ON SOME NUMERICAL TECHNIQUES FOR THE LAPLACIAN IN SOME SIMPLY AND DOUBLY CONNECTED REGIONS

by

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Abstract

The object of this thesis is to use complex variable methods to reduce the solutions of some boundary value problems for simply and doubly connected regions to solutions of linear infinite systems, and indicate computational techniques that can be used to obtain approximate numerical results. The complexity and importance of solving analytically and numerically elliptic partial differential equations in mathematics, physics and engineering are well documented in literature. We are mainly concerned with fully integrable Laplace's, Poisson's and Helmholtz's equations, whose general solutions can be expressed in terms of unknown complex potentials. These complex potentials include relevant logarithmic singularities and Laurent series. For Poisson's equations on doubly connected regions, we have the general solutions satisfying the boundary conditions individually on the two boundaries and then match the unknown coefficients arising from the complex potentials. These procedures lead to equivalent systems of sets of infinite linear algebraic equations with an infinite number of unknowns. Truncation techniques are applied to derive numerical values for the coefficients of the complex potentials. The results are applied to find the rates of flow in the case of slow and steady viscous flow in a pipe-in-a-pipe configuration, whose cross-section is bounded by two eccentric circles. Helmholtz's equations on the simply connected regions with boundaries which can be expressed in the form $z\bar{z} = f(z \pm \bar{z})$ or $z \pm \bar{z} = g(z\bar{z})$,(which includes the well known elliptic case,) are considered. The main contribution of the thesis is in applying complex variable techniques to derive equivalent linear infinite algebraic systems for the eigenvalues of the Laplacian on elliptic regions. The numerical results are briefly compared with those available in literature.

Chapter 1

Introduction

It is well known that the homogeneous boundary value problems associated with the two dimensional Laplace operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is represented by,

$$\nabla^2 u = 0, \qquad \text{(Laplace's equation)} \tag{1.1}$$

$$\nabla^2 u = L \qquad L \text{ being a constant} \qquad \text{(Poisson's equation)} \tag{1.2}$$

and the eigenvalue problem,

$$\nabla^2 u + \lambda^2 u = 0. \qquad \text{(Helmholtz's equation)} \tag{1.3}$$

The problems in two dimensional simply and doubly connected regions are of fundamental importance in various classical and modern fields including vibration systems, antenna analysis, and quantum mechanics.

The ensuing chapters will present solutions of various theoretical problems and numerical algorithms using as our method. The expressions for the equations are in terms of complex variables $z = x + iy$ and its conjugate $\bar{z} = x - iy$, and followed by integration of the equations to derive general solutions in terms of relevant functions of complex variables (including logarithmic singularities, Taylor series and Laurent series). The simply connected regions considered are bounded by closed convex curves which can be expressed in the form $z\bar{z} = f(z \pm \bar{z})$ or $z \pm \bar{z} = g(z\bar{z})$. These expressions include boundaries like circles, ellipses, rectangles, etc. The doubly connected regions are annuli bounded by concentric circles or eccentric circles. In all the problems discussed, the solutions are reduced to solving equivalent infinite linear algebraic equations. Compared to traditional numerical methods which need information regarding solution itself for any meaningful error analysis, the infinite system approach embodies truncations, error analysis and estimates.

Techniques for solving these problems on various domains include separation of variables, complex variables method, and approximation methods of solutions including asymptotic analysis [8], Green function, finite differences, finite elements. For some details of separation of variables see [20] and for an excellent treatment using complex variables see [12]. For some basic properties of these elliptic partial differential equations see [19].

In Chapter 2, we introduce some preliminary information about the equations, the nature of their solutions and description of the regions in terms of complex variables. A brief description of conformal mapping with some examples is also given. In Chapter 3, for the three equations which are fully integrable, the general solutions are derived for both simply and doubly connected regions. Chapter 4 deals with the fluid flow problem for the doubly connected regions bounded by two eccentric circles. We use the technique of satisfying the conditions on the two boundaries individually and then matching the unknown constants in the general solution. This leads to an equivalent infinite system of linear algebraic equations which are then truncated and numerically evaluated. Rate of flow per unit cross-section per unit time is calculated and the results compared with known results in [18]. Chapter 5 deals exclusively with the Helmholtz equation for an elliptic boundary. The analysis again leads to an equivalent infinite system of linear algebraic equations with coefficients of the matrix being known polynomials of λ. The numerical values are then compared to known numerical results in the literature [9]. In chapter 6, problems involving Helmholtz equations for doubly connected regions are stated and their method of solution indicated. Future work would involve establishing the equivalence of solutions of the boundary value problems and the infinite systems, and the establishment of meaningful error analysis where possible.

Chapter 2

Preliminaries

2.1 Some Complex Variable Results

We use the complex variables $z = x + iy$ and $\bar{z} = x - iy$ to express complex potentials, which are analytic in the specified regions, as Taylor and Laurent series. The domain D will either be a bounded simply connected region (i.e. if any simple closed curve γ in D can be shrunk to a point continuously in the set of D.) or a bounded doubly connected region (i.e., domain bounded by two simple closed curves and not simply connected). In the former case, complex potentials expressed by Taylor series for functions of a single variable, have the form

$$\omega(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

and the latter case would involve Laurent series and a logarithmic term to give the multi-value property of the function. I would then have the form

$$\omega(z) = B \ln z + \sum_{n=-\infty}^{\infty} b_n z^n \tag{2.2}$$

with the assumption that the origin is outside the domain.

Stokes' theorem in complex form is given by

$$\int_C f(z, \bar{z}) dz = 2i \int_S \frac{\partial f}{\partial \bar{z}} dS \tag{2.3}$$

for the simply connected region S enclosed by a curve C with a similar expression for doubly connected regions. We will also use the Cauchy Residue Theorem which states that for C, a simple closed positively oriented path, with f analytic inside and on C, except at finitely many isolated singularities $z_1, z_2, \ldots, z_n$ inside C, then

$$\int_C f(z) dz = 2\pi i \sum_{j=1}^{n} \text{Residue}(z_j) \tag{2.4}$$

In this thesis, we need only $n = 1$.

2.2 Conformal Mapping Methods

For solutions of two dimensional elliptic boundary value problems, conformal mapping of regions in the z-plane to circular regions in the ζ-plane, by a mapping $z = f(\zeta)$, is often used. Although the mapping is guaranteed to exist, known specific mappings for specific regions are very few.

Problems in some irregular shaped convex regions that are simply connected or doubly connected can also be solved by conformal mapping. A conformal mapping preserves the magnitude and orientation of the angles between any two curves which intersect at any point in the domain. It can be shown that a mapping $f : \mathbb{C} \to \mathbb{C}$ in the complex plane is conformal if and only if f is a complex holomorphic function and $f'(x) \neq 0$ [10]. From this latter reference, we also have the following remarks:

Remark 2.2.1. *Any simply connected region $\mathfrak{S}$, which has a boundary consisting of a piecewise smooth simple closed curve can be mapped conformally onto the interior of a unit circle.*

Remark 2.2.2. *A doubly-connected region can be mapped one to one and conformally onto a concentric circular annulus, provided that the frontier of the region consists of two disjoint continua and each of them contains more* *than one point.*

Although the existence of such maps are known, the problem of finding explicit expressions of mapping functions for all regions are still open. If a and b are radii of two concentric circles of the annulus, then the modulus of region D given by $\frac{b}{a}$ is a number uniquely determined by D. More precisely, for any doubly connected region with smooth boundaries, there exists a unique real number μ, $0 < \mu < 1$, such that there exists a one to one analytic function f, that maps D onto the annulus $A: \quad \mu < |z'| < 1$. If the outer boundaries correspond to each other, then f is determined up to a rotation of the annulus. By Remark 2.2.1 and 2.2.2, some problems on arbitrary domains can be worked out by conformal mapping.

If C_0 and C_1 are Jordan curves which bound externally and internally a doubly connected region D in the z-plane, which exclude the origin, the mapping function [15]

$$\omega(z) = \exp[\ln z + \phi(z)], \quad z = x + iy = re^{i\theta} \tag{2.5}$$

which is unique except for an arbitrary rotation, maps $D + C_0 + C_1$ onto the annulus $0 < a \leq |\omega(z)| \leq b < \infty$, where the ratio b/a is unique and $\phi(z)$ is analytic in D.

From the above, we get

$$\ln(z\bar{z}) + \phi(z) + \overline{\phi(z)} = \begin{cases} \ln b^2, & z \in C_0 \\ \ln a^2, & z \in C_1 \end{cases} \tag{2.6}$$

We give below some well known mapping functions which can be used to solve the boundary value problems for various regions using some of the methods developed in this thesis:

- $$z = \frac{c}{1-\zeta} \tag{2.7}$$

  where $|z| < 1$, maps two eccentric circles in z-plane to concentric circles in ζ-plane [13].

- $$z = c\zeta\left(1 + \frac{\lambda}{\zeta^n}\right) \tag{2.8}$$

  maps curvilinear polygons in z-plane to circles with radii $\rho \geq 1$ in ζ-plane [15].

- $$z = c\left(\zeta + \frac{\lambda}{\zeta}\right) \tag{2.9}$$

  maps an ellipse in z-plane to a circle in ζ-plane [18]

- $$z = R(\zeta + \frac{m}{\zeta^n}) \tag{2.10}$$

  where $R > 0$ and $0 \leq m \leq \frac{1}{n}$, maps an hypotrochoid in z-plane to a unit disk in ζ-plane [16].

- $$z = \sum_{n=0}^{\infty} a_n \zeta^n \tag{2.11}$$

  can be used for a numerical mapping for arbitrary regions.

Chapter 3

The Three Equations

The three equations and their transformed equations (using the complex variables $z = x + iy$ and its conjugate $\bar{z} = x - iy$) are given by

$$\nabla^2 u = 0, \qquad \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0, \qquad \text{(Laplace)} \quad (3.1a)$$

$$\nabla^2 u = L, \qquad \frac{\partial^2 u}{\partial z \partial \bar{z}} = -\frac{L}{4}, \qquad \text{(Poisson)} \quad (3.1b)$$

$$\nabla^2 u + \lambda^2 u = 0, \qquad \frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\lambda^2}{4} u = 0. \qquad \text{(Helmholtz)} \quad (3.1c)$$

The complex PDEs (partial differential equations) can be solved with the use of holomorphic and analytic functions. The details of these functions and their roles in the methods of mathematical physics, are given more extensively in [4].

In general, the Laplace equations with homogeneous Dirichlet boundary conditions possess only trivial solutions and are obtainable from the maximum principles. The nontrivial solutions exist with nonhomogeneous boundary conditions, and various methods of deriving nontrivial solutions to Laplace's equations have been discussed widely in literatures.

