Pattern Math:

A design experiment of mathematical inquiry

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Abstract

This design experiment research introduces a mathematical inquiry titled Pattern Math. The Pattern Math activities create an atmosphere where students can think mathematically, communicate mathematically and make connections between different mathematical concepts. Based on simple patterns with complex explanations, the Pattern Math activities provide students with the opportunity to develop their conceptual understanding of mathematics. Through reflections on the activities, students are able to reexamine their views of learning mathematics. This design experiment research has a narrative approach and incorporates the teaching and research technique of interactive writing. The research highlights the power of inquiry. By providing students with the opportunity to work within their zone of proximal development, the Pattern Math activities provide students with the opportunity to make mathematical discoveries and come to understand algebra and arithmetic with conceptual understanding.

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I would also like to thank all of my students who participated in this study. You were an amazing group of students who I will always remember. I hope that the Pattern Math activities provided you the opportunity to learn mathematics in a valuable way. Although I hoped that these activities would teach you many things, I have to admit that you were able to teach me many things. I was continually amazed at your abilities to think mathematically and to make mathematical discoveries. Thank-you again for all your hard work and participation in this study.

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Chapter 1: Introduction

I have always been intrigued by patterns in numbers. I remember as a high school student noticing the following pattern: 9 times 11 was one less than 10^2; 7 times 9 was one less than 8^2; and 11 times 13 was one less than 12^2. In other words, one less than a number times one more than a number is always one less than the same number squared. Even though this fact was simply explained by algebra that I had already learned in school, I had not made the connection between the arithmetic and the algebra.

This document describes a design experiment entitled "Pattern Math" which provides students with mathematical patterns and influences them to inquire into these patterns. The introduction will develop the background for Pattern Math. Along with the introduction, the second and third chapters provide the framework for the research that took place. The second chapter includes an extensive literature review that examines various foundations for Pattern Math, from learning mathematics as conceptual understanding and the importance of communication in mathematics, to establishing inquiry as a way to help students communicate and learn mathematics. The third chapter describes the methodologies used in the research. Design experiment research will structure the design of the project, while the Pirie-Kieren model will look at the development of student cognition throughout the project.

Following the framework of the first three chapters, chapters 4 through 6 introduce both the Pattern Math activities as well as the students involved in the research project. Chapter 4 introduces all of the Pattern Math activities while introducing the students. Chapter 5 highlights some of the major discoveries, while Chapter 6 describes the Pattern Math activities through a specific example so the reader can get a sense of the flow of the project.

Chapters 7 through 9 focus on an analysis of the data. The chapters move from an analysis of a particular student in Chapter 7 to more general observations in Chapters 8 and 9. The purpose of these chapters is to answer the research questions that will be presented later in this introduction.

Algebra is the generalization of arithmetic. It is the recognition of patterns in arithmetic and the explanation of these patterns through the use of variables. I believe that much of mathematics instruction is just the organization and summary of patterns.

The problem occurs when mathematics is presented as the summary of these patterns and students are not able to see the patterns themselves. In the example mentioned in the first paragraph, I purposely introduced the pattern that $x^2 - 1 = (x + 1)(x - 1)$ with numbers first. The idea is that algebra can be understood as the process of generalizing arithmetic. Too often, students see algebra as separate from arithmetic. Students view algebra as a construct to be used only in school mathematics whereas arithmetic has uses outside of school.

Ever since I began teaching eight years ago, I have been interested in developing creative ways that I, as a teacher, can help students gain a deep and meaningful conceptual understanding of mathematics. This desire has stemmed from my own experience and enjoyment of learning math with a deep conceptual understanding. My work as a teacher has given me many opportunities to teach mathematics in ways that I feel are meaningful to students. I have been fortunate to conduct formal research into ways of engaging students to think more deeply about mathematics. The first research project I conducted with Dr. Ralph Mason was a design experiment where we used inquiry to invite students to think more deeply about the meaning of circular formulas and the meaning of the coefficient of π. This research project included several phases of design, implementation and modification. I have also been part of a group of teachers and professors who were involved in a research project named “Teaching Through Inquiry”. In this project, I developed an inquiry-based activity where students were given an experience with radical quantities. The inquiry allowed students to see meaning behind radical arithmetic and gave visuals for what the length of a radical is in comparison to other known lengths.

Both of my formal research experiences have led me to believe that mathematical inquiry is a way to teach mathematics for conceptual understanding, at the same time allowing for students to develop procedural fluency in mathematics. Mathematical inquiry can take many forms and can be interpreted in many different ways. Mathematical inquiry may use mathematics to inquire into different real world contexts. Mathematical inquiry may also be a means to inquire into mathematical ideas and concepts. I will refer to mathematical inquiry as an inquiry into mathematical ideas.

In the context of the classroom, inquiry can occur in both formal and informal ways. Informally, a classroom or community of learners may be intrigued by a pattern they see in a question or a unique feature of a question. They may ask, “Why does that happen?” and take the time to explore the idea more deeply. Although this was not part of the planned lesson, a teacher who believes in mathematical inquiry will take time to explore this idea. More formal examples of inquiry can be seen through lesson plans and unit plans. Lesson plans can take the form of guided inquiry where the teacher has created an activity that allows students to explore and learn about mathematical ideas. The inquiry is designed to help students see patterns and learn mathematics in a meaningful way. Full unit plans may extend an inquiry activity over multiple days and include many different opportunities to explore and learn mathematical ideas.

My Professional Goals for the Research

Although inquiry is already a part of my practice as a teacher, I wanted to incorporate inquiry more formally in my mathematics classroom. I wanted to explore ways in which students could understand rich mathematical concepts in deep and meaningful ways. I wanted students to see mathematical patterns and to think mathematically. I also wanted to create materials that I could share with other teachers so that they could incorporate mathematical inquiry into their classrooms as well. In summary, my four research goals are:

1. To formally incorporate inquiry as a way of learning and understanding mathematics.
2. To become aware of student cognition or mathematical thinking in my classroom.
3. To give students the opportunity to learn math with a deeper conceptual understanding and to develop an appreciation for mathematics.
4. To create materials to share with other teachers.

The upcoming research questions will add an academic component to this list.

Context and Description of Project

Before describing how I intend to meet these goals, it is important to provide the context for this study. This section provides an example of mathematical inquiry used in the project. It will describe how the activity fit in the context of a course and how it was situated within lesson plans and unit plans. It will also describe some of the particulars such as the participants in the study and how data was gathered from the participants.

The project involved students participating in activities called "Pattern Math" as seen below. These activities began with three mathematical examples that asked students to solve mathematical problems with which they were familiar. The set of three mathematical questions had a pattern that emerged as students did the arithmetic. In the fourth box, students were asked to create and check their own example that followed the same pattern.

Pattern Math

A: 10^2 B: (9)(11)	A: 16^2 B: (15)(17)
A: 30^2 B: (29)(31)	Your own example:

Communicating
What do you notice? • Write about the pattern that you see. • Communicate as clearly as you can everything that you notice. • Does the pattern you see always work? How would you know?

After completing the first part on patterning, students were asked to communicate their ideas. This began with an explanation of the pattern that they saw in their own words. The examples in the patterning section of the inquiry were designed to invoke responses such as “That’s interesting” or “That’s strange” or “Does that always work?”. After describing the pattern that they saw, the communicating section asked students to verify whether or not the pattern always worked. Their explanation needed to generalize the pattern that they saw in the specifics. This example asked students to move from the arithmetic of numbers to the algebra of ideas. The purpose of communicating was to give students the opportunity to make connections between mathematical ideas on their own and to be able to explain these connections in their own words as well as with proper mathematical terminology. By trying to communicate their own ideas, students were aware of their own mathematical vocabulary and were challenged to improve their communication skills.

