Approximately Local Derivations from Various Classes of Banach Algebras

by

Ebrahim Samei

A Thesis submitted to
the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Department of Mathematics
University of Manitoba
Winnipeg, Manitoba

0-494-08795-1

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

*****

COPYRIGHT PERMISSION

Approximately Local Derivations from Various Classes of Banach Algebras

BY

Ebrahim Samei

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirement of the degree

Of

Doctor of Philosophy

Abstract

We initiate the study of certain linear operators from a Banach algebra A into a Banach A-bimodule X, which we call *approximately local derivations*. We show that when A is a C^*-algebra, a Banach algebra generated by idempotents, a semisimple annihilator Banach algebra, or the group algebra of a SIN or a totally disconnected group, bounded approximately local derivations from A into X are derivations. We also prove that the same result holds if $p \in (1, \infty)$ and A is the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G whose principle component is abelian. Later on, we extend this idea to the space of n-cocycles and we show that, for some of the above algebras, bounded approximately local n-cocycles from $A^{(n)}$ into X are n-cocycles. Finally, we consider the quantization of these results and apply them to the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G for $p \in (1, \infty)$. We show that $A_p(G)$, equipped with an appropriate operator space structure, is operator weakly amenable. We also show that completely bounded approximately local n-cocycles from $A_p(G)^{(n)}$ into any quantized $A_p(G)$-bimodule are n-cocycles.

Acknowledgement

I would like to express my deep gratitude to my advisor Professor Fereidoun Ghahramani for his support during the work of this dissertation. His expertise in mathematics and his constructive comments have been invaluable and is gratefully acknowledged.

I also wish to thank Professors Yong Zhang, Brian Forrest and Dean Slonowsky for having taken the time to carefully read the thesis and providing many helpful suggestions.

I would like to thank members of the Department of Mathematics at the University of Manitoba who have provided a pleasant atmosphere during the time that I spent there as a graduate student.

Last but not least, I would like to thank my family and my friends whose love and support have been a great source of strength and encouragement for me. I love you all!

Contents

Introduction

The motivation for this thesis was sparked by the discovery that *local derivations on C^*-algebras are derivations.* Let A be a Banach algebra, and let X be a Banach A-bimodule. An operator $D\colon A \to X$ is a local derivation if for each $a \in A$, there is a derivation $D_a\colon A \to X$ such that $D(a) = D_a(a)$. This concept was introduced by R. V. Kadison in [**33**] and was motivated by his and J. R. Ringrose's earlier investigation of Hochschild cohomology of various operator algebras. These maps arise naturally when one seeks conditions to ensure that a given map is a derivation. He showed that if A is a von Neumann algebra and X is a dual Banach A-bimodule, bounded local derivations from A into X are derivations. In particular, bounded local derivations from a von Neumann algebra into itself are derivations. V. Shulman extended this result to bounded local derivations from a C*-algebra into itself and showed that they are derivations [**50**]. Finally, B. E. Johnson extended these results and showed that if A is a C^*-algebra, then local derivations from A into any Banach A-bimodule are derivations [**28**].

Local derivations have also appeared in other contexts. They arose from the study of "algebraic reflexivity" of the linear space of derivations. In [**35**], David R. Larson studied reflexivity and algebraic reflexivity of certain subspaces of operators and asked which algebras have an algebraically reflexive derivation space, or equivalently, for which algebras every local derivation on the algebra is a derivation. He, together with A. Sourour, showed that the answer to that question is affirmative for $B(\mathcal{X})$, where $\mathcal{X}$ is a Banach space [**36**]. R. L. Crist also proved that every bounded local derivation on the direct limit of finite dimensional CSL algebras via *-extendable embeddings (e.g. a triangular AF algebra) is a derivation [**9**]. One can also ask for which algebras the linear space of bounded derivations is reflexive.

In this thesis we address these questions for various classes of Banach algebras including C^*-algebras, and group algebras and Figà-Talamanca-Herz algebras of locally compact groups.

In Chapter 1 we give the necessary background material from theories of Banach algebras, Harmonic analysis, C^*-algebras and operator spaces.

In Chapter 2 we study a certain class of commutative semisimple Banach algebras. We call them *hyper-Tauberian algebras*. We first show that the class of hyper-Tauberian algebras forms a proper subclass of weakly amenable Tauberian algebras. Then we investigate the basic and hereditary properties of them in terms of their ideals, tensor products and algebra homomorphisms. In particular, we show that there are some close relationships between hyper-Tauberian algebras and sets of (local) synthesis. Later on, we exploit Johnson's approach and investigate bounded local derivations from hyper-Tauberian algebras and we show that, in most of the cases, they are derivations. On the other hand, we also present other classes of commutative semisimple Banach algebras that are not hyper-Tauberian and show that, even in the most natural cases, they have bounded local derivations which are not derivations.

In Chapter 3 we introduce the concept of an *approximately local derivation*. An operator D from A into X is an approximately local derivation if for each $a \in A$, there is a sequence of derivations $\{D_{a,n}\}$ from A into X such that $D(a) =$

$\lim_{n\to\infty} D_{a,n}(a)$. Our goal here is to see whether (mainly bounded) approximately local derivations are derivations. The advantage of considering these maps rather than local derivations is that they are more general and they allow us to study both reflexivity and algebraic reflexivity at the same time. On the other hand, in the previous studies of local derivations, it has always been helpful first to study the local multipliers (see [**29**]). However, as it is shown in Sections 3.1 and 3.2, we find it more useful to consider approximately local multipliers. For instance, using this idea helps us to extend Johnson's result and show that approximately local derivations from C^*-algebras are derivations. We also show that bounded approximately local derivations are derivations if they are defined from a hyper-Tauberian algebra, a Banach algebra generated by idempotents, a semisimple annihilator Banach algebra, the group algebra of a SIN or a totally disconnected group, or the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G for $p \in (1, \infty)$ when the principle component of G is abelian. Finally, for a non-discrete group G, we provide an example of an essential Banach $M(G)$-bimodule X and a bounded local derivation T from $M(G)$ into X which is not a derivation.

In [**33**], Kadison has raised the question of whether the results of local derivations from von Neumann algebras can be extended to the local higher cohomology, for example, local 2-cocycles. Chapter 4 is devoted to developing a theory for which we can investigate this question. We first generalize the definition of the reflexivity to the linear subspaces of bounded n-linear maps from Banach spaces and consider the question of reflexivity for the space of bounded n-cocycles from $A^{(n)}$ into X. This naturally leads us to the concept of *approximately local n-cocycles* and the question of whether they are n-cocycles. We show that we can reduce the problem to the characterization of certain operators from A into X which we call *hyperlocal* *operators*. Then we use this idea, together with the properties of hyper-Tauberian algebras, to show that bounded approximately local n-cocycles from $A^{(n)}$ into X are n-cocycles when A is a hyper-Tauberian algebra, a C^*-algebra, the group algebra of a SIN or a totally disconnected group, or the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G for $p \in (1, \infty)$ when the principle component of G is abelian.

In Chapter 5 we consider the quantization of the results we obtained in the previous chapters. We first look at quantized hyper-Tauberian algebras and deduce the quantized version of the results we obtained in Chapters 2, 3 and 4. Then we apply them to the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G for $p \in (1, \infty)$. Since the Fourier algebra $A(G) := A_2(G)$ is the predual of the von Neumann algebra $VN(G)$, it has a natural operator space structure which turns it into a "quantized" Banach algebra [**19**]. In [**46**], Ruan showed that a locally compact group G is amenable if and only if $A(G)$ is operator amenable. In [**34**], A. Lambert, M. Neufang, and V. Runde introduced an operator space structure on $A_p(G)$ that turns it into a quantized Banach algebra. As an application, they extended Ruan's result and showed that G is amenable if and only if $A_p(G)$ is operator amenable for all-and equivalently for one- $p \in (1, \infty)$. It was asked in the above cited paper whether other quantized cohomological properties of $A(G)$ can be extended to $A_p(G)$. One of those results, obtained by N. Spronk, states that $A(G)$ is operator weakly amenable [**51**]. We show that the answer to this question is affirmative by proving that, for any locally compact group G, $A_p(G)$ is a quantized hyper-Tauberian algebra. This, in particular, implies that $A_p(G)$ is operator weakly amenable. It also shows that completely bounded approximately local n-cocycles from $A_p(G)$ are n-cocycles.

CHAPTER 1

Preliminaries

1.1. Algebras and Banach algebras

Throughout this thesis, we consider all the vector spaces to be over the complex field $\mathbb{C}$. Terms and concepts of basic real and functional analysis which we have not defined or discussed can be found in [6] and [47].

An algebra is a vector space A together with a multiplication, called an algebra product, $A \times A \to A$; $(a,b) \mapsto ab$, which is associative and respects the vector operations:

$$(ab)c = a(bc), a(b+c) = ab + ac, (b+c)a = ba + ca \quad (a,b,c \in A)$$
$$(a\lambda)b = a(\lambda b) = \lambda(ab) \quad (\lambda \in \mathbb{C}\ ,\ a,b \in A).$$

We say that A is commutative if $ab = ba$ for all $a,b \in A$. We say that A is unital if A has a multiplicative identity i.e. there is an element $1 \in A$ such that $1 = 1a = a1$ for all $a \in A$. we define the unitization of A to be $A^\sharp = A \oplus \mathbb{C}$ with multiplication:

$$(a,\lambda)(b,\mu) = (ab + a\mu + b\lambda, \lambda\mu)(a,b \in A, \lambda,\mu \in \mathbb{C}).$$

Thus $A^\sharp$ is a unital algebra with unit (0,1).

A subalgebra of an algebra A is a linear subspace B of A such that $ab \in B$ for all $a,b \in B$. A left ideal in an algebra A is a subalgebra $I \subseteq A$ such that, if $a \in A$ and $b \in I$, then $ab \in I$. Similarly, we can define a right ideal and a two-sided ideal for A. When A is commutative, there is no distinction between left, right, and two-sided ideals, and so the word "ideal" is used without any such qualification.

A modular left ideal is a left ideal I for which there is an element $e \in A$ such that $a - ae \in I$ for all $a \in A$. A left ideal I is proper if $0 \subsetneq I \subsetneq A$, and maximal if it is proper and not contained in any other proper left ideal. Given a left ideal I of A, the quotient of I is the two-sided ideal $I : A$ defined by

$$I : A = \{a \in A \mid aA \subseteq I\},$$

where $aA = \{ab : b \in A\}$. The quotient of a maximal modular left ideal is called a primitive ideal. The radical of an algebra A, denoted by rad A, is the intersection of all the primitive ideals of A, or equivalently, the intersection of all the maximal modular left ideals of A [**2**, Proposition 24.14]. An algebra A is semisimple if rad $A = \{0\}$.

For algebras A and B, a linear map $T\colon A \to B$ is an *algebra homomorphism* if $T(ab) = T(a)T(b)$ for each $a, b \in A$. If T is a bijection, then T is an isomorphism.

Let X be a vector space, let A be an algebra, and suppose that we have a bilinear map $A \times X \to X; (a, x) \mapsto ax$ such that

$$(ab)x = a(bx) \quad (a, b \in A, x \in X).$$

Then we say that X is a left A-module. We say that Y is a left A-submodule of X if Y is a subspace of X so that $ay \in Y$ for each $a \in A$ and $y \in Y$. Similarly, we have the notions of right A-module and right A-submodule. We say that X is an A-bimodule if X is both a left and a right A-module and

$$(ax)b = a(xb) \quad (a, b \in A, x \in X).$$

Let X and Y be vector spaces, and let $L(X, Y)$ be the linear space of (linear) operators from X into Y. In the case that $X = Y$, we write $L(X)$ instead of

$L(X, X)$ for simplicity. Now let A be an algebra, and let X and Y be left A-modules. A map $T \in L(X, Y)$ is a left A-module morphism if $T(ax) = aT(x)$ for all $a \in A$ and $x \in X$. In the case that $X = A$, we say that T is a right multiplier. For right A-modules X and Y, right A-module morphisms and left multipliers are defined similarly. For left [right] A-modules X and Y, let ${}_AL(X, Y)$ [$L_A(X, Y)$] be the linear space of left [right] A-module morphisms from X into Y.

A Banach algebra is an algebra A with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach space and

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

The definitions of homomorphism, module etc. all follow over, where we insist on bounded maps and Banach spaces, in the appropriate places. For Banach spaces X and Y, let $B(X, Y)$ be the linear space of bounded (linear) operators from X into Y. When A is a Banach algebra, we give $A^\sharp$ the norm

$$\|(a, \lambda)\| = \|a\| + |\lambda| \quad (a \in A, \lambda \in \mathbb{C}),$$

in order to make it a unital Banach algebra. A Banach space X is a Banach left A-module if X is a left A-module and, for some $K \in \mathbb{N}$,

$$\|ax\| \leq K\|a\|\|x\| \quad (a \in A, x \in X).$$

Similarly, we get the notion of a Banach right A-module and a Banach A-bimodule.

Let X be a Banach left A-module. Then X^*, the dual of X, becomes a Banach right A-module by setting

$$\langle fa, x\rangle = \langle f, ax\rangle \quad (a \in A, x \in X, f \in X^*).$$

Similarly, if X is a Banach right A-module, then X^* becomes a Banach left A-module.

A net $\{a_\alpha\}$ in a Banach algebra A is a left approximate identity if $aa_\alpha \to a$ in norm, for each $a \in A$. Similarly, we have the notions of right approximate identity and approximate identity. If $\{a_\alpha\}$ is norm bounded, then we have a bounded (left/right) approximate identity.

Let A be a complex commutative Banach algebra. A multiplicative linear functional on A is a non-zero linear functional ϕ on A such that

$$\phi(xy) = \phi(x)\phi(y) \quad (x, y \in A),$$

i.e. a non-zero algebra homomorphism from A into $\mathbb{C}$. It is well-known that every multiplicative linear functional ϕ on A is bounded and $||\phi|| \leq 1$ [**2**, Proposition 16.3]. Moreover, the maximal modular ideals of A are precisely the kernels of the multiplicative linear functionals on A [**2**, Theorem 16.5]. The set of all multiplicative linear functionals on A is called the carrier space of A; it is denoted by Φ_A. Hence Φ_A is a subset of the dual space A^*. The A-topology on Φ_A is the relative topology on Φ_A induced by the weak* topology on A^*. Thus if $\Phi_A \neq \emptyset$, then a basic of neighborhood of $\phi \in \Phi_A$ is of the form $V(\phi; x_1, \dots, x_n; \epsilon)$ where

$$V(\phi; x_1, \dots, x_n; \epsilon) = \{\psi \in \Phi_A \mid |\psi(x_k) - \phi(x_k)| < \epsilon \quad (k = 1, ..., n)\},$$

for arbitrary positive integers n, elements $x_1, \dots, x_n \in A$, and $\epsilon > 0$. The carrier space for A is the set Φ_A with the A-topology. It can be shown that the carrier space Φ_A is a locally compact Hausdorff space. Moreover, Φ_A is compact if A is unital [**2**, Proposition 17.2].

Given a topological space Ω, we denote by $C(\Omega)$ the algebra of all continuous complex valued functions f on Ω and $C_0(\Omega)$ the uniform algebra of all continuous complex valued functions f on Ω such that it vanishes at infinity i.e. for every $\epsilon > 0$, there is a compact set K such that $|f(x)| < \epsilon$ whenever $x \notin K$. Note that if Ω is compact, $C_0(\Omega) = C(\Omega)$.

Let A be a commutative Banach algebra with the carrier space $\Phi_A \neq \emptyset$. For each $a \in A$, let $a^\wedge$ be the function on Φ_A defined by

$$a^\wedge(\phi) = \phi(a) \quad (\phi \in \Phi_A).$$

It is clear from the definition of the A-topology that $a^\wedge$ is a continuous complex valued function on Φ_A which vanishes at infinity i.e. $a^\wedge \in C_0(\Phi_A)$. The Gelfand representation of A is the mapping $a \mapsto a^\wedge$ of A into $C_0(\Phi_A)$. It can be shown that the Gelfand representation is a bounded algebra homomorphism from A into $C_0(\Phi_A)$. Moreover, it is a monomorphism if and only if A is semisimple.

We say that A is regular if for each closed subset F of Φ_A and each $\phi_0 \in \Phi_A \setminus F$, there exists $a \in A$ with

$$a^\wedge(\phi) = 0 \quad (\phi \in F), \quad a^\wedge(\phi_0) \neq 0.$$

Given Banach spaces X and Y, a norm $\|\cdot\|$ on $X \otimes Y$ is said to be a a cross-norm if $\| x \otimes y \| = \| x \| \, \| y \|$ for all $x \in X$ and $y \in Y$. Given a cross-norm $\|\cdot\|_\mu$ and a linear combination

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y,$$

we have

$$\| u \|_\mu \leq \sum_{i=1}^{n} \| x_i \| \, \| y_i \| .$$

Thus, if we define

$$\| u \|_\gamma = \inf\{\sum \| x_i \| \, \| y_i \| : u = \sum x_i \otimes y_i\},$$

then it follows that $\| u \|_\mu \leq \| u \|_\gamma$. It is a simple matter to verify that $\|\cdot\|_\gamma$ is in fact a cross-norm, which is called the projective tensor product norm on $X \otimes Y$. We let the Banach space projective tensor product of X and Y be the completion of $(X \otimes Y, \|\cdot\|_\gamma)$; we denote it by $X \widehat{\otimes} Y$. If X, Y and Z are normed spaces and

$\phi: X \times Y \to Z$ is a bilinear mapping, then we define

$$\| \phi \| = \sup\{||\phi(x,y)|| \mid ||x||, ||y|| \leq 1\}.$$

We let $B(X \times Y, Z)$ denote the normed space of all such mapping ϕ with $\| \phi \| < \infty$, and the norm $|| \cdot ||$. The linear isomorphisms determine the isometries

$$B(X \widehat{\otimes} Y, Z) \cong B(X \times Y, Z) \cong B(X, B(Y, Z)).$$

Let A be a Banach algebra, and let X and Y be Banach left[right] A-modules. Let ${}_AB(X,Y)$ $[B_A(X,Y)]$ be the linear spaces of bounded left [right] A-module morphisms from X into Y, respectively. In the case that A has a bounded approximate identity, it is known that right [left] multipliers from A into Y are bounded. So ${}_AL(A,Y) ={}_A B(A,Y)$ $[L_A(A,Y) = B_A(A,Y)]$.

Let X be a Banach A-bimodule. An operator $D \in L(A,X)$ is a derivation if for all $a, b \in A$, $D(ab) = aD(b) + D(a)b$. For each $x \in X$, the operator $ad_x \in B(A,X)$ defined by $ad_x(a) = ax - xa$ is a bounded derivation, called an inner derivation. Let $Z^1(A,X)$, $\mathcal{N}^1(A,X)$ and $\mathcal{Z}^1(A,X)$ be the linear spaces of derivations, inner derivations and bounded derivations from A into X, respectively. A is amenable if for every Banach A-bimodule X, every bounded derivation from A into X^* is inner. A is weakly amenable if every bounded derivation from A into A^* is inner. If A is commutative, then zero is the only inner derivation from A into A^*. Hence A is weakly amenable if every bounded derivation from A into A^* is zero. A Banach A-bimodule X is called symmetric if for all $a \in A$ and $x \in X$, $ax = xa$. By [**10**, Theorem 2.8.63], A is weakly amenable if and only if every bounded derivation from A into any symmetric Banach A-module is zero.

For $n \in \mathbb{N}$ and $T \in L^n(A, X)$, define

$$\begin{aligned}\delta^n T : (a_1, \ldots, a_{n+1}) &\mapsto a_1 T(a_2, \ldots, a_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j T(a_1, \ldots, a_{j-1}, a_j a_{j+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} T(a_1, \ldots, a_n) a_{n+1}.\end{aligned}$$

It is clear that δ^n is a linear map from $L^n(A, X)$ into $L^{n+1}(A, X)$; these maps are the *connecting maps*. The elements of $\ker \delta^n$ are the *n-cocycles*; we denote this linear space by $Z^n(A, X)$. If we replace $L^n(A, X)$ with $B^n(A, X)$ in the above, we will have the 'Banach' version of the connecting maps; we denote them with the same symbols δ^n. In this case δ^n is a bounded linear map from $B^n(A, X)$ into $B^{n+1}(A, X)$; these maps are the *bounded connecting maps*. The elements of $\ker \delta^n$ are the *bounded n-cocycles*; we denote this linear space by $\mathcal{Z}^n(A, X)$. It is easy to check that $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ coincide with our previous definition of these spaces.

Let A be a Banach algebra, and let X be a Banach A-bimodule. By [**10**, Section 2.8], for $n \in \mathbb{N}$, the Banach space $B^n(A, X)$ turns into a Banach A-bimodule by the actions defined by:

$$(a \star T)(a_1, \ldots, a_n) = aT(a_1, \ldots, a_n);$$

$$\begin{aligned}(T \star a)(a_1, \ldots, a_n) &= T(aa_1, \ldots, a_n) \\ &+ \sum_{j=1}^{n} (-1)^j T(a, a_1, \ldots, a_j a_{j+1}, \ldots, a_n) \\ &+ (-1)^{n+1} T(a, a_1, \ldots, a_{n-1}) a_n.\end{aligned}$$

In particular, when $n = 1$, $B(A, X)$ becomes a Banach A-bimodule with respect to the products

$$(a \star T)(b) = aT(b) \quad , \quad (T \star a)(b) = T(ab) - T(a)b.$$

Let $\Lambda_n \colon B^{n+1}(A, X) \to B^n(A, B(A, X))$ be the identification given by

$$(\Lambda_n(T)(a_1, \ldots, a_n))(a_{n+1}) = T(a_1, \ldots, a_{n+1}).$$

It is straightforward to check that Λ_n is an A-bimodule isometric isomorphism. If we denote the connecting maps for the complex $B^n(A, (B(A, X), \star))$ by Δ^n, then we can easily show that

$$\Lambda_{n+1} \circ \delta^{n+1} = \Delta^n \circ \Lambda_n.$$

1.2. C^*-algebras and operator spaces

DEFINITION 1.1. Let A be an algebra. A map $* \colon A \to A$, written by $a \mapsto a^*$, is an involution when we have:

1. $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$ for $\alpha \in \mathbb{C}$ and $a, b \in A$;
2. $(ab)^* = b^* a^*$ for $a, b \in A$;
3. $(a^*)^* = a$ for $a \in A$.

When A is a Banach algebra with an involution and $||a^*a|| = ||a||^2$ for every $a \in A$, we say that A is a C^*-algebra. For example, let Ω be a locally compact space. Then $C_0(\Omega)$, with the uniform norm, is a commutative C^*-algebra with respect to the involution $f \mapsto \bar{f}$, where $\bar{f}(t) = \overline{f(t)}$. Conversely, by [**10**, Theorem 3.2.6], every commutative C^*-algebra A is isometrically $*$-isomorphic with $C_0(\Phi_A)$. More generally, let H be a Hilbert space. For $T \in B(H)$, let $T^* \in B(H)$ be the adjoint of T. We can check that $T \mapsto T^*$ is an involution, and that $\| T^*T \| = \| T\|^2$, so that $B(H)$ is a C^*-algebra. In fact, let A be a closed subalgebra of $B(H)$ such that $A = A^* = \{T^* : T \in A\}$. Then A is a C^*-algebra. Moreover, by [**10**, Theorem 3.2.29], every C^*-algebra arises in this way. The weak operator topology on $B(H)$ is the topology that has a basis of neighborhoods of $T \in B(H)$ given by sets of the form

$$\{S \in B(H) \mid |\langle T(x_k) - S(x_k)\ ,\ y_k\rangle| < \epsilon\ ,\ (k = 1, ..., n)\},$$

for arbitrary positive integers n, elements $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$, and $\epsilon > 0$. A von Neumann algebra M is a C^*-algebra that is closed under the weak operator topology.

Let A be a C^*-algebra which is not unital. We can see that, in general, our unitization, $A^\sharp = A \oplus^1 \mathbb{C}$, is not a C^*-algebra (as the norm dose not satisfy the correct condition). However, there is an equivalent norm on $A^\sharp$ that turns it into a C^*-algebra (see [**10**, Definition 3.2.1]).

Let V be a linear space. For $m, n \in \mathbb{N}$, we let $\mathbb{M}_{m,n}(V)$ denote the linear space of m by n matrices whose entries are in V, and we write $\mathbb{M}_n(V) = \mathbb{M}_{n,n}(V)$. If $V = \mathbb{C}$, then we let $\mathbb{M}_{m,n} = \mathbb{M}_{m,n}(\mathbb{C})$ and $\mathbb{M}_n = \mathbb{M}_{n,n}(\mathbb{C})$.

We define a *matrix norm* $||\cdot||$ on a linear space V to be an assignment of a norm $||\cdot||_n$ on the matrix space $\mathbb{M}_n(V)$ for each $n \in \mathbb{N}$. An *abstract operator space* is a linear space V together with a matrix norm $||\cdot||$ for which

M1 $||v \oplus w||_{m+n} = \max\{||v||_m, ||w||_n\}$ and

M2 $||\alpha v \beta||_n \leq ||\alpha|| ||v||_m ||\beta||$,

for all $v \in \mathbb{M}_m(V), w \in \mathbb{M}_n(V)$ and $\alpha \in \mathbb{M}_{n,m}, \beta \in \mathbb{M}_{m,n}$. We let $M_n(V)$ denote $\mathbb{M}_n(V)$ with the given norm $||\cdot|| = ||\cdot||_n$ (we usually omit the given subscript n). We say that a matrix norm is an *operator space matrix norm* if it satisfies the equations **M1** and **M2** of above.

Let V and W be abstract operator spaces and $\phi\colon V \to W$ a linear map. For each $n \in \mathbb{N}$, ϕ induces a linear map $\phi_n\colon M_n(V) \to M_n(W)$ defined by

$$\phi_n([v_{ij}]) = [\phi(v_{ij})]$$

for $[v_{ij}] \in M_n(V)$. The *completely bounded norm* of ϕ is

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| \mid n \in \mathbb{N}\}.$$

Then ϕ is *completely bounded* (resp. *completely contractive, completely isometric*) if $\|\phi\|_{cb} < \infty$ (resp. $\|\phi\|_{cb} \leq 1$, *each* ϕ_n is an isometry).

Let H be a Hilbert space, and let $B(H)$ be the space of all bounded linear operators on H. For each $n \in \mathbb{N}$, there is a natural operator norm $\|\cdot\|_n$ on the $n \times n$ matrix space $M_n(B(H)) \cong B(H^n)$. This family of norms $\{\|\cdot\|_n\}$ is the *operator matrix norm* on $B(H)$. An *operator space* is a linear subspace of $B(H)$ together with the operator matrix norms inherited from $B(H)$.

It is clear that every operator space is an abstract operator space. Moreover, by [**19**, Theorem 2.3.5], the converse is also true i.e. every abstract operator space is completely isometric with an operator space. For this reason, we will not distinguish between these two. We note that all the operator spaces considered in this thesis are normed closed.

We let $CB(V, W)$ denote the space of all completely bounded maps from V into W. It is shown in [**19**] that there is a natural operator space structure on $CB(V, W)$ obtained by the identification

$$M_n(CB(V, W)) \cong CB(V, M_n(W)).$$

Thus for every operator space V, its Banach dual space $V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C})$ is again an operator space and is called the *operator dual* of V.

For operator spaces V and W, we say that an operator space matrix norm $\|\cdot\|_\mu$ on $V \otimes W$ is a cross matrix norm if

$$\| v \otimes w \|_\mu = \| v \| \, \| w \|,$$

for all $v \in M_p(V)$ and $w \in M_q(W)$. Given an element u in $M_n(V \otimes W)$, we define

$$\| u \|_\wedge = \inf\{\| \alpha \| \| v \| \| w \| \| \beta \| : u = \alpha(v \otimes w)\beta\},$$

where the infimum is taken over arbitrary decompositions for all $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,p\times q}$, and $\beta \in M_{p\times q,n}$, with $p, q \in \mathbb{N}$ arbitrary. It is shown in [**19**] that this is an operator space cross matrix norm. We let the operator space projective tensor product of V and W be the completion of $(V \otimes W, \| \cdot \|_\wedge)$; we denote it by $V\widehat{\otimes}_{op}W$. There is a complete isometry $CB(V, W^*) \cong (V\widehat{\otimes}_{op}W)^*$ given by

$$\langle \widetilde{T}(v)\ ,\ w\rangle = \langle T\ ,\ v \otimes w\rangle \quad (v \in V,\ w \in W, T \in (V\widehat{\otimes}_{op}W)^*).$$

Also, if Z is an operator space, then there are natural complete isometric isomorphisms $V\widehat{\otimes}_{op}W \cong W\widehat{\otimes}_{op}V$ and $(V\widehat{\otimes}_{op}W)\widehat{\otimes}_{op}Z \cong V\widehat{\otimes}_{op}(W\widehat{\otimes}_{op}Z)$ (see [**19**]).

