MODULES MAPS AND INVARIANT SUBSETS OF BANACH MODULES OF LOCALLY COMPACT GROUPS

by

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Dedicated to my beloved country, Libya, and every person who is happy for my success, especially my family:

My husband: Omar Hamouda

My father: Alsanousi Hamouda

My mother: Aisha Hamouda

My children: Saleh and Mohamed.

Abstract

For a locally compact group G, the papers [13] and [7] have many results about G-invariant subsets of G-modules, and the relationship between G-module maps, $L^1(G)$-module maps and $M(G)$-module maps. In both papers, the results were given for one specific module action. In this thesis we extended many of their results to arbitrary Banach G-modules. In addition, we give detailed proofs of most of the results found in the first section of the paper [21].

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Contents

Chapter 1

Introduction and Overview

1.1 Introduction

Let G always be a locally compact group with a Haar measure λ. Let $L^p(G)$, $1 \leq p \leq \infty$, be the Banach space of λ-measurable functions $f : G \to \mathbb{C}$, such that $\|f\|_p < \infty$ and when $s \in G$, we let δ_s denotes the Dirac measure at s. In addition we let $M(G)$ denote the set of all complex regular Borel measures on G.

The main purpose of this thesis is to generalize Lau and Ghaffari's results in their papers [13] and [7] respectively. Both Lau and Ghaffari's results were about the relation between G-module maps, $L^1(G)$-module maps and $M(G)$- module maps. In addition to that their papers include results about G-invariant subsets of G-modules. Lau's results were specified for just one module action which is $s \cdot f = \delta_s * f$ where $s \in G, f \in L^p(G), 1 \leq p < \infty$, and Ghaffari's results were specified for just one other module action, that is $s \cdot f(t) = \delta_s \star f(t) = \Delta(s)^{\frac{1}{p}} f(s^{-1}ts)$ whenever $s, t \in G, f \in L^p(G), 1 \leq p < \infty$. In this thesis we will obtain many of their results for any Banach G-module.

First we will introduce essential definitions and basic information which will be used in this thesis. In Chapter 2 we will give basic information about W^*-algebras, and we will give proofs of Stokke's statements in section one of his paper [21] and some other statements that are not contained in Stokke's paper. The last chapter deals with the generalization of Lau and Ghaffari's results in their papers [13] and [7] respectively. Also we will show that $(L^1(G), \star)$ is not associative when G is a non-abelian locally compact group which contradicts the statement in Ghaffari's paper [7] that $(L^1(G), \star)$ is a Banach algebra.

1.2 Preliminaries and Notations

In this section we will provide an overview of some basic and necessary concepts of functional analysis and harmonic analysis which we will use in this Master's thesis. Note that most of the proofs will be omitted in this section but they can be found in several books.

1.2.1 Weak and Weak star topologies on Locally Convex Spaces

For proofs and more details related to this subsection see [2] and [17].

Definition 1.1. *Let X be a normed space. Then a subset C of X is* **convex** *if $\alpha x + (1-\alpha)y \in C$ whenever $x, y \in X$, $\alpha \in [0,1]$. If $A \subseteq X$ is any set, then the convex hull of A, denoted by $co(A)$, is the smallest convex set containing A.*

Definition 1.2. *Let X, Y be normed spaces, and C, D convex subsets of X, Y respectively. A map $f : C \to D$ is called* **affine** *if $f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in C$, $\alpha \in [0,1]$.*

Definition 1.3. *A* **topological vector space** *is a vector space X over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with topology τ, such that*

(i) $X \times X \to X : (x, y) \mapsto x + y$ is continuous, and

(ii) $\mathbb{F} \times X \to : (\alpha, x) \mapsto \alpha x$ is also continuous.

Remark 1. *Let X be a topological vector space and $A \subseteq X$. Then $\overline{co(A)} = \overline{co}(A)$ is defined to be the smallest closed convex set containing A.*

Definition 1.4. *A topological vector space X is a* **locally convex space (LCS)** *if it has a base of neighborhoods $\{B_i : i \in I\}$ of $0 \in X$, such that each B_i is convex.*

Example 1.1. *If X is a normed space, then $\{B_r : \{x \in X : \|x\| < r\}, r > 0\}$ is a base of convex neighborhoods of 0.*

Definition 1.5. *Let X be a normed space. Then the collection of all bounded linear (continuous linear) functionals on X is called* **the dual space** *of X, and it is denoted by X^*.*

Theorem 1.1 (**Hahn-Banach Separation Theorem**). *Let X be a LCS and C, K non-empty convex subsets of X such that C is closed, K is compact and $C \cap K$ is empty. Then there is a continuous linear map $\phi \in X^*, \gamma \in \mathbb{R}$, such that for all $\epsilon > 0$, $k \in K$, $c \in C$ we have,*

$$Re\ \phi(c) \leq \gamma < \gamma + \epsilon \leq Re\ \phi(k).$$

Definition 1.6. The weak topology *on a LCS X is the topology defined by the family of seminorms*

$$\mathcal{P} := \{P_{x^*} : x^* \in X^*\} : p_{x^*}(x) = |\langle x^*, x\rangle| \qquad (x \in X).$$

We denote the weak topology on X by wk, w or $\sigma(X, X^*)$.

Definition 1.7. *Let X be a LCS. Then* **the weak star topology** *on X^* is the topology defined by the family of seminorms*

$$\mathcal{P} := \{P_x : x \in X\} : p_x(x^*) = |\langle x, x^*\rangle| \qquad (x^* \in X^*).$$

We denote the weak star topology on X^* by wk^*, w^* or $\sigma(X^*, X)$.

Note if (x_α) is a net in X, then $x_\alpha \xrightarrow{w} x \iff \langle x_\alpha, x^*\rangle \to \langle x, x^*\rangle$ for all $x^* \in X^*$. Moreover if (x^*_α) is a net in X^* , then $x^*_\alpha \xrightarrow{w^*} x^* \iff \langle x^*_\alpha, x\rangle \to \langle x^*, x\rangle$ for all $x \in X$.

Proposition 1.1. *Let X be a LCS (e.g. a Banach space) and $C \subseteq X$ be convex . Then C is closed if and only if it is weakly closed.*

Theorem 1.2 (**Banach Alaoglu's Theorem**). *Let X be a Banach space. Then $(X^*)_1 = \{x^* \in X^* : \|x^*\| \leq 1\}$ is w^*-compact.*

1.2.2 Adjoint operator

More details about the following can be found in [2].
Let X, Y be Banach spaces and let X^*, Y^* denote the dual spaces of X, Y respectively. Moreover, let $B(X, Y) := \{T : X \to Y : T \text{ is linear and continuous}\}$.

Definition 1.8. *Let X, Y be Banach spaces and $T \in B(X, Y)$. Then the* **adjoint operator** *of T is the operator $T^* : Y^* \to X^*$ defined by*

$$\langle T^*\phi, x\rangle = \langle \phi, Tx\rangle \qquad (x \in X, \phi \in Y^*).$$

Proposition 1.2. *Let X and Y be Banach spaces and $T : X \to Y$ be a linear and bounded map. Then the adjoint operator $T^* : Y^* \to X^*$ is $w^* - w^*$ continuous and linear.*

Proposition 1.3. *Let X and Y be Banach spaces and $S : Y^* \to X^*$ be a linear and bounded map, such that S is $w^* - w^*$ continuous. Then S is the adjoint operator of some linear and bounded map $T : X \to Y$.*

1.2.3 Locally Compact Group

For more details related to locally compact groups see [5], [6],[9], and [22]

Definition 1.9. *Let X be a topological space. Then X is* **locally compact** *if each point $x \in X$ has an open neighborhood with compact closure.*

Definition 1.10. *A* **topological group** *is a group G that is also topological space, such that*

(i) for every $s, t \in G$, the map $G \times G \to G : (s, t) \mapsto st$ is continuous;

(ii) for every $s \in G$, the map $G \to G : s \mapsto s^{-1}$ is also continuous.

Note that in this definition, $G \times G$ has the product topology.

Definition 1.11. *A* **locally compact group** *is a topological group that is both locally compact and Hausdorff.*

1.2.4 Complex Regular Borel Measures, Haar Measures and Strongly Quasi-Invariant Measures

Proofs of the following theorems and more details can be found easily in [5], [6], [9], or [17].

Definition 1.12. *Let X be a topological space. The* **Borel σ-algebra** *on X is the smallest σ-algebra containing all the open subsets of X. We denote the Borel σ-algebra on X by $B(X)$; elements of $B(X)$ are called Borel sets. A* **complex Borel measure** *μ on X is a function $\mu : B(X) \to \mathbb{C}$, such that*

(i) $\mu(\phi) = 0$,

(ii) If E_n is a collection of disjoint sets in $B(X)$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

Moreover, if $\mu : B(X) \to [0, \infty]$ satisfies (i) and (ii) above, then μ is called a **positive Borel measure***; it is called* **regular** *if for every Borel set $E \in B(X)$, we have*

(i) if $K \subseteq B(X)$ is compact, then $\mu(K) < \infty$,

(ii) $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open }\}$,

(iii) $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact }\}$.

A complex Borel measure μ is regular if the total variation of μ,

$$|\mu|(E) := \sup\{\sum_{i=1}^{n} |\mu(E_i)| : \{E_i\}_1^n \text{ is a partition of the Borel set } E \text{ by Borel sets}\},$$

is regular.

We will denote the set of all complex regular Borel measures on X by $M(X)$, which is a Banach space with respect to the norm defined by

$$\|\mu\| = |\mu|(X) < \infty \qquad (\mu \in M(X)).$$

Example 1.2. *Let X be a Hausdorff topological space. Then δ_x, the Dirac measure at $x \in X$, is*

$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \qquad (E \in B(X)).$$

From now on let $C_0(X)$ denote the set of all continuous functions $f : X \to \mathbb{C}$ vanishing at ∞ (i.e. for every $\epsilon > 0$ there is a compact set $K \subseteq X$, such that $|f(x)| < \epsilon$ whenever $x \notin K$). Also let $C_{00}(X)$ be the set of all continuous functions $f : X \to \mathbb{C}$ with compact support (i.e. $\overline{\{x : f(x) \neq 0\}}$ is compact). In addition let $C_{00}^{+}(X) := \{f \in C_{00}(X) : f \geq 0\}$.

Definition 1.13. *Let X be a Hausdorff locally compact topological space. Then $I : C_{00}(X) \to \mathbb{C}$ is positive if $I(f) \geq 0$ whenever $f \geq 0$.*
Note that if μ is a positive regular Borel measure and $I : C_{00}(X) \to \mathbb{C}$ is defined by $I(f) = \int f(x)d\mu(x)$, then I is positive.

Theorem 1.3. *(***Riesz Representation Theorem***) Let X be a locally compact Hausdorff space and $I : C_{00}(X) \to \mathbb{C}$ a positive linear functional. Then there is a unique (up to a positive scalar) positive regular Borel measure μ on X, such that*

$$I(f) = \int_X f(x)d\mu(x) \qquad (f \in C_{00}(X)).$$

Theorem 1.4. *Let X be a locally compact Hausdorff space. Then $M(X) \cong C_0(X)^*$ through the pairing*

$$\langle \mu, f \rangle = \int f d\mu \qquad (\mu \in M(X), f \in C_0(X)).$$

From now on for any function $f : G \to \mathbb{C}$ and $s, x \in G$ let

$$\begin{aligned} (\ell_s f)(x) &= f(sx) \text{ and} \\ (r_s f)(x) &= f(xs). \end{aligned}$$

Theorem 1.5. *Let G be a locally compact group . Then there is a unique non-zero (up to a positive scalar) functional I on $C_{00}^{+}(G)$, such that:*

(i) $I(f) \in [0, \infty)$ *whenever* $f \in C_{00}^{+}(G)$*;*

(ii) $I(\alpha f) = \alpha I(f)$ *whenever* $\alpha \in [0, \infty)$ *and* $f \in C_{00}^{+}(G)$*;*

(iii) $I(f + g) = I(f) + I(g)$ *whenever* $f, g \in C_{00}^{+}(G)$*;*

(iv) $I(\ell_s f) = I(f)$ *whenever* $f \in C_{00}^{+}(G)$ *and* $s \in G$*.*

This I *is called a left Haar integral.*

Definition 1.14. *Let* G *be a locally compact group. A* **left Haar measure** λ *on* G *is a non-zero positive regular Borel measure, such that whenever* E *is a Borel set and* $s \in G$*, we have*

$$\lambda(sE) = \lambda(E).$$

Similarly we can define right Haar measures.

Proposition 1.4. *Let* λ *be a non-zero positive regular Borel measure on* G*. Then the following are equivalent*

(i) λ *is a left Haar measure on* G*;*

(ii) the positive linear function $I : C_{00}(G) \to \mathbb{C}$*;* $I(f) = \int_G f(x)d\lambda(x)$ *is left invariant (i.e.* $I(\ell_s f) = I(f)$ *whenever* $s \in G, f \in C_{00}(G)$*). Moreover, if* $I : C_{00}(G) \to \mathbb{C}$ *is any left invariant positive functional, then* $I(f) = \int f(x)d\lambda(x)$ *where* λ *is a Haar measure on* G*.*

Theorem 1.6. *Every locally compact group* G *has a left Haar measure* λ *which is unique up to a positive constant.*

Theorem 1.7 (**Fubini's Theorem**)**.** *Let* μ *and* λ *be regular Borel measures on locally compact spaces* X *and* Y*. Suppose that* f *is integrable with respect to the product measure* $\mu \times \lambda$*. Then*

$$\int_{X\times Y} f(x, y)d(\mu \times \lambda) = \int_Y \int_X f(x, y)d\mu(x)d\lambda(y) = \int_X \int_Y f(x, y)d\lambda(y)d\mu(x).$$

Definition 1.15. *Let G be a locally compact group and λ a left Haar measure. Let E be a Borel set such that $0 < \lambda(E) < \infty$. Then* **the modular function** *is the function*

$$\Delta : G \to (0, \infty) : \Delta(s) = \frac{\lambda(Es)}{\lambda(E)} \qquad (s \in G).$$

If $\Delta \equiv 1$ (i.e. λ is also a right Haar measure), then G is called unimodular.

Remark 2. *It can be shown that this definition of Δ is independent of our choice of E.*

Example 1.3. *Discrete groups, compact groups and abelian groups are unimodular.*

Proposition 1.5. *The modular function Δ is a continuous homomorphism, when $(0, \infty)$ is viewed as a group with respect to multiplication.*

From now on let G be a locally compact group, λ a left Haar measure.
When $1 \leq p < \infty$, $(L^p(G), \lambda)$ denotes the Banach space of λ-measurable functions $f : G \to \mathbb{R}$, such that

$$\|f\|_p = (\int_G |f|^p d\lambda)^{\frac{1}{p}} < \infty;$$

functions that are equal λ-a.e. are identified. When $p = \infty$, $(L^\infty(G), \lambda)$ denotes the Banach space of essentially bounded λ-measurable functions on G; functions that are equal locally λ-a.e. are identified.

From now on when λ is a Haar measure and $x \in G$, we will denote $d\lambda(x)$ by dx and $(L^p(G), \lambda)$ by $L^p(G), 1 \leq p \leq \infty$.

Proposition 1.6. *Let G be a locally compact group. Then whenever f is a nonnegative λ-measurable function on G, and $s \in G$ we have*

$$\int_G r_s f(t)dt = \int_G \frac{1}{\Delta(s)} f(t)dt.$$

Note that the equality also holds when $f \in L^1(G)$.

Let G be a locally compact group and H a closed subgroup of G. We will let $G/H = \{sH : s \in G\}$ denote the left coset space with quotient topology induced by the natural quotient map $q_H : G \to G/H$. We let the map $P : C_{00}(G) \to C_{00}(G/H)$ be defined by

$$f \mapsto Pf : \ Pf(sH) = \int_H f(sh)dh,$$

where integration is with respect to Haar measure on H.

Definition 1.16. *Let G/H be defined as above, and let ν be a positive regular Borel measure on G/H. Put $(s \cdot \nu)(E) := \nu(s^{-1}E)$ for every $s \in G$ and Borel subset E of G/H. Then ν is called* **strongly quasi-invariant** *if there is a continuous function $\sigma : G/H \times G \to (0, \infty)$, such that $d(s \cdot \nu)(\xi) = \sigma(\xi, s)d\nu(\xi)$.*

Definition 1.17. *Let G be a locally compact group and H be a closed subgroup of G. Also let Δ_G, Δ_H denote the modular functions of G, H respectively. Then a continuous function $\rho : G \to (0, \infty)$ is called a* **rho-function** *for the pair (G, H) if*

$$\rho(sh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(s) \qquad (s \in G, h \in H).$$

Theorem 1.8. *Let G be a locally compact group and H a closed subgroup of G. Then*

(i) (G, H) admits a rho-function on G.

(ii) If ρ is any rho-function for (G,H) on G, then there is a strongly quasi-invariant measure ν_ρ on G/H, such that

$$\int_{G/H} P(f)(sH)d\nu_\rho(sH) = \int_G f(s)\rho(s)ds \qquad (f \in C_{00}(G)).$$

Furthermore, ν_ρ satisfies

$$\left(\frac{d(s \cdot \nu_\rho)}{d\nu_\rho}\right)(tH) = \frac{\rho(st)}{\rho(t)} \qquad (s, t \in G),$$

where $\frac{d(s \cdot \nu_\rho)}{d\nu_\rho}$ is the Radon-Nikodym derivative of $s \cdot \nu_\rho$ with respect to ν_ρ.

(iii) *Every strongly quasi-invariant measure on G/H arises from a rho-function as in (ii), where*

$$\sigma(\xi, s) = \frac{d(s \cdot \nu_\rho)}{d\nu_\rho} \qquad (s \in G, \xi \in G/H).$$

Remark 3. *Let ν be a strongly quasi-invariant measure. Then*

(i) $\sigma(\xi, e) = 1 \qquad (\xi \in G/H)$,

(ii) $\sigma(\xi, st) = \sigma(s^{-1}\xi, t)\sigma(\xi, s) \qquad (s, t \in G, \xi \in G/H)$,

(iii) $\int f(\xi) d\nu(\xi) = \int f(s\xi)\sigma(\xi, s^{-1}) d\nu(\xi) \qquad (f \in L^1(G/H), s \in G)$.