The Pattern Math activity was a small, formal example of what I call “guided inquiry”. It was inquiry because students were exploring a pattern and explaining why the pattern exists. It was guided because it was designed with a specific pattern in mind, but this does not mean that students were not open to explore other patterns or ask a wide variety of questions. Although the inquiry was guided in that it led students towards a certain pattern and a certain relationship, it was still open in that it allowed students to take the inquiry where they wanted.

Unlike lesson plans and unit plans that are designed around inquiry, the Pattern Math activity was a small part of a daily lesson. The activity was incorporated either during the first or last 10 minutes of class. The activity took about 10-15 minutes while the additional time needed to further communicate was done outside of class. This was enough time for students to complete the Patterning section of the activity and begin describing what they noticed. All students were responsible to submit their activity the following class.

I used the activity with a group of 14 Grade 10 students who were taking Grade 11 mathematics. Students were given Pattern Math activities two or three times per week over a period of two months. I read and responded to each of the students’ ideas. The process of students communicating and the teacher responding to each individual is called interactive writing (Mason and McFeetors, 2002). The interactive writing process allowed students to feel that they were communicating to a listening audience. My responses allowed me to communicate to each student directly and these responses often created new opportunities for learning more about the patterns.

Data from this project was gathered from the students' written responses on the Pattern Math sheets. Data was also gathered from teacher responses and teacher field notes during the study. Observations were taken during the activity and any subsequent follow-up activities. Students also reflected on the activity after a series of Pattern Math questions were done.

Revisiting the Goals

The context described in the previous section provides a framework from which the goals can be further explicated. The first goal was to introduce inquiry as a way of learning and understanding mathematics. In Pattern Math, the students were given a model in which they could explore mathematics for understanding. It began by seeing mathematics as looking for patterns. Once a pattern was found, the students tried to explain the pattern and see if the pattern could be generalized or whether the pattern was only a local phenomenon. Being able to communicate a pattern and explaining the significance of the pattern helped build networks and this strengthened students' conceptual understanding. One of the goals of inquiry is to get students interested in asking the question, "Why does that work?" and attempting to explain why it works. The format of the activity asked students not only to attempt to explain why the pattern works, but also to explore connections to previous mathematical concepts they have learned as well as connections to experiences outside of school. Making these connections helped students understand the concepts in a deeper and more meaningful way. This was opposed to isolated experiences that the students saw as only occurring in the classroom. Another aspect of inquiry is the openness to explore ideas that emerge. In the final section of the activity, students were asked to extend the pattern that they found and explore new patterns. The activity introduced inquiry in a formal way by asking students to look for patterns, explain the pattern both specifically and generally, and extend the ideas that they had learned to new ideas.

The second goal of the project was to become more aware of the mathematical thinking in my classroom. As I read through the students' descriptions, explanations and connections in each activity I was given a glimpse into the students' thinking. It gave me an opportunity to assess students' level of understanding and any misconceptions they had. Through part of the process of interactive writing, I was able to communicate back to the students. In addition to providing me with a lens to view students' thinking and misconceptions, it also provided students with a lens to view their own thinking. For example, when students were able to communicate an idea or a concept, they became aware of their understanding and became more confident of their understanding. On the other hand, when students struggled to explain a concept or struggled to make connections, they also became aware of their misconceptions and realized that they had gaps to fill in order to understand the concept more fully. Revealing these gaps for students was the first step in moving towards a deeper understanding. It caused them to ask questions and inquire further into the concept. It was through this inquiry that they could begin to build links and make connections to understand concepts in a deeper way.

The third goal was to provide an opportunity for students to learn mathematics with a deeper conceptual understanding and to develop an appreciation for mathematics. Although this goal has been described already in both of the first two goals, it is important to highlight as a separate goal. I believe that Pattern Math provides students with an opportunity to learn mathematics with a deeper conceptual understanding. By communicating their ideas and the pattern that they saw, students could identify their own misconceptions and move towards filling in the gaps in their understanding. Making connections among other mathematical concepts created a deeper understanding of concepts. I believe that once students have learned mathematics with conceptual understanding and can make connections to other concepts, they will develop an appreciation for mathematics as well. Perhaps it is a lofty or ideal goal, but I hope that once students experience learning mathematics in a meaningful way, they will extend methods of inquiry to other mathematical experiences. For example, imagine if students began to look for patterns and ask questions while they were doing a homework assignment. Or perhaps students begin to look at experiences outside of the classroom and ask how it connects to their mathematical knowledge.

The last goal was to create materials that could be shared with other educators. The Pattern Math activities were small inquiries which could be incorporated into any classroom. I wanted to create activities that other teachers would be willing to try in their classroom. I also wanted to create activities that would encourage communication in the classroom. It is my hope that other teachers who use these activities would gain an appreciation for inquiry in the classroom and see the benefits of communicating and connecting ideas for mathematical understanding.

Research Questions

My research questions can be summarized as follows:

- How did Pattern Math activities contribute to the development of students' abilities to think mathematically, communicate mathematically and connect mathematical concepts?
- How did Pattern Math activities contribute to students' conceptual understanding of the mathematics?
- How did students interpret their experience with the Pattern Math activities with respect to their view of learning mathematics?

It is with these questions and goals in mind that the next chapter looks at the literature to support inquiry as a way to achieve conceptual understanding of mathematical concepts.

Chapter 2: Conceptual Foundations

The following section provides a review of the literature that forms the foundation for my thesis. It begins by looking at the reasons why we teach mathematics. To what extent should teaching focus on students gaining conceptual knowledge or becoming procedurally proficient? After examining conceptual understanding and procedural knowledge, the literature review will turn to Lev Vygotsky to provide a psychological basis for teaching mathematics for conceptual understanding. Vygotsky's sociohistorical psychology emphasizes the importance of communication in learning. After exploring Vygotsky's ideas, the literature review will examine communication and dialogue within the framework of mathematics education. The role of inquiry will be discussed as a way to provide students the opportunity to communicate and to learn mathematics for conceptual understanding. Finally, a discussion of the importance of patterns to the nature of mathematics and to understanding algebra will be discussed.

Conceptual Understanding versus Procedural Knowledge

Why do we teach mathematics? According to Brent Davis (2001), mathematics permeates all levels of our culture and our society. The answer to why we teach mathematics is that it is part of who we are as a culture and as a society. Davis (2001) writes,

> To recap, perhaps the most honest answer to the question 'Why *would we* teach mathematics to all students?' is that we cannot help but do so. It is bound in our collective character; it is part of our systemic being. It is not simply something we do, but something we are. (p. 20)

Davis suggests that we need to get over the question of why we teach mathematics and move towards other philosophical questions in mathematics education. "What I am asserting then, is that we need to get over the question, 'Why teach math?' and replace it with a more temporally appropriate obsession, namely becoming more mindful of what is happening in the name of mathematics education" (p. 22).

As we explore what is happening in mathematics education, questions naturally arise about what is being taught and how it is being taught. What is being taught is often mandated by a curriculum over-burdened with outcomes (Skemp, 1978). What is being taught is also influenced by our view of what mathematics is. Davis (1995) suggests that we have placed too much emphasis on seeing mathematics education as product-driven:

> In the school, for example, with the traditional emphasis on the development of technical proficiency, for the most part mathematical knowledge has been regarded as something “out there” – objectively true, pre-existent, independent of human agency. School mathematics has thus been cast in the same terms as material commodities, and we have tended to be pre-occupied with such matters as the efficiency of production (i.e. acquisition of knowledge), the utility of the product, and the quality of the outcomes. (p. 3)

Because of this product driven mentality, we have focused more on mathematical products rather than mathematical processes. For example, how the curriculum outcomes are taught is often influenced by pressures of time to cover the curriculum and a final exam that tests the outcomes of the curriculum.

