Ramanujan Graphs

by

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Abstract

This thesis explores the area of Ramanujan graphs, gives details of most of the known constructions of Ramanujan graphs, shows examples of these constructions, and explores results related to Ramanujan graphs including Generalized Ramanujan graphs. In addition, it also describes programming many different Ramanujan graph constructions.

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Chapter 1

Introduction

A *Ramanujan graph* is a k-regular graph where the eigenvalues of its adjacency matrix, λ, satisfy $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$. Ramanujan graphs can be thought of as "optimal" expanders. An *expander* is a graph G such that the quantity

$$h(G) = \min \left\{ \frac{|\partial S|}{\min\{|S|, |V \setminus S|\}} \;\middle|\; \varnothing \neq S \subsetneq V \right\},$$

where ∂S is the set of edges with one endpoint in S and the other outside S, is "large". There is a strong connection between this quantity, $h(G)$, and the second largest eigenvalue of a graph's adjacency matrix. And in fact the Alon-Boppana theorem says that $2\sqrt{k-1}$ is the smallest one can hope to achieve in bounding from above the second eigenvalue of any infinite sequence of k-regular graphs.

The first construction for Ramanujan graphs was given independently by Lubotzky, Philips, and Sarnak [66] and Margulis [70]. The name "Ramanujan" derives from the former paper, in which a special case of the Ramanujan conjecture [83] (also called the Ramanujan-Petersson conjecture) was used to prove that the $2\sqrt{k-1}$ bound was achieved (the full conjecture has since been proved). The LPS/Margulis construction produced an infinite sequence of $(k+1)$-regular Ramanujan graphs for any odd prime k. A relatively easy modification by Chiu extends this method to work for all primes [18]. Morgenstern [76] extended the method to work for any prime power k. No other construction is known which produces an infinite sequence of Ramanujan graphs all with the same regularity.

The dearth of explicit constructions would seem to suggest that Ramanujan graphs are rare. However probabilistic results suggest that Ramanujan graphs are plentiful ("almost all graphs are almost Ramanujan"). And, as if to hammer home the point, recently Cioabă and Murty [20] have shown how to construct "almost" Ramanujan graphs of any degree.

It has been said that Ramanujan graphs "fuse diverse branches of pure mathematics, namely, number theory, representation theory and algebraic geometry." [77]

This thesis brings together in one place an overview of most of the known constructions, more than have appeared together before.

This thesis also describes programming carried out to implement many different Ramanujan graph constructions, perform different computations on the graphs, and layout the graphs in a suitable visual manner.

This thesis, all the programs used to construct the graphs as well as high resolution images of the figures, are available on an attached CD for detailed viewing.

Chapter 2

Preliminaries

2.1 Graph Theory

For basic graph theoretic notions one can consult Wilson [89]. Unless otherwise stated graphs are finite, undirected, and without loops or multiple edges.

Definition 1. Let G be a graph. $|G|$ denotes the number of vertices in G. If G is a k-regular graph, then we refer to k as the *degree* of G.

The *diameter* of a graph G, $\mathrm{diam}(G)$, is the maximum distance between two vertices in G.

The *girth* of a graph G, $\mathrm{g}(G)$, is the length of the shortest cycle in G.

The *independence number* of G, $\mathrm{i}(G)$, is the maximum size of a subset of the vertices of G so that no two vertices in the set are adjacent.

The *chromatic number* of G, $\chi(G)$, is the minimum number of colours needed to colour the vertices of G so that no vertices of the same colour are adjacent.

A *matching* of a graph $G = (V, E)$ is a subset of the edges $E' \subseteq E$ such that no two edges in E' are incident to the same vertex.

A *perfect matching* of a graph is a matching in which every vertex is incident to some edge in the matching.

Let G be a graph. A mapping $\phi\colon G \to G$ is an *automorphism* of G if v and w are adjacent iff $\phi(v)$ and $\phi(w)$ are adjacent.

A graph G is *vertex-transitive* if for any two vertices v, w of G there is an automorphism ϕ of G such that $\phi(v) = w$.

Let G be a graph with vertex set V. Let $S \subseteq V$, the set of *neighbours* of S, denoted $\mathrm{N}(S)$, is the set of vertices that are adjacent to at least one vertex in S. For $v \in V$ we also write $\mathrm{N}(v)$ for $\mathrm{N}(\{v\})$.

2.2 Algebra

Definition 2. Define the *trace* of a square $n \times n$ matrix A, $\operatorname{Tr} A$, to be

$$\operatorname{Tr} A = \sum_{i=1}^{n} A_{ii},$$

the sum of the diagonal entries of A.

Definition 3. For b an odd prime, and a an integer, define the *Legendre symbol*, $\left(\frac{a}{b}\right)$, to be

$$\left(\frac{a}{b}\right) = \begin{cases} 0 & a \equiv 0 \pmod{b} \\ 1 & a \text{ is a square modulo } b \\ -1 & a \text{ is not a square modulo } b \end{cases}$$

Definition 4. Left $\mathcal{F}$ be a field. Define $\mathrm{GL}_n(\mathcal{F})$, the *general linear group*, to be the set of all invertible $n \times n$ matrices over $\mathcal{F}$. Define $\mathrm{SL}_n(\mathcal{F})$, the *special linear group*, to be all members of $\mathrm{GL}_n(\mathcal{F})$ with determinant 1. Define $\mathrm{PGL}_n(\mathcal{F})$, the *projective general linear group*, to be $\mathrm{GL}_n(\mathcal{F})$ modulo the equivalence relation where two matrices are considered equal if they are non-zero scalar multiples of each other. $\mathrm{PSL}_n(\mathcal{F})$, the *projective special linear group*, is defined to be the subset of $\mathrm{PGL}_n(\mathcal{F})$ all of whose matrices have determinant 1. As a shorthand we also write $\mathrm{GL}_n(q) = \mathrm{GL}_n(\mathbb{Z}_q)$ and $\mathrm{GL}_n(q^n) = \mathrm{GL}_n(\mathbb{F}_{q^n})$, and similarly for SL, PGL, and PSL. The notation $\mathrm{GL}(V)$, where V is a vector space, denotes the set of all invertible linear transformations of V.

The case $\mathrm{PSL}_2(\mathcal{F})$ will be important for our purposes. In this case, if A is a 2×2 matrix over $\mathcal{F}$, and $\det A = s = t^2 \neq 0$ is a nonzero square in $\mathcal{F}$ then $\det(t^{-1}A) = t^{-2} \det A = s^{-1}s = 1$. And if $A \in \mathrm{PSL}_2(\mathcal{F})$ then $\det A = 1$, which is a square. So $\mathrm{PSL}_2(\mathcal{F})$ is the set of 2×2 matrices over $\mathcal{F}$ with square determinant.

It is necessary to efficiently construct $\mathrm{PGL}_2(p^n)$ and $\mathrm{PSL}_2(p^n)$ in order to construct certain important Ramanujan graphs. It is also helpful to construct them in sorted order, so that time does not need to be wasted sorting them later. For this purpose we represent elements of $\mathbb{F}_{p^n}$ as n-vectors over $\mathbb{Z}_p$ and order them lexicographically; compare the first elements, then if necessary the second elements, and so on. Then we order 2×2 matrices over $\mathbb{F}_{p^n}$ lexicographically as well.

Two different construction methods are needed for $\mathrm{PGL}_2(p^n)$ and $\mathrm{PSL}_2(p^n)$. Let $q = p^n$. We assume that p is odd. As we work through each construction we keep a running tally of the number of matrices to be constructed, and thus determine the number of elements of $\mathrm{PGL}_2(p^n)$ and $\mathrm{PSL}_2(p^n)$.

2.2.1 The Construction of $\mathrm{PGL}_2(p^n)$

We denote a generic element of $\mathrm{PGL}_2(p^n) = \mathrm{PGL}_2(q)$ by

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

We can assume that the first non-zero entry is 1.

- Case 1: $a_{1,1} = 0$

  In order for $\det A \neq 0$ we need $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$. Which simplifies to $-a_{1,2}a_{2,1} \neq 0$, and since $a_{1,2}$ is the first non-zero entry of the matrix, we assume $a_{1,2} = 1$. Thus we get

	Possible values	Number of possibilities
$a_{1,1}$	0	1
$a_{1,2}$	1	1
$a_{2,1}$	anything non-zero	$q-1$
$a_{2,2}$	anything	q
Total number of possibilities		$q(q-1)$

- Case 2: $a_{1,1} \neq 0$ hence $a_{1,1} = 1$

  The determinant condition simplifies to $a_{2,2} - a_{1,2}a_{2,1} \neq 0$.

  - Case 2a: $a_{1,2} = 0$

    The determinant condition becomes $a_{2,2} \neq 0$.

	Possible values	Number of possibilities
$a_{1,1}$	1	1
$a_{1,2}$	0	1
$a_{2,1}$	anything	q
$a_{2,2}$	anything non-zero	$q-1$
Total number of possibilities		$q(q-1)$

- Case 2b: $a_{1,2} \neq 0$

  The determinant condition is $a_{2,2} - a_{1,2}a_{2,1} \neq 0$.

	Possible values	Number of possibilities
$a_{1,1}$	1	1
$a_{1,2}$	anything non-zero	$q-1$
$a_{2,1}$	anything	q
$a_{2,2}$	anything but $a_{1,2}a_{2,1}$	$q-1$
Total number of possibilities		$q(q-1)^2$

Total:

$$q(q-1)+q(q-1)+q(q-1)^2 = (2+q-1)q(q-1) = (q+1)q(q-1) = q(q^2-1)$$

Thus we have shown that $|\mathrm{PGL}_2(q)| = q(q^2-1)$.

Example 5. $\mathrm{PGL}_2(3)$ has 24 elements.

- Case 1: $a_{1,1} \equiv 0 \pmod q$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\begin{pmatrix}0&1\\1&1\end{pmatrix}\quad\begin{pmatrix}0&1\\1&2\end{pmatrix}\quad\begin{pmatrix}0&1\\2&0\end{pmatrix}\quad\begin{pmatrix}0&1\\2&1\end{pmatrix}\quad\begin{pmatrix}0&1\\2&2\end{pmatrix}$$

- Case 2: $a_{1,1} \not\equiv 0 \pmod q$ hence $a_{1,1} \equiv 1 \pmod q$
  - Case 2a: $a_{1,2} \equiv 0 \pmod q$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\begin{pmatrix}1&0\\0&2\end{pmatrix}\quad\begin{pmatrix}1&0\\1&1\end{pmatrix}\quad\begin{pmatrix}1&0\\1&2\end{pmatrix}\quad\begin{pmatrix}1&0\\2&1\end{pmatrix}\quad\begin{pmatrix}1&0\\2&2\end{pmatrix}$$

  - Case 2b: $a_{1,2} \not\equiv 0 \pmod q$

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\begin{pmatrix}1&1\\0&2\end{pmatrix}\quad\begin{pmatrix}1&1\\1&0\end{pmatrix}\quad\begin{pmatrix}1&1\\1&2\end{pmatrix}\quad\begin{pmatrix}1&1\\2&0\end{pmatrix}\quad\begin{pmatrix}1&1\\2&1\end{pmatrix}$$

$$\begin{pmatrix}1&2\\0&1\end{pmatrix}\quad\begin{pmatrix}1&2\\0&2\end{pmatrix}\quad\begin{pmatrix}1&2\\1&0\end{pmatrix}\quad\begin{pmatrix}1&2\\1&1\end{pmatrix}\quad\begin{pmatrix}1&2\\2&0\end{pmatrix}\quad\begin{pmatrix}1&2\\2&2\end{pmatrix}$$

2.2.2 The Construction of $\mathrm{PSL}_2(p^n)$

We denote a generic element of $\mathrm{PGL}_2(p^n) = \mathrm{PGL}_2(q)$ by

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

We can assume that the first non-zero coordinate of the first non-zero entry of A is $\leq \frac{p-1}{2}$, as multiplying by -1 does not change the determinant. We note that there are $\frac{q-1}{2}$ non-zero elements of $\mathbb{F}_p^n$ $(= (\mathbb{Z}_p)^n)$ whose first non-zero coordinate is $\leq \frac{p-1}{2}$, as each non-zero element with first non-zero coordinate $\leq \frac{p-1}{2}$ can be paired with its negative whose first non-zero coordinate is $> \frac{p-1}{2}$.

- Case 1: $a_{1,1} = 0$

  The condition is now $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 1$, which simplifies to $-a_{1,2}a_{2,1} = 1$. $a_{1,2}$ is the first non-zero entry (if it were zero the determinant would be zero), so its first non-zero coordinate must be $\leq \frac{p-1}{2}$.

	Possible values	Number of possibilities
$a_{1,1}$	0	1
$a_{1,2}$	anything non-zero with first coordinate $\leq \frac{p-1}{2}$	$\frac{q-1}{2}$
$a_{2,1}$	$-{a_{1,2}}^{-1}$	1
$a_{2,2}$	anything	q
	Total number of possibilities	$\frac{q(q-1)}{2}$

- Case 2: $a_{1,1} \neq 0$

  $a_{1,1}$ is non-zero so its first non-zero coordinate must be $\leq \frac{p-1}{2}$. We rewrite the condition $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 1$ as $a_{1,1}a_{2,2} = a_{1,2}a_{2,1} + 1$.

	Possible values	Number of possibilities
$a_{1,1}$	anything non-zero with first coordinate $\leq \frac{p-1}{2}$	$\frac{q-1}{2}$
$a_{1,2}$	anything	q
$a_{2,1}$	anything	q
$a_{2,2}$	$\begin{cases} 0 & a_{1,2}a_{2,1}+1=0 \\ {a_{1,1}}^{-1}(a_{1,2}a_{2,1}+1) & a_{1,2}a_{2,1}+1 \neq 0 \end{cases}$	1
	Total number of possibilities	$q^2\frac{q-1}{2}$

Total:

$$\frac{q(q-1)}{2} + q^2\frac{q-1}{2} = \frac{q^2 - q + q^3 - q^2}{2} = \frac{q(q^2-1)}{2}$$

Thus we have shown that $|\mathrm{PSL}_2(q)| = \frac{q(q^2-1)}{2}$, half the number of elements of $\mathrm{PGL}_2(q)$.

Example 6. $\mathrm{PSL}_2(3)$ has 12 elements.

- Case 1: $a_{1,1} \equiv 0 \pmod q$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$$

- Case 2: $1 \leq a_{1,1} \leq \frac{q-1}{2}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

2.3 Algebraic Graph Theory

Definition 7. Let G be a finite graph with vertices $V = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$ matrix $A = (a_{i,j})$ with $a_{i,j} = 1$ if v_i is adjacent to v_j, $a_{i,j} = 0$ otherwise.

The *eigenvalues of a graph* are the eigenvalues of the adjacency matrix of the graph. The *spectrum of a graph* is the set of eigenvalues of the graph along with their multiplicities. The *characteristic polynomial of a graph* is the characteristic polynomial of its adjacency matrix.

When dealing with graphs that may have multiple edges the definition of adjacency matrix is extended so that $a_{i,j}$ is equal to the number of edges between v_i and v_j. There are two different conventions for dealing with self loops in the adjacency matrix: a self loop on vertex v_i either contributes 1 or 2 to the diagonal entry $a_{i,i}$. The usual convention is that it contributes 2, but we will make it clear which convention is used when needed.

When considering a graph with n vertices, unless otherwise stated, we will denote its eigenvalues, counting multiplicities, as $\mu_0, \ldots, \mu_{n-1}$, taken in descending order; so

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}.$$

Some basic results about graph spectra follow. For details see Doob [24].

Theorem 8. *A graph is bipartite iff its spectrum is symmetric about 0.*

Theorem 9 (Principal axis theorem)**.** *Let A be a $n \times n$ symmetric real matrix. Then A has n real eigenvalues (counting multiplicities) and a corresponding orthonormal set of eigenvectors.*

Definition 10. A $n \times n$ matrix A is *decomposable* if there is a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where A and C are square matrices and 0 represents a zero matrix. Otherwise it is said to be *indecomposable*.

If A is the adjacency matrix of a graph G then A is indecomposable precisely when G is connected.

Theorem 11 (Perron-Frobenius [40][74])**.** *Let A be a $n \times n$ indecomposable matrix with non-negative entries. Then there is a simple positive eigenvalue d of A such that $d \geq |\lambda|$ for all other eigenvalues λ of A. The eigenvalue d is called the* Perron-Frobenius eigenvalue *of A.*

Theorem 12. *Let G be a graph with n vertices where d_i is the degree of vertex i in G. Let λ be the largest eigenvalue of G, then*

$$\frac{1}{n}\sum_i d_i \leq \lambda \leq \max_i d_i.$$

If G is a k-regular graph then $\frac{1}{n}\sum_i d_i = \max_i d_i = k$, and k is a simple eigenvalue iff G is connected.

Definition 13. For a k-regular graph G we call the eigenvalues k and $-k$ (if it exists) the *trivial eigenvalues* of G.

Let A be an $n \times n$ matrix. The $(n-r) \times (n-r)$ matrix $(1 \leq r < n)$ obtained by deleting r rows and the corresponding r columns from A is called a *principal submatrix* of A.

Theorem 14 (Interlacing theorem [57][74][80])**.** *Let A be a* self-adjoint $n \times n$ *matrix (that is $A = A^*$ where * denotes the conjugate transpose aka adjoint)* *with eigenvalues $\mu_0 \geq \cdots \geq \mu_{n-1}$ where each eigenvalue appears with multiplicity. Let B be an $(n-r) \times (n-r)$ principal submatrix of A with eigenvalues $\lambda_0 \geq \cdots \geq \lambda_{n-r-1}$ (each eigenvalue appearing with multiplicity). Then for $i = 0, \ldots, n-r-1$*

$$\mu_i \geq \lambda_i \geq \mu_{i+r}.$$

Specifically, when $r = 1$ and B is obtained by deleting one row and the corresponding column from A then

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n-2} \geq \lambda_{n-2} \geq \mu_{n-1}.$$

Corollary 15. *Let G be a graph and v a vertex in G. Let $G - v$ be the graph obtained by removing the vertex v and all edges incident to v from G. Let $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of G where each eigenvalue appears with multiplicity and let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-2}$ be the eigenvalues of $G - v$ (each eigenvalue appearing with multiplicity). Then*

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n-2} \geq \lambda_{n-2} \geq \mu_{n-1}.$$

Definition 16. Let G be a graph then define $\mu_1(G)$ to be the second largest eigenvalue of G.

The difference $\mu_0 - \mu_1$ between the largest and second largest eigenvalue of a graph is called the *spectral gap* and will play a key role in what is to come.

2.3.1 Eigenvalues of Cayley Graphs

Definition 17. Let H be a group. $S \subseteq H$ is a *symmetric* subset of H if for all $s \in S$ we have $s^{-1} \in S$. The *Cayley graph* of H with respect to S is the graph $G = (V, E)$ where $V = H$, and $\{a, b\}$ with $a, b \in H$ is an edge iff there exists an $s \in S$ such that $a = sb$.

The additional structure of Cayley graphs allows easier computation of their eigenvalues. Here we outline a method of Babai [6] who extended work by Lovász [64].

The method we outline makes use of representation theory of groups. For general background on representation theory, one can consult Serre [86] or Fulton and Harris [38]. Let H be a finite group, then a *representation* (also called a *linear representation*, *matrix representation*, or *group representation*) of H is a function $\rho\colon H \to \mathrm{GL}(V)$, where V is a vector space, such that $\rho(a)\rho(b) = \rho(ab)$ for all $a, b \in H$. This implies that $\rho(e) = \mathrm{I}$ and $\rho\left(a^{-1}\right) = \rho\left(a\right)^{-1}$, where e is the group identity and I is the identity matrix. We will only consider representations where the vector space V is $\mathbb{C}^n$, and we naturally identify $\mathrm{GL}\left(\mathbb{C}^n\right)$ with the set of $n \times n$ complex matrices, $\mathrm{M}_{n\times n}\left(\mathbb{C}\right)$.

For a representation ρ, define the associated *character*, $\chi_\rho\colon H \to \mathbb{C}$, of ρ to be the trace of ρ:

$$\chi_\rho = \mathrm{Tr}(\rho).$$

Let $\rho\colon H \to \mathrm{GL}\left(V\right)$ be a representation and suppose that W is a subspace of V that is invariant under every matrix in $\rho\left(H\right)$, then ρ is said to be *reducible* because it can be split into two smaller dimensional representations in the following way. There exists another subspace W' of V that is also invariant under every matrix in $\rho\left(H\right)$ and such that $V = W \oplus W'$ (this is not true for general group representations, but is true under our assumptions). Let $\left.\rho\right|_W : H \to \mathrm{GL}\left(W\right)$ be defined by $\left.\rho\right|_W\left(h\right) = \left.\rho\left(h\right)\right|_W$ and define $\left.\rho\right|_{W'}$ similarly. Then ρ can be written as the direct sum of $\left.\rho\right|_W$ and $\left.\rho\right|_{W'}$ in the following way: if $\rho_1\colon H \to \mathrm{GL}\left(W\right)$ and $\rho_2\colon H \to \mathrm{GL}\left(W'\right)$ are representations then define $\rho_1 \oplus \rho_2\colon H \to \mathrm{GL}\left(V\right)$ by

$$\left(\rho_1 \oplus \rho_2\right)\left(h\right) = \rho_1\left(h\right) \oplus \rho_2\left(h\right).$$

Then $\rho = \left.\rho\right|_W \oplus \left.\rho\right|_{W'}$. It is a result of representation theory that for a finite group there is a finite list of *irreducible representations* of that group. In fact, there are exactly as many irreducible representations of H as conjugacy classes of H ([86]). Let $\rho_1, \ldots, \rho_m$ be the irreducible representations of H with corresponding dimensions $n_1, \ldots, n_m$ and associated characters $\chi_1, \ldots, \chi_m$. Then

$$\sum_{i=1}^{m} {n_i}^2 = \left|H\right|.$$

If H is Abelian, H has $|H|$ conjugacy classes, and so the irreducible representations of H are all 1-dimensional.

Theorem 18 (Babai [6])**.** *Let G be the Cayley graph of H with respect to $S \subseteq H$. Let $\rho_1, \ldots, \rho_m$ be the set of irreducible representations of H. Let $\chi_i = \mathrm{Tr}\,\rho_i$ be the character associated with ρ_i for $i = 1, \ldots, m$, and let n_i be the dimension of the representation ρ_i for $i = 1, \ldots, m,$. Then the eigenvalues of G can be written*

$$\lambda_{ij} \quad \text{for} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n_i$$

where λ_{ij} has multiplicity n_i, and for any natural number t

$$\sum_{j=1}^{n_i} {\lambda_{ij}}^t = \sum_{h_1,\ldots,h_t \in S} \chi_i \left(\prod_{\nu=1}^{t} h_\nu \right).$$

In the case that H is commutative the formula is particularly simple. In that case, every irreducible character has dimension one, there are $|H|$ irreducible characters, and the formula becomes

$$\lambda_i = \sum_{h \in S} \chi_i \left(h \right). \tag{2.1}$$

In the non-commutative case we can use the *Newton identities* to find the eigenvalues from the sums of powers of eigenvalues. Let

$$p(x) = \prod_{i=1}^{n} (x - a_i)$$

and

$$S_t = \sum_{i=1}^{n} {a_i}^t$$

and define the $(n+1) \times (n+1)$ matrix

$$D(x) = \begin{bmatrix} x^n & x^{n-1} & x^{n-2} & x^{n-3} & \ldots & x^2 & x & 1 \\ S_1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ S_2 & S_1 & 2 & 0 & \ldots & 0 & 0 & 0 \\ S_3 & S_2 & S_1 & 3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & \ldots & S_1 & n-1 & 0 \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & \ldots & S_2 & S_1 & n \end{bmatrix}.$$

Then

$$p(x) = \frac{\det D(x)}{n!}.$$

We apply the procedure separately for each irreducible character, so the n used here would be the dimension of the particular irreducible character. And since $\sum_{i=1}^{m} {n_i}^2 = |H|$, each n_i will be at most $\sqrt{|H|}$, thus the matrix $D\left(x\right)$ will be an $r \times r$ matrix where r is no more than $\sqrt{|H|} + 1$, and hence it is much easier to find the eigenvalues of $D\left(x\right)$ than finding the eigenvalues of an $|H| \times |H|$ matrix.

Example 19. The *cycle graph* with n vertices, C_n, is the Cayley graph of $\mathbb{Z}_n$ with respect to $\{-1, 1\}$ under addition. $\mathbb{Z}_n$ is Abelian so every irreducible representation has dimension 1, and so the corresponding characters are also representations. For a character χ we have that for any $h \in C_n$

$$\chi(h)^n = \chi(h^n) = \chi(0) = 1,$$

so the values of a character are nth roots of unity. (Note that 0 is the group identity and not 1.) If $\chi_a(1) = \chi_b(1)$ then

$$\chi_a(k) = \chi_a(\underbrace{1 + \cdots + 1}_{k \text{ times}}) = \underbrace{\chi_a(1) \ldots \chi_a(1)}_{k \text{ times}} = \underbrace{\chi_b(1) \ldots \chi_b(1)}_{k \text{ times}} = \chi_b(\underbrace{1 + \cdots + 1}_{k \text{ times}}) = \chi_b(k),$$

and so $\chi_a = \chi_b$. So

$$\left\{\chi(1) \,\middle|\, \chi \text{ an irreducible character of } C_n\right\} = \left\{e^{\frac{2\pi j}{n}\mathrm{i}} \,\middle|\, j = 0, \ldots, n-1\right\}.$$

Index the irreducible characters so that $\chi_j(1) = e^{\frac{2\pi j}{n}\mathrm{i}}$ for $j = 0, 1, \ldots, n-1$. Then using equation (2.1) we get

$$\lambda_j = \sum_{h \in S} \chi_j(h) = \chi_j(1) + \chi_j(-1) = e^{\frac{2\pi j}{n}\mathrm{i}} + e^{-\frac{2\pi j}{n}\mathrm{i}} = 2\cos\left(\frac{2\pi j}{n}\right).$$

Where we have used the fact that $\chi(a^{-1}) = \chi(a)^{-1}$.

Chapter 3

Expanders

Definition 20. Let $G = (V, E)$ be a graph and for any $S \subseteq V$ define ∂S, the *boundary* of S, to be

$$\partial S = \{\{v, w\} \in E \mid v \in S, w \notin S\}.$$

Definition 21. The *expansion constant* (also known as the *expanding constant*, the *isoperimetric constant*, or the *Cheeger constant*) of a graph $G = (V, E)$ is

$$h(G) = \min \left\{ \frac{|\partial S|}{\min\{|S|, |V \setminus S|\}} \,\middle|\, \varnothing \neq S \subsetneq V \right\}$$

For finite graphs, the symmetry of the formula allows us to only consider subsets S with $\frac{|V|}{2}$ vertices or fewer and the definition simplifies to

$$h(G) = \min \left\{ \frac{|\partial S|}{|S|} \,\middle|\, \varnothing \neq S \subset V,\ |S| \leq \frac{|V|}{2} \right\}$$

Definition 22. Let G be a graph on n vertices. We say G is a (n, d)-expander if $h(G) \geq d$.

We also refer to any graph with "large" expansion constant as an *expander* — obviously this is an informal title. We may also refer to the *expansion* of a graph as a property of the graph that is measured by the expanding constant — as in a graph with "good" expansion is a graph with "large" expanding constant.

Example 23. Let K_n be the *complete graph* on n vertices. Let S be a subset of r vertices of K_n, then $|\partial S| = r(n-r)$ because each vertex in S is adjacent to $n-r$ vertices outside of S. So $h(K_n) = \min\left\{\frac{r(n-r)}{\min\{r,n-r\}}\right\}$. Without loss of generality, we can assume $r \leq \left\lfloor\frac{n}{2}\right\rfloor$, so $h(K_n) = \min\left\{\frac{r(n-r)}{r}\right\} = \min\{n-r\} = \left\lceil\frac{n}{2}\right\rceil$.