From now on when G is a locally compact group and H is a closed subset of G, we let $LUC(G/H)$ be the set of all bounded continuous functions $f : G/H \to \mathbb{C}$, such that $s \mapsto \delta_s * f$ is continuous with respect to $(CB(G/H), \|\cdot\|_\infty)$ where

$$\delta_s * f(\xi) := f(s^{-1}\xi)\sigma(\xi, s) \qquad (s \in G,\ \xi \in G/H).$$

Proposition 1.7. *Let G be a locally compact group and H a closed subgroup of G. Then $C_{00}(G/H) \subseteq LUC(G/H)$.*

1.2.5 General Definitions and Theory

Proposition 1.8. *Let X and Y be topological spaces. Also let $T : X \to Y$ be a continuous map and $K \subseteq X$ compact. Then T(K) is also compact.*

Proof. This is very standard. $\square$

Definition 1.18 (Weak Integral). *Let μ be a positive measure on a measurable space $\mathcal{Q}$, and let X be a topological vector space such that X^* separates points of X. Let $f : \mathcal{Q} \to X$ be a measurable function, such that:*

(i) *Whenever $\phi \in X^*$, we have $\phi f : \mathcal{Q} \to \mathbb{F}$ is integrable with respect to μ. (Note $(\phi f)(q) = \phi(f(q))$; $q \in \mathcal{Q}$).*

(ii) *There is $y \in X$, such that $\langle \phi, y \rangle = \int \phi f d\mu$, whenever $\phi \in X^*$.*

Then $y = \int f d\mu$ *is called a weak integral of* f *over* $\mathcal{Q}$ *with respect to* μ*. So the weak integral has the property*

$$\phi(\int_{\mathcal{Q}} f d\mu) = \int_{\mathcal{Q}} \phi f d\mu \qquad (\phi \in X^*).$$

For more details about the weak integral the reader is referred to [16] and [19]. The following definition and two results appear in [13].

Definition 1.19. *Let* τ *be the locally convex topology on* $M(G)$ *generated by the collection of seminorms* $\{P_f : f \in CB(G)\}$*, such that*

$$P_f(\mu) = |\langle \mu, f \rangle| = |\int f d\mu| \qquad (\mu \in M(G)).$$

So $\mu_\alpha \xrightarrow{\tau} \mu$ *means that whenever* $f \in CB(G), \int f d\mu_\alpha \to \int f d\mu$*.*

Let $M(G)_1^+$ denote the set of probability measures in $M(G)$, and as before $L^1(G)_1^+ := \{f \in L^1(G) : f \geq 0 \text{ and } \|f\|_1 = 1\}$.

Proposition 1.9. *Let* G *be a locally compact group. Then*

$$M(G)_1^+ = \overline{L^1(G)_1^+}^{\tau} = \overline{co}^{\tau}\{\delta_s : s \in G\}.$$

Theorem 1.9. *Let* G *be a locally compact group. Then* G *is non-compact if and only if* $K = \{0\}$ *for every weakly compact convex left (right) translation invariant non-empty subset* K *of* $L^1(G)$*.*

1.2.6 Involution and Convolution

See [5], [6], [9] and [20] for more details related to the following.

Definition 1.20. *Let* $\mathcal{A}$ *be an algebra over* $\mathbb{C}$*. Then an* **involution** *on* $\mathcal{A}$ *is a map* $\mathcal{A} \to \mathcal{A} : x \mapsto x^*$*, such that*

(i) $(x^*)^* = x$,

(ii) $(x + y)^* = x^* + y^*$,

(iii) $(\alpha x)^* = \overline{\alpha} x^*$,

(iv) $(xy)^* = y^*x^*$.

Definition 1.21. *Let $\mathcal{A}$ be a Banach algebra (i.e. a Banach space such that $\mathcal{A}$ is an associative algebra and $\|xy\| \leq \|x\|\|y\|$ whenever $x, y \in \mathcal{A}$) with involution. Then $\mathcal{A}$ is called a C^*-**algebra** if $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$.*

Let G be a locally compact group and $M(G)$ the space of all complex regular Borel measures $\cong C_0(G)^*$. Then the convolution product, $\mu * \nu$, of two measures $\mu, \nu \in M(G) = C_0(G)^*$ is defined as the following:

$$\langle \mu * \nu, f \rangle = \int \int f(xy) d\mu(x) d\nu(y) \qquad (f \in C_0(G)).$$

This convolution makes the Banach space $M(G)$ equipped with the total variation norm into a Banach algebra, called the measure algebra. This measure algebra has an isometric involution $\mu \to \mu^*$ defined by

$$\mu^*(E) = \overline{\mu(E^{-1})}, \text{ for every Borel set } E.$$

Note that although $M(G)$ is an involutive Banach algebra, it is not a C^*-algebra.

Remark 4. *Let G be a locally compact group and δ_s, δ_t be the Dirac measures at $s, t \in G$. Then $\delta_s * \delta_t = \delta_{st}$ and $\delta_s^* = \delta_{s^{-1}}$.*

If $1 \leq p \leq \infty$, $h \in L^p(G)$ and $\mu \in M(G)$, then the convolution product between μ and h is defined as the following:

$$\begin{aligned} \langle \mu * h, \phi \rangle &= \int \int \phi(xy) d\mu(x) h(y) dy \qquad (\phi \in C_{00}(G)) \\ \langle h * \mu, \phi \rangle &= \int \int \phi(xy) h(x) dx d\mu(y) \qquad (\phi \in C_{00}(G)). \end{aligned}$$

From the above formulas we can easily derive that

$$\begin{aligned} \mu * h(x) &= \int_G h(s^{-1}x) d\mu(s), \\ h * \mu(x) &= \int_G \frac{1}{\Delta(s)} h(xs^{-1}) d\mu(s). \end{aligned}$$

a.e. x for $1 \leq p < \infty$ and locally a.e. x for $p = \infty$. Note in this case that $\mu * h \in L^p(G)$ and $\|\mu * h\|_p \leq \|\mu\|\|h\|_p$. Moreover, if $\int \Delta(s)^{-\frac{1}{q}} d|\mu|(s) < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then $h * \mu$ exists and contained in $L^p(G)$. In addition to that $\|h * \mu\|_p \leq \|h\|_p \Delta(s)^{-\frac{1}{q}} d|\mu|$.

If $f \in L^1(G)$, define the convolution product between f, h as the following:

$$\begin{aligned} f * h(x) &= \int_G f(s)h(s^{-1}x)ds = \int_G f(s)\ell_{s^{-1}}h(x)ds, \\ h * f(x) &= \int_G f(s)\frac{1}{\Delta(s)}h(xs^{-1})ds. \end{aligned}$$

a.e. x for $1 \leq p < \infty$ and locally a.e. x for $p = \infty$.

Note that, if $\Delta(s)^{-\frac{1}{q}}f(s) \in L^1(G)$, then $h*f \in L^p(G)$ (eg. when G is unimodular, $f \in C_{00}(G)$ or $p = 1$). Also define the involution of f by

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}.$$

Note that by the Radon-Nikodym Theorem, the Banach algebra $(L^1(G), *)$, which is called the group algebra, is a closed ideal of the measure algebra via $f \mapsto \mu_f$ where $\langle \mu_f, \phi \rangle = \int \phi(s)f(s)ds$ whenever $\phi \in C_0(G)$.

Proposition 1.10. *Let G be a locally compact group and λ be a left Haar measure. Then whenever $f \in L^1(G)$ we have:*

$$\int f(x)dx = \int \frac{1}{\Delta(x)}f(x^{-1})dx.$$

Remark 5. *Let $s \in G$, and $f \in L^p(G)$ where $1 \leq p \leq \infty$. Then*

$$\begin{aligned} \delta_s * f(x) &= f(s^{-1}x) = \ell_{s^{-1}}f(x), \\ f * \delta_s(x) &= \frac{1}{\Delta(s)}f(xs^{-1}) = \frac{1}{\Delta(s)}r_{s^{-1}}f(x). \end{aligned}$$

Proposition 1.11. *Let $h \in L^p(G)$ where $1 \leq p \leq \infty$. Then the maps $s \mapsto \ell_s h$, $s \mapsto r_s h, s \mapsto \delta_s * h$ and $s \mapsto h * \delta_s : G \to L^P(G)$ are continuous.*

1.2.7 G-module and Dual Module

In this subsection we will give some proofs. For the omitted proofs and more details see [3], [5], [11] and [15].

Definition 1.22. *Let X be a vector space and $\mathcal{A}$ an algebra. Then X is a* **left $\mathcal{A}$-module** *if there is an action of $\mathcal{A}$ on X;*

$$\mathcal{A} \times X \to X : (a, x) \mapsto a \cdot x \qquad (a \in \mathcal{A}, x \in X),$$

which behaves nicely with respect to the algebraic operations of both $\mathcal{A}$ and X. More precisely, for every $a, b \in \mathcal{A}$, $x, y \in X$, and $\alpha \in \mathbb{C}$,

(i) $a \cdot (\alpha x + y) = \alpha(a \cdot x) + a \cdot y$,

(ii) $(\alpha a + b) \cdot x = \alpha(a \cdot x) + b \cdot x$ *and*

(iii) $a \cdot (b \cdot x) = (ab) \cdot x$.

Note that if $\mathcal{A}$ has an identity, 1, then X is called a left unital $\mathcal{A}$-module if $1 \cdot x = x$ for every $x \in X$. We can similarly define right (unital) $\mathcal{A}$-modules. And if X is both a left and right $\mathcal{A}$-module and for all $a, b \in \mathcal{A}, x \in X$, $a \cdot (x \cdot b) = (a \cdot x) \cdot b$, then X is called an $\mathcal{A}$-bimodule.

Example 1.4. *If $\mathcal{A}$ is an algebra and $I \lhd \mathcal{A}$ (left ideal), then I is a left $\mathcal{A}$-module with respect to $a \cdot x = ax \in I$ where $a \in \mathcal{A}$ and $x \in I$.*

Definition 1.23. *Let $\mathcal{A}$ be a Banach algebra and X a Banach space. Then X is a* **left Banach $\mathcal{A}$-module** *if X is a left $\mathcal{A}$-module and there is a $k > 0$, such that*

$$\|a \cdot x\| \leq k\|a\|\|x\| \qquad (a \in A, x \in X).$$

We can similarly define right Banach $\mathcal{A}$-modules.

Example 1.5. *If I is a closed left ideal in $\mathcal{A}$, then I is a left Banach $\mathcal{A}$-module.*

Example 1.6. *$L^1(G) \lhd M(G)$ (closed ideal) so $L^1(G)$ is a Banach $M(G)$-bimodule with respect to*

$$\mu \cdot f = \mu * f, \ f \cdot \mu = f * \mu \qquad (f \in L^1(G), \mu \in M(G)).$$

Definition 1.24. *Let $\mathcal{A}$ be a Banach algebra. Then a net $(e_\alpha)_\alpha \subseteq \mathcal{A}$ is called a* **bounded approximate identity (BAI) for $\mathcal{A}$** *if:*

(i) $\sup_\alpha \|e_\alpha\| < \infty$, *and*

(ii) $\|e_\alpha a - a\| \to 0$, $\|a e_\alpha - a\| \to 0 \qquad (a \in \mathcal{A})$.

Definition 1.25. *Let X be a left (right) Banach $\mathcal{A}$-module. Then a bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ is a* **BAI for X in $\mathcal{A}$** *if*

$$\|e_\alpha \cdot x - x\| \to 0 \quad (\|x \cdot e_\alpha - x\| \to 0) \qquad (x \in X).$$

We will always let $L^1(G)_1^+ := \{f \in L^1(G) : f \geq 0 \text{ and } \|f\|_1 = 1\}$.

Remark 6. *Let G be a locally compact group and I be the neighborhood system of e ordered by downward inclusion: $\alpha \geq \beta \Leftrightarrow \alpha \subseteq \beta$. For each α, let v be a compact neighborhood of e, such that $v = v^{-1}$ and $v \subseteq \alpha$, and let $e_\alpha = \frac{1}{\lambda(v)} 1_v$. Then $(e_\alpha)_{\alpha \in I} = (\frac{1}{\lambda(v)} 1_v)_{\alpha \in I} \subseteq L^1(G)_1^+$ is a bounded approximate identity for $L^1(G)$, such that $e_\alpha = \widetilde{e_\alpha}$, where $\widetilde{e_\alpha}(x) = e_\alpha(x^{-1})$, and if we take $u_\alpha = \frac{1}{2}(e_\alpha + e_\alpha^*)$, then $(u_\alpha)_{\alpha \in I} \subseteq L^1(G)_1^+$ is a BAI for $L^1(G)$ with $u_\alpha^* = u_\alpha$.*

Definition 1.26. *Let X be a left (right) Banach $\mathcal{A}$-module. Then X is a* **left (right) neo-unital Banach $\mathcal{A}$-module** *if $\mathcal{A} \cdot X = X$ $(X \cdot \mathcal{A} = X)$.*

Theorem 1.10 (Cohen Factorization Theorem). *Let X be a left (right) Banach $\mathcal{A}$-module, and suppose that there is a BAI for X in $\mathcal{A}$. Then X is a left (right) neo-unital Banach $\mathcal{A}$-module.*

Corollary 1.1. *Let X be a left (right) neo-unital Banach $\mathcal{A}$-module, and $(e_\alpha)_\alpha \subseteq \mathcal{A}$ a BAI for $\mathcal{A}$. Then $(e_\alpha)_\alpha$ is a BAI for X in $\mathcal{A}$.*

Proof. Suppose X is a left neo-unital Banach $\mathcal{A}$-module and let $x \in X$ be given. Then there is an $a \in \mathcal{A}$ and $y \in X$ such that $x = a.y$. Now let $(e_\alpha)_\alpha \subseteq \mathcal{A}$ be a BAI for $\mathcal{A}$. Then

$$\begin{aligned} \|e_\alpha \cdot x - x\| &= \|e_\alpha \cdot (a \cdot y) - a \cdot y\| \\ &= \|(e_\alpha a - a) \cdot y\| \\ &\leq k\|e_\alpha a - a\|\|y\| \to 0 \qquad (k > 0). \end{aligned}$$

A similar proof works when X is a right neo-unital Banach $\mathcal{A}$-module. $\square$

Definition 1.27. *Let X be a Banach space and G be a locally compact group. Then X is a* **left G-module** *if* *(i) G acts on X on the left, and*

(ii) $s \cdot (\alpha x + y) = \alpha(s \cdot x) + s \cdot y$ where $s \in G, x, y \in X, \alpha \in \mathbb{C}$.

Definition 1.28. *Let X be a Banach space. Then X is a* **left Banach G-module** *if*

(i) X is a left G-module,

(ii) there is $k > 0$, such that $\|s \cdot x\| \leq k\|x\|$ whenever $s \in G, x \in X$, and

(iii) for each $x \in X$, $s \mapsto s \cdot x : G \to X$ is continuous.

Similarly we can define right (Banach) G-modules and (Banach) G-bimodules.

We will often refer to the next three examples in this thesis.

Example 1.7. *Let $X = L^p(G)$, $1 \leq p < \infty$, and define $s \cdot f = \delta_s * f = \ell_{s^{-1}} f$ where $s \in G, f \in L^p(G)$. Then X is a left Banach G-module.*

Proof. Let $s, t, x \in G, f, g \in L^p(G)$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} ((st) \cdot f)(x) &= (\delta_{st} * f)(x) \\ &= f((st)^{-1}x) \\ &= f((t^{-1}s^{-1}x) \\ &= f(t^{-1}(s^{-1}x)) \\ &= (\delta_t * f)(s^{-1}x) \\ &= (\delta_s * (\delta_t * f))(x) \\ &= (s \cdot (t \cdot f))(x). \end{aligned}$$

Moreover,

$$\begin{aligned} (s \cdot (\alpha f + g))(x) &= (\delta_s * (\alpha f + g))(x) \\ &= (\alpha f + g)(s^{-1}x) \\ &= \alpha f(s^{-1}x) + g(s^{-1}x) \\ &= \alpha(\delta_s * f)(x) + (\delta_s * g)(x) \\ &= \alpha(s \cdot f)(x) + s \cdot g(x) \\ &= (\alpha(s \cdot f) + (s \cdot g))(x). \end{aligned}$$

Further,

$$\begin{aligned}
\|s \cdot f\|_p &= \|\delta_s * f\|_p \\
&= \left(\int |f(s^{-1}t)|^p dt\right)^{\frac{1}{p}} \\
&= \left(\int |f(t)|^p dt\right)^{\frac{1}{p}} \qquad (t \mapsto st) \\
&= \|f\|_p.
\end{aligned}$$

Finally, if $s_\alpha \to s \in G$, then by Proposition 1.11

$$\|s_\alpha \cdot f - s \cdot f\|_p = \|\delta_{s_\alpha} * f - \delta_s * f\|_p \to 0.$$

Hence $L^p(G)$ is a left Banach G-module. □

Example 1.8. *Let $X = L^p(G),\ 1 \leq p < \infty$, and define $s \cdot f(x) = \Delta^{\frac{1}{p}}(s)f(s^{-1}xs)$ where $s \in G, f \in L^p(G)$. Equivalently, $s \cdot f = \Delta^{\frac{1}{q}}(s^{-1})\delta_s * f * \delta_{s^{-1}}$ where $\frac{1}{p} + \frac{1}{q} = 1$ when $p > 1$; $s \cdot f = \delta_s * f * \delta_{s^{-1}}$ when $p = 1$. Then X is a left Banach G-module.*

Proof. Let $s, t, x \in G, f, g \in L^p(G)$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}
((st) \cdot f)(x) &= \Delta^{\frac{1}{p}}(st)f((st)^{-1}x(st)) \\
&= \Delta^{\frac{1}{p}}(s)\Delta^{\frac{1}{p}}(t)f(t^{-1}(s^{-1}xs)t) \qquad \text{(by Proposition 1.5)} \\
&= \Delta^{\frac{1}{p}}(s)(t \cdot f)(s^{-1}xs) \\
&= s \cdot (t \cdot f)(x).
\end{aligned}$$

Also,

$$\begin{aligned}
s.(\alpha f + g)(x) &= \Delta^{\frac{1}{p}}(s)(\alpha f + g)(s^{-1}xs) \\
&= \alpha\Delta^{\frac{1}{p}}(s)f(s^{-1}xs) + \Delta^{\frac{1}{p}}(s)g(s^{-1}xs) \\
&= (\alpha(s.f) + s.g)(x).
\end{aligned}$$

Next,

$$\begin{aligned}
\|s.f\|_p &= \left(\int_G |\Delta^{\frac{1}{p}}(s)f(s^{-1}xs)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_G \Delta(s)|f(xs)|^p dx\right)^{\frac{1}{p}} \qquad (x \mapsto sx) \\
&= \Delta^{\frac{1}{p}}(s)\left(\int_G \frac{1}{\Delta(s)}|f(x)|^p dx\right)^{\frac{1}{p}} \qquad \text{(by Proposition 1.6)} \\
&= \Delta^{\frac{1}{p}}(s)\Delta^{-\frac{1}{p}}(s)\|f\|_p \\
&= \|f\|_p.
\end{aligned}$$

