The h-p Version of the Finite Element Method in Three Dimensions

by

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A Thesis
Submitted to the Faculty of Graduate Studies
at the University of Manitoba

In Partial Fulfillment of the Requirements for
the Degree
Doctor of Philosophy

Department of Mathematics
University of Manitoba

Acknowledgements

I deeply thank my supervisor Dr. Benqi Guo for invaluable guidance, encouragement, patience and constant support throughout this program.

I would also like to thank the other members of my Advisor Committee: Dr. S. H. Lui, Dr. Ruppa Thulasiram and Dr. Bin Han. They have taken the time to read carefully this thesis and provided many valuable suggestions.

I also thank the Department of Mathematics and Dr. Benqi Guo for their financial support during my graduate studies.

Finally, I would like to acknowledge all my family, especially my wife(Chen Deying), my daughter(Zhang Xinyuan) and my parents for their continuing love and support.

Contents

Abstract

In the framework of the Jacobi-weighted Besov and Sobolev spaces, we analyze the approximation to singular and smooth functions. We construct stable and compatible polynomial extensions from triangular and square faces to prisms, hexahedrons and pyramids, and introduce quasi Jacobi projection operators on individual elements, which is a combination of the Jacobi projection and the interpolation at vertices and on sides of elements. Applying these results we establish the convergence of the h-p version of the finite element method with quasi uniform meshes in three dimensions for elliptic problems with smooth solutions or singular solutions on polyhedral domains in three dimensions. The rate of convergence in terms of h and p is proved to be the best.

CHAPTER 1

Introduction

The finite element method (FEM) has rapidly developed as an important numerical method for partial differential equations in theory, algorithm, and applications since the 1940's, and becomes now the mostly used computational tool to solve large-scale engineering and scientific problems. In the early years, FEM was used in structural mechanics such as civil engineering, automobile industry and aerospace industry, and it has penetrated almost every field of today's engineering and sciences, such as material science, electric-magnetic fields, fluid dynamics, biology, and finance. Numerous softwares of FEM have been successfully used in industry, research and education such as MSC/NASTRAN, ANSYS, ABAQUS, COSMIC, and many others.

According to the structure of finite element solutions, there are three approaches of the finite element method: the h-version, the p-version and the h-p version. In the h-version, the degree p of the elements is fixed at a low level and the accuracy is achieved by properly refining the mesh. In the p-version, the mesh is fixed and the degree p of polynomials is increased uniformly or selectively to achieve the accuracy. The h-p version is the combination of the h-version and p-version, namely, refine meshes and increase polynomial degrees simultaneously and selectively (or uniformly) in order to achieve higher accuracy. The p-version and h-p version are new developments, commercial and research codes based on the p and h-p versions of FEM are now widely used in computational engineering and sciences, for example, the commercial codes Pro/MECHANICA, PolyFEM, ProPHLEX, STRESSCHECK and the research codes STRIPE, HP-2D and HP-3D.

The first theoretical paper on the p -version in two dimensions by Babuška, Szabó and Katz was published in the early 1980s, it was shown in [**9**] that the p-version of FEM converges at least as fast as the classical FEM with quasi-uniform meshes and it converges twice as fast as the classical FEM if the solution has singularity of r^γ-type. Babuška and Suri improved in [**7**] substantially the results of [**9**] and generalized to the h-p version in two dimensions in [**8**]. A detailed analysis of the p and h-p version in one dimension was given by Gui and Babuška in [**23**]. Since then remarkable progresses for the p and h-p version in one and two dimensions were made in the 1980s and 1990s, see e.g. [**7, 8, 2, 19, 29, 30, 31, 41, 42, 44, 45**], and the p and h-p version were implemented in commercial codes and used in practical engineering computation. Despite these progresses, people had struggled for an appropriate mathematical framework which is able to provide a uniform error analysis for the p and h-p version of FEM in one, two and three dimensions and to lead to the optimal convergence of the FEM solutions of the p and h-p version for problems on polygonal and polyhedral domains. After two-decades effort people realize very recently that the most appropriate mathematical framework for error analysis of the p and h-p version is the Jacobi-weighted Besov and Sobolev spaces. In a series of papers by Guo and his collaborators [**3, 4, 5, 6, 34**], a new analysis of the p and h-p version was given in which the approximation theory of the FEM and BEM in two dimensions in this new mathematical framework was systematically developed. It demonstrates that Jacobi-weighted Besov space is the most appropriate tool to obtain optimal upper and lower bounds when dealing with singular solutions on polygons. This framework has been generalized to the p-version and the h-p version of the BEM [**32, 33**]. Thus the approximation theory for the p and h-p version of FEM and BEM in two dimensions has been established in the framework of the Jocobi-weighted Besov and Sobolev spaces.

Although significant progresses for the p and h-p versions FEM in one and two dimensions have been made in the past three decades, the approximation theory of the p and h-p versions of FEM in three dimensions is much less developed due to the complexity of three dimensional problems, and only a few results are available, e.g. [**10, 16, 40**]. There are three fundamental issues or difficulties in the analysis of high-order FEM in three dimensions. First of all, design three types of Jacobi-weighted Besov and Sobolev spaces such that the three types of singularities in three dimensions can be characterized precisely and the Jacobi projections of the singular functions in these spaces lead to the sharpest approximation errors. Secondly, define local projection based operator which is a combination of Jacobi projection and interpolation at vertices and sides of elements remaining the best approximation properties of the Jacobi projection. At the third, establishing stable and compatible polynomial extensions of polynomials from faces of three commonly used elements in three dimensions which realize global continuity of piecewise polynomial and remain the best approximation of local Jacobi projections.

In despite of theoretical difficulties the computation and algorithms of the p and h-p FEM have made remarkable progresses in the past decades. As the computer power grows rapidly in speed and memory, many practical problems in engineering and sciences are modeled and computed in three dimensions, which were not feasible ten or twenty years ago. Because of the limitation on the capacity of computers many three dimensional problems in the real world were reduced or simplified to two dimensional models for the computation, which substantially minimized the reliability of the prediction based on the computation for important engineering, scientific, public health, and financial decision. The high accuracy of computational solutions on original three dimensional model problems significantly increase the knowledge of problems in the real world, and make it possible to validate mathematical models and to verify the computational results. Since the p and h-p versions FEM in three dimensions provide higher accuracy and reduce significantly computational cost, they have been applied to various fields of engineering and sciences such as mechanics, magnetoelectric, biology and material science [**15, 37, 39, 43, 46**], and have been implemented in new codes and enhanced in the existing codes. These codes with three dimensional p and h-p FEM capacity have become very powerful tools to solve large scale engineering and scientific problems and play an important role in computational engineering and sciences. The success of the p and h-p FEM in computation is a great challenge to mathematicians and engineers, i.e., whether the theoretical research of the the p and h-p FEM in three dimensions can provide a solid mathematical foundation and guidance for practical computations, e.g. verification of three dimensional FEM codes (commercial and research) and verification of the numerical results. This challenge has motivated researchers in recent years to establish new mathematical framework for developing new approximation theory of high order FEM in three dimensions, and also provides significant motivation of the thesis, which is a part of the effort to establish a comprehensive understanding of the fundamental issues we are facing now.

In this thesis, we shall develop the approximation theory of the h-p version of FEM with quasi-uniform meshes in three dimensions in the framework of the Jacobi-weighted Besov and Sobolev spaces. The h-p version with quasiuniform meshes is, from methodology and approximation theory, the p-version on scaled meshes. The approach of the p-version gives the p-dependence in the approximation errors, and a proper scaling argument will reveal fully the information of the h-dependence. Hence, the analysis for the best approximation of the h-p version with quasiuniform meshes is not feasible unless the optimal convergence of the p-version in three dimensions is established. Fortunately, a comprehensive analysis of the p-version in the framework of the Jacobi-weighted Besov and Sobolev spaces in three dimensions recently appears in a series of papers [**24, 25**] by Guo, we are now ready to pursue the best error estimation for the h-p version in three dimensions. Here we incorporate the mesh dependence into the analysis for the p-version, and provide optimal estimates for quasi-uniform meshes and quasi-uniform polynomial degrees.

We generalize the Jacobi-weighted Besov and Sobolev spaces on scaled cube $Q_h = (-h, h)^3$, and analyze the properties of Jacobi projection on Q_h. The errors in Jacobi projections with three different Jacobi weights for singular functions with vertex, edge and vertex-edge singularities are investigated in terms of h and p (polynomial degree), which are rigorously proved to be the sharpest.

Next we construct explicitly polynomial extensions on standard elements: cubes, triangular prisms and pyramids which are proved rigorously to be stable and compatible with FEM subspaces on tetrahedrons, cubes, triangular prisms and pyramids. The extensions from a triangular face to a prism and from a square face to a pyramid are of convolution type which realize continuous mappings: $H^{1/2}_{00}(T)($ or $H^{1/2}_{00}(S)) \rightarrow H^1(\Omega_{st})$ where Ω_{st} denotes one of these standard elements and T and S are triangular and square faces. The extension on a cube is constructed by using spectral solutions of the eigenvalue problem of Poisson equation on a square face S and two-point value problem on an interval I. The extension from a square face to prism is quite different from those in other cases, the norm of the extension depends on p , but it is compatible with local quasi-projection operator on prismatic elements and cause no loss of the rate of convergence of the finite element solution of the p and h-p version.

The local quasi projection is based the Jacobi projection and associates with linear or trilinear interpolation at vertices of elements and with the $H^{1/2}$ projection on each side of elements. These projections remain the sharp estimation of Jacobi projection and make the difference of quasi projection on a common face of a pair of elements belong to $H_{00}^{1/2}(T)$(or $H_{00}^{1/2}(S)$), which make it possible to apply the polynomial extensions for continuity across the interfaces of elements.

By utilizing the polynomial extensions and local quasi projections on tetrahedrons, cubes, triangular prisms we proved the best convergence of the h-p FEM for problems with smooth solutions. For the singular solutions for problems on polyhedral domains, we use Jacobi-weighted Besov and Sobolev spaces to characterize the various singularities and derives their best approximabilities. Combining the approximation results for smooth functions and singular functions, we obtain the convergence rate of the h-p version of the finite element method with quasi-uniform meshes for elliptic problems on polyhedral domains, where the singularities of three different types occur and substantially govern the convergence of the finite element solutions.

The rest of this thesis is organized as follows: In Chapter 2, we first review the properties of Jacobi polynomial, then we quote important properties of Jacobi projections in three dimensions, which have been established and will be used in coming chapters. In Chapter 3, the Jacobi-weighted Sobolev spaces $H^{k,\beta}(Q_h)$ and Besov spaces $B_{\nu}^{s,\beta}(Q_h)$ on a scaled cube $Q_h = (-h, h)^3$ and the errors in Jacobi projections with three different Jacobi weights for singular functions with vertex, edge and vertex-edge singularities are given in terms of h and p (polynomial degree), which are rigorously proved to be the sharpest. In Chapter 4, we design polynomial extension on cubes by using spectral solutions of the eigenvalue problem of Poisson equation on a square face S and two-point value problem on an interval I. The extensions from a triangular face to a prism and from a square face to a pyramid are constructed by convolutions. The extension from a square face to prism is of neither convolution type nor spectral solutions, the norm of the extension operator depends on p . In Chapter 5, we introduce quasi projections on tetrahedrons, hexahedrons and triangular prisms. Then we combine these quasi projections and polynomial extensions to derive the convergence of the finite element solution of h-p version. Utilizing the sharp error estimation for singular solutions in the Jacobi-weighted Besov and Sobolev spaces we prove the sharpest rate of convergence of the h-p FEM for elliptic problems on polyhedral domains. The numerical results of model Poisson equation on polyhedral domains and three dimensional elasticity problems on polyhedral domains are presented in Chapter 6. In the last chapter we summarize the major results in the thesis and make concluding comments on open problems we will continue to pursue.

CHAPTER 2

Preliminary

2.1. Jacobi Polynomial

The Jacobi polynomial of degree $n = 0, 1, 2, \dots$ is defined as

$$J_n^{\alpha,\beta}(x) = \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!}\, \frac{d^n\, (1-x)^{\alpha+n}(1+x)^{\beta+n}}{dx^n} \tag{2.1}$$

with $\alpha, \beta > -1$. These polynomials possess important properties, see e.g. [**1, 21**], which are essential to the approximation of the high-order finite element method as the special method.

(J1)

$$J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \quad J_n^{\alpha,\beta}(-1) = \frac{(-1)^n\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)}.$$

(J2)

$$J_n^{\alpha,\beta}(-x) = (-1)^n J_n^{\beta,\alpha}(x).$$

(J3)

$$\frac{d}{dx} J_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1}(x),$$

and for $k \geq 0$,

$$J_{n,k}^{\alpha,\beta}(x) = \frac{d^k}{dx^k} J_n^{\alpha,\beta}(x) = 2^{-k} \frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)} J_{n-k}^{\alpha+k,\beta+k}(x).$$

(J4) $J_n^{\alpha,\beta}(x)$ are orthogonal with Jacobi weight $w_{\alpha,\beta}(x)$

$$\int_I J_m^{\alpha,\beta}(x)\, J_n^{\alpha,\beta}(x)\, w_{\alpha,\beta}(x)\, dx = \begin{cases} \gamma_n^{\alpha,\beta} & m = n \\ 0 & m \neq n \end{cases}, \quad I = (-1,1)$$

with

$$w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$$

and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{2.2}$$

By the Stirling formula [**18**]

$$\Gamma(s+1) = \sqrt{2\pi s}s^s e^{-s}\big(1 + O(s^{-1/5})\big)$$

we have asymptotic estimation

$$\gamma_n^{\alpha,\beta} \simeq \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}. \tag{2.3}$$

(J5) $J_{n,k}^{\alpha,\beta}(x)$ are orthogonal with Jacobi-weight $w_{\alpha+k,\beta+k}(x)$,

$$\int_I J_{m,k}^{\alpha,\beta}(x)\cdot J_{n,k}^{\alpha,\beta}(x)\, w_{\alpha+k,\beta+k}(x)\, dx = \begin{cases} \gamma_{n,k}^{\alpha,\beta} & m = n \geq k \\ 0 & \texttt{otherwise} \end{cases}$$

with

$$w_{\alpha+k,\beta+k}(x) = (1-x)^{\alpha+k}(1+x)^{\beta+k}$$

and

$$\gamma_{n,k}^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+\beta+k+1)\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n+1-k)\,\Gamma^2(n+\alpha+\beta+1)}. \tag{2.4}$$

By the Stirling formula, there holds asymptotically

$$\gamma_{n,k}^{\alpha,\beta} \simeq \frac{2^{\alpha+\beta+1}n^{2k}}{(2n+\alpha+\beta+1)}. \tag{2.5}$$

(J6) $J_n^{\alpha,\beta}(x)$ is the solution of the equation

$$-\widetilde{J_{\alpha,\beta}}v(x) + n\,(n+\alpha+\beta+1)v(x) = 0, \quad x \in (-1,1)$$

where $\widetilde{J_{\alpha,\beta}}$ is the differential operator

$$\widetilde{J_{\alpha,\beta}} = -(1-x)^{\alpha}(1+x)^{-\beta}\frac{d}{dx}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}.$$

Then $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$ and Jacobi polynomials $J_n^{\alpha,\beta}(x)$, $n = 0, 1, 2, \ldots$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\widetilde{J_{\alpha,\beta}}\,v = \lambda v,\ x \in (-1,1).$$

(J7) For $x \in [-1,1]$, there holds

$$|J_n^{\alpha,\beta}(x)| \leq C(n+1)^{\max\{\alpha,\beta,-1/2\}} \tag{2.6}$$

with C independent of α, β, and for $x = \pm 1$, we have more precise estimation

$$|J_n^{\alpha,\beta}(1)| \leq C(\alpha)(n+1)^{\alpha},\ |J_n^{\alpha,\beta}(-1)| \leq C(\beta)(n+1)^{\beta} \tag{2.7}$$

with $C(\alpha) = \frac{C_0}{\Gamma(1+\alpha)}$ and $C(\beta) = \frac{C_0}{\Gamma(1+\beta)}$.

2.2. Jacobi-weighted Sobolev and Besov spaces on $Q = (-1, 1)^3$

2.2.1. Jacobi-weighted Sobolev space $H^{k,\beta}(Q)$ with integer $k \geq 0$. Let $I = (-1,1), \Omega = (-1,1)^2$ and $Q = (-1,1)^3$ be a cube and $\Gamma_i, i = 1, 2, \cdots, 6$ be faces of Q and we denote $\gamma_{ij} = \Gamma_i \cap \Gamma_j, i = 1, 2, \cdots, 6$, and by Γ_2 and Γ_5 we denote the left face ($x_2 = -1$) and the right face ($x_2 = 1$), by Γ_6 and Γ_3 the front face ($x_1 = 1$) and rear face ($x_1 = -1$), by Γ_1 and Γ_4 the bottom face ($x_3 = -1$) and the top face ($x_3 = 1$), respectively. Let

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{3}(1 + x_i)^{\alpha_i+\beta_i}(1 - x_i)^{\alpha_i+\beta_{i+3}}$$

be a weight function on $Q = (-1,1)^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \geq 0$ integer and $\beta = (\beta_i, \beta_{i+3}, 1 \leq i \leq 3)$, $\beta_i, \beta_{i+3} > -1$ real number, which is referred to as Jacobi weight.

The Jacobi-weighted Sobolev space $H^{k,\beta}(Q), k \geq 0$ is defined as a closure of C^∞ functions furnished with the norm

$$\|u\|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=0}^{k} \int_Q |D^\alpha u(x)|^2 w_{\alpha,\beta}(x) dx$$

where $D^\alpha u = u_{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}}, \alpha = (\alpha_1, \alpha_2, \alpha_3), |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\beta = (\beta_1, \cdots, \beta_6)$, and $|u|_{H^{k,\beta}(Q)}$ is the semi norm involving only the k-th derivatives, i.e

$$|u|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k} \int_Q |D^\alpha u|^2 w_{\alpha,\beta}(x) dx.$$

We shall write $L^2_\beta(Q)$ for $H^{0,\beta}(Q)$. $H^{k,\beta}(Q)$ is an inner product space with

$$(u, v)_{H^{k,\beta}(Q)} = \sum_{0 \leq |\alpha| \leq k} \int_Q D^\alpha u \cdot D^\alpha v \, w_{\alpha,\beta} dx.$$

For any function $u \in H^{k,\beta}(Q)$, $k \geq 0$, there is a Jacobi-Fourier expansion

$$u = \sum_{i,j,l=0}^{\infty} c_{i,j,l} J_i^{\beta_4,\beta_1}(x_1) J_j^{\beta_5,\beta_2}(x_2) J_l^{\beta_6,\beta_3}(x_3)$$

where $J_n^{\beta_{m+3},\beta_m}(x_m), n = i, j, l;\ m = 1, 2, 3$ are Jacobi polynomials of degree n with the weights β_{m+3}, β_m in x_m which are defined in (2.1), and

$$c_{i,j,l} = \frac{1}{\gamma_i^{\beta_4,\beta_1} \gamma_j^{\beta_5,\beta_2} \gamma_l^{\beta_6,\beta_3}} \int_Q u(x)\, J_i^{\beta_4,\beta_1}(x_1) J_j^{\beta_5,\beta_2}(x_2) J_l^{\beta_6,\beta_3}(x_3)\, w_{0,\beta}(x) dx$$

with $\gamma_n^{\beta_{m+3},\beta_m}$ given in (2.2).

Due to the orthogonality of the Jacobi polynomials and their derivatives, we have

$$\|u\|^2_{L^2_\beta}(Q) = \sum_{i,j,l=0}^{\infty} |c_{i,j,l}|^2 \gamma_i^{\beta_4,\beta_1}\gamma_j^{\beta_5,\beta_2}\gamma_l^{\beta_6,\beta_3} \tag{2.8}$$

and

$$|u|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k}\ \sum_{i\ge\alpha_1, j\ge\alpha_2, l\ge\alpha_3} |c_{i,j,l}|^2 \gamma_{i,\alpha_1}^{\beta_4,\beta_1}\gamma_{j,\alpha_2}^{\beta_5,\beta_2}\gamma_{l,\alpha_3}^{\beta_6,\beta_3}. \tag{2.9}$$

with $\gamma_{n,\alpha_m}^{\beta_{m+3},\beta_m}(n = i, j, l$ and $m = 1, 2, 3)$ given in (2.4).

Using the asymptotic of $\gamma_{n,\alpha_m}^{\beta_{m+3},\beta_m}$ given in (2.5), we introduce an equivalent semi-norm and norm for $H^{k,\beta}(Q)$,

$$\begin{aligned} |u|^2_{H^{k,\beta}(Q)} &\cong \sum_{|\alpha|=k}\ \sum_{i\ge\alpha_1, j\ge\alpha_2, l\ge\alpha_3} |c_{i,j,l}|^2 \gamma_i^{\beta_4,\beta_1} i^{2\alpha_1}\gamma_j^{\beta_5,\beta_2} j^{2\alpha_2}\gamma_l^{\beta_6,\beta_3} l^{2\alpha_3} \\ &\cong \sum_{i+j+l\ge k} |c_{i,j,l}|^2 \gamma_i^{\beta_4,\beta_1}\gamma_j^{\beta_5,\beta_2}\gamma_l^{\beta_6,\beta_3}\Big(i^2+j^2+l^2\Big)^k = \lceil u\rfloor^2_{H^{k,\beta}(Q)} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \|u\|^2_{H^{k,\beta}(Q)} &\cong \sum_{0\le m\le k}\ \sum_{i+j+k\ge m} |c_{i,j,k}|^2 \gamma_i^{\beta_4,\beta_1}\gamma_j^{\beta_5,\beta_2}\gamma_l^{\beta_6,\beta_3}\Big(i^2+j^2+l^2\Big)^m \\ &\cong \sum_{i,j,l=0}^{\infty} |c_{i,j,l}|^2 \gamma_i^{\beta_4,\beta_1}\gamma_j^{\beta_5,\beta_2}\gamma_l^{\beta_6,\beta_3}\Big(1+i^2+j^2+l^2\Big)^k = |||u|||^2_{H^{k,\beta}(Q)}. \end{aligned} \tag{2.11}$$

It is worth indicating that the equivalent constant of the equivalent norms and semi norms of $H^{k,\beta}(Q)$ depends on k.

To define the projections in the Jacobi-weighted Sobolev spaces we need to introduce polynomial subspaces. By $\mathcal{P}_p^1(Q)$ and $\mathcal{P}_p^2(Q)$ we denote the polynomials on Q with a sum of of degree in all variables $\le p$ (total degree) and with degree $\le p$ in each variable (separate degree), respectively. For $1 < \kappa < 2$, $\mathcal{P}_p^\kappa(Q)$ is a polynomial space such that $\mathcal{P}_p^1(Q) \subset \mathcal{P}_p^\kappa(Q) \subset \mathcal{P}_p^2(Q)$. By $\Pi_{p,\kappa}^\beta$ we denote the Jacobi projection on $\mathcal{P}_p^\kappa(Q)$ in $L^2_\beta(Q)$

$$\Pi_{p,\kappa}^\beta u = \sum_{(i,j,l)\in\mathcal{N}_\kappa^p} c_{i,j,l} J_i^{\beta_4,\beta_1}(x_1) J_j^{\beta_5,\beta_2}(x_2) J_l^{\beta_6,\beta_3}(x_3), \tag{2.12}$$

where

$$\mathcal{N}_1^p = \{(i,j,l),\, i+j+l\le p\},\ \ \mathcal{N}_2^p = \{(i,j,l),\, i,j,l\le p\}, \tag{2.13}$$

and $\mathcal{N}_1^p \subset \mathcal{N}_\kappa^p \subset \mathcal{N}_2^p$ for $1 < \kappa < 2$, for example,

$$\mathcal{N}_{1.5}^p = \{(i,j,l), i+j\le p,\ l\le p\}. \tag{2.14}$$

$u_p \in \mathcal{P}_p^\kappa(Q)$ is the projection of u on $\mathcal{P}_p^\kappa(Q)$ in $H^{k,\beta}(Q)$ if

$$(u-u_p, v)_{H^{k,\beta}(Q)} = \sum_{|\alpha|\le k} \int_Q D^\alpha(u-u_p)\, D^\alpha v\, w_{\alpha,\beta} dx = 0, \quad \forall v \in \mathcal{P}_p^\kappa(Q).$$

It has been proved that Jacobi projection in $L^2_\beta(Q)$ is the Jacobi projection in $H^{\ell,\beta}(Q), 0 \le \ell \le k$, for $u \in H^{k,\beta}(Q)$.

2.2.2. Jacobi-weighted Sobolev and Besov spaces $H^{k,\beta}(Q), B^{s,\beta}(Q)$ and $B^{s,\beta}_\nu(Q)$.

Let $\mathcal{B}^{s,\beta}_{2,q}(Q)$ be the interpolation spaces defined by the K-method

$$\Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,q}$$

where $0<\theta<1, 1\le q\le\infty, s=(1-\theta)\ell+\theta k$, ℓ and k are integers, $\ell<k$,

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,q}(Q)} = \Big(\int_0^\infty t^{-q\theta}\,|K(t,u)|^q \frac{dt}{t}\Big)^{1/q},\ 1\le q<\infty; \tag{2.15}$$

and

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,\infty}(Q)} = \sup_{t>0} t^{-\theta}\, K(t,u) \tag{2.16}$$

where

$$K(t,u) = \inf_{u=v+w}\Big(\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)}\Big).$$

In particular, we are interested in the cases $q=2$ and $q=\infty$. We shall write for $s\ge 0$ and $q=2$

$$H^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,2}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,2}$$

with $0<\theta<1$ and $s=(1-\theta)\ell+\theta k$. This space is called the Jacobi-weighted Sobolev space with fractional order if s is not an integer. It has been proved that $\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{m,\beta}(Q)$ if s is an integer m in two dimensions[**5**], it can be proved analogously in three dimensions.

The equivalent semi norm (2.10) and norm (2.11) for the space $H^{k,\beta}(Q)$ with integer k can be generalized to the the fraction-order Jacobi-weighted space $H^{s,\beta}(Q)$ by replacing k with s.

For $q=\infty$, we shall write

$$B^{s,\beta}(Q) = \mathcal{B}^{s,\beta}_{2,\infty}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty}$$

which is referred as the Jacobi-weighted Besov spaces.

The modified weighted Besov space $B^{s,\beta}_\nu(Q)$ with $\nu\ge 0$ is defined as an interpolation space

$$B^{s,\beta}_\nu(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu}$$

with a modified norm

$$\|u\|_{B^{s,\beta}_\nu(Q)} = \sup_{t>0} K(t,u)\,\frac{t^{-\theta}}{(1+|\log t|)^\nu}. \tag{2.17}$$

Remark 2.1. Since the Jacobi-weighted Besov space $B^{s,\beta}(Q)$ and Sobolev space $H^{s,\beta}(Q)$ are defined by the standard K-method, they are exact of θ-exponent and the reiteration theorem hold for $B^{s,\beta}(Q)$ and $H^{s,\beta}(Q)$ according to [**11**].

By the definition of the exactness of θ-exponent in [**11**], for any operator $T : A_i \to B_i, i = 0,1$ furnished with an operator norm

$$\|T\|_i = \|T\|_{A_i\to B_i}.$$

T is an operator $\bar{A}_\theta \to \bar{B}_\theta$, where $\bar{A}_\theta = (A_0,A_1)_\theta$ and $\bar{B}_\theta = (B_0,B_1)_\theta$ are two interpolation spaces which are exact of θ-exponent, e.g., defined by the K-method, and

$$\|T\|_{\bar{A}_\theta\to\bar{B}_\theta} \le \|T\|_0^{1-\theta}\|T\|_1^\theta. \tag{2.18}$$

This provides us a very powerful and important tool while we generalize the approximation results in integer-order Sobolev spaces to fraction-order Sobolev spaces and Besov spaces, with or without weights.

The reiteration theorem (see, e.g. [**11**]) tells that if $X_i = (A_0,A_1)_{\theta_i}$ with $\theta_i \in (0,1), i = 0,1$ are θ-exact, then for $\eta \in (0,1)$ and $\theta = (1-\eta)\theta_0 + \eta\theta_1$

$$(X_0,X_1)_\eta = (A_0,A_1)_\theta. \tag{2.19}$$

This theorem implies that the Jacobi-weighted Besov space $B^{s,\beta}(Q)$ and Sobolev space $H^{s,\beta}(Q)$ are well defined, which do not depend on the individual value of ℓ and k, but the combination $s = (1-\theta)\ell + \theta k$, and that ℓ and k can be non-integers.

Remark 2.2. Unfortunately the modified Jacobi-weighted Besov space $B^{s,\beta}_\nu(Q)$ with $\nu > 0$ is defined by a modified K-method and is not exact of θ-exponent. Therefore, (2.18) and (2.19) do not hold in general. In [**3**], a weaker exactness, which is called quasi exact of θ-exponent, was proved that

$$\|T\|_{\bar{A}_{\theta,\nu}\to\bar{B}_{\theta,\nu}} \le \Big(1+\log\frac{\|T\|_1}{\|T\|_0}\Big)^\nu \|T\|_0^{1-\theta}\|T\|_1^\theta. \tag{2.20}$$

Also, it was proved in [**6**] that the reiteration theorem holds only for a special case, which is called the partial reiteration theorem,

$$(X_0,X_1)_{\eta,\nu} = (A_0,A_1)_{\theta,\nu} \tag{2.21}$$

if $X_i = (A_0,A_1)_{\theta_i}, i = 0,1$, or

$$(X_0,X_1)_\eta = (A_0,A_1)_{\theta,\nu} \tag{2.22}$$

if $X_i = (A_0, A_1)_{\theta_i,\nu}, i = 0, 1$. This partial reiteration theorem guarantees that the space $B^{s,\beta}_{\nu}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu}$ with $\nu > 0$ is well-defined, and that ℓ and k can be non-integers.

The details of derivation of the partial reiteration theorems and quasi exactness of θ-exponent are included in Appendix of [**28**].

We have the following embedding inequality of the Jacobi-weighted Sobolev spaces. See [**25**].

Theorem 2.1. *If $u \in H^{s,\beta}(Q)$ with $s > 3/2 + \sum_{1\le\ell\le3} \bar\beta_{\ell,\ell+3}$, then $u \in C^0(\bar Q)$, and*

$$\|u\|_{C^0(\bar Q)} \le C\|u\|_{H^{s,\beta}(Q)}. \tag{2.23}$$

Hereafter $\bar\beta_{\ell,\ell+3} = \max\{\beta_\ell + 1/2, \beta_{\ell+3} + 1/2, 0\}$ for $1 \le \ell \le 3$, where the index $\ell + 3$ is modulo by 6, i.e. $\ell + 3 = \ell - 3$ if $\ell + 3 > 6$. In particular, $H^{s,\beta}(Q) \hookrightarrow C^0(\bar Q)$ if $\beta_\ell \le -1/2, 1 \le \ell \le 6$ and $s > \frac{3}{2}$.

2.3. Approximation Properties of Jacobi Projections

We quote important properties of Jacobi projections in three dimensions, which have been established and will be used in coming chapters. We will not elaborate the details of the proof, instead refer to [**28**].

Theorem 2.2. *Let $u \in H^{k,\beta}(Q), k > 0$, and let $\Pi^{\beta}_{p,\kappa}u$ be the Jacobi projection of u on $\mathcal{P}^{\kappa}_{p}(Q), 1 \le \kappa \le 2$ with $p \ge 0$. Then there holds for any integer $l, 0 \le l \le k$*

$$\|u - \Pi^{\beta}_{p,\kappa}u\|_{H^{l,\beta}(Q)} \le C(p+1)^{-(k-l)}\|u\|_{H^{k,\beta}(Q)}. \tag{2.24}$$

Furthermore, if $u \in H^{k,\beta}(Q)$ with $k > 3/2 + \sum_{1\le l\le3} \bar\beta_{l,l+3}$, then

$$\|u - \Pi^{\beta}_{p,\kappa}u\|_{C^0(\bar Q)} \le C(p+1)^{-(k-3/2-\sum_{1\le l\le 3}\bar\beta_{l,l+3})}\|u\|_{H^{k,\beta}(Q)}, \tag{2.25}$$

and on the faces $\Gamma_i, 1 \le i \le 6$

$$\|u - \Pi^{\beta}_{p,\kappa}u\|_{C^0(\bar\Gamma_i)} \le C(p+1)^{-(k-2-\beta_i-\bar\beta_{i+3}-\bar\beta_{i+1,i+4}-\bar\beta_{i+2,i+5})}\|u\|_{H^{k,\beta}(Q)}, \tag{2.26}$$

and on the edges $\Lambda_{ij} = \bar\Gamma_i \cap \bar\Gamma_j, 1 \le i, j \le 6$

$$\|u - \Pi^{\beta}_{p,\kappa}u\|_{C^0(\bar\Lambda_{ij})} \le C(p+1)^{-(k-5/2-\beta_i-\beta_j-\bar\beta_{i+3}-\bar\beta_{j+3}-\bar\beta_{\ell,\ell+3})}\|u\|_{H^{k,\beta}(Q)} \tag{2.27}$$

with $\ell \ne i, j, i + 3, j + 3$, and at the vertices $A_m = \bar\Gamma_i \cap \bar\Gamma_j \cap \bar\Gamma_l, 1 \le i, j, l \le 6$

$$|(u - \Pi^{\beta}_{p,\kappa}u)(A_m)| \le C(p+1)^{-(k-3-\beta_i-\beta_j-\beta_l-\bar\beta_{i+3}-\bar\beta_{j+3}-\bar\beta_{l+3})}|u|_{H^{k,\beta}(Q)}. \tag{2.28}$$

Hereafter $\bar\beta_\ell = \max\{\beta_\ell + \frac{1}{2}, 0\}$. The indices ℓ and m are modulo 6, i.e. ℓ means $\ell - 6$ if $\ell > 6$.

For $p \geq k-1$, there holds

$$|u - \Pi^{\beta}_{p,\kappa} u|_{H^{l,\beta}(Q)} \leq C(p+1)^{-(k-l)} |u|_{H^{k,\beta}(Q)}, \tag{2.29}$$

and in addition if $k > 3/2 + \sum_{1\leq \ell \leq 3} \bar{\beta}_{\ell,\ell+3}$, then

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{Q})} \leq C(p+1)^{-(k-3/2-\sum_{1\leq\ell\leq3}\bar{\beta}_{\ell,\ell+3})} |u|_{H^{k,\beta}(Q)}, \tag{2.30}$$

and on the faces $\Gamma_i, 1 \leq i \leq 6$,

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Gamma}_i)} \leq C(p+1)^{-(k-2-\beta_i-\bar{\beta}_{i+3}-\bar{\beta}_{i+1,i+4}-\bar{\beta}_{i+2,i+5})} |u|_{H^{k,\beta}(Q)}, \tag{2.31}$$

and on the edges $\Lambda_{ij} = \bar{\Gamma}_i \cap \bar{\Gamma}_j, 1 \leq i,j \leq 6$,

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Lambda}_{ij})} \leq C(p+1)^{-(s-5/2-\beta_i-\beta_j-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{\ell,\ell+3})} |u|_{H^{k,\beta}(Q)} \tag{2.32}$$

with $\ell \neq i, j$ and $i+3, j+3$, and at the vertices $A_m = \bar{\Gamma}_i \cap \bar{\Gamma}_j \cap \bar{\Gamma}_l, 1 \leq i,j,l \leq 6$,

$$|(u - \Pi^{\beta}_{p,\kappa} u)(A_m)| \leq C(p+1)^{-(k-5/2-\beta_i-\beta_j-\beta_l-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{l+3})} |u|_{H^{k,\beta}(Q)}. \tag{2.33}$$

Theorem 2.3. *Let $u \in H^{s,\beta}(Q), s > 0$, and let $\Pi^{\beta}_{p,\kappa} u$ be the Jacobi projection of u on $\mathcal{P}^{\kappa}_p(Q)$ with $p \geq 0, 1 \leq \kappa \leq 2$. Then for any integer $l \in [0, s)$ there holds*

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{H^{l,\beta}(Q)} \leq C(p+1)^{-(s-l)} \|u\|_{H^{s,\beta}(Q)}. \tag{2.34}$$

Furthermore, if $u \in H^{s,\beta}(Q)$ with $s > 3/2 + \sum_{1\leq l\leq 3} \bar{\beta}_{l,l+3}$, then

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{Q})} \leq C(p+1)^{-(s-3/2-\sum_{1\leq l\leq 3}\bar{\beta}_{l,l+3})} \|u\|_{H^{s,\beta}(Q)}. \tag{2.35}$$

and on the faces $\Gamma_i, 1 \leq i \leq 6$

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Gamma}_i)} \leq C(p+1)^{-(s-2-\beta_i-\bar{\beta}_{i+3}-\bar{\beta}_{i+1,j+4}-\bar{\beta}_{i+2,i+5})} \|u\|_{H^{s,\beta}(Q)}, \tag{2.36}$$

and on the edges $\Lambda_{ij} = \bar{\Gamma}_i \cap \bar{\Gamma}_j, 1 \leq i,j \leq 6$

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Lambda}_{ij})} \leq C(p+1)^{-(s-5/2-\beta_i-\beta_j-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{\ell,\ell+3}} \|u\|_{H^{s,\beta}(Q)} \tag{2.37}$$

with $\ell \neq i, j, i+3, j+3$, and at the vertices $A_m = \bar{\Gamma}_i \cap \bar{\Gamma}_j \cap \bar{\Gamma}_l, 1 \leq i,j,l \leq 6$

$$|(u - \Pi^{\beta}_{p,\kappa} u)(A_m)| \leq C(p+1)^{-(s-3-\beta_i-\beta_j-\beta_l-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{l+3})} |u|_{H^{s,\beta}(Q)}. \tag{2.38}$$

Theorem 2.4. *Let $u \in B^{s,\beta}_{\nu}(Q), s > 0, \nu \geq 0$, and let $\Pi^{\beta}_{p,\kappa} u$ be the Jacobi projection of u on $\mathcal{P}^{\kappa}_p(Q)$ with $p > 0, 1 \leq \kappa \leq 2$. Then for any integer $l \in [0, s)$, there holds*

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{H^{l,\beta}(Q)} \leq C(p+1)^{-(s-l)} (1 + \log(p+1))^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)}. \tag{2.39}$$

Furthermore, if $u \in B^{s,\beta}_{\nu}(Q)$ with $s > 3/2 + \sum_{1\leq\ell\leq3} \bar{\beta}_{\ell,\ell+3}$, then

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{Q})} \leq C(p+1)^{-(s-3/2-\sum_{1\leq\ell\leq3}\bar{\beta}_{\ell,\ell+3})} (1 + \log(p+1))^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)}, \tag{2.40}$$

and on the faces $\Gamma_i, 1 \leq i \leq 6$,

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Gamma}_i)} \leq C(p+1)^{-(s-2-\beta_i)} (1 + \log(p+1))^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)}, \tag{2.41}$$

and at the edges $\Lambda_{ij} = \bar{\Gamma}_i \cap \bar{\Gamma}_j, 1 \leq i, j \leq 6,$

$$\|u - \Pi^{\beta}_{p,\kappa} u\|_{C^0(\bar{\Lambda}_{ij})} \leq C \frac{(1+\log(p+1))^{\nu}}{(p+1)^{s-5/2-\beta_i-\beta_j-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{\ell,\ell+3}}} \|u\|_{B^{s,\beta}_{\nu}(Q)} \tag{2.42}$$

with $\ell \neq i, j, i+3, j+3$*, and at the vertices* $A_m = \bar{\Gamma}_i \cap \bar{\Gamma}_j \cap \bar{\Gamma}_l, 1 \leq i, j, l \leq 6$

$$|(u - \Pi^{\beta}_{p,\kappa} u)(A_m)| \leq C \frac{(1+\log(p+1))^{\nu}}{(p+1)^{s-3-\beta_i-\beta_j-\beta_l-\bar{\beta}_{i+3}-\bar{\beta}_{j+3}-\bar{\beta}_{l+3}}} \|u\|_{B^{s,\beta}_{\nu}(Q)}. \tag{2.43}$$

CHAPTER 3

Approximation Theory in Jacobi-weighted Spaces on a Scaled Cube $Q_h = (-h, h)^3$

3.1. Jacobi-weighted Sobolev and Besov spaces on $Q_h = (-h, h)^3$

For analyzing the approximation properties for smooth and singular functions on a scaled domain we first introduce the Jacobi-weighted Sobolev spaces $H^{k,\beta}(Q_h)$ and Besov spaces $B^{s,\beta}_{\nu}(Q_h)$ on a scaled cube $Q_h = (-h, h)^3$.

Let $w^h_{\alpha,\beta}(x)$ be a weighted function on $Q_h = (-h, h)^3$,

$$w^h_{\alpha,\beta}(x) = \prod_{i=1}^{3} \left(\frac{h + x_i}{h}\right)^{\alpha_i+\beta_i} \left(\frac{h - x_i}{h}\right)^{\alpha_i+\beta_{i+3}} = \prod_{i=1}^{3} \left(1 + \frac{x_i}{h}\right)^{\alpha_i+\beta_i} \left(1 - \frac{x_i}{h}\right)^{\alpha_i+\beta_{i+3}}$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \geq 0$ integer, and $\beta = (\beta_1, \beta_2, \beta_3), \beta_i > -1, i = 1, 2, 3,$ real.

The Jacobi-weighted Sobolev space $H^{k,\beta}(Q_h)$, $k \geq 0$, is the closure of C^{∞} functions furnished with the norm

$$\|u\|^2_{H^{k,\beta}(Q_h)} = \sum_{0 \leq |\alpha| \leq k} \int_{Q_h} |D^{\alpha} u(x)|^2 w^h_{\alpha,\beta}(x) dx$$

and $|u|_{H^{k,\beta}(Q_h)}$ denotes the semi norm involving only the k-th derivatives.

The Jacobi-weighted Sobolev spaces $H^{s,\beta}(Q_h)$ and Besov spaces $B^{s,\beta}(Q_h)$ can be introduced as usual interpolation spaces by the K-method,

$$H^{s,\beta}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,2}, \quad B^{s,\beta}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,\infty},$$

where $0 < \theta < 1, s = (1 - \theta)l + \theta k, l$ and k are integers, $l < k$, furnished with norms

$$\|u\|_{H^{s,\beta}(Q_h)} = \Big(\int_0^{\infty} t^{-2\theta} |K(t, u)|^2 \frac{dt}{t}\Big)^{1/2}, \quad \|u\|_{B^{s,\beta}(Q_h)} = \sup_{t>0} t^{-\theta} K(t, u) \tag{3.1}$$

with

$$K(t, u) = \inf_{u=v+w} \Big(\|v\|_{H^{l,\beta}(Q_h)} + t\|w\|_{H^{k,\beta}(Q_h)}\Big).$$

The space $H^{s,\beta}(Q_h)$ is called the Jacobi-weighted Sobolev space with fractional order if s is not an integer, and the space $B^{s,\beta}(Q_h)$ is referred as to be the Jacobi-weighted Besov space.

The modified weighted Besov space $B^{s,\beta}_\nu(Q_h)$ with $\nu \geq 0$ is an interpolation space defined by the modified K-method,

$$B^{s,\beta}_\nu(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,\infty,\nu}$$

with a modified norm

$$\|u\|_{B^{s,\beta}_\nu(Q_h)} = \sup_{t>0} K(t,u)\frac{t^{-\theta}}{(1+|\log t|)^\nu}. \tag{3.2}$$

Remark 3.1. The spaces $H^{s,\beta}(Q_h)$ and $B^{s,\beta}(Q_h) = B^{s,\beta}_0(Q_h)$ are exact of θ-exponent, but $B^{s,\beta}_\nu(Q_h)$ with $\nu > 0$ is not, but weakly exact of θ-exponent. Suppose that E realizes a linear operator: $H_l \to H^{m_l,\beta}(Q_h), l = 1,2$ with norms denoted by $\|E\|_l$, where $H_l, l = 1,2$ are Banach spaces. Then E is a linear operator : $(H_1,H_2)_{\theta,q} \to (H^{m_1,\beta}(Q_h), H^{m_2,\beta}(Q_h))_{\theta,q,\nu}$ such that for $\nu = 0$

$$\|E\|_{(H_1,H_2)_{\theta,q}\to(H^{m_1,\beta}(Q_h),H^{m_2,\beta}(Q_h))_{\theta,q,0}} \leq \|E\|_1^{1-\theta}\|E\|_2^\theta \tag{3.3}$$

and for $\nu > 0$

$$\|E\|_{(H_1,H_2)_{\theta,q}\to(H^{m_1,\beta}(Q_h),H^{m_2,\beta}(Q_h))_{\theta,\infty,\nu}} \leq \|E\|_1^{1-\theta}\|E\|_2^\theta\Big(1+\log\frac{\|E\|_2}{\|E\|_1}\Big)^\nu. \tag{3.4}$$

By the definition of interpolation spaces and a simple scaling, we have the following proposition.

Proposition 3.1. *Let $u(x)$ and $U(\xi) = u \circ M_h = u(h\xi)$ be functions defined on Q_h and Q, respectively, where M_h denotes a simple scaling $x = h\xi, \xi \in Q = (-1,1)^3$.*

(i) $u \in H^{k,\beta}(Q_h)$ with integer $k \geq 0$ if $U(\xi) \in H^{k,\beta}(Q)$, vice versa. Furthermore, there holds for $l \leq k$

$$|u|_{H^{\ell,\beta}(Q_h)} = h^{\frac{3}{2}-l}|U|_{H^{l,\beta}(Q)}. \tag{3.5}$$

(ii) $u \in H^{s,\beta}(Q_h)$ with non-integer $s \geq 0$ if $U(\xi) \in H^{s,\beta}(Q)$, vice versa.

(iii) $u \in B^{s,\beta}_\nu(Q_h)$ with real $s > 0$ and integer $\nu \geq 0$ if $U(\xi) \in B^{s,\beta}_\nu(Q)$, vice versa.

3.2. Approximation in the framework of Jacobi-weighted spaces on $Q_h = (-h,h)^3$

Let $\mathcal{P}^\kappa_p(Q_h) = \mathcal{P}^\kappa_p(Q)\circ M_h$ be a set of polynomials of degree (separate) $\leq p$ on the scaled cube Q_h, and let $\prod^\beta_{p,h,\kappa}$ be the Jacobi projection operator on $\mathcal{P}^\kappa_p(Q_h), 1 \leq \kappa \leq 2$. Obviously, for $u \in H^{k,\beta}(Q_h)$ with $k \geq 0$, $u_{hp}(x) = \prod^\beta_{p,h,\kappa} u$ is the Jacobi projection of $u \in H^{k,\beta}(Q_h)$ on $\mathcal{P}^\kappa_p(Q_h)$ if and only if $U_p(\xi) = u_{hp}(h\xi)$ is the Jacobi projection of $U(\xi) = u(h\xi)$ on $\mathcal{P}^\kappa_p(Q)$.

Lemma 3.2. *Let $u \in H^{k,\beta}(Q_h), k \geq 0$, and let $U(\xi) = u \circ M_h = u(h\xi)$. Then*

$$\|U - U_p\|_{H^{k,\beta}(Q)} \leq Ch^{\mu-\frac{3}{2}}\|u\|_{H^{k,\beta}(Q_h)} \tag{3.6}$$

where $\mu = \min\{k, p+1\}$, and C is independent of p, h, k and u.

Proof. For $k = 0$, it holds by (3.5) that

$$\|U - U_p\|_{H^{0,\beta}(Q)} \leq \|U\|_{H^{0,\beta}(Q)} \leq h^{-\frac{3}{2}}\|u\|_{H^{0,\beta}(Q_h)}. \tag{3.7}$$

We now assume that the integer $k \geq 1$. Then we have by (2.29) of Theorem 2.2

$$\begin{aligned}\|U - U_p\|_{H^{k,\beta}(Q)} &\leq \|U - U_p\|_{H^{\mu,\beta}(Q)} + \sum_{m=\mu+1}^{k} (|U|_{H^{m,\beta}(Q)} + |U_p|_{H^{m,\beta}(Q)}) \\ &\leq C\Big(|U|_{H^{\mu,\beta}(Q)} + \sum_{m=\mu+1}^{k} |U|_{H^{m,\beta}(Q)}\Big).\end{aligned}$$

Here $\displaystyle\sum_{m=\mu+1}^{k} |U_p|_{H^{m,\beta}(Q)} = 0$ if $\mu + 1 < k$. By the scaling argument (3.5), we obtain

$$\|U - U_p\|_{H^{k,\beta}(Q)} \leq C\sum_{m=\mu}^{k} h^{m-\frac{3}{2}}|u|_{H^{m,\beta}(Q_h)} \leq Ch^{\mu-\frac{3}{2}}\|u\|_{H^{k,\beta}(Q_h)}.$$

□

Theorem 3.3. *Let $u \in H^{k,\beta}(Q_h)$ and u_{hp} be the Jacobi projection of u on $\mathcal{P}_p^{\kappa}(Q_h), 1 \leq \kappa \leq 2$, with $p \geq 0$. Then for $0 \leq l \leq k$,*

$$\|u - u_{hp}\|_{H^{l,\beta}(Q_h)} \leq C\frac{h^{\mu-l}}{(p+1)^{k-l}}\|u\|_{H^{k,\beta}(Q_h)} \tag{3.8}$$

with $\mu = \min\{k, p+1\}$.

Furthermore, if $k > \frac{3}{2} + \sum_{i=1}^{3}\bar{\beta}_{i,i+3}$, then

$$\|u - u_{hp}\|_{C^0(\bar{Q}_h)} \leq C\frac{h^{\mu-\frac{3}{2}}}{(p+1)^{k-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3}}}\|u\|_{H^{k,\beta}(Q_h)}. \tag{3.9}$$

The constant C is independent of p, h, k and u.

Proof. Let $\xi = \frac{x}{h}$ and $U(\xi) = u(h\xi)$. Then, due to Proposition 3.1, $U(\xi) \in H^{k,\beta}(Q)$, and $U_p = \prod_{p,\beta}^{\kappa} U_p, 1 \leq \kappa \leq 2$, satisfies

$$\begin{aligned}\|U - U_p\|_{H^{l,\beta}(Q)} &= \|U - U_p - \textstyle\prod_{p,\beta}^{\kappa}(U - U_p)\|_{H^{l,\beta}(Q)} \\ &\leq C(p+1)^{-(k-l)}\|U - U_p\|_{H^{k,\beta}(Q)} \\ &\leq C(p+1)^{-(k-l)}h^{\mu-\frac{3}{2}}\|u\|_{H^{k,\beta}(Q_h)}.\end{aligned}$$

The scaling argument (3.5) leads to

$$\begin{aligned} \|u-u_{hp}\|_{H^{l,\beta}(Q_h)} &= h^{\frac{3}{2}-l}\|(U-U_p)\|_{H^{l,\beta}(Q)} \\ &\le C(p+1)^{-(k-l)}h^{\mu-l}\|u\|_{H^{k,\beta}(Q_h)}. \end{aligned}$$

If $k > \frac{3}{2} + \sum_{i=1}^{3}\bar{\beta}_{i,i+3}$, then there holds by Theorem 2.2 and Lemma 3.2

$$\begin{aligned} \|u-u_{hp}\|_{C^0(\bar{Q}_h)} &= \|U-U_p\|_{C^0(\bar{Q})} \le \|(U-U_p-\textstyle\prod_p^{\beta}(U-U_p))\|_{C^0(\bar{Q})} \\ &\le C(p+1)^{-(k-3/2-\sum_{i=1}^{3}\bar{\beta}_{i,i+3})}\|U-U_p\|_{H^{k,\beta}(Q)} \\ &\le C(p+1)^{-(k-3/2-\sum_{i=1}^{3}\bar{\beta}_{i,i+3})}h^{\mu-\frac{3}{2}}\|u\|_{H^{k,\beta}(Q_h)}. \end{aligned}$$

□

By the argument of interpolation spaces, we have the approximation results in the spaces $H^{s,\beta}(Q_h)$ and $B^{s,\beta}_{\nu}(Q_h)$.

Theorem 3.4. *Let $u \in H^{s,\beta}(Q_h)(resp. B^{s,\beta}_{\nu}(Q_h))$ with $s>0, \nu \ge 0$, and u_{hp} be the Jacobi projection of u on $\mathcal{P}^{\kappa}_p(Q_h), 1 \le \kappa \le 2$ with $p \ge 0$. Then for $0 \le l < s$,*

(3.10)
$$\|u-u_{hp}\|_{H^{l,\beta}(Q_h)} \le C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|u\|_{H^{s,\beta}(Q_h)}\Big(resp. \frac{h^{\mu-l}}{(p+1)^{s-l}}(1+\log\frac{p+1}{h})^{\nu}\|u\|_{B^{s,\beta}_{\nu}(Q_h)}\Big)$$

with $\mu = \min\{s, p+1\}$. Furthermore, if $u \in H^{s,\beta}(Q_h)$ $(resp. B^{s,\beta}_{\nu}(Q_h), \nu \ge 0)$ with $s > \frac{3}{2} + \sum_{i=1}^{3}\bar{\beta}_{i,i+3}$, then for $x \in \bar{Q}_h$

$$\begin{aligned} |u-u_{hp}(x)| &\le C\frac{h^{\mu-\frac{3}{2}}}{(p+1)^{s-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3}}}\|u\|_{H^{s,\beta}(Q_h)} \\ &\quad \Big(resp. \frac{h^{\mu-\frac{3}{2}}}{(p+1)^{s-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3}}}(1+\log\frac{p+1}{h})^{\nu}\|u\|_{B^{s,\beta}_{\nu}(Q_h)}\Big). \end{aligned} \tag{3.11}$$

The constant C is independent of p, h and u.

Proof. We will prove the theorem for $u \in B^{s,\beta}_{\nu}(Q_h)$. Let l and k be integers such that $0 \le l \le s < k = l+1$ and $B^{s,\beta}_{\nu}(Q_h) = (H^{l,\beta}(Q_h), H^{k,\beta}(Q_h))_{\theta,\infty,\nu}$ with $\theta = \frac{s-l}{k-l} \in (0,1)$. We have by Theorem 3.3

$$\|u-u_{hp}\|_{H^{l,\beta}(Q_h)} \le Ch^{\mu_1-l}\|u\|_{H^{l,\beta}(Q_h)} \tag{3.12}$$

with $\mu_1 = \min\{p+1, l\}$, and

$$\|u-u_{hp}\|_{H^{l,\beta}(Q_h)} \le C\frac{h^{\mu_2-l}}{(p+1)^{k-l}}\|u\|_{H^{k,\beta}(Q_h)} \tag{3.13}$$

with $\mu_2 = \min\{p+1, l+1\}$. The weak exactness of θ-exponent (3.4) for the modified Jacobi-weighted Besov space $B^{s,\beta}_{\nu}(Q_h)$, together with (3.12) and (3.13) leads to

$$\begin{aligned}
\|u - u_{hp}\|_{H^{l,\beta}(Q_h)} &\leq C\frac{h^{(1-\theta)(\mu_1-l)+\theta(\mu_2-l)}}{(p+1)^{\theta(k-l)}}\Big(1+\log\frac{(p+1)^{-(k-l)}h^{\mu_2-l}}{h^{\mu_1-l}}\Big)^{\nu}\|u\|_{B^{s,\beta}_{\nu}(Q_h)}\\
&= C\frac{h^{\mu-l}}{(p+1)^{s-l}}\Big(1+(k-l)\log(p+1)+(\mu_1-\mu_2)\log\frac{1}{h}\Big)^{\nu}\|u\|_{H^{s,\beta}(Q_h)}\\
&\leq C\frac{h^{\mu-l}}{(p+1)^{s-l}}\,(1+\log\frac{p+1}{h})^{\nu}\,\|u\|_{B^{s,\beta}_{\nu}(Q_h)}.
\end{aligned}$$

Here we used the fact that $(1-\theta)\min(p+1,l)+\theta\min(p+1,l+1) = \min(p+1,s)$.

If $s > \frac{3}{2} + \sum_{i=1}^{3}\bar{\beta}_{i,i+3}$, select l and k such that $1 < l \leq s < k = l+1$. Then by Theorem 3.3 there holds for $x \in \bar{Q}_h$

$$\|u - u_{hp}\|_{C^0(\bar{Q}_h)} \leq C(p+1)^{-(l-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3})}h^{\mu_1-\frac{3}{2}}\|u\|_{H^{l,\beta}(Q_h)} \tag{3.14}$$

and

$$\|u - u_{hp}\|_{C^0(\bar{Q}_h)} \leq C(p+1)^{-(k-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3})}h^{\mu_2-\frac{3}{2}}\|u\|_{H^{k,\beta}(Q_h)}. \tag{3.15}$$

The exactness of θ-exponent (3.3) for $\nu = 0$ and the weak exactness of θ-exponent (3.4) together with (3.14) and (3.15) leads to

$$\begin{aligned}
&\|u - u_{hp}\|_{C^0(\bar{Q}_h)} \leq C\frac{h^{(1-\theta)(\mu_1-\frac{3}{2})+\theta(\mu_2-\frac{3}{2})}}{(p+1)^{\theta(k-\frac{3}{2})+(1-\theta)(l-\frac{3}{2})}}\Big(1+\log\frac{(p+1)^{-(k-\frac{3}{2})}h^{\mu_2-\frac{3}{2}}}{(p+1)^{-(l-\frac{3}{2})}h^{\mu_1-\frac{3}{2}}}\Big)^{\nu}\|u\|_{B^{s,\beta}_{\nu}(Q_h)}\\
&= C\frac{h^{\mu-\frac{3}{2}}}{(p+1)^{s-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3}}}\Big(1+(k-l)\log(p+1)+(\mu_1-\mu_2)\log\frac{1}{h}\Big)^{\nu}\|u\|_{B^{s,\beta}_{\nu}(Q_h)}\\
&\leq C\frac{h^{\mu-\frac{3}{2}}}{(p+1)^{s-\frac{3}{2}-\sum_{i=1}^{3}\bar{\beta}_{i,i+3}}}\,(1+\log\frac{p+1}{h})^{\nu}\,\|u\|_{B^{s,\beta}_{\nu}(Q_h)}.
\end{aligned}$$

Similarly, for $u \in H^{s,\beta}(Q_h)$ we can prove (3.10) and (3.11) by applying (3.3) instead of (3.4). □

3.3. Approximability of singular functions on scaled cube $Q_h = (-h,h)^3$

In this section we will investigate the approximability of singular functions on a scaled cube $Q_h = (-h,h)^3$.

3.3.0.1. *Approximability of vertex-singular functions.* Let (ρ,θ,ϕ) be the spherical coordinates with respect to the vertex $(-h,-h,-h)$ and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -h, x_3 \in (-\infty,\infty)\}$ with $\rho = \{\sum_{i=1}^{3}(x_i+h)^2\}^{1/2}$, $\theta = \arctan\frac{r}{x_3+h} = \arctan\frac{\{(x_1+h)^2+(x_2+h)^2\}^{1/2}}{x_3+h} \in (0,\pi/2)$, and $\phi = \arctan\frac{x_2+h}{x_1+h} \in (0,\pi/2)$.

We now consider the singular functions with $\gamma > 0$

$$u(x) = \rho^\gamma \log^\nu \rho\, \chi_h(\rho)\, \Phi(\theta,\phi) \tag{3.16}$$

with integer $\nu \geq 0$, where $\chi_h(\rho) = \chi(\frac{\rho}{h}), \chi(\cdot)$ and $\Phi(\theta,\phi)$ are C^∞ functions defined on the cube such that for $0 < \rho_0 < 1$

$$\chi_h(\rho) = 1 \quad \text{for } 0 < \rho < \frac{h\rho_0}{2}, \quad \chi_h(\rho) = 0 \quad \text{for } \rho > h\rho_0.$$

and

$$\Phi(\theta,\phi) = 0 \quad \text{for } (\theta,\phi) \notin S_{\kappa_0}.$$

By S_{κ_0} we denote a subset of the intersection of the sphere of radius h and Q_h such that the angles between the radial $A_1 - x$ and the x_i-axis are larger than κ_0. For $0 < \kappa_0 < \pi/4$, let

$$R_0^h = R_{\rho_0,\kappa_0}^h = \{x \in Q_h \mid 0 < \rho < h\rho_0, (\theta,\varphi) \in S_{\kappa_0}\},\ \rho_0 \in (0,1)$$

as shown in Figure 3.1.

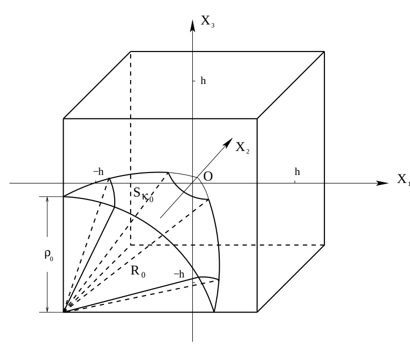


Fig. 3.1 Cubic Domain Q_h and subregion R_{ρ_0,κ_0}^h

We quote the following theorems for $h = 1$ from [**24**].

Theorem 3.5. *Let $u(x) = \rho^\gamma \log^\nu \rho\chi(\rho)\Phi(\theta,\phi)$ with $\nu \geq 0$, and let $\beta = (\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6)$ with $\beta_i > -1, 1 \leq i \leq 6$. Then $u \in H^{s-\varepsilon,\beta}(Q)$, and $u \in B_{\nu^*}^{s,\beta}(Q)$ with $s = 2\gamma + 3 + \sum_{i=1}^3 \beta_i$ and $\varepsilon > 0$ arbitrary, and*

$$\nu^* = \begin{cases} \max\{\nu - 1, 0\} & \text{if } \gamma \text{ is an integer and } \nu \geq 1, \\ \nu & \text{otherwise.} \end{cases} \tag{3.17}$$

Theorem 3.6. *For $u = \rho^\gamma \log^\nu \rho\chi(\rho)\Phi(\theta,\phi)$ with $\nu \geq 0$, there exists $\psi \in \mathcal{P}_p^\kappa(Q), 1 \leq \kappa \leq 2$ with $p \geq 0$ such that*

$$\|u - \psi\|_{L^2(Q)} \leq C(p+1)^{-(2\gamma+3)}(1 + \log(p+1))^{\nu^*}\|u\|_{B_{\nu^*}^{2\gamma+3,\beta}(Q)} \tag{3.18}$$

with $\beta_i = 0, 1 \le i \le 3, \beta_i > -1, 4 \le i \le 6$, *arbitrary. Also, there exists* $\varphi(x) \in \mathcal{P}_p^\kappa(Q), 1 \le \kappa \le 2, p > 1 + 2\gamma$ *such that*

$$\|u - \varphi\|_{H^1(R_0)} \le C(p+1)^{-(2\gamma+1)}(1+\log(p+1))^{\nu^*}\|u\|_{B^{2\gamma+2,\beta}_{\nu^*}(Q)} \tag{3.19}$$

and

$$\|u - \varphi\|_{C^0(\bar{Q})} \le C(p+1)^{-(2\gamma+\frac{1}{2})}(1+\log(p+1))^{\nu^*}\|u\|_{B^{2\gamma+2,\beta}_{\nu^*}(Q)} \tag{3.20}$$

with $\beta_i = -1/3, 1 \le i \le 6$. *In (3.18)-(3.20)* ν^* *is given in (3.17).*

If $u = 0$ *on the plane* $\pi_\ell : \sum_{i=1}^3 a_i^{[\ell]}(x_i+1) = 0, 1 \le \ell \le s, s = 1,$ *or* 2, *or* 3, *then there exist* $\psi \in \mathcal{P}_p^\kappa(Q)$ *and* $\varphi \in \mathcal{P}_p^\kappa(Q), 1 \le \kappa \le 2, p \ge s$ *such that* $\psi = 0$ *and* $\varphi = 0$ *on* $\pi_\ell, 1 \le \ell \le s$, *and*

$$\|u - \psi\|_{L^2(Q)} \le C(p+1)^{-(2\gamma+3)}(1+\log(p+1))^{\nu^*}\|u_s\|_{B^{2\gamma+3,\beta^{[s]}}_{\nu^*}(Q)} \tag{3.21}$$

with $\beta_\ell^{[s]} = s, 1 \le \ell \le 3, \beta_\ell^{[s]} > -1, 4 \le \ell \le 6$, *and*

$$\|u - \varphi\|_{H^1(R_0)} \le C(p+1)^{-(2\gamma+1)}(1+\log(p+1))^{\nu^*}\|u_s\|_{B^{2\gamma+2,\beta^{[s]}}_{\nu^*}(Q)} \tag{3.22}$$

$$\|u - \varphi\|_{C^0(\bar{Q})} \le C(p+1)^{-(2\gamma+\frac{1}{2})}(1+\log(p+1))^{\nu^*}\|u_s\|_{B^{2\gamma+2,\beta^{[s]}}_{\nu^*}(Q)} \tag{3.23}$$

with $\beta_\ell^{[s]} = s - \frac{1}{3}, 1 \le \ell \le 6$ *arbitrary, where*

$$u_s = \frac{u(x)}{\prod_{\ell=1}^s \sum_{i=1}^3 a_i^{[\ell]}(x_i+1)}.$$

Due to Proposition 3.1 and Theorem 3.5, a simple scaling leads to the regularity of u in terms of the Jacobi-weighted Besov spaces $B_\nu^{s,\beta}(Q_h)$.