Let A be a Banach algebra which is additionally an operator space. A is called a *quantized Banach algebra* if the multiplication $m\colon A\widehat{\otimes}_{op}A \to A$, specified by $m(a \otimes b) = ab$, is completely bounded. In the case when m is completely contractive, A is called a *completely contractive Banach algebra.*

Let X be a Banach A-bimodule. Then X is called a *quantized A-bimodule* if X is an operator space and the A-bimodule operations

$$A\widehat{\otimes}_{op}X \to X \ \ ; \ \ a \otimes x \mapsto ax$$

and

$$X\widehat{\otimes}_{op}A \to X \ \ ; \ \ x \otimes a \mapsto xa$$

are completely bounded. It is easy to check that there is a natural quantized A-bimodule structure on X^*. A is operator amenable if for every quantized A-bimodule X, every completely bounded derivation from A into X^* is inner [**46**]. A is called operator weakly amenable if every completely bounded derivation from A

into A^* is inner [**17**]. By [**17**, Proposition 3.2], for A commutative, this is equivalent to saying that every completely bounded derivation from A into any symmetric quantized A-module is zero.

Let A be a quantized Banach algebra, and let X be a quantized A-bimodule. For $n \in \mathbb{N}$, let $CB^n(A, X)$ be the space of completely bounded n-linear maps from $A^{(n)}$ into X. It is easy to see that there is a natural operator quantization of the connecting maps δ^n defined in section 1.1. In this case, δ^n is a completely bounded linear map from $CB^n(A, X)$ into $CB^{n+1}(A, X)$; these maps are the *completely bounded connecting maps.* The elements of $\ker \delta^n$ are the *completely bounded n-cocycles*; we denote this linear space by $\mathcal{O}\mathcal{Z}^n(A, X)$.

1.3. Harmonic analysis

A topological group is a group that is also a Hausdorff topological space in which the multiplication map from $G \times G$ into G and the inversion map from G into G, defined by

$$(u, v) \mapsto uv \quad \text{and} \quad u \mapsto u^{-1},$$

are continuous. A topological group is said to be compact, locally compact, discrete, connected or totally disconnected if it has the corresponding property as a topological space.

Let G be a locally compact group, and let $\mathcal{B}_G$ be the σ-algebra of Borel subsets of G. Then, by [**10**, Theorem 3.3.2], there is a positive, regular Borel measure λ on G such that:

(i) $\lambda(U) > 0$ for each non-empty, open subset U;

(ii) $\lambda(K) < \infty$ for each compact subset K;

(iii) $\lambda(tE) = \lambda(E)$ for each $t \in G$ and $E \in \mathcal{B}_G$.

Moreover, λ is unique up to a positive multiple. It is called the left Haar measure of G.

Let G be a locally compact group with a fixed left Haar measure λ. For $1 \leq p < \infty$, let $L^p(G)$ be the space of all the complex-valued, λ-measurable function f defined λ-almost everywhere on G such that $\int_G |f|^p d\lambda < \infty$. We identify functions f and g in $L^p(G)$ if $f = g$ λ-almost everywhere. In this case, $L^p(G)$ is a Banach space with the norm

$$||f||_p = (\int_G |f|^p d\lambda)^{\frac{1}{p}} \quad (f \in L^p(G)).$$

The measure algebra $M(G)$ is the Banach space of complex-valued, regular Borel measures on G. The space $M(G)$ is identified with the (dual) space of all continuous linear functionals on the Banach space $C_0(G)$, with the duality specified by setting

$$\langle \mu \ , \ f \rangle = \int_G f(t) d\mu(t) \quad (f \in C_0(G), \mu \in M(G)).$$

The convolution multiplication $*$ on $M(G)$ defined by setting

$$\langle \mu * \nu \ , \ f \rangle = \int_G \int_G f(st) d\mu(s) d\nu(t) \quad (f \in C_0(G), \ \mu, \nu \in M(G)).$$

We write δ_s for the point mass at $s \in G$; the element δ_e is the identity of $M(G)$, and $l^1(G)$ is the closed subalgebra of $M(G)$ generated by the point masses. Then $M(G)$ is a unital Banach algebra and $L^1(G)$, the group algebra on G, is a closed ideal in $M(G)$ [**10**, Theorem 3.3.36]. Moreover, $M(G) = L^1(G) = l^1(G)$ if and only if G is discrete.

The *strong operator topology* on $M(G)$ is defined as follows: a net $\{\mu_\alpha\}$ converges to μ ($\mu_\alpha \xrightarrow{s.o.} \mu$) if and only if $\mu_\alpha * f \longrightarrow \mu * f$ and $f * \mu_\alpha \longrightarrow f * \mu$ in norm, for every $f \in L^1(G)$. From [**10**, Theorem 2.9.49 and Theorem 3.3.41], both $L^1(G)$ and $l^1(G)$ are s.o. dense in $M(G)$.

Let G be a locally compact group. A positive-definite function on G is a function $f\colon G \to \mathbb{C}$ such that for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$\sum_{i,j} \lambda_i \overline{\lambda_j} f(x_i x_j^{-1}) \geq 0.$$

Let $P(G)$ be the set of all continuous positive-definite functions on G, and let $B(G)$ be its linear span. The space $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest C^*-norm. With pointwise multiplication and the dual norm, $B(G)$ is a commutative regular semisimple Banach algebra. The Fourier algebra $A(G)$ is the closure of $B(G) \cap C_{00}(G)$ in $B(G)$. It is shown in [**18**] that $A(G)$ is a commutative regular semisimple Banach algebra whose carrier space is G. Also, up to isomorphism, $A(G)$ is the unique predual of $VN(G)$, the von Neumann algebra generated by the left regular representation of G on $L^2(G)$ i.e. the representation $\lambda\colon G \to B(L^2(G))$ given by

$$\lambda(t)(f)(x) = f(t^{-1}x) \quad (t, x \in G, f \in L^2(G)).$$

Let G be a locally compact group, and let $p \in (1, \infty)$. Let $q \in (1, \infty)$ be the dual of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The *Figà-Talamanca-Herz algebra* $A_p(G)$ consists of those functions $f\colon G \to \mathbb{C}$ where there are sequences $\{a_n\}_{n=1}^{\infty}$ in $L^p(G)$ and $\{b_n\}_{n=1}^{\infty}$ in $L^q(G)$ such that

$$\sum_{n=1}^{\infty} ||a_n||_{L^p(G)} ||b_n||_{L^q(G)} < \infty$$

and

$$f = \sum_{n=1}^{\infty} a_n * \check{b}_n,$$

where $\check{u}(x) = u(x^{-1})$ for any functions $u\colon G \to \mathbb{C}$ and $x \in G$. It is easy to see that $A_p(G)$ with the norm

$$||f||_p = \inf\{\sum_{n=1}^{\infty} ||a_n||_{L^p(G)}||b_n||_{L^q(G)} \mid f = \sum_{n=1}^{\infty} a_n * \check{b}_n\}$$

is a Banach space. It was shown by Herz that, with pointwise addition and multiplication, $A_p(G)$ is a commutative regular semisimple Banach algebra whose carrier space is G ([**25**], [**24**]). If $p = q = 2$, then $A_2(G) = A(G)$ [**18**]. Finally, we note that by Herz [**25**], $A_p(G)$ is a Banach $A(G)$-bimodule.

CHAPTER 2

Hyper-Tauberian algebras

In this chapter, we initiate the study of certain commutative regular semisimple Banach algebras, which we call hyper-Tauberian algebras. We first investigate the basic and hereditary properties of these algebras. Then we give some examples. Finally, we look at bounded local derivations which are defined from hyper-Tauberian algebras and we show that, in most of the cases, they are derivations.

2.1. Introduction

Throughout this chapter, A and B are commutative semisimple regular Banach algebras with the carrier spaces Φ_A and Φ_B, respectively. Let I be a closed ideal in A. The *hull* of I is

$$\{t \in \Phi_A \mid a(t) = 0 \text{ for all } a \in I\},$$

and it is denoted by $h(I)$. Let E be a subset of Φ_A. Put

$$I(E) = \{a \in A \mid a = 0 \text{ on } E\},$$

and

$$I_0(E) = \{a \in A \mid a \text{ has a compact support disjoint from } E\}.$$

Let E be a closed set. Then $I(E)$ is the largest and $I_0(E)$ is the smallest ideal in A whose hull is E [**10**, Proposition 4.1.20]. We say that E is *a set of synthesis* for A if there is a unique closed ideal in A whose hull is E. So E is a set of synthesis for A if and only if $I_0(E)$ is dense in $I(E)$. If we let $J(E)$ be the closure of

$$\{a \in I(E) \mid \operatorname{supp} a \text{ is compact}\},$$

then E is *a set of local synthesis* for A if $I_0(E)$ is dense in $J(E)$ (see [24]). It is clear that every set of synthesis is a set of local synthesis. Let A_c be the set of all elements in A with the compact support. If A_c is dense in A ,i.e. A is a *Tauberian algebra* [44], then $J(E)$ is a maximal ideal of A having E as its hull and being essential as a Banach A-bimodule. So if E is a set of local synthesis, then $J(E)$ is the only closed ideal in A with this property.

Let X and Y be Banach spaces. Let $\mathcal{S}$ be a linear subspace of $L(X,Y)$ and for each $x \in X$ let $\mathcal{S}x = \{S(x) \mid S \in \mathcal{S}\}$ and $[\mathcal{S}x]$ be the norm-closure of $\mathcal{S}x$. Put

$$\operatorname{ref}_a(\mathcal{S}) = \{T \in L(X,Y) \mid T(x) \in \mathcal{S}x, x \in X\};$$

and if $\mathcal{S} \subseteq B(X,Y)$ put

$$\operatorname{ref}(\mathcal{S}) = \{T \in B(X,Y) \mid T(x) \in [\mathcal{S}x], x \in X\}.$$

Suppose that $\mathcal{S} \subseteq L(X,Y)$. Then $\mathcal{S}$ is *algebraically reflexive* if $\mathcal{S} = \operatorname{ref}_a(\mathcal{S})$ and when $\mathcal{S} \subseteq B(X,Y)$, it is *reflexive* if $\mathcal{S} = \operatorname{ref}(\mathcal{S})$. For more on these notions see [35] and the references therein.

2.2. Local operators

Let X be a Banach left (right) A-module. For $x \in X$, the annihilator $\operatorname{Ann}_A(x)$ of x is

$$\operatorname{Ann}_A(x) = \{a \in A \mid \ ax = 0 \ (xa = 0)\}.$$

$\operatorname{Ann}_A(x)$ is clearly a closed ideal in A. The hull of $\operatorname{Ann}_A(x)$ is called the *support* of x (in Φ_A), denoted by $\operatorname{supp}_A x$. We will write "$\operatorname{supp} x$" instead of "$\operatorname{supp}_A x$" whenever there is no risk of ambiguity. In the case $X = A$ where we regard A

as a Banach (left or right) A-module on itself, the support of an element $a \in A$ coincides with the usual definition of $\operatorname{supp} a$, namely $\operatorname{cl}\{t \in \Phi_A \mid a(t) \neq 0\}$.

The following lemma is the modification of [**18**, Proposision 4.4].

LEMMA 2.1. *Let X be a Banach left (right) A-module, and let $x \in X$. Then $t \in \operatorname{supp} x$ if and only if for every compact neighborhood V of t, there is an element $a \in A$, with the support in V, such that $ax \neq 0$ $(xa \neq 0)$.*

PROOF. We prove the Lemma in the case of a left module. The other case can be proved similarly. Let $t \in \operatorname{supp} x$ and assume that there is a compact neighborhood V of t such that for every $a \in A$, with $\operatorname{supp} a \subseteq V$, we have $ax = 0$. By the regularity of A [**10**, Proposition 4.1.18], there is $b \in A$ such that $\operatorname{supp} b \subseteq V$ and $b(t) \neq 0$. However, $bx = 0$ and $t \in \operatorname{supp} x$. This implies that $b(t) = 0$, which is a contradiction. For the converse, let $t \in \Phi_A$ with the given property, and let $a \in A$ such that $a(t) \neq 0$. We will show that $ax \neq 0$. There is a compact neighborhood V of t and $\delta > 0$ such that $|a(v)| \geq \delta > 0$ for all $v \in V$. Because of the regularity of A and [**44**, Theorem 3.6.15], there is $b \in A$ such that $ab = 1$ on V. Let $c \in A$ be a function whose support is in V such that $cx \neq 0$. Then $abcx = cx$, therefore $ax \neq 0$. □

The preceding lemma indicates that $t \notin \operatorname{supp} x$ if and only if there is a compact neighborhood V of t in Φ_A such that, for every element $a \in A$, if $\operatorname{supp} a \subseteq V$, then $ax = 0$ $(xa = 0)$. In particular, if $a \in A$ and $x \in X$ are such that $\operatorname{supp} a$ is compact and $\operatorname{supp} a \cap \operatorname{supp} x = 0$, then, by applying a suitable partition of unity on $\operatorname{supp} a$, we have $ax = 0$ $(xa = 0)$.

DEFINITION 2.2. Let X and Y be Banach left (right) A-modules. An operator $T\colon X \to Y$ is *local* with respect to the left (right) A-module action if $\operatorname{supp} T(x) \subseteq \operatorname{supp} x$ for all $x \in X$.

PROPOSITION 2.3. *Let A be a Tauberian algebra, and let X be Banach left (right) A-module. Then a bounded operator $T\colon A \to X$ is local if $\operatorname{supp} T(a) \subseteq \operatorname{supp} a$ for each $a \in A_c$.*

PROOF. We prove the statement in the case of left module. The other case can be shown similarly. Let $a \in A$ and $t \notin \operatorname{supp} a$. There is an open subset V in Φ_A such that $t \in V$, $\overline{V}$ is compact and $V \cap \operatorname{supp} a = \emptyset$. By the regularity of A, there is $e \in A_c$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp} a$. So

$$ae = 0. \tag{1}$$

Since A is Tauberian, there is a sequence $\{a_n\}$ in A_c such that $a_n \to a$ as $n \to \infty$. Put $e_n = a_n - a_n e$. Then

$$e_n \in A_c \text{ and } e_n = 0 \text{ on } V \tag{2}$$

for all n. Moreover, from (1),

$$e_n - a = a_n - a_n e - a = (a_n - a) - (a_n - a)e.$$

Hence $e_n \to a$ as $n \to \infty$. Now let $c \in A$ with $\operatorname{supp} c \subseteq V$. Then $\operatorname{supp} c$ and $\operatorname{supp} e_n$ are compact. Moreover, from (2),

$$\operatorname{supp} c \cap \operatorname{supp} T(e_n) \subseteq \operatorname{supp} c \cap \operatorname{supp} e_n \subseteq V \cap V^c = \emptyset.$$

Therefore $cT(e_n) = 0$, and so, by letting $n \to \infty$, $cT(a) = 0$. Hence $t \notin \operatorname{supp} T(a)$. Therefore $\operatorname{supp} T(a) \subseteq \operatorname{supp} a$ for all $a \in A$. $\square$

Let X and Y be Banach left (right) A-modules. It is easy to see that every left (right) A-module morphism from X into Y is a local operator. We finish this section with the following proposition which indicates that, under some additional assumptions, the converse of the above statement can also be true. We will use this result throughout this chapter.

We recall that a Banach left [right] A-module X is *essential* if it is the closure of $AX = \text{span}\{ax \mid a \in A, x \in X\}$ $[XA = \text{span}\{xa \mid a \in A, x \in X\}]$. A Banach A-bimodule X is *essential* if it is essential both as a Banach left and right A-module.

PROPOSITION 2.4. *If bounded local operators from A into A^* are multipliers, then, for essential Banach left (right) A-modules X and Z, and an essential Banach right (left) A-module Y,*
(i) every bounded local operator T from X into Y^* is a left (right) A-module morphism;
(ii) if A has a bounded approximate identity, then the result in (i) is also true for every bounded local operator T from X into Z.

PROOF. We prove the result for the case when X and Z are left modules and Y is a right module. The proof of the other cases follows similar lines.
(i) Let $x \in X$ and $y \in Y$. Define

$$L_x \colon A \to X \ , \ L_x(a) = ax \ , \ (a \in A);$$

$$K_y \colon Y^* \to A^* \ , \ \langle K_y(y^*) \ , \ a \rangle = \langle y^* \ , \ ya \rangle \quad (a \in A \ , \ y^* \in Y^*).$$

It is easy to see that L_x and K_y are bounded left A-module morphisms. Hence $K_y \circ T \circ L_x$ is a bounded local operator from A into A^*, and so it is a multiplier. Therefore $K_y(T(abx) - aT(bx)) = 0$ for all a, b in A and $x \in X$. So, for each $c \in A$,

$$\langle T(abx) - aT(bx) \ , \ yc \rangle = 0.$$

The final result follows by the essentiality of X and Y.

(ii) Suppose that A has a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. For $z^* \in Z^*$, by replacing K_y with

$$K'_{z^*}: Z \longrightarrow A^* \ , \ \langle K'_{z^*}(z) \ , \ a\rangle = \langle z^* \ , \ az\rangle$$

and applying a similar argument to the one made in (i), we can show that

$$a[T(bx) - bT(x)] = 0. \tag{1}$$

On the other hand, by Cohen's Factorization Theorem, there is $c \in A$ and $z \in Z$ such that $T(bx) - bT(x) = cz$. So we have the final result by putting $a = e_\alpha$ in (1) and letting $\alpha \longrightarrow \infty$. □

2.3. Definition and basic properties of hyper-Tauberian algebras

In Proposition 2.4 we showed that if bounded local operators from A into A^* are multipliers, then we can characterize bounded local operators from essential modules of A into their duals. This, together with the other results of the following two sections, is the motivation behind the following definition.

DEFINITION 2.5. We say that A is a *hyper-Tauberian algebra* if every bounded local operator from A into A^* is a multiplier.

The next theorem shows that the class of hyper-Tauberian algebras is a subclass of weakly amenable Tauberian algebras.

We note that if A is Tauberian, then, by the regularity of A [**10**, Proposition 4.1.18], $A_c = A_c^2$, where A_c is the set of all elements in A with the compact support. Thus $A = \overline{A^2}$, and so A is essential as a Banach A-bimodule on itself.

THEOREM 2.6. *Let A be a hyper-Tauberian algebra. Then:*

(i) A is Tauberian;

(ii) each singleton subset $\{t\}$ of Φ_A is a set of synthesis for A;

(iii) A is weakly amenable.

PROOF. (i) Let $f \in A^*$ be such that $f = 0$ on A_c, and let $\varphi \in A^*$ be such that $\varphi \neq 0$ on A_c. Define the bounded operator $S\colon A \to A^*$ by

$$S(a) = \varphi(a)f \quad (a \in A).$$

Since $f = 0$ on A_c, $af = 0$ for all $a \in A_c$. Thus $\operatorname{supp} f = \emptyset$. Therefore S is local, and so, by hypothesis, it is a multiplier. Take $b \in A_c$ with $\varphi(b) \neq 0$. By the regularity of A, there is $a \in A$ such that $a = 1$ on $\operatorname{supp} b$. So $ab = b$. Hence

$$S(b) = S(ab) = aS(b) = \varphi(b)af = 0.$$

Therefore $\varphi(b)f = 0$, and so $f = 0$. Thus A_c is dense in A.

(ii) Let $F \in A^*$ such that $F = 0$ on $I_0(t)$, and let φ_t be the multiplicative linear functional on A defined by $\varphi_t(a) = a(t)$ for all $a \in A$. Define the bounded operator $T\colon A \to A^*$ by

$$T(a) = F(a)\varphi_t \quad (a \in A).$$

We claim that T is local. We first show that

$$I = \{c \in A \mid t \notin \operatorname{supp} c\} \subseteq \overline{I_0(t)}. \tag{1}$$

Let $c \in I$, and let V be a compact neighborhood of t in Φ_A such that $V \cap \operatorname{supp} c = \emptyset$. By the regularity of A, there is $e \in A$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp} c$. In particular,

$$ce = 0. \tag{2}$$

Since, from (i), A is Tauberian, there is a sequence $\{c_n\}$ in A_c such that $c_n \to c$ as $n \to \infty$. Put $e_n = c_n - c_n e$. Then, since $e = 1$ on V, $e_n = 0$ on V for all n.

Therefore $e_n \in I_0(t)$. Moreover, from (2),

$$e_n - c = c_n - c_n e - c = (c_n - c) - (c_n - c)e.$$

Hence $e_n \to c$ as $n \to \infty$. Thus $c \in \overline{I_0(t)}$. This proves (1). Now let $a \in A$. We consider the following two cases:

Case I: $t \notin \operatorname{supp} a$. Then $a \in I$. However, $F = 0$ on $I_0(t)$, and so, from (1), $F = 0$ on I. Thus $T(a) = 0$. Therefore $\operatorname{supp} T(a) = \emptyset$.

Case II: $t \in \operatorname{supp} a$. Let $s \notin \operatorname{supp} a$. There is a compact neighborhood U of s in Φ_A such that $U \cap \operatorname{supp} a = \emptyset$. Let $b \in A$ with $\operatorname{supp} b \subseteq U$. Since $t \in \operatorname{supp} a$ and $\operatorname{supp} b \cap \operatorname{supp} a = \emptyset$, $t \notin \operatorname{supp} b$. Thus $b(t) = 0$. Hence

$$\langle bT(a)\ ,\ c\rangle = \langle T(a)\ ,\ cb\rangle = F(a)\varphi_t(cb) = 0,$$

for all $c \in A$. Therefore $bT(a) = 0$, and so $s \notin \operatorname{supp} T(a)$.

Therefore, in either of the above cases, we have $\operatorname{supp} T(a) \subseteq \operatorname{supp} a$. Thus T is a bounded local operator, and so it is a multiplier. Now let $a \in I(t)$ and $b \in A_c$. Then, for all $c \in A$,

$$\langle T(ba)\ ,\ c\rangle = \langle T(b)a\ ,\ c\rangle = \langle T(b)\ ,\ ac\rangle = F(b)\varphi_t(ac) = 0.$$

Thus

$$\langle T(a)\ ,\ cb\rangle = \langle bT(a)\ ,\ c\rangle = \langle T(ba)\ ,\ c\rangle = 0.$$

Therefore $T(a)$ vanishes on $A_c \cdot A$. However, $A_c \cdot A$ contains $A_c^2 = A_c$, and so it is dense in A. Thus $T(a) = 0$. Therefore F vanishes on $I(t)$. Hence $I(t) = \overline{I_0(t)}$.

(iii) Let $D\colon A \to A^*$ be a bounded derivation. We first show that D is local. Let $a \in A$, $t \notin \operatorname{supp} a$, and let V be a compact neighborhood of t in Φ_A such that $V \cap \operatorname{supp} a = \emptyset$. Let $c \in A$ with $\operatorname{supp} c \subseteq V$. By the regularity of A, there is $e \in A$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp} a$. Therefore

$$ca = 0, ce = c \text{ and } ae = 0.$$

Thus

$$cD(a) = ceD(a) = cD(a)e = D(ca)e - D(c)ae = 0.$$

Hence $t \notin \operatorname{supp} D(a)$. Therefore D is local, and so it is a multiplier. Thus $D(a)b = D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. Hence $aD(b) = 0$. Therefore $D(b) = 0$ since $\overline{A^2} = A$. □

In [22], N. Groenbaek has given a necessary and sufficient condition for a commutative Banach algebra $\mathfrak{A}$ to be weakly amenable in terms of the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$. As we will see in the following theorem, there is also a parallel characterization for certain hyper-Tauberian algebras.

We note that if $A\widehat{\otimes}B$ is semisimple, then it is a commutative semisimple regular Banach algebra with the carrier space $\Phi_A \times \Phi_B$ [**2**, Proposition 42.19 and Corollary 23.9].

THEOREM 2.7. *Let A be a Tauberian algebra such that $A\widehat{\otimes}A$ is semisimple. Then A is hyper-Tauberian if and only if the diagonal $\Delta = \{(t,t) \mid t \in \Phi_A\}$ is a set of local synthesis for $A\widehat{\otimes}A$.*

PROOF. "$\Longleftarrow$" Let $T\colon A \to A^*$ be a bounded local operator, and let $\tilde{T}\colon A\widehat{\otimes}A \to \mathbb{C}$ be a functional specified by

$$\tilde{T}(a \otimes b) = \langle T(a)\ ,\ b\rangle \quad (a, b \in A).$$

Pick a and b in A_c such that $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$. By the regularity of A, there is $e \in A$ such that $e = 1$ on $\operatorname{supp} b$, so that, $b = eb$. Since T is local,

$$\operatorname{supp} b \cap \operatorname{supp} T(a) \subseteq \operatorname{supp} b \cap \operatorname{supp} a = \emptyset.$$

So $bT(a) = 0$, and hence $\tilde{T}(a \otimes b) = \langle T(a)\ ,\ b\rangle = \langle bT(a)\ ,\ e\rangle = 0$. So $\tilde{T}$ vanishes on J, where

$$J = \overline{\operatorname{span}}\{a \otimes b \in A \widehat{\otimes} A \mid a, b \in A_c \text{ and } \operatorname{supp} a \cap \operatorname{supp} b = \emptyset\}.$$

But J is a closed two-sided ideal whose hull is the diagonal and is also essential as a Banach $A\widehat{\otimes}A$-bimodule. So, by hypothesis, $J = J(\Delta)$. Therefore, for all $a, b, c \in A_c$, $\tilde{T}(ac \otimes b) = \tilde{T}(a \otimes cb)$ since $ac \otimes b - a \otimes cb \in J(\Delta)$. Thus

$$\langle T(ac)\ ,\ b\rangle = \langle T(a)\ ,\ cb\rangle = \langle T(a)c\ ,\ b\rangle.$$

The final result follows from the fact that A_c is dense in A.

"$\Longrightarrow$" Let $\mathcal{T} \in (A\widehat{\otimes}A)^*$ such that $\mathcal{T} = 0$ on $I_0(\Delta)$. We show that $\mathcal{T} = 0$ on $J(\Delta)$. Define the bounded operator $T\colon A \to A^*$ by

$$\langle T(a)\ ,\ b\rangle = \mathcal{T}(a \otimes b) \quad (a, b \in A).$$

We show that T is local. Take $a \in A$ and $t \notin \operatorname{supp} a$. By Proposition 2.3, we can assume that $\operatorname{supp} a$ is compact. There is a compact neighborhood V in Φ_A such that $t \in V$ and $V \cap \operatorname{supp} a = \emptyset$. Now let $c \in A$ with $\operatorname{supp} c \subseteq V$. Then for every $b \in A$, $\operatorname{supp}(bc) \subseteq V$. Thus $\operatorname{supp}(bc) \cap \operatorname{supp} a = \emptyset$. Moreover, $\operatorname{supp} bc$ and $\operatorname{supp} a$ are compact. Therefore $a \otimes bc \in I_0(\Delta)$. Hence

$$\langle cT(a)\ ,\ b\rangle = \langle T(a)\ ,\ bc\rangle = \mathcal{T}(a \otimes bc) = 0.$$

Therefore $t \notin \operatorname{supp} T(a)$. Thus T is a bounded local operator, and so, by hypothesis, it is a multiplier. Now let $c, d \in A$ and $u = \sum_{i=1}^{\infty} a_i \otimes b_i \in I(\Delta)$. If we let $\pi\colon A\widehat{\otimes}A \to A$ be the multiplication operator specified by $\pi(a \otimes b) = ab$, then, with the assumption of the semisimplicity of $A\widehat{\otimes}A$, it is easy to verify that $I(\Delta) = \ker \pi$.

Hence $\sum_{i=1}^{\infty} a_i b_i = 0$, and so

$$\begin{aligned} \mathcal{T}[(c \otimes d)u] &= \sum_{i=1}^{\infty} \mathcal{T}(ca_i \otimes db_i) \\ &= \sum_{i=1}^{\infty} \langle T(ca_i)\ ,\ db_i\rangle \\ &= \sum_{i=1}^{\infty} \langle T(c)a_i\ ,\ b_i d\rangle \\ &= \langle dT(c)\ ,\ \sum_{i=1}^{\infty} a_i b_i\rangle \\ &= 0. \end{aligned}$$

Therefore $\mathcal{T} = 0$ on $\overline{(A\widehat{\otimes}A)I(\Delta)}$. However, $J(\Delta)$ is the closure of

$$\{u \in I(\Delta) \mid \operatorname{supp} u \text{ is compact}\}$$

which is a subset of $(A\widehat{\otimes}A) \cdot I(\Delta)$ since $A\widehat{\otimes}A$ is Tauberian. Hence $\mathcal{T} = 0$ on $J(\Delta)$. This completes the proof. $\square$

2.4. Hereditary properties of hyper-Tauberian algebras

In this section, we give a characterization of hyper-Tauberian algebras in terms of their ideals, tensor products and algebra homomorphisms.

Let I be a closed ideal in A, and let $E = h(I)$. Then, by [**10**, Proposition 4.1.11] and [**44**, Theorem 2.7.2], I is a commutative semisimple regular Banach algebra with the carrier space $\Phi_A \setminus E$. Moreover, $I_0(E)$ is equal to I_c, the set of all elements in I which have compact support in Φ_I. To see this, first we note that $I_0(E) \subseteq I_c$. On the other hand, let $a \in I$ such that $\operatorname{supp}_I a$ is compact. Since $\operatorname{supp}_I a \subseteq \Phi_A \setminus E \subseteq \Phi_A$ and compactness is a topologically invariant property, $\operatorname{supp}_I a$ is also compact as a subset of Φ_A, and so it is closed. Therefore, since $\{t \in \Phi_A \mid a(t) \neq 0\}$ is a subset of $\operatorname{supp}_I a$, its closure, which is $\operatorname{supp} a$, is also a subset of $\operatorname{supp}_I a$. Thus $\operatorname{supp}_A a = \operatorname{supp}_I a$, and so it is compact and disjoint from E. Hence $a \in I_0(E)$.