Finally, if $s_\alpha \to s$ in G, then by Proposition 1.11, we have;
When $p = 1$,

$$\begin{aligned}
\|s_\alpha \cdot f - s \cdot f\|_1 &= \|\delta_{s_\alpha} * f * \delta_{s_\alpha{}^{-1}} - \delta_s * f * \delta_{s^{-1}}\|_1 \\
&\leq \|\delta_{s_\alpha} * f * \delta_{s_\alpha{}^{-1}} - \delta_{s_\alpha} * f * \delta_{s^{-1}}\|_1 + \|\delta_{s_\alpha} * f * \delta_{s^{-1}} - \delta_s * f * \delta_{s^{-1}}\|_1 \\
&\leq \|f * \delta_{s_\alpha{}^{-1}} - f * \delta_{s^{-1}}\|_1 + \|\delta_{s_\alpha} * f - \delta_s * f\|_1 \to 0.
\end{aligned}$$

When $p > 1$,

$$\begin{aligned}
\|s_\alpha \cdot f - s \cdot f\|_p &= \|\Delta(s_\alpha^{-1})^{\frac{1}{q}}\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}} - \Delta(s^{-1})^{\frac{1}{q}}\delta_s * f * \delta_{s^{-1}}\|_p \\
&\leq \|\Delta(s_\alpha^{-1})^{\frac{1}{q}}\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}} - \Delta(s^{-1})^{\frac{1}{q}}\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}}\|_p \\
&+ \|\Delta(s^{-1})^{\frac{1}{q}}\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}} - \Delta(s^{-1})^{\frac{1}{q}}\delta_s * f * \delta_s^{-1}\|_p \\
&= |\Delta(s_\alpha^{-1})^{\frac{1}{q}} - \Delta(s^{-1})^{\frac{1}{q}}|\ \|\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}}\|_p \\
&+ \Delta(s^{-1})^{\frac{1}{q}}\|\delta_{s_\alpha} * f * \delta_{s_\alpha^{-1}} - \delta_s * f * \delta_s^{-1}\|_p \to 0,
\end{aligned}$$

by the same steps found in the $p = 1$ case and Proposition 1.5. Hence $L^p(G)$ is a left Banach G-module. □

Example 1.9. *Let $X = (L^p(G/H), \nu)$, $1 \leq p < \infty$, where G is a locally compact group, H is a closed subgroup of G and ν is a strongly quasi-invariant measure on G/H. Then X is a left Banach G-module with $s \cdot f(\xi) = \sigma(\xi, s)^{\frac{1}{p}} f(s^{-1}\xi)$ $(= \delta_s * f(\xi)$ when $p = 1)$ where $\xi \in G/H$, $s \in G$ and $f \in X$.*

Proof. Let $f, g \in X, s, t \in G, \xi \in G/H, \alpha \in \mathbb{C}$. Then

$$\begin{aligned} s \cdot (\alpha f + g)(\xi) &= (\alpha f + g)(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} \\ &= \alpha f(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} + g(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} \\ &= (\alpha(s \cdot f) + s \cdot g)(\xi). \end{aligned}$$

Moreover,

$$\begin{aligned} s \cdot (t \cdot f)(\xi) &= (t \cdot f)(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} \\ &= f(t^{-1}(s^{-1}\xi))\sigma(s^{-1}\xi, t)^{\frac{1}{p}}\sigma(\xi, s)^{\frac{1}{p}} \\ &= f((st)^{-1}\xi)\sigma(\xi, st)^{\frac{1}{p}} \qquad \text{(by Remark 3)} \\ &= st \cdot f(\xi). \end{aligned}$$

Also

$$\begin{aligned} \|s \cdot f\|_p^p &= \int |f(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}}|^p d\nu(\xi) \\ &= \int |f(s^{-1}\xi)|^p \sigma(\xi, s) d\nu(\xi) \\ &= \int |f(\xi)|^p \sigma(s\xi, s)\sigma(\xi, s^{-1}) d\nu(\xi) \qquad (\xi \mapsto s\xi \text{ and by Remark3}) \\ &= \int |f(\xi)|^p \underbrace{\sigma(\xi, e)}_{=1} d\nu(\xi) \qquad \text{(by Remark 3)} \\ &= \int |f(\xi)|^p d\nu(\xi) \\ &= \|f\|_p^p. \end{aligned}$$

Hence, $\|s \cdot f\|_p = \|f\|_p$. Next we want to show that $s \mapsto s \cdot f$ is continuous. First suppose $f \in C_{00}(G/H)$. Then one can check that $s \mapsto s \cdot f$ is continuous on G with respect to the uniform norm on $CB(G)$; indeed, the proof of Proposition 1.7 which establishes this statement in the case $p = 1$ can readily be adapted to the situation when $p > 1$, so when $s_\alpha \to s$, $s_\alpha \cdot f \to s \cdot f$. Now let $f \in X$ and take $(f_n) \subseteq C_{00}(G/H)$, such that $\|f_n - f\|_p \to 0$. Let $\epsilon > 0$ and suppose $s_\alpha \to s$ in G and take f_{n_0}, such that

$\|f_{n_0} - f\|_p < \frac{\epsilon}{3}$. Take α_0 such that $\alpha \geq \alpha_0$ implies $\|s_\alpha \cdot f_{n_0} - s \cdot f_{n_0}\|_p < \frac{\epsilon}{3}$. Then

$$\begin{aligned}
\|s_\alpha \cdot f - s \cdot f\|_p &\leq \|s_\alpha \cdot f - s_\alpha \cdot f_{n_0}\|_p + \|s_\alpha \cdot f_{n_0} - s \cdot f_{n_0}\|_p \\
&+ \|s \cdot f_{n_0} - s \cdot f\|_p \\
&< \|f - f_{n_0}\|_p + \frac{\epsilon}{3} + \|f_{n_0} - f\|_p \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, $s \mapsto s.f$ is continuous. Hence X is a left Banach G-module. □

Note that if X is a left Banach G-module, then X is a Banach G-bimodule if we define $x \cdot s = x$ whenever $x \in X, s \in G$. So this allows us to view any left Banach G-module as Banach G-bimodule.

Lemma 1.1. *Let X be a Banach space satisfying (i) and (ii) in Definition 1.28. Then X is a left Banach G-module if $s \mapsto s \cdot x$ is weak continuous whenever $s \in G$ and $x \in X$. See [11].*

Proposition 1.12. *Let X be a left (right) Banach G-module, $x \in X$. Then*

(i) X is a left (right) unital Banach $M(G)$-module with respect to

$$\begin{aligned}
\mu \cdot x &= \int s \cdot x d\mu(s) \qquad (\mu \in M(G), s \in G)) \\
(x \cdot \mu &= \int x \cdot s d\mu(s) \qquad (\mu \in M(G), s \in G)).
\end{aligned}$$

Note that the integral here is the weak integral.

(ii) X is a left (right) neo-unital Banach $L^1(G)$-module with respect to the restriction of the equations in (i) to $L^1(G)$:

$$\begin{aligned}
f \cdot x &= \int (s \cdot x) f(s) ds \qquad (f \in L^1(G), s \in G) \\
(x \cdot f &= \int (x \cdot s) f(s) ds \qquad (f \in L^1(G), s \in G)).
\end{aligned}$$

Proposition 1.13. *Let X be a left (right) neo-unital Banach $L^1(G)$-module and let $(e_\alpha)_\alpha \subseteq L^1(G)$ be a BAI for $L^1(G)$, $x \in X$. Then* *(i) X is a left (right) unital Banach $M(G)$-module with respect to:*

$$\begin{aligned} \mu \cdot x &= \lim(\mu * e_\alpha) \cdot x \qquad (\mu \in M(G)) \\ (x \cdot \mu &= \lim x \cdot (e_\alpha * \mu) \qquad (\mu \in M(G))). \end{aligned}$$

(ii) X is a left (right) Banach G-module with respect to:

$$\begin{aligned} s \cdot x &= \delta_s \cdot x \qquad (s \in G, x \in X) \\ ((x \cdot s &= x \cdot \delta_s) \qquad (s \in G, x \in X)). \end{aligned}$$

Definition 1.29. *Let τ_M denote the* **right multiplier (or strict) topology** *on $M(G)$ taken with respect to the ideal $L^1(G)$; that is, τ_M is the locally convex topology on $M(G)$ generated by the seminorms $p_f(\mu) = \|\mu * f\|_1$, $f \in L^1(G)$. (So, $\mu_i \to \mu$ strictly means that $\|\mu_i * f - \mu * f\|_1 \to 0$ for each $f \in L^1(G)$.)*

The correspondence between G-modules, $L^1(G)$-modules and $M(G)$-modules is as follows:
There is a one-to-one correspondence between Banach G-modules X; neo-unital Banach $L^1(G)$-modules X; and unital $M(G)$-modules X such that for each $x \in X$, the map $\mu \mapsto \mu \cdot x : (M(G), \tau_M) \to (X, \|\cdot\|)$ is continuous. Note that the induced $M(G)$-module structure from a Banach G-module X obviously satisfies this property because $X = L^1(G) \cdot X$ by Proposition 1.12. Also, note that by Proposition 1.11 if $s_i \to s$ in G, then $\delta_{s_i} \xrightarrow{\tau_M} \delta_s$ in $M(G)$ so, if the $M(G)$-module action satisfies this property, you will get a Banach G-module by restriction.

Proposition 1.14. *Let X be a left (right) Banach $\mathcal{A}$-module. Then X^* is a right (left) dual Banach $\mathcal{A}$-module with respect to*

$$\begin{aligned} \langle \phi \cdot a, x\rangle &= \langle \phi, a \cdot x\rangle \qquad (a \in \mathcal{A}, \phi \in X^*, x \in X) \\ (\langle a \cdot \phi, x\rangle &= \langle \phi, x \cdot a\rangle \qquad (a \in \mathcal{A}, \phi \in X^*, x \in X)). \end{aligned}$$

This action is called the dual module action.

Proposition 1.15. *Let $L_a : X \to X : L_a(x) = a \cdot x$ where $a \in \mathcal{A}, x \in X$. Then $L_a^* : X^* \to X^*$ is given by $L_a^*(\phi) = \phi \cdot a$. This gives linearity and continuity, in fact w^*-continuity, of the the dual module action.*

Proof. Let $\phi \in X^*, a \in \mathcal{A}, x \in X$, then we observe that

$$\begin{aligned} \langle L_a^*(\phi), x\rangle &= \langle \phi, L_a(x)\rangle \\ &= \langle \phi, a \cdot x\rangle \\ &= \langle \phi \cdot a, x\rangle. \end{aligned}$$

Now let $(\phi_\alpha)_\alpha$ be a net in X^* such that $\phi_\alpha \xrightarrow{w^*} \phi$ (i.e $\langle \phi_\alpha, x\rangle \to \langle \phi, x\rangle$ whenever $x \in X$). Then we have

$$\langle \phi_\alpha \cdot a, x\rangle = \langle \phi_\alpha, a \cdot x\rangle \to \langle \phi, a \cdot x\rangle = \langle \phi \cdot a, x\rangle.$$

□

Proposition 1.16. *Let X be a left (right) Banach G-module. Then X^* is a right (left) G-module with respect to*

$$\begin{aligned} \langle \phi \cdot s, x\rangle &= \langle \phi, s \cdot x\rangle;\ s \in G, \phi \in X^*, x \in X \\ (\langle s \cdot \phi, x\rangle &= \langle \phi, x \cdot s\rangle;\ s \in G, \phi \in X^*, x \in X). \end{aligned}$$

Note that X^* will be called a right (left) dual $G-$module with respect to this action.

In the following examples we will derive the formulas for the $L^1(G)$-module actions and $M(G)$-module actions associated with the Banach G-modules described in Examples 1.7, 1.8 and 1.9. Although these module correspondences are likely known, we are not aware of any references in which they are explicitly derived.

Example 1.10. *Let $f \in L^1(G), s \in G, h \in L^p(G)$ where $1 \leq p < \infty, \mu \in M(G)$, and $\phi \in L^p(G)^* = L^q(G)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Define $s \cdot h = \delta_s * h = \ell_{s^{-1}} h$. Then with* *respect to this action we have:*

$$\begin{aligned}
\langle \phi, \mu \cdot h \rangle &= \int \langle \phi, s \cdot h \rangle d\mu(s) \qquad (\textit{Proposition } 1.12) \\
&= \int \int \phi(t)(\delta_s * h)(t) dt d\mu(s) \\
&= \int \int \phi(t) h(s^{-1}t) dt d\mu(s) \\
&= \int \phi(t) \int h(s^{-1}t) d\mu(s) dt \qquad (\textit{Fubini's Theorem}) \\
&= \int \phi(t)(\mu * h)(t) dt \\
&= \langle \phi, \mu * h \rangle.
\end{aligned}$$

Hence $\mu \cdot h(t) = \mu * h(t)$, *as defined in* §1.2.5, *a.e.* t *[locally a.e. if* $p = 1, q = \infty$*] for all* $\mu \in M(G), h \in L^p(G)$. *By restriction to* $L^1(G)$ *we have* $f \cdot h(t) = f * h(t)$ *a.e.* t *[locally a.e. if* $p = 1, q = \infty$*] for all* $f \in L^1(G), h \in L^p(G)$. *Furthermore,*

$$\begin{aligned}
\langle \phi \cdot s, h \rangle &= \langle \phi, s \cdot h \rangle \\
&= \langle \phi, \delta_s * h \rangle \\
&= \int \phi(t) h(s^{-1}t) dt \\
&= \int \phi(st) h(t) dt \qquad (t \mapsto st) \\
&= \langle \ell_s \phi, h \rangle.
\end{aligned}$$

Thus, $\phi \cdot s(t) = \ell_s \phi(t)$ *a.e.* t *[locally a.e. if* $p = 1, q = \infty$*] for every* $\phi \in L^q(G), s \in G$.

Finally,

$$\begin{aligned}
\langle \phi\cdot\mu, h\rangle &= \langle \phi, \mu\cdot h\rangle \\
&= \int \phi(t)(\mu\cdot h)(t)dt \\
&= \int \phi(t)\int (s\cdot h)(t)d\mu(s)dt \\
&= \int \phi(t)\int h(s^{-1}t)d\mu(s)dt \\
&= \int\int \phi(t)h(s^{-1}t)d\mu(s)dt \\
&= \int\int \phi(t)h(s^{-1}t)dtd\mu(s) \qquad (\textit{Fubini's Theorem}) \\
&= \int\int \phi(st)h(t)dtd\mu(s) \qquad (t\mapsto st) \\
&= \int\int \phi(st)d\mu(s)h(t)dt \qquad (\textit{Fubini's Theorem}).
\end{aligned}$$

Therefore $\phi\cdot\mu(t) = \int \phi(st)d\mu(s)$ *a.e.* t *[locally a.e. if* $p=1, q=\infty$*] for every* $\phi\in L^q(G), \mu\in M(G)$*. By restriction to* $L^1(G)$ *we have* $\phi\cdot f(t) = \int \phi(st)f(s)ds$ *a.e.* t *[locally a.e. if* $p=1, q=\infty$*] for all* $f\in L^1(G), \phi\in L^q(G)$*.*

Example 1.11. *Let* $f\in L^1(G), s\in G, h\in L^p(G)$ *where* $1\le p<\infty, \mu\in M(G)$*, and* $\phi\in L^p(G)^* = L^q(G)$ *where* $\frac{1}{p}+\frac{1}{q}=1$ *. Define* $s\cdot h(x) = \Delta^{\frac{1}{p}}(s)h(s^{-1}xs)$ *where* $s\in G, f\in L^p(G)$*. Then with respect to this action we have:*

$$\begin{aligned}
\langle \phi, \mu\cdot h\rangle &= \int \langle \phi, s\cdot h\rangle d\mu(s) \\
&= \int\int \phi(t)(s\cdot h)(t)dtd\mu(s) \\
&= \int\int \phi(t)\Delta^{\frac{1}{p}}(s)h(s^{-1}ts)dtd\mu(s) \qquad (\textit{Remark } 5) \\
&= \int \phi(t)\int \Delta^{\frac{1}{p}}(s)h(s^{-1}ts)d\mu(s)dt \qquad (\textit{Fubini's Theorem}).
\end{aligned}$$

Hence, $\mu\cdot h(t) = \int \Delta^{\frac{1}{p}}(s)h(s^{-1}ts)d\mu(s)$ *a.e.* t *[locally a.e. if* $p=1, q=\infty$*] for all* $\mu\in M(G), h\in L^p(G)$*. By restriction to* $L^1(G)$*, we have* $f\cdot h(t) = \int \Delta^{\frac{1}{p}}(s)h(s^{-1}ts)f(s)ds$

a.e. t [locally a.e. if $p=1, q=\infty$] for all $f \in L^1(G), h \in L^p(G)$. Furthermore,

$$\begin{aligned}
\langle \phi \cdot s, h \rangle &= \langle \phi, s \cdot h \rangle \\
&= \int \phi(t)(s \cdot h)(t)dt \\
&= \int \phi(t)\Delta^{\frac{1}{p}}(s)h(s^{-1}ts)dt \\
&= \int \phi(st)\Delta^{\frac{1}{p}}(s)h(ts)dt \qquad (t \mapsto st) \\
&= \int \phi(sts^{-1})\Delta^{\frac{1}{p}}(s)\Delta(s^{-1})h(t)dt \qquad (t \mapsto ts^{-1}) \\
&= \int \Delta^{1-\frac{1}{p}}(s^{-1})\phi(sts^{-1})h(t)dt.
\end{aligned}$$

Therefore, $\phi \cdot s(t) = \Delta^{\frac{1}{q}}(s^{-1})\phi(sts^{-1})$ a.e. t [locally a.e. if $p = 1, q = \infty$] for all $\phi \in L^q(G), s \in G$.
Finally,

$$\begin{aligned}
\langle \phi \cdot \mu, h \rangle &= \langle \phi, \mu \cdot h \rangle \\
&= \int \phi(t)(\mu \cdot h)(t)dt \\
&= \int \phi(t) \int (s \cdot h)(t)d\mu(s)dt \\
&= \int \phi(t) \int \Delta^{\frac{1}{p}}(s)h(s^{-1}ts)d\mu(s)dt \\
&= \int \int \phi(t)\Delta^{\frac{1}{p}}(s)h(s^{-1}ts)d\mu(s)dt \\
&= \int \int \phi(st)\Delta^{\frac{1}{p}}(s)h(ts)d\mu(s)dt \qquad (t \mapsto st) \\
&= \int \int \phi(sts^{-1})\Delta^{\frac{1}{p}}(s)\Delta(s^{-1})h(t)d\mu(s)dt \qquad (t \to ts^{-1}) \\
&= \int \int \Delta^{(1-\frac{1}{p})}(s^{-1})\phi(sts^{-1})h(t)d\mu(s)dt \\
&= \int \int \Delta^{(1-\frac{1}{p})}(s^{-1})\phi(sts^{-1})d\mu(s)h(t)dt.
\end{aligned}$$

As a result $\phi \cdot \mu(t) = \int \Delta^{\frac{1}{q}}(s^{-1})\phi(sts^{-1})d\mu(s)$ a.e. t [locally a.e. if $p = 1, q = \infty$] for every $\phi \in L^q(G), \mu \in M(G)$. By restriction to $L^1(G)$, we have $\phi \cdot f(t) = \int \Delta^{\frac{1}{q}}(s^{-1})\phi(sts^{-1})f(s)ds$ a.e. t [locally a.e. if $p = 1, q = \infty$] for all $\phi \in L^q(G), f \in L^1(G)$.