Theorem 3.7. *Let* u *be given in (3.16) with* $\gamma > 0$ *and* $\nu \ge 0$, *and let* $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ *with* $\beta_i > -1, 1 \le i \le 6$. *Then* $u \in B^{s,\beta}_{\nu^*}(Q_h)$ *and* $u \in H^{s-\varepsilon,\beta}(Q_h)$ *with* $s = 2\gamma + 3 + \sum_{i=1}^6 \beta_i$ *and* $\varepsilon > 0$ *arbitrary, and*

$$\nu^* = \begin{cases} \max\{\nu - 1, 0\} & \text{if } \gamma \text{ is an integer and } \nu \ge 1, \\ \nu & \text{otherwise.} \end{cases} \tag{3.24}$$

Proof. Let $\tilde{u}(\xi) = u(h\xi)$. Then for $\nu = 0$

$$\tilde{u}(\xi) = u(h\xi) = h^\gamma \zeta^\gamma \chi(\zeta)\Phi(\theta,\phi) = h^\gamma w(\xi). \tag{3.25}$$

and for $\nu \ge 1$

$$\begin{aligned} \tilde{u}(\xi) &= h^\gamma \zeta^\gamma (\log h + \log \zeta)^\nu \chi(\zeta)\Phi(\theta,\phi) \\ &= h^\gamma \zeta^\gamma \chi(\zeta)\Phi(\theta,\phi) \sum_{m=0}^\nu \binom{\nu}{m} \log^{\nu-m} h \log^m \zeta = h^\gamma \sum_{m=0}^\nu \binom{\nu}{m} \tilde{v}_m(\xi) \log^{\nu-m} h \end{aligned} \tag{3.26}$$

where $\zeta = \sqrt{(\xi_1+1)^2+(\xi_2+1)^2+(\xi_3+1)^2}$, $\tilde{v}_m(\xi) = \zeta^\gamma\chi(\zeta)\Phi(\theta,\phi)\log^m\zeta$ and $w(\xi) = \zeta^\gamma\chi(\zeta)\Phi(\theta,\phi)$. Due to Theorem 3.5, $w(\xi) \in H^{s-\varepsilon,\beta}(Q)$ and $\tilde{v}_m(\xi) \in B^{s,\beta}_{m^*}(Q)$ with $s = 2\gamma+3+\Sigma_{i=1}^6\beta_i, \beta_i > -1, 1 \le i \le 6$, and

$$m^* = \begin{cases} m-1 & \text{if } \gamma \text{ is an integer and } m \ge 1, \\ m & \text{otherwise.} \end{cases} \tag{3.27}$$

The assertions of the theorem follow immediately from Theorem 3.5 and Proposition 3.1. □

A combination of Theorem 3.6 and a proper scaling gives a sharp estimation on the upper bound of approximation error in the Jacobi projections for the singular functions.

Theorem 3.8. *Let $u(x)$ be as given in (3.16). Then there exist polynomials $\psi_{hp}(x)$ and $\varphi_{hp}(x)$ in $\mathcal{P}^\kappa_p(Q_h), 1 \le \kappa \le 2$ with $p \ge 0$ such that*

$$\|u-\psi_{hp}\|_{L^2(Q_h)} \le C\frac{h^{\frac{3}{2}+\gamma}}{(p+1)^{2\gamma+3}}F_\nu(p,h) \tag{3.28}$$

and

$$\|u-\varphi_{hp}\|_{H^1(R^h_0)} \le C\frac{h^{\frac{1}{2}+\gamma}}{(p+1)^{2\gamma+1}}F_\nu(p,h) \tag{3.29}$$

and

$$\|u-\varphi_{hp}\|_{C^0(\bar{Q}_h)} \le C\frac{h^{\gamma}}{(p+1)^{2\gamma+\frac{1}{2}}}F_\nu(p,h). \tag{3.30}$$

The constants C in (3.28)-(3.30) are independent of h and p, where $F_\nu(p,h)$ is a log-polynomial,

$$F_\nu(p,h) = \begin{cases} (1+\log\frac{p+1}{h})^\nu & \text{for non-integer } \gamma, \\ (1+\log\frac{p+1}{h})^{\nu-1} & \text{for integer } \gamma,\ \nu\ge 1 \text{ and } \rho^\gamma\Phi(\theta,\phi)\in\mathcal{P}_\gamma(Q_h), \\ \max\{(1+\log\frac{p+1}{h})^{\nu-1}, \log^\nu\frac{1}{h}\} & \text{for integer } \gamma \text{ and} \\ \qquad \rho^\gamma\Phi(\theta,\phi)\notin\mathcal{P}_\gamma(Q_h). \end{cases} \tag{3.31}$$

If $u = 0$ on the planes $\pi_\ell : \sum_{i=1}^3 a_i^{[\ell]}(x_i+h) = 0, 1 \le \ell \le s, s = 1,$ or 2, or 3, then there exist polynomials $\psi_{hp}(x)$ and $\varphi_{hp}(x)$ in $\mathcal{P}^\kappa_p(Q_h), 1 \le \kappa \le 2$ with $p \ge s$ such that ψ_{hp} and $\varphi_{hp}(x)$ vanish on the planes $\pi_\ell, 1 \le \ell \le s$ and (3.28)-(3.31) hold.

Proof. By (3.25) for $\nu = 0$

$$\tilde{u}(\xi) = u(h\xi) = h^\gamma\zeta^\gamma\chi(\zeta)\Phi(\theta,\phi) = h^\gamma w(\xi).$$

Then (3.28) and (3.29) with $\nu = 0$ follow from Theorem 3.6 and Proposition 3.1 immediately.

Due to (3.26) for $\nu \geq 1$

$$\begin{aligned}\tilde{u}(\xi) &= h^\gamma \zeta^\gamma (\log h + \log\zeta)^\nu \chi(\zeta)\Phi(\theta,\phi)\\ &= h^\gamma\zeta^\gamma\chi(\zeta)\Phi(\theta,\phi)\sum_{m=0}^{\nu}\binom{\nu}{m}\log^{\nu-m} h\log^m\zeta = h^\gamma\sum_{m=0}^{\nu}\binom{\nu}{m}\tilde{v}_m(\xi)\log^{\nu-m}h\end{aligned}$$

By Theorem 3.5, $\tilde{v}_m(\xi) \in B^{s,\beta}_{m^*}(Q)$ with $s = 2\gamma+2, \beta_i = -1/3, 1 \leq i \leq 3, \beta_i > -1, 4 \leq i \leq 6$, arbitrary, and due to Theorem 3.6, $\tilde{\varphi}_m(\xi) = \Pi_p^\beta \tilde{v}_m$ satisfies

$$\|\tilde{v}_m(\xi) - \tilde{\varphi}_m(\xi)\|_{H^1(R_0)} \leq Cp^{-(2\gamma+1)}(1+\log(p+1))^{m^*}\|\tilde{v}_m(\xi)\|_{B^{2\gamma+2,\beta}_{m^*}(Q)} \tag{3.32}$$

with m^* is given in (3.27).

If γ is not an integer, let $\tilde{\varphi}(\xi) = h^\gamma\Sigma_{m=0}^{\nu}\binom{\nu}{m}\tilde{\varphi}_m(\xi)\log^{\nu-m}h$, and let $\varphi(x) = \tilde{\varphi}(x/h) = \Pi^\beta_{p,h}u$ with $\beta_i = -1/3, 1 \leq i \leq 3, \beta_i > -1, 4 \leq i \leq 6$, arbitrary. Then there hold

$$\|\tilde{u}(\xi) - \tilde{\varphi}(\xi)\|_{H^1(R_0)} \leq \frac{Ch^\gamma}{(p+1)^{2\gamma+1}}\sum_{m=0}^{\nu}(1+\log(1+p))^m\log^{\nu-m}\frac{1}{h} \leq C\frac{h^\gamma\Big(1+\log\frac{p+1}{h}\Big)^\nu}{(p+1)^{2\gamma+1}}$$

and for $\ell = 0, 1$

$$\|u(x)-\varphi(x)\|_{H^\ell(R_0^h)} = h^{\frac{3}{2}-\ell}\|\tilde{u}(\xi)-\tilde{\varphi}(\xi)\|_{H^\ell(R_0)} \leq C\frac{h^{\frac{3}{2}+\gamma-\ell}}{(p+1)^{2\gamma+3-2\ell}}\Big(1+\log\frac{p+1}{h}\Big)^\nu.$$

Thus, for non-integer γ, (3.28) and (3.29) are proved.

If γ is an integer, we have by (3.32)

$$\begin{aligned}\|\tilde{u}(\xi) - \tilde{\varphi}(\xi)\|_{H^1(R_0)} &\leq \frac{Ch^\gamma}{(p+1)^{2\gamma+1}}\Big(\log^\nu\frac{1}{h} + \sum_{m=1}^{\nu}\binom{\nu}{m}(1+\log(1+p))^{m-1}\log^{\nu-m}\frac{1}{h}\Big)\\ &\leq \frac{Ch^\gamma}{(p+1)^{2\gamma+1}}\max\Big\{\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}, \log^\nu\frac{1}{h}\Big\}\end{aligned}$$

which implies (3.29) for integer γ.

If γ is an integer and $\rho^\gamma\Phi(\theta,\phi)$ is a polynomial of degree γ in Q_h, then $\tilde{v}_0(\xi) = \zeta^\gamma\Phi(\theta,\phi)$ is a C^∞ function in Q. We rewrite (3.26) as

$$\tilde{u}(\xi) = h^\gamma\Big(\tilde{v}_0(\xi)\log^\nu h + \sum_{m=1}^{\nu}\binom{\nu}{m}\tilde{v}_m(\xi)\log^{\nu-m}h\Big) = h^\gamma\Big(\tilde{v}_0(\xi)\log^\nu h + \tilde{w}(\xi)\Big).$$

By the argument above, there exists a polynomial $\tilde{\varphi}_w(\xi) \in \mathcal{P}^\kappa_p(Q), 1 \leq \kappa \leq 2$ such that

$$\|\tilde{w}(\xi) - \tilde{\varphi}_w(\xi)\|_{H^1(R_0)} \leq C\sum_{m=1}^{\nu}\binom{\nu}{m}\frac{(1+\log(p+1))^{m-1}}{(p+1)^{2\gamma+\frac{3}{2}}}\log^{\nu-m}\frac{1}{h} \leq C\frac{\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}}{(p+1)^{2\gamma+\frac{1}{2}}}.$$

Let $u_0(x) = \tilde{u}(\frac{x}{h}) = \rho^\gamma\chi_h(\rho)\Phi(\theta,\phi)\log^\nu h$ and $w(x) = h^\gamma\tilde{w}(\frac{x}{h})$. Then

$$u(x) = u_0(x) + w(x) \tag{3.33}$$

Since $u_0(x)$ is a C^∞ function, there exists a polynomial $\varphi_0(x) \in \mathcal{P}_p^\kappa(Q_h), 1 \le \kappa \le 2$ with $p \ge 0$ such that

$$\|u_0 - \varphi_0\|_{H^1(R_0^h)} \le C\frac{h^{\frac{1}{2}+\gamma}}{(p+1)^{2\gamma+1}}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}. \tag{3.34}$$

Letting $\varphi(x) = \varphi_0(x) + \varphi_w(x)$ with $\varphi_w(x) = h^\gamma \tilde{\varphi}_w(\frac{x}{h})$. By (3.33)-(3.34), we have

$$\begin{aligned} \|w(x) - \varphi_w(x)\|_{H^1(R_0^h)} &\le Ch^\gamma \|\tilde{w}(\xi) - \tilde{\varphi}_w(\xi)\|_{H^1(R_0)} \\ &\le \frac{Ch^{\frac{1}{2}+\gamma}}{(p+1)^{2\gamma+1}}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1} \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \|u(x) - \varphi(x)\|_{H^1(R_0^h)} &\le \|w(x) - \varphi_w(x)\|_{H^1(R_0^h)} + \|u_0 - \varphi_0\|_{H^1(R_0^h)} \\ &\le \frac{Ch^{\frac{1}{2}+\gamma}}{(p+1)^{2\gamma+1}}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1} \end{aligned}$$

which leads to the estimation (3.29) in the case that $\rho^\gamma \Phi(\theta,\phi)$ is a polynomial.

If $u = 0$ on the planes $\pi_\ell, 1 \le \ell \le s$, $\tilde{v}_m(\xi)$ vanishes on the planes :$\tilde{\pi}_\ell : \sum_{i=1}^3 a_i^{[\ell]}(x_i+1) = 0, 1 \le \ell \le s$, and due to Theorem 3.6 there is a polynomial $\tilde{\varphi}(\xi) \in \mathcal{P}_p^\kappa(Q)$ satisfying (3.32). Consequently, the polynomial $\varphi(x) \in \mathcal{P}_p^\kappa(Q_h)$ vanishes on the planes $\pi_\ell, 1 \le \ell \le s$ and satisfies the estimation (3.29).

Similarly, we can prove (3.28) and (3.30). □

3.3.0.2. *Approximability of edge-singular functions.* Let (r,ϕ,x_3) be the cylindrical coordinates with respect to the vertex $(-h,-h,-h)$ and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -h, x_3 \in (-\infty,\infty)\}$ with $r = \{\sum_{i=1}^2 (x_i+h)^2\}^{1/2}$, and $\phi = \arctan\dfrac{x_2+h}{x_1+h} \in (0,\pi/2)$.

We now consider the singular functions with $\sigma > 0$

$$u(x) = r^\sigma \log^\mu r\, \chi_h(r)\, \Phi(\phi)\Psi(x_3) \tag{3.36}$$

with integer $\mu \ge 0$, where $\chi_h(r) = \chi(\frac{r}{h}), \chi(\cdot)$, $\Phi(\phi)$ and $\Psi(x_3)$ are C^∞ functions such that for $0 < r_0 < h$

$$\chi_h(r) = 1 \quad \text{for } 0 < r < r_0/2, \quad \chi_h(r) = 0 \quad \text{for } r > r_0.$$

and for $0 < \phi_0 < \pi/4$

$$\Phi(\phi) = 0 \quad \text{for } \phi \notin (\phi_0, \pi/2 - \phi_0),$$

and for $0 < z_0 < h/2$

$$\Psi(x_3) = 1 \quad \text{for } x_3 \in (-h+2z_0, h-2z_0), \quad \Psi(x_3) = 0 \quad \text{for } |x_3| \ge h - z_0.$$

Obviously, $u(x)$ and $v(x)$ have a support $R^h_{r_0,\phi_0,z_0} = \{x \in Q_h \mid 0 < r < r_0, \phi_0 \leq \phi \leq \pi/2 - \phi_0, |x_3| \leq h - z_0\} \subset Q_h$. For $0 < \phi_0 < \pi/4$, let

$$R^h_0 = R^h_{r_0,\phi_0,z_0} = \{x \in Q_h \mid 0 < r < r_0, \phi_0 \leq \phi \leq \pi/2 - \phi_0, |x_3| \leq h - z_0\},$$

as shown in Figure 3.2.

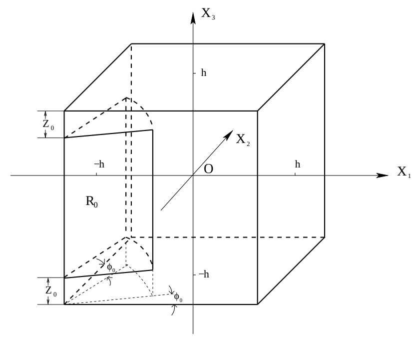


Fig. 3.2 Cubic Domain Q_h and subregion $R^h_{r_0,\phi_0,z_0}$

We quote the following theorems for $h = 1$ from [**24**].

Theorem 3.9. *Let $u(x) = r^\sigma \log^\mu r \chi(r)\Phi(\phi)\Psi(x_3)$, $\mu \geq 0$, and let $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ with $\beta_i > -1, 1 \leq i \leq 6$. Then $u \in H^{s-\varepsilon,\beta}(Q)$, and $u \in B^{s,\beta}_{\mu^*}(Q)$ with $s = 2\sigma + 2 + \beta_1 + \beta_2$ and $\varepsilon > 0$ arbitrary, and*

$$\mu^* = \begin{cases} \max\{\mu - 1, 0\} & \text{if } \sigma \text{ is an integer and } \mu \geq 1, \\ \mu & \text{otherwise.} \end{cases} \tag{3.37}$$

Theorem 3.10. *For $u(x) = r^\sigma \log^\mu r \chi(r)\Phi(\phi)\Psi(x_3)$, $\mu \geq 0$, there exists $\psi \in \mathcal{P}^\kappa_p(Q), 1 \leq \kappa \leq 2$, $p \geq 0$ such that*

$$\|u - \psi\|_{L^2(Q)} \leq C(p+1)^{-(2\sigma+2)}(1 + \log(p+1))^{\mu^*}\|u\|_{B^{2\sigma+2,\beta}_{\mu^*}(Q)} \tag{3.38}$$

with $\beta_1 = \beta_2 = 0$ and $\beta_i > -1, 3 \leq i \leq 6$, arbitrary. Also, there exists $\varphi(x) \in \mathcal{P}^\kappa_p(Q), 1 \leq \kappa \leq 2, p \geq 0$ such that

$$\|u - \varphi\|_{H^1(R_0)} \leq C\|u - \varphi\|_{H^{1,\beta}(Q)} \leq C(p+1)^{-2\sigma}(1 + \log(p+1))^{\mu^*}\|u\|_{B^{1+2\sigma,\beta}_{\mu^*}(Q)} \tag{3.39}$$

and

$$\|u - \varphi\|_{C^0(\bar{Q})} \leq C(p+1)^{-2\sigma}(1 + \log(p+1))^{\mu^*}\|u\|_{B^{1+2\sigma,\beta}_{\mu^*}(Q)} \tag{3.40}$$

with $\beta_i = -1/2, 1 \leq i \leq 6$. In (3.38)-(3.40) μ^* is given in (3.37).

If $u = 0$ on the plane $\pi_\ell : \sum_{i=1}^2 a_i^{[\ell]}(x_i + 1) = 0, 1 \leq \ell \leq s, s = 1$, or 2, then there exist $\psi \in \mathcal{P}^\kappa_p(Q)$ and $\varphi \in \mathcal{P}^\kappa_p(Q), 1 \leq \kappa \leq 2, p \geq s$ such that $\psi = 0$ and $\varphi = 0$ on $\pi_\ell, 1 \leq \ell \leq s$, *and*

$$\|u-\psi\|_{L^2(Q)} \le C(p+1)^{-(2\sigma+2)}(1+\log(p+1))^{\mu^*}\|u_s\|_{B^{2\sigma+2,\beta^{[s]}}_{\mu^*}(Q)} \tag{3.41}$$

with $\beta^{[s]}_\ell = s, 1\le \ell\le 2, \beta^{[s]}_\ell > -1, 3\le\ell\le 6$ *arbitrary, and*

$$\|u-\varphi\|_{H^1(R_0)} \le C(p+1)^{-2\sigma}(1+\log(p+1))^{\mu^*}\|u_s\|_{B^{2\sigma+1,\beta^{[s]}}_{\mu^*}(Q)} \tag{3.42}$$

and

$$\|u-\varphi\|_{C^0(\bar Q)} \le C(p+1)^{-2\sigma+1/2}(1+\log(p+1))^{\mu^*}\|u_s\|_{B^{1+2\sigma,\beta}_{\mu^*}(Q)} \tag{3.43}$$

with $\beta^{[s]}_\ell = s-1/2, 1\le\ell\le 6$, *where*

$$u_s = \frac{u(x)}{\prod_{\ell=1}^s\sum_{i=1}^2 a_i^{[\ell]}(x_i+1)}.$$

For singularity with logarithmic terms we need to use the modified Jacobi-weighted Besov spaces for the best approximation. Due to Proposition 3.1 and Theorem 3.9, a simple scaling leads to the regularity of u in terms of the modified Jacobi-weighted Besov spaces.

Theorem 3.11. *Let* $u(x) = r^\sigma \log^\mu r\, \chi_h(r)\,\Phi(\phi)\Psi(x_3), \mu\ge 0$ *as given in (3.36), and let* $\beta = (\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6)$ *with* $\beta_i > -1, 1\le i\le 6$, *arbitrary. Then* $u\in B^{s,\beta}_{\mu^*}(Q_h)$ *and* $u\in H^{s-\epsilon,\beta}(Q_h)$ *with* $s = 2\sigma+2+\beta_1+\beta_2$ *and* $\varepsilon>0$ *arbitrary and* μ^* *as given in (3.37).*

Proof. Let $\tilde u(\xi) = u(h\xi)$. Then for $\mu = 0$

$$\tilde u(\xi) = u(h\xi) = h^\sigma r^\sigma\chi(r)\Phi(\phi)\Psi(\xi_3) = h^\sigma w(\xi). \tag{3.44}$$

and for $\mu\ge 1$

$$\begin{aligned}\tilde u(\xi) &= h^\sigma r^\sigma(\log h+\log r)^\mu\chi(r)\Phi(\phi)\Psi(\xi_3)\\ &= h^\sigma r^\sigma\chi(r)\Phi(\phi)\Psi(\xi_3)\sum_{m=0}^{\mu}\binom{\mu}{m}\log^{\mu-m}h\log^m r = h^\sigma\sum_{m=0}^{\mu}\binom{\mu}{m}\tilde v_m(\xi)\log^{\mu-m}h\end{aligned} \tag{3.45}$$

where $r = \sqrt{(\xi_1+1)^2+(\xi_2+1)^2}$, $\tilde v_m(\xi) = r^\sigma\chi(r)\Phi(\phi)\log^m r$ and $w(\xi) = r^\sigma\chi(r)\Phi(\phi)\Psi(\xi_3)$. Due to Theorem 3.9, $w(\xi)\in H^{s-\varepsilon,\beta}(Q)$ and $\tilde v_m(\xi)\in B^{s,\beta}_{m^*}(Q)$ with $s = 2\sigma+2+\beta_1+\beta_2$, and

$$m^* = \begin{cases} m-1 & \text{if } \sigma \text{ is an integer and } m\ge 1,\\ m & \text{otherwise.}\end{cases} \tag{3.46}$$

The assertions of the theorem follow immediately from Theorem 3.9 and Proposition 3.1. □

Theorem 3.11 and Theorem 3.4 lead to the best approximation of the singular function u.

Theorem 3.12. *Let $u(x) = r^\sigma \log^\mu r\, \chi_h(r)\, \Phi(\phi)\Psi(x_3), \mu \geq 0$ as given in (3.36). Then there exists a polynomial $\psi(x)$ in $\mathcal{P}_p^\kappa(Q_h), 1 \leq \kappa \leq 2, p \geq 0$ such that*

$$\|u-\psi\|_{L^2(Q_h)} \leq C\frac{h^{\frac{3}{2}+\sigma}}{(p+1)^{2(\sigma+1)}}F_\mu(p,h). \tag{3.47}$$

Also there exists $\varphi(x) \in \mathcal{P}_p^\kappa(Q_h), 1 \leq \kappa \leq 2$, $p \geq 0$ such that

$$\|u-\varphi\|_{H^1(R_0^h)} \ \leq\ C\frac{h^{\frac{1}{2}+\sigma}}{(p+1)^{2\sigma}}F_\mu(p,h) \tag{3.48}$$

and

$$\|u-\varphi\|_{C^0(\bar{Q}_h)} \ \leq\ C\frac{h^{\sigma}}{(p+1)^{2\sigma-1/2}}F_\mu(p,h) \tag{3.49}$$

where

$$F_\mu(p,h) = \begin{cases} (1+\log\frac{p+1}{h})^\mu \ \ \textit{for non-integer } \sigma, \\[2ex] (1+\log\frac{p+1}{h})^{\mu-1} \ \textit{for integer } \sigma,\ \mu \geq 1 \textit{ and} \\ \qquad r^\sigma\Phi(\phi) \textit{ is a polynomial of degree } \sigma \textit{ in } x_1 \textit{ and } x_2, \\[2ex] \max\{(1+\log\frac{p+1}{h})^{\mu-1}, \log^\mu\frac{1}{h}\} \ \textit{for integer } \sigma \textit{ and} \\ \qquad r^\sigma\Phi(\phi)\Psi(x_3) \textit{ is not a polynomial of degree } \sigma \textit{ in } x_1 \textit{ and } x_2. \end{cases} \tag{3.50}$$

If $u = 0$ on the plane $\pi_\ell : \sum_{i=1}^2 a_i^{[\ell]}(x_i+h) = 0, 1 \leq \ell \leq s, s = 1$, or 2, then there exist $\psi \in \mathcal{P}_p^\kappa(Q)$ and $\varphi \in \mathcal{P}_p^\kappa(Q), 1 \leq \kappa \leq 2, p \geq s$ such that $\psi = 0$ and $\varphi = 0$ on $\pi_\ell, 1 \leq \ell \leq s$ and (3.47)-(3.49) hold.

Proof. By (3.44) for $\mu = 0$

$$\tilde{u}(\xi) = u(h\xi) = h^\sigma r^\sigma \chi(r)\Phi(\phi)\Psi(\xi_3) = h^\sigma w(\xi).$$

Then (3.47) and (3.48) with $\mu = 0$ follow from Theorem 3.10 immediately.

Due to (3.45) for $\nu \geq 1$

$$\begin{aligned}\tilde{u}(\xi) &= h^\sigma r^\sigma (\log h + \log r)^\mu \chi(r)\Phi(\phi)\Psi(\xi_3) \\ &= h^\sigma r^\sigma \chi(r)\Phi(\phi)\Psi(\xi_3)\sum_{m=0}^{\mu}\binom{\mu}{m}\log^{\mu-m}h\log^m r = h^\sigma\sum_{m=0}^{\mu}\binom{\mu}{m}\tilde{v}_m(\xi)\log^{\mu-m}h\end{aligned}$$

By Theorem 3.9, $\tilde{v}_m(\xi) \in B_{m^*}^{s,\beta}(Q)$ with $s = 2\sigma+2+\beta_1+\beta_2$, and due to Theorem 3.10, there exists a polynomial $\tilde{\varphi}_m(\xi) \in \mathcal{P}_P^\kappa(Q)$ satisfying

$$\|\tilde{v}_m(\xi) - \tilde{\varphi}_m(\xi)\|_{H^1(R_0)} \leq Cp^{-2\sigma}(1+\log(p+1))^{m^*}\|\tilde{v}_{m,\xi_3}(\xi)\|_{B_{m^*}^{1+2\sigma,\beta}(Q)} \tag{3.51}$$

with $\beta_i = -1/2, 1 \leq i \leq 4, \beta_i = 2, i = 3,6$ and m^* is given in (3.46).

If σ is not an integer, let $\tilde{\varphi}(\xi) = h^\sigma \Sigma_{m=0}^{\mu}\binom{\mu}{m}\tilde{\varphi}_m(\xi)\log^{\mu-m} h$, and let $\varphi(x) = \tilde{\varphi}(x/h)$. Then there hold

$$\|\tilde{u}(\xi)-\tilde{\varphi}(\xi)\|_{H^1(R_0)} \le \frac{Ch^\sigma}{(p+1)^{2\sigma}}\sum_{m=0}^{\mu}(1+\log(1+p))^m \log^{\mu-m}\frac{1}{h} \le C\frac{h^\sigma\Big(1+\log\frac{p+1}{h}\Big)^\mu}{(p+1)^{2\sigma}}$$

and

$$\|u(x)-\varphi(x)\|_{H^1(R_0^h)} = h^{\frac{1}{2}}\|\tilde{u}(\xi)-\tilde{\varphi}(\xi)\|_{H^1(R_0)} \le C\frac{h^{\frac{1}{2}+\sigma}}{(p+1)^{2\sigma}}\Big(1+\log\frac{p+1}{h}\Big)^\mu.$$

Thus, for non-integer σ, (3.48) is proved.

If σ is an integer, we have by (3.51)

$$\begin{aligned}\|\tilde{u}(\xi)-\tilde{\varphi}(\xi)\|_{H^1(R_0)} \;&\le\; \frac{Ch^\sigma}{(p+1)^{2\sigma}}\Big(\log^\mu\frac{1}{h}+\sum_{m=1}^{\mu}\binom{\mu}{m}(1+\log(1+p))^{m-1}\log^{\mu-m}\frac{1}{h}\Big)\\ &\le\; \frac{Ch^\sigma}{(p+1)^{2\sigma}}\max\Big\{\Big(1+\log\frac{p+1}{h}\Big)^{\mu-1},\log^\mu\frac{1}{h}\Big\}\end{aligned}$$

which implies (3.48) for integer σ.

If σ is an integer and $r^\sigma\Phi(\phi)$ is a polynomial of degree σ in x_1 and x_2, then $\tilde{v}_0(\xi) = \zeta^\sigma\Phi(\phi)$ is a polynomial of degree σ in in ξ_1 and ξ_2. We rewrite (3.45) as

$$\tilde{u}(\xi) = h^\sigma\Big(\tilde{v}_0(\xi)\log^\mu h+\sum_{m=1}^{\mu}\binom{\mu}{m}\tilde{v}_m(\xi)\log^{\mu-m} h\Big) = h^\sigma\Big(\tilde{v}_0(\xi)\log^\mu h+\tilde{w}(\xi)\Big).$$

By the argument above, there exists a polynomial $\tilde{\varphi}_w(\xi)\in\mathcal{P}_p^\kappa(Q), 1\le\kappa\le 2$ such that

$$\|\tilde{w}(\xi)-\tilde{\varphi}_w(\xi)\|_{H^1(R_0)} \le Ch^\sigma\sum_{m=1}^{\mu}\binom{\mu}{m}\frac{(1+\log(p+1))^{m-1}}{(p+1)^{2\sigma}}\log^{\mu-m}\frac{1}{h} \le C\frac{h^\sigma\Big(1+\log\frac{p+1}{h}\Big)^{\mu-1}}{(p+1)^{2\sigma}}.$$

Let $u_0(x) = \tilde{u}(\frac{x}{h}) = \zeta^\sigma\chi_h(\zeta)\Phi(\phi)\log^\mu h$ and $w(x) = h^\sigma\tilde{w}(\frac{x}{h})$. Then

$$u(x) = u_0(x)+w(x) \tag{3.52}$$

Since $u_0(x)$ is a C^∞ function, there exists a polynomial $\varphi_0(x)\in\mathcal{P}_p^\kappa(Q_h), 1\le\kappa\le 2$ such that

$$\|u_0-\varphi_0\|_{H^1(R_0^h)} \le C\frac{h^{\frac{1}{2}+\sigma}}{(p+1)^{2\sigma}}\Big(1+\log\frac{p+1}{h}\Big)^{\mu-1}. \tag{3.53}$$

Letting $\varphi(x) = \varphi_0(x)+\varphi_w(x)$ with $\varphi_w(x) = h^\sigma\tilde{\varphi}_w(\frac{x}{h})$. By (3.52)-(3.53), we have

$$\begin{aligned}\|u(x)-\varphi(x)\|_{H^1(R_0^h)} \;&\le\; \|w(x)-\varphi_w(x)\|_{H^1(R_0^h)}+\|u_0-\varphi_0\|_{H^1(R_0^h)}\\ &\le\; \frac{Ch^{\frac{1}{2}+\sigma}}{(p+1)^{2\sigma}}\Big(1+\log\frac{p+1}{h}\Big)^{\mu-1}\end{aligned}$$

which leads to the estimation (3.48) in the case that $r^\sigma \Phi(\phi)$ is a polynomial of degree σ in x_1 and x_2.

If $u = 0$ on the planes $\pi_\ell, 1 \le \ell \le s$, $\tilde{v}_m(\xi)$ vanishes on the planes $:\tilde{\pi}_\ell : \sum_{i=1}^2 a_i^{[\ell]}(x_i+1) = 0, 1 \le \ell \le s$. Due to Theorem 3.10 there is a polynomial $\tilde{\varphi}_m(\xi) \in \mathcal{P}_p^\kappa(Q)$ satisfying (3.32). Consequently, the polynomial $\varphi(x) \in \mathcal{P}_p^\kappa(Q_h)$ vanishes on the planes $\pi_\ell, 1 \le \ell \le s$ and satisfies the estimation (3.48).

Similarly, we can prove (3.47) and (3.49). □

3.3.0.3. *Approximability of vertex-edge singular functions.* Let (ρ, θ, ϕ) be the spherical coordinates with respect to the vertex $(-h,-h,-h)$ and the vertical line $L = \{x = (x_1,x_2,x_3) \mid x_1 = x_2 = -h, x_3 \in (-\infty,\infty)\}$ as in previous section.

We now consider the singular functions with real $\gamma, \sigma > 0$ and integers $\nu, \mu \ge 0$,

$$u(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta \, \chi_h(\rho) \, \Phi(\phi)\Psi(\theta) \tag{3.54}$$

where $\rho = \{(x_1+h)^2 + (x_2+h)^2 + (x_3+h)^2\}^{1/2}, \chi_h(\rho) = \chi(\frac{\rho}{h}), \chi(\cdot)$ is defined as in previous subsection, $\Phi(\phi)$ and $\Psi(\theta)$ are C^∞ cut-off functions such that for $\theta_0 \in (0,\pi/2)$

$$\Psi(\theta) = 1 \quad \text{for } 0 \le \theta \le \theta_0/2, \quad \Psi(\theta) = 0 \quad \text{for } \theta \ge \theta_0.$$

and for $0 < \phi_0 < \pi/4$

$$\Phi(\phi) = 0 \quad \text{for } \phi \notin (\phi_0, \pi/2 - \phi_0).$$

Let

$$R_0^h = R_{\rho_0,\theta_0,\phi_0}^h = \{x \in Q_h \mid 0 < \rho < \rho_0, \theta \in (0,\theta_0), \phi \in (\phi_0, \pi/2 - \phi_0)$$

as shown in Figure 3.3.

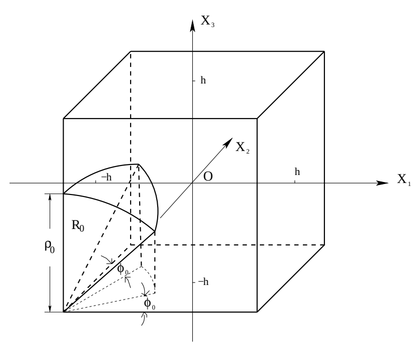


Fig. 3.3 Cubic Domain Q_h and sub region $R_{\rho_0,\theta_0,\phi_0}^h$

We quote the following theorems for $h = 1$ from [**24**].

Theorem 3.13. *Let* $u(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta \chi(\rho)\Phi(\phi)\Psi(\theta), \nu \ge 0, \mu \ge 0,$ *and let* $\beta_i > -1, 1 \le i \le 6$. *Then* $u(x) \in H^{s-\varepsilon,\beta}(Q)$ *and* $u(x) \in B_\lambda^{s,\beta}(Q)$ *with* $s = 2 + 2\min\{\gamma + (1 +$

$\beta_3)/2, \sigma\} + \beta_1 + \beta_2$, $\varepsilon > 0$ *arbitrary, and*

$$\lambda = \begin{cases} \mu & \text{if } \sigma < \gamma + (1+\beta_3)/2, \\ \mu + \nu + 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2, \\ \mu + \nu & \text{if } \sigma > \gamma + (1+\beta_3)/2. \end{cases} \tag{3.55}$$

Theorem 3.14. *Let* $u(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta \chi(\rho)\Phi(\phi)\Psi(\theta), \nu \geq 0, \mu \geq 0$, *then there exists* $\psi(x) \in \mathcal{P}_p^\kappa(Q), 1 \leq \kappa \leq 2$ *with* $p \geq 0$ *such that for* $\beta_i = 0, 1 \leq i \leq 3, \beta_i > -1$ *arbitrary,* $4 \leq i \leq 6$

$$\|u - \psi\|_{L^2(Q)} \leq C \frac{(1+\log(p+1))^\lambda}{(p+1)^{2+2\min\{\sigma,\gamma+1/2\}}} \|u\|_{B_\lambda^{2+2\min\{\sigma,\gamma+1/2\},\beta}(Q)}. \tag{3.56}$$

Also, there exists $\varphi(x) \in \mathcal{P}_p^\kappa(Q), 1 \leq \kappa \leq 2$ *with* $p \geq 0$ *such that for* $\beta_i = -1/2, i = 1,2,4,5, \beta_3 = \beta_6 = 0$

$$\|u - \varphi\|_{H^1(R_0)} \leq C \frac{(1+\log(p+1))^\lambda}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}} \|u\|_{B_\lambda^{1+2\min\{\sigma,\gamma+1/2\},\beta}(Q)} \tag{3.57}$$

and

$$\|u - \varphi\|_{C^0(\bar{Q})} \leq C \frac{(1+\log(p+1))^\lambda}{(p+1)^{2\min\{\sigma,\gamma+1/2\}-1/2}} \|u\|_{B_\lambda^{1+2\min\{\sigma,\gamma+1/2\},\beta}(Q)} \tag{3.58}$$

with λ *in (3.56)-(3.58) given in (3.55).*

If $u = 0$ *on the plane* $\pi_\ell : \sum_{i=1}^2 a_i^{[\ell]}(x_i + 1) = 0, 1 \leq \ell \leq s, s = 1$, *or* 2, *or* 3, *then there exist* $\psi \in \mathcal{P}_p^\kappa(Q)$ *and* $\varphi \in \mathcal{P}_p^\kappa(Q), 1 \leq \kappa \leq 2, p \geq s$ *such that* $\psi = 0$ *and* $\varphi = 0$ *on* $\pi_\ell, 1 \leq \ell \leq s$, *and*

$$\|u - \psi\|_{L^2(Q)} \leq \frac{C(1+\log(p+1))^\lambda}{(p+1)^{2+2\min\{\sigma,\gamma+1/2\}}} \|u_s\|_{B_\lambda^{2+2\min\{\sigma,\gamma+1/2\},\beta^{[s]}}(Q)}$$

with $\beta_\ell^{[s]} = s, 1 \leq \ell \leq 3, \beta_\ell^{[s]} > -1, 4 \leq \ell \leq 6$ *arbitrary, and*

$$\|u - \varphi\|_{H^1(R_0)} \leq \frac{C(1+\log(p+1))^\lambda}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}} \|u_s\|_{B_\lambda^{1+2\min\{\sigma,\gamma+1/2\},\beta^{[s]}}(Q)}$$

and

$$\|u - \varphi\|_{C^0(\bar{Q})} \leq \frac{C(1+\log(p+1))^\lambda}{(p+1)^{2\min\{\sigma,\gamma+1/2\}-1/2}} \|u_s\|_{B_\lambda^{1+2\min\{\sigma,\gamma+1/2\},\beta^{[s]}}(Q)} \tag{3.59}$$

with $\beta_\ell^{[s]} = s - \frac{1}{2}, \ell = 1,2,4,5, \beta_\ell^{[s]} = 0, \ell = 3,6$, *where*

$$u_s = \frac{u(x)}{\prod_{\ell=1}^s \sum_{i=1}^2 a_i^{[\ell]}(x_i+1)}.$$

Due to Proposition 3.1 and Theorem 3.13, a simple scaling leads to the following theorem.

Theorem 3.15. *Let* $u(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta\, \chi_h(\rho)\, \Phi(\phi)\Psi(\theta), \nu \geq 0, \mu \geq 0$ *as given in (3.54), and let* $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ *with* $\beta_i > -1, 1 \leq i \leq 6$, *arbitrary. Then* $u \in H^{s-\varepsilon,\beta}(Q_h)$ *and* $u \in B^{s,\beta}_\lambda(Q_h)$ *with* $s = 2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2, \varepsilon > 0$ *arbitrary, and*

$$\lambda = \begin{cases} \mu & \text{if } \sigma < \gamma + (1+\beta_3)/2, \\ \mu + \nu + 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2, \\ \mu + \nu & \text{if } \sigma > \gamma + (1+\beta_3)/2. \end{cases} \tag{3.60}$$

Proof. Let $\tilde{u}(\xi) = u(h\xi)$. Then for $\nu = \mu = 0$

$$\tilde{u}(\xi) = u(h\xi) = h^\gamma \zeta^\gamma \sin^\sigma \theta\, \chi(\zeta)\Phi(\phi)\Psi(\theta) = h^\gamma w(\xi). \tag{3.61}$$

and for $\nu \geq 1, \mu \geq 1$

$$\begin{aligned} \tilde{u}(\xi) &= h^\gamma \zeta^\gamma (\log h + \log\zeta)^\nu \sin^\sigma\theta \log^\mu \sin\theta\, \chi(\zeta)\Phi(\phi)\Psi(\theta) \\ &= h^\gamma \zeta^\gamma \sin^\sigma\theta \log^\mu \sin\theta\, \chi(\zeta)\Phi(\phi)\Psi(\theta) \sum_{m=0}^{\nu} \binom{\nu}{m} \log^{\nu-m} h \log^m \zeta \\ &= h^\gamma \sum_{m=0}^{\nu} \binom{\nu}{m} \tilde{v}_m(\xi) \log^{\nu-m} h \end{aligned} \tag{3.62}$$

where $\zeta = \sqrt{(\xi_1+1)^2 + (\xi_2+1)^2 + (\xi_3+1)^2}$, $\tilde{v}_m(\xi) = \zeta^\gamma \log^m \zeta \sin^\sigma\theta \log^\mu \sin\theta\, \chi(\zeta)\Phi(\phi)\Psi(\theta)$ and $w(\xi) = \zeta^\gamma \sin^\sigma\theta\, \chi(\zeta)\Phi(\phi)\Psi(\theta)$. Due to Theorem 3.13, $w(\xi) \in H^{s-\varepsilon,\beta}(Q)$ and $\tilde{v}_m(\xi) \in B^{s,\beta}_{\lambda_m}(Q)$ with $s = 2 + 2\min\{\gamma + (1+\beta_3)/2, \sigma\} + \beta_1 + \beta_2, \varepsilon > 0$, and

$$\lambda_m = \begin{cases} \mu & \text{if } \sigma < \gamma + (1+\beta_3)/2, \\ \mu + m + 1/2 & \text{if } \sigma = \gamma + (1+\beta_3)/2, \\ \mu + m & \text{if } \sigma > \gamma + (1+\beta_3)/2. \end{cases} \tag{3.63}$$

The assertions of the theorem follow immediately from Theorem 3.13 and Proposition 3.1. □

By using Theorem 3.15 and the approximation property described in Theorem 3.4, we obtain the approximability of $u(x)$.

Theorem 3.16. *Let* $u(x) = \rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta\, \chi_h(\rho)\, \Phi(\phi)\Psi(\theta), \nu \geq 0, \mu \geq 0$ *as given in (3.54). Then there exists a polynomial* $\psi(x)$ *in* $\mathcal{P}^\kappa_p(Q_h), 1 \leq \kappa \leq 2$ *with* $p \geq 0$ *such that*

$$\|u - \psi\|_{L^2(Q_h)} \leq C \frac{h^{\frac{3}{2}+\gamma}}{(p+1)^{2(1+\min\{\gamma+1/2,\sigma\})}} F_{\nu,\mu}(p,h). \tag{3.64}$$

Also there exists $\varphi(x) \in \mathcal{P}_p^\kappa(Q_h), 1 \le \kappa \le 2$, $p \ge 0$ *such that*

$$\|u-\varphi\|_{H^1(R_0^h)} \le C\frac{h^{\frac{1}{2}+\gamma}}{(p+1)^{2\min\{\gamma+1/2,\sigma\}}}F_{\nu,\mu}(p,h). \tag{3.65}$$

and

$$\|u-\varphi\|_{C^0(\bar{Q}_h)} \le C\frac{h^{\gamma}}{(p+1)^{2\min\{\gamma+1/2,\sigma\}-1/2}}F_{\nu,\mu}(p,h), \tag{3.66}$$

where

$$F_{\nu,\mu}(p,h) = \begin{cases} (1+\log(p+1))^{\bar\mu}(1+\log\frac{p+1}{h})^{\nu-1} \textit{ for integer } \gamma,\sigma,\ \nu\ge 1, \mu=0, \\ \qquad\qquad \textit{and } \rho^\gamma\sin^\sigma\theta\Phi(\phi)\Psi(\theta)\in\mathcal{P}_\gamma(Q_h), \\ (1+\log(p+1))^{\bar\mu}(1+\log\frac{p+1}{h})^{\nu},\ \textit{otherwise.} \end{cases} \tag{3.67}$$

and

$$\bar\mu = \begin{cases} \mu & \textit{if } \sigma\ne\gamma+1/2 \\ \mu+1/2 & \textit{if } \sigma=\gamma+1/2. \end{cases} \tag{3.68}$$

If $u=0$ *on the plane* $\pi_\ell : \sum_{i=1}^2 a_i^{[\ell]}(x_i+1)=0, 1\le\ell\le s, s=1,$ *or* $2,$ *or* $3,$ *then there exist* $\psi\in\mathcal{P}_p^\kappa(Q)$ *and* $\varphi\in\mathcal{P}_p^\kappa(Q), 1\le\kappa\le 2, p\ge s$ *such that* $\psi=0$ *and* $\varphi=0$ *on* $\pi_\ell, 1\le\ell\le s$ *and (3.64)-(3.66) hold.*

Proof. By (3.61) for $\nu=\mu=0$

$$\tilde u(\xi) = u(h\xi) = h^\gamma\zeta^\gamma\,\sin^\sigma\theta\,\chi(\zeta)\Phi(\phi)\Psi(\theta) = h^\gamma w(\xi).$$

Then (3.64) and (3.65) with $\nu=\mu=0$ follow from Theorem 3.14 and Proposition 3.1 immediately.

Due to (3.62) for $\nu\ge 1, \mu\ge 1$

$$\begin{aligned} \tilde u(\xi) &= h^\gamma\zeta^\gamma(\log h+\log\zeta)^\nu\,\sin^\sigma\theta\,\log^\mu\sin\theta\,\chi(\zeta)\Phi(\phi)\Psi(\theta) \\ &= h^\gamma\zeta^\gamma\,\sin^\sigma\theta\,\log^\mu\sin\theta\,\chi(\zeta)\Phi(\phi)\Psi(\theta)\sum_{m=0}^{\nu}\binom{\nu}{m}\log^{\nu-m}h\log^m\zeta \\ &= h^\gamma\sum_{m=0}^{\nu}\binom{\nu}{m}\tilde v_m(\xi)\log^{\nu-m}h \end{aligned}$$

By Theorem 3.13, $\tilde v_m(\xi)\in B^{s,\beta}_{\lambda_m}(Q)$ with $s=1+2\min\{\gamma+1/2,\sigma\}$, and due to Theorem 3.14, $\tilde\varphi_m(\xi)=\Pi_p^\beta\tilde v_m$ satisfies

$$\begin{aligned} &\|\tilde v_m(\xi)-\tilde\varphi_m(\xi)\|_{H^1(R_0)} \\ &\le C(p+1)^{-2\min\{\sigma,\gamma+1/2\}}(1+\log(p+1))^{\lambda_m}\|\tilde v_m(\xi)\|_{B^{s,\beta}_\lambda(Q)} \end{aligned} \tag{3.69}$$

with λ given in (3.60).

If γ is not an integer, let $\tilde\varphi(\xi) = h^\gamma \Sigma_{m=0}^{\nu}\binom{\nu}{m}\tilde\varphi_m(\xi)\log^{\nu-m} h$, and let $\varphi(x) = \tilde\varphi(x/h) = \Pi^{\beta}_{p,h}u$ with $\beta_i = -1/3, 1 \le i \le 3, \beta_i > -1, 4 \le i \le 6$, arbitrary. Then there hold

$$\begin{aligned}\|\tilde u(\xi) - \tilde\varphi(\xi)\|_{H^1(R_0)} &\le \frac{Ch^\gamma}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}\sum_{m=0}^{\nu}\binom{\nu}{m}(1+\log(p+1))^{\lambda_m}\log^{\nu-m} h \\ &\le C\frac{h^\gamma(1+\log(p+1))^{\bar\mu}\Big(1+\log\frac{p+1}{h}\Big)^\nu}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}},\end{aligned}$$

and

$$\|u(x)-\varphi(x)\|_{H^1(R_0^h)} = h^{\frac12}\|\tilde u(\xi)-\tilde\varphi(\xi)\|_{H^1(R_0)} \le C\frac{h^{\frac12+\gamma}(1+\log(p+1))^{\bar\mu}}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}\Big(1+\log\frac{p+1}{h}\Big)^\nu$$

with $\bar\mu$ as given in (3.68). Thus, (3.65) for non integer γ is proved.

If γ and σ are integers, $\mu = 0$ and $\rho^\gamma \sin^\sigma\theta\Phi(\phi)\Psi(\theta)$ is a polynomial of degree γ in Q_h, then $\tilde v_0(\xi) = \zeta^\gamma\sin^\sigma\theta\Phi(\phi)\Psi(\theta)$ is a polynomial of degree γ in Q. We rewrite (3.62) as

$$\tilde u(\xi) = h^\gamma\Big(\tilde v_0(\xi)\log^\nu h + \sum_{m=1}^{\nu}\binom{\nu}{m}\tilde v_m(\xi)\log^{\nu-m} h\Big) = h^\gamma\Big(\tilde v_0(\xi)\log^\nu h + \tilde w(\xi)\Big).$$

By the argument above, there exists a polynomial $\tilde\varphi_w(\xi)\in\mathcal{P}_p^\kappa(Q), 1\le\kappa\le 2$ such that

$$\begin{aligned}\|\tilde w(\xi)-\tilde\varphi_w(\xi)\|_{H^1(R_0)} &\le Ch^\gamma\sum_{m=1}^{\nu}\binom{\nu}{m}\frac{(1+\log(p+1))^{\lambda_m-1}}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}\log^{\nu-m} h \\ &\le C\frac{h^\gamma(1+\log(p+1))^{\bar\mu}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}.\end{aligned}$$

Let $u_0(x) = \tilde u(\frac{x}{h}) = \rho^\gamma\chi_h(\zeta)\sin^\sigma\theta\Phi(\phi)\Psi(\theta)\log^\nu h$ and $w(x) = h^\gamma\tilde w(\frac{x}{h})$. Then

$$u(x) = u_0(x) + w(x) \tag{3.70}$$

Since $u_0(x)$ is a C^∞ function, there exists a polynomial $\varphi_0(x)\in\mathcal{P}_p^\kappa(Q_h), 1\le\kappa\le 2$ such that

$$\|u_0-\varphi_0\|_{H^1(R_0^h)} \le C\frac{h^{\frac12+\gamma}(1+\log(p+1))^{\bar\mu}}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}. \tag{3.71}$$

Letting $\varphi(x) = \varphi_0(x)+\varphi_w(x)$ with $\varphi_w(x) = h^\gamma\tilde\varphi_w(\frac{x}{h})$. By (3.70)-(3.71), we have

$$\begin{aligned}\|u(x)-\varphi(x)\|_{H^1(R_0^h)} &\le \|w(x)-\varphi_w(x)\|_{H^1(R_0^h)} + \|u_0-\varphi_0\|_{H^1(R_0^h)} \\ &\le \frac{Ch^{\frac12+\gamma}(1+\log(p+1))^{\bar\mu}}{(p+1)^{2\min\{\sigma,\gamma+1/2\}}}\Big(1+\log\frac{p+1}{h}\Big)^{\nu-1}\end{aligned}$$

which leads to the estimation (3.65) in the case that $\rho^\gamma\sin^\sigma\theta\Phi(\phi)\Psi(\theta)$ is a polynomial.

If $u = 0$ on the planes $\pi_\ell, 1\le\ell\le s$, $\tilde v_m(\xi)$ vanishes on the planes :$\tilde\pi_\ell : \sum_{i=1}^{2} a_i^{[\ell]}(x_i+1) = 0, 1\le\ell\le s$. Due to Theorem 3.14 there is a polynomial $\tilde\varphi_m(\xi)\in\mathcal{P}_p^\kappa(Q)$ satisfying (3.69).

Consequently, the polynomial $\varphi(x) \in \mathcal{P}_p^{\kappa}(Q_h)$ vanishes on the planes $\pi_\ell, 1 \leq \ell \leq s$ and satisfies the estimation (3.65).

Similarly, we can prove (3.64) and (3.66). $\square$

CHAPTER 4

Polynomial Extensions in Three Dimensions

4.1. Extension on a standard triangular prisms

4.1.1. Polynomial extension on a tetrahedron. For the construction of polynomial extensions on a triangular prism, we need quote results on the extension on a tetrahedron from [**40**]. We denote, by K, a standard tetrahedron $\{(x_1, x_2, x_3)|x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}$ in $\mathbb{R}^3$ shown in Fig. 4.1, and ∂K denotes the boundary of K. Let $T = \{(x_1, x_2)|x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$ be a standard triangle in $\mathbb{R}^2$, and let $\Gamma_i, 1 \leq i \leq 3$ be faces of K contained in the plane $x_i = 0$ and Γ_4 be the oblique face.

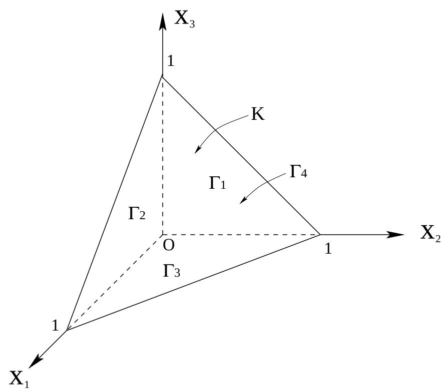


Fig. 4.1 The tetrahedron K

Muñoz-Sola introduced the following operators [**40**]

$$F_K f(x_1, x_2, x_3) = \frac{2}{x_3^2} \int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_1+x_2+x_3-\xi_1} f(\xi_1, \xi_2) d\xi_2, \tag{4.1}$$

and

$$R_K f(x_1, x_2, x_3) = (1 - x_1 - x_2 - x_3) x_1 x_2 F_K \tilde{f}(x_1, x_2, x_3) \tag{4.2}$$

with

$$\tilde{f}(x_1, x_2) = \frac{f(x_1, x_2)}{x_1 x_2 (1 - x_1 - x_2)}.$$

The operator R_K has the following decomposition:

$$R_K f(x_1,x_2,x_3) = (1-x_1-x_2-x_3)R_{12}f(x_1,x_2,x_3) + x_2 R_{13} f(x_1,x_2,x_3) + x_1 R_{23} f(x_1,x_2,x_3), \tag{4.3}$$

where

$$R_{12}f(x_1,x_2,x_3) = x_1 x_2 F_K \tilde{f}_{12}(x_1,x_2,x_3), \quad \tilde{f}_{12}(x_1,x_2) = \frac{f(x_1,x_2)}{x_1 x_2}, \tag{4.4}$$

$$R_{i3}f(x_1,x_2,x_3) = (1-x_1-x_2-x_3)x_i F_K \tilde{f}_{i3}(x_1,x_2,x_3) \tag{4.5}$$

with

$$\tilde{f}_{i3}(x_1,x_2) = \frac{f(x_1,x_2)}{x_i(1-x_1-x_2)}, \; i=1,2.$$

The following theorems were proved in [**40**].

Theorem 4.1. *Let R_K be the operator defined by (4.2). Then $R_K f(x) \in \mathcal{P}_p^1(K)$ for all $f \in \mathcal{P}_p^{1,0}(\Gamma_3)$, and*

$$\|R_K f\|_{H^1(K)} \leq C \|f\|_{H_{00}^{\frac{1}{2}}(\Gamma_3)}, \tag{4.6}$$

$$R_K f \,|_{\Gamma_3} = f, \quad R_K f \,|_{\Gamma_i} = 0, \; i=1,2,4, \tag{4.7}$$

where C is a constant independent of f and p.

Theorem 4.2. *For $f \in \mathcal{P}_p^1(\partial K) = \{f \in C^0(\partial K) \mid f\,|_{\Gamma_i} \in \mathcal{P}_p^1(\Gamma_i), 1 \leq i \leq 4\}$, there exists a polynomial $E_K f \in \mathcal{P}_p^1(K)$ such that $E_K f|_{\partial K} = f$ and*

$$\|E_K f\|_{H^1(K)} \leq C \|f\|_{H^{1/2}(\partial K)}, \tag{4.8}$$

where C is a constant independent of f and p.

4.1.2. Polynomial extension on prisms from a triangular face. Let $G = T \times I$ be a triangular prism with faces $\Gamma_i, 1 \leq i \leq 5$ shown in Fig. 4.2 where $T = \{(\tilde{x}_1,\tilde{x}_2) \mid \tilde{x}_1 \geq 0, \tilde{x}_2 \geq 0, \tilde{x}_1 + \tilde{x}_2 \leq 1\}$ and $I = [0,1]$. $\Gamma_i, 1 \leq i \leq 3$ are on the planes $\tilde{x}_i = 0$, Γ_5 is the face of G contained in the plane $\tilde{x}_3 = 1$ and Γ_4 is the face of G contained in the plane $\tilde{x}_1 + \tilde{x}_2 = 1$. Then $\Gamma_3 = T$ and $\Gamma_2 = S = I \times I$. By $\mathcal{P}_p^1(T) \times \mathcal{P}_p(I)$ we denote a set of polynomials with the sub-total degree in $\tilde{x}_1$ and $\tilde{x}_2 \leq p$ and with the degree $\leq p$ in $\tilde{x}_3$. Obviously $\mathcal{P}_p^1(G) \subset \mathcal{P}_p^1(T) \times \mathcal{P}_p(I) \subset \mathcal{P}_p^2(G)$, it is denoted by $\mathcal{P}_p^{1.5}(G)$.

We shall establish polynomial extensions from the triangle T to the prism G.

Since the mapping M:

$$x_1 = \tilde{x}_1(1-H\tilde{x}_3), \quad x_2 = \tilde{x}_2(1-H\tilde{x}_3), \quad x_3 = H\tilde{x}_3 \tag{4.9}$$

maps the prism G onto a truncated tetrahedron $K_H = \{(x_1,x_2,x_3) | x_1 \geq 0, x_2 \geq 0, H \geq x_3 \geq 0, x_1+x_2+x_3 \leq 1\}$ with $H \in (0,1)$ shown in Fig. 4.2. $\tilde{\Gamma}_i, i=1,2,3,4,5$ are the faces of K_H, $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_5$ contained in the planes $x_3 = 0$ and $x_3 = H$, respectively, and $\tilde{\Gamma}_i, i=1,2,4$ are portions of the faces of the tetrahedron K. Hence, we need to construct a polynomial

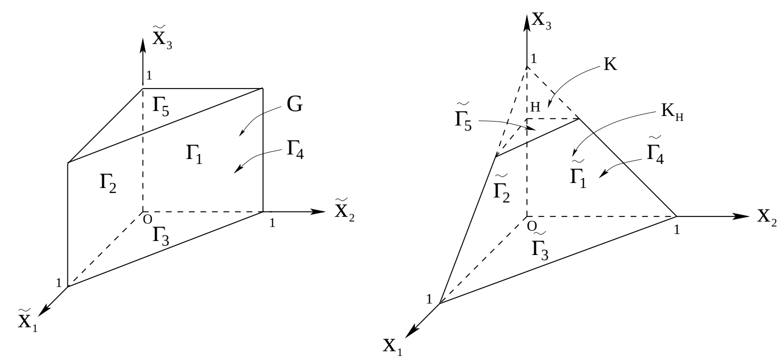


Fig. 4.2 The prism G and truncated tetrahedron K_H

extension operator $R_H : \mathcal{P}_p^{1,0}(T) \to \mathcal{P}_p^1(K_H) \bigoplus \mathcal{P}_p^{1,0}(T) \times \mathcal{P}_1(I_H)$ with desired properties, where $I_H = (0, H)$, which can lead to a polynomial extension from a triangular face to the whole prism.

We now introduce polynomial lifting operator R_H on K_H defined by

$$R_H f(x_1, x_2, x_3) = R_K f(x_1, x_2, x_3) - \frac{x_3}{H} R_K f(x_1, x_2, H), \tag{4.10}$$

where R_K is the lifting operator on K given in (4.2).

Theorem 4.3. *Let R_H be the operator given in (4.10). Then, $R_H f(x) \in \mathcal{P}_p^1(K_H) \bigoplus \mathcal{P}_p^{1,0}(T) \times \mathcal{P}_1(I_H)$ for $f \in \mathcal{P}_p^{1,0}(T)$ such that $R_H f(x) |_{\tilde{\Gamma}_3} = f, R_H f |_{\tilde{\Gamma}_i} = 0, i = 1, 2, 4, 5,$ and*

$$\|R_H f\|_{H^1(K_H)} \leq C \|f\|_{H_{00}^{\frac{1}{2}}(\tilde{\Gamma}_3)}, \tag{4.11}$$

where $I_H = (0, H)$ and $T_H = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1 - H\}$, and C is a constant independent of f and p.

Incorporating R_H and the mapping M, we construct an extension R_G by

$$R_G^T f(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = R_H f \circ M = U(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) - \tilde{x}_3 U(\tilde{x}_1, \tilde{x}_2, 1). \tag{4.12}$$

where $U(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = R_K f \circ M$. Suppose that $R_K f(x_1, x_2, x_3) = \sum_{i+j+k \leq p} a_{ijk} x_1^i x_2^j x_3^k$, then

$$\begin{aligned} R_K f \circ M(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) &= U(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \\ &= \sum_{i+j+k \leq p} a_{ijk} H^k \tilde{x}_1^i \tilde{x}_2^j \tilde{x}_3^k (1 - H\tilde{x}_3)^{i+j} \in \mathcal{P}_p^{1,0}(T) \times \mathcal{P}_p(I). \end{aligned}$$

and

$$\frac{x_3}{H} R_K f(x_1, x_2, H) \circ M = \tilde{x}_3 U(\tilde{x}_1, \tilde{x}_2, 1) \in \mathcal{P}_p^{1,0}(T) \times \mathcal{P}_1(I).$$

Therefore $R_G^T f(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = R_H f \circ M \in \mathcal{P}_p^{1,0}(T) \times \mathcal{P}_p(I)$ if $f \in \mathcal{P}_p^{1,0}(T)$. We are able to establish the polynomial extension from a triangular face to a prism.

