t-blocking sets and semi-ovals in the Witt designs

by

Xiaomin Bao

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

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T-BLOCKING SETS AND SEMICVALS IN THE WITT DESIGNS

BY

XIAOMIN BAO

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

We analyze the Witt designs and obtain a number of new results concerning their structures. We characterize all \( t \)-blocking sets and semiovals in the Witt designs, and determine the possible sizes of a \( t \)-blocking set, classifying them by their frequency vectors. We prove that there are only three types of semiovals in \( S(3,6,22) \), one of them has size nine, the other two have size ten; \( S(5,6,12) \) and \( S(4,7,23) \) each have only one type of semioval; but both \( S(4,5,11) \) and \( S(5,8,24) \) have no semioval at all.


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1. INTRODUCTION

First of all we introduce some terminology and notations which will be used in the sequel.

**Definition 1.1.** A $t$-design $\mathcal{D}$ is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a $v$-set of elements called points. $\mathcal{B}$ is a family of $k$-sets of $\mathcal{V}$ called blocks, such that any fixed $t$-set of $\mathcal{V}$ is contained in exactly $\lambda$ elements of $\mathcal{B}$. We also say that $\mathcal{D}$ is a $t-(v, k, \lambda)$ design.

When $\lambda = 1$, the design is called a Steiner system and is written $S(t, k, v)$.

A $t$-design is also a $s$-design for $s \leq t$ [56. pp. 2].

Let $b$ denote the number of blocks in a $t$-design. Then for any 2-design we have $v \leq b$ (Fisher's inequality [56. pp. 4]).

A design with $v = b$ is called a symmetric design.

Suppose $\mathcal{D}$ is a $t-(v, k, \lambda)$ design with blocks $B_1, B_2, \ldots, B_b$. The number $|B_i \cap B_j|, i \neq j$, is called an intersection number of $\mathcal{D}$. Assume that $x_1, x_2, \ldots, x_s$ are the distinct intersection numbers of $\mathcal{D}$. The $x_i$'s and the number $s$ sometimes provide very useful information about the design.

A symmetric 2-design can be characterized by the property that the design has precisely one intersection number [13. pp. 87].

A quasi-symmetric design is just a design with at most two intersection numbers $x$ and $y$ ($x < y$).

**Definition 1.2.** Let $\Gamma$ be a finite graph on $v$ vertices. The degree or valency of a vertex $x$ is the number of edges on $x$. If each vertex $x$ has the same degree $d$, then the graph is said to be regular of degree $d$. 
Let $\Gamma$ be a regular graph of degree $d$, with $v$ vertices. If

1. any two adjacent vertices are simultaneously adjacent to a other vertices.
2. any two non-adjacent vertices are simultaneously adjacent to b other vertices.

then $\Gamma$ is called a strongly regular graph with parameters $(v, b, a, d)$.

**Definition 1.3.** Let $\mathcal{D}$ be a design, and $V$ and $\mathcal{B}$ be the point set and block set of $\mathcal{D}$ respectively. Let $x \in V$. Define:

$$B_x = \{ B - \{x\} : x \in B, B \in \mathcal{B}\}, V_x = V \cap \bigcup_{X \in B} X.$$  

The pair $(V_x, B_x)$ is called the contraction of $\mathcal{D}$ at $x$ and we denote it by $\mathcal{D}_x$.

If $\mathcal{D}$ is a $t$-design, and $\xi$ is a $(t - 1)$-design such that $\xi \supseteq \mathcal{D}_x$, then $\mathcal{D}$ is called an extension of $\xi$.

Suppose $\mathcal{D}$ is a design, $B$ is a block of $\mathcal{D}$. Define:

$$\mathcal{D}^B = (V - B, B - \{B\}).$$

where $B - \{B\} := \{ A - B : A \in B \}$. $\mathcal{D}^B$ is called the residual of $\mathcal{D}$ with respect to the block $B$. If $\mathcal{D}_1$ is a design, and $\mathcal{D}_1 \cong \mathcal{D}^B$, then $\mathcal{D}_1$ is said to be embedded in $\mathcal{D}$.

Suppose $\mathcal{D}$ is a quasi-symmetric design with two intersection numbers $x$ and $y (x < y)$. The block graph of $\mathcal{D}$ is the graph whose vertices are the blocks of $\mathcal{D}$; two vertices are adjacent whenever the corresponding blocks intersect in $y$ points.

The following five Steiner systems are called Witt designs:

$$S(4,5,11), S(5,6,12), S(3,6,22), S(4,7,23), S(5,8,24).$$

Witt designs are very important in combinatorial design theory. They provide good examples of the extensions of designs, quasi-symmetric designs, and strongly regular graphs.
Example 1. [52, pp. 35] $S(3.6.22)$ is an extension of $PG(2,4)$. $S(3.6.22)$ is a quasi-symmetric design with $x = 0$, $y = 2$.

Example 2. [52, pp. 35] $S(4.7.23)$ is an extension of $S(3.6.22)$, and it is a two-fold extension of $PG(2,4)$. It is a quasi-symmetric design with $x = 1$ and $y = 3$.

By theorem 3.2 in [19] we know that both the block graphs of $S(3.6.22)$ and $S(4.7.23)$ are strongly regular graphs.

Witt designs also provide examples of Steiner systems with $t \geq 4$. Beside the Witt designs, there are only eight known examples of Steiner systems with $t > 4$: $S(5.6.24)$, $S(5.7.28)$, $S(5.6.48)$, $S(5.6.84)$, $S(5.6.72)$, $S(5.6.108)$, $S(5.6.132)$ and $S(5.6.168)$ [22]. Up to now there are no examples of Steiner systems with $t \geq 6$ [13]. From [29, 37, 17] we know that the only non-trivial quasi-symmetric 4-designs are $S(4.7.23)$ and its complement. Shrikhande and Sane [52] conjectured that the only non-trivial quasi-symmetric 3-designs (other than the Hadamard 3-designs) are the 3-designs related to the Witt designs or their complements. Witt designs have close relation with group theory. The automorphism groups of $S(4.5.11)$, $S(5.6.12), S(3.6.22), S(4.7.23)$ and $S(5.8.24)$ are the famous Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$ respectively. The five Mathieu groups were the first discovered sporadic simple groups. Historically, statisticians were the first to make a systematic and exhaustive study of block designs, particularly from the point of view of construction. In Shrikhande and Sane’s words [52] “From the combinatorialist’s point of view, a substantial portion of the research work in design theory centers around various characterizations of the Witt designs by their properties.”

The attempt to prove the existence and uniqueness of the Witt designs produced
many techniques and methods, which also enhanced the theory of combinatorial designs.

The Witt designs were considered to be first constructed by Witt [57] and Carmichael [21].

Witt constructed the five Witt designs from the five Mathieu groups. He proved that $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$ are the automorphism groups of $S(4.5.11), S(5.6.12), S(3.6.22), S(4.7.23)$ and $S(5.8.24)$ respectively. Witt [58] also proved the uniqueness of these designs.

To construct $S(5.6.12)$, Carmichael considered the linear fractional group $G$ modulo 11 of order $12 \cdot 11 \cdot 5 = 660$. $G$ is a doubly transitive group of degree 12 on $S = \{ \infty, 0, 1, 2, \ldots, 10 \}$. Select $B \subseteq S$. $|B| = 6$, such that $B$ is transformed into itself by just five elements of $G$. Let

$$B = \{ g(B) \mid g \in G \}.$$ 

then $|B| = 132$ and $(S, B)$ is an $S(5.6.12)$. Let $B_\infty = \{ A - \{ \infty \} \mid \infty \in A \subseteq B \}$. then $|B_\infty| = 66$ and $(S - \{ \infty \}, B_\infty)$ is an $S(4.5.11)$.

Now let $G$ be the linear fractional group modulo 23 of order $24 \cdot 23 \cdot 11 = 6072$ and let $S = \{ \infty, 0, 1, 2, \ldots, 22 \}$. $G$ contains a subgroup $H$ of order 8, such that $h(B) = B$ for all $h \in H$, where $B = \{ \infty, 0, 1, 12, 15, 21, 22 \}$. Let

$$B = \{ g(B) \mid g \in G \}.$$ 

Then $(S, B)$ is an $S(5.8.24)$.

In 1969, Lüneburg [47] constructed the Witt designs by extending known structures. In his proof, Lüneburg considered the geometry of the affine plane over $GF(3)$ (for $S(4.5.11)$ and $S(5.6.12)$) and the geometry of the projective plane
over $GF(4)$ (for $S(3.6.22)$, $S(4.7.23)$ and $S(5.8.24)$). Other proofs of the uniqueness of $S(3.6.22)$, $S(4.7.23)$ and $S(5.8.24)$ were given by Jónsson [40] and Iwasaki [39]. Jónsson's proof was based on the geometric aspect of an elementary abelian subgroup of order 16 and a knowledge of the geometries associated with certain subgroups of the alternating group $A_8$. Iwasaki's proof was based on the fact that any two blocks intersect in 0 or 2 points, in 1 or 3 points or in 0, 2 or 4 points, respectively. From this the blocks can be determined explicitly.

Before Lüneburg, L.J. Paige [50] constructed $S(4.7.23)$ and $S(4.5.11)$. His construction of $S(4.7.23)$ is as follows.

Let $V(23)$ be a vector space of dimension 23 over $GF(2)$, let

$$v_1 = (0,0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,0,0,0,0).$$
$$v_2 = (1,1,1,1,1,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0).$$
$$v_3 = (1,1,1,0,0,0,1,1,1,0,0,0,1,0,0,0,0,0,0,0).$$
$$v_4 = (1,0,0,1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0).$$
$$v_5 = (0,1,0,1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0).$$
$$v_6 = (0,0,1,0,1,1,0,1,1,0,0,0,0,0,0,1,0,0,0,0).$$
$$v_7 = (1,1,0,0,1,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0).$$
$$v_8 = (1,0,1,0,0,1,1,0,0,1,1,0,0,0,0,0,0,0,0,0).$$
$$v_9 = (0,1,1,1,0,0,0,1,0,1,1,0,0,0,0,0,0,0,0,0).$$
$$v_{10} = (1,0,0,1,0,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0).$$
$$v_{11} = (0,1,0,0,1,1,1,0,0,1,0,0,0,0,0,0,0,0,0,0).$$
$$v_{12} = (0,0,1,1,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,0).$$
Let $T = \langle v_1, v_2, \ldots, v_{12} \rangle$. Then $T$ is a subspace of $V(23)$. For any given vector $v = (t_1, t_2, \ldots, t_{23}) \in T$, define:

$$H_v = \{ i \mid 1 \leq i \leq 23, t_i = 0 \}.$$ 

let $\mathcal{B} = \{ H_v \mid v \in T, H_v = 7 \}, \mathcal{V} = \{1, 2, \ldots, 23\}$. Then $(\mathcal{V}, \mathcal{B})$ is an $S(4.7.23)$.

$S(4.5.11)$ can be constructed similarly by considering $V(11)$ over $GF(3)$.

In order to prove the existence of $S(5.8.24)$, Curtis [26] considered the power set $\mathcal{P}(\Omega)$ of a 24-set $\Omega$ as a linear space of dimension 24 over $GF(2)$, where the sum of two subsets is defined to be their symmetric difference. Choose a subspace (dimension 12) $\varphi$ of $\mathcal{P}(\Omega)$, such that the smallest cardinality of the elements in $\varphi$ is 8 (Curtis called this kind of element an octad) and $\varphi$ contains exactly 759 octads. The set of all octads is the set of blocks of $S(5.8.24)$. In his paper Curtis also introduced the so called “Miracle Octad Generator” or MOG. The MOG provided a convenient computational device for finding the block containing any five points.

Around 1980, starting from the designs $L_{11}$ and $L_{12}$ associated with the Hadamard matrix of order 12, Hughes [34] constructed $S(4.5.11)$ and $S(5.6.12)$ by an elementary technique. Before long, based on Hughes' ideas, Beth [11] gave a simpler method to construct $S(5.6.12).$ He also gave a proof of the uniqueness of $S(5.6.12).$ In 1981, Beth and Jungnickel [12] gave another method to construct $S(4.5.11)$ and $S(5.6.12)$.

Cameron [18], using some of the properties of $S(4.7.23)$ and $S(5.8.24)$, proved the uniqueness of these two designs.

Cameron and Van Lint [20] discussed the structures of $S(5.6.12)$ and $S(5.8.24)$. 
Using the binary code of $PG(2, 4)$, Lander [43] constructed $S(3, 6, 22)$. Then from $S(3, 6, 22)$, constructed $S(5, 8, 24)$.

To study a design $D$ by specifying some of the intersection numbers $x_i$'s or the number $s$ of $D$ is an interesting research direction in design theory. We already know that if $s = 1$, then $D$ is a symmetric design; if $s = 2$, then $D$ is a quasi-symmetric design. Calderbank and Morton [17] classified all quasi-symmetric 3-designs with two intersection numbers $x, y$ and $x = 1$. The only nontrivial examples are $S(4, 7, 23)$ and its residual, a 3-(22, 7, 4) design. Ionin and Shrikhande [36] got the following characterizations of $S(4, 7, 23)$ and $S(5, 8, 24)$:

1. A $(2s - 1)$-design with $s$ intersection numbers is $S(5, 8, 24)$ if and only if $s \geq 3$ and $\sum_{i=1}^{s} x_i \leq s(s - 1)$.
2. A $2s$-design with $s$ intersection numbers is $S(4, 7, 23)$ if and only if $s \geq 2$ and $\sum_{i=1}^{s} x_i \leq s^2$.

The usual geometric construction of $S(4, 7, 23)$ starts from $PG(2, 4)$. Starting from $PG(3, 2)$ Baartmans [1] constructed $S(4, 7, 23)$.

Let $G = PSL(2, q)$ with $q = 11$ or 13. Iwasaki [38] considered the action of $G$ on a set $P$, and took all $G$-images of a subset $A$ in $P$ as blocks. To construct $S(5, 6, 12)$, he selected $A = Q := \{x^2 \mid x \in GF(11)\}$. To construct $S(5, 8, 24)$, he selected $A$ as the symmetric difference of the sets $Q$, $Q + 1$ and $Q - 4$ in $GF(23)$.

Lenz [45] gives a short uniqueness proof for $S(5, 8, 24)$, which immediately gives the order of $M_{24}$.

Let $PSU_6(2^2)$ act on the unitary polar geometry consisting of 693 points, 6237 totally isotropic lines and 891 totally isotropic planes. Jónsson and Mckay [41] proved that $M_{22}$ is a subgroup of $PSU_6(2^2)$ leaving invariant some configuration of
22 planes any two having a point in common, no three having a point in common.

$S(3, 6, 22)$ is obtained from this configuration of 22 planes as a set of 77 other planes called blocks, a block being called incident to one of the 22 planes if the corresponding planes have a line in common.

The monographs [13, 35, 52] all have complete self-contained discussions of Witt designs.

In [42] the action of the Mathieu groups $M_n, n = 22, 23, 24,$ on the power sets of the point set of respective Witt designs have been studied. The orbits of all point subsets of $S(3, 6, 22), S(4, 7, 23),$ and $S(5, 8, 24)$ together with the numbers $t_i$ of blocks which meet a point subset in a orbit at $i$ $(0 \leq i \leq 8)$ points in $S(5, 8, 24)$ and the formulas to calculate these numbers for the orbits in $S(3, 6, 22)$ and $S(4, 7, 23)$ have been given. The vectors $(t_0, \cdots, t_k)$, called frequency vectors, for $S(3, 6, 22), S(4, 7, 23),$ and $S(5, 8, 24)$ are listed in appendix A, B and C, respectively.

Witt designs also have relations with other interesting objects, such as Golay codes and the Leech lattice. Just as Hughes and Piper [35] described, they "are a fundamental feature of combinatorics and algebra".

Let $F$ be a finite field.

A code $C$ is a subset of $F^n$, the vectors in $C$ are called codewords. If $C$ is a subspace of dimension $k$, then $C$ is called a linear $[n, k]$ code. If $F = GF(2)$, then $C$ is called a binary code. If $F = GF(3)$, then $C$ is called a ternary code. The (Hamming) weight of a vector $v$ in $F^n$, denoted by $w(v)$, is the number of non-zero coordinates of $v$. The (Hamming) distance $d(x, y)$ between two codewords $x$ and $y$ is the number of coordinate positions in which they differ. Let $C$ be an $[n, k]$ code.
The (Hamming) distance $d$ of the code $C$ is

$$d = \min \{ d(x, y) \mid x, y \in C, x \neq y \}.$$  

An $[n,k]$ code with distance $d$ will be denoted by $[n,k,d]$. The support of a codeword is the set of coordinate positions where its entries are non-zero.

With these definitions we can see that Paige's construction of $S(4,7,23)$ mentioned above is actually to use the supports of the code $T$ to construct the blocks of $S(4,7,23)$.

Let $G_{24}$ be the code generated by the row vectors of the matrix $[12, B][33]$, where $12$ is the $12$ by $12$ identity matrix and

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Let $G_{23}$ be the code generated by the row vectors of the matrix $[12, \hat{B}]$, where $\hat{B}$ is the matrix obtained from $B$ by deleting the last column of $B$. 
Let $G_{12}$ be the code generated by the row vectors of the matrix $[I_6, C]$ [56], where $I_6$ is the 6 by 6 identity matrix, and

$$C = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & -1 & -1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
-1 & 1 & 0 & 1 & -1 & 1 \\
-1 & -1 & 1 & 0 & 1 & 1 \\
1 & -1 & -1 & 1 & 0 & 1
\end{bmatrix}$$

Let $G_{11}$ be the code generated by the row vectors of the matrix $[I_6, \hat{C}]$, where $\hat{C}$ is the matrix obtained from $C$ by deleting the last column of $C$.

The codes $G_{24}$ and $G_{23}$ are binary [24, 12, 8] and [23, 12, 7] codes respectively [33]; the codes $G_{12}$ and $G_{11}$ are ternary [12, 6, 6] and [11, 6, 5] codes respectively [56].

The codes $G_{24}$ and $G_{12}$ are called Extended Golay codes, while $G_{23}$ and $G_{11}$ are called Golay codes. Golay codes $G_{23}$ and $G_{11}$ were discovered by Golay [31]. Golay codes are very important. They have far-reaching implications for sphere packing and simple groups. The extended Golay code $G_{24}$ was used in the Voyager spacecraft program to transmit the colour pictures of Jupiter and Saturn.

The close relation between Witt designs and Golay codes can be illustrated by the following facts:

- Let $A$ be an incidence matrix of $S(5.8.24)$. Then $A^T$ is a generating matrix for $G_{24}$[35. pp. 222-229]. The supports of the codewords of minimum weight (weight 8) of $G_{24}$ form an $S(5.8.24)$[48. pp. 634].
• The supports of the codewords of minimum weight (weight 7) of $G_{23}$ form an $S(4.7.23)$. $G_{23}$ may be obtained by deleting any coordinate of $G_{24}$.[48, pp. 634-635]

• The supports of the codewords of minimum weight (weight 6) of $G_{12}$ form an $S(5.6.12)[48, pp. 635]$.

• The supports of the codewords of minimum weight (weight 5) of $G_{11}$ form an $S(4.5.11)[48, pp. 635]$.

The Leech lattice $\mathcal{L}_{24}$ consists of the vectors [25]

$$\frac{1}{\sqrt{8}}(0 + 2c - 4x)$$

$$\frac{1}{\sqrt{8}}(1 + 2c + 4y)$$

where $0 = (0, 0, \ldots, 0), 1 = (1, 1, \ldots, 1), c \in G_{24}$ (the components of $c$ are regarded as real 0's and 1's) and $x, y \in \mathbb{Z}^{24}$. satisfy $\sum x_i \equiv 0(\text{mod } 2), \sum y_i \equiv 1(\text{mod } 2)$.

The Leech lattice $\mathcal{L}_{24}$ was discovered by Leech in 1965 [44]. The Leech lattice $\mathcal{L}_{24}$ has a significant impact in the theory of finite simple groups. Conway [23] constructed three simple groups $Co_1$, $Co_2$ and $Co_3$ by using the Leech lattice. $Co_1$, $Co_2$ and $Co_3$ are all subgroups of $Co_0$, the group of automorphisms of the Leech lattice $\mathcal{L}_{24}$. The Leech lattice also provides a lattice packing in $\mathbb{R}^{24}$, which still stands as the densest known packing in $\mathbb{R}^{24}$ [25].

The purpose of this thesis is to characterize all $t$-blocking sets (see definition 3.1) and all semi-ovals (see definition 4.2) in the Witt designs up to the frequency vectors. In chapter 2 we obtain some results which correct some mistakes in [7, 8]. We also improve the results in [9]. In chapter 3 we thoroughly determine the structure of a Fano set - a fundamental structure used to characterize blocking sets. $t$-blocking
sets and semiovals in $S(3.6.22)$, and obtain a construction method for a Fano set. Then we characterize the $t$-blocking sets in Witt designs. We determine the possible size of a $t$-blocking set, classifying them by their frequency vectors. We also analyze the Witt designs and obtain a number of new results. In chapter 4, we characterize all semiovals in the Witt designs. We prove that there are only three types of semiovals in $S(3.6.22)$, one of them has size nine, the other two have size ten. one of them is also a $t$-blocking set: $S(5.6.12)$ and $S(4.7.23)$ each have only one type of semioval, which are also $t$-blocking sets; but both $S(4.5.11)$ and $S(5.8.24)$ have no semioval at all.
2. NOTES ON $S(3.6.22), S(4.7.23) AND S(5.8.24)$

2.1. Some definitions and notations.

Definition 2.1. A set of points of a Steiner system is called a blocking set if it contains no block, but intersects every block.

Definition 2.2. A blocking set $C$ is said to be of index $t$ if $C$ is contained in $t$ blocks, but can not be contained in $t - 1$ blocks. The index of $C$ is denoted by $i(C)$.

Definition 2.3. A blocking set $C$ is said to be irreducible if for any $x \in C$, the set $C - \{x\}$ is not a blocking set. Otherwise, $C$ is said to be reducible.

Definition 2.4. Let $X$ be a set of points of $S(t, k, \nu)$. Let $t_i$ be the number of blocks that are $i$-secant to $X$. $i = 0, 1, \ldots, k$. If $\{i_1, \ldots, i_n\}$ is the set of all those $i_j$'s ($0 \leq i_j \leq k$) such that $t_{i_j} \neq 0$ and $0 \leq i_1 < i_2 < \cdots < i_n$, then we say that $X$ is of type $(i_1, i_2, \ldots, i_n)$ and call $(t_0, t_1, \ldots, t_k)$ the frequency vector of $X$, denote it by $FV(X)$.

Definition 2.5. Let $X$ and $Y$ be two point subsets of $S(t, k, \nu)$. If $|X| = |Y|$ and $FV(X) = FV(Y)$, then we say that $X$ and $Y$ are the same type.

In the sequel discussion, $\mathcal{S}$ will be used to denote the point set of $S(t, k, \nu)$. Given any $t$ points $p_1, \ldots, p_t$ in $\mathcal{S}$, the unique block $B$ which contains $p_1, \ldots, p_t$ is also referred to as determined by $p_1, \ldots, p_t$.

Let $r_s(s = 0, 1, \ldots, t)$ be the number of blocks containing a fixed $s$-set, then

$$r_s = \frac{\binom{\nu - s}{t - s}}{\binom{k - s}{t - s}}$$
We have the following identities:

\[
\sum_{i=0}^{k} \binom{k}{i} t_i = r_i \binom{k}{s}, s = 0, 1, \ldots, t.
\]

If \( k - t \) of \( t_0, \ldots, t_k \) are given, then the rest can be determined by solving the above linear system.

We recall the following:

**Result 2.1.** [7. result 2.1] Let \( B \) and \( B' \) be two blocks in \( S(3, 6, 22) \). Then either \( B \cap B'' = 0 \) or \( B \cap B'' = 2 \).

**Result 2.2.** [7. lemma 2.2, lemma 2.7] Let \( B \) and \( B' \) be two blocks in \( S(3, 6, 22) \). If \( B \cap B'' = 2 \), then the type of \( B \cup B' \) is \( (0, 2, 3, 4, 6) \) with

\[
t_0 = 4, t_2 = 27, t_4 = 32, t_4 = 12, t_6 = 2.
\]

If \( B \cap B'' = 0 \), then the type of \( B \cup B' \) is \( (2, 4, 6) \) with

\[
t_2 = 30, t_4 = 45, t_6 = 2.
\]

**Result 2.3.** [7. lemma 5.1] Let \( B, B' \) be two blocks of \( S(3, 6, 22) \) with \( B \cap B'' = 2 \).

Denote by \( a, b \) two points of \( B' \cup B \). Let \( \mathcal{E} = \{E_1, E_2, E_3, E_4\} \) be the set of the four blocks external to \( B \cup B' \).

1. There exist two blocks \( S_1, S_2 \) which are 2-secant to \( B \cup B' \) at \( a \) and \( b \).

2. For every \( x \in S_1 \) (or \( S_2 \)), with \( x \neq a, b \), the two blocks \( E_i, E_j \) of \( \mathcal{E} \) through \( x \) have the other common point \( y \) on \( S_2 \) (or \( S_1 \)).

3. One of the two points outside \( B \cup B' \cup E_1 \cup E_3 \) is in \( S_1 \) and the other in \( S_2 \).

**Result 2.4.** [8. result 2.1] Every block in \( S(4, 7, 23) \) is a 7-set of type \( (1, 3, 7) \).
Result 2.5. [8. lemma 2.5] Let $B$, $B'$ be two blocks of $S(4.7.23)$ with $|B \cap B'| = 3$. Fix $x \in B - B'$ and $y \in B' - B$. There are exactly three blocks $D_1$, $D_2$ and $D_3$ intersecting $B \cup B'$ only at $x$ and $y$. Moreover $D_1 \cap D_2 \cap D_3 = \{x, y\}$.