THEOREM 2.8. *Let A be a hyper-Tauberian algebra. Then:*
(i) a closed ideal I in A is hyper-Tauberian if and only if $I = \overline{I_0(E)}$ for some closed subset E of Φ_A;
(ii) a closed subset E of Φ_A is a set of synthesis (local synthesis) for A if and only if $I(E)$ $(J(E))$ is hyper-Tauberian.

PROOF. (i) Let I be hyper-Tauberian. Then, from Theorem 2.6(i), I is a Tauberian algebra. Hence I_c is dense in I. However $I_c = I_0(E)$. Therefore $I = \overline{I_0(E)}$. Conversely, let $I = \overline{I_0(E)}$ for some closed subset E of Φ_A, and let $T\colon I \to I^*$ be a bounded local operator with respect to I-module actions. We first show that T is local with respect to A-module actions. Let $a \in I$, $t \notin \operatorname{supp}_A a$, and let V be a compact neighborhood of t in Φ_A such that $V \cap \operatorname{supp}_A a = \emptyset$. Let $c \in A$ with $\operatorname{supp}_A c \subseteq V$, and let $v \in I_0(E)$. By the regularity of A, there is $u \in A_c$ such that $u = 1$ on $\operatorname{supp}_A v$ and $u = 0$ on an open set containing E. Thus

$$u \in I_0(E) \quad \text{and} \quad vu = v. \tag{1}$$

Hence $uc \in I_0(E) = I_c$. Moreover, since T is local with respect to I-module actions,

$$\begin{aligned} \operatorname{supp}_I (uc) \cap \operatorname{supp}_I T(a) &\subseteq \operatorname{supp}_A c \cap \operatorname{supp}_I a \\ &\subseteq V \cap \operatorname{supp}_A a \\ &= \emptyset. \end{aligned}$$

Therefore $ucT(a) = 0$, and so, by (1)

$$\langle cT(a)\ ,\ v\rangle = \langle cT(a)\ ,\ vu\rangle = \langle ucT(a)\ ,\ v\rangle = 0.$$

Hence $cT(a)$ vanishes on $\overline{I_0(E)} = I$. Thus $cT(a) = 0$. Therefore $t \notin \operatorname{supp}_A T(a)$. Hence T is local with respect to A-module actions, and so, by hypothesis and Proposition 2.4, it is a multiplier. Thus $T(ab) = aT(b)$ for all $a \in A$ and $b \in I$. In particular, T is a multiplier. Therefore I is hyper-Tauberian. The statement in (ii) follows immediately from (i) and the definition of set of synthesis and local synthesis. □

COROLLARY 2.9. *Let A be a hyper-Tauberian algebra. Then every finite subset of Φ_A is a set of synthesis for A.*

PROOF. Let E be a finite subset of A, and let n be the cardinality of E. We prove the statement by induction on n. For $n = 1$, the result follows from Theorem 2.6(ii). Now assume that the result is true for $n = k$. Let $t \in E$ and put $F = E \setminus \{t\}$. By induction, F is a set of synthesis, and so, by Theorem 2.8(ii), $I(F)$ is hyper-Tauberian. Moreover,

$$I(E) = \{a \in I(F) \mid a(t) = 0\}, \tag{1}$$

where the right hand side in (1) is exactly the largest ideal in $I(F)$ whose hull is the singleton $\{t\}$. However, by Theorem 2.6(ii), $\{t\}$ is a set of synthesis for $I(F)$. Therefore, from Theorem 2.8(ii), $I(E)$ is hyper-Tauberian. Hence E is a set of synthesis for A. This completes the proof. □

Let I be a closed ideal in A, and let $E = h(I)$. Then A/I is semisimple if and only if $I = I(E)$ [**10**, p. 412]. In this case, A/I is a commutative semisimple regular Banach algebra with the carrier space E (see [**10**, Proposition 4.1.11] and [**44**, Theorem 2.7.2]).

THEOREM 2.10. *Let E be a closed subset of Φ_A. If $I(E)$ and $A/I(E)$ are hyper-Tauberian, then A is hyper-Tauberian.*

PROOF. Put $I = I(E)$. We first note that A is Tauberian since I and A/I are Tauberian. Let $T\colon A \to A^*$ be a bounded local operator, and let $\iota\colon I \to A$ be the inclusion map. Then $\iota^* \circ T \circ \iota\colon I \to I^*$ is local with respect to I-module actions, and so it is a multiplier. Thus

$$\iota^*(T(ab)) = a\iota^*(T(b)) \quad (a, b \in I). \tag{1}$$

Let $a \in A$ and $b \in I_0(E)$. Then, since T is local,

$$\operatorname{supp}_A(T(ab) - aT(b)) \subseteq \operatorname{supp}_A b \subseteq \Phi_A \setminus E.$$

On the other hand, if V is a compact neighborhood in $\Phi_A \setminus E$ and $c \in A$ with $\operatorname{supp}_A c \subseteq V$, then $c \in I_0(E)$ and there is $e \in I_0(E)$ such that $ec = c$. Thus, for all $d \in A$,

$$\begin{aligned}
\langle c[T(ab) - aT(b)]\ ,\ d\rangle &= \langle ec[T(ab) - aT(b)]\ ,\ d\rangle \\
&= \langle c[T(ab) - aT(b)]\ ,\ de\rangle \\
&= \langle \iota^*[cT(ab) - caT(b)]\ ,\ de\rangle \\
&= 0,
\end{aligned}$$

where the last equality follows from (1). Thus $\operatorname{supp}_A(T(ab) - aT(b)) = \emptyset$, and so $T(ab) - aT(b) = 0$ since A is Tauberian. Hence, by hypothesis,

$$T(ab) = aT(b) \quad (a \in A, b \in I). \tag{2}$$

Now pick $a \in A$ and define the bounded operator $D\colon A \to A^*$ by

$$D(b) = T(ab) - aT(b) \quad (b \in A).$$

From (2), D vanishes on I. Moreover, for each $b \in A$, $D(b) \in I^\perp$. In order to see this, by hypothesis, it suffices to show that $D(b)$ vanishes on $I_0(E)$. Let $c \in I_0(E)$ and take $e \in I_0(E)$ such that $e = 1$ on a neighborhood containing $\operatorname{supp}_A c$. Then, from (2),

$$\begin{aligned} cD(b) &= c[T(a(b-be))-aT(b-be)]+c[T(abe)-aT(be)] \\ &= cT(a(b-be))-caT(b-be) \\ &= 0. \end{aligned}$$

where the last equality follows since T is local, and $\operatorname{supp}_A c$ is compact and disjoint from $\operatorname{supp}_A(b-be)$. Hence $\langle D(b)\ ,\ c\rangle = \langle cD(b)\ ,\ e\rangle = 0$. Therefore we can define the bounded operator $\widetilde{D}\colon A/I \longrightarrow I^{\perp} \cong (A/I)^*$ by

$$\widetilde{D}(\tilde{b}) = D(b) \quad (b \in A).$$

We show that $\widetilde{D}$ is local with respect to A/I-module actions. Let $b \in A$ and put $K = \operatorname{supp}_{A/I}\tilde{b}$. By Proposition 2.3, we can assume that K is compact. Let $t \in E$ such that $t \notin K$. Since K is closed in Φ_A, there is a compact neighborhood V of t in Φ_A such that $V \cap K = \emptyset$. By the regularity of A, there is $e \in A$ such that $e = 1$ on K and $e = 0$ on V. Thus $be - b = 0$ on K. However, K is the closure of $\{s \in E \mid b(s) \neq 0\}$. Hence $b = 0$ on $E \setminus K$. Therefore $be - b = 0$ on E, and so $\tilde{b} = \widetilde{be}$. Thus $\widetilde{D}(\tilde{b}) = \widetilde{D}(\widetilde{be}) = D(be)$. On the other hand, be vanishes on a neighborhood containing t. So $t \notin \operatorname{supp}_A D(be)$ since D is local with respect to A-module actions. Hence $t \notin \operatorname{supp}_{A/I}\widetilde{D}(\tilde{b})$. Thus $\widetilde{D}$ is local, and so, by hypothesis, it is a multiplier. Therefore $\widetilde{D}(\widetilde{bc}) = \widetilde{D}(\tilde{b})\tilde{c}$ for all $b, c \in A$. Hence

$$T(abc) - aT(bc) = T(ab)c - aT(b)c \quad (a, b, c \in A). \tag{3}$$

Define the bounded operator $\mathcal{D}\colon A \longrightarrow \mathcal{B}_A(A, A^*)$ by

$$\mathcal{D}(a)(b) = T(ab) - aT(b) \quad (a, b \in A).$$

From (3), it is easy to verify that D is well-defined. Moreover, upon setting

$$\langle a \cdot S\ ,\ b\rangle = \langle S \cdot a\ ,\ b\rangle = \langle S\ ,\ ab\rangle,$$

the space $\mathcal{B}_A(A, A^*)$ becomes a symmetric Banach A-module and $\mathcal{D}$ becomes a bounded derivation from A into $\mathcal{B}_A(A, A^*)$. However, from Theorem 2.6(iii), A is weakly amenable. Hence $\mathcal{D} = 0$. Thus T is a multiplier. □

Let $A^\sharp$ be the unitalization of A. Then $A^\sharp$ is a commutative semisimple regular Banach algebra with the carrier space $\Phi_A \cup \{\infty\}$, where $\Phi_A \cup \{\infty\}$ is the one-point compactification of Φ_A [**10**, p. 412].

COROLLARY 2.11. *A is hyper-Tauberian if and only if $A^\sharp$ is hyper-Tauberian.*

PROOF. "$\Longrightarrow$" Follows immediately from Theorem 2.10.
"$\Longleftarrow$" Since $A = I(\{\infty\})$, the result follows from Theorem 2.6(ii) and Theorem 2.8(ii). □

Let $A \widehat{\otimes} B$ be the projective tensor product of A and B. There is a symmetric Banach A-module action on $A\widehat{\otimes}B$ specified by

$$c \cdot (a \otimes b) = (a \otimes b) \cdot c = ca \otimes b \quad (a, c \in A, b \in B).$$

Similarly, we can define a symmetric Banach B-module action on $A\widehat{\otimes}B$ specified by

$$d \diamond (a \otimes b) = (a \otimes b) \diamond d = a \otimes db \quad (a \in A, b, d \in B).$$

Moreover, it is straightforward to check that for $c \in A$, $d \in B$ and $x \in A\widehat{\otimes}B$

$$(c \otimes d)x = c \cdot (d \diamond x) = d \diamond (c \cdot x).$$

THEOREM 2.12. *Let A and B be hyper-Tauberian algebras such that $A\widehat{\otimes}B$ is semisimple. Then $A\widehat{\otimes}B$ is hyper-Tauberian.*

PROOF. Let $T\colon A\widehat{\otimes}B \to (A\widehat{\otimes}B)^*$ be a bounded local operator. First we show that T is local with respect to A-module actions. Let $x \in A\widehat{\otimes}B$ and $t \in \Phi_A$ such that $t \notin \operatorname{supp}_A x$. There is an open set V in Φ_A such that $t \in V$, $\overline{V}$ is compact and $\overline{V} \cap \operatorname{supp}_A x = \emptyset$. Let U be an open subset of Φ_B such that $\overline{U}$ is compact. We claim that

$$(V \times U) \cap \operatorname{supp}_{A\widehat{\otimes}B} x = \emptyset. \tag{1}$$

To this end, it suffices to show that for each $y = \sum_{i=1}^{\infty} a_i \otimes b_i \in A\widehat{\otimes}B$ with $\operatorname{supp}_{A\widehat{\otimes}B} y \subseteq V \times U$, $yx = 0$. By the regularity of A and B, there are $c \in A_c$ and $d \in B$ such that $c = 1$ on V, $c = 0$ on a neighborhood containing $\operatorname{supp}_A x$ and $d = 1$ on U. Thus $c \otimes d = 1$ on $V \times U$, and so

$$y = (c \otimes d)y = \sum_{i=1}^{\infty} ca_i \otimes db_i. \tag{2}$$

However, for each i, $\operatorname{supp}_A ca_i$ is compact and disjoint from $\operatorname{supp}_A x$. Hence $(ca_i) \cdot x = 0$, and so, from (2),

$$yx = \sum_{i=1}^{\infty} [db_i \diamond (ca_i \cdot x)] = 0.$$

This shows that the equation (1) holds. Therefore, from the locality of T, $V \times U$ is disjoint from $\operatorname{supp}_{A\widehat{\otimes}B} T(x)$, where U can be any open subset of Φ_B with compact closure. Hence, for each $a \in A$ with $\operatorname{supp}_A a \subseteq V$, $a \cdot T(x)$ vanishes on $A \otimes B_c$. Thus $a \cdot T(x) = 0$. This means that $t \notin \operatorname{supp}_A T(x)$. Therefore T is local with respect to A-module action, and so it is a left A-module morphism. Hence $T(a \cdot u) = a \cdot T(u)$ for all $a \in A$ and $u \in A\widehat{\otimes}B$. Similarly, we can show that $T(b \diamond u) = b \diamond T(u)$ for all $b \in B$. Therefore

$$T[(a \otimes b)u] = T[a \cdot (b \diamond u)] = a \cdot T(b \diamond u) = (a \otimes b)T(u).$$

Hence T is a multiplier. □

Let $\varphi \colon A \to B$ be a bounded algebra homomorphism with dense range. Then φ^* induces a continuous map $\sigma \colon \Phi_B \to \Phi_A$.

THEOREM 2.13. *Let A be hyper-Tauberian, and let $\varphi\colon A \to B$ be a bounded algebra homomorphism with dense range. Then B is hyper-Tauberian.*

PROOF. First assume that both A and B are unital. In this case, Φ_B is compact and homeomorphic to $\sigma(\Phi_B)$. In particular, $\sigma(\Phi_B)$ is a closed subset of Φ_A. It is easy to see that B becomes an essential symmetric Banach A-module for the action defined by

$$a \cdot b = b \cdot a = \varphi(a)b \quad (a \in A, b \in B).$$

Moreover, φ is a bounded A-module morphism. Let $T\colon B \to B^*$ be a bounded local operator with respect to B-module actions. We claim that T is local with respect to A-module actions. Let $c \in B$ and $t \notin \operatorname{supp}_A c$. Consider the following two cases:

Case I: $t \notin \sigma(\Phi_B)$. Hence there is a compact neighborhood V of t in Φ_A such that $V \cap \sigma(\Phi_B) = \emptyset$. Let $a \in A$ with $\operatorname{supp}_A a \subseteq V$. Then $\varphi(a) = 0$, and so, for all $b \in B$, $b \cdot a = 0$. Therefore $a \cdot [T(c)] = 0$. Hence $t \notin \operatorname{supp}_A T(c)$.

Case II: $t \in \sigma(\Phi_B)$. So $t \notin \operatorname{supp}_B c$. Thus $t \notin \operatorname{supp}_B T(c)$ since T is local. Therefore there is a compact neighborhood V in Φ_A such that $t \in V \cap \sigma(\Phi_B)$ and, for every $e \in B$ with $\operatorname{supp}_B e \subseteq V \cap \sigma(\Phi_B)$, we have

$$eT(c) = 0. \tag{1}$$

Now let $a \in A$ with $\operatorname{supp}_A a \subseteq V$. Then $\operatorname{supp}_B \varphi(a) \subseteq V \cap \sigma(\Phi_B)$, and so, from (1), $a \cdot T(c) = \varphi(a)T(c) = 0$. Therefore $t \notin \operatorname{supp}_A T(c)$.

Hence T is local with respect to A-module actions. Thus, by hypothesis and Proposition 2.4, T is a bounded A-module morphism. Therefore, for all $a \in A$ and $b \in B$, $T(\varphi(a)b) = T(a \cdot b) = a \cdot T(b) = \varphi(a)T(b)$. The final result follows from the fact that the range of φ is dense in B.

Now consider the general case. Since A is hyper-Tauberian, $A^\sharp$ is hyper-Tauberian from Corollary 2.11. On the other hand, we can extend φ to a bounded algebra homomorphism from $A^\sharp$ into $B^\sharp$ by defining $\varphi(1) = 1$. Moreover, $\varphi(A^\sharp)$ is dense in $B^\sharp$. Thus, by the first part, $B^\sharp$ is hyper-Tauberian. Therefore, from Corollary 2.11, B is hyper-Tauberian.

□

LEMMA 2.14. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of commutative semisimple regular Banach algebras, and let $\mathcal{A}$ be the l^1-direct sum of $\{A_\gamma\}_{\gamma\in\Gamma}$. Then $\mathcal{A}$ is a commutative semisimple regular Banach algebra whose carrier space, $\Phi_\mathcal{A}$, is the disjoint union of all Φ_{A_γ}. Moreover, for each $\gamma \in \Gamma$, Φ_{A_γ} is an open-closed subset of $\Phi_\mathcal{A}$.*

PROOF. It is clear that $\mathcal{A}$ is a commutative semisimple Banach algebra. For each $\gamma \in \Gamma$, let $P_\gamma \colon \mathcal{A} \to A_\gamma$ be the canonical projection of $\mathcal{A}$ onto A_γ, and let

$$\Psi(A_\gamma) = \{P_\gamma^{-1}(I) \mid I \in \Phi_{A_\gamma}\}.$$

We first claim that

$$\Phi_\mathcal{A} = \bigcup_{\gamma\in\Gamma} \Psi(A_\gamma). \tag{1}$$

It is clear that $\Psi(A_\gamma) \subseteq \Phi_\mathcal{A}$ for each $\gamma \in \Gamma$. For the converse, let M be a maximal modular ideal in $\mathcal{A}$. For $\gamma \in \Gamma$, put $M_\gamma = P_\gamma^{-1}(P_\gamma(M))$. It is easy to see that M_γ is an ideal in $\mathcal{A}$ and it contains M. Thus, since M is maximal,

$$M_\gamma = M \quad \text{or} \quad M_\gamma = \mathcal{A} \quad (\gamma \in \Gamma). \tag{2}$$

On the other hand, since $M \neq \mathcal{A}$, there is $\gamma_0 \in \Gamma$ such that

$$A_{\gamma_0} \nsubseteq M. \tag{3}$$

We show that

$$M = M_{\gamma_0}. \tag{4}$$

Let $e \in \mathcal{A}$ be a modular unit for M. If $M \neq M_{\gamma_0}$, then, from (2), $M_{\gamma_0} = \mathcal{A}$. Therefore, $P_{\gamma_0}(e) \in P_{\gamma_0}(M)$, and so, there is $m \in M$ such that $P_{\gamma_0}(m) = P_{\gamma_0}(e)$. Hence, for each $a \in A_{\gamma_0}$,

$$ae = aP_{\gamma_0}(e) = aP_{\gamma_0}(m) = P_{\gamma_0}(am) = am \in M.$$

Thus $A_{\gamma_0} = A_{\gamma_0}(1-e) + A_{\gamma_0}e \subseteq M$ which, from (3), is impossible. Thus (4) holds. This, together with the fact that P_{γ_0} is onto, implies that $P_{\gamma_0}(M)$ is a proper modular ideal in A_{γ_0}. Therefore there is $I \in \Phi_{A_{\gamma_0}}$ such that $P_{\gamma_0}(M) \subseteq I$, and so, $M \subseteq P_{\gamma_0}^{-1}(I)$. Hence, from the maximality of M, $M = P_{\gamma_0}^{-1}(I)$ i.e. $M \in \Psi(A_{\gamma_0})$. This proves (1).

It is straightforward to check that the sets $\Psi(A_\gamma)$ are mutually disjoint. Also, for every $\gamma \in \Gamma$, Φ_{A_γ} and $\Psi(A_\gamma)$ are homeomorphic, and so Φ_{A_γ} can be viewed as an open-closed subset of $\Phi_{\mathcal{A}}$. This, in particular, implies that $\mathcal{A}$ is regular. □

COROLLARY 2.15. *Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of hyper-Tauberian subalgebras of A such that $A = (\oplus_{\gamma\in\Gamma} A_\gamma)^-$. Then A is hyper-Tauberian.*

PROOF. Let $\mathcal{A}$ be the l^1-direct sum of $\{A_\gamma\}_{\gamma\in\Gamma}$, and let $\gamma \in \Gamma$. By the preceding lemma, Φ_{A_γ} is an open-closed subset of $\Phi_{\mathcal{A}}$. Thus $\Phi_{\mathcal{A}} \setminus \Phi_{A_\gamma}$ is a closed subset of $\Phi_{\mathcal{A}}$. Moreover, $A_\gamma = I(\Phi_{\mathcal{A}} \setminus \Phi_{A_\gamma})$. Therefore, if $T\colon \mathcal{A} \to \mathcal{A}^*$ is a bounded local operator with respect to $\mathcal{A}$-module action, then, by a similar argument to the one made in the proof of Theorem 2.10 and the fact that A_γ is hyper-Tauberian, we have

$$T(ab) = aT(b) \quad (a \in \mathcal{A}, b \in A_\gamma, \gamma \in \Gamma).$$

Thus, from the assumption on $\mathcal{A}$, T is a multiplier. Hence $\mathcal{A}$ is hyper-Tauberian. The final result follows from Theorem 2.13 and the fact that map $\varphi\colon \mathcal{A} \to A$ defined by $\varphi(\{a_\gamma\}) = \Sigma_\gamma a_\gamma$ is a bounded algebra homomorphism with dense range. □

2.5. Some examples of hyper-Tauberian algebras

It follows from B. E. Johnson's work that $C_0(\mathbb{R})$ is hyper-Tauberian [**28**, Proposition 3.1]. We extend this result by showing that $C_0(\Omega)$ is hyper-Tauberian for every locally compact topological space Ω. To this end, we first prove it for the special case when Ω is a compact subset of $\mathbb{R}$.

LEMMA 2.16. *Let K be a compact subset of $\mathbb{R}$. Then $C(K)$ is a hyper-Tauberian algebra.*

PROOF. Let R be the restriction map from $C_0(\mathbb{R})$ onto $C(K)$. Then R is a bounded algebra homomorphism. However, by [**28**, Proposition 2.1], the diagonal is a set of synthesis for $C_0(\mathbb{R})\widehat{\otimes}C_0(\mathbb{R})$, and so, by Theorem 2.7 (see also [**28**, Proposition 3.1]), $C_0(\mathbb{R})$ is hyper-Tauberian. Thus, from Theorem 2.13, $C(K)$ is hyper-Tauberian. $\square$

THEOREM 2.17. *Let Ω be a locally compact topological space. Then $C_0(\Omega)$ is a hyper-Tauberian algebra.*

PROOF. First consider the case when Ω is compact. Let $T\colon C(\Omega) \to C(\Omega)^*$ be a bounded local operator. First we show that T satisfies the following condition:

$$ab = 0 \text{ implies } aT(b) = 0. \tag{$\star$}$$

Let $a, b \in C(\Omega)$ with $ab = 0$. So $a \in I(E)$ where $E = \operatorname{supp} b$. Since E is a closed subset of Ω, E is a set of synthesis [**10**, Theorem 4.2.1]. Thus there is a sequence $\{a_n\}$ in $I_0(E)$ such that $a_n \to a$ as $n \to \infty$. On the other hand, since T is local and $\operatorname{supp} a_n$ is disjoint from E,

$$\begin{aligned} \operatorname{supp} a_n \cap \operatorname{supp} T(b) &\subseteq \operatorname{supp} a_n \cap \operatorname{supp} b \\ &= \operatorname{supp} a_n \cap E \\ &= \emptyset. \end{aligned}$$

Therefore, since $\operatorname{supp} a_n$ is compact, $a_n T(b) = 0$. Thus, by letting $n \to \infty$, we have $aT(b) = 0$. This proves $(\star)$. Now let $a \in C(\Omega)$ be a self-adjoint element, and let $A(a)$ be the C^*-subalgebra of $C(\Omega)$ generated by $\{a, 1\}$. It is well-known that there is a compact subset K of $\mathbb{R}$ such that $A(a)$ is isometrically isomorphic to $C(K)$. In particular, $C(\Omega)$ is an essential and symmetric Banach $C(K)$-module. Let $d \in C(\Omega)$ and $c \in C(K)$ with $cd = 0$. Then, since $c \in A$ and T satisfies condition $(\star)$, $cT(d) = 0$. Hence $\operatorname{Ann}_{C(K)} d \subseteq \operatorname{Ann}_{C(K)} T(d)$, and so $\operatorname{supp}_{C(K)} T(d) \subseteq \operatorname{supp}_{C(K)} d$. Therefore T is local with respect to $C(K)$-module actions. Thus, from Lemma 2.16 and Proposition 2.4, T is a $C(K)$-module morphism. Hence, for each $b \in C(\Omega)$, $T(ab) = aT(b)$. The final result follows since $C(\Omega)$ is the linear span of its self-adjoint elements.

We now consider the general case. Let Ω be a locally compact space, and let $\Omega \cup \{\infty\}$ be its one point compactification. Then, from the first case, $C(\Omega \cup \{\infty\})$ is hyper-Tauberian. On the other hand,

$$C_0(\Omega) = \{a \in C(\Omega \cup \{\infty\}) \mid a(\infty) = 0\} = I(\{\infty\})$$

and $\{\infty\}$ is a set of synthesis for $C(\Omega \cup \{\infty\})$. Thus, from Theorem 2.8(ii), $C_0(\Omega)$ is hyper-Tauberian. $\square$

COROLLARY 2.18. *Let Ω be a locally compact space. Then the diagonal is a set of synthesis for $C_0(\Omega)\widehat{\otimes}C_0(\Omega)$.*

PROOF. Since $C_0(\Omega)$ has the Grothendieck approximation property, $C_0(\Omega)\widehat{\otimes}C_0(\Omega)$ is semisimple [**52**]. Hence the result follows from Theorem 2.7 and the preceding theorem. □

Let G be a locally compact group. We recall that the *principal component* of G is the component (the largest connected set) containing the identity; we denote it by G_e. It is easy to see that G_e is a closed normal subgroup of G. B. Forrest and V. Runde have shown in [**16**] that if G is a locally compact group such that G_e is abelian, then the Fourier algebra $A(G)$ is weakly amenable. In Theorem 2.22, we prove a stronger statement that for this class of groups, $A(G)$ is hyper-Tauberian. To do this, we will follow similar steps to those taken in [**16**]; the tools developed in Sections 2.3 and 2.4 will be used to modify the approach in [**16**]. Finally, we prove a similar result for $A_p(G)$, $p \in (1, \infty)$.

PROPOSITION 2.19. *Let G be a locally compact abelian group. Then $A(G)$ is hyper-Tauberian.*

PROOF. It is well-known that $A(G)\widehat{\otimes}A(G) \cong A(G \times G)$ (e.g. [**39**]). Thus, since the diagonal Δ is a closed subgroup of $G \times G$, by [**53**, Theorem 3], Δ is a set of synthesis for $A(G)\widehat{\otimes}A(G)$. So we have the result from Theorem 2.7. □

LEMMA 2.20. *Let G be a locally compact group, and let H be a closed subgroup of G. Then:*

(i) If $A(G)$ is hyper-Tauberian, then $A(H)$ is hyper-Tauberian.

(ii) If H is open and $A(H)$ is hyper-Tauberian, then $A(G)$ is hyper-Tauberian.

PROOF. (i) Let $\varphi\colon A(G) \to A(H)$ be the restriction map of elements of $A(G)$ to H, i.e. $\varphi(u) = u|_H$ for $u \in A(G)$. By [**14**, Lemma 3.8], φ is a bounded algebra epimorphism. Hence the result follows from Theorem 2.13.

(ii) Let $T\colon A(G) \to A(G)^*$ be a bounded local operator, and let $a, b, c \in A(G)$ have compact support. Since H is open, there are $\{x_i\}_{i=1}^n \subseteq G$ such that

$$\operatorname{supp} a \cup \operatorname{supp} b \cup \operatorname{supp} c \subseteq \cup_{i=1}^n x_i H, \tag{1}$$

where the union on the right side is disjoint. For $i = 1, \ldots, n$, let χ_i be the characteristic function of $x_i H$. Since H is open, each χ_i belongs to $B(G)$ [**18**, Proposition 2.31]. Hence $A_i = \chi_i A(G)$ is a closed subalgebra of $A(G)$. Moreover, from [**18**, Corollary 2.19 and Proposition 3.21], A_i is isometrically isomorphic to $A(H)$. Since $A(H)$ is hyper-Tauberian, A_i is hyper-Tauberian, and so, by Corollary 2.15, $A = A_1 \oplus^1 \ldots \oplus^1 A_n$ is hyper-Tauberian. Now let $\iota\colon A \to A(G)$ be the inclusion map. Then $\iota^* \circ T \circ \iota\colon A \to A^*$ is local with respect to A-module actions, and so, it is a multiplier. Therefore

$$\langle T(uv)\ ,\ w\rangle = \langle uT(v)\ ,\ w\rangle \quad (u, v, w \in A). \tag{2}$$

On the other hand, if u is an element in $A(G)$ with $\operatorname{supp} u \subseteq \cup_{i=1}^n x_i H$, then $u = \chi_1 u + \ldots + \chi_n u \in A$. Hence, from (1) and (2),

$$\langle T(ab)\ ,\ c\rangle = \langle aT(b)\ ,\ c\rangle.$$

The final result follows since $A(G)$ is a Tauberian algebra. □

LEMMA 2.21. *Let G be a locally compact group such that G_e is abelian, and let K be a compact normal subgroup of G such that G/K is a Lie group. Then $A(G/K)$ is hyper-Tauberian.*

PROOF. Let $\pi\colon G \to G/K$ be the quotient map. By [**27**, Theorem 7.12], the principle component of G/K is $\pi(G_e)^-$. Thus $(G/K)_e$ is abelian, and so, by Proposition 2.19, $A((G/K)_e)$ is hyper-Tauberian. However, $(G/K)_e$ is open, since

G/K is a Lie group [**41**, 12.2.4 Definition]. Hence $A(G/K)$ is hyper-Tauberian from Lemma 2.20(ii). □

We recall from [**41**, 12.2.14 Definition] that a locally compact group G is called a *pro-Lie group* if every neighborhood of the identity contains a compact normal subgroup K such that G/K is a Lie group.