Example 24. Let C_n be the cycle graph with n vertices. The minimum value of $|\partial S|$ for nonempty $S \subsetneqq V$ is 2 and occurs when S is a chain of vertices. In which case $\frac{|\partial S|}{\min\{|S|,|V\setminus S|\}}$ takes the minimum value of $\frac{2}{\left\lfloor\frac{n}{2}\right\rfloor} \approx \frac{4}{n}$ when S is a chain of $\left\lfloor\frac{n}{2}\right\rfloor$ vertices. Hence $h(C_n) = \frac{2}{\left\lfloor\frac{n}{2}\right\rfloor} \to 0$ as $n \to \infty$.

K_n and C_n provide two extremes of regular connected graphs, with K_n being "maximally connected" and C_n being "minimally connected", and the differing amount of connection shows up in the expansion constant. (C_n) is a family of 2-regular graphs, whereas K_n is a $(n-1)$-regular graph. The challenge is to find a family of graphs with fixed regularity (like (C_n)) but with asymptotically non-zero expansion (like (K_n)). This motivates the definition of expander family.

Definition 25. The sequence $G_n = (V_n, E_n)$ of connected graphs is an *expander family* if each G_n is a k-regular graph for some fixed k, $|V_n| \to \infty$, and there exists $\delta > 0$ such that $h(G_n) \geq \delta$ for all $n \in \mathbb{N}$.

One way of thinking of graphs with "good" expansion is graphs without "bottlenecks". We can think of a bottleneck of a graph G as a "small" subset of the edges of G whose removal disconnects G into two "large" components (see Figure 3.1).

3.1 Some Constructions

The first construction of expander families was given by Margulis [72]. This construction was improved by Gabber and Galil [39]. And it was further improved by Alon, Galil, and Milman [3].

In this section we allow graphs to have multiple edges and self loops, where each self loop contributes 2 to the degree of a vertex.

Most of the early constructions of expanders actually constructed structures closely related to expanders that we will call *b-expanders*, which can

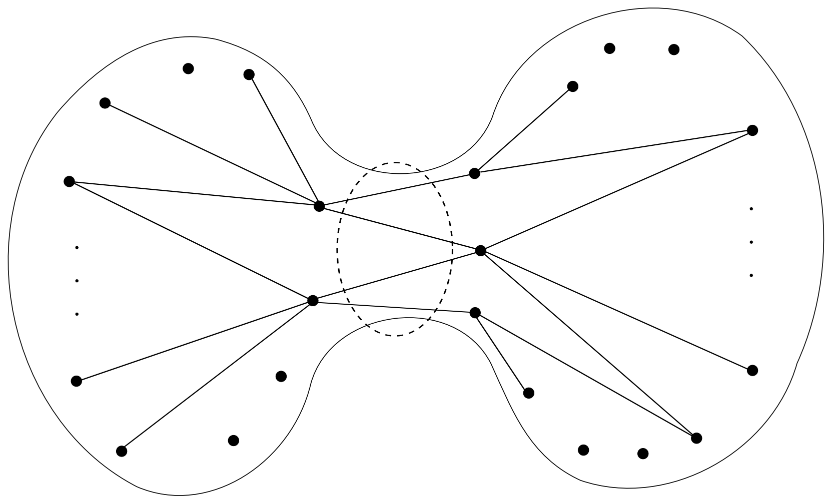


Figure 3.1: A bottleneck.

be thought of as bipartite analogues of expanders. In the literature of the time b-expanders were called "expanders", and our definition of expanders was not used. A bipartite graph $G = (V = A\dot{\cup}B, E)$ is a (n, k, d)-b-expander if $|A| = |B| = n$, G has no more than kn edges and for every $X \subseteq A$

$$\frac{|\mathrm{N}(X)|}{|X|} \geq 1 + d\left(1 - \frac{|X|}{n}\right).$$

To make the relationship between expanders and b-expanders clear we introduce the notion of the *double cover* of a graph. The *double cover* (sometimes the *extended double cover*) of a graph $G = (V, E)$ is the graph

$$\overline{G} = \left((V \times \{0\}) \bigcup (V \times \{1\}), \overline{E}\right)$$

where $(v, 0)$ is adjacent to $(v, 1)$ for all $v \in V$ and $(v, 0)$ is adjacent to $(w, 1)$ iff v is adjacent to w in G. If G is (n, c)-expander with maximal degree d then $\overline{G}$ is a $(n, d + 1, c)$-b-expander (Alon [1]). Further, if G is an (n, d, c)-b-expander then G is a $\left(n, \frac{c}{16}\right)$-expander (Alon [1]). So the expanders and b-expanders are two different manifestations of one idea.

Margulis [72] defines the sequence of graphs (M_i). M_i has vertex bipartition (A_i, B_i), where $A_i = B_i = \mathbb{Z}_i \times \mathbb{Z}_i$, and each $(x, y) \in A_i$ is adjacent to the elements of B_i

$$(x, y), (x + 1, y), (x, y + 1), (x, x + y), (-y, x).$$

Margulis proves that there exists a $d > 0$ such that for $i = 1, 2, \ldots$ M_i is a $(i^2, 5, d)$-b-expander. However, Margulis does not determine any specific d that would make this statement true.

Gabber and Galil [39] modify Margulis' construction and determine a constant explicitly. They define a sequence of graphs (G_i) where $G_i = (V_i = A_i \dot{\cup} B_i, E_i)$, $A_i = B_i = \mathbb{Z}_i \times \mathbb{Z}_i$, and each element $(x, y) \in A_i$ is adjacent to the elements of B_i

$$(x, y), (x, x + y), (x, x + y + 1), (x + y, y), (x + y + 1, y).$$

They showed that for $i = 1, 2, \ldots$ G_i is a $\left(i^2, 5, \frac{2-\sqrt{3}}{4}\right)$-b-expander ($\frac{2-\sqrt{3}}{4} \approx 0.067$). They further define graphs $G_i{}'$ on the same vertices as G_i but with more edges and with an improved constant. Each element $(x, y) \in A_i{}'$ is adjacent to the elements of $B_i{}'$

$$\begin{array}{lll} (x, y) & (x, y + 2x) & (x + 2y, y) \\ & (x, y + 2x + 1) & (x + 2y + 1, y) \\ & (x, y + 2x + 2) & (x + 2y + 2, y). \end{array}$$

They show that for $i = 1, 2, \ldots$ $G_i{}'$ is a $\left(i^2, 7, \frac{2-\sqrt{3}}{2}\right)$-b-expander ($\frac{2-\sqrt{3}}{2} \approx 0.134$).

Alon, Galil, and Milman [3] improve on these results. They define graphs $H_i = (V_i = C_i \dot{\cup} D_i, E_i)$ where $C_i = D_i = \mathbb{Z}_i \times \mathbb{Z}_i$ and each element $(x, y) \in C_i$ is adjacent to the elements of D_i (x, y) and $\sigma_i (x, y)$ and $\sigma_i{}^{-1} (x, y)$ for $i = 1, 2, 3, 4, 5, 6$, where the σ_i are permutations of $\mathbb{Z}_i \times \mathbb{Z}_i$ defined as follows:

$$\begin{aligned} \sigma_1 (x, y) &= (x, y + 2x) \\ \sigma_2 (x, y) &= (x, y + 2x + 1) \\ \sigma_3 (x, y) &= (x, y + 2x + 2) \\ \sigma_4 (x, y) &= (x + 2y, y) \\ \sigma_5 (x, y) &= (x + 2y + 1, y) \\ \sigma_6 (x, y) &= (x + 2y + 2, y). \end{aligned}$$

They also define a different, but related, sequence of graphs (T_i). Let $T_i = (V = C_i{}' \dot{\cup} D_i{}', E_i{}')$, where $C_i{}' = D_i{}' = \mathbb{Z}_i \times \mathbb{Z}_i$ and $(x, y) \in C_i{}'$ is adjacent to the elements of $D_i{}'$ (x, y) and $\sigma_i (x, y)$ and $\sigma_i{}^{-1} (x, y)$ for $i = 1, 2, 4, 5$. They used results of Jimbo and Maruoka [51] to show that for $i = 1, 2, \ldots$ H_i is a $(i^2, 13, c)$-b-expander where $c \approx 0.466$; and that for $i = 1, 2, \ldots$ T_i is a $(i^2, 9, c')$-b-expander where $c' \approx 0.412$. Specifically

$$c = \frac{4}{\alpha + \sqrt{1 + \alpha^2}}, \qquad \alpha = \frac{1 + 2d}{4d}, \qquad d = \frac{2 - \sqrt{3}}{4}$$

and

$$c' = \frac{4}{\alpha' + \sqrt{1 + \alpha'^2}}, \qquad \alpha' = \frac{1 + 2d'}{4d'}, \qquad d' = \frac{8 - 5\sqrt{2}}{16}.$$

3.2 Zig-Zag Graph Product

Reingold, Vadhan, and Wigderson [84] defined the *zig-zag graph product*, as a way of combining two expander graphs to get a larger expander. We will indicate the zig-zag product using the notation "ⓩ". In this section we allow graphs to have multiple edges and loops. Given a D_1-regular graph G_1 with N_1 vertices and a D_2-regular graph G_2 with D_1 vertices, the *zig-zag product* of G_1 and G_2, $G_1 ⓩ G_2$, is a ${D_2}^2$-regular graph with $N_1 D_1$ vertices. The actual definition of the zig-zag product is intricate and will be describe later. Reingold, Vadhan, and Wigderson [84] show that if $\mu_1(G_1) \leq \lambda_1$ and $\mu_1(G_2) \leq \lambda_2$ then $\mu_1(G_1 ⓩ G_2) \leq \lambda_1 + \lambda_2 + {\lambda_2}^2$ (in this section $\mu_1(G)$ refers to the normalized second largest eigenvalue of the graph G, which is the second largest eigenvalue of the graph divided by the regularity of the graph). Although we have not yet made clear the connection between the expansion properties of a graph and its second largest eigenvalue, for now we will make do with the informal notion that "small second largest eigenvalue implies large expansion constant" and vice versa; we will make this explicit in Chapter 4.

There are connections between the zig-zag graph product and the semi-direct product of groups. Specifically, Alon, Lubotzky, and Wigderson [4] showed that the for groups A and B the Cayley graph of $A \rtimes B$ (the semi-direct product of A and B) with certain generators is equal to the zig-zag graph product of the Cayley graph of A with certain generators and the Cayley graph of B with certain generators.

The zig-zag graph product can be used in a concrete recursive construction (provided a base graph), we also need the concept of the square of a graph. If G is a graph with adjacency matrix A then by G^2 we mean the graph whose adjacency matrix is A^2. If G is a D-regular graph with N vertices that satisfies $\mu_1(G) \leq \lambda$ then G^2 is a D^2-regular graph with N vertices that satisfies $\mu_1(G^2) \leq \lambda^2$.

We start with a D-regular graph H with D^4 vertices that satisfies $\mu_1(H) \leq \frac{1}{5}$. Then define a sequence of graphs (G_i) as follows. $G_1 = H^2$ and

$$G_{i+1} = {G_i}^2 \, ⓩ \, H.$$

Then G_i is a D^2-regular graph with D^{4i} vertices that satisfies $\mu_1(G_i) \leq \frac{2}{5}$.

To describe the zig-zag product and recursive construction of Reingold, Vadhan, and Wigderson in detail we need to first introduce a number of preliminary concepts. We use the notation $[N]$ to denote the set $\{1, \ldots, N\}$ of natural numbers. We assume that all graphs are regular and have vertex set $[N]$ for some N. For a D-regular graph we assume that the edges incident any vertex are numbered by the set $[D]$; these numbers are local to each vertex, so each edge is numbered twice: once for each vertex it is incident to (or in the case of a loop, once for each "end" of the edge). For a D-regular graph G with vertex set $[N]$ we define the *rotation map* of G, $\mathrm{Rot}_G\colon [N] \times [D] \to [N] \times [D]$, by

$$\mathrm{Rot}_G(v, i) = (w, j)$$

if v and w are connected by an edge, and that edge is numbered i relative to v and numbered j relative to w. (Note that this is different from the similar concept of rotation systems.) It is useful to think of each edge as two directed half edges: one in each direction. If $\mathrm{Rot}_G(v, i) = (w, j)$, then we label the directed half edge from v to w with i and the directed half edge from w to v with j.

The *tensor product* of the graphs G and H, denoted $G \otimes H$, is the graph whose adjacency matrix is the tensor product of the adjacency matrices of G and H respectively, or

$$A_{G \otimes H} = A_G \otimes A_H.$$

The *tensor product* of two matrices is best defined visually. Let A be an $m \times n$ matrix, and let B be a matrix, then

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{bmatrix}.$$

This is also known as the Kronecker product, consult Dummit and Foote [30] for details. For a D_1-regular graph G_1 on $[N_1]$ and a D_2-regular graph G_2 on $[N_2]$, $G_1 \otimes G_2$ is a $(D_1 \cdot D_2)$-regular graph on $[N_1] \times [N_2]$, if the edge from v to v' in G_1 is numbered i with reverse edge numbered i' and the edge from w to w' in G_2 is numbered j with reverse edge numbered j' then there is an edge from (v, w) to (v', w') in $G_1 \otimes G_2$ numbered (i, j) with reverse edge numbered (i', j'). For two square matrices A and B, the eigenvalues of $A \otimes B$ are all of the pairwise products of eigenvalues of A and eigenvalues of B (counted according to multiplicity).

Let G be a D-regular graph and r a natural number. Define the r^{th} *power of* G, denoted G^r, to be the graph with adjacency matrix $(A_G)^r$, where A_G is the adjacency matrix of G. G^r is a D^r-regular graph. In G^r there is an edge between v and w for every path of length r in G from v to w. Let $v = v_0, v_1, \ldots, v_r = w$ be a path of length r in G, then we number the associated edge from v to w in G^r with the r-tuple $(i_1, \ldots, i_r) \in [D]^r$ where the path used the edge numbered i_α in going from $v_{\alpha-1}$ to v_α. The reverse edge from w to v is numbered with the r-tuple $(j_1, \ldots, j_r)$, where j_α is the number of the reverse edge from $v_{r-\alpha}$ to $v_{r-\alpha+1}$ numbered $i_{r-\alpha+1}$. Observe that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G, then the eigenvalues of G^r are ${\lambda_1}^r, \ldots, {\lambda_n}^r$, since

$$A_{G^r} = (A_G)^r .$$

We are now in a position to define the zig-zag product. Let G be a D-regular graph on $[N]$, and let H be a K-regular graph on $[D]$. (Note that the regularity of G is the same as the number of vertices of H.) In order to define the zig-zag product of G and H, $G \textcircled{z} H$, we consider a graph F defined on $[N] \times [D]$. Refer to Figure 3.2. For each $v \in [N]$ we call the set $\{(v, k) \mid k \in [D]\}$ the v-*cloud* in F. The v-cloud contains D vertices, one for each edge of v. So for each edge of G, say an edge between v and w, we connect (v, i) to (w, j) with an edge where $\mathrm{Rot}_G(v, i) = (w, j)$. Finally we add edges within each cloud to make each cloud a "copy" of H: for each v in G we add edges between (v, i) and (v, j) whenever i and j have an edge between them in H.

Now we are ready to define $G \textcircled{z} H$. $G \textcircled{z} H$ has vertex set $[N] \times [D]$ (the same vertex set as F), and two vertices are adjacent in $G \textcircled{z} H$ if there is a certain type of path of length 3 between those same two vertices in the graph F.

Let (v, k) be a vertex in F. We consider paths with the following three

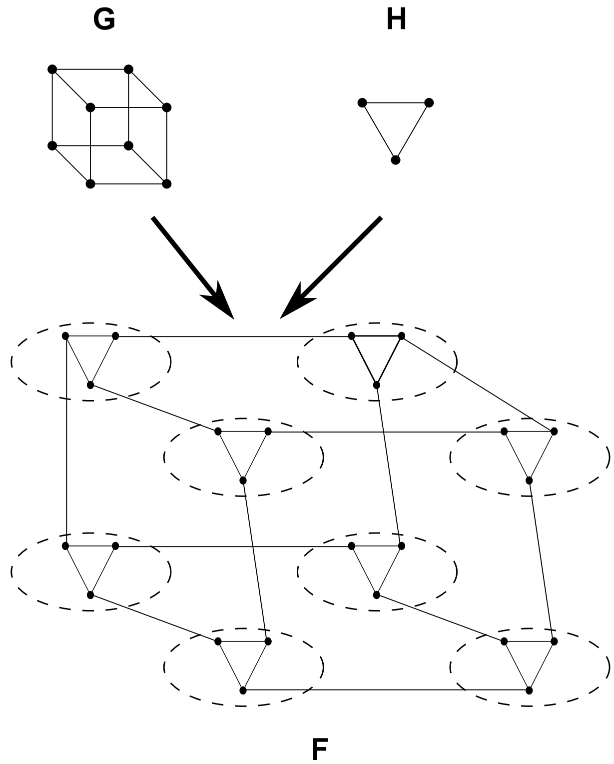


Figure 3.2: One example of the graph F produced from the graphs G and H.

steps. Refer to Figure 3.3.

- Move from (v, k) to (v, k') inside the v-cloud. There are K choices here.
- Move from the v-cloud to another cloud. There is no choice here, there is only one edge linking (v, k') to another cloud, say the edge is to (w, ℓ'). So we move from the v-cloud to the w-cloud, specifically: $(v, k') \to (w, \ell')$.
- Move from (w, ℓ') to (w, ℓ) inside the w-cloud. There are K choices here.

Such a path is called a *zig-zag path*. In order to define the rotation map of $G \textcircled{z} H$ we index the edges of $G \textcircled{z} H$ by $[D] \times [D]$ (instead of $[D^2]$). When we move from (v, k) to (v, k') within the v-cloud we record the edge number

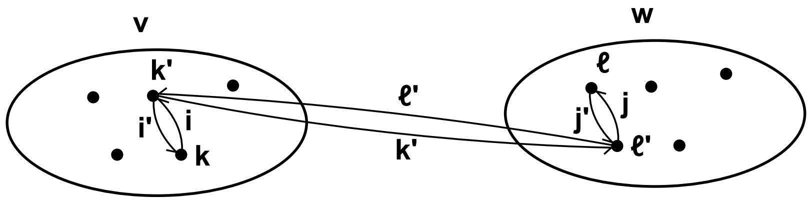


Figure 3.3: Illustration of a zig-zag path.

in H we used to get from k to k', call it i (and let i' be the number of the reverse edge from k' to k). And when we move from (w, ℓ') to (w, ℓ) within the w-cloud we record the edge number in H we used to get from ℓ' to ℓ, call it j (and let j' be the number of the reverse edge from ℓ to ℓ'). Then we label the edge from (v, k) to (w, ℓ) in $G \textcircled{z} H$ with number (i, j). The reverse edge, from (w, ℓ) to (v, k), is labelled (j', i'). In essence, (i, j) records the path taken in F, and (j', i') records the reverse path. An example of a zig-zag product is shown in Figure 3.4.

We now consider the effect of the above operations on the eigenvalues of the graphs. When working with the eigenvalues of regular graphs of different regularity it is convenient to define the *normalized adjacency matrix* of a graph. The normalized adjacency matrix of a D-regular graph G is $N_G = \frac{1}{D} A_G$ where A_G is the adjacency matrix of G. This has the effect of dividing each eigenvalue by D, hence 1 is the largest eigenvalue of N_G, and all the eigenvalues of N_G lie in $[-1, 1]$.

Let G_1 be a D_1-regular graph and G_2 be a D_2-regular graph. Let λ_1 be the second largest normalized eigenvalue of G_1, λ_2 the second largest normalized eigenvalue of G_2. Then the following are easy to see given the comments about eigenvalues and powers/tensoring of graphs above.

- The second largest normalized eigenvalue of ${G_1}^r$ is ${\lambda_1}^r$.
- The second largest normalized eigenvalue of $G_1 \otimes G_2$ is $\max\{\lambda_1, \lambda_2\}$.

Reingold, Vadhan, and Wigderson [84] show that the following holds for the zig-zag product: the second largest normalized eigenvalue of $G_1 \textcircled{z} G_2$ is less than or equal to $\lambda_1 + \lambda_2 + {\lambda_2}^2$.

Reingold, Vadhan, and Wigderson [84] used these operations to define an infinite sequence of constant degree graphs with expansion bounded away

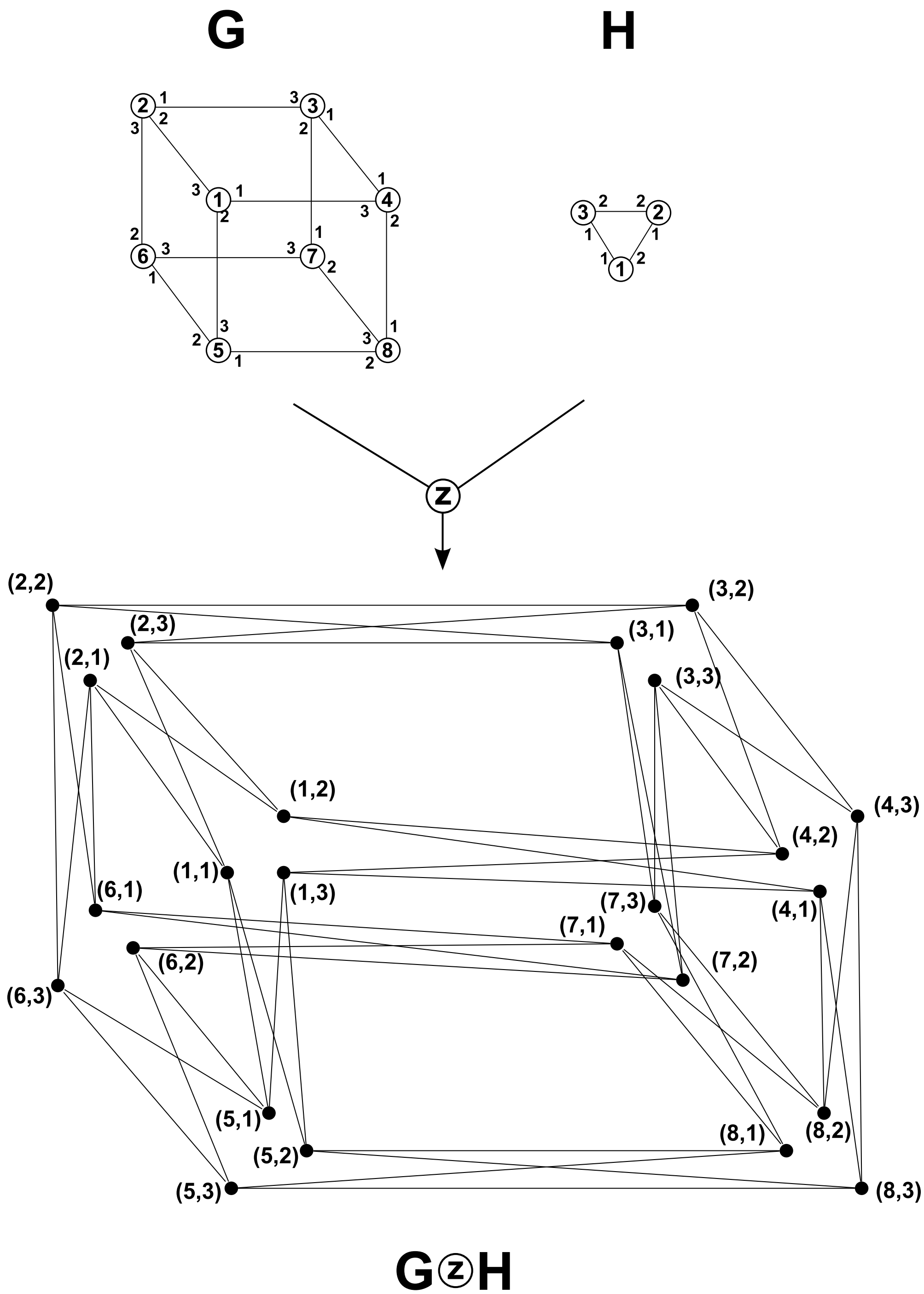


Figure 3.4: An example of a zig-zag product.

from zero. Let H be a D-regular graph with D^8 vertices with second largest normalized eigenvalue λ. Define G_1 to be H^2 and G_2 to be $H \otimes H$. We recursively define G_n as follows:

$$G_n = \left(G_{\lceil \frac{n-1}{2} \rceil} \otimes G_{\lfloor \frac{n-1}{2} \rfloor}\right)^2 ⓩ H.$$

G_n is a D^2-regular graph with D^{8n} vertices. Using the link between the second largest eigenvalue and the expansion constant of a graph which will be made explicit in Chapter 4 and the remarks above about eigenvalues allow one to show that (G_n) is a expander family provided that λ is small enough (say, $\lambda < \frac{1}{5}$) [84].

Chapter 4

Ramanujan Graphs

In this chapter we seek to explore the connection between the eigenvalues of a graph and its expansion.

Example 26. The eigenvalues of K_n are $n-1$ and -1 with respective multiplicities 1 and $n-1$. From Example 19 we know that the eigenvalues of C_n are $2\cos\left(\frac{2k\pi}{n}\right)$ for $k = 0, 1, \ldots, (n-1)$ (listed according to multiplicity). Notice that in the "more connected" K_n the gap between the largest eigenvalue and the second largest eigenvalue is n and for C_n this gap is $2 - 2\cos\left(\frac{2\pi}{n}\right)$ which goes to 0 as $n \to \infty$. This is no coincidence as the following theorem shows.

Theorem 27 (Alon-Milman [2], Dodziuk [29], Cheeger [17]-Buser [14])**.** *Let G be a k-regular connected graph then*

$$\frac{k-\mu_1}{2} \leq h(G) \leq \sqrt{2k(k-\mu_1)}$$

This tells us that for a fixed k, purely in terms of the spectrum, in order to get a large expansion constant we want to minimize the second largest eigenvalue of the graph.

Definition 28. For a k-regular graph the quantity $k-\mu_1$ is called the *spectral gap*. (This quantity is also sometimes called the *algebraic connectivity*.)

So the above theorem relates the expansion constant of the graph to the spectral gap of the graph.

In order to prove this theorem we first need the following definition and a small lemma pertaining to it.

