking $u = p$ in the relation (*) we have:

$$\lim_i(\sum_{pk=p} c^i_{k,e} - \sum_{kp=e} c^i_{p,k}) = 1$$

Using condition (v), statement i) from the above Proposition we have that:

$$\lim_i \sum_{v\in A_p} c^i_{vp,e} = 0$$

Taking $u = v$ in the relation (*) we have:

$$\lim_i(\sum_{vk=e} c^i_{k,v} - \sum_{kv=v} c^i_{e,k}) = -1$$

Using condition (v), statement ii) from the above Proposition we have that:

$$\lim_i c^i_{e,vp} = 0 \qquad (v \in A_p)$$

So, we get $(\mid A_p \mid +1)(-1) = 0$ which is a contradiction. $\square$

The result can be generalized at such kind of semigroups:

$$S_n = \langle\, e, p, v_0, v_1, ..., v_{n-1} \mid pv_0 = pv_1 = ... = pv_{n-1} = e \,\rangle$$

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