One question that arises when looking at how we should teach mathematics is to what extent should mathematics education focus on conceptual understanding and to what extent should mathematics education focus on procedural knowledge. The National Council of Teachers of Mathematics (NCTM, 2000) suggests that students need to develop both conceptual understandings as well as procedural fluency. So the question becomes how do we teach mathematics to accomplish the dual goals of conceptual understanding and procedural fluency? Gerald Kulm (1994) writes,

> One of the real challenges of teaching mathematics is to find a proper balance between conceptual understanding and procedural skills. Without a sound understanding of concepts, skills may be used mechanically and easily forgotten. At the same time, strong mathematical skills and computation can help students build understanding of new concepts (p. 18).

To begin to understand how to balance conceptual understanding and procedural knowledge in the classroom we must first define the difference between conceptual understanding and procedural knowledge. The distinction between the two has generated much discussion over the years. The discussion has also created many different labels for the two ways of knowing mathematics and knowing in general. Piaget (1978) differentiated between conceptual understanding and successful action. Hiebert and Lefevre (1986) describe the historical debate as one between understanding (advocated over the years by McLellan and Dewey, (1895); Brownell, (1935); and Bruner, (1960)) versus skill learning (advocated over the years by Thorndike, (1922) and Gagné, (1977)). Anderson (1983) examines the differences between declarative knowledge and procedural knowledge. Baroody and Ginsburg (1986) distinguish between meaningful knowledge and mechanical knowledge. As previously mentioned, NCTM (2000) creates a distinction between conceptual understanding and procedural fluency. Many authors simply refer to the two as different types of knowledge, that is, conceptual knowledge and procedural knowledge (Kulm, 1994; Hiebert and Lefevre, 1986; Carpenter, 1986; Silver, 1986; Hiebert and Wearne, 1986; Davis, 1986).

Although my definitions of conceptual understanding and procedural knowledge are consistent with the ideas of conceptual knowledge and procedural knowledge, I have chosen to use the word *understanding* instead of just *knowledge*. The word understanding implies not only a *"know how to"* but also a *"know why"*. The word knowledge itself does not convey the deeper meanings involved when one talks about having an understanding. It can be argued that having the word *conceptual* as an adjective in front of the word *knowledge* is enough to emphasize the *know why* along with the *know how to,* but I wanted to further emphasis the *know why* by using the more powerful word *understanding*. Conceptual understanding in mathematics therefore involves a richness of *knowing why* coupled with a *knowing how to*. The *knowing why* implies that the new concept or idea being learned is not learned in isolation, but is connected to an array of other concepts and ideas that are already known. These connections to other conceptual understandings make a new concept or idea make sense and thus create an understanding of the concept. Hiebert and Lefevre (1986) describe conceptual knowledge as "knowledge that is rich in relationships" (p. 3). For Kulm (1994), conceptual knowledge consists of "a rich network of relationships between pieces of information and that permits flexibility in accessing and using the information" (p. 19). A new idea or concept only becomes part of one's conceptual understanding once it is connected to a network of relationships in one's existing conceptual understanding. Carpenter (1986) argues that once an idea becomes part of one's conceptual knowledge, it "permits flexibility in accessing and using the information" (p. 113). I would argue that conceptual understanding consists of ideas that are accessible, not in the same way as an idea that is memorized, but as ideas that can be recreated from a rich network of concepts.

In contrast, procedural knowledge can be defined as the knowledge of steps or the knowledge of an algorithm to complete an assigned task. Procedural knowledge involves many rules for completing mathematical tasks. A common example of procedural knowledge used in a math class is when students need to divide a fraction by another fraction such as $\frac{2}{3} \div \frac{4}{5}$. Students learn the procedure to complete this task. They flip or take the reciprocal of the second fraction and multiply. Of course, multiplying fractions also has its procedure: multiply the numerators (the top of the fraction) and multiply the denominators (the bottom of the fraction). Procedures are steps that students can follow in a linear manner to complete a mathematical task. In fact, many mathematical tasks presented in schools test whether or not the student can recognize the correct procedure to use and can perform all the steps in the algorithm flawlessly to come to the correct answer. Unlike conceptual understandings, procedural knowledge can stand alone as a discrete memorized fact that can be reiterated when called upon. For example, a student can divide two fractions and arrive at the correct answer without connecting the problem to concepts of division or concepts of fractions or what the final answer means in relation to the problem. This is not to say that procedural knowledge is not useful or not important. Many times a student may have to divide fractions as part of a larger mathematical problem they are trying to solve. Using the efficient procedure allows the student to quickly move along within the problem without losing focus of the main mathematical task which is being considered. Silver (1986) writes that "there is nothing wrong with efficient, automatized procedures, but students need to recognize the limits of procedures and when to draw upon conceptual knowledge" (p. 130).

Both conceptual understanding and procedural knowledge are important for students to succeed in mathematics. Although I have defined each as a separate concept, the two are not so easily separated. If a student knows a procedure and each of the steps in the procedure can be connected to other mathematical concepts that the student understands so that each of the steps make sense to the student, is the procedure part of the students' procedural knowledge or part of the student's conceptual understanding?

The answer is both. The fact that the student connects the procedure to a rich set of conceptual understandings means that it is part of the student's conceptual understanding. The student may also use the procedure at times to solve problems and not necessarily think about the underlying relationships within the procedure, so it is also part of the student's procedural knowledge.

The key aspect is that conceptual understanding and procedural knowledge are not mutually exclusive. Many relationships exist between conceptual understanding and procedural knowledge (Hiebert and Lefevre, 1986; Silver, 1986; Carpenter, 1986). Procedural knowledge may be taught based on conceptual understanding providing a strong foundation for the procedures which are taught. On the other hand, procedures may be learned first without conceptual understanding, but form a basis for the concepts at a later time (Silver, 1986). A student may become fluent with a procedure before they are developmentally able to really understand the procedure. However, at a time when they are developmentally able to understand the procedure, they are able to understand it more easily because they have fluency with the procedure.

Hiebert and Lefevre (1986) describe the benefits of linking procedural knowledge to conceptual understanding. Procedures connected to understanding are remembered better because they are part of a rich and connected network of ideas. As a result, these procedures are more easily retrieved. Furthermore, students will be able to select correct procedures more often or modify procedures when conceptual understanding is linked with procedural knowledge. Silver (1986) writes, "systematic bugs in procedures can often be traced to flaws in conceptual knowledge or to the lack of conceptual/procedural knowledge linkages" (p. 187).

Students need both procedural knowledge and conceptual understanding to succeed in mathematics. However, it is clear that the focus in traditional mathematics classrooms has been on providing procedures and testing these procedures. Little emphasis is placed on conceptual understanding especially on assessments such as tests or final exams where questions largely test whether or not the student knows a particular procedure. Kulm (1994) explains that we "have often assumed that if a student has learned a procedure, the related conceptual knowledge has also been acquired" (p. 19). This however is not the case as students often show competency with completing a procedure void of any conceptual knowledge. The result is that students come to know mathematics as a set of procedures to be memorized and reiterated. Even when teachers provide students with explanations about the conceptual knowledge behind a procedure, the questions the students are required to answer on the final exams do not value the conceptual knowledge. Magdalene Lampert (1990) explains that when teachers provide explanations they do not “invite students to examine the mathematical assumptions behind the explanation” (p. 32). In fact, a student may come to know through experience that when the teacher is explaining the concepts behind a procedure that they can ‘tune out’ since they will soon be given an algorithm that will be sufficient to ‘get the right answer.’ Kulm (1994) avers that “procedures should be built on related conceptual knowledge rather than preformed from rote memory” (p. 20). Students need to be invited to think about the relationships and concepts behind why procedures work.