Theorem 4.4. *Let R_G^T be the extension defined in (4.12). Then $R_G^T f \in \mathcal{P}_p^{1,0}(T)\times\mathcal{P}_p(I)$ for $f\in\mathcal{P}_p^{1,0}(T)$, $R_G^T f\,|_{\Gamma_3}= f$ and vanishes on $\partial G\backslash\Gamma_3$, and*

$$\|R_G^T f\|_{H^1(G)} \le C\|f\|_{H^{\frac{1}{2}}_{00}(\Gamma_3)}, \tag{4.13}$$

where C is a constant independent of f and p.

Proof. Obviously, $R_G^T:\mathcal{P}_p^{1,0}(T)\to\mathcal{P}_p^{1,0}(T)\times\mathcal{P}_p(I)$, and $R_G^T f = f$ for all $f\in\mathcal{P}_p^{1,0}(T)$, $R_G^T f\,|_{\Gamma_i}= 0, i=1,2,4,5$. Since the mapping M is trilinear,

$$\|R_G^T f\|_{H^1(G)} \le C\|R_H f\|_{H^1(K_H)}.$$

Then (4.13) follows from (4.11) easily. □

It remained to prove Theorem 4.3. To this end, we need the following lemmas.

Lemma 4.5. *For $0<h<a$ and any function $g\in L^2(0,a)$, it holds that*

$$\int_0^{a-h}\Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \int_0^a |g(x)|^2 dx. \tag{4.14}$$

Also there hold

$$\int_0^{a-h}\Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \frac{1}{h}\int_0^a x|g(x)|^2 dx \tag{4.15}$$

and

$$\int_0^{a-h}\Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \frac{1}{h}\int_0^a (a-x)|g(x)|^2 dx. \tag{4.16}$$

Proof. By Schwarz inequality, we have

$$\int_0^{a-h}\Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \int_0^{a-h}\Big|\frac{1}{h}\int_x^{x+h} |g(\xi)|d\xi\Big|^2 dx \;\;\le\;\; \int_0^{a-h}dx\int_x^{x+h}\frac{|g(\xi)|^2}{h}d\xi.$$

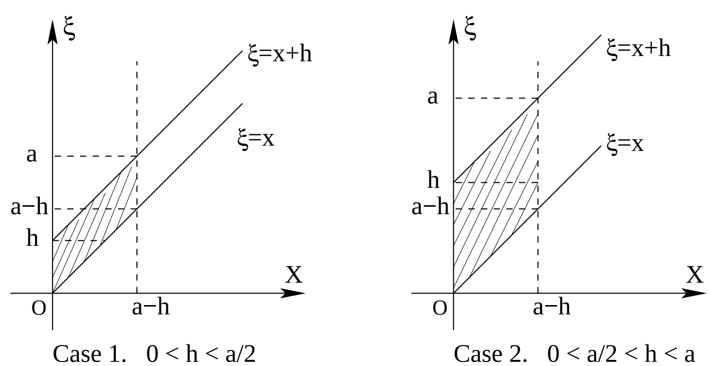


Fig. 4.3 Case 1 and Case 2

- **Case 1 :** $0 < h \le a/2$. There holds

$$\begin{aligned}
&\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \int_0^{a-h} dx \int_x^{x+h} \frac{|g(\xi)|^2}{h} d\xi \\
=& \int_0^h d\xi \int_0^{\xi} \frac{|g(\xi)|^2}{h} dx + \int_h^{a-h} d\xi \int_{\xi-h}^{\xi} \frac{|g(\xi)|^2}{h} dx + \int_{a-h}^{a} d\xi \int_{\xi-h}^{a-h} \frac{|g(\xi)|^2}{h} dx \\
=& \int_0^h \frac{\xi|g(\xi)|^2}{h} d\xi + \int_h^{a-h} \frac{h|g(\xi)|^2}{h} d\xi + \int_{a-h}^{a} \frac{(a-\xi)|g(\xi)|^2}{h} d\xi.
\end{aligned}$$

Hence, we have

$$\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \ \le\ \int_0^a |g(\xi)|^2 d\xi$$

and

$$\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \ \le\ \frac{1}{h}\int_0^a \xi|g(\xi)|^2 d\xi.$$

- **Case 2 :** $a/2 < h < a$. Similarly, there holds

$$\begin{aligned}
&\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \le \int_0^{a-h} dx \int_x^{x+h} \frac{|g(\xi)|^2}{h} d\xi \\
=& \int_0^{a-h} d\xi \int_0^{\xi} \frac{|g(\xi)|^2}{h} dx + \int_{a-h}^{h} d\xi \int_0^{a-h} \frac{|g(\xi)|^2}{h} dx + \int_h^{a} d\xi \int_{\xi-h}^{a-h} \frac{|g(\xi)|^2}{h} dx \\
=& \int_0^{a-h} \frac{\xi|g(\xi)|^2}{h} d\xi + \int_{a-h}^{h} \frac{(a-h)|g(\xi)|^2}{h} d\xi + \int_h^{a} \frac{(a-\xi)|g(\xi)|^2}{h} d\xi,
\end{aligned}$$

which implies

$$\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \ \le\ \int_0^a |g(\xi)|^2 d\xi$$

and

$$\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx \ \le\ \frac{1}{h}\int_0^a \xi|g(\xi)|^2 d\xi.$$

Therefore we always have (4.14) and (4.15) for $0 < h \le a/2$ or $a/2 < h < a$.

Letting $\eta = a - \xi$ and $\hat{x} = a - h - x$ and using (4.15) we obtain

$$\begin{aligned}
&\int_0^{a-h} \Big|\frac{1}{h}\int_x^{x+h} g(\xi)d\xi\Big|^2 dx = \int_0^{a-h} \Big|\frac{1}{h}\int_{\hat{x}}^{\hat{x}+h} g(a-\eta)d\eta\Big|^2 d\hat{x} \\
\le\ & \frac{1}{h}\int_0^a \hat{x}|g(a-\hat{x})|^2 d\hat{x} = \frac{1}{h}\int_0^a (a-z)|g(z)|^2 dz,
\end{aligned}$$

which yields (4.16). $\square$

Lemma 4.6. *Let $R_{12}(x_1,x_2,H)$ and $R_{i3}(x_1,x_2,H)$ be the operators given in (4.4) and (4.5) with $x_3 = H$. Then*

$$\|R_{12}f(x_1,x_2,H)\|_{L^2(K_H)} \le C\|(x_1x_2)^{\frac{1}{2}}f(x_1,x_2)\|_{L^2(T)}, \tag{4.17}$$

and for $i = 1,2$

$$\|R_{i3}f(x_1,x_2,H)\|_{L^2(K_H)} \le C\|x_i^{\frac{1}{2}}(1-x_1-x_2)^{\frac{1}{2}}f(x_1,x_2)\|_{L^2(T)}, \tag{4.18}$$

where C is a constant independent of f.

Proof. Note that

$$\|R_{12}f(x_1,x_2,H)\|^2_{L^2(K_H)} \le \frac{4}{H^2}\int_0^H dx_3\int_0^{1-x_3}dx_2\int_0^{1-x_2-x_3}\Big|\frac{1}{H}\int_{x_1}^{x_1+H}g_1(\xi_1)d\xi_1\Big|^2dx_1$$

with $g_1(\xi_1) = \displaystyle\int_{x_2}^{x_2+H}|\tilde f(\xi_1,\xi_2)|d\xi_2$. Hereafter $\tilde f$ denotes the extension of f by zero outside T. We apply here Lemma 4.5 to $g_1(\xi_1)$ with $a = 1-x_2-x_3, h = H, x = x_1, \xi = \xi_1$, then we get

$$\int_0^{1-x_2-x_3}\Big(\frac{1}{H}\int_{x_1}^{x_1+H}g(\xi_1)d\xi_1\Big)^2dx_1 \le \frac{1}{H}\int_0^{1-x_2-x_3+H}x_1|g_1(x_1)|^2dx_1,$$

which implies

$$\begin{aligned}
&\int_0^{1-x_3}dx_2\int_0^{1-x_2-x_3}\Big|\frac{1}{H}\int_{x_1}^{x_1+H}g_1(\xi_1)d\xi_1\Big|^2dx_1 \\
\le\ &\frac{1}{H}\int_0^{1-x_3}dx_2\int_0^{1-x_2-x_3+H}x_1\Big|\int_{x_2}^{x_2+H}|\tilde f(x_1,\xi_2)|d\xi_2\Big|^2dx_1 \\
=\ &H\Big\{\int_0^H x_1dx_1\int_0^{1-x_3}\Big|\frac{1}{H}\int_{x_2}^{x_2+H}|\tilde f(x_1,\xi_2)|d\xi_2\Big|^2dx_2 \\
&+\int_H^{1-x_3+H}x_1dx_1\int_0^{1-x_1-x_3+H}\Big|\frac{1}{H}\int_{x_2}^{x_2+H}|\tilde f(x_1,\xi_2)|d\xi_2\Big|^2dx_2\Big\}.
\end{aligned}\tag{4.19}$$

Applying Lemma 4.5 again, we have

$$\int_0^{1-x_3}\Big|\frac{1}{H}\int_{x_2}^{x_2+H}|\tilde f(x_1,\xi_2)|d\xi_2\Big|^2dx_2 \le \frac{1}{H}\int_0^{1-x_3+H}x_2|\tilde f(x_1,x_2)|^2dx_2,$$

and

$$\int_0^{1-x_1-x_3+H}\Big|\frac{1}{H}\int_{x_2}^{x_2+H}|\tilde f(x_1,\xi_2)|d\xi_2\Big|^2dx_2 \ \le\ \frac{1}{H}\int_0^{1-x_1-x_3+2H}x_2|\tilde f(x_1,x_2)|^2dx_2,$$

which together with (4.19) yields

$$
\begin{aligned}
&\int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3} \Big(\frac{1}{H}\int_{x_1}^{x_1+H} d\xi_1 \int_{x_2}^{x_2+H} |\tilde{f}(\xi_1,\xi_2)| d\xi_2\Big)^2 dx_1 \\
\leq\ & \Big(\int_0^H dx_1 \int_0^{1+H} + \int_H^{1-x_3+H} dx_1 \int_0^{1-x_1+2H}\Big) x_2 x_1 |\tilde{f}(x_1,x_2)|^2 dx_2 \\
\leq\ & \Big(\int_0^H dx_1 \int_0^{1+H} + \int_H^{1+H} dx_1 \int_0^{1-x_1+2H}\Big) x_2 x_1 |\tilde{f}(x_1,x_2)|^2 dx_2 \leq 2\|(x_1x_2)^{\frac{1}{2}} f\|^2_{L^2(T)}.
\end{aligned}
$$

Therefore (4.17) follows immediately.

Let Q_1 be the mapping:

$$x_1 = \hat{x}_2, \quad x_2 = 1-\hat{x}_1-\hat{x}_2-\hat{x}_3, \quad \hat{x}_3 = x_3, \tag{4.20}$$

which maps K_H onto itself, and let W_1 be the mapping:

$$\xi_1 = \hat{\xi}_2, \quad \xi_2 = 1-\hat{\xi}_1-\hat{\xi}_2, \tag{4.21}$$

which maps T onto itself. Then $\hat{f}(\hat{\xi}_1,\hat{\xi}_2) = f(\xi_1,\xi_2)\circ W_1 = f(\hat{\xi}_2, 1-\hat{\xi}_1-\hat{\xi}_2)$ and $R_{12}f(\hat{x}_1,\hat{x}_2,H) = R_{13}f(x_1,x_2,x_3)\circ Q_1 \mid_{x_3=H}$. Therefore

$$
\begin{aligned}
\|R_{13}f(x_1,x_2,H)\|_{L^2(K_H)} &\leq \|R_{12}\hat{f}(\hat{x}_1,\hat{x}_2,H)\|_{L^2(K_H)} \leq C\|(\hat{\xi}_1\hat{\xi}_2)^{\frac{1}{2}}\hat{f}\|_{L^2(T)} \\
&\leq C\|\xi_1^{\frac{1}{2}}(1-\xi_1-\xi_2)^{\frac{1}{2}} f\|_{L^2(T)}.
\end{aligned}
$$

For $R_{23}f$, we introduce mapping Q_2 and W_2:

$$Q_2: \quad x_1 = 1-\hat{x}_1-\hat{x}_2-\hat{x}_3, \quad x_2 = \hat{x}_1, \quad x_3 = \hat{x}_3, \tag{4.22}$$

which maps K_H onto itself, and

$$W_2: \quad \xi_1 = 1-\hat{\xi}_1-\hat{\xi}_2, \quad \xi_2 = \hat{\xi}_1, \tag{4.23}$$

which maps T onto itself. Similarly

$$
\begin{aligned}
\|R_{23}f(x_1,x_2,H)\|_{L^2(K_H)} &\leq \|R_{12}\hat{f}(\hat{x}_1,\hat{x}_2,H)\|_{L^2(K_H)} \leq C\|(\hat{\xi}_1\hat{\xi}_2)^{\frac{1}{2}}\hat{f}\|_{L^2(T)} \\
&\leq C\|\xi_2^{\frac{1}{2}}(1-\xi_1-\xi_2)^{\frac{1}{2}} f\|_{L^2(T)}.
\end{aligned}
$$

$\square$

Lemma 4.7. *Let $R_{12}(x_1,x_2,H)$ and $R_{i3}(x_1,x_2,H)$ be the operators given in (4.4) and (4.5) with $x_3 = H$. Then for $i = 1,2$*

$$\Big\|\frac{\partial R_{12}f(x_1,x_2,H)}{\partial x_i}\Big\|_{L^2(K_H)} \leq C\|x_i^{-\frac{1}{2}} f\|_{L^2(T)}, \tag{4.24}$$

and $t = 1,2$

$$\Big\|\frac{\partial R_{i3}f(x_1,x_2,H)}{\partial x_t}\Big\|_{L^2(K_H)} \leq C\Big(\|x_t^{-\frac{1}{2}} f\|_{L^2(T)} + \|(1-x_1-x_2)^{-\frac{1}{2}} f\|_{L^2(T)}\Big), \tag{4.25}$$

where C is a constant independent of f.

PROOF. Note that

$$\frac{\partial R_{12}f(x_1,x_2,H)}{\partial x_1} = \frac{2x_2}{H^2}\int_{x_1}^{x_1+H} d\xi_1 \int_{x_2}^{x_1+x_2+H-\xi_1} \frac{f(\xi_1,\xi_2)}{\xi_1\xi_2} d\xi_2$$
$$-\frac{2x_2}{H^2}\int_{x_2}^{x_2+H} \frac{f(x_1,\xi_2)}{\xi_2} d\xi_2 + \frac{2x_1x_2}{H^2}\int_{x_1}^{x_1+H} \frac{f(\xi_1,x_1+x_2+H-\xi_1)}{\xi_1(x_1+x_2+H-\xi_1)} d\xi_1$$

and

$$\Big|\frac{\partial R_{12}f(x_1,x_2,H)}{\partial x_1}\Big| \le I_1+I_2+I_3, \tag{4.26}$$

where

$$I_1 = \frac{2}{H^2}\int_{x_1}^{x_1+H} d\xi_1 \int_{x_2}^{x_2+H} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_2, \quad I_2 = \frac{2}{H^2}\int_{x_2}^{x_2+H} |f(x_1,\xi_2)| d\xi_2$$
$$I_3 = \frac{2}{H^2}\int_{x_1}^{x_1+H} |f(\xi_1,x_1+x_2+H-\xi_1)| d\xi_1.$$

Note that

$$\|I_1\|^2_{L^2(K_H)} = \frac{4}{H^2}\int_0^H dx_3 \int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3} \Big(\frac{1}{H}\int_{x_1}^{x_1+H} g_1(\xi_1)d\xi_1\Big)^2 dx_1$$

with $g_1(\xi_1) = \int_{x_2}^{x_2+H} \frac{|\tilde f(\xi_1,\xi_2)|}{\xi_1} d\xi_2$. Applying Lemma 4.5 to $g_1(\xi_1)$ with $a = 1-x_2-x_3, h = H, x = x_1, \xi = \xi_1$, we have

$$\int_0^{1-x_2-x_3} \Big|\frac{1}{H}\int_{x_1}^{x_1+H} g_1(\xi_1)d\xi_1\Big|^2 dx_1 \le \frac{1}{H}\int_0^{1-x_2-x_3+H} x_1\Big|\int_{x_2}^{x_2+H} \frac{\tilde f(x_1,\xi_2)}{x_1} d\xi_2\Big|^2 dx_1,$$

which implies

$$\begin{aligned}
&\int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3} \Big|\frac{1}{H}\int_{x_1}^{x_1+H} g_1(\xi_1)d\xi_1\Big|^2 dx_1 \\
\le\ & \frac{1}{H}\int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3+H} \frac{1}{x_1}\Big|\int_{x_2}^{x_2+H} \tilde f(x_1,\xi_2)d\xi_2\Big|^2 dx_1 \\
\le\ & H\Big\{\int_0^H \frac{1}{x_1}dx_1 \int_0^{1-x_3} \Big|\frac{1}{H}\int_{x_2}^{x_2+H} \tilde f(x_1,\xi_2)d\xi_2\Big|^2 dx_2 \\
&+ \int_H^{1-x_3+H} \frac{1}{x_1}dx_1 \int_0^{1-x_1-x_3+H} \Big|\frac{1}{H}\int_{x_2}^{x_2+H} \tilde f(x_1,\xi_2)d\xi_2\Big|^2 dx_2\Big\}.
\end{aligned}$$

Applying Lemma 4.5 again to the function $g_2(\xi_2) = \tilde{f}(x_1, \xi_2)$, we have

$$
\begin{aligned}
(4.27) \quad \|I_1\|^2_{L^2(K_H)} &\leq \frac{4}{H^2}\int_0^H dx_3 \int_0^H \frac{1}{x_1}dx_1 \int_0^{1-x_3+H} |\tilde{f}(x_1,x_2)|^2 dx_2 \\
&\quad + \frac{4}{H^2}\int_0^H dx_3 \int_H^{1-x_3+H} \frac{1}{x_1}dx_1 \int_0^{1-x_1-x_3+2H} |\tilde{f}(x_1,x_2)|^2 dx_2 \\
&\leq \frac{4}{H}\Big(\int_0^H dx_1 \int_0^{1+H} + \int_H^{1+H} dx_1 \int_0^{1-x_1+2H}\Big)\frac{|\tilde{f}(x_1,x_2)|^2}{x_1}dx_2 \\
&\leq \frac{8}{H}\|{x_1}^{-\frac{1}{2}} f\|^2_{L^2(T)}.
\end{aligned}
$$

Similarly we have by Lemma 4.5,

$$
\begin{aligned}
(4.28) \|I_2\|^2_{L^2(K_H)} &= \frac{4}{H^2}\int_0^H dx_3 \int_0^{1-x_3} dx_1 \int_0^{1-x_1-x_3} \Big|\frac{1}{H}\int_{x_2}^{x_2+H} |f(x_1,\xi_2)| d\xi_2\Big|^2 dx_2 \\
&\leq \frac{4}{H^3}\int_0^H dx_3 \int_0^{1-x_3} dx_1 \int_0^{1-x_1-x_3+H} x_2 |f(x_1,x_2)|^2 dx_2 \\
&\leq \frac{4}{H^3}\int_0^H dx_3 \int_0^{1} dx_1 \int_0^{1-x_1+H} x_2 |\tilde{f}(x_1,x_2)|^2 dx_2 = \frac{4}{H^2}\|x_2^{\frac{1}{2}} f\|^2_{L^2(T)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|I_3\|^2_{L^2(K_H)} &= \frac{4}{H^2}\int_0^H dx_3 \int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3} \Big|\frac{1}{H}\int_{x_1}^{x_1+H} |\tilde{f}(\xi_1, x_1+x_2+H-\xi_1)| d\xi_1\Big|^2 dx_1 \\
&\leq \frac{4}{H^3}\int_0^H dx_3 \int_0^{1-x_3} dx_2 \int_0^{1-x_2-x_3+H} x_1 |\tilde{f}(x_1, x_2+H)|^2 dx_1 \\
&\leq \frac{4}{H^3}\int_0^H dx_3 \int_0^{1} dx_2 \int_0^{1-x_2+H} x_1 |\tilde{f}(x_1, x_2+H)|^2 dx_1 \\
&= \frac{4}{H^2}\int_0^1 dx_2 \int_0^{1-x_2+H} x_1 |\tilde{f}(x_1, x_2+H)|^2 dx_1.
\end{aligned}
$$

Letting $z = x_2 + H$, we have

$$
\begin{aligned}
&\frac{4}{H^2}\int_0^1 dx_2 \int_0^{1-x_2+H} x_1|\tilde{f}(x_1,x_2+H)|^2 dx_1 = \frac{4}{H^2}\int_H^{1+H} dz \int_0^{1-z+2H} x_1|\tilde{f}(x_1,z)|^2 dx_1 \\
&= \frac{4}{H^2}\int_H^1 dz \int_0^{1-z} x_1 |\tilde{f}(x_1,z)|^2 dx_1 \leq \frac{4}{H^2}\|x_1^{\frac{1}{2}} f\|^2_{L^2(T)},
\end{aligned}
$$

which implies

$$
(4.29) \qquad \|I_3\|^2_{L^2(K_H)} \leq \frac{4}{H^2}\|x_1^{\frac{1}{2}} f\|^2_{L^2(T)}.
$$

Combining (4.26)-(4.29), we have

$$\Big\|\frac{\partial R_{12}f(x_1,x_2,H)}{\partial x_1}\Big\|_{L^2(K_H)} \leq C\|x_1^{-\frac{1}{2}}f\|_{L^2(T)}.$$

Similarly, we can prove

$$\Big\|\frac{\partial R_{12}f(x_1,x_2,H)}{\partial x_2}\Big\|_{L^2(K_H)} \leq C\|x_2^{-\frac{1}{2}}f\|_{L^2(T)}.$$

Let Q_i and W_i (i=1,2) be the mapping as defined in (4.20)-(4.23). Then, for $t=1,2$,

$$\begin{aligned}\Big\|\frac{\partial R_{13}f(x_1,x_2,H)}{\partial x_t}\Big\|_{L^2(K_H)} &\leq \sum_{i=1,2}\Big\|\frac{\partial R_{12}\hat{f}(\hat{x}_1,\hat{x}_2,H)}{\partial \hat{x}_i}\Big\|_{L^2(K_H)} \leq C\sum_{i=1,2}\|\hat{\xi}_i^{-\frac{1}{2}}\hat{f}\|_{L^2(T)}\\ &\leq C(\|\xi_1^{-\frac{1}{2}}f\|_{L^2(T)}+\|(1-\xi_1-\xi_2)^{-\frac{1}{2}}f\|_{L^2(T)}).\end{aligned}$$

Similarly, we have for $t=1,2$

$$\begin{aligned}\Big\|\frac{\partial R_{23}f(x_1,x_2,H)}{\partial x_t}\Big\|_{L^2(K_H)} &\leq \sum_{i=1,2}\Big\|\frac{\partial R_{12}\hat{f}(\hat{x}_1,\hat{x}_2,H)}{\partial \hat{x}_i}\Big\|_{L^2(K_H)} \leq C\sum_{i=1,2}\|\hat{\xi}_i^{-\frac{1}{2}}\hat{f}\|_{L^2(T)}\\ &\leq C(\|\xi_2^{-\frac{1}{2}}f\|_{L^2(T)}+\|(1-\xi_1-\xi_2)^{-\frac{1}{2}}f\|_{L^2(T)}).\end{aligned}$$

□

Proof of Theorem 4.3 Obviously, $R_Hf(x)\in\mathcal{P}_p^1(K_H)\bigoplus\mathcal{P}_p^{1,0}(T)\times\mathcal{P}_1(I_H)$ for $f\in\mathcal{P}_p^{1,0}(T)$. Due to (4.10), we have

$$\begin{aligned}&\|R_Hf(x_1,x_2,x_3)\|_{H^1(K_H)}\\ &\leq \|R_Kf(x_1,x_2,x_3)\|_{H^1(K_H)}+\Big\|\frac{x_3}{H}R_Kf(x_1,x_2,H)\Big\|_{H^1(K_H)}.\end{aligned}\tag{4.30}$$

By Theorem 4.1, there holds

$$\|R_Kf(x_1,x_2,x_3)\|_{H^1(K_H)}\leq\|R_Kf(x_1,x_2,x_3)\|_{H^1(K)}\leq C\|f(x_1,x_2)\|_{H_{00}^{\frac{1}{2}}(T)},\tag{4.31}$$

and by (4.3) and Lemma 4.6- Lemma 4.7, it holds that

$$\begin{aligned}&\Big\|\frac{x_3}{H}R_Kf(x_1,x_2,H)\Big\|_{H^1(K_H)}\leq C(\|R_{12}f(x_1,x_2,H)\|_{H^1(K_H)}+\sum_{i=1,2}\|R_{i3}f(x_1,x_2,H)\|_{H^1(K_H)})\\ &\leq C(\|f\|_{H^{\frac{1}{2}}(T)}+\sum_{i=1,2}\|x_i^{-\frac{1}{2}}f\|_{L^2(T)}+\|(1-x_1-x_2)^{-\frac{1}{2}}f\|_{L^2(T)})\leq C\|f\|_{H_{00}^{\frac{1}{2}}(T)},\end{aligned}$$

which together with (4.30)-(4.31) leads to (4.11) immediately. □

4.1.3. Polynomial extension on prisms from a square face. We shall construct a polynomial extension on prisms from a square face $S = \{x = (x_1, x_2, x_3) \mid 0 \le x_1, x_3 \le 1\}$, which is as important as the extension from a triangular face for error analysis and preconditioning of high-order finite element methods in three dimensions [**25**].

Lemma 4.8. *Let $T = \{(x_1, x_2) | 0 < x_2 < 1 - x_1, 0 \le x_1 < 1\}$ be the standard triangle and $I = (0, 1)$. Then there is a polynomial extension operator $R_T^* : H_0^1(I) \to H^1(T)$ such that $R_T^* f \in \mathcal{P}_p^1(T)$ if $f(x_1) \in \mathcal{P}_p^0(I)$, and*

$$R_T^* f|_I = f(x_1),\ R_T^* f|_{\partial T \setminus I} = 0, \tag{4.32}$$

$$\|R_T^* f\|_{H^t(T)} \le C\Big(p^{t-\frac{3}{2}}\|f\|_{H^1(I)} + p^{t-\frac{1}{2}}\|f\|_{L^2(I)}\Big),\ t = 0, 1. \tag{4.33}$$

with C independent of f and p.

Proof. Let $\psi(x_2) = (1 - x_2)^p$. Then for $t \ge 0$

$$\|\psi\|_{H^t(I)} \le Cp^{t-\frac{1}{2}}. \tag{4.34}$$

We introduce a function $\Psi \in \mathcal{P}_{2p+1}^1(T)$ by

$$\Psi(x_1, x_2) \;=\; \psi(x_2)\Big((1 - x_1 - x_2)f(x_1) + x_1 f(x_1 + x_2)\Big).$$

Then $\Psi(x_1, 0) = f(x_1), \Psi(1, x_2) = \Psi(x_1, 1 - x_1) = 0$, and

$$\|\Psi\|_{L^2(T)} \le Cp^{-\frac{1}{2}}\|f\|_{L^2(I)}, \tag{4.35}$$

$$\|\Psi\|_{H^1(T)} \le C\Big(p^{-\frac{1}{2}}\|f\|_{H^1(I)} + p^{\frac{1}{2}}\|f\|_{L^2(I)}\Big). \tag{4.36}$$

By the lifting theorem on the triangle T [**35**], there exists a lifting operator $R_T : H_{00}^{\frac{1}{2}}(I) \to H^1(T)$

$$R_T f = \frac{x_1(1 - x_1 - x_2)}{x_2^2} \int_{x_1}^{x_1 + x_2} \frac{f(\xi)}{\xi(1 - \xi)} d\xi$$

such that $R_T f \in \mathcal{P}_p^1(T), R_T f\ |_I = f$ and $R_T f\ |_{\partial T \setminus I} = 0$, and

$$\|R_T f\|_{H^1(T)} \le C\|f\|_{H_{00}^{\frac{1}{2}}(I)},$$

which implies that R_T satisfies (4.33) with $t = 1$. Unfortunately, the extension does not give a precise information on $\|R_T f\|_{L^2(T)}$, and the desired estimation (4.33) with $t = 0$ may not be true for R_T. Therefore we have to construct a new extension operator R_T^*.

Note that $\Psi - R_T f = 0$ on ∂T. By Π_T we denote the orthogonal projection operator: $H_0^1(T) \to \mathcal{P}_p^{1,0}(T)$, and let

$$w_p = R_T f + \Pi_T(\Psi - R_T f).$$

Then $w_p(x_1, 0) = f(x_1)$ and $w_p(1, x_2) = w_p(x_1, 1 - x_1) = 0$, and

$$\Psi - w_p = \big(I - \Pi_T\big)(\Psi - R_T f). \tag{4.37}$$

Due to the continuity of operator R_T and a trace theorem, we obtain

$$
\begin{aligned}
\|w_p\|_{H^1(T)} &\le \|\Psi\|_{H^1(T)} + \|\Psi - w_p\|_{H^1(T)} \le \|\Psi\|_{H^1(T)} + \|\Psi - R_T f\|_{H^1(T)} \\
&\le 2\|\Psi\|_{H^1(T)} + \|R_T f\|_{H^1(T)} \le C(\|\Psi\|_{H^1(T)} + \|f\|_{H^{\frac{1}{2}}_{00}(I)}) \\
&\le C(\|\Psi\|_{H^1(T)} + \|\Psi\|_{H^{\frac{1}{2}}(\partial T)}) \le C\|\Psi\|_{H^1(T)}.
\end{aligned} \tag{4.38}
$$

Let $R_T^* f = w_p$. Then (4.36) and (4.38) leads to (4.32) and (4.33) with $t = 1$. Note that $\Pi_T(\Psi - R_T f)$ is the finite element solution in $\mathcal{P}_p^{1,0}(T)$ for the the boundary value problem:

$$
-\Delta u + u = \tilde{f} \qquad \text{in } \ T
$$

$$
u\,|_{\partial T} = 0
$$

with $\tilde{f} = -\Delta(\Psi - R_T f) + \Psi - R_T f$. By the Nitsche's trick, we have

$$
\|\big(I - \Pi_T\big)(\Psi - R_T f)\|_{L^2(T)} \le Cp^{-1}\|\big(I - \Pi_T\big)(\Psi - R_T f)\|_{H^1(T)} \le Cp^{-1}\|\Psi\|_{H^1(T)},
$$

which implies

$$
\|\Psi - w_p\|_{L^2(T)} = \|\big(I - \Pi_T\big)(\Psi - R_T f)\|_{L^2(T)} \le Cp^{-1}\|\Psi\|_{H^1(T)}. \tag{4.39}
$$

Combining (4.39) and (4.36) we have (4.33) for $t = 0$. $\square$

We construct a polynomial extension from a square face to the prism G with help of the extension R_T^* in triangle T:

$$
R_G^S f(x_1, x_2, x_3) = R_T^* f(\cdot, x_3) \tag{4.40}
$$

Theorem 4.9. *Let $\Gamma_2 = S$ be a square face of the prism G as shown in Fig. 4.2, and let R_G^S be the extension operator defined as in (4.40). Then $R_G^S f \in \mathcal{P}_p^1(T) \times \mathcal{P}_p(I)$ for $f \in \mathcal{P}_p^{2,0}(\Gamma_2)$, and*

$$
R_G^S f = f \ \text{ on } \ \Gamma_2, \quad R_G^S f = 0 \ \text{ on } \ \partial G \backslash \Gamma_2, \tag{4.41}
$$

$$
\|R_G^S f\|_{H^1(G)} \le C\Big(p^{-\frac{3}{2}}|f_{x_3}|_{H^1(\Gamma_2)} + p^{-\frac{1}{2}}|f|_{H^1(\Gamma_2)} + p^{\frac{1}{2}}\|f\|_{L^2(\Gamma_2)}\Big), \tag{4.42}
$$

$$
\|R_G^S f\|_{L^2(G)} \le C\Big(p^{-\frac{3}{2}}\|f\|_{H^1(\Gamma_2)} + p^{-\frac{1}{2}}\|f\|_{L^2(\Gamma_2)}\Big). \tag{4.43}
$$

Proof. Obviously, $R_G^S f \in \mathcal{P}_p^1(T) \times \mathcal{P}_p(I)$ and (4.41) holds. Due to (4.40)

$$
\begin{aligned}
\|R_G^S f\|_{L^2(G)}^2 &= \int_0^1 \Big(\int_T |\ R_G^S f\ |^2 \, dx_1 dx_2\Big) dx_3 \le \int_0^1 \|R_T^* f\|_{L^2(T)}^2 dx_3 \\
&\le C \int_0^1 \Big(p^{-3}\|f(\cdot, x_3)\|_{H^1(I)}^2 + p^{-1}\|f(\cdot, x_3)\|_{L^2(I)}^2\Big) dx_3 \\
&\le C\Big(p^{-3}\|f\|_{H^1(S)}^2 + p^{-1}\|f\|_{L^2(S)}^2\Big)
\end{aligned}
$$

which leads to (4.43).

Applying (4.40) to $f(x_1,x_3)$ and $f_{x_3}(x_1,x_3)$, respectively, we have

$$\begin{aligned}|R_G^S f|^2_{H^1(G)} &\le \int_0^1 \Big(|R_T^* f|^2_{H^1(T)} + |R_T^* f_{x_3}|^2_{L^2(T)}\Big)dx_3\\ &\le C\int_0^1 \Big(p^{-1}\|f\|^2_{H^1(I)} + p^1\|f\|^2_{L^2(I)} + p^{-3}\|f_{x_3}\|^2_{L^2(I)}\Big)dx_3\\ &\le C\Big(p^{-3}|f_{x_3}|^2_{H^1(S)} + p^{-1}|f|^2_{H^1(S)} + p\|f\|^2_{L^2(S)}\Big),\end{aligned}$$

which implies (4.42). □

Remark 4.1. It is an open problem whether there exists a polynomial extension operator R_G^S such that

$$\|R_G^S f\|_{H^1(G)} \le C\|f\|_{H_{00}^{1/2}(\Gamma_2)}. \tag{4.44}$$

Although (4.42) is not strong as (4.44), it gives the dependence of $\|R_G^S f\|_{H^1(G)}$ on $\|f\|_{H^t(S)}, t = 2,1,0$ furnished precisely with a weight $p^{-3/2}, p^{-1/2}$ and $p^{1/2}$, respectively. This estimation is sufficient while applying the extension to a pair of elements sharing a common square face for constructing a continuous piecewise polynomial in $\mathcal{P}_p^{1.5}(G)$. Hence, Theorem 4.9 plays an important role in error analysis for the p and h-p versions of finite element method in three dimensions on meshes containing triangular prism elements.

4.2. Extension on a standard pyramid

We denote, by Λ, a standard pyramid $\{(x_1,x_2,x_3) \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1, x_1+x_3 \le 1, x_2+x_3 \le 1\}$in $\mathbb{R}^3$ shown in Fig. 4.4 . Let $\Gamma_i, 1 \le i \le 3$ be the faces of Λ contained in the plane $x_i = 0$ and $\Gamma_i, i = 4,5$ be the oblique faces, and let $S = \{(x_1,x_2) \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ be a square in $\mathbb{R}^2$. Obviously, $\Gamma_3 = S$. For f in $L^1(S)$, we introduce an extension operator $F_\Lambda f$ on Λ by

$$F_\Lambda f(x_1,x_2,x_3) = \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} f(\xi_1,\xi_2)d\xi_2. \tag{4.1}$$

It is easy to verify that if $f \in \mathcal{P}_p^1(S)$, then $F_\Lambda f \in \mathcal{P}_p^1(\Lambda)$ and it holds that $F_\Lambda f(x_1,x_2,0) = f(x_1,x_2)$ and we have the following theorem.

Theorem 4.10. *Let F_Λ be the operator defined by (4.1). Then for any f in $H^{\frac{1}{2}}(S)$, there hold*

$$\|F_\Lambda f\|_{L^2(\Lambda)} \le C\|f\|_{L^2(S)} \tag{4.2}$$

and

$$\|F_\Lambda f\|_{H^1(\Lambda)} \le C\|f\|_{H^{\frac{1}{2}}(S)}. \tag{4.3}$$

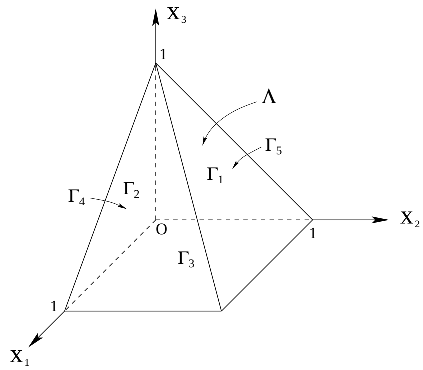


Fig. 4.4 The pyramid Λ

PROOF. We first extend f to a function defined on the entire plane $\mathbb{R}^2$ so that

$$\|f(x_1,x_2)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \le C\|f(x_1,x_2)\|_{H^{\frac{1}{2}}(S)},$$

where we have used the same notation f to denote the extended function as well. Then, we have

$$\begin{aligned} F_\Lambda f(x_1,x_2,x_3) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi_1,\xi_2)H(x_1-\xi_1,x_2-\xi_2,x_3)d\xi_1 d\xi_2 \\ &= (f * H(\cdot,x_3))(x_1,x_2,x_3) \end{aligned} \tag{4.4}$$

with $H(x_1,x_2,x_3) = H_1(x_1,x_3)H_2(x_2,x_3)$, and for $i=1,2$

$$H_i(x_i,x_3) = \begin{cases} \dfrac{1}{x_3}, & -x_3 \le x_i \le 0, \\ 0 & otherwise. \end{cases} \tag{4.5}$$

Let $\tilde{f}(\xi_1,\xi_2)$ and $\tilde{H}(\xi_1,\xi_2,x_3)$ represent the Fourier transform of the function $f(x_1,x_2)$ and $H(x_1,x_2,x_3)$ in the x_1 and x_2 direction. Then by (4.4)

$$\tilde{F}_\Lambda f(\xi_1,\xi_2,x_3) = \tilde{f}(\xi_1,\xi_2)\tilde{H}(\xi_1,\xi_2,x_3) = \tilde{f}(\xi_1,\xi_2)\tilde{H}_1(\xi_1,x_3)\tilde{H}_2(\xi_2,x_3), \tag{4.6}$$

where

$$\tilde{H}_1(\xi_j,x_3) = \frac{1}{\sqrt{2\pi}}\frac{1}{x_3}\int_{-x_3}^{0} e^{-i\xi_j x_j}dx_j = \frac{1}{\sqrt{2\pi}}\frac{e^{i\xi_j x_3}-1}{i\xi_j x_3}, j=1,2.$$

Let $\Omega = \{(x_1,x_2,x_3)| -\infty < x_1 < \infty, -\infty < x_2 < \infty, 0 < x_3 < 1\}$ (resp., $\widetilde{\Omega} = \{(\xi_1,\xi_2,x_3)| -\infty < \xi_1 < \infty, -\infty < \xi_2 < \infty, 0 < x_3 < 1\}$). By Parseval's equality, we have using (4.6)

$$\|F_\Lambda\|^2_{H^1(\Lambda)} \le \|F_\Lambda\|^2_{H^1(\Omega)} = \|\tilde{F}_\Lambda\|^2_{H^1(\widetilde{\Omega})} = \sum_{i=1,2}\int_{\widetilde{\Omega}} |\tilde{f}(\xi_1,\xi_2)|^2|\xi_i\tilde{H}(\xi_1,\xi_2,x_3)|^2 d\xi_1 d\xi_2 dx_3$$

$$+\int_{\widetilde{\Omega}} |\tilde{f}(\xi_1,\xi_2)|^2\Big|\frac{\partial}{\partial x_3}\tilde{H}(\xi_1,\xi_2,x_3)\Big|^2 d\xi_1 d\xi_2 dx_3 + \int_{\widetilde{\Omega}} |\tilde{f}(\xi_1,\xi_2)|^2|\tilde{H}(\xi_1,\xi_2,x_3)|d\xi_1 d\xi_2 dx_3.$$

Noting that $|\tilde{H}_i(\xi_i, x_3)| \le C$ for $i = 1, 2$, we have

$$(4.7)\quad \int_{\widetilde{\Omega}} |\tilde{f}(\xi_1, \xi_2)|^2 |\tilde{H}(\xi_1, \xi_2, x_3)|^2 d\xi_1 d\xi_2 dx_3 \le C \int_{\mathbb{R}^2} |\tilde{f}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \le C\|f\|^2_{L^2(\mathbb{R}^2)},$$

which implies (4.2).

Letting $u_j = x_3|\xi_j|$ for $j = 1, 2$, we obtain

$$\int_0^1 |\tilde{H}_j(\xi_j, x_3)|^2 dx_3 = \frac{1}{2\pi|\xi_j|} \int_0^{|\xi_j|} \Big|\frac{e^{iu_j} - 1}{iu_j}\Big|^2 du_j = \frac{1}{\pi|\xi_j|} \int_0^{|\xi_j|} \frac{1 - \cos u_j}{u_j^2} du_j \le \frac{C}{|\xi_j|},$$

which yields for $j = 1, 2$

$$(4.8)\quad \begin{aligned} &\int_{\widetilde{\Omega}} |\tilde{f}|^2 |\xi_j \tilde{H}(\xi_1, \xi_2, x_3)|^2 d\xi_1 d\xi_2 dx_3 \le C \int_{\mathbb{R}^2} |\xi_j||\tilde{f}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \\ &\le C\|\tilde{f}\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} \le C\|f\|^2_{H^{\frac{1}{2}}(S)}. \end{aligned}$$

Note that

$$\frac{\partial}{\partial x_3}\widetilde{H}(\xi_1, \xi_2, x_3) = \frac{1}{2\pi}\Big(\frac{i\xi_1 x_3 e^{i\xi_1 x_3} - e^{i\xi_1 x_3} + 1}{i\xi_1 x_3^2}\widetilde{H}(\xi_2, x_3) + \frac{i\xi_2 x_3 e^{i\xi_2 x_3} - e^{i\xi_2 x_3} + 1}{i\xi_2 x_3^2}\widetilde{H}(\xi_1, x_3)\Big)$$

and

$$\Big|\frac{\partial}{\partial x_3}\widetilde{H}(\xi_1, \xi_2, x_3)\Big|^2 \le \frac{C}{2\pi}\sum_{j=1}^{2} |\xi_j|^2 \frac{2(1 - \cos u_j) + u_j^2 - 2u_j \sin u_j}{u_j^4},$$

which imply

$$\int_0^1 \Big|\frac{\partial}{\partial x_3}\widetilde{H}(\xi_1, \xi_2, x_3)\Big|^2 dx_3 \le \frac{C}{2\pi}\sum_{j=1}^{2} \int_0^{|\xi_j|} |\xi_j| \frac{2(1 - \cos u_j) + u_j^2 - 2u_j \sin u_j}{u_j^4} du_j \le C(|\xi_1| + |\xi_2|)$$

and

$$(4.9)\quad \begin{aligned} \int_{\widetilde{\Omega}} |\tilde{f}(\xi_1, \xi_2)|^2 \Big|\frac{\partial}{\partial x_3}\widetilde{H}(\xi_1, \xi_2, x_3)\Big|^2 d\xi_1 d\xi_2 dx_3 \;&\le\; C\int_{\mathbb{R}^2} (|\xi_1| + |\xi_2|)|\tilde{f}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \\ &\le\; C\|\tilde{f}\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} \le C\|f\|^2_{H^{\frac{1}{2}}(S)}. \end{aligned}$$

A combination of (4.7)-(4.9) leads to (4.3). □

Remark 4.2. Theorem 4.10 can be generalized to high order, $F_\Lambda f$ realizes a continuous mapping $H^{m-\frac{1}{2}}(S) \to H^m(\Lambda)$ for $m \ge 1$.

We now introduce an extension operator $R_\Lambda : H^{\frac{1}{2}}_{00}(S) \to H^1(\Lambda)$

$$(4.10)\qquad R_\Lambda f(x_1, x_2, x_3) = x_1 x_2 (1 - x_1 - x_3)(1 - x_2 - x_3) F_\Lambda \tilde{f}(x_1, x_2, x_3)$$

with $\tilde{f}(x_1,x_2) = \frac{f(x_1,x_2)}{x_1x_2(1-x_1)(1-x_2)}$. Due to the identity

$$\begin{aligned}
&\frac{1}{x_1x_2(1-x_1)(1-x_2)}\\
&=\frac{1}{2}\left\{\frac{1}{x_1x_2(1-x_1)}+\frac{1}{x_1x_2(1-x_2)}+\frac{1}{x_1(1-x_1)(1-x_2)}+\frac{1}{x_2(1-x_1)(1-x_2)}\right\},
\end{aligned}$$

R_Λ has the decomposition:

$$\begin{aligned}
R_\Lambda f(x_1,x_2,x_3) &= \frac{1}{2}\left\{x_2R_{145}f(x_1,x_2,x_3)+x_1R_{245}f(x_1,x_2,x_3)\right.\\
&\left.+(1-x_2-x_3)R_{124}f(x_1,x_2,x_3)+(1-x_1-x_3)R_{125}f(x_1,x_2,x_3)\right\},
\end{aligned}\tag{4.11}$$

where

$$R_{i45}f(x_1,x_2,x_3) = x_i(1-x_1-x_3)(1-x_2-x_3)F_\Lambda\tilde{f}_{i45}(x_1,x_2,x_3)\tag{4.12}$$

with $\tilde{f}_{i45}(x_1,x_2) = \frac{f(x_1,x_2)}{x_i(1-x_1)(1-x_2)}$, $i = 1,2$, and for $s = 4,5$

$$\begin{aligned}
R_{12s}f(x_1,x_2,x_3) &= x_1x_2(1-x_{s-3}-x_3)F_\Lambda\tilde{f}_{12s}(x_1,x_2,x_3),\\
\tilde{f}_{12s}(x_1,x_2) &= \frac{f(x_1,x_2)}{x_1x_2(1-x_{s-3})}.
\end{aligned}\tag{4.13}$$

We shall prove the desired polynomial extension theorem on the pyramid Λ.

Theorem 4.11. *Let R_Λ be the operator as given in (4.10). Then $R_\Lambda f(x_1,x_2,x_3) \in \mathcal{P}^1_p(\Lambda)$ for all $f \in \mathcal{P}^{1,0}_p(S)$, and*

$$\|R_\Lambda f(x_1,x_2,x_3)\|_{H^1(\Lambda)} \le C\|f(x_1,x_2)\|_{H^{\frac{1}{2}}_{00}(S)},\tag{4.14}$$

$$R_\Lambda f\,|_{\Gamma_3} = f, \qquad R_\Lambda f\,|_{\Gamma_i} = 0, \qquad i = 1,2,4,5,\tag{4.15}$$

where C is a constant independent of f and p.

In order to prove this theorem, we need following lemmas.

Lemma 4.12. *Let*

$$g_1 = \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} |f(x_1+x_3,\xi_2)-f(x_1,\xi_2)|d\xi_2\tag{4.16}$$

and

$$g_2 = \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2+x_3)-f(\xi_1,x_2)|d\xi_1.\tag{4.17}$$

Then there holds

$$\|g_s\|_{L^2(\Lambda)} \le C\|f\|_{H^{\frac{1}{2}}(S)}, \; s = 1,2.\tag{4.18}$$

PROOF. We first extend f to square $S_2 = \{(x_1, x_2) \mid 0 \le x_1 \le 2, 0 \le x_2 \le 2\}$ and denoted the extension of f by $\tilde{f}$ such that

$$\|\tilde{f}\|_{H^{\frac{1}{2}}(S_2)} \le C\|f\|_{H^{\frac{1}{2}}(S)}.$$

Let $Q = \{(x_1, x_2, x_3) \mid 0 \le x_1 \le 1, 0 \le x_3 \le 1, 0 \le x_2 \le 1 - x_3\}$ be a prism containing Λ, there holds

$$\begin{aligned}
\|g_1\|^2_{L^2(\Lambda)} &\le \Big\|\frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} \mid \tilde{f}(x_1+x_3,\xi_2) - \tilde{f}(x_1,\xi_2) \mid d\xi_2\Big\|^2_{L^2(Q)} \\
&\le \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_2} \Big|\frac{1}{x_3}\int_{x_2}^{x_2+x_3} \frac{\mid \tilde{f}(x_1+x_3,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{x_3} d\xi_2\Big|^2 dx_3.
\end{aligned}$$

Letting $z = x_2 + x_3$, we have

$$\begin{aligned}
&\int_0^{1-x_2} \Big|\frac{1}{x_3}\int_{x_2}^{x_2+x_3} \frac{\mid \tilde{f}(x_1+x_3,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{x_3} d\xi_2\Big|^2 dx_3 \\
= &\int_{x_2}^1 \Big|\frac{1}{z-x_2}\int_{x_2}^{z} \frac{\mid \tilde{f}(x_1+z-x_2,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{z-x_2} d\xi_2\Big|^2 dz.
\end{aligned}$$

Applying Hardy inequality of [**40**],

$$\int_a^b \Big|\frac{1}{b-x}\int_x^b g(\xi)d\xi\Big|^2 dx \le 4\int_a^b |g(x)|^2 dx,$$

we have

$$\begin{aligned}
&\int_0^1 dx_2 \int_0^{1-x_2} \Big(\frac{1}{x_3}\int_{x_2}^{x_2+x_3} \frac{\mid \tilde{f}(x_1+x_3,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{x_3} d\xi_2\Big)^2 dx_3 \\
= &\int_0^1 dx_2 \int_{x_2}^1 \frac{1}{(z-x_2)^2}\Big(\int_{x_2}^{z} \frac{\mid \tilde{f}(x_1+z-x_2,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{z-x_2} d\xi_2\Big)^2 dz \\
= &\int_0^1 dz \int_0^{z} \frac{1}{(z-x_2)^2}\Big(\int_{x_2}^{z} \frac{\mid \tilde{f}(x_1+z-x_2,\xi_2) - \tilde{f}(x_1,\xi_2) \mid}{z-x_2} d\xi_2\Big)^2 dx_2 \\
\le\ &4\int_0^1 dz \int_0^{z} \frac{\mid \tilde{f}(x_1+z-x_2,x_2) - \tilde{f}(x_1,x_2) \mid^2}{(z-x_2)^2} dx_2,
\end{aligned}$$

which implies

$$\|g_1\|^2_{L^2(\Lambda)} \le 4\int_0^1 dx_1 \int_0^1 dz \int_0^z \frac{\mid \tilde f(x_1+z-x_2,x_2)-\tilde f(x_1,x_2)\mid^2}{(z-x_2)^2}dx_2$$
$$= 4\int_0^1 dx_1 \int_0^1 dx_2 \int_{x_2}^1 \frac{\mid \tilde f(x_1+z-x_2,x_2)-\tilde f(x_1,x_2)\mid^2}{(z-x_2)^2}dz$$
$$= 4\int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_2} \frac{\mid \tilde f(x_1+w,x_2)-\tilde f(x_1,x_2)\mid^2}{w^2}dw$$
$$\le \int_0^2 dw \int_0^{2-w} dx_1 \int_0^2 \frac{\mid \tilde f(x_1+w,x_2)-\tilde f(x_1,x_2)\mid^2}{w^2}dx_2 \le C\|\tilde f\|^2_{H^{\frac12}(S_2)} \le C\|f\|^2_{H^{\frac12}(S)}.$$

Here we have used the equivalence of the norm [**20**] between $\|u\|^2_{H^{\frac12}(S)}$ and

$$\|u\|^2_{L^2(S)} + \int_0^2 dw \int_0^{2-w} dx_1 \int_0^2 \Big|\frac{u(x_1+w,x_2)-u(x_1,x_2)}{w}\Big|^2 dx_2$$
$$+ \int_0^2 dw \int_0^2 dx_1 \int_0^{2-w} \Big|\frac{u(x_1,x_2+w)-u(x_1,x_2)}{w}\Big|^2 dx_2.$$

Similarly, (4.18) for $s=2$ can be proved. □

Lemma 4.13. *Let*

$$R_1 f(x_1,x_2,x_3) = x_1 F_\Lambda \tilde f_1(x_1,x_2,x_3),\ \tilde f_1(x_1,x_2) = \frac{f(x_1,x_2)}{x_1}. \tag{4.19}$$

Then, if $f\in H^{\frac12}(S)$ and $x_1^{-\frac12}f\in L^2(S)$, it holds that

$$\|R_1 f\|_{H^1(G)} \le C(\|f\|_{H^{\frac12}(S)} + \|x_1^{-\frac12}f\|^2_{L^2(S)}) \tag{4.20}$$

and

$$R_1 f\mid_{\Gamma_3} = f, \quad R_1 f\mid_{\Gamma_1} = 0, \tag{4.21}$$

where C is a constant independent of f and p.

Proof. Obviously (4.21) holds. Due to the definition of $R_1 f$ and $F_\Lambda|f|$, there holds

$$|R_1 f(x_1,x_2,x_3)| \ \le\ F_\Lambda|f|(x_1,x_2,x_3)$$

which together with (4.2) leads to

$$\|R_1 f\|_{L^2(\Lambda)} \le C\|f\|_{L^2(S)}. \tag{4.22}$$

From (4.1) and (4.19), we get

$$\frac{\partial F_\Lambda f}{\partial x_1}(x_1,x_2,x_3) = \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} (f(x_1+x_3,\xi_2)-f(x_1,\xi_2))d\xi_2,$$

and

$$\begin{aligned}\frac{\partial R_1 f}{\partial x_1}(x_1,x_2,x_3) &= \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{f(\xi_1,\xi_2)}{\xi_1} d\xi_2 - \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} f(x_1,\xi_2)d\xi_2 \\ &+ \frac{x_1}{x_3^2(x_1+x_3)}\int_{x_2}^{x_2+x_3} f(x_1+x_3,\xi_2)d\xi_2;\end{aligned}$$

Hence,

$$\Big|\frac{\partial R_1 f}{\partial x_1} - \frac{\partial F_\Lambda f}{\partial x_1}\Big| \;\leq\; I_1 + I_2 \tag{4.23}$$

with

$$I_1 = \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_2, \quad I_2 = \frac{1}{x_3(x_1+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1+x_3,\xi_2)|d\xi_2.$$

Note that

$$\begin{aligned}\|I_1\|^2_{L^2(\Lambda)} &= \int_0^1 dx_3 \int_0^{1-x_3} dx_1 \int_0^{1-x_3} \Big(\frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} d\xi_2 \int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_1\Big)^2 dx_2 \\ &= \int_0^1 dx_3 \int_0^{1-x_3} dx_1 \int_0^{1-x_3} \Big\{\frac{1}{x_3}\int_{x_2}^{x_2+x_3} d\xi_2 \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_1\Big)\Big\}^2 dx_2.\end{aligned}$$

By using the Lemma 4.5, we have

$$\int_0^{1-x_3}\int_{x_2}^{x_2+x_3} d\xi_2 \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_1\Big)^2 dx_2 \leq \int_0^1 \Big(\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,x_2)|}{\xi_1} d\xi_1\Big)^2 dx_2,$$

and

$$\|I_1\|^2_{L^2(\Lambda)} \;\leq\; \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_1} \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,x_2)|}{\xi_1} d\xi_1\Big)^2 dx_3.$$

Letting $z = x_1 + x_3$, we get

$$\int_0^{1-x_1} \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,x_2)|}{\xi_1} d\xi_1\Big)^2 dx_3 = \int_{x_1}^1 \frac{1}{(z-x_1)^2}\Big(\int_{x_1}^{z} \frac{|f(\xi_1,x_2)|}{\xi_1} d\xi_1\Big)^2 dz.$$

Using Hardy's inequality 327 of [**36**],

$$\int_a^b \Big|\frac{1}{x-a}\int_a^x g(\xi)d\xi\Big|^2 dx \leq 4\int_a^b |g(x)|^2 dx, \tag{4.24}$$

we obtain

$$\int_{x_1}^1 \frac{1}{(z-x_1)^2}\Big|\int_{x_1}^{z} \frac{|f(\xi_1,x_2)|}{\xi_1} d\xi_1\Big|^2 dz \leq 4\int_{x_1}^1 \frac{|f(z,x_2)|^2}{z^2} dz, \tag{4.25}$$

which leads to

$$\begin{aligned}\|I_1\|^2_{L^2(\Lambda)} &\leq 4\int_0^1 dx_2\int_0^1 dx_1\int_{x_1}^1 \frac{|f(z,x_2)|^2}{z^2}dz \\ &= 4\int_0^1 dx_2\int_0^1 dz\int_0^z \frac{|f(z,x_2)|^2}{z^2}dx_1 = 4\|x_1^{-\frac{1}{2}}f\|^2_{L^2(S)}.\end{aligned}\tag{4.26}$$

We further note that by Lemma 4.5

$$\|I_2\|^2_{L^2(\Lambda)} = \int_0^1 dx_1\int_0^{1-x_1} dx_3\int_0^{1-x_3}\Big|\frac{1}{x_3}\int_{x_2}^{x_2+x_3}\frac{|f(x_1+x_3,\xi_2)|}{x_1+x_3}d\xi_2\Big|^2 dx_2$$

$$\leq \int_0^1 dx_1\int_0^{1-x_1} dx_3\int_0^1\Big|\frac{f(x_1+x_3,x_2)}{x_1+x_3}\Big|^2 dx_2 = \int_0^1 dx_1\int_0^1 dx_2\int_0^{1-x_1}\Big|\frac{f(x_1+x_3,x_2)}{x_1+x_3}\Big|^2 dx_3.$$

Letting $z = x_1 + x_3$, we have

$$\begin{aligned}\|I_2\|^2_{L^2(\Lambda)} &\leq \int_0^1 dx_1\int_0^1 dx_2\int_{x_1}^1 \frac{|f(z,x_2)|^2}{z^2}dz = \int_0^1 dx_2\int_0^1 dx_1\int_{x_1}^1\frac{|f(z,x_2)|^2}{z^2}dz \\ &= \int_0^1 dx_2\int_0^1 dz\int_0^z\frac{|f(z,x_2)|^2}{z^2}dx_1 = \int_0^1 dx_2\int_0^1\frac{|f(z,x_2)|^2}{z}dz = \|x_1^{-\frac{1}{2}}f\|^2_{L^2(\Lambda)}.\end{aligned}\tag{4.27}$$

From (4.23)-(4.27) and Theorem 4.10, we obtain

$$\Big\|\frac{\partial R_1 f}{\partial x_1}\Big\|_{L^2(\Lambda)} \leq \Big\|\frac{\partial F_\Lambda f}{\partial x_1}\Big\|_{L^2(\Lambda)} + \sum_{l=1,2}\|I_l\|_{L^2(\Lambda)} \leq C(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_1^{-\frac{1}{2}}f\|_{L^2(\Lambda)}).\tag{4.28}$$

We next bound the term $\Big\|\frac{\partial R_1 f}{\partial x_2}\Big\|_{L^2(\Lambda)}$. From (4.19) we have

$$\frac{\partial R_1 f}{\partial x_2}(x_1,x_2,x_3) = \frac{x_1}{x_3^2}\int_{x_1}^{x_1+x_3}\frac{(f(\xi_1,x_2+x_3)-f(\xi_1,x_2))}{\xi_1}d\xi_1.$$

For $0\leq x_1\leq \xi_1$, there holds

$$\Big|\frac{\partial R_1 f}{\partial x_2}(x_1,x_2,x_3)\Big| \leq \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} |\,(f(\xi_1,x_2+x_3)-f(\xi_1,x_2))\,|\,d\xi_1.$$

Applying Lemma 4.12, we get

$$\Big\|\frac{\partial R_1 f}{\partial x_2}\Big\|_{L^2(\Lambda)} \leq C\|f\|_{H^{\frac{1}{2}}(S)}.\tag{4.29}$$

We now bound the term $\Big\|\frac{\partial R_1 f}{\partial x_3}\Big\|_{L^2(\Lambda)}$. Note that

$$\begin{aligned}\frac{\partial R_1 f}{\partial x_3}(x_1,x_2,x_3) &= -\frac{2x_1}{x_3^3}\int_{x_1}^{x_1+x_3}d\xi_1\int_{x_2}^{x_2+x_3}\frac{f(\xi_1,\xi_2)}{\xi_1}d\xi_2 \\ &+ \frac{x_1}{x_3^2}\int_{x_1}^{x_1+x_3}\frac{f(\xi_1,x_2+x_3)}{\xi_1}d\xi_1 + \frac{x_1}{x_3^2}\int_{x_2}^{x_2+x_3}\frac{f(x_1+x_3,\xi_2)}{x_1+x_3}d\xi_2;\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_\Lambda f}{\partial x_3}(x_1,x_2,x_3) &= -\frac{2}{x_3^3}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} f(\xi_1,\xi_2)d\xi_2 \\ &+ \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} f(\xi_1,x_2+x_3)d\xi_1 + \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} f(x_1+x_3,\xi_2)d\xi_2.\end{aligned}$$

Therefore

$$\Big|\frac{\partial R_1 f}{\partial x_3} - \frac{\partial F_\Lambda f}{\partial x_3}\Big| \;\le\; J_1+J_2+J_3 \tag{4.30}$$

with

$$\begin{aligned} J_1 &= \frac{2}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_1} d\xi_2, \quad J_2 = \frac{1}{x_3(x_1+x_3)}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2+x_3)|d\xi_1, \\ J_3 &= \frac{1}{x_3(x_1+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1+x_3,\xi_2)|d\xi_2.\end{aligned}$$

Since $0\le 1-\frac{x_1}{\xi_1}\le\frac{x_3}{\xi_1}$ and $0\le 1-\frac{x_1}{\xi_1}\le\frac{x_3}{x_1+x_3}$ for $0\le x_1\le\xi_1\le x_1+x_3$, by the inequality (4.26), we obtain

$$\|J_1\|_{L^2(\Lambda)} \le \|2I_1\|_{L^2(\Lambda)} \le C\|x_1^{-\frac{1}{2}}f\|_{L^2(S)}, \tag{4.31}$$

and

$$\begin{aligned}|J_2| &\le \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2+x_3)-f(\xi_1,x_2)|d\xi_1 + \frac{1}{x_3(x_1+x_3)}\int_{x_1}^{x_1+x_3}|f(\xi_1,x_2)|d\xi_1 \\ &= J_{2,1}+J_{2,2}.\end{aligned}$$

By Lemma 4.12, there holds

$$\|J_{2,1}\|_{L^2(\Lambda)} \le C\|f\|_{H^{\frac{1}{2}}(S)}, \tag{4.32}$$

and by Lemma 4.5, we have

$$\begin{aligned}\|J_{2,2}\|^2_{L^2(\Lambda)} &= \int_0^1 dx_2 \int_0^{1-x_2} dx_3 \int_0^{1-x_3} \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3}\frac{|f(\xi_1,x_2)|}{x_1+x_3}d\xi_1\Big)^2 dx_1 \\ &\le \int_0^1 dx_2\int_0^{1-x_2}dx_3\int_0^1 \frac{|f(x_1,x_2)|^2}{(x_1+x_3)^2}dx_1 = \int_0^1 dx_2\int_0^1 dx_1\int_0^{1-x_2}\frac{|f(x_1,x_2)|^2}{(x_1+x_3)^2}dx_3 \\ &\le \int_0^1 dx_2\int_0^1\frac{|f(x_1,x_2)|^2}{x_1}dx_1 = \|x_1^{-\frac{1}{2}}f\|^2_{L^2(S)}.\end{aligned} \tag{4.33}$$

A combination of (4.32) and (4.33) gives

$$\|J_2\|_{L^2(\Lambda)} \le C(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_1^{-\frac{1}{2}}f\|_{L^2(S)}). \tag{4.34}$$

Similarly, there holds

$$\begin{aligned}|J_3| &\leq \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3}|f(x_1+x_3,\xi_2)-f(x_1,\xi_2)|d\xi_2+\frac{1}{x_3(x_1+x_3)}\int_{x_2}^{x_2+x_3}|f(x_1,\xi_2)|d\xi_2\\ &= J_{3,1}+J_{3,2}.\end{aligned}$$

Due to Lemma 4.12, there hold

$$\|J_{3,1}\|_{L^2(\Lambda)} \leq C\|f\|_{H^{\frac{1}{2}}(S)}, \tag{4.35}$$

and by Lemma 4.5, we have

$$\begin{aligned}\|J_{3,2}\|^2_{L^2(\Lambda)} &= \int_0^1 dx_3\int_0^{1-x_3}dx_1\int_0^{1-x_3}\Big(\frac{1}{x_3}\int_{x_2}^{x_2+x_3}\frac{|f(x_1,\xi_2)|}{x_1+x_3}d\xi_2\Big)^2dx_2\\ &\leq \int_0^1 dx_3\int_0^{1-x_3}dx_1\int_0^1\frac{|f(x_1,x_2)|^2}{(x_1+x_3)^2}dx_2=\int_0^1 dx_2\int_0^1 dx_1\int_0^{1-x_1}\frac{|f(x_1,x_2)|^2}{(x_1+x_3)^2}dx_3\\ &\leq \int_0^1 dx_2\int_0^1\frac{|f(x_1,x_2)|^2}{x_1}dx_1\leq \|x_1^{-\frac{1}{2}}f\|^2_{L^2(S)}.\end{aligned} \tag{4.36}$$

Hence,

$$\|J_{3,2}\|_{L^2(\Lambda)} \leq C\|x_1^{-\frac{1}{2}}f\|_{L^2(S)},$$

which together with (4.35) implies

$$\|J_3\|_{L^2(\Lambda)} \leq C(\|f\|_{H^{\frac{1}{2}}(S)}+\|x_1^{-\frac{1}{2}}f\|_{L^2(S)}). \tag{4.37}$$

Combining (4.30),(4.31),(4.34),(4.37) and Theorem 4.10, we obtain

$$\Big\|\frac{\partial R_1 f}{\partial x_3}\Big\|_{L^2(\Lambda)} \leq \Big\|\frac{\partial F_\Lambda f}{\partial x_3}\Big\|_{L^2(\Lambda)}+\sum_{l=1}^{3}\|J_l\|_{L^2(\Lambda)} \leq C(\|f\|_{H^{\frac{1}{2}}(S)}+\|x_1^{-\frac{1}{2}}f\|^2_{L^2(S)}). \tag{4.38}$$

A combination of (4.22),(4.28)-(4.29) and (4.38) leads to (4.20). □

Lemma 4.14. *Let R_{12} be the operator defined by*

$$R_{12}f(x_1,x_2,x_3)=x_1x_2F_\Lambda\tilde{f}_{12}(x_1,x_2,x_3), \quad \tilde{f}_{12}(x_1,x_2)=\frac{f(x_1,x_2)}{x_1x_2}. \tag{4.39}$$

Then for all $f\in H^{\frac{1}{2}}(S)$ and $x_i^{-\frac{1}{2}}f\in L^2(S)$, i=1,2, it holds that

$$\|R_{12}f\|_{H^1(\Lambda)} \leq C(\|f\|_{H^{\frac{1}{2}}(S)}+\sum_{i=1,2}\|x_i^{-\frac{1}{2}}f\|_{L^2(S)}) \tag{4.40}$$

and

$$R_{12}f\,|_{\Gamma_3}=f, \quad R_{12}f\,|_{\Gamma_i}=0, \quad i=1,2, \tag{4.41}$$

where the constant C is independent of f and p.