Definition 29. Let A be an $n \times n$ complex matrix that is self-adjoint and let $\vec{x}$ be a complex n-dimensional vector. Then the *Rayleigh quotient* of A and $\vec{x}$ is the quantity

$$\frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle}.$$

Lemma 30. *Let A be a real symmetric $n \times n$ matrix with eigenvalues $\mu_0 \geq \cdots \geq \mu_{n-1}$. If μ_0 is simple and $\vec{e}_0$ is an eigenvector of μ_0, and if $\vec{x}$ is any vector with $\langle \vec{x}, \vec{e}_0\rangle = 0$ then the Rayleigh quotient of A and x satisfies*

$$\mu_1 \geq \frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle} \geq \mu_{n-1}.$$

Analogously if μ_{n-1} is simple and $\vec{e}_{n-1}$ is an eigenvector of μ_{n-1}, then if $\vec{x}$ is any vector with $\langle \vec{x}, \vec{e}_{n-1}\rangle = 0$ then the Rayleigh quotient of A and x satisfies

$$\mu_0 \geq \frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle} \geq \mu_{n-2}.$$

Proof. Let $\vec{e}_0, \ldots, \vec{e}_{n-1}$ be an orthonormal basis of eigenvectors of A with $\vec{e}_i$ corresponding to μ_i (by the Principal axis theorem, Theorem 9, these exist). Let $\vec{x} = \sum_{i=0}^{n-1} x_i \vec{e}_i$ be a vector, and consider the Rayleigh quotient.

$$\begin{aligned}
\frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle} &= \frac{\langle A\left(\sum x_i \vec{e}_i\right), \sum x_i \vec{e}_i\rangle}{\langle \sum x_i \vec{e}_i, \sum x_i \vec{e}_i\rangle} \\
&= \frac{\langle \sum x_i A\vec{e}_i, \sum x_i \vec{e}_i\rangle}{\langle \sum x_i \vec{e}_i, \sum x_i \vec{e}_i\rangle} \\
&= \frac{\langle \sum x_i \mu_i \vec{e}_i, \sum x_i \vec{e}_i\rangle}{\sum x_i^2} \\
&= \frac{\sum {x_i}^2 \mu_i}{\sum {x_i}^2} \\
&= \sum \alpha_i \mu_i
\end{aligned}$$

where $\alpha_i = \frac{{x_i}^2}{\sum {x_i}^2}$. Then $\sum \alpha_i = 1$ and $0 \leq \alpha_i \leq 1$. So $\frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle}$ is a convex combination of the eigenvalues of A, hence

$$\mu_0 \geq \frac{\langle A\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle} \geq \mu_{n-1} \tag{4.1}$$

If μ_0 is a simple eigenvalue, then its eigenspace is spanned by $\vec{e}_0$, so if $\langle \vec{x}, \vec{e}_0 \rangle = 0$ then $x_0 = 0$, and $\dfrac{\langle A\vec{x}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle}$ is a convex combination of the eigenvalues $\mu_1, \ldots, \mu_{n-1}$, hence

$$\mu_1 \geq \frac{\langle A\vec{x}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} \geq \mu_{n-1}.$$

Similarly if μ_{n-1} is simple. □

We note that Equation 4.1 (which is true for any vector $\vec{x}$) can be used to show one half of Theorem 12. Indeed, let $\vec{x}$ be the all 1 vector, then

$$\mu_0 \geq \frac{\langle A\vec{x}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} = \frac{\langle (\deg(1), \ldots, \deg(n)), (1, \ldots, 1) \rangle}{n} = \frac{1}{n} \sum_i \deg(i).$$

Proof of Theorem 27. This proof is based on the one in Davidoff, Sarnak, and Valette [25].

First we prove the left inequality.

Let G be a k-regular connected graph with n vertices and adjacency matrix A, with eigenvalues $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}$.

Now let F be a subset of the vertices of G and define a vector $\vec{x} = (x_1, \ldots, x_n)$ by

$$x_i = \begin{cases} |V| - |F| & i \in F \\ -|F| & i \notin F \end{cases}.$$

Since G is a connected regular graph $\mu_0 = k$ is a simple eigenvalue with eigenspace spanned by $(1, \ldots, 1)$ and

$$\langle \vec{x}, (1, \ldots, 1) \rangle = \sum x_i = |F| (|V| - |F|) + (|V| - |F|)(-|F|) = 0, \quad (4.2)$$

so Lemma 30 applies to $\vec{x}$.

Let K be the *signed vertex-edge incidence matrix* — the matrix with rows indexed by V and columns indexed by E so that the row corresponding to the vertex i has a 1 or -1 in each column corresponding to the edges vertex i is incident to, each column having exactly one 1 and exactly one -1, for convenience we assume that the 1 corresponds to the vertex of lowest index. We give an example of one such matrix K which shows two columns corresponding to two edges that share one vertex in common.

$$K = \begin{bmatrix} \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \vdots \\ -1 & \vdots \\ 0 & \vdots \\ \vdots & \vdots \\ \vdots & 0 \\ \vdots & -1 \\ \vdots & 0 \\ \vdots & \vdots \end{bmatrix}$$

Then $KK^t = k\,\mathrm{I} - A = \Delta$, which is called the *Laplacian* of the graph. In general the Laplacian has $\deg(v_i)$ as the ith diagonal entry and -1 in the i, j entry if v_i is adjacent to v_j. Observe that for e an edge we have

$$\left(K^t \vec{x}\right)_e = \begin{cases} \pm\,|V| & e \in \partial F \\ 0 & e \notin \partial F \end{cases} .$$

Now

$$\left\|K^t \vec{x}\right\|^2 = \left\langle K^t \vec{x}, K^t \vec{x} \right\rangle = |\partial F|\, |V|^2 .$$

Consider the Rayleigh quotient of the Laplacian at $\vec{x}$.

$$\begin{aligned} \frac{\langle \Delta \vec{x}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} &= \frac{\langle K K^t \vec{x}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} \\ &= \frac{\langle K^t \vec{x}, K^t \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} \\ &= \frac{|\partial F|\, |V|^2}{(|V| - |F|)\, |F|\, |V|} \\ &= \frac{|\partial F|\, |V|}{|F|\, (|V| - |F|)}. \end{aligned} \tag{4.3}$$

The eigenvalues of $\Delta = k\,\mathrm{I} - A$ are (in ascending order) $0 = k - \mu_0, k - \mu_1, \ldots, k - \mu_{n-1}$. So the smallest eigenvalue of Δ, 0, is a simple eigenvalue of Δ since its corresponding eigenvalue of A (k) is simple. In both cases (0 an eigenvalue of Δ and k an eigenvalue of A) the eigenspace is spanned by $(1, \ldots, 1)$. Using Equation 4.3 and Lemma 30 on the eigenvalue $k - \mu_0 = 0$ of Δ, the smallest eigenvalue of Δ, we get

$$k - \mu_1 \leq \frac{\langle \Delta\vec{x}, \vec{x}\rangle}{\langle \vec{x}, \vec{x}\rangle} = \frac{|\partial F|\,|V|}{|F|\,(|V| - |F|)} \leq k - \mu_{n-1}.$$

We focus only on the left hand inequality. When $0 < |F| \leq \frac{n}{2}$ we get

$$\frac{|\partial F|}{|F|} \geq (k - \mu_1)\,\frac{|V| - |F|}{|V|} \geq (k - \mu_1)\,\frac{n/2}{n} = \frac{k - \mu_1}{2}.$$

This holds for all subsets F with $0 < |F| \leq \frac{n}{2}$, so

$$\frac{k - \mu_1}{2} \leq h(G).$$

Now we prove the right side of the inequality.

Let $\vec{x}$ be an eigenvector of the eigenvalue $k - \mu_1$ of the Laplacian (so it is also an eigenvector of the eigenvalue μ_1 of A). Let $V^+ = \{i \mid x_i > 0\}$. By multiplying by -1 if necessary we can assume $|V^+| \leq \frac{n}{2}$. Let $\vec{y} = (y_1, \ldots, y_n)$ be defined by

$$y_i = \left\{ \begin{array}{ll} x_i & x_i > 0 \\ 0 & \text{otherwise} \end{array} \right. .$$

For $i \in V^+$

$$\begin{aligned}
(\Delta\vec{y})_i &= ky_i - (A\vec{y})_i \\
&= kx_i - \sum_{j \in V^+} a_{ij}x_j \\
&= kx_i - \sum_{j \in V} a_{ij}x_j + \sum_{j \in V \setminus V^+} a_{ij}x_j \\
&\leq kx_i - \sum_{j \in V} a_{ij}x_j \\
&= kx_i - \mu_1 x_i = (k - \mu_1)x_i. \qquad (4.4)
\end{aligned}$$

Using this inequality we obtain the following inequality.

$$\begin{aligned}
\left\|K^t\vec{y}\right\|^2 &= \left\langle K^t\vec{y}, K^t\vec{y}\right\rangle = \left\langle KK^t\vec{y}, \vec{y}\right\rangle = \langle \Delta\vec{y}, \vec{y}\rangle \\
&= \sum_{i\in V}(\Delta\vec{y})_i y_i = \sum_{i\in V^+}(\Delta\vec{y})_i y_i \\
&\leq \sum_{i\in V^+}(k-\mu_1)\,x_i y_i \qquad\qquad \text{(using Equation 4.4)} \\
&= \sum_{i\in V^+}(k-\mu_1)\,{y_i}^2 \\
&= (k-\mu_1)\,\|\vec{y}\|^2 . \qquad\qquad (4.5)
\end{aligned}$$

Define B to be $\sum_{e=\{i,j\}\in E}\left|{y_i}^2 - {y_j}^2\right|$. We will bound B from above and below and use those two inequalities to derive the required inequality.

$$\begin{aligned}
B &= \sum_{e=\{i,j\}\in E}\left|{y_i}^2 - {y_j}^2\right| \\
&= \sum_e |y_i - y_j|\,|y_i + y_j| \\
&\leq \left(\sum_e |y_i - y_j|^2\right)^{\frac{1}{2}}\left(\sum_e |y_i + y_j|^2\right)^{\frac{1}{2}} \qquad \text{(Cauchy-Schwarz)} \\
&= \left\|K^t\vec{y}\right\|\left(\sum_e (y_i + y_j)^2\right)^{\frac{1}{2}} \\
&\leq \left\|K^t\vec{y}\right\|\left(\sum_e 2({y_i}^2 + {y_j}^2)\right)^{\frac{1}{2}} \qquad \left((a+b)^2 \leq 2\left(a^2+b^2\right)\right) \\
&= \left\|K^t\vec{y}\right\|\left(2\sum_{i\in V} k{y_i}^2\right)^{\frac{1}{2}} \\
&= \sqrt{2k}\left\|K^t\vec{y}\right\|\|\vec{y}\| \\
&\leq \sqrt{2k}(k-\mu_1)^{\frac{1}{2}}\|\vec{y}\|\,\|\vec{y}\| \qquad \text{(by (4.5))} \\
&= \sqrt{2k(k-\mu_1)}\,\|\vec{y}\|^2 \qquad\qquad (4.6)
\end{aligned}$$

Now we bound B from below. Let $\beta_0 < \cdots < \beta_r$ be the distinct values of the entries of $\vec{y}$ and for $j = 0, \ldots, r$ let $L_j = \{i \in V \mid y_i \geq \beta_j\}$. Define a function $v\colon V \to [0,\dots,r]$ so that $\beta_{v(i)} = y_i$, and arrange each edge in the sum $B = \sum_e \left|{y_i}^2 - {y_j}^2\right|$ to be written so that $v(i) > v(j)$, then

$$
\begin{aligned}
B &= \sum_e \left|{y_i}^2 - {y_j}^2\right| \\
&= \sum_e {\beta_{v(i)}}^2 - {\beta_{v(j)}}^2 \\
&= \sum_e {\beta_{v(i)}}^2 - {\beta_{v(i)-1}}^2 + {\beta_{v(i)-1}}^2 - \cdots - {\beta_{v(j)+1}}^2 + {\beta_{v(j)+1}}^2 - {\beta_{v(j)}}^2 \\
&= \sum_e \sum_{p=v(j)+1}^{v(i)} {\beta_p}^2 - {\beta_{p-1}}^2.
\end{aligned}
$$

${\beta_p}^2 - {\beta_{p-1}}^2$ occurs in the overall sum once for every edge $\{i,j\}$ such that $v(i) \geq p$ and $p-1 \leq v(j)$. Which is the same as saying $y_i \geq \beta_p$ and $\beta_{p-1} \leq y_j$. Which is equivalent to $i \in L_p$ and $j \notin L_p$; which is another way of saying $\{i,j\} \in \partial L_p$. So

$$B = \sum_{p=1}^{r} |\partial L_p| \left({\beta_p}^2 - {\beta_{p-1}}^2\right).$$

Since $\vec{x}$ is an eigenvector of $k - \mu_1$, it must have zero inner product with the eigenvector $(1,\dots,1)$ of $k - \mu_0$, so $\vec{x}$ must have some entry less than or equal to 0, so $\vec{y}$ is 0 in that coordinate by its definition. So $\beta_0 = 0$. Further note that $\vec{x}$ was chosen so that it had no more $\frac{n}{2}$ coordinates positive. Thus $\vec{y}$ is 0 in at least $\frac{n}{2}$ coordinates, so $|L_0| \geq \frac{n}{2}$, and thus $|L_p| \leq \frac{n}{2}$ for $p = 1,\dots,r$. Thus

$$
\begin{aligned}
B &= \sum_{p=1}^{r} |\partial L_p| \left({\beta_p}^2 - {\beta_{p-1}}^2\right) \\
&\geq \sum_{p=1}^{r} h(G)\,|L_p| \left({\beta_p}^2 - {\beta_{p-1}}^2\right) \qquad \text{(definition of } h\,(G)\,\text{).}
\end{aligned}
$$

We set $L_{r+1} = \varnothing$ for convenience, and continue.

$$\begin{aligned}
B &\geq \sum_{p=1}^{r} h(G)\,|L_p|\left({\beta_p}^2 - {\beta_{p-1}}^2\right) \\
&= h(G)\left(|L_1|\left({\beta_1}^2 - {\beta_0}^2\right) + |L_2|\left({\beta_2}^2 - {\beta_1}^2\right) + \cdots + |L_r|\left({\beta_r}^2 - {\beta_{r-1}}^2\right)\right) \\
&= h(G)\big({\beta_0}^2\,(-|L_1|) + {\beta_1}^2\,(|L_1| - |L_2|) + {\beta_2}^2\,(|L_2| - |L_3|) + \cdots \\
&\qquad\qquad + {\beta_{r-1}}^2\,(|L_{r-1}| - |L_r|) + {\beta_r}^2\,|L_r|\,\big) \\
&= h(G)\big({\beta_0}^2\,(|L_0| - |L_1|) + {\beta_1}^2\,(|L_1| - |L_2|) + {\beta_2}^2\,(|L_2| - |L_3|) + \cdots \\
&\qquad\qquad + {\beta_{r-1}}^2\,(|L_{r-1}| - |L_r|) + {\beta_r}^2\,(|L_r| - |L_{r+1}|)\,\big) \\
&\qquad ({\beta_0}^2\,(-|L_1|) = {\beta_0}^2\,(|L_0| - |L_1|), \text{ since } \beta_0 = 0) \\
&= h(G)\sum_{i=0}^{r} {\beta_i}^2\,(|L_i| - |L_{i+1}|) \\
&= h(G)\,\|\vec{y}\|^2 \qquad (4.7)
\end{aligned}$$

Where the last equality comes from the fact that the β_i's are the values of the entries of $\vec{y}$ and $|L_i| - |L_{i+1}|$ is exactly the number of entries taking on the value β_i.

Combining the bounds from (4.6) and (4.7) we get

$$h(G)\,\|\vec{y}\|^2 \leq B \leq \sqrt{2k(k-\mu_1)}\,\|\vec{y}\|^2$$

$$h(G) \leq \sqrt{2k(k-\mu_1)}$$

□

4.1 Alon-Boppana Theorem

The Alon-Boppana theorem is fundamental to the study of Ramanujan graphs.

Theorem 31 (Alon-Boppana [1], Burger [13], Serre [87])**.** *Let (G_n) be a sequence of connected k-regular graphs with $|G_n| \to \infty$. Then*

$$\liminf_{n\to\infty} \mu_1(G_n) \geq 2\sqrt{k-1}.$$

Outline of Proof. This is an outline of the proof in Davidoff, Sarnak, and Valette [25].

Let $G = (V, E)$ be a k-regular graph with d vertices. Let A be the adjacency matrix of G. We define matrices A_r, associated with G, by

$$\begin{aligned}(A_r)_{ij} \quad = \quad & \text{number of paths of length } r \text{ from vertex } i \text{ to vertex } j \text{ that don't} \\ & \text{traverse the same edge twice in a row in opposite directions.}\end{aligned}$$

We consider each vertex to have one such path to itself of length 0, so A_0 is the identity matrix and $A_1 = A$ is the adjacency matrix of G. Then (A_r) satisfies the following recurrence relation.

$$\begin{aligned} {A_1}^2 &= A_2 + k\,\mathrm{I} \\ A_1 A_r = A_r A_1 &= A_{r+1} + (k-1)\,A_{r-1} \qquad\qquad \text{for } r \geq 2. \end{aligned}$$

Using this it can be shown that associated formal power series $\sum_{r=0}^{\infty} A_r t^r$ is equal to

$$\sum_{r=0}^{\infty} A_r t^r = \frac{1-t^2}{1 - At + (k-1)\,t^2}.$$

What this really means is that

$$\left(\sum_{r=0}^{\infty} A_r t^r\right) \left(\mathrm{I} - At + (k-1)\,t^2\,\mathrm{I}\right) = \left(1-t^2\right)\mathrm{I}\,.$$

We aim to eliminate the factor of $1 - t^2$, and we do this by defining matrices T_m:

$$T_m = \sum_{0 \leq r \leq \frac{m}{2}} A_{m-2r}.$$

As engineered, the formal power series $\sum_{m=0}^{\infty} T_m t^m$ is equal to

$$\sum_{m=0}^{\infty} T_m t^m = \frac{1}{1 - At + (k-1)\,t^2}.$$

Let (U_m) be the *Chebyshev polynomials* of the 2nd kind defined by

$$U_m(\cos\theta) = \frac{\sin{(m+1)\theta}}{\sin\theta}.$$

It is easily seen that $U_0(x) = 1$ and $U_1(x) = 2x$, and using basic trigonometric identities it is easy to prove that the Chebyshev polynomials of the second kind satisfy the recurrence relation

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x).$$

This justifies calling them polynomials and not just functions. Using this recurrence it is straightforward to show that (U_m) has the generating function

$$\sum_{m=0}^{\infty} U_m(x) t^m = \frac{1}{1 - 2xt + t^2}.$$

Performing a change of variables we can massage this into the same form as the power series for (T_m). Specifically, substitute $t\sqrt{k-1}$ for t and substitute $\frac{x}{2\sqrt{k-1}}$ for x, to get

$$\sum_{m=0}^{\infty} (k-1)^{\frac{m}{2}} U_m\left(\frac{x}{2\sqrt{k-1}}\right) t^m = \frac{1}{1 - xt + (k-1)t^2}.$$

Since they have the same generating functions, we conclude

$$T_m = (k-1)^{\frac{m}{2}} U_m\left(\frac{A}{2\sqrt{k-1}}\right).$$

Let $\mu_0 \geq \cdots \geq \mu_{d-1}$ be the eigenvalues of G. Then, by the above equality,

$$\operatorname{Tr} T_m = (k-1)^{\frac{m}{2}} \sum_{i=0}^{d-1} U_m\left(\frac{\mu_i}{2\sqrt{k-1}}\right). \tag{4.8}$$

T_m is defined as a sum of A_r's, which are matrices with integral non-negative entries, so T_m has integral non-negative entries, and thus $\operatorname{Tr} T_m$ is a non-negative integer. In the above equality, it is perhaps surprising that the right hand side is a non-negative integer. This is the key for using the following technical result about the Chebyshev polynomials and measure theory. We use this insight after we prove the result about the Chebyshev polynomials and measure theory.

Claim. For $L \geq 2$ and $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, L)$ so that for every probability measure ν on $[-L, L]$ that satisfies

$$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu(x) \geq 0$$

for every $m \geq 0$, then

$$\nu\left[2-\varepsilon, L\right] \geq C.$$

Outline of Proof of Claim. Define

$$X_m\left(x\right) = U_m\left(\frac{x}{2}\right).$$

Then X_m satisfies

$$X_m\left(2\cos\theta\right) = \frac{\sin\left(m+1\right)\theta}{\sin\theta},$$

so the roots of X_m are $2\cos\frac{j\pi}{m+1}$ for $j = 1, \ldots, m$. Let $\alpha_m = 2\cos\frac{\pi}{m+1}$, the largest root of X_m. The X_m also satisfy a similar recurrence to the U_m, namely

$$X_{m+1}\left(x\right) = xX_m\left(x\right) - X_{m-1}\left(x\right),$$

with $X_0\left(x\right) = 1$ and $X_1\left(x\right) = x$. From this it is elementary to deduce that $X_m\left(x\right) > 0$ for $x > \alpha_m$ for all $m \geq 0$.

For $j \leq k$, it can be proved by induction that

$$X_j X_k = \sum_{m=0}^{j} X_{j+k-2m}, \tag{4.9}$$

by expanding X_j according to the recurrence, then applying the induction hypothesis twice to the resulting expressing, and finally by gathering terms and using the recurrence in reverse.

It is elementary to show that

$$\frac{X_m\left(x\right)}{x-\alpha_m} = \sum_{j=0}^{m-1} X_{m-1-j}\left(\alpha_m\right) X_i\left(x\right). \tag{4.10}$$

This can be checked by multiplying the sum on the right hand side by $x-\alpha_m$, replacing $xX_i\left(x\right)$ by $X_{i+1}\left(x\right)+X_{i-1}\left(x\right)$ using the recurrence, then gathering terms that share the factor $X_i\left(x\right)$ in common (for $i = 0, \ldots, m$). Applying the recurrence to the coefficients of the $X_i\left(x\right)$ will show them all to be 0 except for $i = m$.

Let

$$Y_m\left(x\right) = \frac{\left(X_m\left(x\right)\right)^2}{x-\alpha_m}.$$

Using the above two facts we show that

$$Y_m = \sum_{j=0}^{2m-1} y_j X_j,$$

with $y_j \geq 0$. First we expand Y_m using Equation 4.10 to get

$$Y_m(x) = \sum_{j=0}^{m-1} X_{m-1-j}(\alpha_m) X_j(x) X_m(x).$$

$\alpha_m = 2\cos\frac{\pi}{m+1}$ is the largest root of $X_m(x)$ and (α_m) is an increasing sequence, so for $\ell < m$ we have $X_\ell(\alpha_m) > 0$. So the coefficients of $X_j(x) X_m(x)$ in this sum are all positive. To finish we use Equation 4.9 to write each $X_j(x) X_m(x)$ as a linear combination of $X_0(x), \ldots, X_{2m-1}(x)$ with non-negative coefficients.

Let $L \geq 2$ and $\varepsilon > 0$ be given. Let ν be a probability measure on $[-L, L]$ such that $\int_{-L}^{L} X_m(x)\, d\nu(x) \geq 0$ for every $m \geq 0$. We first show that $\nu[2-\varepsilon, L] > 0$. Suppose towards contradiction that $\nu[2-\varepsilon, L] = 0$. We have that $\alpha_m = 2\cos\frac{\pi}{m+1} \to 2$ as $m \to \infty$, so choose m large enough so that $2 - \varepsilon < \alpha_m$. For $x < \alpha_m$ we have $Y_m(x) \leq 0$, so

$$\begin{aligned}
\int_{-L}^{L} Y_m(x)\, d\nu(x) &= \int_{-L}^{2-\varepsilon} Y_m(x)\, d\nu(x) + \int_{2-\varepsilon}^{L} Y_m(x)\, d\nu(x) \\
&= \int_{-L}^{2-\varepsilon} Y_m(x)\, d\nu(x) \leq 0.
\end{aligned}$$

However, $Y_m = \sum_{j=0}^{2m-1} y_j X_j$ where $y_j \geq 0$, so

$$\int_{-L}^{L} Y_m(x)\, d\nu(x) = \sum_{j=0}^{2m-1} y_j \int_{-L}^{L} X_j(x)\, d\nu(x) \geq 0,$$

since by assumption $\int_{-L}^{L} X_m(x)\, d\nu(x) \geq 0$ for every $m \geq 0$. So $\int_{-L}^{L} Y_m(x)\, d\nu(x) = 0$. Since Y_m is continuous, except on a set of measure 0, it must be that the support of ν is contained in $F_m = \left\{2\cos\frac{j\pi}{m+1} \mid j = 1, \ldots, m\right\}$, the set of zeroes of Y_m. However this analysis holds for any m large enough, in particular, the support of ν must also be contained in F_{m+1}, but $F_m \cap F_{m+1} = \varnothing$, and thus $\nu[-L, L] = 0$, and so ν is not a probability measure as assumed, a contradiction. So $\nu[2-\varepsilon, L] > 0$.

Let $f\colon [-L,L] \to \mathbb{R}$ be the function with $f(x) = 0$ on $[-L, L-\varepsilon]$, $f(x) = 1$ on $\left[L - \frac{\varepsilon}{2}, L\right]$, and on $\left(L-\varepsilon, L-\frac{\varepsilon}{2}\right)$ define f to linearly interpolate between 0 and 1, so that f is continuous. Then

$$\int_{-L}^{L} f(x)\, d\nu(x) \geq \int_{L-\frac{\varepsilon}{2}}^{L} f(x)\, d\nu(x) = \nu\left[L - \frac{\varepsilon}{2}, L\right],$$

and

$$\int_{-L}^{L} f(x)\, d\nu(x) \leq \int_{-L}^{L} \chi_{[L-\varepsilon,L]} d\nu(x) = \nu[L-\varepsilon, L].$$

For topological space X, let $C_0(X)$ be the set of all functions $f\colon X \to \mathbb{C}$ such that for every $\delta > 0$, the set $\left\{x \in X \mid |f(x)| \geq \delta\right\}$ is compact. Then $C_0(X)^*$, the *dual* of $C_0(X)$, is the set of all continuous linear functions $C_0(X) \to \mathbb{C}$. For X a locally compact topological space let Ω be the smallest σ-algebra of subsets of X which contains all of the open sets of X. Let $M(X)$ be the set of all complex-valued regular Borel measures on (X, Ω). Then $C_0(X)^* = M(X)$ under the correspondence $\mu \mapsto F_\mu$ where $F_\mu\colon C_0(X) \to \mathbb{C}$ is defined by

$$F_\mu(f) = \int_X f d\mu.$$

$M(X)$ has a topology defined by the norm $\|\mu\| = |\mu|(X)$ where $|\mu|$ is the variation of μ. We define another topology on $M(X)$ called the *weak* topology* (or *weak-star topology*). The weak* topology on $M(X)$ is the weakest topology on $M(X)$ such that the function $\mu \mapsto \int_X f d\mu$ is continuous for every $f \in C_0(X)$. One of the reasons the weak* topology is important is Alaoglu's Theorem. Alaoglu's Theorem states that if X is a normed space then

$$\operatorname{ball}(X^*) = \left\{x^* \in X^* \mid \|x^*\| \leq 1\right\}$$

is compact in the weak* topology. See Conway [23] for details on these matters.

Let $\mathcal{P}$ be the set of all probability measures, ν, on $[-L, L]$ such that $\int_{-L}^{L} X_m(x)\, d\nu(x) \geq 0$ for every $m \geq 0$. $\mathcal{P}$ is a closed subset of ball $M[-L, L]$, hence it is a compact set in the weak* topology. And the map $\mathfrak{E}\colon M[-L,L] \to \mathbb{C}$ defined by $\mathfrak{E}(\nu) = \int_{-L}^{L} f d\nu$ is continuous in the weak* topology, so $\mathfrak{E}(\mathcal{P})$ is compact. From above we have that for $\nu \in \mathcal{P}$

$$\nu[2-\varepsilon, L] \geq \mathfrak{E}(\nu) = \int_{-L}^{L} f(x)\, d\nu(x) > 0.$$

So by the compactness of $\mathfrak{E}(\mathcal{P})$ there exists a constant $C > 0$ such that

$$\nu[2-\varepsilon, L] \geq \mathfrak{E}(\nu) \geq C$$

for every $\nu \in \mathcal{P}$. $\square$

To finish the proof we apply the result of the claim to a carefully chosen measure. Let δ_a be the Dirac point measure at a. Let $L = \frac{k}{\sqrt{k-1}} \geq 2$ and let

$$\nu = \frac{1}{d}\sum_{j=0}^{d-1} \delta_{\frac{\mu_j}{\sqrt{k-1}}}.$$

Then ν is a probability measure and

$$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu(x) = \frac{1}{d}\sum_{j=0}^{d-1} U_m\left(\frac{\mu_j}{2\sqrt{k-1}}\right) = \frac{1}{d(k-1)^{\frac{m}{2}}} \operatorname{Tr} T_m \geq 0$$

by Equation 4.8. This is where we have used our insight from before. So the assumptions of the claim are satisfied, therefore for $\varepsilon > 0$ there is a constant C such that

$$\nu[2-\varepsilon, L] \geq C.$$

But $\nu[2-\varepsilon, L]$ is $\frac{1}{d}$ times the number of $\frac{\mu_j}{\sqrt{k-1}}$ in the interval $[2-\varepsilon, L]$. Hence there are at least $C \cdot d$ eigenvalues of G in the interval $\left[(2-\varepsilon)\sqrt{k-1}, k\right]$.