THEOREM 2.22. *Let G be a locally compact group such that G_e is abelian. Then $A(G)$ is hyper-Tauberian.*

PROOF. First consider the case where G is a pro-Lie group. Let $T\colon A(G) \to A(G)^*$ be a bounded local operator, and let $\epsilon > 0$. As it is shown in [**16**, Theorem 3.3], there is a compact, normal subgroup K of G such that G/K is a Lie group, and there is a projection $P\colon A(G) \to A(G:K)$ such that

$$||u - Pu|| \leq \epsilon \quad (u \in A(G)), \tag{1}$$

where $A(G:K)$ denotes the (closed) subalgebra of $A(G)$ consisting of those functions that are constant on cosets of K. From [**18**, Proposition 3.25], $A(G:K)$ is isometrically isomorphic to $A(G/K)$, and so, by Lemma 2.21, $A(G:K)$ is hyper-Tauberian. Now let $\iota\colon A(G:K) \to A(G)$ be the inclusion map. Then $\iota^* \circ T \circ \iota\colon A(G:K) \to A(G:K)^*$ is local with respect to $A(G:K)$-module actions, and so it is a multiplier. Therefore

$$\langle T(PaPb)\ ,\ Pc\rangle = \langle PaT(Pb)\ ,\ Pc\rangle \quad (a,b,c \in A(G)). \tag{2}$$

Now let $a,b,c \in A(G)_1$ where $A(G)_1 = \{u \in A(G) : ||u|| \leq 1\}$. Then, from (1),

$$\begin{aligned} |\langle T(ab) - aT(b)\ ,\ c - Pc\rangle| &\leq ||T(ab) - aT(b)||\ ||c - Pc|| \\ &\leq 2||T||\epsilon. \end{aligned}$$

Consequently

$$\begin{aligned}|\langle T(ab) - aT(b)\ ,\ c\rangle| &= |\langle T(ab) - aT(b)\ ,\ (c - Pc) + Pc\rangle| \\ &\leq 2||T||\epsilon + |\langle T(ab) - aT(b)\ ,\ Pc\rangle|.\end{aligned}$$

Similarly, we can show that

$$|\langle T(ab) - aT(b)\ ,\ Pc\rangle| \leq 2||T||\epsilon + |\langle T[(Pa)b] - PaT(b)\ ,\ Pc\rangle|,$$

and so,

$$|\langle T(ab) - aT(b)\ ,\ c\rangle| \leq 4||T||\epsilon + |\langle T[(Pa)b] - PaT(b)\ ,\ Pc\rangle|.$$

Finally, an argument similar to the above yields

$$\begin{aligned}|\langle T(ab) - aT(b)\ ,\ c\rangle| &\leq 6||T||\epsilon + |\langle T(PaPb) - PaT(Pb)\ ,\ Pc\rangle| \\ &= 6||T||\epsilon,\end{aligned}$$

where the last equality follows from (2). Hence

$$\langle T(ab) - aT(b)\ ,\ c\rangle = 0,$$

since ϵ was arbitrary. The final result follows since $A(G)_1$ spans $A(G)$.

For the general case, we note that by [**41**, 12.2.15 Theorem], G has an open subgroup H such that H is a pro-Lie group. In particular, $G_e \subseteq H$ since H is an open-closed subset of G and $G_e \cap H \neq \emptyset$. Hence H_e is abelian, and so, by the preceding case, $A(H)$ is hyper-Tauberian. Therefore $A(G)$ is hyper-Tauberian from Lemma 2.20(ii). □

THEOREM 2.23. *Let G be a locally compact group such that G_e is abelian, and let $p \in (1, \infty)$. Then $A_p(G)$ is hyper-Tauberian. In particular, $A_p(G)$ is weakly amenable.*

PROOF. Let $T\colon A_p(G) \to A_p(G)^*$ be a bounded local operator. Since $A_p(G)$ is a Banach $A(G)$-module (Section 1.3), we can consider the locality for T with respect to $A(G)$-module actions. Let $a \in A_p(G)$ and $t \notin \operatorname{supp}_{A(G)} a$. It is easy to see that

$$\operatorname{supp}_{A(G)} a = \operatorname{supp}_{A_p(G)} a = \operatorname{cl}\{s \in G \mid a(s) \neq 0\}.$$

Therefore there is a compact neighborhood V of t in G such that $V \cap \operatorname{supp}_{A_p(G)} a = \emptyset$. Let $c \in A(G)$ with $\operatorname{supp}_{A(G)} c \subseteq V$. By the regularity of $A_p(G)$, there is an element $e \in A_p(G)$ with compact support such that $e = 1$ on V and $\operatorname{supp}_{A_p(G)} e \cap \operatorname{supp}_{A_p(G)} a = \emptyset$. Thus, since T is local,

$$\operatorname{supp}_{A_p(G)} e \cap \operatorname{supp}_{A_p(G)} T(a) \subseteq \operatorname{supp}_{A_p(G)} e \cap \operatorname{supp}_{A_p(G)} a = \emptyset.$$

Hence $eT(a) = 0$, and so,

$$cT(a) = ceT(a) = 0.$$

Therefore $t \notin \operatorname{supp}_{A(G)} T(a)$. Thus T is local with respect to $A(G)$-module actions. On the other hand, for all $q \in (1, \infty)$, the linear span of $C_{00}(G) * C_{00}(G)$ is a dense subset of $A_q(G)$ in $\|\cdot\|_q$-norm. Hence $A_p(G)$ is essential as a Banach $A(G)$-module. Consequently, by Theorem 2.22 and Proposition 2.4, T is an $A(G)$-module morphism. Hence

$$T(ab) = aT(b) \quad (a \in A(G), b \in A_p(G)).$$

Therefore T is a multiplier. $\square$

REMARK 2.24. (i) Let G be the group of rotations of $\mathbb{R}^3$. Then, by [**30**, Corollary 7.3], $A(G)$ is not weakly amenable. Therefore $A(G)$ is not hyper-Tauberian.
(ii) In Theorem 2.6 we showed that a hyper-Tauberian algebra is a weakly amenable Tauberian algebra. However, the converse is not true. To see this, let $\mathbb{T}$ be the unit circle. It is shown in [**10**, Corollary 5.6.45] and its proof that there is a closed

subset E of $\mathbb{T}$ such that E is a set of non-synthesis for $A(\mathbb{T})$ but $I(E) = \overline{I(E)^2}$, so that $I(E)$ is weakly amenable [**10**, Theorem 2.8.69(ii)]. Hence $I(E)^\sharp$ is a weakly amenable Tauberian algebra [**10**, Corollary 2.8.70]. However, by Proposition 2.19, $A(\mathbb{T})$ is a hyper-Tauberian algebra. Therefore, by Theorem 2.8(ii), $I(E)$ is not hyper-Tauberian. Thus, by Corollary 2.11, $I(E)^\sharp$ is not hyper-Tauberian.

It is well-known that one sufficient condition for a commutative Banach algebra $\mathfrak{A}$ to be weakly amenable is that $\mathfrak{A}$ is closed linear span of idempotents. The following theorem shows that the same assumption forces a Tauberian algebra to be hyper-Tauberian.

THEOREM 2.25. *Let A be a Tauberian algebra. If A is closed linear span of idempotents, then A is hyper-Tauberian.*

PROOF. Let $T\colon A \longrightarrow A^*$ be a bounded local operator, and let $p \in A$ be an idempotent. Since $p^2 = p$, $\operatorname{supp} p$ is an open-closed subset of Φ_A and $p = 1$ on $\operatorname{supp} p$. Let $a \in A$. Then $\operatorname{supp} p \cap \operatorname{supp}(a - pa) = \emptyset$. Hence

$$\begin{aligned}\operatorname{supp}[pT(a-pa)] &\subseteq \operatorname{supp} p \cap \operatorname{supp} T(a-pa)\\ &\subseteq \operatorname{supp} p \cap \operatorname{supp}(a-pa)\\ &= \emptyset.\end{aligned}$$

Therefore $pT(a - pa) = 0$, since A is Tauberian. Thus

$$pT(a) = pT(pa). \tag{1}$$

Now let $b \in A_c$ and take $e \in A$ such that $eb = b$. Since T is local and supp $(bp - b)$ is compact and disjoint from supp pa, $(bp - b)T(pa) = 0$. Thus

$$\begin{aligned}\langle pT(pa) - T(pa)\ ,\ b\rangle &= \langle pT(pa) - T(pa)\ ,\ eb\rangle \\ &= \langle (bp - b)T(pa)\ ,\ e\rangle \\ &= 0.\end{aligned}$$

Therefore $pT(pa) - T(pa) = 0$, since A_c is dense in A. Together with (1), this shows that $T(pa) = pT(a)$. The final result follows since A is the closed linear span of idempotents. $\square$

EXAMPLE 2.26. Let Γ be a non-empty set, and let $p \in [1, \infty)$. Then $l^p(\Gamma)$ and $c_0(\Gamma)$, with pointwise addition and multiplication, are Tauberian algebras that are closed linear span of idempotents. Therefore they are hyper-Tauberian. This result also follows from Corollary 2.15.

2.6. Local multipliers and local derivations from hyper-Tauberian algebras

B. E. Johnson in **[28]** showed that every local derivation from a C^*-algebra $\mathfrak{A}$ into any Banach $\mathfrak{A}$-bimodule is a derivation. He showed that it is enough to establish the result for the commutative regular semisimple Banach algebra $C_0(\mathbb{R})$. For $C_0(\mathbb{R})$, he first studied local operators from this algebra and then deduced results about local derivations. However, $C_0(\mathbb{R})$ is very well-behaved; it is a commutative C^*-algebra so that it is amenable and all the derivations from it into any Banach $C_0(\mathbb{R})$-bimodule are automatically continuous. In this section, we exploit Johnson's approach and investigate local derivations from certain hyper-Tauberian algebras which do not necessarily have the above properties. To compensate for this, we look more into the "local structure" of these algebras.

DEFINITION 2.27. Let X be a Banach left A-bimodule. An operator $T\colon A \to X$ is a *local right multiplier* if for each $a \in A$, there is a right multiplier $T_a\colon A \to X$ such that $T(a) = T_a(a)$. Similarly, we can define *local left multipliers* for Banach right A-modules.

THEOREM 2.28. *Let A be a hyper-Tauberian algebra, and let X be a Banach (right or left) A-module. Then every bounded local multiplier T from A into X is a multiplier.*

PROOF. We prove this for the local right multipliers. The other case can be proved similarly. Let X be a Banach left A-module, and let $T\colon A \to X$ be a bounded local right multiplier. Then, for each $a \in A$, there is a right multiplier $T_a\colon A \to X$ such that $T(a) = T_a(a)$. Hence $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(T_a(a)) = \mathrm{Ann}(T(a))$, and so $\operatorname{supp} T(a) \subseteq \operatorname{supp} a$. Thus T is a local operator. Therefore, by a similar argument to what we have made in the proof of Proposition 2.4(ii),

$$cT(ab) - caT(b) = 0 \qquad (a, b, c \in A).$$

Take $a \in A_c$ and $c \in A$ such that $c = 1$ on $\operatorname{supp} a$. Hence $ca = a$. Since T is a local multiplier, there is a right multiplier M from A into X such that $T(ab) = M(ab)$. Therefore

$$\begin{aligned} T(ab) - aT(b) &= M(ab) - aT(b) \\ &= M(cab) - aT(b) \\ &= cM(ab) - aT(b) \\ &= cT(ab) - caT(b) \\ &= 0. \end{aligned}$$

The final result follows by the density of A_c in A. $\square$

For any two subsets E_1 and E_2 of Φ_A, let $V_0(E_1, E_2)$ be the closed linear span in $A\widehat{\otimes}A$ of the elements $a_1 \otimes a_2$ where $a_i \in J(\Phi_A \setminus E_i)$, $i = 1, 2$. It is easy to check that $V_0(E_1, E_2)$ is a Banach A-submodule of $A\widehat{\otimes}A$. The following lemma is a modification of [**28**, Lemma 5.2]. We include the proof for the sake of completeness.

LEMMA 2.29. *Let E_1 and E_2 be subsets of Φ_A, and let $\theta \in (A\widehat{\otimes}A)^*$.*
(i) If $a \in I(E_1)$, then $\theta a \in V_0(E_1, E_2)^\perp$.
(ii) If $a \in I(E_2)$, then $a\theta \in V_0(E_1, E_2)^\perp$.
(iii) If $a \in A$ and $a = 1$ on E_1, then $\theta - \theta a \in V_0(E_1, E_2)^\perp$.
(iv) If $a \in A$ and $a = 1$ on E_2, then $\theta - a\theta \in V_0(E_1, E_2)^\perp$.

PROOF. Let $c_i \in J(\Phi_A \setminus E_i)$, $i = 1, 2$. For (i) we have

$$\langle \theta a, c_1 \otimes c_2 \rangle = \langle \theta, ac_1 \otimes c_2 \rangle = 0.$$

For (iii) we have

$$\langle \theta - \theta a, c_1 \otimes c_2 \rangle = \langle \theta, (c_1 - ac_1) \otimes c_2 \rangle = 0,$$

since $c_1 = ac_1$. The other two statements can be proved similarly. $\square$

Let X be a Banach A-bimodule. We recall that an operator D from A into X is a local derivation if for each $a \in A$, there is a derivation D_a from A into X such that $D(a) = D_a(a)$. We are now ready to present the main result of this section.

THEOREM 2.30. *Let A be a hyper-Tauberian algebra such that $A\widehat{\otimes}A$ is semi-simple, and let X be an essential Banach A-bimodule. Then every bounded local derivation D from A into X^* is a derivation. Moreover, if A has a bounded approximate identity, then the above statement of theorem is true for all Banach A-bimodules.*

PROOF. Consider first the case $X = A\widehat{\otimes}A$. Let $D\colon A \to (A\widehat{\otimes}A)^*$ be a bounded local derivation, let E_1 and E_2 be disjoint compact subsets of Φ_A, and let

$$q\colon (A\widehat{\otimes}A)^* \to (A\widehat{\otimes}A)^*/V_0(E_1,E_2)^\perp$$

be the natural quotient map. Put $\tilde{D} = q \circ D$. Since q is a bounded A-module morphism, $\tilde{D}$ is a bounded local derivation. Now let $b_1 \in I(E_1)$, and define

$$T_1\colon A \to (A\widehat{\otimes}A)^*/V_0(E_1,E_2)^\perp \;,\;\; T_1(a) = \tilde{D}(ab_1) \qquad (a \in A).$$

Since $\tilde{D}$ is a local derivation, for each $a \in A$, there is a derivation $S\colon A \to (A\widehat{\otimes}A)^*/V_0(E_1,E_2)^\perp$ such that $\tilde{D}(ab_1) = S(ab_1)$. So, by Lemma 2.29(i),

$$T_1(a) = S(ab_1) = aS(b_1) + S(a)b_1 = aS(b_1).$$

Thus T_1 is a bounded local right multiplier, and so, by Theorem 2.28, it is a right multiplier. Hence, for all $a, c \in A$,

$$\tilde{D}(acb_1) = a\tilde{D}(cb_1). \tag{1}$$

Similarly, we can show that for all $a, c \in A$ and $b_2 \in I(E_2)$,

$$\tilde{D}(acb_2) = \tilde{D}(cb_2)a. \tag{2}$$

Let $a, c \in A_c$, and let U be a compact neighborhood in Φ_A such that $E_2 \subseteq U$ and $U \cap E_1 = \emptyset$. By the regularity of A, there are b, e and b_1 in A such that $b = 1$ on $\operatorname{supp} a \cup \operatorname{supp} c \cup U \cup E_1$, $e = 1$ on E_2 and $e = 0$ outside of U, and finally $b_1 = 0$ on E_1 and $b_1 = 1$ on E_2. Put $b_2 = b - b_1$. Then

$$b_i \in I(E_i) \;\;,\;\; ab = a \;\;,\;\; bc = c \;\;,\;\; \textit{and}\;\; eb = e. \tag{3}$$

Since $\tilde{D}$ is a local derivation, there is a derivation δ from A into $(A\widehat{\otimes}A)^*/V_0(E_1,E_2)^\perp$ such that $\tilde{D}(b^2) = \delta(b^2)$. So, by Lemma 2.29 (iii) and (iv),

$$\tilde{D}(b^2) = \delta(b^2) = e\delta(b^2) = \delta(eb^2) - \delta(e)b^2 = \delta(e) - \delta(e) = 0.$$

On the other hand, from (1), (2) and (3),

$$\begin{aligned}\tilde{D}(a) &= \tilde{D}(ab^2)\\ &= \tilde{D}(ab(b_1+b_2))\\ &= \tilde{D}(abb_1)+\tilde{D}(abb_2)\\ &= a\tilde{D}(bb_1)+\tilde{D}(bb_2)a.\end{aligned}$$

But $\tilde{D}(bb_1)+\tilde{D}(bb_2)=\tilde{D}(b^2)=0$. So, if we put $\theta=\tilde{D}(bb_1)$, then

$$\tilde{D}(a)=a\theta-\theta a. \tag{4}$$

Similarly, we can show that (4) holds with the same θ if we replace a by either c or ac. Therefore $\tilde{D}(ac)=a\tilde{D}(c)+\tilde{D}(a)c$. Since a and c were arbitrary elements in A_c, by the density of A_c, we can conclude that $\tilde{D}$ is a derivation into $(A\widehat{\otimes}A)^*/V_0(E_1,E_2)^{\perp}$.

Consider the connecting map $\delta^1 D$ given by $\delta^1 D(a,b)=aD(b)-D(ab)+D(a)b$. It is a 2-cocycle from A with the values in $(A\widehat{\otimes}A)^*$. However, since $\tilde{D}=q\circ D$ is a derivation, we have

$$q\circ\delta^1 D(a,b)=0 \quad (a,b\in A).$$

Thus $\delta^1 D$ maps into $V_0(E_1,E_2)^{\perp}$ and since this holds for all the choices of E_1 and E_2, $\delta^1 D$ maps into

$$(\overline{\operatorname{span}}\{V_0(E_1,E_2)\mid E_1 \text{ and } E_2 \text{ are disjoint compact sets}\})^{\perp}$$

which is $J(\Delta)^{\perp}$, by the assumption that the diagonal Δ is a set of local synthesis for $A\widehat{\otimes}A$. On the other hand, $J(\Delta)^{\perp}\cong(A\widehat{\otimes}A/J(\Delta))^*$, and so δD maps into $(A\widehat{\otimes}A/J(\Delta))^*$ which is the dual of an essential Banach A-module. Moreover, $A\widehat{\otimes}A/J(\Delta)$ is symmetric. To see this, let $u\in A\widehat{\otimes}A$ and $a\in A$. Take a sequence $\{u_n\}$ in $A\widehat{\otimes}A$ such that each u_n has a compact support and $u_n\to u$. It is clear that $au_n - u_n a \in J(\Delta)$ and $au - ua = \lim_{n\to\infty} au_n - u_n a$. Thus $au - ua \in J(\Delta)$. Now fix $b \in A$ and define the bounded operator $\mathcal{D}\colon A \to (A\widehat{\otimes}A/J(\Delta))^*$ by

$$\mathcal{D}(a) = \delta D(a,b) \quad (a \in A).$$

We claim that $\mathcal{D}$ is a local operator. Let $a \in A$ and $t \notin \operatorname{supp} a$. There is a compact neighborhood V of t such that $\operatorname{supp} a \cap V = \emptyset$. Take $c \in A$ with $\operatorname{supp} c \subseteq V$. By the regularity of A, there is $e \in A$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp} a$. Then, since $(A\widehat{\otimes}A/J(\Delta))^*$ is symmetric and $ca = 0$,

$$c\mathcal{D}(a) = ce\mathcal{D}(a) = c\delta D(a,b)e = -cD(ab)e + cD(a)eb. \tag{5}$$

On the other hand, let $h \in A$ be any element such that $ch = eh = 0$, and let $\delta\colon A \to (A\widehat{\otimes}A)^*$ be a derivation such that $D(h) = \delta(h)$. Then

$$cD(h)e = c\delta(h)e = \delta(ch)e - \delta(c)he = 0.$$

Thus, from (5), $c\mathcal{D}(a) = 0$, and so, by Lemma 2.1, $t \notin \operatorname{supp} \mathcal{D}(a)$ i.e. $\mathcal{D}$ is a bounded local operator. Therefore, by Proposition 2.4, it is a multiplier. Hence, for all $a, b, c \in A$, $\delta D(ac, b) = a\delta D(c, b)$. So

$$D(acb) - D(ac)b = aD(cb) - aD(c)b. \tag{6}$$

Now take $a, b \in A_c$ and $c \in A$ such that $c = 1$ on $\operatorname{supp} a \cup \operatorname{supp} b$. Then from (6)

$$D(ab) - D(a)b - aD(b) = -aD(c)b. \tag{7}$$

However, D is a local derivation so that there is a derivation N from A into $(A\widehat{\otimes}A)^*$ such that $D(c) = N(c)$. So

$$aD(c)b = aN(c)b = N(ac)b - N(a)cb = N(a)b - N(a)b = 0.$$

Hence, from (7), $\delta D(a,b) = 0$ for all $a, b \in A_c$. Therefore, by the density of A_c, $\delta D = 0$, and so D is a derivation.

We now consider the general case. Let $x \in X$ and define $L_x \colon X^* \longrightarrow (A\widehat{\otimes}A)^*$ by

$$\langle L_x(x^*)\ ,\ a \otimes b\rangle = \langle x^*\ ,\ axb\rangle \quad (a, b \in A\ ,\ x^* \in X^*).$$

It is easy to check that L_x is a bounded A-bimodule homomorphism, and hence $L_x \circ D$ is a bounded local derivation into $(A\widehat{\otimes}A)^*$. Thus $L_x(\delta D(c, e)) = 0$ for all $c, e \in A$ and $x \in X$. So

$$\langle \delta D(c, e)\ ,\ axb\rangle = 0 \quad (a, b \in A, x \in X).$$

Thus, by the essentiality of X, $\delta D = 0$, showing that D is a derivation.

Finally, suppose that A has a bounded approximate identity, X is a Banach A-bimodule, and $D \colon A \longrightarrow X$ is a bounded local derivation. By a similar argument to the one made above (by replacing X with X^{**}), we can show that for all $a, b, c, d \in A$

$$c\, \delta D(a, b)\, d = 0. \tag{8}$$

Put $Y = XA$. By Cohen's factorization theorem [**2**, Theorem 11.10], Y is a closed submodule of X. Let q be the natural quotient map from X onto X/Y. Let $\{e_\alpha\}_{\alpha \in \Lambda}$ be a bounded approximate identity for A. For each $\alpha \in \Lambda$, define $T_\alpha \colon A \longrightarrow X/Y$ by

$$T_\alpha(a) = q(D(ae_\alpha)) \quad (a \in A).$$

It is easy to see that T_α is a bounded local right multiplier, and so, by Theorem 2.28, it is a right multiplier. Hence, for all $a, b \in A$ and $\alpha \in \Lambda$, we have $q(D(abe_\alpha)) = aq(D(be_\alpha))$. By letting $\alpha \longrightarrow \infty$, we see that $q(D(ab) - aD(b)) = 0$. So $D(ab) - aD(b) \in XA$. Hence $\delta D(a, b) \in XA$. So, by Cohen's Factorization Theorem, there is $e \in A$ and $x \in X$ such that $\delta D(a, b) = xe$. If we put $d = e_\alpha$ in (8) and let $\alpha \longrightarrow \infty$, then $c\delta D(a, b) = 0$. Similarly, by letting $Y = AX$, we can show that $\delta D(a, b) = 0$ for all $a, b \in A$. Therefore D is a derivation.

□

COROLLARY 2.31. *Let A be a hyper-Tauberian algebra such that $A\widehat{\otimes}A$ is semisimple, and let X be a symmetric Banach A-module. Then every bounded local derivation D from A into X is zero.*

PROOF. First consider the case $X = A^*$. Since X is the dual of an essential Banach A-bimodule, by Theorem 2.30, D is a bounded derivation. Therefore, by weak amenability of A [Theorem 2.6(iii)], $D = 0$. For the general case, by a similar argument to the one made in the proof of Proposition 2.4(ii), we have

$$bD(a) = 0 \quad (a, b \in A). \tag{1}$$

Now let $a, b \in A_c$ and take $c \in A$ such that $c = 1$ on $\operatorname{supp} a \cup \operatorname{supp} b$. So $ac = a$ and $bc = b$. Since D is a local derivation, there is a derivation S from A into X such that $D(ab) = S(ab)$. Hence

$$D(ab) = S(abc) = abS(c) + S(ab)c = aS(c)b + D(ab)c. \tag{2}$$

However,

$$aS(c)b = S(ac)b - S(a)cb = S(a)b - S(a)b = 0.$$

Thus, from (1) and (2), $D(ab) = 0$. Hence $D = 0$ on A_c^2. Therefore $D = 0$ since A_c^2 is dense in A. □

2.7. Commutative semisimple Banach algebras with local derivations which are not derivations

In this final section, we give examples of classes of commutative semisimple Banach algebras for which the question of the reflexivity of the space of derivations fails to be true even in most natural setting i.e. from the algebra into its dual.

Let Ω be an open connected subset of the complex plane, and let $H(\Omega)$ be the algebra of analytic functions on Ω. A *Banach algebra of analytic functions* on Ω is a subalgebra A of $H(\Omega)$ such that it is a Banach algebra with respect to some norm and it contains a non-constant function. For each $t \in \Omega$, let φ_t be the character on A specified by

$$\varphi_t(u) = u(t) \quad (u \in A).$$

By [**10**, Theorem 2.1.29(ii)], φ_t is bounded and $||\varphi_t|| \leq 1$ $(t \in \Omega)$. Therefore $||u||_\infty \leq ||u||_A$ $(u \in A)$, where $||\cdot||_\infty$ and $||\cdot||_A$ are the supremum norm and the A-norm on A, respectively. This, in particular, implies that A is a commutative semisimple Banach algebra, and so, by a result of Johnson [**2**, Theorem 18.21], 0 is the only derivation on A. Therefore (bounded) approximately local derivations on A are derivations. However, as we will see in the following theorem, this is not always the case even for the bounded local derivations from A into A^*.

THEOREM 2.32. *Let Ω be an open connected subset of the plane, and let A be a Banach algebra of analytic functions on Ω. Then there is a bounded local derivation from A into A^* which is not a derivation. Moreover, $\mathcal{Z}^1(A, A^*)$ is not reflexive.*

PROOF. Let K be a closed disk in Ω, and, for $i = 0, 1, 2$, let

$$K_i = \{t \in K \mid a^{(i)}(t) = 0 \text{ for all } a \in A\}.$$

We claim that for each i, K_i is finite. Otherwise, K_i has a limit point in K, and so, in Ω. Thus, by [**5**, Theorem 3.7], $a^{(i)} = 0$ for all $a \in A$. Therefore the degree of each element in A is at most 1. However, if a is a non-constant element in A, then a^2 has the degree of at least 2. This contradiction shows that K_i is finite. Hence there is $t \in \Omega \setminus (K_0 \cup K_1 \cup K_2)$ and $a, b, c \in A$ such that

$$a(t) \neq 0 \ , \ b'(t) \neq 0 \ , \ c''(t) \neq 0. \tag{1}$$

Now consider the operator $D\colon A \to A^*$ defined by

$$D(u) = u''(t)\varphi_t \quad (u \in A).$$

From (1), $\langle D(c) \ , \ a\rangle = c''(t)a(t) \neq 0$. Thus D is non-zero. We claim that D is a bounded local derivation but it is not a derivation. We first show that D is bounded. Take $r > 0$ such that $\overline{B_r(t)}$, the closed disk with the center t and the radius r, is a subset of Ω. Let $u \in A$. Since u is analytic and bounded on Ω, by [**5**, Cauchy's Estimate 2.14], we have

$$|u''(t)| \leq \frac{2||u||_\infty}{r^2} \leq \frac{2||u||_A}{r^2}.$$

Thus $||D(u)|| \leq \dfrac{2||u||_A}{r^2}$, and so D is bounded.