Part of inviting students to think about relationships and concepts behind procedures may involve inviting students to rethink what it means to understand mathematics. Richard Skemp (1978) describes understanding in mathematics as a *faux amis*, a term that has dual and often contradictory meanings. For Skemp, there are two types of understanding mathematics. Relational understanding is “knowing what to do and why” (Skemp, 1978, p. 9). Relational understanding is built on conceptual understanding. When a student has a relational understanding of a concept they are able to not only know what method to use to solve a problem and why it works, but will possibly be able to adapt a method or procedure to a novel situation or problem. In this sense, a student with a relational understanding can enact procedures to solve problems, but the procedures are not preformed from rote memory. Instead they are built on a foundation of conceptual understanding.

The other type of understanding in mathematics is what Skemp terms instrumental understanding. Skemp (1978) describes instrumental understanding as “rules without reasons” and that “the possession of such a rule, and ability to use it” is what understanding means in mathematics (p. 9). Although most teachers would argue that they would rather teach relational understanding over instrumental understanding, many students (and teachers) see positive results when learning mathematics with an instrumental understanding. Students see instrumental understanding as only having to learn to memorize a few steps to be able to get a correct answer to a question. Examining only one type of question and comparing what is needed to have either an instrumental understanding or a conceptual understanding, it would likely appear that an instrumental understanding is easier to obtain. For example, to answer a question about adding fractions, all students need to remember are two steps: get a common denominator and add the numerators. However, on a macro level the opposite is true: instrumental understanding becomes more difficult than relational understanding. Skemp (1978) argues that instrumental understanding “usually involves a multiplicity of rules rather than fewer principles of more general application” (p. 10). With only an instrumental understanding, students may have difficulties when approaching similar problems that are slightly different. Often we ask students to use previous mathematical procedures when examining a new concept. If students have learned procedures with a relational understanding, it is likely that they can see a new concept as an extension of an old concept. The result will be that students will understand (know what to do and why) a new procedure as perhaps one new step to a previous procedure that made sense conceptually. If students have learned procedures with an instrumental understanding, it is likely that they will not see the new concept as an extension of an old procedure. The result is that students will memorize a new procedure with many “new” steps since a connection is not made to a previous procedure.

Lampert (2001), in her extensive study of a group of grade five students explains that mathematics needs to be more than just instrumental understanding. She writes,

> I could teach them a rule and hope they remember it until they finish fifth grade (when they become the responsibility of some other teacher). But mathematics is so much more than rules, and that was something I also wanted all students to learn. It seems important to teach all students that mathematics is complex, and also to teach them that they are people who can learn it. (p. 388)

She provided students with problems that invoked thinking and caused the students to inquire into the nature behind mathematics they were studying. The students had to come up with their own reasoning and explain their hypotheses on how they would solve a problem. Through classroom discourse, students validated mathematical procedures and developed a relational understanding of the mathematics they were studying. Lampert

(2001) writes, “For students to study the discourse of mathematics, I needed to teach them to reason about why the ideas and processes that are known to the field are legitimate” (p. 438).

A challenge of mathematics education is how to provide students with an opportunity to become mathematically proficient. How do we ensure that students can learn and apply mathematical skills and have the conceptual understanding behind these skills? Jeremy Kilpatrick et al. (2001) describe mathematical proficiency as more than just procedural fluency and conceptual understanding. They suggest five interwoven and interdependent strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. For Kilpatrick et al. (2001), conceptual understanding refers to a comprehension of mathematical ideas where new ideas can be connected to ideas that are already known. Procedural fluency refers to a knowledge of procedures, when to use particular procedures and how to perform them not only accurately, but efficiently and flexibly as well. Strategic competence “refers to the ability to formulate mathematical problems, represent them, and solve them” (Kilpatrick et. al., 2001, p. 124). The capacity to think about relationships among situations and concepts in a logical way is what is referred to as adaptive reasoning. Finally, productive disposition brings an attitudinal strand to mathematical proficiency. It “refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics” (Kilpatrick et. al., 2001, p. 131).

I suggest that the strands of mathematical proficiency as presented by Kilpatrick et al. (2001) can be accomplished through inquiry. Using inquiry we can create activities, lessons and units where students develop conceptual understandings of mathematical relationships and procedures used to solve mathematical questions. Inquiry provides students with the opportunity to ask questions and explore mathematical ideas and relationships. Formulating questions develops the strand of strategic competence, while exploring mathematical relationships develops the strand of adaptive reasoning. Inquiry also develops a productive disposition towards mathematics as students value the conceptual understanding and the mathematical connections they have made while exploring mathematics. Through inquiry, teachers can develop curricula, units and lessons that meet the needs of students.

Before describing inquiry more fully, I would like to establish a psychological basis for my research using Vygotsky's sociohistorical psychology and his concept of the zone of proximal development. This will establish the need for communicating in mathematics and the place of inquiry where communication and learning of mathematics can be done for conceptual understanding.

Vygotsky: Sociohistorical Psychology and the Zone of Proximal Development

Lev Semenovich Vygotsky was a Russian psychologist whose work focused on the social and historical contexts in the psychology of learning. Although Vygotsky's works are now over 70 years old, current educational contexts have created a renewed interest in Vygotsky (Rosa & Montero, 1990). Two areas of interest connected to my thesis are the ideas of sociohistorical psychology and the zone of proximal development. Sociohistorical psychology places emphasis on the social nature of development and the use of tools, such as language, to mediate this development. The zone of proximal development establishes an area where instruction should occur that will create the best learning for a student. The idea of the social nature of learning helps form a basis for examining communication in mathematics education which will be reviewed after explicating Vygotsky's views more extensively.

When Vygotsky entered the academic scene in 1924, it did not take him long to embark on a task that would greatly affect psychology—creating a new psychology based on Marxist views of philosophy and the social sciences. This new psychology focused on the social nature of development and placed great importance on social, cultural and historical influence on development. The basic tenet of his approach was that human behavior was too complex to isolate and study in its separate parts (Dixon-Krauss, 1996a). It must be studied in its historical and social context. Vygotsky focused on how language—a fundamentally social, cultural and historical construct—affects development. Dixon-Krauss (1996b) asserts, "Vygotsky's primary objective was to create a unified psychological science by restoring the concept of consciousness to a field dominated by strict behaviorism" (p. 8). Vygotsky (1962) describes consciousness as "the awareness of the activity of the mind—the consciousness of being conscious" (p. 91). This consciousness occurs as enculturation of children occurs through social and historical activities such as the acquisition of language and other psychological tools. Describing Vygotsky's ideas, Karpov and Bransford (1995) explain, "the child's mind develops in the course of acquisition of social experiences, which are presented to the child in the form of special psychological tools: language, mnemonic techniques, formulae, concepts, symbols, signs, on so on" (p. 61). For Vygotsky, two genetically identical children, reared in different cultural and historical contexts, would each develop differently.

Three themes characterize Vygotsky's sociohistorical psychology (Wertsch, 1985; Penuel & Wertsch, 1995; John-Steiner & Mahn, 1996; Tappan, 1998). First, understanding higher mental functions is only possible through a genetic and developmental analysis and interpretation. Tappan (1998) explains, "a reliance on a genetic or developmental method means that it is impossible to come to a full understanding of any aspect of individual mental functioning without first analyzing and exploring its developmental history—focusing, specifically, both on its origins and on the transformations it undergoes from earlier to later forms" (p. 24). For example, Vygotsky believed that thought and speech have different genetic roots. Studying the processes and transformations of genetic lines of both thought and speech, Vygotsky (1962) concludes, "at a certain point these lines meet, whereupon thought becomes verbal and speech rational" (p. 44).

Second, higher psychological functions are mediated by signs and psychological tools, in particular, language. "Mediation is the means by which newer forms of higher order development occur" (Robbins, 2001, p. 26). A word, as a form of sign, is not only a label for a concept that is understood, but is also a way to transform that concept. Penuel and Wertsch (1995) write, "signs and tools are not simply servants of those individuals' purposes, but in important ways, transform those purposes and mediate mental functioning" (p. 86). John-Steiner and Mahn (1996) write, "knowledge is not internalized directly, but through the use of psychological tools" (p. 193). The process of transforming natural behaviors to higher psychological behaviors through signs is called semiotic mediation.