The constant C depends only on k and ε, so this C is valid for all k-regular connected graphs. So we apply this to our original sequence of k-regular graphs (G_n): there are $C \cdot |G_n|$ eigenvalues of G_n in the interval $\left[(2-\varepsilon)\sqrt{k-1}, k\right]$. Since $|G_n| \to \infty$ as $n \to \infty$, there is N so that for all $n \geq N$ we have $C \cdot |G_n| \geq 2$, so for $n \geq N$ G_n has a non-trivial eigenvalue λ_n with $\lambda_n \geq (2-\varepsilon)\sqrt{k-1}$. Hence

$$\liminf_{n\to\infty} \mu_1(G_n) \geq (2-\varepsilon)\sqrt{k-1}.$$

Since ε was arbitrary the theorem follows. $\square$

The Alon-Boppana theorem gives us a limit for how small the second largest eigenvalue can be for an expander family. We call graphs that meet or exceed this optimal bound Ramanujan graphs.

Definition 32. A *Ramanujan graph* is a connected k-regular graph G such that for all eigenvalues λ of G we have $\lambda = \pm k$ or $|\lambda| \leq 2\sqrt{k-1}$.

Some authors give a more relaxed definition of Ramanujan graph that would seem more natural based on the Alon-Boppana theorem: that only $\mu_1(G) \leq 2\sqrt{k-1}$. Most of the important constructions and theorems of Ramanujan graphs however use the stricter definition requiring all non-trivial eigenvalues to obey the $2\sqrt{k-1}$ bound (for example the LPS/M and Morgenstern constructions of Section 5.1 and Section 5.2 respectively, and Friedman's result on the prevalence of Ramanujan graphs of Section 6.1). Cioabă [22] has some results on the eigenvalues of k-regular graphs being between $-k$ and $-2\sqrt{k-1}$.

The ultimate goal is to be able to construct an infinite sequence of k-regular Ramanujan graphs with an increasing number of vertices (which implies, for expanders, increasing diameter) for every value of $k \geq 3$. As we will see, this goal has only been partially achieved for $k = p^n + 1$ for p prime. It is not known if this goal is possible or not for all k.

Chapter 5

Constructions

There are only two main constructions that produce infinite families of fixed degree Ramanujan graphs. The Lubotzky-Phillips-Sarnak/Margulis construction produces families with regularity $p + 1$ for any odd prime p, and the Morgenstern construction, which produces families with regularity $q + 1$ for any prime power q. (There is also a small extension of the LPS/M construction by Chiu to the case $p = 2$.) There are a host of constructions that produce only a finite number of Ramanujan graphs for various specific degrees.

5.1 Lubotzky-Phillips-Sarnak/Margulis

Lubotzky, Phillips and Sarnak (LPS) [66] and Margulis [70] independently provided the first explicit construction of Ramanujan graphs. For describing the Lubotzky-Phillips-Sarnak/Margulis (LPS/M)graphs we take the expository approach of LPS. We define graphs $X_{p,q}$ for p, q unequal odd primes with $q \equiv 1 \pmod 4$ and $p < q^2$. $X_{p,q}$ is a $(p+1)$-regular graph with $q(q^2-1)$ vertices if $\left(\frac{p}{q}\right) = -1$ and $\frac{q(q^2-1)}{2}$ vertices if $\left(\frac{p}{q}\right) = 1$. LPS only defined the construction for case $p \equiv 1 \pmod 4$, but the construction is easily extended to the case $p \equiv 3 \pmod 4$ with minimal modifications — for example, Davidoff, Sarnak, and Valette [25] describe the extension. We give a unified presentation of both cases.

The graphs are defined as Cayley graphs of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$ using $(p + 1)$ generators. The generator matrices are derived from representations of p as the sum of four squares. That every number is the sum of four squares is known as Lagrange's Theorem; see Hardy and Wright [46] for a proof. And in fact a well known theorem of Jacobi [49] states that there are

$$8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

ways to represent n as a sum of four squares. When $p \equiv 1 \pmod 4$ let

$$\begin{aligned} \mathcal{S}_p \;=\; & \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p \text{ with } x_1 \text{ odd and} \\ & \qquad \text{strictly positive, and } x_2,\ x_3,\ x_4 \text{ all even}\}. \end{aligned}$$

When $p \equiv 3 \pmod 4$ let

$$\begin{aligned} \mathcal{S}_p \;=\; & \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p \text{ with } x_1 \text{ even,} \\ & \qquad x_2,\ x_3,\ x_4 \text{ all odd, and the first } x_i \neq 0 \text{ has } x_i > 0\}. \end{aligned}$$

We show that $|\mathcal{S}_p| = p + 1$ in both cases. By Jacobi's theorem there are $8(p + 1)$ total solutions to the equation

$${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p.$$

First we consider only the case $p \equiv 1 \pmod 4$. Suppose ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p$. If x is odd then $x^2 \equiv 1 \pmod 4$, and if x is even then $x^2 \equiv 0 \pmod 4$. Since $p \equiv 1 \pmod 4$, exactly one of x_1, x_2, x_3, or x_4 will be odd. Thus enforcing the condition that x_1 be odd on the solutions reduces the number of solutions by a factor of 4. Further, enforcing the condition that x_1 be positive on the solutions removes another factor of 2 from the number of solutions. Thus, in this case, $|\mathcal{S}_p| = p + 1$.

Now we consider the case $p \equiv 3 \pmod 4$. Suppose ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p$. From above we know that squares of integers are either 0 or 1 mod 4. Since $p \equiv 3 \pmod 4$, exactly three of x_1, x_2, x_3, and x_4 will be odd. Thus enforcing the condition that x_1 be even on the solutions reduces the number of solutions by a factor of 4. We also have $(-x_1)^2 + (-x_2)^2 + (-x_3)^2 + (-x_4)^2 = p$, so enforcing the condition that the first non-zero x_i be positive on the solutions reduces the number of solutions by a factor of 2. Thus, in this case also, $|\mathcal{S}_p| = p + 1$.

We define a mapping, ϕ_q, from $\mathcal{S}_p$ to $\mathrm{M}_{2\times 2}(\mathbb{Z}_q)$, the set of 2×2 matrices over $\mathbb{Z}_q$, by

$$\phi_q(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

where $i \in \mathbb{Z}_q$ is such that $i^2 \equiv -1 \pmod{q}$ (such i exists because $q \equiv 1 \pmod{4}$).

For $(x_1, x_2, x_3, x_4) \in \mathcal{S}_p$

$$\begin{aligned}
\det \phi_q (x_1, x_2, x_3, x_4) &= \det \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \\
&= (x_1 + ix_2)(x_1 - ix_2) - (-x_3 + ix_4)(x_3 + ix_4) \\
&= {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = p.
\end{aligned}$$

So if $\left(\frac{p}{q}\right) = 1$ and $p < q$, p is a square mod q and so (by the comments following Definition 4) $\phi_q(\mathcal{S}_p) \subseteq \mathrm{PSL}_2(q)$. Otherwise if $\left(\frac{p}{q}\right) = -1$ and $p < q$ then $\phi_q(\mathcal{S}_p) \subseteq \mathrm{PGL}_2(q)$.

If $\left(\frac{p}{q}\right) = 1$ then $X_{p,q}$ is defined to be the Cayley graph of $\mathrm{PSL}_2(q)$ with respect to $\phi_q(\mathcal{S}_p)$. If $\left(\frac{p}{q}\right) = -1$ then $X_{p,q}$ is defined to be the Cayley graph of $\mathrm{PGL}_2(q)$ with respect to $\phi_q(\mathcal{S}_p)$.

We check that $\phi_q(\mathcal{S}_p)$ is symmetric (in the sense of Definition 17). If $(x_1, x_2, x_3, x_4) \in \mathcal{S}_p$ then $(x_1, -x_2, -x_3, -x_4) \in \mathcal{S}_p$ also, and

$$\begin{aligned}
&\phi_q (x_1, -x_2, -x_3, -x_4) \cdot \phi_q (x_1, x_2, x_3, x_4) = \\
&\qquad = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \begin{pmatrix} x_1 - ix_2 & -x_3 - ix_4 \\ x_3 - ix_4 & x_1 + ix_2 \end{pmatrix} \\
&\qquad = \begin{pmatrix} {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 & 0 \\ 0 & {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 \end{pmatrix} \\
&\qquad \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

So $(\phi_q (x_1, x_2, x_3, x_4))^{-1} = \phi_q (x_1, -x_2, -x_3, -x_4)$ and $\phi_q(\mathcal{S}_p)$ is symmetric.

In the case $\left(\frac{p}{q}\right) = 1$ and $p < q$, $X_{p,q}$ is the Cayley graph of $\mathrm{PSL}_2(q)$ with respect to $\phi_q(\mathcal{S}_p)$. If instead we form the Cayley graph of $\mathrm{PGL}_2(q)$ with respect to $\phi_q(\mathcal{S}_p)$ then the resulting Cayley graph will have two components of equal size: each component a copy of $X_{p,q}$ and one component made up of the vertices $\mathrm{PSL}_2(q)$ and the other made up of the vertices $\mathrm{PGL}_2(q) \backslash \mathrm{PSL}_2(q)$.

Example 33. We construct in detail $X_{3,5}$. Refer to Figures 5.1 and 5.2. For a different visual example see Figure 5.3. The notation $(a\ ,\ b\ ;\ c\ ,\ d)$ in the

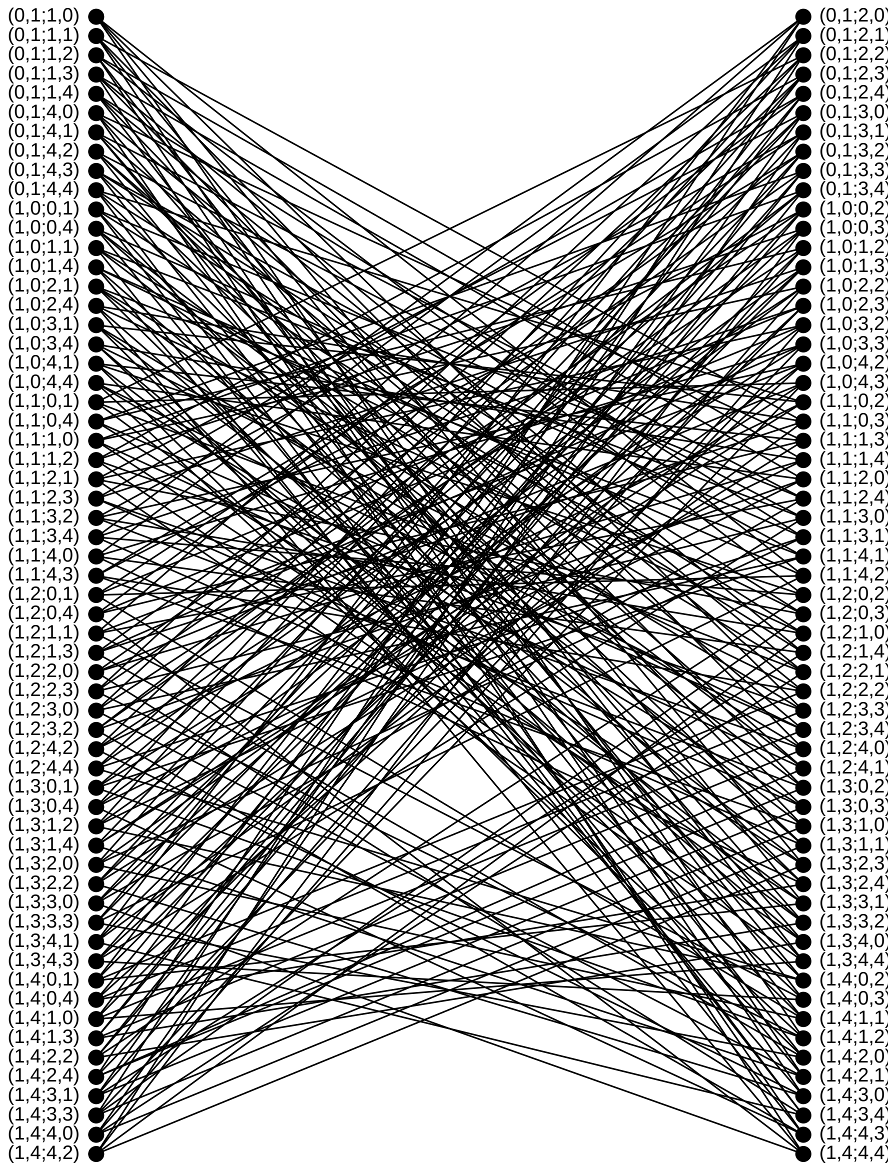


Figure 5.1: The graph $X_{3,5}$ drawn in bipartite form. It has eigenvalues 4, -4, 3, -3, 2, -2, 1, -1, 0 with respective multiplicities 1, 1, 12, 12, 28, 28, 4, 4, 30. For more on the programming work to construct and draw these graphs see Chapter 8.

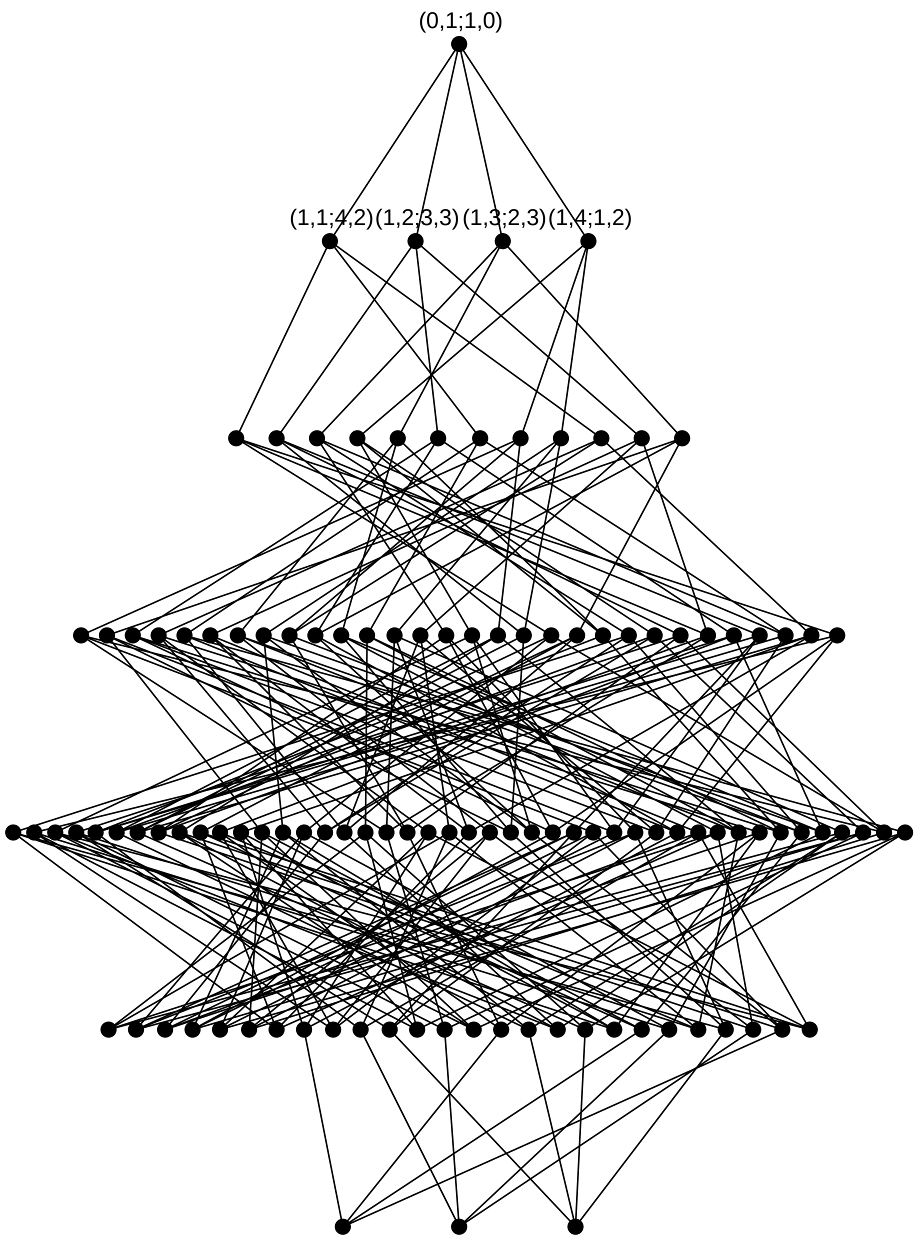


Figure 5.2: The graph $X_{3,5}$ drawn in *tree form* (see Section 5.5 for a precise definition of "tree form"). For more on the programming work to construct and draw these graphs see Chapter 8.

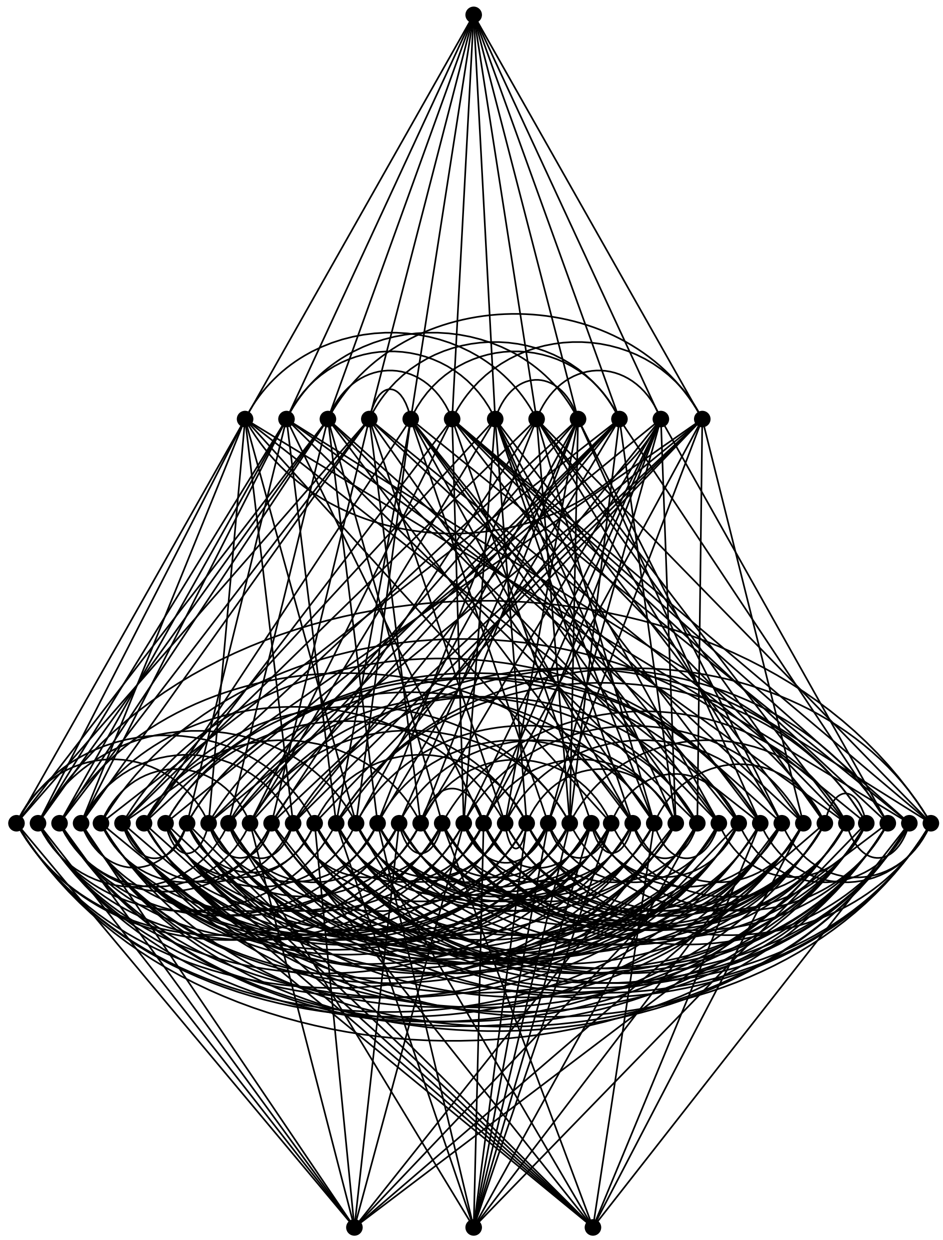


Figure 5.3: The graph $X_{11,5}$ drawn in tree form. The eigenvalues of $X_{11,5}$ are 12, 4, 1, 0, -3, -4 with respective multiplicities 1, 15, 12, 10, 4, 18.

figures represents the 2×2 matrix over $\mathbb{Z}_q$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have

$$\mathcal{S}_3 = \{(0,1,1,1), (0,1,1,-1), (0,1,-1,1), (0,1,-1,-1)\}.$$

$2^2 \equiv -1 \pmod 5$, so 2 is a square root of -1 in $\mathbb{Z}_5$, so we let $i = 2$, and then we get

$$\begin{aligned} \phi_5(\mathcal{S}_3) = \Bigg\{ & \begin{pmatrix} 0+i & 1+i \\ -1+i & -i \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}, \\ & \begin{pmatrix} 0+i & 1-i \\ -1-i & -i \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}, \\ & \begin{pmatrix} 0+i & -1+i \\ 1+i & -i \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}, \\ & \begin{pmatrix} 0+i & -1-i \\ 1-i & -i \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix} \Bigg\}. \end{aligned}$$

$\left(\dfrac{3}{5}\right) = -1$, so we form the Cayley graph of $\mathrm{PGL}_2(5)$ with respect to $\phi_5(\mathcal{S}_3)$. Multiplying each element of $\phi_5(\mathcal{S}_3)$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{PGL}_2(5)$ (the first matrix/vertex in Figure 5.2) we get:

$$\begin{aligned} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}. \end{aligned}$$

The reason the condition $p < q^2$ is necessary is that if $p \geq q^2$ then there is the possibility that one of the solutions to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = p$ will have $x_i \geq q$ for some i. Then the mapping of these solutions to $\mathrm{PGL}_2(q)$ by ϕ_q may not be a one-to-one mapping. Thus we will obtain fewer than $p + 1$

generators in the set $\phi_q(\mathcal{S}_p)$. For example if $p = 29$ and $q = 5$, then $\mathcal{S}_p$ is

x_1	x_2	x_3	x_4
5	0	0	± 2
5	0	± 2	0
5	± 2	0	0
3	0	± 2	± 4
3	0	± 4	± 2
3	± 2	0	± 4
3	± 4	0	± 2
3	± 2	± 4	0
3	± 4	± 2	0

.

The first two solutions listed, $(5, 0, 0, 2)$ and $(5, 0, 0, -2)$, produce the (same) matrices (with $i = 2$):

$$\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} = \begin{pmatrix} 5 + 2 \cdot 0 & 0 + 2 \cdot 2 \\ -0 + 2 \cdot 2 & 5 - 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} = \begin{pmatrix} 5 + 2 \cdot 0 & 0 + 2 \cdot (-2) \\ -0 + 2 \cdot (-2) & 5 - 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

respectively. It remains to be seen whether or not the resulting graphs are Ramanujan. Experimental results produced no counterexample. The author knows of no place in the literature where this is discussed.

Open Question. *When $p \geq q^2$ does the LPS/M construction still yield Ramanujan graphs?*

The graphs $X_{p,q}$ also have interesting properties other than being Ramanujan. If $\left(\frac{p}{q}\right) = -1$ then $X_{p,q}$ is a bipartite graph with $q(q^2 - 1)$ vertices and the girth of $X_{p,q}$ satisfies

$$\mathrm{g}(X_{p,q}) \geq 4 \log_p(q) - \log_p(4). \tag{5.1}$$

If $\left(\frac{p}{q}\right) = 1$ then $X_{p,q}$ is not a bipartite graph. $X_{p,q}$ has $\frac{q(q^2-1)}{2}$ vertices and its girth satisfies

$$\mathrm{g}(X_{p,q}) \geq 2 \log_p(q). \tag{5.2}$$

It is easy to show that for $k \geq 3$ a sequence of k-regular graphs (G_n) with $|G_n| \to \infty$ must have

$$\mathrm{g}\left(G_n\right) \leq \left(2 + \mathrm{o}\left(1\right)\right) \log_{k-1} \left|G_n\right|,$$

where $\mathrm{o}\left(1\right)$ is a function that goes to 0 as $n \to \infty$ (see Biggs [11]). Erdős and Sachs [33] showed using a nonconstructive method (the probabilistic method) the existence of a k-regular family (G_n) with

$$\mathrm{g}\left(G_n\right) \geq \left(1 + \mathrm{o}\left(1\right)\right) \log_{k-1} \left|G_n\right|.$$

Previous to the LPS/M graphs all constructive k-regular families (G_n) only satisfied

$$\mathrm{g}\left(G_n\right) \geq \left(C + \mathrm{o}\left(1\right)\right) \log_{k-1} \left|G_n\right|$$

for some $C < 1$. Using the above inequalities in the $\left(\frac{p}{q}\right) = -1$ case we see that since $|X_{p,q}| = q(q^2 - 1) \approx q^3$ we have

$$\mathrm{g}\left(X_{p,q}\right) \geq 4 \log_p(q) - \log_p(4) \approx 4 \log_p \left(\left|X_{p,q}\right|^{\frac{1}{3}}\right) = \frac{4}{3} \log_p \left|X_{p,q}\right|.$$

So the LPS/M graphs in the case $\left(\frac{p}{q}\right) = -1$ have $C = \frac{4}{3}$. Thus the LPS/M construction not only produced the best known constructive family with high girth, they also beat the best known nonconstructive family with high girth. In the original paper Margulis [71] described an extension of the construction of $(p+1)$-regular graphs with high girth to a construction of $\left(p^\ell + 1\right)$-regular graphs with high girth ($C = \frac{4}{3}$).

Margulis [71] and Biggs & Boshier [10] improve on the above inequalities for the girth of the LPS/M graphs by obtaining an exact equation for the girth. In the $\left(\frac{p}{q}\right) = -1$ case it is shown that

$$4 \log_p(q) - \log_p(4) \leq \mathrm{g}(X_{p,q}) < 4 \log_p(q) + \log_p(4) + 2.$$

The difference between the upper and lower bound is only $2\log_p(4) + 2$, which goes to 2 very quickly as p grows large; when $p > 4$ we have $2\log_p(4) + 2 < 4$ and when $p > 16$ we have $2\log_p(4) + 2 < 3$. When $\left(\frac{p}{q}\right) = -1$, $X_{p,q}$ is bipartite, so it has even girth. This, combined with the tightness of the bound allows an exact result for the girth to be given. It is shown that if $p^{\left\lceil 2\log_p(q)\right\rceil} - q^2$

is of the form $4^\nu(8\omega+7)$ then $\mathrm{g}(X_{p,q}) = 2\left\lceil 2\log_p(q)+\log_p(2)\right\rceil$, otherwise $\mathrm{g}(X_{p,q}) = 2\left\lceil 2\log_p(q)\right\rceil$. (The condition on the form is the same as that of Legendre's Theorem for an integer not being the sum of three squares.) This also shows that $C=\frac{4}{3}$ is the largest possible constant for the girth of the LPS/M graphs.