We now show that D is a local derivation. Define the operators $D_i\colon A \to A^*$ $(i = 1, 2)$ by

$$\langle D_1(v) \ , \ w\rangle = v'(t)w(t);$$

$$\langle D_2(v) \ , \ w\rangle = v''(t)w(t) + v'(t)w'(t).$$

It is straightforward to check that D_1 and D_2 are derivations. Now let $D_u = \dfrac{u''(t)}{u'(t)}D_1$ whenever $u'(t) \neq 0$, and $D_u = D_2$ whenever $u'(t) = 0$. Then, for each $u \in A$, $D(u) = D_u(u)$. Hence D is a local derivation. Moreover, by applying a similar argument to the one made to prove that D is bounded, we can show that D_1 and D_2 are bounded. Therefore $D \in \mathrm{ref}[\mathcal{Z}^1(A, A^*)]$. Finally a simple calculation shows that

$$D(b^2) - 2bD(b) = 2[b'(t)]^2\varphi_t.$$

However, by (1), $b'(t) \neq 0$. Thus D is not a derivation. $\square$

EXAMPLE 2.33. Let

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\},$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc. Then $(A(\overline{\mathbb{D}}), ||\cdot||_\infty)$ is a Banach algebra of analytic functions; it is called the *disc algebra*. Thus $\mathcal{Z}^1(A(\overline{\mathbb{D}}), A(\overline{\mathbb{D}})^*)$ is not reflexive.

CHAPTER 3

Approximately local multipliers and approximately local derivations

In this section, we primarily define approximately local multipliers and approximately local derivations from Banach algebras, and then address the question of when (mostly in the bounded case) they are multipliers and derivations, respectively. To do this, we investigate the relationship between these two families of operators. One of our main results (Theorem 3.5) states that, for unital Banach algebras, if bounded approximately local multipliers are multipliers, then bounded approximately local derivations are derivations. Later on, we extend this result to a considerably larger class of Banach algebras that we call approximately locally unital Banach algebras. By applying these ideas to the various classes of Banach algebras, we show that bounded approximately local derivations are derivations if they are defined from a hyper-Tauberian algebra, a C^*-algebra, a Banach algebra generated by idempotents, a semisimple annihilator Banach algebra, or the group algebra $L^1(G)$ when G is a SIN or a totally disconnected group.

3.1. Definitions and basic properties of approximately local multipliers and approximately local derivations

DEFINITION 3.1. Let A be a Banach algebra, and let X be a Banach right A-module. An operator T from A into X is an *approximately local left multiplier* if for each $a \in A$, there a sequence of left multipliers $\{T_{a,n}\}$ from A into X such that $T(a) = \lim_{n\to\infty} T_{a,n}(a)$. If, in addition, T is bounded we say that T is a *bounded approximately local left multiplier.* Similarly, for a Banach left A-module X, we can define *approximately local right multipliers* and *bounded approximately local right multipliers* from A into X.

It is clear that local multipliers (respectively, bounded local multipliers) are approximately local multipliers (respectively, bounded approximately local multipliers). We are interested in determining when bounded approximately local multipliers are multipliers. This seems to be more general than determining the reflexivity of the linear space of bounded multipliers (since, in Definition 3.1, we are not assuming that each $T_{a,n}$ is bounded), but the following theorem shows that these two notions are closely related.

THEOREM 3.2. *Let A be a Banach algebra with $A = \overline{A^2}$, and let Y be a Banach left (respectively, right) A-module. Then the following statements are equivalent:*
(i) Every bounded approximately local right (respectively, left) multiplier from A into Y is a right (respectively, left) multiplier.
(ii) For each essential Banach left (respectively, right) A-module X, ${}_AB(X,Y)$ [respectively, $B_A(X,Y)$] is reflexive.
(iii) ${}_AB(A,Y)$ [respectively, $B_A(A,Y)$] is reflexive.

PROOF. We prove the theorem for the case of a left module. The other case can be proved similarly.
(i) $\Longrightarrow$ (ii) Let $T \in \mathrm{ref}[{}_AB(X,Y)]$. For each $x \in X$, define:

$$T_x\colon A \to Y \quad , \quad T_x(a) = T(ax) \quad (a \in A). \tag{1}$$

It is easy to check that T_x is a bounded approximately local right multiplier and so it is a right multiplier. Thus, $T(acx) = aT(cx)$ for all $a, c \in A$. Now the result follows by essentiality of X.

(ii)$\Longrightarrow$(iii) Clear.

(iii)$\Longrightarrow$(i) Let $T\colon A \to Y$ be a bounded approximately local right multiplier. For $b \in A$, we define T_b as in (1). Then $T_b \in \text{ref}[{}_AB(A,Y)] = {}_A B(A,Y)$. So $T(acb) = aT(cb)$ for all $a, c \in A$. The result now follows from the fact that $\overline{A^2} = A$. □

The following theorem shows that if a Banach algebra A has a bounded approximate identity, then the reflexivity of the linear space of bounded multipliers from A into A^* implies that every bounded approximately local multiplier from A is a multiplier.

PROPOSITION 3.3. *Let A be a Banach algebra with a bounded approximate identity such that ${}_AB(A, A^*)$ [respectivly, $B_A(A, A^*)$] is reflexive. Then for any Banach left [respectively, right] A-module X, every bounded approximately local right [respectively, left] multiplier from A into X is a right [respectively, left] multiplier.*

PROOF. We prove the theorem for the case of a left module. The other case can be treated similarly. In view of Theorem 3.2, it suffices to prove that ${}_AB(A,X)$ is reflexive. Let $T \in \text{ref}[{}_AB(A,X)]$ and $F \in X^*$. Consider

$$K_F\colon X \to A^* \ , \ \langle K_F(x) \ , \ a\rangle = F(ax) \quad (a \in A, x \in X).$$

Then K_F is a bounded left A-module morphism and so,

$$K_F \circ T \in \text{ref}[{}_AB(A, A^*)] = {}_A B(A, A^*).$$

Thus, for all $a, b \in A$, $K_F(T(ab) - aT(b)) = 0$. Hence, if we put $u = T(ab) - aT(b)$, then $F(cu) = 0$ for all $c \in A$ and $F \in A^*$ so that $cu = 0$. On the other hand, there is a sequence $\{T_n\}$ of bounded right multipliers from A into X such that $T(ab) = \lim_{n \to \infty} T_n(ab)$. So

$$u = \lim_{n \to \infty} T_n(ab) - aT(b) = \lim_{n \to \infty} a[T_n(b) - T(b)].$$

Therefore, $u \in \overline{AX}$. So, by Cohen's factorization theorem, there are $e \in A$ and $x \in X$ such that $u = ex$. This, together with the fact that $Au = 0$, implies that $u = 0$.

□

DEFINITION 3.4. Let A be a Banach algebra, and let X be a Banach A-bimodule. An operator D from A into X is called an *approximately local derivation* if for each $a \in A$, there is a sequence of derivations $\{D_{a,n}\}$ from A into X such that $D(a) = \lim_{n \to \infty} D_{a,n}(a)$. If, in addition, D is bounded, we say that D is a *bounded approximately local derivation.*

It is clear that local derivations from A into X are exactly members of $\mathrm{ref}_a[Z^1(A, X)]$. Also, any element of $\mathrm{ref}_a[Z^1(A, X)]$ or $\mathrm{ref}[\mathcal{Z}^1(A, X)]$ is an approximately local derivation (bounded, in the latter case). But the converse may not be true, since the derivations $D_{a,n}$ considered in Definition 3.4 need not be bounded.

The following theorem which determines the relationship between bounded approximately local multipliers and bounded approximately local derivations is critical in our study.

THEOREM 3.5. *Let A be a Banach algebra such that every bounded approximately local multiplier from A into any Banach A-module is a multiplier. Let X be a Banach A-bimodule, and let D be a bounded approximately local derivation* *from A into X. Then for all $a, b, c, d \in A$ we have*

$$D(acdb) - D(acd)b - aD(cdb) + aD(cd)b = 0$$

If, in addition, A is unital, then D is a derivation.

PROOF. Take $a \in A$, and let Y be the norm-closure of aX. Then Y is a closed right A-submodule of X. So X/Y is a Banach right A-module. Define the operator $\tilde{D}$ by

$$\tilde{D}\colon A \to X/Y \quad , \quad \tilde{D}(b) = D(ab) + Y \quad (b \in A).$$

Since D is a bounded approximately local derivation, for each $b \in A$, there is a sequence of derivations $\{D_n\}$ from A into X such that $D(ab) = \lim_{n\to\infty} D_n(ab)$. Hence

$$\begin{aligned}
\tilde{D}(b) &= \lim_{n\to\infty} D_n(ab) + Y \\
&= \lim_{n\to\infty} [aD_n(b) + D_n(a)b] + Y \\
&= \lim_{n\to\infty} [D_n(a) + Y]b.
\end{aligned}$$

Hence $\tilde{D}$ is a bounded approximately local left multiplier from A into X/Y. Thus, by the hypothesis, $\tilde{D}$ is a left multiplier. Therefore, for all $b, d \in A$, $\tilde{D}(db) = \tilde{D}(d)b$. Hence $D(adb) - D(ad)b \in Y$. Thus there is a sequence $\{x_n\}$ in X (possibly depending upon a, b and d) such that

$$D(adb) - D(ad)b = \lim_{n\to\infty} ax_n. \tag{1}$$

Fix $b, d \in A$ and define

$$T\colon A \to X \quad , \quad T(a) = D(adb) - D(ad)b \quad (a \in A).$$

By (1), T is a bounded approximately local right multiplier and so it is a right multiplier. Therefore, for all $a, c \in A$, $T(ac) = aT(c)$. So $D(acdb) - D(acd)b = aD(cdb) - aD(cd)b$.

Let A be unital with identity e. Put $c = d = e$. Then $D(ab) - D(a)b - aD(b) = -aD(e)b$. On the other hand, if $\Delta \colon A \to X$ is a derivation then

$$a\Delta(e)b = \Delta(ae)b - \Delta(a)eb = 0.$$

Therefore $aD(e)b = 0$, since $aD(e)b = \lim_{n\to\infty} aD_n(e)b$ for a sequence $\{D_n\}$ of derivations from A into X. So $D(ab) - D(a)b - aD(b) = 0$. Thus D is a derivation. $\square$

The next theorem provides useful criteria for determining whether a bounded approximately local derivation from a Banach algebra with a bounded approximate identity is a derivation.

THEOREM 3.6. *Let A be a Banach algebra with a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive. Let X be a Banach A-bimodule, and let D be a bounded approximately local derivation from A into X^*. Then there is a bounded derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. Moreover, if X is essential, then $\mathcal{D}$ and T are uniquely determined by this property and the following statements are equivalent:*
(i) D is a derivation.
(ii) T is zero.
(iii) weak* $- \lim_{\alpha\to\infty} D(e_\alpha) = 0$.
(iv) For each $\alpha \in \Lambda$, there is a sequence of bounded derivations $\{D_{\alpha,n}\}$ such that $D(e_\alpha) = \lim_{n\to\infty} D_{\alpha,n}(e_\alpha)$ and sup$\{||D_{\alpha,n}|| \mid \alpha \in \Lambda, n \in \mathbb{N}\}$ is finite.

PROOF. By Proposition 3.3 and Theorem 3.5 (considering the fact that $A = A^2$) for all $a, b, c \in A$, we have

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0.$$

By putting $c = e_\alpha$ and letting $\alpha \longrightarrow \infty$ we obtain

$$D(ab) - D(a)b - aD(b) + \lim_{\alpha \longrightarrow \infty} aD(e_\alpha)b = 0. \tag{1}$$

Since $\{D(e_\alpha)\}$ is bounded, there is $x^* \in X^*$ and a subnet $\{D(e_{\alpha_i})\}$ such that $D(e_{\alpha_i}) \longrightarrow x^*$ in the weak* topology. So

$$D(ab) - D(a)b - aD(b) + ax^*b = 0.$$

Define $T\colon A \longrightarrow X^*$ by $T(a) = ax^*$ and put $\mathcal{D} = D - T$. It is straightforward to check that T is a right multiplier and $\mathcal{D}$ is a bounded derivation.

(i) $\Longrightarrow$ (iv) and (ii)$\Longrightarrow$(i) are obvious and (iii)$\Longrightarrow$(ii) and the uniqueness follows from the essentiality of X. For (iv)$\Longrightarrow$(iii), put $K = \sup\{||D_{\alpha,n}|| \mid \alpha \in \Lambda, n \in \mathbb{N}\}$. Then for each $a, b \in A, x \in X, \alpha \in \Lambda$ and $n \in \mathbb{N}$,

$$\begin{aligned} \langle D_{\alpha,n}(e_\alpha)\ ,\ bxa\rangle &= \langle aD_{\alpha,n}(e_\alpha)\ ,\ bx\rangle \\ &= \langle D_{\alpha,n}(ae_\alpha)\ ,\ bx\rangle - \langle D_{\alpha,n}(a)\ , e_\alpha bx\rangle \\ &= \langle D_{\alpha,n}(ae_\alpha - a)\ ,\ bx\rangle - \langle D_{\alpha,n}(a)\ ,\ e_\alpha bx - bx\rangle. \end{aligned}$$

So

$$|\langle D_{\alpha,n}(e_\alpha)\ ,\ bxa\rangle| \leq K||ae_\alpha - a||\,||bx|| + K||a||\,||x||\,||e_\alpha b - b||.$$

Therefore, first by letting $n \longrightarrow \infty$ and then $\alpha \longrightarrow \infty$, we obtain

$$\langle D(e_\alpha)\ ,\ bxa\rangle \longrightarrow 0.$$

The final result follows since, by Cohen's factorization theorem and the essentiality of X, every element in X can be written as bxa for some $a, b \in A$ and $x \in X$. □

We note that the statements of the preceding theorem hold if we replace "right multiplier" with "left multiplier". In fact, since A has a bounded approximate identity, we can write every right multiplier from A into X^* as a sum of a left multiplier and an inner derivation from A into X^*.

We recall that a Banach algebra is approximately weakly amenable if every bounded derivation D from A into A^* is approximately inner [**21**].

COROLLARY 3.7. *Let A be an approximately weakly amenable Banach algebra with a bounded approximate identity. If ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive, then $\mathcal{Z}^1(A,A^*)$ is reflexive.*

PROOF. Let $D \in \text{ref}[\mathcal{Z}^1(A,A^*)]$. By Theorem 3.6, there are $\Delta \in \mathcal{Z}^1(A,A^*)$ and $T \in_A B(A,A^*)$ such that $D = \Delta + T$. So $T \in \text{ref}[\mathcal{Z}^1(A,A^*)]$. Hence, since A is approximately weakly amenable, for every $a \in A$, there is a sequence $\{x_{a,n}\} \in A^*$ such that

$$T(a) = \lim_{n\to\infty} ax_{a,n} - x_{a,n}a. \tag{1}$$

Now, let $\{e_\alpha \mid \alpha \in \Lambda\}$ be a bounded approximate identity for A and E be a weak*-cluster point of $\{e_\alpha \mid \alpha \in \Lambda\}$ in A^{**}. Then E is a two-sided identity for A in A^{**}. In particular, for all $a \in A$ and $x \in A^*$, $\langle E\ ,\ ax - xa\rangle = \langle Ea - aE\ ,\ x\rangle = 0$. Therefore, from (1),

$$\langle E\ ,\ T(a)\rangle = 0 \quad (a \in A). \tag{2}$$

But T is a (bounded) right multiplier and so, by taking a weak*-cluster point of $T(e_\alpha)$, there is $y \in A^*$ such that $T(a) = ay$. Thus

$$\langle E\ ,\ T(a)\rangle = \langle Ea\ ,\ y\rangle = \langle y\ ,\ a\rangle.$$

So, from (2), $y = 0$, and hence $T = 0$. Therefore, D is a derivation. $\square$

Let φ be a character on the Banach algebra A (i.e. φ is a non-zero multiplicative linear functional on A). A functional d on A is called a point derivation at φ if

$$d(ab) = d(a)\varphi(b) + \varphi(a)d(b) \quad (a, b \in A).$$

It is shown in [**21**, Proposition 2.1] that if a Banach algebra has a non-zero bounded point derivation, then it is not approximately weakly amenable. As we see in the next proposition and the following corollary, non-zero point derivations also generate local multipliers which are not multipliers. This is noted for a particular example in [**28**, p.319]. We modify it for the general case.

PROPOSITION 3.8. *Let A be a Banach algebra, let φ be a character on A, and let d be a point derivation at φ. Then the operator $D\colon A \longrightarrow A^*$ defined by $D(a) = d(a)\varphi$ is a derivation which is also both a local left and right multiplier. Moreover, if d is bounded, then so is D.*

PROOF. It is straightforward to check that D is a derivation, and, if d is bounded, then D is bounded. So it just remains to show that D is both a local left and right multiplier. For that, it suffices to show that for every $a \in A$ there is $F_a \in A^*$ such that for every $b \in A$,

$$\langle D(a)\ ,\ b\rangle = \langle F_a\ ,\ ba\rangle = \langle F_a\ ,\ ab\rangle. \tag{1}$$

Take $a \in A$. We consider two cases:

Case I: $\varphi(a) \neq 0$. Put $F_a = D(a)/\varphi(a)$. Then

$$\begin{aligned}\langle F_a\ ,\ ab\rangle &= 1/\varphi(a)\langle D(a)\ ,\ ab\rangle \\ &= [1/\varphi(a)]d(a)\varphi(ab) \\ &= [1/\varphi(a)]d(a)\varphi(a)\varphi(b) \\ &= \langle D(a)\ ,\ b\rangle.\end{aligned}$$

Similarly, we have $\langle F_a\ ,\ ba\rangle = \langle D(a)\ ,\ b\rangle$.

Case II: $\varphi(a) = 0$. Put $F_a = d$. Then it is easy to see that (1) holds. $\square$

COROLLARY 3.9. *Let A be a Banach algebra such that $A = \overline{A^2}$. Suppose that every bounded local left multiplier from A into A^* is a left multiplier. Then A does not have a non-zero bounded point derivation. The same result is true if we replace "left" with "right" in the above statement.*

PROOF. Let d, φ and D be as considered in Proposition 3.8. Then, by hypothesis and the same proposition, D is both a bounded derivation and a bounded multiplier. Therefore, a simple calculation shows that D vanishes on $\{ab : a, b \in A\}$, and so does d. Hence $d = 0$. $\square$

Let A be a Banach algebra, let X and Y be left (right) Banach A-modules, and let $T\colon X \to Y$ be an operator. Then T is *left-intertwining* (*right-intertwining*) over A if the operator $x \mapsto T(ax) - aT(x)$; $X \to Y$ ($x \mapsto T(xa) - T(x)a$; $X \to Y$) is bounded for all $a \in A$. Suppose that X and Y are Banach A-bimodules. Then T is *intertwining* over A if it is both left-intertwining and right-intertwining. It is easy to see that each derivation is an intertwining map. A classical approach for determining whether an operator $T\colon X \to Y$ is bounded is first to see whether it is intertwining. As we see in the next two results, this approach will help us to determine whether an approximately local derivation is bounded.

THEOREM 3.10. *Let A be a Banach algebra with a bounded approximate identity such that every approximately local multiplier from A into any Banach A-module is a multiplier. Then every approximately local derivation from A into any Banach A-bimodule is an intertwining map over A.*

PROOF. Let X be a Banach A-bimodule, let $D\colon A \to X$ be an approximately local derivation, and let $a \in A$. Let $\{b_n\}$ be a sequence in A such that $b_n \to 0$. By [**2**, Corollaries 11.11 and 11.12], there are $c, d \in A$ and a sequence $\{t_n\} \subseteq A$ such that for each $n \in \mathbb{N}$, $b_n = cdt_n$ and $t_n \to 0$. An argument similar to the one found in Theorem 3.5 shows that the operator $T\colon A \to X$ defined by

$$T(b) = D(acb) - aD(cb) \quad (b \in A)$$

is an approximately local left multiplier, and so, by hypothesis, T is a left multiplier. Thus

$$[D \times a - a \cdot D](b_n) = T(dt_n) = T(d)t_n \to 0.$$

So D is left-intertwining over A. Similarly, we can show that D is right-intertwining over A. □

COROLLARY 3.11. *Let A be a Banach algebra with a bounded approximate identity such that every approximately local multiplier from A into any Banach A-module is a multiplier. If every derivation from A into any Banach A-bimodule is bounded, then every approximately local derivation from A into any Banach A-bimodule is bounded.*

PROOF. The result follows from Theorem 3.10 and [**10**, Corollary 2.7.7]. □

3.2. Approximately locally unital Banach algebras

In Theorem 3.5 we showed that for unital Banach algebras, the reflexivity of the linear space of bounded multipliers is sufficient for bounded approximately local derivations to be derivations. In this section, we extend that result to a considerably larger family of Banach algebras that we call approximately locally unital.

DEFINITION 3.12. Suppose that A is a Banach algebra and $a \in A$. We say that a has a *left* (*right*) *identity* in A if for some $b \in A$, $ba = a$ ($ab = a$). We say that A is *approximately locally unital* if there are subsets A_l and A_r of A such that A

is the closed linear span of both A_l and A_r and each element of A_l and A_r has a left identity and a right identity in A, respectively. From the definition, it is clear that if A is approximately locally unital then $A = \overline{A^2}$.

EXAMPLE 3.13. Let A be a Tauberian algebra, and let A_c be the set of those elements in A with compact support. By the regularity of A, for each $a \in A_c$, there is an element $b \in A$ such that $b = 1$ on $\operatorname{supp} a$, and so $ab = ba = a$. Thus A is approximately locally unital since A_c is dense in A.

PROPOSITION 3.14. *Let A be an approximately locally unital Banach algebra such that ${}_AB(A, A^*)$ [respectively, $B_A(A, A^*)$] is reflexive. Then every bounded approximately local right [respectively, left] multiplier from A into any Banach left [respectively, right] A-module is a right [respectively, left] multiplier.*

PROOF. Let X be a Banach left A-module. By Theorem 3.2, it suffices to show that ${}_AB(A, X)$ is reflexive. Let $T \in \operatorname{ref}[{}_AB(A, X)]$. By a similar argument to the one made in the proof of Proposition 3.3, we can show that for all $a, b, c \in A$, $aT(bc) = abT(c)$. Now let A_l be as in Definition 3.12 and take $b \in A_l$. There is an element $a \in A$ such that $ab = b$. Since $T \in \operatorname{ref}[{}_AB(A, X)]$, for each $c \in A$ there is a sequence $\{T_n\}$ in ${}_AB(A, X)$ such that $T(bc) = \lim_{n\to\infty} T_n(bc)$. So

$$\begin{aligned} T(bc) - bT(c) &= \lim_{n\to\infty} T_n(bc) - bT(c) \\ &= \lim_{n\to\infty} T_n(abc) - abT(c) \\ &= a \lim_{n\to\infty} T_n(bc) - abT(c) \\ &= aT(bc) - abT(c) \\ &= 0. \end{aligned}$$

The final result follows because A is the closed linear span of A_l. The proof in the case of right modules follows similar lines. $\square$

THEOREM 3.15. *Let A be an approximately locally unital Banach algebra such that ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive. Then every bounded approximately local derivation D from A into a Banach A-bimodule X is a derivation. In particular, $\mathcal{Z}^1(A,X)$ is reflexive and $ref_a[Z^1(A,X)] \cap B(A,X) = \mathcal{Z}^1(A,X)$.*

PROOF. From Proposition 3.14 and Theorem 3.5, for all $a,b,c,d \in A$,

$$D(acdb) - D(acd)b - aD(cdb) + aD(cd)b = 0. \tag{1}$$

Let A_l and A_r be as in Definition 3.12. Take $a \in A_r$ and $b \in A_l$. There are $e,f \in A$ such that $ae = a$ and $fb = b$. Let $g = e + f - ef$. Then it is easy to see that $ag = a$ and $gb = b$. Now, if, in (1), we put $c = d = g$ then

$$D(ab) - D(a)b - aD(b) = -aD(g^2)b.$$

On the other hand, if Δ is any derivation from A into X then,

$$a\Delta(g^2)b = \Delta(ag^2)b - \Delta(a)g^2b = \Delta(a)b - \Delta(a)b = 0.$$

Therefore, $aD(g^2)b = 0$, since $aD(g^2)b = \lim_{n\to\infty} aD_n(g^2)b$ for a sequence $\{D_n\}$ of derivations from A into X. So $D(ab) = D(a)b + aD(b)$. The final result follows because A is the closed linear span of both A_l and A_r. $\square$

The following is an application of Theorem 3.15 which will be used later.

THEOREM 3.16. *Let A be a Banach algebra, and let $\{A_i\}_{i\in I}$ be a family of Banach subalgebras of A such that $(\oplus_{i\in I}A_i)^- = A$. Suppose that, for each $i \in I$, A_i is approximately locally unital and ${}_{A_i}B(A_i,A_i^*)$ and $B_{A_i}(A_i,A_i^*)$ are reflexive. Then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation.*

PROOF. By hypothesis, for each $i \in I$, there are subsets A_i^l and A_i^r of A_i such that A_i is the closed linear span of both A_i^l and A_i^r, each element in A_i^l has a left identity and each element in A_i^r has a right identity in A_i. So if we put $A_l = \cup_{i\in I} A_i^l$ and $A_r = \cup_{i\in I} A_i^r$, then A_l and A_r satisfy the assumption of Definition 3.12 and it follows that A is an approximately locally unital Banach algebra. Therefore, by Theorem 3.15, it suffices to show that ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive. Let $T \in \mathrm{ref}[{}_AB(A, A^*)]$, and let T_i be the restriction of T to A_i. By hypothesis and Proposition 3.14, each T_i is a right multiplier. On the other hand, if $a \in A_i$ and $b \in A_j$ where $i \neq j$ then $aT(b) = 0$. To see this, let $\{T_n\}$ be a sequence in ${}_AB(A, A^*)$ such that $T(b) = \lim_{n\to\infty} T_n(b)$. Then

$$aT(b) = \lim_{n\to\infty} aT_n(b) = \lim_{n\to\infty} T_n(ab) = 0.$$

Thus, for each $a_i, b_i \in A_i$ and $a_j, b_j \in A_j$ with $i \neq j$,

$$\begin{aligned} T[(a_i + a_j)(b_i + b_j)] &= T(a_i a_j + b_i b_j) \\ &= a_i T(a_j) + b_i T(b_j) \\ &= (a_i + a_j)T(b_i + b_j). \end{aligned}$$

Therefore T acts as a right multiplier on the direct sum of $\{A_i\}_{i\in I}$ which is dense in A. Since T is bounded, it follows that $T \in_A B(A, A^*)$. Similarly, we can show that $B_A(A, A^*)$ is reflexive. □

Let A and B be Banach algebras. There are Banach A-bimodule actions on $A\widehat{\otimes}B$ specified by

$$c \cdot (a \otimes b) = ca \otimes b \quad , \quad (a \otimes b) \cdot c = ac \otimes b \quad (a, c \in A, b \in B).$$

Similarly, we can define Banach B-bimodule actions on $A\widehat{\otimes}B$ specified by

$$d \diamond (a \otimes b) = a \otimes db \quad , \quad (a \otimes b) \diamond d = a \otimes bd \quad (a \in A, b, d \in B).$$

Moreover, it is straightforward to check that for $c \in A$, $d \in B$ and $x \in A\widehat{\otimes}B$

$$(a \otimes b)x = a \cdot (b \diamond x) = b \diamond (a \cdot x).$$

THEOREM 3.17. *Let $\{A_i\}_{i=1}^n$ be a finite set of Banach algebras, and let $\mathcal{A} = \widehat{\bigotimes}_{i=1}^n A_i$. Suppose that, for each i, A_i is approximately locally unital and ${}_{A_i}B(A_i, A_i^*)$ and $B_{A_i}(A_i, A_i^*)$ are reflexive. Then every bounded approximately local derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is a derivation.*

PROOF. First consider the case $n = 2$. For $i = 1, 2$, there are subsets A_i^l and A_i^r of A_i such that A_i is the closed linear span of both A_i^l and A_i^r, each element in A_i^l has a left identity and each element in A_i^r has a right identity in A_i. Put

$$\mathcal{A}_l = \{a_1 \otimes a_2 \mid a_i \in A_i^l\} \quad , \quad \mathcal{A}_r = \{b_1 \otimes b_2 \mid b_i \in A_i^r\}.$$

Then $\mathcal{A}_l$ and $\mathcal{A}_r$ satisfy the assumption of Definition 3.12 and so $\mathcal{A}$ is an approximately locally unital Banach algebra. Therefore, by Theorem 3.15, it suffices to show that ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ and $B_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^*)$ are reflexive. We show that ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)$ is reflexive. The other case can be treated similarly. Let $T \in_{\mathcal{A}} B(\mathcal{A}, \mathcal{A}^*)$. We claim that $T(c \cdot u) = c \cdot T(u)$ for all $c \in A_1$ and $u \in \mathcal{A}$. To this end, it suffices to show that, for all $c \in A_1$ and $a_i, b_i \in A_i$ $(i = 1, 2)$,

$$\langle T(ca_1 \otimes a_2) \ , \ b_1 \otimes b_2\rangle = \langle T(a_1 \otimes a_2) \ , \ b_1c \otimes b_2\rangle. \tag{1}$$

Let $a_2 \in A_2^l$ and $b_2 \in A_2^r$. By a similar argument to the one made in Theorem 3.15, there is $g \in A_2$ such that $ga_2 = a_2$ and $b_2g = b_2$. Hence

$$\begin{aligned}
\langle T(ca_1 \otimes a_2) \ , \ b_1 \otimes b_2\rangle &= \langle T(ca_1 \otimes ga_2) \ , \ b_1 \otimes b_2\rangle \\
&= \langle (c \otimes g)T(a_1 \otimes a_2) \ , \ b_1 \otimes b_2\rangle \\
&= \langle T(a_1 \otimes a_2) \ , \ b_1c \otimes b_2g\rangle \\
&= \langle T(a_1 \otimes a_2) \ , \ b_1c \otimes b_2\rangle.
\end{aligned}$$

So equation (1) holds since A_2 is the closed linear span of both A_2^l and A_2^r. Therefore, ${}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*) \subseteq_{A_1} B(\mathcal{A}, \mathcal{A}^*)$. On the other hand, $\mathcal{A}$ is an essential Banach A_1-bimodule, and so, from Proposition 3.14 and Theorem 3.2, ${}_{A_1}B(\mathcal{A}, \mathcal{A}^*)$ is reflexive. Thus

$$\mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)] \subseteq_{A_1} B(\mathcal{A}, \mathcal{A}^*). \tag{2}$$

Similarly, we can show that

$$\mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)] \subseteq_{A_2} B(\mathcal{A}, \mathcal{A}^*). \tag{3}$$

Now let $T \in \mathrm{ref}[{}_{\mathcal{A}}B(\mathcal{A}, \mathcal{A}^*)]$. Then for every $a_1 \in A_1$,$a_2 \in A_2$, and $u \in \mathcal{A}$, from (2) and (3) we have

$$T[(a_1 \otimes a_2)u] = T[a_1 \cdot (a_2 \diamond u)] = a_1 \cdot T(a_2 \diamond u) = (a_1 \otimes a_2)T(u).$$

Hence $T \in_{\mathcal{A}} B(\mathcal{A}, \mathcal{A}^*)$. For $n > 2$, put $\mathcal{B} = \widehat{\bigotimes}_{i=1}^{n-1} A_i$. Then $\mathcal{A} = \mathcal{B}\widehat{\otimes}A_n$. So the result follows by induction. $\square$

3.3. Approximately local multipliers and approximately local derivations from hyper-Tauberian algebras

Section 2.6 was concerned with the question of when the bounded local derivations from hyper-Tauberian algebras are derivations. Although we showed that the answer is affirmative in some cases, we were not able to obtain a general result. In this section, by using the theory of approximately locally unital Banach algebras that we developed in section 3.2, we answer the question completely in an even more general setting:

THEOREM 3.18. *Let A be a hyper-Tauberian algebra. Then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation.*

PROOF. By Example 3.13, A is approximately locally unital. Hence, from Theorem 3.15, it suffices to show that ${}_A\mathcal{B}(A, A^*)$ is reflexive. Let $T \in \text{ref}[{}_A\mathcal{B}(A, A^*)]$. For each $a \in A$, there is a sequence $\{T_n\}$ in ${}_A B(A, A^*)$ such that $T(a) = \lim_{n\to\infty} T_n(a)$. In particular, since each T_n is a multiplier, $ba = 0$ implies that $bT(a) = 0$. Therefore T is a local operator, and so, by hypothesis, it is a multiplier. This completes the proof. □

Let A be a Tauberian algebra such that every closed ideal of finite codimension in A has a bounded approximate identity. Then, by [**10**, Corollary 5.3.5], each intertwining map from A into any Banach A-bimodule is bounded. As we see in the following theorem, this will help us to show that approximately local derivations from this type of hyper-Tauberian algebras are bounded, and so they are derivations. Some ideas in the proof have been extracted from [**28**, p. 324-325].