In the following example we will derive the formulas of G-module actions, $L^1(G)$-module actions and $M(G)$-module action for $L^p(G/H)$ and the corresponding dual module actions on $L^q(G/H)$, where $\frac{1}{p}+\frac{1}{q}=1$ and H is a closed subgroup of G.

Example 1.12. *Let H be a closed subgroup of a locally compact group G and Y=G/H. Also let $f \in L^1(G)$, $s \in G, h \in (L^p(Y),\nu)$ where $1 \leq p < \infty$ and ν is a strongly quasi-invariant measure on G/H. Moreover let $\mu \in M(G)$ and $\phi \in (L^p(Y),\nu)^* = (L^q(Y),\nu)$ where $\frac{1}{p}+\frac{1}{q}=1$. Define $s \cdot h(\xi) = h(s^{-1}\xi)\sigma(\xi,s)^{\frac{1}{p}}$. Then with respect to this action we have:*

$$\begin{aligned}
\langle \mu \cdot h, \phi \rangle &= \int_G \langle s \cdot h, \phi \rangle d\mu(s) \\
&= \int_Y \left(\int_G h(s^{-1}\xi)\sigma(\xi,s)^{\frac{1}{p}} d\mu(s) \right) \phi(\xi) d\nu(\xi) \qquad (Fubini's Theorem).
\end{aligned}$$

Thus, $\mu \cdot h(\xi) = \int_G h(s^{-1}\xi)\sigma(\xi,s)^{\frac{1}{p}} d\mu(s)$ ν-a.e. ξ [locally a.e. if $p=1, q=\infty$] for all $\mu \in M(G), h \in (L^p(Y),\nu)$. By restriction to $L^1(G)$ we have $f \cdot h(\xi) = \int_G h(s^{-1}\xi)\sigma(\xi,s)^{\frac{1}{p}} f(s) ds$ ν-a.e. ξ [locally ν-a.e. if $p=1, q=\infty$] whenever $f \in L^1(G), h \in (L^p(Y),\nu)$. Furthermore,

$$\begin{aligned}
\langle \phi \cdot s, h \rangle &= \langle \phi, s \cdot h \rangle \\
&= \int_Y \phi(\xi) h(s^{-1}\xi)\sigma(\xi,s)^{\frac{1}{p}} d\nu(\xi) \\
&= \int_Y \phi(s\xi) h(\xi)\sigma(s\xi,s)^{\frac{1}{p}} \sigma(\xi,s^{-1}) d\nu(\xi) \qquad (\text{by Remark } 3,\ \xi \mapsto s\xi) \\
&= \int_Y \phi(s\xi) h(\xi)\sigma(s\xi,s)^{\frac{1}{p}-1} \underbrace{\sigma(\xi,s^{-1}s)}_{=1} d\nu(\xi) \qquad (\text{by Remark } 3).
\end{aligned}$$

Hence, $\phi \cdot s(\xi) = \phi(s\xi)\sigma(s\xi,s)^{-\frac{1}{q}}$ ν-a.e. ξ [locally ν-a.e. when $p=1$ and $q=\infty$] for *all* $\phi \in (L^q(Y), \nu), s \in G$. *Also*

$$\begin{aligned}
\langle \phi \cdot \mu, h \rangle &= \langle \phi, \mu \cdot h \rangle \\
&= \int_Y \phi(\xi) \int_G h(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} d\mu(s) d\nu(\xi) \\
&= \int_G \int_Y \phi(\xi) h(s^{-1}\xi)\sigma(\xi, s)^{\frac{1}{p}} d\nu(\xi) d\mu(s) \qquad (\textit{by Fubini's Theorem}) \\
&= \int_G \int_Y \phi(s\xi) h(\xi)\sigma(s\xi, s)^{\frac{1}{p}} \sigma(\xi, s^{-1}) d(\nu)(\xi) d\mu(s) \qquad (\textit{by Remark } 3,\ \xi \mapsto s\xi) \\
&= \int_Y h(\xi) \int_G \phi(s\xi)\sigma(s\xi, s)^{\frac{1}{p}} \sigma(\xi, s^{-1}) d\mu(s) d\nu(\xi) \qquad (\textit{by Fubini's Theorem}) \\
&= \int_Y h(\xi) \int_G \phi(s\xi)\sigma(s\xi, s)^{\frac{1}{p}-1} \underbrace{\sigma(\xi, ss^{-1})}_{=1} d\mu(s) d\nu(\xi) \qquad (\textit{by Remark } 3).
\end{aligned}$$

As a result $\phi \cdot \mu(\xi) = \int_G \phi(s\xi)\sigma(s\xi, s)^{-\frac{1}{q}} d\mu(s)$ ν*-a.e.* ξ *[locally* ν*-a.e. if* $p = 1, q = \infty$*] for all* $\phi \in (L^q(Y), \nu), \mu \in M(G)$. *By restriction to* $L^1(G)$ *we have* $\phi \cdot f(\xi) = \int_G \phi(s\xi)\sigma(s\xi, s)^{-\frac{1}{q}} f(s) ds$ ν*-a.e.*ξ *[locally* ν*-a.e. if* $p = 1, q = \infty$*] for all* $\phi \in L^q(Y, \nu), f \in L^1(G)$.

From now on let $LUC(G)$ denote the set of all bounded continuous functions $f \in L^\infty(G)$, such that $s \mapsto \ell_s\phi : G \mapsto (L^\infty(G), \|\cdot\|_\infty)$ is continuous.

Lemma 1.2. *The following statements hold:*

(i) With respect to the right dual module action of G *on* $L^\infty(G) = L^1(G)^*$, $LUC(G)$ *is a right Banach* G*-module.*

(ii) $LUC(G) = L^\infty(G) \cdot L^1(G)$.

Proof. See the proof of Theorem 3.3 in Chapter 3 which is a general result that includes this statement. □

Note: If X is a Banach G-module, then X^* is not necessarily a Banach G-module.

Example 1.13. $L^\infty(G)$ *is not always a right Banach* G*-module.*

Proof. Suppose that $L^\infty(G)$ is a right Banach G-module. Then by Definition 1.28, given $\phi \in L^\infty(G)$, the map $s \mapsto \phi \cdot s = \ell_s\phi : G \to (L^\infty(G), \|\cdot\|_\infty)$ is continuous, and therefore $\phi \in LUC(G)$. But we can have $LUC(G) \subsetneqq L^\infty(G)$. So $L^\infty(G)$ is not always a Banach G-module. $\square$

Observe that the dual of a unital Banach $M(G)$-module is also unital (e.g. $L^\infty(G)$ as a dual $M(G)$-module) even though it is not necessarily a Banach G-module. Thus, being a unital Banach $M(G)$-module is not sufficient to give you a Banach G-module.

Chapter 2

Amenable Actions on Preduals of W^*-algebras

2.1 W^*-algebra

This brief section includes the definition of a W^*-algebra and a theorem, with an omitted proof, that we need for this thesis. For more information you can see [20] and [24].

Definition 2.1. *A Banach algebra M is a* W^***-algebra** *if M is a C^*-algebra such that there is a Banach space M_* (necessarily unique up to isometric isomorphism), such that $M = (M_*)^*$.*

Definition 2.2. *An element $x \in M$ is positive if there is $y \in M$, such that $x = yy^*$. Let $M_+ := \{x \in M : x \text{ positive}\}$. Then $\phi \in M^*$ is positive if $\phi(x) \geq 0$ whenever $x \in M_+$.*

Example 2.1. *If $M = L^\infty(G)$, then $f \in M_+$ if and only if $f = g\overline{g} = |g|^2$ for some $g \in L^\infty(G)$. (i.e. if and only if $f \geq 0$ λ-locally almost every where).*

Definition 2.3. *$\phi \in M^*$ is a* **state** *if*

(i) ϕ is positive and

(ii) $\|\phi\| = 1$.

Note: Any W^*-algebra has an identity. We denote this identity by e_M so

$$e_M x = x e_M = x \qquad (x \in M).$$

Theorem 2.1. *Let $\phi \in M^*$. Then ϕ is a state if and only if $\|\phi\| = \phi(e_M) = 1$.*

Note that a state on $L^\infty(G)$ is called a mean.

Definition 2.4. *A locally compact group G is* **amenable** *if there is a mean m on $L^\infty(G)$, such that $m(\ell_x \phi) = m(\phi)$ whenever $x \in G,\ \phi \in L^\infty(G)$.*

Theorem 2.2 (Day's Fixed Point Theorem). *Let G be a locally compact group. Then G is amenable if and only if any separately continuous affine action of G on a compact convex subset K of a locally convex topological vector space, (i.e.*

$$s \cdot (\alpha x + (1-\alpha)y) = \alpha(s \cdot x) + (1-\alpha)(s \cdot y) \qquad (s \in G,\ x, y \in K,\ \alpha \in [0,1]).$$

such that

$$G \times K \to K : \ (s, x) \mapsto s.x$$

is separately continuous) has a fixed point (i.e. there is $x \in K$ such that $s.x = x$ for all $s \in G$).

Proof. See proof of Theorem 1.3.1 in [18]. □

We will let $S(M)$ denote the set of all states on M (i.e. $\phi \in S(M)$ if and only if $\|\phi\| = \phi(e_M) = 1$). The collection of all w^*-continuous states on M_* - the so-called normal states on M - is the set $(M_*)_1^+ = S(M) \cap M_*$.

2.2 Amenable Action on the Predual of a W^*-algebra

There are several notions of amenability corresponding to group actions. These include the classical notion of amenability for groups, inner amenability, amenable actions on coset spaces, and amenable unitary representations as defined in [1]. In [21], the author provided a unified approach to these various types of amenability in which the basic theory, up to and including Reiter's theorem, is developed. For the sake of brevity, proofs were not included in [21], however, providing detailed proofs somewhere, for the record, of these results seems like a useful endeavour. This is the purpose of this chapter in addition to providing some additional results. The proofs are based on those found in the classical situation as found for example in [8], [14] and [15].

Definition 2.5. *A locally compact group G has a* **positive action** *on M_* if M_* is a left Banach G-module, such that*

(i) $\|s.\phi\| \leq \|\phi\|$ *whenever* $\phi \in M_*, s \in G$ *and*

(ii) $s.\phi \in (M_*)_1^+$ *whenever* $s \in G, \phi \in (M_*)_1^+$.

Example 2.2. *Let $M = L^\infty(G), M_* = L^1(G)$. Then the actions in Examples 1.10, 1.11 and 1.12 are positive actions.*

From now on let M be a W^*-algebra with predual M_*, G be a locally compact group with positive action on M_*, $e \in G$ the identity of G, $e_M \in M$ the identity of M. Dual module actions on M and M_* are defined in the usual way (see Chapter 1). Note that M_* is a unital Banach $M(G)$-module and neo-unital Banach $L^1(G)$-module.

Lemma 2.1. *The following statements hold:*

(i) $(M_*)_1^+$ *is w^*-dense in* $S(M)$.

(ii) $e_M \cdot \mu = e_M$ *whenever* $\mu \in M(G)_1^+$.

(iii) $(M_*)_1^+ = G \cdot (M_*)_1^+ = M(G)_1^+ \cdot (M_*)_1^+$.

(iv) $S(M) = G \cdot S(M) = M(G)_1^+ \cdot S(M)$.

Proof.

(i) For example see [23].

(*ii*) Let $\mu \in M(G)_1^+$, and $\phi \in (M_*)_1^+$. Then we have

$$\begin{aligned}\langle \phi, e_M \cdot \mu \rangle &= \langle \mu \cdot \phi, e_M \rangle \\ &= \int \langle s \cdot \phi, e_M \rangle d\mu(s) \qquad \text{(by Proposition 1.12).} \qquad (*)\end{aligned}$$

Since the action is positive, $s \cdot \phi \in (M_*)_1^+ = S(M) \cap M_*$ and so $\langle s \cdot \phi, e_M \rangle = 1$ for each $s \in G$. Now $(*)$ gives us,

$$\begin{aligned}\langle \phi, e_M \cdot \mu \rangle &= \int 1 d\mu(s) \\ &= \mu(G) \\ &= 1 = \langle \phi, e_M \rangle.\end{aligned}$$

Hence $e_M \cdot \mu = e_M$ since $(M_*)_1^+$ separates points of M.

(*iii*) Claim I: $(M_*)_1^+ = G \cdot (M_*)_1^+$.
$G \cdot (M_*)_1^+ \subseteq (M_*)_1^+$ (by the definition of positive action). Now let $\phi \in (M_*)_1^+$. Then $\phi = e \cdot \phi \in G \cdot (M_*)_1^+$, so $(M_*)_1^+ \subseteq G \cdot (M_*)_1^+$.
Claim II: $(M_*)_1^+ = M(G)_1^+ \cdot (M_*)_1^+$.
Let $\mu \in M(G)_1^+$, and $\phi \in (M_*)_1^+ = M_* \cap S(M)$. To show $\mu \cdot \phi \in (M_*)_1^+$, it is enough to show $\mu \cdot \phi \in S(M)$ (i.e.$\|\mu \cdot \phi\| = \mu \cdot \phi(e_M) = 1$) since $\mu \cdot \phi \in M_*$. We have

$$\begin{aligned}\|\mu \cdot \phi\| &= \sup\{|\langle \mu \cdot \phi, \psi \rangle| : \psi \in (M)_1\}. \\ &= \sup\{|\int \langle s \cdot \phi, \psi \rangle d\mu(s)| : \psi \in (M)_1\}. \\ &\leq \sup\{\int |\langle s \cdot \phi, \psi \rangle| d\mu(s) : \psi \in (M)_1\} \\ &\leq \sup\{\|s \cdot \phi\| \|\psi\| \|\mu\| : \psi \in (M)_1\} = 1 \qquad (\text{since } s \cdot \phi \in (M_*)_1^+).\end{aligned}$$

Also

$$\begin{aligned}\mu \cdot \phi(e_M) &= \langle \mu \cdot \phi, e_M \rangle \\ &= \langle \phi, e_M \cdot \mu \rangle \\ &= \langle \phi, e_M \rangle \qquad \text{(by part } (ii)) \\ &= 1.\end{aligned}$$

Hence $\|\mu \cdot \phi\| = \mu \cdot \phi(e_M) = 1$.

(iv) Claim I: $S(M) = G \cdot S(M)$.

Let $m \in S(M)$. Then $m = e \cdot m \in G \cdot S(M)$, so $S(M) \subseteq G \cdot S(M)$.

Now let $s \in G, m \in S(M)$. Then

$$\begin{aligned} \|s \cdot m\| &= \sup\{|\langle s \cdot m, \psi\rangle| : \psi \in (M)_1\} \\ &\leq \sup\{\|s \cdot m\|\|\psi\| : \psi \in (M)_1\} \\ &\leq \sup\{\|m\|\} = 1, \end{aligned}$$

and

$$\begin{aligned} s \cdot m(e_M) &= \langle s \cdot m, e_M\rangle \\ &= \langle m, e_M \cdot s\rangle \\ &= \langle m, e_M \cdot \delta_s\rangle \\ &= \langle m, e_M\rangle \qquad \text{(by part } (ii)) \\ &= 1. \end{aligned}$$

Hence, $\|s \cdot m\| = s \cdot m(e_M) = 1$.

Claim II: $S(M) = M(G)_1^+ \cdot S(M)$.

By claim I, $S(M) = G \cdot S(M) \subseteq M(G)_1^+ \cdot S(M)$. Now let $\mu \in M(G)_1^+$, and $m \in S(M)$. To show that $\mu \cdot m \in S(M) = \overline{(M_*)_1^+}^{w^*}$, let (ϕ_i) be a net in $(M_*)_1^+$ such that $\phi_i \xrightarrow{w^*} m$ (i.e. $\langle \phi_i, x\rangle \to \langle m, x\rangle$ whenever $x \in M$). By part (iii) $(\mu \cdot \phi_i)$ is a net in $(M_*)_1^+$. For $x \in M$,

$$\langle \mu \cdot \phi_i, x\rangle = \langle \phi_i, x \cdot \mu\rangle \to \langle m, x \cdot \mu\rangle = \langle \mu \cdot m, x\rangle,$$

so $\mu \cdot \phi_i \xrightarrow{w^*} \mu \cdot m$. Hence $\mu \cdot m \in \overline{(M_*)_1^+}^{w^*} = S(M)$.

□

Definition 2.6. *A state m on M is a G-**invariant mean** (G-**IM**) if*

$$m(x \cdot s) = m(x) \qquad (s \in G, x \in M).$$

Definition 2.7. *A locally compact group G* **acts amenably on** M_* *(or has amenable G-action) if there is a G-invariant mean (G-IM) on M.*

Note that when this definition is applied to any of the actions in Chapter 1 we obtain a well-studied notion of amenability:

(i) When $M = L^\infty(G)$, then with respect to the action in Example 1.10, G acts amenably on $L^1(G)$ if and only if G is amenable.

(ii) When $M = L^\infty(G)$, then with respect to the action in Example 1.11, G acts amenably on $L^1(G)$ if and only if G is inner amenable.

(iii) When $M = L^\infty(G/H)$, then with respect to the action in Example 1.12 G acts amenably on $L^1(G/H)$ if and only if G acts amenably on the coset space G/H.

Other classical notions of amenability encompassed by this definition are described in [21].