Solutions to Poisson's equations can be expressed explicitly on some regular domains [7] and [19], and in addition, we can write out the analytic solutions to Poisson's equation defined in simply connected regions bounded by rectangles and circles. For the first of these latter cases, the solutions can be obtained by separation of the variables, while in the latter one, we can assume that the simply connected domain denoted Ω is bounded by a circle of radius a, and then we have the polar-coordinate boundary value problem,

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \Phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2} = f(r,\theta) & \text{in } \Omega, \\ \Phi(r,\theta) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The solution to (3.2) is given by,

$$u(r,\theta) = \frac{1}{4\pi}\int_0^a \rho\, d\rho \int_0^{2\pi} \ln \frac{\rho^2 r^2 + a^4 - 2r\rho a^2 \cos(\theta-\gamma)}{a^2\left[r^2+\rho^2-2r\rho\cos(\theta-\gamma)\right]} f(\rho,\gamma)d\gamma \tag{3.3}$$

The two dimensional Poisson equation with complex variables (3.1b) with

$L = -P/\mu$, is given by

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = -\frac{P}{4\mu},$$

which can be formally integrated to give

$$u = -\frac{P}{4\mu} z\bar{z} + \omega(z) + \overline{\omega(z)}, \tag{3.4}$$

where the complex potential $\omega(z)$ has the form

$$\omega(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{3.5}$$

This general form can be used for any convex and simply connected region including an elliptic region. For doubly connected regions, the potentials can be expressed by a Laurent series combined with a logarithmic term, which is given by,

$$\omega(z) = B \ln z + \sum_{n=-\infty}^{\infty} b_n z^n \tag{3.6}$$

Here we assume that the origin is outside of the region. The unknown coefficients b_n and B in general solutions are to be determined by the boundary conditions.

The Helmholtz equation is also known as the eigenvalue problem of the Laplacian. For a given eigenvalue of λ, we can solve Helmholtz equations analytically on some regular domains. Among the regions, the simply connected region bounded by a circle and the doubly connected region bounded by concentric circles both have radially symmetric properties. Thus we can just consider the radical part of the Helmholtz equation in these regions. The equation can be expressed in polar coordinates and thereafter solved as an ordinary differential equation. The radical part of Helmholtz equation is given by

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \lambda^2 \Phi = 0.$$

This is a typical Sturm-Liouville system with eigenvalues λ_n, where the quantities λ_n are the roots of $J_0(\lambda\, a) = 0$ with J_0 the Bessel's function of the first kind, and a is the radius of the circle. The bounded solutions to the Helmholtz equation for this simply connected region is given by, $\Phi = J_0(\lambda r)$ with J_0 the Bessel function of first kind and order zero. The solution to the Helmholtz equation in the doubly connected region bounded by two concentric circles is given by,

$$\phi(z, \bar{z}) = AJ_0(\lambda\sqrt{z\bar{z}}) + BY_0(\lambda\sqrt{z\bar{z}})$$

where A, B are arbitrary constants and J_0, Y_0 are Bessel functions of the first and second kind respectively. These can be expressed as the series,

$$\begin{aligned} J_0(\lambda\sqrt{z\bar{z}}) &= \sum_{k=0}^{\infty}\left(\frac{\lambda^2}{4}\right)^k \frac{z^k\bar{z}^k}{k!k!}, \\ Y_0(\lambda\sqrt{z\bar{z}}) &= \frac{2}{\pi}\left\{\left[\ln\frac{\lambda\sqrt{z\bar{z}}}{2} + \gamma\right] J_0(\lambda\sqrt{z\bar{z}}) + \sum_{k=1}^{\infty}(-1)^{k+1}H_k\left(\frac{\lambda^2}{4}\right)^k \frac{z^k\bar{z}^k}{k!k!}\right\} \end{aligned}$$

where γ is the Euler-Mascheroni constant and $H_k = 1+1/2+1/3+\cdots+1/k$ is a harmonic number.

For the other simply connected regions without such symmetric properties, we cannot just consider the radical part. However, it is still possible to solve the eigenvalue problem in elliptic regions, on which the problem can be formulated by using separation of variables and an elliptic coordinate system [8].

The elliptical coordinates $(0 \leq u < \infty, 0 \leq v \leq 2\pi)$, are defined through the transformation equations to Cartesian coordinates:

$$\begin{aligned} x &= c\cosh u \cos v; \\ y &= c\sinh u \sin v. \end{aligned}$$

Helmholtz equations in these coordinates (u, v) take the form

$$\left\{\frac{2}{c^2(\cosh 2u - \cos 2v)}\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial u^2}\right] + k^2\right\}\psi(u,v) = 0$$

$\Psi(u, v)$ is separable and admits product solutions of the form

$$\psi(u,v) = U(u)V(v).$$

The functions U and V then satisfy the ordinary differential equations,

$$\begin{aligned}
\frac{\partial^2 U}{\partial u^2} + (a - 2q\cos 2u)U(u) &= 0 \text{ (Mathieu function)}, && (3.7)\\
\frac{\partial^2 V}{\partial v^2} - (a - 2q\cosh 2v)V(v) &= 0 \text{ (modified Mathieu function)}. && (3.8)
\end{aligned}$$

Discussion of approximate solutions to Mathieu equations and modified Mathieu equations can be found in [1].

We now consider the two dimensional Helmholtz equation (3.1c) and integrate. For real ϕ, we assume the formal solution in the form of

$$\phi = \sum_{n=0}^{\infty} z^n \overline{f_n(z)} + \text{ conjugate}, \qquad (3.9)$$

where f_n are chosen to have no terms of order lower than z^n, without loss of generality. Substituting equation (3.9) into equation (3.1c), we get

$$\left\{ \sum_{n=1}^{\infty} n\bar{z}^{(n-1)} f_n'(z) + \frac{\lambda^2}{4} \sum_{n=0}^{\infty} \bar{z}^n f_n(z) \right\} + \text{ conjugate } = 0, \qquad (3.10)$$

yielding,

$$\sum_{n=0}^{\infty} \bar{z}^n \left[(n+1) f_{n+1}'(z) + \frac{\lambda^2}{4} f_n(z) \right] + \text{conjugate} = 0.$$

Equating the coefficients of $\bar{z}$ to be zero, gives the iteration relation for f_n and f_{n+1},

$$(n+1) f_{n+1}'(z) + \frac{\lambda^2}{4} f_n(z) = 0 \qquad \text{for } n = 0, 1, 2, \cdots$$

From the iteration, f_n can be expressed as

$$f_n = \frac{1}{n!}\left(-\frac{\lambda^2}{4}\right)^n \underbrace{\int_0^z \cdots \int_0^z}_{n} f_0(t) \underbrace{dt \cdots dt}_{n}. \tag{3.11}$$

Here we assume that 0 is in the domain of all functions f_n. Letting f_0 be any holomorphic function in the region, means that from Cauchy's theorem [20], the multi-integral can be simplified as

$$f_n = \frac{1}{n!}\frac{1}{(n-1)!}\left(-\frac{\lambda^2}{4}\right)^n \int_0^z f_0(t)(z-t)^{n-1}dt. \tag{3.12}$$

Now we substitute back f_n in (3.9), to give,

$$\begin{aligned}
\phi &= \sum_{n=0}^{\infty} a_n \bar{z}^n f_n(z) + \text{ conjugate} \\
&= f_0 + \sum_{n=1}^{\infty} \frac{a_n \bar{z}^n}{n!(n-1)!}\left(-\frac{\lambda^2}{4}\right)^n \int_0^z f_0(t)(z-t)^{n-1}dt + \text{ conjugate} \\
&= f_0 + \int_0^z f_0(t) \sum_{n=1}^{\infty} \frac{a_n \bar{z}^n}{n!(n-1)!}\left(-\frac{\lambda^2}{4}\right)^n (z-t)^{n-1}dt + \text{ conjugate} \\
&= f_0(z) - \int_0^z f_0(t)\frac{\partial}{\partial t} J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) dt + \text{ conjugate,}
\end{aligned}$$

and after some simplification, the general form of the solution is

$$\phi = 2Re\left\{f_0(z) - \int_0^z f_0(t)\frac{\partial}{\partial t} J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) dt\right\}, \tag{3.13}$$

which is the same as the general solution in simply connected regions given in [12], where $f_0(z)$ is an arbitrary holomorphic function on the whole elementary domain. For simply connected regions, we can assume

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{3.14}$$

while for doubly connected regions with the origin outside, we can assume

$$f_0(z) = B \ln z + \sum_{n=-\infty}^{\infty} b_n z^n. \tag{3.15}$$

where a_n, B and b_n are arbitrary real numbers.

The Helmholtz equation is also known as the eigenvalue problem of the Laplacian. For the eigenvalues of the Laplacian, we have the following theorem

Theorem 3.0.3. *All the eigenvalues of the Laplacian are positive if the eigenvalue problem satisfies the Dirichlet boundary condition.*

Proof: Let v and λ be an eigenpair, ν is the unit normal. Then

$$\begin{aligned}\lambda \int_\Omega v^2 dx &= -\int_\Omega (\Delta v) v dx \\ &= \int_\Omega |\nabla v|^2 dx - \int_{\partial\Omega} v \frac{\partial v}{\partial \nu} ds(x) \\ &= \int_\Omega |\nabla v|^2 dx\end{aligned}$$

Therefore

$$\lambda \int_\Omega v^2 dx = \int_\Omega |\nabla v|^2 dx \geq 0.$$

Now, we only need to prove

$$\int_\Omega |\nabla v|^2 dx \neq 0$$

Suppose $\int_{\Omega} |\nabla v|^2 dx = 0$, then $|\nabla v| = 0$. i.e. v is a constant on domain Ω. Since we have the homogeneous Dirichlet boundary condition, and $v = 0$ on $\partial\Omega$, then $v \equiv 0$ on the whole domain, which is contrary to $v \neq 0$ as an eigenfunction.□

It is also well known that, eigenvalues of the Laplacian are positive real numbers satisfying,

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots .$$

Chapter 4

Flow of Fluids

Poisson equation occurs frequently in classical theoretical physics and in engineering applications [21], such as the steady state of some diffusion equations, the steady state of heat distribution, astronomy, fluid dynamics and electrostatics, because it describes the behavior of electric, gravitational, and fluid potentials. One of the most important applications of Poisson equation which will discussed in the thesis is in the field of the slow and steady flow of an incompressible and irrotational ideal fluid through pipes whose cross-sections are simply or doubly connected regions.