Let $n=|X_{p,q}|$. Then

$$\log_p(n) \leq \mathrm{diam}(X_{p,q}) \leq 2\log_p(n)+2\log_p(2)+1.$$

In addition, for the case $\left(\frac{p}{q}\right)=1$ we have

$$\mathrm{i}(X_{p,q}) \leq \frac{2\sqrt{p}}{p+1}n$$

and

$$\chi(X_{p,q}) \geq \frac{p+1}{2\sqrt{p}}. \tag{5.3}$$

These bounds are proved in the original LPS paper and are direct consequences of the graphs being Ramanujan and the following two facts.

- $\mathrm{i}(G)\,\chi(G) \geq n$ [25]
- $\mathrm{i}(G) \leq \dfrac{n}{k}\max\{|\mu_1|,\ldots,|\mu_{n-1}|\}$ where G is a connected k-regular graph with n vertices with eigenvalues $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}$. [25]

Adding an edge to a graph may increase its chromatic number, but it will not decrease it. As well, adding an edge to a graph may decreases its girth, but it will not increase it. Thus, achieving a graph with both "large" girth and "large" chromatic number would seem to be a contradictory pursuit. However, Erdős [32] showed nonconstructively (using the probabilistic method) the existence of graphs with arbitrarily large chromatic number and arbitrarily large girth. Combining Equation 5.2 for the girth and Equation 5.3 for the chromatic number in the $\left(\frac{p}{q}\right)=1$ case we see that the LPS/M graphs provide a constructive proof that there are graphs with arbitrarily large chromatic number and arbitrarily large girth.

In order to prove some of the important properties of the LPS/M graphs (the Ramanujan eigenvalue bound, for example) both LPS and Margulis used an equivalent representation of the LPS/M graphs. This representation is less concrete but allows easier access to the number theoretic properties of the LPS/M graphs.

We assume $p \equiv 1 \pmod 4$. Let $\mathbb{H}(\mathbb{Z})$ be the set of integral quaternions, with the typical quaternion, α, represented as $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ with $a_0, a_1, a_2, a_3 \in \mathbb{Z}$. Define $\bar{\alpha} = a_0 - a_1 i - a_2 j - a_3 k$ and $\mathrm{N}(\alpha) = \alpha\bar{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Let

$$\mathcal{T}_p = \{\alpha \in \mathbb{H}(\mathbb{Z}) \mid \mathrm{N}(\alpha) = p,\, \alpha \equiv 1 \pmod 2, \text{ and } a_0 > 0\},$$

where the mod notation is extended to quaternions by applying it to each component of the quaternion individually. There are $p+1$ elements of $\mathcal{T}_p$, since it is just a representation of the set $\mathcal{S}_p$ in $\mathbb{H}(\mathbb{Z})$. Let

$$\Lambda'(2) = \{\alpha \in \mathbb{H}(\mathbb{Z}) \mid \alpha \equiv 1 \pmod 2 \text{ and } \mathrm{N}(\alpha) = p^n \text{ for some } n \geq 0\}.$$

Let $\Lambda(2)$ be $\Lambda'(2)$ modulo the equivalence relation

$$\alpha \sim \beta \text{ if } \pm p^{n_1}\alpha = p^{n_2}\beta \text{ for some } n_1, n_2 \geq 0.$$

Then the Cayley graph of $\Lambda(2)$ with respect to $\mathcal{T}_p$ is the infinite $(p+1)$-regular tree. Let

$$\Lambda(2q) = \{[\alpha] \in \Lambda(2) \mid 2q \text{ divides } a_j,\ j = 1, 2, 3\}.$$

$\Lambda(2q)$ is a normal subgroup of $\Lambda(2)$. Then $X_{p,q}$ is the Cayley graph of $\Lambda(2)/\Lambda(2q)$ with respect to $\{\alpha\Lambda(2q) \mid \alpha \in \mathcal{T}_p\}$. In this way $X_{p,q}$ can be thought of as a quotient of the infinite $(p+1)$-regular tree. And in fact, every $X_{p,q}$ starts out the same was as the $(p+1)$-regular tree, but at some level its finite nature forces it to collapse and no longer expand out with p new branches on each level. See also Section 5.5.

LPS also define graphs $Y_{p,q}$ which are related to $X_{p,q}$ and also Ramanujan, however they are not simple graphs — they may have loops and multiple edges (a loop contributing 2 to the degree of a vertex). Consider $\mathbb{P}^1(\mathbb{Z}_q) = \{0, 1, \ldots, q-1, \infty\}$; the "projectivized" $\mathbb{Z}_q$. We let $\mathrm{PGL}_2(\mathbb{Z}_q)$ act on $\mathbb{P}^1(\mathbb{Z}_q)$ in the conventional manner: by identifying k with $\begin{bmatrix} k \\ 1 \end{bmatrix}$ and ∞ with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and using regular matrix multiplication. We define $Y_{p,q}$ to be the graph with vertices $\mathbb{P}^1(\mathbb{Z}_q)$ and an edge joining $\begin{bmatrix} a \\ b \end{bmatrix}$ to $M\begin{bmatrix} a \\ b \end{bmatrix}$ for every $M \in \phi_q(\mathcal{S}_p)$.

Recently Lubotzky, Samuels, and Vishne [67][68] extended the definition of "Ramanujan" and the LPS/M construction to *Ramanujan complexes*. These can be thought of as higher dimensional analogues of Ramanujan graphs. The Ramanujan complexes are quotients of Bruhat-Tits buildings.

5.1.1 Chiu

Chiu [18] adapted the LPS/M construction to the case $p = 2$. Chiu does this by using a different quaternion algebra than the usual quaternion algebra that LPS/M uses. Chiu constructs 3-regular Ramanujan graphs $X_{2,q}$ for every prime $q \neq 2, 13$ such that -2 and 13 are squares in $\mathbb{Z}_q$. Every prime of the form $104m + 1$ satisfies these conditions, and the prime number theorem for arithmetic progressions gives that there are almost as many of these as regular primes. We define $X_{2,q}$ to be the Cayley graph of $\mathrm{PGL}_2(q)$ if $\left(\frac{2}{q}\right) = -1$ or $\mathrm{PSL}_2(q)$ if $\left(\frac{2}{q}\right) = 1$ with respect to

$$\mathcal{C}_{2,q} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2+\sqrt{-2} & \sqrt{-26} \\ \sqrt{-26} & 2-\sqrt{-2} \end{pmatrix}, \begin{pmatrix} 2-\sqrt{-2} & -\sqrt{-26} \\ -\sqrt{-26} & 2+\sqrt{-2} \end{pmatrix} \right\}.$$

Example 34. We construct the Chiu graph for $q = 3$. Refer to Figure 5.4. The vertex label $(a, b; c, d)$ represents the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We determine the neighbours of the vertex $(0, 1; 2, 0)$, in the top left of the figure. We have $\sqrt{-2} \equiv 1 \pmod 3$ and $\sqrt{13} \equiv 1 \pmod 3$, so the generating set, $\mathcal{C}_{2,3}$, is

$$\begin{aligned} \mathcal{C}_{2,3} = \Bigg\{ & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \\ & \begin{pmatrix} 2+\sqrt{-2} & \sqrt{-26} \\ \sqrt{-26} & 2-\sqrt{-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 2-\sqrt{-2} & -\sqrt{-26} \\ -\sqrt{-26} & 2+\sqrt{-2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \Bigg\}. \end{aligned}$$

$\left(\dfrac{2}{3}\right) = -1$, so we form the Cayley graph of $\mathrm{PGL}_2(3)$ with respect to $\mathcal{C}_{2,3}$. And so $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \in \mathrm{PGL}_2(3)$ is adjacent to

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

as in the figure.

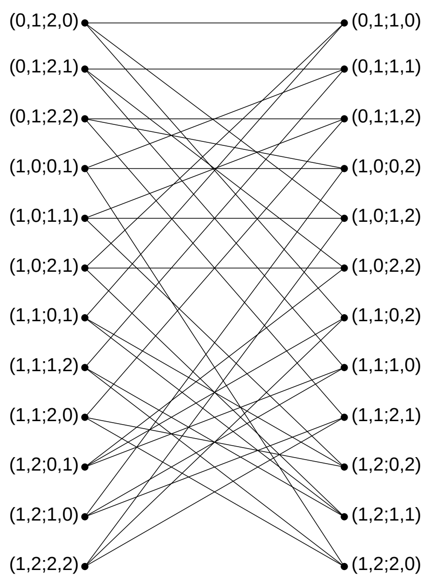


Figure 5.4: The Chiu graph for $q = 3$.

5.2 Morgenstern

Morgenstern [76] extended the LPS/M construction of $(k+1)$-regular Ramanujan graphs where k is prime to the case where k is a prime power. These graphs also satisfy similar properties to the LPS/M graphs. Morgenstern used Drinfel′d's [28] result on Petersson's conjecture instead of Deligne's [27] proof of the Weil conjectures which implied the Ramanujan conjecture which was used by the LPS/M construction.

There are two similar constructions for odd prime powers and even prime powers; we describe the construction for odd prime powers first. Let q be an odd prime power and let d be even. Represent $\mathbb{F}_{q^d}$ as $\mathbb{F}_q\left[x\right]/\left(g\left(x\right)\right)$ where $g(x) \in \mathbb{F}_q[x]$ is an irreducible polynomial of degree d. Let ε be a non-square element of $\mathbb{F}_q$, and let

$$\mathcal{M}_g = \left\{(\delta, \gamma) \in \mathbb{F}_q \times \mathbb{F}_q \mid \delta^2\varepsilon - \gamma^2 = 1\right\}.$$

$\mathcal{M}_g$ has $q+1$ elements. Let $i \in \mathbb{F}_{q^d}$ be such that $i^2 = \varepsilon$ and define a mapping $\psi_g \colon \mathbb{F}_q \times \mathbb{F}_q \to \mathrm{PGL}_2\left(\mathbb{F}_{q^d}\right)$ by

$$\psi_g\left(\delta, \gamma\right) = \begin{pmatrix} 1 & \gamma - \delta i \\ (\gamma + \delta i)(x-1) & 1 \end{pmatrix}.$$

Note that if $(\delta,\gamma)\in\mathcal{M}_g$ then

$$\begin{aligned}\det\psi_g(\delta,\gamma) &= \det\begin{pmatrix}1 & \gamma-\delta i\\ (\gamma+\delta i)(x-1) & 1\end{pmatrix}\\ &= 1-(\gamma-\delta i)(\gamma+\delta i)(x-1)\\ &= 1-\left(\gamma^2-\delta^2 i^2\right)(x-1)\\ &= 1-\left(\gamma^2-\delta^2\varepsilon\right)(x-1)\\ &= 1+(x-1)=x.\end{aligned}$$

So $\psi_g(\mathcal{M}_g)\subseteq \mathrm{PSL}_2\left(\mathbb{F}_{q^d}\right)$ if x is a square in $\mathbb{F}_{q^d}$ and $\psi_g(\mathcal{M}_g)\subseteq \mathrm{PGL}_2\left(\mathbb{F}_{q^d}\right)\setminus \mathrm{PSL}_2\left(\mathbb{F}_{q^d}\right)$ if x is not square in $\mathbb{F}_{q^d}$.

If x is square in $\mathbb{F}_{q^d}$ we define the graph Ω_g to be the Cayley graph of $\mathrm{PSL}_2\left(\mathbb{F}_{q^d}\right)$ with respect to $\psi_g(\mathcal{M}_g)$. If x is not square in $\mathbb{F}_{q^d}$ we define the graph Ω_g to be the Cayley graph of $\mathrm{PGL}_2\left(\mathbb{F}_{q^d}\right)$ with respect to $\psi_g(\mathcal{M}_g)$.

Morgenstern [76] showed that the following facts are true about these graphs. If x is a square in $\mathbb{F}_{q^d}$ then

1. $|\Omega_g|=\frac{q^{3d}-q^d}{2}$ and Ω_g is not bipartite.
2. $\mathrm{g}(\Omega_g)\geq\frac{2}{3}\log_q|\Omega_g|+1$
3. $\mathrm{diam}(\Omega_g)\leq 2\log_q|\Omega_g|+2$
4. $\chi(\Omega_g)\geq\frac{q+1}{2\sqrt{q}}+1$
5. $\mathrm{i}(\Omega_g)\leq\frac{2\sqrt{q}}{q+1}|\Omega_g|$.

If x is not a square in $\mathbb{F}_{q^d}$ then

1. $|\Omega_g|=q^{3d}-q^d$ and Ω_g is bipartite.
2. $\mathrm{g}(\Omega_g)\geq\frac{4}{3}\log_q|\Omega_g|$
3. $\mathrm{diam}(\Omega_g)\leq 2\log_q|\Omega_g|+2$.

Now we define the graphs for even prime powers. Let q be an even prime power (i.e. a power of 2) and let d be even. Let $\varepsilon\in\mathbb{F}_q$ be such that $f(x)=x^2+x+\varepsilon$ is irreducible in $\mathbb{F}_q[x]$. Represent $\mathbb{F}_{q^d}$ as $\mathbb{F}_q[x]/(g(x))$ where $g(x)\in\mathbb{F}_q[x]$ is an irreducible polynomial of degree d. Let $i\in\mathbb{F}_{q^d}$ be a root of $f(x)$. Let

$$\mathcal{M}'_g=\left\{(\delta,\gamma)\in\mathbb{F}_q\times\mathbb{F}_q\mid\gamma^2+\gamma\delta+\delta^2\varepsilon=1\right\}.$$

$\mathcal{M}'_g$ has $q+1$ elements. Define a mapping $\psi'_g \colon \mathbb{F}_q \times \mathbb{F}_q \to \mathrm{PGL}_2\left(\mathbb{F}_{q^d}\right)$ by

$$\psi'_g\left(\delta, \gamma\right) = \begin{pmatrix} 1 & \gamma + \delta i \\ (\gamma + \delta i + \delta)x & 1 \end{pmatrix}.$$

Let Γ_g be the Cayley graph of $\mathrm{PSL}_2\left(\mathbb{F}_{q^d}\right)$ with respect to $\psi'_g\left(\mathcal{M}'_g\right)$. Then Morgenstern [76] showed that the following properties hold.

1. $|\Gamma_g| = \frac{q^{3d}-q^d}{2}$ and Γ_g is not bipartite.
2. $\mathrm{g}(\Gamma_g) \geq \frac{2}{3}\log_q |\Gamma_g| + 1$
3. $\mathrm{diam}(\Gamma_g) \leq 2\log_q |\Gamma_g| + 2$
4. $\chi(\Gamma_g) \geq \frac{q+1}{2\sqrt{q}} + 1$
5. $\mathrm{i}(\Gamma_g) \leq \frac{2\sqrt{q}}{q+1} |\Gamma_g|$.

Example 35. We construct Γ_g with $q=2$, $d=2$, and $g(x) = x^2+x+1$. Refer to Figure 5.5. The notation $(a, b\ ;\ c, d\ ;\ e, f\ ;\ g, h)$ in the figure represents the 2×2 matrix over $\mathbb{F}_{2^2}$

$$\begin{pmatrix} bx+a & dx+c \\ fx+e & hx+g \end{pmatrix}.$$

We have $\varepsilon = 1$, $i = x$, $f(x) = x^2+x+1$, and $g(x) = x^2+x+1$. We have

$$\mathcal{M}'_g = \left\{(1,0), (0,1), (1,1)\right\}.$$

Then

$$\psi'_g\left(\mathcal{M}'_g\right) = \left\{ \begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x+1 \\ x+1 & 1 \end{pmatrix} \right\}.$$

Multiplying each element of $\psi'_g\left(\mathcal{M}'_g\right)$ by $\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \in \mathrm{PSL}_2\left(\mathbb{F}_{2^2}\right)$ (the first matrix in Figure 5.5) we get:

$$\begin{aligned}
\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} &= \begin{pmatrix} x & 1 \\ x & x \end{pmatrix} \cong \begin{pmatrix} 1 & x+1 \\ 1 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} &= \begin{pmatrix} x+1 & 1 \\ x & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & x \\ x+1 & x \end{pmatrix} \\
\begin{pmatrix} 1 & x+1 \\ x+1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ x & x+1 \end{pmatrix} \cong \begin{pmatrix} 1 & x+1 \\ 1 & x \end{pmatrix}.
\end{aligned}$$

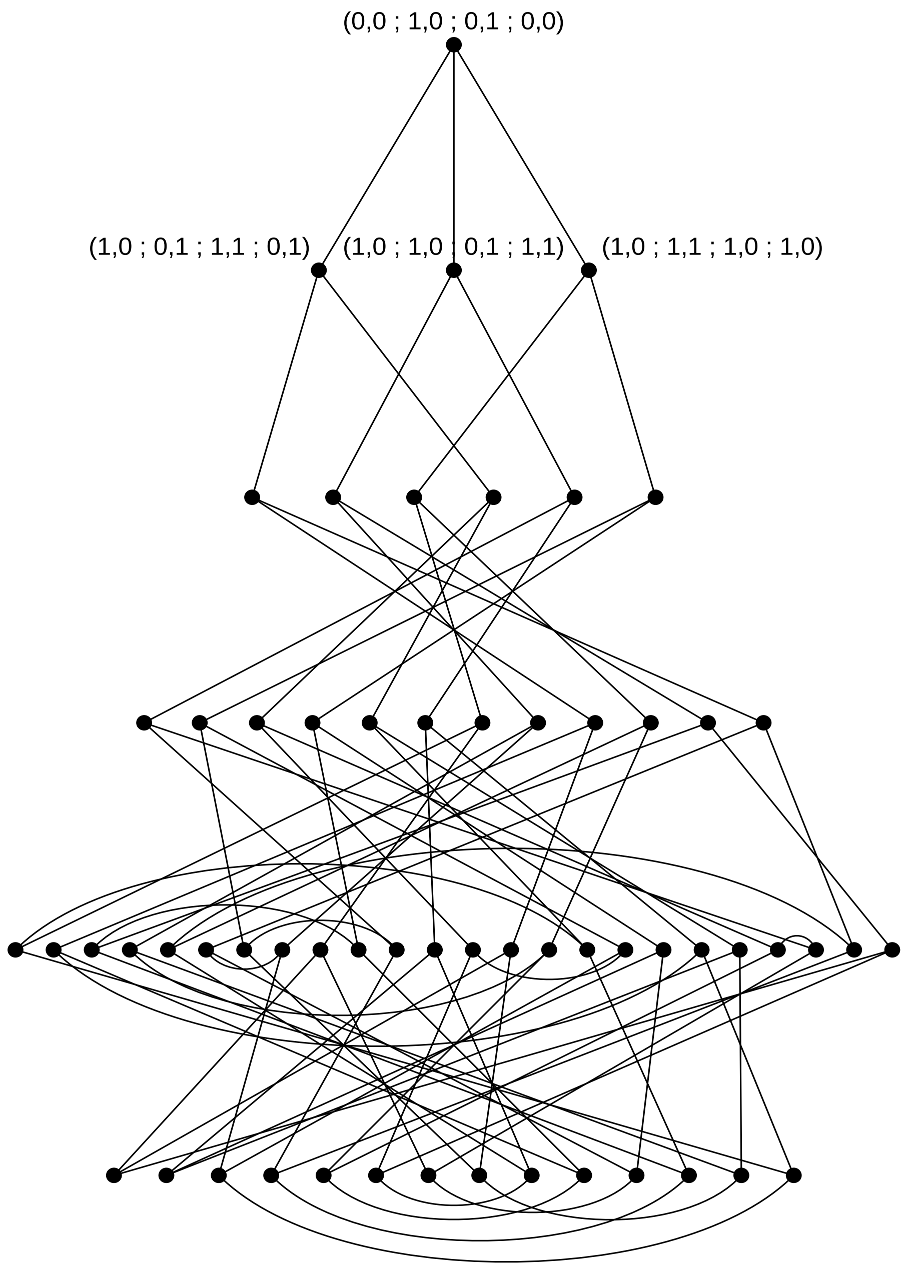


Figure 5.5: The Morgenstern graph Γ_g with $q = 2$, $d = 2$, $g(x) = x^2 + x + 1$ drawn in tree form. It has eigenvalues 3, $\frac{1+\sqrt{13}}{2}$, $\sqrt{5}$, 1, 0, $\frac{-3+\sqrt{5}}{2}$, $\frac{1-\sqrt{13}}{2}$, -2, $-\sqrt{5}$, $\frac{-3-\sqrt{5}}{2}$ with respective multiplicities 1, 10, 3, 13, 4, 6, 10, 4, 3, 6.

5.3 Pizer

Pizer [82] constructs $(p+1)$-regular Ramanujan multigraphs for p prime, based on Hecke operators on modular forms. In this section graphs may have multiple edges and self loops, with a self loop contributing 2 to the degree of a vertex.

One can represent the Hecke operator T_p on modular forms of weight 2 on $\Gamma_0(N)$ by a matrix $B(p)$, called the Brandt matrix, with integer entries. Pizer showed that under certain conditions the multigraph with adjacency matrix $B(p)$ for p prime is a $(p+1)$-regular Ramanujan graph. Pizer extends this method to any positive integer m and gets what he refers to as "almost Ramanujan" multigraphs. Let $\omega(m)$ be the number of positive integer divisors of m and let $\sigma(m)$ be the sum of the positive integer divisors of m. Pizer uses the Brandt matrices $B(m)$ to construct $\sigma(m)$-regular multigraphs whose non-trivial eigenvalues λ satisfy $|\lambda| \leq \omega(m)\sqrt{m}$. The reason for the descriptive "almost Ramanujan" is clear when m is prime — in which case, $\sigma(m) = p+1$, $\omega(m) = 2$, and thus this method gives a $(p+1)$-regular multigraph whose non-trivial eigenvalues λ satisfy $|\lambda| \leq 2\sqrt{p}$.

5.4 Finite-Family Constructions

5.4.1 Gunnells

Gunnells [45] proposes constructions for two different types of Ramanujan graphs.

Fix the vector space $(\mathbb{F}_q)^n$ for q a prime power. Let $\mathcal{V}_k$ denote the set of all k-dimensional subspaces of $(\mathbb{F}_q)^n$. Define $[n]_q = \frac{q^n-1}{q-1}$ and $[n]_q! = \prod_{i=1}^{n}[i]_q$. Let

$$G_{k,n} = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

and set $G_{0,n} = G_{n,n} = 1$ by convention. Then $|\mathcal{V}_k| = G_{k,n}$.

Gunnells defines a graph $B_q(k, n-k)$ with vertex set $V = \mathcal{V}_k \cup \mathcal{V}_{n-k}$ and vertices are joined by an edge iff one is a subspace of the other. This is a $(G_{k,n-k})$-regular bipartite graph with $2G_{k,n}$ vertices.

So $B_q(1, n-1)$ is a bipartite graph with regularity $G_{1,n-1} = \frac{q^{n-1}-1}{q-1}$ and

$2G_{1,n} = 2\frac{q^n-1}{q-1}$ vertices. Gunnells shows that $B_q(1, n-1)$ has spectrum

$$\pm\frac{q^{n-1}-1}{q-1},\ \pm\sqrt{q^{n-2}}$$

and hence is Ramanujan. Gunnells also shows that for $k \geq 2$ $B_q(k, n-k)$ can only be Ramanujan for finitely many values of q.

To show this Gunnells constructs eigenvectors, $\Phi^+_{a,b}$ and $\Phi^-_{a,b}$, of the graph $B_q(k, n-k)$ for any two distinct elements $a, b \in \mathcal{V}_1$. Let $H_k(a, b)$ be the set of all k-dimensional subspaces that contain a and b, and let $H_k(a)$ be the set of all k-dimensional subspaces containing a. $\Phi^+_{a,b}$ is defined on the vertices, v, of $B_q(k, n-k)$ by

$$\Phi^+_{a,b}(v) = \begin{cases} 0 & \text{if } v \in H_k(a,b) \cup H_{n-k}(a,b) \\ 1 & \text{if } v \in H_k(a) \setminus H_k(a,b) \\ -1 & \text{if } v \in H_k(b) \setminus H_k(a,b) \\ q^{-\frac{n-2k}{2}} & \text{if } v \in H_{n-k}(a) \setminus H_{n-k}(a,b) \\ -q^{-\frac{n-2k}{2}} & \text{if } v \in H_{n-k}(b) \setminus H_{n-k}(a,b) \\ 0 & \text{otherwise} \end{cases}.$$

$\Phi^-_{a,b}$ is defined the same except the signs of the $q^{-\frac{n-2k}{2}}$ in the fourth and fifth condition are interchanged.

Gunnells shows that $\Phi^+_{a,b}$ has corresponding eigenvalue

$$\lambda = (G_{n-2k,n-k} - G_{n-2k-1,n-k-l})\, q^{-\frac{n-2k}{2}},$$

and $\Phi^-_{a,b}$ has corresponding eigenvalue $-\lambda$. Asymptotically, if $k \geq 2$, this eigenvalue does not admit $B_q(k, n-k)$ to being Ramanujan. When $k = 1$, by considering $\Phi^+_{a,b}$ for all $a, b \in \mathcal{V}_k$, Gunnells shows that we get $\frac{q^n-1}{q-1} - 1$ linearly independent eigenvectors with eigenvalue λ. Combining these with the eigenvectors for $-\lambda$ and the two trivial eigenvectors (since $B_q(k, n-k)$ is bipartite and regular), this accounts for all eigenvalues of the graph.

Gunnells also defines another type of graph $G(N)$ which has $\frac{N^2-1}{2}$ vertices and is N-regular. Gunnells shows that $G(p)$ is Ramanujan for all primes $p \geq 3$.

Gunnells then goes on to define a simple technique to produce more Ramanujan graphs from the $B_q(1, n-1)$ and $G(p)$ graphs. Let $G = (V, E)$ be a graph, we define a new graph $G^k = (V^k, E^k)$ as follows.

$$V^k = V \times \{1, \ldots, k\}$$

$$\{(u,i),(v,j)\} \in E^k \Leftrightarrow (u,v) \in E$$

Gunnells shows that if the spectrum of G is $\lambda_1, \ldots, \lambda_n$ then the spectrum of G^k is $k\lambda_1, \ldots, k\lambda_n, 0$ and the multiplicity of $k\lambda_i$ in the spectrum of G^k is equal to the multiplicity of λ_i in the spectrum of G and 0 has multiplicity $|V|(k-1)$.

Gunnells then uses this to show that $B_q^2(1,n-1)$, $B_q^3(1,n-1)$, $B_q^4(1,n-1)$ and $G^2(p)$ are Ramanujan.

For one example of the Gunnells construction see Figure 5.6.

5.4.2 Li

Li [60] constructs $(q+1)$-regular Ramanujan graphs with q^2 vertices for q a prime power, and also presents other constructions for similar graphs.

Let $q = p^m$ be a prime power (p a prime) and consider the finite field $\mathbb{F}_{q^n}$. Define the *norm* of an element a of $\mathbb{F}_{q^n}$, $\eta_n(a)$, to be

$$\eta_n(a) = \prod_{\Psi \in \mathrm{Aut}\left(\mathbb{F}_{q^n}\right)} \Psi(a)$$

where $\mathrm{Aut}(\mathbb{F}_{q^n})$ is the *automorphism group* of field automorphisms of $\mathbb{F}_{q^n}$. $\eta_n(a)$ is always an element of $\mathbb{F}_q$. The automorphism group of $\mathbb{F}_{q^n} = \mathbb{F}_{p^{nm}}$ is cyclic of order nm — it is generated by the *Frobenius automorphism*, Φ, defined by $\Phi(b) = b^p$. Φ fixes (element-wise) the base field $\mathbb{F}_p$, and is both multiplicative and linear. Let

$$N_n = \mathrm{Ker}(\eta_n) = \{a \in \mathbb{F}_{q^n} \mid \eta_n(a) = 1\},$$

be the kernel of the normal map. Then $|N_n| = \frac{q^n-1}{q-1}$.