Recall from chapter (1) that M, with respect to the dual module action, is not necessarily a Banach G-module (see Example 1.13). Suppose however $Z \preceq M$ is a right Banach G-submodule of M with respect to

$$\phi \bullet s = \phi \cdot s \qquad (\phi \in Z, s \in G),$$

where $(\cdot)$ is the dual action on M and $(\bullet)$ is the submodule action on Z. By Proposition 1.12, Z has corresponding neo-unital Banach $L^1(G)$-module and unital Banach $M(G)$-module structures $\phi \bullet f$ and $\phi \bullet \mu$ where $f \in L^1(G)$, $\mu \in M(G)$ and $\phi \in Z$, given by

$$\phi \bullet \mu = \int_{\sigma(Z,Z^*)} \phi \bullet s d\mu(s) = \int_{\sigma(Z,Z^*)} \phi \cdot s d\mu(s). \qquad (1)$$

Also for $\phi \in Z$ and $\mu \in M(G)$, we have $\phi \cdot \mu \in Z$ given by

$$\langle \phi \cdot \mu, x \rangle = \langle \phi, \mu \cdot x \rangle \qquad (x \in M_*). \qquad (2)$$

Proposition 2.1. *Let Z be defined as above, $\phi \in Z$, and $\mu \in M(G)$. Then $\phi \bullet \mu$ and $\phi \cdot \mu$ as they are defined in* (1) *and* (2) *are equal.*

Proof. Let $\phi \in Z$, $\mu \in M(G)$ and $x \in M_*$. Then

$$\begin{aligned}
\langle \phi \bullet \mu, x \rangle &= \langle \hat{x}, \underbrace{\phi \bullet \mu}_{\in Z} \rangle \\
&= \langle \underbrace{\hat{x} \restriction_Z}_{\in Z^*}, \phi \bullet \mu \rangle \\
&= \int \langle \hat{x} \restriction_Z, \phi \bullet s \rangle d\mu(s) \\
&= \int \langle \hat{x} \restriction_Z, \phi \cdot s \rangle d\mu(s) \\
&= \int \langle \phi \cdot s, x \rangle d\mu(s) \\
&= \int \langle \phi, s \cdot x \rangle d\mu(s) \\
&= \langle \phi, \int_{\sigma(X, X^*)} s \cdot x d\mu(s) \rangle \\
&= \langle \phi, \mu \cdot x \rangle = \langle \phi \cdot \mu, x \rangle.
\end{aligned}$$

□

Definition 2.8. *Let $x \in M$. Then x is called* **uniformly continuous** *if $s \mapsto x \cdot s : G \to (M, \|\cdot\|)$ is continuous.*

We will always let $UC(M) := \{x \in M : x \text{ is uniformly continuous}\}$.

Lemma 2.2. *$UC(M)$ is a right Banach G-submodule of M containing e_M.*

Proof. As M_* is a left Banach G-module, by Proposition 1.14, M is a right dual G-module with respect to

$$\langle \phi \cdot s, x \rangle = \langle \phi, s \cdot x \rangle \qquad (\phi \in M, s \in G, x \in M_*).$$

Note, if $\phi \in UC(M)$, then $\phi \cdot s \in UC(M)$. To see this, let $(t_i) \subseteq G$ be a net such that $t_i \to t \in G$. Then $(st_i) \subseteq G$ is a net that converges to st in G. Since $\phi \in UC(M)$, $\phi \cdot (st_i) \to \phi \cdot (st)$. And as $\phi \in UC(M) \subseteq M$, we have

$$(\phi \cdot s) \cdot t_i = \phi \cdot (st_i) \to \phi \cdot (st) = (\phi \cdot s) \cdot t.$$

Thus $\phi \cdot s \in UC(M)$. Furthermore it is easy to see that $UC(M)$ is a closed subspace of M and $e_M \in UC(M)$. Therefore $UC(M)$ is a right G-submodule of M. Now if $\phi \in UC(M)$, then $s \mapsto \phi \cdot s : G \to (UC(M), \|\cdot\|)$ is continuous by definition of $UC(M)$. Hence $UC(M)$ is a right Banach G-submodule of M. □

Corollary 2.1. *The dual $M(G)$-module structure and the induced $M(G)$-module structure on $UC(M)$ as they are defined in* (1) *and* (2) *before Proposition 2.1, when $Z := UC(M)$, agree.*

Note that by the above Corollary whenever $\phi \in UC(M)$ and $\mu \in M(G)$, we can write $\phi \cdot \mu$ without causing any confusion.

Lemma 2.3. $UC(M) = M \cdot L^1(G)$.

Proof. Let $\phi \in M$ and $f \in L^1(G)$. Suppose that $s_i \to s$ in G. Then by Proposition 1.11,

$$\|(\phi \cdot f) \cdot s_i - (\phi \cdot f) \cdot s\| \leq \|\phi\| \|f * \delta_{s_i} - f * \delta_s\| \to 0,$$

so $\phi \cdot f \in UC(M)$. Hence $M \cdot L^1(G) \subseteq UC(M)$. Moreover since $UC(M)$ is a Banach G-module, it is a neo-unital $L^1(G)$-module by Proposition 1.12. So $UC(M) = UC(M) \cdot L^1(G) \subseteq M \cdot L^1(G)$. Therefore, $UC(M) = M \cdot L^1(G)$.

□

Definition 2.9. *A state m on M is a* **topological invariant mean (TIM)** *if*

$$m(x \cdot u) = m(x) \qquad (x \in M, u \in L^1(G)^+_1).$$

Definition 2.10. *An element $m \in UC(M)^*$* **is called a mean** *if $\|m\| = m(e_M) = 1$. The set of all means on $UC(M)$ is denoted by $S(UC(M))$. A mean m on $UC(M)$ is called a G-IM if*

$$m(x \cdot s) = m(x) \qquad (s \in G, x \in UC(M)),$$

and it is called a TIM if

$$m(x \cdot u) = m(x) \qquad (x \in UC(M), u \in L^1(G)^+_1).$$

Lemma 2.4. *The following statements hold:*

(i) If m is a TIM on M (respectively $UC(M)$), then m is a G-IM on M (respectively $UC(M)$).

(ii) If m is a G-IM on $UC(M)$, then m is a TIM on $UC(M)$.

(iii) If m is a G-IM on M (or $UC(M)$), and $u \in L^1(G)^+_1$, then m_u is a TIM on M where

$$m_u(x) = m(x \cdot u) \qquad (x \in M).$$

Proof. (i) Suppose that $m(x \cdot u) = m(x)$ whenever $x \in M, u \in L^1(G)^+_1$. Then for $x \in M, s \in G, u \in L^1(G)^+_1$, we see that

$$\begin{aligned} m(x \cdot s) &= m((x \cdot s) \cdot u) \\ &= m(x \cdot \underbrace{(\delta_s * u)}_{\in L^1(G)^+_1}) \\ &= m(x). \end{aligned}$$

(ii) Suppose that $m(x \cdot s) = m(x)$ whenever $x \in UC(M), s \in G$. Then for $x \in UC(M)$, $u \in L^1(G)^+_1$, and $s \in G$, we have

$$m(x \cdot u) = m(x \cdot u \cdot s) \qquad \text{(since } x \cdot u \in UC(M) \text{ by Lemma 2.2).}$$

Define $I_x : L^1(G) \to \mathbb{C}$ by $I_x(f) = \overline{m(x \cdot f^*)}$. Let $f, g \in C_{00}(G), \alpha \in \mathbb{C}$. Then

(a)

$$\begin{aligned} I_x(\ell_s f) = I_x(\delta_{s^{-1}} * f) &= \overline{m(x \cdot (\delta_{s^{-1}} * f)^*)} \\ &= \overline{m(x \cdot (f^* * \delta_s))} \\ &= \overline{m(\underbrace{(x \cdot f^*)}_{\in\ UC(M)} \cdot s)} \\ &= \overline{m(x \cdot f^*)} \\ &= I_x(f); \end{aligned}$$

(b)

$$\begin{aligned} I_x(\alpha f) &= \overline{m(x\cdot(\alpha f)^*)} \\ &= \overline{m(x\cdot(\overline{\alpha} f^*))} \\ &= \overline{m(\overline{\alpha}(x\cdot f^*))} \\ &= \overline{\overline{\alpha} m(x\cdot f^*)} \\ &= \alpha\overline{m(x\cdot f^*)} \\ &= \alpha I_x(f); \end{aligned}$$

(c)

$$\begin{aligned} I_x(f+g) &= \overline{m(x\cdot(f+g)^*)} \\ &= \overline{m(x\cdot f^* + x\cdot g^*)} \\ &= \overline{m(x\cdot f^*) + m(x\cdot g^*)} \\ &= \overline{m(x\cdot f^*)} + \overline{m(x\cdot g^*)} \\ &= I_x(f) + I_x(g); \text{ and} \end{aligned}$$

(d)

$$\begin{aligned} |I_x(f)| &= |\overline{m(x\cdot f^*)}| \\ &= |m(x\cdot f^*)| \\ &\leq \|x\cdot f^*\| \\ &\leq \|x\|\|f^*\|_1 \\ &= \|x\|\|f\|_1. \end{aligned}$$

So I_x is a left invariant bounded linear functional defined on $L^1(G)$. Thus I_x is Haar integral on $L^1(G)$ or $I_x = 0$, and hence there is $c \geq 0$, such that

$$I_x(f) = \overline{m(x\cdot f^*)} = c\int f(s)d\lambda(s).$$

If $u \in L^1(G)_1^+$, then $\int u(s)d\lambda(s) = 1$. Therefore

$$\overline{m(x\cdot u^*)} = c \text{ and so } m(x\cdot u^*) = c. \qquad (*)$$

Now let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$ such that $e_\alpha = e_\alpha^*$, so by Proposition 1.1, $\|x \cdot e_\alpha - x\| \to 0$. Since $m \in UC(M)^*$,

$$|m(x \cdot e_\alpha) - m(x)| \leq \|m\|\|x \cdot e_\alpha - x\| \to 0. \qquad (**)$$

But since $e_\alpha = e_\alpha^*$, by $(*)$ we have

$$m(x \cdot e_\alpha) = c$$

and by $(**)$, we see that

$$m(x) = c. \qquad (***)$$

Hence

$$m(x \cdot u^*) = m(x) \qquad (\text{by } (*), (***)).$$

Therefore,

$$\begin{aligned} m(x \cdot u) &= m(x \cdot (\underbrace{u^*}_{\in L^1(G)_1^+})^*) \\ &= m(x). \end{aligned}$$

(iii) Suppose m is a G-IM on M. Then m is a TIM on $UC(M)$ by part (ii). Let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then for $x \in M, u_0 \in L^1(G)_1^+$ we have

$$\begin{aligned} |m(x \cdot e_\alpha) - m(x \cdot u_0)| &= |m((x \cdot e_\alpha) \cdot u_0) - m(x \cdot u_0)| \\ &= |m(x \cdot (e_\alpha * u_0 - u_0))| \\ &\leq \|m\|\|x\|\|e_\alpha * u_0 - u_0\| \to 0 \end{aligned}$$

since $x \cdot e_\alpha \in UC(M)$ for each α. Hence,

$$m(x \cdot u_1) = m(x \cdot u_2) \qquad (x \in M, u_1, u_2 \in L^1(G)_1^+)). \qquad (*)$$

Fix $u \in L^1(G)_1^+$, and define

$$m_u(x) = m(x \cdot u) \qquad (x \in M).$$

Then we have

$$\begin{aligned}|m_u(x)| &= |m(x \cdot u)| \\ &\leq \|m\|\|x \cdot u\| \\ &\leq \|x\|\|u\|_1.\end{aligned}$$

So

$$\|m_u\| \leq \|u\|_1 = 1.$$

Moreover,

$$\begin{aligned}m_u(e_M) &= m(e_M \cdot u) \\ &= m(e_M) \qquad (\text{ by Lemma 2.1}) \\ &= 1.\end{aligned}$$

Since $\|m_u\| = \sup\{|m_u(x)| : x \in (M)_1\}$, $\|m_u\| = m_u(e_M) = 1$. Hence $m_u \in S(M)$.
Claim: m_u is a TIM on M (i.e. $m_u(x \cdot v) = m_u(x)$ where $v \in L^1(G)_1^+, x \in M$).
Let $v \in L^1(G)_1^+, x \in M$. Then $v * u \in L^1(G)_1^+$ so

$$\begin{aligned}m_u(x \cdot v) &= m((x \cdot v) \cdot u) \\ &= m(x \cdot (v * u)) \\ &= m(x \cdot u) \qquad (\text{by } (*)) \\ &= m_u(x).\end{aligned}$$

□

Proposition 2.2. *The following statements are equivalent:*

(i) G acts amenably on M_*.

(ii) There is a TIM on M.

(iii) There is a G-IM on $UC(M)$.

(iv) There is a TIM on $UC(M)$.

Proof. $(i) \Rightarrow (ii)$ Suppose G acts amenably on M_*. Then there is a G-IM on M. Hence by Lemma 2.4 (iii), there is a TIM on M.

$(ii) \Rightarrow (iii)$ Suppose m is a TIM on M. Then by Lemma 2.4 (i), m is a G-IM on M. Letting $m_0 = m \restriction_{UC(M)}$, m_0 is a G-IM on $UC(M)$.

$(iii) \Rightarrow (iv)$ This is Lemma 2.4 (ii).

$(iv) \Rightarrow (i)$ Suppose there is a TIM m on $UC(M)$. Then by Lemma 2.4 (i), m is a G-IM on $UC(M)$. So by Lemma 2.4 (iii), there is a TIM on M. By Lemma 2.4 (i), G acts amenably on M_* □

We now apply Day's fixed point Theorem (Theorem 2.2) to obtain the following result:

Corollary 2.2. *The following statements are equivalent:*

(i) G is amenable.

(ii) Every positive action of G on any M_* is amenable.

Proof. $(ii) \Rightarrow (i)$ Suppose every positive action on any M_* is amenable. Then when $M_* = L^1(G)$, the action in Example 1.10 is amenable. Therefore G is amenable.

$(i) \Rightarrow (ii)$ Suppose G is amenable, and let G have a positive action on M_*.
Claim: There is a G-IM on $UC(M) = M \cdot L^1(G)$
Since G has a positive action on M_*, G acts on $UC(M) = M \cdot L^1(G)$;

$$(x, s) \mapsto x \cdot s \qquad (x \in UC(M), s \in G).$$

Therefore G acts on $UC(M)^*$,

$$(s, m) \mapsto s \cdot m \text{ where } \langle s \cdot m, x \rangle = \langle m, x \cdot s \rangle.$$

In fact, G acts on $S(UC(M))$, the set of means on $UC(M)$. To see this we will show that for $m \in S(UC(M))$, and $s \in G$, $s \cdot m \in S(UC(M))$. Since the action on M is contractive, the action on $S(UC(M))$ is also contractive. Therefore

$$\|s \cdot m\| \leq \|m\| = 1.$$

Also

$$\begin{aligned} \langle s \cdot m, e_M \rangle &= \langle s \cdot m, e_M \rangle \\ &= \langle m, e_M \cdot s \rangle \\ &= \langle m, e_M \cdot \delta_s \rangle \\ &= \langle m, e_M \rangle \qquad \text{(by Lemma 2.1)} \\ &= 1. \end{aligned}$$

We will show that this action satisfies the hypotheses of Day's fixed point theorem with respect to the compact convex subset $S(UC(M))$ of the locally convex space $(UC(M)^*, w^*)$.

(a) Claim: $S(UC(M))$ is a w^*-compact convex non-empty subset of $UC(M)^*$. $S(UC(M)) \neq \phi$ since $e_M \in UC(M)$ and $S(M) \neq \phi$. Now let $m_\alpha \subseteq S(UC(M))$ and $m \in UC(M)^*$, such that

$$m_\alpha \xrightarrow{w^*} m \ (i.e. \langle m_\alpha, x \rangle \to \langle m, x \rangle \text{ whenever } x \in UC(M)).$$

Then

$$\begin{aligned} \langle m, e_M \rangle &= \lim \langle m_\alpha, e_M \rangle \\ &= \lim 1 = 1. \end{aligned}$$

Also $\|m\| \leq 1$ because $(UC(M^*))_{\|\cdot\| \leq 1}$ is a w^*-closed, so $\|m\| = m(e_M) = 1$. Hence, $m \in S(UC(M))$. Since $S(UC(M)$ is a w^*-closed subset of $(UC(M)^*)_1$ which is w^*-compact by Banach Alaoglu's Theorem, $S(UC(M))$ is w^*-compact. For convexity let $0 \leq t \leq 1, m_1, m_2 \in S(UC(M))$. Then

$$\begin{aligned} \langle tm_1 + (1-t)m_2, e_M \rangle &= \langle tm_1, e_M \rangle + \langle (1-t)m_2, e_M \rangle \\ &= t\langle m, e_M \rangle + (1-t)\langle m_2, e_M \rangle \\ &= t + 1 - t = 1. \end{aligned}$$

Moreover,

$$\|tm_1 + (1-t)m_2\| \le t\|m_1\| + (1-t)\|m_2\| = 1.$$

So $\|tm_1 + (1-t)m_2\| = 1$. Thus $S(UC(M)$ is convex.

(b) Since $UC(M)$ is Banach G-module, for each $s \in G$ the map $T_s : UC(M) \to UC(M) : x \mapsto x \cdot s$ is linear (by the definition of action) and continuous.
Let $s \in G$ be fixed. Then

$$(T_s)^* : UC(M)^* \;\; \to \;\; UC(M)^* : \; \phi \mapsto s \cdot \phi, \; \langle s \cdot \phi, x \rangle = \langle \phi, x \cdot s \rangle$$

is $w^* - w^*$ continuous and linear by Proposition 1.2.
Therefore $L_s = (T_s)^* \restriction_{S(UC(M))}: S(UC(M)) \to S(UC(M)) : m \mapsto s.m$ is $w^* - w^*$ continuous. Since ${T_s}^*$ is linear on $S(UC(M))$, the action is affine on $S(UC(M))$. Fix $m \in S(UC(M))$ and suppose $(s_i)_i$ is a net in G, such that $s_i \to s$ in G. Then for $x \in UC(M)$ we have

$$\langle s_i \cdot m - s \cdot m, x \rangle = \langle m, x \cdot s_i - x \cdot s \rangle \to \langle m, 0 \rangle$$

by our definition of UC(M). Hence $s_i.m \xrightarrow{w^*} s \cdot m$ in $S(UC(M))$. By Day's fixed point Theorem, there is an $m_0 \in S(UC(M))$, such that

$$s \cdot m_0 = m_0 \qquad (s \in G).$$

Hence for $\phi \in UC(M)$,

$$\begin{aligned} m_0(\phi \cdot s) &= \langle m_0, \phi \cdot s \rangle \\ &= \langle s \cdot m_0, \phi \rangle \\ &= \langle m_0, \phi \rangle \\ &= m_0(\phi). \end{aligned}$$

Hence m_0 is a G-IM on $UC(M)$.

□

Corollary 2.3. *The following statements are equivalent:*

(i) G acts amenably on M_*.

(ii) There is a net $(\phi_\alpha) \subseteq (M_*)_1^+$ such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0$; $s \in G$.

(iii) There is a net $(\psi_\alpha) \subseteq (M_*)_1^+$ such that $\|u \cdot \psi_\alpha - \psi_\alpha\| \to 0$; $u \in L^1(G)_1^+$.