4.1 Mathematical Model of the Problem

The velocity of the fluid satisfies the Naiver Stokes equation,

$$\begin{aligned} \frac{\partial q}{\partial t} - \mu\Delta q + \langle q, \nabla\rangle q &= f - \nabla p \\ \operatorname{div} q &= 0 \end{aligned} \tag{4.1}$$

in a bounded domain $D \subseteq \mathbb{R}^3$ with the boundary of class C^2 and q is the velocity vector, p is the pressure in an incompressible fluid; $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^3$. Also $f : D \times [0, \infty) \to \mathbb{R}^3$ represents the external force [3]. The system contains four equations and the unknown functions are the three components of the velocity field and the pressure p. In our case, we consider slow and steady viscous flow under a constant pressure gradient $P = \nabla p$ along the pipe. The fluid in our problem is supposed to be viscous and steady, and therefore it is reasonable to ignore the velocity in the vertical direction. In another words, there is no rotation in the pipe. With the assumption of slow fluid, we have $\dfrac{\partial q}{\partial t} = 0$. Then the problem can be modeled easily by Poisson equation with homogeneous Dirichlet boundary conditions. In the published paper [18], Shivakumar gives some analysis of flow through a pipe, whose cross-section is bounded by an ellipse and a circle. The conformal mapping (2.9), is introduced which maps the outer ellipse in the z-plane to a circle of radius b in the ζ-plane. In his later work with Ji in [13], they also considered the problem on the doubly connected region bounded by concentric circles and eccentric circles. In this thesis, we follow the mathematical model of the problem set up by Shivakumar by introducing the complex variables $z = x + iy$ and $\bar{z} = x - iy$ [13], and we can use the two dimensional Poisson equation of complex form (3.1b), given by,

$$\begin{cases} \dfrac{\partial^2 u}{\partial z \partial \bar{z}} = -\dfrac{P}{4\mu} & \\ u = 0 & \text{on } \partial D \end{cases} \tag{4.2}$$

where P is the constant pressure gradient along the pipe in the direction of the flow, $u(z, \bar{z})$ is the velocity of the fluid in the direction of the axis of the pipe, and μ represents the viscosity of the fluid. Assume that the general form of the solution to (4.2) is given by,

$$u(z, \bar{z}) = -\frac{P}{4\mu} z\bar{z} + \omega(z) + \overline{\omega(z)} \tag{4.3}$$

where $\omega(z)$ represents the complex potentials given by (2.1) in simply connected regions and (2.2) in doubly connected regions. We will be concerned with the rate of flow, calculated by Stokes formula (2.3) with outer boundary C_1 and inner boundary C_2,

$$\begin{aligned} R &= \int\int_D u(x, y) dx\, dy = \int\int_S u(z, \bar{z}) dS \\ &= \frac{1}{2i} \int_{C_1 - C_2} z\bar{z} \left(\frac{P}{8\mu} \bar{z} - \omega'(z) \right) dz. \end{aligned} \tag{4.4}$$

The procedure to derive the formula of the rate of flow is described below,

$$\begin{aligned}
R &= \iint_D u dS \\
&= \iint_D \left(\frac{\partial}{\partial z}(uz) - z\frac{\partial u}{\partial z}\right) dS \\
&= \iint_D \left[\frac{\partial}{\partial z}(uz) - \frac{\partial}{\partial \bar{z}}\left(z\bar{z}\frac{\partial u}{\partial z}\right) + \frac{\partial^2 u}{\partial z \partial \bar{z}} z\bar{z}\right] dS \\
&= -\frac{1}{2i}\int_{C_1-C_2} uz d\bar{z} - \frac{1}{2i}\int_{C_1-C_2} z\bar{z}\frac{\partial u}{\partial z} dz - \frac{P}{4\mu}\iint_D z\bar{z} dS.
\end{aligned}$$

On both boundary curves we have $u = 0$ and therefore

$$\int_{C_1-C_2} uz d\bar{z} = 0.$$

From the general form of the solution, we obtain the expression,

$$\frac{\partial u}{\partial z} = -\frac{P}{4\mu}\bar{z} + \omega'(z).$$

Substituting the above equations in the expression of R, then we get,

$$\begin{aligned}
R &= -\frac{1}{2i}\int_{C_1-C_2} z\bar{z}\left(-\frac{P}{4\mu}\bar{z} + \omega'(z)\right) dz - \frac{1}{2i}\int_{C_1-C_2} z\bar{z}^2 \frac{P}{8\mu} \\
&= -\frac{1}{2i}\int_{C_1-C_2} z\bar{z}\left(\frac{P}{8\mu}\bar{z} - \frac{P}{4\mu}\bar{z} + \omega'(z)\right) dz \\
&= \frac{1}{2i}\int_{C_1-C_2} z\bar{z}\left(\frac{P}{8\mu}\bar{z} - \omega'(z)\right) dz.
\end{aligned}$$

4.2 The Concentric Annulus

We consider here the doubly connected region of an annulus bounded by concentric circles and the simply connected region of a disk bounded by a circle. Because the concentric annulus region that we consider here possesses radial symmetry, provided that we set the centers of the circles to be the origin of the coordinate system, the solution to the Poisson equation on such a region will contain only powers of $r = \sqrt{z\bar{z}}$. For a simply connected region bounded by the circle $z\bar{z} = l^2$, the complex potential is given by (2.1), and here in our case it is given by,

$$\omega(z) = a_0, \qquad a_0 \text{ is an arbitrary real number.} \tag{4.5}$$

By referring to the equation (4.3) and the boundary $z\bar{z} = l^2$, on the boundary, we have,

$$\begin{aligned} u(z, \bar{z}) &= -\frac{P}{4\mu} z\bar{z} + 2a_0 \\ &= -\frac{P}{4\mu} l^2 + 2a_0 \\ &= 0 \end{aligned}$$

from the which, we find $a_0 = \frac{P}{8\mu} l^2$ and the solution is $u = -\frac{P}{4\mu}(z\bar{z} - l^2)$.

On applying the formula for the rate of flow, we have that R is given by

$$R = \int\int_S u(z, \bar{z}) dS = \frac{P\pi l^4}{8\mu}. \tag{4.6}$$

Referring to (2.2) and the symmetric properties, we can assume the potential for the concentric annulus to be

$$\omega(z) = B \ln z + b_0, \qquad B \text{ and } b_0 \text{ are arbitrary.} \tag{4.7}$$

Then the real solution to (4.2) on such a region becomes

$$u(z, \bar{z}) = -\frac{P}{4\mu} z\bar{z} + B \ln z\bar{z} + 2b_0. \tag{4.8}$$

Assuming that the region is bounded by two circles C_1 and C_2 of radii a and b respectively, we have boundary conditions given by

$$u = 0, \qquad \text{on } C_1 : \quad z\bar{z} = a^2 \qquad \text{and} \qquad \text{on } C_2 : \quad z\bar{z} = b^2.$$

Substituting the boundary conditions in (4.8), we have a linear system of two equations in the two unknowns B and b_0, given by,

$$-\frac{P}{4\mu} a^2 + B \ln a^2 + 2b_0 = 0, \tag{4.9a}$$

$$-\frac{P}{4\mu} b^2 + B \ln b^2 + 2b_0 = 0. \tag{4.9b}$$

Now we can obtain the solution to (4.2) from (4.9a) and (4.9b) on the region bounded by the concentric circles, and find that,

$$\begin{aligned} B &= \frac{P}{8\mu} \frac{a^2 - b^2}{\ln a - \ln b}, \\ 2b_0 &= \frac{P}{4\mu} \frac{a^2 \ln b - b^2 \ln a}{\ln b - \ln a}, \end{aligned}$$

and

$$u = -\frac{P}{4\mu} \left(z\bar{z} - \frac{a^2 - b^2}{2 \ln a - 2 \ln b} \ln z\bar{z} - \frac{a^2 \ln b - b^2 \ln a}{\ln b - \ln a} \right). \tag{4.10}$$

Substituting the above equations in (4.4) we get the rate of flow through the pipe whose cross section is the concentric annulus bounded by C_1 and C_2, and this is just

$$R = \frac{P\pi}{8\mu}(a^4 - b^4) + \frac{P\pi}{8\mu}\frac{(a^2 - b^2)^2}{\ln b - \ln a}. \tag{4.11}$$

4.3 The Eccentric Annulus

4.3.1 Mathematical Model

Consider the cross-section of the pipe when it is bounded by two eccentric circles C_1 and C_2, expressed in the Cartesian coordinate system by,

$$C_1 : \quad (x-h)^2 + y^2 = a^2, \qquad C_2 : \quad (x-k)^2 + y^2 = b^2 \tag{4.12}$$

Referring to Remark 2.2.2, there exists a conformal mapping, which maps the annulus bounded by eccentric circles to the one bounded by concentric circles. As in [13], the mapping function (2.9),

$$z = \frac{c}{\zeta - 1} \qquad z = x + i\,y, \qquad \zeta = \xi + i\,\eta, \qquad c \text{ real} \tag{4.13}$$

with $z'(\zeta) \neq 0$ can be used. For the transformation to be conformal, the ring space excludes the critical point $\zeta = 1$. In Cartesian coordinate, the mapping (4.13) has the form:

$$(x^2 + y^2)\zeta\bar{\zeta} = x^2 + y^2 + 2cx + c^2$$

and the eccentric circles showing in (4.12) are concentric in the ζ-plane, and $|\zeta| = \rho$, $\rho = \rho_1, \rho_2$ and $\rho_1 > \rho_2$. The following relations are satisfied:

$$\rho_1 = \frac{a}{h}; \qquad \rho_2 = \frac{b}{k}$$
$$c = \frac{a^2}{h} - h = \frac{b^2}{k} - k$$
$$k - h = \frac{b^2}{k^2} - \frac{a^2}{h^2}$$

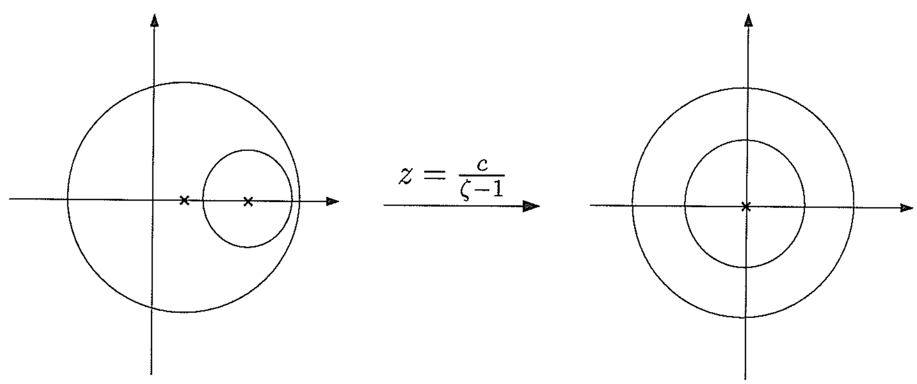

Figure 4.1: Conformal mapping of doubly connected region

The details of the method and numerical results can be found in Shivakumar and Ji's paper [13].