Define the *sum graph* $X_s(H,S)$ and *difference graph* $X_d(H,S)$ to be the graphs with vertex set H and a vertex x is adjacent to $-x+s$ for $s \in S$ in the sum graph, and is adjacent to $x+s$ for $s \in S$ in the difference graph. The reason for the names is that x and y are adjacent in the sum graph if $x+y = s \in S$ and are adjacent in the difference graph if $x-y = s \in S$.

Li shows that if λ is an eigenvalue of $X_s(\mathbb{F}_{q^n}, N_n+c)$ or $X_d(\mathbb{F}_{q^n}, N_n+c)$, where c is any element of $\mathbb{F}_{q^n}$, then either $\lambda = |N_n| = \frac{q^n-1}{q-1}$ or

$$|\lambda| \leq nq^{\frac{n-1}{2}}.$$

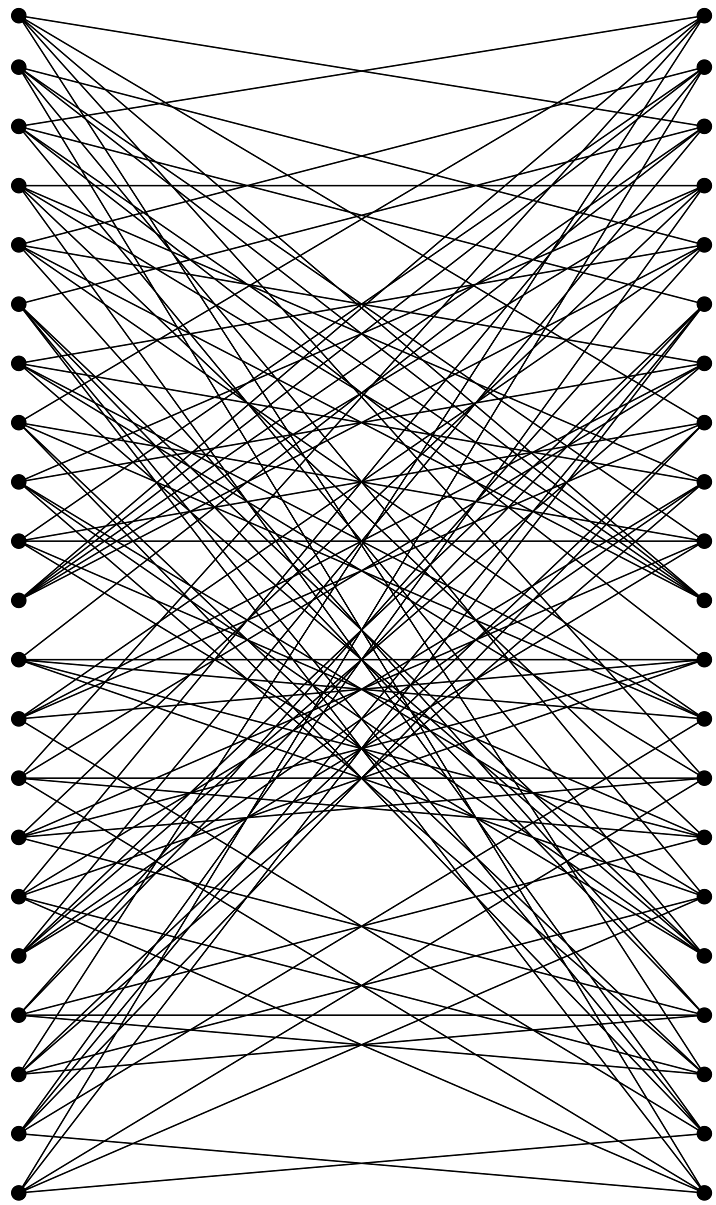


Figure 5.6: The Gunnells graph $B_4(1,2)$ over $(\mathbb{F}_4)^3$ using extension polynomial x^2+x+1 drawn in bipartite form. It is a 5-regular graph with 42 vertices and has eigenvalues 5, 2, -2, -5 with respective multiplicities 1, 20, 20, 1.

Hence, for $n = 2$, the bound on the non-trivial eigenvalues is $2\sqrt{q}$, and $|N_n| = q+1$ and the sum and differences graphs $X_s(\mathbb{F}_{q^2}, N_2 + c)$ and $X_d(\mathbb{F}_{q^2}, N_2 + c)$ are Ramanujan for any $c \in \mathbb{F}_{p^2}$.

Janwa and Moreno [50] give a more elementary proof of the Ramanujan property of Li's graphs for even prime powers.

For one example of a Li graph see Figure 5.7.

5.4.3 Chung

Chung [19] constructs p-regular Ramanujan graphs with $p^2 - 1$ vertices for p prime.

In this section we allow graphs to have self loops but not multiple edges. Each self loop contributes 1 (and not 2 as is more usual) to the degree of a vertex.

Let $g(x)$ be an irreducible polynomial of degree n in $\mathbb{Z}_p[x]$ and realize $\mathbb{F}_{p^n}$ as $\mathbb{Z}_p[x]/(g(x))$. ${\mathbb{F}_{p^n}}^* = \mathbb{F}_{p^n} \setminus \{0\}$ is a cyclic group under the field multiplication, say a generates it, then for $0 \le i < p$, $x + i = a^{d_i}$ for some d_i. Let $S_{p^n} = \{d_0, \ldots, d_{p-1}\}$. Define C_{p^n} to be the sum graph (see Section 5.4.2 above) $X_s((\mathbb{Z}_{p^n-1}, +), S_{p^n})$ — there is an edge in C_{p^n} between x and y iff $x + y \equiv d_i \pmod{p^n - 1}$ for some $i \in \{0, \ldots, p-1\}$. Then C_{p^2} for p prime is a Ramanujan graph.

Another, perhaps simpler, way of looking at the graphs C_{p^n} is as graphs with vertex set ${\mathbb{F}_{p^n}}^*$, where $u, v \in {\mathbb{F}_{p^n}}^*$ are adjacent iff $uv = x + i$ for some $i \in \{0, \ldots, p-1\}$.

For one example of a Chung graph see Figure 5.8.

5.4.4 de Reyna

Ramanujan graphs derived from matrices that satisfy certain constraints are defined by de Reyna [26].

Let $\mathbb{T}_d$ be the set of all d-roots or unity. Let $C = (c_{i,j})$ be a $q \times q$ matrix with entries coming from $\mathbb{T}_d \cup \{0\}$. Let $X = \{x_1, \ldots, x_q\}$ and $Y = \{y_1, \ldots, y_q\}$ be two disjoint sets of q elements. We define a bipartite graph G on the partition of vertices $(X \times \mathbb{T}_d, Y \times \mathbb{T}_d)$ where (x_i, θ_1) is adjacent to (y_j, θ_2) iff $c_{i,j} \neq 0$ and $\theta_2 = c_{i,j}\theta_1$. If we let $C_m = ({c_{i,j}}^m)$, then de Reyna shows that if C_m is normal for all m, then the eigenvalues of G are $\pm|\lambda|$ for each eigenvalue λ of the matrices C_m.

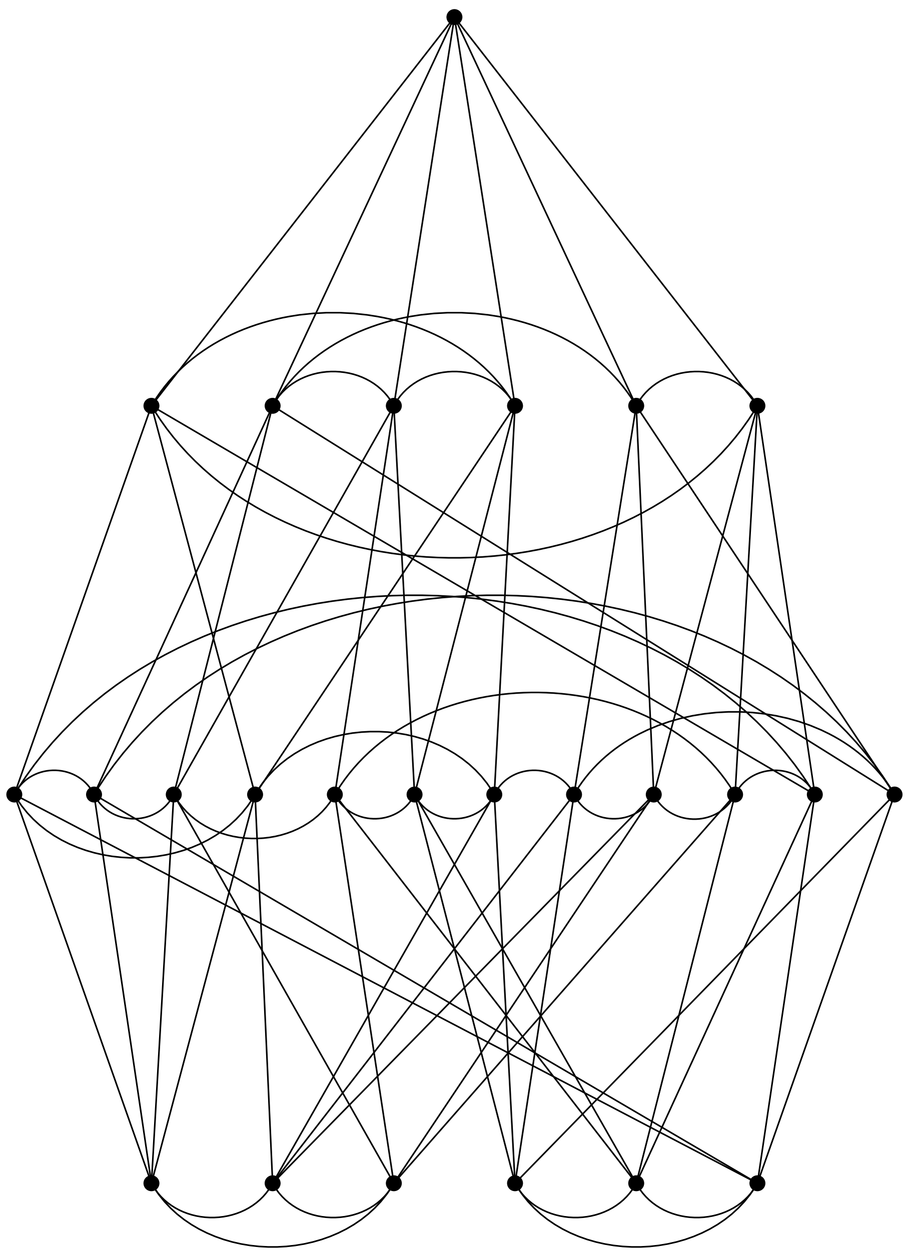


Figure 5.7: The Li graph $X_d\left(\mathbb{F}_{5^2}, N_2\right)$ using extension polynomial x^2+x+1. It is a 6-regular graph with 25 vertices and has eigenvalues 6, $1+\sqrt{5}$, $\frac{-3-\sqrt{5}}{2}$, $\frac{-3+\sqrt{5}}{2}$, $1-\sqrt{5}$ with respective multiplicities 1, 6, 6, 6, 6.

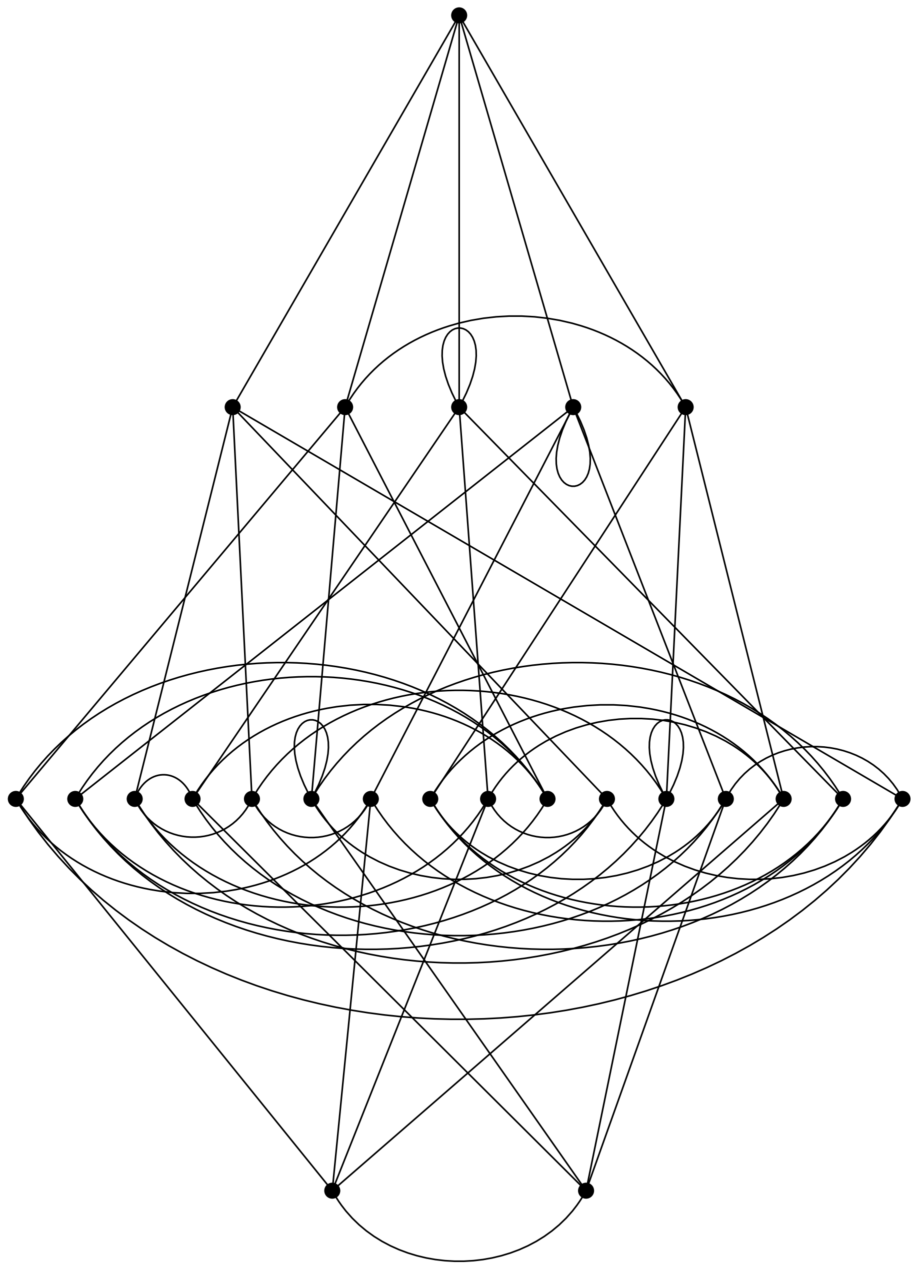


Figure 5.8: The Chung graph C_{5^2} using extension polynomial $x^2 + x + 1$. It is a 5-regular graph with 24 vertices and has eigenvalues 5, -1, 1, $\sqrt{5}$, $-\sqrt{5}$ with respective multiplicities 1, 3, 2, 9, 9.

If we apply this to the specific matrix C with $c_{i,j} = {\zeta_p}^{ij}$ ($\zeta_p = e^{\frac{2\pi}{p}i}$ the principle pth root of unity), then the spectrum of C_m can be computed, and thus the spectrum of the graph. The graph is regular of degree p and the second largest eigenvalue is $\sqrt{p}$, hence it is a Ramanujan graph.

Essentially the roots of unity in this construction are playing the part of a finite field of prime order. So this construction can be extended to finite field of prime power order using a non-trivial multiplicative character of order d, ψ, where the condition that two vertices are adjacent is $\theta_2 = \psi(x+y)\theta_1$.

For examples of the first few de Reyna graphs refer to Figures 5.9 and 5.10.

5.4.5 Friedman

Friedman [35] constructs $2(p-1)$-regular graphs with $p(p-1)$ vertices that have second largest eigenvalue $\leq 2\sqrt{p} \leq 2\sqrt{2(p-1)}$ (for $p > 2$) for primes $p \equiv 3 \pmod 4$.

Let AFFINE(p) be the group of affine linear transformations of $\mathbb{Z}_p$ with composition being the group operation. This is the group ${\mathbb{Z}_p}^* \times \mathbb{Z}_p$ with group operation $(a, b) \cdot (c, d) = (ac, b + ad)$. Let H_p be the set $\{(\pm r^2, r) \mid r \in {\mathbb{Z}_p}^*\}$, and define SQRT($p$) to be the Cayley graph of AFFINE(p) with respect to H_p. Then SQRT(p) is a $2(p-1)$-regular graph with $p(p-1)$ vertices. Friedman proves that the second largest eigenvalue is less than $2\sqrt{p}$ which is less than the Ramanujan bound of $2\sqrt{2(p-1)}$ for $p > 2$.

For two examples of the Friedman graphs see Figure 5.11.

5.5 Unimodality

By the *tree form* of a graph we mean that one vertex of the graph is distinguished as the root and is on level 1; the remaining vertices are partitioned into levels where the vertices on level $i+1$ have distance i from the root. For visual examples see Figures 5.2, 5.3, 5.5, 5.7, 5.8, and 5.11. For a graph with diameter d let l_i ($1 \leq i \leq d$) be the number of vertices on level i of the graph with some vertex distinguished as the root. We call the sequence $l_1, \ldots, l_d$ the *distance sequence* of the graph. For graphs that are vertex-transitive this distance sequence does not depend on which vertex is distinguished as the root. More specifically, Cayley graphs are vertex-transitive, and so all of the

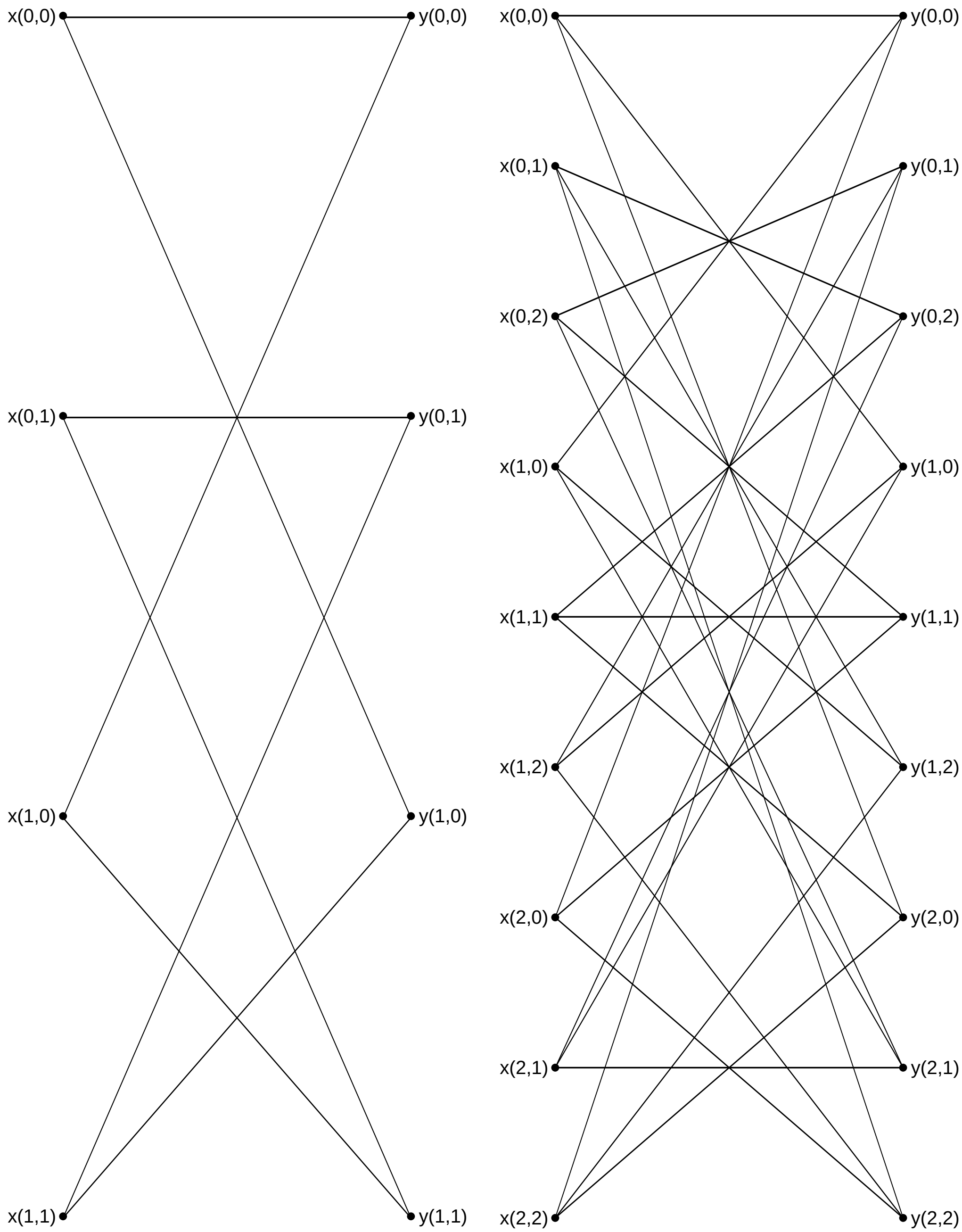


Figure 5.9: The de Reyna graph for $p = 2, 3$ respectively.

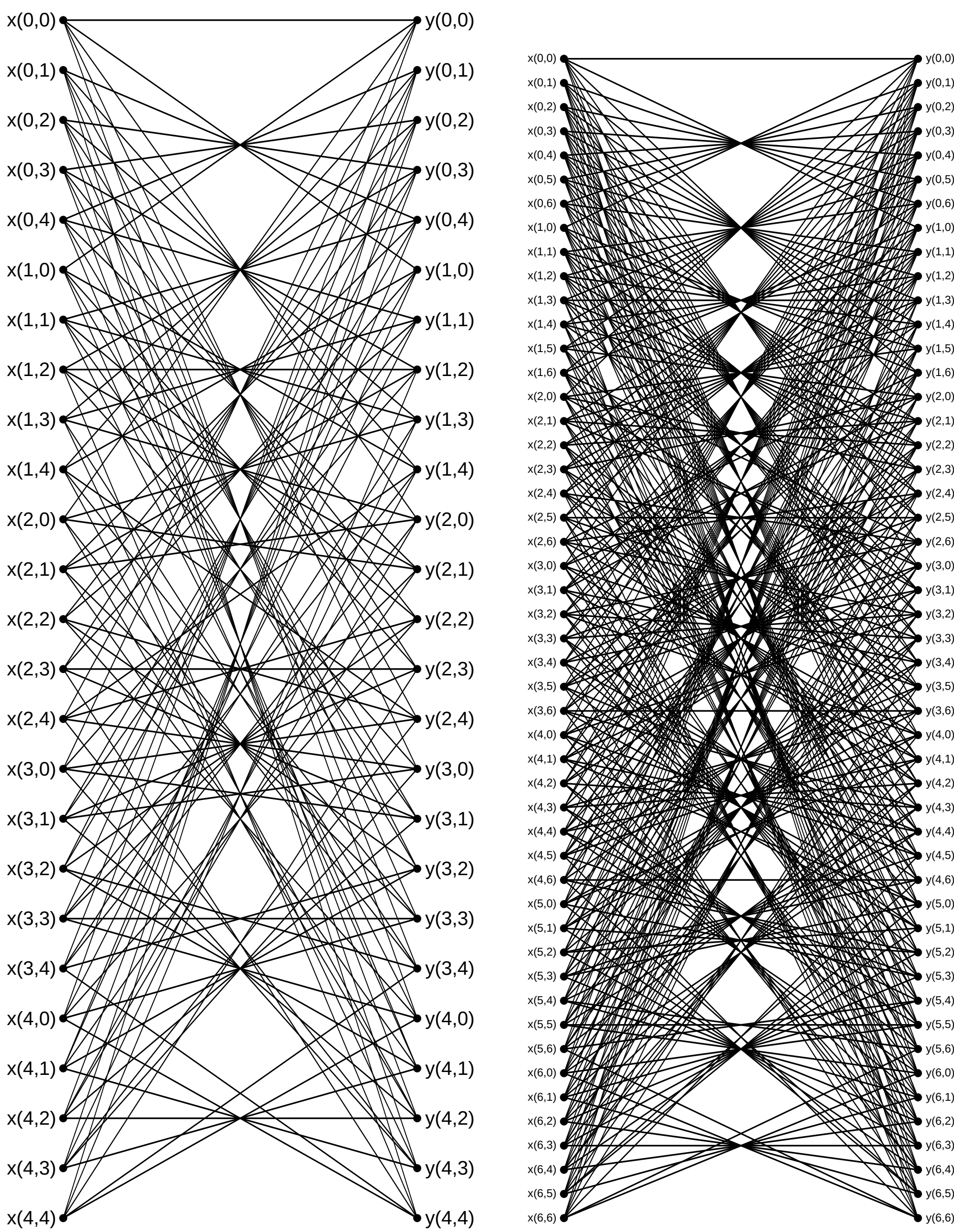


Figure 5.10: The de Reyna graph for $p = 5, 7$ respectively.

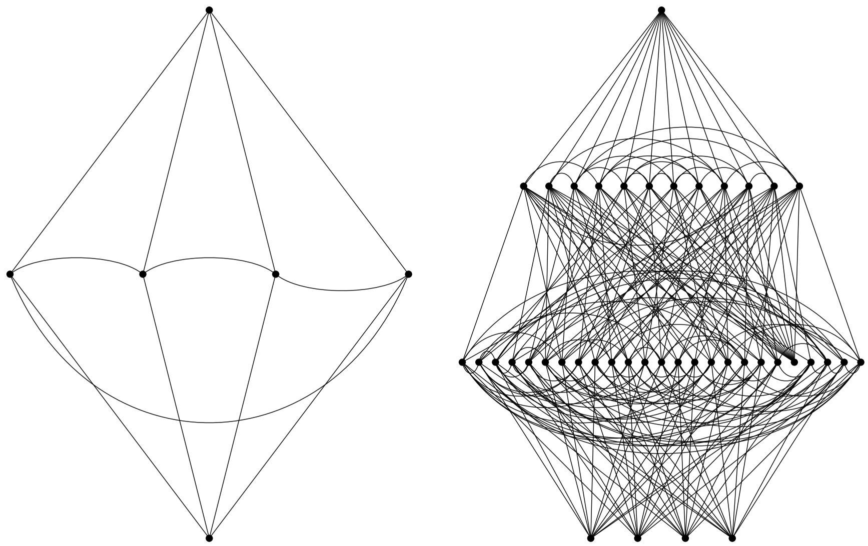


Figure 5.11: The Friedman graph SQRT(p) for $p = 3, 7$ respectively.

known constructions of infinite families of Ramanujan graphs (LPS/M, Chiu, Morgenstern) are vertex-transitive.

We say that a sequence $a_1, \ldots, a_n$ is *unimodal* if there is some m, $1 \leq m \leq n$, such that $a_1, \ldots, a_m$ is an increasing sequence and $a_m, \ldots, a_n$ is a decreasing sequence. Are the distance sequences of the LPS/M, Chiu, or Morgenstern graphs unimodal?

For k-regular vertex-transitive graphs we know that the distance sequence will start 1, k, $(k-1)k$, $(k-1)^2k$, $(k-1)^3k, \ldots$. We refer to this as "the pattern". The pattern will stop when one of the vertices in a level is adjacent to a vertex in the same level, or it is adjacent to two vertices in the level above. In either situation we get a cycle in the graph, hence the girth of the graph determines at which level this happens. For the LPS/M graphs Equation 5.1 gives us a tight bound on the girth. If the girth is even then we must get the case that a vertex is adjacent to two vertices on the level above, and if the girth is odd then we must get the case that two vertices in the same level are adjacent. If the girth is even, then a vertex on level $\frac{\mathrm{g}(G)}{2} + 1$ is adjacent to two vertices on the level above, if the girth is odd, then two vertices in level $\frac{\mathrm{g}(G)+1}{2}$ are adjacent. In either case, the pattern stops at level $\left\lfloor \frac{\mathrm{g}(G)}{2} \right\rfloor + 1$.