Berardi [7. 8] defined the following sets:

1. Use the same notations of result 2.3. Fix $u \in B - B'$ and let $z$ be the only point of $S_2$ outside $B \cup B' \cup E_1 \cup E_2$. Define

   $$N_1 := (B \cup B' \setminus \{a, b, u\}) \cup \{x, z\}.$$ 

2. Let $B$, $B'$ and $B''$ be three blocks in $S(3.6.22)$ with $|B \cap B' \cap B''| = 1$. Fix a point $z$ of $B'' - (B \cup B')$. Let $y$ be one of the joint points of the two external blocks to $B \cup B' \cup \{z\}$. Define:

   $$N_0 := [(B \cup B') - B''] \cup \{y, z\}.$$ 

3. $E_0 := S - (B \triangle B')$, where $B$ and $B'$ are two blocks of $S(4.7.23)$ with $|B \cap B'| = 1$.

4. $E := (B - \{x\}) \cup (B' - \{y\} \cup \{a, b\})$, where $B$ and $B'$ are two blocks of $S(4.7.23)$ with $|B \cap B'| = 3$, $x \in B - B'$, $y \in B' - B$ and

   $$\{a, b\} \subset \{x_1, x_2, x_3\}, D_1 \cap D_j - (B \cup B'), \{i, j, k\} = \{1, 2, 3\}.$$ 

   Here $D_1$, $D_2$ and $D_3$ are the three blocks intersecting $B \cup B'$ only at $x$ and $y$.

5. $E_1 := B \cup B' - \{a, w\}$, where $B$ and $B'$ are two blocks of $S(4.7.23)$ with $B \cap B' = \{a\}$ and $w \in B' - B$.

Berardi [7] proved that $N_1$ and $N_0$ are the only blocking sets of size nine in $S(3.6.22)$. It is also implied that $N_1$ and $N_0$ are different types of blocking sets in $S(3.6.22)$. In [8] Berardi proved that $E_0$, $E$ and $E_1$ are blocking sets of size
eleven in $S(4.7.23)$. He also claims that $E_0$, $E$ and $E_1$ are three different types of blocking sets in $S(4.7.23)$. Here we prove that $N_1$ and $N_0$ are the same type of blocking set in $S(3.6.22)$; while $E$ and $E_1$ are the same type of blocking set in $S(4.7.23)$.

For convenience, we will use $X = Y$ to denote $X$ and $Y$ are the same type.

2.2. The proofs of $N_1 = N_0$ and $E = E_1$.

**Lemma 2.1.** There exists a block which is 5-secant to $N_0$ in $S(3.6.22)$.

**Proof.** Let $B = \{o.r.1.2.3.4\}$, $B' = \{o.r.a.b.c.d\}$, $B'' = \{o.4.d.z.u.v\}$.

$E_1 = \{x.y.m.n.p.q\}$, $E_2 = \{x.y.s.u.v.w\}$; then

$$N_0 = \left( (B \cup B') - B'' \right) \cup \{y.z\} = \{r.1.2.3.a.b.c.y.z\}.$$

It is not difficult to prove the following conclusion:

Let $A(i, \alpha)$ be the blocks determined by $\{y.i, \alpha\}$, where $1 \leq i \leq 3$, $\alpha \in \{a, b, c\}$. If one of $4, d$ is in $A(i, \alpha)$, then the other one is also in $A(i, \alpha)$.

Suppose there is no block which is 5-secant to $N_0$, then $A(1, \alpha) \neq A(1, J)$ if $\alpha \neq J$, so each one of $\{o\}$, $\{r\}$ and $\{4, d\}$ is contained in exactly one of $A(1, a)$, $A(1, b)$ and $A(1, c)$. Without loss of generality, we may assume $\{o\} \subset A(1, a)$, $\{r\} \subset A(1, b)$ and $\{4, d\} \subset A(1, c)$. Since $A(2, c) \neq A(1, c) \neq A(3, c)$ and $4, d \notin A(2, c) \cup A(3, c)$, then either $o \in A(2, c)$, $r \in A(3, c)$ or $r \in A(2, c)$, $o \in A(3, c)$.

If $o \in A(2, c)$, $r \in A(3, c)$, then since $A(2, b) \neq A(1, a)$, $A(1, b)$, $A(1, c)$, $A(2, c)$, $A(2, d) \notin A(2, b)$, therefore $|A(2, b) \cap B'| = |\{b\}| = 1$, which is a contradiction.

If $r \in A(2, c)$ and $o \in A(3, c)$, similarly we can also get a contradiction. \qed
It is not difficult to prove that a blocking set of size 9 in $S(3.6.22)$ has at most one 5-secant block [7, pp. 39]. By lemma 2.1 and the proof of lemma 5.6 of [7] we can prove

**Theorem 2.1.** The set $N_0$ is the unique blocking set of size nine in $S(3.6.22)$ and $FV(N_0) = (0, 18, 20, 26, 12, 1, 0)$.

Now we prove

**Theorem 2.2.** The blocking sets $E$ and $E_1$ are the same type of blocking set in $S(4.7.23)$.

**Proof.** Let $E$ be defined as in 4 of section 2.1. We only need to prove that there exist two blocks $A$ and $A'$ such that $E = A - A' - \{a, w\}$ with $A - A' = \{a\}$ and $w \in A' - A$.

Without loss of generality, we may assume $a = x_1$, $b = x_2$ and $c = x_3$; then we have $a, b \in D_3$. Consider the blocks determined by $\{x, a, b, c\}$ and $\{y, a, b, c\}$, respectively. If any one of the blocks meets $B \cap B'$, then it meets $B \cap B'$ at exactly two points. So at most one meets $B \cap B'$. Let $A$ be one which does not meet $B \cap B'$.

We may assume that $A$ is determined by $\{x, a, b, c\}$. Since $x, a, b \in D_3$, $x, a, c \in D_2$ and $x, b, c \in D_1$, we have $A - \{x, a, b, c\} \subset (B \cap B') - \{x\}$. By result 2.4, we can obtain that $(A - \{x, a, b, c\}) \cap (B' - B) = 1$ or 3.

If $(A - \{x, a, b, c\}) \cap (B' - B) = 1$, then $(A - \{x, a, b, c\}) \cap |B - (B' \cup \{x\})| = 2$. Let

\[(A - \{x, a, b, c\}) \cap (B' - B) = \{z\}.

\[(A - \{x, a, b, c\}) \cap |B - (B' \cup \{x\})| = \{u, v\}.

By lemma 2.4 in [8], there is a block \( U \) through \( u, z \) and \( a \) such that
\[ U \cap (B \cup B') = \{u, z\}. \]
Since \( U \cap A = \{u, z, a\} \), we have \( b, c \notin U \). Therefore,
\[ |U \cap (D_1 - \{x, y, b, c\})| = 1 \text{ or } 3, \]
so \( |U \cap D_2| = 2 \) or \( |U \cap D_3| = 2 \). a contradiction.

So \( |(A - \{x, a, b, c\}) \cap (B' - B)| = 3. \)
Therefore \( |A \cap E| = 5. \)
\[ A \cap B = \{x\} \] and \[ E = (A \cup B) - \{x, c\}. \]

By remark 2.8 in [8] and theorem 2.2 we know that \( E_0 \) and \( E_1 \) are the only two different types of blocking sets of size eleven in \( S(4.7.23) \). So there are four different types of blocking sets in \( S(4.7.23) \), not six as indicated in [8].
2.3. **Notes on** $S(5.8.24)$. We use the same notations and terminologies as in [9].

Let $B, B'$ be two blocks in $S(5.8.24)$ with $|B \cap B'| = 2$. We have

$$M := B \triangle B'; \quad M_0 := B \triangle B' - \{a\}; \quad I := B \cup B' - \{u, v\}; \quad R := B \cup B' - \{z, a\}$$

where $u \in B - B', v \in B' - B, a \in B \triangle B', z \in B \cap B' = \{x, y\}$.

In [9] the following theorem had been proved:

**Theorem 2.3.** Let $C$ be a blocking set in $S(5.8.24)$. Then $11 \leq |C| \leq 13$. Moreover.

1. $|C| = 11$ implies that $C = M_0$ and $i(M_0) = 2$.
2. $|C| = 12$ and $C$ irreducible imply that $C = I$ and $i(I) = 2$.
3. $|C| = 12$ and $C$ reducible imply that $C = M_0 - \{x\}, x \notin M_0$. Moreover, if $i(C) = 2$, then either $C = M$ or $C = R$.
4. $|C| = 13$ implies that $C$ is reducible and $C$ is the complement of $M_0$. Moreover, if $i(C) = 2$, then $C = B \cup B' - \{a\}$, where $B, B'$ are two blocks with $|B \cap B'| = 2$ and $a \in B \cap B'$.

In this section we prove the following theorem, which has improved the results in the above theorem.

**Theorem 2.4.** Let $C$ be a blocking set in $S(5.8.24)$. Then $11 \leq |C| \leq 13$ and $i(C) = 2$. Moreover.

1. If $|C| = 11$, then $C = M_0$.
2. If $|C| = 12$ and $C$ is irreducible, then $C = I$.
3. If $|C| = 12$ and $C$ is reducible, then $C = M$ or $R$.
4. If $|C| = 13$, then $C = S - M_0 = B \cup B' - \{z\}$ is reducible, where $B, B'$ are two blocks with $|B \cap B'| = 2$ and $z \in B \cap B'$. 
2.3.1. **Some known results on** $S(5,8,24)$. In the case of $S(5,8,24)$, if $E$ is a blocking set, then (1) implies:

\[
t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 = 759
\]
\[
t_2 - 2t_3 - 3t_4 - 4t_5 - 5t_6 - 6t_7 = g_1
\]
\[
t_3 - 3t_4 - 6t_5 - 10t_6 - 15t_7 = g_2
\]
\[
t_4 - 4t_5 - 10t_6 - 20t_7 = g_3
\]
\[
t_5 - 5t_6 - 15t_7 = g_4
\]
\[
t_6 - 6t_7 = g_5
\]

where

\[
g_1 = 253c - 759
\]
\[
2g_2 = 77c(c - 1) - 2g_1
\]
\[
6g_3 = 21c(c - 1)(c - 2) - 6g_2
\]
\[
24g_4 = 5c(c - 1)(c - 2)(c - 3) - 24g_3
\]
\[
120g_5 = c(c - 1)(c - 2)(c - 3)(c - 4) - 120g_4
\]

The following lemmas are quoted from [8, 9].

**Lemma 2.2.** [9] *Let $B, B'$ be two blocks in $S(5,8,24)$. Then*

1. *The type of $B$ is* $(0, 2, 4, 8)$ *with*

   \[
t_0 = 30, t_2 = 448, t_4 = 280, t_8 = 1.
\]

2. *If* $|B \cap B'| = 4$, *then* $B \Delta B'$ *is a block.*
3. If \( B \cap B' = 2 \), then \( M = B \triangle B' \) is a set of type \((2, 4, 6)\) with

\[
t_2 = t_6 = 132, t_1 = 495.
\]

4. If \( B \cap B' = 0 \), then \( B \triangle B' = B \setminus B' \) is a set of type \((0, 4, 6, 8)\) with

\[
t_0 = 1, t_4 = 280, t_6 = 448, t_8 = 30.
\]

5. Let \( E \) be a set. Then \( S - E \) is a block if and only if \( E = B \cup B' \), \( B \cap B' = \emptyset \).

6. Let \( F \) be a 4-set, \( F \cap B = \emptyset \); then there exists a block \( B' \) such that \( F \subseteq B' \)

and \( B \cap B' = \emptyset \).

By 1, 3 and 4 of lemma 2.2 we get the following corollaries, respectively.

**Corollary 2.1.** No blocking set can be contained in one block.

**Corollary 2.2.** The sets \( M, M_0 \) are blocking sets in \( S(5, 8, 24) \).

**Corollary 2.3.** Let \( C \) be a blocking set. If \( C \subseteq B \setminus B' \), then \( B \setminus B' = \emptyset \).

Fix a point \( x \) in \( S(t, k, v) \), the contraction of \( S(t, k, v) \) at \( x \) is an \( S(t - 1, k - 1, v - 1) \).

For \( S(4, 7, 23) \) we have

**Lemma 2.3.** [8] Let \( B, B' \) be two blocks in \( S(4, 7, 23) \) with \( B \cap B' = \{x\} \), then for any \( u \in B - B' \) and \( v \in B' - B \), there exists a block \( B'' \) in \( S(4, 7, 23) \) such that \( B'' \cap (B \cup B') = \{u, v\} \).

**Corollary 2.4.** Let \( B, B' \) be two blocks in \( S(5, 8, 24) \) with \( B \cap B' = \{x, y\} \) and let \( u \in B - B' \), \( v \in B' - B \). Then \( (B \cup B') - \{y, u, v\} \) is not a blocking set.
2.3.2. **Proof of the theorem 2.4.** From now on, $C$ will be used to denote a blocking set in $S(5,8,24)$.

Lemma 2.4. proposition 2.1, proposition 2.2 and proposition 2.3 had been proved in [9], we quote them here for our convenience.

**Lemma 2.4.** $11 \leq |C| \leq 13$.

**Proposition 2.1.** If $|C| = 11$, then $C = M_0$ and $C$ has no 7-secant block.

**Proposition 2.2.** $I$ is an irreducible blocking set.

**Proposition 2.3.** $R$ is a reducible blocking set.

By (2), if $|C| = 12$, then

\begin{equation}
(4) \quad t_1 = t_7, t_2 = t_6 = 132 - 6t_7, t_4 = t_7 = 15t_7, t_4 = 495 - 20t_7.
\end{equation}

The following proposition plays a crucial role in our proof.

**Proposition 2.4.** Let $|C| = 12$.

1. If $C$ has a 7-secant block, then $C = R$ or $I$.
2. If $C$ has no 7-secant block, then $C = M$.

**Proof.** Let $B$ be a block 7-secant to $C$, $B'$ be a block containing the five points in $C - B$: then $|B \cap B'| = 2$. Let $B \cap B' = \{x, y\}$. Since $|B \cap C| = 7$. \{x, y\} \cap C \neq \emptyset.

If $x \in C$, $y \notin C$, then $C = R$. If $x, y \in C$, then $C = I$.

If $C$ has no 7-secant block, then by (4). $C$ is of type (2.4.6). Let $B$ be a block 6-secant to $C$, let five of the six points in $C - B$ be contained in block $B'$: then $B'$ contains another point in $C$. We claim that this point must be the remaining point in $C - B$. Suppose this point is in $B$: then by lemma 2.2.1, $|B \cap B'| = 2$. 

Let $B \cap B' = \{x, y\}, x \in C, y \notin C, u \in B - B', v \in B' - B, w \in C - (B \cup B')$.

Since $C$ is of type $(2, 4, 6)$, the set $C - \{w\} = (B \cup B') - \{y, u, v\}$ is a blocking set. a contradiction. So $B, B'$ are blocks 6-secant to $C$ and $C = B \Delta B' = M$. □

Proposition 2.5 and proposition 2.7 are proved in [9], but using proposition 2.4. we can simplify the proofs.

**Proposition 2.5.** If $|C| = 12$ and $C$ is irreducible. then $C = I$.

**Proof.** Since $C$ is irreducible. $t_7 = t_1 \geq 12$. By proposition 2.4. $C = I$. □

**Proposition 2.6.** If $|C'| = 12$ and $C$ is reducible. then $C = M$ or $R$.

**Proof.** If $C$ has a 7-secant block. then $C = R$; if $C$ has no 7-secant block. then $C = M$. □

Since $M_0$ has no 7-secant block. we have

$$FV(M_0) = (0, 22, 110, 165, 330, 66, 60, 0, 0).$$

In theorem 3.17 we will prove that $R$ has eleven 7-secant blocks. so

$$FV(R) = (0, 11, 66, 165, 275, 165, 66, 11, 0).$$

From the fact that $I$ is an irreducible blocking set we know that $I$ has at least twelve 7-secant blocks. Checking the appendix C we know that

$$FV(I) = (0, 12, 60, 180, 255, 180, 60, 12, 0).$$

Lemma 2.2 tells us that $FV(M) = (0, 0, 132, 0, 495, 0, 132, 0, 0)$.

**Proposition 2.7.** Let $A$ be one of the 12-sets $I, M$ and $R$. Then $S - A = A$. 

Proof. Let $A = M$. Since $M$ is of type (2.4.6), so is $S - M$. By proposition 2.4, $S - M = M$.

Let $A = R = B \cup B' - \{z, a\}$, where $B \cap B' = 2$, $a \in B \cup B'$, and $z \in B \cap B'$.

Since $R$ has a 7-secant block and $R - \{a\}$ is a blocking set, $S - R$ is reducible and also has a 7-secant block. By proposition 2.6 and proposition 2.4, $S - R = R$.

Let $A = I$. Suppose $S - I$ is reducible, then $S - I = R$, but $S - (S - I) = I$, so $I = S - R = R$, a contradiction. Therefore $S - I$ is irreducible and $S - I = I$. □

**Proposition 2.8.** If $|C| = 13$, then $C = S - M_0 = B \cup B' - \{z\}$ is reducible, where $B, B'$ are blocks with $B \cap B' = 2$ and $z \in B \cap B'$.

*Proof.* Since $|S - C| = 11$, we have $S - C = M_0$. But $M_0$ has no 7-secant block, so $C$ has no 1-secant block. This means that $C$ is reducible.

The fact that $M_0$ has a 1-secant block means that $C$ has 7-secant blocks. Let $B$ be a 7-secant block to $C$ and let five of the six points in $C - B$ be contained in block $B'$.

We claim that the remaining one point $w \in C - B$ is still in $B'$.

Suppose $w \not\in B'$: we may assume that $B \cap B' = \emptyset$. If $B \cap B' = \emptyset$, then $B'$ contains three points in $S - (C \cup B)$. Since there are six blocks that contain five points in $C - B$, any two of these only intersect at four points in $C - B$, and there are only ten points in $S - (C \cup B)$, so at least one of these blocks will intersect $B$: we can label this block as $B'$. Then $B \cap B' = 2$. Let $B \cap B' = \{x, y\}$; since $B$ is 7-secant to $C$, $\{x, y\} \cap C \neq \emptyset$.

If $x, y \in C$, then $C - \{w\} = I$. But on the other hand, $M_0 \cup \{w\}$ is reducible, so $S - I = M_0 \cup \{w\} = R$. A contradiction.
If \( x \in C \), \( y \notin C \), let \( v \in B' - (C \cap B) \). \( B = \{x, y, a_1, a_2, a_3, a_4, a_5, a_6\} \). By lemma 2.3 we know that there is a block \( B_i \) that contains \( a_i, v, y \) such that 
\( B_i \cap (C - \{w, a_i\}) = \emptyset \) for \( i = 1, 2, 3, 4, 5, 6 \). Since \( C \) has no 1-secant block, \( w \in B_i \).

for \( i = 1, 2, 3, 4, 5, 6 \) let \( D_i = B_i - \{v, y, w, a_i\} \); then \( D_i \sim 4 \). \( D_i \subseteq S - (C - \{x, y\}) \)
and \( D_i \cap D_j = 1 \) for \( i \neq j \). Since \( S - (C - \{v, y\}) \sim 9 \), we have \( D_i \cap D_j \neq D_i \cap D_j \).

if \( i \neq j \). Hence \( D_i \geq 5 \), a contradiction.

Now we have proved that \( w \in B' \). From \( C \subseteq (B \cup B') \) we know that \( B - B' \sim 2 \).

Let \( B - B' = \{x, y\} \); since \( B - C' = 7 \), this means that \( (B - B') - C' = 1 \), so \( C = B \cup B' - \{z\} \) with \( z \notin B - B' \).

\( \square \)

From time to time we apply results of \( 7, 8, 9 \) in this thesis. In order to ensure our proofs are not affected by the errors in \( 7, 8, 9 \), those we use here have been thoroughly checked.
3.1. Preliminaries.

Definition 3.1. A set $C_t$ ($t \geq 1$) is called a $t$-blocking set if $C_t$ meets every block in at least $t$ points and meets at least one block in exactly $t$ points.

Our definition of $t$-blocking set is different from that in [14]. Here we require that at least one block meets $C_t$ in exactly $t$ points, while in [14] this is not required.

In recent years, more and more papers dealing with $t$-blocking sets have been published. Not only is the $t$-blocking set of theoretic importance by itself (see [3, 16]), it also has applications in other areas. For example, it is known that there is a link between optimal linear codes and $t$-blocking sets in a projective plane [2]. Batten [5] has pointed out that critical systems are connected with $t$-blocking sets and Batten [6] has presented a private key cryptosystem which is based on $t$-blocking sets.

The automorphism groups of the Witt designs $S(4, 5, 11)$, $S(5, 6, 12)$, $S(3, 6, 22)$, $S(4, 7, 23)$ and $S(5, 8, 24)$ are respectively Mathieu's five sporadic simple groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$ and $M_{24}$; the Mathieu groups $M_{11}$, $M_{12}$, $M_{23}$ and $M_{24}$ are the automorphism group of the Golay codes $G_{11}$, $G_{12}$, $G_{23}$ and $G_{24}$, respectively [56, pp. 111-112]. In [7, 8, 9, 10] Berardi, Eugeni and Ferri characterized the blocking sets in Witt designs. In this chapter we characterize all $t$-blocking sets in Witt designs.

3.2. $t$-blocking sets in $S(4, 5, 11)$ and $S(5, 6, 12)$.

3.2.1. $t$-blocking sets in $S(4, 5, 11)$. 


Theorem 3.1. Let $C_t$ be a $t$-blocking set in $S(4.5.11)$.

1. If $t = 1$, then $5 \leq |C_t| \leq 7$, and if $|C_t| = 4 - t$, $t \in \{1, 2, 3\}$, then we have $C_t = B \cup X$, where $B$ is a block, $X$ is an $(i-1)$-subset of $S$ such that $X \cap B = \emptyset$. When $|C_t| = 5$, $FV(C_t) = (0.15, 0.20, 0.30, 0.11)$; when $|C_t| = 6$, $FV(C_t) = (0.5, 0.20, 0.30, 0.11)$; when $|C_t| = 7$, $FV(C_t) = (0.1, 0.12, 0.30, 0.20)$.

2. If $5 \geq t \geq 2$, then $C_t = X$, where $X$ is a $(6 - t)$-subset of $S$, and

$$FV(C_2) = (0.0, 0.4, 0.24, 0.30, 0.8), \quad FV(C_4) = (0.0, 0.0, 0.12, 0.36, 0.18).$$

$$FV(C_4) = (0.0, 0.0, 0.30, 0.36), \quad FV(C_5) = (0.0, 0.0, 0.0, 0.36).$$

Proof. Since the type of the block in $S(4.5.11)$ is $(1, 2, 3, 5)$ and any 4-set is contained in exactly one block, we conclude that $5 \leq C_1 \leq 7$.

Because $S(4.5.11)$ does not have any blocking set, $C_1$ contains a block $B$. If $|C_1| = 5$, then $C_1 = B$; if $|C_1| = 6$, then $C_1 = B \cup \{x\}$, where $x \in B$; and if $|C_1| = 7$, then $C_1 = B \cup \{x, y\}$, where $x, y \in B$.

It is easy to prove that if $5 \geq t \geq 2$, then $C_t = X$, where $X$ is a $(6 - t)$-subset of $S$. The frequency vector can be obtained by letting $t_0 = 0$ and solve the linear system (1).

3.2.2. $t$-blocking sets in $S(5.6.12)$.

Theorem 3.2. Let $C_t$ be a $t$-blocking set in $S(5.6.12)$.

1. If $C_t$ does not contain any block, then $t = 1$, $|C_t| = 6$ and $C_t = (B \setminus \{a\}) \cup \{x\}$ and $FV(C_t) = (0.6, 0.30, 0.30, 0.6, 0.3)$, where $B$ is a block and $a \in B$, $x \notin B$. 


2. If $C_t$ contains a block, then $C_t = B \cup X_t$, where $B$ is a block and $X_t$ is a $t$-set such that $X_t \cap B = \emptyset$. The frequency vectors are:

$$FV(C_1) = (0, 1.15, 50, 50, 15, 1), \quad FV(C_2) = (0, 0.4, 32, 60, 32, 4).$$

$$FV(C_3) = (0, 0.0, 12, 54, 54, 12), \quad FV(C_4) = (0, 0.0, 0.30, 72, 30).$$

$$FV(C_5) = (0, 0.0, 0.0, 0.66, 66), \quad FV(C_6) = (0, 0.0, 0.0, 0.132).$$

Proof. The type of a block in $S(5, 6, 12)$ is $(0, 2, 3, 4, 6)$. Noticing the fact that any five points are contained in exactly one block we can easily prove this theorem. \[\square\]

3.3. Some known results on $S(3, 6, 22)$. The following results are quoted from [7] for our reference convenience:

Result 3.1. Let $B$ and $B'$ be two blocks in $S(3, 6, 22)$ with $B \sim B' = 2$. Let $x$ be a point with $x \notin B \cup B'$. Then the set $B \cup B' \cup \{x\}$ has two external blocks.

Result 3.2. Let $B$ and $B'$ be two blocks in $S(3, 6, 22)$ with $B \sim B' = 2$. Denote by $W$ the point set of the complement of $B \cup B'$, and by $R$ the set of external blocks of $B \cup B'$. Then the pair $(W, R)$ is a 1-$(12, 6, 2)$ design.

Result 3.3. Each point in $S(3, 6, 22)$ is on 21 blocks, each pair of points is on 5 blocks.

3.4. The structure of a Fano set in $S(3, 6, 22)$.

Definition 3.2. A Fano set in $S(3, 6, 22)$ is a 7-set $F$ of type $(1, 3)$.

The frequency vector of a Fano set $F$ is $(0, 42, 0, 35, 0, 0, 0)$. Fano sets played a very important role in Berardi's characterization of the blocking sets in $S(3, 6, 22)$. 
We will also use Fano sets in our characterization and calculation of the frequency vectors. So in this section we study the structure of a Fano set thoroughly, and give a construction method for a Fano set. Our main result is theorem 3.3.

Definition 3.3. Let \( B \) be a block in \( S(3,6,22) \). We define \( R(B) \) to be the set of blocks which are disjoint from \( B \).