Third, higher psychological functions originate in social relations. The primary purpose of language is social. Through social interactions, higher psychological functions become individualized. Vygotsky (1962) posits, "In our conception, egocentric speech is a phenomenon of the transition from interpsychic to intrapsychic functioning, i.e., from the social, collective activity of the child to his more individualised activity—a pattern of development common to all the higher psychological functions" (p. 133). This process is characterized by internalization where an "external activity is reconstructed and begins to occur internally" (Vygotsky, 1978, pp. 56-57). The ideas of socialization and egocentric speech becoming internalized into inner speech will be developed later in this section.

In the formation of his new psychology, Vygotsky criticized other psychologists for not considering the social implications in the development of thinking. In particular, Vygotsky critiqued the psychology of Piaget and his developmental stages. For Piaget, development occurred impervious to instruction, experience and other social interactions. The individual (child) was the most important source for development and cognition. The child extended his or her ideas to the world. Development of language and thought began with nonverbal autistic thought, then moved to egocentric speech, and finally to socialized speech and logical thinking. Piaget claimed that egocentric speech—verbalization by the child of the child's actions—was evidence of the individualistic nature of the child's development. As the child develops to be more socialized, egocentric speech simply disappears. Thus, the development of the child, according to Piaget, began with the individual and moved to the social.

Vygotsky, in contrast to Piaget, believed that the social and cultural context of the child was the greatest determining factor in development. "In our conception, the true direction of the development of thinking is not from the individual to the socialized, but from the social to the individual" (Vygotsky, 1962, p. 20). The world, through cultural institutions such as language, develops the child. Salkind (2004) writes, Vygotsky "believed that the child is part of the world and that ideas originate and develop as a dialectical process" (p. 288). With language, Vygotsky demonstrated that the nature of egocentric speech was not individual but social. Through cleverly designed experiments—children with other children who did not speak the same language, children with deaf-mute children, and children in situations with loud music interrupting their egocentric speech (Vygotsky, 1962)—Vygotsky showed that egocentric speech diminishes when the social context is removed. Vygotsky hypothesized that egocentric speech still emphasized the social nature of development and in fact turns into inner speech (thought) as the child develops higher levels of thinking. Thus, Vygotsky's sociohistorical psychology sees development from the social, cultural and historical context of the child.

Vygotsky (1962) describes two different types of concepts: spontaneous concepts and scientific concepts. Spontaneous concepts are concepts, such as "brother", that develop naturally through the social experiences of the child. The child is able to use spontaneous concepts correctly, but may not actually be able to fully explain these concepts because they are not fully conscious of their meaning. Scientific concepts are concepts, such as "exploitation", that are taught through social interactions and social institutions such as schools. Vygotsky (1962) claims, "the very rudiments of systemization first enter the child's mind by way of his contact with scientific concepts and are then transferred to everyday concepts, changing their psychological structure from the top down" (p. 93). Spontaneous concepts develop from the ground up, whereas scientific concepts develop from the top down. Spontaneous and scientific concepts "*develop in reverse directions*" (Vygotsky, 1962, p. 108). As they develop towards each other, they complement each other by providing experiences and structures. The scientific concepts provide the systems for understanding spontaneous concepts. The child becomes aware and conscious of the meaning of spontaneous concepts through the development of scientific concepts. For example, Vygotsky explains that one cannot truly understand one's own language until one learns another language.

The importance of the role of scientific concepts on the development of the child has great pedagogical implications. That scientific concepts provide systems for which students can become conscious of everyday concepts asserts the need for properly designed curriculum to enhance development. "As long as the curriculum supplies the necessary material, the development of scientific concepts runs ahead of the development of spontaneous concepts" (Vygotsky, 1962, p. 106). According to Vygotsky, this curriculum should be designed at the child's zone of proximal development.

Zone of Proximal Development

The zone of proximal development describes the functions that a child cannot yet do individually, but can do with the help of an adult or more capable peer. In Vygotsky's (1978) words, "It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). The following example helps clarify the idea of the zone of proximal development.

> Having found that the mental age of two children was, let us say, eight, we gave each of them harder problems than he could manage on his own and provided some slight assistance: the first step in a solution, a leading question, or some other form of help. We discovered that one child could, in co-operation, solve problems designed for twelve-year-olds, while the other could not go beyond problems intended for nine-year-olds." (Vygotsky, 1962, p. 103)

The two children mentioned above have the same mental age. Their actual development is equal. Vygotsky examined what the children could do when they were given some assistance by an adult or more capable peer. The one child could complete problems of a nine-year-old with the assistance and instruction of the adult while the other child could complete problems of a twelve-year-old with the assistance of the adult. Vygotsky claimed that the zone of proximal development for the one child was larger than the other child. This level of potential development was considered even more indicative of development than the actual development according to Vygotsky. For Vygotsky (1978), "the zone of proximal development defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embryonic state" (p. 86).

Working within the zone of proximal development entails that instruction precedes development. One of Vygotsky's main critiques of Piaget was his failure to see the effect that instruction has on development. Vygotsky (1962) explains the importance of instruction preceding development, “What the child can do in cooperation today he can do alone tomorrow. Therefore the only good kind of instruction is that which marches ahead of development and leads it; it must be aimed not so much at the ripe as at the ripening functions” (104).

The child who is working in the zone of proximal development is in fact solving a problem that is beyond his or her capabilities. The instruction given by the teacher enables this to happen. Levykh (2008) states two results of the child working within the zone of proximal development. The first result is that the educator is able to catch a glimpse into the near future of the child’s development. The second result is that the instruction given to the child in the zone of proximal development “speeds up the process of the child’s development of higher psychological functions” (p. 90). Vygotsky believes that all instruction should be focused at stretching the child to learn something new. The zone of proximal development is in the space where the child is still connected with the concepts that he or she knows, but is learning something beyond their individual capabilities. A child whose instruction is only aimed at what they know—their current development—will soon become bored and disinterested in the instruction. Instruction that is too far removed from the developmental capabilities of the child will result in frustration in the child. The zone of proximal development is the space in between. It acknowledges what the child already knows, but stretches their understanding and gives them tools to solve problems that are beyond their development. The instruction is always ahead of development in the area of the child’s potential development.

Figure 1 below shows how the zone of proximal development works and transitions a student to higher psychological developments. In both sets of circles, the