4.3.2 A New Approach

We consider here another approach to the problem as an alternative to using conformal mapping. We introduce the formal solutions (4.3) to the

Poisson equation (3.1b) with unknown potential $\omega(z)$ on doubly connected regions, given by (3.6). Substituting these in (3.4), the solution has the form,

$$\begin{aligned} u &= -\frac{P}{4\mu}z\bar{z} + \omega(z) + \overline{\omega(z)} \qquad (4.14)\\ &= -\frac{P}{4\mu}z\bar{z} + \left\{ B\ln z + \sum_{n=-\infty}^{\infty} b_n z^n \right\} + \text{conjugate} \\ &= -\frac{P}{4\mu}z\bar{z} + B\ln z\bar{z} + 2b_0 + \sum_{n=1}^{\infty} b_n(z^n + \bar{z}^n) + \sum_{n=1}^{\infty} b_{-n}(z^{-n} + \bar{z}^{-n}) \end{aligned}$$

Then applying the two boundary conditions respectively and separately to the formal solutions, and equating the coefficients, we obtain an infinite system of linear equations $A\vec{b} = \vec{f}$ with infinitely many unknowns $B, \quad b_0, \quad b_1 \cdots, b_n, \cdots$. The coefficient matrix A is truncated to an $n \times n$ matrix, and the approximate values of $B, \quad b_0, \quad b_1 \cdots, b_n, \cdots$ obtained by solving the truncated linear system. Focusing on the eccentric circles problem and examining the change of the rate of flow when the inner boundary approaches to the outer one and finally almost touches it, means the doubly connected region tends to a simply connected one. Figure [figure4.2] illustrates this change, but note that here we fix the center of the inner circle at the origin and keep the origin outside the doubly connected regions.

Denote the radii of the boundary circles by a and b, and the distance

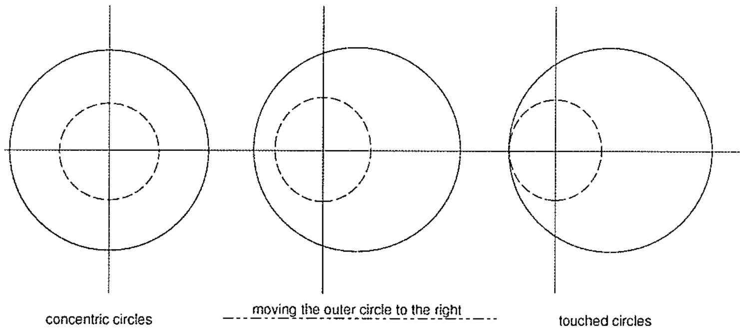

Figure 4.2: Moving the circles to touch

between the two centers by h. The two eccentric circles can then be described by,

$$x^2 + y^2 = a^2,$$

$$(x-h)^2 + y^2 = b^2, \tag{4.15}$$

where $a < b$ and $b - a > h$. In the z-plane, the boundary can be expressed as

$$C_1: \quad z\bar{z} = a^2, \qquad C_2: \quad (z-h)\overline{(z-h)} = b^2. \tag{4.16}$$

and so C_2 in the z-plane can be then mapped conformally to a circle in the ζ-plane of radius b with its center at the origin. This is given by,

$$\zeta\bar{\zeta} = b^2, \qquad \text{where } \zeta = z - h \tag{4.17}$$

So finally the new boundary value problem of complex form would be written as,

$$\begin{cases} \dfrac{\partial^2 u}{\partial z \partial \bar{z}} = -\dfrac{P}{4\mu} & \text{in } \Omega \\ u = 0 & \text{on } C_1 \\ u = 0 & \text{on } C_2 \end{cases} \tag{4.18}$$

Using the homogeneous boundary condition, we have $u = 0$ on C_1 in z-plane. So applying the boundary condition to the formal solution (4.14), we obtain,

$$\begin{aligned} u &= -\frac{Pa^2}{4\mu} + B\ln a^2 + \sum_{n=-\infty}^{\infty} b_n(z^n + \bar{z}^n) \\ &= -\frac{Pa^2}{4\mu} + B\ln a^2 + 2b_0 \\ &\quad + \sum_{n=1}^{\infty} b_n(z^n + a^{2n}z^{-n}) + \sum_{n=1}^{\infty} b_{-n}(z^{-n} + a^{-2n}z^n) \\ &= -\frac{Pa^2}{4\mu} + B\ln a^2 + 2b_0 \\ &\quad + \sum_{n=1}^{\infty}\left(b_n + \frac{b_{-n}}{a^{2n}}\right) z^n + \sum_{n=1}^{\infty}(b_n\, a^{2n} + b_{-n})z^{-n} = 0. \end{aligned}$$

By equating coefficients on both sides, we have,

$$B\ln a^2 + 2b_0 = \frac{Pa^2}{4\mu}, \tag{4.19a}$$

$$b_{-n} = -b_n a^{2n} \qquad n = 1, 2, \cdots, \infty. \tag{4.19b}$$

On the outer boundary $\zeta\bar{\zeta} = b^2$, the equation (4.14) becomes,

$$\begin{aligned} u &= -\frac{P}{4\mu}(\zeta + h)(\bar{\zeta} + h) + \omega(\zeta) + \overline{\omega(\zeta)} \\ &= -\frac{P}{4\mu}(\zeta + h)(\bar{\zeta} + h) + B\ln(\zeta + h)(\bar{\zeta} + h) + \sum_{n=-\infty}^{\infty} b_n \left[(\zeta + h)^n + (\bar{\zeta} + h)^n\right] \\ &= 0. \end{aligned}$$

On substituting (4.19a) and the boundary condition $\zeta\bar{\zeta} = b^2$ in the above equation, we have

$$\begin{aligned} u &= -\frac{P}{4\mu}\left[h^2 + (\zeta + \bar{\zeta})h + b^2\right] + B\ln b^2 + B\ln\left(1 + \frac{h}{\zeta}\right) + B\ln\left(1 + \frac{h}{\bar{\zeta}}\right) + 2b_0 \\ &\quad + \sum_{n=1}^{\infty} b_n \left[(\zeta + h)^n + (\bar{\zeta} + h)^n\right] - \sum_{n=1}^{\infty} a^{2n} b_n \left[(\zeta + h)^{-n} + (\bar{\zeta} + h)^{-n}\right] \\ &= -\frac{P}{4\mu}(b^2 + h^2) + B\ln b^2 + 2b_0 \\ &\quad + \left\{B\ln\left(1 + \frac{h}{\zeta}\right) + \sum_{n=1}^{\infty} b_n(\zeta + h)^n - \sum_{n=1}^{\infty} a^{2n} b_n(\zeta + h)^{-n} - \frac{Ph\zeta}{4\mu}\right\} \\ &\quad + \text{conjugate}. \end{aligned}$$

By the assumption $b > a$ and $h < b - a$, and the fact that $|\zeta| = b$ on C_2, we have $h/\zeta < 1$. Therefore the Taylor expansion of $\ln\left(1 + \frac{h}{\zeta}\right)$ is convergent, which can be expressed as,

$$\ln\left(1 + \frac{h}{\zeta}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} h^n}{\zeta^n}.$$

Then every term in the solution is in terms of coefficients of ζ^n and $\left(\frac{b^2}{\zeta}\right)^n$, and u is given by,

$$\begin{aligned} u &= -\frac{P}{4\mu}(b^2+h^2)+B\ln b^2+2b_0+2\sum_{n=1}^{\infty}b_nh^n \\ &\quad -\frac{P}{4\mu}\zeta+B\sum_{n=1}^{\infty}\frac{(-1)^{n-1}h^n}{nb^{2n}}\zeta^n \\ &\qquad +\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}b_k\binom{k}{n}h^{k-n}\zeta^n-\sum_{n=1}^{\infty}\sum_{k=1}^{n}a^{2k}b_k\binom{-k}{n-k}h^{n-k}\frac{\zeta^n}{b^{2n}} \\ &\quad -\frac{P}{4\mu}\frac{b^2}{\zeta}+B\sum_{n=1}^{\infty}\frac{(-1)^{n-1}h^n}{nb^{2n}}\left(\frac{b^2}{\zeta}\right)^n \\ &\qquad +\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}b_k\binom{k}{n}h^{k-n}\left(\frac{b^2}{\zeta}\right)^n-\sum_{n=1}^{\infty}\sum_{k=1}^{n}a^{2k}b_k\binom{-k}{n-k}h^{n-k}\frac{1}{\zeta^n} \\ &= 0. \end{aligned} \tag{4.20}$$

Equating the coefficients on both sides of the above system (4.20), we obtain the set of equations

$$\begin{aligned} &B(\ln b^2-\ln a^2)+2\sum_{n=0}^{\infty}b_nh^n = -\frac{P}{4\mu}(b^2+h^2) \\ &\frac{B\,h}{b^2}+\sum_{k=1}^{\infty}kh^{k-1}b_k-\frac{a^2b_1}{b^2}=\frac{Ph}{4\mu} \\ &B\frac{(-1)^{n-1}h^n}{nb^{2n}}+\sum_{k=n}^{\infty}b_k\binom{k}{n}h^{k-n}-\sum_{k=1}^{n}\frac{a^{2k}b_k}{b^{2n}}\binom{-k}{n-k}h^{n-k}=0 \end{aligned}$$

which is combined with (4.19a) to replace $2b_0$ with $\left(\frac{Pa^2}{4\mu}-B\ln a^2\right)$. The new system in terms of coefficients of ζ^n, $n=0,1,2,\cdots,\infty$, is thus given by,

$$\zeta^0: \quad B(\ln b^2 - \ln a^2) + 2\sum_{n=1}^{\infty} b_n h^n = -\frac{P}{4\mu}(b^2 + h^2 - a^2) \tag{4.21}$$

$$\zeta: \quad B\,h + \sum_{k=1}^{\infty} k h^{k-1} b^2 b_k - a^2 b_1 = \frac{Phb^2}{4\mu}$$

$$\zeta^n: \quad B\frac{(-1)^{n-1}h^n}{n} + \sum_{k=n}^{\infty} b_k \binom{k}{n} h^{k-n} b^{2n} - \sum_{k=1}^{n} a^{2k} b_k \binom{-k}{n-k} h^{n-k} = 0, n = 2, 3, \cdots, \infty$$

The expression $\binom{-k}{n-k}$ can be written as,

$$\binom{-k}{n-k} = (-1)^{n-k}\binom{n-1}{k-1}.$$

So finally the system (4.21) can be written in matrix notation as $A\vec{b} = \vec{f}$,

where

$$\vec{b} = (B, b_1, b_2, \ldots, b_n \ldots)^T$$

and

$$\vec{f} = \left(\frac{P}{4\mu}(a^2 - b^2 - h^2), \frac{Phb^2}{4\mu}, 0, \ldots, 0 \ldots\right)^T$$

and the coefficient matrix A is given by,

$$A = \begin{bmatrix} \ln(b/a) & h & h^2 & h^3 & \cdots & h^n & \cdots \\ h & b^2 - a^2 & 2b^2h & 3b^2h^2 & \cdots & nb^2h^{n-1} & \cdots \\ \frac{-h^2}{2} & a^2h & b^4 - a^4 & 3b^4h & \cdots & \binom{n}{2} b^4 h^{n-2} & \cdots \\ \frac{h^3}{3} & -a^2h^2 & a^4h & b^6 - a^6 & \cdots & \binom{n}{3} b^6 h^{n-3} & \cdots \\ \vdots & \vdots & \cdots & & \ddots & \vdots & \cdots \\ \frac{(-1)^{n+1}h^n}{n} & (-1)^n a^2 h^{n-1} & \cdots & & & b^{2n} - a^{2n} & \cdots \\ \cdots & & \cdots & & & & \end{bmatrix}$$

For the coefficient matrix A, all the entries above the diagonal are positive and those below the diagonal are of alternating sign, The sign matrix for this system would then be,

$$\begin{pmatrix} + & + & + & + & + & + & + & \cdots \\ + & + & + & + & + & + & + & \cdots \\ - & + & + & + & + & + & + & \cdots \\ + & - & + & + & + & + & + & \cdots \\ - & + & - & + & + & + & + & \cdots \\ + & - & + & - & + & + & + & \cdots \\ - & + & - & + & - & + & + & \cdots \\ \vdots & \vdots & & & & & \ddots & \end{pmatrix}$$

This sign pattern is following one of the patterns in [14], but the infinite matrix A does not fully satisfy all the conditions of non-singularity in [14]. However, the matrix A possesses some useful properties when h is smaller than 1; for example, the magnitudes of the entries tend to zero along each row and each column. In addition, when $h << 1$, all the conditions of non-singularity presented in [14] are satisfied.