By result 2.2 we can prove

Lemma 3.1. The pair \( \mathcal{S} - B, R(B) \) is a \( 2-(16,6,2) \) design.

Lemma 3.2. Let \( B \) and \( R(B) \) be as above, let \( B' = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\} \) and

\[ U = \{o, a, b, u, v, w\} \] be two distinct elements of \( R(B) \). Let \( V, W \subset R(B) \) be the other blocks on \( \{o, a\} \) and \( \{o, b\} \), respectively. If \( V \cap B' = \{a, c\} \), then \( W \cap B' = \{b, c\} \).

Proof. By lemma 3.1 we only need to prove that the block \( X \) determined by \( \{b, c, o\} \) is in \( R(B) \).

Suppose \( X \notin R(B) \); then \( X \cap B' = 2 \). Let \( B = \{x, y, z, \bar{x}, \bar{y}, \bar{z}\} \) and

\[ X = \{o, b, c, x, \bar{x}, d\} \] let \( B_x = \{o, b, \bar{a}, \bar{y}, \bar{g}, \bar{e}\} \) be the block determined by \( \{o, b, g\} \), let \( B_z = \{o, b, \bar{z}, \bar{z}, f\} \) be the block determined by \( \{o, b, z\} \). Then the block \( B'' \) determined by \( \{o, b, \bar{c}\} \) would be in \( R(B) \); so let \( B'' = \{o, b, \bar{c}, p, q, r\} \). Because \( V \cap U = \{o, a\}, V \cap X = \{o, c\} \), so \( u, v, w, d \notin V \). As \( o \notin V \cap B'' \), so \( V \) contains only one of \( p, q \) and \( r \). We may assume that \( p \notin V \); then \( V = \{o, a, c, e, f, p\} \).

By lemma 3.1 there exists a block \( A \subset R(B) \) on \( \{o, c\} \) such that \( A \neq V \). Then \( a, b, e, f \notin A \), so \( \bar{a}, \bar{b} \in A \) and \( A \cap B'' \geq 3 \), a contradiction.

This lemma guarantees that \( o \) is uniquely determined by \( a, b, c \) and any two of \( U, V, W \).
Use the notation of lemma 3.2 and let \( x \in B \). Let \( B_1, B_2 \) and \( B_3 \) be the blocks determined by \( \{o, a, x\}, \{o, b, x\} \) and \( \{o, c, x\} \), respectively. Then \( B_i = B_j \) if \( i \neq j \) (otherwise \( B_i \) would be one of \( U, V, W \), and \( B_i \notin \mathbb{R}(B) \)). Let

\[
B_1 \setminus B - \{x\} = \{\bar{x}\}, \quad B_2 \setminus B - \{x\} = \{\bar{y}\}, \quad B_3 \setminus B - \{x\} = \{\bar{z}\}.
\]

\[
B_1 \setminus B' - \{a\} = \{\bar{a}\}, \quad B_2 \setminus B' - \{b\} = \{\bar{b}\}, \quad B_3 \setminus B' - \{c\} = \{\bar{c}\}.
\]

These produce partitions of \( B \) and \( B' \) into two parts, say \( \{x, y, z\}, \{\bar{x}, \bar{y}, \bar{z}\} \) and \( \{a, b, c\}, \{\bar{a}, \bar{b}, \bar{c}\} \).

**Lemma 3.3.** The above partition does not depend on the choice of \( x \), that is for any element of \( B \), we get the same partition of \( B \).

In the same way, the blocks \( A_1, A_2, A_3 \) determined by \( \{a, b, x\}, \{a, c, x\}, \{b, c, x\} \), respectively also produce a partition of \( B \). It can be proved that this partition does not depend on \( x \) either. The next lemma shows that these are actually the same partition.

**Lemma 3.4.**

\[
\bigcup_{i=1}^{3}(A_i \setminus B) - \{x\} = \bigcup_{i=1}^{3}(B_i \setminus B) - \{x\}.
\]

**Proof.** It suffices to show \((A_i - \{x\}) \cap \{x, y, z\} = \emptyset \) for \( i = 1, 2, 3 \).

Let \( U, V \) and \( W \) be as in lemma 3.2.

Now we prove \((A_1 - \{x\}) \cap \{x, y, z\} = \emptyset \). Suppose \((A_1 - \{x\}) \cap \{x, y, z\} \neq \emptyset \); then \( A_1 \cap B = \{x, \alpha\} \), where \( \alpha \in \{y, z\} \). Let \( A \) be the block determined by \( \{o, c, \alpha\} \); then from

\[
A \cap V = A \cap W = A \cap B_3 = \{o, c\}
\]
and result 3.1 we know that \( a, b, x \notin A \) and \( |A \cap (U - B')| = 2 \). From \( a, b \in A_1 \) we know that \( A_1 \cap (U - B') = \emptyset \). Therefore, \( A \cap A_1 = \{a\} \), contradicting result 3.1. Thus, \( (A_1 - \{x\}) \cap \{x, y, z\} = \emptyset \).

That \( (A_2 - \{x\}) \cap \{x, y, z\} = \emptyset \) and \( (A_3 - \{x\}) \cap \{x, y, z\} = \emptyset \) can be proved similarly.

For the remainder of this section, we assume

\[ B = \{x, y, z, \bar{x}, \bar{y}, \bar{z}\}, \quad B' = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\} \]

and \( o \) is as above.

**Lemma 3.5.** Let \( D_1, D_2, D_3 \in R(B) \) be the blocks on \( \{\bar{a}, \bar{b}\}, \{\bar{a}, \bar{c}\}, \{\bar{b}, \bar{c}\} \), respectively, and \( D_i \neq B', i = 1, 2, 3 \). Then \( o \in D_i, i = 1, 2, 3 \).

**Proof.** If \( o \notin D_i \), then \( D_i \) would be disjoint from one of \( U, V, W \), say \( U \), and then the union of the two disjoint blocks \( D_i \) and \( U \) would have an external block \( B \), contradicting result 3.4.

**Lemma 3.6.** Any block determined by three points from \( \{\bar{a}, \bar{b}, \bar{c}, \bar{x}, \bar{y}, \bar{z}\} \) contains exactly three points of this set.

**Proof.** Suppose there exists a block \( A \) containing four points from \( \{\bar{a}, \bar{b}, \bar{c}, \bar{x}, \bar{y}, \bar{z}\} \): then \( A \) contains two of \( \bar{x}, \bar{y}, \bar{z} \), say \( \bar{x}, \bar{y} \). Then one of the blocks on \( \bar{x}, \bar{y} \) would contain two points \( \alpha, \beta \in \{a, b, c\} \); consequently, one of the blocks on \( \{\alpha, \beta\} \) would contain two of \( x, y, z \), which is a contradiction.

**Lemma 3.7.** Any block determined by three points from either \( \{a, b, c, x, y, z\} \) or \( \{\bar{a}, \bar{b}, \bar{c}, \bar{x}, \bar{y}, \bar{z}\} \) does not contain \( o \).
Proof. Since three points uniquely determine a block, this lemma is just a consequence of lemmas 3.1, 3.2, 3.4 and 3.5. □

Lemma 3.8. Let $X, Y, Z \in \mathcal{R}(B') - \{B\}$ be three blocks on $\{x, y\}, \{x, z\}$ and $\{y, z\}$, respectively. $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{R}(B') - \{B\}$ be three blocks on $\{\tilde{x}, \tilde{y}\}, \{\tilde{x}, \tilde{z}\}, \{\tilde{y}, \tilde{z}\}$, respectively. Then $\alpha \in X - Y - Z - \tilde{X} - \tilde{Y} - \tilde{Z}$.

Proof. Since each block meets $B - B'$, and there are totally 21 blocks on $\alpha$, the conclusion is obtained by using the above lemmas to count the blocks which do not contain $\alpha$. □

By summarizing what we have already got, we obtain:

Theorem 3.3. Let $F$ be a 7-set of the point set $S$ of $S(3, 6, 22)$. Then $F$ is a Fano set if and only if no block determined by three points from $F$ contains more than three points from $F$.

The above results give a construction method for a Fano set $F$:

1. choose $\{a, b, c, x\} \subset S$ such that $\{a, b, c, x\}$ is not contained in any block:

2. let $B'$ be the block determined by $\{a, b, c\}$, let $B \in \mathcal{R}(B')$ be a block on $x$:

3. let $U, V \in \mathcal{R}(B) - B'$ be the blocks on $\{a, b\}$ and $\{a, c\}$, respectively: let $\alpha \in (U - V) - \{a\}$:

4. let $B_1, B_2, B_3$ be the blocks determined by $\{a, b, x\}, \{a, c, x\}, \{b, c, x\}$, respectively, and let

$$\{x, y, z\} = \left[ B - \left( \bigcup_{i=1}^{3} (B_i - B) \right) \right] - \{x\}.$$ 

Then $F = \{a, b, c, x, y, z, o\}$ is a Fano set.
Example 3. Let the point set of $S(3.6.22)$ be \{1, 2, \ldots, 22\}. the blocks be those listed on the next page. Let $a = 10, b = 11, c = 13, x = 4$: then $B' = \{10, 11, 13, 16, 18, 20\}$. Now we choose $B = \{4, 21, 22, 5, 17, 19\}$: then $U = \{10, 11, 9, 12, 14, 3\}, V = \{10, 13, 1, 6, 8, 3\}, o = 3, B_1 = \{10, 11, 4, 2, 5, 6\}$.

$B_2 = \{10, 13, 4, 7, 9, 19\}, B_1 = \{11, 13, 4, 8, 14, 17\}, \{x, y, z\} = \{4, 21, 22\}$ and $F = \{3, 4, 10, 11, 13, 21, 22\}$ is a Fano set.

Note 3.1. In the above mentioned construction, there are six choices of $B$, but some of them produce the same Fano set. For instance, in the above example, if we choose $B = \{4, 3, 21, 2, 8, 9\}$: then $U = \{10, 11, 1, 7, 17, 22\}, V = \{10, 13, 5, 14, 15, 22\}$ and $o = 22, \{x, y, z\} = \{4, 3, 21\}$, and we get the same Fano set as in the above example. It can be proved that there are two different Fano sets which contain \{a, b, c, x\}.

Note 3.2. Let $F_1$ be one of \{x, y, z\}, \{x, z, y\}, let $F_2$ be one of \{a, b, c\}, \{a, b, c\}; then $F = F_1 \sim F_2 \sim \{o\}$ is again a Fano set.

Theorem 3.4. Let $F$ be a Fano set: then there are two disjoint blocks $B', B''$ and a point $o \in B \cup B'$ such that $F$ is just the set which consists of three points from each of $B', B''$ and $o$.

Proof. Let $F = \{x, y, z, a, b, c, o\}$: let $B, B'$ be the blocks determined by $\{x, y, z\}$ and $\{a, b, c\}$, respectively. Then $B \cap B' = \emptyset$. \hfill $\square$

The Fano sets all have frequency vector $(0, 42, 0, 35, 0, 0, 0)$. In appendix A there are two orbits which have the same frequency vector as a Fano set. This means that the Fano sets are divided into two orbits under the action of $M_{22}$. 


The blocks of $S(3.6.22)$.  

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3.5. $t$-blocking sets in $S(3.6.22)$.  

3.5.1. 1-blocking sets in $S(3.6.22)$. In this section we characterize the 1-blocking sets in $S(3.6.22)$.  

From the definitions of blocking set and $t$-blocking set we know that a blocking set is a $t$-blocking set with $t \geq 1$, which contains no block. Hence, when characterizing the $t$-blocking sets, we can make use of the results in $T$. Let $C$ be a blocking set in $S(3, 6, 22)$; then the type of $C$ can be determined by checking the results in $T$ when $|C| \neq 9, 10, 12, 13$.

In order to determine the types of other blocking sets, we need some more detailed results. We first consider the blocking sets with ten points.

**Lemma 3.9.** The blocking set $F \cup \{u, v, w\}$, where $F$ is a Fano set, $x, y, z \notin F$ and the block determined by $\{u, v, w\}$ is 1-secant to $F$, has three 5-secant blocks. So $FV(\{u, v, w\}) = (0, 11, 21, 26, 16, 3, 0)$.

**Proof.** Let $F = \{x, y, z, a, b, c, o\}$, let $B''$ be the block determined by $\{u, v, w\}$, and $o \in B''$. On $B'' - \{u, v, w, o\}$ there are four blocks besides $B''$, so there exists a block $B$ on $B'' - \{u, v, w, o\}$ such that $B$ contains at least two points in $\{x, y, z, a, b, c\}$, say $x, y$. Since $F$ is a Fano set, we have $B = B''$. Therefore, $B$ contains another point in $\{x, y, z, a, b, c\}$, say $z$. Let $B'$ be the block determined by $\{u, b, c\}$, then $B' = B'$.

By lemma 3.1, there exist three blocks $U, V$ and $W$ on $\{u, v\}, \{v, w\}$ and $\{u, w\}$, respectively, such that $U \neq B'', V \neq B''$, $W \neq B''$. $U, V$ and $W$ are disjoint from $B'$. By lemma 3.2 we know that $U \cap V \cap W = 1$. Since $B \cap B' = \emptyset$ and $(U \cup V \cup W) \cap B' = \emptyset$, we have

$$|U \cap B| = |V \cap B| = |W \cap B| = 2.$$

But it is easy to see that $o \notin U \cup V \cup W$, so

$$|U \cap \{x, y, z\}| = |V \cap \{x, y, z\}| = |W \cap \{x, y, z\}| = 1.$$
From $[B'' \cap (B - \{x, y, z\})] - (U - V - W) = \emptyset$, we know that the only point $\hat{x}$ in
$(B - \{x, y, z\}) - (B'' - B)$ is contained in $U - V - W$.

Now we consider the blocks $A_i$ ($i = 1, 2, 3$) determined by $\{u, v, a\}$, $\{v, w, \beta\}$
and $\{u, w, \theta\}$, respectively, where

\[
\alpha = \{x, y, z\} - (U - \{x, y, z\}).
\]

\[
\beta = \{x, y, z\} - (V - \{x, y, z\}).
\]

\[
\theta = \{x, y, z\} - (W - \{x, y, z\}).
\]

It is easy to see that $A_1 \subseteq U$, $B'' \neq V$, $B'' \neq W$. So $A_1 - B' = \emptyset$.
$A_2 - B' = \emptyset$. $A_3 - B' = \emptyset$ and $\hat{x} \notin A_1 \cup A_2 \cup A_3$. Therefore,

\[
\{x, y, z\} - (U - \{x, y, z\}) \subseteq A_1.
\]

\[
\{x, y, z\} - (V - \{x, y, z\}) \subseteq A_2.
\]

\[
\{x, y, z\} - (W - \{x, y, z\}) \subseteq A_3.
\]

Consequently, $A_i \cap \{a, b, c\} = 1$, $i = 1, 2, 3$. Hence $A_i \cap (F - \{u, v, w\}) = 5$, $i = 1, 2, 3$.

From the above proof we can see that any 5-secant block contains two points in
$\{u, v, w\}$ and three points in $F$, of which one is from $\{a, b, c\}$, two are from $\{x, y, z\}$.

So it must be one of $A_1$, $A_2$ and $A_3$.

The fact that $F - \{u, v, w\}$ is a blocking set means that $t_0 = t_6 = 0$. So
$FV(F - \{u, v, w\}) = (0, 11, 21, 26, 16, 3, 0)$.

The proof of the above lemma actually gives out a method to find the three 5-secant blocks of $F \cup \{u, v, w\}$. Now we consider the blocking sets with eleven points.
Lemma 3.10. The blocking set $C = D \cup \{w\}$, where $D = (B \setminus \{u\}) \cup (B' \setminus \{v\})$. $B$ and $B'$ are disjoint blocks. $u \in B$, $v \in B'$ and $w \not\in B \setminus B'$. has six 5-secant blocks, and $FV(C) = (0, 6, 20, 25, 20, 6, 0)$.

Proof. We can construct a Fano set $F$ based on $B$ and $B'$ such that $u, v, w \in F$. Any 5-secant block of $C$ besides $B$ and $B'$ contains $w$. It meets $B \setminus \{u\}$ and $B' \setminus \{v\}$. So it contains $w$. one point in $B \setminus F = \{u\}$ and one point in $B' \setminus F = \{v\}$. But

$$B \setminus F = \{u\} = 2 = B' \setminus F = \{v\}.$$ 

so we have four different blocks of this kind. Hence $C$ has six 5-secant blocks and $FV(C) = (0, 6, 20, 25, 20, 6, 0)$.

Lemma 3.11. The blocking set $F = A \setminus \{u\}$, where $F$ is a Fano set. $A$ is a block. $A \setminus F = 1$ and $u \not\in A \setminus F$. has seven 5-secant blocks. So its frequency vector is $FV(C) = (0, 7, 16, 31, 16, 7, 0)$.

Proof. Let $F = \{x, y, z, a, b, c, o\}$. $A = \{a, u, v, p, q, r, s\}$. We can properly choose two disjoint blocks $B$ and $B'$ such that $B \setminus F = 3 = B' \setminus F$. $\{x, y, z\} \subseteq B$, $\{a, b, c\} \subseteq B'$ and $B \setminus A = \{u, v\}$. Using the method used in the proof of lemma 3.9 we can prove that on any two points in $\{p, q, r\}$, there exists one 5-secant block to $F = A \setminus \{u\}$. Now we consider the blocks $A_a$, $A_b$ and $A_c$ determined by $\{p, v, a\}$, $\{p, v, b\}$ and $\{p, v, c\}$, respectively. It is obvious that $o, u \in A_a \cup A_b \cup A_c$. If there exists a $i \in \{a, b, c\}$ such that $A_i \cap F' = 1$, then $A_i$ contains the unique point in $B \setminus \{x, y, z, u, v\}$. So there exists at most one block of this kind. Therefore two of $A_a$, $A_b$ and $A_c$. say $A_a$ and $A_b$, contain one point in $\{x, y, z\}$, respectively. Therefore,

$$|A_a \cap \{a, b, c\}| = 2 = |A_b \cap \{a, b, c\}|.$$
So $A_a = A_b$, and $|A_a \cap (F \cup A - \{u\})| = 5$.

We can similarly prove that there exist a block determined by $q.r$ and a point in $\{a,b,c\}$, a block determined by $r.u$ and a point in $\{a,b,c\}$ such that these two blocks are both 5-secant to $F \cup A - \{u\}$.

Any 5-secant block to $F \cup A - \{u\}$, other than $A$, meets $F$ at three points, therefore contains two points in $A$. So it must be one of the above mentioned blocks, and so $FV(F \cup A - \{u\}) = (0,7,16,16,7,0)$. □

**Lemma 3.12.** Let $C = F \cup \{x,y,z,w\}$, where $F$ is a Fano set, $x,y,z,w \notin F$, any block determined by three points in $\{x,y,z,w\}$ is 1-secant to $F$. If $\{x,y,z,w\}$ is not contained in any block, then $C$ has six 5-secant blocks and the frequency vector of $C$ is $(0,6,20,25,20,6,0)$.

**Proof.** It is easy to see that $C$ is a blocking set. Now we only need to prove that $C$ has six 5-secant blocks. But this can be proved by lemma 3.9 and the following fact:

Let $u,v$ be two points not in the union of two disjoint blocks $B$ and $B'$, then there exists only one block $B''$ on $\{u,v\}$ such that $B'' \cap B' = 2 = B'' \cap B'$.

In [7] the blocking sets in lemma 3.11 and lemma 3.12 are considered the same type, but are different from the blocking set in lemma 3.10. Here we see that the blocking set in lemma 3.12 and lemma 3.10 are the same type: while the blocking set in lemma 3.11 is a different type.

Let $C$ be a blocking set with ten points: then

$$t_1 = 8 + t_5, \ t_2 = 33 - 4t_5, \ t_3 = 8 - 6t_5, \ t_4 = 28 - 4t_5.$$  

**Lemma 3.13.** Let $C$ be a blocking set in $S(3.6.22)$ with $|C| = 10$. Then $t_5 < 5$.  

Proof. Suppose \( t_5 \geq 5 \) let \( B, B' \) be two 5-secant blocks to \( C \). Then \( B \cap B' \neq \emptyset \) (otherwise \( t_5 = 2 \) and \( B \cap B' \subset C \)(otherwise \( C \) would have two external blocks).

So \( |C - (B \cup B')| = 2 \). Let \( C - (B \cup B') = \{x, y\} \), by result 3.2 there are two blocks \( A \) and \( A' \) such that

\[ A \cap (B \cup B' \cup \{x\}) = 0 = A' \cap (B \cup B' \cup \{x\}) \]

So \( y \in A \cap A' \). Let \( B_1, B_2 \) and \( B_3 \) be another three 5-secant blocks to \( C \). Then

\[ |B_i \cap [(B \cup B') \cap C]| = 4 \times i = 1, 2, 3 \]  

Hence \( B_i \) contains either \( x \) or \( y \). So one of \( x \) and \( y \), say \( x \), is contained in at least two of \( B_1, B_2 \) and \( B_3 \), say \( B_1, B_2 \). Therefore \( |B_1 \cap B_2| \geq 3 \), this is a contradiction. \( \square \)

By lemma 3.9 we know that the type of \( C \) is \( 1.2.3.4.5 \); therefore the type of \( S - C \) is also \( 1.2.3.4.5 \). This strengthens the result of \( \overline{1.2.3.4.5} \). The following theorem is a slight improvement of the results of \( \overline{1.2.3.4.5} \).

**Theorem 3.5.** If \( C_1 \) does not contain any block, then \( 7 \leq C_1 \leq 13 \). and

1. If \( |C_1| = 7 \), then \( C_1 = F \) and \( FV(F) = (0.42, 0.35, 0, 0, 0) \). where \( F \) is a Fano set.

2. If \( |C_1| = 8 \), then \( C_1 = F \cup \{x\} \) and \( FV(C_1) = (0.28, 14, 28, 7, 0, 0) \). where \( F \) is a Fano set and \( x \notin F \).

3. If \( |C_1| = 9 \), then \( C_1 = F \cup \{x, y\} \) and \( FV(C_1) = (0.18, 20, 26, 12, 1, 0) \). where \( F \) is a Fano set and \( x, y \notin F \).

4. If \( |C_1| = 10 \), then \( C_1 \) is one of the following:

   (a) \( C_1 = D := (B - \{u\}) \cup (B' - \{v\}) \) and \( FV(D) = (0.10, 25, 20, 20, 20, 0) \). where \( B, B' \) are two blocks with \( B \cap B' = \emptyset \). \( u \in B, v \in B' \).
(b) $C_1 = F \cup \{x, y, z\}$ and $FV(C_1) = (0.11.21.26.16.3.0)$, where $F$ is a Fano set. $x, y, z \notin F$ and the block determined by $\{x, y, z\}$ is 1-secant to $F$.

5. If $|C_1| = 11$, then $C_1$ is one of the following:

   (a) $C_1 = (F \cup B) - \{x\}$. $FV(C_1) = (0.11.0.55.0.11.0)$, where $F$ is a Fano set. $B$ is a block and $F \cap B = \{x\}$.

   (b) $C_1 = D \cup \{w\}$ and $FV(C_1) = (0.6.20.25.20.6.0)$, where $D$ is defined as in 4 and $w \notin B \cup B$.

   (c) $C_1 = F \cup A - \{u\}$. $FV(C_1) = (0.7.16.31.16.7.0)$, where $F$ is a Fano set. $A$ is a block and $|A \cap F| = 1$. $u \in A - F$.

6. If $|C_1| = 12$, then $C_1$ is one of the following:

   (a) $C_1 = S - (F \cup \{x, y, z\})$ and $FV(C_1) = (0.3.16.26.21.11.0)$, where $F$ is a Fano set. $x, y, z \notin F$, and the block determined by $\{x, y, z\}$ is 1-secant to $F$.

   (b) $C_1 = S - D$ and $FV(C_1) = (0.2.20.25.10.0)$.

7. If $|C_1| = 13$, then $C_1 = S - (F \cup \{x, y\})$, where $F$ is a Fano set. $x, y \notin F$, and $FV(C_1) = (0.1.12.26.20.18.0)$.

Proof. Let $C$ be a blocking set; then $7 \leq |C| \leq 15$.

1. If $|C| = 7$, then the type of $C$ is $(1.3)$. Solve the linear system 1 and let $t_0 = t_2 = t_4 = t_5 = t_6 = 0$. we get $t_1 = 42$ and $t_3 = 35$. So we have $FV(C) = (0.42.0.35.0.0.0)$.

2. If $|C| = 8$, then the type of $C$ is $(1.2.3.4)$. $FV(C) = (0.28.14.28.7.0.0)$.

3. If $|C| = 9$, then the type of $C$ is $(1.2.3.4.5)$. $FV(C) = (0.18.20.26.12.1.0)$.

4. If $|C| = 10$, then the type of $C$ is $(1.2.3.4.5)$.
5. If \(|C| = 11\), then the type of \(C\) is \((1.3.5)\) or \((1.2.3.4.5)\).

6. If \(|C| = 12\), then the type of \(C\) is \((1.2.3.4.5)\).

7. If \(|C| = 13\), then the type of \(C\) is \((1.2.3.4.5)\).

8. If \(|C| = 14\), then the type of \(C\) is \((2.3.4.5)\).

9. If \(|C| = 15\), then the type of \(C\) is \((3.5)\).

Since \(C_1\) does not contain any block, it is a blocking set. By lemmas 3.10, 3.11, 3.12 and theorem 7.4 in [7] the theorem is proved. \(\Box\)

**Theorem 3.6.** If \(C_1\) contains at least one block, then \(10 \leq C_1 \leq 17\), and

1. If \(|C_1| = 10\), then \(C_1 = F \setminus \{x, y, z\}\). \(FV(C_1) = (0.12, 18.28, 18.0.1)\), where \(F\) is a Fano set. \(x, y, z \notin F\) and the block determined by \(\{x, y, z\}\) is 3-secant to \(F\).