Example 36. The distance sequence of $X_{3,5}$ is $1, 4, 12, 30, 44, 26, 3$. See Figure 5.2. $X_{3,5}$ is 4-regular, so the first three levels follow the pattern.

The distance sequence of $X_{11,5}$ is $1, 12, 44, 3$. See Figure 5.3. $X_{11,5}$ is 12-regular so only the first two levels follow the pattern.

See Section 5.5.1 for more examples of distance sequences.

Open Question. *Are the distance sequences of all Ramanujan graphs unimodal?*

To the author's knowledge this question has not appeared in the literature. *Distance regular* graphs do have the unimodal property (for example, see Brouwer, Cohen, and Neumaier [12]). The informal "no bottlenecks" property of expanders (see Chapter 3) provides some intuitive reasoning to suggest that Ramanujan graphs are unimodal. Suppose a Ramanujan graph is not unimodal, then there are three levels in a row, say $d, d+1, d+2$ with $l_d > l_{d+1}$ and $l_{d+1} < l_{d+2}$, and so level $d+1$ would literally look like a bottleneck in the graph.

5.5.1 Table of Distance Sequences

The following tables display the number of vertices in each level of the tree form of a variety of Ramanujan graphs. The rows labelled k and n represent the regularity and number of vertices of the graph respectively. The rows labelled "level n" list the number of vertices on level n. For the first two tables, which consist of LPS/M graphs only, the rows labelled p and q represent the parameters for the graph $X_{p,q}$. For third table, which contains a variety of constructions other than LPS/M, the table caption describes what each parameter represents for each construction. A * indicates the first level where the pattern fails.

Construction	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M
p	3	7	11	13	17	19	23
q	5	5	5	5	5	5	5
k	4	8	12	14	18	20	24
n	120	120	60	120	120	60	120
level 0	1	1	1	1	1	1	1
level 1	4	8	12	14	18	20	24
level 2	12	*44	*44	*59	*59	*39	*59
level 3	*30	52	3	46	42		36
level 4	44	15					
level 5	26						
level 6	3						

Table 5.1: Number of vertices on each level for various constructions 1

Construction	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M	LPS/M
p	3	5	7	11	17	19	23
q	13	13	13	13	13	13	13
k	4	6	8	12	18	20	24
n	1092	2184	2184	2184	1092	2184	1092
level 0	1	1	1	1	1	1	1
level 1	4	6	8	12	18	20	24
level 2	12	30	56	132	*276	*352	*408
level 3	36	150	*326	*688	760	1072	659
level 4	108	*550	902	955	37	739	
level 5	*216	890	758	392			
level 6	378	511	133	4			
level 7	307	46					
level 8	30						

Table 5.2: Number of vertices on each level for various constructions 2

Construction	Morg.	Morg.	Morg.	Gunnells	Chung	Li	de Reyna	Chiu
p_1	2	3	5	4	5	5	11	2
p_2	4	9	25	3	2	2	11	17
k	3	4	6	5	5	6	11	3
n	60	720	15600	42	24	25	242	2448
level 0	1	1	1	1	1	1	1	1
level 1	3	4	6	5	5	6	11	3
level 2	6	12	30	20	*16	*12	110	6
level 3	12	36	150	*16	2	6	*110	12
level 4	24	*104	*726				10	24
level 5	*14	216	2958					*46
level 6		236	6328					90
level 7		104	4677					169
level 8		7	714					290
level 9			9					497
level 10			1					634
level 11								521
level 12								138
level 13								13
level 14								3
level 15								1

Table 5.3: Number of vertices on each level for various constructions 3 (Morg. = Morgenstern). For Morg. $p_1 = q$, $p_2 = q^d$. For Gunnells $p_1 = q$, $p_2 = n$. For Chung $p_1 = p$, $p_2 = n$. For Li $p_1 = q$, $p_2 = n$. For de Reyna $p_1 = p$, $p_2 = p$. For Chiu $p_1 = p$, $p_2 = q$. Refer to the specific section for each construction for the meaning of the parameters.

Chapter 6

Asymptotic Results

6.1 Friedman

A result of Friedman [34] shows that "almost all" connected k-regular graphs are "almost" Ramanujan.

In this section we allow graphs to have self-loops and multiple edges. Each self-loop contributes 2 to the degree of a vertex, and also contributes 2 to the diagonal of the adjacency matrix. We also allow *half-loops*, which are loops that contribute only 1 to the degree of a vertex, and also appears as a 1 on the diagonal of the adjacency matrix. We first need to introduce a number of models of random regular graphs.

Let d, n be positive integers, d even. We define a probability space $\mathcal{G}_{n,d}$ on the set of d-regular graphs with n vertices by choosing $\frac{d}{2}$ permutations from the set of all $n!$ permutations of the n vertices, each permutation chosen uniformly and independently. Call these chosen $\frac{d}{2}$ permutations $\pi_1, \ldots, \pi_{d/2}$. Add an edge $\{i, \pi_j(i)\}$ to the graph for $i = 1, \ldots, n$ and $j = 1, \ldots, \frac{d}{2}$.

For d, n positive integers, d even, let $\mathcal{G}'_{n,d}$ be the restriction of $\mathcal{G}_{n,d}$ to the graphs that do not have self-loops.

For d, n positive integers, d even, let $\mathcal{H}_{n,d}$ be the probability space that is defined like $\mathcal{G}_{n,d}$ but each π_i is instead chosen from the set of $(n-1)!$ permutations whose cycle decompositions consist of a single cycle of length n.

For d, n positive integers n even, let $\mathcal{I}_{n,d}$ be the probability space given by choosing d perfect matchings on n vertices (each choice uniform and independent among all such perfect matchings) and adding all of the edges of the perfect matchings into one graph.

For d, n positive integers, $\mathcal{J}_{n,d}$ is defined the same as $\mathcal{I}_{n,d}$ for n even; when n is odd instead choose d *near perfect matchings*, which are perfect matchings on $n-1$ vertices and a half-loop on the one remaining vertex.

For d, n positive integers, let $\mathcal{K}_{n,d}$ be the probability space where each d-regular simple graph with n vertices has equal probability.

Theorem 37 (Friedman)**.** *Let $\varepsilon > 0$. For the models $\mathcal{G}_{n,d}$, $\mathcal{G}'_{n,d}$, $\mathcal{H}_{n,d}$, $\mathcal{I}_{n,d}$, $\mathcal{J}_{n,d}$, and $\mathcal{K}_{n,d}$ of random regular graphs, the probability that a random k-regular graph G satisfies*

$$|\lambda| \leq 2\sqrt{k-1} + \varepsilon$$

for all eigenvalues $\lambda \neq k$ (note that $\lambda = -k$ is a possibility) of G goes to 1 as the number of vertices goes to infinity.

Although this would seem to show that within ε Ramanujan graphs are plentiful, there are a limited number of explicit constructions of them.

Friedman only directly proves the theorem for the models $\mathcal{G}_{n,d}$, $\mathcal{H}_{n,d}$, $\mathcal{I}_{n,d}$, and $\mathcal{J}_{n,d}$; the result for $\mathcal{G}'_{n,d}$ and $\mathcal{K}_{n,d}$ follows based on some results about probability spaces. Let (X_n, Ω_n) be a sequence of measure spaces and let (μ_n) and (ν_n) be sequences of measures on (X_n, Ω_n). The sequence (μ_n) is said to *dominate* the sequence (ν_n) if for any sequence of measurable sets (E_n) with $E_n \in \Omega_n$ for all n we have that $\mu_n(E_n) \to 0$ as $n \to \infty$ implies $\nu_n(E_n) \to 0$ as $n \to \infty$. We say (μ_n) and (ν_n) are *contiguous* if (μ_n) dominates (ν_n) and (ν_n) dominates (μ_n).

Friedman's theorem is also true for any random graph model that is dominated by one of $\mathcal{G}_{n,d}$, $\mathcal{H}_{n,d}$, $\mathcal{I}_{n,d}$, and $\mathcal{J}_{n,d}$. Since $\mathcal{G}_{n,d}$ dominates $\mathcal{G}'_{n,d}$ and $\mathcal{I}_{n,d}$ is contiguous with $\mathcal{K}_{n,d}$, the theorem is proved for the models $\mathcal{G}'_{n,d}$ and $\mathcal{K}_{n,d}$ too.

Friedman uses a variant of "the trace method". The trace method is a general method that looks at the expected value of the trace of "large" powers of the adjacency matrix of a random graph. For a graph G and A is its adjacency matrix, we have

$$\operatorname{Tr} A^r = \sum {\mu_i}^r,$$

where the μ_i's are the eigenvalues of A. However, the i, j entry of A^r is the number of walks from vertex i to vertex j in G that traverse r edges. And so $\operatorname{Tr} A^r$ is the number of closed walks of length r in G. The equating of these two ways of computing $\operatorname{Tr} A^r$ is the essence of the trace method. The standard trace method does not work to prove this theorem, so Friedman modifies it to work around several infinite sums that arise which do not converge.

6.2 Cioabă and Murty

Recently Cioabă and Murty [20] showed how to construct "almost" Ramanujan graphs of any degree. They did this by taking known Ramanujan graphs and attaching or removing perfect matchings to/from them. They show that for any fixed $\varepsilon > 0$ and for almost every value of d there are infinitely many d-regular graphs such that all non-trivial eigenvalues of the graphs are less than $(2 + \varepsilon)\sqrt{d-1}$ in absolute value.

Chapter 7

Generalized Ramanujan Graphs

Greenberg [43] introduced the notion of the Ramanujan property for any finite graph, not just regular finite graphs (see Lubotzky and Nagnibeda [65] for an overview in English). As we will see, the concept of generalized Ramanujan graph, as well as the theory we go through to get there, allows some insight into regular Ramanujan graphs.

Using the Perron-Frobenius theorem and the Principal axis theorem we know that all the eigenvalues of a connected graph are real, and the largest eigenvalue is the largest eigenvalue in absolute value and the largest eigenvalue is simple; so there is a non-zero gap between the largest and second largest eigenvalue of a connected graph. This is a generalization of k being both a simple eigenvalue and the largest eigenvalue in absolute value of a connected k-regular graph. Greenberg compared the second largest eigenvalue with a quantity derived from the graph itself. In order to define this quantity we need to take a short detour into algebraic topology and operators on Hilbert spaces. In this chapter, unless otherwise stated, graphs can be infinite but with bounded finite degree.

7.1 Covering Maps

Let X, Y be a topological spaces. The map $\rho\colon Y \to X$ is called a *covering map* if ρ is a continuous onto map such that for every $x \in X$ there exists an open neighbourhood, U, of x such that $\rho^{-1}(U) = \dot{\bigcup}_{i \in I} V_i$, where the V_i are mutually disjoint open sets in Y and $\rho|_{V_i}$ is a homeomorphism of V_i onto U for each $i \in I$. It can be shown that the cardinality of the index set I

in the definition is independent of the point x and the neighbourhood U. We call this fixed cardinality the number of *sheets* of the covering map ρ (it is also sometimes called the *index* of the cover). We will also say that ρ is an $|I|$*-fold* cover. Intuitively, it is sometimes helpful to think of the covering space Y as being $|I|$ copies of X glued together in some fashion; although it is not true that the covering map induces a homeomorphism of X with some subspace of Y. When talking about covering maps we are only interested in connected covering spaces. For it is trivially easy to construct many uninteresting covers of a space by taking disjoint disconnected copies of the base space as the covering space. From now on we add connectivity to the list of requirements for a cover space.

Example 38. Let $S^1 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$, and let $\rho\colon \mathbb{R} \to S^1$ be defined by $\rho(x) = e^{ix}$. Then ρ is a covering map of $|\mathbb{Z}|$ sheets.

A space can "cover" itself in a non-trivial way. Let $\nu\colon S^1 \to S^1$ be defined by $\nu(e^{ix}) = e^{2ix}$. Then ν is a covering map of two sheets.

If $\rho\colon Y \to X$ is a covering map and Y is *simply connected* (informally, every loop can be continuously shrunk to a point) then ρ is said to be a *universal cover*. The reason for the term "universal" is that if $\rho\colon Y \to X$ is a universal cover and $\nu\colon C \to X$ is any other cover then there exists a covering map $\sigma\colon Y \to C$ such that $\nu \circ \sigma = \rho$. So the universal cover of a space covers every cover of that space.

$$\begin{tikzcd}
Y \arrow[rr, bend left, "\rho"] \arrow[dr, dashed, "\sigma"'] & & X \\
& C \arrow[ur, bend right, "\nu"'] &
\end{tikzcd}$$

It is easy to see that the universal cover (if it exists) is essentially unique. With a few topological assumptions about a space it can be proved that it has a universal cover. Namely, every *path-connected*, *locally path-connected*, and *semi-locally simply connected* (every point has a neighbourhood for which all loops in the neighbourhood can be shrunk to a point in the main space) space has a universal cover. The proof of this builds the universal cover as equivalence classes of paths starting at an arbitrary fixed point in the base space [73]. The covering map is then defined to send an equivalence class of paths to their common endpoint. With this construction of the universal cover it becomes clearer why simply connected is the defining property of universal covers: in a simply connected space there is only one equivalence class of paths with any given starting and ending point.

Example 39. From above, it is clear that the universal cover of S^1 is $\mathbb{R}$, since $\mathbb{R}$ is simply connected.

7.1.1 Graph Covers

The topological definition of covering spaces as it applies to graphs as topological spaces can be simplified and stated in terms of graph theory. A covering map, ρ, from the graph Y to the graph X is a map of vertices of Y to vertices of X, that is onto, and for each vertex y in Y, $\rho|_{\mathrm{N}(y)}$ is a bijection with $\mathrm{N}(\rho(y))$. ($\mathrm{N}\,(y)$ is the set of neighbours of y, see Definition 1.) For an example of a graph cover see Figure 7.1.

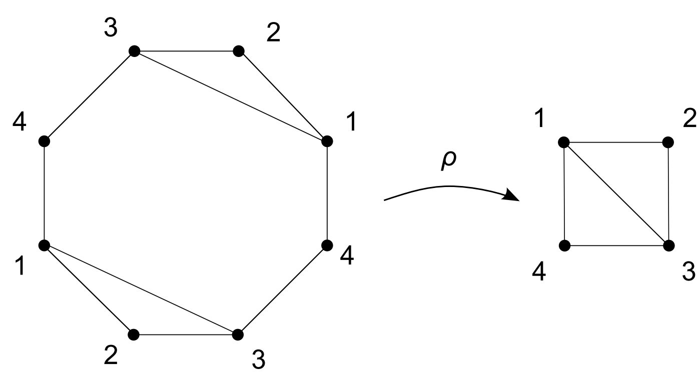


Figure 7.1: A cover of a 4 vertex graph.

Every connected graph with finite maximum degree satisfies the sufficient conditions above for a universal cover. For a graph, simply connected, means that the graph does not have a cycle, i.e. is a tree. So universal covers of graphs are trees. If G is a tree, then the universal cover of G is G itself. Otherwise, if G is not a tree, then the universal cover of G is an infinite tree. Denote the universal cover of G by $\widetilde{G}$. The universal cover of a graph can be defined abstractly in a more explicit manner [59] using the proof of existence of universal covers. Let G is a graph with universal cover $\widetilde{G} = \left(\widetilde{V}, \widetilde{E}\right)$. Pick any vertex v of G, then $\widetilde{G}$ can be described as follows. $\widetilde{V}$ is the set of all walks starting at v that do not traverse the same edge in opposite directions twice in a row. The edges are defined by

$$\widetilde{E} = \{(\omega_1, \omega_2) \mid \omega_1 \text{ is a one edge extension of } \omega_2 \text{ or vice versa}\}.$$

The covering map then sends a walk to its terminating vertex. This definition is independent of the choice of starting vertex v. The universal cover of a k-regular graph is the infinite k-regular tree. See Figure 7.2 for a diagram of the 3-regular tree, and see Figure 7.3 for an example of a finite graph and its infinite universal cover.

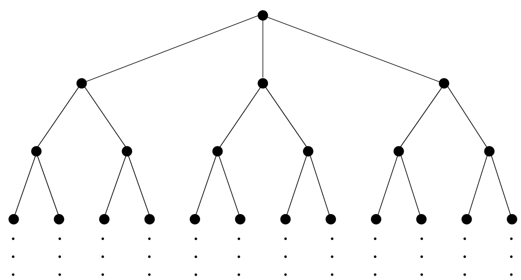


Figure 7.2: The infinite 3-regular tree.

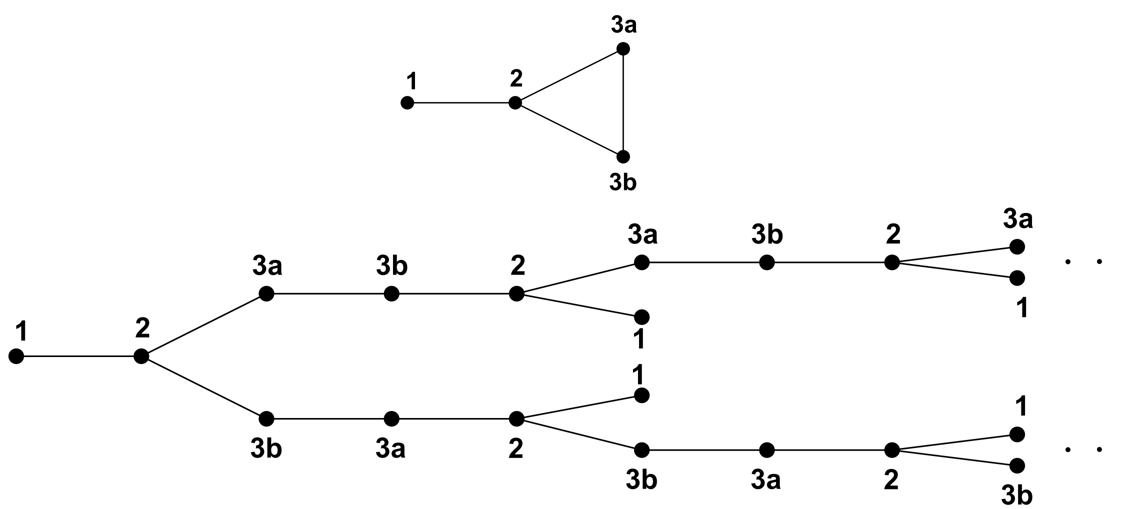


Figure 7.3: A graph and its universal cover.

7.2 Adjacency Operator

As we saw in the previous section, the universal cover of a graph may be an infinite graph. So we extend the concept of adjacency matrix for finite graphs to infinite graphs by defining the *adjacency operator* of an infinite graph. The *Hilbert space* $\ell^2(X)$ is the space of all functions $h\colon X \to \mathbb{C}$ such that

$$\sum_{x\in X} |h(x)|^2 < \infty.$$

We define the *norm* of an element of a Hilbert space, $h \in \ell^2(X)$, by

$$\|h\| = \left(\sum_{x\in X} |h(x)|^2\right)^{\frac{1}{2}}.$$

A *bounded linear operator* on $\ell^2(X)$ is a linear function $\mathcal{B}\colon \ell^2(X) \to \ell^2(X)$ such that $\left\{\|\mathcal{B}x\| \;\middle|\; \|x\| = 1\right\}$ is a bounded set. In this case we define $\|\mathcal{B}\|$, the *norm* of $\mathcal{B}$, to be

$$\|\mathcal{B}\| = \sup\left\{\|\mathcal{B}x\| \;\middle|\; \|x\| = 1\right\}.$$

We can also define eigenvalues of bounded linear operators on Hilbert space. An *eigenvalue* of $\mathcal{B}$ is a scalar λ such that $\mathcal{B} - \lambda$ does not have a bounded inverse (this generalizes the concept in the finite dimensional vector spaces, since all finite dimensional linear operators are bounded). The *spectral radius* of $\mathcal{B}$, denoted $\rho(\mathcal{B})$, is then the supremum of the absolute values of the eigenvalues of $\mathcal{B}$:

$$\rho(\mathcal{B}) = \sup\left\{|\lambda| \;\middle|\; \lambda \text{ is an eigenvalue of } \mathcal{B}\right\}.$$

The spectrum is actually a compact set [23], so the supremum is attained. If $\mathcal{B}$ is a bounded linear operator then it can be shown that

$$\rho(\mathcal{B}) = \lim_{n\to\infty} \|\mathcal{B}^n\|^{\frac{1}{n}}.$$

For a graph $G = (V, E)$ we define the adjacency operator of G, $\mathcal{A}_G\colon \ell^2(V) \to \ell^2(V)$, by

$$(\mathcal{A}_G h)(x) = \sum_{\{x,y\}\in E} h(y).$$

In the finite case this definition reduces to the usual notions. It is clear that $\mathcal{A}_G$ is bounded (since the maximum degree is finite), so this permits us to define the concepts of *eigenvalue*, *spectrum*, and *spectral radius* for infinite graphs with finite maximum degree. In particular the expression for the spectral radius ($\rho(\mathcal{A}_G) = \lim_{n\to\infty} \|\mathcal{A}_G{}^n\|^{\frac{1}{n}}$) has a graph theoretic analogue: $\rho(\mathcal{A}_G) = \lim\limits_{n\to\infty} a_n{}^{\frac{1}{n}}$ where a_n is the number of closed paths of length n starting at some fixed vertex in the graph.

7.3 Generalized Ramanujan Graph

We are now ready to define Ramanujan for general graphs.

A finite graph G is a *generalized Ramanujan graph* if for all eigenvalues λ of G except the Perron-Frobenius one and its negative (if it exists) we have $|\lambda| \le \rho\left(\widetilde{G}\right)$, where $\widetilde{G}$ is the universal covering tree of G.

To see that this definition makes sense we consider it in the regular case. The universal covering tree of a k-regular graph is the infinite k-regular tree. The spectral radius of the infinite k-regular tree is $2\sqrt{k-1}$ [54][55]. Thus this definition generalizes the regular definition of Ramanujan graphs.

It can be shown that if the graph B covers the graph A, then $\rho(B) \le \rho(A)$ [81]. Further if B is finite (and hence A is finite) then the spectrum of B contains the spectrum of A. Indeed, suppose $\sigma\colon B \to A$ is a covering map. Let $\mathcal{A}$ and $\mathcal{B}$ be the adjacency matrices of A and B respectively and let λ be an eigenvalue of A with eigenvector $\vec{v}$ ($\vec{v}$ is indexed by the vertices of A, $\vec{v}(a)$ being the value of $\vec{v}$ for vertex a of A). Define the vector $\vec{w}$, indexed by the vertices of B, by

$$\vec{w}(b) = \vec{v}(\sigma(b))$$

($\vec{w}$ replicates the value of $\vec{v}(a)$ for each "copy" of a in B). Let $b \in B$, and let $a = \sigma(b)$, then

$$\begin{aligned}
(\mathcal{B}\vec{w})(b) &= \sum_{b'\in \mathrm{N}(b)} \vec{w}(b') \\
&= \sum_{b'\in \mathrm{N}(b)} \vec{v}(\sigma(b')) \\
&= \sum_{a'\in \mathrm{N}(a)} \vec{v}(a') \qquad\qquad (\text{since } \mathrm{N}(b) \overset{\sigma}{\simeq} \mathrm{N}(a)) \\
&= (\mathcal{A}\vec{v})(a) = \lambda\vec{v}(a) = \lambda\vec{w}(b).
\end{aligned}$$

So $\mathcal{B}\vec{w} = \lambda\vec{w}$, and $\vec{w}$ is an eigenvector of B with eigenvalue λ. Further, if we let $\mathrm{E}_\alpha(\mathcal{G})$ be the eigenspace of the eigenvalue α for the operator $\mathcal{G}$, then the above proof also shows that $\dim \mathrm{E}_\lambda(\mathcal{B}) \geq \dim \mathrm{E}_\lambda(\mathcal{A})$. To see this let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of linearly independent eigenvectors in $\mathrm{E}_\lambda(\mathcal{A})$, it is easy to see that the set of associated vectors indexed by the vertices of B, $\{\vec{w}_1, \ldots, \vec{w}_k\}$, is also linearly independent. And by Theorem 9, the geometric and algebraic multiplicity of each eigenvalue of a graph are the same, so we have also shown that the characteristic polynomial of A divides the characteristic polynomial of B. Applying this result to the spectral radius we now know that for finite graphs A and B with B covering A that $\rho(B) \geq \rho(A)$ and hence $\rho(B) = \rho(A)$.

Theorem 40. *Let A and B be finite graphs with B covering A, then*

$$\rho(B) = \rho(A)$$

and the characteristic polynomial of A divides the characteristic polynomial of B.

There is a generalization of the Alon-Boppana theorem for general finite graphs (Alon-Boppana only dealt with regular graphs) due to Greenberg [43].

Theorem 41 (Greenberg [43])**.** *Let (G_n) be a sequence of finite graphs with $|V_n| \to \infty$ that are all covered by the same universal cover $\widetilde{G}$, then*

$$\liminf_{n\to\infty} \mu_1(G_n) \geq \rho\left(\widetilde{G}\right).$$

In order to generalize the theorem we must replace the condition that every graph in the sequence has the same regularity with the condition that every graph in the sequence is covered by the same graph. This theorem also shows that it is useful to consider the class of all finite graphs covered by a given graph.

For a graph G, define $\Omega_f(G)$ to be the set of all finite graphs that are covered by G. We have already shown that the spectral radius of two finite graphs is equal when one covers the other, but we can't conclude that all graphs in $\Omega_f(G)$ have the same spectral radius because there may be two graphs in $\Omega_f(G)$ which (by definition) are covered by G but neither one covers the other and they share no common *finite* cover. Leighton [59] proved that this cannot happen. Specifically he showed that if two finite graphs share a common cover, then they share a common finite cover. Thus, for any $H_1, H_2 \in \Omega_f(G)$ we have $\rho(H_1) = \rho(H_2)$, and so any graph in $\Omega_f(G)$ has the same Perron-Frobenius eigenvalue. Thus we have a complete analogy with k-regular Ramanujan graphs. The class of all k-regular graphs is replaced by $\Omega_f(G)$, k by $\rho(\Omega_f(G))$, and $2\sqrt{k-1}$ by $\rho(G)$.

$$\begin{array}{ccc} k\text{−regular graphs} & \Longleftrightarrow & \Omega_f(G) \\ k & \Longleftrightarrow & \rho(\Omega_f(G)) \\ 2\sqrt{k-1} & \Longleftrightarrow & \rho(G) \end{array}$$

We already know that $\rho(T_k) = 2\sqrt{k-1}$ where T_k is the infinite k-regular tree, but what about $\rho(G)$ for other graphs? We are only interested in $\rho(G)$ for infinite graphs (the finite case is trivial), and trees appear as universal covers, so the following theorem gives us a good foot hold in this area.