The (i,j)th entry of A is denoted as $A_{i,j}$ and given by the following ,

$$A_{i,j} = \begin{cases} \ln b/a, & i = j = 1; \\ h^{j-1}, & i = 1, j > 1; \\ \dfrac{(-1)^i h^{i-1}}{i-1}, & i > 1, j = 1; \\ b^{2(i-1)} - a^{2(i-1)}, & i = j \neq 1; \\ \binom{j-1}{i-1} h^{j-i} b^2 i - 1, & 1 < i < j; \\ -a^{2(j-1)}(-1)^{i-j}\binom{i-2}{i-j} h^{i-j}, & i > j > 1. \end{cases} \tag{4.22}$$

Truncating the infinite system, we can solve for the unknowns $\vec{b}$, and thereafter we can approximate the solution u of the differential equation (4.2) defined on the eccentric annulus.

With the approximate solution u, we can find the rate of flow from the formula for R (4.4), and we have

$$\begin{aligned} R &= \int\int_D u dS \\ &= \frac{1}{2i}\int_{C_1-C_2} z\bar{z}\left(\frac{P}{8\mu}\bar{z} - \omega'(z)\right) dz \end{aligned}$$

where C_1 and C_2 are the eccentric circles, $\omega(z)$ can be expressed by $B, \quad b_n, \quad n = 1, 2, \cdots$. Substitute the solution in the formula for R, and the rate of flow in this case is given by,

$$R = \frac{P}{8\mu}\pi(b^4 - a^4) + B\pi(a^2 - b^2) + \frac{P}{4\mu}\pi h^2 b^2 - \pi a^2 b_1 h + \pi b^2 \sum_{n=1}^{\infty} n b_n h^n \quad (4.23)$$

4.3.3 Numerical Results

In the following tables, we compare the numerical results of the rate of flow with previous results using conformal mapping given in [13]. Let $\tilde{h}$ be the distance between the centers of the circles, which corresponds to the value of $h - k$ in [13], and let a and b be the radii of the circles. We can take $b = 1$ and $\dfrac{P}{8\mu} = 1$. R_1 denotes the rate of flow computed by conformal mapping and R_2 denotes the rate of flow computed by our approach.

The tables show the rates of flow for different radii of the eccentric circles. We can observe the change in R, when the centers of circles diverge.

b=1.000	a=0.05	area=3.134	R_c=2.09812	
distance between centers	R_1	R_2 n=2	R_2 n=10	R_2 n=15
0.01	2.098296	2.098293	2.098293	2.098293
0.06	2.104425	2.104313	2.104311	2.104311
0.11	2.119244	2.118886	2.118863	2.118863
0.16	2.141784	2.141890	2.141783	2.141783
0.21	2.172815	2.173126	2.172814	2.172814
0.26	2.211598	2.212315	2.211597	2.211597
0.31	2.257676	2.259091	2.257674	2.257674
0.36	2.310489	2.312989	2.310485	2.310485
0.41	2.369373	2.373433	2.369367	2.369367
0.46	2.433556	2.439718	2.433545	2.433545
0.51	2.502152	2.510988	2.502132	2.502132
0.56	2.574153	2.586211	2.574120	2.574119
0.61	2.648427	2.664136	2.648371	2.648370
0.66	2.723702	2.743254	2.723611	2.723607
0.71	2.798397	2.821731	2.798413	2.798396
0.76	2.871129	2.897328	2.871188	2.871127
0.81	2.939994	2.967297	2.940177	2.939993
0.86	3.002909	3.028233	3.003439	3.002965
0.91	3.057576	3.075877	3.058720	3.057786

Table 4.1: $b = 1, \quad a = 0.05$

b=1.000 distance between centers	a=0.2 R_1	area=3.3.016 R_2 n=2	R_c=1.338 R_2 n=10	 R_2 n=15
0.01	1.337857	1.337857	1.337857	1.337857
0.06	1.346150	1.346155	1.346150	1.346150
0.11	1.366210	1.366262	1.366210	1.366210
0.16	1.397838	1.398071	1.397838	1.397838
0.21	1.440719	1.441400	1.440719	1.440719
0.26	1.494418	1.495990	1.494418	1.494418
0.31	1.558385	1.561486	1.558385	1.558385
0.36	1.631946	1.637419	1.631946	1.631946
0.41	1.714312	1.723177	1.714312	1.714312
0.46	1.804572	1.817963	1.804572	1.804572
0.51	1.901696	1.920738	1.901696	1.901696
0.56	2.004535	2.030134	2.004535	2.004535
0.61	2.111826	2.144330	2.111826	2.111826
0.66	2.222197	2.260857	2.222197	2.222197
0.71	2.334175	2.376302	2.334175	2.334175
0.76	2.446191	2.485819	2.446194	2.446192

Table 4.2: $b = 1, \quad a = 0.2$

b=1.000	a=0.3	area=2.859	R_c=0.955	
distance between centers	R_1	R_2 n=5	R_2 n=10	R_2 n=20
0.01	0.955586	0.955586	0.955586	0.955586
0.06	0.964233	0.964233	0.964233	0.964233
0.11	0.985154	0.985154	0.985154	0.985154
0.16	1.018156	1.018156	1.018156	1.018156
0.21	1.062930	1.062930	1.062930	1.062930
0.26	1.119054	1.119054	1.119054	1.119054
0.31	1.185996	1.185996	1.185996	1.185996
0.36	1.263111	1.263112	1.263111	1.263111
0.41	1.349648	1.349650	1.349648	1.349648
0.46	1.444751	1.444758	1.444751	1.444751
0.51	1.547465	1.547483	1.547466	1.547466
0.56	1.656741	1.656782	1.656741	1.656740
0.61	1.771440	1.771530	1.771441	1.771440
0.66	1.890348	1.890524	1.890350	1.890348

Table 4.3: $b = 1, \quad a = 0.3$

b=1.000 distance between centers	a=0.5 R_1	area=2.356 R_2 n=5	R_c=0.395791 R_2 n=10	R_2 n=20
0.01	0.396014	0.396014	0.396014	0.396014
0.06	0.403840	0.403840	0.403840	0.403840
0.11	0.422783	0.422783	0.422783	0.422783
0.16	0.452693	0.452693	0.452693	0.452693
0.21	0.493331	0.493331	0.493331	0.493331
0.26	0.544372	0.544372	0.544372	0.544372
0.31	0.605412	0.605412	0.605412	0.605412
0.36	0.675965	0.675965	0.675965	0.675965
0.41	0.755475	0.755476	0.755475	0.755475
0.46	0.843315	0.843316	0.843315	0.843315

Table 4.4: $b = 1, \quad a = 0.5$

b=1.000 distance between centers	a=0.75 R_1	area=2.356 R_2 n=5	R_c=0.395791 R_2 n=10	R_2 n=20
0.01	0.057484	0.057484	0.057484	0.057484
0.06	0.062247	0.062247	0.062247	0.062247
0.11	0.073784	0.073784	0.073784	0.073784
0.16	0.092024	0.092024	0.092024	0.092024
0.21	0.116849	0.116849	0.116849	0.116849

Table 4.5: $b = 1, \quad a = 0.75$

In tables [4.3 to 4.5], the numerical results are obtained by truncating the infinite systems at $n = 5$ $n = 10$ and $n = 20$ for R_2; Table 4.1 and Table 4.2 give the numerical results are obtained by truncating the infinite systems at $n = 2$ $n = 10$ and $n = 15$ for R_2. The advantage of the new approach is that we can estimate the rate of flow through the pipe, whose cross-section is bounded by two nearly touching circles (see figure [4.2] touched circles) and therefore the region is nearly simply-connected. It is clear that the rate of flow becomes larger while the distance between the two circles is smaller. i.e., the rate of flow gets larger while the inner circle moves closer to the outer one. We may finally obtain the greatest rate when the circles are almost touching.

In addition, even if the areas of the regions are the same, the rates of flow vary depending on the eccentricity. We can see this by comparing the rate of fluid through the simply connected regions of the same area with different shapes.

Table 4.6 shows the comparison of the rates of flow through the pipes, whose cross-sections are of the same area.

a	area	concentric circles	one circle	almost touching eccentric circles
0.05	3.13374	2.09821	3.12596	3.09349
0.2	2.85885	0.95534	2.60155	1.98764
0.3	2.35619	0.39579	1.76715	0.91912
0.5	2.09531	1.33762	2.89529	2.53472
0.75	1.37445	0.05735	0.60132	0.14135

Table 4.6: Comparison of rates of flows through the pipes of different cross-sectional area when $b = 1$

In the case where the circles touch, the coefficient matrix A is almost singular, and it causes larger errors in the approximation. The values for nearly-touching case given in Table 4.6 are approximate values obtained by computing rates of flow when the two circles are very close to each other and the system is truncated at larger n.