2. If \(|C_1| = 11\), then \(C_1\) is one of the following:
   (a) \(C_1 = B \setminus (B' - \{a\})\). Where \(B, B'\) are two blocks. \(B \cap B' = \emptyset\). \(a \in B'\): \(FV(C_1) = (0.5.25, 15.30.1.11)\).
   (b) \(C_1 = (B \cup B' - \{a\}) \cup \{u, v\}\). \(FV(C_1) = (0.7.17, 27.22.3.1)\). Where \(B, B'\) are two blocks. \(B \cap B' = 2\). \(a \notin B' - B\), \(u \notin B \cup B'\). \(v\) is one of the two joint points of the two external blocks of \(B \cup B' - \{u\}\).

3. If \(|C_1| = 12\), then \(C_1\) is one of the following:
   (a) \(C_1 = B \cup B' \cup \{u, v\}\). \(FV(C_1) = (0.4.14, 24.29.4.2)\). Where \(B\) and \(B'\) are blocks. \(|B \cap B'| = 2\). \(u \notin B \cup B'\). \(v\) is one of the two joint points of the two external blocks of \(B \cup B' \cup \{u\}\).
   (b) \(C_1 = B \cup (B' - \{u\}) \cup \{x\}\). \(FV(C_1) = (0.3.17, 22.27.7.1)\). Where \(B, B'\) are disjoint blocks. \(u \in B'\) and \(x \notin B \cup B'\).
(c) \( C_1 = B \cup (B' \triangle B'' - \{u, v, w\}) \cup \{x\} \). \( FV(C_1) = (0.4, 13, 28, 23, 8, 1) \).

where \( B, B' \) and \( B'' \) are blocks. \( B \cap B' = B \cap B'' = \emptyset. |B' \cap B''| = 2. \)

\( u, v, w \in B'' - B' \) and \( x \notin B \cup B' \cup B'' \).

(d) \( C_1 = F \cup B \cup B' - \{o\} \). \( FV(C_1) = (0.6, 5, 49, 15, 10, 1) \). where \( F \) is a

Fano set. \( B, B' \) are blocks. \( |F \cap B \cap B'| = 2 \) and \( o \in B' \cap F - B \).

4. If \( |C_1| = 13 \), then \( C_1 \) is one of the following:

(a) \( C_1 = B \cup B' \cup \{u, v, w\} \). \( FV(C_1) = (0.2, 12, 16, 40, 3, 4) \). where \( B, B' \)

are two blocks. \( |B \cap B'| = 2 \) and \( u, v, w \notin B \cup B' \) and \( v, w \) are the joint

points of the two external blocks of \( B \cup B' \cup \{u\} \).

(b) \( C_1 = B \cup B' \cup \{u, v, w\} \). \( FV(C_1) = (0.2, 10, 24, 28, 11, 2) \). where \( B, B' \)

are two blocks. \( |B \cap B'| = 2 \) and \( u, v, w \notin B \cup B' \). \( v \) and \( w \) are contained

in the two external blocks of \( B \cup B' \cup \{u\} \). but only one of \( v, w \) is in

the join.

(c) \( C_1 = B \cup B' \cup \{u, v, w\} \). \( FV(C_1) = (0.3, 6, 30, 24, 12, 2) \). where \( B, B' \) are

two blocks. \( |B \cap B'| = 2 \) and \( u, v, w \notin B \cup B' \). each of the two external

blocks of \( B \cup B' \cup \{u\} \) contains only one of \( v \) and \( w \) and \( u, v \) and \( w \) are

contained in an external block of \( B \cup B' \).

(d) \( C_1 = B \cup [B' \cup B'' - \{a\}] \). \( FV(C_1) = (0.1, 13, 22, 26, 14, 1) \). where \( B, B' \)

and \( B'' \) are blocks with \( B \cap B' = \emptyset. |B' \cap B''| = |B \cap B''| = 2 \) and

\( a \in B' \cap B'' \).

5. If \( |C_1| = 14 \), then \( C_1 \) is one of the following:

(a) \( C_1 = S - [(B \cup B') - \{u, v\}] \). where \( B \) and \( B' \) are two blocks. \( |B \cap B'| = 2. \)

\( u \in B \setminus B' \) and \( v \in B' \setminus B \). \( FV(C_1) = (0.2, 3, 24, 30, 14, 4) \).

(b) \( C_1 = S - [(B \cup B') - \{x, u\}] \). where \( B \) and \( B' \) are two blocks. \( x \in B \cap B' \)

and \( u \in B \triangle B' \); \( FV(C_1) = (0.1, 7, 18, 34, 13, 4) \).
(c) \( C_1 = \mathcal{S} - [(B \cup B') - \{x, a, b, c\}] \), where \( B \) and \( B' \) are two disjoint blocks.

\[ x \in B \text{ and } a, b, c \in B': \quad FV(C_1) = (0, 1, 6, 22, 28, 17, 3). \]

6. If \(|C_1| = 15\), then \( C_1 = \mathcal{S} - (B \cup \{a, b\}) \cup \{x\} \), where \( B \) is a block. \( x \in B \) and \( a, b \notin B': \quad FV(C_1) = (0, 1, 2, 17, 32, 19, 6). \)

7. If \(|C_1| = 16\), then \( C_1 = \mathcal{S} - (B \cup \{a\}) \cup \{x\} \), where \( B \) is a block. \( x \in B \) and \( z \notin B': \quad FV(C_1) = (0, 1, 0, 10, 35, 21, 10). \)

8. If \(|C_1| = 17\), then \( C_1 = (\mathcal{S} - B) \cup \{x\} \), where \( B \) is a block and \( x \notin B': \quad FV(C_1) = (0, 1, 0, 0, 40, 20, 16). \)

**Proof.** Checking the appendix A we obtain that \( 10 \leq |C_1| \leq 17. \)

Let \( B \) be a block and \( B \subset C_1. \)

1. If \(|C_1| = 10\), let \( C_1 - B = \{a, b, c, o\} \). Let \( B' \) be the block determined by \( \{a, b, c\} \). Since \( C_1 \) is a 1-blocking set, we have \( B' \cap B = \emptyset \). According to our construction method for a Fano set in section 3.4, we only need to prove that if \( B_1, B_2 \) are two blocks on \( \{a, o\} \) which are disjoint from \( B \), then \( b \in B_1, c \in B_2 \) or \( c \in B_1, b \in B_2. \)

Suppose \( B' = \{a, b, c, d, e, f\} \). Since on each of \( \{d, e\}, \{d, f\} \) and \( \{e, f\} \) there are two blocks respectively, which are disjoint from \( B \), and \( B' \) is one of these, let \( U, V, W \) be the other ones on \( \{d, e\}, \{d, f\} \) and \( \{e, f\} \), respectively. Because \( C_1 \) is a 1-blocking set, so \( o \in U, V, W \); therefore, either \( b \in B_1 \) and \( c \in B_2 \) or \( c \in B_1 \) and \( b \in B_2. \)

It is easy to see that \( C_1 \) has only one block, and has no 2-secant block. So \( FV(C_1) = (0, 12, 18, 28, 18, 0, 1). \)

2. If \(|C_1| = 11\), let \( C_1 - B = \{a, b, c, d, e\} \).
If \( \{a, b, c, d, e\} \) is contained in a block \( B' \), then \( B' \cap B = \emptyset \) and \( C_1 \) has five 1-secant blocks. So \( C_1 = B \cup B' - \{x\} \), and \( FV(C_1) = (0, 5, 25, 15, 30, 1, 1) \), where \( x \in B' \).

If \( \{a, b, c, d, e\} \) is not contained in any block, then we claim that there exists a block \( B' \) such that \( B' \) is determined by three points from \( \{a, b, c, d, e\} \) and \( |B' \cap B| = 2 \).

If there exist four points from \( \{a, b, c, d, e\} \), say \( a, b, c, d \), which are contained in a block \( B'' \), then \( B'' \cap B = \emptyset \). Since there are only two blocks on \( \{a, e\} \) which are disjoint from \( B \), one of the blocks determined by \( \{a, b, e\} \), \( \{a, c, e\} \) and \( \{a, d, e\} \), respectively meets \( B \). Let \( B' \) be this block.

If no four points from \( \{a, b, c, d, e\} \) are contained in any block and suppose the block determined by \( \{a, b, c\} \) does not meet \( B \); then one of the blocks determined by \( \{a, b, d\} \) and \( \{a, b, e\} \) respectively meets \( B \). Let this block be \( B' \).

Without loss of generality we may assume \( B' \) is determined by \( \{a, b, c\} \). Since \( B \cup B' \cup \{d\} \) has two external blocks, \( e \) must be in the intersection of the two external blocks.

It not difficult to prove that on each of \( d \) and \( e \) there exists only one block which is 5-secant to \( C_1 \). So \( C_1 \) has three 5-secant blocks. Therefore, \( FV(C_1) = (0, 7, 17, 27, 22, 3, 1) \).

3. Let \( |C_1| = 12 \). First we consider the case when \( C_1 \) contains another block \( B' \).

In this case \( |B \cap B'| = 2 \). Let \( C_1 = (B \cup B') - \{u, v\} \); then \( v \) must lie on the intersection of the two external blocks of \( B \cup B' \cup \{u\} \) and \( C_1 \) has four 5-secant blocks. So \( FV(C_1) = (0, 4, 14, 24, 29, 4, 2) \).
Now suppose $C_1$ contains no block other than $B$; then $C_1$ has more than two 5-secant blocks, hence there exists a block $A$ such that $A$ is 5-secant to $C_1$ and $B - A = 2$. Let $C_1 - (B - A) = \{a, b, c\}$.

If the block $A'$ determined by $\{a, b, c\}$ is not disjoint from $B - A$, then there exists a block $B'$ disjoint from $B$ which contains at least two points in $\{a, b, c\}$ and meets $A$ at the two points in $A - C_1$.

If $B' \cap \{a, b, c\} = 3$, then $C_1 = B - (B' - \{u\} - \{x\}$, where $u \in B'$.

$x \notin B - B'$. It is easy to see that $C_1$ has three 1-secant blocks. So we have $FV(C_1) = (0.3.17.22.27.7.1)$.

If $B' \cap \{a, b, c\} = 2$, then the block $B'' \neq B'$ on $B' - C_1$, which is disjoint from $B$, contains only one of the two points in $C_1 - (B - B')$ (otherwise $C_1$ would contain at least two blocks), so $C_1 = B - (B' - B'' - \{u, v, w\} - \{x\}$, where $u, v, w \in B'' - B'$ and $x \notin B - B' - B''$.

From the structure of a Fano set we can see that $C_1$ has four 1-secant blocks, therefore $FV(C_1) = (0.4.13.28.23.8.1)$.

Now suppose $A'$ is disjoint from $B - A$. We may even assume that any block disjoint from $B$ on two points in $\{a, b, c\}$ contains the unique point in $A - C_1$ (otherwise, we would have a block which contains four points in $C_1$ and is disjoint from $B$). Then the three points in $\{a, b, c\}$ and the point in $A - C_1$ form a Fano set $F$ with three points in $B$. The block on two points in $A' - \{a, b, c\}$ which is disjoint from $B$ is a 1-secant block to $C_1$. There are three of this kind of 1-secant blocks to $C_1$. For any point $x \in \{a, b, c\}$, we have three different blocks on $x$ and a point in $A' - \{a, b, c\}$ which are disjoint from $B$. But only one of them is disjoint from $A$, so it is a 1-secant block.
There are three of this kind of 1-secant blocks to $C_1$. So $C_1$ has six 1-secant blocks. $FV(C_1) = \{0.6.5.40.15.10.1\}$.

4. Let $|C_1| = 13$. If $C_1$ contains another block $B'$, then $|B \cap B'| = 2$ and $C_1 = B \cup B' \cup \{u, v, w\}$, where $u \notin B \cup B'$ and each of the two external blocks $E_1$ and $E_2$ of $B \cup B' \cup \{u\}$ contains at least one of $v, w$.

If $v, w \in E_1 \cap E_2$, then the two external blocks $E_1$ and $E_4$ of $B \cup B'$ which contain $u$ are the only two 1-secant blocks of $C_1$. $B$ and $B'$ are two of the four external blocks of $E_3 \cup E_4$; the other two all contain $\{v, w\}$, so they are contained in $C_1$, thus $FV(C_1) = \{0.2.12.16.40.3.4\}$.

If $v \in E_1 \cap E_2$, $w \in E_1 \triangle E_2$, then $C_1$ has only two 1-secant blocks and contains only two blocks. So $FV(C_1) = \{0.2.10.24.28.11.2\}$.

If $v \in E_1 - E_2$, $w \in E_2 - E_1$, we can even assume that $u, v$ and $w$ are contained in another external block of $B \cup B'$ (otherwise, we reduce the case to the above case). then $C_1$ has only three 1-secant blocks, and contains only two blocks. So $FV(C_1) = \{0.3.6.30.24.12.2\}$.

Now suppose $C_1$ contains only one block $B$, then there exists a block $B'$ which is 5-secant to $C_1$. So $B \cap B' = \emptyset$. Let $C_1 - (B \cup B') = \{u, v\}$ and $a \in B' - C_1$; then the block $B''$ determined by $\{a, u, v\}$ meets $B$ (otherwise, $C_1$ would contain another block). So $C_1 = B \cup [(B' \cap B'') \setminus \{a\}]$. It is easy to prove that $B$ is the only block contained in $C_1$: the block which contains $B'' \cap B'$, and is disjoint from $B$, is the only block which meets $C_1$ at one point.

So $FV(C_1) = \{0.1.13.22.26.14.1\}$.

5. Here we only prove the case of $|C_1| = 14$; the other cases can be proved similarly.
Since $C_1$ is a 1-blocking set, there exists a block $B$ such that $B$ is 1-secant to $C_1$. Let $x \in B \cap C_1$, $S - (C_1 \cup B) = X$, and let $B'$ be the block determined by $X$. If $B' \cap B \neq \emptyset$, then the four external blocks of $B \cup B'$ are the only blocks contained in $C_1$. If $x \not\in B' \cap B$, then $B$ and $B'$ are the only 1-secant blocks of $C_1$, so $FV(C_1) = (0, 2, 3, 24, 30, 14, 4)$. If $x \in B' \cap B$, then $B$ is the only 1-secant block of $C_1$. $FV(C_1) = (0, 1, 7, 34, 13, 4)$. If $B' \cap B = \emptyset$, then $|B' \cap C_1| = 3$. The three blocks which contain two points in $B' \cap C_1$ and are disjoint from $B$ are the only blocks contained in $C_1$: $B$ is the only 1-secant block of $C_1$. So $FV(C_1) = (0, 1, 6, 22, 28, 17, 3)$. 

3.5.2. 2-blocking sets in $S(3,6,22)$.

**Lemma 3.14.** Let $B$ be a block in $S(3,6,22)$. $x, y \in B$, and $A_i, i = 1, 2, 3, 4$, be another four blocks on $\{x, y\}$. Let $a, b \in A_1 - B$, and let $U_1, U_2 \neq A_1$ be another two blocks on $\{a, b\}$ which meet $B$. Then $U_1$ meets one of $A_j, j = 2, 3, 4$, and if $U_1 \cap A_j \neq \emptyset$, then $U_2 \cap A_j \neq \emptyset$ and $(U_1 \cap A_j) \cup (U_2 \cap A_j) = A_j - \{x, y\}$.

**Proof.** Because $|U_1 \cap B| = 2$, $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup B| = 22$ and $|U_1| = 6$, so $U_1$ meets one of $A_2, A_3$, and $A_4$. Suppose $U_1 \cap A_2 \neq \emptyset$. Since $U_2$ also meets one of $A_2, A_3$, and $A_4$, if $U_2 \cap A_2 = \emptyset$, and $U_2$ meets one of $A_3, A_4$, say $A_3$, then $|U_2 \cap A_3| = 2$. Let $V_1, V_2$ be the two blocks on $\{a, b\}$ which do not meet $B$: then neither $V_1$ nor $V_2$ can meet both $A_2$ and $A_3$, while both $V_1$ and $V_2$ can only meet two of $A_2, A_3$, and $A_4$. Suppose $V_1 \cap A_2 \neq \emptyset, V_1 \cap A_4 \neq \emptyset, V_2 \cap A_3 \neq \emptyset$. $V_2 \cap A_4 \neq \emptyset$: then $(V_1 \cap A_2) \cup (U_1 \cap A_2) = A_2 - B$. $(V_2 \cap A_3) \cup (U_2 \cap A_3) = A_3 - B$ and $(V_1 \cup V_2) \cap A_4 = A_4 - B$. Now consider $A_2 \cup B$. The block $V_2$ is external to $A_2 \cup B$ and $a \in V_2$. Let $W$ be another external block to $A_2 \cup B$ which contains $a$:
then \( W \) contains at most one point in each of \((V_2 \cap A_3) \cup (V_2 \cap A_4), V_1 \cap A_4, U_2 \cap A_4\), respectively, so \( W \) meets \( A_3 \cup A_4 \) in at most three points. Thus, \( W \) contains at most five points, which contradicts \(|W| = 6\). So \( U_1 \cap A_2 \neq \emptyset \). Since \( \{a, b\} \subseteq U_1 \cap U_2 \) and \( U_1 \neq U_2 \), we must have \((U_1 \cap A_2) \cup (U_2 \cap A_2) = A_2 \neq \{x, y\}\). \( \square \)

**Lemma 3.15.** Let \( B, x, y, A_i, i = 1, 2, 3, 4 \) be as in lemma 3.14. Let \( a, b, c \in A_1 \setminus \{x, y\} \) and let \( U, V, W \) be blocks on \( \{a, b\}, \{a, c\} \) and \( \{b, c\} \), respectively, which meet \( B \). Then

\[
\{j \mid 2 \leq j \leq 4, A_j \cap (U \cup V \cup W) \neq \emptyset\} = \{2, 3, 4\}.
\]

**Proof.** Suppose \( U \cap A_2 \neq \emptyset, V \cap A_2 \neq \emptyset \); then by lemma 3.14, \( a \) would lie on at least four external blocks of \( A_1 \cup A_4 \), which contradicts result 3.2. \( \square \)

**Lemma 3.16.** Let \( B, B' \) be two disjoint blocks, \( E \) a 13-subset with \( B \subset E \) and \(|E \cap B'| = 5\); then \( E \) is not a 2-blocking set.

**Proof.** Let \( E - (B \cup B') = \{u, v\}, B' = \{a, b, c, d, e, x\}, \) where \( a, b, c, d, e, x \in E \) and \( x \notin E \); let \( r \) be the number of the blocks on any one of \( \{x, u\}, \{x, v\} \), which are disjoint from \( B \). Then by lemma 3.1, \( r \leq 4 \). Because the five blocks on \( \{x, u\}, \{x, v\}, \{x, b\}, \{x, c\}, \{x, d\} \) and \( \{x, e\} \), respectively, which are disjoint from \( B \) and are not equal to \( B' \), are all different, so at least one of them contains neither \( u \) nor \( v \), and this block is 1-secant to \( E \). Therefore \( E \) is not a 2-blocking set. \( \square \)

Similarly we can prove

**Lemma 3.17.** If \( B, B' \) are two disjoint blocks, \( E \) is a 12-subset, \( B \subset E \) and \( 4 \leq |E \cap B'| \leq 5 \), then \( E \) is not a 2-blocking set.
Lemma 3.18. Let $C_2$ be a 2-blocking set in $S(3,6,22)$. Let $U$, $V$ be blocks such that $\emptyset \neq U \cap B = V \cap B \neq B$. If $B \subset C_2$ and $C_2$ contains at least five points in each of $U$ and $V$, then $C_2$ contains a block $B'$ such that $B' \cap B = 2$.

Proof. This is a consequence of lemmas 3.14 and 3.15.

Theorem 3.7. Let $C_2$ be a 2-blocking set in $S(3,6,22)$.

1. If $C_2$ does not contain any block, then $|C_2| = 14$ and $C_2 = S - (F - \{x\})$, $FV(C_2) = (0,0,7,28,14,28,0)$, where $F$ is a Fano set and $x \notin F$.

2. If $C_2$ contains a block, then $12 \leq |C_2| \leq 15$.

(a) If $|C_2| = 12$, then $C_2 = B - B'$, $FV(C_2) = (0,0,30,0,45,0,21)$, where $B$, $B'$ are blocks and $B' \cap B' = \emptyset$.

(b) If $|C_2| = 13$, then $C_2 = B - B' - \{x\}$, $FV(C_2) = (0,0,18,12,36,9,2)$, where $B$ and $B'$ are two disjoint blocks and $x \notin B - B'$.

(c) If $|C_2| = 14$, then $C_2$ is one of the following:

(i) $C_2 = B - B' - B''$, $FV(C_2) = (0,0,14,0,56,0,7)$, where $B$, $B'$ and $B''$ are blocks with $B \cap B' - B'' = 2$.

(ii) $C_2 = B \cup B' \cup B''$, $FV(C_2) = (0,0,10,16,32,16,31)$, where $B$, $B'$ and $B''$ are blocks with $B \cap B' = 2 = B' \cap B''$, and $(B' \cap B'') \cap (B' - B'') = \emptyset$.

(iii) $C_2 = B \cup B' \cup (B'' - \{u\}) \cup \{x\}$, $FV(C_2) = (0,0,9,20,26,20,2)$, where $B$, $B'$ and $B''$ are blocks with $B \cap B' = 2 = B' \cap B''$, and $(B \cap B') \cap (B' \cap B'') = \emptyset$. $u \in B'' - B'$ and $x \notin B - B' \cup B''$.

(d) If $|C_2| = 15$, then $C_2$ is one of the following:

(i) $C_2 = S - (B \Delta B') - \{u\}$, $FV(C_2) = (0,0,7,7,42,14,7)$, where $B$ and $B'$ are blocks, $u \in B \Delta B'$.
(ii) \( C_2 = \mathcal{S} - [(F - \{v\}) \cup \{u\}] \). \( FV(C_2) = (0.0.4.19.24.26.4) \). where \( F \)
is a Fano set. \( o \in F \) and \( u \notin F \).

(iii) \( C_2 = \mathcal{S} - [(F - \{o,v\}) \cup \{u,v\}] \). \( FV(C_2) = (0.0.5.15.30.22.5) \).

where \( F \) is a Fano set \( o,x \in F, u,v \notin F \).

(e) If \( |C_2| = 16 \), then \( C_2 \) is one of the following:

(i) \( C_2 = \mathcal{S} - (B \triangle B' - \{u,v\}) \). \( FV(C_2) = (0.0.3.8.33.24.9) \). where \( B \)
and \( B' \) are two blocks. \( |B \cap B'| = 2 \) and \( u,v \in B' - B \).

(ii) \( C_2 = \mathcal{S} - (B \cup B' - \{u,v,w,o\}) \). \( FV(C_2) = (0.0.2.12.27.28.8) \).

where \( B \) and \( B' \) are two blocks. \( |B \cap B'| = 2 \), \( o \in B \cap B', u,v \in B' - B \)
and \( w \in B - B' \).

(f) If \( |C_2| = 17 \), then \( C_2 = \mathcal{S} - (B \triangle B' - \{u,v,w\}) \), and

\( FV(C_2) = (0.0.1.6.26.31.13) \). where \( B \) and \( B' \) are two blocks with

\( |B \cap B'| = 2 \) and \( u,v,w \in B' - B \).

(g) If \( |C_2| = 18 \), then \( C_2 = \mathcal{S} - (B - \{u,v\}) \). \( FV(C_2) = (0.0.1.0.24.32.20) \).

where \( B \) is a block and \( u,v \in B \).

Proof. We prove the theorem case by case:

1. Since \( C_2 \) does not contain any block, \( C_2 \) actually is a blocking set. By checking
the type of the blocking sets we obtain \( |C_2| = 14 \) and

\( C_2 = \mathcal{S} - (F \cup \{x\}) \). where \( F \) is a Fano set. \( x \notin F \).

2. Let \( B \subseteq C_2 \) be a block. By checking the appendix A. we can obtain that
\( 12 \leq |C_2| \leq 18 \).

(a) \( |C_2| = 12 \).

Since both \( B \cup B' \cup \{u,v\} \) and \( (B \cup B' - \{x\}) \cup \{u,v,w\} \), where \( |B \cap B'| = 2 \).

\( x \in B' - B, u,v,w \notin B \cup B' \), are not 2-blocking sets. any block determined
by three points in $C_2 - B = \{a, b, c, d, e, f\}$ is disjoint from $B$. Let $B'$ be the block determined by $\{a, b, c\}$; we claim that $d, e, f \notin B'$. Otherwise, by lemma 3.17 $d, e, f \notin B'$ and one of the blocks determined by $\{e, d, f\}$, $\{e, d, a\}$, $\{e, d, b\}$ and $\{e, d, c\}$ would meet $B$. Thus $C_2 = B \cup B'$ and $B \cap B' = \emptyset$. It is easy to see that $C_2$ contains only two blocks, so we have $FV(C_2) = (0.0, 0.30, 0.45, 0.2)$.

(b) $|C_2| = 13$.

If $C_2$ contains another block $B'$, then $B \cap B' = \emptyset$. Otherwise each of the four external blocks of $B \cup B'$ would contain at least two of the three points in $C_2 - (B \cup B')$: this is not possible since each point outside of $B \cup B'$ lies on exactly two of the four external blocks. So $B \cap B' = \emptyset$.

Let $\{x\} = C_2 - (B \cup B')$: then we are done.

Now we prove that $C_2$ does really contain another block.

Suppose $C_2$ does not contain another block.

It is not difficult to prove that there exists a block $B''$ which is 5-secant to $C_2$. By lemma 3.16 $|B \cap B''| = 2$. Let $B'' - B = \{a, b, c, o\}$, where $a, b, c \in C_2$ and $o \notin C_2$. Let $u, v, x, y \in C_2 - (B \cup B'')$ and let $E_1, E_2$ be two external blocks of $B \cup B'' \cup \{u\}$. Then each of $E_1, E_2$ contains at least two of $v, x$ and $y$: therefore $E_1 \cap E_2$ contains at least one of $v, x$ and $y$, say $v$. Since in $\mathcal{R}(B)$ there are two blocks on $\{x, y\}$, one of them, say $U$, meets $B''$ at two of $a, b$ and $c$, say $a$ and $b$. Let $V$ be the block determined by $\{u, v, o\}$; then $V \in \mathcal{R}(B)$ and $|V \cap B''| = 2$. Since by lemma 3.16 there is no block in $\mathcal{R}(B)$ which is 5-secant to $C_2$, $\{x, y\}$ is not contained in $V$.