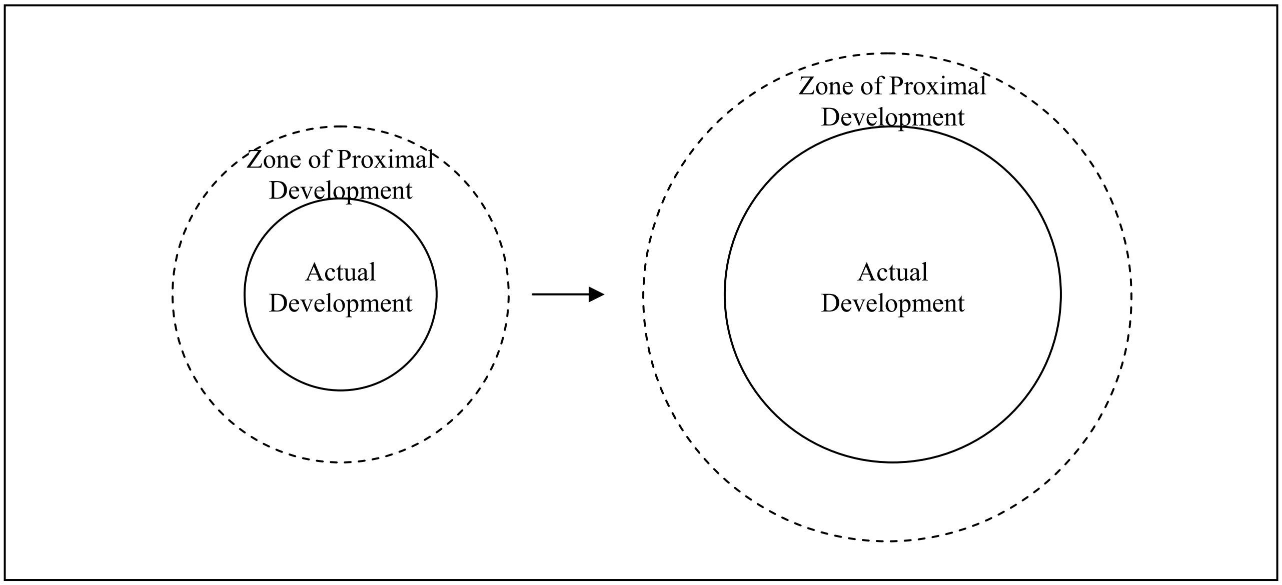


Figure 1: The student's initial zone of proximal development leads the student to increase their actual development thus creating a new zone of proximal development.

solid line represents the student's actual development. The student's actual development is all of the concepts and tasks that the student is able to solve individually. The outer, dotted circle in both sets represents the student's zone of proximal development—those tasks that they are only able to solve with the aid of the teacher. The student and teacher both work within the zone of proximal development. The student imitates while the teacher instructs. Over time and through imitation, what were once areas that were part of a student's zone of proximal development become part of the student's actual development and the zone of proximal development is expanded to new areas of potential development. The role of the teacher is to provide students with instruction that is in advance of their development.

Vygotsky (1978) summarizes his discussion on the zone of proximal development with two essential features. First, he stresses "that the developmental processes do not coincide with learning processes. Rather, the developmental process lags behind the learning process; this sequence then results in zones of proximal development" (p. 90). Figure 1 above attempted to show the relationship between instruction in the zone of proximal development and actual development. The visual is of course an over-simplification. Vygotsky warns about this over-simplification. His second essential feature asserts that, despite the correlation between learning and development, we cannot presuppose that development will fall naturally and in a predictable manner following learning. He writes,

> although learning is directly related to the course of child development, the two are never accomplished in equal measure or in parallel. Development in children never follows school learning the way a shadow follows the object that casts it. In actuality, there are highly complex dynamic relations between developmental and learning processes that cannot be encompassed by an unchanging hypothetical formulation. (Vygotsky, 1978, p. 91).

Connecting with Vygotsky's Sociohistorical Psychology

Vygotsky's sociohistorical psychology is very relevant today and resounds with my personal views on teaching and learning. Focusing on the social nature of learning, Vygotsky highlights the importance of communication in learning and the learning process. The zone of proximal development establishes purposes for instruction and a new way of looking at assessment. In particular, Vygotsky's ideas are concurrent with my philosophy of learning when one looks at the role of instruction and the role of the teacher in the classroom.

For Vygotsky, instruction leads development. As a teacher, this idea resonates with me because it gives purpose to my profession. The teacher, through social contact with the child, leads the child to higher levels of thinking. For example, as a teacher of mathematics, I am often asked the question, "Why are we learning this?" The students often do not see the relevance of algebraic systems or problem solving using graphical representations to their personal lives. For Vygotsky, the learning of higher academic or "scientific" concepts helps students to better understand "spontaneous" concepts that they are able to use. A child may know how to multiply and divide numbers, but only through learning the higher levels of algebra may they truly master these techniques. Vygotsky (1962) writes, "The new higher concepts in turn transform the meaning of the lower. The adolescent who has mastered algebraic concepts has gained a vantage point from which he sees arithmetical concepts in a broader perspective" (p. 115). Instruction in new and abstract areas helps children strengthen their skills in everyday concepts. To answer the question of "why are we doing this?," I can now tell students that I firmly believe that learning the material will not only develop their abilities in new and abstract ways, but will also strengthen and enhance their skills that have already developed. Thus all mathematical skills, like all concepts or words, are always in a stage of developing. Each new instruction enhances previous developments and creates new developments.

Vygotsky's zone of proximal development is the area where the teacher can provide students with instruction. Salkind (2004) explains that the zone of proximal development "forms the link between the psychological basis for development and the pedagogical basis for instruction" (p. 280). Instruction in the zone of proximal development focuses on areas where the student is in the process of developing. The teacher can provide students with aides that will enable them to solve questions that they could not solve on their own. By modeling an approach to a question, the student can gain skills in answering questions beyond their development. This in turn leads to students developing new concepts. After working in the zone of proximal development, they will now be able to do tasks on their own where they previously needed assistance.

The importance of instruction in development means that the role of the teacher is central. Gredner and Shields (2008) outline five roles that the teacher takes in creating instruction aimed at the child's potential for development. First, the teacher must determine the cognitive processes needed for the lesson. In what ways will the lessons stretch the child to think in new ways? Second, the teacher must assess the child's potential for understanding. Vygotsky believes that if the child only works on ideas that are already developed, the child will become bored. If the instruction is too advanced for the child and does not connect at all to the child's development, the child will become frustrated and unable to learn. The teacher must take the child's needs into account to ensure that instruction is aimed at the child's zone of proximal development. Once the instruction is developed within the child's zone of proximal development, the third role is that the teacher collaborates with the child. This collaboration is key as learning is based on socialization. The teacher's fourth role is to make the child aware or conscious of his or her thinking. Only through awareness of concepts will concept development occur and subsequently affect all earlier development. Finally, the teacher helps the child's thinking to move from concrete ideas to logical ideas. The child moves from everyday concepts to higher level abstract concepts. I firmly believe that a teacher should follow

Vygotsky's ideas. By creating instruction in the zone of proximal development, the teacher is considering the needs of the students and designing instruction accordingly. Instruction is always purposeful and has positive implications—the development of the child.

Communication in Mathematics

Vygotsky's sociohistorical perspective provides a foundation for communication in mathematics classrooms. His sociohistorical perspective emphasizes the importance of social and cultural contexts in learning. Recent views on mathematics describe mathematics as an inherently social and cultural activity (Gravemeijer et. al., 2000; van Oers, 2000; Kieran, 2002; Zack and Graves, 2002). As a social activity, learning mathematics is seen as being a participant in a collective doing (Sfard, Forman & Kieran, 2002; Lerman, 2002) or as part of a community of practice (van Oers, 2002; Forman, 2002; Lave and Wenger, 1991). Communication plays a vital role in becoming a participant in a community of practice.

Communication has been an integral part of curriculum documents over the last decade. Manitoba Education (2001; 2009) lists communication as one of seven mathematical processes. These processes are deemed "critical aspects of learning, doing, and understanding mathematics" (Manitoba Education, 2009, p. 8). NCTM (2000) lists communication as one of five process standards. Manitoba Education (2001; 2009) stresses that students need opportunities to interact with mathematical ideas in a variety of ways. This includes talking about, listening to and writing about mathematical ideas. By communicating, students are able to create links between their own personal language, the language of others and the formal language of mathematics. By communicating, students can clarify, reinforce and modify ideas, attitudes and beliefs about mathematics. Communication helps students make connections between various representations of mathematical ideas.

NCTM (2000) identifies four components of the communication standard that all students should experience through schooling. The first component is that all students should be able to "organize and consolidate their mathematical thinking through communication" (p. 60). By communicating their own thoughts, students can better understand their own thinking. Communicating their mathematical thinking can help students create questions to further their thinking. It can also provide students with an opportunity to view their own misconceptions. Communication helps students learn new mathematical concepts as they write, explain verbally, draw and use other methods to communicate their ideas.