First let us recall that for Banach spaces X and Y, and an operator $T\colon X \to Y$, the *separating space* of T is $\mathfrak{S}(T)$, where

$$\mathfrak{S}(T) = \{y \in Y \mid \exists\{x_n\} \subseteq X \text{ such that } x_n \to 0 \text{ and } T(x_n) \to y\}.$$

So, by Closed Graph Theorem, T is bounded if and only if $\mathfrak{S}(T) = \{0\}$.

THEOREM 3.19. *Let A be a hyper-Tauberian algebra such that every closed ideal with finite codimension in A has a bounded approximate identity. Then every approximately local derivation from A into any Banach A-bimodule is a derivation.*

PROOF. By the preceding theorem, it suffices to show that every approximately local derivation from A into any Banach A-module is bounded. To this end, from Theorem 3.10 and the remark made after Theorem 3.18, it suffices to show that every approximately local multiplier from A into any Banach A-module is a multiplier. Let Z be a Banach left A-module, and let $T\colon A \to Z$ be an approximately local right multiplier. For each open subset U of Φ_A put

$$X(U) = I(U)\ ,\ Y(U) = \{z \in Z \mid \operatorname{supp} z \subseteq \Phi_A \setminus U\},$$

where our notations follow that of [**49**, Theorem 2.3]. Therefore, from the result referred to, there is a finite subset E of Φ_A such that for every $t \in \Phi_A \setminus E$ there is an open neighborhood U_t of t for which $Q(U_t) \circ T$ is bounded where $Q(U_t)\colon Z \to Z/Y(U_t)$ is the natural quotient map. Now let $z \in \mathfrak{S}(T)$ and $\{a_n\} \subseteq A$ such that $a_n \to 0$ and $T(a_n) \to z$. Thus $Q(U_t)z = 0$, and so $\operatorname{supp} z \subseteq \Phi_A \setminus U_t$. Hence

$$\operatorname{supp} z \subseteq E \quad (z \in \mathfrak{S}(T)). \tag{1}$$

On the other hand, every element in $I_0(E)$ has a compact support which is disjoint from E. Thus, by (1), $I_0(E)z = 0$. This means that $I(E)z = 0$ since, from Corollary 2.9, E is a set of synthesis. Therefore, from the fact that $I(E)$ has a bounded approximate identity, we have

$$\mathfrak{S}(T) \cap I(E)Z = \{0\}. \tag{2}$$

On the other hand, let T_0 be the restriction of T to $I(E)$ and $a \in I(E)$. By hypothesis, there is a sequence $\{T_n\}$ of right multipliers from A into Z such that $T(a) = \lim_{n\to\infty} T_n(a)$. Also, from Cohen's factorization theorem, there are $b, c \in I(E)$ such that $a = bc$. So

$$T(a) = \lim_{n\to\infty} T_n(bc) = \lim_{n\to\infty} bT_n(c).$$

Hence $T(a) \in \overline{I(E)Z}$. However, a simple application of Cohen's Factorization Theorem shows that $I(E)Z$ is closed, and so $T(a) \in I(E)Z$. Thus, from (2),

$$\mathfrak{S}(T_0) \subseteq \mathfrak{S}(T) \cap I(E)Z = \{0\}.$$

Therefore T_0 is bounded, and so is T since $I(E)$ is a closed ideal of finite codimension in A. A similar result can be obtained for approximately local left multipliers. $\square$

COROLLARY 3.20. *Let G be a locally compact group such that G_e is abelian, and let $p \in (1,\infty)$. Then every bounded approximately local derivation from $A_p(G)$ into any Banach $A_p(G)$-bimodule is a derivation. If, in addition, G is amenable, then the result is true for all approximately local derivations.*

PROOF. By Theorem 2.23, $A_p(G)$ is hyper-Tauberian. Hence the result follows from Theorem 3.18. If G is amenable, then, by [**15**, Theorem 4.2], every closed ideal with finite codimension in $A_p(G)$ has a bounded approximate identity. Therefore, by Theorem 3.19, approximately local derivations from $A_p(G)$ are derivations. $\square$

3.4. Approximately local multipliers and approximately local derivations from C^*-algebras

Let A be a C^*-algebra, and let X be a Banach A-bimodule. It is known that derivations from A into X are bounded [**10**, Theorem 5.3.7], and so $\mathcal{Z}^1(A,X) = Z^1(A,X)$. In [**28**], Johnson has shown that local derivations from A into X are derivations i.e. $Z^1(A,X)$ is algebraically reflexive. In this section we prove the stronger statement that approximately local derivations from A into X are derivations. In particular, $Z^1(A,X)$ is also reflexive. First, we need the following lemma.

LEMMA 3.21. *Let A be a unital C^*-algebra. Then every bounded approximately local multiplier from A into any Banach A-module is a multiplier.*

PROOF. By Proposition 3.14, it suffices to show that ${}_AB(A,A^*)$ and $B_A(A,A^*)$ are reflexive. First assume that $A = C(K)$ for a compact subset K of $\mathbb{R}$ and R

is the restriction map from $C_0(\mathbb{R})$ onto $C(K)$. Then $C(K)$ becomes an essential Banach $C_0(\mathbb{R})$-bimodule for the actions defined by

$$ba = R(b)a\ ,\ ab = aR(b)\quad (a \in C(K), b \in C_0(\mathbb{R})).$$

On the other hand, $C_0(\mathbb{R})$ is hyper-Tauberian (e.g. Theorem 2.17). Hence, by Theorem 3.18 (see also Proposition 2.4), $B_{C_0(\mathbb{R})}(C(K), C(K)^*)$ and ${}_{C_0(\mathbb{R})}B(C(K), C(K)^*)$ are reflexive. The final result follows from the fact that

$$B_{C(K)}(C(K), C(K)^*) = B_{C_0(\mathbb{R})}(C(K), C(K)^*);$$

$${}_{C(K)}B(C(K), C(K)^*) =_{C_0(\mathbb{R})} B(C(K), C(K)^*).$$

We now consider the general case. Let $T \in \mathrm{ref}[{}_AB(A, A^*)]$, let a be a self-adjoint element in A, and let $A(a)$ be the C^*-subalgebra of A generated by a and 1. It is well-known that there is a compact subset K of $\mathbb{R}$ such that $A(a)$ is isometrically isomorphic to $C(K)$. On the other hand, A is an essential Banach left $A(a)$-module so, by the preceding case and Theorem 3.2, ${}_{A(a)}B(A, A^*)$ is reflexive. But ${}_AB(A, A^*) \subseteq_{A(a)} B(A, A^*)$ so $T \in_{A(a)} B(A, A^*)$ i.e. $T(ab) = aT(b)$ for all $b \in A$. The final result follows since A is the linear span of its self-adjoint elements. The reflexivity of $B_A(A, A^*)$ can be proved similarly. $\square$

THEOREM 3.22. *Let A be a C^*-algebra, and let X be a Banach A-bimodule. Then every approximately local derivation D from A into X is a derivation.*

PROOF. Without loss of generality, we can assume that A is unital, for otherwise we may consider the unitalization $A^\sharp$ of A (see [**10**, Definition 3.2.1]) and extend X to a Banach $A^\sharp$-bimodule by defining $1x = x1 = x$ and D to an approximately local derivation from $A^\sharp$ into X by defining $D(1) = 0$. By Theorem 3.5 and Lemma 3.21, in order to show that D is a derivation it suffices to show that D is bounded. To this end, from [**7**, Corollary 1.2], it suffices to show that the restriction of D to any commutative unital C^*-subalgebra $C(\Omega)$ of A is bounded. But this follows immediately from the remark made after Theorem 3.18 since every closed ideal in a C^*-algebra has a bounded approximate identity [**10**, Theorem 3.2.21]. This completes the proof. □

3.5. Approximately local multipliers and approximately local derivations from Banach algebras generated by idempotents

Let A be a Banach algebra, let E be the set of idempotents in A, and let $A(E)$ be the subalgebra of A generated by E. We say that A *is generated by idempotents* if $A(E)$ is dense in A. It is easy to see that $A(E)$ is the linear span of $\Sigma = \{e_1...e_n \mid n \in N, e_i \in E\}$. Moreover, for each element $u = e_1...e_n$ in Σ, e_1 and e_n are left and right identities for u, respectively.

THEOREM 3.23. *Let A be a Banach algebra generated by idempotents. Then every bounded approximately local derivation from A into any Banach A-bimodule is a derivation.*

PROOF. By the remark made before the theorem, A is approximately locally unital. So from Theorem 3.15, it suffices to show that ${}_AB(A, A^*)$ and $B_A(A, A^*)$ are reflexive. Let $T \in \text{ref}[{}_AB(A, A^*)]$, and $a \in A$. Let e be an idempotent in A. By the assumption, there is a sequence $\{T_n\} \subseteq_A B(A, A^*)$ such that $T(ea) = \lim_{n\to\infty} T_n(ea)$. Hence

$$eT(ea) = \lim_{n\to\infty} eT_n(ea) = \lim_{n\to\infty} T_n(e^2a) = \lim_{n\to\infty} T_n(ea) = T(ea). \qquad (1)$$

On the other hand, there is a sequence $\{S_n\} \subseteq_A B(A, A^*)$ such that $T(a - ea) = \lim_{n\to\infty} S_n(a - ea)$. So

$$eT(a - ea) = \lim_{n\to\infty} eS_n(a - ea) = \lim_{n\to\infty} S_n(ea - e^2a) = 0. \tag{2}$$

Therefore, from (1) and (2), $T(ea) = eT(a)$. So $T \in_A B(A, A^*)$ because A is generated by idempotents. The reflexivity of $B_A(A, A^*)$ can be proved similarly. □

Let X be a Banach space, let $F(X)$ be the algebra of all finite rank operators in $B(X)$, and let Y be an (algebraic) $F(X)$-bimodule. M. Brešar and P. Šemrl have shown in [**3**, Theorem 3.6] that every linear mapping $D\colon F(X) \to Y$ satisfying $D(P) = PD(P) + D(P)P$ for every projection P in $F(X)$ is a derivation. Now assume that Y is, in addition, a normed $F(X)$-bimodule and $D\colon F(X) \to Y$ is an approximately local derivation (the definition would be similar to Definition 3.4). Then it is straightforward to check that D satisfies the above condition of Brešar and Šemrl so D is a derivation. Hence we can state the following:

THEOREM 3.24. *Let X be a Banach space, let $F(X)$ be the algebra of all finite rank operators in $B(X)$, and let Y be a normed $F(X)$-bimodule. Then every approximately local derivation D from $F(X)$ into Y is a derivation.*

We recall that a *Banach operator algebra* on a Banach space X is a subalgebra of $B(X)$ containing $F(X)$ such that it is a Banach algebra with respect to some norm.

The following corollary follows immediately from the preceding theorem and Corollary 3.9.

COROLLARY 3.25. *Let X be a Banach space, and let A be a Banach operator algebra on X. If $F(X)$ is dense in A then every bounded approximately local* *derivation from A into any Banach A-bimodule is a derivation. In particular, A does not have a non-zero bounded point derivation.*

EXAMPLE 3.26. Let X be a Banach space. Then $\mathcal{A}(X)$, the space of approximable operators, $\mathcal{N}(X)$, the space of Nuclear operators and $\mathcal{C}_p(X)$ for $1 \leq p < \infty$ when X is a Hilbert space are Banach operator algebras having $F(X)$ as a dense subalgebra. See [**10**, Definitions A.3.55 and A.3.57] and [**45**, Chapter 2] for the details.

3.6. Approximately local multipliers and approximately local derivations from semisimple annihilator Banach algebras

Let A be Banach algebra and $E \subseteq A$. The left and right annihilator of E are the sets $\mathrm{lan}(E)$ and $\mathrm{ran}(E)$ given by

$$\mathrm{lan}(E) = \{a \in A \mid aE = \{0\}\} \quad , \quad \mathrm{ran}(E) = \{a \in A \mid Ea = \{0\}\}.$$

A is an *annihilator Banach algebra* if for every closed left ideal L and closed right ideal R,

$$\mathrm{ran}(L) = \{0\} \text{ if and only if } L = A;$$

$$\mathrm{lan}(R) = \{0\} \text{ if and only if } R = A.$$

EXAMPLE 3.27. (i) Let G be a compact group. Then $L^p(G)$ for $1 \leq p < \infty$ and $C(G)$ with the convolution product are semisimple annihilator Banach algebras [**1**, Section 4].

(ii) Let G be a locally compact group. Then $AP(G)$, the algebra of almost periodic functions on G with convolution product and uniform norm, is a semisimple annihilator Banach algebra [**1**, Section 4].

LEMMA 3.28. *Let A be a semisimple annihilator Banach algebra. Then A is approximately locally unital.*

PROOF. Let

$$A_l = \cup \{I \mid I \text{ is a minimal right ideal}\};$$

and

$$A_r = \cup \{I \mid I \text{ is a minimal left ideal}\}.$$

From [**2**, Proposition 32.17], there is a two-sided ideal of A, denoted by $\mathrm{soc}(A)$, such that the linear span of both A_l and A_r are dense in $\mathrm{soc}(A)$ and $A = \overline{\mathrm{soc}(A)}$. On the other hand, by [**2**, Proposition 30.6], every minimal right ideal and minimal left ideal has the form eA and Af for some minimal idempotents e and f in A, respectively. In particular, e is a left identity for elements of eA and f is a right identity for the elements of Af. Therefore, A_l and A_r satisfy the condition considered in Definition 3.12 and so A is approximately locally unital. □

THEOREM 3.29. *Let A be a semisimple annihilator Banach algebra, and let X be a Banach A-bimodule. Then every bounded approximately local derivation from A into X is a derivation.*

PROOF. By [**2**, Proposition 32.17], A is the closure of the direct sum of the minimal closed bi-ideals of A. Also, by [**2**, Corollary 32.12], each minimal closed bi-ideal of A is a semisimple annihilator Banach algebra. So, from Lemma 3.28 and Theorem 3.16, it suffices to show that for each minimal closed bi-ideal I, ${}_IB(I, I^*)$ and $B_I(I, I^*)$ are reflexive. Let $T \in \mathrm{ref}[{}_IB(I, I^*)]$. By [**2**, Theorem 32.20], there is a Banach space X such that $\mathrm{soc}(I)$ and $F(X)$ are algebraically isomorphic. Therefore, from a similar argument to the one made is Theorem 3.24, the restriction of T to $\mathrm{soc}(I)$ is a right multiplier. So $T \in_I B(I, I^*)$ because of the fact that $I = \overline{\mathrm{soc}(I)}$ and T is bounded. The reflexivity of $B_I(I, I^*)$ can be proved similarly. $\square$

COROLLARY 3.30. *Let A be a semisimple annihilator Banach algebra. Then there is no non-zero bounded point derivation on A.*

PROOF. This is a result of the preceding theorem and Corollary 3.9. $\square$

3.7. Approximately local multipliers and approximately local derivations from group algebras

In this section, we investigate bounded approximately local multipliers and bounded approximately local derivations from the group algebras. We start with the following essential theorem which states that bounded approximately local multipliers from the group algebras are multipliers.

THEOREM 3.31. *Let G be a locally compact group. Then every bounded approximately local multiplier from $L^1(G)$ into any Banach $L^1(G)$-module is a multiplier.*

PROOF. Since $L^1(G)$ has a bounded approximate identity, from Proposition 3.3, it is enough to show that ${}_{L^1(G)}B(L^1(G), L^1(G)^*)$ and $B_{L^1(G)}(L^1(G), L^1(G)^*)$ are reflexive. First we consider the case when G is discrete. Let

$$T \in \mathrm{ref}[{}_{l^1(G)}B(l^1(G), l^1(G)^*)].$$

Take $x \in G$ and let H be the subgroup generated by x. Then H is an abelian subgroup of G, $l^1(H)$ is a closed subalgebra of $l^1(G)$ and $l^1(G)$ is an essential Banach $l^1(H)$-bimodule. Also, it is easy to see that the restriction of T to $l^1(H)$ is a bounded approximately local right multiplier from $l^1(H)$ into $l^1(G)^*$. Thus, by Proposition 2.19 and Theorem 3.18, $T|_{l^1(H)}\colon l^1(H) \to l^1(G)^*$ is a right multiplier.

In particular, if e is the identity in G then $T(\delta_x) = \delta_x T(\delta_e)$. Therefore $T(f) = fT(\delta_e)$ where f is in the linear span of $\Sigma = \{\delta_x \mid x \in G\}$. The final result follows from the fact that T is bounded and $l^1(G)$ is the closed linear span of Σ.

We now consider the general case. First we note that

$$\begin{aligned} {}_{L^1(G)}B(L^1(G), L^1(G)^*) &= {}_{l^1(G)}B(L^1(G), L^1(G)^*) \\ &= {}_{M(G)}B(L^1(G), L^1(G)^*). \qquad (1) \end{aligned}$$

To see this, let $T \in_{L^1(G)} B(L^1(G), L^1(G)^*)$ and $\mu \in M(G)$. Since $L^1(G)$ is s.o. dense in $M(G)$ [**10**, Theorem 2.9.49] there is a net $\{f_i\}_{i\in I}$ in $L^1(G)$ such that for all $g \in L^1(G)$, $f_i * g \longrightarrow \mu * g$ and $g * f_i \longrightarrow g * \mu$ in the L^1-norm. Hence

$$\begin{aligned} T(\mu * g) &= norm - \lim_{i\to\infty} T(f_i * g) \\ &= w^* - \lim_{i\to\infty} T(f_i * g) \\ &= w^* - \lim_{i\to\infty} f_i * T(g) \\ &= \mu * T(g). \end{aligned}$$

So $T \in_{M(G)} B(L^1(G), L^1(G)^*)$. Since $l^1(G)$ is also s.o. dense in $M(G)$, the other equality in (1) follows by the same argument. On the other hand, from Proposition 3.3, Theorem 3.2 and the result we obtained in the first part,

$${}_{l^1(G)}B(L^1(G), L^1(G)^*)$$

is reflexive. So from (1), ${}_{L^1(G)}B(L^1(G), L^1(G)^*)$ is reflexive. The reflexivity of $B_{L^1(G)}(L^1(G), L^1(G)^*)$ can be proved similarly. □

COROLLARY 3.32. *Let G be a locally compact group, and let J be an ideal in $L^1(G)$ with finite codimension. Then every bounded approximately local multiplier from J into any Banach $L^1(G)$-module is a multiplier. If, in addition, G is amenable, then the result is true for all Banach J-modules.*

PROOF. We give the proof for the left module case. The other case can be proved similarly. From [54], $J = J^2$ so J is an essential Banach $L^1(G)$-bimodule. Also, for every Banach left $L^1(G)$-module Y, $T \in_J B(J,Y)$, $a \in A$ and $b, c \in J$ we have

$$T(abc) = abT(c) = aT(bc).$$

Now, since $J = J^2$,

$${}_JB(J,Y) =_{L^1(G)} B(J,Y).$$

Hence we have the result from Theorem 3.31 and Theorem 3.2. If G is amenable, then, by Johnson's theorem, $L^1(G)$ is amenable, and so, by [8, Corollary 3.8], J has a bounded approximate identity. On the other hand, by the preceding part, ${}_JB(J,J^*)$ is reflexive. Therefore the result follows from Proposition 3.3. □

REMARK 3.33. Let G be a locally compact group, and let J be an ideal in $L^1(G)$ with finite codimension. Since J^* is a Banach $L^1(G)$-bimodule, from Corollary 3.32, bounded local multipliers from J into J^* are multipliers. Hence, from Corollary 3.9, there is no non-zero bounded point derivation on J. On the other hand, if $G = SL(2,\mathbb{R})$, then ideals of codimension 1 are not be weakly amenable [32].

The next theorem follows immediately from the preceding theorem and Theorem 3.6.

THEOREM 3.34. *Let G be a locally compact group, let X be a Banach $L^1(G)$-bimodule, and let D be a bounded approximately local derivation from $L^1(G)$ into X^*. Then there is a bounded derivation $\mathcal{D}$ and a right multiplier T from $L^1(G)$ into X^* such that $D = \mathcal{D} + T$. Moreover, if X is essential, then $\mathcal{D}$ and T are uniquely determined by this property and the following statements are equivalent for each bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ for $L^1(G)$:*

(*i*) *D is a derivation.*

(*ii*) *T is zero.*

(*iii*) *weak** $- \lim_{\alpha \to \infty} D(e_\alpha) = 0$.

COROLLARY 3.35. *Let G be a locally compact group. Then*

$$\mathcal{Z}^1(L^1(G), L^1(G)^*)$$

is reflexive.

PROOF. By [**10**, Theorem 5.6.48], $L^1(G)$ is weakly amenable. Therefore, the result follows from Theorem 3.34 and Corollary 3.7. □

REMARK 3.36. Since $L^1(G)$ is semisimple, derivations on $L^1(G)$ are bounded [**31**], and so bounded approximately local derivations on $L^1(G)$ are exactly the members of ref$[Z^1(L^1(G), L^1(G))]$. On the other hand, $L^1(G)$ is a submodule of $M(G) = C_0(G)^*$ and $C_0(G)$ is an essential Banach $L^1(G)$-bimodule. Hence, by the preceding theorem, a bounded approximately local derivation D on $L^1(G)$ is a derivation if and only if weak$^*-\lim_{\alpha \to \infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha \in \Lambda}$ for $L^1(G)$. This provides a useful criterion for determining whether D is a derivation. For example, let G be a SIN group. Then $L^1(G)$ has a central bounded approximate identity $\{e_\alpha\}_{\alpha \in \Lambda}$. However, if $\mathcal{D}$ is a derivation on $L^1(G)$, then by [**10**, Theorem 5.6.53], there is a measure $\mu \in M(G)$ such that $\mathcal{D}(f) = f * \mu - \mu * f$ for all $f \in L^1(G)$. Thus, for each $\alpha \in \Lambda$, $\mathcal{D}(e_\alpha) = 0$, and so $D(e_\alpha) = 0$, since $D(e_\alpha) = \lim_{n \to \infty} D_{\alpha,n}(e_\alpha)$ for a sequence of derivations $\{D_{\alpha,n}\}$ on $L^1(G)$. Hence D is a derivation.

The following corollary indicates another application of Theorem 3.34.

COROLLARY 3.37. *Let G be a locally compact group such that every derivation from $L^1(G)$ into $M(G)$ is inner. Then every bounded local derivation from $L^1(G)$ into $M(G)$ is an inner derivation. In particular, the result is true if G is amenable or connected.*

PROOF. By [**10**, Theorem 3.3.15], there are right and left actions of $M(G)$ on $C_0(G)$ which turns $C_o(G)$ into a Banach $M(G)$-bimodule and the duals of these actions agree with the convolution product in $M(G)$. Moreover, by [**10**, Theorem 3.3.23], there is a net $\{e_\alpha\}$ in $C_{00}(G)$ which is a bounded approximate identity in $L^1(G)$ for both $L^1(G)$ and the Banach $L^1(G)$-bimodule $(C_0(G), \cdot\,)$. Let $D\colon L^1(G) \to M(G)$ be a bounded local derivation. Hence, by Theorem 3.34, there is a derivation $\mathcal{D}$ and a right multiplier T from $L^1(G)$ into $M(G)$ such that $D = \mathcal{D} + T$. Thus $T = \mathcal{D} - D$, and so

$$T \in \mathrm{ref}_a[Z^1(L^1(G), M(G)] = \mathrm{ref}_a[\mathcal{N}^1(L^1(G), M(G)].$$

Therefore, for every $a \in L^1(G)$, there is a measure $\mu_a \in M(G)$ such that

$$T(a) = a * \mu_a - \mu_a * a.$$

Now let $a \in C_{00}(G)$. Then

$$\begin{aligned} \langle T(a)\ ,\ e_\alpha\rangle &= \langle a * \mu_a - \mu_a * a\ ,\ e_\alpha\rangle \\ &= \langle \mu_a\ ,\ e_\alpha \cdot a - a \cdot e_\alpha\rangle. \end{aligned}$$

However, $e_\alpha \cdot a$ and $a \cdot e_\alpha$ approach a as $\alpha \to \infty$. Therefore

$$\langle T(a)\ ,\ e_\alpha\rangle \to 0 \text{ as } \alpha \to \infty. \tag{1}$$

On the other hand, since T is a right multiplier, there is $\mu \in M(G)$ such that $T(b) = b * \mu$ for all $b \in L^1(G)$. Hence

$$\begin{aligned}\langle T(a)\ ,\ e_\alpha\rangle &= \langle a * \mu\ ,\ e_\alpha\rangle \\ &= \langle \mu\ ,\ e_\alpha \cdot a\rangle.\end{aligned}$$

Thus

$$\langle T(a)\ ,\ e_\alpha\rangle \longrightarrow \langle \mu\ ,\ a\rangle \text{ as } \alpha \longrightarrow \infty. \tag{2}$$

From (1) and (2), we have $\langle \mu\ ,\ a\rangle = 0$. Hence μ vanishes on $C_{00}(G)$. Therefore $T = 0$, since $C_{00}(G)$ is dense in $C_0(G)$. Thus D is a derivation, and so, by hypothesis, it is an inner derivation. Finally, we note that if G is amenable, then, by Johnson's theorem [**10**, Theorem 5.6.42], every derivation on $L^1(G)$ is inner. From [**29**, Corollary 4.4], the same result is true for G being connected . □

REMARK 3.38. Recently, V. Losert has been able to proved that for every locally compact group G, every derivation from $L^1(G)$ into $M(G)$ is inner [**40**]. Hence, by the preceding corollary, every bounded local derivation from $L^1(G)$ into $M(G)$ is an inner derivation.

We can extend the result we have obtained in Remark 3.36. In order to do that, we will use the theory of approximately locally unital Banach algebras that we developed in section 3.2. First we need the following Lemma.