If $V \cap B'' = \{c, o\}$, then $U \cap V$ contains a point $w \notin C_2$. Then the block in $\mathcal{R}(B)$, which goes through $\{w, o\}$ and is different from $V$, can not contain
$u, v, c$ and contains at most one of $a, b, x, y$. So this block contains at most one point in $C_2$. If $V \cap B'' = \{a, o\}$, let $V' \in R(B)$ be the other block on $\{u, v\}$. Then $b, c \in V'$ and therefore by lemma 3.16 $x, y \notin V'$. So there are two points $p, q \in V'$ such that $p, q \notin C_2, p \in V' \cap U$ and $q \in V' - U$. The second block in $R(B)$ which goes through $\{p, q\}$ cannot contain $u, v, b, c$ and contains at most one of $x, y, a$, so this block contains at most one point in $C_2$.

(c) $|C_2| = 14.$

We prove that $C_2$ contains two blocks $A, A'$ such that $|A \cap A'| = 2$.

Let $T$ be a block which is 2-secant to $C_2$, let $C_2 \cap T = \{x, y\}$ and let $A_i (i = 1, 2, 3, 4)$ be the other four blocks on $\{x, y\}$.

If at least two of $A_i (i = 1, 2, 3, 4)$ are contained in $C_2$, then we are done: if only one of them is contained in $C_2$, then two of the rest are 5-secant to $C_2$, and so by lemma 3.18, the conclusion is also true.

If $|T \cap B| = 2$, then we are done.

If $|T \cap B| = 0$ we may even assume that none of $A_i (i = 1, 2, 3, 4)$ is contained in $C_2$, so $A_i (i = 1, 2, 3, 4)$ are all 5-secant to $C_2$, and one of $A_i (i = 1, 2, 3, 4), say A_1,$ is disjoint from $B$. Let $a, b$ and $c$ be the three points contained in $C_2 - (B \cup T \cup A_1); let U, V$ and $W$ be the other three external blocks of $T \cup A_1, a \in U \cap V$.

(i) If $U \cap V \subset C_2$, then the block determined by $(W \cap B) \cup \{a\}$ contains $U \cap V, T \cap A_1$ and $W \cap B$. Let this block and $B$ be $A$ and $A'$, respectively: then we are done.

(ii) If $U \cap V$ is not contained in $C_2$, then $b, c \in W$. If $b, c \in U$ or $b, c \in V$, then the block determined by $(B \cap V) \cup \{b\}$ or $(B \cap U) \cup \{b\}$ contains
Let $A, A'$ be two blocks contained in $C_2$ and $A \cap A' = 2$; then

$C_2 - (A \cup A') = \{a, b, c, d\}$. Since $C_2$ is a 2-blocking set, each of the four external blocks of $A \cup A'$ contains exactly two of $a, b, c$ and $d$. Now we consider the block $A''$ determined by $\{a, b, c\}$. $A''$ contains another point outside $A \cup A$. If $d \notin A''$, then either $A \cap A' \subseteq A''$ or $A''$ meets only one of $A$ and $A'$. If $A \cap A' \subseteq A''$, then $C_2 = A \cup A' \cup A''$ and $C_2$ contains seven blocks. So $FV(C_2) = (0, 0, 14, 0, 56, 0, 7)$. If $A''$ meets only one of $A$ and $A'$, say $A'$, then $C_2 = A \cup A' \cup A''$, $A'' \cap A = \emptyset$. In this case $C_2$ contains only three blocks. So $FV(C_2) = (0, 0, 16, 32, 16, 3)$. If $d \notin A''$, then $A''$ meets only one of $A$ and $A'$, say $A'$. In this case $C_2$ contains only two blocks, so $FV(C_2) = (0, 0, 9, 20, 26, 20, 2)$.

(d) $C_{2, 1} = 15$.

Let $B$ be a block which is 2-secant to $C_2$, let $B'$ be the block determined by the three points in $S - (C_2 \cup B)$; then either $B' \cap (B \cap C_2) \neq \emptyset$ or $B' \cap B = \emptyset$. If $B' \cap (B \cap C_2) \neq \emptyset$, we may even assume that $B \cap C_2 \subseteq B'$ (otherwise it could reduced to the other case), then $C_2$ contains seven blocks. So $FV(C_2) = (0, 0, 7, 7, 42, 14, 7)$. If $B' \cap B = \emptyset$, then we choose a point $o$ in $C_2$ such that $\{o\} \cup (B' - C_2)$ and a point in $B - C_2$ forms a Fano.
set $F$. If $F$ contains no point in $B \cap C_2$, then $C_2 = S - (F - \{o\} \cup \{u\}]$. where $u \notin F$. From the structure of a Fano set we know that $C_2$ contains four blocks. So $FV(C_2) = (0.0.4.19.24.26.4)$. If $F$ contains one point $x$ in $B \cap C_2$, then $C_2 = S - [(F - \{o,x\} \cup \{u,v\}]$, where $o,x \in F$. In this case $C_2$ contains five blocks. So $FV(C_2) = (0.0.5.15.30.22.5)$.

(e) $|C_2| = 16$.

Let $B$ be a block which is 2-secant to $C_2$, then $|S - (B \cup C_2)| = 2$. Let $B'$ be a block determined by $S - (B \cup C_2)$ and one point in $B \cap C_2$. If $B'$ contains the other point in $B \cap C_2$, then $C_2 = S - (B \cap B' - \{u,v\}]$. where $u,v \in B' - B$. It is easy to see that $C_2$ contains nine blocks. So $FV(C_2) = (0.0.3.8.33.24.9)$. If $B$ does not contain the other point in $B \cap C_2$, then $C_2 = [S - (B \cup B' - \{u,v,w,o\}]$, $o \in B \cap B'$. $u,v \in B' - B$ and $w \in B - B'$. It is easy to see that there exist only two 2-secant blocks to $C_2$. So $FV(C_2) = (0.0.2.12.27.28.8)$.

The cases of $|C_2| = 17$ and $|C_2| = 18$ can be proved similarly.

\[\square\]

3.5.3. 3-blocking sets in $S(3.6.22)$.

**Theorem 3.8.** We have $15 \leq |C_3| \leq 19$.

1. If $|C_3| = 15$. then $C_3 = S - F$. $FV(C_3) = (0.0.0.35.0.42.0)$. where $F$ is a Fano set.

2. If $|C_2| = 16$. then $C_3 = (S - F) \cup \{x\}$. $FV(C_3) = (0.0.0.20.15.36.6)$. where $F$ is a Fano set and $x \in F$.

3. If $|C_3| = 17$, then $C_3 = (S - F) \{x,y\}$. $FV(C_3) = (0.0.0.10.20.35.12)$. where $F$ is a Fano set. $x,y \in F$. 


4. If $|C_3| = 18$, then $C_3 = (S - F) \cup \{x, y, z\}$. $FV(C_3) = (0, 0, 0, 4, 18, 36, 19)$.
   where $F$ is a Fano set. $x, y, z \in F$.

5. If $|C_3| = 19$, then $C_3 = (S - F) \cup \{x, y, z, w\}$. $FV(C_3) = (0, 0, 0, 1, 12, 36, 28)$.
   where $F$ is a Fano set. $x, y, z, w \in F$.

Proof. Let $C = (B \cup B' - A) \cup X$, where $B, B'$ are blocks. $B \cap B'' = 2$. $A \subseteq B' - B$.

0 \leq |A| \leq 1. |X| = 4 + |A|$. $X \cap (B \cup B') = \emptyset$. Since $B \cup B'$ has four external blocks.
it is not difficult to prove that $C$ is not a 3-blocking set.

Let $C = (B \cup B' - A) \cup X$, where $B, B'$ are two disjoint blocks. $A \subseteq B'$.

0 \leq |A| \leq 3. |X| = 2 + |A|$. $X \cap (B \cup B') = \emptyset$. If $|A| = 3$, let $a, b \in A$, let $B''$ be
another block on $\{a, b\}$ which is disjoint from $B$. If $B'' \cap X \geq 3$, then at least one
of the four external blocks of $B'' \cup B'$ contains no more than two points from $C$.
so $C$ is not a 3-blocking set. When $|A| = 0, 1$ or 2, we can similarly prove that $C$
is not a 3-blocking set either.

So $15 \leq |C_3|$. It is easy to see that $|C_3| \leq 19$.

Suppose $|C_3| = 15$; then $C_3 = S - F$, where $F = \{a, b, c, x, y, z, o\}$. Since $C_3$
is a 3-blocking set, any block determined by three points of $F$ contains exactly
three points of $F$. Let $B$ and $B'$ be the blocks determined by $\{x, y, z\}$ and $\{a, b, c\}$,
respectively; then $B \cap B' = \emptyset$ (otherwise two of the blocks determined by $\{x, y, a\}$,
$\{x, y, b\}$ and $\{x, y, c\}$ would be the same block, and this block would contain four
points from $\{a, b, c, x, y \subseteq F\}$. So $F$ is a Fano set.

Now suppose $|C_3| = 15 + r$. $1 \leq r \leq 4$; then $C_3 = S - (\{a, b, c\} \cup A)$, where
$|A| = 4 - r$. $A \cap \{a, b, c\} = \emptyset$. Let $B'$ be the block determined by $\{a, b, c\}$. let
$B \in R(B')$ be a block on $A$ (if $A = \emptyset$; then let $B \in R(B')$). let $x \in A$ (if $A = \emptyset$;
then choose any point $x \in B$). Denote by $B_1, B_2$ and $B_3$ the blocks determined by
\{a, b, x\}, \{a, c, x\} and \{b, c, x\}, respectively: then \(B_i \neq B_j\) if \(i \neq j\). Let
\[
\begin{align*}
x & \in B_1 \cap B - \{x\}, \\
y & \in B_2 \cap B - \{x\}, \\
z & \in B_3 \cap B - \{x\} \text{ and } \{x, y, z\} = B - \{x, y, z\};
\end{align*}
\]
then any block determined by three points from \(\{a, b, c, x, y, z\}\) contains exactly three points in \(\{a, b, c, x, y, z\}\). By theorem 3.3, there exists a point \(o\) such that
\[F = \{a, b, c, x, y, z, o\}\] is a Fano set. Let \(X = F - (\{a, b, c\} \cup \mathcal{A})\); then \(X = r\) and \(C_3 = (S - F) \cup X\).

3.5.4. **t-blocking sets, \(t = 4.5\), in \(S(3, 6, 22)\).** The following theorem is easy to prove, so we omit the proof.

**Theorem 3.9.**
\[
C_1 = S - \{x, y\}, \quad FV(C_1) = (0, 0, 0, 0, 5.32, 40); \\
C_3 = S - \{x\}, \quad FV(C_3) = (0, 0, 0, 0, 21.56).
\]

3.6. **t-blocking sets in \(S(4, 7, 23)\).**

3.6.1. **Some known results about \(S(4, 7, 23)\).** We recall the following:

**Result 3.4.** ([40], 5.5) Every block in \(S(4, 7, 23)\) is of type \((1, 3, 7)\).

Check the appendix B we know that the frequency vector of a block is
\[(0, 112, 0, 140, 0, 0, 0, 1).\]

**Result 3.5.** (L. Berardi [8] lemma 2.3.) Let \(B, B'\) be two blocks in \(S(4, 7, 23)\) with \(|B \cap B'| = 3\). Then \(FV(B \cup B') = (0, 12, 48, 75, 80, 36, 0, 2)\).

**Result 3.6.** ([8], lemma 2.4) Let \(B, B'\) be two blocks in \(S(4, 7, 23)\) with \(|B \cap B'| = 3\).
Fix \(x \in B - B', y \in B' - B\) and \(u \notin B \cup B'\). Then there exists at least one block through \(x, y, u\) intersecting \(B \cup B'\) only at \(x\) and \(y\).
Result 3.7. [8. lemma 2.5] Let \( B, B' \) be two blocks in \( S(4.7.23) \) with \( |B \cap B'| = 3 \).

Fix \( x \in B - B' \) and \( y \in B' - B \). There are exactly three blocks \( E_1, E_2 \) and \( E_3 \) intersecting \( B \cup B' \) only at \( x \) and \( y \). Moreover, \( E_1 \cap E_2 \cap E_3 = \{x, y\} \).

Result 3.8. [8. lemma 2.7] Let \( B, B' \) be two blocks in \( S(4.7.23) \) with \( B \cap B' = 1 \). The symmetric difference \( B \triangle B' \) is a 12-set of type \((2.4.6)\), so it is a reducible blocking set. \( FV(B \triangle B') = (0.0.66.0.165.0.22.0) \).

Result 3.9. [8. lemma 2.10] Let \( B, B' \) be two blocks in \( S(4.7.23) \) with \( B \cap B' = 1 \).

Then \( FV(B \cup B') = (0.0.36.30.120.45.20.2) \).

Result 3.10. [8. lemma 2.11] Let \( B, B' \) be two blocks in \( S(4.7.23) \) with \( B \cap B' = 1 \).

Fix \( x \in B - B' \) and \( y \in B' - B \). There exists exactly one block \( B'' \) intersecting \( B \cup B' \) at \( x, y \) exactly.

Result 3.11. [8. 2.12] Let \( B, B' \) be two blocks in \( S(4.7.23) \) with \( B \cap B' = \{w\} \). Fix \( x \in B - \{w\}, y \in B' - \{w\} \) and \( z \notin B \cup B' \). Then the block \( B'' \) that is 2-secant to \( B \cup B' \) at \( x, y \) contains \( z \) iff \( z \) is on a block through \( x, y \) and 6-secant to \( B \cup B' \).

Here we use the same terminology and notations as in [8]. Let \( B \) and \( B' \) be two blocks. Define:

\[
E_0 := S - (B \triangle B'), \text{ where } |B \cap B'| = 1;
\]

\[
E_1 := B \cup B' - \{x, u\}, \text{ where } B \cap B' = \{x\} \text{ and } u \in B' - B.
\]

By [8] and theorem 2.2 we have

Lemma 3.19. Let \( C \) be a blocking set in \( S(4.7.23) \); then \( |C| = 11 \) or \( 12 \) and

1. if \( |C| = 11 \), then \( C = E_0 \) or \( E_1 \);

2. if \( |C| = 12 \), then \( C = S - E_0 \) or \( S - E_1 \).
**Lemma 3.20.** Let $B$ be a block in $S(4,7,23)$, and $x,1,2,3,4$ five points not in $B$. Then there is a block $B'$ such that $B'$ contains at least four of $x,1,2,3,4$ and $|B \cap B'| = 1$.

**Proof.** If $x,1,2,3,4$ are contained in a block, then we are done; otherwise the five different 4-subsets of $\{x,1,2,3,4\}$ determine five different blocks, and at least four of them meet $B$ at only one point. \qed

Let $B$ and $B'$ be two blocks in $S(4,7,23)$ with $B \cap B' = 1$, and let

$B - B' = \{a_1,a_2,a_3,a_4,a_5,a_6\}$, $x \notin B' - B$. Denote by $U_i$ the unique block that intersects $B \cup B'$ only at $\{x,a_i\}$, $i = 1, \ldots , 6$.

**Lemma 3.21.** The six blocks $U_i$, $i = 1, \ldots, 6$, satisfy the following conditions:

1. $U_i \cap U_j \geq 3$, and $U_i \cap U_j = 3$ if $i \neq j$;

2. Any point in $S - (B \cup B')$ lies on exactly three of $U_i$, $i = 1, \ldots, 6$. Any two points in $S - U_i \cup B'$ are contained in at least one of $U_i$, $i = 1, \ldots, 6$.

**Proof.** 1. Suppose $U_1 \cap U_2 \cap U_3 \leq 3$: then $U_1 \cap U_2 \cap U_3 = 1$, hence $U_1 \cap U_2 = 1$.

2. If there exists $w \in S - (B \cup B')$ such that $w \in U_1 \cap U_2 \cap U_3$, then one of $U_i$, $i = 1, \ldots, 6$, contains at most six points.

If any point in $S - (B \cup B')$ lies on one or two of $U_i$, $i = 1, \ldots, 6$, then there must exist another point which lies on four or more than four blocks.

Let $a, b \in S - (B \cup B')$. Suppose $a$ lies on $U_1, U_2$ and $U_3$. Since $S - (B \cup B') = 10$.

$$|\left((S - (B \cup B')) \cap \left(\bigcap_{i=1}^{3} U_i\right)\right)| = 1,$$

and $a$ is the only point in $S - B \cup B'$ that lies on $U_1 \cap U_2 \cap U_3$. Since

$$|U_i \cap U_j| = 3, \ i \neq j, \ 1 \leq i, j \leq 3.$$
there are three points in
\[
\left( \bigcup_{i=1}^{3} U_i \right) - B \cup B' \cup \{a\}.
\]
every one of which lies on two of \( U_i \), \( i = 1, 2, 3 \); each one of the remaining points
lies on only one of \( U_i \), \( i = 1, 2, 3 \). Therefore
\[
3 \times 2 + \left| \left( \bigcup_{i=1}^{3} U_i \right) - B \cup B' \cup \{a\} \right| = 3 \times 4 = 12.
\]
so
\[
\left| \left( \bigcup_{i=1}^{3} U_i \right) - B \cup B' \cup \{a\} \right| = 9.
\]
and \( b \) lies on one of \( U_i \), \( i = 1, 2, 3 \).

Let \( B \) and \( B' \) be two blocks in \( S(4, 7, 23) \) with \( B \cap B' = \{a, b, c\} \).

**Lemma 3.22.** There are exactly three blocks \( T_1, T_2, \) and \( T_3 \) on \( \{a, b, c\} \) such that
\( T_i \cap (B \triangle B') = \emptyset \) for \( i = a, b, c \).

**Proof.** There are five blocks on \( \{a, b, c\} \). \( B \) and \( B' \) being two of them. So the
remaining three can not contain any point in \( B \triangle B' \).

**Lemma 3.23.** For \( i \in \{a, b, c\} \), there are exactly four blocks \( T_{ij} \) (\( j = 1, 2, 3, 4 \))
such that \( T_{ij} \cap (B \cup B') = \{i\} \), \( |T_{ij} \cap T_{ik}| = 3 \) and each point in \( S - (B \cup B') \) lies
on only two of them.

**Proof.** There are twelve 1-secant blocks to \( B \cup B' \), so each one of them meets \( B \cup B' \)
at one point in \( B \cap B' \). If there is a point \( x \in S - (B \cup B') \) such that \( x \) lies on
three blocks \( T_{i1}, T_{i2} \) and \( T_{i3} \), which meet \( B \cup B' \) at \( i \), then one of \( T_{i1}, T_{i2} \) and \( T_{i3} \)
would contain at most six points, a contradiction. So the twelve 1-secant blocks to
\( B \cup B' \) can be divided into three groups, each group with four blocks, all of which
meet $B \cap B'$ at the same point; each point in $S - (B \cup B')$ lies on exactly two blocks of the group. Let $T_{ij}$ ($j = 1, 2, 3, 4$) be the four blocks which meet $B \cap B'$ at $i$. Then $|T_{ij} \cap T_{ik}| = 3$.

**Lemma 3.24.** Fix $i \in \{a, b, c\}$. For any $l \in \{a, b, c\}$ we have $T_{ij} \cap T_l - \{i\} = 2$ ($j = 1, 2, 3, 4$) and either

$$T_{ij} \cap T_l - \{i\} = T_{ik} \cap T_l - \{i\}$$

or

$$(T_{ij} \cap T_l - \{i\}) \cup (T_{ik} \cap T_l - \{i\}) = T_l - \{a, b, c\}.$$

**Proof.** If $|T_{ij} \cap T_l - \{i\}| \neq 2$, then $T_{ij} \cap T_l - \{i\} = \emptyset$ and one of the blocks $T_{ik}$ ($j \neq k \in \{1, 2, 3, 4\}$) would contain at most five points.

If there exist $j, k \in \{1, 2, 3, 4\}$ and $j \neq k$ such that

$$T_{ij} \cap T_l - \{i\} \neq T_{ik} \cap T_l - \{i\}$$

and

$$(T_{ij} \cap T_l - \{i\}) \cup (T_{ik} \cap T_l - \{i\}) = T_l - \{a, b, c\}.$$

then $|T_{ij} \cap T_l - \{i\} \cap (T_{ik} \cap T_l - \{i\})| = 1$. Let $(T_{ij} \cap T_l - \{i\}) \cap (T_{ik} \cap T_l - \{i\}) = \{x\}$ and $T_{ij} \cap T_{ik} - \{i, x\} = \{a\}$, where $a \in T_m$, $m \neq l$. Then the block determined by $(T_{ij} \cup T_{ik}) \cap T_l$ and $T_{ij} \cap T_m - \{a, i\}$ would either not meet at least one of $B$ and $B'$ or meet one of them at two points. □

Fix $u \in B - B'$, $v \in B' - B$. Let $B''$ be the block determined by $\{i, j, u, v\}$, where $i, j \in \{a, b, c\}$.

**Lemma 3.25.** 1. $|B'' - (B \cup B')| \cap T_l| = 1$ for $l = a, b, c$. 


2. For a fixed \( l \in \{1, 2, 3, 4\} \). \( B'' - (T_{lk} - \{l\}) = 0 \) or \( 2 \) for \( k = 1, 2, 3, 4 \). Moreover, there exists only one \( k \in \{1, 2, 3, 4\} \) such that \( B'' - (T_{lk} - \{l\}) = 0 \).

3. For \( x \in \{a, b, c\} - \{1, 2\} \) we have \( B'' - T_{xj} = 1 \) or \( 3 \) (\( j = 1, 2, 3, 4 \)) and there exists only one \( k \in \{1, 2, 3, 4\} \) such that \( B'' - T_{jk} = 3 \).

Proof. The results are the consequence of lemma 3.23 and the fact that any two blocks meet either at one point or three points. \( \square \)

**Lemma 3.26.** Let \( A_i \ (i = 1, 2, 3) \) be the three blocks which meet \( B \cap B' \) only at \( x \) and \( x' \) where \( x \in B - B' \).

1. If \( x = u \) then

\[
A_i - (B'' - B \cup B') = 1, \ i = 1, 2, 3.
\]

and

\[
[A_2 - A_1] - (B'' - B \cup B') = B'' - B - B'.
\]

2. If \( x = u \) then one of \( A_1, A_2, A_3 \) does not meet \( B'' - B \cup B' \) and the remaining two meet \( B'' - B \cup B' \) at two points.

Proof. Since \( A_i - (B - B') = 5 \), \( A_i \) meets one of \( T_1 = \{a, b, c\} \), \( T_2 = \{a, b, c\} \) and \( T_3 = \{a, b, c\} \) at three points, and meets each of the remaining two at one point.

Let

\[
T_1 = \{a, b, c, \alpha, J, \gamma, \theta\}
\]

\[
T_2 = \{a, b, c, x, y, z, w\}
\]

\[
T_3 = \{a, b, c, p, q, s, t\}
\]

and \( B'' = \{a, b, u, v, \alpha, x, p\} \).
1. If \( x = u \), then from \( u, v \in A_i \cap B'' \) we know that \( |A_i \cap (B'' - B \cup B')| = 1 \) for \( i = 1, 2, 3 \).

In order to prove \( (A_1 \cup A_2 \cup A_3) \cap (B'' - B \cup B') = B'' - B \cup B' \), we only need to prove that \( A_i \cap (B'' - B \cup B') \neq A_j \cap (B'' - B \cup B') \) if \( i \neq j \). Suppose

\[ A_1 \cap (B'' - B \cup B') = A_2 \cap (B'' - B \cup B') \]

without loss of generality we assume

\[ A_1 \cap (B'' - B \cup B') = \{ \alpha \} \]. Then one of \( A_1 \) and \( A_2 \) meets one of \( T_2 - \{ a, b, c \} \) and \( T_3 - \{ a, b, c \} \) at three points, say \( A_1 \) meets \( T_2 - \{ a, b, c \} \) at three points. Then either \( x \in A_1 \) or \( x \in A_2 \). So either \( |A_1 \cap B''| \geq 4 \) or \( |A_2 \cap B''| \geq 4 \). contradiction.

2. Now suppose \( x \neq u \). Then \( |A_i \cap (B'' - B \cup B')| = 0 \) or \( 2 \) for \( i = 1, 2, 3 \).

If \( |A_1 \cap (B'' - B \cup B')| = 0 \), suppose \( |A_1 \cap (T_1 - \{ a, b, c \})| = 3 \), then

\[ A_1 \cap (T_1 - \{ a, b, c \}) = \{ \beta, \gamma, \theta \} \]

so

\[ |A_2 \cap (T_1 - \{ a, b, c \})| = 1 = |A_3 \cap (T_1 - \{ a, b, c \})| \]

Therefore.

\[ |A_2 \cap (T_2 - \{ a, b, c \})| = 3 = |A_3 \cap (T_2 - \{ a, b, c \})| \]

or

\[ |A_2 \cap (T_3 - \{ a, b, c \})| = 3 = |A_3 \cap (T_3 - \{ a, b, c \})| \]

Assume \( |A_2 \cap (T_2 - \{ a, b, c \})| = 3 = |A_3 \cap (T_3 - \{ a, b, c \})| \).