Chapter 5

The Helmholtz Equation on Elliptic Domains

One of the applications of the Helmholtz equation is in the determination of the eigenvalues and eigenfunctions of the Laplacian. This problem is very important in vibrating systems and quantum mechanics. In addition, the research on the Helmholtz equations has been extensive as it is associated with hyperbolic equations or wave equations via Fourier Transform with respect to time [6]. In an elliptic region, the classical way has been to get approximate eigenvalues of the Helmholtz equations with the help of Mathieu equations (3.7) and (3.8). The eigenvalues λ are obtained by evaluating $q = (\lambda^2 c^2/4)$. The details of solving the Helmholtz equation through variable separation and then solving the resulting Mathieu equations have been introduced earlier in Chapter 3. Some notable works are using the WKB expansion to estimate the eigenvalues [2]; developing a visualization of a vibrating elliptical membrane [9]; using matrix evaluation of Mathieu equations [5]. Note that (3.8) is derived from (3.7) by replacing x with ix. As an example, if attention is concentrated on solutions of (3.7), which are even with period π, denoted usually by $Ce_{2n}(x, q)$, then an intricate method used to derive approximate eigenvalues with an included error analysis, is given in [17]. All the eigenvalues can be isolated in disjoint intervals so that the eigenvalues can be calculated to any required degree of accuracy, using error estimates and bisection techniques. In [8], a defect-minimization method for ellipse is proposed to give approximate eigenvalues and eigenfunctions. An excellent account of visualization of special eigenmodes is given by [9] with approximate numerical values for eigenvalues given. Asymptotic solutions for large eigenvalues are discussed in [2] and for this procedure, separation of variables is unnecessary. The region in this case is a two dimensional convex domain and the boundary is an arbitrary piecewise smooth curve.

5.1 Solution of the Problem

5.1.1 General solutions

Using the complex variables $z = x + iy$ and $\bar{z} = x - iy$, we can write the eigenvalue problem of the Laplacian by referring to (3.1c), which is given by,

$$\begin{cases} \dfrac{\partial^2 u}{\partial z \partial \bar{z}} + \dfrac{\lambda^2}{4} u = 0, & \text{in } D, \\ u = 0, & \text{on } C, \end{cases} \tag{5.1}$$

where u is a real function of complex variable $z, \bar{z}$ and we denote

$$u(x,y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = u(z,\bar{z}).$$

Correspondingly, the boundary condition in (5.1) needs to be expressed in terms of the complex variables z and $\bar{z}$

The general real solution of (5.1) is given by ([12] page 58) and (3.13) as,

$$\begin{aligned} u &= 2Re\left(f(z) - \int_0^z f(t)\frac{\partial}{\partial t}J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right)dt\right) \\ &= \left\{f(z) - \int_0^z f(t)\frac{\partial}{\partial t}J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right)dt\right\} + \text{conjugate} \end{aligned} \tag{5.2}$$

where $f(z)$ is an arbitrary holomorphic function in the simply connected domain D bounded by the curve C. Integrating (5.2) by parts then we have,

$$u = \left\{f(0) + \int_0^z \frac{\partial}{\partial t} f(t) J_0(\lambda\sqrt{\bar{z}(z-t)})\right\} + \text{conjugate.} \tag{5.3}$$

For a simply connected region, there are no singularities and we can assume a Taylor series expansion for $f(z)$ given by,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{5.4}$$

where the coefficients a_n are arbitrary real numbers and in the above J_0 is the Bessel's function of the first kind of order 0, namely,

$$J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) = \sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)\frac{\bar{z}^k(z-t)^k}{k!\,k!}. \tag{5.5}$$

On substituting (5.4) and (5.5) in (5.3), we obtain:

$$\begin{aligned} u &= \left[a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k \frac{n\,a_n\bar{z}^k}{k!k!}\int_0^z t^{n-1}(z-t)^k dt\right] \\ &\quad + \text{conjugate} \\ &= 2a_0 J_0(\lambda\sqrt{z\bar{z}}) \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k \frac{n\,a_n\,(z\bar{z})^k}{k!k!} B(n,k+1)\,(z^n+\bar{z}^n)\,, \end{aligned} \tag{5.6}$$

where, $B(n, k+1)$ denotes the Beta function with integer parameters n and $k+1$, given by

$$B(n,k+1) = \frac{(n-1)!k!}{(n+k)!}.$$

We will rewrite (5.6) to give :

$$u = 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} A_{n,k}\,(z^n+\bar{z}^n)\,a_n(z\bar{z})^k, \tag{5.7}$$

where

$$A_{n,k} = \left(-\frac{\lambda^2}{4}\right)^k \frac{n}{k!k!} B(n, k+1), \quad n = 1, 2, \cdots, \infty, \quad k = 0, 1, 2, \cdots, \infty. \tag{5.8}$$

Equation (5.7) gives the general solution of Helmholtz equation on an arbitrary simply connected domain. The problem (5.1) now reduces to finding the coefficients a_n $n = 0, 1, 2, \cdots \infty$, which will be determined by the boundary condition. Since the eigenvalue problems all possess homogeneous Dirichlet boundary conditions, the solution depends of the shape of the domain.

5.1.2 An identity for $z^n + \bar{z}^n$

The solution (5.7) consists of terms of $(z\bar{z})^n$ and powers of $(z+\bar{z})$. On the boundary, the power series representation for u can be simplified considerably, and this will be a key element in the determination of the coefficients of the power series . If $z^n + \bar{z}^n$ can be expressed in terms of $z\bar{z}$ using the expression of the boundary, then we can equate the coefficients of $z\bar{z}$ in (5.7) to obtain equations for $a_0, a_1, a_2, \cdots$ in (5.7).

Using the identity ([11] page 27):

$$\cos(n\theta) = 2^{n-1}\cos^n\theta - 2^{n-3}\frac{n}{1}\cos^{n-2}\theta + 2^{n-5}\frac{n}{2}\binom{n-3}{1}\cos^{n-4}\theta\cdots \quad (5.9)$$

and using the polar form of the complex variable $z = r(\cos\theta + i\sin\theta)$, we can write $z^n + \bar{z}^n$ in terms of $z + \bar{z}$ and $z\bar{z}$, as,

$$\begin{aligned} z^n + \bar{z}^n &= 2r^n\cos(n\theta) \\ &= r^n\left[(2\cos\theta)^n + \sum_{m=1}^{[n]}(-1)^m\frac{n}{m}\binom{n-m-1}{m-1}(2\cos\theta)^{n-2m}\right] \\ &= (z+\bar{z})^n + \sum_{m=1}^{[n]}(-1)^m\frac{n}{m}\binom{n-m-1}{m-1}(z+\bar{z})^{n-2m}(z\bar{z})^m. \end{aligned}$$

5.2 Elliptic Boundary

An example of this method is our case of an elliptic boundary. We consider the two dimensional domain bounded by an ellipse given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

which can also be expressed in the complex plane as,

$$(z+\bar{z})^2 = a + bz\bar{z}, \quad (5.10)$$

where $a = \dfrac{4\alpha^2\beta^2}{\beta^2-\alpha^2} \quad b = \dfrac{4\alpha^2}{\alpha^2-\beta^2}$.

Due to the symmetry of the ellipse with respect to both of the axes, the arbitrary holomorphic function $f(z)$ in (5.6) can be assumed as

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$$

and correspondingly, (5.7) becomes,

$$u = 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} A_{2n,k}\left(z^{2n} + \bar{z}^{2n}\right) a_n (z\bar{z})^k. \tag{5.11}$$

Now, we will express $z^{2n} + \bar{z}^{2n}$ as a function of $z\bar{z}$ and $(z + \bar{z})$ using (5.10).

From the identity (5.9) we can write

$$\begin{aligned} z^{2n} + \bar{z}^{2n} &= 2r^{2n}\cos(2n\theta) \\ &= r^{2n}\left[(2\cos\theta)^{2n} + \sum_{m=1}^{n}(-1)^m \frac{2n}{m}\binom{2n-m-1}{m-1}(2\cos\theta)^{2n-2m}\right] \\ &= (z+\bar{z})^{2n} + \sum_{m=1}^{n}(-1)^m \frac{2n}{m}\binom{2n-m-1}{m-1}(z+\bar{z})^{2n-2m}(z\bar{z})^m. \end{aligned}$$

Substituting for $(z+\bar{z})^2$ in terms of $z\bar{z}$, using equation (5.10), for a given n and z on the boundary curve, we get,

$$\begin{aligned} z^{2n} + \bar{z}^{2n} &= \sum_{l=0}^{n}\sum_{m=0}^{l} \frac{(-1)^m(2n-m-1)!\,2n}{m!\,(2n-2m)!}\binom{n-m}{l-m} a^{n-l} b^{l-m} (z\bar{z})^l \\ &= \sum_{l=0}^{n} b_{l,n}\,(z\bar{z})^l, \end{aligned} \tag{5.12}$$

where

$$b_{l,n} = \sum_{m=0}^{l} \frac{(-1)^m(2n-m-1)!\,2n}{m!\,(2n-2m)!}\binom{n-m}{l-m} a^{n-l} b^{l-m} \tag{5.13}$$

$l = 1, 2, \cdots n$ and $b_{0,n} = a^n$.

Now reverting to (5.7), we can rewrite, formally interchanging summations and rearranging

$$\begin{aligned}
u &= 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} A_{2n,k}(z\bar{z})^k \sum_{l=0}^{n} b_{l,n}\,(z\bar{z})^l a_n \\
&= 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{n} A_{2n,k} b_{l,n}\,(z\bar{z})^{l+k} a_n \\
&= 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} A_{2n,k} b_{0,n} (z\bar{z})^k a_n \\
&\quad + \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\left[\sum_{n=l}^{\infty} A_{2n,k} b_{l,n}\right](z\bar{z})^{l+k} a_n \\
&= 2a_0 J_0(\lambda\sqrt{z\bar{z}}) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} A_{2n,k} b_{0,n} (z\bar{z})^k a_n \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=1}^{k}\left[\sum_{n=l}^{\infty} A_{n,k-l} b_{l,n}\right](z\bar{z})^k a_n.
\end{aligned}$$

Substituting the series for J_0 from (5.5), on the elliptic boundary, $u(z, \bar{z})$ can be expressed in terms of a power series in $(z\bar{z})$ given by

$$\begin{aligned}
u &= 2a_0 + \sum_{n=1}^{\infty} A_{2n,0} b_{0,n} a_n \\
&+ \sum_{k=1}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k \frac{2a_0}{k!k!}(z\bar{z})^k \\
&+ \sum_{k=1}^{\infty}\left[\sum_{n=1}^{k}\left(A_{2n,k} b_{0,n} + \sum_{l=1}^{n} A_{n,k-l} b_{l,n}\right) a_n\right](z\bar{z})^k \\
&+ \sum_{k=1}^{\infty}\left[\sum_{n=k+1}^{\infty}\left(A_{2n,k} b_{0,n} + \sum_{l=1}^{k} A_{n,k-l} b_{l,n}\right) a_n\right](z\bar{z})^k. \qquad (5.14)
\end{aligned}$$

5.3 Boundary Value Problem

We need $u = 0$ on the bounding ellipse given by (5.10) to solve the boundary value problem (5.1). This implies equating powers $(z\bar{z})^k$ to zero for $k = 0, 1, 2\cdots$ in (5.14).