Proof. From (4.39) and (4.1), we get for $0 \le x_i \le \xi_i, i = 1, 2$

$$| \ R_{12}f(x_1, x_2, x_3) \ | \le \ F_\Lambda |f|(x_1, x_2, x_3).$$

By (4.2), it holds that

$$\|R_{12}f\|_{L^2(\Lambda)} \le C\|f\|_{L^2(S)}. \tag{4.42}$$

We next prove that the first order derivatives of $R_{12}f$ are in $L^2(\Lambda)$ by comparing them with the corresponding first order derivatives of R_1f. From (4.39) we obtain

$$\begin{aligned} \frac{\partial R_{12}f}{\partial x_1}(x_1, x_2, x_3) &= \frac{x_2}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{f(\xi_1, \xi_2)}{\xi_1\xi_2} d\xi_2 - \frac{x_2}{x_3^2}\int_{x_2}^{x_2+x_3} \frac{f(x_1, \xi_2)}{\xi_2} d\xi_2 \\ &\quad + \frac{x_1x_2}{x_3^2(x_1+x_3)}\int_{x_2}^{x_2+x_3} \frac{f(x_1+x_3, \xi_2)}{\xi_2} d\xi_2; \end{aligned}$$

Since $0 \le 1 - \frac{x_2}{\xi_2} \le 1$ and $0 \le 1 - \frac{x_2}{\xi_2} \le \frac{x_3}{x_2+x_3}$, for $x_2 \le \xi_2 \le x_2 + x_3$

$$\Big|\frac{\partial R_{12}f}{\partial x_1} - \frac{\partial R_1 f}{\partial x_1}\Big| \ = \ I_1 + I_2 + I_3, \tag{4.43}$$

where

$$\begin{aligned} I_1 &= \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{|f(\xi_1, \xi_2)|}{\xi_1} d\xi_2, \quad I_2 = \frac{1}{x_3(x_2+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1, \xi_2)| d\xi_2 \\ I_3 &= \frac{1}{x_3(x_2+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1+x_3, \xi_2)| d\xi_2. \end{aligned}$$

Due to (4.26), we have

$$\|I_1\|^2_{L^2(\Lambda)} \le 4\|x_1^{-\frac{1}{2}} f\|^2_{L^2(S)}. \tag{4.44}$$

By the argument similar to that for $J_{3,2}$, there holds

$$\begin{aligned} \|I_2\|^2_{L^2(\Lambda)} &\le \int_0^1 dx_2 \int_0^1 dx_1 \int_0^{1-x_1} \frac{|f(x_1, x_2)|^2}{(x_2+x_3)^2} dx_3 \\ &\le \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_1} \frac{|f(x_1, x_2)|^2}{(x_2+x_3)^2} dx_3 \le \int_0^1 dx_1 \int_0^1 \frac{|f(x_1, x_2)|^2}{x_2} dx_2 = \|x_2^{-\frac{1}{2}} f\|^2_{L^2(S)}. \end{aligned} \tag{4.45}$$

For the third term, we have

$$\begin{aligned} |I_3| &\le \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} |f(x_1+x_3, \xi_2) - f(x_1, \xi_2)| d\xi_2 + \frac{1}{x_3(x_2+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1, \xi_2)| d\xi_2 \\ &= I_{3,1} + I_2. \end{aligned}$$

By Lemma 4.12, there holds

$$\|I_{3,1}\|_{L^2(\Lambda)} \le C\|f\|_{H^{\frac{1}{2}}(S)},$$

which together with (4.45) imply

$$\|I_3\|_{L^2(\Lambda)} \leq \quad C(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_2^{-\frac{1}{2}} f\|_{L^2(S)}). \tag{4.46}$$

By (4.44)-(4.46) and (4.28), we obtain

$$\Big\|\frac{\partial R_{12} f}{\partial x_1}\Big\|_{L^2(\Lambda)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|x_i^{-\frac{1}{2}} f\|_{L^2(S)}\Big). \tag{4.47}$$

Since x_1 and x_2 are symmetric, we have

$$\Big\|\frac{\partial R_{12} f}{\partial x_2}\Big\|_{L^2(\Lambda)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|x_i^{-\frac{1}{2}} f\|_{L^2(S)}\Big). \tag{4.48}$$

We next bound the term $\Big\|\frac{\partial R_{12} f}{\partial x_3}\Big\|_{L^2(\Lambda)}$. From (4.39) we obtain

$$\begin{aligned}
&\frac{\partial R_{12} f}{\partial x_3}(x_1,x_2,x_3) = -\frac{2x_1x_2}{x_3^3}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{f(\xi_1,\xi_2)}{\xi_1\xi_2} d\xi_2 \\
&+\frac{x_1x_2}{x_3^2(x_2+x_3)}\int_{x_1}^{x_1+x_3} \frac{f(\xi_1,x_2+x_3)}{\xi_1} d\xi_1 + \frac{x_1x_2}{x_3^2(x_1+x_3)}\int_{x_2}^{x_2+x_3} \frac{f(x_1+x_3,\xi_2)}{\xi_2} d\xi_2;
\end{aligned}$$

Since $0 \leq 1 - \frac{x_2}{\xi_2} \leq \frac{x_3}{\xi_2}$ and $0 \leq 1 - \frac{x_2}{\xi_2} \leq \frac{x_3}{x_2+x_3}$ for $x_2 \leq \xi_2 \leq x_2 + x_3$,

$$\Big|\frac{\partial R_{12} f}{\partial x_3} - \frac{\partial R_1 f}{\partial x_3}\Big| \quad \leq \quad J_1 + J_2 + J_3, \tag{4.49}$$

where

$$\begin{aligned}
J_1 &= \frac{2}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{|f(\xi_1,\xi_2)|}{\xi_2} d\xi_2, \quad J_2 = \frac{1}{x_3(x_2+x_3)}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2+x_3)| d\xi_1 \\
J_3 &= \frac{1}{x_3(x_2+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1+x_3,\xi_2)| d\xi_2.
\end{aligned}$$

By the symmetry and (4.26), we have

$$\|J_1\|_{L^2(\Lambda)} \leq 2\|x_2^{-\frac{1}{2}} f\|_{L^2(S)}. \tag{4.50}$$

Note that

$$\begin{aligned}
|J_2| \quad &\leq \quad \frac{1}{x_3^2}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2+x_3) - f(\xi_1,x_2)| d\xi_1 + \frac{1}{x_3(x_2+x_3)}\int_{x_1}^{x_1+x_3} |f(\xi_1,x_2)| d\xi_1 \\
&= \quad J_{2,1} + \tilde{I}_2.
\end{aligned}$$

By (4.17) and

$$\|\tilde{I}_2\|^2_{L^2(\Lambda)} = \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \int_0^{1-x_3} \Big(\frac{1}{x_3}\int_{x_1}^{x_1+x_3} \frac{|f(\xi_1,x_2)|}{x_2+x_3} d\xi_1\Big)^2 dx_1 \tag{4.51}$$

$$\le \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \int_0^1 \frac{|f(x_1,x_2)|^2}{(x_2+x_3)^2} dx_1 = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{1-x_2} \frac{|f(x_1,x_2)|^2}{(x_2+x_3)^2} dx_3$$

$$\le \int_0^1 dx_1 \int_0^1 \frac{|f(x_1,x_2)|^2}{x_2} dx_2 \le \|x_2^{-\frac{1}{2}} f\|^2_{L^2(S)},$$

we have

$$\|J_2\|_{L^2(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_2^{-\frac{1}{2}} f\|_{L^2(S)}\Big). \tag{4.52}$$

Similarly, for the third term of (4.49), as $0 \le 1 - \frac{x_2}{\xi_2} \le \frac{x_3}{x_2+x_3} \le 1$,

$$|J_3| \le \frac{1}{x_3^2}\int_{x_2}^{x_2+x_3} |f(x_1+x_3,\xi_2) - f(x_1,\xi_2)| d\xi_2 + \frac{1}{x_3(x_2+x_3)}\int_{x_2}^{x_2+x_3} |f(x_1,\xi_2)| d\xi_2.$$

By (4.16) and (4.45),

$$\|J_3\|_{L^2(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_2^{-\frac{1}{2}} f\|_{L^2(S)}\Big). \tag{4.53}$$

By (4.49)-(4.53) and (4.38), we obtain

$$\Big\|\frac{\partial R_{12} f}{\partial x_3}\Big\|_{L^2(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|x_i^{-\frac{1}{2}} f\|_{L^2(S)}\Big). \tag{4.54}$$

A combination of (4.42),(4.47),(4.48) and (4.54) leads to (4.40), and (4.41) follows easily. □

Lemma 4.15. *Let R_{24} be the operator defined by*

$$\begin{aligned} R_{24} f(x_1,x_2,x_3) &= x_2(1-x_1-x_3) F_\Lambda \tilde{f}_{24}(x_1,x_2,x_3), \\ \tilde{f}_{24}(x_1,x_2) &= \frac{f(x_1,x_2)}{x_2(1-x_1)}. \end{aligned} \tag{4.55}$$

Then for all $f \in H^{\frac{1}{2}}(S)$, $x_2^{-\frac{1}{2}} f \in L^2(S)$ and $(1-x_1-x_3)^{-\frac{1}{2}} f \in L^2(S)$, it holds that

$$\|R_{24} f\|_{H^1(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \|x_2^{-\frac{1}{2}} f\|_{L^2(S)} + \|(1-x_1-x_3)^{-\frac{1}{2}} f\|_{L^2(S)}\Big) \tag{4.56}$$

and

$$R_{24} f\,|_{\Gamma_3} = f, \quad R_{24} f\,|_{\Gamma_i} = 0, \quad i = 2,4, \tag{4.57}$$

where C is a constant independent of f and p.

Proof. Let the mapping M:

$$x_1 = 1 - \hat{x}_1 - \hat{x}_3, \quad x_2 = \hat{x}_2, \quad x_3 = \hat{x}_3 \tag{4.58}$$

which maps Λ and S onto itself respectively, maps Γ_4 and Γ_5 onto Γ_1 and Γ_5, and maps Γ_1, Γ_2 and Γ_3 onto Γ_4, Γ_2 and Γ_3, respectively. Letting $f^*(\xi_1,\xi_2) = f(1-\xi_1,\xi_2)$ and $\eta_1 = 1-\xi_1, \eta_2 = \xi_2$, we have

$$\begin{aligned} R_{24}f &= \frac{(1-x_1-x_3)x_2}{x_3^2}\int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{f(\xi_1,\xi_2)}{\xi_2(1-\xi_1)}d\xi_2 \\ &= \frac{(1-x_1-x_3)x_2}{x_3^2}\int_{1-x_1-x_3}^{1-x_1} d\eta_1 \int_{x_2}^{x_2+x_3} \frac{f^*(\eta_1,\eta_2)}{\eta_2\eta_1}d\eta_2 \\ &= \frac{\hat{x}_1\hat{x}_2}{\hat{x}_3^2}\int_{\hat{x}_1}^{\hat{x}_1+\hat{x}_3} d\eta_1 \int_{\hat{x}_2}^{\hat{x}_2+\hat{x}_3} \frac{f^*(\eta_1,\eta_2)}{\eta_1\eta_2}d\eta_2 = R_{12}f^*\circ M. \end{aligned}$$

By Lemma 4.14, we obtain

$$\begin{aligned} \|R_{24}f\|_{H^1(\Lambda)} &\leq C\Big(\|f^*\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2}\|\hat{x}_i^{-\frac{1}{2}}f^*\|_{L^2(S)}\Big) \\ &= C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \|(1-x_1-x_3)^{-\frac{1}{2}}f\|_{L^2(S)} + \|x_2^{-\frac{1}{2}}f\|_{L^2(S)}\Big), \end{aligned}$$

and

$$R_{24}f\,|_{\Gamma_3} = f, \quad R_{24}f\,|_{\Gamma_i} = 0, \quad i = 2, 4.$$

$\square$

Lemma 4.16. *Let R_{124} be the operator defined by (4.13),then for all $f \in H^{\frac{1}{2}}(S)$, $x_i^{-\frac{1}{2}}f \in L^2(S)$, i=1,2, and $(1-x_1-x_3)^{-\frac{1}{2}}f \in L^2(S)$, it holds that*

$$\|R_{124}f\|_{H^1(\Lambda)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2}\|x_i^{-\frac{1}{2}}f\|_{L^2(S)} + \|(1-x_1-x_3)^{-\frac{1}{2}}f\|_{L^2(S)}\Big) \tag{4.59}$$

and

$$R_{124}f\,|_{\Gamma_3} = f, \quad R_{124}f\,|_{\Gamma_i} = 0, \quad i = 1, 2, 4, \tag{4.60}$$

where C is a constant independent of f and p.

Proof. Due to the identity

$$\frac{1}{\xi_1\xi_2(1-\xi_1)} = \frac{1}{\xi_1\xi_2} + \frac{1}{\xi_2(1-\xi_1)},$$

we have the following decomposition

$$R_{124}f(x_1,x_2,x_3) = (1-x_1-x_3)R_{12}f + x_1R_{24}f,$$

by Lemma 4.14 and Lemma 4.15, we obtain

$$\begin{aligned} \|R_{124}f\|_{H^1(\Lambda)} &\leq \|R_{12}f\|_{H^1(\Lambda)} + \|R_{24}f\|_{H^1(\Lambda)} \\ &\leq C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2}\|x_i^{-\frac{1}{2}}f\|_{L^2(S)} + \|(1-x_1-x_3)^{-\frac{1}{2}}f\|_{L^2(S)}\Big). \end{aligned}$$

$\square$

Similar to Lemma 4.16, we can prove following lemma.

Lemma 4.17. *Let R_{125} be the operator defined by (4.13), then for all $f \in H^{\frac{1}{2}}(S)$, $x_i^{-\frac{1}{2}} f \in L^2(S)$, i=1,2, and $(1-x_2-x_3)^{-\frac{1}{2}} f \in L^2(S)$, it holds that*

$$\|R_{125} f\|_{H^1(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|x_i^{-\frac{1}{2}} f\|_{L^2(S)} + \|(1-x_2-x_3)^{-\frac{1}{2}} f\|_{L^2(S)}\Big) \tag{4.61}$$

and

$$R_{125} f \mid_{\Gamma_3} = f, \quad R_{125} f \mid_{\Gamma_i} = 0, \quad i = 1, 2, 5, \tag{4.62}$$

where C is a constant independent of f and p.

Lemma 4.18. *Let R_{i45} be the operator defined by (4.12), then for all $f \in H^{\frac{1}{2}}(S)$, $x_i^{-\frac{1}{2}} f \in L^2(S)$ and $(1-x_i-x_3)^{-\frac{1}{2}} f \in L^2(S)$, i=1,2, it holds that*

$$\|R_{i45} f\|_{H^1(\Lambda)} \le C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|(1-x_i-x_3)^{-\frac{1}{2}} f\|_{L^2(S)} + \|x_i^{-\frac{1}{2}} f\|_{L^2(S)}\Big), \tag{4.63}$$

and

$$R_{i45} f \mid_{\Gamma_3} = f, \quad R_{i45} f \mid_{\Gamma_i} = 0, \quad i = 1, 4, 5, \tag{4.64}$$

where C is a constant independent of f and p.

Proof. Let the mapping M:

$$x_1 = 1 - \hat{x}_1 - \hat{x}_3, \quad x_2 = 1 - \hat{x}_2 - \hat{x}_3, \quad x_3 = \hat{x}_3, \tag{4.65}$$

which maps Λ and S onto itself respectively, maps Γ_4 and Γ_5 onto Γ_1 and Γ_2, and maps Γ_1 and Γ_2 onto Γ_4 and Γ_5, respectively. Letting $f^*(\xi_1, \xi_2) = f(1-\xi_1, 1-\xi_2)$, we have

$$\begin{aligned}
R_{145} f &= \frac{(1-x_1-x_3)(1-x_2-x_3)x_1}{x_3^2} \int_{x_1}^{x_1+x_3} d\xi_1 \int_{x_2}^{x_2+x_3} \frac{f(\xi_1,\xi_2)}{(1-\xi_1)(1-\xi_2)\xi_1} d\xi_2 \\
&= \frac{(1-x_1-x_3)(1-x_2-x_3)x_1}{x_3^2} \int_{1-x_1-x_3}^{1-x_1} d\eta_1 \int_{1-x_2-x_3}^{1-x_2} \frac{f^*(\eta_1,\eta_2)}{\eta_1\eta_2(1-\eta_1)} d\eta_2 \\
&= \frac{\hat{x}_1\hat{x}_2(1-\hat{x}_1-\hat{x}_3)}{\hat{x}_3^2} \int_{\hat{x}_1}^{\hat{x}_1+\hat{x}_3} d\eta_1 \int_{\hat{x}_2}^{\hat{x}_2+\hat{x}_3} \frac{f^*(\eta_1,\eta_2)}{\eta_1\eta_2(1-\eta_1)} d\eta_2 \\
&= R_{124} f^* \circ M.
\end{aligned}$$

By Lemma 4.16, we obtain

$$\begin{aligned}
\|R_{145} f\|_{H^1(\Lambda)} &\le C\Big(\|f^*\|_{H^{\frac{1}{2}}(S)} + \|(1-\hat{x}_1-\hat{x}_3)^{-\frac{1}{2}} f^*\|_{L^2(S)} + \sum_{i=1,2} \|\hat{x}_i^{-\frac{1}{2}} f^*\|_{L^2(S)}\Big) \\
&= C\Big(\|f\|_{H^{\frac{1}{2}}(S)} + \sum_{i=1,2} \|(1-x_i-x_3)^{-\frac{1}{2}} f\|_{L^2(S)} + \|x_1^{-\frac{1}{2}} f\|_{L^2(S)}\Big),
\end{aligned}$$

and

$$R_{145}f \mid_{\Gamma_3} = f, \quad R_{145}f \mid_{\Gamma_i} = 0, \quad i = 1, 4, 5.$$

An analogous result holds for $R_{245}f$ because of the symmetry. □

Proof of Theorem 4.11 . Obviously, $R_\Lambda f \in \mathcal{P}_p^1(\Lambda), R_\Lambda f \mid_{\Gamma_3} = f$, and $R_\Lambda f \mid_{\partial\Lambda\setminus\Gamma_3} = 0$. By (4.11), there holds

$$\|R_\Lambda f\|_{H^1(\Lambda)} \leq C\Big(\sum_{i=1,2} \|R_{i45}f(x_1,x_2,x_3)\|_{H^1(\Lambda)} + \sum_{j=4,5} \|R_{12j}f(x_1,x_2,x_3)\|_{H^1(\Lambda)}\Big).$$

Applying Lemma 4.16-Lemma 4.18, we obtain

$$\|R_\Lambda f\|_{H^1(\Lambda)} \leq C\|f\|_{H_{00}^{\frac{1}{2}}(S)}.$$

Thus we complete the proof of the theorem. □

Remark 4.3. Analogue to the extension on a prism we may define, as proposed in [**12**], an extension operator $\tilde{R}_D$ on a cube D via a mapping from a cube onto a truncated pyramid Λ_H, shown in Fig. 4.5.

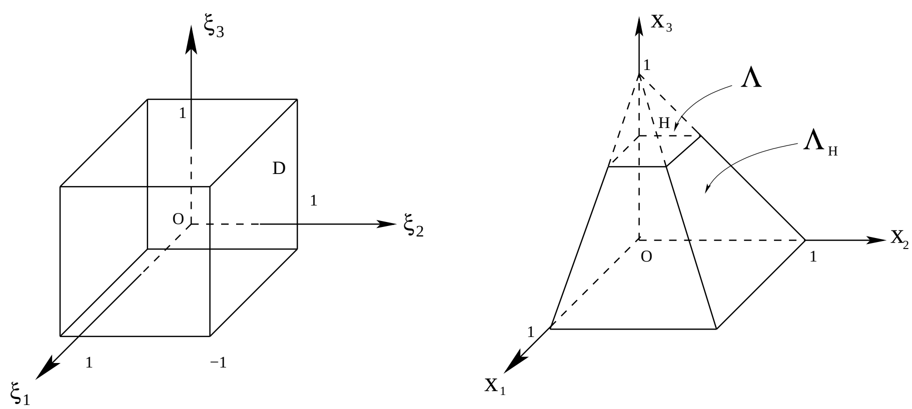


Fig. 4.5 A cube and a truncated pyramid Λ_H

$$\begin{aligned}\tilde{R}_D f &= R_{\Lambda_H} f \circ M,\\ \tilde{R}_{\Lambda_H} f(x_1,x_2,x_3) &= \tilde{R}_\Lambda f(x_1,x_2,x_3) - \frac{x_3}{H} R_\Lambda f(x_1,x_2,H),\end{aligned}$$

where the mapping

$$M : x_i = \frac{\xi_i + 1}{2}\Big(1 - \frac{H(\xi_3+1)}{2}\Big),\ i = 1,2,\ x_3 = \frac{H(\xi_3+1)}{2}$$

maps the cube D onto the pyramid Λ_H as an analogue of R_G on a prism. It can be proved analogously that

$$\|\tilde{R}_D f\|_{H^1(D)} \leq C\|f\|_{H^{\frac{1}{2}}_{00}(\tilde{S})}.$$

It is easy to verify that $\tilde{R}_D f \in \mathcal{P}_p^2(D)$ if $f \in \mathcal{P}_p^{1,0}(S)$, and $\tilde{R}_D f \in \mathcal{P}_p^{2,0}(S) \times \mathcal{P}_{2p}(I)$ if $f \in \mathcal{P}_p^{2,0}(S)$. Note that $\mathcal{P}_p^{1,0}(S)$ is not a trace of $\mathcal{P}_p^2(D)$ and $\mathcal{P}_p^{2,0}(S) \times \mathcal{P}_{2p}(I) \not\subset \mathcal{P}_p^2(D)$. Therefore, $\tilde{R}_D f$ is not compatible with the FEM subspace on cubic elements, the polynomial extension $\tilde{R}_D$ of convolution-type is not useful for error analysis of the p and h-p FEM on meshes containing hexahedral elements.

4.3. Extension on a standard cube

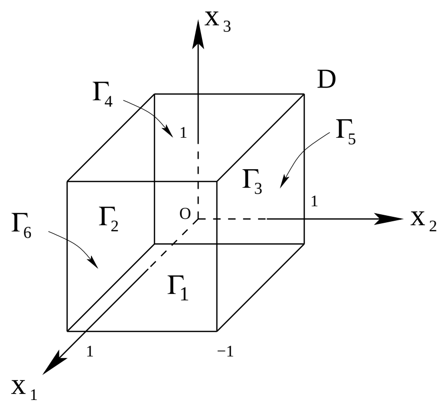


Fig. 4.6 A cube D

Let D be a cube and $\Gamma_i, i = 1, 2, \cdots, 6$ be faces of D shown in Fig. 4.6, and let $\gamma_{ij} = \Gamma_i \cap \Gamma_j, i = 1, 2, \cdots, 6$. As usual, $I = [-1, 1]$ and $S = [-1, 1]^2$.

4.3.1. Polynomial extension from a face. Let $J_j^{\alpha,\beta}(x)$ be the Jacobi polynomial of degree j;

$$J_j^{\alpha,\beta}(x) = \frac{(-1)^j(1-x)^{-\alpha}(1+x)^{-\beta}}{2^j j!}\frac{d^j(1-x)^{j+\alpha}(1+x)^{j+\beta}}{dx^j}, \quad j \geq 0 \tag{4.1}$$

with weights $\alpha, \beta > -1$, and let

$$\varphi_i(x) = \frac{1-x^2}{\sqrt{\gamma_{i-1}^{2,2}}} J_{i-1}^{2,2}(x), \quad i = 1, 2, 3, \cdots, \tag{4.2}$$

where $\gamma_{i-1}^{2,2} = \dfrac{2^5 i(i+1)}{(2i+3)(i+2)(i+3)}$.

Proposition 4.19. $\varphi_i(x), i = 1, 2, \cdots, p-1$ *form an orthogonal basis of* $\mathcal{P}_p^0(I)$,

$$< \varphi_i(x), \varphi_j(x) >_{L^2(I)} = \delta_{ij}, \quad 1 \leq i, j \leq p-1. \tag{4.3}$$

PROOF. Due to the orthonormality of Jacobi polynomials

$$< \varphi_i(x), \varphi_j(x) >_{L^2(I)} = \frac{1}{\sqrt{\gamma_{i-1}^{2,2}}\sqrt{\gamma_{j-1}^{2,2}}} \int_I (1-x^2)^2 J_{i-1}^{2,2}(x) J_{j-1}^{2,2}(x) dx = \delta_{ij}.$$

□

We introduce

$$\varphi_n(x_1,x_2) = \varphi_i(x_1)\varphi_j(x_2) = \frac{(1-x_1^2)(1-x_2^2)}{\sqrt{\gamma_{i-1}^{2,2}\gamma_{j-1}^{2,2}}} J_{i-1}^{2,2}(x_1) J_{j-1}^{2,2}(x_2), \quad 1 \le i,j \le p-1 \tag{4.4}$$

with $n = (p-1)(i-1)+j$.

Proposition 4.20. $\{\varphi_n(x_1,x_2), n = 1,2,\cdots,(p-1)^2\}$ *forms an orthonormal basis of* $\mathcal{P}_p^{2,0}(S)$ *in* $L^2(S)$*, i.e.*

$$< \varphi_n, \varphi_m >_{L^2(S)} = \delta_{nm}, \quad 1 \le n,m \le N_p = (p-1)^2. \tag{4.5}$$

PROOF. Let $n = (p-1)(i-1)+j$, and $m = (p-1)(i'-1)+j'$, then

$$\begin{aligned} < \varphi_n, \varphi_m >_{L^2(S)} &= \int_I \frac{(1-x_1^2)^2}{\sqrt{\gamma_{i-1}^{2,2}\gamma_{i'-1}^{2,2}}} J_{i-1}^{2,2}(x_1) J_{i'-1}^{2,2}(x_1) dx_1 \int_I \frac{(1-x_2^2)^2}{\sqrt{\gamma_{j-1}^{2,2}\gamma_{j'-1}^{2,2}}} J_{j-1}^{2,2}(x_2) J_{j'-1}^{2,2}(x_2) dx_2 \\ &= \delta_{i,i'}\delta_{j,j'} = \delta_{nm}. \end{aligned}$$

□

We consider an eigenvalue problem

$$\begin{cases} -\triangle u = \lambda u, & \text{in } S = (-1,1)^2 \\ u\big|_\Gamma = 0 \end{cases} \tag{4.6}$$

and its spectral solution (λ_p, ψ_p) with $\psi_p \in \mathcal{P}_p^{2,0}(S)$ which satisfies

$$\int_S \nabla\psi_p \nabla q dx_1 dx_2 = \lambda_p \int_S \psi_p q dx_1 dx_2, \quad \forall q \in \mathcal{P}_p^{2,0}(S). \tag{4.7}$$

Selecting the basis $\{\varphi_n(x_1,x_2), n = 1,2,\cdots,N_p\}$ as in (4.4) with $N_p = (p-1)^2$, and letting $\psi_p(x_1,x_2) = \sum_{i=1}^{N_p} c_i\varphi_i(x_1,x_2)$, we have the corresponding system of linear algebraic equations

$$K\overrightarrow{C} = \lambda M\overrightarrow{C} = \lambda\overrightarrow{C},$$

where $\overrightarrow{C} = (c_1,c_2,\cdots,c_{N_p})^T, K = (k_{ij})_{i,j=1}^{N_p}$ with $k_{ij} = \int_S \nabla\varphi_i\nabla\varphi_j dx_1dx_2$. Here we used the orthonormality of $\varphi_n(x_1,x_2)$ in $L^2(S)$ which implies the matrix $M = I$. Therefore the spectral solution of eigenvalue problem (4.7) is equivalent to the eigenvalue problem of matrix

K. Since K is symmetric and positive definite, the eigenvalues $\lambda_{p,k} > 0, k = 1, 2 \cdots, N_p$ and the corresponding eigenvectors $\overrightarrow{C}^{(k)}$ are orthonormal, i.e

$$< \overrightarrow{C}^{(k)}, \overrightarrow{C}^{(l)} >= \sum_{i=1}^{N_p} c_i^{(k)} c_i^{(l)} = \delta_{k,l}, 1, \leq k, l \leq N_p.$$

The corresponding eigen polynomial $\psi_{p,k} = \sum_{n=1}^{N_p} c_n^{(k)} \varphi_n(x_1, x_2)$. Then, due to the properties of eigenvalues and vectors of K, we have the following theorem.

Theorem 4.21. *The problem (4.7) has N_p real eigenvalues, and the corresponding eigen-polynomials $\{\psi_{p,k}(x_1, x_2), 1 \leq k \leq N_p\}$ are orthogonal in $L^2(S)$ and $H^1(S)$, which form a L^2-orthonormal basis of $\mathcal{P}_p^{2,0}(S)$.*

Proof. The problem (4.7) has N_p real eigenvalues because the corresponding stiffness matrix K is positive definite, and there hold for $1 \leq k, k' \leq N_p$

$$< \psi_{p,k}, \psi_{p,k'} >_{L^2(S)} = \sum_{j=1}^{N_p} \sum_{i=1}^{N_p} c_i^{(k)} c_j^{(k')} < \varphi_i, \varphi_j >_{L^2(S)} = < \overrightarrow{C}^{(k)}, \overrightarrow{C}^{(k')} >= \delta_{k,k'}$$

and

$$\int_S \nabla\psi_{p,k} \nabla\psi_{p,k'} dx_1 dx_2 = \lambda_k \int_S \psi_{p,k} \psi_{p,k'} dx_1 dx_2 = \lambda_k \delta_{k,k'}.$$

Therefore, $\{\psi_{p,k}, k = 1, 2, \cdots, N_p\}$ is orthogonal in $L^2(S)$ and $H^1(S)$ and forms an orthonormal basis in $L^2(S)$. □

We next consider a two-points boundary value problem

$$\begin{cases} -v''_{p,k}(x_3) + \lambda_{p,k} v_{p,k}(x_3) = 0, & x_3 \in I = (-1, 1), \\ v_{p,k}(-1) = 1, \quad v_{p,k}(1) = 0, \end{cases} \tag{4.8}$$

and its spectral solution $\phi_{p,k} \in \mathcal{P}_p(I)$ such that $\phi_{p,k}(-1) = 1, \phi_{p,k}(1) = 0$ and

$$\int_I (\phi'_{p,k} q' + \lambda_{p,k} \phi_{p,k} q) dx_3 = 0 \tag{4.9}$$

which is equivalent to find $\phi_{p,k} = \tilde{\phi}_{p,k} + \dfrac{1 - x_3}{2}$ with $\tilde{\phi}_{p,k} \in P_p^0(I)$ satisfying

$$\int_I (\widetilde{\phi}'_{p,k}(x_3) q'(x_3) + \lambda_{p,k} \widetilde{\phi}_{p,k}(x_3) q(x_3)) dx_3 = \frac{1}{2} \int_I (q'(x_3) - \lambda_{p,k}(1 - x_3) q(x_3)) dx_3. \tag{4.10}$$

Since the corresponding bilinear form is coercive and continuous on $H_0^1(I) \times H_0^1(I)$, the solution $\tilde{\phi}_{p,k}(x_3)$ uniquely exists in $P_p^0(I)$ for each $\lambda_{p,k}$.

Lemma 4.22. *(Inverse inequality)*

$$\int_S |\nabla\psi_{p,k}|^2 dx_1 dx_2 \leq C p^4 \int_S |\psi_{p,k}|^2 dx_1 dx_2, \tag{4.11}$$

where C is a constant independent of p and k.

PROOF. Since $\psi_{p,k}(x_1, x_2) \in \mathcal{P}_p^{2,0}(S)$

$$\psi_p(x_1, x_2) = \sum_{n=0}^{p} \sum_{m=0}^{p} a_{nm} L_m(x_1) L_n(x_2) = \sum_{n=0}^{p} b_n(x_1) L_n(x_2)$$

where $b_n(x_1) = \sum_{m=0}^{p} a_{nm} L_m(x_1)$ and $\{L_n(x_1), n = 0, 1, \cdots, p\}$ is an orthogonal basis of $\mathcal{P}_p^0(I)$.

Due to inverse inequality for a polynomial of degree n on I, we have

$$\begin{aligned}
\int_S \Big|\frac{\partial \psi_{p,k}}{\partial x_1}\Big|^2 dx_1 dx_2 &= \int_S \Big(\sum_{n=0}^{p} b_n^{'}(x_1) L_n(x_2)\Big)^2 dx_1 dx_2 = \sum_{n=0}^{p} \int_S (b_n^{'}(x_1))^2 L_n^2(x_2) dx_1 dx_2 \\
&= \sum_{n=0}^{p} \int_{-1}^{1} (b_n^{'}(x_1))^2 dx_1 \int_{-1}^{1} L_n^2(x_2) dx_2 \leq Cp^4 \int_{-1}^{1} b_n^2(x_1) dx_1 \int_{-1}^{1} L_n^2(x_2) dx_2 \\
&= Cp^4 \sum_{n,m=0}^{p} a_{nm}^2 \int_{-1}^{1} L_m^2(x_1) dx_1 \int_{-1}^{1} L_n^2(x_2) dx_2 = Cp^4 \sum_{n,m=0}^{p} a_{nm}^2.
\end{aligned}$$

Similarly,

$$\int_S \Big|\frac{\partial \psi_{p,k}}{\partial x_2}\Big|^2 dx_1 dx_2 \leq Cp^4 \sum_{n,m=0}^{p} a_{nm}^2 \int_{-1}^{1} L_m^2(x_1) dx_1 \int_{-1}^{1} L_n^2(x_2) dx_2 = Cp^4 \sum_{n,m=0}^{p} a_{nm}^2,$$

which leads to

$$\begin{aligned}
\int_S |\nabla \psi_{p,k}|^2 dx_1 dx_2 &= \int_S \Big[\Big|\frac{\partial \psi_{p,k}}{\partial x_1}\Big|^2 + \Big|\frac{\partial \psi_{p,k}}{\partial x_2}\Big|^2\Big] dx_1 dx_2 \\
\leq \; Cp^4 \sum_{m=0}^{p} a_{nm}^2 &= Cp^4 \int_S |\psi_{p,k}|^2 dx_1 dx_2 = Cp^4.
\end{aligned}$$

□

Lemma 4.23. *Let $\lambda_{p,k}$ be an eigenvalue of the problem (4.7), and let $\phi_{p,k}^{'}(x_3)$ be the corresponding solution of two-point value problem (4.8). Then*

$$\int_{-1}^{1} (|\phi_{p,k}^{'}|^2 + \lambda_{p,k} |\phi_{p,k}|^2) dx_3 \leq C\sqrt{\lambda_{p,k}}, \quad k = 1, 2, \cdots, N_p. \tag{4.12}$$

PROOF. Since $\lambda_{p,k}$ is an eigenvalue of the problem (4.7), then

$$\lambda_{p,k} = \int_S (\nabla \psi_{p,k})^2 dx_1 dx_2.$$

By Lemma 4.22, there exists a constant $\eta > 0$ independent of p and k, such that $0 < \lambda_{p,k} \leq \eta p^4$. Then for each k, we always can find a unique integer $1 \leq M_k \leq p$ satisfying

$$\eta(M_k - 1)^4 \leq \lambda_{p,k} \leq \eta M_k^4. \tag{4.13}$$

For each k, correspondingly we introduce the knots and the weights $\xi_i, \omega_i (i = 0, 1, \cdots, M_k)$ of the Gauss-Legendre-Lobatto quadrature formula of order M_k on the interval $[-1, 1]$. We assume that the knots are ordered in such a way that $\xi_0 = -1$. Let χ_k be the Lagrange interpolation polynomial of degree M_k such that

$$\chi_k(\xi_i) = \begin{cases} 1, & \text{if i=0 ,} \\ 0, & \text{otherwise.} \end{cases}$$

By the equivalence of discrete and continuous L^2 norms over $\mathcal{P}_{M_k}(-1, 1)$ (see [**14**]), there exists a constant $c_1 > 0$ independent of M_k such that

$$\int_{-1}^{1} |\chi_k(x_1)|^2 dx_1 \leq c_1 \sum_{i=0}^{M_k} \chi_k^2(\xi_i)\omega_i = c_1\omega_0.$$

Since $\omega_0 = \frac{2}{M_k(M_k+1)}$ (see [**17**]) we obtain

$$\int_{-1}^{1} |\chi_k(x_1)|^2 dx_1 \leq \frac{c_2}{M_k^2},$$

and by the inverse inequality, we have

$$\int_{-1}^{1} |\chi_k'(x_1)|^2 dx_1 \leq c_2 \eta M_k^2.$$

Setting $q = \phi_{p,k} - \chi_k$ in (4.10) and by using the Cauchy-Schwarz inequality, we obtain

$$\int_{-1}^{1} ((\phi_{p,k}^{'})^2 + \lambda_{p,k}(\phi_{p,k})^2) dx_3 \leq C M_k^2.$$

Lemma 4.23 follows immediately by this inequality and (4.13). $\square$

Since $f(x_1, x_2) \in \mathcal{P}_p^{2,0}(S)$ and $\{\psi_{p,k}(x_1, x_2), 1 \leq k \leq N_p\}$ is an orthonormal basis of $\mathcal{P}_p^{2,0}(S)$,

$$f(x_1, x_2) = \sum_{k=1}^{N_p} \beta_k \psi_{p,k}(x_1, x_2)$$

with $\beta_k = \int_S f(x_1, x_2)\psi_{p,k}(x_1, x_2) dx_1 dx_2$. Let

$$R_D f = \sum_{k=1}^{N_p} \beta_k \psi_{p,k}(x_1, x_2)\phi_{p,k}(x_3). \tag{4.14}$$

Obviously,

$$R_D f|_{\Gamma_1} = \sum_{k=1}^{N_p} \beta_k \psi_{p,k}(x_1, x_2) = f(x_1, x_2),$$

where $\Gamma_1 = \{(x_1, x_2, -1) | -1 < x_1, x_2 < 1\}$.

Theorem 4.24. *Let $D = (-1,1)^3$, and $\Gamma_1 = \{(x_1, x_2, -1) | -1 < x_1, x_2 < 1\}$, then for $f \in \mathcal{P}_p^{2,0}(\Gamma_1)$, there exists $R_D f \in \mathcal{P}_p^2(D)$ such that $R_D f|_{\Gamma_1} = f, R_D f|_{\partial D \setminus \Gamma_1} = 0,$ and*

$$\|R_D f\|_{H^1(D)} \leq C \|f\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)}, \tag{4.15}$$

where C is a constant, which is independent of p and f.

Proof. Let $\psi_{p,k}$ and $\phi_{p,k}$ be defined as in (4.7) and (4.10), and let $R_D f$ be given in (4.14), then

$$R_D f|_{\Gamma_1} = f, \quad R_D f|_{\partial D \setminus \Gamma_1} = 0.$$

Due to the orthogonality of the $\psi_{p,k}$ in $L^2(S)$ and $H^1(S)$, and by using (4.7) and Lemma 4.23 we have

$$\|R_D f\|^2_{L^2(D)} = \sum_{k=1}^{N_p} \beta_k^2 \frac{1}{\sqrt{\lambda_{p,k}}}$$

and

$$\begin{aligned}
|R_D f|^2_{H^1(D)} &= \int_D \Big(\Big| \frac{\partial R_D f}{\partial x_1} \Big|^2 + \Big| \frac{\partial R_D f}{\partial x_2} \Big|^2 + \Big| \frac{\partial R_D f}{\partial x_3} \Big|^2 \Big) dx_1 dx_2 dx_3 \\
&= \sum_{k=1}^{N_p} \beta_k^2 \Big(\int_S |\psi_{p,k}|^2 dx_1 dx_2 \int_I |\phi'_{p,k}|^2 dx_3 + \int_S |\nabla \psi_{p,k}|^2 dx_1 dx_2 \int_I |\phi_{p,k}|^2 dx_3 \Big) \\
&= \sum_{k=1}^{N_p} \beta_k^2 \int_I (|\phi'_{p,k}|^2 + \lambda_{p,k} |\phi_{p,k}|^2) dx_3 \leq C \sum_{k=1}^{N_p} \beta_k^2 \sqrt{\lambda_{p,k}}.
\end{aligned}$$

Therefore

$$\|R_D f\|^2_{H_0^1(D)} \leq C \sum_{k=1}^{N_p} \beta_k^2 (1 + \sqrt{\lambda_{p,k}}). \tag{4.16}$$

Note that

$$\|f\|^2_{L^2(\Gamma_1)} = \sum_{k=1}^{N_p} \beta_k^2, \quad \|f\|^2_{H_0^1(\Gamma_1)} = \sum_{k=1}^{N_p} \beta_k^2 (1 + \lambda_{p,k}).$$

By interpolation space theory (see [**38**], Chap.1)

$$\|f\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_1)} \approx \sum_{k=1}^{N_p} \beta_k^2 (1 + \lambda_{p,k})^{\frac{1}{2}} \approx \sum_{k=1}^{N_p} \beta_k^2 (1 + \sqrt{\lambda_{p,k}}),$$

which together with (4.16) implies (4.15). □

Remark 4.4. Theorem 4.24 can be proved on a cube $(0,1)^3$ by a simple mapping. Hereafter, $D = (0,1)^3$ shall be the standard cube for the convenience in following sections.

Remark 4.5. This polynomial extension without using convolution was first proposed by Canuto and Funaro for the extension in square [**13**]. Since the polynomial extension of convolution-type is sufficient on triangle and square elements, the generalization of this approach to a cube is much more significant because it is only polynomial extension compatible to FEM subspace on a cube.

Remark 4.6. In [**10**] a similar extension was proposed by using spectral solutions of two eigenvalue problems in one dimension and one boundary value problem on an interval without detailed proof. Recently the same approach was developed in [**16**]. A genuine generalization of Canuto and Funaro's approach from a square to a cube should be based on the spectral solution of an eigenvalue problem on a square, which is much better than the spectral solutions of two eigenvalue problems on an interval. Moreover, this approach can be used for a prism with non-square bases on which the eigenvalue problem can not be decomposed into two one dimensional problems, e.g. a prism with triangular bases.

4.3.2. Polynomial extension from all faces. We shall construct a polynomial extension E which lifts a polynomial on a whole boundary of a cube D in three steps, which is proved to be a continuous operator: $H^{\frac{1}{2}}(\partial D) \to H^1(T)$.

Besides the trace space $H^{\frac{1}{2}}_{00}(\Gamma_i)$, we need introduce $H^{\frac{1}{2}}_{00}(\Gamma_i, \gamma_{il} \cup \gamma_{im})$ with the norm:

$$\|u\|^2_{H^{\frac{1}{2}}_{00}(\Gamma_i, \gamma_{il} \cup \gamma_{im})} = \|u\|^2_{H^{\frac{1}{2}}(\Gamma_i)} + \int_{\Gamma_i} \frac{|u|^2}{dist(x, \gamma_{il})} dS_x + \int_{\Gamma_i} \frac{|u|^2}{dist(x, \gamma_{im})} dS_x.$$

Theorem 4.25. *Let* $D = [0,1]^3$, *and* $\Gamma_1 = \{(x_1, x_2, 0) | 0 < x_1, x_2 < 1\}$, *then for* $f \in \mathcal{P}^2_p(\Gamma_1)$, *there exists* $U \in \mathcal{P}^2_p(D)$ *such that* $U|_{\Gamma_1} = f, U|_{\Gamma_4} = 0$, *and*

$$\|U\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}, \tag{4.17}$$

where C *is a constant, which is independent of* p *and* f.

Proof. The proof is very similar with the proof of Theorem 4.24, except adopting the basis

$$\Big\{ \frac{\sqrt{(2i+1)(2j+1)}}{2} L_i(x_1) L_j(x_2), 0 \le i, j \le p \Big\}$$

for $\mathcal{P}^2_p(\Gamma_1)$ instead of the basis $\{\frac{(1-x_1^2)(1-x_2^2)}{\sqrt{\gamma^{2,2}_{i-1}\gamma^{2,2}_{j-1}}} J^{2,2}_{i-1}(x_1) J^{2,2}_{j-1}(x_2), 1 \le i, j \le p-1\}$ for $\mathcal{P}^{2,0}_p(\Gamma_1)$, where $L_i(x_1)$ and $J^{2,2}_{i-1}(x_1)$ denote the Legendre and the Jacobi polynomials, respectively. □

Theorem 4.26. *Let* $D = [0,1]^3$, *and* $\Gamma_1 = \{(x_1, x_2, 0) \mid 0 < x_1, x_2 < 1\}$, *then for* $f \in \mathcal{P}^2_p(\Gamma_1)$, $f|_{\gamma_{12}} = 0$, $f|_{\gamma_{15}} = 0$, *there exists* $U \in \mathcal{P}^2_p(D)$ *such that* $U|_{\Gamma_1} = f$, $U|_{\Gamma_4} = 0$, $U|_{\Gamma_2} = 0$, $U|_{\Gamma_5} = 0$, *and*

$$\|U\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}_{00}(\Gamma_1, \gamma_{12} \cup \gamma_{15})}, \tag{4.18}$$

where C *is a constant, which is independent of* p *and* f.

PROOF. The arguments in the proof of Theorem 4.24 can be carried out here except replacing the basis of $\mathcal{P}_p^{2,0}(\Gamma_1)$ by the basis $\{\frac{\sqrt{2j+1}(1-x_1^2)}{\sqrt{2\gamma_{i-1}^{2,2}}}J_{i-1}^{2,2}(x_1)L_j(x_2), 1 \le i \le p-1, 0 \le j \le p\}$ for ${}^0P_p^2(\Gamma_1) = \{\varphi \in \mathcal{P}_p^2(\Gamma_1) \mid \varphi(\pm 1, x_2) = 0\}$. □

Theorem 4.27. *Let $D = [0,1]^3$ be the cube and $f \in \mathcal{P}_p^2(\partial D) = \{f \in C^0(\partial D), f\mid_{\Gamma_i} = f_i \in \mathcal{P}_p^2(\Gamma_i), i = 1, \cdots, 6\}$, where Γ_i be the faces of cube D. Then there exists $E_D f \in \mathcal{P}_p^2(D)$ such that $E_D f\mid_{\partial D} = f$ and*

$$\|E_D f\|_{H^1(D)} \le C\|f\|_{H^{\frac{1}{2}}(\partial D)}, \tag{4.19}$$

where C is a constant independent of p and f, ∂D is the boundary of D.

PROOF. By Theorem 4.25, there exist $U_1, U_4 \in \mathcal{P}_p^2(D)$, such that $U_1|_{\Gamma_1} = f_1, U_1|_{\Gamma_4} = 0$; $U_4|_{\Gamma_4} = f_4, U_4|_{\Gamma_1} = 0$ and

$$\|U_1\|_{H^1(D)} \le C\|f_1\|_{H^{\frac{1}{2}}(\Gamma_1)}, \quad \|U_4\|_{H^1(D)} \le C\|f_4\|_{H^{\frac{1}{2}}(\Gamma_4)}. \tag{4.20}$$

Let $g_2 = f_2 - U_1|_{\Gamma_2} - U_4|_{\Gamma_2}$ and $g_5 = f_5 - U_1|_{\Gamma_5} - U_4|_{\Gamma_5}$, then g_2 vanishes at the sides γ_{12} and γ_{24} on Γ_2 and g_5 vanishes at the sides γ_{15} and γ_{45} on Γ_5. By Theorem 4.26, there exist $U_2, U_5 \in \mathcal{P}_p^2(D)$, such that $U_2|_{\Gamma_2} = g_2, U_2|_{\Gamma_i} = 0, i = 1,4,5$; $U_5|_{\Gamma_5} = g_5, U_5|_{\Gamma_j} = 0, j = 1,2,4$ and

$$\|U_2\|_{H^1(D)} \le C\|g_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2,\gamma_{12}\cup\gamma_{24})}, \quad \|U_5\|_{H^1(D)} \le C\|g_5\|_{H_{00}^{\frac{1}{2}}(\Gamma_5,\gamma_{15}\cup\gamma_{45})}. \tag{4.21}$$

Let

$$g_3 = f_3 - \sum_{i=1,2,4,5} U_i|_{\Gamma_3}, \quad g_6 = f_6 - \sum_{i=1,2,4,5} U_i|_{\Gamma_6},$$

then

$$\begin{aligned} g_3|_{\gamma_{13}} &= -U_2|_{\gamma_{13}} - U_5|_{\gamma_{13}}, \quad g_3|_{\gamma_{23}} = 0, \quad g_3|_{\gamma_{34}} = -U_2|_{\gamma_{34}} - U_5|_{\gamma_{34}}, \quad g_3|_{\gamma_{35}} = 0, \\ g_6|_{\gamma_{16}} &= -U_2|_{\gamma_{16}} - U_5|_{\gamma_{16}}, \quad g_6|_{\gamma_{26}} = 0, \quad g_6|_{\gamma_{46}} = -U_2|_{\gamma_{46}} - U_5|_{\gamma_{46}}, \quad g_6|_{\gamma_{56}} = 0. \end{aligned}$$

By Theorem 4.26, there exist $U_3, U_6 \in \mathcal{P}_p^2(D)$, such that $U_3|_{\Gamma_3} = g_3, U_3|_{\Gamma_i} = 0, i = 2,5,6$; $U_6|_{\Gamma_6} = g_6, U_6|_{\Gamma_j} = 0, j = 2,3,5$ and

$$\|U_3\|_{H^1(D)} \le C\|g_3\|_{H_{00}^{\frac{1}{2}}(\Gamma_3,\gamma_{23}\cup\gamma_{35})}, \quad \|U_6\|_{H^1(D)} \le C\|g_6\|_{H_{00}^{\frac{1}{2}}(\Gamma_6,\gamma_{26}\cup\gamma_{56})}. \tag{4.22}$$

Let $U = U_1 + U_2 + U_3 + U_4 + U_5 + U_6$, then $U|_{\Gamma_i} = f_i, i = 2,3,5,6$, and let $g_1 = f_1 - U|_{\Gamma_1}$, $g_4 = f_4 - U|_{\Gamma_4}$, then it is easy to verify that $g_1 = 0$ on $\partial\Gamma_1$ and $g_4 = 0$ on $\partial\Gamma_4$. In fact, since $U_1\mid_{\Gamma_1} = f_1$ and $U_2\mid_{\Gamma_1} = U_4\mid_{\Gamma_1} = U_5\mid_{\Gamma_1} = U_3\mid_{\Gamma_2} = U_6\mid_{\Gamma_2} = 0$, there holds

$$\begin{aligned} g_1|_{\gamma_{12}} &= (f_1 - U|_{\Gamma_1})|_{\gamma_{12}} = f_1|_{\gamma_{12}} - \Big((U_1 + U_2 + U_3 + U_4 + U_5 + U_6)|_{\Gamma_1}\Big)\,|_{\gamma_{12}} \\ &= f_1|_{\gamma_{12}} - \Big(f_1 + U_2\mid_{\Gamma_1} + U_3|_{\Gamma_2} + U_4\mid_{\Gamma_1} + U_5\mid_{\Gamma_1} + U_6\mid_{\Gamma_2}\Big)|_{\gamma_{12}} = 0 \end{aligned}$$

and since $U_3 \mid_{\gamma_{13}} = g_3 \mid_{\gamma_{13}} = \Big(f_3 - U_1 + U_2 + U_4 + U_5\Big) \mid_{\gamma_{13}}$ and $U_6 \mid_{\Gamma_3} = 0$, there holds

$$g_1|_{\gamma_{13}} \;=\; (f_1 - U|_{\Gamma_1})|_{\gamma_{13}} = f_1|_{\gamma_{13}} - (U|_{\Gamma_3})|_{\gamma_{13}} = f_1|_{\gamma_{13}} - f_3|_{\gamma_{13}} = 0.$$

Similarly, we have $g_1|_{\gamma_{15}} = 0$ and $g_1|_{\gamma_{16}} = 0$. By the symmetry we have $g_4 \mid_{\partial\Gamma_4} = 0$.

By Theorem 4.24, there exist $V_1 \in \mathcal{P}_p^{2,0}(\Gamma_1)$ and $V_4 \in \mathcal{P}_p^{2,0}(\Gamma_4)$ such that

$$V_1|_{\Gamma_1} = g_1, \quad V_1|_{\Gamma_i} = 0, \quad i = 2,3,4,5,6,$$
$$V_4|_{\Gamma_4} = g_4, \quad V_4|_{\Gamma_i} = 0, \quad i = 1,2,3,5,6,$$

and

$$\|V_1\|_{H^1(D)} \le C\|g_1\|_{H^{\frac{1}{2}}_{00}(\Gamma_1)}, \quad \|V_4\|_{H^1(D)} \le C\|g_4\|_{H^{\frac{1}{2}}_{00}(\Gamma_4)}.$$

Let $E_D f = U + V_1 + V_4$, then we have $E_D f|_{\Gamma_i} = f_i$, $i = 1,2,3,4,5,6$, and

$$\begin{aligned}
\|E_D f\|_{H^1(S)} &\le \|U\|_{H^1(S)} + \|V_1\|_{H^1(S)} + \|V_4\|_{H^1(S)} \\
&\le \sum_{i=1}^{6} \|U_i\|_{H^1(S)} + \|V_1\|_{H^1(S)} + \|V_4\|_{H^1(S)} \\
&\le C\Big(\|f_1\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|f_4\|_{H^{\frac{1}{2}}(\Gamma_4)} + \|g_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} + \|g_5\|_{H^{\frac{1}{2}}_{00}(\Gamma_5,\gamma_{15}\cup\gamma_{45})} \\
&\quad + \|g_3\|_{H^{\frac{1}{2}}_{00}(\Gamma_3,\gamma_{23}\cup\gamma_{35})} + \|g_6\|_{H^{\frac{1}{2}}_{00}(\Gamma_6,\gamma_{26}\cup\gamma_{56})} + \|g_1\|_{H^{\frac{1}{2}}_{00}(\Gamma_1)} + \|g_4\|_{H^{\frac{1}{2}}_{00}(\Gamma_4)}\Big).
\end{aligned} \tag{4.23}$$

First, we prove that

$$\|g_2\|_{H^{\frac{1}{2}}_{00}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} \le C\|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)}. \tag{4.24}$$

Due to (4.20), there holds

$$\begin{aligned}
\|g_2\|_{H^{\frac{1}{2}}(\Gamma_2)} &= \|f_2 - U_1|_{\Gamma_2} - U_4|_{\Gamma_2}\|_{H^{\frac{1}{2}}(\Gamma_2)} \le \|f_2\|_{H^{\frac{1}{2}}(\Gamma_2)} + \|U_1\|_{H^{\frac{1}{2}}(\Gamma_2)} + \|U_4\|_{H^{\frac{1}{2}}(\Gamma_2)} \\
&\le \|f_2\|_{H^{\frac{1}{2}}(\Gamma_2)} + C\|U_1\|_{H^1(D)} + C\|U_4\|_{H^1(D)} \le C(\|f_2\|_{H^{\frac{1}{2}}(\Gamma_2)} + \|f_1\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|f_4\|_{H^{\frac{1}{2}}(\Gamma_4)}).
\end{aligned} \tag{4.25}$$

By the definition of $H^{\frac{1}{2}}_{00}(\Gamma_2, \gamma_{12} \cup \gamma_{24})$, we need to prove that

$$\int_S \frac{|f_2 - U_1|_{\Gamma_2} - U_4|_{\Gamma_2}|^2}{x_3} dx_1 dx_3 \le C\|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)}, \tag{4.26}$$

and

$$\int_S \frac{|f_2 - U_1|_{\Gamma_2} - U_4|_{\Gamma_2}|^2}{1 - x_3} dx_1 dx_3 \le C\|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)}. \tag{4.27}$$

Noting that $U_1(x_1, x_3, 0) = f_1(x_1, x_3)$ and $U_4(x_1, x_3, 0) = 0$, there holds

$$\begin{aligned}
g_2(x_1, x_3) &= f_2(x_1, x_3) - \sum_{i=1,4} U_i(x_1, x_2, x_3)|_{\Gamma_2} = f_2(x_1, x_3) - \sum_{i-1,4} U_i(x_1, 0, x_3) \\
&= (f_2(x_1, x_3) - f_1(x_1, x_3)) + (U_1(x_1, x_3, 0) - U_1(x_1, 0, x_3)) + (U_4(x_1, x_3, 0) - U_4(x_1, 0, x_3)).
\end{aligned}$$

Since the following norm is an equivalent norm for the space $H^{\frac{1}{2}}(\Gamma_2 \cup \Gamma_1)$ [**2, 22**],

$$\|f\|_{H^{\frac{1}{2}}(\Gamma_2\cup\Gamma_1)} \approx \Big(\|f_2\|^2_{H^{\frac{1}{2}}(\Gamma_2)} + \|f_1\|^2_{H^{\frac{1}{2}}(\Gamma_1)} + D(f_2, f_1)\Big)^{\frac{1}{2}},$$

where

$$D(f_2, f_1) = \int_S \frac{|f_2(t_1, 0, t_2) - f_1(t_1, t_2, 0)|^2}{t_2} dt_1 dt_2,$$

we have

$$\int_S \frac{|f_2(x_1, x_3) - f_1(x_1, x_3)|^2}{x_3} dx_1 dx_3 \le \|f\|^2_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2)},$$

$$\begin{aligned} & \int_S \frac{|U_1(x_1, x_3, 0) - U_1(x_1, 0, x_3)|^2}{x_3} dx_1 dx_3 = D(U_1|_{\Gamma_1}, U_1|_{\Gamma_2}) \\ \le\ & C\|U_1\|^2_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2)} \le C\|U_1\|^2_{H^1(D)} \le C\|f_1\|^2_{H^{\frac{1}{2}}(\Gamma_1)}, \end{aligned}$$

and

$$\begin{aligned} & \int_S \frac{|U_4(x_1, x_3, 0) - U_4(x_1, 0, x_3)|^2}{x_3} dx_1 dx_3 = D(U_4|_{\Gamma_1}, U_4|_{\Gamma_2}) \\ \le\ & C\|U_4\|^2_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2)} \le C\|U_4\|^2_{H^1(D)} \le C\|f_4\|^2_{H^{\frac{1}{2}}(\Gamma_4)}. \end{aligned}$$

Therefore we obtain (4.26). Similarly, noting that $U_4(x_1, x_3, 1) = f_4(x_1, x_2)$ and $U_1(x_1, x_3, 1) = 0$ we have

$$\begin{aligned} & g_2(x_1, x_3) = f_2(x_1, x_3) - \sum_{i=1,4} U_i(x_1, x_2, x_3)|_{\Gamma_2} = f_2(x_1, x_3) - \sum_{i=1,4} U_i(x_1, 0, x_3) \\ =\ & (f_2(x_1, x_3) - f_4(x_1, x_3)) + (U_4(x_1, x_3, 1) - U_4(x_1, 0, x_3)) + (U_1(x_1, x_3, 1) - U_1(x_1, 0, x_3)), \end{aligned}$$

and

$$\int_S \frac{|f_2(x_1, x_3) - f_4(x_1, x_3)|^2}{1 - x_3} dx_1 dx_3 \le \|f\|^2_{H^{\frac{1}{2}}(\Gamma_2\cup\Gamma_4)},$$

$$\begin{aligned} & \int_S \frac{|U_4(x_1, x_3, 1) - U_4(x_1, 0, x_3)|^2}{1 - x_3} dx_1 dx_3 = D(U_4|_{\Gamma_4}, U_4|_{\Gamma_2}) \\ \le\ & C\|U_4\|^2_{H^{\frac{1}{2}}(\Gamma_2\cup\Gamma_4)} \le C\|U_4\|^2_{H^1(D)} \le C\|f_4\|^2_{H^{\frac{1}{2}}(\Gamma_4)}, \end{aligned}$$

and

$$\begin{aligned} & \int_S \frac{|U_1(x_1, x_3, 1) - U_1(x_1, 0, x_3)|^2}{1 - x_3} dx_1 dx_3 = D(U_1|_{\Gamma_4}, U_1|_{\Gamma_2}) \\ \le\ & C\|U_1\|^2_{H^{\frac{1}{2}}(\Gamma_2\cup\Gamma_4)} \le C\|U_1\|^2_{H^1(D)} \le C\|f_1\|^2_{H^{\frac{1}{2}}(\Gamma_1)}. \end{aligned}$$

Then, we obtain (4.27), which together with (4.25) and (4.26) lead to (4.24). Analogously, we have

$$\|g_5\|_{H_{00}^{\frac{1}{2}}(\Gamma_5,\gamma_{15}\cup\gamma_{45})} \leq C\|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_4\cup\Gamma_5)}. \tag{4.28}$$

Furthermore, we shall prove that

$$\|g_3\|_{H_{00}^{\frac{1}{2}}(\Gamma_3,\gamma_{23}\cup\gamma_{35})} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D\setminus\Gamma_6)} \tag{4.29}$$

and

$$\|g_6\|_{H_{00}^{\frac{1}{2}}(\Gamma_6,\gamma_{26}\cup\gamma_{56})} \leq C\|f\|_{H^{\frac{1}{2}}(\partial D\setminus\Gamma_3)}. \tag{4.30}$$

Due to (4.21),(4.24) and (4.28)

$$\begin{aligned}
\|g_3\|_{H^{\frac{1}{2}}(\Gamma_3)} &= \|f_3 - \sum_{i=1,2,4,5} U_i|_{\Gamma_3}\|_{H^{\frac{1}{2}}(\Gamma_3)} \leq \|f_3\|_{H^{\frac{1}{2}}(\Gamma_3)} + \sum_{i=1,2,4,5}\|U_i\|_{H^{\frac{1}{2}}(\Gamma_3)} \\
&\leq \|f_3\|_{H^{\frac{1}{2}}(\Gamma_3)} + C\sum_{i=1,2,4,5}\|U_i\|_{H^1(D)} \\
&\leq \|f_3\|_{H^{\frac{1}{2}}(\Gamma_3)} + C\Big(\|f_1\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|f_4\|_{H^{\frac{1}{2}}(\Gamma_4)} + \|g_2\|_{H_{00}^{\frac{1}{2}}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} + \|g_5\|_{H_{00}^{\frac{1}{2}}(\Gamma_5,\gamma_{15}\cup\gamma_{45})}\Big) \\
&\leq \|f_3\|_{H^{\frac{1}{2}}(\Gamma_3)} + C\Big(\|f_1\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|f_4\|_{H^{\frac{1}{2}}(\Gamma_4)} + \|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)} + \|f\|_{H^{\frac{1}{2}}(\Gamma_1\cup\Gamma_4\cup\Gamma_5)}\Big) \\
&\leq C\|f\|_{H^{\frac{1}{2}}(\partial D\setminus\Gamma_6)}.
\end{aligned}$$

By the definition of $H_{00}^{\frac{1}{2}}(\Gamma_3,\gamma_{23}\cup\gamma_{35})$, we need to show that

$$\int_S \frac{|f_3 - U_1|_{\Gamma_3} - U_4|_{\Gamma_3} - U_2|_{\Gamma_3} - U_5|_{\Gamma_3}|^2}{x_2}dx_1dx_3 \leq C\|f\|_{H^{\frac{1}{2}}(\partial D\setminus\Gamma_6)}, \tag{4.31}$$

and

$$\int_S \frac{|f_3 - U_1|_{\Gamma_3} - U_4|_{\Gamma_3} - U_2|_{\Gamma_3} - U_5|_{\Gamma_3}|^2}{1-x_2}dx_1dx_3 \leq C\|f\|_{H^{\frac{1}{2}}(\partial D\setminus\Gamma_6)}. \tag{4.32}$$

Noting that $U_2(x_2,0,x_3)=g_2(x_2,x_3)$ and $U_2(x_2,0,x_3)=0$, we have

$$\begin{aligned}
& g_3(x_2,x_3) = f_3(x_2,x_3) - \sum_{i=1,2,4,5} U_i(x_1,x_2,x_3)|_{\Gamma_3} = f_3(x_2,x_3) - \sum_{i=1,2,4,5} U_i(0,x_2,x_3) \\
= \;& f_3(x_2,x_3) - g_2(x_2,x_3) + g_2(x_2,x_3) - U_1(0,x_2,x_3) - U_4(0,x_2,x_3) \\
& -U_2(0,x_2,x_3) - U_5(0,x_2,x_3) \\
= \;& f_3(x_2,x_3) - \Big(f_2(x_2,x_3) - U_1(x_2,0,x_3) - U_4(x_2,0,x_3)\Big) + U_2(x_2,0,x_3) \\
& -U_1(0,x_2,x_3) - U_4(0,x_2,x_3) - U_2(0,x_2,x_3) - U_5(0,x_2,x_3) \\
= \;& \Big(f_3(x_2,x_3) - f_2(x_2,x_3)\Big) + \Big(U_1(x_2,0,x_3) - U_1(0,x_2,x_3)\Big) \\
& +\Big(U_4(x_2,0,x_3) - U_4(0,x_2,x_3)\Big) + \Big(U_2(x_2,0,x_3) - U_2(0,x_2,x_3)\Big) \\
& +\Big(U_5(x_2,0,x_3) - U_5(0,x_2,x_3)\Big)
\end{aligned}$$

and

$$\int_S \frac{|f_3(x_2,x_3) - f_2(x_2,x_3)|}{x_2} dx_2 dx_3 \le \|f\|^2_{H^{\frac12}(\Gamma_3\cup\Gamma_2)},$$

$$\begin{aligned}
& \int_S \frac{|U_1(x_2,0,x_3) - U_1(0,x_2,x_3)|}{x_2} dx_2 dx_3 = D(U_1|_{\Gamma_2}, U_1|_{\Gamma_3}) \\
\le \;& C\|U_1\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_3)} \le C\|U_1\|^2_{H^1(D)} \le C\|f_1\|^2_{H^{\frac12}(\Gamma_1)},
\end{aligned}$$

$$\begin{aligned}
\int_S \quad & \frac{|U_4(x_2,0,x_3) - U_4(0,x_2,x_3)|}{x_2} dx_2 dx_3 = D(U_4|_{\Gamma_2}, U_4|_{\Gamma_3}) \\
\le \;& C\|U_4\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_3)} \le C\|U_4\|^2_{H^1(D)} \le C\|f_4\|^2_{H^{\frac12}(\Gamma_4)},
\end{aligned}$$

$$\begin{aligned}
& \int_S \frac{|U_2(x_2,0,x_3) - U_2(0,x_2,x_3)|}{x_2} dx_2 dx_3 = D(U_2|_{\Gamma_2}, U_2|_{\Gamma_3}) \le C\|U_2\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_3)} \\
\le \;& C\|U_2\|^2_{H^1(D)} \le C\|g_2\|^2_{H^{\frac12}_{00}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} \le C\|f\|^2_{H^{\frac12}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)},
\end{aligned}$$

and

$$\begin{aligned}
& \int_S \frac{|U_5(x_2,0,x_3) - U_5(0,x_2,x_3)|}{x_2} dx_2 dx_3 = D(U_5|_{\Gamma_2}, U_5|_{\Gamma_3}) \le C\|U_5\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_3)} \\
\le \;& C\|U_5\|^2_{H^1(D)} \le C\|g_5\|^2_{H^{\frac12}_{00}(\Gamma_5,\gamma_{12}\cup\gamma_{24})} \le C\|f\|^2_{H^{\frac12}(\Gamma_1\cup\Gamma_4\cup\Gamma_5)}.
\end{aligned}$$

Hence, we obtain (4.31).