The second component is that all students should be able to "communicate their mathematical thinking coherently and clearly to peers, teachers, and others" (p. 61). This involves creating classrooms where discourse is an integral part. Students need to feel comfortable and free to express their ideas. NCTM (2000) explains, "students need opportunities to test their ideas on the basis of shared knowledge in the mathematical community of the classroom to see whether they can be understood and if they are sufficiently convincing" (p. 61).

The third component is that all students should be able to "analyze and evaluate the mathematical thinking and strategies of others" (p. 62). As part of a community of discourse, students need to question each other's thinking and ask for clarification. Questions are not seen as critiques of an individual's abilities, but as invitations to extend learning and increase dialogue. Through questioning and discourse, students can help each other clarify underdeveloped ideas and advance the learning of all participants in the classroom.

The fourth component is that all students should be able to "use the language of mathematics to express mathematical ideas precisely" (p. 63). This component stresses that students need to become comfortable using their own language as well as the language of mathematics. Through discursive classrooms, where students are encouraged to communicate their thoughts, the need for formal mathematical language will emerge as students seek words to best describe their thoughts.

Both Manitoba Education and NCTM stress communication as a fundamental mathematical process. Researchers in mathematics education have also stressed the importance of communication in mathematics education. Kieran, Forman, and Sfard (2002) advocate for a discursive approach to research and learning in mathematics. The discursive approach is based on Vygotsky's views of learning as a sociocultural phenomenon. Vygotsky's book *Thought and Language* (1962) (later translated as

Thinking and Speech (Vygotsky, 1987)) created the link between what we think and what we communicate. Keiran, Forman and Sfard (2002) and Sfard (2002) use this link to describe *thinking-as-communicating*. Kieran, Forman, and Sfard (2002) write that "within the discursive framework, thinking is conceptualized as a special case of the activity of communication and learning mathematics means becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors" (p. 5). Communication does not just mean verbal acts between people. It also includes one's thinking with oneself (communicating to oneself) as well as communicating through writing, diagrams and symbols.

It is within these forms of communication that we can see mathematics as discourse and the learning of mathematics as an enculturation into mathematical discourse (Cobb, 2000; Sfard, 2000; Dörfler, 2000). As students communicate mathematically they learn mathematics. Students need opportunities to communicate in order to learn mathematics with deeper conceptual understanding.

Communicating in the mathematics classroom has benefits for both students and teachers. For students, communication clarifies thinking and understanding (Greenes and Schulman, 1996; McCoy, Baker and Little, 1996) and allows them to formulate their own ideas (Pimm, 1996). It also gives students the opportunity to see the cognition of others (Pimm, 1996). Communication gives students the opportunity to explore their thinking, organize their thinking and make connections to other mathematical concepts (Masingila and Prus-Wisniowska, 1996). Communication generates and directs students' reflection (Peressini and Bassett, 1996). This reflection is invaluable as it gives students opportunities to make connections and organize their ideas. During this organization and reflection students are made aware of their own misconceptions. Classrooms based on communities of mathematical discourse give students a reason to clarify these misconceptions. Becoming aware of student misconceptions is also a benefit of communication for teachers. It allows teachers to see the concept development of individual students. For example, if a teacher uses writing as a way for students to communicate with each other and with the teacher, the teacher can gather information on each student about their level of understanding. Masingila and Prus-Wisnioska (1996) describe four benefits of using writing to communicate in the mathematics classroom.

First, it allows teachers to have direct communication with all students in the classroom. Second, it gives teachers information about student cognition, attitudes and misconceptions. Third, a teacher is able to see the variety of student conceptions of the same idea. Finally, a teacher is able to see evidence of a student's progress. Shield and Swinson (1996) explain that writing also helps students by encouraging reflection, increasing understanding, and acting as a catalyst for discussion.

Inquiry

The word inquiry has powerful implications for education. At the core of the word is the verb, to inquire. To inquire means to examine or to investigate something that is unknown. Inquiry implies that one may not know exactly what the answer will be or what the result of the inquiry will all entail. In inquiry, the investigation itself provides the means to experience and develop conceptual knowledge. In this section of the thesis, I will explore what it means to do inquiry and more specifically what it means to do mathematical inquiry.

Inquiry has often been compared to discovery learning or exploratory learning. Barbara Jaworski (2006) defines inquiry in contrast to discovery. For Jaworski (2006), inquiry is "seeking to know through creative exploration" while discovery is "trying to find out what is" (p. 197). In inquiry, one may raise many questions and find results that may be learned only because one took the time to question in the first place. The idea of discovery implores the metaphor of knowledge as an entity that can be found. Inquiry asserts that knowledge is gained through the process of experience with a concept. It is through the experience and through the subsequent questions that naturally arise from the experience that knowledge is developed.

Unlike a question that presupposes an answer or a problem that suggests a solution, inquiry does not assert a final solution. An inquiry creates an environment where questions and problems may be posed. These questions and problems may have answers and solutions which may be ascertained, but these answers and solutions do not necessarily mean that the inquiry is finished. Whereas questions and problems are largely product driven (in the search of the solution), inquiry purports that it is the process where learning can occur. Often certain understandings can be learned in the process and certain products may be obtained, but new explorations may surface which can create new inquiries.

Traditional mathematics has often focused on mathematics as product-driven and not process oriented. This can be seen through testing techniques where students are asked numerous questions and marked on their efficiency to provide a correct answer using a correct method. When the focus is on getting the correct answer (mathematical product), often the means of getting the correct answer (mathematical processes) are reduced to a set of procedures or algorithms to be enacted by the student. Focusing primarily on mathematical products has the danger to reduce mathematical proficiency to merely procedural fluency without conceptual understanding. A student may be rewarded with high test marks by performing a procedure correctly even though the student doesn't understand the procedure they are using. Thus a product-driven mentality may also lead to students and teachers valuing an instrumental understanding of mathematics (Skemp, 1978).

In contrast, inquiry-based teaching focuses on mathematics as process oriented. The processes need to be understood and explained in order to show mathematical understanding. Focusing on the processes stresses that students need to have a conceptual understanding of the procedures they are enacting. Inquiry-based teaching also invites students to develop their strategic competence as they formulate questions and try to develop their reasoning to solve their questions. Another benefit of inquiry-based teaching is that students are given the opportunity to examine mathematical relationships and hence to develop their adaptive reasoning. When the focus is on the mathematical processes over the mathematical products, students come to value mathematics as something that makes sense. As a result, mathematical products have meaning and are based on conceptual understanding.

If we consider critical thinking to be one of the main goals of school (Heymann, 2003) then it is imperative that we promote critical thinking in mathematics classrooms as well. As opposed to learning mathematics as a set of algorithms to be enacted as recipes, inquiry presupposes thinking. To inquire into an idea or a subject or a question involves thinking. Smith (2007) writes that "one of the cornerstones of improving students' abilities to think and reason mathematically is engaging them in high-level tasks" (559).

Inquiry naturally involves students in high-level tasks because students are investigating a situation where there isn't a procedure or algorithm to get the correct answer. When investigating a novel mathematics problem, one has to examine which pieces of information are important. One has to keep in mind the larger goal or question of the problem to be solved. In problem solving, students have to use thinking or heuristics (Polya, 1945/1988) to solve the problem. Jaworski (2006) claims that the use of inquiry as a classroom tool "can lead to developing inquiry as a way of being when practiced as part of a community, in which members collaborate, as learners, to develop their practice" (p. 187). If students are able to engage in the process of inquiry, students will develop skills of thinking mathematically. As they are prompted to inquire into mathematical ideas they will develop the skills to inquire into mathematics on their own and in turn further develop their own mathematical thinking.