LEMMA 3.39. *Let G be a locally compact group, and let H be an open subgroup of G. If $L^1(H)$ is approximately locally unital, then $L^1(G)$ is approximately locally unital.*

PROOF. Since H is an open subgroup of G, we can consider $L^1(H)$ as a closed subalgebra of $L^1(G)$ where elements in $L^1(H)$ are exactly those in $L^1(G)$ which vanishes off H. By hypothesis, $L^1(H)$ is approximately locally unital and so it has subsets B_l and B_r satisfying the conditions in Definition 3.12. Put

$$A_l = \{f * \delta_x \mid f \in B_l, x \in G\}\ ,\ A_r = \{\delta_x * f \mid f \in B_r, x \in G\}.$$

We show that A_l and A_r satisfy the assumption in Definition 3.12 for $L^1(G)$. Let $f \in B_l$ and $x \in G$. By the assumption, there is $g \in L^1(H)$ such that $g * f = f$. So $g * (f * \delta_x) = f * \delta_x$. Hence each element in A_l has a left identity. Since $C_c(G)$ is dense in $L^1(G)$, in order to show that $L^1(G)$ is the closed linear span of A_l, it suffices to show that $C_c(G) \subseteq \overline{\operatorname{span}}\, A_l$. Let $\varphi \in C_c(G)$. Since $\operatorname{supp}\varphi$ is compact, H is open and $G = \cup_{x\in G} Hx$, there is a finite subset $\{x_1, \ldots, x_n\}$ of G such that $\operatorname{supp}\varphi \subseteq \cup_{i=1}^n Hx_i$, where the union is disjoint. Thus, $\varphi = \Sigma_{i=1}^n \varphi_i$ where $\varphi_i = \varphi\chi_{Hx_i}$. It is easy to see that $\varphi_i \in L^1(G)$ and $\operatorname{supp}\varphi_i \subseteq Hx_i$. Therefore, $\varphi_i * \delta_{x_i^{-1}} \in L^1(H) = \overline{\operatorname{span}}\, B_l$. However, $\varphi_i = (\varphi_i * \delta_{x_i^{-1}}) * \delta_{x_i}$. So $\varphi_i \in \overline{\operatorname{span}}\, A_l$ for each $i = 1, \ldots, n$. Therefore, $\varphi \in \overline{\operatorname{span}}\, A_l$. A similar result can be obtained for A_r.

□

THEOREM 3.40. *Let G be a locally compact group, and let X be a Banach $L^1(G)$-bimodule. Then, in either of the following cases, every bounded approximately local derivation from $L^1(G)$ into X is a derivation:*

(i) G is a SIN-group,

(ii) G is a totally disconnected group.

PROOF. From Theorem 3.31 and Theorem 3.15, it suffices to show that $L^1(G)$ is approximately locally unital whenever G is a SIN or a totally disconnected group. To this end, we first consider the following two cases:

Case I: G is abelian. Then $L^1(G)$ is a Tauberian algebra, and so, by Example 3.13, it is approximately locally unital.

Case II: G is compact. Then $L^1(G)$ is a semisimple annihilator Banach algebra by Example 3.27(i). So it is approximately locally unital by Lemma 3.28.

Now let G be a SIN-group. Then, by [**23**, Theorem 2.13], G has an open normal subgroup $V \times K$, where $V \cong \mathbb{R}^n$ for some integer n and K is compact. Since $L^1(V \times K) \cong L^1(V)\widehat{\otimes}L^1(K)$, from the above two cases and the proof of Theorem 3.17, $L^1(V \times K)$ is approximately locally unital. Therefore, $L^1(G)$ is approximately locally unital by Lemma 3.39. A similar argument applies when G is a totally disconnected group because in this case the identity in G has a basis consisting of open compact subgroups. □

REMARK 3.41. We note that there is a solvable connected Lie group G such that $L^1(G)$ is not approximately locally unital. This is the semidirect product $G = H \ltimes_\rho \mathbb{R}$, where H is the Heisenberg group and $\rho\colon \mathbb{R} \to \mathrm{Aut}H$ is a continuous morphism. See [**10**, p. 404] for more details.

We conclude this section by the following theorem which provides necessary and sufficient condition for all the bounded approximately local derivations from $M(G)$ to be derivations.

THEOREM 3.42. *Let G be a locally compact group. Then every bounded approximately local derivation from $M(G)$ into any Banach $M(G)$-bimodule is a derivation if and only if G is discrete. Moreover, if G is non-discrete, then there is a unital Banach $M(G)$-bimodule X and a bounded local derivation from $M(G)$ into X which is not a derivation.*

PROOF. If G is discrete, then $M(G) = l^1(G)$, and so the result follows from Theorem 3.40. Now assume that G is non-discrete. By [**11**, Theorem 3.2], there is a character φ on $M(G)$ and a non-zero bounded point derivation d on $M(G)$ at

φ. So $d \in M(G)^*$ and

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b) \quad (a, b \in A).$$

In particular, if we put $I = \ker d$ and $J = \ker \varphi$, then J is a maximal ideal and I is a (proper) linear subspace of $M(G)$ such that $J^2 \subseteq I$. Moreover, by Proposition 3.8, the bounded operator $D\colon M(G) \to M(G)^*$ defined by $D(a) = d(a)\varphi$ is both a derivation and a local left multiplier. Let ψ be the augmentation character on $M(G)$ defined by

$$\psi(\mu) = \mu(G) \quad (\mu \in M(G)).$$

Put $M = \ker \psi$. Then M is a maximal ideal in $M(G)$. If T is the restriction of D to M, then T is a bounded local left multiplier from M. We claim that T is not a left multiplier. Otherwise, a simple calculation shows that $aD(b) = 0$ and so $d(b)\varphi(a) = 0$ for all $a, b \in M$. Therefore, $M \subset I$ or $M \subset J$. On the other hand, by [**10**, Theorem 3.3.30], $M = M^2$. This means that there is no non-zero point derivation on $M(G)$ at ψ, and so $\psi \neq \varphi$. Hence $M \not\subseteq J$. Also, $M \subsetneqq I$, for otherwise, since $M(G) = J + M$, we would have

$$M(G) = M(G)^2 \subseteq J^2 + M \subseteq I + I = I,$$

which is impossible. Hence T is not a left multiplier. Now if we let $X = M(G)^*$ become a Banach M-bimodule by defining the left action to be 0 and the right action to be the one induced from $M(G)^*$, when it is viewed as the dual module of $M(G)$, then $T\colon M \to X$ is a bounded local derivation which is not a derivation. Finally, because $M(G) = M \oplus \mathbb{C}\delta_e$, by defining $\delta_e x = x\delta_e = x$ $(x \in X)$ and $T(\delta_e) = 0$, X becomes an essential Banach $M(G)$-bimodule and T extends to a bounded local derivation from $M(G)$ into X which it is not a derivation. $\square$

CHAPTER 4

Approximately local n-cocycles

In this chapter we study reflexivity of higher cohomology of various Banach algebras. We first generalize the definition of reflexivity to the linear subspaces of bounded n-linear maps from Banach spaces. Then, for a Banach algebra A and a Banach A-bimodule X, we consider the question of reflexivity for the space of bounded n-cocycles from $A^{(n)}$ into X. Using the similar approach as the one introduced in Chapter 3, we consider the concept of *approximately local n-cocycles* and the question of whether they are n-cocycles. We show that we can reduce the problem to the characterization of certain operators from A into X which we call *hyperlocal operators*. Then we use this idea, together with the properties of hyper-Tauberian algebras, to show that bounded approximately local n-cocycles from $A^{(n)}$ into X are n-cocycles when A is a hyper-Tauberian algebra, a C^*-algebra, the group algebra of a SIN or a totally disconnected group, or the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G, for $p \in (1, \infty)$, when the connected component of G is abelian.

4.1. Definition of approximately local n-cocycles

Let X and Y be Banach spaces. For $n \in \mathbb{N}$, let $X^{(n)}$ be the Cartesian product of n copies of X, and let $L^n(X, Y)$ and $B^n(X, Y)$ be the spaces of n-linear maps and bounded n-linear maps from $X^{(n)}$ into Y, respectively. Let $\mathcal{S}$ be a linear subspace of $L^n(X, Y)$, for each $\tilde{x} = (x_1, \ldots, x_n) \in X^{(n)}$, let $\mathcal{S}(\tilde{x}) = \{S(\tilde{x}) \mid S \in \mathcal{S}\}$, and let

$[\mathcal{S}(\tilde{x})]$ be the norm-closure of $\mathcal{S}(\tilde{x})$. Put

$$\mathrm{ref}_a(\mathcal{S}) = \{T \in L^n(X,Y) \mid T(\tilde{x}) \in \mathcal{S}(\tilde{x}), \text{for each } \tilde{x} \in X^{(n)}\};$$

and if $\mathcal{S} \subseteq B^n(X,Y)$, put

$$\mathrm{ref}(\mathcal{S}) = \{T \in B^n(X,Y) \mid T(\tilde{x}) \in [\mathcal{S}(\tilde{x})], \text{for each } \tilde{x} \in X^{(n)}\}.$$

Suppose that $\mathcal{S} \subseteq L^n(X,Y)$. Then $\mathcal{S}$ is *algebraically reflexive* if $\mathcal{S} = \mathrm{ref}_a(\mathcal{S})$ and when $\mathcal{S} \subseteq B^n(X,Y)$, it is *reflexive* if $\mathcal{S} = \mathrm{ref}(\mathcal{S})$.

DEFINITION 4.1. Let A be a Banach algebra, and let X be a Banach A-bimodule. For $n \in \mathbb{N}$, an n-linear map T from $A^{(n)}$ into X is called an *approximately local n-cocycle* if, for each $\tilde{a} = (a_1, \ldots, a_n) \in A^{(n)}$, there is a sequence $T_{\tilde{a},n}$ of n-cocycles from $A^{(n)}$ into X such that $T_{\tilde{a},n}(\tilde{a}) = \lim\limits_{n\to\infty} T_{\tilde{a},n}(\tilde{a})$. If, in addition, T is bounded, we say that T is a *bounded approximately local n-cocycle.*

It is clear that each element of $\mathrm{ref}[\mathcal{Z}^n(A,X)]$ is a bounded approximately local n-cocycle but the converse may not be true, since the n-cocycles $T_{\tilde{a},n}$ considered above need not be bounded.

4.2. N-hyperlocal operators

Let A and B be Banach algebras, and let X be both a Banach left A-module and a Banach right B-module such that for all $a \in A$, $b \in B$ and $x \in X$, $a(xb) = (ax)b$. Then we write $X \in A - \mathrm{mod} - B$. If, in addition, X is essential both as a Banach left A-module and a Banach right B-module, then we write $X \in \mathrm{ess.}\ A - \mathrm{mod} - B$.

Let A and B be Banach algebras, and let $X, Y \in A - \mathrm{mod} - B$. An operator $D\colon X \to Y$ is *hyperlocal* with respect to A-mod-B actions if, for all $a \in A$, $b \in B$ and $x \in X$,

$$ax = xb = 0 \quad \text{implies} \quad aD(x)b = 0.$$

Let A be a Banach algebra, and let X be a Banach A-bimodule. For $n \in \mathbb{N}$, an n-linear map T from $A^{(n)}$ into X is n-*hyperlocal* if, for $a_0, \dots, a_{n+1} \in A$,

$$a_0 a_1 = a_1 a_2 = \cdots = a_n a_{n+1} = 0 \text{ implies } a_0 T(a_1, \dots, a_n) a_{n+1} = 0.$$

For $n = 1$, it is clear that 1-hyperlocal operators are exactly hyperlocal operators.

The following proposition states some sufficient conditions for a bounded n-linear map to be an n-cocycle. This is critical for us to obtain our result.

PROPOSITION 4.2. *Let A be a unital Banach algebra with unit 1 which satisfies the following two conditions:*
(i) For every unital Banach A-bimodule X, a bounded operator $D\colon A \to X$ is a left multiplier if and only if $ba = 0$ implies $D(b)a = 0$.
(ii) For every unital Banach A-bimodule X, a bounded operator $D\colon A \to X$ is hyperlocal if and only if

$$D(acb) - aD(cb) - D(ac)b + aD(c)b = 0$$

for all $a, b, c \in A$.
Let X be a unital Banach A-bimodule, let $n \in \mathbb{N}$, and let $T \in B^n(A, X)$ be n-hyperlocal such that $T(a_1, \dots, a_n) = 0$ if any one of $a_1, \dots, a_n$ is 1. Then $T \in \mathcal{Z}^n(A, X)$.

PROOF. We prove the statement by induction on n. For $n = 1$, by hypothesis,

$$T(acb) - aT(cb) - T(ac)b + aT(c)b = 0$$

for all $a, b, c \in A$. Since $T(1) = 0$, by putting $c = 1$ we get the result. Now suppose that the result is true for $n = k$ $(k \geq 1)$. We show that it is also true for $n = k + 1$. Let $T \in B^{k+1}(A, X)$ be $k + 1$-hyperlocal such that $T(a_1, \dots, a_{k+1}) = 0$ if any one of $a_1, \dots, a_{k+1}$ is 1. We first show that $\Lambda_k(T) \in B^k(A, B(A, X))$ is k-hyperlocal.

Let $a_0, \ldots, a_{k+1} \in A$ such that $a_0 a_1 = \cdots = a_k a_{k+1} = 0$, and put

$$S = a_0 \star \Lambda_k(T)(a_1, \ldots, a_k) \star a_{k+1}.$$

Then $S\colon A \to X$ is a bounded operator. We claim that S satisfies the following condition:

$$bc = 0 \text{ implies } S(b)c = 0. \tag{1}$$

Let $b, c \in A$ such that $bc = 0$. Then

$$\begin{aligned} S(b)c &= [a_0 \star \Lambda_k(T)(a_1, \ldots, a_k) \star a_{k+1}](b)c \\ &= a_0(\Lambda_k(T)(a_1, \ldots, a_k))(a_{k+1}b)c - a_0(\Lambda_k(T)(a_1, \ldots, a_k))(a_{k+1})bc \\ &= a_0 T(a_1, \ldots, a_k, a_{k+1}b)c - a_0 T(a_1, \ldots, a_k, a_{k+1})bc \\ &= a_0 T(a_1, \ldots, a_k, a_{k+1}b)c. \end{aligned}$$

However, $a_0 a_1 = \cdots = a_k(a_{k+1}b) = (a_{k+1}b)c = 0$, and T is $k+1$-hyperlocal. Hence

$$a_0 T(a_1, \ldots, a_k, a_{k+1}b)c = 0.$$

Thus (1) holds, and so, by hypothesis, S is a left multiplier. Therefore $S(a) = S(1)a$ for all $a \in A$. However,

$$\begin{aligned} S(1) &= [a_0 \star \Lambda_k(T)(a_1, \ldots, a_k) \star a_{k+1}](1) \\ &= a_0(\Lambda_k(T)(a_1, \ldots, a_k))(a_{k+1}1) - a_0(\Lambda_k(T)(a_1, \ldots, a_k))(a_{k+1})1 \\ &= a_0 T(a_1, \ldots, a_k, a_{k+1}) - a_0 T(a_1, \ldots, a_k, a_{k+1}) \\ &= 0. \end{aligned}$$

Thus $S = 0$. Hence $\Lambda_k(T)$ is k-hyperlocal. Let q be the natural quotient mapping from $B(A, X)$ into $B(A, X)/B_A(A, X)$, where $B_A(A, X)$ is the space of right multipliers. Since $\Lambda_k(T)$ is k-hyperlocal and q is an A-bimodule morphism with the $\star$ actions, $q \circ \Lambda_k(T)$ is k-hyperlocal. Moreover, because of the assumption on T,

$q \circ \Lambda_k(T)(a_1, \ldots, a_k) = 0$ if any one of $a_1, \ldots, a_k$ is 1. On the other hand, for every $T \in B(A, X)$,

$$1 \star T = T \quad \text{and } T \star 1 - T \in B_A(A, X).$$

Thus $B(A, X)/B_A(A, X)$ is a unital Banach A-bimodule. Therefore, by the inductive hypothesis, $q \circ \Lambda_k(T)$ is a k-cocycle. This means that for $a_1, \ldots, a_{k+1} \in A$,

$$\Delta^k(q \circ \Lambda_k(T))(a_1, \ldots, a_{k+1}) = 0.$$

Hence, from the equation $\Lambda_{k+1} \circ \delta^{k+1} = \Delta^k \circ \Lambda_k$,

$$\Lambda_{k+1}(\delta^{k+1}(T))(a_1, \ldots, a_{k+1}) = \Delta^k(\Lambda_k(T))(a_1, \ldots, a_{k+1}) \in B_A(A, X).$$

Thus, for every $a_{k+2} \in A$,

$$\begin{aligned} \delta^{k+1}(T)(a_1, \ldots, a_{k+1}, a_{k+2}) &= [\Lambda_{k+1}(\delta^{k+1}(T))(a_1, \ldots, a_{k+1})](a_{k+2}) \\ &= [\Lambda_{k+1}(\delta^{k+1}(T))(a_1, \ldots, a_{k+1})](1)a_{k+2} \\ &= \delta^{k+1}(T)(a_1, \ldots, a_{k+1}, 1)a_{k+2}. \end{aligned}$$

On the other hand, by the assumption on T,

$$a_1 T(a_2, \ldots, a_{k+1}, 1) + \sum_{j=1}^{k} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_{k+1}, 1) = 0.$$

Also,

$$\delta^{k+1}(T)(a_1, \ldots, a_k, a_{k+1}1) - \delta^{k+1}(T)(a_1, \ldots, a_k, a_{k+1})1 = 0.$$

Hence $\delta^{k+1}(T)(a_1, \ldots, a_{k+1}, 1) = 0$. Therefore $\delta^{k+1}(T) = 0$, and so $T \in B^{k+1}(A, X)$. This completes the proof. □

We are now ready to state the main result of this section:

THEOREM 4.3. *Let A be a Banach algebra such that $A^\sharp$ satisfies conditions (i) and (ii) of Proposition 4.2. Then, for any Banach A-bimodule X and $n \in \mathbb{N}$,* *every bounded approximately local n-cocycle T from $A^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{Z}^n(A,X)$ is reflexive.*

PROOF. We can extend X to a Banach $A^\sharp$-bimodule by defining $1x = x1 = x$. Let $\sigma\colon L^n(A,X) \to L^n(A^\sharp, X)$ be a linear map defined by

$$\sigma(T)(a_1+\lambda_1,\ldots,a_n+\lambda_n) = T(a_1,\ldots,a_n),$$

for $a_1,\ldots,a_n \in A$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$. It is straightforward to check that $T \in L^n(A,X)$ is an n-cocycle if and only if $\sigma(T)$ is an n-cocycle. Now let $T \in B^n(A,X)$ be a bounded approximately local n-cocycle, and let $(a_1+\lambda_1,\ldots,a_n+\lambda_n) \in A^{\sharp(n)}$. By the assumption on T, for $\tilde{a} = (a_1,\ldots,a_n) \in A^{(n)}$, there is a sequence $T_{\tilde{a},n}$ of n-cocycles from $A^{(n)}$ into X such that $T_{\tilde{a},n}(a_1,\ldots,a_n) = \lim_{n\to\infty} T_{\tilde{a},n}(a_1,\ldots,a_n)$. Thus

$$\begin{aligned}
\sigma(T)(a_1+\lambda_1,\ldots,a_n+\lambda_n) &= T(a_1,\ldots,a_n) \\
&= \lim_{n\to\infty} T_{\tilde{a},n}(a_1,\ldots,a_n) \\
&= \lim_{n\to\infty} \sigma(T_{\tilde{a},n})(a_1+\lambda_1,\ldots,a_n+\lambda_n).
\end{aligned}$$

Hence $\sigma(T)$ is a bounded approximately local n-cocycle. Moreover, $\sigma(T)(a_1,\ldots,a_n) = 0$ if any one of $a_1,\ldots,a_n$ is 1. Thus, by Proposition 4.2, $\sigma(T)$ is an n-cocycle. Therefore T is an n-cocycle. □

4.3. Approximately local n-cocycles from hyper-Tauberian algebras

In this section, we use the properties of hyper-Tauberian algebras, together with the results of the preceding section, to show bounded approximately local n-cocycles from these algebras are n-cocycles. We start with the following critical theorem:

THEOREM 4.4. *Let A and B be hyper-Tauberian algebras. Then, for all $X, Z \in$ ess. $A - mod - B$ and $Y \in$ ess. $B - mod - A$,*

(i) a bounded operator $D\colon X \to Y^*$ is hyperlocal if and only if

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0$$

for all $a \in A$, $b \in B$ and $x \in X$.

(ii) If A and B have bounded approximate identities, then the result in (i) is also true for all bounded hyperlocal operators from X into Z.

PROOF. (i) First assume that $Y = B\widehat{\otimes}A$, where the B-mod-A actions on $B\widehat{\otimes}A$ are specified by

$$d(b \otimes a) = db \otimes a \quad , \quad (b \otimes a)c = b \otimes ac \quad (a, c \in A, b, d \in B).$$

Let $D\colon X \to (B\widehat{\otimes}A)^*$ be a bounded hyperlocal operator, and let $x \in X$ and $a \in A$. Define the bounded operator $\widetilde{D}\colon B \to (B\widehat{\otimes}A)^*$ by

$$\widetilde{D}(b) = D(axb) - aD(xb) \quad (b \in B).$$

We claim that D is local with respect to right B-module action. Let $b \in A$ and $t \notin \operatorname{supp}_B b$. There is a compact neighborhood V of t (in Φ_B) such that $V \cap \operatorname{supp}_B b = \emptyset$. Let $c \in B$ with $\operatorname{supp}_B c \subseteq V$. By the regularity of B, there is $e \in B$ such that $e = 1$ on V and $e = 0$ on $\operatorname{supp}_B b$. So

$$ec = c \text{ and } eb = 0. \tag{1}$$

Put

$$K_0(V) = \overline{\operatorname{span}}\{n \otimes m \mid m \in A\ , n \in B \text{ and } n = 0 \text{ on } \Phi_B \setminus V\}.$$

Since $e = 1$ on V, for all $\theta \in (B\widehat{\otimes}A)^*$,

$$\theta e - \theta = 0 \text{ on } K_0(V). \tag{2}$$

Let $z \in X$, and define the bounded operator $T\colon A \to (B\widehat{\otimes}A)^*/K_0(V)^\perp$ by

$$T(u) = D(uzb) + K_0(V)^\perp \quad (u \in A).$$

Let $h \in A$ such that $hu = 0$. Then, from (1), $huzb = 0 = uzbe$. Since D is hyperlocal, $hD(uzb)e = 0$. Hence

$$\begin{aligned} hT(u) &= hD(uzb) + K_0(V)^\perp \\ &= hD(uzb)e + K_0(V)^\perp \\ &= 0. \end{aligned}$$

In particular, T is local with respect to left A-module action. Since

$$(B\widehat{\otimes}A)^*/K_0(V)^\perp \cong K_0(V)^*,$$

and $K_0(V)$ is an essential Banach right A-module, from Proposition 2.4, it follows that T is a right multiplier. Therefore $T(uv) = uT(v)$ for all $u, v \in A$. Hence, if we put $u = a$, then $D(avzb) - aD(vzb) \in K_0(V)^\perp$. Thus, from essentiality of X, we have

$$\widetilde{D}(b) = D(axb) - aD(xb) \in K_0(V)^\perp.$$

Therefore $\widetilde{D}(b)c = 0$, since $\operatorname{supp}_B c \in V$. This means that $t \notin \operatorname{supp}_B \widetilde{D}(b)$, and so $\widetilde{D}$ is a bounded local operator. Hence, from Proposition 2.4, $\widetilde{D}$ is a left multiplier. Thus $\widetilde{D}(bd) = \widetilde{D}(b)d$ for all $b, d \in B$. Therefore

$$D(axbd) - aD(xbd) = D(axb)d - aD(xb)d.$$

The final result follows from the essentiality of X.

Now consider the general case. Let $y \in Y$ and define $S_y\colon Y^* \to (B\widehat{\otimes}A)^*$ by

$$\langle S_y(y^*)\ ,\ b \otimes a\rangle = \langle y^*\ ,\ bya\rangle \quad (a \in A\ ,\ b \in B\ ,\ y^* \in Y^*).$$

It is easy to see that S_y is both a bounded left A-module morphism and a bounded right B-module morphism, and so $S_x \circ D$ is a bounded hyperlocal operator from

X into $(B\widehat{\otimes}A)^*$. Thus, for all $a \in A$, $b \in B$, $x \in X$ and $y \in Y$,

$$S_y[D(axb) - aD(xb) - D(ax)b + aD(x)b] = 0.$$

Hence, for all $c \in A$ and $d \in B$,

$$\langle D(axb) - aD(xb) - D(ax)b + aD(x)b\ ,\ dyc\rangle = 0.$$

The final results follows from the essentiality of Y,

(ii) Let $\{e_\alpha\}_{\alpha\in\Lambda}$ and $\{f_\beta\}_{\beta\in\Omega}$ be bounded approximate identities for A and B, respectively. Similar to the argument made in (i) (by replacing Z with Z^{**}), we can show that

$$c[D(axb) - aD(xb) - D(ax)b + aD(x)b]d = 0 \tag{3}$$

for all $a, c \in A$, $b, d \in B$ and $x \in X$. On the other hand, since A and B have bounded approximate identities, by Cohen's factorization theorem [**2**, Theorem 11.10], there are $e \in A$, $f \in B$ and $z \in Z$ such that

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = ezf.$$

So we have the final result if we put $c = e_\alpha$ and $d = f_\beta$ in (3), and let $\alpha, \beta \to \infty$. $\square$

THEOREM 4.5. *Let A be a hyper-Tauberian algebra, and let X be a Banach A-bimodule. Then, for $n \in \mathbb{N}$, every bounded approximately local n-cocycle T from $A^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{Z}^n(A, X)$ is reflexive.*

PROOF. Let $A^\sharp$ be the unitalization of A. By Corollary 2.11, $A^\sharp$ is hyper-Tauberian. Therefore, by Proposition 2.4 and Theorem 4.4, $A^\sharp$ satisfies the conditions (i) and (ii) of Proposition 4.2. Hence the result follows from Theorem 4.3. $\square$

COROLLARY 4.6. *Let A be a hyper-Tauberian algebra with a bounded approximate identity, and let X be an essential Banach A-bimodule. Then a bounded* *operator $D\colon A \to X^*$ is hyperlocal if and only if there is a bounded derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. In particular, D is a derivation if and only if weak* $- \lim_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in A.*

PROOF. It is easy to see that all derivations and multipliers are hyperlocal. On the other hand, let $D\colon A \to X^*$ be a bounded hyperlocal operator. By Theorem 4.4, for all $a, b, c \in A$,

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0.$$

The final result follows from a similar argument to the one made in the proof of Theorem 3.6.

□

COROLLARY 4.7. *Let A be a hyper-Tauberian algebra with a bounded approximate identity. Then A is amenable if and only if for any essential Banach A-bimodule X and every bounded hyperlocal operator $D\colon A \to X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. Let A be amenable, let X be an essential Banach A-bimodule, and let $D\colon A \to X^*$ be a bounded hyperlocal operator. By Corollary 4.6, there is a derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. Since A is amenable, there are y^* and z^* in X^* such that $\mathcal{D}(a) = ay^* - y^*a$ and $T(a) = az^*$ for all $a \in A$. Thus $D(a) = a(y^* + z^*) - y^*a$. The converse follows immediately from Corollary 4.6 and [**10**, Corollary 2.9.27]. □

4.4. Approximately local n-cocycles from C*-algebras

In this section, we characterized bounded approximately local n-cocycles from C^*-algebras.

THEOREM 4.8. *Let A be a C^*-algebra, let X be an essential Banach A-bimodule, and let Y be an essential or the dual of an essential Banach A-bimodule. Then a bounded operator $D\colon X \to Y$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0$$

for all $a, b \in A$ and $x \in X$.

PROOF. Since A has a bounded approximate identity, by similar arguments to the ones made in the proof of Theorem 4.4, it suffices to prove the result for $Y = (A\widehat{\otimes}A)^*$. Let $A^\sharp$ be the unitalization of A [**10**, Definition 3.2.1]. We show that D is hyperlocal with respect to $A^\sharp$-module actions. Let $u, v \in A^\sharp$ and $x \in X$ such that $ux = xv = 0$. So, for all $a, b \in A$, $(au)x = x(vb) = 0$. Thus $auD(x)vb = 0$. Hence $uD(x)v = 0$ on $A^2 \otimes A^2$ which is dense in $A\widehat{\otimes}A$. So $uD(x)v = 0$. Now let c and d be self-adjoint elements in A, and let $A(c)$ and $A(d)$ be the C^*-subalgebras of $A^\sharp$ generated by $\{c, 1\}$ and $\{d, 1\}$, respectively. It is well-known that there are compact subsets E and K of $\mathbb{R}$ such that $A(c)$ and $A(d)$ are isometrically isomorphic to $C(E)$ and $C(K)$, respectively. Moreover, $D\colon X \to (A\widehat{\otimes}A)^*$ is a bounded hyperlocal operator with respect to $C(E) - \mathrm{mod} - C(K)$ actions. Thus, from Theorem 4.4 and Lemma 2.16, for every $x \in X$,

$$D(cxd) - cD(xd) - D(cx)d + cD(x)d = 0.$$

The final result follows since A is the linear span of its self-adjoint elements. $\square$

REMARK 4.9. In the preceding theorem, if we replace the locality condition that we used in the definition of a hyperlocal operator with the following condition:

$$xa = 0 \ \text{ implies } \ D(x)a = 0,$$

then, by a similar argument and using Proposition 2.4 instead of Theorem 4.4, we can show that D is a right A-module morphism. We can also have a similar result regarding bounded left A-module morphisms.