Similarly, we can decompose $g_3(x_2,x_3)$ as follows

$$\begin{aligned} g_3(x_2,x_3) &= (f_3(x_2,x_3)-f_5(x_2,x_3))+(U_1(x_2,1,x_3)-U_1(0,x_2,x_3)) \\ &\quad +(U_4(x_2,1,x_3)-U_4(0,x_2,x_3))+(U_5(x_2,1,x_3)-U_5(0,x_2,x_3)) \\ &\quad +(U_2(x_2,1,x_3)-U_2(0,x_2,x_3)), \end{aligned}$$

and we have the estimates

$$\int_S \frac{|f_3(x_2,x_3)-f_5(x_2,x_3)|}{1-x_2}dx_2dx_3 \le \|f\|^2_{H^{\frac12}(\Gamma_3\cup\Gamma_5)},$$

$$\int_S \frac{|U_1(x_2,1,x_3)-U_1(0,x_2,x_3)|}{1-x_2}dx_2dx_3 \le C\|U_1\|^2_{H^{\frac12}(\Gamma_5\cup\Gamma_3)} \le C\|f_1\|^2_{H^{\frac12}(\Gamma_1)},$$

$$\int_S \frac{|U_4(x_2,1,x_3)-U_4(0,x_2,x_3)|}{1-x_2}dx_2dx_3 \le C\|U_4\|^2_{H^{\frac12}(\Gamma_5\cup\Gamma_3)} \le C\|f_4\|^2_{H^{\frac12}(\Gamma_4)},$$

$$\begin{aligned} &\int_S \frac{|U_5(x_2,1,x_3)-U_5(0,x_2,x_3)|}{1-x_2}dx_2dx_3 \le C\|U_5\|^2_{H^{\frac12}(\Gamma_5\cup\Gamma_3)} \\ \le\ & C\|g_5\|^2_{H^{\frac12}_{00}(\Gamma_5,\gamma_{15}\cup\gamma_{45})} \le C\|f\|^2_{H^{\frac12}(\Gamma_1\cup\Gamma_4\cup\Gamma_5)}, \end{aligned}$$

and

$$\begin{aligned} &\int_S \frac{|U_2(x_2,1,x_3)-U_2(0,x_2,x_3)|}{1-x_2}dx_2dx_3 \le C\|U_2\|^2_{H^{\frac12}(\Gamma_5\cup\Gamma_3)} \\ \le\ & C\|g_2\|^2_{H^{\frac12}_{00}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} \le C\|f\|^2_{H^{\frac12}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)}. \end{aligned}$$

Then we obtain (4.32). A combination of (4.31)-(4.32) and (4.25) leads to (4.29). By the symmetry, we have (4.30).

Finally, we prove that

$$\|g_i\|_{H^{\frac12}_{00}(\Gamma_1)} \le C\|f\|_{H^{\frac12}(\partial D)}, \quad i=1,4. \tag{4.33}$$

By (4.20)-(4.22) and (4.24), (4.28)-(4.30), we obtain

$$\|g_1\|_{H^{\frac12}(\Gamma_1)} = \|f_1-U|_{\Gamma_1}\|_{H^{\frac12}(\Gamma_1)} \le \|f_1\|_{H^{\frac12}(\Gamma_1)} + \sum_{i=1}^{6}\|U_i\|_{H^{\frac12}(\Gamma_1)} \le C\|f\|_{H^{\frac12}(\partial D)}. \tag{4.34}$$

By the definition of $H^{\frac12}_{00}(\Gamma_1)$, we need to show that

$$\int_S \frac{|g_1(x_1,x_2)|^2}{x_2}dx_1dx_2 \le C\|f\|_{H^{\frac12}(\partial D)}, \ \int_S \frac{|g_1(x_1,x_2)|^2}{1-x_2}dx_1dx_2 \le C\|f\|_{H^{\frac12}(\partial D)}, \tag{4.35}$$

and

$$\int_S \frac{|g_1(x_1,x_2)|^2}{x_1}dx_1dx_2 \le C\|f\|_{H^{\frac12}(\partial D)}, \ \int_S \frac{|g_1(x_1,x_2)|^2}{1-x_1}dx_1dx_2 \le C\|f\|_{H^{\frac12}(\partial D)}. \tag{4.36}$$

Since $U_2|_{\Gamma_2} = g_2$, $U_5|_{\Gamma_1} = 0$, $U_3|_{\Gamma_2} = 0$ and $U_6|_{\Gamma_2} = 0$, we have

$$\begin{aligned}
& g_1(x_1,x_2) = f_1(x_1,x_2) - U(x_1,x_2,x_3)|_{\Gamma_1} \\
= \;& f_1(x_1,x_2) - g_2(x_1,x_2) + g_2(x_1,x_2) - U(x_1,x_2,x_3)|_{\Gamma_1} \\
= \;& f_1(x_1,x_2) - \Big(f_2(x_1,x_2) - U_1(x_1,0,x_2) - U_4(x_1,0,x_2)\Big) + U_2(x_1,0,x_2) \\
& -U_1(x_1,x_2,0) - U_2(x_1,x_2,0) - U_3(x_1,x_2,0) - U_4(x_1,x_2,0) \\
& -U_5(x_1,x_2,0) - U_6(x_1,x_2,0) \\
= \;& \Big(f_1(x_1,x_2) - f_2(x_1,x_2)\Big) + \Big(U_1(x_1,0,x_2) - U_1(x_1,x_2,0)\Big) \\
& +\Big(U_4(x_1,0,x_2) - U_4(x_1,x_2,0)\Big) + \Big(U_2(x_1,0,x_2) - U_2(x_1,x_2,0)\Big) \\
& +\Big(U_3(x_1,0,x_2) - U_3(x_1,x_2,0)\Big) + \Big(U_6(x_1,0,x_2) - U_6(x_1,x_2,0)\Big).
\end{aligned}$$

Note that there hold

$$\int_S \frac{|f_1(x_1,x_2) - f_2(x_1,x_2)|^2}{x_2} dx_1 dx_2 \le \|f\|_{H^{\frac12}(\Gamma_1\cup\Gamma_2)},$$

$$\begin{aligned}
& \int_S \frac{|U_1(x_1,0,x_2) - U_1(x_1,x_2,0)|^2}{x_2} dx_1 dx_2 = D(U_1|_{\Gamma_2}, U_1|_{\Gamma_1}) \\
\le \;& C\|U_1\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_1)} \le C\|U_1\|^2_{H^1(D)} \le C\|f_1\|^2_{H^{\frac12}(\Gamma_1)},
\end{aligned}$$

$$\begin{aligned}
& \int_S \frac{|U_4(x_1,0,x_2) - U_4(x_1,x_2,0)|^2}{x_2} dx_1 dx_2 = D(U_4|_{\Gamma_2}, U_4|_{\Gamma_1}) \\
\le \;& C\|U_4\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_1)} \le C\|U_4\|^2_{H^1(D)} \le C\|f_4\|^2_{H^{\frac12}(\Gamma_4)},
\end{aligned}$$

$$\begin{aligned}
& \int_S \frac{|U_2(x_1,0,x_2) - U_2(x_1,x_2,0)|^2}{x_2} dx_1 dx_2 = D(U_2|_{\Gamma_2}, U_2|_{\Gamma_1}) \le C\|U_2\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_1)} \\
\le \;& C\|U_2\|^2_{H^1(D)} \le C\|g_2\|^2_{H^{\frac12}_{00}(\Gamma_2,\gamma_{12}\cup\gamma_{24})} \le C\|f\|^2_{H^{\frac12}(\Gamma_1\cup\Gamma_2\cup\Gamma_4)},
\end{aligned}$$

$$\begin{aligned}
& \int_S \frac{|U_3(x_1,0,x_2) - U_3(x_1,x_2,0)|^2}{x_2} dx_1 dx_2 = D(U_3|_{\Gamma_2}, U_3|_{\Gamma_1}) \le C\|U_3\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_1)} \\
\le \;& C\|U_3\|^2_{H^1(D)} \le C\|g_3\|^2_{H^{\frac12}_{00}(\Gamma_3,\gamma_{23}\cup\gamma_{35})} \le C\|f\|^2_{H^{\frac12}(\partial D\setminus\Gamma_6)},
\end{aligned}$$

and

$$\begin{aligned}
& \int_S \frac{|U_6(x_1,0,x_2) - U_6(x_1,x_2,0)|^2}{x_2} dx_1 dx_2 = D(U_6|_{\Gamma_2}, U_6|_{\Gamma_1}) \le C\|U_6\|^2_{H^{\frac12}(\Gamma_2\cup\Gamma_1)} \\
\le \;& C\|U_6\|^2_{H^1(D)} \le C\|g_6\|^2_{H^{\frac12}_{00}(\Gamma_6,\gamma_{26}\cup\gamma_{56})} \le C\|f\|^2_{H^{\frac12}(\partial D\setminus\Gamma_3)}.
\end{aligned}$$

Above inequalities lead to (4.35).

Noting that $U_5|_{\Gamma_5} = g_5$, $U_2|_{\Gamma_5} = 0$, $U_3|_{\Gamma_5} = 0$ and $U_6|_{\Gamma_5} = 0$, we can rewrite $g_1 = f_1 - U|_{\Gamma_1}$ as follows:

$$\begin{aligned} g_1(x_1,x_2) &= \Big(f_1(x_1,x_2) - f_5(x_1,x_2)\Big) + \Big(U_1(x_1,1,x_2) - U_1(x_1,x_2,0)\Big) \\ &\quad + \Big(U_4(x_1,1,x_2) - U_4(x_1,x_2,0)\Big) + \Big(U_5(x_1,1,x_2) - U_5(x_1,x_2,0)\Big) \\ &\quad + \Big(U_2(x_1,1,x_2) - U_2(x_1,x_2,0)\Big) + \Big(U_3(x_1,1,x_2) - U_3(x_1,x_2,0)\Big) \\ &\quad + \Big(U_6(x_1,1,x_2) - U_6(x_1,x_2,0)\Big). \end{aligned}$$

Arguing as above we can get the second inequality of (4.35).

Since $U_3|_{\Gamma_3} = g_3$, and $U_6|_{\Gamma_3} = 0$, we can rewrite $g_1 = f_1 - U|_{\Gamma_1}$ as follows:

$$\begin{aligned} g_1(x_1,x_2) &= \Big(f_1(x_1,x_2) - f_3(x_1,x_2)\Big) + \Big(U_1(0,x_1,x_2) - U_1(x_1,x_2,0)\Big) \\ &\quad + \Big(U_4(0,x_1,x_2) - U_4(x_1,x_2,0)\Big) + \Big(U_2(0,x_1,x_2) - U_2(x_1,x_2,0)\Big) \\ &\quad + \Big(U_5(0,x_1,x_2) - U_5(x_1,x_2,0)\Big) + \Big(U_3(x_1,1,x_2) - U_3(x_1,x_2,0)\Big) \\ &\quad + \Big(U_6(0,x_1,x_2) - U_6(x_1,x_2,0)\Big), \end{aligned}$$

it leads to (4.36).

Since $U_6|_{\Gamma_6} = g_6$, and $U_3|_{\Gamma_6} = 0$, we can rewrite $g_1 = f_1 - U|_{\Gamma_1}$ as follows:

$$\begin{aligned} g_1(x_1,x_2) &= \Big(f_1(x_1,x_2) - f_6(x_1,x_2)\Big) + \Big(U_1(1,x_1,x_2) - U_1(x_1,x_2,0)\Big) \\ &\quad + \Big(U_4(1,x_1,x_2) - U_4(x_1,x_2,0)\Big) + \Big(U_2(1,x_1,x_2) - U_2(x_1,x_2,0)\Big) \\ &\quad + \Big(U_5(1,x_1,x_2) - U_5(x_1,x_2,0)\Big) + \Big(U_3(1,x_1,x_2) - U_3(x_1,x_2,0)\Big) \\ &\quad + \Big(U_6(1,x_1,x_2) - U_6(x_1,x_2,0)\Big), \end{aligned}$$

it leads to the second inequality of (4.36). Combining (4.35), (4.36) and (4.34), we obtain (4.33) for $i = 1$. By the symmetry, we have (4.33) for $i = 4$, which together with (4.23)-(4.24) and (4.28)-(4.30) lead to (4.19). $\square$

CHAPTER 5

The convergence of the h-p version of the Finite Element Method

In this chapter we will investigate performance of the *h-p* version of the finite element method in three dimensions in the framework of Jacobi-weighted Besov and Sobolev spaces. We first deal with the problems with smooth solutions, then focus on the optimal convergence for the problems with singular solutions.

5.1. A model boundary value problem

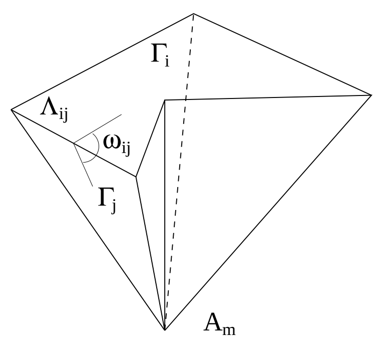


Fig. 5.1 A polyhedral domain Ω

Consider a boundary value problem

$$
\begin{aligned}
&-\Delta u + u = f && \text{in } \Omega \subset \mathbb{R}^3 \\
&u\,|_{\Gamma_D} = 0, && \frac{\partial u}{\partial n}\Big|_{\Gamma_N} = g.
\end{aligned}
\tag{5.1}
$$

where Ω is Lipschitz domain in $\mathbb{R}^3$, and $\Gamma_D = \bigcup_{i\in\mathcal{D}} \bar{\Gamma}_i$ and $\Gamma_N = \bigcup_{i\in\mathcal{N}} \Gamma_i$ are referred as the Dirichlet boundary and the Neumann boundary where the Dirichlet and Neumann boundary conditions are imposed. $\mathcal{D}$ is a subset of $\mathcal{I} = \{1, 2, \cdots, I\}$ and $\mathcal{N} = \mathcal{I} \setminus \mathcal{D}$.

The variational form of (5.1) is to seek $u(x) \in H^1_{\mathrm{D}}(\Omega) = \{u \in H^1_{\mathrm{D}} \mid u = 0 \text{ on } \Gamma_D\}$ such that

$$
B(u, v) = F(v), \quad \forall\, v \in H^1_{\mathrm{D}}(\Omega) \tag{5.2}
$$

where B is a bilinear form on $H^1_{\mathrm{D}}(\Omega) \times H^1_D(\Omega)$ and F is a linear functional on $H^1_D(\Omega)$, given by

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + u\, v)\, dx \tag{5.3}$$

and

$$F(v) = \int_\Omega f\, v\, dx + \int_{\Gamma_{\mathrm{N}}} g\, v\, ds. \tag{5.4}$$

Let $\Delta = \{\Omega_j, 1 \leq j \leq J\}$ be a partition of Ω. $\Omega_j' s$ are shape-regular tetrahedral, triangular prism and hexahedral elements with plane or surface faces. We shall assume that $\overline{\Omega}_i \cap \overline{\Omega}_j$ is either the empty set, or an entire side,or a whole face, or a vertex of Ω_i and Ω_j, and assume that all vertices of Ω are vertices of some Ω_i. By M_j we denote a mapping of standard element Ω_{st} onto Ω_j, where Ω_{st} is a standard tetrahedron K, or a standard triangular-prism G, or a standard hexahedron D, shown in Fig. 5.2. By V_l, F_i and $\gamma_{i,m} = \bar{F}_i \cap \bar{F}_m$ we denote the vertices, faces and edges of the standard elements, which are mapped onto vertices $V_l^{[j]}$, faces $F_i^{[j]}$ and edges $\gamma_{i,m}^{[j]}$ of the element Ω_j under the mapping M_j.

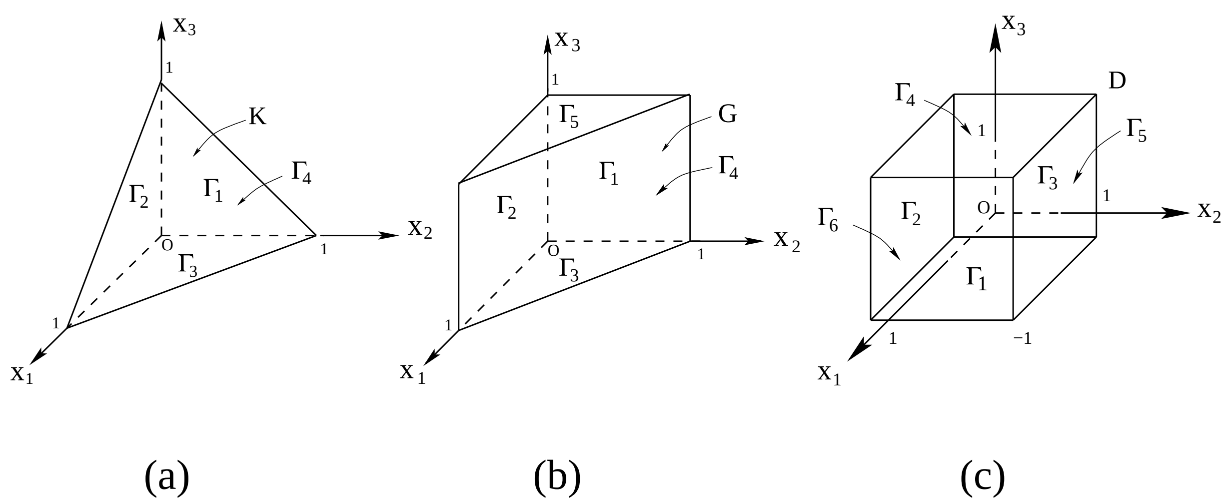


Fig. 5.2 Standard Elements

(a) Tetrahedron K, (b) Triangular prism G, (c) Hexahedron D

Let $\mathcal{P}^\kappa_{p_j}(\Omega_j)$ denote a set of pull-back polynomials φ on Ω_j such that $\varphi \circ M_j \in \mathcal{P}^\kappa_{p_j}(\Omega_{st})$ with $\kappa = 1$ if Ω_{st} is the tetrahedron K, $\mathcal{P}^{1.5}_p(\Omega_{st}) = \mathcal{P}^1_p(T) \times \mathcal{P}_p(I)$ if Ω_{st} is the triangular prism G, and $\kappa = 2$ if Ω_{st} is the hexahedron D . For sake of simplicity we use $\mathcal{P}_{p_j}(\Omega_j)$, which is understood as $\mathcal{P}^\kappa_{p_j}(\Omega_j)$ with $\kappa = 1$, or 1.5, or 2 according to the shape of the element Ω_j, if it causes no confusion. By P we denote the distribution of element degrees $\{p_1, p_2, \cdots, p_J\}$.

A finite element subspace of continuous and piecewise pull-back polynomials is defined as

$$S_D^{P,1}(\Omega;\Delta;\mathcal{M}) = S^P(\Omega;\Delta;\mathcal{M}) \cap H_D^1(\Omega), \tag{5.5}$$
$$S^P(\Omega;\Delta;\mathcal{M}) = \{\varphi \big| \, \varphi|_{\Omega_j} \in \mathcal{P}_{p_j}(\Omega_j), 1 \le j \le J\}.$$

Let $I_h = (-h,h)$, we have the following one-dimensional Jacobi projection theorem.

Lemma 5.1. *Let $u \in H^{k,\beta}(I_h)$, and let $u_{hp} = \Pi^{\beta}_{p,h} u$ be the Jacobi projection of u on $\mathcal{P}_p(I_h)$ with $p \ge 0$. Then for $0 \le l \le k$,*

$$\|u - u_{hp}\|_{H^{l,\beta}(I_h)} \le C \frac{h^{\mu-l}}{(p+1)^{k-l}} \|u\|_{H^{k,\beta}(I_h)}, \tag{5.6}$$

with $\mu = \min\{k, p+1\}$.

Furthermore, if $p \ge k-1$ and $k > \frac{1}{2} + \bar{\beta}_{1,2} = \frac{1}{2} + \max\{\beta_1 + \frac{1}{2}, \beta_2 + \frac{1}{2}, 0\}$, there holds

$$\|u - u_{hp}\|_{C^0(\bar{I}_h)} \le C \frac{h^{\mu-\frac{1}{2}}}{(p+1)^{k-\frac{1}{2}}} \|u\|_{H^{k,\beta}(I_h)}. \tag{5.7}$$

Proof. Similar to the proof of Theorem 3.3. □

Lemma 5.2. *Let $\gamma_h = (a,b)$ with $h = b-a$ and $u \in H^s(\gamma_h), s > 1/2$. Then there exists an operator $\pi_{\gamma_h} : H^s(\gamma_h) \to \mathcal{P}_p(\gamma_h)$ such that $u(a) = \pi_{\gamma_h} u(a), u(b) = \pi_{\gamma_h} u(b)$ and there holds for $0 \le l \le s$*

$$\|u - \pi_{\gamma_h} u\|_{H^l(\gamma_h)} \le C \frac{h^{\mu-l}}{(p+1)^{s-l}} \|u\|_{H^s(\gamma_h)} \tag{5.8}$$

with $\mu = \min\{s, p+1\}$, a constant C independent of p and u, and $\pi_{\gamma_h} u = u$ if $u \in \mathcal{P}_p(\gamma_h)$.

Proof. We may assume without loss of generality that $\gamma_h = (a,b) = I_{\frac{h}{2}} = (-\frac{h}{2}, \frac{h}{2})$. For $u \in \mathcal{P}_p(\gamma_h)$, let $\pi_{\gamma_h} u = u$. For $u \notin \mathcal{P}_p(\gamma_h)$, we extend u on $I_h = (-h,h)$ preserving the norm with a support in $I_{\frac{3}{4}h} = (-\frac{3}{4}h, \frac{3}{4}h)$, and denote the extended function by $\tilde{u}$. Then $\tilde{u} \in H^{s,-1/2}(I_h)$, and

$$C_1 \|u\|_{H^s(\gamma_h)} \le \|\tilde{u}\|_{H^{s,-\frac{1}{2}}(I_h)} \le C_2 \|u\|_{H^s(\gamma_h)}. \tag{5.9}$$

By Lemma 5.1 the Jacobi projection $\psi_{hp} = \Pi^{\beta}_{p,h} \tilde{u}$ of $\tilde{u}$ on $\mathcal{P}_p(I_h)$ with $\beta = (-\frac{1}{2}, -\frac{1}{2})$ satisfies

$$\|\tilde{u} - \psi_{hp}\|_{H^{l,-\frac{1}{2}}(I_h)} \le C \frac{h^{\mu-l}}{(p+1)^{s-l}} \|\tilde{u}\|_{H^{s,-\frac{1}{2}}(I_h)}$$

with $\mu = \min\{s, p+1\}$.

Let

$$\pi_{\gamma_h} u = \psi_{hp} + \Big(u(-\frac{h}{2}) - \psi_{hp}(-\frac{h}{2})\Big) g_1(x) + \Big(u(\frac{h}{2}) - \psi_{hp}(\frac{h}{2})\Big) g_2(x) \tag{5.10}$$

with $g_1(x) = \left(\frac{h-2x}{2h}\right)^p$ and $g_2(x) = \left(\frac{h+2x}{2h}\right)^p$. Then $u(\pm\frac{h}{2}) = \pi_\gamma u(\pm\frac{h}{2})$. Note that for $0 \le l \le p$

$$\|g_i(x)\|_{H^l(I_{\frac{h}{2}})} \le Ch^{\frac{1}{2}-l}(p+1)^{l-\frac{1}{2}},\ i=1,2, \tag{5.11}$$

and by (5.7) there holds

$$\left|u(\pm\frac{h}{2}) - \psi_{hp}(\pm\frac{h}{2})\right| = \left|\tilde{u}(\pm\frac{h}{2}) - \psi_{hp}(\pm\frac{h}{2})\right| \le Ch^{\mu-\frac{1}{2}}(p+1)^{-(s-\frac{1}{2})}\|\tilde{u}\|_{H^{s,-\frac{1}{2}}(I_h)}.$$

Therefore

$$\begin{aligned}\|u-\pi_{\gamma_h}u\|_{H^l(\gamma_h)} &\le \|\tilde{u}-\psi_{hp}\|_{H^{l,-\frac{1}{2}}(I_h)} + \left|u(-\frac{h}{2}) - \psi_{hp}(-\frac{h}{2})\right|\|g_1\|_{H^l(I_{\frac{h}{2}})} \\ &\quad + \left|u(\frac{h}{2}) - \psi_{hp}(\frac{h}{2})\right|\|g_2\|_{H^l(I_{\frac{h}{2}})} \\ &\le C\Big(h^{\mu-l}(p+1)^{-(s-l)} + h^{\mu-\frac{1}{2}}(p+1)^{-(s-\frac{1}{2})}h^{\frac{1}{2}-l}(p+1)^{l-\frac{1}{2}}\Big)\|\tilde{u}\|_{H^{s,-\frac{1}{2}}(I)} \\ &\le C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|\tilde{u}\|_{H^{s,-\frac{1}{2}}(I_h)} \le C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|u\|_{H^s(\gamma_h)}.\end{aligned}$$

$\square$

Lemma 5.3. *Let $T_h = \{(x_1,x_2)|0 < x_2 < h - x_1, 0 \le x_1 < h\}$ be the standard triangle with size h and $f(x_1) \in \mathcal{P}^0_p(I_h), I_h = (0,h)$. Then there is a polynomial $w_p \in \mathcal{P}^1_p(T_h)$ such that $w_p(x_1,0) = f(x_1)$ and $w_p(0,x_2) = w_p(x_1, h-x_1) = 0$, and there holds for $t = 0,1$*

$$\|w_p\|_{H^t(T_h)} \le C\Big(h^{\frac{3}{2}-t}p^{t-\frac{3}{2}}\|f\|_{H^1(I_h)} + h^{\frac{1}{2}-t}p^{t-\frac{1}{2}}\|f\|_{L^2(I_h)}\Big) \tag{5.12}$$

with C independent of f, p and h.

Proof. Let the mapping M_1: $\xi = \frac{x}{h}$ maps $f \in \mathcal{P}^0_p(I_h)$ onto $\tilde{f} \in \mathcal{P}^0_p(I)$, $I = (0,1)$, then

$$\|\tilde{f}\|_{L^2(I)} = h^{-\frac{1}{2}}\|f\|_{L^2(I_h)},\ \|\tilde{f}\|_{L^2(I)} \le C|\tilde{f}|_{H^1(I)} = Ch^{\frac{1}{2}}|f|_{H^1(I_h)}. \tag{5.13}$$

By Lemma 4.8, there is an extension $\tilde{w}_p \in \mathcal{P}^1_p(T)$ defined on T such that $\tilde{w}_p\,|_I = \tilde{f}$ and $\tilde{w}_p\,|_{\partial T\setminus I} = 0$, and there holds for $t = 0,1$

$$\|\tilde{w}_p\|_{H^t(T)} \le C\big(p^{t-\frac{3}{2}}\|\tilde{f}\|_{H^1(I)} + p^{t-\frac{1}{2}}\|\tilde{f}\|_{L^2(I)}\big), \tag{5.14}$$

where $T = \{(x_1,x_2)|0 < x_2 < 1 - x_1, 0 \le x_1 < 1\}$.

Let the mapping $M_2 : x_1 = h\xi_1, x_2 = h\xi_2$ maps $\tilde{w}_p \in \mathcal{P}^1_p(T)$ onto $w_p \in \mathcal{P}^1_p(T_h)$, $w_p = \tilde{w}_p \circ M_2$, then we have

$$\begin{aligned}\|w_p\|_{L^2(T_h)} &= h\|\tilde{w}_p\|_{L^2(T)} \le Ch\big(p^{-\frac{3}{2}}\|\tilde{f}\|_{H^1(I)} + p^{-\frac{1}{2}}\|\tilde{f}\|_{L^2(I)}\big) \\ &\le C\Big(h^{\frac{3}{2}}p^{-\frac{3}{2}}\|f\|_{H^1(I_h)} + h^{\frac{1}{2}}p^{-\frac{1}{2}}\|f\|_{L^2(I_h)}\Big),\end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \|w_p\|_{H^1(T_h)} &\leq C\|\tilde{w}_p\|_{H^1(T)} \leq C\big(p^{-\frac{1}{2}}\|\tilde{f}\|_{H^1(I)} + p^{\frac{1}{2}}\|\tilde{f}\|_{L^2(I)}\big) \\ &\leq C\Big(h^{\frac{1}{2}}p^{-\frac{1}{2}}\|f\|_{H^1(I_h)} + h^{-\frac{1}{2}}p^{\frac{1}{2}}\|f\|_{L^2(I_h)}\Big). \end{aligned} \tag{5.16}$$

□

5.2. Adjustment of local projection polynomials at the vertices and edges

Theorem 5.4. *Let $u \in H^k(\Omega_j), k > \frac{3}{2}$, where Ω_j is a tetrahedron, or a triangular prism, or a hexahedron with plane surfaces or non-plane surfaces. Then there exists a pull-back polynomial $\phi \in \mathcal{P}_p^{\kappa}(\Omega_j)$ with $p \geq 1$, where $\kappa = 1$, or 1.5, or 2 if Ω_j is a tetrahedron, or a triangular prism, or a hexahedron, respectively, such that for $0 \leq \ell \leq k$*

$$\|u - \phi\|_{H^\ell(\Omega_j)} \leq C\frac{h^{\mu-\ell}}{p^{k-\ell}}\|u\|_{H^k(\Omega_j)}, \tag{5.17}$$

with $\mu = \min\{p+1, k\}$, and $u = \phi$ at vertices V_l of Ω_j, $1 \leq l \leq L, L = 4$ or 6 or 8, respectively.

Proof. We may assume without loss of generality that Ω_j is a standard element, which is a standard tetrahedron, or a standard triangular prism, or a standard hexahedron contained in $Q_{\frac{h}{2}} = (-\frac{h}{2}, \frac{h}{2})^3$ shown as in Fig. 5.3. Then u can be extended to $Q_h = (-h, h)^3$ such that the extended function (denoted by u again for simplicity) has a support $\subseteq Q_{\frac{3h}{4}}$ and preserves the norm,

$$\|u\|_{H^k(\Omega_j)} \leq C\|u\|_{H^k(Q_h)} \leq C\|u\|_{H^k(\Omega_j)}.$$

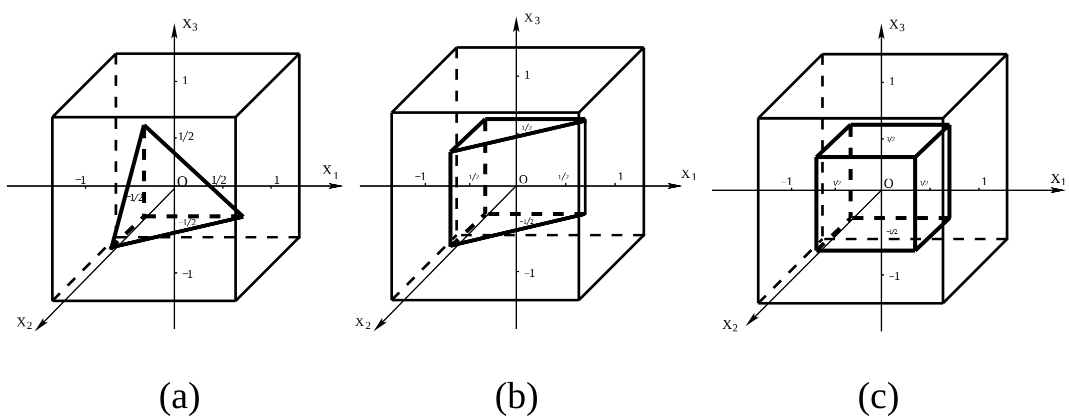


Fig. 5.3 Standard Elements inserted in $Q_h = (-h, h)^3$

(a) Inserted tetrahedron, (b) Inserted triangular prism, (c) Inserted hexahedron

Since u has a compact support in Q_h, $u \in H^{k,\beta}(Q_h)$ with Jacobi weight $\beta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, and

$$\tilde{C}\|u\|_{H^{k,\beta}(Q_h)} \le \|u\|_{H^k(Q_h)} \le C\|u\|_{H^k(D)}.$$

By Theorem 3.3 the Jacobi projection $\tilde{\phi} = \Pi^{\beta}_{p,\kappa} u$ of u on $\mathcal{P}^{\kappa}_p(Q_h)$ satisfies for $0 \le \ell \le k$,

$$\|u - \tilde{\phi}\|_{H^\ell(\Omega_j)} \le C\|u - \tilde{\phi}\|_{H^{\ell,\beta}(Q_h)} \le C\frac{h^{\mu-1}}{p^{k-\ell}}\|u\|_{H^{k,\beta}(Q_h)} \le C\frac{h^{\mu-1}}{p^{k-\ell}}\|u\|_{H^k(\Omega_j)} \tag{5.18}$$

and

$$|u(V_\ell) - \tilde{\phi}(V_\ell)| \le C\frac{h^{\mu-\frac{3}{2}}}{p^{k-\frac{3}{2}}}\|u\|_{H^{k,\beta}(Q)} \le C\frac{h^{\mu-\frac{3}{2}}}{p^{k-\frac{3}{2}}}\|u\|_{H^k(\Omega_j)}, \tag{5.19}$$

where $V_l^{'}s$ are vertices of Ω_j, namely, $V_1 = (-\frac{h}{2}, -\frac{h}{2}, -\frac{h}{2}), V_2 = (\frac{h}{2}, -\frac{h}{2}, -\frac{h}{2}), V_3 = (-\frac{h}{2}, \frac{h}{2}, -\frac{h}{2})$, $V_4 = (-\frac{h}{2}, -\frac{h}{2}, \frac{h}{2})$, $V_5 = (\frac{h}{2}, -\frac{h}{2}, \frac{h}{2}), V_6 = (-\frac{h}{2}, \frac{h}{2}, \frac{h}{2}), V_7 = (\frac{h}{2}, \frac{h}{2}, \frac{h}{2})$ and $V_8 = (\frac{h}{2}, \frac{h}{2}, -\frac{h}{2})$.

Let $g_1(x_1) = (\frac{h-2x_1}{2h})$ and $g_2(x_1) = (\frac{h+2x_1}{2h})$, and let $g_i^p(x_1) = \big(g_i(x_1)\big)^p, i = 1, 2$. Then

$$\|g_i^p\|_{H^\ell(I)} \le Ch^{\frac{1}{2}-\ell}(p+1)^{\ell-\frac{1}{2}}, \quad \text{for } i = 1,2 \ \text{ and } \ 0 \le \ell \le k, \tag{5.20}$$

where $I = (-\frac{h}{2}, \frac{h}{2})$.

If Ω_j is a hexahedron and $\kappa = 2$, let $\phi = \tilde{\phi} + \sum_{1\le l\le 8}(u - \tilde{\phi})(V_l)\chi_l(x)$ with

$$\begin{cases} \chi_1(x) = g_1^p(x_1)g_1^p(x_2)g_1^p(x_3), & \chi_2(x) = g_2^p(x_1)g_1^p(x_2)g_1^p(x_3), \\ \chi_3(x) = g_1^p(x_1)g_2^p(x_2)g_1^p(x_3), & \chi_4(x) = g_1^p(x_1)g_1^p(x_2)g_2^p(x_3), \\ \chi_5(x) = g_2^p(x_1)g_1^p(x_2)g_2^p(x_3), & \chi_6(x) = g_1^p(x_1)g_2^p(x_2)g_2^p(x_3), \\ \chi_7(x) = g_2^p(x_1)g_2^p(x_2)g_2^p(x_3), & \chi_8(x) = g_2^p(x_1)g_2^p(x_2)g_1^p(x_3). \end{cases} \tag{5.21}$$

Obviously, $\phi \in \mathcal{P}^2_p(Q_h)$ and $u(V_l) = \phi(V_l), 1 \le l \le 8$. By (5.18)-(5.20), we have for $0 \le \ell \le k$

$$\begin{aligned} \|u - \phi\|_{H^\ell(\Omega_j)} &\le \|u - \tilde{\phi}\|_{H^\ell(\Omega_j)} + h^{\frac{3}{2}-\ell}(p+1)^{\ell-\frac{3}{2}}\sum_{1\le l\le L}|(u-\phi)(V_l)| \\ &\le C\Big((p+1)^{-(k-\ell)} + h^{\frac{3}{2}-\ell}(p+1)^{\ell-\frac{3}{2}}\frac{h^{\mu-\frac{3}{2}}}{(p+1)^{k-\frac{3}{2}}}\Big)\|u\|_{H^k(\Omega_j)} \\ &\le C\frac{h^{\mu-\ell}}{(p+1)^{k-\ell}}\|u\|_{H^k(\Omega_j)}. \end{aligned} \tag{5.22}$$

If Ω_j is a tetrahedron and $\kappa = 1$, $\chi_\ell(x), 1 \le \ell \le 4$ are similar to the first four functions in (5.21) except replacing $g_1^p(x_i)$ and $g_2^p(x_i)$ by $g_1^{[\frac{p}{3}]}(x_i)$ and $g_2^{[\frac{p}{3}]}(x_i), i = 1, 2, 3$ in (5.21) for $p \ge 3$. Hereafter $[\alpha]$ denotes the smallest integer $\ge \alpha$ for a real number α. For $p < 3$, $\chi_l(x), 1 \le l \le 4$ are selected such that $\chi_\ell \in \mathcal{P}^1_1(\Omega_j)$ and $\chi_l(V_m) = \delta_{l,m}, 1 \le l, m \le 4$, i.e.,

$$\chi_1 = \frac{(-x_1 - x_2 - x_3 - \frac{h}{2})}{h}, \ \chi_2(x) = g_2(x_1), \ \chi_3(x) = g_2(x_2), \ \chi_4(x) = g_2(x_3). \tag{5.23}$$

Therefore $\phi = \tilde{\phi} + \sum_{1\le l\le 4}(u-\tilde{\phi})(V_l)\chi_l(x) \in \mathcal{P}_p^1(\Omega_j)$ and $\phi(V_l) = u(V_l), 1 \le l \le 4$, and (5.22) holds.

If Ω_j is a triangular prism and $\kappa = 1.5$, $\chi_l(x), 1 \le l \le 6$ are defined similarly as in (5.21) except replacing $g_1^p(x_i)$ and $g_2^p(x_i), i = 1,2$ by $g_1^{[\frac{p}{2}]}$ and $g_2^{[\frac{p}{2}]}(x_2)$ for $p \ge 2$. For $p = 1$, $\chi_l(x), 1 \le l \le 6$ are selected as

$$\left\{\begin{array}{ll} \chi_1(x) = -\frac{(x_1+x_2)}{h}g_1(x_3), & \chi_2(x) = g_2(x_1)g_1(x_3), \\ \chi_3(x) = g_2(x_2)g_1(x_3), & \chi_4(x) = -\frac{(x_1+x_2)}{h}g_2(x_3), \\ \chi_5(x) = g_2(x_1)g_2(x_3), & \chi_6(x) = g_2(x_2)g_2(x_3). \end{array}\right. \tag{5.24}$$

Therefore $\phi = \tilde{\phi} + \sum_{1\le l\le 6}(u-\tilde{\phi})(V_l)\chi_l(x) \in \mathcal{P}_p^{1.5}(\Omega_j)$ and $\phi(V_l) = u(V_l), 1 \le l \le 6$, and (5.22) follows easily. $\square$

Let K be a standard tetrahedron with faces F_i on the planes $x_i = 0, 1 \le i \le 3$ and F_4 on the planes $x_1 + x_2 + x_3 = 1$, shown in Fig. 5.2(a). By $\gamma_{ij} = \bar{F}_i \cap \bar{F}_j$ we denote edges of K. Let $G = T \times I$ be a triangular prism with faces $F_i, 1 \le i \le 5$ shown in Fig. 5.2(b), where $T = \{(x_1,x_2) \mid x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 1\}$ and $I = [0,1]$. $F_i, 1 \le i \le 3$ are on the planes $x_i = 0$, F_5 be the face of G contained in the plane $x_3 = 1$ and F_4 be the face of G contained in the plane $x_1 + x_2 = 1$. Then $F_3 = T$ and $F_2 = S = I \times I$. Let $\gamma_{ij} = \bar{F}_i \cap \bar{F}_j, 1 \le i < j \le 5$ be the edges of G. Let $D = (0,1)^3$ be a standard hexahedron.

Theorem 5.5. *Let $u \in H^k(D_h), k \ge 2$, where D_h is a standard hexahedron with size h. Then there exists a polynomial $\psi \in \mathcal{P}_p^2(D_h)$ satisfying*

$$\|u-\psi\|_{H^l(D_h)} \le C\frac{h^{\mu-l}}{p^{k-l}}\|u\|_{H^k(D_h)}, \quad 0 \le l \le k, \tag{5.25}$$

$$\|u-\psi\|_{H^t(F_i)} \le C\frac{h^{\mu-t-\frac{1}{2}}}{p^{k-t-\frac{1}{2}}}\|u\|_{H^k(D_h)}, \quad 1 \le i \le 6,\ 0 \le t \le k-\frac{1}{2}, \tag{5.26}$$

$$\psi(V_l) = u(V_l), \qquad 1 \le l \le 8, \tag{5.27}$$

$$\psi|_{\gamma_{ij}} = \pi_\gamma(u|_{\gamma_{ij}}), \quad 1 \le i < j \le 6, \tag{5.28}$$

where $\mu = \min\{p+1,k\}$, V_l are the vertices of $D_h = (0,h)^3$, $\gamma_{ij}, 1 \le i < j \le 6$ are edges of D_h, $F_i, 1 \le i \le 6$ are faces of D_h, and π_γ is the operator defined in Lemma 5.2.

Proof. In order to construct a polynomial $\psi \in \mathcal{P}_p^2(D_h)$ satisfying (5.25)-(5.28), let $\psi = \phi + v_p$ where ϕ defined as Theorem 5.4 such that $\phi(V_l) = u(V_l), 1 \le l \le 8$ and satisfies (5.17). It suffices to select the correction $v_p \in \mathcal{P}_p^2(D_h)$ satisfying

$$v_p(V_l) = 0, \quad 1 \le l \le 8, \tag{5.29}$$

$$v_p|_{\gamma_{ij}} = \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}}, \quad 1 \le i < j \le 6, \tag{5.30}$$

$$\|v_p\|_{H^\ell(D_h)} \le Ch^{\mu-\ell}p^{\ell-k}\|u\|_{H^k(D_h)}, \ 0 \le \ell \le k. \tag{5.31}$$

Let $g_{ij} = \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}} \in \mathcal{P}_p^0(\gamma_{ij})$. By Lemma 5.2 and trace theorem, there hold for $\ell \le k-1, 1 \le i < j \le 6$

$$\begin{aligned}
\|u - \pi_\gamma(u|_{\gamma_{ij}})\|_{H^\ell(\gamma_{ij})} &\le C\frac{h^{\min(p+1,k-1)-\ell}}{p^{k-1-\ell}}\|u\|_{H^{k-1}(\gamma_{ij})} \\
&\le C\frac{h^{\min(p+1,k-1)-\ell}}{p^{k-1-\ell}}\|u\|_{H^k(D_h)} \le C\frac{h^{\mu-\ell-1}}{p^{k-1-\ell}}\|u\|_{H^k(D_h)},
\end{aligned} \tag{5.32}$$

and

$$\|u-\phi\|_{H^\ell(\gamma_{ij})} \le C\|u-\phi\|_{H^{\ell+1}(D_h)} \le C\frac{h^{\mu-\ell-1}}{p^{k-\ell-1}}\|u\|_{H^k(D_h)}. \tag{5.33}$$

Hence, we have

$$\begin{aligned}
\|g_{ij}\|_{H^\ell(\gamma_{ij})} &= \|\pi_\gamma(u|_{\gamma_{ij}}) - u + u - \phi\|_{H^\ell(\gamma_{ij})} \\
&\le \|u - \pi_\gamma(u|_{\gamma_{ij}})\|_{H^\ell(\gamma_{ij})} + \|u-\phi\|_{H^\ell(\gamma_{ij})} \le Ch^{\mu-1-\ell}p^{\ell+1-k}\|u\|_{H^k(D_h)}.
\end{aligned} \tag{5.34}$$

Let $v_{12} = g_{12}(x_1)\varphi(x_2)\psi(x_3)$ with $\varphi(x_2) = \Big(1-\dfrac{x_2}{h}\Big)^p, \psi(x_3) = \Big(1-\dfrac{x_3}{h}\Big)^p$. Then $v_{12}\,|_{\gamma_{12}}= g_{12}$ and $v_{12}\,|_{\gamma_{ij}}=0$ for $(ij)\ne(12)$. Obviously

$$\|\varphi\|_{H^t(I_h)} \le Ch^{\frac12-t}p^{t-\frac12}, \quad \|\psi\|_{H^t(I_h)} \le Ch^{\frac12-t}p^{t-\frac12}, \quad \ell \ge 0$$

and

$$|\varphi\psi|_{H^t(S_h)} \le Ch^{1-t}p^{t-1},\ S_h = I_h \times I_h$$

which together with (5.34) imply for $\ell \ge 0$

$$\begin{aligned}
|v_{12}|^2_{H^\ell(D_h)} &= \le C\Big(|g_{12}|^2_{H^\ell(I_h)}\|\varphi\psi\|^2_{L^2(S_h)} + \|g_{12}\|^2_{L^2(I_h)}|\varphi\psi|^2_{H^\ell(S_h)}\Big) \\
&\le C\Big(h^{2(\mu-1-\ell)}p^{2(\ell+1-k)}h^2p^{-2}\|u\|^2_{H^k(D_h)} + h^{2(\mu-1)}p^{2(1-k)}h^{2(1-\ell)}p^{2(\ell-1)}\|u\|^2_{H^k(D_h)}\Big) \\
&\le Ch^{2(\mu-\ell)}p^{2(\ell-k)}\|u\|^2_{H^k(D_h)}.
\end{aligned}$$

Similarly, we can construct $v_{ij} \in \mathcal{P}_p^2(D_h)$ for $1 \le i < j \le 6$ such that

$$v_{ij}(V_l) = 0, \quad 1 \le l \le 8, \tag{5.35}$$

$$v_{ij}|_{\gamma_{ij}} = g_{ij}, \quad v_{ij}|_{\gamma_{mn}} = 0 \text{ for } (m,n)\ne(i,j),\ 1\le m<n\le 6, \tag{5.36}$$

$$\|v_{ij}\|_{H^\ell(D_h)} \le Ch^{\mu-\ell}p^{\ell-k}\|u\|_{H^k(D_h)}. \tag{5.37}$$

Let $v_p = \sum_{1\le i<j\le 6} v_{ij}$. Then v_p satisfies (5.29)-(5.31), and ψ satisfies (5.25)-(5.28). $\square$

Theorem 5.6. *Let $u \in H^k(G_h), k \geq 2$, where G_h is a standard triangular prism with size h. Then there exists a polynomial $\psi \in \mathcal{P}_p^{1.5}(G_h)$ satisfying*

$$\|u-\psi\|_{H^1(G_h)} \leq C\frac{h^{\mu-1}}{p^{k-1}}\|u\|_{H^k(G_h)}, \tag{5.38}$$

$$\|u-\psi\|_{H^t(F_i)} \leq C\frac{h^{\mu-t-\frac{1}{2}}}{p^{k-t-\frac{1}{2}}}\|u\|_{H^k(G_h)}, \quad 1 \leq i \leq 5,\, t=0,1, \tag{5.39}$$

$$\psi(V_l) = u(V_l), \qquad 1 \leq l \leq 6, \tag{5.40}$$

$$\psi|_{\gamma_{ij}} = \pi_\gamma(u|_{\gamma_{ij}}), \quad 1 \leq i < j \leq 5. \tag{5.41}$$

If $u \in H^k(G_h), k \geq 3$, there holds for $1 \leq i \leq 5$

$$\Big\|\frac{\partial(u-\psi)}{\partial x_3}\Big\|_{H^1(F_i)} \leq C\frac{h^{\mu-\frac{5}{2}}}{p^{k-\frac{5}{2}}}\|u\|_{H^k(G_h)}, \tag{5.42}$$

where $\mu = \min\{p+1, k\}$, $V_l, 1 \leq l \leq 6$ are the vertices of G_h, $\gamma_{ij}, 1 \leq i < j \leq 5$ are edges of G_h, $F_i, 1 \leq i \leq 5$ are faces of G_h, and π_γ is the operator defined in Lemma 5.2.

Proof. By Theorem 5.4 there is $\phi \in \mathcal{P}_p^{1.5}(G_h)$ such that $\phi(V_l) = u(V_l), 1 \leq l \leq 6$ and satisfies (5.17). Analogously, we construct a polynomial extension $\psi \in \mathcal{P}_p^{1.5}(G_h)$ by

$$\psi = \phi + \sum_{1 \leq i < j \leq 5} v_{ij}.$$

It suffices to select $v_{ij} \in \mathcal{P}_p^{1.5}(G_h)$ satisfying

$$v_{ij}(V_l) = 0, \quad 1 \leq l \leq 6, \tag{5.43}$$

$$\begin{aligned} v_{ij}|_{\gamma_{ij}} &= g_{ij} = \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}},\ 1 \leq i < j \leq 5, \\ v_{ij}|_{\gamma_{mn}} &= 0 \text{ for } (mn) \neq (ij), 1 \leq m < n \leq 5, \end{aligned} \tag{5.44}$$

$$\|v_{ij}\|_{H^1(G_h)} \leq Ch^{\mu-1}p^{1-k}\|u\|_{H^k(G_h)}, \tag{5.45}$$

$v_{ij}\,|_{F_m} = 0$ for $m \neq i, j$, and for $t = 0, 1, m = i, j$

$$\|v_{ij}\|_{H^t(F_m)} \leq Ch^{\mu-t-\frac{1}{2}}p^{t+\frac{1}{2}-k}\|u\|_{H^k(G_h)}. \tag{5.46}$$

Note that $g_{ij} = \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}} \in \mathcal{P}_p^0(\gamma_{ij})$, and due to Lemma 5.2 and trace theorem, there holds for $0 \leq \ell \leq k-1$ and $1 \leq i < j \leq 5$,

$$\begin{aligned} \|g_{ij}\|_{H^\ell(\gamma_{ij})} &= \|\pi_\gamma(u|_{\gamma_{ij}}) - u + u - \phi\|_{H^\ell(\gamma_{ij})} \\ &\leq \|u - \pi_\gamma(u|_{\gamma_{ij}})\|_{H^\ell(\gamma_{ij})} + \|u-\phi\|_{H^\ell(\gamma_{ij})} \leq Ch^{\mu-1-\ell}p^{\ell+1-k}\|u\|_{H^k(G_h)}. \end{aligned} \tag{5.47}$$

There are two types of edges: an interface of two square faces such as $\gamma_{12}, \gamma_{14}, \gamma_{24}$ and an interfaces of a triangular face and a square face such as $\gamma_{23}, \gamma_{13}, \gamma_{34}, \gamma_{25}, \gamma_{15}, \gamma_{45}$.

For $(ij) = (12)$, let

$$v_{12} = g_{12}(x_3)\varphi(x_1)\psi(x_2) \in \mathcal{P}_p^{1.5}(G_h)$$

where $\varphi(x_1) = (1 - \frac{x_1}{h})^{[\frac{p}{2}]}, \psi(x_2) = (1 - \frac{x_2}{h})^{[\frac{p}{2}]}$. Then $v_{12}(V_l) = 0$ at the vertices of G_h, and $v_{12} = g_{12}$ on γ_{12} and vanishes on other edges of G_h. Obviously,

$$\|\varphi\|_{H^\ell(I_h)} \le Ch^{\frac{1}{2}-\ell}p^{\ell-\frac{1}{2}},\ \|\psi\|_{H^\ell(I_h)} \le Ch^{\frac{1}{2}-\ell}p^{\ell-\frac{1}{2}},\ \|\varphi\psi\|_{H^\ell(T_h)} \le Ch^{1-\ell}p^{\ell-1},\ 0 \le \ell \le k-1.$$

The arguments for edges of a cube in previous lemma can be carried out here except replacing p by $\frac{p}{2}$, which implies for $0 \le \ell \le k-1$

$$\|v_{12}\|_{H^\ell(G_h)} \ \le\ Ch^{\mu-\ell}p^{\ell-k}\|u\|_{H^k(G_h)}.$$

Furthermore, $v_{12}|_{F_i} = 0$ for $i = 3, 4, 5$, and for $i = 1, 2$ and $0 \le \ell \le k-1$

$$\|v_{12}\|_{H^\ell(F_i)} \le C\|v_{12}\|_{H^{\ell+\frac{1}{2}}(G_h)} \le Ch^{\mu-\ell-\frac{1}{2}}p^{-(k-\ell-\frac{1}{2})}\|u\|_{H^k(G_h)}.$$

For $(ij) = (23)$, it suffices to construct v_{23} satisfying (5.43)-(5.46). Let the mapping M: $x = h\xi$, which maps $g_{23}(x) \in \mathcal{P}_p^0(\gamma_{23})$ to $\tilde{g}_{23}(\xi) \in \mathcal{P}_p^0(\tilde{\gamma}_{23})$, where $\tilde{\gamma}_{23}$ is an edge of the standard triangular prism G, then

$$\|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})} = h^{-\frac{1}{2}}\|g_{23}\|_{L^2(\gamma_{23})},\ \|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})} \le C|\tilde{g}_{23}|_{H^1(\tilde{\gamma}_{23})} = Ch^{\frac{1}{2}}|g_{23}|_{H^1(\gamma_{23})}. \tag{5.48}$$

By the arguments of Lemma 8.4 of [**28**], there exists $\tilde{v}_{23} \in \mathcal{P}_p^{1.5}(G)$ such that $\tilde{v}_{23}\ |_{\tilde{\gamma}_{23}} = \tilde{g}_{23}$ and $\tilde{v}_{23}\ |_{\tilde{F}_i} = 0$ for $i = 1, 4, 5$, and

$$\|\tilde{v}_{23}\|^2_{H^1(G)} \le C(p^{-2}\|\tilde{g}_{23}\|^2_{H^1(\tilde{\gamma}_{23})} + \|\tilde{g}_{23}\|^2_{L^2(\tilde{\gamma}_{23})}),$$

$$\|\tilde{v}_{23}\|_{H^t(\tilde{F}_3)} \le C\big(p^{t-\frac{3}{2}}\|\tilde{g}_{23}\|_{H^1(\tilde{\gamma}_{23})} + p^{t-\frac{1}{2}}\|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})}\big),\ \text{for } t = 0, 1$$

$$\|\tilde{v}_{23}\|_{H^t(\tilde{F}_2)} \ \le\ C\Big(p^{t-\frac{1}{2}}\|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})} + p^{-1/2}\|\tilde{g}_{23}\|_{H^t(\tilde{\gamma}_{23})}\Big),\ \text{for } t = 0, 1.$$

Let $v_{23} = \tilde{v}_{23} \circ M^{-1}$, then $v_{23}\ |_{\gamma_{23}} = g_{23}$ and $v_{23}\ |_{F_i} = 0$ for $i = 1, 4, 5$. Note that

$$\|v_{23}\|^2_{H^1(G_h)} \le Ch\|\tilde{v}_{23}\|^2_{H^1(G)}, \quad \|v_{23}\|^2_{H^t(\tilde{F}_i)} \le Ch^{1-t}\|\tilde{v}_{23}\|^2_{H^t(\tilde{F}_i)}, i = 2, 3$$

and

$$\|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})} = h^{-\frac{1}{2}}\|g_{23}\|_{L^2(\gamma_{23})},\ \|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})} \le C|\tilde{g}_{23}|_{H^1(\tilde{\gamma}_{23})} = Ch^{\frac{1}{2}}|g_{23}|_{H^1(\gamma_{23})}$$

which together with (5.47) lead to

$$\begin{aligned} \|v_{23}\|^2_{H^1(G_h)} &\le Ch(p^{-2}\|\tilde{g}_{23}\|^2_{H^1(\tilde{\gamma}_{23})} + \|\tilde{g}_{23}\|^2_{L^2(\tilde{\gamma}_{23})}) \\ &\le\ C(p^{-2}h^2\|g_{23}\|^2_{H^1(\gamma_{23})} + \|g_{23}\|^2_{L^2(\gamma_{23})}) \le Ch^{2(\mu-1)}p^{2(1-k)}\|u\|^2_{H^k(G_h)}. \end{aligned} \tag{5.49}$$

and for $t = 0, 1$

$$\|v_{23}\|_{H^t(F_3)} \le C\big(h^{\frac{3}{2}-t}p^{t-\frac{3}{2}}\|g_{23}\|_{H^1(\gamma_{23})} + h^{\frac{1}{2}-t}p^{t-\frac{1}{2}}\|g_{23}\|_{L^2(\gamma_{23})}\big) \le C\frac{h^{\mu-t-\frac{1}{2}}}{p^{k-t-\frac{1}{2}}}\|u\|_{H^k(G_h)},$$

$$\|v_{23}\|_{H^t(F_2)} \leq C\Big(h^{\frac{1}{2}-t}p^{t-\frac{1}{2}}\|g_{23}\|^2_{L^2(\gamma_{23})} + h^{1/2}p^{-1/2}\|g_{23}\|_{H^t(\gamma_{23})}\Big) \leq C\frac{h^{\mu-t-\frac{1}{2}}}{p^{k-t-\frac{1}{2}}}\|u\|_{H^k(G_h)}.$$

Similarly, we can construct v_{ij} on all edges of G_h satisfying (5.43)-(5.46). Thus ψ satisfies (5.38)-(5.41).