Theorem 42. *If T is an infinite tree with the degree of every vertex bounded by k, then*

$$\rho(T) \leq \rho(T_k) = 2\sqrt{k-1}.$$

Proof. This proof is based on the one in [65]. We already know that $\rho(T) = \lim_{n\to\infty} a_n{}^{\frac{1}{n}}$ where a_n is the number of closed paths of length n starting from some fixed vertex in T. There are $\leq k$ edges incident to every vertex in T, so there are at least as many such paths in T_k, which has k edges incident to every vertex, as in T. So $a_n \leq a'_n$ where a'_n is the number of closed paths of length n starting from some fixed vertex in T_k. Hence

$$\rho(T) = \lim_{n\to\infty} a_n{}^{\frac{1}{n}} \leq \lim_{n\to\infty} a'_n{}^{\frac{1}{n}} = \rho(T_k) = 2\sqrt{k-1}.$$

□

For any k-regular finite graph G we have $\rho(G) = k$, but for the k-regular infinite tree T_k we have $\rho(T_k) = 2\sqrt{k-1}$. It may seem strange that every finite graph that T_k covers has largest eigenvalue k, but yet when going from finite graphs to the infinite graph T_k the largest eigenvalue suddenly drops to $2\sqrt{k-1}$. We aim to provide some justification for this jump. An unsatisfying explanation is that the all 1 vector, which is the eigenvector for the eigenvalue k for all k-regular graphs, is not an element of $\ell^2(V)$ for V infinite. A more satisfying explanation goes as follows. For $k \geq 2$ and $m \geq 1$ define the tree $T_{k,m}$ to be the first m levels of T_k (with some vertex identified

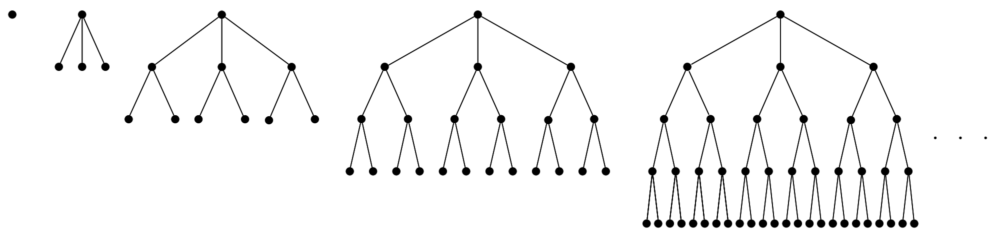


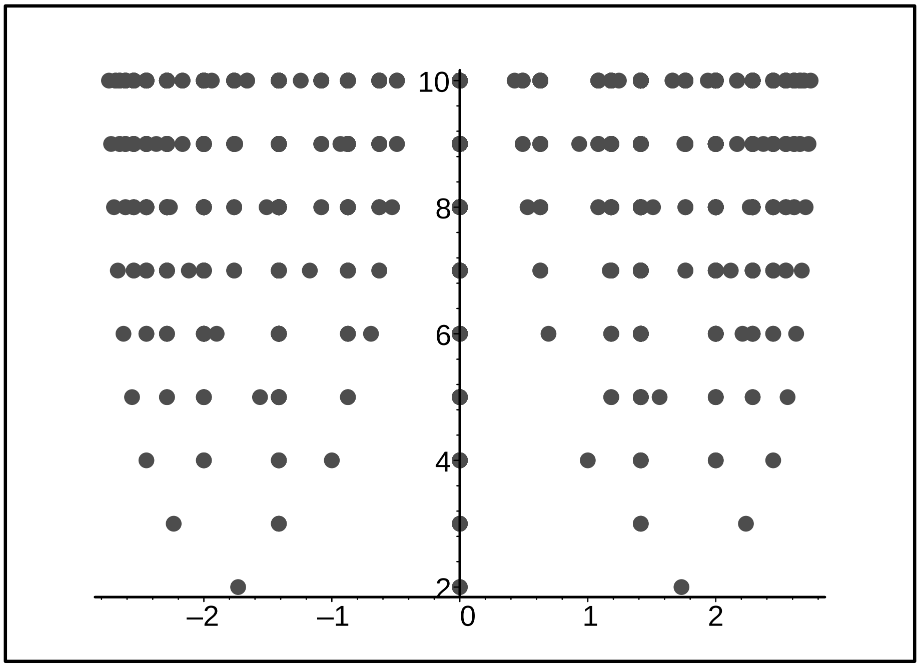
Figure 7.4: The graphs $T_{3,1}$, $T_{3,2}$, $T_{3,3}$, $T_{3,4}$, and $T_{3,5}$ respectively.

as the root in order to define the levels). Refer to Figure 7.4 for examples. Intuitively, T_k is the limit of $T_{k,m}$ as $m \to \infty$. We will formalize this notion, but first we consider the eigenvalues of $T_{k,m}$. Figure 7.5 gives some evidence that $\rho(T_{k,m}) \to 2\sqrt{k-1}$ as $m \to \infty$. Mohar [75] formalizes these concepts. We say that a sequence of graph (G_i) converges to the graph G if each G_i is a subgraph of G and for every edge e of G there exists an N such that for all $i > N$, G_i contains the edge e. Mohar proves that if $(G_i) \to G$ then $\rho(G_i) \to \rho(G)$. So

$$\rho(T_{k,m}) \xrightarrow[m\to\infty]{} 2\sqrt{k-1},$$

and we have some intuitive reason to believe that $\rho(T_k) = 2\sqrt{k-1}$.

The Ramanujan graphs of LPS/M and Morgenstern can be recast in the light of generalized Ramanujan graphs. These constructions provide infinite families of Ramanujan graphs for the class of graphs $\Omega_f\left(T_{p^k+1}\right)$ for every prime p and $k \geq 1$. The results of Friedman in Section 6.1 and Cioabă and Murty in Section 6.2 seem to show that Ramanujan graphs are plentiful. Which might lead one to conjecture that $\Omega_f(T_n)$ contains infinitely many Ramanujan graphs (in other words, there is an infinite family of n-regular Ramanujan graphs for every n). A natural extension of this conjecture to generalized Ramanujan graphs is: for every infinite tree T for which $\Omega_f(T) \neq \varnothing$, $\Omega_f(T)$ contains infinitely many Ramanujan graphs. (The condition $\Omega_f(T) \neq \varnothing$ is necessary because not every infinite tree covers a finite graph; Bass and Kulkarni [7] give a necessary and sufficient condition for this to be the case.) However, Lubotzky and Nagnibeda [65] show that the answer to this is a strong no. In fact, they give many examples of finite graphs G such that $\Omega_f\left(\widetilde{G}\right)$ contains no Ramanujan graph at all (see Figure 7.6 for one example of such G). Let $\mathrm{Aut}(G)$ be the automorphism group of the graph



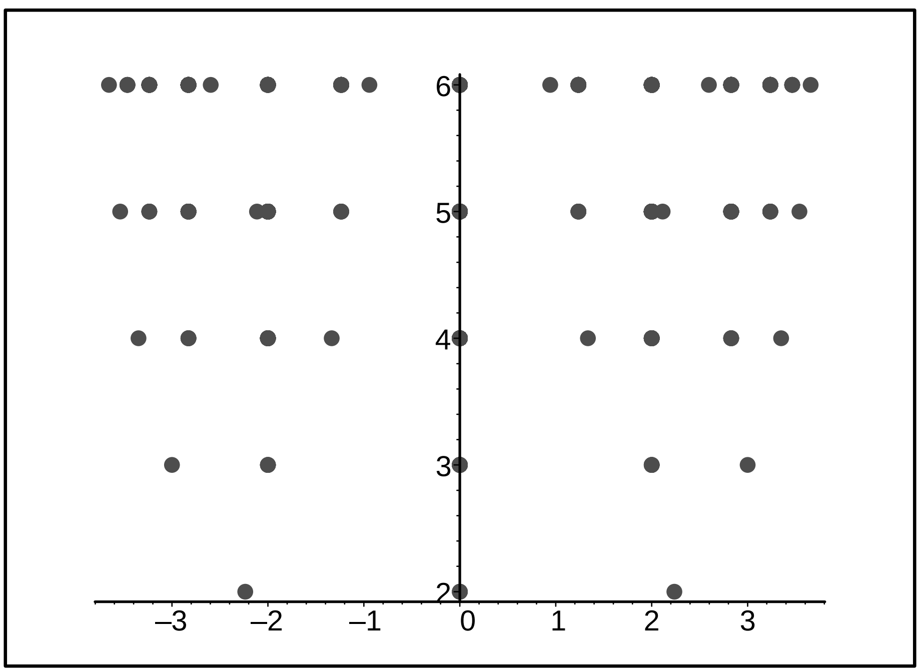


Figure 7.5: The eigenvalues of trees $T_{k,m}$ for $k = 3, 5$ respectively. The y-axis represents the value of m and the x-axis shows the distribution of the eigenvalues.

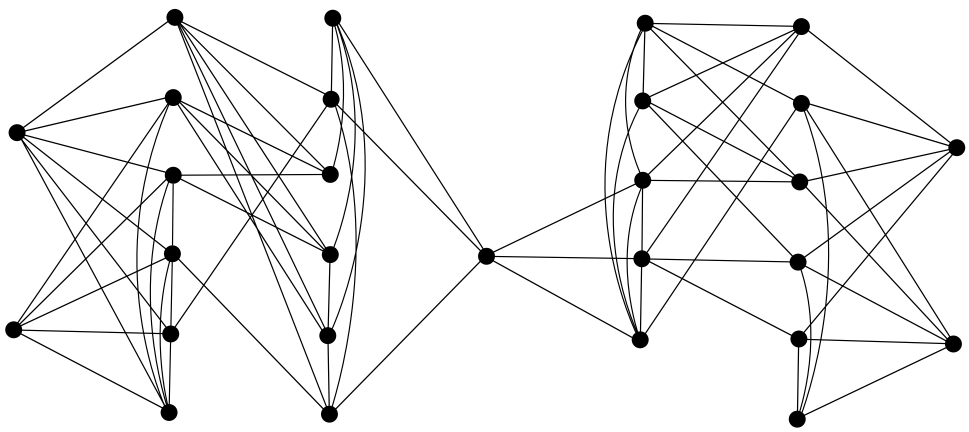


Figure 7.6: An example of a graph G such that $\Omega_f\left(\widetilde{G}\right)$ contains no Ramanujan graph at all. (Lubotzky and Nagnibeda [65])

G (all isomorphisms of G with itself). We say that a graph G is *minimal* if

$$G = \widetilde{G}/\operatorname{Aut}\left(\widetilde{G}\right).$$

One must be careful in understanding this definition, as the quotient of an infinite tree T by its automorphism group may not be a graph. For example, for T_k, the infinite k-regular tree, and any vertices v, w vertices in T_k, there is a $\sigma \in \operatorname{Aut}(T_k)$ with $\sigma(v) = w$. So $T_k/\operatorname{Aut}(T_k)$ is one vertex with self loops. Friedman [36] relaxes the definition of a graph and gets a structure called a *pregraph*, which allows one to consider the minimal pregraph for a class of finite graphs covered by an infinite tree. The reason for the usage of the term "minimal" is that minimal graphs are minimal in their covering class under the relation of "is covered by". Formally, Bass and Tits [8] show that if a finite graph X is minimal then for every $X' \in \Omega_f\left(\widetilde{X}\right)$, X' covers X.

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, bend left] \arrow[dr, bend right] & & X \\
& X' \arrow[ur, dashed] &
\end{tikzcd}$$

Bass and Tits also give a procedure to determine if a finite graph is minimal. For a vertex v in a graph G we define the notation $\deg_{k_0,k_1,\ldots,k_n}(v)$ to denote the number of paths $(a = a_0, a_1, \ldots, a_n)$ of length n with $\deg(a_i) = k_i$.

We define an equivalence relation, R_d, on the vertices of G: $v\ R_d\ w$ iff $\deg_{k_0,k_1,\ldots,k_n}(v) = \deg_{k_0,k_1,\ldots,k_n}(w)$ for all tuples $(k_0, k_1, \ldots, k_n)$ and all $n \leq d$. It is only necessary to consider tuples with each k_i between the minimum and maximum degree of G, so these condition can be checked in a finite amount of time. Two vertices are R_1 equivalent iff they have the same degree. Two vertices are R_2 equivalent iff they have the same degree and for every degree they have the same number of neighbours of that degree. Notice that $v\ R_1\ w$ and $v\ R_2\ w$ are both necessary conditions for v and w to be mapped to the same vertex by some covering map. And in general for each n, $v\ R_n\ w$ is a necessary condition for v and w to be mapped to the same vertex by some covering map. The idea is to take the "limit" of these relations in order to obtain a relation that gives necessary and sufficient conditions for two vertices of X to be mapped to the same vertex by some covering map.

It is clear that if $d_2 \geq d_1$ then $v\ R_{d_2}\ w \implies v\ R_{d_1}\ w$. So the set of R_{d_2}-equivalence classes is a refinement of the set of R_{d_1}-equivalence classes. Since G is a finite graph, this process of refining equivalence classes must terminate at some point, so there must be a D such that $R_d = R_D$ for all $d \geq D$. Denote this common final value of the relation by R. Let X^* be the graph with vertices the R-equivalence classes and two classes, A, B are adjacent if any $a \in A$ is adjacent in X to any $b \in B$. Bass and Tits showed that if $X = X^*$ then X is minimal. Thus X can be tested for minimality by constructing X^* and seeing if it is the same as X.

We are now ready to describe the graphs that Lubotzky and Nagnibeda [65] used.

Theorem 43. *Let X be a minimal graph with maximum degree k. Let X have a* cut vertex *v, so deleting v from X results in a graph with two or more connected components. Let $X_1, X_2 \subset X - v$ be two induced subgraphs of $X - v$ that are disjoint and disconnected from each other. If the average degrees in X_1 and X_2 are both strictly greater than $2\sqrt{k-1}$, then $\Omega_f\left(\widetilde{X}\right)$ contains no Ramanujan graph. In fact, there exists $\varepsilon > 0$ such that*

$$\inf_{Y \in \Omega_f\left(\widetilde{X}\right)} \mu_1(Y) \geq \rho\left(\widetilde{X}\right) + \varepsilon.$$

Proof. We aim to show that X is not Ramanujan, and hence show that no finite cover of X is Ramanujan.

Since the maximum degree of X is k, $\widetilde{X}$ is an infinite tree with the degree of every vertex bounded by k, so by Theorem 42, $\rho\left(\widetilde{X}\right) \leq 2\sqrt{k-1}$. Let λ_1

and λ_2 be the largest eigenvalues of X_1 and X_2 respectively. Then λ_1 and λ_2 are also eigenvalues of $X - v$. By Theorem 12, λ_i is bounded from below by the average degree of X_i, so $\lambda_i > 2\sqrt{k-1}$ $(i = 1, 2)$. By the interlacing theorem for graphs (Corollary 15), $\mu_1(X)$ is bounded between the two largest eigenvalues of $X - v$, which are both at least as big as $\min\{\lambda_1, \lambda_2\} > 2\sqrt{k-1}$. Hence $\mu_1(X) > 2\sqrt{k-1} \geq \rho\left(\widetilde{X}\right)$, and X is not Ramanujan.

Let $X' \in \Omega_f\left(\widetilde{X}\right)$, then since X is minimal, there is a covering map $\nu\colon X' \to X$. By Theorem 40, $\rho(X') = \rho(X)$ and every eigenvalue of X is an eigenvalue of X'. So the largest eigenvalue of X and X' are the same, and $\mu_1(X') \geq \mu_1(X)$. So X' is not Ramanujan either. Thus $\Omega_f\left(\widetilde{X}\right)$ contains no Ramanujan graph. In fact, we can find an absolute fixed constant ε $\left(= \mu_1(X) - \rho\left(\widetilde{X}\right)\right)$ such that $\mu_1(X') \geq \rho\left(\widetilde{X}\right) + \varepsilon$ for all $X' \in \Omega_f\left(\widetilde{X}\right)$. $\square$

As was remarked earlier (Chapter 3) graphs with good expansion don't have bottlenecks, and this proof requires us to start out with a graph that has a cut vertex—the quintessential example of a bottleneck.

Hoory [48] shows that for a general graph with average degree d the spectral radius of its universal cover is $\geq 2\sqrt{d-1}$. Hoory also proves a generalization of the Alon-Boppana bound for irregular graphs that applies to graphs that do not share the same universal cover (as was the case for Greenberg's generalization of the Alon-Boppana bound). For a graph G let $\mathrm{B}_r(v)$ be the *ball* of radius r around the vertex v in G, i.e. the set of all vertices of distance $\leq r$ from v in G. Then we say that G has *r-robust average degree* d if for every vertex v of G the induced subgraph $G \setminus \mathrm{B}_r(v)$ of G has average degree d. Hoory shows that if (G_n) is a sequence of graphs and r is a function such that $r(n) \geq 2$ and $r(n) \to \infty$ as $n \to \infty$ and G_n has $r(n)$-robust average degree $d \geq 2$ for every n, then

$$\liminf_{n\to\infty} \mu_1(G_n) \geq 2\sqrt{d-1}.$$

Hoory also shows that the following, perhaps more natural, generalization does not hold: let (G_n) be a sequence of graphs such that $\operatorname{diam}(G_n) \to \infty$ and the average degree of every G_n is at least d, then $\liminf_{n\to\infty} \mu_1(G_n) \geq 2\sqrt{d-1}$.

It can be shown that two graphs share the same universal cover iff they have the same *degree refinement* [5]. The degree refinement of a graph X is closely related to the graph X^* defined above by Bass and Tits. Specifically, if

A and B are R-equivalence classes from above, then for any $a \in A$ the number of edges incident to a and a vertex in B is independent of the choice of a. Number the R-equivalence classes $A_1, \ldots A_\ell$, and denote by $d_{i,j}$ $(1 \leq i, j \leq \ell)$ the number of edges incident to some $a_i \in A_i$ and to any vertex in A_j. The degree refinement of X is then the $\ell \times \ell$ matrix $D = (d_{i,j})$. This allows one to determine when two graphs belong to the same "covering-class", a necessary tool when dealing with Generalized Ramanujan graphs in a concrete form.

Nagnibeda [78] outlines an algorithm to calculate the spectral radius of the universal cover of a non-regular graph, thus giving a way, in practice, to determine if a general finite graph is Ramanujan or not.

A *lift* of a graph G is a graph H along with a covering map $\nu \colon H \to G$. If ν is a covering of n sheets (n not necessarily finite) we say that is it a n-lift of G. If G is a k-regular graph, and H is a lift of G then H is a k-regular graph with more vertices than G. And we have already shown there is a strong relationship between the eigenvalues of a graph and its cover. Along these lines Bilu and Linial [63] have pursued finding a construction of infinite families of k-regular Ramanujan graphs for any k using lifts. For the specific case that H is a 2-lift of G we can be even more explicit about the eigenvalue relationship between the two graphs. For H a cover of G, we follow Friedman [37] and call the eigenvalues of G the *old eigenvalues* of H and the remaining eigenvalues the *new eigenvalues* of H. Starting with a base graph $G = (V, E)$ we define a *signing* of the edges of G to be a function $s \colon E \to \{-1, 1\}$. Using the graph G and the signing s we can construct a 2-lift of G. The 2-*lift of G associated with the signing s* is the graph H on vertices $(V \times \{0\}) \dot\cup (V \times \{1\})$ and for each edge $\{v, w\} \in E$ we have two edges in H: if $s(v, w) = 1$ we have the edges $\{(v, 0), (w, 0)\}$ and $\{(v, 1), (w, 1)\}$; if $s(v, w) = -1$ we have the edges $\{(v, 0), (w, 1)\}$ and $\{(v, 1), (w, 0)\}$. It is easy to see that every 2-lift must arise as the 2-lift associated with some signing. We define the *signed adjacency matrix* of G and s to be the matrix A_s with the (v, w) entry of A_s being equal to $s(v, w)$ if (v, w) is an edge and 0 otherwise. We say that the *spectral radius* of the signing s is the spectral radius of A_s. Bilu and Linial [63] shows that if H is the 2-lift associated with a signing s of G, A the adjacency matrix of G, A_s the signed adjacency matrix of G and s, then the eigenvalues of H are exactly the eigenvalues of A combined with the eigenvalues of A_s counting eigenvalues by multiplicity. Hence the new eigenvalues of H are the eigenvalues of A_s. This motivates the following idea for a construction of sequences of graphs with "large" spectral gap: start with a graph G_1 with "large" spectral gap, find a signing of G_1 with small spectral radius, let G_2 be the 2-lift of G_1 with the associated signing, G_2 will also have "large" spectral gap, and continue on in this manner. Bilu and Linial [63] shows that for a graph with maximal degree d there is a signing with spectral radius $\mathcal{O}\left(\sqrt{d \cdot log^3 d}\right)$. Further, they conjecture, backed up by numerical evidence, that this can be reduced to the best possible (the Alon-Boppana bound) spectral radius: $2\sqrt{d-1}$.

Friedman [37] presents some results along these lines that contrast with the results of Lubotzky and Nagnibeda above. Let G be a fixed graph with largest eigenvalue λ and let $\rho = \rho\left(\widetilde{G}\right)$. Friedman shows that for H a random n-fold covering of G, there is a function α with $\alpha(n) \to 0$ as $n \to \infty$ such that the probability that the new eigenvalues of the covering H are bounded in absolute value by $\sqrt{\lambda\rho} + C\alpha(n)$ goes to 1 as $n \to \infty$. Friedman also proves another version of the Alon-Boppana bound. Let G be a fixed graph, then there exists a function α with $\alpha(n) \to 0$ as $n \to \infty$ such that for any n-fold cover H of G, H has a new eigenvalue with absolute value as least $\rho\left(\widetilde{G}\right) - \alpha(n)$.

Chapter 8

Computations

C code was written to construct each of the main types of graphs presented here, as well as perform various computations on them. The GNU Scientific Library [44] and Maple [69] were used to compute eigenvalues of graphs. The GNU Compiler Collection (gcc) [41] was the main compiler used. The total amount of code written is approximately 5000 lines. The full code is on the attached CD.

Each construction had a function associated with it that implemented the details that were specific to that construction. There was supporting code that was more general and shared between the different constructions. The general purpose code included an implementation of finite fields and various operations within them, code to generate all matrices in PSL and PGL over $\mathbb{Z}_p$ and $\mathbb{F}_{p^n}$ in a simple linear order for easier searching later on, code to output graphs in various formats suitable for graphical viewing or computations (i.e. computing the eigenvalues with Maple), code to use the GSL to calculate the eigenvalues directly, code to layout a graph in tree form.

The typical form of the construction function goes something like

- Calculate generators
- Construct graph using generators
- Output graphs, calculate eigenvalues, etc

The following is a list of each function that implements the named construction.

```
void lpsm(int p, int q);
```

```
void morgenstern(int q, int n, ffe poly);
void gunnells(int q, int n, ffe poly, int w);
void chung(int q, int n, ffe poly);
void li(ffe k, int q, int n, ffe poly);
void dereyna(int p);
void chiu(int p);
void friedman(int p);
```

Below is sanitized for clarity C code for a typical such function, the one for the LPS/M construction.

```
void lpsm(int p, int q) {
    Matrix *PGSL;
    Matrix *generators;

    if (legendre_symbol(p,q) == 1) {
        //PSL
        N = (q*(q*q-1))/2;
        mode = 0;
    }
    if (legendre_symbol(p,q) == -1) {
        //PGL
        N = (q*(q*q-1));
        mode = 1;
    }


    generators = //allocate space for p+1 Matrix's

    int thei = 0;
    for (int i = 1; i < q; i++) {
        if (mod(i*i,q) == mod(-1,q)) {
            thei = i;
            break;
        }
    }

    int curr = 0;
    if (mod(p,4) == 1) {
        int a4;
        for (int a1 = 1; a1*a1 <= p; a1+=2) {
```

```
            for (int a2 = 0; a1*a1 + a2*a2 <= p; a2+=2) {
                for (int a3 = 0; a1*a1 + a2*a2 + a3*a3 <= p; a3+=2) {
                    a4 = round(sqrt(p - a1*a1 - a2*a2 - a3*a3));
                    if (a4 * a4 == p - a1*a1 - a2*a2 - a3*a3) {
                        //add all +/- combination of solutions
                        //to the list of generators
                        curr = addall(generators, a1, a2, a3, a4, curr);
                    }
                }
            }
        }
    } else {
        for (int a1 = 0; a1 <= p; a1+=2) {
            int a2;
            if (a1 > 0) {
                a2 = -p;
            } else {
                a2 = 1;
            }
            for ( ; a2 <= p; a2+=2) {
                for (int a3 = -p; a3 <= p; a3+=2) {
                    for (int a4 = -p; a4 <= p; a4+=2) {
                        if (a1*a1 + a2*a2 + a3*a3 + a4*a4 == p) {
                            generators[curr].a11 = mod(a1 + thei*a2, q);
                            generators[curr].a12 = mod(a3 + thei*a4, q);
                            generators[curr].a21 = mod(-a3 + thei*a4, q);
                            generators[curr].a22 = mod(a1 - thei*a2, q);
                            curr++;
                        }
                    }
                }
            }
        }
    }

    for (int i = 0; i < p+1; i++) {
        normalize(&(generators[i]), q);
    }

    if (mode == 0) {
```

```
        for (int i = 0; i < p+1; i++) {
            int thedet = det(&(generators[i]), q);
            for (int j = 1; j < q; j++) {
                if (mod(j*j,q) == thedet) {
                    int jinv = modularinverse(j, q);
                    scalarmultiply(jinv, &(generators[i]));
                    break;
                }
            }
        }

        PGSL = generatePSL(q);
    }

    if (mode == 1) {
        PGSL = generatePGL(q);
    }

    Matrix dest;
    int result;
    int *adjacencylist = //allocate space for (p+1)*N int's
    for (int i = 0; i < N; i++) {
        for (int j = 0; j < p+1; j++) {
            multiply(&(PGSL[i]), &(generators[j]), &dest);
            result = findMatrix(&dest, PGSL);
    }

    calculate_eigenvalues(adjacencylist, N, p+1);

    calculatelevels(adjacencylist, N, p+1);
}
```

Each construction needs various tools, the main ones being:

```
//partition the graph into levels
//determine the number of nodes on each level
void levelscalculations(int *grlist, const int nverts,
    const int regularity);
//calculate the eigenvalues of the graph using GSL
void print_gsl_eigs(int *grlist, int nverts, int regularity);
//output the graph for Maple to compute the eigenvalues
```

```
void mapleoutput(int *grlist, const int nverts, const int regularity);
//output the graph to a Postscript file
void psoutput(coord *verts, int nverts, edge *edges, int nedges);
//generate PSL/PGL over Z_q
Mtrix* generatePSL(int q);
Mtrix* generatePGL(int q);
//generate PSL/PGL over F_q^n
//where poly is the irreducible extension polynomial of F_q to F_q^n
ffeMtrix* generatePSL_Fqn(int q, int n, ffe poly);
ffeMtrix* generatePGL_Fqn(int q, int n, ffe poly);
```

And a variety of functions to deal with finite fields in order to add, divide, multiply, and iterate finite field elements and find roots of finite field polynomials.

Code was also programmed to layout graphs nicely in a “tree form”. The code picks an arbitrary vertex as the “root” on level 1 and partitions all of the vertices of the graph into levels where the vertices on level $i+1$ have distance i from the root. The code places the levels so that there is an equal amount of vertical space between each level. The horizontal space between vertices on the same level is more complicated. If one were to space the vertices in each level evenly, each level using the full width, then the diagram would not give a true sense of the size of each level. Whereas if one were to space the largest level evenly using the full width, and then space the remaining levels using the same amount of horizontal space between vertices then the smaller levels would be compressed to a space smaller than is necessary, making it harder to distinguish individual features of the diagram. The code compromises between these two approaches. When placing a level the code first determines the total amount of unused space to the left and to the right if the vertices in the level were spaced with the minimum horizontal space between vertices of the largest level. Call this amount of unused space u. The code determines the new amount of unused space u' by multiplying u by a linear function of u that is 0.8 when u is the full width, and is 0.3 when $u = 0$. This ensures that a level with more vertices will always be wider then a level with fewer vertices. The code then places the vertices of the level evenly so that there is u' total unused space on both sides. Edges that are between vertices on different levels are drawn as straight lines. Edges between vertices in the same level are drawn as arcs. The arcs curve either up or down; the code attempts the make the following equation true:

$$\frac{\text{curves upward}}{\text{curves downward}} = \frac{\text{vertices in level below}}{\text{vertices in level above}}.$$

This is done so that if the level above has fewer vertices than the level below, there will be fewer edges curving upwards than downwards, so more of the arcs will be placed where there are fewer edges; reducing the overall clutter.

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