Proof. $(i) \Rightarrow (iii)$ Suppose G acts amenably on M_*. Then by Proposition 2.2 there is a TIM m on M (i.e.$\langle m, x \cdot u\rangle = \langle m, x\rangle$ for $u \in L^1(G)_1^+, x \in M$)). Since $(M_*)_1^+$ is w^*-dense in $S(M)$, we can find a net $(\phi_i) \subseteq (M_*)_1^+$, such that $\phi_i \xrightarrow{w^*} m$. Thus

$$\langle \phi_i, x\rangle \to \langle m, x\rangle \text{ whenever } x \in M. \qquad (*)$$

Then we have

$$\begin{aligned} \langle u \cdot \phi_i - \phi_i, x\rangle &= \langle u \cdot \phi_i, x\rangle - \langle \phi_i, x\rangle \\ &= \langle \phi_i, x \cdot u\rangle - \langle \phi_i, x\rangle \to \langle m, x \cdot u\rangle - \langle m, x\rangle = 0 \qquad (by (*)). \end{aligned}$$

Hence, $u \cdot \phi_i - \phi_i \to 0$ with respect to $\sigma(M_*, M) = wk$ whenever $u \in L^1(G)_1^+$. $\quad (**)$
Let $E = \prod_{u \in L^1(G)_1^+} (E_u, \|\cdot\|)$ where whenever $u \in L^1(G)_1^+$, $(E_u, \|\cdot\|) = (M_*, \|\cdot\|)$
(Note: The elements in E look like $(f_u)_{u \in L^1(G)_1^+} = (f(u))_{u \in L^1(G)_1^+}$).
Then E is a LCS with respect to the product topology, and

$$\alpha(f_u)_u + (g_u)_u = (\alpha f_u + g_u)_u \qquad ((f_u)_u, (g_u)_u \in E,\ \alpha \in \mathbb{C}).$$

Also by [12, 17.13],

$$\sigma(E, E^*) = (E, wk) = \prod_{u \in L^1(G)_1^+} (E_u, wk).$$

Define $T : M_* \to E$, by $f \mapsto Tf = (Tf(u))_u$ where $Tf(u) = u \cdot f - f$.
We know $(\phi_i) \subseteq (M_*)_1^+$, so by $(**)$ we see that

$$T\phi_i(u) = u \cdot \phi_i - \phi_i \xrightarrow{wk} 0 \ in\ E_u \text{ (i.e. } T\phi_i \xrightarrow{wk} 0 \ in\ E).$$

Since $T\phi_i \in T((M_*)_1^+) = C$, and C is convex (since $(M_*)_1^+$ is convex and T is linear), so by Proposition 1.1,

$$0 \in \overline{C}^{(E,\ wk)} = \overline{C}^{(E,\ original\ prod.\ top)}.$$

Hence there is a net $(\psi_\alpha) \subseteq (M_*)_1^+$, such that $T_{\psi_\alpha} \to 0$ in $E = \prod_{u \in L^1(G)_1^+} (E_u, \|\cdot\|)$. That is $\|u \cdot \psi_\alpha - \psi_\alpha\| \to 0$ whenever $u \in L^1(G)_1^+$.

$(iii) \Rightarrow (ii)$ Suppose $(\psi_\alpha) \subseteq (M_*)_1^+$, satisfies condition (iii) and choose $u \in L^1(G)_1^+$. Since $(\psi_\alpha) \subseteq (M_*)_1^+$, $(u \cdot \psi_\alpha) \subseteq (M_*)_1^+$ by Lemma 2.1. Let $s \in G$ and $\phi_\alpha = u \cdot \psi_\alpha$. Then

$$\begin{aligned}
\|s \cdot \phi_\alpha - \phi_\alpha\| &= \|s \cdot (u \cdot \psi_\alpha) - u \cdot \psi_\alpha\| \\
&\leq \|(\delta_s * u) \cdot \psi_\alpha - \psi_\alpha\| + \|\psi_\alpha - u \cdot \psi_\alpha\| \to 0 \qquad (\text{since } \delta_s * u \in L^1(G)_1^+).
\end{aligned}$$

$(ii) \Rightarrow (i)$ Let $(\phi_\alpha) \subseteq (M_*)_1^+$ be such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0$ for $s \in G$. Since $(M_*)_1^+ \subseteq S(M)$, $(\phi_\alpha) \subseteq S(M)$. So by w^*-compactness of $S(M)$, there is $m \in S(M)$, such that (by perhaps passing to a subnet of (ϕ_α))

$$\phi_\alpha \xrightarrow{w^*} m.$$

Claim: $m(x \cdot s) = m(x) \qquad (x \in M,\ s \in G)$.
Let $x \in M$ and $s \in G$. Then we have

$$\begin{aligned}
m(x \cdot s) - m(x) &= \langle m, x \cdot s\rangle - \langle m, x\rangle \\
&= \lim(\langle \phi_\alpha, x \cdot s\rangle - \langle \phi_\alpha, x\rangle) \\
&= \lim(\langle s \cdot \phi_\alpha, x\rangle - \langle \phi_\alpha, x\rangle) \\
&= \lim\langle s \cdot \phi_\alpha - \phi_\alpha, x\rangle = 0.
\end{aligned}$$

Hence, $m(x \cdot s) = m(x)$. □

Proposition 2.3. *(Reiter's Condition) The following statements are equivalent:*

(i) *G acts amenably on M_*.*

(ii) *For every $\epsilon > 0$ and every compact subset K of G there is $\phi \in (M_*)_1^+$, such that $\|s \cdot \phi - \phi\| < \epsilon$ whenever $s \in K$.*

(iii) *There is a net $(\phi_\alpha) \subseteq (M_*)_1^+$, such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0$ uniformly on compact subsets of G.*

(iv) There is a net $(\phi_\alpha) \subseteq (M_*)_1^+$*, such that* $\|\mu\cdot\phi_\alpha - \phi_\alpha\| \to 0$ *whenever* $\mu \in M(G)_1^+$.

Proof. $(i) \Rightarrow (ii)$ Suppose G acts amenably on M_*. By Corollary 2.3 there is a net $(\phi_\alpha)_{\alpha\in I} \subseteq (M_*)_1^+$, such that

$$\|u\cdot\phi_\alpha - \phi_\alpha\| \to 0 \qquad (u \in (L^1(G)_1^+).$$

Let $\epsilon > 0$, let K be a compact subset of G and let v be a fixed element in $(L^1(G))_1^+$. Then $s \mapsto \delta_s * v$ is continuous (see Proposition 1.11). By continuity at e, the unit in G, we may choose a compact neighborhood E of e such that

$$\|\delta_s * v - v\|_1 < \epsilon \text{ for each } s \in E. \qquad (1)$$

Letting $\phi_u = \frac{1}{\lambda(u)}1_u$, then $(\phi_u) \subseteq L^1(G)_1^+$ is a BAI for $L^1(G)$, so we can further assume E is chosen so that

$$\|\phi_E * v - v\|_1 < \epsilon. \qquad (2)$$

By compactness of K we can select $\{x_1, x_2, ..., x_N\} \subseteq G$ such that $\bigcup_{k=1}^{N} x_k E \supseteq K$, and let $\psi_k = \phi_{x_k E}$ $[= \underbrace{\delta_{x_k} * \phi_E}_{\in L^1(G)_1^+}]$ for $k = 1, 2, ..., N$; assume $x_1 = e$. For each k there is α_k, such that

$$\alpha \geq \alpha_k \Rightarrow \|\psi_k.\phi_\alpha - \phi_\alpha\| < \epsilon \qquad (k = 1, 2, ..., N+1) \qquad (3)$$

where $\psi_{N+1} = v$. Take $\alpha_0 \in I$, where I is the directed set associated with $(\phi_\alpha)_{\alpha\in I}$, such that $\alpha_0 \geq \alpha_k$ for $k = 1, 2, ..., N+1$. Then (3) gives us

$$\left.\begin{array}{l} \|\psi_k\cdot\phi_{\alpha_0} - \phi_{\alpha_0}\| < \epsilon;\ k = 1, 2, ..., N \\ \|v\cdot\phi_{\alpha_0} - \phi_{\alpha_0}\| < \epsilon. \end{array}\right\}. \qquad (4)$$

Now let $\phi = v\cdot\phi_{\alpha_0} \in (M_*)_1^+$. As $K \subseteq \bigcup_{k=1}^{N} x_k E$ we can complete the proof of this implication by showing that $\|x_i t.\phi - \phi\| < 5\epsilon$ for $i = 1, 2, ..., N$, and $t \in E$. To see this first note that

$$\begin{aligned} \|\phi_E\cdot\phi - t\cdot\phi\| &= \|\phi_E\cdot(v\cdot\phi_{\alpha_0}) - t\cdot(v\cdot\phi_{\alpha_0})\| \\ &\leq \|(\phi_E * v)\cdot\phi_{\alpha_0} - v\cdot\phi_{\alpha_0}\| + \|v\cdot\phi_{\alpha_0} - (\delta_t * v)\cdot\phi_{\alpha_0}\| \\ &\leq \underbrace{\|\phi_E * v - v\|_1}_{<\epsilon \text{ by } (2)} \underbrace{\|\phi_{\alpha_0}\|}_{=1} + \underbrace{\|v - \delta_t * v\|_1}_{<\epsilon\ by\ (1)} \|\phi_{\alpha_0}\| < 2\epsilon. \end{aligned}$$

Hence,

$$\|\phi_E \cdot \phi - t \cdot \phi\| < 2\epsilon,$$

and therefore,

$$\begin{aligned}
\|\phi_{x_iE} \cdot \phi - x_i t \cdot \phi\| &= \|(\delta_{x_i} * \phi_E) \cdot \phi - x_i t \cdot \phi\| \\
&= \|x_i \cdot \underbrace{(\phi_E \cdot \phi)}_{\in (M_*)_1^+} - x_i \cdot (t \cdot \phi)\| \\
&= \|\delta_{x_i} \cdot (\phi_E \cdot \phi) - \delta_{x_i} \cdot (t \cdot \phi)\| \\
&\le \underbrace{\|\delta_{x_i}\|}_{=1} \|\phi_E \cdot \phi - t \cdot \phi\| < 2\epsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_i t \cdot \phi - \phi\| &\le \|x_i t \cdot \phi - \phi_{x_iE} \cdot \phi\| + \|\phi_{x_iE} \cdot \phi - \phi\| \\
&< 2\epsilon + \|\phi_{x_iE} \cdot (v \cdot \phi_{\alpha_0}) - v \cdot \phi_{\alpha_0}\| \\
&\le 2\epsilon + \|\phi_{x_iE} \cdot (v \cdot \phi_{\alpha_0}) - \phi_{x_iE} \cdot \phi_{\alpha_0}\| + \underbrace{\|\phi_{x_iE} \cdot \phi_{\alpha_0} - \phi_{\alpha_0}\|}_{<\epsilon \ by \ (4)} + \underbrace{\|\phi_\alpha - v \cdot \phi_{\alpha_0}\|}_{<\epsilon \ by \ (4)} \\
&\le 4\epsilon + \underbrace{\|\phi_{x_iE}\|}_{=1} \|v \cdot \phi_{\alpha_0} - \phi_{\alpha_0}\| < 5\epsilon \qquad \text{(by (4))}.
\end{aligned}$$

$(ii) \Rightarrow (iii)$ Suppose that for each $\epsilon > 0$ and compact subset K of G there is $\phi \in (M_*)_1^+$, such that

$$\|s \cdot \phi - \phi\| < \epsilon \qquad (s \in K). \qquad (*)$$

Let $D = \{\alpha = (\epsilon, K) : \epsilon > 0 \text{ and } K \subseteq G \text{ is compact}\}$ directed by
$(\epsilon_0, K_0) = \alpha_0 \preceq \alpha_1 = (\epsilon_1, K_1) \iff \epsilon_1 \le \epsilon_0$, and $K_1 \supseteq K_0$.
Claim I: $(D, \preceq)$ is a directed set.

(a) Suppose $(\epsilon_0, K_0) = \alpha_0 \in D$. Then $\alpha_0 \preceq \alpha_0$ since $\epsilon_0 \le \epsilon_0$ and $K_0 \supseteq K_0$.

(b) Suppose $(\epsilon_0, K_0) = \alpha_0,\ (\epsilon_1, K_1) = \alpha_1 \in D$ such that $\alpha_0 \preceq \alpha_1$ and $\alpha_1 \preceq \alpha_0$. Then $\epsilon_1 \le \epsilon_0,\ \epsilon_0 \le \epsilon_1$, so $\epsilon_0 = \epsilon_1$, and $K_1 \supseteq K_0,\ K_0 \le K_1$, so $K_0 = K_1$. Thus $\alpha_0 = \alpha_1$.

(c) Suppose $(\epsilon_0, K_0) = \alpha_0,\ (\epsilon_1, K_1) = \alpha_1,\ (\epsilon_2, K_2) = \alpha_2 \in D$, such that $\alpha_0 \preceq \alpha_1$ and $\alpha_1 \preceq \alpha_2$. Then $\epsilon_1 \leq \epsilon_0,\ \epsilon_2 \leq \epsilon_1$, so $\epsilon_2 \leq \epsilon_0$, and $K_1 \supseteq K_0, K_2 \supseteq K_1$, so $K_2 \supseteq K_0$. Therefore $\alpha_0 \preceq \alpha_2$.

(d) Let $(\epsilon_0, K_0) = \alpha_0,\ (\epsilon_1, K_1) = \alpha_1 \in D$, and $\epsilon = \min\{\epsilon_0, \epsilon_1\}$, and suppose $K = K_0 \cup K_1$. Then $\epsilon \leq \epsilon_0$ and $K \supseteq K_0$, so $\alpha_0 \preceq \alpha = (\epsilon, K)$, $\epsilon \leq \epsilon_1$, and $K \supseteq K_1$. Thus $\alpha_1 \preceq \alpha$ and $\alpha_0 \preceq \alpha$.

Hence, D is a directed set.

Now by $(*)$ for every $\alpha = (\epsilon, K) \in D$ there is $\phi_\alpha \in (M_*)_1^+$, such that

$$\|s \cdot \phi_\alpha - \phi_\alpha\| < \epsilon \qquad (s \in K).$$

Claim II: $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0$ uniformly on compact subsets of G. (That is for each compact subset K of G and $\epsilon > 0$, there is α_0, such that

$$\alpha \geq \alpha_0 \Rightarrow \|s \cdot \phi_\alpha - \phi_\alpha\| < \epsilon \qquad (s \in K).)$$

Let $K_0 \subseteq G$ be compact, $\epsilon_0 > 0$, and $\alpha_0 = (\epsilon_0, K_0) \in D$. Suppose $(\epsilon, K) = \alpha \geq \alpha_0 = (\epsilon_0, K_0)$, (so $K \supseteq K_0$ and $\epsilon \leq \epsilon_0$). Then for $s \in K_0$, $s \in K$, so

$$\|s \cdot \phi_\alpha - \phi_\alpha\| < \epsilon \leq \epsilon_0.$$

$(iii) \Rightarrow (iv)$ Suppose there is a net $(\phi_\alpha) \subseteq (M_*)_1^+$, such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0$ uniformly on compact subsets of G. Let $\mu \in M(G)_1^+$ with compact support K_μ. We will show that $\|\mu \cdot \phi_\alpha - \phi_\alpha\| \to 0$. To see this, let $\epsilon > 0$ and take α_0, such that

$$\alpha \geq \alpha_0 \Rightarrow \|s \cdot \phi_\alpha - \phi_\alpha\| < \frac{\epsilon}{2} \qquad (s \in K_\mu).$$

Let $\psi \in M = (M_*)^*,\ \|\psi\| \leq 1$, and suppose $\alpha \geq \alpha_0$. Then

$$\begin{aligned}
|\langle\psi,\mu\cdot\phi_\alpha-\phi_\alpha\rangle| &= |\langle\psi,\int s\cdot\phi_\alpha d\mu(s)\rangle-\langle\psi,\phi_\alpha\rangle| \\
&= |\int\langle\psi,s\cdot\phi_\alpha\rangle d\mu(s)-\int\langle\psi,\phi_\alpha\rangle d\mu(s)| \\
&= |\int\langle\psi,s\cdot\phi_\alpha-\phi_\alpha\rangle d\mu(s)| \\
&\leq \int|\langle\psi,s\cdot\phi_\alpha-\phi_\alpha\rangle|d\mu(s) \\
&\leq \int_{K_\mu}\underbrace{\|\psi\|}_{\leq 1}\underbrace{\|s\cdot\phi_\alpha-\phi_\alpha\|}_{<\frac{\epsilon}{2}}d\mu(s) \\
&= \frac{\epsilon}{2}|\mu|(G)=\frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$\|\mu\cdot\phi_\alpha-\phi_\alpha\|=\sup\{|\langle\psi,\mu\cdot\phi_\alpha-\phi_\alpha\rangle|:\psi\in(M)_1\}\leq\frac{\epsilon}{2}<\epsilon.$$

Now let $\mu\in M(G)_1^+$ be arbitrary. Then by regularity of μ there is a sequence $(\mu_n)\subseteq M(G)_1^+$ with supp μ_n is compact, such that

$$\|\mu_n-\mu\|\to 0.$$

Then

$$\begin{aligned}
\|\mu\cdot\phi_\alpha-\phi_\alpha\| &\leq \|\mu\cdot\phi_\alpha-\mu_n\cdot\phi_\alpha\|+\|\mu_n\cdot\phi_\alpha-\phi_\alpha\| \\
&\leq \|\mu-\mu_n\|\underbrace{\|\phi_\alpha\|}_{=1}+\|\mu_n\cdot\phi_\alpha-\phi_\alpha\|\to 0,
\end{aligned}$$

by the previous case.

$(iv)\Rightarrow(i)$ Suppose there is a net $(\phi_\alpha)\subseteq(M_*)_1^+$, such that

$$\|\mu\cdot\phi_\alpha-\phi_\alpha\|\to 0 \qquad (\mu\in M(G)_1^+).$$

Since if $s\in G$, then $\delta_s\in M(G)_1^+$, we have

$$\|s\cdot\phi_\alpha-\phi_\alpha\|=\|\delta_s\cdot\phi_\alpha-\phi_\alpha\|\to 0.$$

Hence by Corollary 2.3, G acts amenably on M_*. $\square$

Chapter 3

G-module Maps and G-invariant Subsets of Left Banach G-modules

In this section we will extend many of the statements which Ghaffari in [7] and Lau in [13] proved for specific actions to arbitrary Banach G-modules and dual G-modules. The ideas of our proofs combine those of Ghaffari, Lau and our own. Lau's paper is concerned with the action described in Examples 1.7 and 1.10 while Ghaffari's paper is concerned only with the action described in Examples 1.8 and 1.11.