If \( A_2 \cap (T_1 - \{ a, b, c \}) \not\subset B'' - B \cup B' \) or \( A_3 \cap (T_1 - \{ a, b, c \}) \not\subset B'' - B \cup B' \),

then \( A_2 \cap (T_2 - \{ a, b, c \}) \) or \( A_3 \cap (T_3 - \{ a, b, c \}) \) would contain a point in


\(B'' = B \cup B'\). So

\[
A_2 \cap (B'' - B \cup B') = 2 = A_1 \cap (B'' - B \cup B') .
\]

If \(|A_i \cap (B'' - B \cup B')| = 2\) for \(i = 1, 2, 3\), then

\[
\bigcup_{i=1}^3 (A_i - B'') = (A_1 - A_2 - A_3) - B'' = (T_1 \cup T_2 \cup T_3) - (B'' - \{a, b, c\}).
\]

\(\bigcup_{i=1}^3 (A_i - B'')\) is a partition of \((T_1 \cup T_2 \cup T_3) - (B'' - \{a, b, c\})\). Let \(T_{ik}\) be the block in lemma 3.25, which satisfies \(B'' - (T_{ik} - \{i\}) = 0\). then

\[
T_{ik} - \{i\} \subset (T_1 \cup T_2 \cup T_3) - (B'' - B \cup B').
\]

so

\[
(T_{ik} - \{i\}) \cap (A_1 - A_2 - A_3) = T_{ik} - \{i\}
\]

and

\[
\sum_{j=1}^3 (T_{ik} - \{i\}) \cap A_j = 6.
\]

This is impossible since \((T_{ik} - \{i\}) \cap A_j = 1\) or \(3\) for \(j = 1, 2, 3\). Therefore,

\[
\sum_{j=1}^3 (T_{ik} - \{i\}) \cap A_j = 3, 5, 7, 9.
\]

\(\square\)

Let \(B\) and \(B'\) be two blocks in \(S(4, 7, 23)\) with \((B \cap B') = 1\). We know that there are twenty 6-secant blocks to \(B \cup B'\). Moreover, we have

**Lemma 3.27.** On each point in \(S - (B \cup B')\), there are two 6-secant blocks to \(B \cup B'\) and these two blocks meet only at this point.
Proof. We only need to prove the following:

Suppose \( x \notin B \cup B' \). If \( A \) is a 6-secant block to \( B \cup B' \) which contains \( x \), then \[ \{ x \} \cup \left[ B - (A \cap B) \cup (B \cap B') \right] \cup \left[ B' - (A \cap B) \cup (B \cap B') \right] \] is a block.

Let \( A' \) be the block determined by \( \{ x \} \cup \left[ B - (A \cap B) \cup (B \cap B') \right] \). We prove \( A' \) contains \( B' - \left[ (A \cap B') \cup (B \cap B') \right] \).

First we prove \( |A' \cap B'| = 3 \). If \( |A' \cap B'| = 1 \), then \( (A' \cap B') \cap A = \emptyset \). The block determined by \( x \), the point in \( A' \cap B' \), a point in \( A \cap B' \) and a point in \( B' - \left[ (B \cap B') \cup (A \cap B') \cup (A' \cap B') \right] \) would meet \( B \) at two points, a contradiction.

If \( |A' \cap B'| = 3 \), but \( (A' \cap B') \cap A \neq \emptyset \), then \( (A' \cap B') \cap A \cap B = \emptyset \). The block determined by \( x \), the two points in \( B' - \left[ (B \cap B') \cup (A \cap B') \cup (A' \cap B') \right] \) and a point in \( A \cap A' \cap B' \) would also meet \( B \) at two points.

So \( A' \cap B' = B' - \left[ (A \cap B') \cup (B \cap B') \right] \).

3.6.2. 1-blocking sets in \( S(4.7.23) \).

**Theorem 3.10.** Let \( C_1 \) be a 1-blocking set in \( S(4.7.23) \). Then \( 7 \leq |C_1| \leq 17 \).

1. If \( |C_1| = 7 \), then \( C_1 = B \) with \( FV(C_1) = (0.112, 0.140, 0.0.0.0.1) \), where \( B \) is a block.

2. If \( |C_1| = 7 - i \), \( 1 \leq i \leq 3 \), then \( C_1 = B \cup X \) with frequency vectors

\[
(0.70, 0.42, 1.05.35, 0.0.1), (0.42, 0.56, 0.91, 0.56, 7.0.1) \text{ and } (0.24, 0.54, 0.85, 0.70, 1.1.1)
\]

respectively: where \( B \) is a block, \( |X| = i \) and \( X \cap B = \emptyset \).

3. If \( |C_1| = 11 \), then \( C_1 \) is one of the following:

(a) \( C_1 = E_0 \) with \( FV(C_1) = (0.22, 0.165, 0.66, 0.0) \):

(b) \( C_1 = E_1 \) with \( FV(C_1) = (0.11, 0.55, 0.55, 0.11, 0.11) \):

(c) \( C_1 = B \cup B' \) with \( FV(C_1) = (0.12, 0.48, 0.75, 0.80, 0.36, 0.2) \), where \( B \) and \( B' \)

are blocks with \( |B \cap B'| = 3 \).
(d) $C_1 = B \cup B' - \{x, y\}$ with $FV(C_1) = (0.13, 44.80, 80, 31.4, 11)$. where $B$ and $B'$ are blocks with $B \cap B' = 1$ and $x, y \in B' - B$.

4. If $|C_1| = 12$, then $C_1$ is one of the following:

(a) $C_1 = S - E_1$ with $FV(C_1) = (0.11, 11.110, 55, 55, 11, 0)$:

(b) $C_1 = B \cup B' - \{a\}$ with $FV(C_1) = (0.6, 34.70, 85.50, 6.2)$. where $B$ and $B'$ are blocks with $B \cap B' = 3$ and $a \notin B \cup B'$.

(c) $C_1 = B \cup B' - \{x\}$ with $FV(C_1) = (0.6, 35.65, 95.40, 11.1)$. where $B$ and $B'$ are blocks with $B \cap B' = 1$ and $x \in B' - B$.

(d) $C_1 = B \cup B' \cup B'' - \{a, u\}$. $FV(C_1) = (0.7, 30.75, 85.45, 10.11)$. where $B$, $B'$ and $B''$ are blocks with $B \cap B' \cap B'' = 2$ and $a \notin B' \cap B'' - B \cap B' - B''$.

5. If $|C_1| = 13$, then $C_1$ is one of the following:

(a) $C_1 = B \cup B' \cup B'' - \{x, y\}$. $FV(C_1) = (0.2, 24.60, 80, 75, 8, 41)$. where $B$, $B'$ and $B''$ are blocks with $B \cap B' \cap B'' = 3$. $x, y \in B'' - (B \cup B')$.

(b) $C_1 = B \cup B' \cup B'' - \{u\} - \{x, y, z\}$. $FV(C_1) = (0.3, 21.60, 90, 60, 17, 2)$. where $B$, $B'$ and $B''$ are blocks. $B \cap B' \cap B'' = 3$. $x, y, z \in B'' - (B \cup B')$.

(c) $C_1 = B \cup B' \cup B'' - \{x\}$. $FV(C_1) = (0.5, 12.75, 80, 60, 20, 1)$. where $B$, $B'$ and $B''$ are blocks. $B \cap B' \cap B'' = 2$ and $x \in B' - B'' - B$.

6. If $|C_1| = 14$, then $C_1$ is one of the following:

(a) $C_1 = S - (B \cup B' - \{a, u\})$. $FV(C_1) = (0.1, 13, 46, 86, 77, 26, 4)$. where $B$ and $B'$ are blocks with $|B \cap B'| = 3$ and $u \in B \cap B'$. $u \in B \cap B'$.

(b) $C_1 = S - (B \cup B' - \{u, v\})$. $FV(C_1) = (0.2, 9, 51, 86, 72, 30, 3)$. where $B$ and $B'$ are blocks with $|B \cap B'| = 3$. $u \in B - B'$ and $v \in B' - B$. 
7. If \(|C_1| = 17 - n\) (1 \(\leq n \leq 3\)), then \(C_1 = S - (B \cup X - \{a\})\) with frequency vectors
\((0.1, 0.4, 0.35, 0.30, 0.33, 0.44, 0.6), (0.1, 0.0, 0.75, 0.81, 0.66, 0.10)\) and \((0.1, 0.0, 0.80, 0.60, 0.96, 0.16)\),
respectively, where \(B\) is a block, \(a \subseteq B\), \(X| = n\) and \(X - B = \emptyset\).

Proof. By lemma 3.19 every blocking set in \(S(4, 7, 23)\) contains at least eleven points, so \(C_1 \geq 7\). By definition there exists at least one block such that \(C_1\) meets this block at only one point, so \(C_1 \leq 23 - 6 = 17\).

1. If \(C_1 = 7\), let \(B\) be a block contained in \(C_1\); then \(C_1 = B\).

2. If \(C_1 = 7 - 1\) (1 \(\leq i \leq 3\)), let \(B\) be a block contained in \(C_1\); then we have \(C_1 = B \cup X\), where \(X| = i\) and \(X - B = \emptyset\). It is easy to see that when \(C_1 = 8\) or 9, \(C_1\) has no 6-secant block, so \(FV(C_1) = (0.70, 0.42, 0.105, 0.35, 0.0, 0.1)\)
or \((0.42, 0.56, 0.91, 0.56, 0.7, 0.1)\); when \(C_1 = 10\), \(C_1\) has only one 6-secant block and so \(FV(C_1) = (0.24, 0.54, 0.85, 0.70, 0.18, 0.1)\).

3. Let \(C_1 = 11\). If \(C_1\) contains no block, then \(C_1\) is a blocking set, so \(C_1 = E_0\) or \(E_1\). If \(C_1\) contains a block \(B\), let \(B'\) be the block determined by the four points of \(C_1 - B\), then either \(B \cap B' = 3\) or \(B - B' = 1\). If \(B - B' = 3\), then \(C_1 = B \cup B'\) with \(FV(C_1) = (0.12, 0.48, 0.80, 0.36, 0.2); \) if \(B - B' = 1\), then \(C_1 = B \cup B' - \{x, y\}\), \(FV(C_1) = (0.13, 0.44, 0.80, 0.31, 0.1, 0.1)\), where \(x, y \subseteq B' - B\).

4. Let \(C_1 = 12\). If \(C_1\) contains no block, then \(C_1\) is a blocking set, so we have \(C_1 = S - E_1, FV(C_1) = (0.11, 0.11, 0.11, 0.55, 0.55, 0.10)\).

If \(C_1\) contains two blocks \(B, B'\), then \(B \cap B' = 3\). \(C_1 = B \cup B' - \{a\}\), where \(a \notin B \cup B'\). By lemma 3.25 we know that \(C_1\) has six 1-secant blocks. Since \(B \cup B'\) has no 6-secant block, \(C_1\) contains only two blocks. So we have \(FV(C_1) = (0.24, 0.34, 0.70, 0.85, 0.50, 0.62)\).

Now suppose \(C_1\) contains only one block \(B\).
If there is a block $B'$ which is 6-secant to $C_1$ and meets $B$ at only one point, then $C_1 = B \cup B' - \{x\}$, where $x \in B' - B$. In this case $C_1$ has six 1-secant blocks. So $FV(C_1) = (0.6, 35.65, 95, 40, 11, 1)$.

If there is no block which is 6-secant to $C_1$ and meets $B$ at one point, then let $B'$ be a block such that $B'$ is 6-secant to $C_1$ and $|B' \cap B| = 3$, and let $x, y \in C_1 - (B \cup B')$, $a \in B' - C_1$. Let $B''$ be the block on $\{x, y, a\}$ such that $|B'' \cap B| = 3$. We claim that $B'' \cap B \not\subseteq B - B'$. Otherwise, let $B - B' = \{u, v, w, z\}$ and $B'' \cap B = \{u, v\}$; then consider the block $A$ on $\{x, y, z\}$ which meets $B'$ at three points. If $A \cap B' \not\subseteq B' - B$, then there would exist two blocks on $\{x, y, a\}$ or $B'' - (B \cup B')$ which meet $B$ at three points, this is impossible. So $A \cap B' \subseteq B' - B$. Since there are five blocks on $\{x, y, a\}$, we have $a \notin A$. So $|A \cap B| = 1$, $|A \cap C_1| = 6$, a contradiction. If $|B'' \cap (B \cup B')| = 1$, then $|B'' \cap (B - B')| = 2 = |B'' \cap (B' - B)|$. We consider the block $X$ determined by $x$, $y$ and the two points in $B' - [B \cup (B'' \cap B')]$. $X$ can not contain $B'' \cap (B' - B) - \{a\}$. If $a \in X$, then $X$ would contain a point in $B - [B' \cup (B'' \cap B)]$. The block determined by $x, y, a$ and a point in $B \cap B' - B''$ would contain $B \cap B' - B''$ and the remaining point in $B - [B' \cup (B'' \cap B)]$. So either there are only four blocks on $\{x, y, a\}$ or the fifth block on $\{x, y, a\}$ would not meet $B$, contradiction. Therefore, we have $|B'' \cap (B \cap B')| = 2$, and $C_1 = B \cup B' \cup B'' - \{a, u\}$, where $u \in B'' - (B \cup B')$. By lemmas 3.23, 3.24, 3.25 and 3.26 we know that for a point $p$ in $B \cap B'$, there is only one block which meets $B \cup B'$ at $p$ but does not meet $B'' - (B \cup B')$: for a point $q$ in $B - B'$, there is also only one block which meets $B \cup B'$ at $q$ and $a$, but does not meet $B'' - (B \cup B' \cup \{u\})$. So $C_1$ has seven 1-secant blocks and $FV(C_1) = (0.7, 30.75, 85, 45, 10, 1)$. 
5. Let $|C_1| = 13$.

If $C_1$ contains two blocks $B$ and $B'$, then $|B \cap B'| = 3$. Let $B''$ be a block which contains one point in $C_1 - (B \cup B')$ and meets $B \cup B'$ at $B \cap B'$. If $B''$ contains the other point in $C_1 - (B \cup B')$, then $C_1 = B \cup B' \cup B'' - \{x, y\}$.

where $x, y \in B'' - (B \cup B')$. By lemma 3.24, we know that $C_1$ has two 1-secant blocks. these are two of the blocks which meet $B \cup B'$ only at one point $o$ in $B \cap B'$. The block determined by $o$, the two points in $B'' - (B \cup B'\cup \{x, y\})$ and a point in $B - B'$ can not contain any point in $S - (B \cup B'\cup B'' - \{x, y\})$. So it is contained in $C_1$. We have two of this kind of block. Any block contained in $C_1$ other than $B$ and $B'$ must be one of the two blocks. So $C_1$ contains four blocks, and $FV(C_1) = (0.2.24.60.80.75.8.4)$.

If $B''$ does not contain the other point in $C_1 - (B \cup B')$, then we have $C_1 = B \cup B' \cup B'' - \{u\} - \{x, y, z\}$, where $x, y, z \in B'' - (B \cup B')$ and $u \notin B \cup B' \cup B''$. By lemma 3.24 we can see that $C_1$ has three 1-secant blocks and contains only two blocks. So $FV(C_1) = (0.3.21.60.90.60.17.2)$.

Now we assume $C_1$ contains only one block $B$. Let $B'$ be a block which contains three points in $C_1 - B$ and meets $B$ at three points. Let $B''$ be the block on $C_1 - (B \cup B')$ and meets $B$ at three points. If $B'' \cap B \subset B - B'$, then since $C_1$ contains only one block $B$, we have $a \in B''$. Consider the four blocks which meet $B \cup B'$ at a fixed point in $B \cap B'$. By lemma 3.23, one of them contains $C_1 - (B \cup B')$. So there exist three blocks on $C_1 - (B \cup B')$ which meet $B \cup B'$ at one point in $B \cap B'$. Therefore, the fifth block on $C_1 - (B \cup B')$ would contain $B - (B' \cup B'')$ and $B' - (B \cup \{a\})$, so it is contained in $C_1$, a contradiction. Let $B'' \cap B \not\subset B - B'$. If $B'' \cap (B' - B) = \emptyset$, then $|B'' \cap (B \cap B')| = 1$, so one of the blocks which meet $B \cup B'$ only at
$B \cap B'$ would contain two of the three points in $C_1 \setminus (B \cup B')$. Using the same argument developed in the first paragraph, we obtain that $C_1$ would contain two blocks, a contradiction. So $B'' \setminus (B' - B) \neq \emptyset$, hence $B'' \setminus (B' - B) = \{a\}$, and $C_1 = B \cup B' \cup B'' - \{a\}$. By lemmas 3.25 and 3.26 we know that $C_1$ has five 1-secant blocks. Therefore, $FV(C_1) = \{0, 5, 12, 75, 80, 60, 20, 1\}$.

6. Let $|C_1| = 14$. Let $B$ be a 1-secant block to $C_1$: then $S - (C_1 \cup B) | = 3$.

Let $B'$ be the block on $S - (C_1 - B)$ which meets $B$ at three points. If $B \cap C_1 \subset B'$, then $C_1 = S - (B \cup B' \cup \{a, u\})$, where $u \in B \cap B'$. If $B' - B$, by lemma 3.23 we know that there exist blocks which meet $B \cup B'$ at $u$. So $C_1$ contains four blocks. It is easy to see that $C_1$ has only one 1-secant block to $B$. Therefore,

$$FV(C_1) = \{0, 1, 13, 46, 86, 77, 26, 4\}.$$

If $B \cap C_1 \subset B'$, then $C_1 = S - (B \cup B' \cup \{u, v\})$, where $u \in B - B'$ and $v \in B' - B$. It is easy to prove that $C_1$ contains three blocks, and has two 1-secant blocks. So

$$FV(C_1) = \{0, 2, 9, 51, 86, 72, 30, 2\}.$$

The cases of $|C_1| = 15, 16$ and 17 can be proved similarly.

\hfill \square

3.6.3. 2-blocking sets in $S(4.7.23)$.

**Theorem 3.11.** Let $C_2$ be a 2-blocking set in $S(4.7.23)$. Then $12 \leq |C_2| \leq 18$.

and
1. If \(|C_2| = 12\), then \(C_2 = B \triangle B'\). \(FV(C_2) = (0.0.066.0.165.0.22.0)\), where \(B\) and \(B'\) are blocks with \(B \cap B' = 1\).

2. If \(|C_2| = 13\), then \(C_2 = B \cup B'\). \(FV(C_2) = (0.0.36.30.120.45.20.2)\), where \(B\) and \(B'\) are blocks with \(B \cap B' = 1\).

3. If \(|C_2| = 14\), then \(C_2\) is one of the following:
   (a) \(C_2 = B \cup B' \setminus \{a\}\). \(FV(C_2) = (0.0.0.18.36.96.72.27.4)\), where \(B\) and \(B'\) are blocks with \(B \cap B' = 1\) and \(a \in B \cup B'\).
   (b) \(C_2 = B \cup B' \cup B'' \setminus \{x, y, z\}\) and \(FV(C_2) = (0.0.14.56.56.112.7.8)\), where \(B\), \(B'\) and \(B''\) are blocks with \(B \cap B' = B \cap B'' = B \cap B' \cap B''\). \(B' \setminus B'' - B = \{x, y\}\) and \(z \in B'' - B'\).

4. If \(|C_2| = 15\), then \(C_2\) is one of the following:
   (a) \(C_2 = S \setminus (B \cup B') - \{x, y, u\}\). \(FV(C_2) = (0.0.0.7.35.70.98.35.8)\), where \(B\) and \(B'\) are blocks with \(B \cap B' = 3\), \(x, y \in B \cap B'\) and \(u \in B \cup B'\).
   (b) \(C_2 = S \setminus (B \cup B') - \{x, u, v\}\). \(FV(C_2) = (0.0.0.8.30.80.88.40.7)\), where \(B\) and \(B'\) are blocks with \(B \cap B' = 3\), \(x \in B \cap B'\), \(u \in B - B'\) and \(v \in B' - B\).

5. If \(|C_2| = 16\), then \(C_2 = S \setminus \{a, b, c, d, e, x, y\}\). \(FV(C_2) = (0.0.0.3.20.65.96.57.12)\), where \(a, b, c, d, e \in B\) and \(x, y \notin B\).

6. If \(|C_2| = 17\), then \(C_2 = S \setminus \{a, b, c, d, e, x\}\). \(FV(C_2) = (0.0.0.1.10.50.95.77.20)\), where \(a, b, c, d, e \in B\) and \(x \notin B\).

7. If \(|C_2| = 18\), then \(C_2 = S \setminus \{a, b, c, d, e\}\). \(FV(C_2) = (0.0.0.0.40.80.100.32)\), where \(a, b, c, d, e \in B\).

**Proof.** Since the blocking sets in \(S(4.7.23)\) all have size greater than or equal to eleven, and the blocking sets of size eleven are all 1-blocking sets, by lemma 3.20.
result 3.9 and result 3.10. we know $|C_2| \geq 12$. By definition, $C_2$ meets a block at two points, so $|C_2| \leq 23 - 5 = 18$.

1. Since $|C_2| = 12$, $C_2$ contains no block, otherwise $C_2$ can not be a 2-blocking set by lemma 3.20. result 3.10 and lemma 3.21. So $C_2$ is a blocking set: therefore $C_2 = B \cap B'$. $FV(C_2) = (0.0, 0.0, 0.065, 0.22, 0.0)$, where $B$ and $B'$ are blocks with $|B \cap B'| = 1$.

2. If $|C_2| = 13$, then since there is no blocking set of size thirteen in $S(4,7,23)$, $C_2$ contains a block $B$. By lemma 3.20 there exists a block $B'$ such that $|B' \cap (C_2 - B)| = 4.5$ or 6 and $|B \cap B'| = 1$. If $|B' \cap (C_2 - B)| = 4$ or 5, then $C_2$ would not be a 2-blocking set, so $|B' \cap (C_2 - B)| = 6$ and $C_2 = B \cup B'$ with $|B \cap B'| = 1$ and $FV(C_2) = (0.0, 0.36, 0.12, 0.45, 0.20)$.

3. When $|C_2| = 14$, let $B$ be a block. $B \subset C_2$.

If there exists a block $B'$ which contains six points in $C_2 - B$, then $C_2 = B \cup B' \cup \{a\}$, where $a \notin B \cup B'$ and $|B \cap B'| = 1$. By lemma 3.27, $C_2$ contains four blocks, so $FV(C_2) = (0.0, 0.18, 0.36, 0.96, 0.72, 0.27, 0.4)$.

Suppose no six points in $C_2 - B$ are contained in any block. Since $C_2$ is a 2-blocking set, no block contains only five points in $C_2 - B$. By lemma 3.20, there exists a block $B'$ such that $B'$ contains only four points in $C_2 - B$ and $|B' \cap B| = 1$. Let $C_2 - (B \cup B') = \{a, b, c\}$. $B' - C_2 = \{x, y\}$. We prove that there exists a block $B''$ such that

$$a, b, c \in B'', B \cap B' \subset B''. B'' \cap B' - (B \cap B') = \{x, y\}.$$  

Consider the five blocks on $\{a, b, c\}$. One of them meets $B'$ at three points. Let $B''$ be this block. We prove $B''$ has the above mentioned properties.
Let \( B \cap B' = \{0\} \). \( B = \{0, 1, 2, 3, 4, 5, 6\} \). First we prove \( 0 \in B' \cap B'' \). If \( 0 \notin B' \cap B'' \), then \( \{x, y\} \subseteq B'' \cap B' \). Let the other point in \( B'' \cap B' \) be \( u \) and let \( B'' \cap B = \{6\} \); then the block determined by \( \{a, b, u, v\} \), where \( v \in B' - (B'' \cup B) \), would contain five points in \( C_2 - B \).

So we have proved that \( 0 \in B'' \cap B' \).

We claim \( B'' \cap B' - \{0\} = \{x, y\} \). Otherwise we would have \( x \in B'' \cap B' \) and \( y \notin B'' \cap B' \) or \( x \notin B'' \cap B' \) and \( y \in B'' \cap B' \). In each case we would obtain a block which contains five points in \( C_2 - B \). Thus \( C_2 = B \cup B' \cup B'' - \{x, y, z\} \), where \( x, y \in B' \cap B'' - B \) and \( z \in B'' - (B \cup B') \).

4. When \( |C_2| = 15 \), let \( B \) be a block with \( |B \cap C_2| = 2 \) and let \( S - (C_2 \cup B) = \{p, q, r\} \). We consider the block \( B' \) on \( \{p, q, r\} \) which meets \( B \) at three points. It is easy to see that \( B' \cap (B \cap C_2) \neq \emptyset \). If \( |B' \cap (B \cap C_2)| = 2 \), then \( C_2 = S - [(B \cup B') - \{x, y, u\}] \), where \( x, y \in B \cap B' \) and \( u \in B' - B \). By lemma 3.23 \( C_2 \) contains eight blocks. So \( FV(C_2) = (0, 0, 7, 35, 70, 98, 35, 8) \).

If \( |B' \cap (B \cap C_2)| = 1 \), then \( C_2 = S - [(B \cup B') - \{x, u, v\}] \), where \( x \in B \cap B' \), \( u \in B - B' \) and \( v \in B' - B \). It can be proved that \( C_2 \) contains seven blocks. So \( FV(C_2) = (0, 0, 8, 30, 80, 88, 40, 7) \).

5. If \( |C_2| = 16 \) then since \( C_2 \) meets a block \( B \) at two points, we have \( C_2 = S - \{a, b, c, d, e, x, y\} \), where \( a, b, c, d, e \in B \) and \( x, y \notin B \). By lemma 3.23 there are eight blocks which meet \( B \cup B' \) at one point in \( B \cap C_2 \). It can be proved that on \( (B' - B) \cap C_2 \) and a point in \( B \cap C_2 \) there are two blocks which meet \( B \cup B' \) only at the three points. So \( C_2 \) contains twelve blocks and \( FV(C_2) = (0, 0, 3, 20, 65, 96, 57, 12) \).

The set \( S - \{a, b, c, d, e, x, y\} \), where \( a, b, c, d, e \in B \) and \( x, y \notin B \) is indeed a 2-blocking set. since any block \( B'' \neq B \) meets \( B \) in at most three points;
in the extreme case, if $x, y \in B''$. $B''$ still has at least two points in common with $S - \{a, b, c, d, e, x, y\}$.