Equating the constant term in (5.14) to zero, we get

$$2\,a_0 + \sum_{n=1}^{\infty} A_{2n,0} b_{0,n} a_n = 0, \tag{5.15}$$

while equating the coefficients of $(z\bar{z})^k$ $(k \neq 0)$ to zero, yields

$$\begin{aligned}\left(-\frac{\lambda^2}{4}\right)^k \frac{2a_0}{k!k!} \quad &+ \quad \sum_{n=1}^{k} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{n} A_{n,k-l} b_{l,n} \right) a_n \\ &+ \sum_{n=k+1}^{\infty} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{k} A_{n,k-l} b_{l,n} \right) a_n = 0, \\ &\qquad k = 1, 2, \cdots. \end{aligned} \tag{5.16}$$

Using the expressions for coefficients in (5.8) and (5.13) and letting $\dfrac{\lambda^2}{4} = T$, equations (5.15) and (5.16) become, after some simplification,

$$2\,a_0 + \sum_{n=1}^{\infty} a_n\, a^n = 0, \tag{5.17}$$

and for the coefficients of $(z\bar{z})^k$, $k = 1, 2, \cdots$,

$$\frac{2a_0(-T)^k}{k!k!} + \sum_{n=1}^{k}\left(\frac{(2n)!(-T)^k a^n}{k!(2n+k)!} + \sum_{l=1}^{n}\frac{(2n)!(-T)^{k-l}a^n}{(k-l)!(2n+k-l)!}\right)a_n$$
$$+\sum_{n=k+1}^{\infty}\left(\frac{(2n)!(-T)^k a^n}{k!(2n+k)!} + \sum_{l=1}^{n}\frac{(2n)!(-T)^{k-l}a^n}{(k-l)!(2n+k-l)!}\right)a_n = 0,$$
$$k = 1, 2, \cdots. \quad (5.18)$$

Using equation (5.17), we can eliminate a_0 from the set of equations (5.18) to derive the linear algebraic infinite system in matrix form as,

$$D\vec{a} = 0, \qquad \vec{a} = \{a_1, a_2, \dots\}$$

where $D = (d_{k,n})$ is an infinite matrix and the coefficients are given by,

$$d_{k,n} = \begin{cases} \displaystyle\sum_{l=1}^{n}\frac{(-T)^{k-l}(2n)!}{(2n+k-l)!(k-l)!}b_{l,n} + \frac{a^n(-T)^k}{k!}\left(\frac{(2n)!}{(2n+k)!} - \frac{1}{k!}\right) & n \le k, \\[2ex] \displaystyle\sum_{l=1}^{k}\frac{(-T)^{k-l}(2n)!}{(2n+k-l)!(k-l)!}b_{l,n} + \frac{a^n(-T)^k}{k!}\left(\frac{(2n)!}{(2n+k)!} - \frac{1}{k!}\right) & n > k, \end{cases} \quad (5.19)$$

where $k = 1, 2, \cdots, \infty$ and $n = 1, 2, \cdots, \infty$.

5.3.1 Numerical Solution of Elliptic Boundary Problem

To get numerical values for eigenvalues of the boundary value problem (5.1), we truncate the infinite matrix in (5.19) to give $D^{(n)}$ an $n \times n$ matrix with n

equations and n unknowns $a_1, a_2, \cdots, a_n$. We now find the zeros of $\det D^{(n)}$ using Maple. Notice that, in the expression of $d_{k,n}$(5.19), we can factor $(-T)^k$ from each row of the matrix and denote the new matrix by $E^{(n)}$, then the determinant of the matrix $D^{(n)} = 0$ can be expressed as

$$\det D^{(n)} = \det E^{(n)} = 0.$$

The (k, n)th component of the matrix $E = (e_{k,n})$ and its $n \times n$ truncation $E^{(n)}$ is given by,

$$e_{k,n} = \begin{cases} \displaystyle\sum_{l=1}^{n} \frac{(-T)^{-l}(2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n}{k!}\left(\frac{(2n)!}{(2n+k)!} - \frac{1}{k!}\right) & n \leq k, \\ \\ \displaystyle\sum_{l=1}^{k} \frac{(-T)^{-l}(2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n}{k!}\left(\frac{(2n)!}{(2n+k)!} - \frac{1}{k!}\right) & n > k, \end{cases} \tag{5.20}$$

$k, n = 1, 2, 3, \cdots$. When $n = 2$, the explicit form of $E^{(2)}$

$$E^{(2)} = \begin{bmatrix} -\frac{2}{3}a + \frac{2-b}{T}, & -\frac{4a^2}{5} + \frac{4a-2ab}{T} \\ \\ -\frac{5a}{24} + \frac{2-b}{3T} & -\frac{7a^2}{30} + \frac{4a-2ab}{5T} + \frac{b^2-4b+2}{T^2} \end{bmatrix}$$

Computing the determinant of $D^{(2)}$, we have the polynomial of T is

$$H = -\frac{56Tab}{15} + \frac{12aT}{5} - 4 - 6b^2 + 10b - \frac{11a^2T^2}{30} + \frac{14Tab^2}{15} + b^3 + \frac{11ba^2T^2}{60} + \frac{a^3T^3}{90}$$

For purposes of comparison, we follow the values chosen in [9], where $\alpha = \sinh 2$ and $\beta = \cosh 2$. Substituting these values in $E^{(2)}$, we have,

$$E^{(2)} = \begin{bmatrix} \frac{54.61646568}{T} - 496.4930539, & \frac{81350.08751}{T} - 443709.6346T \\ \frac{18.20548856}{T} - 155.1540794 & \frac{16270.01750}{T} + \frac{2980.958323}{T^2} - 129415.3101 \end{bmatrix},$$

and setting $\det E^{(2)}$ to zero, the real eigenvalue λ is 0.6513444659. Compared with the result in [9], the first value of C_{e_0} is 0.65123129; the difference is 1.13×10^{-4}. For $n = 4$, the real eigenvalues are 0.6512312879; 1.388920047, 1.496159845 and 2.454121936. Compared with the first values of C_{e_0}, 0.65123129 and C_{e_2}, 1.38747716 in [9], the differences are 2×10^{-9} and 1.4×10^{-3}.

EIGENVALUE	OUR RESULT	[9]RESULT	DIFFERENCE
1	0.6512312879	0.65123129	10^{-9}
2	1.388920047	1.38747716	10^{-3}
3	1.496159845	1.49784709	10^{-3}

Table 5.1: Comparison of the results when $n = 4$ for λ

EIGENVALUE	OUR RESULT	[9]RESULT	DIFFERENCE
1	0.6512312876	0.65123129	10^{-9}
2	1.3874937049	1.38747716	10^{-5}
3	1.4978369769	1.49784709	10^{-5}

Table 5.2: Comparison of the results when $n = 5$ for λ

EIGENVALUE	OUR RESULT	[9]RESULT	DIFFERENCE
1	0.6512312882	0.65123129	10^{-9}
2	1.3874803904	1.38747716	10^{-6}
3	1.4977714457	1.49784709	10^{-5}

Table 5.3: Comparison of the results when $n = 6$ for λ

From the above tables, we can see that the first three eigenvalues in our case match with the result in [9] very well. The values in Table 5.4 are given using the method of bisection. We compare the values here with the values given by [9] for C_{e_0}.

C_{e_0} in [9]	Our result	truncation at n
0.65123129	0.65123129	5
1.49784709	1.49784709	8
2.35503473	2.35503473	10
3.21749843	3.12671218	15
4.08203626	4.08204227	15
4.94732655	4.94036498	20
5.81293420	5.92055836	25

Table 5.4: Values by using the bisection method

Since we choose the same value of α and β as the values in [9], the elliptic region is close to the shape of a circle. Therefore the infinite matrix D is almost singular. Numerically, we solve the truncated system, and the real solution is further refined by using the bisection method to compare with the given value in [9]. Table 5.4 shows the comparison of our results by the bisection method with those values given in [9].

Chapter 6

Helmholtz's Equation on a Doubly Connected Domain

The eigenvalue problem of the negative Laplacian is important in electromagnetism. However there are few methods for obtaining the solutions of Helmholtz's equations on doubly connected regions either numerically or analytically. Yu references some famous work in his paper, and also introduces a modified perturbation method to find the fundamental frequency of a doubly connected membrane [22]. The process described in this chapter for finding eigenvalues of the Laplacian on doubly connected regions is similar to the method we discussed in Chapter 5, which we used to find eigenvalues on elliptic regions. We now establish general solutions (involving an infinite number of unknowns) for doubly connected regions and apply boundary conditions to the general solution.

6.1 General Solutions

We assume that the doubly connected domain D is bounded by two closed curves C_1 and C_0. The complement of D consists of two continuums D_0 and D_∞, where D_0 contains the point of origin and D_∞ contains the point of ∞.

The solutions consist of four parts, the elementary part containing only terms of $(z\bar{z})^n$ where $n \geq 0$; holomorphic solutions on the whole elementary region, containing only terms of $z^n\bar{z}^m$, $m \neq n$ and $m, n \geq 0$; holomorphic solutions on the infinite D_∞, containing only terms of $\bar{z}^m/z^n$, where $m, n \geq 0$ and the solutions represented by terms of $1/(z^n\bar{z}^m)$. The four parts are,

1. Solutions in terms of $z\bar{z}$ or $r = \sqrt{z\bar{z}}$ in polar coordinates.

$$u = AJ_0(\lambda r) + BY_0(\lambda r),$$

where A and B are arbitrary and real;

2.

$$u = \left\{ \sum_{n=0}^{\infty} \bar{z}^n f_n(z) \right\} + \text{ conjugate },\tag{6.1}$$

where f_n are holomorphic functions in $D + D_0$, and chosen to have no terms of order lower than z^n. and the Taylor series of an arbitrary holomorphic function f_0 is given by,

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n;\tag{6.2}$$

3.

$$u = \left\{ \sum_{n=0}^{\infty} \bar{z}^n g_n(z) \right\} + \text{ conjugate },\tag{6.3}$$

where g_n are holomorphic functions in $D+D_\infty$ and g_0 can be expanded by Taylor series in $\frac{1}{z}$, which is then given by,

$$g_0(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n};\tag{6.4}$$

4.

$$u = \left\{ \sum_{n=1}^{\infty} \frac{h_n(z)}{\bar{z}^n} \right\} + \text{ conjugate },\tag{6.5}$$

where h_n are holomorphic functions in $D + D_\infty$ and

$$h_1(z) = \sum_{n=1}^{\infty} \frac{c_n}{z^n}.\tag{6.6}$$

In order to obtain the solutions described in part 2, we substitute the expression (6.1) in (5.1), and then we have,

$$\sum_{n=1}^{\infty} n\bar{z}^{(n-1)} f_n'(z) + \frac{\lambda^2}{4}\sum_{n=0}^{\infty} \bar{z}^n f_n(z) = 0 \tag{6.7}$$

$$\text{or } \sum_{n=0}^{\infty} \bar{z}^n \left\{ (n+1) f_{n+1}'(z) + \frac{\lambda^2}{4} f_n(z) \right\} = 0 \tag{6.8}$$

$$\text{hence, } \quad (n+1) f_{n+1}'(z) + \frac{\lambda^2}{4} f_n(z) = 0 \quad \text{for } n = 0, 1, 2, \cdots \tag{6.9}$$