3.1 G-Module Maps Between Left Banach G-Modules

Definition 3.1. *Let X and Y be left Banach G-modules, and let $T : X \to Y$ be an operator from X into Y. Then we will say that T is a* G**-module map** *if*

$$T(s \cdot x) = s \cdot Tx \qquad (s \in G, x \in X).$$

Moreover, if $T : Y^* \to X^*$ is an operator from Y^* into X^*, then T is a right G-module map if with respect to the dual actions of G on Y^* and X^*,

$$T(\phi \cdot s) = T\phi \cdot s \qquad (s \in G, \phi \in Y^*).$$

Similarly, we can define (right) $M(G)$-module maps and (right) $L^1(G)$-module maps.

The following two theorems generalizes [7, Theorem 2.1].

Theorem 3.1. *Let X and Y be left Banach G-modules and $T : X \to Y$ a linear and bounded map. Then the following statements are equivalent:*

(i) T is a G-module map.

(ii) T is an $M(G)$-module map.

(iii) T is an $L^1(G)$-module map.

Proof. $(i) \Rightarrow (ii)$ Let X and Y be left Banach G-modules. Suppose that $T : X \to Y$ is a bounded linear G-module map, and $T^* : Y^* \to X^*$ is its adjoint operator. For $\mu \in M(G), x \in X$ and $\phi \in Y^*$,

$$\begin{aligned} \langle T(\mu \cdot x), \phi \rangle &= \langle \mu \cdot x, T^* \phi \rangle \\ &= \int \langle s \cdot x, T^* \phi \rangle d\mu(s) \\ &= \int \langle T(s \cdot x), \phi \rangle d\mu(s) \\ &= \int \langle s \cdot Tx, \phi \rangle d\mu(s) \\ &= \langle \mu \cdot Tx, \phi \rangle . \end{aligned}$$

Since Y^* separates points of Y, $T(\mu \cdot x) = \mu \cdot Tx$.

$(ii) \Rightarrow (iii)$ This is obvious since $L^1(G) \subset M(G)$.

$(iii) \Rightarrow (i)$ Suppose that $T : X \to Y$ is a bounded linear $L^1(G)$-module map. $\quad (*)$
Let $s \in G, x \in X$, and $f \in L^1(G)$. Also let $(e_\alpha)_\alpha$ be a BAI for $L^1(G)$. Then by Corollary 1.1, $(e_\alpha)_\alpha$ is a BAI for both the neo-unital $L^1(G)$-module X and the neo-untal

$L^1(G)$-module Y. So

$$\begin{aligned}
s \cdot Tx &= \delta_s \cdot Tx \\
&= \lim \delta_s \cdot (e_\alpha \cdot Tx) \\
&= \lim \underbrace{(\delta_s * e_\alpha)}_{\in L^1(G)} \cdot Tx. \\
&= \lim T((\delta_s * e_\alpha) \cdot x) \qquad \text{(by } (*)) \\
&= T(\lim(\delta_s * e_\alpha) \cdot x) \\
&= T(\lim \delta_s \cdot (e_\alpha \cdot x)) \\
&= T(\delta_s \cdot x) \\
&= T(s \cdot x).
\end{aligned}$$

□

Note that because X^* and Y^* are not necessarily right Banach G-modules, (see Example 1.13), the next result is not immediately contained in the right module version of Theorem 3.1.

Theorem 3.2. *Let X and Y be left Banach G-modules. Suppose that $T : Y^* \to X^*$ is linear, bounded and $w^* - w^*$ continuous. Then the following statements are equivalent:*

(i) T is a right G-module map.

(ii) T is a right $M(G)$-module map.

(iii) T is a right $L^1(G)$-module map.

Proof. Suppose X and Y are left Banach G-modules, so that X^* and Y^* have right dual module structures and suppose $T : Y^* \to X^*$ is linear, bounded and $w^* - w^*$ continuous. Then by Proposition 1.3, T is an adjoint operator for some $L : X \to Y$.
$(i) \Rightarrow (ii)$ Suppose that T is a right G-module map . $\qquad (*)$
Claim I: L is a G-module map.

Let $s \in G, x \in X$ and $\phi \in Y^*$. Then

$$\begin{aligned}
\langle \phi, L(s \cdot x)\rangle &= \langle T\phi, s \cdot x\rangle \\
&= \langle (T\phi) \cdot s, x\rangle \\
&= \langle T(\phi \cdot s), x\rangle \qquad \text{(by } (*))\\
&= \langle \phi \cdot s, Lx\rangle \\
&= \langle \phi, s \cdot (Lx)\rangle.
\end{aligned}$$

Therefore $L(s \cdot x) = s \cdot Lx$ (i.e. L is a G-module map). Hence by Theorem 3.1, L is an $M(G)$-module map (i.e. $L(\mu \cdot x) = \mu \cdot Lx$ whenever $\mu \in M(G), x \in X$).
Claim II: T is an $M(G)$-module map.
Let $x \in X, \phi \in Y^*$, and $\mu \in M(G)$. Then

$$\begin{aligned}
\langle x, T(\phi \cdot \mu)\rangle &= \langle Lx, \phi \cdot \mu\rangle \\
&= \langle \mu \cdot Lx, \phi\rangle \\
&= \langle L(\mu \cdot x), \phi\rangle \\
&= \langle \mu \cdot x, T\phi\rangle \\
&= \langle x, T\phi \cdot \mu\rangle.
\end{aligned}$$

Hence $T(\phi \cdot \mu) = T\phi \cdot \mu$.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Suppose $T : Y^* \to X^*$ is a right $L^1(G)$-module map. $\qquad (*)$
Claim I: $L : X \to Y$ is an $L^1(G)$-module map.
Let $f \in L^1(G), x \in X$, and $\phi \in Y^*$. Then

$$\begin{aligned}
\langle \phi, L(f \cdot x)\rangle &= \langle T\phi, f \cdot x\rangle \\
&= \langle T\phi \cdot f, x\rangle \\
&= \langle T(\phi \cdot f), x\rangle \qquad \text{(by } (*))\\
&= \langle \phi \cdot f, Lx\rangle \\
&= \langle \phi, f \cdot Lx\rangle.
\end{aligned}$$

Therefore $L(f \cdot x) = f \cdot Lx$ (i.e. L is an $L^1(G)$-module map). Hence L is a G-module map by Theorem 3.1.

Claim II: T is a G- module map.

Let $x \in X, \phi \in Y^*$ and $s \in G$. Then

$$\begin{aligned} \langle x, T(\phi \cdot s\rangle &= \langle Lx, \phi \cdot s\rangle \\ &= \langle s \cdot Lx, \phi\rangle \\ &= \langle L(s \cdot x), \phi\rangle \qquad \text{(by } (*)) \\ &= \langle s \cdot x, T\phi\rangle \\ &= \langle x, T\phi \cdot s\rangle. \end{aligned}$$

Hence $T(\phi \cdot s) = T\phi \cdot s$. □

The following corollary [10, Theorem 35.5] is an immediate consequence of Theorem 3.1 applied to the Banach G-module action $s \cdot h = \delta_s * h$ of G on $L^1(G)$; see Examples 1.7 and 1.10 where this action and the corresponding $L^1(G)$, $M(G)$-module actions -given by the convolution $*$ - are discussed.

Corollary 3.1. *Let G be a locally compact group and $T : L^1(G) \to L^1(G)$ be a bounded linear operator. If $s \cdot h = \delta_s * h = h(s^{-1}x)$, where $s \in G$, and $h \in L^1(G)$, then the following are equivalent:*

(i) $T(\delta_s * h) = \delta_s * Th$ whenever $s \in G$, and $h \in L^1(G)$.

(ii) $T(\mu * h) = \mu * Th$ whenever $\mu \in M(G)$, and $h \in L^1(G)$.

(iii) $T(f * h) = f * Th$ whenever $f, h \in L^1(G)$.

Let X be a left Banach G-module, so by Proposition 1.14, X^* is a right dual G-module but it is not necessarily a right Banach G-module (see Example 1.13). Suppose however that $Z \preceq X^*$ is a right Banach G-submodule of X^* with respect to

$$\phi \bullet s = \phi \cdot s \qquad (\phi \in Z, s \in G),$$

where $(\cdot)$ is the dual action on X^* and $(\bullet)$ is the submodule action on Z. By Proposition 1.12, Z has corresponding neo-unital Banach $L^1(G)$-module and unital

Banach $M(G)$-module operations $\phi \bullet f$ and $\phi \bullet \mu$ where $f \in L^1(G)$, $\mu \in M(G)$ and $\phi \in Z$ given by

$$\phi \bullet \mu = \int_{\sigma(Z,Z^*)} \phi \bullet s d\mu(s) = \int_{\sigma(Z,Z^*)} \phi . s d\mu(s). \qquad (1)$$

Also for $\phi \in Z$ and $\mu \in M(G)$, we have $\phi \cdot \mu \in Z$ given by

$$\langle \phi \cdot \mu, x \rangle = \langle \phi, \mu \cdot x \rangle \qquad (x \in X). \qquad (2)$$

Proposition 3.1. *Let Z be defined as above, $\phi \in Z$, and $\mu \in M(G)$. Then $\phi \bullet \mu$ and $\phi \cdot \mu$ as they are defined in* (1) *and* (2) *are equal.*

Proof. The proof of Proposition 2.1 works by replacing $Z \preceq M$ with $Z \preceq X^*$. □

Let X be a left Banach G-module and define $UC(X^*)$ by the following:

$$UC(X^*) := \{\phi \in X^* : s \mapsto \phi \cdot s : G \to (X^*, \| \cdot \|) \text{ is continuous}\},$$

it is easy to see that $UC(X^*)$ is a closed linear subspace of X^*. The next theorem includes [7, Lemma 2.3], among other results concerning specific G-module actions. The next two results will also appear as lemmas in a forthcoming paper by Y. Choi, E. Samei and R. Stokke.

Lemma 3.1. *Let X be a left Banach G-module. Then $UC(X^*)$ is a right Banach G-submodule of X^*.*

Proof. Suppose X is a left Banach G-module. Then by Proposition 1.14, X^* is a right dual G-module with respect to

$$\langle \phi \cdot s, x \rangle = \langle \phi, s \cdot x \rangle \qquad (\phi \in X^*, s \in G, x \in X).$$

Note that if $\phi \in UC(X^*)$, then $\phi . s \in UC(X^*)$. To see this let $(t_i) \subseteq G$ be a net, such that $t_i \to t$ in G. Then $(st_i) \subseteq G$ is a net converging to st in G. Since $\phi \in UC(X^*) \subseteq X^*$,

$$(\phi \cdot s) \cdot t_i = \phi \cdot (st_i) \to \phi \cdot (st) = (\phi \cdot s) \cdot t.$$

Thus $\phi \cdot s \in UC(X^*)$ and hence $UC(X^*)$ is a right G-submodule of X^*. Moreover

$$\begin{aligned} |\langle \phi \cdot s, x \rangle| &= |\langle \phi, s \cdot x \rangle| \\ &\leq \|\phi\| \|s \cdot x\| \\ &\leq k\|\phi\|\|x\|. \end{aligned}$$

Therefore $\|\phi \cdot s\| \leq k\|\phi\|$.
Finally if $\phi \in UC(X^*)$, then $s \mapsto \phi \cdot s : G \to (UC(X^*), \| \cdot \|)$ is continuous by definition. Hence $UC(X^*)$ is a right Banach G-submodule of X^*. □

Corollary 3.2. *The dual $M(G)$-module structure and the induced $M(G)$-module structure on $UC(X^*)$ as they are defined in* (1) *and* (2) *before Proposition 3.1, when $Z := UC(X^*)$, agree.*

Note that by the above Corollary whenever $\phi \in UC(X^*)$ and $\mu \in M(G)$, we can write $\phi \cdot \mu$ without causing any confusion.

Lemma 3.2. *Let X be a left Banach G-module. Then $UC(X^*) = X^* \cdot L^1(G)$.*

Proof. For $\phi \in X^*$ and $f \in L^1(G)$, $s_i \to s$ in G implies

$$\|(\phi \cdot f) \cdot s_i - (\phi \cdot f) \cdot s\| \leq \|\phi\| \|f * \delta_{s_i} - f * \delta_s\|_1 \to 0 \qquad \text{(by Proposition 1.11)},$$

so $X^* \cdot L^1(G) \subseteq UC(X^*)$. But by Proposition 1.12, $UC(X^*)$ is a neo-unital $L^1(G)$-module so by Corollary 3.2,

$$X^* \cdot L^1(G) \subseteq UC(X^*) = UC(X^*) \cdot L^1(G) \subseteq X^* \cdot L^1(G),$$

hence $UC(X^*) = X^* \cdot L^1(G)$. □

Now by Corollary 3.2, we can obtain the following corollary, which includes [7, Theorem 2.4], as an immediate corollary to the right module version of Theorem 3.1.

Corollary 3.3. *Let X and Y be left Banach G-modules, and let $T : UC(X^*) \to UC(Y^*)$ be a bounded linear operator. Then the following statements are equivalent:*

(i) $T(\phi \cdot s) = T\phi \cdot s$ *where* $\phi \in UC(X^*), s \in G$.

(ii) $T(\phi \cdot \mu) = T\phi \cdot \mu$ *where* $\phi \in UC(X^*), \mu \in M(G)$.

(iii) $T(\phi \cdot f) = T\phi \cdot f$ *where* $\phi \in UC(X^*), f \in L^1(G)$.

3.2 Closed Convex G-Invariant Subsets of Left Banach G-modules

Definition 3.2. *Let X be a left Banach G-module. If C is a convex subset of X, then C is called G-**invariant** if $s \cdot x \in C$ whenever $s \in G, x \in C$. Similarly, we can define $L^1(G)$-invariant, and $M(G)$-invariant convex sets.*

The following theorem includes [13, Theorem 4.1(a)]. Note that by Proposition 1.1, a convex subset of a Banach space is closed if and only if it is weakly closed.

Theorem 3.3. *Let X be a left Banach G-module and C a closed convex subset of X. Then the following are equivalent:*

(i) C is G-invariant.

(ii) C is $M(G)_1^+$-invariant.

(iii) C is $L^1(G)_1^+$-invariant.

Proof. $(i) \Rightarrow (ii)$ Let $\mu \in M(G)_1^+, x \in C$ and suppose that C is G-invariant.
Claim: $\mu \cdot x \in C$.
Suppose $\mu \cdot x \notin C$. Then by the Hahn-Banach Separation Theorem there is $x^* \in X^*, \gamma \in \mathbb{R}$, and $\epsilon > 0$, such that

$$Re\langle x^*, c\rangle \leq \gamma < \gamma + \epsilon \leq Re\langle x^*, \mu \cdot x\rangle \qquad (c \in C).$$

So

$$Re\langle x^*, s \cdot x\rangle \leq \gamma < \gamma + \epsilon \leq Re\langle x^*, \mu \cdot x\rangle \qquad (s \in G).$$

But

$$\begin{aligned} Re\langle x^*, \mu \cdot x\rangle &= Re\int \langle x^*, s \cdot x\rangle d\mu(s) \\ &= \int Re\langle x^*, s \cdot x\rangle d\mu(s) \qquad \text{(since } \mu \text{ is positive)} \\ &\leq \int \gamma d\mu(s) = \gamma\mu(G) = \gamma, \end{aligned}$$

a contradiction. Hence, $\mu \cdot x \in C$.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, x \in C$ and suppose that C is $L^1(G)^+_1$-invariant. $\quad (*)$
For each compact neighborhood α of e, the identity of G, let $e_\alpha = \frac{1}{\lambda(\alpha)}1_\alpha$. Then $(e_\alpha)_\alpha \subseteq L^1(G)^+_1$ is a BAI for $L^1(G)$, and

$$\begin{aligned} e_\alpha \cdot (s \cdot x) &= e_\alpha \cdot (\delta_s \cdot x) \\ &= \underbrace{(e_\alpha * \delta_s)}_{\in L^1(G)^+_1} \cdot x \in C \qquad \text{(by } (*)). \end{aligned}$$

Since $s \cdot x \in X$, and X is a neo-unital Banach $L^1(G)$-module, by Corollary 1.1,

$$e_\alpha \cdot (s \cdot x) \to s \cdot x.$$

As C is closed, $s \cdot x \in C$. $\square$

The following theorem includes [7, Theorem 2.5] and [13, Theorem 4.1(b)]. The proof is similar to that of Theorem 3.3.

Theorem 3.4. *Let X be a left Banach G-module, L a w^*-closed convex subset of X^*. Then the following are equivalent:*

(i) L is G-invariant.

(ii) L is $M(G)^+_1$-invariant.

(iii) L is $L^1(G)^+_1$-invariant.

Proof. $(i) \Rightarrow (ii)$ Let $\mu \in M(G)^+_1, \phi \in L$ and suppose L is G-invariant. Suppose $\phi \cdot \mu \notin L$. Then by the Hahn-Banach Separation Theorem there is an $x \in (X^*, w^*)^* = X, \gamma \in \mathbb{R}$, and an $\epsilon > 0$, such that

$$Re\langle \ell, x\rangle \leq \gamma < \gamma + \epsilon \leq Re\langle \phi \cdot \mu, x\rangle \qquad (\ell \in L).$$

So

$$Re\langle \phi \cdot s, x\rangle \le \gamma < \gamma + \epsilon \le Re\langle \phi \cdot \mu, x\rangle \qquad (s \in G).$$

But we have

$$\begin{aligned} Re\langle \phi \cdot \mu, x\rangle &= Re\langle \phi, \mu \cdot x\rangle \\ &= Re \int \langle \phi, s \cdot x\rangle d\mu(s) \\ &= \int Re\langle \phi, s \cdot x\rangle d\mu(s) \qquad \text{(since } \mu \text{ is positive)} \\ &= \int Re\langle \phi \cdot s, x\rangle d\mu(s) \\ &\le \int \gamma d\mu(s) = \gamma\mu(G) = \gamma, \end{aligned}$$

which is a contradiction. Hence $\phi \cdot \mu \in L$.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, \phi \in L$ and suppose that L is $L^1(G)^+_1$-invariant. Let $(e_\alpha)_\alpha \subseteq L^1(G)^+_1$ be a BAI for $L^1(G)$. Then we have

$$\begin{aligned} (\phi \cdot s) \cdot e_\alpha &= (\phi \cdot \delta_s) \cdot e_\alpha. \\ &= \phi \cdot \underbrace{(\delta_s * e_\alpha)}_{\in L^1(G)^+_1} \in L, \end{aligned}$$

by our assumption. For $x \in X$,

$$\begin{aligned} |\langle (\phi \cdot s) \cdot e_\alpha - \phi \cdot s, x\rangle| &= |\langle (\phi \cdot s) \cdot e_\alpha, x\rangle - \langle \phi \cdot s, x\rangle| \\ &= |\langle \phi \cdot s, e_\alpha \cdot x\rangle - \langle \phi \cdot s, x\rangle \\ &= |\langle \phi \cdot s, e_\alpha \cdot x - x\rangle| \\ &\le \|\phi \cdot s\|\|e_\alpha \cdot x - x\| \to 0 \qquad \text{(by Corollary1.1).} \end{aligned}$$

Hence, $(\phi \cdot s) \cdot e_\alpha \xrightarrow{w^*} \phi \cdot s$, and as L is w^*-closed, $\phi \cdot s \in L$. $\square$

The next corollary includes [13, Corollary 4.2].