If $u \in H^k(G_h), k \geq 3$, (5.47) with $\ell = 2$ gives

$$\|g'_{12}(x_3)\|_{H^1(\gamma_{12})} \leq Ch^{\mu-3}p^{3-k}\|u\|_{H^k(G_h)}. \tag{5.50}$$

By the arguments of Lemma 8.4 of [**28**], we have

$$\Big\|\frac{\partial \tilde{v}_{12}}{\partial \xi_3}\Big\|^2_{H^1(\tilde{F}_1)} \leq p^{-1}\|\tilde{g}'_{12}(\xi_3)\|^2_{H^1(\tilde{\gamma}_{12})}\| + p\|\tilde{g}_{12}(\xi_3)\|^2_{H^1(\tilde{\gamma}_{12})}\|, \tag{5.51}$$

$$\Big\|\frac{\partial \tilde{v}_{23}}{\partial \xi_3}\Big\|_{H^1(\tilde{F}_3)} \leq Cp\Big(p^{-\frac{1}{2}}\|\tilde{g}_{23}\|_{H^1(\tilde{\gamma}_{23})} + p^{\frac{1}{2}}\|\tilde{g}_{23}\|_{L^2(\tilde{\gamma}_{23})}\Big), \tag{5.52}$$

and

$$\Big\|\frac{\partial \tilde{v}_{23}}{\partial \xi_3}\Big\|^2_{H^1(\tilde{F}_2)} \leq p\|\tilde{g}_{23}\|^2_{H^1(\tilde{\gamma}_{23})} + p^3\|\tilde{g}_{23}\|^2_{L^2(\tilde{\gamma}_{23})}. \tag{5.53}$$

Combining (5.47) and (5.48)-(5.51) we obtain

$$\begin{aligned}\Big\|\frac{\partial v_{12}}{\partial x_3}\Big\|^2_{H^1(F_1)} &\leq Ch^{-2}\|\frac{\partial \tilde{v}_{12}}{\partial \xi_3}\Big\|^2_{H^1(\tilde{F}_1)} \leq Ch^{-2}(p^{-1}\|\tilde{g}_{12}(\xi_3)\|^2_{H^2(\tilde{\gamma}_{12})}\| + p\|\tilde{g}_{12}(\xi_3)\|^2_{H^1(\tilde{\gamma}_{12})}\|) \\ &\leq Ch^{-2}(p^{-1}h^3\|g'_{12}(x_3)\|^2_{H^1(\gamma_{12})}\| + ph\|\tilde{g}_{12}(x_3)\|^2_{H^1(\gamma_{12})}\|) \leq Ch^{2(\mu-5/2)}p^{2(5/2-k)}\|u\|_{H^k(G_h)},\end{aligned} \tag{5.54}$$

$$\begin{aligned}\Big\|\frac{\partial v_{23}}{\partial x_3}\Big\|_{H^1(F_3)} &\leq Ch^{-1}\Big\|\frac{\partial \tilde{v}_{23}}{\partial \xi_3}\Big\|_{H^1(\tilde{F}_3)} \\ &\leq Ch^{-1}p\Big(h^{\frac{1}{2}}p^{-\frac{1}{2}}\|g_{23}\|_{H^1(\gamma_{23})} + h^{-\frac{1}{2}}p^{\frac{1}{2}}\|g_{23}\|_{L^2(\gamma_{23})}\Big) \\ &\leq Ch^{\mu-\frac{5}{2}}p^{\frac{5}{2}-k}\|u\|_{H^k(G_h)},\end{aligned} \tag{5.55}$$

and

$$\begin{aligned}\Big\|\frac{\partial v_{23}}{\partial x_3}\Big\|^2_{H^1(F_2)} &\leq Ch^{-2}\Big\|\frac{\partial \tilde{v}_{23}}{\partial \xi_3}\Big\|_{H^1(\tilde{F}_2)} \\ &\leq Ch^{-1}(p\|\tilde{g}_{23}\|^2_{H^1(\tilde{\gamma}_{23})} + p^3\|\tilde{g}_{23}\|^2_{L^2(\tilde{\gamma}_{23})}) \\ &\leq Ch^{-2}((ph\|g_{23}\|^2_{H^1(\gamma_{23})} + p^3h^{-1}\|g_{23}\|^2_{L^2(\gamma_{23})}) \\ &\leq Ch^{2\mu-5}p^{5-2k}\|u\|^2_{H^k(G_h)}.\end{aligned} \tag{5.56}$$

Similarly there holds

$$\Big\|\frac{\partial v_{12}}{\partial x_3}\Big\|_{H^1(F_2)} \leq Ch^{\mu-\frac{5}{2}}p^{\frac{5}{2}-k}\|u\|_{H^k(G_h)}. \tag{5.57}$$

Combining (5.54)-(5.56) we have (5.42) for $u \in H^k(G_h), k \geq 3$. ☐

Theorem 5.7. *Let $u \in H^k(K_h), k \geq 2$, where K_h is a standard tetrahedron with size h. Then there exists a polynomial $\psi \in \mathcal{P}_p^1(K_h)$ satisfying*

$$\|u-\psi\|_{H^1(K_h)} \leq C\frac{h^{\mu-1}}{p^{k-1}}\|u\|_{H^k(K_h)}, \tag{5.58}$$

$$\|u-\psi\|_{H^t(F_i)} \leq C\frac{h^{\mu-t-\frac{1}{2}}}{p^{k-t-\frac{1}{2}}}\|u\|_{H^k(K_h)}, \quad 1 \leq i \leq 4,\, t = 0, 1 \tag{5.59}$$

$$\psi(V_l) = u(V_l), \qquad 1 \leq l \leq 4, \tag{5.60}$$

$$\psi|_{\gamma_{ij}} = \pi_\gamma(u|_{\gamma_{ij}}),\ 1 \leq i < j \leq 4, \tag{5.61}$$

where $\mu = \min\{p+1, k\}$, $V_l, 1 \leq l \leq 4$ are the vertices of K_h, $\gamma_{ij}, 1 \leq i < j \leq 4$ are edges of K_h, $F_i, 1 \leq i \leq 4$ are faces of K_h, and π_γ is the operator defined in Lemma 5.2.

Proof. By Theorem 5.4 there is $\phi \in \mathcal{P}_p^1(K_h)$ such that $\phi(V_l) = u(V_l), 1 \leq l \leq 4$ and satisfies (5.17). Analogously, we construct a polynomial $\psi \in \mathcal{P}_p^1(K_h)$ by

$$\psi = \phi + \sum_{1\leq i<j\leq 4} v_{ij}.$$

It suffices to select $v_{ij} \in \mathcal{P}_p^1(K_h)$ satisfying

$$v_{ij}(V_l) = 0, \quad 1 \leq l \leq 4, \tag{5.62}$$

$$\begin{aligned} v_{ij}|_{\gamma_{ij}} &= \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}},\ 1 \leq i < j \leq 4, \\ v_{ij}|_{\gamma_{mn}} &= 0 \text{ for } (mn) \neq (ij), 1 \leq m < n \leq 4, \end{aligned} \tag{5.63}$$

$$\|v_{ij}\|_{H^1(K_h)} \leq Ch^{\mu-1}p^{1-k}\|u\|_{H^k(K_h)}, \tag{5.64}$$

$v_{ij} \mid_{F_m} = 0$ for $m \neq i, j$, and for $t = 0, 1$

$$\|v_{ij}\|_{H^t(F_i)} \leq Ch^{\mu-t-\frac{1}{2}}p^{t+\frac{1}{2}-k}\|u\|_{H^k(K_h)},\ \|v_{ij}\|_{H^t(F_j)} \leq Ch^{\mu-t-\frac{1}{2}}p^{t+\frac{1}{2}-k}\|u\|_{H^k(K_h)}. \tag{5.65}$$

Note that $g_{ij} = \pi_\gamma(u|_{\gamma_{ij}}) - \phi|_{\gamma_{ij}} \in \mathcal{P}_p^0(\gamma_{ij})$ and (5.34) holds for $0 \leq \ell \leq k-1$ and $1 \leq i < j \leq 4$. It suffices to construct v_{12} satisfying (5.62)-(5.65). Let M be the mapping : $x = h\xi$ which maps $g_{ij}(x) \in \mathcal{P}_p^0(\gamma_{ij})$ to $\tilde{g}_{ij}(\xi) \in \mathcal{P}_p^0(\tilde{\gamma}_{ij})$, where $\tilde{\gamma}_{ij}$ is an edge of standard tetrahedron K, and it holds that

$$\|\tilde{g}_{ij}\|_{L^2(\tilde{\gamma}_{ij})} = h^{-\frac{1}{2}}\|g_{ij}\|_{L^2(\gamma_{ij})},\ \|\tilde{g}_{ij}\|_{L^2(\tilde{\gamma}_{ij})} \leq C|\tilde{g}_{ij}|_{H^1(\tilde{\gamma}_{ij})} = Ch^{\frac{1}{2}}|g_{ij}|_{H^1(\gamma_{ij})}. \tag{5.66}$$

By Lemma 4.8, there are extensions $\tilde{w}_m \in \mathcal{P}_p^1(\tilde{F}_m)$ on $\tilde{F}_m, m = 1, 2$ such that $\tilde{w}_m \mid_{\tilde{\gamma}_{12}} = \tilde{g}_{12}$ and $\tilde{w}_m \mid_{\partial\tilde{F}_m\setminus\tilde{\gamma}_{12}} = 0$, and there holds for $t = 0, 1$

$$\|\tilde{w}_m\|_{H^t(\tilde{F}_m)} \leq C\big(p^{t-\frac{3}{2}}\|\tilde{g}_{12}\|_{H^1(\tilde{\gamma}_{12})} + p^{t-\frac{1}{2}}\|\tilde{g}_{12}\|_{L^2(\tilde{\gamma}_{12})}\big). \tag{5.67}$$

We further extend $\tilde{w}_m, m=1,2$ on whole boundary of K by zero extension on $\tilde{F}_3$ and $\tilde{F}_4$ and denote the extended polynomial by $\tilde{w}$. Then for $\tilde{w}\in\mathcal{P}_p^1(\partial K)$ there is a polynomial extension $\widetilde{W}$ in K [**40**] such that $\widetilde{W}|_{\partial K}=\tilde{w}$, i.e.,$\widetilde{W}|_{\tilde{F}_m}=\tilde{w}_m, m=1,2$ and $\widetilde{W}|_{\tilde{F}_m}=0, m=3,4$, and

$$(5.68)\quad \|\widetilde{W}\|^2_{H^1(K)}\le C\|\tilde{w}\|^2_{H^{\frac{1}{2}}(\partial K)}=\Big(\|\tilde{w}_1\|^2_{H_{00}^{\frac{1}{2}}(\tilde{F}_1,\tilde{\gamma}_{13}\cup\tilde{\gamma}_{14})}+\|\tilde{w}_2\|^2_{H_{00}^{\frac{1}{2}}(\tilde{F}_2,\tilde{\gamma}_{23}\cup\tilde{\gamma}_{24})}\Big)^{\frac{1}{2}}.$$

Due to (5.67) there holds for $m=1,2$

$$(5.69)\,\|\tilde{w}_m\|_{H_{00}^{\frac{1}{2}}(\tilde{F}_m,\tilde{\gamma}_{m3}\cup\tilde{\gamma}_{m4})}\le C\|\tilde{w}_m\|^{\frac{1}{2}}_{H^1(\tilde{F}_m)}\|\tilde{w}_m\|^{\frac{1}{2}}_{L^2(\tilde{F}_m)}\le C(p^{-1}\|\tilde{g}_{12}\|_{H^1(\tilde{\gamma}_{12})}+\|\tilde{g}_{12}\|_{L^2(\tilde{\gamma}_{12})}).$$

Let $W=\widetilde{W}\circ M^{-1}$. Then $W|_{\gamma_{12}}=g_{12}, W|_{F_m}=0, m=3,4$, and

$$(5.70)\qquad \begin{aligned}\|W\|_{H^1(K_h)} &\le Ch^{1/2}\|\tilde{W}\|_{H^1(K)}\le Ch^{1/2}(p^{-1}\|\tilde{g}_{12}\|_{H^1(\tilde{\gamma}_{12})}+\|\tilde{g}_{12}\|_{L^2(\tilde{\gamma}_{12})})\\ &\le Ch^{1/2}(p^{-1}h^{1/2}\|g_{12}\|_{H^1(\gamma_{12})}+h^{-1/2}\|g_{12}\|_{L^2(\gamma_{12})})\\ &\le Ch^{\mu-1}p^{1-k}\|u\|_{H^k(K_h)},\end{aligned}$$

and for $m=1,2,t=0,1$

$$(5.71)\ \begin{aligned}\|W\|_{H^t(F_m)} &\le Ch^{1-t}\|\tilde{W}\|_{H^t(\tilde{F}_m)}\le Ch^{1-t}\big(p^{t-\frac{3}{2}}\|\tilde{g}_{12}\|_{H^1(\tilde{\gamma}_{12})}+p^{t-\frac{1}{2}}\|\tilde{g}_{12}\|_{L^2(\tilde{\gamma}_{12})}\big)\\ &\le Ch^{1-t}\big(p^{t-\frac{3}{2}}h^{1/2}\|g_{12}\|_{H^1(\gamma_{12})}+p^{t-\frac{1}{2}}h^{-1/2}\|g_{12}\|_{L^2(\gamma_{12})}\big)\\ &\le Ch^{\nu-t-1/2}p^{t+1/2-k}\|u\|_{H^k(K_h)}.\end{aligned}$$

Let $v_{12}=W$. Then v_{12} satisfies (5.62)-(5.65) with $(ij)=(12)$. The polynomial v_{ij} can be similarly constructed for $1\le i<j\le 4$ satisfying (5.62)-(5.65). Consequently, ψ satisfies (5.58)-(5.61). $\square$

5.3. The convergence for elliptic problems with smooth solution in $H^k(\Omega)$

We shall analyze the approximation error in finite element solutions for the boundary value problem (5.1) with a smooth solution $u\in H^k(\Omega)$.

Let $\Delta_h=\big\{\Omega_j,\ 1\le j\le J\big\}$ be a quasi-uniform mesh over the domain Ω. The elements Ω_j are shape regular (non-planed) tetrahedron, prism and cubic elements with diameters h_j. We shall assume as in previous chapter that $\overline{\Omega}_i\cap\overline{\Omega}_j$ is either the empty set, or an entire side, or an entire face, or a vertex of Ω_i and Ω_j, and assume that all vertices of Ω are vertices of some Ω_i. For a family $\mathcal{T}$ of quasi-uniform meshes Δ_h, there exists a constant C_ω such that for all meshes in $\mathcal{T}$

$$1\le\frac{\max_j h_j}{\min_j h_j}\le C_\omega,$$

and $h=\max\limits_j h_j$ denotes the element size for a quasi-uniform meshes Δ_h.

By M_j we denote a mapping of standard element Ω_{st} onto Ω_j, where Ω_{st} is a standard tetrahedron K, or a standard prism G or a standard cube D. Let $\mathcal{P}_{p_j}(\Omega_j)$ denote a set of pull-back polynomials φ on Ω_j such that $\varphi\circ M_j \in \mathcal{P}^{\kappa}_{p_j}(\Omega_{st})$ with $\kappa = 1$ if Ω_{st} is the standard tetrahedron K, $\kappa = 1.5$ if Ω_{st} is the standard prism G and $\kappa = 2$ if Ω_{st} is the standard cube D. By P and $\mathcal{M}$ we denote the distribution of the element degrees p_j and the union of mappings $M_j, 1 \leq j \leq J$. As usual, the finite element subspaces of piecewise and continuous (pull-back) polynomials are defined as

$$\begin{aligned} S^{P,1}_D(\Omega;\Delta;\mathcal{M}) &= S^P_D(\Omega;\Delta;\mathcal{M})\cap H^1_D(\Omega), \\ S^P(\Omega;\Delta;\mathcal{M}) &= \{\varphi \mid \varphi|_{\Omega_j} \in \mathcal{P}_p(\Omega_j), 1\leq j\leq J\}. \end{aligned} \tag{5.72}$$

The h-p version finite element solution $u_{hp} \in S^{P,1}_D(\Omega;\Delta_h;\mathcal{M})$ is such that

$$B(u_{hp}, v) = F(v), \qquad \forall\, v \in S^{P,1}_D(\Omega;\Delta_h;\mathcal{M}). \tag{5.73}$$

Using the coercivity and continuity of the bilinear form (5.3), one can show that

$$\|u-u_{hp}\|_{H^1(\Omega)} \leq C \inf_{w\in S^{p,1}_D(\Omega;\Delta;\mathcal{M})} \|u-w\|_{H^1(\Omega)}. \tag{5.74}$$

5.3.1. Elliptic problems with homogeneous Dirichlet condition.

Lemma 5.8. *Let Γ_h be a face of K_h which is a tetrahedron or a hexahedron with size h, and let ψ be a polynomial or pull-back polynomial of degree p on Γ_h vanishing at all sides of Γ_h.*

If K_h is a tetrahedron or a parallelepiped and $\psi \in \mathcal{P}^{\kappa,0}_p(\Gamma_h), \kappa = 1$ or 2, then there exists an extension $\Psi(x) \in \mathcal{P}^{\kappa}_p(K_h), \kappa = 1,2$ such that $\Psi(x)\mid_{\Gamma_h} = \psi$ and vanishes at other faces of K_h, and

$$\|\Psi\|_{H^1(K_h)} \leq C\|\psi\|_{H^{1/2}_{00}(\Gamma_h)}. \tag{5.75}$$

with C independent of p and h.

If K_h is a hexahedron and $\psi \in \mathcal{P}^{2,0}_p(\Gamma_h)$, then the extension is a pull-back polynomial $\Psi = \tilde{\Psi}\circ M^{-1}$ with $\tilde{\Psi} \in \mathcal{P}^2_p(Q)$, $\Psi\mid_{\Gamma_h} = \psi$ and vanishes at other faces of K_h, and (5.75) holds.

If K_h is a tetrahedron or a hexahedron with non-plane surfaces, and $\tilde{\psi} = \psi\circ M$ is a polynomial of degree p on $\tilde{\Gamma} = \Gamma_h\circ M$ and vanishes on the sides of Γ_h where M is a mapping of standard tetrahedron or cube K onto K_h, then there exists an extension $\Psi = \tilde{\Psi}\circ M^{-1}$ with $\Psi \in \mathcal{P}^{\kappa}_p(K), \kappa = 1,2$, $\Psi\mid_{\Gamma} = \psi$ and vanishes at other faces of K_h, and (5.75) holds.

Proof. Let M be the mapping of K onto K_h and Γ onto Γ_h, and $\tilde{\psi} = \psi\circ M \in \mathcal{P}^{\kappa}_p(\Gamma)$. Note that

$$|\tilde{\psi}|^2_{H^{\frac{1}{2}}(\Gamma)} = \frac{1}{h}|\psi|^2_{H^{\frac{1}{2}}(\Gamma_h)},$$

and

$$\|\tilde{\psi}\|^2_{L^2(\Gamma)} \leq C\int_{\Gamma}\frac{|\tilde{\psi}|^2}{dist.(\tilde{\tau},\partial\Gamma)}d\tilde{\tau} = \frac{1}{h}\int_{\Gamma_h}\frac{|\psi|^2}{dist.(\tau,\partial\Gamma_h)}d\tau$$

which implies

$$\|\tilde{\psi}\|^2_{H^{1/2}_{00}(\Gamma)} \leq \frac{C}{h}\|\psi\|^2_{H^{1/2}_{00}(\Gamma_h)}.$$

Due to Lemma 8 of [**40**] and Theorem 4.24, there exists an extension $\tilde{\Psi} \in \mathcal{P}^{\kappa}_p(K), \kappa = 1, 2$ (K is a standard tetrahedron or a cube) such that

$$\|\tilde{\Psi}\|_{H^1(K)} \leq C\|\tilde{\psi}\|_{H^{1/2}_{00}(\Gamma)} \leq Ch^{-\frac{1}{2}}\|\psi\|_{H^{1/2}_{00}(\Gamma_h)}.$$

Let $\Psi = \tilde{\Psi} \circ M^{-1}$. Then $\Psi \in \mathcal{P}_p(K_h)$, and by a simple scaling there holds

$$\|\Psi\|_{H^1(K_h)} \leq Ch^{\frac{1}{2}}\|\tilde{\Psi}\|_{H^1(K)} \leq Ch^{\frac{1}{2}}\|\tilde{\psi}\|_{H^{1/2}_{00}(\Gamma)} \leq C\|\psi\|_{H^{1/2}_{00}(\Gamma_h)}$$

which leads to (5.75). □

Lemma 5.9. *Let Γ_h be a face of a triangular prism G_h with size h, and let ψ be a polynomial or pull-back polynomial of degree p on Γ_h vanishing at all sides of Γ_h.*

If G_h is a triangular prism and $\psi \in \mathcal{P}^{1,0}_p(\Gamma_h)$, where Γ_h is a triangular face, then there exists an extension $\Psi(x) \in \mathcal{P}^{1.5}_p(G_h)$, such that $\Psi(x) \mid_{\Gamma_h} = \psi$ and vanishes at other faces of G_h, and

$$\|\Psi\|_{H^1(G_h)} \leq C\|\psi\|_{H^{1/2}_{00}(\Gamma_h)}. \tag{5.76}$$

with C independent of p and h.

If G_h is a triangular prism and $\psi \in \mathcal{P}^{2,0}_p(\Gamma_h)$, where Γ_h is a square face, then there exists an extension $\Psi(x) \in \mathcal{P}^{1.5}_p(G_h)$, such that $\Psi(x) \mid_{\Gamma_h} = \psi$ and vanishes at other faces of G_h, and

$$\|\Psi\|_{H^1(G_h)} \leq C\Big(h^{\frac{3}{2}}p^{-\frac{3}{2}}|\psi_{x_3}|_{H^1(\Gamma_h)} + h^{\frac{1}{2}}p^{-\frac{1}{2}}|\psi|_{H^1(\Gamma_h)} + h^{-\frac{1}{2}}p^{\frac{1}{2}}\|\psi\|_{L^2(\Gamma_h)}\Big). \tag{5.77}$$

If G_h is a triangular prism with non-plane surfaces, and $\tilde{\psi} = \psi \circ M$ is a polynomial of degree p on $\tilde{\Gamma} = \Gamma_h \circ M$ and vanishes on the sides of Γ_h where M is a mapping of standard triangular prism G onto G_h, then there exists an extension $\Psi = \tilde{\Psi} \circ M^{-1}$ with $\Psi \in \mathcal{P}^{\kappa}_p(G_h), \kappa = 1.5$, $\Psi \mid_{\Gamma_h} = \psi$ and vanishes at other faces of G_h, and (5.76) holds if Γ_h is a triangle, and (5.77) holds if Γ_h is a square.

Proof. If Γ_h is a triangular face, then the proof of the assertion is the same as one for the extension from a face to tetrahedron and hexahedron K_h and D_h in Lemma 5.8.

We concentrate the proof of the extension from a square face Γ_h. Let M be the mapping of G onto G_h and Γ onto Γ_h, and $\tilde{\psi} = \psi \circ M \in \mathcal{P}^2_p(\Gamma)$. By the Theorem 4.9 and a simple scaling we have

$$\begin{aligned}\|\Psi\|_{H^1(G_h)} &\leq Ch^{\frac{1}{2}}\|\tilde{\Psi}\|_{H^1(G)} \leq Ch^{\frac{1}{2}}\Big(p^{-\frac{3}{2}}|\tilde{\psi}_{x_3}|_{H^1(\tilde{\Gamma})} + p^{-\frac{1}{2}}|\tilde{\psi}|_{H^1(\tilde{\Gamma})} + p^{\frac{1}{2}}\|\tilde{\psi}\|_{L^2(\tilde{\Gamma})}\Big)\\ &\leq C\Big(h^{\frac{3}{2}}p^{-\frac{3}{2}}|\psi_{x_3}|_{H^1(\Gamma_h)} + h^{\frac{1}{2}}p^{-\frac{1}{2}}|\psi|_{H^1(\Gamma_h)} + h^{-\frac{1}{2}}p^{\frac{1}{2}}\|\psi\|_{L^2(\Gamma_h)}\Big).\end{aligned}$$

□

Theorem 5.10. *Let $\Delta_h = \left\{\Omega_j,\ 1 \le j \le J\right\}$ be a quasi-uniform mesh with element size h over Ω containing hexahedron, prism and tetrahedron elements, and let $S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ be the finite element space defined as above. The data functions f and g are assumed such that the solution u of (5.1) is in $H^k(\Omega)$ with $k \ge 1$. Then the finite element solution $u_{hp} \in S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ with $p \ge 0$ for the problem (5.1) satisfies*

$$\|u - u_{hp}\|_{H^1(\Omega)} \le C\,\frac{h^{\mu-1}}{(p+1)^{k-1}}\|u\|_{H^k(\Omega)} \tag{5.78}$$

where $\mu = \min\{p+1,k\}, p = \min_j p_j$ and the constant C is independent of p, k and u.

Proof. Due to (5.74) it is sufficient to construct a piecewise polynomial $\varphi \in S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ such that

$$\|u - \varphi\|_{H^1(\Omega)} \le C\frac{h^{\mu-1}}{(p+1)^{k-1}}\|u\|_{H^k(\Omega)}. \tag{5.79}$$

We first assume that $k \ge 3$ and $p \ge 1$. Due to Theorems 5.5-5.7, we have polynomials $\varphi_j \in \mathcal{P}_p^\kappa(\Omega_j)(\kappa = 1,$ or $1.5,$ or 2 if Ω_j is a tetrahedron, or a prism or a hexahedron respectively) in each element Ω_j such that $u = \varphi_j$ at each vertex V of Ω_j and $\varphi_j = \pi_\gamma u$ on each edge γ of Ω_j where π_γ is the operator defined as in Lemma 5.2 and

$$\|u - \varphi_j\|_{H^l(\Omega_j)} \le C\frac{h^{\mu-1}}{(p_j+1)^{k-l}}\|u\|_{H^k(\Omega_j)}, \tag{5.80}$$

and for $t = 0, 1$

$$\|u - \varphi_j\|_{H^t(F_i)} \le C\frac{h^{\mu-t-\frac{1}{2}}}{(p_j+1)^{k-t-\frac{1}{2}}}\|u\|_{H^k(\Omega_j)}, \tag{5.81}$$

where F_i are faces of Ω_j.

Suppose that $F = \bar{\Omega}_j \cap \bar{\Omega}_i$ is a common face of two neighboring elements Ω_j and Ω_i. We may assume without loss of generality that Ω_i and Ω_j are standard elements with size h.

If F is a standard triangle T, there are three possible cases as shown in Fig. 5.4 :

(T1) both Ω_j and Ω_i are tetrahedrons;

(T2) both Ω_j and Ω_i are triangular prisms;

(T3) Ω_j is a tetrahedron and Ω_i is a triangular prism.

We shall modify φ_i and φ_j accordingly in the above three cases.

In the case (T1): Ω_i and Ω_j are tetrahedrons, and we assume here that $p_j \ge p_i$. $\psi = (\varphi_i - \varphi_j)\big|_F \in \mathcal{P}_{p_j}^{1,0}(F)$. By Lemma 5.8 there is a polynomial extension $\Psi \in \mathcal{P}_{p_j}^1(\Omega_j)$ such that $\Psi\,|_F = \psi$ and $\Psi\,|_{\partial\Omega_j \setminus F} = 0$, and

$$\|\Psi\|_{H^1(\Omega_j)} \le C\|\psi\|_{H_{00}^{\frac{1}{2}}(F)} = C\|\varphi_i - \varphi_j\|_{H_{00}^{\frac{1}{2}}(F)}. \tag{5.82}$$

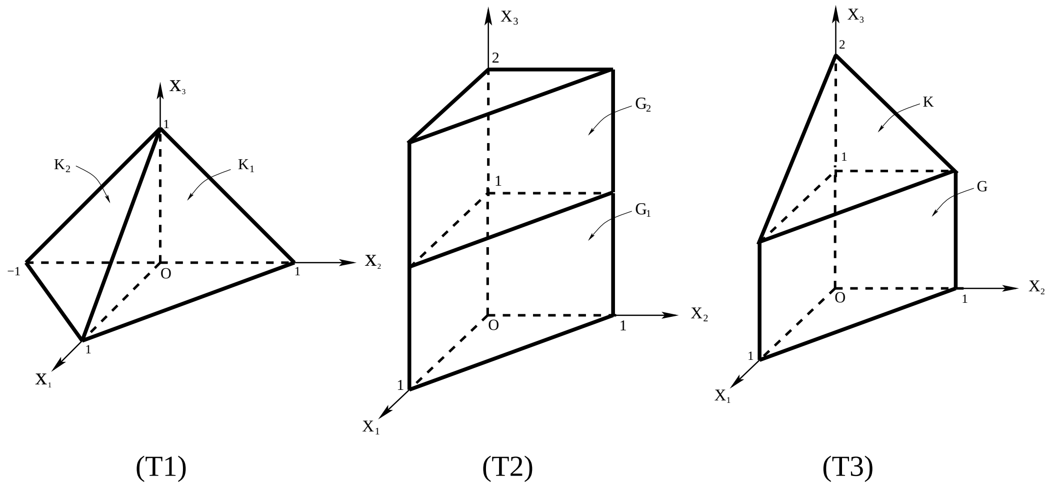


Fig. 5.4 A pair of elements sharing a common triangular face

(T1) A pair of tetrahedrons, (T2) A pair of prisms, (T3) A tetrahedron and a prism

Note that $(\varphi_i - \varphi_j)\big|_F \in H_{00}^{\frac{1}{2}}(F) = \Big(H^0(F), H_0^1(F)\Big)_{\frac{1}{2},2}$ and that for $t = 0, 1$

$$\begin{aligned}\|\varphi_i - \varphi_j\|_{H^t(F)} &\leq C\Big(\|\varphi_i - u\|_{H^t(F)} + \|\varphi_j - u\|_{H^t(F)}\Big)\\ &\leq C\frac{h^{\mu-t-\frac{1}{2}}}{(p_{ij}+1)^{k-t-\frac{1}{2}}}\Big(\|u\|_{H^k(\Omega_j)} + \|u\|_{H^k(\Omega_i)}\Big),\end{aligned}$$

where $p_{ij} = \min\{p_i, p_j\}$, which implies

$$\|\Psi\|_{H^1(\Omega_j)} \leq C\|\psi\|_{H_{00}^{\frac{1}{2}}(F)} \leq C\frac{h^{\mu-1}}{(p_{ij}+1)^{k-1}}\Big(\|u\|_{H^k(\Omega_j)} + \|u\|_{H^k(\Omega_i)}\Big). \tag{5.83}$$

In the case (T2): Ω_j and Ω_i are triangular prisms, the argument can be carried out here except $\Psi \in \mathcal{P}_{p_j}^{1.5}(\Omega_j)$, $\varphi_i \in \mathcal{P}_{p_i}^{1.5}(\Omega_i)$ and $\varphi_j \in \mathcal{P}_{p_j}^{1.5}(\Omega_j)$.

In the case (T3): Adjust φ_j on the tetrahedron Ω_j as in the case (T1) if $p_j \geq p_i$ or φ_i on the triangular prism Ω_i as in the case (T2) if $p_j < p_i$.

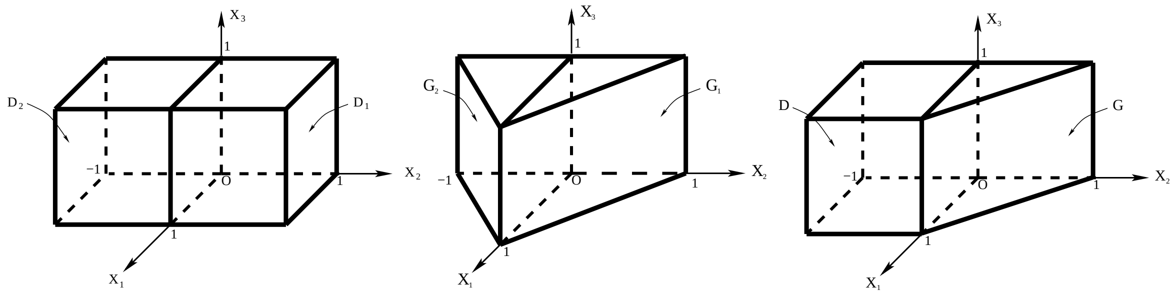
We next consider the modification of φ_j and φ_i if F is a standard square face S. Similarly, there are three possible cases:

(S1) both Ω_j and Ω_i are hexahedrons;

(S2) both Ω_j and Ω_i are triangular prisms;

(S3) Ω_j is a hexahedron and Ω_i is a triangular prism.

In the case (S1): The arguments in the case (T1) can be carried out here except $\psi = (\varphi_i - \varphi_j)\big|_F \in \mathcal{P}_{p_j}^{2,0}(F)$ instead of $\mathcal{P}_{p_j}^{1,0}(F)$, and $\Psi \in \mathcal{P}_{p_j}^{2}(\Omega_j)$ instead of $\mathcal{P}_{p_j}^{1}(\Omega_j)$.



(S1) **(S2)** **(S3)**

Fig. 5.5 A pair of elements sharing a common square face

(S1) A pair of hexahedrons, (S2) A pair of prisms, (S3) A hexahedron and a prism

In the case (S2): By Lemma 5.6, there are $\varphi_i \in \mathcal{P}^{1.5}_{p_i}(\Omega_i)$ and $\varphi_j \in \mathcal{P}^{1.5}_{p_j}(\Omega_j)$ satisfying (5.39)-(5.42). Suppose that $p_j \geq p_i$ and $F = \{x = (x_1, 0, x_3) \mid 0 \leq x_1, x_3 \leq 1\}$ as shown in Fig. 5.5. Then $\psi(x_1, x_3) = (\varphi_i - \varphi_j)\big|_F \in \mathcal{P}^{2,0}_{p_j}(F)$, and by Lemma 5.9 there exists a polynomial extension Ψ on Ω_j such that $\Psi \in \mathcal{P}^{1.5}_{p_j}(\Omega_j)$, $\Psi|_F = \psi$ and $\Psi|_{\partial\Omega_j \setminus F} = 0$, and

$$\|\Psi\|_{H^1(\Omega_j)} \leq C(h^{\frac{3}{2}}(p_j+1)^{-\frac{3}{2}}\|\psi_{x_3}\|_{H^1(F)} + h^{\frac{1}{2}}(p_j+1)^{-\frac{1}{2}}\|\psi\|_{H^1(F)} + h^{-\frac{1}{2}}(p_j+1)^{\frac{1}{2}}\|\psi\|_{L^2(F)}).$$

Due to (5.41) and (5.42) there hold for $t = 0, 1$

$$\|\psi\|_{H^t(F)} \leq \|u - \varphi_j\|_{H^t(F)} + \|u - \varphi_i\|_{H^t(F)} \leq C\frac{h^{\mu-t-\frac{1}{2}}}{(p_{ij}+1)^{k-t-\frac{1}{2}}}(\|u\|_{H^k(\Omega_j)} + \|u\|_{H^k(\Omega_i)})$$

and

$$\|\psi_{x_3}\|_{H^1(F)} \leq \left\|\frac{\partial(u-\varphi_j)}{\partial x_3}\right\|_{H^1(F)} + \left\|\frac{\partial(u-\varphi_i)}{\partial x_3}\right\|_{H^1(F)} \leq C\frac{h^{\mu-\frac{5}{2}}}{(p_{ij}+1)^{k-\frac{5}{2}}}(\|u\|_{H^k(\Omega_j)} + \|u\|_{H^k(\Omega_i)}),$$

which implies (5.83).

In the case (S3): Ω_j is a standard hexahedron and Ω_i is a standard triangular prism. If $p_j \geq p_i$, modify φ_j as in the case (S1), and if $p_j < p_i$, modify φ_i as in the case (S2).

Let $\tilde{\varphi}_j = \varphi_j + \Psi$ and $\tilde{\varphi}_i = \varphi_i$. Then $\tilde{\varphi}_j = \tilde{\varphi}_i$ on F, and by (5.80) and (5.83)

$$\begin{aligned} \|u - \tilde{\varphi}_j\|_{H^1(\Omega_j)} &\leq \|u - \varphi_j\|_{H^1(\Omega_j)} + \|\Psi\|_{H^1(\Omega_j)} \\ &\leq C\frac{h^{\mu-1}}{(p_{ij}+1)^{k-1}}\Big(\|u\|_{H^k(\Omega_j)} + \|u\|_{H^k(\Omega_i)}\Big) \end{aligned} \tag{5.84}$$

and

$$\|u - \tilde{\varphi}_i\|_{H^1(\Omega_i)} = \|u - \varphi_i\|_{H^1(\Omega_i)} \leq C\frac{h^{\mu-1}}{(p_{ij}+1)^{k-1}}\|u\|_{H^k(\Omega_i)}. \tag{5.85}$$

Adjusting φ_j on each face of Ω_j by the lifting polynomial Ψ, we achieve the continuity across interfaces of elements. For the homogeneous Dirichlet boundary condition, we can adjust φ_j in similar way such that $\tilde{\varphi}_j|_{\partial\Omega_j\cap\Gamma_D} = 0$. Let $\varphi = \tilde{\varphi}_j$ in $\Omega_j, 1 \le j \le J$. Then $\varphi \in S_D^{p,1}(\Omega;\Delta_h;\mathcal{M})$, and satisfies (5.79).

We next prove (5.79) for $1 < k < 3$ and $p \ge 2$. It was shown in, e.g., [**2**] that $H^k(\Omega) \cap H_D^1(\Omega) = \Big(H_D^1(\Omega), H^3(\Omega)\cap H_D^1(\Omega)\Big)_{\theta,2} \subset \Big(H^1(\Omega), H^3(\Omega)\Big)_{\theta,2} \cap H_D^1(\Omega)$ with $\theta = \frac{k-1}{2} \in (0,1)$ for $1 < k < 3$. Since $\Big(H^1(\Omega), H^3(\Omega)\Big)_{\theta,2} \subset \Big(H^1(\Omega), H^3(\Omega)\Big)_{\theta,\infty} = B^k(\Omega)$, $H^k(\Omega)\cap H_D^1(\Omega) \subset B^k(\Omega)\cap H_D^1(\Omega)$. Suppose that $v \in H_D^1(\Omega)$ and $w \in H^3(\Omega)\cap H_D^1(\Omega)$ form a decomposition of $u \in B^k(\Omega)\cap H_D^1(\Omega)$. Applying (5.79) for $k = 3$, we have a polynomial $\varphi \in S_D^{p,1}(\Omega;\Delta_h;\mathcal{M})$ with $p \ge 2$ such that

$$\|w - \varphi_p\|_{H^1(\Omega)} \le C\ \frac{h^2}{(p+1)^2}\|w\|_{H^3(\Omega)}.$$

Therefore, we have for any decomposition v and w of u

$$\begin{aligned} &\|u-\varphi\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)} + \|w-\varphi\|_{H^1(\Omega)} \\ \le\ & C\left(\|v\|_{H^1(\Omega)} + \frac{h^2}{(p+1)^2}\|w\|_{H^3(\Omega)}\right) = C\ \left(\|v\|_{H^1(\Omega)} + t_1\|w\|_{H^3(\Omega)}\right) \end{aligned}$$

with $t_1 = \frac{h^2}{(p+1)^2}$ and C independent of v and w. Due to the definition of the Besov space $B^k(\Omega)$, we have

$$\|u-\varphi\|_{H^1(\Omega)} \le CK(t_1,u) \le Ct_1^\theta \sup_{t>0} t^{-\theta}K(t,u) \le Ct_1^\theta\|u\|_{B^k(\Omega)} \le C\frac{h^{k-1}}{(p+1)^{k-1}}\|u\|_{H^k(\Omega)},$$

which is (5.78) for $p \ge 2$ and $1 < k < 3$.

For $p = 1$ and $k \ge 2$, by Theorem 5.4 there exist polynomials $\phi_j \in \mathcal{P}_p^\kappa(\Omega_j)(\kappa = 1$, or 1.5, or 2 if Ω_j is a tetrahedron, or a prism or a hexahedron respectively) in each element Ω_j such that $u = \varphi_j$ at each vertex V of Ω_j, and

$$\|u - \phi_j\|_{H^l(\Omega_j)} \le C\frac{h^{\mu-1}}{(p_j+1)^{k-l}}\|u\|_{H^k(\Omega_j)}, \tag{5.86}$$

Let $\varphi = \phi_j$ in $\Omega_j, 1 \le j \le J$. Then $\varphi \in S_D^{p,1}(\Omega;\Delta_h;\mathcal{M})$, and satisfies (5.79).

For $p = 1$ and $1 < k < 2$, similarly, it holds that $H^k(\Omega)\cup H_D^1(\Omega) = \Big(H_D^1(\Omega), H^2(\Omega)\cap H_D^1(\Omega)\Big)_{\theta,2} \subset \Big(H^1(\Omega), H^3(\Omega)\Big)_{\theta,2} \cap H_D^1(\Omega)$ with $\theta = k-1 \in (0,1)$ for $1 < k < 3$. Since $\Big(H^1(\Omega), H^2(\Omega)\Big)_{\theta,2} \subset \Big(H^1(\Omega), H^2(\Omega)\Big)_{\theta,\infty} = B^k(\Omega)$, $H^k(\Omega)\cap H_D^1(\Omega) \subset B^k(\Omega)\cap H_D^1(\Omega)$. Suppose that $v \in H_D^1(\Omega)$ and $w \in H^2(\Omega)\cap H_D^1(\Omega)$ form a decomposition of $u \in B^k(\Omega)\cap H_D^1(\Omega)$. Applying (5.79) for $k = 2$, we have a polynomial $\varphi \in S_D^{p,1}(\Omega;\Delta_h;\mathcal{M})$ with $p = 1$

such that

$$\|w-\varphi_p\|_{H^1(\Omega)} \le C\,\frac{h}{(p+1)}\|w\|_{H^2(\Omega)}.$$

Arguing as for $1<k<3$ and $p\ge 2$ we can shown that

$$\|u-\varphi\|_{H^1(\Omega)} \le CK(t_1,u) \le Ct_1^\theta \sup_{t>0} t^{-\theta}K(t,u) \le Ct_1^\theta\|u\|_{B^k(\Omega)} \le C\frac{h^{k-1}}{(p+1)^{k-1}}\|u\|_{H^k(\Omega)},$$

which is (5.78) for $p=2$ and $1<k<2$.

For $p=0$ or $k=1$, (5.79) is trivial by selecting $\varphi=0$. Thus we complete the proof of the theorem. □

5.3.2. Elliptic problems with non-homogeneous Dirichlet condition.

Consider a boundary value problem with non-homogeneous Dirichlet boundary condition

$$\begin{aligned} &-\Delta u + u = f \qquad \text{in } \Omega\subset\mathbb{R}^3 \\ &u\,|_{\Gamma_D} = q, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma_N} = g, \end{aligned} \tag{5.87}$$

where Ω is Lipschitz domain in $\mathbb{R}^3$, and Γ_D and Γ_N are referred as the Dirichlet boundary and the Neumann boundary where the Dirichlet and Neumann boundary conditions are imposed.

Lemma 5.11. *Let $u\in H^s(S_h)$, $S_h=(-h,h)^2$ with $s>1$. Then there exists a polynomial $\varphi\in\mathcal{P}_p^2(S_h)$ satisfying*

$$\|u-\varphi\|_{H^l(S_h)} \le C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|u\|_{H^s(S_h)},\ 0\le l\le 1 \tag{5.88}$$

$$\varphi(V_i)=u(V_i),\ 1\le i\le 4 \tag{5.89}$$

$$\varphi|_{\gamma_i} = \pi_{\gamma_i}u,\ 1\le i\le 4 \tag{5.90}$$

where $\mu=\min(p+1,s)$, V_i and γ_i are vertices and sides of S_h and π_{γ_i} is the operator defined as in Lemma 5.2.

Proof. We first prove the lemma for integers s and l. By Lemma 4.2 of [**34**] there exists a polynomial $\phi\in\mathcal{P}_p^2(S_h)$ such that $\phi(V_i)=u(V_i)$, $1\le i\le 4$, and

$$\|u-\phi\|_{H^l(S_h)} \le C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|u\|_{H^s(S_h)},\ 0\le l\le s,\ \mu=\min\{p+1,s\}.$$

To construct a desired polynomial $\psi\in\mathcal{P}_p^2(S_h)$, let $\varphi=\phi+v$ with $v\in\mathcal{P}_p^2(S_h)$ satisfying

$$v(V_i)=0, 1\le i\le 4, \tag{5.91}$$

$$v|\gamma_i = \pi_{\gamma_i}u - \phi|\gamma_i, 1\le i\le 4, \tag{5.92}$$

$$\|v\|_{H^l(S_h)} \leq C\frac{h^{\mu-l}}{p^{s-l}}\|u\|_{H^s(S_h)}, 0 \leq l \leq s. \tag{5.93}$$

For $s > 1$, $u \in H^{s-\frac{1}{2}}(\gamma_i)$. By Lemma 5.2 $\pi_{\gamma_i}u = u$ at ending points V_i and V_{i+1}, and

$$\|u-\pi_{\gamma_i}u\|_{H^l(\gamma_i)} \leq C\frac{h^{\mu-\frac{1}{2}-l}}{(p+1)^{s-\frac{1}{2}-l}}\|u\|_{H^{s-\frac{1}{2}}(\gamma_i)},\ 0 \leq l \leq s-\frac{1}{2}.$$

Then $g_i = \pi_{\gamma_i}u - \phi|_{\gamma_i} \in \mathcal{P}_p^0(\gamma_i)$, and

$$\begin{aligned} \|g_i\|_{H^t(\gamma_i)} &\leq \|u-\pi_{\gamma_i}u\|_{H^t(\gamma_i)} + \|u-\phi\|_{H^t(\gamma_i)} \\ &\leq C\Big(\frac{h^{\mu-t-\frac{1}{2}}}{(p+1)^{s-t-\frac{1}{2}}}\|u\|_{H^{s-\frac{1}{2}}(\gamma_i)} + \|u-\phi\|_{H^{t+\frac{1}{2}}(S)}\Big) \\ &\leq C\frac{h^{\mu-t-\frac{1}{2}}}{(p+1)^{s-t-\frac{1}{2}}}\|u\|_{H^s(S)}. \end{aligned} \tag{5.94}$$

Let $v_1 = g_1(x_1)\psi(x_2)$ with $\psi(x_2) = \Big(\dfrac{h-x_2}{2h}\Big)^p$, then we have

$$\|\psi\|_{H^t(I)} \leq Ch^{\frac{1}{2}-t}p^{t-\frac{1}{2}}, \quad t = 0, 1. \tag{5.95}$$

Then $v_1|_{\gamma_1} = g_1$ and $v_1 = 0$ on γ_i, $2 \leq i \leq 4$, and due to (5.95) and (5.94) there holds for $0 \leq l \leq s$

$$\begin{aligned} |v_1|^2_{H^l(S)} &\leq C(\|g_1\|^2_{H^l(I)}\|\psi\|^2_{L^2(I)} + \|g\|^2_{L^2(I)}\|\psi\|^2_{H^l(I)}) \\ &\leq C\Big(\frac{h^{2(\mu-l-\frac{1}{2})}}{(p+1)^{2(s-l-\frac{1}{2})}}hp^{-1} + \frac{h^{2(\mu-\frac{1}{2})}}{(p+1)^{2(s-\frac{1}{2})}}h^{2(\frac{1}{2}-l)}p^{2(l-\frac{1}{2})}\Big)\|u\|^2_{H^s(S)} \\ &\leq \frac{h^{2(\mu-l)}}{(p+1)^{-2(s-l)}}\|u\|^2_{H^s(S)}. \end{aligned}$$

Similarly we can construct $v_i \in \mathcal{P}_p^2(\gamma_i)$, $1 \leq i \leq 4$ such that $v_i(V_j) = 0$

$$v_i|_{\gamma_i} = g_i,\ v_i|_{\gamma_j} = 0 \text{ for } j \neq i, 1 \leq j \leq 4$$

and

$$\|v_i\|_{H^l(S)} \leq C\frac{h^{\mu-l}}{(p+1)^{s-l}}\|u\|_{H^s(S)}.$$

Let $v = \sum_{i=1}^4 v_i$. Then v satisfies (5.91)-(5.93), and (5.88)-(5.90) follow for integer l and s. The results can be generalized to non-integer l and s by the standard argument of interpolation spaces. □

Lemma 5.12. *Let $u \in H^s(T_h)$ with $s > 1$, where T_h is a triangle $\{(x_1, x_2)|0 \leq x_2 \leq h - x_1, 0 \leq x_1 \leq h\}$. Then there exists a polynomial $\varphi(x) \in \mathcal{P}_p^1(T_h)$ such that*

$$\|u - \varphi\|_{H^l(T_h)} \leq C \frac{h^{\mu-l}}{(p+1)^{s-l}} \|u\|_{H^s(T_h)}, \qquad l = 0, 1, \tag{5.96}$$

$$\varphi(V_i) = u(V_i), \qquad 1 \leq i \leq 3, \tag{5.97}$$

$$\varphi|\gamma_i = \pi_{\gamma_i} u, \qquad 1 \leq i \leq 3 \tag{5.98}$$

where $\mu = \min\{p+1, s\}$, V_i and γ_i are vertices and sides of T_h and π_{γ_i} is the operator defined as in Lemma 5.2.

Proof. We may prove the lemma for integer s and l because the desired results for non-integer l and s follow easily by the standard argument of interpolation space.

By Lemma 4.2 of [**34**] there exists a polynomial $\phi \in \mathcal{P}_p^1(T_h)$ such that $\phi(V_i) = u(V_i)$, $1 \leq i \leq 3$ and

$$\|u - \phi\|_{H^l(T_h)} \leq C \frac{h^{\mu-l}}{(p+1)^{s-l}} \|u\|_{H^s(T_h)}, \quad 0 \leq l \leq s, \quad \mu = \min\{p+1, s\}. \tag{5.99}$$

By Lemma 5.2, $\pi_{\gamma_i} u = u$ at the ending points V_i and V_{i+1}, and

$$\begin{aligned} \|u - \pi_{\gamma_i} u\|_{H^t(\gamma_i)} &\leq C \frac{h^{\mu-\frac{1}{2}-t}}{(p+1)^{s-\frac{1}{2}-t}} \|u\|_{H^{s-\frac{1}{2}}(\gamma_i)} \\ &\leq C \frac{h^{\mu-\frac{1}{2}-t}}{(p+1)^{s-\frac{1}{2}-t}} \|u\|_{H^s(T_h)} \end{aligned} \tag{5.100}$$

Then $g_i = (\pi_{\gamma_i} u - \phi|_{\gamma_i}) \in \mathcal{P}_p^0(\gamma_i)$, and (5.99)-(5.100) lead to the following estimates which is an analogue to (5.94)

$$\|g_i\|_{H^t(\gamma_i)} \leq C \frac{h^{\mu-\frac{1}{2}-t}}{(p+1)^{s-\frac{1}{2}-t}} \|u\|_{H^s(T_h)}, \qquad 0 \leq t \leq s - \frac{1}{2}. \tag{5.101}$$

By Lemma 5.3, there exists an extension $v_i \in \mathcal{P}_p^1(T_h)$ such that for $t = 0, 1$

$$v_i(V_j) = 0, \qquad 1 \leq j \leq 3, \tag{5.102}$$

$$v_i|_{\gamma_i} = g_i, \quad v_i|_{\gamma_j} = 0, \quad j \neq i, \quad 1 \leq j \leq 3, \tag{5.103}$$

$$\begin{aligned} \|v_i\|_{H^l(T_h)} &\leq C(h^{\frac{3}{2}-l}(p+1)^{l-\frac{3}{2}} \|g_i\|_{H^1(\gamma_i)} + h^{\frac{1}{2}-l}(p+1)^{l-\frac{1}{2}} \|g_i\|_{L^2(\gamma_i)}) \\ &\leq C \frac{h^{\mu-l}}{(p+1)^{s-l}} \|u\|_{H^s(T_h)}. \end{aligned} \tag{5.104}$$

Let $\varphi = \phi + \sum_{i=1}^{3} v_i$. Then $\varphi \in \mathcal{P}_p^1(T_h)$ and satisfies (5.96)-(5.98). □

The partition Δ_h on Ω induces a partition $\Delta_{\gamma_i} = \Gamma_{i,j}, j \in \mathcal{J}_i$ on Γ_i, $i \in \mathcal{D}$ where $\mathcal{J}_i$ is a subset of $\mathcal{J} = \{1, 2, \cdots, J\}$, and a partition $\Delta_{\Gamma_D} = \{\Delta_{\Gamma_i}, i \in \mathcal{D}\}$ on Γ_D. $\Gamma_{i,j}$'s are (curved) triangles and quadrilaterals. By $V_{i,j,l}$ and $\gamma_{i,j,l}$ we denote vertices and sides of $\Gamma_{i,j}$, then Δ_{Γ_i} and Δ_{Γ_D} are the restrictions of the partition Δ_h of Ω on Γ_i and Γ_D. For $i \in \mathcal{D}$, Let $S^{P,1}(\Gamma_i, \Delta_{\Gamma_i}, \mathcal{M}_{\Gamma_{i,j}})$ be the restriction on Δ_{Γ_i} of polynomials in $S^{P,1}(\Omega; \Delta_h; \mathcal{M})$, ie.

$$S^{P,1}(\Gamma_i; \Delta_i; \mathcal{M}_{\Gamma_i}) = \{\varphi \mid \varphi|_{\Gamma_{i,j}} \circ M_{i,j} \in P^1_{p_j}(T_h) \text{ or } P^2_{p_j}(S_h)\} \cap H^1(\Gamma_i)$$

where $\mathcal{M}_{\Gamma_i} = \{M_{i,j}, 1 \le j \le J(i), i \in \mathcal{D}\}$ and $M_{i,j}$ is the restriction of the map M_j on $\Gamma_{i,j}$ which maps a standard triangle or a square with size h onto $\Gamma_{i,j}$. By $S^{P,1}(\Gamma_D; \Delta_{\Gamma_D}; \mathcal{M}_{\Gamma_D})$ we denote the union of $S^{P,1}(\Gamma_i; \Delta_{\Gamma_i}; \mathcal{M}_{\Gamma_i})$, $i \in \mathcal{D}$.

Lemma 5.13. *Let q_i be the restriction of g on Γ_i, $i \in \mathcal{D}$ and let $q_i \in H^s(\Gamma_i)$, $s > 1$. Then there exist a polynomial $q_{i,p} \in S^{P,1}(\Gamma_i; \Delta_{\Gamma_i}; \mathcal{M}_{\Gamma_i})$ such that for integer l, $0 \le l \le s$*

$$\|q_i - q_{i,p}\|_{H^l(\Gamma_D)} \le C \frac{h^{\mu-l}}{(\overline{p}_i + 1)^{s-l}} \|q_i\|_{H^s(\Gamma_i)} \tag{5.105}$$

with $\mu = \min\{\overline{p}+1, s\}, \overline{p}_i = \min_{j \in \mathcal{J}_i} p_j$.

Proof. By Lemma 5.11 and Lemma 5.12, there exists a polynomial $\psi_{i,j} \in \mathcal{P}^{\kappa}_{p_j}(\Gamma_{i,j}), \kappa = 1$ or 2, respectively, such that

$$\begin{aligned} \|q_i - \psi_{i,j}\|_{H^l(\Gamma_{i,j})} &\le C \frac{h^{\mu-l}}{(p_j+1)^{s-l}} \|q_i\|_{H^s(\Gamma_{i,j})}, \\ q_i(V_{i,j,l}) &= \psi_{i,j}(V_{i,j,l}), 1 \le l \le 3 \quad \text{or} \quad 4, \\ \psi_{ij}|_{\gamma_{i,j,l}} &= \pi_{\gamma_{i,j,l}} q_i, \ 1 \le l \le 3 \quad \text{or} \quad 4. \end{aligned}$$

Let $q_{i,p} = \psi_{i,j}$ on $\Gamma_{i,j}, j \in \mathcal{J}_i$. Then $q_{i,p} \in S^{P,1}(\Gamma_i; \Delta_{\Gamma_i}; \mathcal{M}_{\Gamma_i})$ and

$$\begin{aligned} \|q_i - q_{i,p}\|^2_{H^l(\Gamma_i)} &\le \sum_{j \in \mathcal{J}_i} \|q_i - \psi_{i,j}\|^2_{H^l(\Gamma_{i,j})} \le C \sum_{j \in \mathcal{J}_i} \frac{h^{2(\mu-l)}}{(p_j+1)^{2(s-l)}} \|q_i\|^2_{H^s(\Gamma_{i,j})} \\ &\le C \frac{h^{2(\mu-l)}}{(\overline{p}_i+1)^{2(s-l)}} \|q_i\|^2_{H^s(\Gamma_i)}. \end{aligned}$$

□

Corollary 5.14. *Let $w_{i,p} \in S^{P,1}(\Gamma_i; \Delta_i; \mathcal{M}_{\Gamma_i})$ be the finite element solution of the boundary value problem*

$$\begin{aligned} &\int_{\Gamma_i} (\nabla w \nabla v + wv) dS = \int_{\Gamma_i} (\nabla q_i \nabla v + q_i v) dS, \quad \forall v \in S^{P,1}(\Gamma_i; \Delta_i; \mathcal{M}_{\Gamma_i}) \cap H^1_0(\Gamma_i), \\ &w \mid_{\partial \Gamma_i} = \pi_{\gamma_{i,j,l}} q, \ \gamma_{i,j,l} = \partial \Gamma_{i,j} \cap \partial \Gamma_i, \ j \in \mathcal{J}_i. \end{aligned} \tag{5.106}$$

Then

$$\|q_i - w_{i,p}\|_{H^l(\Gamma_i)} \le C \frac{h^{\mu-l}}{(\overline{p}_i+1)^{s-l}} \|q_i\|_{H^s(\Gamma_i)},$$

where $\mu = \min\{\overline{p}_i, s\}$.

Remark 5.1. Lemma 5.13 indicates the existence of the polynomial $q_{i,p} \in S^{P,1}(\Gamma_i; \Delta_{\Gamma_i}; \mathcal{M}_{\Gamma_i})$ satisfying (5.105). Notice that such a polynomial is not unique. Corollary 5.14 gives a practical way to construct the polynomial $q_{i,p}$ by compute a finite element solution $w_{i,p}$.

Define q_p on Γ_D such that $q_p = q_{i,p}$ on Γ_i, $i \in \mathcal{D}$. Then $q_p \in C^0(\Gamma_D)$ and $q_p \in S^{P,1}(\Gamma_D; \Delta_{\Gamma_D}; \mathcal{M}_{\Gamma_D})$. Due to (5.105) there holds for $t = 0, 1$

$$
\begin{aligned}
\|q - q_p\|_{H^t(\Gamma_D)} &\le \sum_{i\in\mathcal{D}} \|q_i - q_{i,p}\|_{H^t(\Gamma_i)} \le C \sum_{i\in\mathcal{D}} \frac{h^{\mu-\frac{1}{2}-t}}{(\overline{p}_i+1)^{k-\frac{1}{2}-t}} \|q_i\|_{H^{k-\frac{1}{2}}(\Gamma_i)} \\
&\le C \frac{h^{\mu-\frac{1}{2}-t}}{(p+1)^{k-\frac{1}{2}-t}} \sum_{i\in\mathcal{D}} \|q_i\|_{H^{k-\frac{1}{2}}(\Gamma_i)},
\end{aligned} \tag{5.107}
$$

where $p = \min_{j\in\mathcal{J}} p_j$. By the standard argument of interpolation spaces (5.107) can be proved for $0 < t < 1$.