Therefore, following the iteration given by (6.9), we have

$$f_n = \left(-\frac{\lambda^2}{4}\right)^n \frac{1}{n!} \underbrace{\int_0^z \int_0^z \cdots \int_0^z}_{n} f_0(t) dt dt \cdots dt \tag{6.10}$$

According to Cauchy formula, the multiple integrals in(6.10) can be written as,

$$f_n = \left(-\frac{\lambda^2}{4}\right)^n \frac{1}{n!} \frac{1}{(n-1)!} \int_0^z f_0(t)(z-t)^{n-1} dt \tag{6.11}$$

where $n = 1, 2, \cdots$. Substituting (6.11) to the general form of solution, we have

$$\begin{aligned} u &= \left\{ \sum_{n=0}^{\infty} \bar{z}^n f_n(z) \right\} + \text{conjugate} \\ &= 2Re\left\{ f_0(z) + \sum_{n=1}^{\infty} \bar{z}^n \left(-\frac{\lambda^2}{4}\right)^n \frac{1}{n!} \frac{1}{(n-1)!} \int_0^z f_0(t)(z-t)^{n-1} dt \right\} \\ &= 2Re\left\{ f_0(z) + \int_0^z f_0(t) \sum_{n=1}^{\infty} \left(-\frac{\lambda^2}{4}\right)^n \frac{\bar{z}^n}{n!} \frac{(z-t)^{n-1}}{(n-1)!} dt \right\} \\ &= 2Re\left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) dt \right\} \end{aligned} \tag{6.12}$$

where u is the general solution in any simply connected domain with the origin in the domain. It is of the same form as equation (5.3) given in [12].

The procedure for finding the third part of the solutions is very similar, except that the singular point is 0 rather than ∞. Referring to (6.3), the iteration for g_n is given by,

$$(n+1)g'_{n+1} + \frac{\lambda^2}{4} g_n = 0,$$

from which the expression of g_n is obtained as,

$$g_n = \frac{1}{n!}\left(\frac{\lambda^2}{4}\right)^n \underbrace{\int_z^\infty \cdots \int_z^\infty}_{n} g_0(t)dt \cdots dt.$$

The solution is obtained thereafter as,

$$\begin{aligned} u &= \left\{\sum_{n=0}^{\infty} \bar{z}^n g_n(z)\right\} + \text{ conjugate} \\ &= 2Re\left\{g_0(z) + \sum_{n=1}^{\infty} \bar{z}^n \left(\frac{\lambda^2}{4}\right)^n \frac{1}{n!}\frac{1}{(n-1)!}\int_z^\infty g_0(t)(z-t)^{n-1}dt\right\}, \end{aligned}$$

where $g_0(t)$ is given by (6.4). The solution exists if and only if for any n the coefficients of t^{-n} are zeros, i.e., $g_0 \equiv 0$. In another word, the non-trivial solution of the form (6.3) with the assumption of $g_0(z) = \sum\limits_{n=1}^{\infty} \frac{b_n}{z^n}$ doesn't exist.

Now, we come to the fourth part. Substituting back the expression

$$u = \left\{ \sum_{n=0}^{\infty} \frac{h_n(z)}{\bar{z}^n} \right\} + \text{ conjugate },$$

we have

$$\sum_{n=1}^{\infty} (-n) \frac{h_n'(z)}{\bar{z}^{n+1}} + \frac{\lambda^2}{4} \sum_{n=0}^{\infty} \frac{h_n(z)}{\bar{z}^n} + \text{ conjugate } = 0$$

or

$$\frac{\lambda^2}{4} \left(h_0(z) + \frac{h_1(z)}{\bar{z}} + \overline{h_0(z)} + \frac{\overline{h_1(z)}}{z} \right)$$
$$+ \left\{ \sum_{n=2}^{\infty} (1-n) \frac{h_{n-1}'(z)}{\bar{z}^n} + \frac{\lambda^2}{4} \frac{h_n(z)}{\bar{z}^n} \right\} + \text{ conjugate } = 0$$

Hence we have the iteration

$$(1-n)h_{n-1}'(z) + \frac{\lambda^2}{4} h_n(z) = 0, \qquad n = 2, 3, \cdots \tag{6.13}$$

Following the iteration (6.13), we obtain the expression of h_n, which is given by

$$h_n(z) = \left(\frac{4}{\lambda^2} \right)^{(n-1)} (n-1)! h_1^{(n-1)}(z). \tag{6.14}$$

In addition, $h_0(z) + \dfrac{h_1(z)}{\bar{z}} + \overline{h_0(z)} + \dfrac{\overline{h_1(z)}}{z} = 0$. However $h_0(z) + \overline{h_0(z)}$ has already been included in case 3, hence here we can ignore these two terms in case 4. To consider $h_1(z) = \dfrac{c_1}{z}$, the solution of form $\dfrac{1}{\bar{z}} \dfrac{c_1}{z}$ is also included in case 1. Hence here we can take it away and assume that

$$h_1(z) = \sum_{n=2}^{\infty} \frac{c_n}{z^n}.$$

Putting this term back into the solution then we have

$$u = \left\{ \sum_{n=1}^{\infty} (n-1)! \frac{h_1^{(n-1)}(z)}{\bar{z}^n} \left(\frac{4}{\lambda^2} \right)^{(n-1)} \right\} + \text{ conjugate} \tag{6.15}$$

Combining the solutions obtained from the above four cases, we have the general solution of Helmholtz equations in a doubly connected region,

$$\begin{aligned} u &= AJ_0(\lambda r) + BY_0(\lambda r) \\ &\quad +2Re\left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0(\lambda\sqrt{\bar{z}(z-t)})dt \right\} \\ &\quad +\sum_{n=1}^{\infty} \left(\frac{4}{\lambda^2} \right)^{n-1} \left(\frac{h_1^{(n-1)}(z)}{\bar{z}^n} + \frac{\overline{h_1^{(n-1)}(z)}}{z^n} \right). \end{aligned} \tag{6.16}$$

If the domain is simply connected, then in (6.16), $B = 0$ and $h_1 = 0$, and therefore we have the same form of solution as the one given in [12]. If the domain is bounded by concentric circles with the origin the centers, that is, the solution only contains the terms of r or $z\bar{z}$, then the solution would be $u = AJ_0(\lambda r) + BY_0(\lambda r)$.

6.2 Applications

Substituting the expansions of f_0, g_0 and h_1 (6.2), (6.4), (6.6) in the solution (6.16), we can get the solution in the form of summations. For example,

$$\begin{aligned} J_0(\lambda r) &= \sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k \frac{r^{2k}}{k!k!}; \\ Y_0(\lambda r) &= \frac{2}{\pi}\left\{\left[\ln\left(\frac{\lambda r}{2}\right)+\gamma\right] J_0(\lambda r)+\sum_{k=1}^{\infty}(-1)^{(k+1)} H_k \frac{\left(\frac{1}{4}\lambda^2 r^2\right)^k}{(k!)^2}\right\}; \end{aligned}$$

where γ is Euler-Mascheroni constant and H_k is a harmonic number, the logarithm term can be expanded by Taylor series. In chapter 5, we developed the procedure which expresses the second part of the solution

$$2Re\left\{f_0(z)-\int_0^z f_0(t)\frac{\partial}{\partial t}J_0(\lambda\sqrt{\bar{z}(z-t)})dt\right\},$$

as a series shown in equation (5.6)

$$2a_0 J_0(\lambda\sqrt{z\bar{z}})+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k \frac{n\, a_n\,(z\bar{z})^k}{k!k!}B(n,k+1)\,(z^n+\bar{z}^n)\,.$$

For the last part of the solution, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(\frac{4}{\lambda^2}\right)^{n-1}\left(\frac{h_1^{(n-1)}(z)}{\bar{z}^n}+\frac{\overline{h_1^{(n-1)}(z)}}{z^n}\right) \\ =\ &\sum_{n=1}^{\infty}\left(\frac{4}{\lambda^2}\right)^{n-1}\frac{1}{(z\bar{z})^n}\left(\sum_{k=2}^{\infty}\frac{c_k(-1)^{n-1}(k+n-2)!}{(k-1)!z^{k-1}}+\frac{c_k(-1)^{n-1}(k+n-2)!}{(k-1)!\bar{z}^{k-1}}\right) \\ =\ &\sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\left(-\frac{4}{\lambda^2}\right)^{n-1}\frac{c_k(k+n-2)!}{(k-1)!(z\bar{z})^{n+k-1}}\left(z^{k-1}+\bar{z}^{k-1}\right). \end{aligned}$$

Now, the solutions are in the form of series and with terms of $z\bar{z}$ and $(z^n+\bar{z}^n)$. If $(z^n+\bar{z}^n)$ can be expressed in terms of $z\bar{z}$ when applying boundary conditions, then it is easy to find the solutions and eigenvalues of the Laplacian on a doubly connected region. By equalizing the coefficients, the eigenvalue problem can be worked out by solving an infinite system of linear equations.

Chapter 7

Conclusions and Future Work

We have mainly examined two kinds of PDEs, Poisson's and the Helmholtz equations. These are both related to the Laplace operator. In Chapter 4, one of the applications of Poisson equation on the region bounded by eccentric circles has been studied. The numerical results showed us the effectiveness of our method by comparing the values of rates of the flow with those in existing published work. The solution to Poisson's equation is obtained by solving an infinite system of linear equations $A\vec{b} = \vec{f}$. Similar methods could be applied to other doubly connected regions. The formal solution always holds for any doubly connected region with the origin outside. However, the infinite matrix A in our case is definitely nonsingular only under the condition that h, the distance between the centers of the two circles, is much smaller than 1. Hence more work is needed to find out additional conditions for non-singularity when h is relatively larger.

In Chapter 5 and Chapter 6, we discussed the eigenvalue problem of the Laplacian on both simply connected and doubly connected regions. The general solutions derived in Chapter 5 can be applied to other simply connected regions, whose boundaries are of the form $z+\bar{z} = f(z\bar{z})$ or $g(z+\bar{z}) = z\bar{z}$. For example, the rectangular domain is bounded by $x = \pm a/2$ and $y = \pm b/2$. The boundary can be written as,

$$(z + \bar{z} - a)(z + \bar{z} + a)(z - \bar{z} - bi)(z - \bar{z} + bi) = 0$$

or

$$[(z + \bar{z})^2 - a^2][(z + \bar{z})^2 - 2z\bar{z} + b^2] = 0$$

The eigenvalues of the Laplacian are obtained by seeking zeros of the determinant of the coefficient matrix D. In this way, we are able to control the accuracy of the results, i.e. the larger the truncated matrix is, the accurate the eigenvalues will be. By comparing with the results in existing published research paper with our results, the method we have developed appears very promising. A similar method can also be applied to solving eigenvalue problems on doubly connected regions. Chapter 6 gives the procedure for deducing the general solution of Helmholtz equations on doubly connected region.

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