Corollary 3.4. *Let X be a left Banach G-module. Then the following statements hold:*

(i) $\overline{co}\{s\cdot x : s\in G\} = \overline{\{f\cdot x : f\in L^1(G)_1^+\}} = \overline{\{\mu\cdot x : \mu\in M(G)_1^+\}}$ *where $x\in X$.*

(ii) $\overline{co}^{w^*}\{\phi\cdot s : s\in G\} = \overline{\{\phi\cdot f : f\in L^1(G)_1^+\}}^{w^*} = \overline{\{\phi\cdot\mu : \mu\in M(G)_1^+\}}^{w^*}$ *where $\phi\in X^*$.*

Proof. (i) Let $x\in X$, $C_1 = \overline{co}\{s\cdot x : s\in G\}$, $C_2 = \overline{\{\mu\cdot x : \mu\in M(G)_1^+\}}$, and $C_3 = \overline{\{f\cdot x : f\in L^1(G)_1^+\}}$. Obviously $co\{s\cdot x : s\in G\}$ is G-invariant, and for each $s\in G$, $y\mapsto s\cdot y : X\to X$ is continuous. Let $z\in C_1$. Then there is a net (ϕ_i) in $co\{s\cdot x : s\in G\}$ such that $\phi_i\to z$. As $co\{s\cdot x : s\in G\}$ is G-invariant, $(s\cdot\phi_i)$ is a net in it. By continuity, $s\cdot\phi_i\to s\cdot z$, thus $s\cdot z\in C_1$. Hence C_1 is G-invariant, so by Theorem 3.3 C_1 is $M(G)_1^+$-invariant and $L^1(G)_1^+$-invariant. Since $x = e\cdot x\in\{s\cdot x : s\in G\}\subseteq C_1$, $\{\mu\cdot x : \mu\in M(G)_1^+\}\subseteq C_1$. As C_1 is closed,

$$C_2 = \overline{\{\mu\cdot x : \mu\in M(G)_1^+\}}\subseteq C_1, \text{ and clearly } C_3\subseteq C_2. \qquad (*)$$

Now let $(e_\alpha)_\alpha\subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then by Corollary 1.1,

$$e_\alpha\cdot x\to x,$$

so $x\in C_3$. But C_3 is closed, convex and $L^1(G)_1^+$-invariant so by Theorem 3.3, C_3 is G-invariant. As $x\in C_3$, it is clear that $C_1\subseteq C_3$. This and $(*)$ give $C_1 = C_2 = C_3$.

(ii) Suppose that $\phi\in X^*$. Let $L_1 = \overline{co}^{w^*}\{\phi\cdot s : s\in G\}$, $L_2 = \overline{\{\phi\cdot\mu : \mu\in M(G)_1^+\}}^{w^*}$ and $L_3 = \overline{\{\phi\cdot f : f\in L^1(G)_1^+\}}^{w^*}$. Obviously $co\{\phi\cdot s : s\in G\}$ is G-invariant. Let $\psi\in L_1$. Then there is a net (ψ_i) in $co\{\phi\cdot s : s\in G\}$ such that $\psi_i\xrightarrow{w^*}\psi$. Since $co\{\phi\cdot s : s\in G\}$ is G-invariant, $(\psi_i\cdot s)$ is a net in it, and for $x\in X$

$$\langle x, \psi_i\cdot s\rangle = \langle s\cdot x, \psi_i\rangle\to\langle s\cdot x, \psi\rangle = \langle x, \psi\cdot s\rangle.$$

Thus $\psi \cdot s \in L_1$ and therefore L_1 is G-invariant, so by Theorem 3.4 L_1 is $M(G)_1^+$-invariant and $L^1(G)_1^+$-invariant. Since $\phi = \phi \cdot e \in \{\phi \cdot s : s \in G\} \subseteq L_1$, $\{\phi \cdot \mu : \mu \in M(G)_1^+\} \subseteq L_1$. As L_1 is w^*-closed,

$$L_2 = \overline{\{\phi \cdot \mu : \mu \in M(G)_1^+\}}^{w^*} \subseteq L_1, \text{ and clearly } L_3 \subseteq L_2. \qquad (*)$$

Now let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then by Corollary 1.1,

$$\langle \phi \cdot e_\alpha, x\rangle = \langle \phi, e_\alpha \cdot x\rangle \to \langle \phi, x\rangle \qquad (x \in X),$$

so

$$\phi \cdot e_\alpha \xrightarrow{w^*} \phi.$$

Hence, $\phi \in L_3$. But L_3 is w^*-closed, convex and $L^1(G)_1^+-$invariant so, by Theorem 3.4, L_3 is G-invariant. As $\phi \in L_3$, it is therefore clear that $L_1 \subseteq L_3$. This and $(*)$ give $L_1 = L_2 = L_3$.

□

3.3 G-Module Maps Between Closed Convex G-Invariant Subsets of Left Banach G-Modules

The following theorem contains [13, Theorem 5.1].

Theorem 3.5. *Let X, Y be left Banach G-modules, and B, C be closed G-invariant convex subsets of X and Y respectively. If $T : B \to C$ is continuous and affine, then the following are equivalent:*

(i) $T(s \cdot x) = s \cdot Tx$ *whenever* $s \in G, x \in B$.

(ii) $T(\mu \cdot x) = \mu \cdot Tx$ *whenever* $\mu \in M(G)_1^+, x \in B$.

(iii) $T(f \cdot x) = f \cdot Tx$ *whenever* $f \in L^1(G)_1^+, x \in B$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in B$ and suppose $T(s \cdot x) = s \cdot Tx$; $s \in G$. $\quad (*)$
Let $\mu \in M(G)_1^+, (\mu_\alpha) = (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha})$ where $\lambda_i^\alpha \in \mathbb{R}^+$, $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1$, be a net in $co\{\delta_s : s \in G\}$ converging to μ in $\tau-$topology; see Definition 1.19 and Proposition 1.9. Let $\phi \in X^*$. Then, noting that if $f(s) = \langle \phi, s \cdot x \rangle$ where $s \in G$, then $f \in CB(G)$, we obtain

$$\langle \phi, \mu_\alpha \cdot x \rangle = \int \langle \phi, s \cdot x \rangle d\mu_\alpha(s) \to \int \langle \phi, s \cdot x \rangle d\mu(s) = \langle \phi, \mu \cdot x \rangle.$$

Therefore,

$$\mu_\alpha \cdot x \xrightarrow{w} \mu \cdot x.$$

Since A and B are closed and convex, they are weakly closed by Proposition 1.1. Note as well that by [4, Remark 2], T is continuous when A and B have their respective weak topologies. Hence,

$$\begin{aligned} T(\mu \cdot x) = \lim T(\mu_\alpha \cdot x) &= \lim T((\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha}) \cdot x) \\ &= \lim T(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (\delta_{s_i^\alpha} \cdot x)) \\ &= \lim T(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (s_i^\alpha \cdot x)) \\ &= \lim(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha T(s_i^\alpha \cdot x)) \\ &= \lim(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (s_i^\alpha \cdot Tx)) \qquad \text{(by } (*)\text{)} \\ &= \lim(\sum_{i=1}^{n_\alpha} (\lambda_i^\alpha \delta_{s_i^\alpha}) \cdot Tx) \\ &= \lim((\sum_{i=1}^{n_\alpha} (\lambda_i^\alpha \delta_{s_i^\alpha})) \cdot Tx) = \lim \mu_\alpha \cdot Tx = \mu \cdot Tx. \end{aligned}$$

where in each case "lim" is a weak limit.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, x \in B$ and suppose $T(f \cdot x) = f \cdot Tx$ for every $f \in L^1(G)^+_1$. $\quad (*)$
Since B and C are G-invariant, $s \cdot x \in B$ and $s \cdot Tx \in C$. Thus by Theorem 3.3 $f \cdot x \in B$ and $f \cdot Tx \in C$, for all $f \in L^1(G)^+_1$. Now let $(e_\alpha)_\alpha \subseteq L^1(G)^+_1$ be a BAI for $L^1(G)$. Then by Corollary 1.1, $(e_\alpha)_\alpha$ ia a BAI for both the $L^1(G)$-module X and the $L^1(G)$-module Y, so

$$\begin{aligned} s \cdot Tx = \delta_s \cdot Tx &= \lim(\delta_s \cdot (e_\alpha \cdot Tx)) \\ &= \lim(\underbrace{(\delta_s * e_\alpha)}_{L^1(G)^+_1} \cdot Tx) \\ &= \lim T((\delta_s * e_\alpha) \cdot x) \qquad \text{(by } (*)) \\ &= T\left(\lim(\delta_s \cdot (e_\alpha \cdot x))\right) = T(\delta_s \cdot x) = T(s \cdot x). \end{aligned}$$

□

The next theorem contains [7, Theorem 2.6] and [13, Theorem 5.2]. The proof is similar to the proof of Theorem 3.5.

Theorem 3.6. *Let X, Y be left Banach G-modules and let L, K be w^*-closed G-invariant convex subsets of X^* and Y^* respectively. If $T : L \to K$ is $w^* - w^*$ continuous and affine, then the following are equivalent:*

(i) $T(\phi \cdot s) = T\phi \cdot s$ whenever $s \in G, \phi \in L$.

(ii) $T(\phi \cdot \mu) = T\phi \cdot \mu$ whenever $\mu \in M(G)^+_1, \phi \in L$.

(iii) $T(\phi \cdot f) = T\phi \cdot f$ whenever $f \in L^1(G)^+_1, \phi \in L$.

Proof. $(i) \Rightarrow (ii)$ Let $\phi \in L$ and suppose $T(\phi \cdot s) = T\phi \cdot s$; $s \in G$. $\quad (*)$
Let $\mu \in M(G)^+_1, \mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha} \subseteq co\{\delta_s : s \in G\}$ be a net converging to μ in $\tau-$topology; see Definition 1.19 and Proposition 1.9. Let $x \in X$. Then noting that

if $f(s) = \langle \phi, s \cdot x \rangle$ where $s \in G$, then $f \in CB(G)$, we obtain

$$\begin{aligned}
\langle \phi \cdot \mu_\alpha, x \rangle &= \langle \phi, \mu_\alpha \cdot x \rangle \\
&= \int \langle \phi, s \cdot x \rangle d\mu_\alpha(s) \\
&\to \int \langle \phi, s \cdot x \rangle d\mu(s) \\
&= \langle \phi, \mu \cdot x \rangle \\
&= \langle \phi \cdot \mu, x \rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T(\phi \cdot \mu) &= w^* - \lim T(\phi \cdot \mu_\alpha) \\
&= w^* - \lim T(\phi \cdot (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha})) \\
&= w^* - \lim T(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (\phi \cdot \delta_{s_i^\alpha})) \\
&= w^* - \lim T(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (\phi \cdot s_i^\alpha)) \\
&= w^* - \lim (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha T(\phi \cdot s_i^\alpha)) \\
&= w^* - \lim (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (T\phi \cdot s_i^\alpha)) \qquad \text{(by } (*)) \\
&= w^* - \lim (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha (T\phi \cdot \delta_{s_i^\alpha})) \\
&= w^* - \lim (\sum_{i=1}^{n_\alpha} T\phi \cdot (\lambda_i^\alpha \delta_{s_i^\alpha})) \\
&= w^* - \lim (T\phi \cdot (\sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{s_i^\alpha})) \\
&= w^* - \lim T\phi \cdot \mu_\alpha = T\phi \cdot \mu.
\end{aligned}$$

$(ii) \Rightarrow (iii)$This is obvious.

$(iii) \Rightarrow (i)$ Let $s \in G, \phi \in L$ and suppose that $T(\phi \cdot f) = T\phi \cdot f$ whenever

$f \in L^1(G)_1^+$. $\qquad (*)$

Since L, K are G-invariant, $\phi \cdot s \in L$ and $T\phi \cdot s \in K$. Thus by Theorem 3.4, $\phi \cdot f \in L$ and $T\phi \cdot f \in K$ for all $f \in L^1(G)_1^+$. Let $(e_\alpha)_\alpha \subseteq L^1(G)_1^+$ be a BAI for $L^1(G)$. Then, as shown in the proof of Corollary 3.4 (ii), $\psi \cdot e_\alpha \xrightarrow{w^*} \psi$ in X^* (respectively Y^*) for any $\psi \in X^*$ (respectively Y^*). Therefore,

$$\begin{aligned}
T\phi \cdot s &= T\phi \cdot \delta_s \\
&= w^* - \lim((T\phi \cdot e_\alpha) \cdot \delta_s) \\
&= w^* - \lim(T\phi \cdot \underbrace{(e_\alpha * \delta_s)}_{L^1(G)_1^+}) \\
&= w^* - \lim T(\phi \cdot (e_\alpha * \delta_s)) \qquad \text{(by } *) \\
&= T\left(w^* - \lim(\phi \cdot (e_\alpha * \delta_s))\right) \\
&= T\left(w^* - \lim((\phi \cdot e_\alpha) \cdot \delta_s)\right) \\
&= T(\phi \cdot \delta_s) \\
&= T(\phi \cdot s).
\end{aligned}$$

□

In the next two theorems $L^1(G)$ is viewed as a left Banach G-module via $s \cdot f = \delta_s * f$ (see example 1.7). The following theorem contains [13, Theorem 5.3].

Theorem 3.7. *Let G be a locally compact non-compact group. Let B be a non-empty closed convex left G-invariant subset of $L^1(G)$, and C a non-empty weakly compact closed convex left G-invariant subset of a left Banach G-module X. If $T : C \to B$ is a continuous affine G-module map, then whenever $f \in C$, $T(f) = 0$.*

Proof. Since C and B are closed and convex, they are weakly closed. Therefore by [4, Remark 2], T is affine continuous when C and B have their respective weak topologies. As C is weakly compact and convex, $T(C)$ is a weakly compact convex left G-invariant subset of $L^1(G)$. Hence by Theorem 1.9, $T(C) = \{0\}$. □

The next theorem includes [13, Theorem 5.5].

Theorem 3.8. *Let G be any locally compact group, let C be a weakly compact closed bounded left G-invariant subset of a left Banach G-module X, and let $T : L^1(G)_1^+ \to C$ be a continuous affine map. Then the following are equivalent:*

(i) T is a G-module map.

(ii) There is $x \in C$, such that $T(u) = u \cdot x$ whenever $u \in L^1(G)_1^+$.

Proof. $(i) \Rightarrow (ii)$ Let $s \in G$ and suppose $T(s \cdot u) = s \cdot Tu$ whenever $u \in L^1(G)_1^+$. Observe that $L^1(G)_1^+$ is a G-invariant weakly closed convex subset of $L^1(G)$, so by Theorem 3.5, we have

$$T(f * u) = f \cdot Tu \qquad (f, u \in L^1(G)_1^+). \qquad (*)$$

Suppose $(u_\alpha)_\alpha \subseteq L^1(G)_1^+$ is a BAI for $L^1(G)$. Since $T(u_\alpha) \in C$ for each α and C is weakly compact, there is an $x \in C$, such that, by passing to a subnet if necessary, $T(u_\alpha) \xrightarrow{w} x$.
Now let $\phi \in X^*$. Then

$$\begin{aligned}
\langle u \cdot x, \phi \rangle &= \langle x, \phi \cdot u \rangle \\
&= \lim \langle T u_\alpha, \phi \cdot u \rangle \\
&= \lim \langle u \cdot T u_\alpha, \phi \rangle \\
&= \lim \langle T \underbrace{(u * u_\alpha)}_{\in L^1(G)_1^+}, \phi \rangle \qquad (\text{by } (*)) \\
&= \langle Tu, \phi \rangle \qquad (\text{by Definition 1.24 and continuity of } T).
\end{aligned}$$

$(ii) \Rightarrow (i)$ Let $s \in G$ and suppose there is $x \in C$, such that $T(u) = u \cdot x$ for all $u \in L^1(G)_1^+$. Then

$$T(s \cdot u) = (s \cdot u) \cdot x = s \cdot (u \cdot x) = s \cdot Tu.$$

□

Ghaffari's action of $L^1(G)$ on $L^p(G)$ $(1 \leq p < \infty)$ in his paper [7] is the action in Example 1.11;

$$f \star h(t) = f \cdot h(t) = \int \Delta(s)^{\frac{1}{p}} h(s^{-1}ts) f(s) ds \qquad (f, \in L^1(G), h \in L^p(G), s \in G).$$

In that paper he says that $(L^1(G), \star)$ is a Banach algebra. But is $\star$ associative for any locally compact group?

Theorem 3.9. *Let G be any non-abelian discrete group. Then $\star$ is not associative on $L^1(G) = \ell^1(G)$.*

Proof. Observe that $\delta_x \star \delta_y = \delta_{xyx^{-1}}$ whenever $x, y \in G$. Suppose G is a non-abelian discrete group, and choose $s, t, r \in G$, such that $tr \neq rt$. Suppose $\star$ is associative on $L^1(G)$. Then

$$\delta_{rst(rs)^{-1}} = \delta_r \star (\delta_s \star \delta_t) = (\delta_r \star \delta_s) \star \delta_t = \delta_{(rsr^{-1})t(rsr^{-1})^{-1}},$$

so

$$rsts^{-1}r^{-1} = rsr^{-1}trs^{-1}r^{-1} \Rightarrow t = r^{-1}tr \Rightarrow rt = tr,$$

a contradiction. Hence $\star$ is not associative. $\square$

Proposition 3.2. *Let G be a locally compact group. If G is abelian, then $\star$ is associative on $L^1(G)$.*

Proof. Let G be an abelian locally compact group, and let $f, g \in L^1(G)$. Then

$$\begin{aligned} f \star g(t) &= \int \underbrace{\Delta(s)}_{=1} g(\underbrace{s^{-1}ts}_{=t}) f(s)ds \\ &= \int g(t)f(s)ds \\ &= \left(\int f(s)ds\right) g(t). \end{aligned}$$

Now let $h \in L^1(G)$. Then

$$\begin{aligned} (f \star g) \star h(t) &= \left(\int (f \star g)(s)ds\right) h(t) \\ &= \left(\int \left(\int f(r)dr\right) g(s)ds\right) h(t) \\ &= \int f(r)dr \left(\int g(s)dsh(t))\right) \\ &= \int f(r)dr(g \star h(t)) \\ &= f \star (g \star h)(t). \end{aligned}$$

$\square$

Corollary 3.5. *Let G be a discrete group. Then $\star$ is associative on $\ell^1(G)$ if and only if G is abelian.*

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