Now we formulate the finite element solution for boundary value problem (5.87) with non-homogeneous Dirichlet boundary condition: Find u_p such that

$$
\begin{gathered}
\int_\Omega (\nabla u_p \nabla v + u_p\, v) dx = \int_\Omega f v dx + \int_{\Gamma_N} g v dS, \quad \forall v \in S_D^{P,1}(\Omega; \Delta_h; \mathcal{M}) \\
u_p \in S^{P,1}(\Omega; \Delta_h; \mathcal{M}),\ u_p \,|_{\Gamma_D} = q_p.
\end{gathered} \tag{5.108}
$$

Obviously u_p is the finite element approximation to the solution $\tilde{u}$ of the following boundary value problem

$$
\begin{gathered}
\int_\Omega (\nabla \tilde{u} \nabla v + \tilde{u}\, v) dx = \int_\Omega f v dx + \int_{\Gamma_N} g v dS, \quad \forall v \in H_D^1(\Omega), \\
\tilde{u} \in H^1(\Omega),\ \tilde{u} \,|_{\Gamma_D} = q_p.
\end{gathered} \tag{5.109}
$$

Lemma 5.15. *Let $\tilde{u}$ and u_p be the solution of the problem (5.109) and the finite element solution satisfying (5.108). Then*

$$
B(\tilde{u} - u_p, v) = \int (\nabla(\tilde{u} - u_p)\nabla v + (\tilde{u} - u_p) v) dx = 0, \qquad \forall v \in S^{P,1}(\Omega; \Delta_h; \mathcal{M}) \tag{5.110}
$$

and

$$
\|\tilde{u} - u_p\|_{H^1(\Omega)} = \inf_{w \in S^{P,1}(\Omega;\Delta_h;\mathcal{M}) w|_{\Gamma_D} = q_p} \|\tilde{u} - w\|_{H^1(\Omega)}. \tag{5.111}
$$

Proof. (5.110) follows from (5.108) and (5.109) directly. For any $w \in S^{P,1}(\Omega;\Delta_h;\mathcal{M})$ with $w|_{\Gamma_D} = q_p$ we have by (5.107)

$$\begin{aligned} B(\tilde u - w, \tilde u - w) &= B(\tilde u - u_p, \tilde u - u_p) + B(u_p - w, u_p - w) + 2B(\tilde u - u_p, u_p - w)\\ &= B(\tilde u - u_p, \tilde u - u_p) + B(u_p - w, u_p - w)\\ &\geq B(\tilde u - u_p, \tilde u - u_p), \end{aligned}$$

which implies (5.111). □

Theorem 5.16. *Let $u \in H^k(\Omega)$ be the solution of the problem of (5.87) and $q \in C^0(\Gamma_D)$, $q_i = q|_{\Gamma_i} \in H^{k-1}(\Gamma_i)$, $i \in \mathcal{D}$, and let $u_p \in S^{P,1}(\Omega;\Delta_h;\mathcal{M})$ with $p \geq 1$ be the finite element solution of (5.108) with $u_p|_{\Gamma_D} = q_p \in S^{P,1}(\Gamma_D;\Delta_{\Gamma_D};\mathcal{M}_{\Gamma_D})$ satisfying (5.107). Then*

$$\|u - u_p\|_{H^1(\Omega)} \leq C\frac{h^{\mu-1}}{(p+1)^{k-1}}\|u\|_{H^k(\Omega)}, \quad \mu = \min\{p+1, k\}. \tag{5.112}$$

Proof. Note that

$$\|u - u_p\|_{H^1(\Omega)} \leq \|u - \tilde u\|_{H^1(\Omega)} + \|\tilde u - u_p\|_{H^1(\Omega)}. \tag{5.113}$$

Since $u - \tilde u$ satisfies

$$\begin{aligned} \Delta(u - \tilde u) &= 0\\ \frac{\partial}{\partial n}(u - \tilde u)|_{\Gamma_N} &= 0, \qquad (u - \tilde u)|_{\Gamma_D} = q - q_p. \end{aligned}$$

By the continuous dependence of solution on boundary condition and (5.107), we have

$$\|u - \tilde u\|_{H^1(\Omega)} \leq C\|q - q_p\|_{H^{\frac12}(\Gamma_D)} \leq C\frac{h^{\mu-1}}{(p+1)^{k-1}}\sum_{i\in\mathcal{D}}\|q_i\|_{H^{k-\frac12}(\Gamma_i)}. \tag{5.114}$$

Due to (5.111) it is sufficient to construct a polynomial $\varphi \in S^{P,1}(\Omega;\Delta_h;\mathcal{M})$ such that $\varphi|_{\Gamma_D} = q_p$ and

$$\|\tilde u - \varphi\|_{H^1(\Omega)} \leq C\frac{h^{\mu-1}}{(p+1)^{k-1}}\|\tilde u\|_{H^k(\Omega)}. \tag{5.115}$$

By the argument in proof of Theorem 5.10, there is a polynomial $\tilde\varphi \in S^{P,1}(\Omega;\Delta_h;\mathcal{M})$ such that

$$\|\tilde u - \tilde\varphi\|_{H^t(\Omega)} \leq C\frac{h^{\mu-t}}{(p+1)^{k-t}}\|u\|_{H^k(\Omega)}, \qquad t = 0,1,2, \tag{5.116}$$

where $\mu = \{p+1, k\}, p = \min\limits_{j\in\mathcal{J}} p_j$, and $\tilde u = \tilde\varphi$ at vertices of Ω_j's and $\tilde u|_\gamma = \pi_\gamma u$ on each side of elements. It remains to adjust $\tilde\varphi$ on Γ_D. Suppose that $\Gamma_{i,j} = \partial\Omega_j \bigcap \Gamma_i$ for some $i \in \mathcal{D}$ is a face of Ω_j. Then $\phi = (q_{i,p} - \tilde\varphi)\,|_{\Gamma_{i,j}} \in \mathcal{P}^{\kappa,0}_{p_j}(\Gamma_{i,j})$ with $\kappa = 1$ or 2. Hence there exists an extension $\Phi \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j)$ such that $\Phi|_{\Gamma_{i,j}} = \phi$ and $\Phi|_{\partial\Omega_j\setminus\Gamma_{i,j}} = 0$, and by Lemma 5.8

$$\|\Phi\|_{H^1(\Omega_j)} \leq C\|\phi\|_{H^{\frac12}_{00}(\Gamma_{i,j})}$$

if Ω_j is a tetrahedron or hexahedron with $\kappa = 1$ or 2 or $\Gamma_{i,j}$ is a triangular face of prism with $\kappa = 1.5$, and

$$(5.117)\; \|\Phi\|_{H^1(\Omega_j)} \leq C\Big(\sum_{0\leq t\leq 1}\frac{h^{t-\frac{1}{2}}}{(p_j+1)^{t-\frac{1}{2}}}\|\phi\|_{H^t(\Gamma_{i,j})} + h^{\frac{3}{2}}(p_j+1)^{-\frac{3}{2}}\Big\|\frac{\partial\phi}{\partial x_3}\Big\|_{H^1(\Gamma_{i,j})}\Big)$$

if Ω_j is a triangular prism with $\kappa = 1.5$ and all $\Gamma_{i,j}$ is its square face.

If Ω_j is tetrahedron or hexahedron or $\Gamma_{i,j}$ is a triangular face of a prism it holds that by Lemma 5.13 and Lemma 5.6 for $t = 0, 1$

$$\begin{aligned}\|q_{i,p} - \tilde{\varphi}\|_{H^t(\Gamma_{i,j})} &\leq \|q_{i,p} - q_i\|_{H^t(\Gamma_{i,j})} + \|\tilde{u} - \tilde{\varphi}\|_{H^t(\Gamma_{i,j})}\\ &\leq C\frac{h^{\mu-\frac{1}{2}-t}}{(p_j+1)^{k-\frac{1}{2}-t}}(\|q\|_{H^{k-\frac{1}{2}}(\Gamma_{i,j})} + \|\tilde{u}\|_{H^k(\Omega_j)}),\end{aligned}$$

by the standard argument of interpolation space for $H^{\frac{1}{2}}_{00}(\Gamma_{ij}) = (L^2(\Gamma_{i,j}, H^1_0(\Gamma_{i,j}))_{\frac{1}{2}}$ there holds

$$\begin{aligned}\|\Phi\|_{H^1(\Omega_j)} &\leq \|q_{i,p} - \tilde{\varphi}\|_{H^{\frac{1}{2}}_{00}(\Gamma_{i,j})} \leq C\frac{h^{\mu-1}}{(p_j+1)^{k-1}}(\|q_i\|_{H^{k-\frac{1}{2}}(\Gamma_{i,j})} + \|\tilde{u}\|_{H^k(\Omega_j)})\\ &\leq C\frac{h^{\mu-1}}{(p_j+1)^{k-1}}\|\tilde{u}\|_{H^k(\Omega_j)}.\end{aligned}$$

If Ω_j is a prism and $\Gamma_{i,j}$ is a square face, we have by Lemma 5.6 and Lemma 5.13 for $t = 0, 1$

$$\begin{aligned}(5.118)\quad \|\Phi\|_{H^1(\Omega_j)} &\leq C\sum_{t=0}^{1}\frac{h^{\mu-\frac{1}{2}}}{(p_j+1)^{t-\frac{1}{2}}}(\|\tilde{u} - \tilde{\varphi}\|_{H^t(\Gamma_{i,j})} + \|q_i - q_{i,p}\|_{H^t(\Gamma_{i,j})})\\ &\quad + h^{\frac{3}{2}}(p_j+1)^{-\frac{3}{2}}\Big\|\frac{\partial\tilde{u} - \tilde{\varphi}}{\partial x_3}\Big\|_{H^1(\Gamma_{i,j})} \leq C\frac{h^{\mu-1}}{(p_j+1)^{k-1}}\|\tilde{u}\|_{H^k(\Omega_j)}.\end{aligned}$$

Let $\varphi_j = \tilde{\varphi}_j + \Phi$ on Ω_j. Then $\varphi_j|_{\Gamma_{i,j}} = q_{i,p}|_{\Gamma i,j}$, and by (5.116) and (5.117)

$$\|\tilde{u} - \varphi_j\|_{H^1(\Omega_j)} \leq C\frac{h^{\mu-1}}{(p_j+1)^{k-1}}\|\tilde{u}\|_{H^k(\Omega_j)}.$$

The treatment $\Gamma_{i,j}$ can be carried out on each face of Ω_j which is on Γ_D. Let $\varphi = \tilde{\varphi}_j + \Phi_j$ on Ω_j if $\partial\Omega_j \cap \Gamma_D \neq \emptyset$ and $\varphi = \tilde{\varphi}_j$ on Ω_j if $\partial\Omega_j \cap \Gamma_D = \emptyset$. Then $\varphi \in S^{P,0}(\Omega;\Delta_h;\mathcal{M})$ and $\varphi|_{\Gamma_D} = q_p$, and satisfies (5.115). A combination of (5.113)-(5.115) leads to (5.112). □

5.4. The convergence for problems on polyhedral domains

In this section we will investigate the optimal convergence of the h-p version of the finite element method for elliptic problems on polyhedral domains by combining the previous results on smooth and singular functions in the framework of Jacobi-weighted Besov and Sobolev spaces.

Let Ω be a polyhedral domain in $\mathbb{R}^3$, shown in Fig. 5.1, and let $\Gamma_i, i \in \mathcal{I} = \{1, 2, 3, \ldots, I\}$, be the faces(open), Λ_{ij} be the edges, which are the intersections of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$, and $A_m, m \in \mathcal{M} = \{1, 2, \ldots, M\}$, be the vertices of Ω. By $\mathcal{J}_m$ we denote a subset $\{j \in \mathcal{J} | A_m \in \bar{\Gamma}_j\}$ of $\mathcal{J}$ for $m \in \mathcal{M}$. Let $\mathcal{L} = \{ij | i, j \in \mathcal{J}, \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$, and let $\mathcal{L}_m$ denote a subset of $\mathcal{L}$ such that $\mathcal{L}_m = \{ij \in \mathcal{L} | A_m \in \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$. We denote by ω_{ij} the interior angle between Γ_i and Γ_j for $ij \in \mathcal{L}$. Let $\Gamma^0 = \cup_{i \in \mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^1 = \cup_{i \in \mathcal{N}} \Gamma_i$, where $\mathcal{D}$ is a subset of $\mathcal{J}$ and $\mathcal{N} = \mathcal{J} \backslash \mathcal{D}$. For $m \in \mathcal{M}, \mathcal{D}_m = \mathcal{D} \cap \mathcal{J}_m$ and $\mathcal{N}_m = \mathcal{N} \cap \mathcal{J}_m$.

In order to study effectively the regularity of the solutions of elliptic problems on a polyhedral domain, we shall decompose the domain into various neighborhoods of low dimensional manifolds.

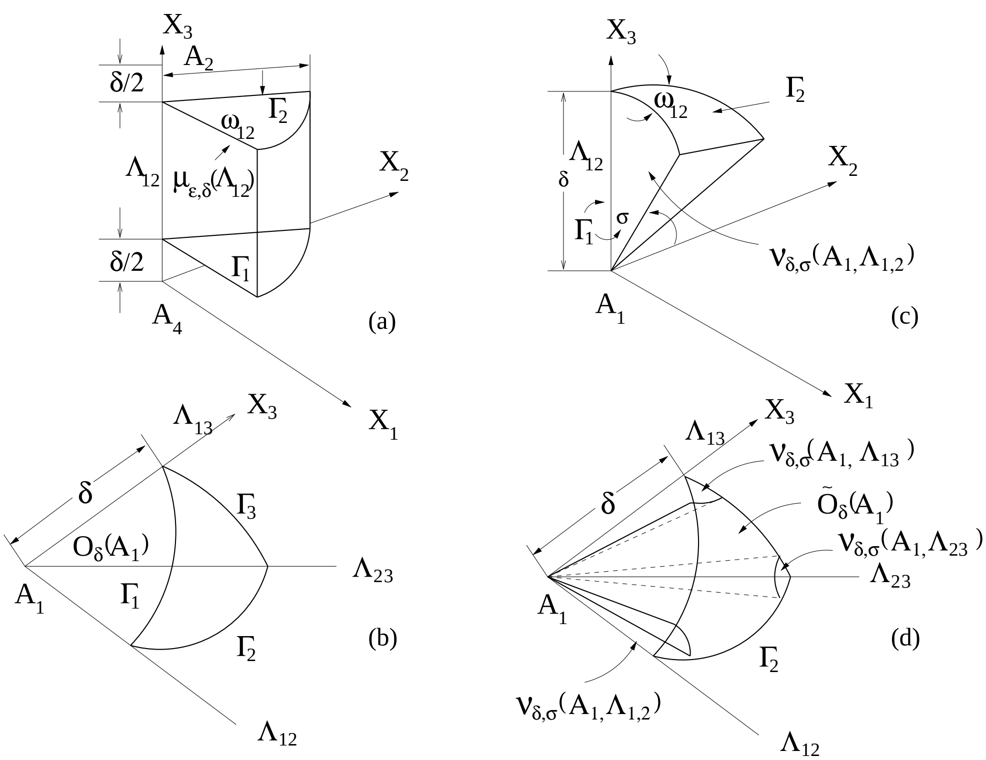


Fig. 5.6 Neighborhoods of edges and vertices:

(a) the neighbourhood $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$; (b) the neighborhood $\mathcal{O}_{\delta_m}(A_m)$;

(c) the neighborhood $\mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij})$; (d) the inner-neighborhood $\tilde{\mathcal{O}}_{\delta_{m(A_m)}}$.

We define a neighborhood $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$ of the edge Λ_{ij}, shown in Fig. 5.6(a), and assume that $\Lambda_{ij} = \{x = (x_1, x_2, x_3) | x_1 = x_2 = 0, a \le x \le b\}$, as follows:

$$\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) = \{x \in \Omega | 0 \le r = dist(x, \Lambda_{ij}) < \varepsilon_{ij}, a + \delta_{ij} < x_3 < b - \delta_{ij}\}.$$

It can be written as $Q_{\varepsilon_{ij}} \times I_{\delta_{ij}}$ with $Q_{\varepsilon_{ij}} = \{(r,\phi)|0 < r < \varepsilon, 0 < \phi < \omega_{ij}\}$ and $I_{\delta_{ij}} = (a+\delta_{ij}, b-\delta_{ij})$, where (r,ϕ,x_3) are cylindrical coordinates with respect to the edge $\Lambda_{ij}, \varepsilon_{ij}$ and δ_{ij} are selected such that $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \cap \bar{\Gamma}_\ell = \emptyset$ for $\ell \in \mathcal{J}, \ell \neq i,j$.

By $\mathcal{O}_{\delta_m}(A_m)$ we denote a neighborhood of the vertex A_m, shown in Fig. 5.6(b),

$$\mathcal{O}_{\delta_m}(A_m) = \{x \in \Omega | 0 < \rho = dist(x, A_m) < \delta_m\}.$$

Here we assume that A_m is in the origin and $0 < \delta_m < 1$ such that $\mathcal{O}_{\delta_m} \cap \bar{\Gamma}_\ell = \emptyset$ for any $\ell \in (\mathcal{J} \backslash \mathcal{J}_m)$. We need further to decompose $\mathcal{O}_{\delta_m}(A_m)$ into several neighborhoods of vertex-edges and an inner-neighborhood of a vertex.

We denote a neighborhood $\mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij})$, shown in Fig. 5.6(c), by

$$\mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}) = \{x \in \mathcal{O}_{\delta_\ell}(A_\ell) | 0 < \phi < \sigma_{ij}\},$$

where ϕ is the angle between the edge $\Lambda_{ij}, ij \in \mathcal{L}_\ell$, and the radial from A_ℓ to the point x. We always assume that the vertex A_ℓ is at the origin and the edge Λ_{ij} lies along the positive x_3- axis. Let (ρ,θ,ϕ) be the spherical coordinates with respect to the vertex A_ℓ and the edge Λ_{ij}, then $\mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}) = S_{\sigma_{ij}} \times I_{\delta_\ell}$ with $I_{\delta_\ell} = (0, \delta_\ell)$ and $S_{\sigma_{ij}} = \{(\phi,\theta)|0 < \theta < \sigma_{ij}, 0 < \phi < \omega_{ij}\}, \sigma_{ij} \in (0, \pi/6)$, is selected such that

$$\bar{\mathcal{V}}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}) \cap \bar{\mathcal{V}}_{\delta_\ell,\sigma_{kl}}(A_\ell, \Lambda_{kl}) = A_\ell, \text{ for all } kl \in \mathcal{L}_\ell, kl \neq ij.$$

Next we define an inner neighborhood $\tilde{\mathcal{O}}_{\delta_\ell}(A_\ell)$ of the vertex A_ℓ by

$$\tilde{\mathcal{O}}_{\delta_\ell}(A_\ell) = \mathcal{O}_{\delta_\ell}(A_\ell) \backslash \bigcup_{ij \in \mathcal{L}_\ell} \bar{\mathcal{V}}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}),$$

which is shown in Fig. 5.6(d).

Let $\Omega_0 = \Omega \backslash \{\cup_{\ell \in \mathcal{M}} \{\mathcal{O}_{\delta_{\ell/2}}(A_\ell) \cup_{ij \in \mathcal{L}_\ell} \mathcal{U}_{\varepsilon_{ij/2},\delta_{ij/2}}(\Lambda_{ij})\}\}$ contains no vertices and edges of Ω, and $\Omega_0 \cap \mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \neq \emptyset$, $\Omega_0 \cap \tilde{\mathcal{O}}_{\delta_\ell}(A_\ell) = \emptyset$, $\Omega_0 \cap \mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}) \neq \emptyset$ for any $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$. Ω_0 is called the regular part of Ω. Meanwhile, we note that $\mathcal{U}_{\varepsilon_{ij/2},\delta_{ij/2}}(\Lambda_{ij}) \cap \mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}) \neq \emptyset$ for $ij \in \mathcal{L}_\ell$ and $\ell \in \mathcal{M}$. For the sake of simplicity, we shall drop the index and denote $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}), \mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij})$ and $\tilde{\mathcal{O}}_{\delta_\ell}(A_\ell)$ by $\mathcal{U}_{ij}$ or $\mathcal{U}(\Lambda_{ij})$, $\mathcal{V}_{\ell,ij}$ or $\mathcal{V}(A_\ell, \Lambda_{ij})$, $\tilde{\mathcal{O}}_\ell$ or $\tilde{\mathcal{O}}(A_\ell)$.

In a neighborhood of vertex A_ℓ, u has an expansion in terms of singular functions of $\rho^\gamma \log^\nu \rho$ -type

$$\begin{aligned} u &= \sum_{m \geq 1, 0 < \gamma_m^{[\ell]} \leq k - \frac{3}{2}} \sum_{j=1}^{L_m^{[\ell]}} C_m^{[\ell]} \rho_\ell^{\gamma_m^{[\ell]}} \log^{\nu_{j,m}^{[\ell]}} \rho_\ell \Phi_m^{[\ell]}(\theta_\ell, \phi_\ell)\, \chi(\rho_\ell) + u_0^{[\ell]} \\ &= u_1^{[\ell]} + u_0^{[\ell]} \end{aligned} \tag{5.119}$$

where $(\rho_\ell, \theta_\ell, \phi_\ell)$ are spherical coordinates with the vertex A_ℓ. $\gamma_m^{[\ell]} > 0$, and $\nu_{j,m}^{[\ell]} \geq 0$ are integers, $u_0^{[\ell]} \in H^k(\tilde{O}_{\delta_\ell})$ is the smooth part of u, $\chi(\rho_\ell)$ and $\Phi_m^{[\ell]}(\theta_\ell, \phi_\ell)$ are C^∞ functions such that $\chi(\rho_\ell) = 1$ for $0 < \rho_\ell < \delta_\ell/2$ and $\chi(\rho_\ell) = 0$ for $\rho_\ell > \delta_\ell$. We assume that $\gamma_m^{[\ell]} < \gamma_{m+1}^{[\ell]}$ and $\nu_{j,m}^{[\ell]} > \nu_{j+1,m}^{[\ell]}$.

In a neighborhood of edge $\Lambda_{ij}, ij \in \mathcal{L}$, u has an expansion in terms of singular functions of $r^\sigma \log^\mu r$ -type

$$
\begin{aligned}
u &= \sum_{m\geq 1, 0<\sigma_m^{[ij]} \leq k-\frac{3}{2}} C_m^{[ij]} r_{ij}^{\sigma_m^{[ij]}} \sum_{l=1}^{L_m^{[ij]}} \log^{\mu_{l,m}^{[ij]}} r_{ij}\ \chi(r_{ij}) \Phi_m^{[ij]}(\phi_{ij}) \Psi^{[ij]}(x_3) + u_0^{[ij]} \\
&= u_1^{[ij]} + u_0^{[ij]}
\end{aligned}
\tag{5.120}
$$

where (r_{ij}, ϕ_{ij}, x_3) are cylindrical coordinates with the edge Λ_{ij}. $\sigma_m^{[ij]} > 0$, and $\mu_{l,m}^{[ij]} \geq 0$ are integers, $u_0^{[ij]} \in H^k(\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}))$ is the smooth part of u, $\chi(r_{ij})$, $\Phi_m^{[ij]}(\phi_{ij})$ and $\Psi^{[ij]}(x_3)$ are C^∞ functions such that $\chi(r_{ij}) = 1$ for $0 < r_{ij} < \varepsilon_{ij}/2$ and $\chi(r_{ij}) = 0$ for $r_{ij} \geq \varepsilon_{ij}$. We assume that $\sigma_m^{[ij]} < \sigma_{m+1}^{[ij]}$ and $\mu_{l,m}^{[ij]} > \mu_{l+1,m+1}^{[ij]}$.

In a neighborhood of vertex-edge A_ℓ-$\Lambda_{ij}, \ell \in \mathcal{M}, ij \in \mathcal{L}_\ell$, u has an expansion in terms of singular functions of $\rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta$ -type

$$
\begin{aligned}
u &= \sum_{0<\gamma_m^{[\ell]} \leq k-\frac{3}{2}} C_{m,j,s,t}^{[\ell,ij]} \rho_m^{\gamma_m^{[\ell]}} \sum_{j=1}^{L_\ell^{[ij]}} \log^{\nu_{j,m}^{[\ell]}} \rho_m \sum_{0<\sigma_s^{[ij]} \leq k-1} (\sin\theta_{ij})^{\sigma_s^{[ij]}} \\
&\quad \sum_{t=1}^{T_s} \log^{\mu_{t,s}^{[ij]}} \sin\theta_{ij}\ \chi(\rho_\ell) \Phi_m^{[\ell,ij]}(\theta_{ij}) \Psi_m^{[\ell,ij]}(\phi_{ij}) + u_0^{[\ell,ij]} \\
&= u_1^{[\ell,ij]} + u_0^{[\ell,ij]}.
\end{aligned}
\tag{5.121}
$$

where $(\rho_m, \theta_{ij}, \phi_{ij})$ are spherical coordinates with the vertex-edge A_ℓ-Λ_{ij}. $\gamma_m^{[\ell]} > 0$, $\nu_{j,m}^{[\ell]} \geq 0$, $\sigma_s^{[ij]} > 0$ and $\mu_{t,s}^{[ij]}$ are integers, $u_0^{[\ell,ij]} \in H^k(\mathcal{V}_{\delta_\ell,\sigma_{ij}}(A_\ell, \Lambda_{ij}))$ is the smooth part of u, $\chi(\rho_\ell)$, $\Phi_m^{[\ell,ij]}(\theta_{ij})$ and $\Psi_m^{[\ell,ij]}(\phi_{ij})$ are C^∞ functions such that $\chi(\rho_\ell) = 1$ for $0 < \rho_\ell < \delta_\ell/2$ and $\chi(\rho_\ell) = 0$ for $\rho_\ell > \delta_\ell$. We assume that $\gamma_m^{[\ell]} < \gamma_{m+1}^{[\ell]}$, $\nu_{j,m}^{[\ell]} > \nu_{j+1,m}^{[\ell]}$, $\sigma_s^{[ij]} < \sigma_{s+1}^{[ij]}$ and $\mu_{t,s}^{[ij]} > \mu_{t+1,s+1}^{[ij]}$.

Let

$$
\gamma = \min_{\ell\in\mathcal{M}} \gamma_1^{[\ell]}, \quad \nu_\gamma = \max_{\ell\in\mathcal{M}, \gamma_1^{[\ell]}=\gamma} \nu_{1,1}^{[\ell]}, \quad \sigma = \min_{ij\in\mathcal{L}} \sigma_1^{[ij]}, \quad \mu_\sigma = \max_{ij\in\mathcal{L}, \sigma_1^{[ij]}=\sigma} \mu_{1,1}^{[ij]}.
\tag{5.122}
$$

We shall analyze the convergence of the *h-p* version of the finite element method with quasi-uniform meshes for problems on polyhedral domains.

Lemma 5.17. *Let Ω_j is a hexahedron or a triangular prism or a tetrahedron element containing a vertex A_ℓ of a polyhedron Ω with size h_j, and the function u has the expansion (5.119) with $u_0^{[\ell]} \in H^k(\mathcal{O}_{\delta_\ell})$ with $k \geq 2 + 2\gamma$. Then there exists a polynomial*

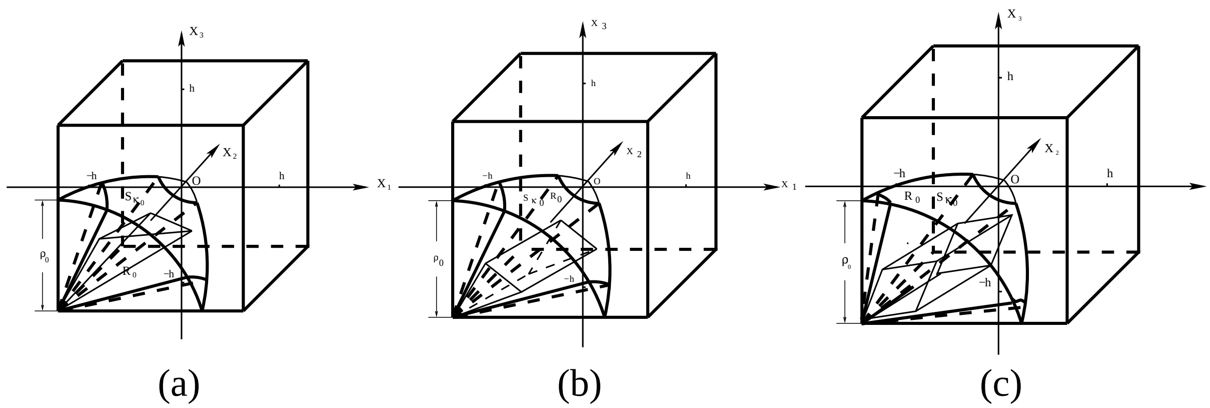


Fig. 5.7 3d elements contained in $R^h_{\rho_0,\kappa_0+\varepsilon} \subset R^h_0$.
(a) A tetrahedron contained in $R^h_{\rho_0,\kappa_0+\varepsilon}$;
(b) A parallel prism contained in $R^h_{\rho_0,\kappa_0+\varepsilon}$; (c) A parallelepiped contained in $R^h_{\rho_0,\kappa_0+\varepsilon}$.

$\varphi_j \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j), p_j \geq 1/2+\gamma, 1 \leq \kappa \leq 2$ such that $\varphi_j = u$ at the vertices of Ω_j and vanishes on $\partial\Omega \cap \Gamma_D$ and and $\varphi_j = 0$ on $\partial\Omega_j \cap \Gamma_D$, and

$$\|u-\varphi_j\|_{H^1(\Omega_j)} \leq C\frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}}\, F_{\nu_\gamma}(p_j,h_j) \tag{5.123}$$

with the constant C depending on u, γ and ν_γ, but not on p_j and h_j, where γ and ν_γ are given in (5.122) and $F_{\nu_\gamma}(p_j,h_j)$ is given as in (3.31).

Proof. We may assume that $\ell = 1$. Then the expansion (5.119) holds in Ω_j. By Theorem 5.4 and Theorem 5.10, there exists a polynomial $\psi_0 \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j), 1 \leq \kappa \leq 2$ such that $\psi_0 = u_0^{[1]}$ at the vertices of Ω_j and vanishes on $\partial\Omega_j \cap \Gamma_D$, and

$$\|u_0^{[1]} - \psi_0\|_{H^1(\Omega_j)} \leq C\,\frac{h_j^{\mu_j-1}}{(p_j+1)^{k-1}}$$

with $\mu_j = \min\{p_j+1, k\} \geq \frac{3}{2}+\gamma$. For a sharp approximation to $u_1^{[1]}$, we map Ω_j into $R^h_0 = R^h_{\rho_0,\kappa_0} \subset Q_h$ by an affine mapping F_j such that $A_1 \circ M_j = (-h,-h,-h)$ and that $\Omega_j \circ M_j$ is contained in R^h_0, as shown in Fig. 5.7. Without losing generality we may assume that $A_1 = (-h,-h,-h)$ and Ω_j is a tetrahedron, or a parallelepiped, or a parallel prism which is contained in $R^h_{\rho_0,\kappa_0+\varepsilon} \subset R^h_0$. We extend the function $\Phi(\theta,\phi)$ analytically on $(0,\pi/2;0,\pi/2)$ such that $[\theta_0, \pi/2-\theta_0; \phi_0, \pi/2-\phi_0]$ is its support. Then $u_1^{[1]} \in B^{2+2\gamma_1^{[1]}}_{\nu^{[1]*}_{1,1}}(Q_h)$ by Theorem 3.7

with

$$\nu_{1,1}^{[1]*} = \begin{cases} \nu_{1,1}^{[1]} \ \text{ if } \gamma_1^{[1]} \text{ is not an integer or } \ \nu_{1,1}^{[1]} = 0, \\ \\ \nu_{1,1}^{[1]} - 1 \ \text{ if } \gamma_1^{[1]} \text{ is an integer and } \ \nu_{1,1}^{[1]} \geq 1, \end{cases}$$

and there exists polynomials $\tilde{\psi}_1^{[1]} \in \mathcal{P}_{p_j}^{\kappa}(\Omega_j), 1 \leq \kappa \leq 2$ due to Theorem 3.8 such that $\tilde{\psi}_1^{[1]} = 0$ on $\partial\Omega_m \cap \Gamma_D$ and

$$\|u_1^{[1]} - \tilde{\psi}_1^{[1]}\|_{H^1(\Omega_j)} \leq C \frac{h_j^{\frac{1}{2}+\gamma_1^{[1]}}}{(p_j+1)^{2\gamma_1^{[1]}+1}} \, F_{\nu_{1,1}^{[1]}}(p_j, h_j) \leq C \frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}} \, F_{\nu_\gamma}(p_j, h_j).$$

We now adjust $\tilde{\psi}_1^{[1]}$ to $\psi_1^{[1]}$ such that $\psi_1^{[1]}$ satisfies (5.124) and $u_1^{[\ell]} = \psi_1^{[1]}$ at the vertices of Ω_j.

We will adjust the polynomial for a hexahedron Ω_j since the adjustment on tetrahedron and prism elements is similar to what follows. To this end, we may assume that Ω_j standard shaped hexahedron $(-h/2, h/2)^3$. Let $\psi_1^{[1]} = \tilde{\psi}_1^{[1]} + \chi(x)$ with $\chi(x) = \sum_{1\leq l\leq 8}(u_1^{[1]} - \tilde{\psi}_1^{[1]})(V_l)\chi_l(x)$ where $V_l, 1 \leq l \leq 8$ are the vertices of Ω_j and $\chi_l(x)$ are defined in (5.21).

Obviously, $\psi_1^{[1]} \in \mathcal{P}_{p_j}^2(\Omega_j)$ and $u_1^{[1]}(V_l) = \psi_1^{[1]}(V_l), 1 \leq l \leq 8$. By (5.20) and (3.30), we have

$$\|\chi_l(x)\|_{H^1(\Omega_j)} \leq C h_j^{\frac{1}{2}}(p_j+1)^{-\frac{1}{2}} \sum_{1\leq l\leq 8} |(u_1^{[\ell]} - \psi)(V_l)| \leq C \frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}} F_{\nu_\gamma}(p_j, h_j),$$

which implies

$$\begin{aligned} \|u_1^{[\ell]} - \psi_1^{[1]}\|_{H^1(\Omega_j)} &\leq \|u_1^{[\ell]} - \tilde{\psi}_1^{[1]}\|_{H^1(\Omega_j)} + \|\chi_l(x)\|_{H^1(\Omega_j)} \\ &\leq C \frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}} F_{\nu_\gamma}(p_j, h_j). \end{aligned} \tag{5.124}$$

Let $\varphi_j = \psi_1^{[1]} + \psi_0^{[1]}$. Then $u_1^{[\ell]} = \varphi_j$ at the vertices of Ω_j and vanishes on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u - \varphi_j\|_{H^1(\Omega_j)} \leq C\Big(\frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}} \, F_{\nu_\gamma}(p_j, h_j) + \frac{h_j^{\mu_j - 1}}{(p_j+1)^{k-1}} \Big) \leq C \frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma+1}} \, F_{\nu_\gamma}(p_j, h_j).$$

$\square$

Lemma 5.18. *Let Ω_m is a hexahedron or a triangular prism or a tetrahedron element of size h_j with a side γ_h on a portion of an edge Λ_{ij} not including its two ending points and the function u has the expansion (5.120) with $u_0^{[ij]} \in H^k(\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}), k \geq 1 + 2\sigma$. Then there exists a polynomial $\varphi_m \in \mathcal{P}_{p_j}^{\kappa}(\Omega_j), p_j \geq 1/2 + \sigma, 1 \leq \kappa \leq 2$ such that $\varphi_m = u$ at the vertices of Ω_m*

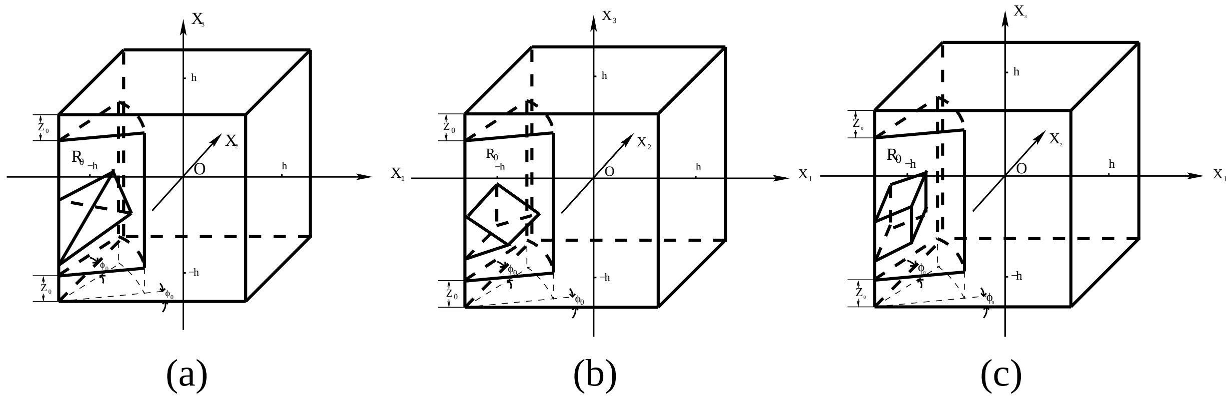


Fig. 5.8 3d elements contained in $R^h_{r_0,\phi_0+\varepsilon,z_0+\delta} \subset R^h_0$.
(a) A tetrahedron contained in $R^h_{r_0,\phi_0+\varepsilon,z_0+\delta}$;
(b) A parallel prism contained in $R^h_{r_0,\phi_0+\varepsilon,z_0+\delta}$; (c) A parallelepiped contained in $R^h_{r_0,\phi_0+\varepsilon,z_0+\delta}$.

and $\varphi_m = 0$ on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u - \varphi_m\|_{H^1(\Omega_m)} \leq C \frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma-1/2}} \, F_{\mu_\sigma}(p_m, h_m) \tag{5.125}$$

with the constant C depending on u, σ and μ_σ, but not on p_m and h_m, where σ and μ_σ are given in (5.122) and $F_{\mu_\sigma}(p_m, h_m)$ is given as in (3.50).

Proof. We may assume that $ij = 12$. (5.120) holds with $ij = 12$ in Ω_m. By Theorem 5.4 and Theorem 5.10, there exists a polynomial $\psi_0^{[12]} \in \mathcal{P}^{\kappa}_{p_m}(\Omega_m), p_m \geq 1/2 + \sigma, 1 \leq \kappa \leq 2$ such that $\psi_0^{[12]} = u_0^{[12]}$ at the vertices of Ω_m and vanishes on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u_0^{[12]} - \psi_0^{[12]}\|_{H^1(\Omega_m)} \leq C \, \frac{h_m^{\mu_j - 1}}{(p_m+1)^{k-1}}$$

with $\mu_m = \min\{p_m + 1, k\} \geq \frac{3}{2} + \sigma$. For a sharp approximation to $u_1^{[12]}$, we map Ω_j into $R^h_0 = R^h_{r_0,\phi_0,z_0} \subset Q_h$ by an affine mapping M_j such that $\gamma_h \circ M_m = \{(-h, -h, x_3), x_3 \in (h - z_0, -h + z_0)\}$ and $\Omega_m \circ M_m$ is contained in R^h_0, as shown in Fig. 5.8. Without loss of generality we may assume that $\gamma_h = \{(-h, -h, x_3), x_3 \in (-h + z_0, h - z_0)\}$ and Ω_j is a tetrahedron, or a parallelepiped, or a parallel prism which is contained in $R^h_{r_0,\phi_0+\varepsilon,z_0+\delta} \subset R^h_0$. Then we extend the function $\Phi(\phi)$ and $\Psi(x_3)$ analytically on $(0, \pi/2)$ and $(0, h)$ such that $[\phi_0, \pi/2 - \phi_0]$ and $[-h + z_0, h - z_0]$ are their supports, respectively. Due to Theorem 3.11

and Theorem 3.12, $u_1^{[ij]} \in B^{1+2\sigma_1^{[1]}}_{\mu_{1,1}^{[1]*}}(Q_h)$ with

$$\mu_{1,1}^{[1]*} = \begin{cases} \mu_{1,1}^{[1]} & \text{if } \sigma_1^{[1]} \text{ is not an integer or } \mu_{1,1}^{[1]} = 0, \\ \\ \mu_{1,1}^{[1]} - 1 & \text{if } \sigma_1^{[1]} \text{ is an integer and } \mu_{1,1}^{[1]} \geq 1, \end{cases}$$

and there exists a polynomial $\tilde{\psi}_1^{[ij]} \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j), p_j \geq 1/2+\sigma, 1 \leq \kappa \leq 2$ such that $\tilde{\psi}_1^{[12]} = 0$ on $\partial\Omega_m \cap \Gamma_D$ and

$$\|u_1^{[12]} - \tilde{\psi}_1^{[12]}\|_{H^1(\Omega_m)} \leq C \, \frac{h_m^{\frac{1}{2}+\sigma_1^{[12]}}}{(p_m+1)^{2\sigma_1^{[12]}}} \, F_{\mu_{1,1}^{[1]}}(p_m, h_m) \leq C \frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma}} \, F_{\mu_\sigma}(p_m, h_m). \tag{5.126}$$

We next adjust $\tilde{\psi}_1^{[12]}$ to $\psi_1^{[12]}$ such that $u_1^{[\ell]} = \psi_1^{[1]}$ at the vertices of Ω_j and $\psi_1^{[1]}$ and satisfies

$$\|u_1^{[12]} - \psi_1^{[12]}\|_{H^1(\Omega_m)} \leq C \frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma}} \, F_{\mu_\sigma}(p_m, h_m). \tag{5.127}$$

We will adjust the polynomial for a prism Ω_j since the adjustment on tetrahedron and hexahedron elements is similar to what follows. To this end, we may assume that Ω_j standard shaped prism $x = (x_1, x_2, x_3) : x_1+x_2 \leq 0, x_i \in (-h/2, h/2), i = 1, 2, 3$. Let $\psi_1^{[1]} = \tilde{\psi}_1^{[1]} + \chi(x)$ with $\chi(x) = \sum_{1\leq l\leq 6}(u_1^{[\ell]} - \phi)(V_l)\chi_l(x)$ where $V_l, 1 \leq l \leq 6$ are the vertices of Ω_m and $\chi_l(x)$ are defined in (5.21) for $p_m \geq 2$ and in (5.24) for $p_m = 1$.

Obviously, $\psi_1^{[12]} \in \mathcal{P}^2_{p_m}(\Omega_m)$, and $u_1^{[12]}(V_l) = \psi_1^{[12]}(V_l), 1 \leq l \leq 6$. By (5.20) and (3.49), we have

$$\|\chi_l(x)\|_{H^1(\Omega_j)} \leq C h_m^{\frac{1}{2}}(p_m+1)^{-\frac{1}{2}} \sum_{1\leq l\leq 6} |(u_1^{[12]} - \tilde{\psi}_1^{[12]})(V_l)| \leq C \frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma+1/2}} F_{\mu_\sigma}(p_m, h_m),$$

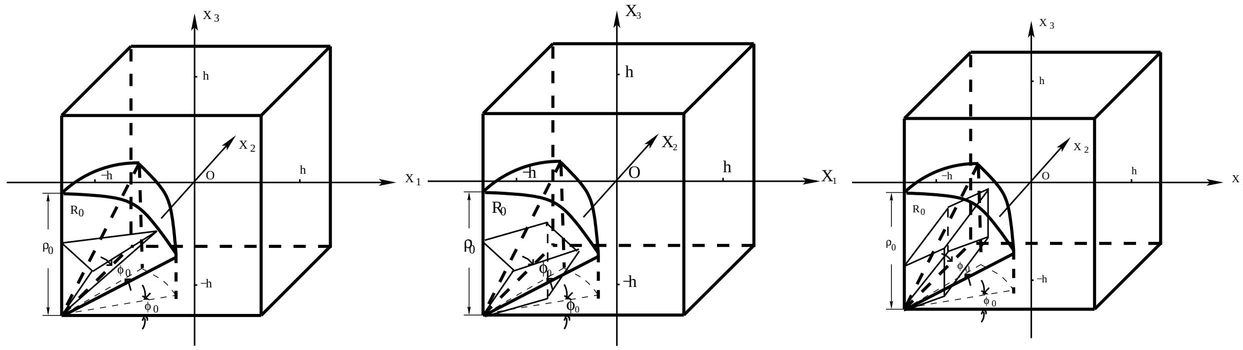
which together with (5.126) implies (5.127).

Let $\varphi_m = \psi_1^{[12]} + \psi_0^{[12]}$. Then $u = \varphi_m$ at the vertices of Ω_m and vanishes on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u - \varphi_m\|_{H^1(\Omega_m)} \leq C\Big(\frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma}} \, F_{\mu_\sigma}(p_m, h_m) + \frac{h_j^{\mu_m - 1}}{(p_m+1)^{k-1}}\Big) \leq C \frac{h_m^{\frac{1}{2}+\sigma}}{(p_m+1)^{2\sigma - 1/2}} \, F_{\mu_\sigma}(p_m, h_m).$$

$\square$

Lemma 5.19. *Let Ω_m is a hexahedron or a triangular prism or a tetrahedron element of size h_m with a vertex A_ℓ and a portion of an edge $\Lambda_{ij}, ij \in \mathcal{L}_\ell$(including the ending point A_ℓ) of Ω as one of its vertex and a side γ_h. Suppose that the function u has the expansion (5.121) with $u_0^{[\ell,ij]} \in H^k(\mathcal{V}_{\delta_\ell,\sigma_{ij}}), k \geq 1 + 2\min\{\gamma + 1/2, \sigma\}$. Then there exists a polynomial*



(a) (b) (c)

Fig. 5.9 3d elements contained in $R^h_{\rho_0,\theta_0+\varepsilon,\phi_0+\varepsilon} \subset R^h_0$.
(a) A tetrahedron contained in $R^h_{\rho_0,\theta_0+\varepsilon,\phi_0+\varepsilon}$;
(b) A parallel prism contained in $R^h_{\rho_0,\theta_0+\varepsilon,\phi_0+\varepsilon}$; (c) A parallelepiped contained in $R^h_{\rho_0,\theta_0+\varepsilon,\phi_0+\varepsilon}$.

$\varphi_m \in \mathcal{P}^\kappa_{p_m}(\Omega_j), p_m \geq 1/2+\gamma, 1 \leq \kappa \leq 2$ such that $\varphi_m = u$ at the vertices of Ω_m and $\varphi_m = 0$ on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u - \varphi_m\|_{H^1(\Omega_j)} \leq C \frac{h_m^{\frac{1}{2}+\gamma}}{(p_m+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}-1/2}} F_{\nu_\gamma,\mu_\sigma}(p_m, h_m) \tag{5.128}$$

with the constant C depending on $u, \gamma, \sigma, \nu_\gamma$ and μ_σ, but not on p_m and h_m, where γ, σ, ν_γ and μ_σ are given in (5.122) and $F_{\nu_\gamma,\mu_\sigma}(p_m, h_m)$ is given as in (3.67).

Proof. We may assume that $\ell = 1, ij = 23$. Then (5.121) holds with $m = 1, ij = 23$ in Ω_m. By Theorem 5.4 and Theorem 5.10, there exists a polynomial $\psi_0^{[1,23]} \in \mathcal{P}^\kappa_{p_m}(\Omega_m), p_m \geq 1/2+\gamma, 1 \leq \kappa \leq 2$ such that $\psi_0^{[1,23]} = u_0^{[1,23]}$ at the vertices of Ω_m and vanishes on $\partial\Omega_m \cap \Gamma_D$, and

$$\|u_0^{[1,23]} - \psi_0^{[1,23]}\|_{H^1(\Omega_m)} \leq C\ \frac{h_j^{\mu_m-1}}{(p_m+1)^{k-1}}$$

with $\mu_m = \min\{p_m+1, k\} \geq \frac{3}{2}+\gamma$. For a sharp approximation to $u_1^{[1,23]}$, we map Ω_m into $R^h_0 = R^h_{\rho_0,\theta_0,\phi_0} \subset Q_h$ by an affine mapping M_j such that $A_1 \circ M_j = (-h,-h,-h), \gamma_h \circ M_j = \{x = (-h,-h,x_3), x_3 \in (-h, h-\delta_1)\}$ as shown in Fig. 5.9. Without loss of generality we may assume that $A_1 = (-h,-h,-h)$, $\gamma_h = \{x = (-h,-h,x_3), x_3 \in (-h, h-\delta_1)\}$, and Ω_j is a tetrahedron, or a parallelepiped, or a parallel prism which is contained in $R^h_{\rho_0,\theta_0+\varepsilon,\phi_0+\varepsilon} \subset R^h_0$. Then we extend the function $\Phi(\phi)$ and $\Psi(\theta)$ analytically on $(0,\pi/2)$ and $(0,\pi/2)$ such that $[\theta_0, \pi/2-\theta_0]$ and $[\phi_0, \pi/2-\phi_0]$ are their support. Due to Theorem 3.15 and Theorem 3.16,

$u_1^{[1,23]} \in B_\lambda^{1+2\min\{\gamma_1^{[1]}+1/2,\sigma_1^{[23]}\},\beta}$ with $\beta = (-1/2,-1/2,0,-1/2,-1/2,0)$ and

$$\lambda = \begin{cases} \mu_{1,1}^{[23]} & \text{if } \sigma < \gamma_1^{[1]} + 1/2, \\ \mu_{1,1}^{[23]} + \nu_{1,1}^{[1]} + 1/2 & \text{if } \sigma = \gamma_1^{[1]} + 1/2, \\ \mu_{1,1}^{[23]} + \nu_{1,1}^{[1]} & \text{if } \sigma_1^{[23]} > \gamma_1^{[1]} + 1/2, \end{cases} \tag{5.129}$$

and there exists a polynomial $\tilde{\psi}_1^{[1,23]} \in \mathcal{P}_{p_m}^{\kappa}(\Omega_j), p_m \geq 1/2+\gamma, 1 \leq \kappa \leq 2$ such that $\tilde{\psi}_1^{[1,23]} = 0$ on $\partial\Omega_m \cap \Gamma_D$ and

$$\begin{aligned} \|u_1^{[1,23]} - \tilde{\psi}_1^{[1,23]}\|_{H^1(\Omega_m)} &\leq C \frac{h_m^{\frac{1}{2}+\gamma_1^{[1]}}}{(p_m+1)^{2\min\{\gamma_1^{[1]}+\frac{1}{2},\sigma_1^{[23]}\}}} F_{\nu_{1,1}^{[1]},\mu_{1,1}^{[23]}}(p_m,h_m) \\ &\leq C \frac{h_m^{\frac{1}{2}+\gamma}}{(p_m+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}}} F_{\nu_\gamma,\mu_\sigma}(p_m,h_m). \end{aligned} \tag{5.130}$$

We need to adjust $\tilde{\psi}_1^{[12]}$ to $\psi_1^{[12]}$ such that $u_1^{[1,23]} = \psi_1^{[1,23]}$ at the vertices of Ω_m and satisfies

$$\|u_1^{[1,23]} - \psi_1^{[1,23]}\|_{H^1(\Omega_m)} \leq C \frac{h_m^{\frac{1}{2}+\gamma}}{(p_m+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}-1/2}} F_{\nu_\gamma,\mu_\sigma}(p_m,h_m). \tag{5.131}$$

We will adjust the polynomial for on a tetrahedron Ω_m since the adjustment on prism and hexahedral elements is similar. To this end, we may assume that Ω_j is a standard shaped tetrahedron $\{x = (x_1,x_2,x_3) : x_1+x_2+x_3 \leq 0, x_i \in (-h/2,h/2), i = 1,2,3\}$. Let $\psi_1^{[1,23]} = \tilde{\psi}_1^{[1,23]} + \chi(x)$ with $\chi(x) = \sum_{1\leq l\leq 4}(u_1^{[1,23]} - \tilde{\psi}_1^{[1,23]})(V_l)\chi_l(x)$ where $V_l, 1 \leq l \leq 4$ are the vertices of Ω_m and $\chi_l(x)$ are defined as in (5.21) with replacing p by $[\frac{p}{3}]$ for $p_m \geq 3$ and in (5.23) for $p_m < 3$.

Obviously, $\psi_1^{[1,23]} \in \mathcal{P}_{p_m}^2(\Omega_m)$ and $u_1^{[1,23]}(V_l) = \psi_1^{[1,23]}(V_l), 1 \leq l \leq 4$. By (5.20) and (3.66), we have

$$\begin{aligned} \|\chi_l(x)\|_{H^1(\Omega_j)} &\leq C h_m^{\frac{1}{2}}(p_m+1)^{-\frac{1}{2}} \sum_{1\leq l\leq 4} |(u_1^{[12]} - \tilde{\psi}_1^{[12]})(V_l)| \\ &\leq C \frac{h_m^{\frac{1}{2}+\gamma}}{(p_m+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}-1/2}} F_{\nu_\gamma,\mu_\sigma}(p_m,h_m), \end{aligned}$$

which together with (5.130) implies (5.131).

Let $\varphi_m = \psi_1^{[1,23]} + \psi_0^{[1,23]}$. Then $u = \varphi_m$ at the vertices of Ω_m and vanishes on $\partial\Omega_m \cap \Gamma_D$, and

$$\begin{aligned}\|u-\varphi_m\|_{H^1(\Omega_m)} &\leq C\Big(\frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}-1/2}}\,F_{\nu_\gamma,\mu_\sigma}(p_m,h_m) + \frac{h_m^{\mu_m-1}}{(p_m+1)^{k-1}}\Big)\\ &\leq C\frac{h_m^{\frac{1}{2}+\gamma}}{(p_m+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}-1/2}}\,F_{\nu_\gamma,\mu_\sigma}(p_m,h_m).\end{aligned}$$

□

Theorem 5.20. *Let $\Delta_h = \{\Omega_j,\ 1 \leq j \leq J\}$ be in a family of quasi-uniform meshes with element size h over Ω containing hexahedral, triangular prism and tetrahedral elements, and let $S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ be the finite element space associated with quasi-uniform element degree distributions $P = \{p_1, p_2, \cdots, p_J\}$. The data functions f and g are assumed such that the solution u of (5.1) is in $H^k(\Omega_0)$ with $k \geq 1 + 2\min\{\gamma+\frac{1}{2},\sigma\}$, and u has the expansion (5.119),(5.120) and (5.121) in vertex neighborhoods $\tilde{\mathcal{O}}(A_\ell)$, in edge neighborhoods $\mathcal{U}(\Lambda_{ij})$ and in vertex-edge neighborhoods $\mathcal{V}(A_\ell,\Lambda_{ij})$ with $u_0^{[\ell]} \in H^k(\tilde{\mathcal{O}}(A_\ell)$, $u_0^{[ij]} \in H^k(\mathcal{U}(\Lambda_{ij}))$, and $u_0^{[\ell,ij]} \in H^k(\mathcal{V}(A_\ell,\Lambda_{ij}))$. Then the finite element solution $u_{hp} \in S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ with $p = \min_{1\leq j\leq J} p_j \geq 1/2 + \max\{\gamma,\sigma\}$ and $h = \max_{1\leq j\leq J} h_j$ for the problem (5.1) satisfies*

$$\|u-u_{hp}\|_{H^1(\Omega)} \leq C\frac{h^{\min\{\gamma+\frac{1}{2},\sigma\}}}{(p+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}}}\,F_{\nu_\gamma,\mu_\sigma}(p,h) \tag{5.132}$$

with the constant C depending on $\gamma, \sigma, \nu_\gamma, \mu_\sigma$ and u, but not on p and h, where $\gamma, \sigma, \nu_\gamma, \mu_\sigma$ are given in (5.122) and $F_{\nu_\gamma,\mu_\sigma}(p,h)$ is given as in (3.67).

Proof. Due to (5.74), it suffices to construct a piecewise polynomial $\varphi \in S_D^{P,1}(\Omega;\Delta_h;\mathcal{M})$ with $p \geq \frac{1}{2} + \max\{\gamma,\sigma\}$ such that

$$\|u-\varphi\|_{H^1(\Omega)} \leq C\,\frac{h^{\min\{\gamma+\frac{1}{2},\sigma\}}}{(p+1)^{2\min\{\gamma+\frac{1}{2},\sigma\}}}\,F_{\nu_\gamma,\mu_\sigma}(p,h). \tag{5.133}$$

If elements Ω_j contain no vertices and edges of Ω, then $\Omega_j \subset \Omega_0$ and $u \in H^k(\Omega_j)$ with $k \geq 1 + 2\min\{\gamma+\frac{1}{2},\sigma\}$. By Theorem 5.4 and the argument on homogeneous Dirichlet boundary condition of Theorem 5.10 there exists a polynomial $\varphi_j \in \mathcal{P}_{p_j}^{\kappa}(\Omega_j)$ with $p_j \geq \frac{1}{2} + \max\{\gamma,\sigma\}$ and $\kappa = 1$, or 2, or 1.5 such that $u = \varphi_j$ at the vertices of Ω_j and vanishes on $\partial\Omega_j \cap \Gamma_D$, and

$$\|u-\varphi_j\|_{H^1(\Omega_j)} \leq C\,\frac{h_j^{\mu_j-1}}{(p_j+1)^{k-1}} \leq C\,\frac{h^{\min\{\gamma+\frac{1}{2},\sigma\}}}{(p+1)^{2\min(\gamma+\frac{1}{2},\sigma)}}. \tag{5.134}$$

If elements Ω_j contain a vertex A_ℓ of Ω as one of its vertices, by Lemma 5.17, there exists a polynomial $\varphi_j \in \mathcal{P}_{p_j}^{\kappa}(\Omega_j)$ with $p_j \geq \frac{1}{2} + \gamma$ and $\kappa = 1$, or 2, or 1.5 such that $u = \varphi_j$ at the vertices of Ω_j and vanishes on $\partial\Omega_j \cap \Gamma_D$, and

$$\|u-\varphi_j\|_{H^1(\Omega_j)} \leq C\,\frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\gamma}} F_{\nu_\gamma}(h_j,p_j) \leq C\,\frac{h^{\frac{1}{2}+\gamma}}{(p+1)^{2\gamma}} F_{\nu_\gamma}(h,p). \tag{5.135}$$

If elements Ω_j contain a potion of an edge Λ_{ij} (not including its ending points) as one of sides, then by Lemma 5.18, there exists a polynomial $\varphi_j \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j)$ with $p_j \geq \frac{1}{2}+\sigma$ and $\kappa = 1$, or 2, or 1.5 such that $u = \varphi_j$ at the vertices of Ω_j and vanishes on $\partial\Omega_j \cap \Gamma_D$, and

$$\|u-\varphi_j\|_{H^1(\Omega_j)} \leq C\,\frac{h_j^{\frac{1}{2}+\sigma}}{(p_j+1)^{2\sigma}}\, F_{\mu_\sigma}(h_j,p_j) \leq C\,\frac{h^{\frac{1}{2}+\sigma}}{(p+1)^{2\sigma}}\, F_{\mu_\sigma}(h,p). \tag{5.136}$$

Note that the number of elements containing a portion of edges of the polyhedron is of $O\left(\frac{1}{h}\right)$. Hence

$$\sum_{\Omega_i \text{ containing edges}} \|u-\varphi_j\|^2_{H^1(\Omega_j)} \leq C\,\frac{h^{2\sigma}}{(p+1)^{4\sigma}}\, F^2_{\mu_\sigma}(h,p). \tag{5.137}$$

If elements Ω_j contain a vertex A_ℓ and a potion of an edge Λ_{ij} (including its ending point A_ℓ), then by Lemma 5.19, there exists a polynomial $\varphi_j \in \mathcal{P}^{\kappa}_{p_j}(\Omega_j)$ with $p_j \geq \frac{1}{2}+\gamma$ and $\kappa = 1$, or 2, or 1.5 such that $u = \varphi_j$ at the vertices of Ω_j and vanishes on $\partial\Omega_j \cap \Gamma_D$, and

$$\begin{aligned} \|u-\varphi_j\|_{H^1(\Omega_j)} &\leq C\,\frac{h_j^{\frac{1}{2}+\gamma}}{(p_j+1)^{2\min(\gamma+\frac{1}{2},\sigma)}}\, F_{\nu_\gamma,\mu_\sigma}(h_j,p_j) \\ &\leq C\,\frac{h^{\frac{1}{2}+\gamma}}{(p+1)^{2\min(\gamma+\frac{1}{2},\sigma)}}\, F_{\nu_\gamma,\mu_\sigma}(h,p). \end{aligned} \tag{5.138}$$

We can further adjust φ_j as in the proof of Theorems 5.5-5.7 and Theorem 5.10 to achieve the continuity across the interfaces of elements.

Let $\varphi = \varphi_j$ on each Ω_j, $1 \leq j \leq J$. Then $\varphi \in S^{P,1}_D(\Omega;\Delta_h;\mathcal{M})$ and vanishes on Γ_D, and a combination of (5.134)-(5.138) leads to (5.133) immediately.

Remark 5.2. The theorem can be generalized for elliptic problem (5.1) with non-homogeneous Dirichlet boundary condition $q = u\,|_{\Gamma_D}$ if $q \in H^{k-\frac{1}{2}}(\Gamma_i) \cap C^0(\Gamma_D)$, $i \in \mathcal{D}$. The arguments for Theorem 5.16 can be carried out here easily.

□

CHAPTER 6

Computation and Applications of the h-p version of Finite Element Solutions

In this chapter, we will present and analyze numerical results for problems with singular solutions in three dimensions. We selected the Poission equation and linear elasticity problems on polyhedral domains as the models problems. We will compare the performance of the h, p and h-p version of the finite element method for these problems, and verify the theoretical predictions given in previous chapters by the numerical computation.

6.1. Poission equation on polyhedral domains

In this section, we will solve the three dimensional Poission problems with edge and vertex-edge singularities by the finite element method. We denote the exact solution and the finite element solution of the problem (6.1) (or (6.6)) by u and u_{FE}, respectively. Let $e = u - u_{FE}$ be the error in the finite element solution. Then the energy norm of error

$$\|e\|_E^2 = E(u) - E(u_{FE}) = \frac{1}{2}B(u,u) - \frac{1}{2}B(u_{FE},u_{FE})$$

and the relative error in energy norm is defined as

$$\|e\|_{E,R} = \frac{\|e\|_E}{\|u\|_E}100\% = \frac{\|e\|_E}{E(u)^{\frac{1}{2}}}100\%.$$

6.1.1. Poission equation with edge singularity. We solve first the following Poission equation with edge singularity:

$$(6.1)\qquad \left\{\begin{array}{rcl} -\triangle u &=& 0 \qquad \text{in } D = [-1,1]\times[0,1]\times[0,1], \\ u\big|_{\Gamma_2} &=& 0, \\ \frac{\partial u}{\partial n}\big|_{\Gamma_1} &=& \frac{\partial u}{\partial n}\big|_{\Gamma_4} = \frac{\partial u}{\partial n}\big|_{\Gamma_7} = 0, \\ \frac{\partial u}{\partial n}\big|_{\Gamma_3} &=& \frac{1}{2}r^{-\frac{3}{2}}(\sin\frac{\theta}{2} + r\sin\theta\cos\frac{\theta}{2}), \\ \frac{\partial u}{\partial n}\big|_{\Gamma_6} &=& \frac{1}{2}r^{-\frac{3}{2}}(\sin\frac{\theta}{2} - r\sin\theta\cos\frac{\theta}{2}), \\ \frac{\partial u}{\partial n}\big|_{\Gamma_5} &=& \frac{1}{2}r^{-\frac{3}{2}}(\sin\frac{\theta}{2} + r\cos\theta\cos\frac{\theta}{2}), \end{array}\right.$$

where (r,θ,x_3) is the cylindric

$$r = \sqrt{x_1^2 + x_2^2},\ \theta = \arcsin\frac{x_2}{r},$$

the domain D is a cuboid shown in Fig. 6.1 , and $\Gamma_1, \Gamma_2, \cdots, \Gamma_7$ are faces of D, $\Gamma_1 : x_3 = 0$, $\Gamma_4 : x_3 = 1$; $\Gamma_2 : x_2 = 0, 0 \le x_1 \le 1$, $\Gamma_7 : x_2 = 0, -1 \le x_1 \le 0$, $\Gamma_5 : x_2 = 1$; $\Gamma_3 : x_1 = -1$, $\Gamma_6 : x_1 = 1$.

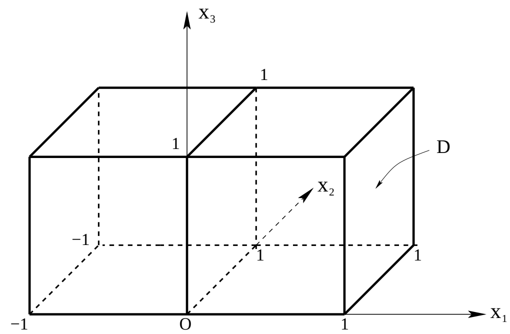


Fig. 6.1 Domain D for 3d Poisson problem

It is known that

$$u = r^{\frac{1}{2}} \sin \frac{\theta}{2}.$$

is the exact solution of the problem (6.1), and the strain energy of u.

$$\begin{aligned} E(u) &= \frac{1}{2} B(u, u) = \frac{1}{2} \int_D \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 + \left(\frac{\partial u}{\partial x_3} \right)^2 \right] dx_1 dx_2 dx_3 \\ &= \frac{1}{8} \int_D r^{-1} r dr d\theta \approx 0.4406868. \end{aligned}$$

The finite element solution of the h, p and h-p version computed on uniform meshes $A_n, 1 \le n \le 3$ with $h = 1, 1/2$ and $1/4$ shown in Fig. 6.2.

The solution has the singularity of r^σ-type with $\sigma = 1/2$. According to Theorem 5.20, there holds for the solution of h-p version

$$\|e\|_{E,R} \le C \frac{h^{\frac{1}{2}}}{p}. \tag{6.2}$$

The p version is a special case of the h-p version with fixed h. Then error estimation is deduced to

$$\|e\|_{E,R} \le C p^{-1} \approx C N^{-\frac{1}{3}}. \tag{6.3}$$

Hereafter N denotes the number of degrees of freedom for finite element spaces, which the the number of unknown of the resulting system of linear algebraic equations.