THEOREM 4.10. *Let A be a C^*-algebra, and let X be a Banach A-bimodule. Then, for $n \in \mathbb{N}$, every bounded approximately local n-cocycle T from $A^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{Z}^n(A, X)$ is reflexive.*

PROOF. Since $A^\sharp$ is a C^*-algebra, it satisfies the conditions (i) and (ii) of Proposition 4.2 from Theorem 4.8 and Remark 4.9. Hence the result follows from Theorem 4.3. $\square$

THEOREM 4.11. *Let A be a C^*-algebra, and let X be an essential Banach A-bimodule. Then a bounded operator $D\colon A \to X^*$ is hyperlocal if and only if there is a derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. In particular, D is a derivation if and only if weak$^*-\lim\limits_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in A.*

PROOF. The proof is similar to the one made in Corollary 4.6. $\square$

COROLLARY 4.12. *Let A be a C^*-algebra. Then A is amenable if and only if for any essential Banach A-bimodule X and every bounded hyperlocal operator $D\colon A \to X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. The proof is similar to the one made in Corollary 4.7. $\square$

COROLLARY 4.13. *Let A be a C^*-algebra. Then, for every bounded hyperlocal operator $D\colon A \to A^*$, there are $x^*, y^* \in A^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. The result follows from an argument similar to the one made in Corollary 4.7 together with the fact that every C^*-algebra is weakly amenable [**10**, Theorem 5.6.77].

$\square$

4.5. Approximately local n-cocycles from group algebras

In this section, we characterized bounded approximately local n-cocycles from the group algebra $L^1(G)$ when G is a SIN or a totally disconnected group.

THEOREM 4.14. *Let G be a locally compact group, let X be an essential Banach $L^1(G)$-bimodule, and let Y be an essential or the dual of an essential Banach $L^1(G)$-bimodule. Then a bounded operator $D\colon X \to Y$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0$$

for all $a, b \in L^1(G)$ and $x \in X$.

PROOF. Since $L^1(G)$ has a bounded approximate identity, by arguments similar to the ones made in the proof of Theorem 4.4, it suffices to prove the result for $Y = (L^1(G)\widehat{\otimes}L^1(G))^*$. Let $h, k \in G$, and let H and K be the closed subgroups in G generated by h and k, respectively. We claim that D is hyperlocal with respect to $l^1(H) - \text{mod} - l^1(K)$ actions. Let $f \in L^1(H)$,$g \in L^1(K)$ and $x \in X$ such that $fx = xg = 0$, and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a bounded approximate identity for $L^1(G)$. For all $\alpha, \beta \in \Lambda$, $(e_\alpha f)x = x(ge_\beta) = 0$. Hence $e_\alpha f D(x) g e_\beta = 0$. Therefore, by taking w^*-limit and letting $\alpha, \beta \to \infty$, we have $fD(x)g = 0$, and so, D is hyperlocal with respect to $l^1(H) - \text{mod} - l^1(K)$ actions. Thus, by Proposition 2.19 and Theorem 4.4, D is a generalized derivation. In particular,

$$D(\delta_h x \delta_k) - \delta_h D(x\delta_k) - D(\delta_h x)\delta_k + \delta_h D(x)\delta_k = 0 \quad (h, k \in G, x \in X).$$

Therefore, since $l^1(G)$ is the closed linear span of $\{\delta_t \mid t \in G\}$,

$$D(fxg) - fD(xg) - D(fx)g + fD(x)g = 0 \quad (f, g \in l^1(G), x \in X). \tag{1}$$

Let $a, b \in L^1(G)$ and $x \in X$. Since $l^1(G)$ is s.o. dense in $M(G)$ [**10**, Theorem 3.3.41] there are $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ in $l^1(G)$ such that for all $c \in L^1(G)$, $a_i * c \to a * c$, $c * a_i \to c * a$, $b_j * c \to b * c$ and $c * b_j \to c * b$ in the L^1-norm. On the other hand, by Cohen's Factorization Theorem [**2**, Theorem 11.10], there are $e \in L^1(G)$ and $z \in X$ such that $x = ze$. Now if we put $f = a_i$ and $g = b_j$ in (1), then, by letting $j \to \infty$, we have

$$\begin{aligned}
D(a_ixb) - a_iD(xb) &= D[a_iz(e * b)] - a_iD[z(e * b)] \\
&= norm - \lim_{j\to\infty} D[a_iz(e * b_j)] - a_iD[z(e * b_j)] \\
&= norm - \lim_{j\to\infty} D(a_ixb_j) - a_iD(xb_j) \\
&= norm - \lim_{j\to\infty} D(a_ix)b_j - a_iD(x)b_j \\
&= w^* - \lim_{j\to\infty} D(a_ix)b_j - a_iD(x)b_j \\
&= D(a_ix)b - a_iD(x)b.
\end{aligned}$$

Hence

$$D(a_ixb) - a_iD(xb) - D(a_ix)b + a_iD(x)b = 0. \tag{2}$$

Similarly, by applying Cohen's factorization theorem and letting $i \to \infty$ in (2), we have

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0.$$

This completes the proof. □

REMARK 4.15. In the preceding theorem, if we replace the locality condition that we used in the definition of a hyperlocal operator with the following condition:

$$xa = 0 \text{ implies } D(x)a = 0,$$

then, by a similar argument and using Proposition 2.4 instead of Theorem 4.4, we can show that D is a right $L^1(G)$-module morphism. We can also have a similar result regarding bounded left $L^1(G)$-module morphisms.

LEMMA 4.16. *Let G be a locally compact group such that $L^1(G)$ is approximately locally unital, and let X be a unital Banach $L^1(G)^\sharp$-bimodule. Then:*
(i) a bounded operator $T\colon L^1(G)^\sharp \to X$ is a right multiplier if and only if $ab = 0$ implies $aT(b) = 0$;
(ii) a bounded operator $T\colon L^1(G)^\sharp \to X$ is a left multiplier if and only if $ba = 0$ implies $T(b)a = 0$;
(iii) a bounded operator $D\colon L^1(G)^\sharp \to X$ is hyperlocal with respect to $L^1(G)^\sharp$-bimodule actions if and only if

$$D(acb) - aD(cb) - D(ac)b + aD(c)b = 0$$

for all $a, b, c \in L^1(G)^\sharp$.

PROOF. (i) Let T a bounded operator from $L^1(G)^\sharp$ into X that satisfies the following condition:

$$ab = 0 \ \text{ implies } \ aT(b) = 0 \quad (a, b \in L^1(G)^\sharp).$$

We will show that T is a right multiplier. First we note that if we consider the restriction of T to $L^1(G)$, then, by Remark 4.15 and an argument similar to the one made in the proof of Proposition 2.4(ii),

$$aT(bc) = abT(c) \quad (a, b, c \in L^1(G)). \tag{1}$$

Since $L^1(G)$ is approximately locally unital, there are subsets A_l and A_r of $L^1(G)$ such that $L^1(G)$ is the closed linear span of both A_l and A_r and each element of A_l and A_r has a left identity and a right identity in $L^1(G)$, respectively. Now take

$b \in A_l$ and $e \in L^1(G)$ with $eb = b$. Then $(e-1)bc = 0$, and so, by the assumption on T, $(e-1)T(bc) = 0$. Thus $eT(bc) = T(bc)$. This, together with (1), shows that

$$T(bc) = eT(bc) = eb(Tc) = bT(c) \quad (b \in A_l, c \in L^1(G)).$$

Therefore the restriction of T to $L^1(G)$ is a right multiplier, since $L^1(G)$ is the closed linear span of A_l. That is

$$T(bc) = bT(c) \quad (b, c \in L^1(G)). \tag{2}$$

Now let $c \in L^1(G)^\sharp$, $b \in A_r$ and $e \in L^1(G)$ with $be = b$. Since $b(ec - c) = 0$, by the assumption on T, $bT(ec - c) = 0$. Therefore, from (2),

$$\begin{aligned} T(bc) &= T(bec) \\ &= bT(ec) \\ &= bT(ec-c) + bT(c) \\ &= bT(c). \end{aligned}$$

Hence, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} T[(b+\lambda)c] &= T(bc + \lambda c) \\ &= bT(c) + \lambda T(c) \\ &= (b+\lambda)T(c). \end{aligned}$$

The final result follows from the fact that $L^1(G)$ is the closed linear span of A_r.

(ii) follows by a similar argument in (i).

(iii) Let $D\colon L^1(G)^\sharp \to X$ be a bounded hyperlocal operator with respect to $L^1(G)^\sharp$-bimodule actions. It is clear that the restriction of D to $L^1(G)$ is hyperlocal with respect to $L^1(G)$-bimodule actions. Thus, by Theorem 4.14 and a similar argument to the one made in the proof of Theorem 4.4(ii), we have

$$e[D(acb) - D(ac)b - aD(cb) + aD(c)b]d = 0 \ (a, b, c, d, e \in L^1(G)). \tag{3}$$

Fix $a \in A_l$ and $e \in L^1(G)$ with $ea = a$. Define the bounded operator $K\colon L^1(G)^\sharp \to X$ by

$$K(c) = (e-1)D(ac).$$

Take $c, d \in L^1(G)^\sharp$ such that $cd = 0$. Since $(e-1)ac = eac - ac = 0$ and D is hyperlocal, we have

$$K(c)d = (e-1)D(ac)d = 0.$$

Hence, from (ii), K is a left multiplier. In particular, $K(cb) = K(c)b$ for all $c, b \in L^1(G)$. Therefore,

$$\begin{aligned} eD(acb) - D(acb) &= K(cb) \\ &= K(c)b \\ &= eD(ac)b - D(ac)b. \end{aligned}$$

Consequently

$$D(acb) - D(ac)b = e[D(acb) - D(ac)b].$$

This, together with (3), shows that for every $d \in L^1(G)$,

$$\begin{aligned} [D(acb) - D(ac)b]d &= e[D(acb) - D(ac)b]d \\ &= e[aD(cb) - aD(c)b]d \\ &= [aD(cb) - aD(c)b]d. \end{aligned}$$

However, $L^1(G)$ is the closed linear span of A_l. Hence

$$[D(acb) - D(ac)b - aD(cb) + aD(c)b]d = 0 \ (a, b, c, d \in L^1(G)). \tag{4}$$

Fix $b \in A_r$ and $d \in L^1(G)$ with $bd = b$. Define the bounded operator $K'\colon L^1(G)^\sharp \to X$ by

$$K'(c) = D(cb)(1-d).$$

By going through similar steps as in the preceding part, with (ii) instead of (i) and K' instead of K, and by using the fact that $L^1(G)$ is the closed linear span of A_r, we can show that the factor d in (4) can be deleted as well. Therefore

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0 \quad (a, b, c \in L^1(G)). \tag{5}$$

Now let $a \in A_r$ and $e \in L^1(G)$ with $ae = a$. Define the bounded operator $S\colon L^1(G)^\sharp \to X$ by

$$S(c) = aD(ec - c).$$

Let $c, d \in L^1(G)^\sharp$ such that $cd = 0$. Then $a(ec - c) = (ec - c)d = 0$, and so, since D is hyperlocal, $S(c)d = aD(ec - c)d = 0$. Therefore, by part (ii), S is a left multiplier. In particular, $S(cb) = S(c)b$ for every $b \in L^1(G)$ and $c \in L^1(G)^\sharp$. Hence

$$\begin{aligned} aD(ecb) - aD(cb) &= S(cb) \\ &= S(c)b \\ &= aD(ec)b - aD(c)b. \end{aligned}$$

Consequently

$$aD(ecb) - aD(ec)b = aD(cb) - aD(c)b.$$

This, together with (5), shows that

$$\begin{aligned} D(acb) - D(ac)b &= D(aecb) - D(aec)b \\ &= aD(ecb) - aD(ec)b \\ &= aD(cb) - aD(c)b. \end{aligned}$$

However, $L^1(G)$ is the closed linear span of A_r. Hence, for all $a, b \in L^1(G)$ and $c \in L^1(G)^\sharp$,

$$D(acb) - D(ac)b - aD(cb) + aD(c)b = 0. \tag{6}$$

Finally, we note that the equality in (6) holds if we let a or b be any scalar $\lambda \in \mathbb{C}$. □

THEOREM 4.17. *Let G be a SIN or a totally disconnected group. Then, for any Banach $L^1(G)$-bimodule X and $n \in \mathbb{N}$, every bounded approximately local n-cocycle from $L^1(G)^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{Z}^n(A,X)$ is reflexive.*

PROOF. From the proof of Theorem 3.40, $L^1(G)$ is approximately locally unital whenever G is a SIN or a totally disconnected group. Therefore the result follows from Lemma 4.16 and Theorem 4.3. □

THEOREM 4.18. *Let G be a locally compact group, and let X be an essential Banach $L^1(G)$-bimodule. Then a bounded operator D from $L^1(G)$ into X^* is hyperlocal if and only if there is a derivation $\mathcal{D}$ and a right multiplier T from $L^1(G)$ into X^* such that $D = \mathcal{D} + T$. In particular, D is a bounded derivation if and only if weak* $- \lim_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in $L^1(G)$.*

PROOF. The proof is similar to the one made in Corollary 4.6. □

COROLLARY 4.19. *Let G be a locally compact group. Then G is amenable if and only if for any essential Banach $L^1(G)$-bimodule X and every bounded hyperlocal operator $D\colon L^1(G) \to X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in L^1(G))$.*

PROOF. By Johnson's theorem, G is amenable if and only if $L^1(G)$ is amenable. Hence the result follows from a similar argument to the one made in Corollary 4.7. □

Corollary 4.20. *Let G be a locally compact group. Then for every bounded hyperlocal operator $D\colon L^1(G) \to L^1(G)^*$, there are $x^*, y^* \in L^1(G)^*$ such that $D(a) = ax^* - y^*a$ $(a \in L^1(G))$.*

Proof. The result follows from a similar argument to the one made in Corollary 4.7 together with the fact that $L^1(G)$ is weakly amenable [**10**, Theorem 5.6.48]. □

4.6. Approximately local n-cocycles from Figà-Talamanca-Herz algebras

In the final section of this chapter, we state some of the major results that we can obtain for approximately local n-cocycles and hyperlocal operators from Figà-Talamanca-Herz algebras.

Theorem 4.21. *Let G be a locally compact group such that G_e is abelian, let $p \in (1, \infty)$, and let X be a Banach $A_p(G)$-bimodule. Then, for $n \in \mathbb{N}$, every bounded approximately local n-cocycle T from $A_p(G)^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{Z}^n(A_p(G), X)$ is reflexive.*

Proof. It follows immediately from Theorem 2.23 and Theorem 4.5. □

Corollary 4.22. *Let G be a locally compact amenable group such that G_e is abelian, let $p \in (1, \infty)$, and let X be a Banach $A_p(G)$-bimodule. Then a bounded operator $D\colon A_p(G) \to X^*$ is hyperlocal if and only if there is a bounded derivation $\mathcal{D}$ and a right multiplier T from $A_p(G)$ into X^* such that $D = \mathcal{D} + T$. In particular, D is a derivation if and only if weak$^*-\lim\limits_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in $A_p(G)$.*

PROOF. If G is amenable, then $A_p(G)$ has a bounded approximate identity [**42**, Theorem 4.10]. Hence the result follows from Theorem 2.23 and Corollary 4.6. □

COROLLARY 4.23. *Let G be a locally compact group such that it has an abelian subgroup of finite index, and let $p \in (1, \infty)$. Then, for any essential Banach $A_p(G)$-bimodule X and every bounded hyperlocal operator $D\colon A_p(G) \to X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in A_p(G))$.*

PROOF. It is well-known that if G has an abelian subgroup of finite index, then $A_p(G)$ is amenable (e.g. [**16**] or [**34**]). Thus the result follows from Theorem 2.23 and Corollary 4.7. □

CHAPTER 5

Quantized hyper-Tauberian algebras

In this chapter, we consider the quantization of the results we obtained in the previous chapters. We first look at quantized hyper-Tauberian algebras and deduce the quantized results we obtained in Chapters 2, 3 and 4. Then we apply them to the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G for $p \in (1, \infty)$. We show that $A_p(G)$ is a quantized hyper-Tauberian algebra. This, in particular, shows that $A_p(G)$ is operator weakly amenable. It also shows that every finite subset of G is a set of synthesis for $A_p(G)$ and completely bounded approximately local n-cocycles from $A_p(G)$ are n-cocycles.

5.1. Definition and basic properties

In this section we study the quantized theory of hyper-Tauberian algebras. We start with the following definition which is the natural quantization of Definition 2.5.

DEFINITION 5.1. Let A be a commutative semisimple regular quantized Banach algebra. We say that A is a *quantized hyper-Tauberian algebra* if every completely bounded local operator from A into A^* is a multiplier.

It is straightforward to verify that the analogue properties of hyper-Tauberian algebras and hyperlocal operators which we studied in Section 2.3, Section 2.4 and Section 4.3 hold also for quantized hyper-Tauberian algebras. We summarize some of them in the following theorems and corollaries:

THEOREM 5.2. *Let A be a quantized hyper-Tauberian algebra. Then:*

(i) A is Tauberian;

(ii) every finite subset of Φ_A is a set of synthesis for A;

(iii) A is operator weakly amenable;

(iv) a closed ideal I in A is quantized hyper-Tauberian if and only if $I = \overline{I_0(E)}$ for some closed subset E of Φ_A;

(v) a closed subset E of Φ_A is a set of synthesis (local synthesis) for A if and only if $I(E)$ $(J(E))$ is quantized hyper-Tauberian;

(vi) $A^\sharp$ is quantized hyper-Tauberian.

PROOF. We note that if $\varphi \in A^*$, then, by [**19**, Corollary 2.2.3], $||\varphi||_{cb} = ||\varphi||$. Thus, for every $f \in A^*$, the operator $S\colon A \longrightarrow A^*$ defined by $S(a) = \varphi(a)f$ $(a \in A)$ is completely bounded. Hence (i)-(iv) follow from the quantized versions of Theorem 2.6, Theorem 2.8, and Corollary 2.9. Finally, it is shown in [**19**, Chapter 3] that there is an operator space structure on $A^\sharp$ such that the inclusion map $\iota\colon A \longrightarrow A^\sharp$ is completely bounded. Therefore a simple calculation shows that $A^\sharp$ is a quantized Banach algebra, and so, by the quantized version of Corollary 2.11, $A^\sharp$ is quantized hyper-Tauberian. $\square$

THEOREM 5.3. *Let A be a quantized hyper-Tauberian algebra, and let X and Y be essential quantized A-bimodules. Then a completely bounded operator $D\colon X \longrightarrow Y^*$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0$$

for all $a, b \in A$ and $x \in X$. If A has a bounded approximate identity, then the result is also true for all completely bounded hyperlocal operators from X into Y.

PROOF. It follows from the quantized version of Theorem 4.4. $\square$

In Theorem 4.5 we showed that bounded approximately local n-cocycles from hyper-Tauberian algebras are n-cocycles. The following theorem states the quantized version of that result. So, in particular, completely bounded approximately local derivations from quantized hyper-Tauberian algebras are derivations.

THEOREM 5.4. *Let A be a quantized hyper-Tauberian algebra, and let X be a quantized A-bimodule. Then, for $n \in \mathbb{N}$, every completely bounded approximately local n-cocycle T from $A^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{OZ}^n(A, X)$ is reflexive.*

PROOF. It follows from Theorem 5.2(v), Theorem 5.3 and the quantized version of Theorem 4.5. □

COROLLARY 5.5. *Let A be a quantized hyper-Tauberian algebra with a bounded approximate identity, and let X be an essential quantized A-bimodule. Then a completely bounded operator $D\colon A \longrightarrow X^*$ is hyperlocal if and only if there is a completely bounded derivation $\mathcal{D}$ and a right multiplier T from A into X^* such that $D = \mathcal{D} + T$. In particular, D is a derivation if and only if weak$^* - \lim_{\alpha\to\infty} D(e_\alpha) = 0$ for a bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ in A.*

PROOF. It follows from Theorem 5.3 and the quantized version of Corollary 4.6. □

COROLLARY 5.6. *Let A be a quantized hyper-Tauberian algebra with a bounded approximate identity. Then A is operator amenable if and only if for any essential quantized A-bimodule X and every completely bounded hyperlocal operator $D\colon A \longrightarrow X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ $(a \in A)$.*

PROOF. It follows from Theorem 5.3 and the quantized version of Corollary 4.7. □

5.2. Figà-Talamanca-Herz algebras as quantized hyper-Tauberian algebras

Let G be a locally compact group. Since $VN(G) \subset B(L^2(G))$ is an operator space, $A(G)$, regarded as the operator predual of $VN(G)$, has a natural operator space structure which makes it a completely contractive Banach algebra [**19**, Chapter 16]. Now let $p \in (1, \infty)$ and suppose that there is an operator space structure on the Figà-Talamanca-Herz algebra $A_p(G)$ such that it turns $A_p(G)$ into a quantized Banach algebra and $A_p(G)$ becomes a quantized $A(G)$-module (In [**34**], one such operator space structure has been constructed on $A_p(G)$). In this case, we can show that $A_p(G)$ is a quantized hyper-Tauberian algebra.

THEOREM 5.7. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then $A_p(G)$ is a quantized hyper-Tauberian algebra.*

PROOF. We first show that $A(G)$ is quantized hyper-Tauberian. It is shown in [**20**] that there is a complete isometry

$$A(G)\widehat{\otimes}_{op}A(G) \cong A(G \times G).$$

This map is also an algebraic isomorphism. Thus $A(G)\widehat{\otimes}_{op}A(G)$ is semisimple, and so, since the diagonal Δ is a closed subgroup of $G \times G$, by [**53**, Theorem 3], Δ is a set of synthesis for $A(G)\widehat{\otimes}_{op}A(G)$. Hence the result follows from the quantized version of Theorem 2.7. The final result follows from the quantization of the argument made in the proof of Theorem 2.23. □

COROLLARY 5.8. *Let G be a locally compact group, and let $p \in (1,\infty)$. Then $A_p(G)$ is operator weakly amenable.*

PROOF. The result follows from Theorem 5.7 and part (iii) of Theorem 5.2. □

THEOREM 5.9. *Let G be a locally compact group, let $p \in (1,\infty)$, and let X and Y be essential quantized $A_p(G)$-bimodules. Then a completely bounded operator $D\colon X \to Y^*$ is hyperlocal if and only if*

$$D(axb) - aD(xb) - D(ax)b + aD(x)b = 0$$

for all $a, b \in A_p(G)$ and $x \in X$. Moreover, if G is amenable, then the result is also true for all completely bounded hyperlocal operators from X into Y.

PROOF. It follows from Theorem 5.7 and Theorem 5.3. □

THEOREM 5.10. *Let G be a locally compact group, let $p \in (1,\infty)$, and let X be a quantized $A_p(G)$-bimodule. Then, for $n \in \mathbb{N}$, every completely bounded approximately local n-cocycle T from $A_p(G)^{(n)}$ into X is an n-cocycle. In particular, $\mathcal{OZ}^n(A_p(G), X)$ is reflexive.*

PROOF. It follows from Theorem 5.7 and Theorem 5.4. □

COROLLARY 5.11. *Let G be a locally compact amenable group, and let $p \in (1,\infty)$. Then, for every essential quantized $A_p(G)$-bimodule X and every completely bounded hyperlocal operator $D\colon A_p(G) \to X^*$, there are $x^*, y^* \in X^*$ such that $D(a) = ax^* - y^*a$ for all $a \in A_p(G)$.*

PROOF. It is shown in [34] that G is amenable if and only if $A_p(G)$ is operator amenable. Therefore the result follows from Corollary 5.6. □

Bibliography

[1] B. A. Barnes, *Examples of modular annihilator algebras*, Rocky. Mountain J. Math. **1** (1971) 657-665.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, New York, Springer-Verlag 1973.

[3] M. Brešar and P. P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivatioins*, Can. J. Math. **45** (1993) 483-496.

[4] J. T. Burnham, *Closed ideals in subalgebras of Banach algebras*, I. Proc. Amer. Math. Soc. **32** (1972), 551-555.

[5] John B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York 1973.

[6] John B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1990.

[7] J. Cuntz, *On the continuity of seminorms on operator algebras.* Math. Ann. **220** (1976), 171-183.

[8] P. C. Curtis and R. J. Loy, *The structure of amenable Banach algebras*, J. London Math. Soc. (2) **40** (1989) 89-104.

[9] Randel L. Crist, *Local derivations on operator algebras*, Journal of Functional Analysis, **135** (1996) 76-92.

[10] H. G. Dales, *Banach algebras and automatic continuity*, New York, Oxford University Press, 2000.

[11] H. G. Dales, F. Ghahramani and A. Ya. Helemskii, *The amenability of measure algebras*, J. London Math. Soc. (2) **66** (2002) 213-226.

[12] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964).

[13] B. Forrest, *Fourier analysis on coset spaces*, Rocky Mountain J. Math. **28** (1998), no. 1, 173-190.

[14] B. Forrest, *Amenability and bounded approximate identities in ideals of $A(G)$*, Illinois J. Math. **34** (1990) 1-25.

[15] B. Forrest, *Amenability and the structure of the algebras $A_p(G)$*, Trans. Amer. Math. Soc, **343**(1994), 233-243.

[16] B. E. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra*, Math. Z. to appear.

[17] B. E. Forrest and P. J. Wood, *Cohomology and the operator space structure of the Fourier algebra and its second dual*, Indiana Math. J. **50** (2001), 1217-1240.

[18] P. Eymard, *L'algebra de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236.

[19] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Math. Soc. Monograghs, New series, vol. 23, Oxford University Press, New York, 2000.

[20] E. G. Effros and Z.-J. Ruan, *On approximation properties for operator spaces*, Internat. J. Math. **1** (1990), 163-187.

[21] F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Functional Analysis **208** (2004), 229-260.

[22] N. Groenbaek, *A characterization of weakly amenable Banach algebras*, Studia Math. **94** (1989), 149-162.

[23] S. Grosser and M. Moskowitz, *Compactness condition in topological groups*, J. Reine Angew. Math. **246** (1971), 1-40.

[24] C. S. Herz, *Harmonic synthesis for subgroups.* Ann. Inst. Fourier, Grenoble **23** (1973), no. 3, 91-123.

[25] C. S. Herz, *The theorem of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. **154** (1971) 69-82.

[26] D. Hadwin and J. Li, *Local derivations and local automorphism*, J. Math. Anal. Appl. **290** (2004) 702-714.

[27] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, I. Springer Verlage, 1963.

[28] B. E. Johnson, *Local derivations on C^*-algebras are derivations*, Trans. Amer. Math. Soc, **353** (2000), 313-325.

[29] B. E. Johnson, *The derivation problem for group algebras of connected locally compact groups*, J. London Math. Soc. (2) **63** (2001) 441-452.

[30] B. E. Johnson, *Non-amenability of the Fourier algebra of a compact group*, J. London Math. Soc. (2) **50** (1994) 361-374.

[31] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math., **90** (1968) 1067-1073.

[32] B. E. Johnson and M. C. White, *A non-weakly amenable augementation ideal*, preprint.

[33] R. V. Kadison, *Local derivations*, J. Algebra **130** (1990), 494-509.

[34] A. Lambert, M. Neufang and V. Runde, *Operator space structure and amenability for Figà-Talamanca-Herz algebras*, J. Functional Analysis **211** (2004) 245-269.

[35] David R. Larson, *Reflexivity, algebraic reflexivity and linear interpolation*, Amer. J. Math. **110** (1988), 283-299.

[36] D. Larson and A. Sourour, *Local derivations and local automorphisims of $B(X)$*, Proc. Sympos. Pure Math. **51** (1990), 187-194.

[37] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), 1180-1182.

[38] L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, Van Nostrand, 1953.

[39] V. Losert, *On tensor products of Fourier algebras*, Arch. Math. **43** (1984), 370-372.

[40] V. Losert, *The derivation problem for group algebras*, preprint.

[41] T. W. Palmer, *Banach algebras and general theorem of *-algebras*, II. Cambrige University Press, 2001.

[42] J. P. Pier, *Amenable Locally Compact Groups*. Wiley-Interscience, New York, 1984.

[43] H Reiter, *L^1-Algebras and Segal Algebras*, Lecture Notes in Mathematics 231, Springer-Verlag, (1971).

[44] C. E. Rickart, *General theory of Banach algebra*, Princeton, N.J., Van Nostrand, 1960.

[45] John R. Ringrose, *Compact non-self-adjoint operators*, London, Van Nostrand, 1971.

[46] Zhong-Jin Ruan, *The operator amenability of $A(G)$*, Amer. J. Math. **117** (1995), no. 6, 1449–1474.

[47] W. Rudin, *Real and complex analysis*, New York, McGraw-Hill, 1987.

[48] J. Schweizer, *An analogue of Peetr's theorem in non-commutative topology*, Quart. J. Math. **52** (2001) 499-506.

[49] A. M. Sinclair, *Automatic continiuity of linear operators*, London Mathemtical Society Lecture Note Series, NO 21, Cambridge University Press, 1976.

[50] V. Shulman, *Operators preserving ideals in C^*-algebras*, Studia Math. **109** (1994) 67-72.

[51] N. Spronk, *Operator weak amenability of the Fourier algebra*, Proc. Amer. Math. Soc, **130** (2002), 3609-3617.

[52] J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J. (**2**) **12** (1960), 143-154.

[53] M. Takesaki and N. Tatsuuma, *Duality and subgroups. II*, J. Func. Anal. **11** (1972) 184-190.

[54] George A. Willis, *Factorization in finite-codimensional ideals of group algebras*, Proc. London Math. Soc. (3) **82** (2001), no. 3, 676-700.