6. Similarly we can prove that if $|C_2| = 17$, then $C_2 = S - \{a, b, c, d, e, x\}$, where $a, b, c, d, e$ are in a block $B$ and $x \not\in B$. $FV(C_2) = \{0, 0, 1, 10, 50, 95, 77, 20\}$.

7. If $|C_2| = 18$, then $C_2 = S - \{a, b, c, d, e\}$, where $a, b, c, d, e$ are in a block, and $FV(C_2) = \{0, 0, 1, 0, 40, 80, 100, 32\}$.

\[ \square \]

3.6.4. $t$-blocking sets, $t \geq 3$, in $S(4, 7, 23)$.

**Theorem 3.12.** Let $C_1$ be a 3-blocking set in $S(4, 7, 23)$: then $15 \leq C_1 \leq 19$, and

1. If $|C_1| = 15$, then $C_1 = S - (B \triangle B')$. $FV(C_1) = \{0, 0, 0.70, 0.168, 0.15\}$.

where $B, B'$ are blocks with $|B \cap B'| = 3$;

2. If $|C_1| = 16$, then $C_1 = S - (B \triangle B' - \{a\})$. $FV(C_1) = \{0, 0, 0.35, 0.35, 126, 42, 15\}$.

where $B$ and $B'$ are blocks with $|B \cap B'| = 3$ and $a \in B' - B$;

3. If $|C_1| = 17$, then $C_1 = S - \{a, b, c, d, e, f\}$. $FV(C_1) = \{0, 0, 0.15, 0.40, 105, 72, 21\}$.

where no five points in $\{a, b, c, d, e, f\}$ are contained in any block;

4. If $|C_1| = 18$, then $C_1 = S - \{a, b, c, d, e\}$. $FV(C_1) = \{0, 0, 0.5, 30, 90, 95, 33\}$.

where $\{a, b, c, d, e\}$ is not contained in any block;

5. If $|C_1| = 19$, then $C_1 = S - \{a, b, c, d\}$ and $FV(C_1) = \{0, 0, 0, 1, 16, 72, 112, 52\}$.

**Proof.** First we prove that $X = S - [(B \cup B') - \{o, x, y, a, b\}]$ is not a 3-blocking set in $S(4, 7, 23)$, where $B$ and $B'$ are blocks with $|B \cap B' = \{o\}, a, b \in B - \{o\}$ and $x, y \in B' - \{o\}$.

Let $B - \{o, a, b\} = \{c, d, e, f\}$, $B' - \{o, x, y\} = \{u, v, w, z\}$. We consider the blocks on $\{u, v, w\}$ and $\{u, v, z\}$, respectively, which meet $B - \{o\}$ at three points. One of them contains at most one point in $\{a, b\}$, so it meets $X$ at most 2 points.
By lemma 3.20 we know that \(|C_3| \geq 15\). Obviously \(|C_4| \leq 19\).

If \(|C_3| = 15\), let \(B\) be a block with \(B \cap C_3 = \{x, y, z\}\). \(S - (C_3 \cup B) = \{a, b, c, d\}\).

Let \(B'\) be the block determined by \(\{a, b, c, d\}\); then we have \(B \cap B' = \{x, y, z\}\).

\(C_3 = S - (B \triangle B')\) and \(FV(C_3) = (0.0.0.35, 0.0.15, 126, 120, 0.0.21, 112, 120)\).

If \(|C_3| = 16\), let \(B\) be a block with \(B \cap C_3 = \{x, y, z\}\), let \(S - (C_3 \cup B) = \{a, b, c\}\)
and \(B'\) the block on \(\{a, b, c\}\) and meets \(B\) at three points; then \(B \cap B'\) can not contain
two points in \(B - \{x, y, z\}\). Furthermore, we can prove that \(B' - (B - \{x, y, z\}) = \emptyset\).

So \(B \cap B' = \{x, y, z\}\) and \(C_4 = S - (B \triangle B' - \{a\})\), where \(a \in B' - B\), and
\(FV(C_3) = (0.0.0.35, 0.0.15, 126, 120, 0.0.21, 112, 120)\). 3. 4 and 5 can be proved similarly. □

Theorem 3.13. Let \(C_4\) be a 4-blocking set in \(S(4, 7, 23)\); then \(C_4 = 20\) and
\(C_4 = S - \{a, b, c\}\) and \(FV(C_4) = (0.0.0.0.48, 120, 0.0.5.35, 0.0.21, 112, 120)\).

Theorem 3.14. Let \(C_5\) be a 5-blocking set in \(S(4, 7, 23)\); then \(C_5 = 21\) and
\(C_5 = S - \{a, b\}\) and \(FV(C_5) = (0.0.0.0.21, 112, 120, 0.0.0.0.5.35, 0.0.0.0.48, 120)\).

Theorem 3.15. Let \(C_5\) be a 6-blocking set in \(S(4, 7, 23)\); then \(C_5 = 22\) and
\(C_6 = S - \{a\}\) and \(FV(C_6) = (0.0.0.0.0.77, 176, 0.0.0.0.21, 112, 120, 0.0.0.0.48, 120)\).

Theorem 3.16. The 7-blocking set in \(S(4, 7, 23)\) is \(S\).

3.7. \(t\)-blocking sets in \(S(5, 8, 24)\).

3.7.1. A general result on \(S(5, 8, 24)\). Now we characterize the \(t\)-blocking sets
in \(S(5, 8, 24)\). We use the same terminology and notations as in section 2.3.

First we prove the following

Lemma 3.28. If \(C_t\) contains a block, then \(|C_t| > 12\).
Proof. Let $B$ be a block. $B \subseteq C_t$ and let $x, y, z, w \not\in B$. Then there exists a block $B'$ such that $x, y, z, w \in B'$ and $B \cap B' = \emptyset$. Let $B''$ be the block such that $B'' \cap (B \cup B') = \emptyset$. If $|C_t| \leq 12$, then $B'' - C_t = \emptyset$. 

3.7.2. 1-blocking sets in $S(5, 8, 24)$.

**Theorem 3.17.** For the size of a 1-blocking set $C_1$ we have $11 \leq C_1 \leq 17$.

1. If $|C_1| = 11$, then $C_1 = M_0$ and $\text{FV}(M_0) = (0.22, 110, 165, 330, 66, 66, 0, 0)$.

2. If $|C_1| = 12$, then $C_1 = I$ and $\text{FV}(I) = (0.12, 60, 180, 255, 180, 60, 12, 0)$: or $C_1 = R$. $\text{FV}(R) = (0.11, 66, 165, 275, 165, 66, 11, 0)$.

3. If $|C_1| = 13$, then $C_1 = B \cup B' - \{a\}$, where $B$ and $B'$ are blocks.

4. If $|C_1| = 17 - r$. $0 \leq r \leq 3$), then $C_1 = B \cup B' \cup B'' - \{a, b, c, d, e, f, x_0, x_1, \ldots, x_r\}$.

where $B$, $B'$ and $B''$ are blocks, and $B_1 \cap B' = B \cap B'' = B' \cap B'' = \emptyset$. $\{a, b, c, d, e, f, x_0\} \subseteq B'$. $\{x_1, \ldots, x_r\} \subseteq B''$.

The frequency vectors of $C_1$ are

\[(0.2, 19, 96, 215, 250, 138, 36, 3)\]

\[(0.1, 7, 63, 175, 259, 189, 58, 7)\]

\[(0.1, 0, 35, 140, 231, 252, 85, 15)\]

and

\[(0.1, 0, 0, 140, 140, 336, 112, 30)\]

respectively.
Proof. If \( C_1 \) contains a block, then by lemma 3.28, \( |C_1| > 12 \); if \( C_1 \) does not contain a block, then \( C_1 \) is a blocking set, so \( |C_1| \geq 11 \). Since \( C_1 \) is a 1-blocking set, there exists a block \( B \) such that \( |C_1 \cap B| = 1 \); therefore \( |C_1| \leq 24 - 5 = 17 \).

1. If \( |C_1| = 11 \), then by lemma 3.28 \( C_1 \) is a blocking set. So

\[
C_1 = M_0. \quad \text{It is no difficult to see that there is no 7-secant block to } M_0. \text{So}
\]

\[
FV(M_0) = (0.22.110.165.330.66.66.0.0).
\]

2. If \( |C_1| = 12 \), then \( C_1 \) is a blocking set. But in \( S(5.8.24) \) there are three blocking sets of size twelve: \( I, R \) and \( M \), of which \( M \) is a 2-blocking set. \( I \) and \( R \) are 1-blocking sets, so \( C_1 = I \) or \( R \).

Let \( R = B \cup B' - \{z,a\} \), where \( B \cap B' = \{z,w\}, a \in B \Delta B' \). Consider the contraction of \( S(5.8.24) \) at \( z \). By lemma 3.27 we know that on each point in \( S - (B \cup B') \), there exists a block in \( S(5.8.24) \) which meets \( B \Delta B' - \{a\} \) at six points, and meets \( B \cap B' \) at \( z \). We have ten of this kind of blocks. By lemma 2.2 we know that \( B \cup B' \) contains only two blocks, \( B \) and \( B' \). So \( R \) has eleven 7-secant blocks, therefore, \( FV(R) = (0.11.66.165.275.165.16.5.66.11.0) \).

We already know that \( FV(R) = (0.11.66.165.275.165.66.11.0) \).

3. If \( |C_1| = 13 \), then because \( M_0 \subset M \), \( M \) contains no 7-secant block, so \( S - M_0 \)

is a 2-blocking set. Therefore \( C_1 \) contains a block \( B \). Let \( B' \) be the block that contains the 5 points of \( C_1 - B \). Since \( C_1 \) is a 1-blocking set and \( B' \cap B \neq \emptyset \), then \( |B \cap B'| = 2 \). Let \( a \in B' - C_1 \); then \( C_1 = B \cup B' - \{a\} \).

Now we prove that \( B \cup B' - \{a\} \) is a 1-blocking set.

Since \( M \subset B \cup B' \) and the type of \( M \) is \((2,4,6)\), every block meets the set \( B \cup B' - \{a\} \). Let \( B' - B = \{a,b,c,d,e,f\} \), and let \( B'' \) be a block that contains \( c,d,e,f \) and is disjoint from \( B \); then \( S - (B \cup B'') \) is a block, and
this block meets \( B \cup B' \) only at \( \{a,b\} \), so is 1-secant to \( B \cup B' \setminus \{a\} \). So
\[
FV(C_1) = (0.5, 37, 135, 245, 220, 95, 21, 1).
\]

4. If \( C_1 = 17 - r \), then by lemma 3.28, there exists a block \( B \) such that \( B \subseteq C_1 \). Since \( C_1 \) is a 1-blocking set, there is a block \( B' \) such that \( B' \setminus C_1 = 1 \). Therefore, \( B \setminus B' = \emptyset \) and \( B'' = S - B \setminus B' \) is a block.

Let \( B' - C_1 = \{a,b,c,d,e,f,x_0\} \). \( B'' - C_1 = \{x_1, \ldots, x_r\} \). If \( r = 0 \), then \( B'' - C_1 = \emptyset \); then \( C_1 = B \setminus B' \setminus B'' = \{a,b,c,d,e,f,x_0,x_1,\ldots, x_r\} \), and \( B \setminus B' = B \cap B'' = B' \setminus B'' = \emptyset \).

When \( C_1 = 14 \), \( r = 3 \) and \( C_1 = B \cup B' \setminus B'' = \{a,b,c,d,e,f,x_0,x_1,x_2,x_3\} \).

It can be proved that on \( \{x_1,x_2,x_3\} \) there are three blocks which are disjoint from \( B \). So besides \( B' \) there is only one 1-secant block to \( C_1 \), and there are only two blocks which are contained in \( C_1 \). So
\[
FV(C_1) = (0.2, 19, 96, 215, 250, 138, 36, 3).\]

When \( C_1 = 15 \), it can be proved that besides \( B \) there are six blocks which are contained in \( C_1 \). It is easy to see that \( B' \) is the only 1-secant block to \( C_1 \). So in this case \( FV(C_1) = (0.1, 7, 63, 175, 259, 189, 58, 7) \). While \( C_1 = 16 \) and \( 17 \), there is only one 1-secant block to \( C_1 \), and there is no 2-secant block to \( C_1 \). So the frequency vectors of \( C_1 \) are \((0, 1, 0, 35, 140, 231, 252, 85, 15) \) and \((0, 1, 0, 140, 140, 336, 112, 30) \), respectively.

\[\blacksquare\]

### 3.7.3. 2-blocking sets in \( S(5, 8, 24) \).

**Theorem 3.18.** For the size of the 2-blocking set \( C_2 \) we have \( 12 \leq |C_2| \leq 18 \).

1. If \(|C_2| = 12\), then \( C_2 = M \). \( FV(M) = (0.0, 132, 0, 95, 0, 132, 0, 0) \).

2. If \(|C_2| = 13\), then \( C_2 = S - M_0 \). \( FV(C_2) = (0.0, 66, 66, 330, 165, 110, 22, 0) \).
3. If \( |C_2| = 14 \), then \( C_2 = B \cup B' \). \( FV(C_2) = (0.0.30.72.240.240.135.40.2) \), where \( B \) and \( B' \) are blocks. \( B \cap B' = 2 \).

4. If \( |C_2| = 15 \), then \( C_2 = B \cup B' \cap \{a\} \). \( FV(C_2) = (0.0.12.54.180.264.180.63.6) \), where \( B \) and \( B' \) are blocks. \( B \cap B' = 2 \), \( a \notin B \cup B' \).

5. If \( |C_2| = 18 - r \), \( 0 \leq r \leq 2 \), then \( C_2 = S - (B \cup X) - \{a, b\} \). \( |C_2| = 18 - r \). \( B \) is a block, \( a, b \notin B \) and \( X = r \) with \( X \cap B = \emptyset \). The frequency vectors of \( C_2 \) are \((0.0.4.32.130.256.225.141.141.5) \) and \((0.1.15.85.225.267.141.25) \) and \((0.0.1.0.30.192.170.192.46) \), respectively.

Proof. By the proof of lemma 3.28 we know that if \( C_2 \) contains a block, then \( C_2 \geq 13 \). By theorem 3.17, \( M_0 \) is a 1-blocking set. Consequently \( C_2 \geq 12 \). Since \( C_2 \) is a 2-blocking set, there exists a block \( B' \) such that \( B' \cap C_2 \geq 2 \), therefore \( C_2 \leq 24 - 6 = 18 \).

1. If \( C_2 \geq 12 \), then \( C_2 = M \) and \( FV(M) = (0.0.132.0.495.0.132.0.0) \).

2. If \( C_2 \geq 13 \), then \( C_2 = S - M_0 \). \( FV(C_2) = (0.0.66.66.330.165.110.22.0) \).

3. If \( C_2 \geq 14 \), then since there is no blocking set of size 14, \( C_2 \) must contain at least one block. Let \( B \) be one of them. Assume \( B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\} \).

and \( C_2 - B = \{a_1, a_2, a_3, a_4, a_5, a_6\} \): let \( B_i \) be the unique block determined by \( (C_2 - B) \). \( i = 1, \ldots, 6 \). We claim that \( B_1 = \cdots = B_6 \), as otherwise we would have six different blocks: \( B_1, \ldots, B_6 \). Since \( C_2 \) is a 2-blocking set. \(|B \cap B_i| = 2 \). But \( B_1 \neq B_j \) \((i \neq j)\) implies that \( (B \cap B_i) \cap (B \cap B_j) = \emptyset \), and therefore \( B \) contains at least 12 points. This contradicts the fact \(|B| = 8 \). Let \( B' = B_1 = \cdots = B_6 \), then \( C_2 = B \cup B' \). Since \( B \triangle B' \) is of type \((2, 4, 6)\), \( C_2 \) contains only two blocks. So \( FV(C_2) = (0.0.30.72.240.240.135.40.2) \).
4. If $|C_2| = 15$, let $B$ be a block such that $B \subseteq C_2$. We claim that there exists at least one block $B'$ such that $B'$ contains six points in $C_2 - B$. Otherwise, let $C_2 - B = \{a_1, a_2, a_3, a_4, a_5, a, b\}$. Consider the six blocks determined by the six 5-subsets of $\{a_1, a_2, a_3, a_4, a_5, a\}$. None of these blocks contains $b$, while each of these blocks contains only five points in $\{a_1, a_2, a_3, a_4, a_5, a\}$. If at least two of these blocks do not meet $B$, then each of them would contain three points in $S - C_2$; the other four blocks, each containing at least one point in $S - C_2$, would contain four points in $S - C_2$; the six blocks would contain ten points in $S - C_2$. This is impossible, because $S - C_2$ has only nine points. So among the six blocks, there exists at most one block which does not meet $B$. Any block which meets $B$ meets $B$ at two points, so five blocks would meet $B$ at ten points, this again is a contradiction. Since $C_2$ is a 2-blocking set, we have $B' \cap B = 2$. So $C_2 = B \cup B' - \{a\}$, where $a \in B \cup B'$.

$B \cup B' - \{a\}$ indeed is a 2-blocking set. In fact, let $B''$ be a block which contains $a$ and three points in $B' - B$, and is disjoint from $B$: then $S - (B \cup B'')$ is a block which is 2-secant to $B \cup B' - \{a\}$.

5. Let $B$ be a block that meets $C_2$ at two points, and $X = S - (C_2 - B)$; then $X = r$ and $C_2 = S - (B \cup X) - \{a, b\}$, where $a, b \in B$.

\[\Box\]

### 3.7.4. $t$-blocking sets, $t \geq 3$, in $S(5, 8, 24)$.

#### Theorem 3.19. Let $C_3$ be a 3-blocking set in $S(5, 8, 24)$: then $|C_3| = 18$ or 19.

1. If $|C_3| = 18$, then $C_3 = S - \{a, b, c, d, e, f\}$, where $\{a, b, c, d, e, f\}$ is not contained in any block, and $FV(C_3) = \{0, 0, 0, 6, 45, 180, 285, 198, 45\}$.
2. If $|C_3| = 19$, then $C_3 = \mathcal{S} - \{a, b, c, d, e\}$ and 
$$FV(C_4) = (0.0.0.1.20.120.280.260.78).$$

Proof. Since $C_3$ is a 3-blocking set, it meets at least one block at only three points; therefore $|C_3| \leq 24 - 5 = 19.$

If $|C_3| \leq 17$, let $\{a_1, a_2, a_3, a_4, x, y\} \subseteq \mathcal{S} - C_3$, and let $B$ be the block determined by $\{a_1, a_2, a_3, a_4\}$. Since $C_3$ is a 3-blocking set, $B$ contains three points of $C_3$. Let $a_6, a_7, a_8$ be the three points. Let $B_1, B_2$ and $B_3$ be the blocks determined by $\{x, y, a_1, a_2, a_3\}, \{x, y, a_1, a_2, a_4\}$ and $\{x, y, a_1, a_3, a_5\}$, respectively. Since $C_3$ is a 3-blocking set and $B_i$ has at least 3 points in common with $B_i$ ($i = 1, 2, 3$), so $B_i$ contains one of $a_6, a_7, a_8$. Assume $a_6 \in B_1, a_7 \in B_2$ and $a_8 \in B_3$. The block $B_4$ determined by $\{x, y, a_1, a_4, a_5\}$ has at least three points $a_1, a_4, a_5$ in common with $B$, so it must contain one and only one of $a_1, a_2, a_6, a_7, a_8$. If $a_6 \in B_4$, let $A$ be the block determined by $\{x, y, a_2, a_3, a_4\}$; then $a_1, a_6, a_7, a_8 \notin A$. So $A \cap B^c = 3$, a contradiction. Therefore, $a_6 \notin B_4$. Similarly we can prove that $a_7, a_8 \notin B_4$. So $B$ contains either $a_1$ or $a_2$, therefore $B_4$ contains at most two points of $C_3$, this contradicts $C_3$ is a 3-blocking set. So $|C_3| > 17$.

1. If $|C_3| = 18$, then $\mathcal{S} - C_3 = 6$. Let $\mathcal{S} - C_3 = \{a, b, c, d, e, f\}$; then $\{a, b, c, d, e, f\}$ can not be contained in any block.

2. If $|C_3| = 19$, then $C_3 = \mathcal{S} - \{a, b, c, d\}. \quad \square$

Using the same method as above, we can prove

**Theorem 3.20.** Let $C_4$ be a 4-blocking set; then $|C_4| = 20$, and 
$$C_4 = \mathcal{S} - \{a, b, c, d\} \text{ and } FV(C_4) = (0.0.5.64.240.320.130).$$
Theorem 3.21. Let $C_5$ be a 5-blocking set: then $|C_5| = 21$ and $C_5 = S - \{a, b, c\}$.
$FV(C_5) = (0.0.0.0.21.168.360.210)$.  

Theorem 3.22. Let $C_6$ be a 6-blocking set: then $|C_6| = 22$ and $C_6 = S - \{a, b\}$ and $FV(C_6) = (0.0.0.0.0.77.352.330)$.  

Theorem 3.23. Let $C_7$ be a 7-blocking set: then $|C_7| = 23$ and $C_7 = S - \{a\}$ and $FV(C_7) = (0.0.0.0.0.253.506)$.  
4. SEMIOVALS IN THE WITT DESIGNS

4.1. The definitions.

Definition 4.1. A block $B$ of a $t$-design $D$ is called a tangent to a set $X$ of points of $D$ if $B$ contains only one point in $X$.

If $B \cap X = \{x\}$, we also say that $B$ is a tangent on $x$ to $X$.

Definition 4.2. A semioval $O$ is a set of points such that on every point of $O$ there exists only one tangent to $O$.

Semiovals have been studied by many authors (see 15, 30, 54). But much of the previous work was focused on the semiovals of projective planes. In this section we study the semiovals in the Witt designs and characterize all semiovals in the Witt designs up to the frequency vectors.

4.2. Semiovals in $S(4,5,11)$ and $S(5,6,12)$. In $S(4,5,11)$ we have

$$r_0 = 66, r_1 = 30, r_2 = 12, r_3 = 4, r_4 = 1.$$ 

Lemma 4.1. The type of a block in $S(4,5,11)$ is $(1,2,3,5)$ with

$$t_1 = 15, t_2 = 20, t_3 = 30, t_5 = 1.$$ 

Lemma 4.2. Let $B, B'$ be two blocks in $S(4,5,11)$ with $B \cap B' = 1$. Then the type of $B \cup B'$ is $(3,4,5)$ with $t_3 = 12, t_4 = 36, t_5 = 18$.

By lemma 4.1 and lemma 4.2 we can prove the following
Lemma 4.3. Let $B$ be a block in $S(4.5.11)$. Let $B_1, B_2$ and $B_3$ be three blocks which meet $B$ only at $a$. Then $|B_i \cap B_j - \{a\}| = 2$ $(i \neq j)$ and $B_1 \cap B_2 \cap B_3 = \{a\}$.

By the above lemmas we can prove

Lemma 4.4. Let $B$ be a block in $S(4.5.11)$. Then for every $a \in B$, there exist exactly three blocks which meet $B$ only at $a$.

Now we can prove

Theorem 4.1. There exists no semierval in $S(4.5.11)$.

Proof. Suppose there exists a semierval $O$ in $S(4.5.11)$. Then since there is no blocking set in $S(4.5.11)$, there must exist a block $B$ such that either $B \subseteq O$ or $B \cap O = \emptyset$.

If $B \subseteq O$, then by lemma 4.4, $O - B \neq \emptyset$. Let $b \in O - B$. By lemma 4.1 any block which contains $b$ meets $B$. So there is no tangent on $b$.

If $B \cap O = \emptyset$, then by lemma 4.1, $|O| \neq 6$ and $|O| \geq 5$. So $|O| = 5$. Let $o \notin O \cup B$, let $u,v,w \in O$ and let $B'$ be the block determined by $\{o,u,v,w\}$. Then $|B' \cap B| = 1$. Let $B' \cap B = \{a\}$. By lemma 4.3 there is a block $B''$ such that $B'' \cap B = \{a\}$ and $B''$ contains $o$ and one of $u,v$ and $w$. say $u$. There are four blocks on $\{a,o,u\}$ and $B'$. $B''$ are two of them. The other two can not contain any points in $B' \triangle B''$. So both of them are tangents on $u$. contradiction.

In $S(5.6.12)$ we have

$$r_0 = 132, r_1 = 66, r_2 = 30, r_3 = 12, r_4 = 4, r_5 = 1.$$
Lemma 4.5. The type of a block in $S(5,6,12)$ is $(0,2,3,4,6)$ with

$$t_0 = 1, t_2 = 45, t_3 = 40, t_4 = 45, t_6 = 1.$$ 

Lemma 4.6. Let $B$, $B'$ be two disjoint blocks in $S(5,6,12)$. Then for any two points $a, b \in B$, there exist exactly three blocks $B_1$, $B_2$ and $B_3$ such that

1. $B_1 \cap B = B_2 \cap B = B_3 \cap B = \{a, b\}$ and $|B_1 \cap B'| = |B_2 \cap B'| = |B_3 \cap B'| = 4$;
2. $|B_1 \cap B_2 \cap B'| = 2$ and $B_1 \cap B_2 \cap B_3 \cap B' = \emptyset$.

Lemma 4.7. Let $B$, $B'$ be two disjoint blocks in $S(5,6,12)$. Then for any four points $a, b, c, d \in B$, there exist exactly three blocks $A_1$, $A_2$ and $A_3$ such that

$$A_1 \cap B = A_2 \cap B = A_3 \cap B = \{a, b, c, d\} \text{ and } (A_1 \cap B') \cup (A_2 \cap B') \cup (A_3 \cap B') = B'.$$

Lemma 4.8. The blocking set in $S(5,6,12)$ is a semioval.

Proof. The blocking set $C$ in $S(5,6,12)$ has the structure $C = (B - \{a\}) \cup \{x\}$, where $a \in B$ and $x \notin B$.

Let $B'$ be a block disjoint from $B$: then $x \in B'$ and $B'$ is a tangent on $x$ to $C$. By lemma 4.5 the other blocks which contain $x$ meet $B$ in at least two points.