Table 6.1−Table 6.3 show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α for the p version on each fixed mesh A_n. The relationship are plotted in Fig. 6.5. Due to (6.3) the

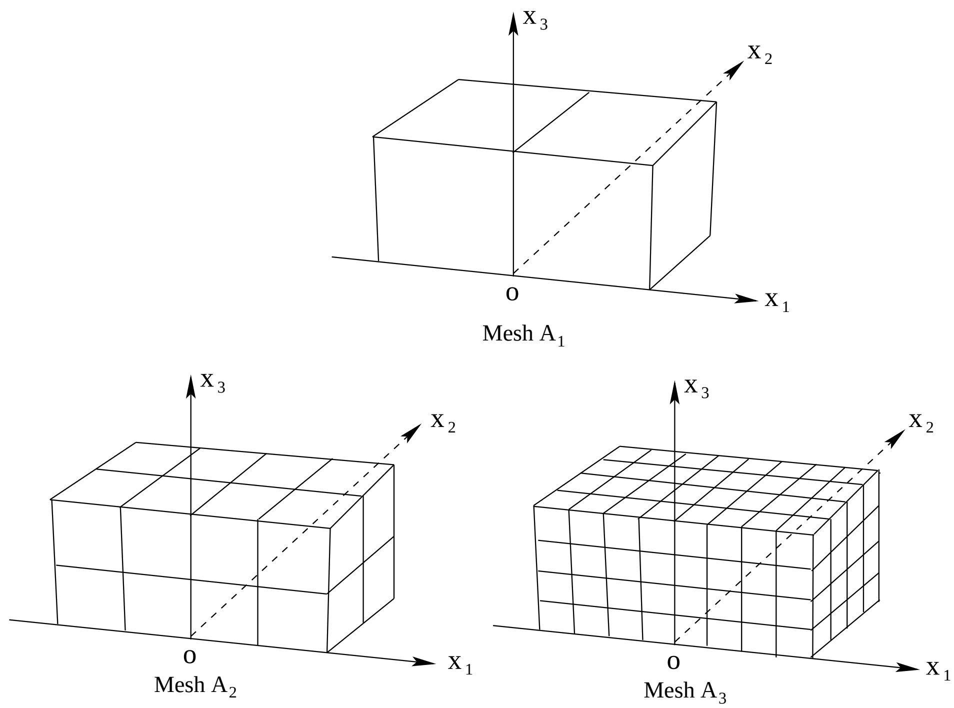


Fig. 6.2 Uniform meshes $A_n, 1 \leq n \leq 3$ with $h = 1, 1/2, 1/4$

rate of convergence is $\frac{1}{3}$. We calculate the convergence rate for each finite element of the p version by the extrapolation [**45**],

$$\alpha = \frac{\log \frac{E(u)-E(u_p)}{E(u)-E(u_{p+1})}}{2 \log \frac{N_{p+1}}{N_p}}. \tag{6.4}$$

where u_p is finite element solution of the p version with degree p and N_p denotes the corresponding total number of the freedom.

From Table 6.1−Table 6.3 we can see that α asymptotically approaches to the theoretical value 1/3 as the degree p increases and the meshes are refined. The computation indicates that the convergence rate of the p version with the degree $1 \leq p \leq 8$ on these uniform meshes associated is in pre-asymptotical range and higher than asymptotical value.

Table 6.1 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity on Mesh $A_1, h = 1$

P	1	2	3	4	5	6	7	8
N	8	24	40	66	102	150	212	290
$E(u)$	0.3772212	0.4074706	0.4107365	0.4192947	0.4256659	0.4296458	0.4322513	0.4340413
α		0.2946719	0.10130499	0.33599666	0.40611584	0.39908853	0.38902423	0.38064655
$\|e\|_{E,R}\%$	37.94934	27.45427	26.06967	22.03239	18.46218	15.82848	13.83536	12.28001

Table 6.2 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity on Mesh $A_2, h = 1/2$

P	1	2	3	4	5	6	7	8
N	36	120	204	352	564	856	1244	1744
$E(u)$	0.4043461	0.4221298	0.4234627	0.4292477	0.4328495	0.4350002	0.4363744	0.4373069
α		0.2791140	0.0702352	0.3751273	0.4010654	0.3844313	0.3699912	0.3605875
$\|e\|_{E,R}\%$	28.71651	20.52054	19.76984	16.11131	13.33577	11.35956	9.892237	8.757636

Table 6.3 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity on Mesh $A_3, h = 1/4$

P	1	2	3	4	5	6	7	8
N	200	720	1240	2208	3624	5616	8312	11840
$E(u)$	0.4215796	0.4312682	0.4319313	0.4349163	0.4367471	0.437833	0.4385247	0.4389933
α		0.27611845	0.0671471	0.3613006	0.3851270	0.3680625	0.3539725	0.3452432
$\|e\|_{E,R}\%$	20.82253	14.61935	14.09534	11.44305	9.455109	8.047236	7.004431	6.199085

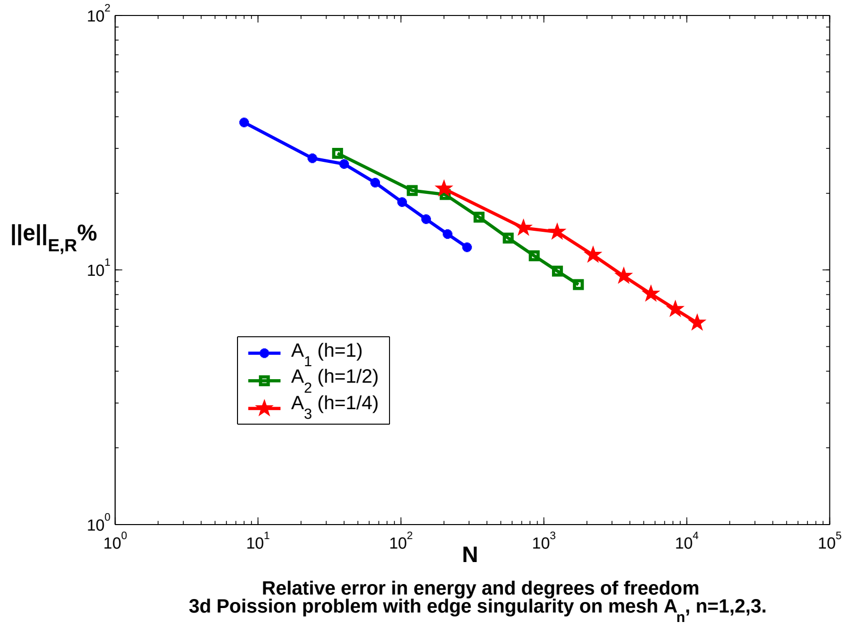


Fig. 6.3 Relative error in the p version between energy norm and degrees of freedom for 3d Poission problem with edge singularity on uniform meshes

The h version is a special case of the h-p version with fixed p. Then error estimation (6.2) is deduced to

$$\|e\|_{E,R} \leq Ch^{\frac{1}{2}} \approx CN^{-\frac{1}{6}}. \tag{6.5}$$

The convergence rate α for the h version is $\frac{1}{6}$. We calculate the rate for each solution of the h version by the extrapolation similar with (6.4) except replacing u_p by u_n and N_p by N_n, $1 \leq n \leq 3$.

The Table 6.4−Table 6.11 describe the performance of the h version on the uniform meshes $A_n, 1 \leq n \leq 3$ with fixed degree p, $1 \leq p \leq 8$, which show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α for the h version. The relationship is plotted in Fig. 6.4. From Table 6.4−Table 6.11 we can see that α asymptotically approaches its theoretical value 0.1666667 as the mesh is refined and the degree increases.

Table 6.4 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 1$

h	1	1/2	1/4
N	8	36	200
$E(u)$	0.3772212	0.4043461	0.4215796
α		0.1853494	0.187448591
$\|e\|_{E,R}\%$	37.94934	28.71651	20.82253

Table 6.5 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 2$

h	1	1/2	1/4
N	24	120	720
$E(u)$	0.4074706	0.4221298	0.4312682
α		0.1808677	0.1892443
$\|e\|_{E,R}\%$	27.45427	20.52054	14.61935

Table 6.6 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 3$

h	1	1/2	1/4
N	40	204	1240
$E(u)$	0.4107365	0.4234627	0.4319313
α		0.1697815	0.1874577
$\|e\|_{E,R}\%$	26.06967	19.76984	14.09534

Table 6.7 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 4$

h	1	1/2	1/4
N	66	352	2208
$E(u)$	0.4192947	0.4292477	0.4349163
α		0.1869753	0.1863289
$\|e\|_{E,R}\%$	22.03239	16.11131	11.44305

Table 6.8 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 5$

h	1	1/2	1/4
N	102	564	3624
$E(u)$	0.4256659	0.4328495	0.4367471
α		0.19020971	0.1848620
$\|e\|_{E,R}\%$	18.46218	13.33577	9.455109

Table 6.9 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 6$

h	1	1/2	1/4
N	150	856	5616
$E(u)$	0.4296458	0.4350002	0.437833
α		0.1904828	0.1832596
$\|e\|_{E,R}\%$	15.82848	11.35956	8.047236

Table 6.10 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 7$

h	1	1/2	1/4
N	212	1244	8312
α		0.1895886	0.1817485
$E(u)$	0.4322513	0.4363744	0.4385247
$\|e\|_{E,R}\%$	13.83536	9.892237	7.004431

Table 6.11 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d Poisson problem with edge singularity for $p = 8$

h	1	1/2	1/4
N	290	1744	11840
$E(u)$	0.4340413	0.4373069	0.4389933
α		0.1884261	0.1804020
$\|e\|_{E,R}\%$	12.28001	8.757636	6.199085

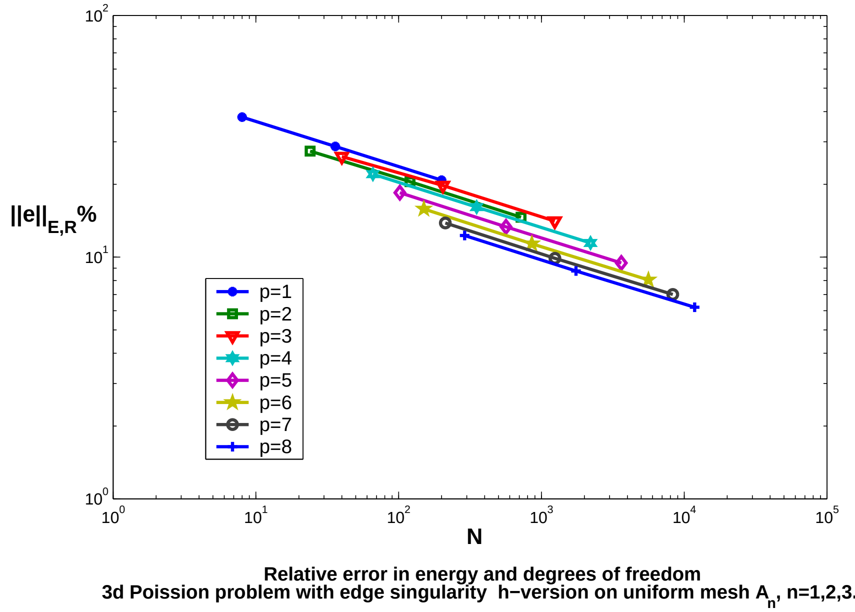


Fig. 6.4 Relative error in the h version between energy norm and degrees of freedom for 3d Poission problem with edge singularity on uniform meshes A_n, $1 \leq n \leq 3$.

The h-p version reduces the element size h and increase the degree p simultaneously to reduce quickly the error by less computation. The performance of the h-p version on uniform meshes $A_n, n = 1, 2, 3$ associated with $p = 2, 3, 4$ is given in Table 6.12 and Fig. 6.7.

Table 6.12 Relationship in the h-p version between $\|e\|_{E,R}, p, N, E(u)$ for 3d Poisson problem with edge singularity on uniform mesh A_n.

Mesh	h	p	N	$E(u)$	α	$\|e\|_{E,R}\%$
A_1	1	2	24	0.4074706		27.45
A_2	$\frac{1}{2}$	3	204	0.4234627	0.1534364	19.77
A_3	$\frac{1}{4}$	4	2208	0.4349163	0.2295714	11.44

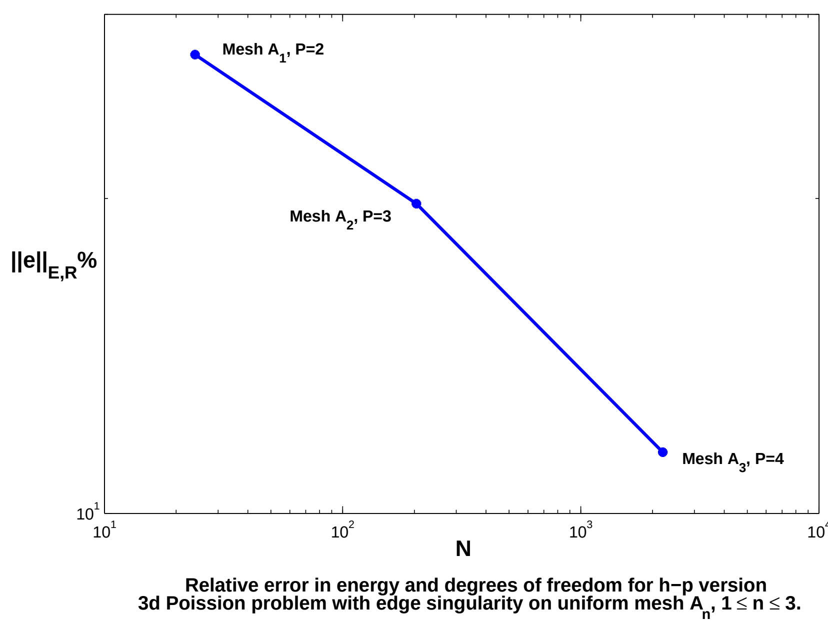


Fig. 6.5 Relative error in the h-p version between energy norm and degrees of freedom for 3d Poission problem with edge singularity on uniform meshes

The computation results in Table 6.1−Table 6.12 show that the theory of finite element method for edge-singular functions is very reliable and coincides with the computation.

The computations are done also on geometric meshes $B_n, 1 \leq n \leq 6$ with mesh factor $\sigma = 0.15$ shown in Fig. 6.3. and Fig. 6.4.

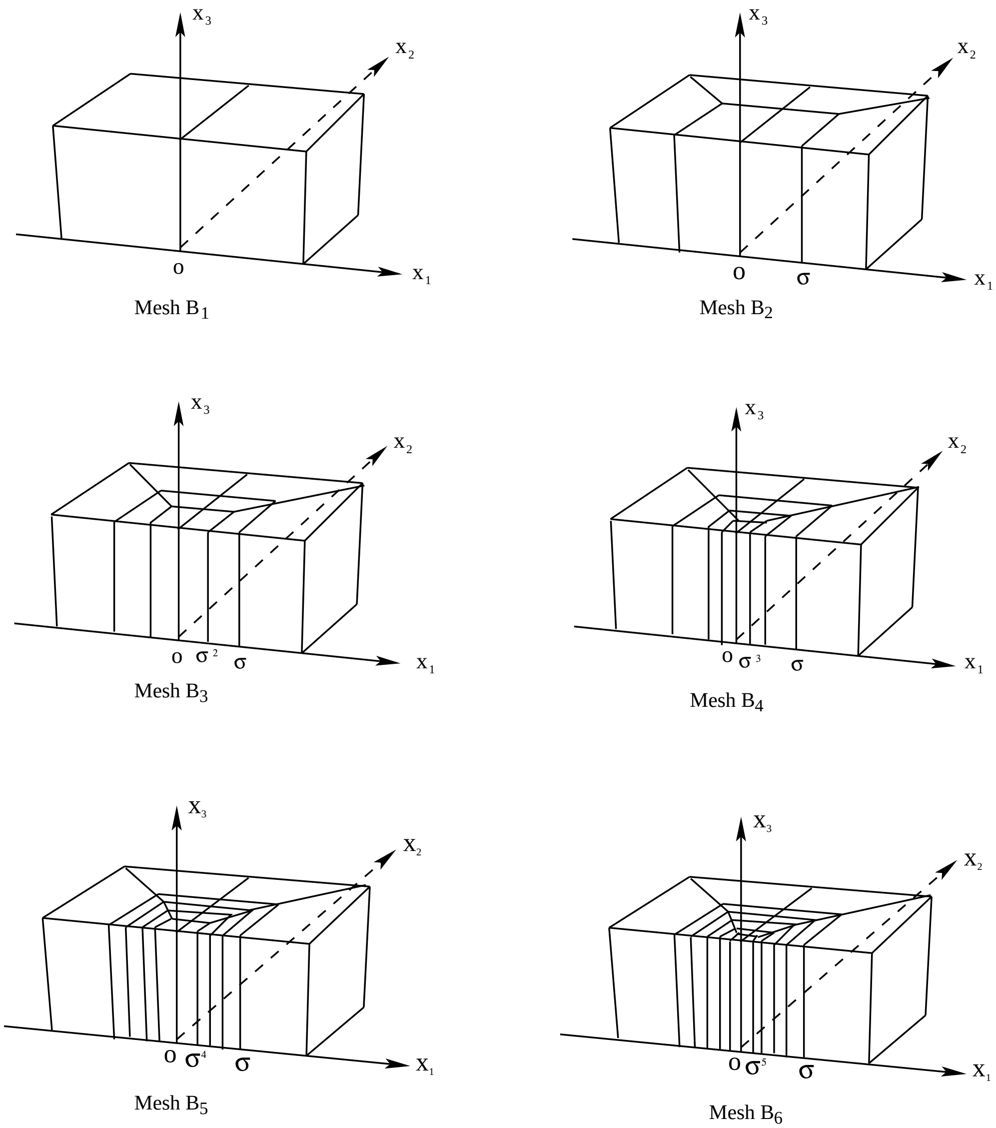


Fig. 6.6 Geometric meshes $B_n,\ 1 \leq n \leq 6$

Table 6.13−Table 6.17 show the relationship between $\|e\|_{E,R}$ and p,N,$E(u)$,α for p version. The relationship are plotted in Fig. 6.8. Due to (6.3), the rate of convergence is asymptotically $\alpha = \frac{1}{3} \approx 0.3333333$. Table 6.13−Table 6.17 and Fig. 6.8. indicate that the p version on the geometric meshes $B_n, 1 \leq n \leq 6$ with increasing p perform in pre-asymptotic range and α calculated by (6.4) is bigger than the asymptotic value, which make the p

version on the geometric meshes B_n more effective than on uniform meshes A_n. The over-refined meshes balance the strong edge singularity on the x_3-axis where the elements size is reduced dramatically to 0.15^{n-1} and delay significantly entering the asymptotical range of the p version.

Table 6.13 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_2

P	1	2	3	4	5	6	7	8
N	16	52	88	150	238	358	516	718
$E(u)$	0.4127698	0.4330268	0.4349613	0.4371067	0.4382714	0.4389508	0.4393763	0.4396627
α		0.5486015	0.2766451	0.4402217	0.4262301	0.4044964	0.3845665	0.3732178
$\|e\|_{E,R}\%$	25.1692	13.18407	11.39834	9.013274	7.403371	6.276389	5.453226	4.820656

Table 6.14 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_3

P	1	2	3	4	5	6	7	8
N	24	80	136	234	374	566	820	1146
$E(u)$	0.4191359	0.4379563	0.4396005	0.4401144	0.4403182	0.4404250	0.4404898	0.440533
α		0.8579649	0.8685056	0.5903211	0.4692828	0.4128646	0.3835571	0.3697782
$\|e\|_{E,R}\%$	22.11402	7.871474	4.964893	3.604	2.892096	2.437361	2.114307	1.868156

Table 6.15 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_4

P	1	2	3	4	5	6	7	8
N	32	108	184	318	510	774	1124	1574
$E(u)$	0.4204377	0.4387308	0.4403132	0.4405695	0.4406265	0.4406467	0.4406571	0.4406637
α		0.9607110	1.5535445	1.0586893	0.7043395	0.4889666	0.4023689	0.3731730
$\|e\|_{E,R}\%$	21.43571	6.662227	2.911645	1.631488	1.169752	0.953909	0.820943	0.724004

Table 6.16 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_5

P	1	2	3	4	5	6	7	8
N	40	136	232	402	646	982	1428	2002
$E(u)$	0.4206717	0.4388525	0.440421	0.4406379	0.4406728	0.44068	0.4406822	0.4406833
α		0.9764145	1.8084042	1.5398594	1.3183587	0.8621643	0.5219361	0.4044335
$\|e\|_{E,R}\%$	21.31149	6.451641	2.45591	1.053391	0.563636	0.392816	0.323083	0.281818

Table 6.17 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_6

P	1	2	3	4	5	6	7	8
N	48	164	280	486	782	1190	1732	2430
$E(u)$	0.4207152	0.4388714	0.4404373	0.4406481	0.4406797	0.4406849	0.4406860	0.4406862
α		0.9758578	1.8550348	1.6898395	1.7825698	1.5698807	1.1523358	0.4247930
$\|e\|_{E,R}\%$	21.28832	6.418317	2.379415	0.93711	0.401388	0.20764	0.134735	0.116684

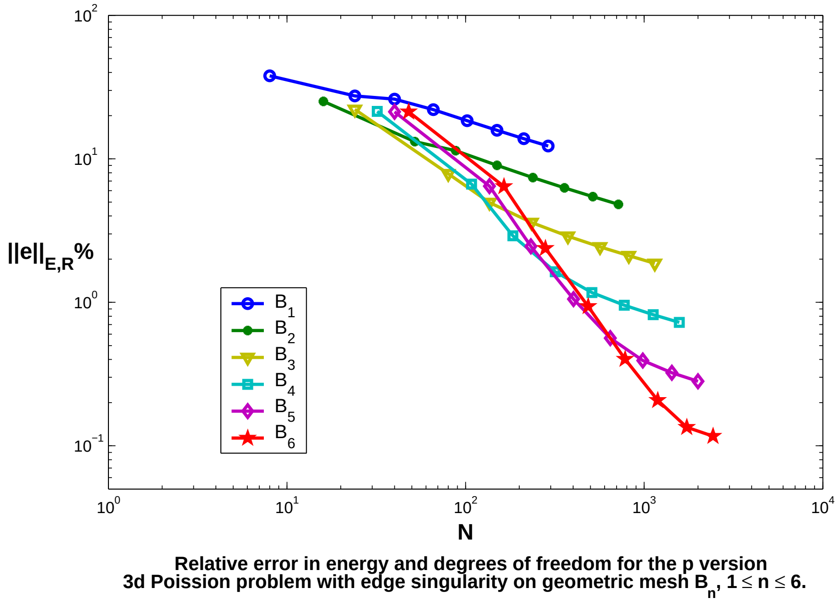


Fig. 6.7 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh B_n, $1 \leq n \leq 6$

The h-p version on the geometric meshes simultaneously refines the meshes by a mesh factor $\sigma = 0.15$ and increases the degree p to achieve the high accuracy with less total number of freedom N. It was indicated in [**26, 27**] that the finite element solution of h-p version converges exponentially,

$$\|e\|_E \leq c\, e^{-bN^{\frac{1}{4}}}$$

where constants b and c depend on the singularity of u and mesh factor σ but not on N. The performances of the h-p version on the meshes B_n with the degree $p = n, 1 \leq n \leq 6$ is illustrated in Table 6.18 and Fig. 6.9, which show that the exponential rate is achieved as predicted by the theory.

Table 6.18 Relationship in the h-p version between $\|e\|_{E,R}, p, N, N^{1/4}, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh $B_n, 1 \leq n \leq 6$

Mesh	P	N	$N^{1/4}$	$E(u)$	$\|e\|_{E,R}\%$	b	c
B_1	1	8	1.6818	0.3772212	37.94934		
B_2	2	52	2.6853	0.4330268	13.18407	2.1144860	0.5980102
B_3	3	136	3.4150	0.4396005	4.964893	1.9532346	0.4438187
B_4	4	318	4.2229	0.4405695	1.631488	2.2257980	0.9672330
B_5	5	646	5.0415	0.4406728	0.563636	1.99858	0.6564106
B_6	6	1190	5.8734	0.4406849	0.20764	1.8513516	0.3518381

In Table 6.18 b and c are calculated by extrapolation, i.e.,

$$b = \ln \frac{\Delta(p)}{\Delta(p+1)}, \qquad c = \frac{\Delta(p)\left(\frac{\Delta(p)}{\Delta(p+1)}\right)}{\left(1 - \frac{\Delta(p+1)}{\Delta(p)}\right)}$$

$$\Delta(p) \quad = \quad E(u_{p+1}) - E(u_p).$$

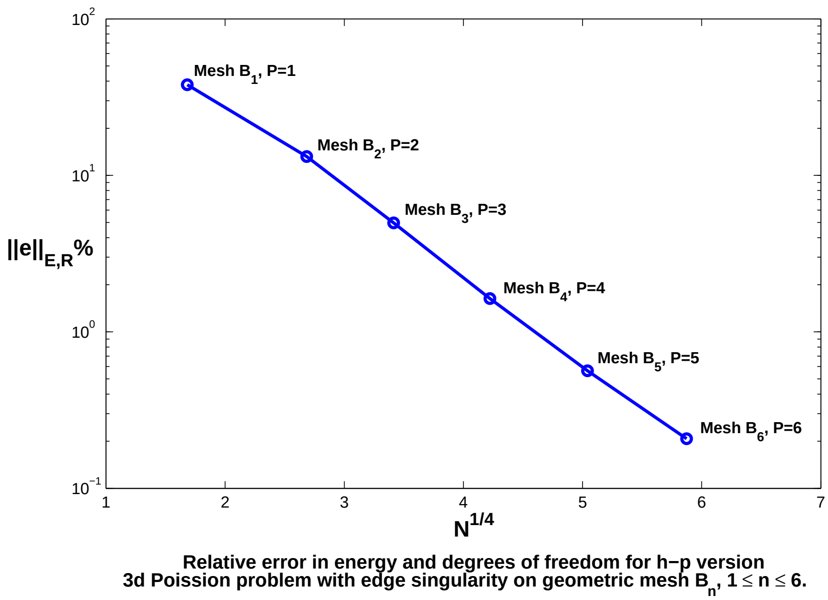


Fig. 6.8 Relationship in the h-p version between $\|e\|_{E,R}, p, N, N^{1/4}, E(u)$ for 3d Poisson problem with edge singularity on geometric mesh $B_n, 1 \leq n \leq 6$

6.1.2. Poisson equation with vertex-edge singularity. Next we solve the following Poisson problem:

$$(6.6) \qquad \left\{ \begin{array}{lcl} -\triangle u & = & -2\rho^{\frac{1}{2}} \sin^{\frac{1}{2}} \theta \sin \frac{\phi}{2} \quad \text{in } D = [-1,1] \times [0,1] \times [0,1], \\ u\big|_{\Gamma_2} & = & u\big|_{\Gamma_1} = u\big|_{\Gamma_4} = 0, \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_1} & = & \dfrac{\partial u}{\partial n}\Big|_{\Gamma_4} = \dfrac{\partial u}{\partial n}\Big|_{\Gamma_7} = 0, \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_3} & = & \frac{1}{2}\rho^{\frac{1}{2}} \sin^{-\frac{1}{2}} \theta \cos\theta(\rho\cos\theta - 1) \sin\frac{\phi}{2}, \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_6} & = & -\frac{1}{2}\rho^{\frac{1}{2}} \sin^{-\frac{1}{2}} \theta \cos\theta(\rho\cos\theta - 1) \sin\frac{\phi}{2}, \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_5} & = & \frac{1}{2}\rho^{\frac{1}{2}} \sin^{-\frac{1}{2}} \theta \cos\theta(\rho\cos\theta - 1) \cos\frac{\phi}{2} \end{array} \right.$$

where the domain D for the problem is shown in Fig. 6.1 , and $\Gamma_1, \Gamma_2, \cdots, \Gamma_7$ are faces of D, $\Gamma_1 : x_3 = 0$, $\Gamma_4 : x_3 = 1$; $\Gamma_2 : x_2 = 0, 0 \leq x_1 \leq 1$, $\Gamma_7 : x_2 = 0, -1 \leq x_1 \leq 0$, $\Gamma_5 : x_2 = 1$; $\Gamma_3 : x_1 = -1$, $\Gamma_6 : x_1 = 1$,

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2},\ r = \sqrt{x_1^2 + x_2^2},\ \ \theta = \arctan\frac{r}{x_3},\ \ \phi = \arcsin\frac{x_2}{r}.$$

It is known that

$$u = \rho^{\frac{1}{2}} \sin^{\frac{1}{2}} \theta \sin\frac{\phi}{2}.$$

is the exact solution of the problem (6.6), and the strain energy of u.

$$
\begin{aligned}
E(u) &= \frac{1}{2}B(u,u) = \frac{1}{2}\int_D \Big[\Big(\frac{\partial u}{\partial x_1}\Big)^2 + \Big(\frac{\partial u}{\partial x_2}\Big)^2 + \Big(\frac{\partial u}{\partial x_3}\Big)^2\Big] dx_1 dx_2 dx_3 \\
&\approx 0.14222217919263.
\end{aligned}
$$

The finite element solution of the h, p and h-p version computed on uniform meshes $A_n, 1 \le n \le 3$ with $h = 1, 1/2$ and $1/4$ shown in Fig. 6.2.

The solution has the singularity of $\rho^\gamma \sin^\sigma \theta$-type with $\gamma = \sigma = 1/2$. According to Theorem 5.20, there holds for the solution of h-p version

$$
\|e\|_{E,R} \le C \frac{h^{\frac{1}{2}}}{p}. \tag{6.7}
$$

The p version is a special case of the h-p version with fixed h. Then error estimation is deduced to

$$
\|e\|_{E,R} \le Cp^{-1} \approx CN^{-\frac{1}{3}}. \tag{6.8}
$$

Hereafter N denotes the number of degrees of freedom for finite element spaces, which the the number of unknown of the resulting system of linear algebraic equations.

Table 6.19−Table 6.21 show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α for the p version on each fixed mesh A_n. The relationship are plotted in Fig. 6.9. Due to (6.8) the rate of convergence is $\frac{1}{3}$. We calculate the convergence rate for each finite element of the p version by the extrapolation [**45**],

$$
\alpha = \frac{\log \frac{E(u)-E(u_p)}{E(u)-E(u_{p+1})}}{2\log \frac{N_{p+1}}{N_p}}. \tag{6.9}
$$

where u_p is finite element solution of the p version with degree p and N_p denotes the corresponding total number of the freedom.

From Table 6.19−Table 6.21 we can see that α asymptotically approaches to the theoretical value 1/3 as the degree p increases and the meshes are refined. The computation indicates that the convergence rate of the p version with the degree $1 \le p \le 8$ on these uniform meshes associated is in pre-asymptotical range and higher than asymptotical value.

Table 6.19 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity on Mesh $A_1, h = 1$

P	3	4	5	6	7	8
N	8	18	34	58	92	138
$E(u)$	0.1382483	0.1408121	0.1410113	0.1414079	0.1416734	0.1418290
α		0.6388323	0.1197342	0.3714769	0.4276698	0.4111706
$\|e\|_{E,R}\%$	16.71567	9.95722	9.227132	7.566639	6.211767	5.257891

Table 6.20 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity on Mesh $A_2, h = 1/2$

P	1	2	3	4	5	6	7	8
N	12	56	100	192	332	536	820	1200
$E(u)$	0.1084475	0.1414734	0.1415308	0.1418040	0.1419372	0.1420201	0.1420714	0.1421052
α		1.2363383	0.0687764	0.3853741	0.3501334	0.3588278	0.3443835	0.3332966
$\|e\|_{E,R}\%$	48.73174	7.255932	6.972275	5.422475	4.476338	3.769443	3.25602	2.867944

Table 6.21 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity on Mesh $A_3, h = 1/4$

P	1	2	3	4	5	6	7	8
N	120	496	872	1632	2776	4432	6728	9792
$E(u)$	0.1335454	0.1418894	0.1419145	0.1420234	0.1420877	0.1421254	0.1421492	0.1421652
α		1.1489546	0.069496323	0.3485023	0.3678291	0.3515919	0.3380923	0.3297312
$\|e\|_{E,R}\%$	24.69992	4.837204	4.651204	3.738538	3.074991	2.608599	2.265249	2.001587

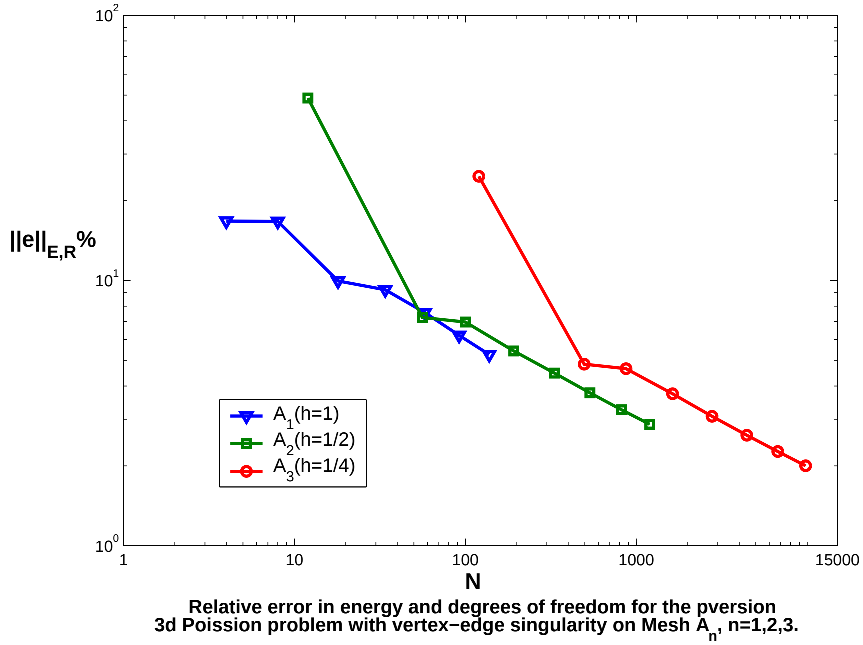


Fig. 6.9 Relative error in the p version between energy norm and degrees of freedom for 3d Poission problem with vertex-edge singularity on uniform meshes

The h version is a special case of the h-p version with fixed p. Then error estimation (6.7) is deduced to

$$\|e\|_{E,R} \leq Ch^{\frac{1}{2}} \approx CN^{-\frac{1}{6}}. \tag{6.10}$$

The convergence rate α for the h version is $\frac{1}{6}$. We calculate the rate for each solution of the h version by the extrapolation similar with (6.9) except replacing u_p by u_n and N_p by N_n, $1 \leq n \leq 3$.

The Table 6.22−Table 6.28 describe the performance of the h version on the uniform meshes $A_n, 1 \leq n \leq 3$ with fixed degree p, $1 \leq p \leq 8$, which show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α for the h version. The relationship is plotted in Fig. 6.10. From Table 6.22−Table 6.28 we can see that α asymptotically approaches its theoretical value 0.1666667 as the mesh is refined and the degree increases.

Table 6.22 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 2$

h	1	1/2	1/4
N	4	56	496
$E(u)$	0.1382354	0.1414734	0.1418894
α		0.3168355	0.1858964
$\|e\|_{E,R}\%$	16.74277	7.255932	4.837204

Table 6.23 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 3$

h	1	1/2	1/4
N	8	100	872
$E(u)$	0.1382483	0.1415308	0.1419145
α		0.3461988	0.1869279
$\|e\|_{E,R}\%$	16.71567	6.972275	4.651204

Table 6.24 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poission problem with vertex-edge singularity for $p = 4$

h	1	1/2	1/4
N	18	192	1632
$E(u)$	0.1408121	0.141804	0.1420234
α		0.2567440	0.1737594
$\|e\|_{E,R}\%$	9.95722	5.422475	3.738538

Table 6.25 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 5$

h	1	1/2	1/4
N	34	332	2776
$E(u)$	0.1410113	0.1419372	0.1420877
α		0.3174258	0.1768206
$\|e\|_{E,R}\%$	9.227132	4.476338	3.074991

Table 6.26 Relationship the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 6$

h	1	1/2	1/4
N	58	536	4432
$E(u)$	0.1414079	0.1420201	0.1421254
α		0.3133620	0.1742563
$\|e\|_{E,R}\%$	7.566639	3.769443	2.608599

Table 6.27 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 7$

h	1	1/2	1/4
N	92	820	6728
$E(u)$	0.1416734	0.1420714	0.1421492
α		0.2952838	0.1723824
$\|e\|_{E,R}\%$	6.211767	3.25602	2.265249

Table 6.28 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d Poisson problem with vertex-edge singularity for $p = 8$

h	1	1/2	1/4
N	138	1200	9792
$E(u)$	0.141829	0.1421052	0.1421652
α		0.2802505	0.1713242
$\|e\|_{E,R}\%$	5.257891	2.867944	2.001587

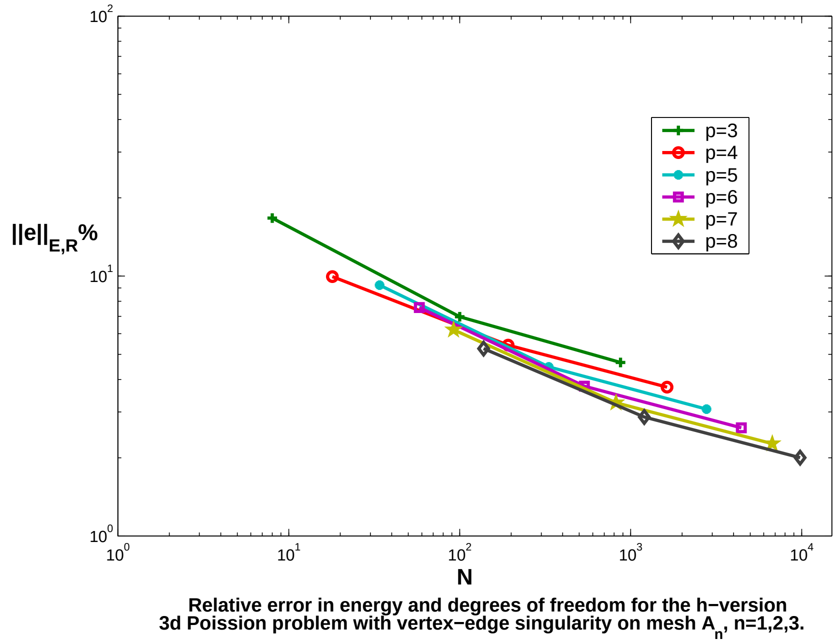


Fig. 6.10 Relative error in the h version between energy norm and degrees of freedom for 3d Poission problem with vertex-edge singularity on uniform meshes

The h-p version reduces the element size h and increase the degree p simultaneously to reduce quickly the error by less computation. The performance of the h-p version on uniform meshes $A_n, n = 1, 2, 3$ associated with $p = 2, 3, 4$ is given in Table 6.29 and Fig. 6.11.

Table 6.29 Relationship in the h-p version between $\|e\|_{E,R}, p, N, N^{1/4}, E(u)$ for 3d Poisson problem with edge singularity on uniform mesh A_n.

Mesh	h	p	N	$N^{1/4}$	$E(u)$	α	$\|e\|_{E,R}\%$
A_1	1	2	4	1.4142	0.1382354		16.74277
A_2	$\frac{1}{2}$	3	100	3.1623	0.1415308	0.2721524	6.972275
A_3	$\frac{1}{4}$	4	1632	6.3559	0.1420234	0.2231942	3.738538

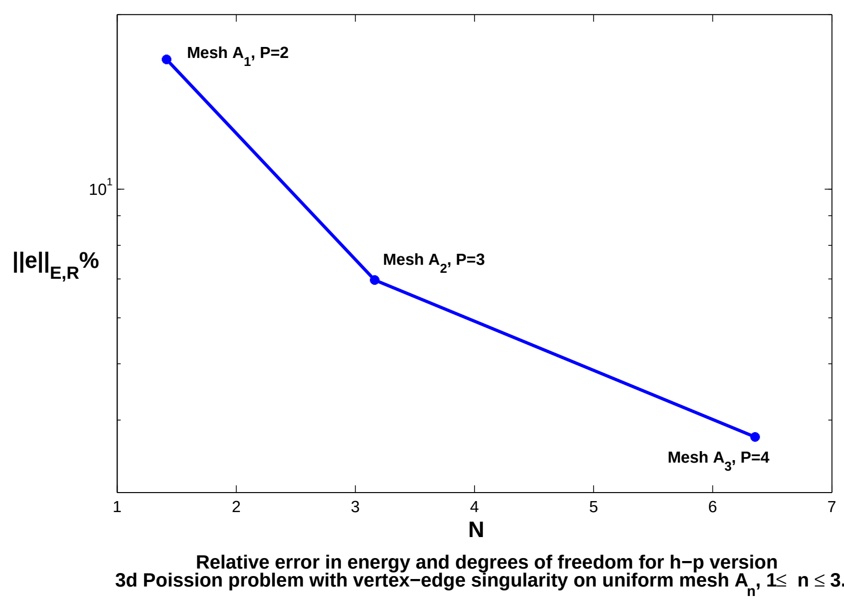


Fig. 6.11 Relative error in the h-p version between energy norm and degrees of freedom for 3d Poisson problem with vertex-edge singularity on uniform mesh $A_n, 1 \le n \le 3$.

The computation results in Table 6.19−Table 6.29 show that the theory of finite element method for vertex-edge singular functions is very reliable and coincides with the computation.

6.2. Three dimensional elasticity problems on polyhedral domains

In this section, we will solve three dimensional elasticity problems(homogeneous isotropic material) on polyhedral domains. We denote the Young's modulus of elasticity and Poisson ratio respectively by E and ν. The domain Ω is shown in Fig. 6.12.

Let $u = (u_1, u_2, u_3)$ be the displacement vector, and let σ be the stress tensor and T be the traction vector.

$$\begin{aligned}
\sigma_{11} &= \lambda\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + 2\mu\frac{\partial u_1}{\partial x_1} \\
\sigma_{22} &= \lambda\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + 2\mu\frac{\partial u_2}{\partial x_2} \\
\sigma_{33} &= \lambda\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + 2\mu\frac{\partial u_3}{\partial x_3}
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{23} &= \mu\Big(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\Big) \\
\sigma_{31} &= \mu\Big(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\Big) \\
\sigma_{12} &= \mu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big).
\end{aligned}$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$, μ is the conventional shear modulus.

We consider the three dimensional elasticity problem for $E = 100$ and $\nu = 0.3$:

$$\left\{\begin{aligned}
(\lambda+\mu)\frac{\partial}{\partial x_1}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + \mu\nabla^2 u_1 &= 0, \\
(\lambda+\mu)\frac{\partial}{\partial x_2}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + \mu\nabla^2 u_2 &= 0, \\
(\lambda+\mu)\frac{\partial}{\partial x_3}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\Big) + \mu\nabla^2 u_3 &= 0
\end{aligned}\right. \tag{6.11}$$

with boundary conditions:

$$\left\{\begin{array}{rcl}
u_1 = 0,\ T_2 = T_3 = 0 \text{ on } \Gamma_1 &=& \{(0, x_2, x_3) |\, 0 \le x_2 \le 1,\, 0 \le x_3 \le 1\}, \\
u_2 = 0,\ T_1 = T_3 = 0 \text{ on } \Gamma_2 &=& \{(x_1, 0, x_3) |\, 0 \le x_1 \le 1,\, 0 \le x_3 \le 1\}, \\
u_3 = 0,\ T_1 = T_2 = 0 \text{ on } \Gamma_3 &=& \{(x_1, x_2, 0) |\, 0 \le x_1 \le 1,\, 0 \le x_2 \le 1\}, \\
\Gamma_4 &=& \{(x_1, x_2, \frac{1}{2}) |\, 0 \le x_1 \le \frac{1}{2},\, 0 \le x_2 \le \frac{1}{2}\}, \\
T_1 = T_2 = T_3 = 0 \quad \text{on } \Gamma_5 &=& \{(\frac{1}{2}, x_2, x_3) |\, 0 \le x_2 \le \frac{1}{2},\, \frac{1}{2} \le x_3 \le 1\}, \\
\Gamma_6 &=& \{(x_1, \frac{1}{2}, x_3) |\, 0 \le x_1 \le \frac{1}{2},\, \frac{1}{2} \le x_3 \le 1\}, \\
\Gamma_7 &=& \{(1, x_2, x_3) |\, 0 \le x_2 \le 1,\, 0 \le x_3 \le 1\}, \\
\Gamma_8 &=& \{(x_1, 1, x_3) |\, 0 \le x_1 \le 1,\, 0 \le x_3 \le 1\}, \\
T_1 = T_2 = 0,\ T_3 = 2 \text{ on } \Gamma_9 &=& \{(x_1, x_2, 1) |\, \frac{1}{2} \le x_2 \le 1 \text{ for } 0 \le x_1 \le \frac{1}{2}, \\
&& \text{and } 0 \le x_2 \le 1 \text{ for } \frac{1}{2} \le x_1 \le 1\}.
\end{array}\right. \tag{6.12}$$

In this problem, we do not know the exact solution and the strain energy of the problem.

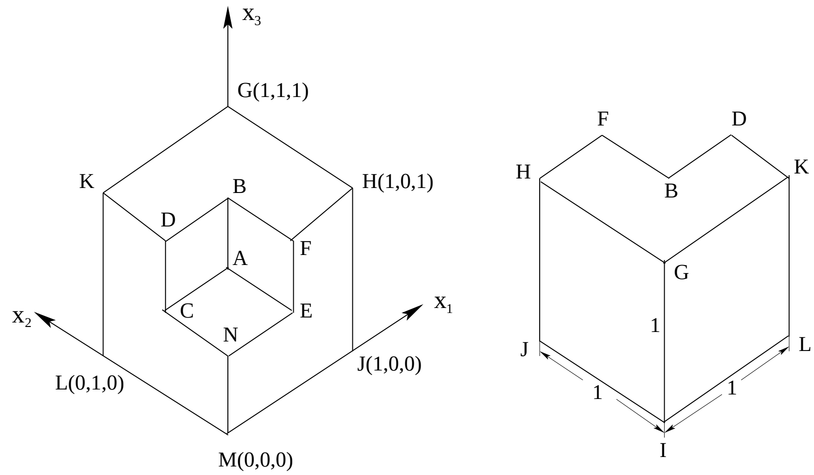


Fig. 6.12 L-shape domain Ω

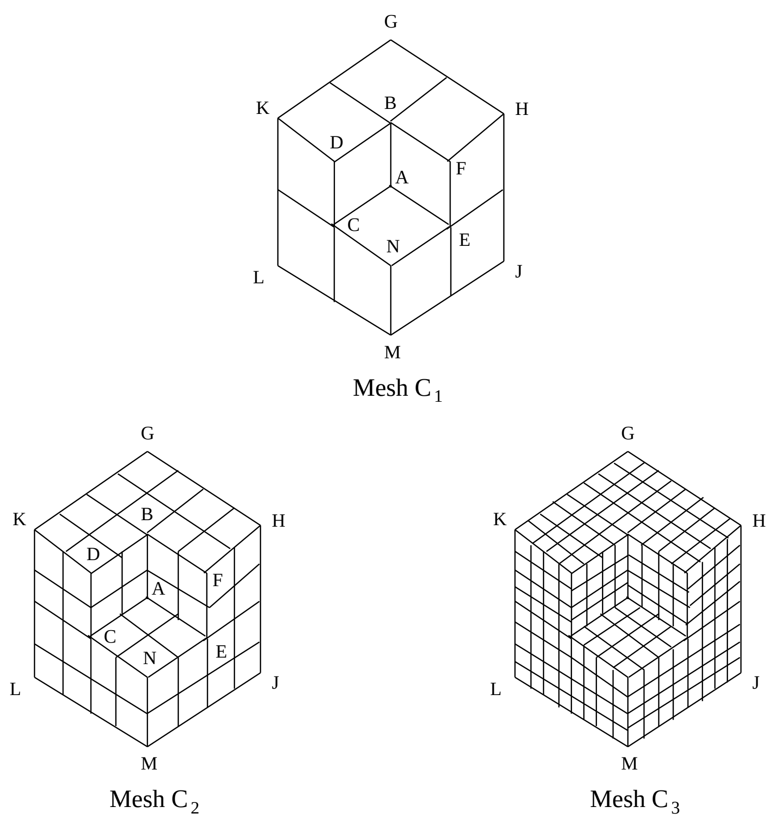


Fig. 6.13 Uniform meshes $C_n, 1 \leq n \leq 3$ with $h = 1/2, 1/4, 1/8$

The computation of the h, p and h-p version have been done on uniform mesh C_n with square elements, $1 \leq n \leq 3$, shown on Fig. 6.13, by the commercial codes StressCheck V7.1 developed by ESRD Inc. and ProPHLEX developed by Altair Engineering Inc.

Table 6.30−Table 6.32 show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α in the p version on each fixed mesh C_n, $1 \leq n \leq 3$. The relationship are plotted in Fig. 6.14.

We calculate the convergence rate for each finite element solution by the extrapolation [**45**].

$E(u)$ is the root of the equation

$$F(E(u)) = \frac{E(u) - E(u_2)}{E(u) - E(u_3)} - \Big(\frac{E(u) - E(u_1)}{E(u) - E(u_2)}\Big)^Q = 0$$

where $Q = \frac{\log \frac{N_3}{N_2}}{\log \frac{N_2}{N_1}}$ with $N_i, i = 1, 2, 3$ denotes the degree of freedom corresponding to the finite element space. $E(u)$ can be solved by Newton's iteration method or other numerical method, and denoted by $E(\tilde{u})$ to distinguish the energy $E(u)$ of exact solution.

Once $E(\tilde{u})$ is solved, we can determine the exponent α by

$$\alpha \approx \frac{\log F_{2,3}}{2 \log \frac{N_3}{N_2}} = \frac{\log \frac{E(\tilde{u})-E(u_2)}{E(\tilde{u})-E(u_3)}}{2 \log \frac{N_3}{N_2}}.$$

In Fig. 6.14 the error curves for the p-version on the mesh $C_n, n = 1, 2, 3$ plotted in log-log scale are very close to straight line which are parallel and the slope is the rate of the convergence. Obviously, the solution has singularity at the vertex $A = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and on the edge AB, AC and AE. The strongest singularity occurs along the edge $AC = \{(x_1, \frac{1}{2}, \frac{1}{2}) | 0 \leq x_1 \leq \frac{1}{2}\}$ and $AE = \{(\frac{1}{2}, x_2, \frac{1}{2}) | 0 \leq x_2 \leq \frac{1}{2}\}$. It seems the rate α of the convergence is around $0.3 \sim 0.4$, which is predicted as $\frac{2}{3} \min(\gamma + \frac{1}{2}, \sigma)$, where γ is the vertex singularity and σ is the edge singularity.

Table 6.30 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem on Mesh $C_1, h = 1/2$

P	1	2	3	4	5	6	7	8
N	49	275	803	1759	3269	5459	8455	12383
$E(u)$	0.01193771	0.01287213	0.01313627	0.01325604	0.01332475	0.01336881	0.01339907	0.01342105
α		0.2531353	0.5469898	0.5096197	0.4536066	0.4483468	0.4552743	0.4457332
$\|e\|_{E,R}\%$	34.40	22.20	17.26	14.47	12.60	11.23	10.19	9.36

Table 6.31 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem on Mesh $C_2, h = 1/4$

P	1	2	3	4	5	6	7	8
N	275	1759	5459	12383	23539	39935	62579	92479
$E(u)$	0.01262429	0.01314234	0.01329410	0.01336622	0.01340775	0.01343445	0.01345291	0.01346635
α		0.2239465	0.4775917	0.4256249	0.4201082	0.4199877	0.4207112	0.4219232
$\|e\|_{E,R}\%$	26.02	17.17	13.52	11.38	9.94	8.89	8.09	7.45

Table 6.32 Relationship in the p version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem on Mesh $C_3, h = 1/8$

P	1	2	3	4	5	6
N	1746	12351	39876	92385	177942	304611
$E(u)$	0.01300402	0.01329891	0.01338862	0.01343203	0.01345726	0.01347357
α		0.2029820	0.4216856	0.3865125	0.3913336	0.3973223
$\|e\|_{E,R}\%$	19.94	13.40	10.65	9.02	7.92	7.12

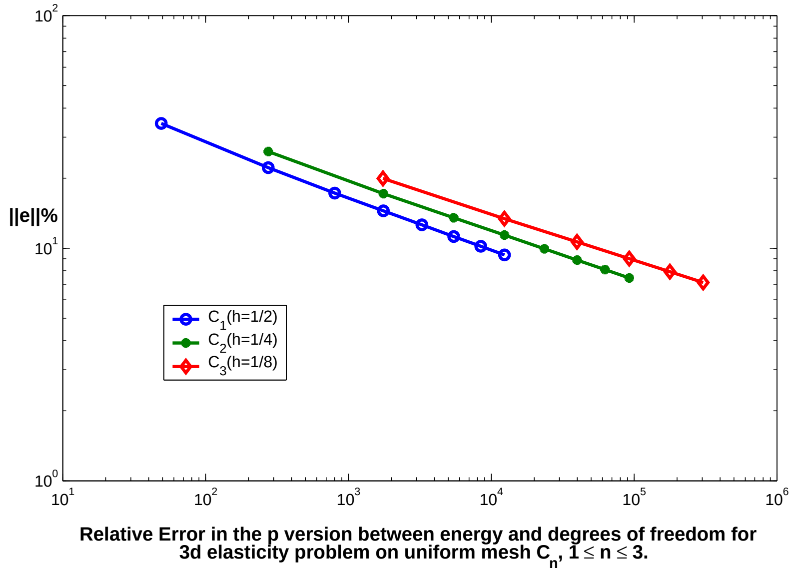


Fig. 6.14 Relative error in the p version between energy norm and degrees of freedom for 3d elasticity problem on uniform meshes

The Table 6.33−Table 6.38 describe the performance of the h version on the uniform meshes $C_n, 1 \leq n \leq 3$ with fixed degree p, $1 \leq p \leq 8$, which show the relationship between $\|e\|_{E,R}$ and p, N, $E(u)$, α in the h version. The relationship is plotted in Fig. 6.15.

Table 6.33 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 1$

h	1/2	1/4	1/8
N	49	275	1746
$E(u)$	0.01193771	0.01262429	0.01300402
α		0.3237904	0.3715672
$\|e\|_{E,R}\%$	34.40	26.02	19.94

Table 6.34 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 2$

h	1/2	1/4	1/8
N	275	1759	12351
$E(u)$	0.01287213	0.01314234	0.01329891
α		0.2782727	0.3133864
$\|e\|_{E,R}\%$	22.20	17.17	13.40

Table 6.35 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 3$

h	1/2	1/4	1/8
N	803	5459	39876
$E(u)$	0.01313627	0.01329410	0.01338862
α		0.2569841	0.2818973
$\|e\|_{E,R}\%$	17.26	13.52	10.65

Table 6.36 Relationship the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 4$

h	1/2	1/4	1/8
N	1759	12383	92385
$E(u)$	0.01325604	0.01336622	0.01343203
α		0.2492676	0.2753614
$\|e\|_{E,R}\%$	14.47	11.38	9.02

Table 6.37 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 5$

h	1/2	1/4	1/8
N	3269	23539	177942
$E(u)$	0.01332475	0.01340775	0.01345726
α		0.2437201	0.2709999
$\|e\|_{E,R}\%$	12.60	9.94	7.92

Table 6.38 Relationship in the h version between $\|e\|_{E,R}$ and $p, N, E(u), \alpha$ for 3d elasticity problem for $p = 6$

h	1/2	1/4	1/8
N	5459	39935	304611
$E(u)$	0.01336881	0.01343445	0.01347357
α		0.2392914	0.2679601
$\|e\|_{E,R}\%$	11.23	8.89	7.12

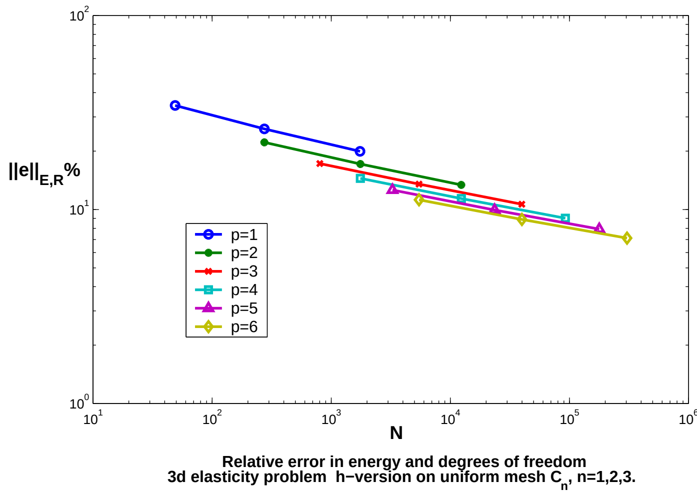


Fig. 6.15 Relative error in the h version between energy norm and degrees of freedom for 3d elasticity problem on uniform meshes

The h-p version reduces the element size h and increase the degree p simultaneously to reduce quickly the error by less computation. The performance of the h-p version on uniform meshes $C_n, n = 1, 2, 3$ associated with $p = 2, 3, 4$ is given in Table 6.39 and Fig. 6.16.

Table 6.39 Relationship in the h-p version between $\|e\|_{E,R}, p, N, E(u), \alpha$ for 3d elasticity problem on uniform mesh C_n.

Mesh	h	p	N	$E(u)$	α	$\|e\|_{E,R}\%$
C_1	$\frac{1}{2}$	2	275	0.01287213		22.24208
C_2	$\frac{1}{4}$	3	5459	0.01329410	0.1662981	13.53185
C_3	$\frac{1}{8}$	4	92385	0.01343203	0.1436109	9.014316

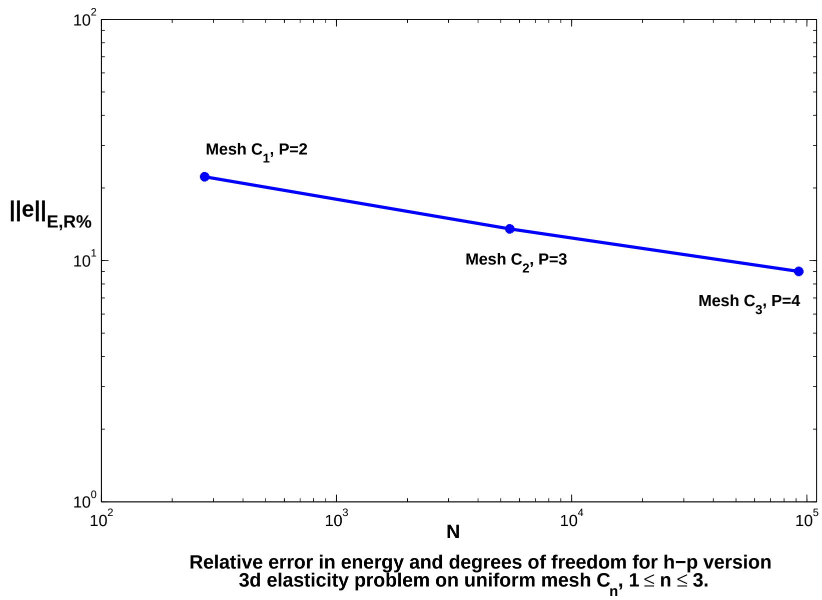


Fig. 6.16 Relative error in the h-p version between energy norm and degrees of freedom for 3d elasticity problem on uniform meshes

In the above computation, the product polynomial space $\mathcal{P}_p^2(\Omega_i)$ are adopted on each square element. We also conducted the computation on trunk space $\mathcal{P}_p^\kappa(\Omega_i), 1 < \kappa < 2$. Table 6.40−Table 6.42 show that the degree of freedom reduced significantly without compromising the convergence as predicted by the theorems in previous chapter.

Table 6.40 Relationship in the p version (trunk space) between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d elasticity problem on Mesh $C_1, h = 1/2$

P	1	2	3	4	5	6	7	8
N	49	166	283	488	781	1183	1715	2398
$E(u)$	0.01193771	0.01269697	0.01279953	0.01294672	0.01307518	0.01316207	0.01322776	0.01327642
$\|e\|_{E,R}\%$	34.14	24.57	22.97	20.46	17.99	16.10	14.52	13.22

Table 6.41 Relationship in the p version (trunk space) between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d elasticity problem on Mesh $C_2, h = 1/4$

P	1	2	3	4	5	6	7	8
N	275	987	1699	3015	4935	7627	11259	15999
$E(u)$	0.01262429	0.01303359	0.01308542	0.01318434	0.01325984	0.01331346	0.01335256	0.01338169
$\|e\|_{E,R}\%$	25.85	19.12	18.09	15.94	14.08	12.60	11.39	10.40

Table 6.42 Relationship in the p version (trunk space) between $\|e\|_{E,R}$ and $p, N, E(u)$ for 3d elasticity problem with edge singularity on Mesh $C_3, h = 1/8$

P	1	2	3	4	5	6	7	8
N	1757	6608	11459	20747	34472	53978	80609	115709
$E(n)$	0.01359820	0.01386188	0.01389421	0.01395551	0.01400238	0.01403526	0.01405908	0.01407670
$\|e\|_{E,R}\%$	19.96	14.57	13.76	12.09	10.63	9.48	8.54	7.78

CHAPTER 7

Conclusion

In this thesis, we have introduced the Jacobi-weighted Besov and Sobolev spaces on scaled cube $Q_h = (-h, h)^3$ and analyzed the approximabilities of smooth and singular functions in the framework of these spaces. In particular, the Jacobi-weighted Besov and Sobolev spaces with three different weights are defined to precisely characterize the natures of the vertex singularity, the edge singularity, and the vertex-edge singularity, and to explore their best approximabilities in terms of these spaces. We also have constructed explicitly polynomial extensions on standard elements: cubes, triangular prisms and pyramids, which together with the extensions on tetrahedrons are applied to the h-p finite element method in three dimensions. The local quasi projection operators on tetrahedral, hexahedral and prismatic elements have been well designed which preserve the best approximation on each element. Based on three fundamental issues which are raised in the introduction and responded in the thesis we have proved the convergence of the h-p version of FEM with quasi uniform degrees and meshes with tetrahedral, hexahedral and prismatic elements for elliptic problems with smooth solutions and singular solutions on polyhedral domain, where singularities of $\rho^\gamma \log^\nu \rho$-type, $r^\sigma \log^\mu r$-type and $\rho^\gamma \log^\nu \rho \sin^\sigma \theta \log^\mu \sin\theta$-type occur. The rate of convergence in terms of h and p is the best in the literatures.

The concepts, methods and techniques in the thesis can be generalized to three dimensional elliptic problems such as elasticity problems on polyhedral domains, the heat problems and magnetic-electric problems on smooth and un-smooth domains in three dimensions. Certainly it will influence the boundary element method in three dimensions as well.

It is worth indicating that although we have constructed a polynomial extension on pyramid from a square face of the pyramid, how to construct a polynomial extension on pyramid from a triangular face of the pyramid, which is stable and compatible with FEM subspace on pyramid, is still an open problem. The construction of a stable and compatible polynomial extension on prism in H^1, H(curl) and H(div)-spaces for magnetic-electric problems is another open problem. Also we have not established the optimal convergence of the h-p version of the finite element method with quasi-uniform meshes for elliptic problems on polyhedral domains by analyzing the lower bound of the error. The convergence for the h-p version of the finite element method with geometric meshes has not been addressed in the thesis and in the literatures of FEM yet. There are a lot of open problems in the h-p FEM in three dimensions which we should work out in future.

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