Let $b \in B - \{a\}$. By lemma 4.6 there are exactly three blocks $B_1$, $B_2$ and $B_3$ such that $B_i \cap B = \{a, b\}$ (i = 1, 2, 3), and $x$ is contained in only two of $B_1$, $B_2$ and $B_3$. So one of these blocks is a tangent on $b$ to $C$. The other blocks which contain $b$ meet $B$ in at least three points. \[\square\]

Theorem 4.2. If $O$ is a semioval in $S(5,6,12)$, then $O$ is a blocking set.

Proof. Suppose $O$ is not a blocking set. If $O$ contains a block $A$, then any block which contains a point in $A$ contains at least two points in $A$. So $O$ contains no block: therefore there exists a block $B$ and a point $x$ such that $B \cap O = \emptyset$.
and \( x \notin B \cup O \). By lemma 4.6 for \( u \in O \) there are three tangents on \( u \) to \( O \). contradiction.

\[ \square \]

4.3. **Semiovals in** \( S(3,6,22) \). In the sequel sometimes we will use the results about \( S(3,6,22) \) in section 3.3 without explicit reference.

**Lemma 4.9.** A semioval in \( S(3,6,22) \) can not contain a Fano set.

**Lemma 4.10.** A semioval in \( S(3,6,22) \) contains no block.

**Lemma 4.11.** A semioval in \( S(3,6,22) \) can not contain \( B \triangle B' \), where \( B \) and \( B' \) are blocks.

**Lemma 4.12.** The set \( D := (B - \{ u \}) \cup (B' - \{ v \}) \), where \( B \) and \( B' \) are disjoint blocks, \( u \in B \) and \( v \in B' \), is a semioval.

**Theorem 4.3.** Let \( O \) be a semioval in \( S(3,6,22) \). Then \( 9 \leq |O| \leq 10 \).

**Proof.** First we prove that \( O \) contains at most 10 points.

Let \( u, v \in O \), let \( B \) and \( B' \) be the tangents on \( u \) and \( v \) respectively.

If \( B \cap B' = \emptyset \), then the points not in \( B \cup B' \) can not all be contained in \( O \).

If there is only one point \( x \notin O \), then the nine tangents on the other nine points not in \( B \cup B' \) would all contain \( x \) and meet both \( B \) and \( B' \); on \( x \) there are twelve blocks which meet only one of \( B, B' \). None of these mentioned blocks contains all of \( u, v, x \). So there are at least twenty-two blocks on \( x \), a contradiction.

If \( B \cap B' \neq \emptyset \), then consider the four external blocks \( E_1, E_2, E_3 \) and \( E_4 \) of \( B \cup B' \). By the proof of lemma 2.2 in [7] we know that \( |E_i \cap E_j| = 2 \) if \( i \neq j \), and \( (E_i \cap E_j) \cap (E_k \cap E_h) = \emptyset \) if \( \{i, j\} \neq \{k, h\} \). Let \( E_1 \cap E_2 = \{x, y\}, E_3 \cap E_4 = \{z, w\} \).

By lemma 4.10 and lemma 4.11 we only need to prove that the following sets are
The set $X_1$ is not a semiport. Otherwise the tangents on the two points in $E_2 \cap E_1$ would all contain $\{x, z, p\}$.

The set $X_2$ is not a semiport. Otherwise the tangents on the points in $E_1 \cap E_1$ would all contain $\{x, p, q\}$.

Suppose $X_3$ is a semiport. Then the tangent on the point in $E_1 \cap E_1 \setminus \{p\}$ contains $p$, but can not meet $(E_1 \cup E_2 \cup E_3 \cup E_4 \setminus (E_1 \cup E_3))$. So it meets $(E_1 \cup E_2) \setminus \{u, v\}$ at four points. Let $A_1$ be the other block on $E_1 \cap E_1$ which does not meet $E_2$; then $\{u, v\} \subseteq A_1$ and $A_1 \setminus (B \Delta B') = 4$. We can similarly prove that there is a block $A_2$ on $E_1 \cap E_1$ such that $A_2 \cap E_2 = \emptyset$. $\{u, v\} \subseteq A_2$ and $A_2 \setminus (B \Delta B') = 4$. Let $A_3$ and $A_4$ be the blocks on $\{u, v\}$ which are disjoint from $E_1$; then

$$E_1 \cap E_1 \cap A_i \cap E_2 \cap E_3 \cap A_i \cap E_2 \cap E_4 \cap A_i, (i = 3, 4).$$

So

$$A_i \cap (E_1 \cap E_1) = A_i \cap (E_2 \cap E_1) = A_i \cap (E_3 \cap E_1) = 1.$$

Now, the block $A_5$ determined by $\{u, v, y\}$ would contain $x$ and two points in $(B \Delta B') \setminus \{u, v\}$. The blocks $B$, $B'$ and $A_5$ are three external blocks to $E_3 \cap E_4$. Let the forth external block of $E_3 \cup E_4$ be $T$; then $u, v \notin T$ and $x, y \in T$. So $T$ and $E_2$ are two tangents on $y$.

Now we need to show that a semiport contains at least nine points. This can be done by checking the frequency vectors in appendix A.
Let $B$ and $B'$ be two disjoint blocks of $S(3, 6, 22)$. Define

$$P := \mathcal{S} \setminus (B \cup B' \cup \{x\}).$$

**Theorem 4.4.** The set $P$ is a semioval of size 9 and $FV(P) = (2, 9, 36, 12, 18, 0, 0)$.

**Proof.** Let $y \in P$. Since on $\{x, y\}$ there are four blocks which are all disjoint from $B$ or $B'$ and there are a total of five blocks on $\{x, y\}$, thus there is a block on $\{x, y\}$ which meets both $B$ and $B'$, so this block is a tangent on $y$. The other blocks on $y$ all contain at least two points in $P$. It is easy to see that $B$ and $B'$ are the only two blocks which do not meet $P$. $P$ contains no block. So we have $FV(P) = (2, 9, 36, 12, 18, 0, 0)$. \(\square\)

**Theorem 4.5.** If $O$ is a semioval of size 9, then $O = P$.

**Proof.** First we prove that there is no 5-secant block to $O$.

By solving the linear system (1) we know that the possible values for $t_5$ are 0 and 4.

If $t_5 = 4$, let $B, B'$ be two blocks which are 5-secant to $O$: then $B \cap B' = \emptyset$. If $B \cap B' \subseteq O$, then $O \setminus (B \cup B')$ has only one point, and on this point there are at least two tangents. If $B \cap B'$ is not contained in $O$, then $O = B \cup B' \setminus \{x\}$, where $x \in B \cap B'$. By lemma 4.11 this is impossible.

So $t_5 = 0$ and therefore $t_0 = 2$. Let $B, B'$ be the two blocks which do not meet $O$. We claim that $B \cap B' = \emptyset$. Otherwise, consider $B \cup B' \setminus \{x\}$, where $x \notin B \cup B' \cup O$. Then there are two external blocks $U$ and $V$ to $B \cup B' \setminus \{x\}$ such that $U \cup V$ contains only 8 points of $O$. The other point in $O$ would lie on two of the four external blocks of $U \cup V$, a contradiction. Therefore $O = P$. \(\square\)
Let \( B = \{x, y, z, \bar{x}, \bar{y}, \bar{z}\} \) and \( B' = \{a, b, \bar{a}, \bar{b}, \bar{c}\} \) be two disjoint blocks, and let \( o \) be the point such that
\[ F = \{a, b, c, x, y, z, o\} \] forms a Fano set. Let \( U' \) be the block on \( x, y \) which does not meet \( B' \). Then \( o \in U' \). We define
\[ M := S - (B \cup B' \cup U' - \{a, b, c, o\}). \]

**Theorem 4.6.** The set \( M \) is a semiwal of size 10 with \( \{1, 10, 15, 40, 5, 6, 0\} \) as its frequency vector.

**Proof.** We only need to prove that on every point of \( M \), there exists only one tangent.

Let \( V, W \neq B \) be the blocks on \( x, z \) and \( y, z \) respectively which are disjoint from \( B' \). Let
\[ U = \{y, o, 1, 2, 3\}, \ V = \{x, o, y, 4, 5, 6\}, \ W = \{y, z, a, 7, 8, 9\}; \]
then \( M = \{a, b, c, o, 4, 5, 6, 7, 8, 9\} \).

It is easy to see that \( U' \) is a tangent on \( o \).

Let \( X \neq U' \) be a block which contains \( o \). Then \( |X \cap F'| = 1 \) or 3. If \( |X \cap F'| = 1 \), then \( |X \cap \{a, \bar{b}, \bar{c}\}| = 2 \) or \( |X \cap \{\bar{x}, \bar{y}, \bar{z}\}| = 2 \), so \( X \) contains two points in \( \{4, 5, 6, 7, 8, 9\} \). Therefore, \( X \) is not a tangent to \( M \). If \( |X \cap F'| = 3 \), then
\( |X \cap \{a, b, c\}| = 1 \) or \( |X \cap \{x, y, z\}| = 1 \) or \( |X \cap \{a, b, c\}| = 2 \) and \( |X \cap \{x, y, z\}| = 0 \) or \( |X \cap \{a, b, c\}| = 0 \) and \( |X \cap \{x, y, z\}| = 2 \). In the first two cases, \( X \) is not a tangent to \( M \). In the last case, \( X = W \), so is not a tangent to \( M \).

Let \( u \in \{a, b, c\} \).

Consider the blocks determined by \( \{u, 1, 2\}, \{u, 1, 3\} \) and \( \{u, 2, 3\} \), respectively. None of the three blocks contains any of \( x, y, z, o \). So at most two of them meet \( V \).
and $W$ at two points in $\{4.5.6\}$ and two points in $\{7.8.9\}$; therefore at least one of them, say $Y$, meets $B - \{x,y,z\} = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ at two points. So $Y$ is a tangent to $M$.

Let $X \neq Y$ be a block on $u$. If $X \cap F' = 3$, then $X$ is not a tangent; if $X \cap F' = 1$, then $X \cap \{\tilde{x}, \tilde{y}, \tilde{z}\} = 2$. Since $X \neq Y$, then $X \cap \{Y \cap U\} = \emptyset$. Thus $X \cap U = \emptyset$, so $X$ contains two points in $\{4.5.6.7.8.9\}$ and is not a tangent to $M$.

Let $v \in \{4.5.6\}$.

Consider the blocks determined by $\{v,x,1\}$, $\{v,x,2\}$ and $\{v,x,3\}$ respectively. Since $o$ is not in any of these blocks, none of them contains $y$ or $z$. So at most two of them can meet either $W$ or $\{a,b,c\}$ at two points; therefore at least one of them, say $Z$, meets $B' - \{a,b,c\}$. $Z$ is a tangent to $M$.

Let $X \neq Z$ be a block on $v$. If $X \cap F' = 3$, then $X$ contains $o$ or at least one point in $\{a,b,c\}$. If $X \cap F' = 1$, since $X = Z$, we have $x \notin X$ (otherwise $y,z,o \notin X$, so $X \cap \{\tilde{a}, \tilde{b}, \tilde{c}\} = \emptyset$ and $X$ has at least three points in common with $Z$, and so $X = Z$). If $z \notin X$, then $y \notin X$, so $X$ contains one point in $\{7.8.9\}$. If $z \notin X$, then $X$ contains one point in $\{o,4.5.6\} - \{v\}$. So in either case $X$ is not a tangent to $M$.

If $v \in \{7.8.9\}$, we can similarly prove that there is only one tangent on $v$.

From the structure of a Fano set we can conclude that $B$ is the only block which does not meet $M$, and $M$ contains no block. So $FV(M) = \{1,10,15,40,5,6,9\}$.

Theorem 4.7. Let $O$ be a semioval of size 10. If $O \neq D$, then $O = M$.

Proof. Checking the appendix B we can see that there is a block $B$ such that $O \cap B = \emptyset$. 


Choose a point $o \in O$ and let $U$ be the unique tangent on $o$. Then using a similar argument as in the third paragraph of the proof of theorem 4.3, we can prove that $U \cap B \neq \emptyset$.

We claim that there exists a block $B'$ which contains three points in $O$ and is disjoint from $B \cup U$.

Let $S = (B \cup U \cup O) = \{d, e, f\}$. Consider the two external blocks $X$, $Y$ of $B \cup U \cup \{d\}$. By lemma 4.10 and lemma 4.11, $X \cap Y$ contains eight points in $O$, and $|(X \cap Y) \cap O| \neq \emptyset$. If $|(X \cap Y) \cap O| = 1$, then suppose $e \in (X \cap Y) - O$; then $f$ lies on only one of $X$ and $Y$, say $X$. In this case, there exist two points on $Y - X$, the tangents of which would all contain $\{d, e, f\}$. This is impossible. So $|(X \cap Y) \cap O| = 2$. The three points $d$, $e$ and $f$ must all be contained in one of the other two external blocks of $B \cup U$ (otherwise, the tangents on the two points in $X \cap Y$ would all contain $d$, $e$ and $f$). Let this block be $B'$, then $B' \cap O = 3$ and $B' \cap (B \cup U') = \emptyset$.

Let $B' \cap O = \{a, b, c\}$. Let $V, W \in R(B)$ with $V \neq B' = W$, be the blocks on $\{a, b\}$ and $\{a, c\}$ respectively and let $o' \in (V - W) - \{a\}$.

Now we prove $o' = o$.

If $o' \neq o$, then we claim $o' \notin U$. Otherwise, let $U' = \{x, y, o, o', u, v\}$. where $\{x, y\} = U \cap B$, let $V = \{a, b, o, u, 1, 2\}$, $W = \{a, c, o', v, 3, 4\}$. Let $Z \in R(B)$, $Z \neq B'$. be the block on $\{b, c\}$. Then $Z = \{b, c, o', 5, 6\}$. The tangent $T$ on 5 would contain $o', u, v$, a contradiction. So $o' \notin U$ and therefore $o' \in O$.

Using the construction method of a Fano set to $o'$. \{a, b, c\} and $B$ (see section 3.4), we can partition $B$ into two parts: $B_1$ and $B_2$. Since $o' \notin U$, then $x$ and $y$ cannot be in the same part.

Let $X$ be a block on $o'$; then $|X \cap (\{a, b, c, o'\} \cup B_1)| = 1$ or $3$. 
If \(|X \cap (\{a,b,c\} \cup B_1)| = 1\), then \(|X \cap [B_2 \cup (B' - \{a,b,c\})]| = 2\). So \(X\) contains at least one point in \(O - \{a,b,c\}\); hence \(|X \cap O| \geq 2\).

If \(|X \cap (\{a,b,c\} \cup B_1)| = 3\), then either \(|X \cap \{a,b,c\}| \geq 1\) or \(|X \cap B_1| = 2\). If \(|X \cap B_1| = 2\), then \(|X \cap B'| = 0\); so \(X\) contains at least one point in \(O - (B' \cup B' - \{o'\})\).

So we have proved \(o' = a\). Therefore \(O = M\). 

In the definition of \(M\), there are six choices for \(U\), so there are six semiovals which do not contain \(B\).

4.4. **Semiovals in \(S(4,7,23)\).** By checking the frequency vectors in appendix B we can see that the only set that could be a semioval is \(E_1\).

**Lemma 4.13.** The set \(E_1\) is a semioval.

*Proof.* By lemma 2.11 in [8] we know that on every point in \(B'\), there exists at least one tangent.

Let \(a \in B' - \{o,w\}\). Of the five blocks through \(\{o,w,a\}\), only three meet \(B - \{o\}\). So there is one which meets \(B \cup B'\) only at \(o, w\) and \(a\). This block is a tangent on the point \(a\).

Of the five blocks through three fixed points in \(B' - \{o,w\}\), four meet \(B - \{o\}\). So one meets \(B - \{o\}\) at three points. This block is a 6-secant to \(E_1\). There are \(\binom{5}{3} = 10\) of these kind of blocks. So there are at least eleven 6-secant blocks to \(E_1\).

Let \(c = 11\) and solve (2.1) in [8] to get \(t_1 = 22 - t_6\). So \(t_1 = t_6 = 11\) and there are only eleven tangents to \(E_1\). Hence on every point of \(E_1\) there is only one tangent to \(E_1\), and therefore \(E_1\) is a semioval. 

4.5. **The non-existence of semiovals in \(S(5,8,24)\).** In this section we use the same notations and terminology as in [9].
By checking the frequency vectors in appendix C we can see that the only set in $S(5.8.24)$ that could be a semi-oval is $I$.

**Lemma 4.14.** The set $I$ is not a semi-oval.

**Proof.** Suppose $I$ is a semi-oval: then since $S - I$ is the same type of blocking set as $I$, $S - I$ is also a semi-oval. Let $I = (B - \{u\}) \cup (B' - \{v\})$, where $B$ and $B'$ are blocks with $|B \cap B'| = 2$, $u \in B - B'$, $v \in B' - B$ and let $U$ be the tangent on $u$ to $S - I$; then $U$ is 7-secant to $I$. So $|U \cap \{B - \{u\}\}| = 3$ and $|U \cap \{B' - \{v\}\}| = 4$. Now $|U \cap B| = 4$, so by lemma 2.1(b) in [9], $U \cap B$ is a block. But $(U \cap B) - (B' - \{v\}) = 6$, a contradiction. \qed

So we have

**Theorem 4.8.** There exists no semi-oval in $S(5.8.24)$.

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Appendix

We use $n$ to denote the size of a point subset of $S(t,k,r)$. The number of vectors under each size $n$ is the number of orbits under the action of the Mathieu group $M_{12}$.

APPENDIX A. THE FREQUENCY VECTORS OF THE POINT SUBSETS IN $S(3,6,22)$

$n = 0$

(77, 0, 0, 0, 0, 0, 0).

$n = 1$

(56, 21, 0, 0, 0, 0, 0).

$n = 2$

(40, 32, 5, 0, 0, 0, 0).

$n = 3$

(28, 36, 12, 1, 0, 0, 0).

$n = 4$

(20, 32, 24, 0, 1, 0, 0), (18, 36, 18, 4, 0, 0, 0).

$n = 5$

(16, 20, 40, 0, 0, 1, 0), (12, 35, 20, 10, 0, 0, 0), (12, 35, 20, 10, 0, 0, 0), (13, 31, 26, 6, 1, 0, 0).

$n = 6$

(16, 0, 60, 0, 0, 0, 1), (6, 36, 15, 20, 0, 0, 0), (6, 36, 15, 20, 0, 0, 0), (10, 21, 35, 10, 0, 1, 0).

$n = 7$

(9, 24, 33, 8, 3, 0, 0), (8, 28, 27, 12, 2, 0, 0).

$n = 8$

(0, 42, 0, 35, 0, 0, 0), (0, 42, 0, 35, 0, 0, 0), (10, 6, 45, 15, 0, 0, 1), (7, 14, 42, 7, 7, 0, 0), (4, 26, 24, 19, 4, 0, 0), (4, 26, 24, 19, 4, 0, 0), (6, 19, 32, 17, 2, 1, 0), (5, 22, 30, 15, 5, 0, 0).
\[(7.0, 56.0, 14.0, 0.0), (0.28, 14.28, 7.0, 0.0), (0.28, 14.28, 7.0, 0.0), (6.8, 35.24, 3.0, 1.0).\]
\[(4.13, 34.18, 7.1, 0.0), (2.20, 26.20, 9.0, 0.0), (3.17, 28.22, 6.1, 0.0), (3.17, 28.22, 6.1, 0.0).\]
\[(4.14, 30.24, 3.2, 0.0), (3.16, 32.16, 10.0, 0.0).\]
\[n = 9\]
\[(4.3, 40.16, 12.2, 0.0), (0.18, 20.26, 12.1, 0.0), (0.18, 20.26, 12.1, 0.0), (4.6, 29.30, 6.1, 1.1).\]
\[(3.9, 27.28, 9.0, 1.1), (2.11, 28.24, 10.2, 0.0), (1.14, 26.22, 13.1, 0.0), (2.12, 24.30, 5.3, 0.0).\]
\[(2.12, 24.30, 6.3, 0.0), (2.9, 36.12, 18.0, 0.0).\]
\[n = 10\]
\[(4.0, 37.32, 12.0, 2.0), (0.12, 18.28, 18.0, 0.1), (2.4, 29.24, 14.0, 0.0), (0.11, 21.26, 16.3, 0.0).\]
\[(0.11, 21.26, 16.3, 0.0), (2.6, 22.32, 12.2, 1.0), (1.8, 24.24, 19.0, 1.0), (1.8, 23.28, 13.4, 0.0).\]
\[(0.10, 25.20, 20.2, 0.0), (1.7, 27.22, 17.3, 0.0), (1.10, 15.40, 5.6, 0.0), (1.10, 15.40, 5.6, 0.0).\]
\[(2.0, 45.0, 30.0, 0.0).\]
\[n = 11\]
\[(0.6, 20.25, 20.6, 0.0), (0.6, 20.25, 20.6, 0.0), (2.2, 18.33, 18.2, 2.2), (0.7, 17.27, 22.3, 1.1).\]
\[(1.3, 22.27, 17.7, 0.0), (0.6, 20.25, 20.6, 0.0), (0.7, 16.31, 16.7, 0.0), (0.6, 20.25, 20.6, 0.0).\]
\[(1.4, 19.29, 19.4, 1.1), (1.5, 15.35, 15.5, 1.1), (0.7, 16.31, 16.7, 0.0), (1.5, 15.35, 15.5, 1.1).\]
\[(0.5, 25.15, 30.1, 1.1), (1.1, 30.15, 25.5, 0.0), (0.11, 0.55, 0.11, 0.0), (0.11, 0.55, 0.11, 0.0).\]

**Appendix B. The frequency vectors of the point subsets in \(S(4, 7, 23)\)**

\[n = 0\]
\[(253.0, 0.0, 0.0, 0.0, 0.0).\]
\[n = 1\]
\[(176.77, 0.0, 0.0, 0.0, 0.0).\]
\[n = 2\]
\[(120.112.21, 0.0, 0.0, 0.0, 0.0).\]
\[
\begin{align*}
n &= 3 \\
&s = (80, 120, 48, 5, 0, 0, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 4 \\
&s = (52, 112, 72, 16, 1, 0, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 5 \\
&s = (32, 100, 80, 40, 0, 1, 0, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 6 \\
&s &= (16, 96, 60, 80, 0, 0, 1, 0, 0), (21, 72, 105, 40, 15, 0, 0, 0), (20, 77, 95, 50, 10, 1, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 7 \\
&s &= (0, 112, 0, 140, 0, 0, 0, 1, 0), (15, 42, 126, 35, 0, 0, 0), (10, 66, 81, 75, 20, 0, 1, 0), (12, 57, 96, 65, 20, 3, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 8 \\
&s &= (15, 0, 168, 0, 70, 0, 0, 0), (0, 70, 42, 105, 35, 0, 0, 1), (8, 35, 98, 70, 35, 7, 0, 0), (6, 44, 83, 80, 35, 4, 1, 0), (7, 40, 88, 80, 30, 8, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 9 \\
&s &= (8, 7, 112, 56, 56, 14, 0, 0), (0, 42, 56, 91, 56, 7, 0, 1), (4, 26, 77, 86, 46, 13, 1, 0), (3, 30, 72, 86, 51, 9, 2, 0), (4, 27, 72, 96, 36, 18, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 10 \\
&s &= (4, 8, 75, 80, 60, 24, 2, 0), (0, 24, 54, 85, 70, 18, 1, 1), (2, 17, 60, 90, 60, 21, 3, 0), (1, 20, 60, 80, 75, 12, 5, 0), (2, 20, 45, 120, 30, 36, 0, 0).
\end{align*}
\]

\[
\begin{align*}
n &= 11 \\
&s &= (0, 12, 48, 75, 80, 36, 0, 2), (2, 6, 50, 85, 70, 34, 6, 0), (0, 13, 44, 80, 80, 31, 4, 1), (1, 10, 45, 85, 75, 30, 7, 0), (1, 11, 40, 95, 65, 35, 6, 0), (0, 11, 55, 55, 110, 11, 11, 0), (0, 22, 0, 165, 0, 66, 0, 0).
\end{align*}
\]
Appendix C. The frequency vectors of the point subsets in $S(5, 8, 24)$

$n = 0$
$(759, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 1$
$(506, 253, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 2$
$(330, 352, 77, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 3$
$(210, 360, 168, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 4$
$(130, 320, 240, 64, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 5$
$(78, 260, 280, 120, 20, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

$n = 6$
$(46, 192, 300, 160, 60, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
$(45, 198, 285, 180, 45, 6, 0, 0, 0, 0, 0)$.

$n = 7$
$(30, 112, 336, 140, 140, 0, 0, 0, 1, 0)$.
$(25, 141, 267, 225, 85, 15, 1, 0, 0)$.

$n = 8$
$(30, 0, 448, 90, 280, 0, 0, 0, 0, 1)$.
$(15, 85, 252, 381, 140, 35, 0, 1, 0)$.

$(13, 96, 228, 256, 130, 32, 4, 0, 0)$.

$n = 9$
$(15, 15, 280, 168, 210, 70, 0, 0, 0, 1, 0)$.
$(7, 58, 189, 259, 175, 63, 7, 1, 0, 0)$.

$6, 63, 180, 264, 180, 54, 12, 0, 0)$.

$n = 10$
\[(7, 16, 175, 224, 210, 112, 14, 0, 1), (3, 36, 138, 250, 215, 96, 19, 2, 0)\].
\[(2, 40, 135, 240, 240, 72, 30, 0, 0)\].

\(n = 11\)
\[(3, 12, 108, 219, 230, 148, 36, 2, 1), (1, 21, 95, 220, 245, 135, 37, 5, 0)\].
\[(0, 22, 110, 165, 330, 66, 66, 0, 0)\].

\(n = 12\)
\[(3, 0, 72, 192, 225, 192, 72, 0, 3), (1, 8, 64, 184, 245, 184, 64, 8, 1)\].
\[(0, 12, 60, 180, 255, 180, 60, 12, 0), (0, 11, 66, 165, 275, 165, 66, 11, 0)\].
\[(0, 0, 132, 0, 495, 0, 132, 0, 0)\].
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