Although I have talked broadly about the term inquiry, it is my intent to clarify inquiry within a mathematics classroom. The word inquiry can have a broad range of meanings and as such it is often clarified through the use of an adjective. Thus, one may hear of ideas such as guided inquiry, mathematical inquiry, narrative inquiry, critical inquiry and so on. Even the adjective does not truly define what type of inquiry is taking place. For example, does mathematical inquiry imply that one is inquiring into mathematics or does it imply that one is using mathematics to inquire into the world? Mathematical inquiry for me means an inquiry into the patterns and structures of mathematical content as opposed to using mathematics to inquire into other phenomenon.

Mathematical inquiry can be a very powerful tool for engaging students in mathematical content. Mathematical inquiry provides students with experiences where they can develop conceptual knowledge about mathematical concepts. The goal is to have students develop mathematical competencies in mathematical objectives. Inquiry provides this opportunity as students are allowed to explore mathematical concepts and question their own conceptual understanding as they try to make sense of the mathematics they are investigating. Lampert (1990) asserts that students come to know mathematics through mathematical discourse that oscillates between inductive observations and deductive generalizations. She writes that "students learn about how the truth of a mathematical assertion gets established in mathematical discourse as they zig-zag between their own observations and generalizations – their own proofs and refutations – revealing and testing their own definitions and assumptions as they go along" (p. 42). A well-structured, inquiry-based mathematics lesson or classroom invites students to zig-zag among mathematical ideas in a manner where they can think, communicate and behave mathematically.

The previous paragraph alludes to the idea that an inquiry activity is more than just an open-ended free-for-all investigation where the students can do or think or act as they please. The structure of the inquiry itself provides students with time to engage with the mathematics. At times an inquiry may provide students with time to explain their ideas or share them with other students or in a larger class discussion. The important characteristic of an inquiry is that the students are given time to actively engage in thinking about the mathematics. Instead of watching and seeing procedures which have to be reproduced at a later time, students can experiment, communicate and make sense of the mathematics as they experience it. Jaworski (2006) writes,

> the notion of inquiry relates particularly to perspectives in mathematics education dealing with cognition in terms of the active 'construction' of mathematical knowledge, where 'active' implies personal involvement. Inquiry, or investigative methods in mathematics teaching are seen to fit with a constructivist view of knowledge and learning: they demand activity, offer challenges to stimulate mathematical thinking and create opportunities for critical reflection of mathematical understanding (Cobb, Wood, & Yackel, 1990; Glasersfeld, 1984, Jaworski, 1994) leading to the development of conceptual, relational and principled understanding of mathematics (e.g., Skemp 1976). (Jaworski, 2006, p. 199)

It is through the active involvement of thinking mathematically that students can develop mathematical proficiency. Mathematical inquiry invites students to critically examine their own conceptions of mathematics. It provides an opportunity for students to ask questions. As inquiry becomes one of the norms of practice, a community of inquiry can develop. Jaworski (2006) defines a community of inquiry to consist of "two or more people working together to embody an inquiry approach to learning and teaching, developing inquiry as a way of being within the community" (p. 206). As students explore a mathematical concept through an inquiry, they create questions and provide reasoning to develop their own understanding. But inquiry is more than just creating an understanding for an individual. As students work on a mathematical inquiry in a classroom, they are situated in a context where explaining their own ideas and developing their own questions naturally extends to the community of inquirers. As students attempt to clarify their own understandings, they question and explain their thoughts to others trying to reach the same goals. As a result, a community of inquirers can develop conceptual understandings that may not be possible at an individual level.

In order to establish communities of inquiry, it is important to create activities which promote inquiry in the classroom. These activities could be developed within the framework of the existing curriculum, or these activities could be used to help develop new curriculums. A classroom where inquiry is used as a tool can become a classroom where students behave mathematically and inquire mathematically.

As Azita Manouchehri (2007) writes, "natural consequences of using an inquiry-based approach to teaching include the emergence of unexpected mathematical results and the articulation of novel and different strategies by students" (p. 290). Rafaella Borasi (1992) relates a similar experience in her inquiry into definitions that "produced some new learning not only for the students, but for the teacher as well" (p. 31). Allowing time to explore novel ideas shows the students that the teacher values the exploration into mathematical ideas. The role of the teacher in inquiry-based teaching is to be open to new ways of thinking and new ways of doing mathematics. Unlike traditional mathematics based on teaching the most efficient procedure, a teacher who promotes inquiry allows students to explore other ways of thinking. Although these new ways of thinking may not present the correct answer as quickly or efficiently as an algorithm, the fact that a teacher is open to thinking about mathematical ideas in different ways shifts the focus in mathematics classroom. Instead of valuing a correct answer using a correct method, an inquiry-based teacher values mathematical thinking that can lead to a correct conception of a problem. Ultimately, students will come to a procedure for learning a mathematical concept, but they will be able to understand why it is that the procedure works. The varied experiences and explorations of a single mathematical concept also allow students to compare and contrast different procedures to produce the same answer. It is through varied rich experiences that students can develop a deeper conceptual understanding of a mathematical concept. A community of inquiry creates a classroom that values mathematical processes.

Patterns

An integral part of my research with inquiry involves the recognition and explanation of mathematical patterns. This section of the thesis will examine the importance of patterns in learning mathematics. It will focus on the extent to which curriculum documents establish patterns as part of the nature of mathematics. The search for patterns and the generalization of patterns is closely linked to algebraic thinking. The connection between patterns and algebra will be explored fully. Finally a case will be made that the study of patterns is intrinsically linked to thinking mathematically and learning mathematics with a conceptual understanding.

Manitoba Education (2009) describes patterns as one of seven characteristics that define the nature of mathematics. “Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns” (p. 14). By studying patterns, students can make connections between different mathematical concepts. Manitoba Education (2009) asserts that “students need to learn to recognize, extend, create, and apply mathematical patterns” (p. 14). NCTM (2000) writes that “instructional programs from prekindergarten through grade 12 should enable all students to understand patterns, relations, and functions” (p. 37).

Patterns can be generalized and described using algebra. NCTM (2000) discusses patterns in the content standard of algebra. As students progress through the grade levels the connection between patterns and algebra becomes more apparent. In the middle grades, “students will learn that patterns can be represented and analyzed mathematically” (p. 297). The introduction of algebra to generalize patterns in the middle years provides a context for which students can understand patterns better. Karen Koeliner, Mary Pittman and Jeffrey Frykholm (2009) write that “algebra provides a structure and a language with which to talk about patterns” (p. 304).

In Pattern Math, students are reintroduced to the notion that algebra can be used to describe patterns. Darin Beigie (2011) writes, “the transition to thinking mathematically using variables has many layers, and for all students an abstraction that is clear in one setting may be opaque in another" (p. 329). Although they have been introduced to algebra as a way to generalize in previous grades, the Pattern Math activities invite students to rediscover the power of algebra. By moving from arithmetic observations to algebraic generalizations, they once again are given an opportunity to view the power of algebra to generalize. Tina Rapke (2009) writes, "the difficulties students experience with algebra are related not to the numerical context but to the transition from the numerical context to the algebraic context" (p. 376).

The Pattern Math activities also provide students with the opportunity to reflect on algebraic operations. Seeing the algebraic operations generalize the arithmetic patterns helps provide meaning for the algebraic operations. Manitoba Education (2009) explains that "learning to work with patterns helps students' algebraic thinking, which is foundational for working with more abstract mathematics" (p. 14). Koeliner, Pittman, and Frykholm (2009) describe "the essence of what it means to generalize—to use the tools and language of algebra to articulate mathematical relationships in ways that are clear, concise, and powerful" (p. 304). NCTM (2000) describes the algebra experience of high school students; "High school students' algebra experience should enable them to create and use tabular, symbolic, graphical, and verbal representations and to analyze and understand patterns, relations, and functions with more sophistication than in the middle grades" (p. 297).

Studying and understanding patterns is part of the nature of mathematics. Looking for and explaining patterns can help students conceptually understand mathematics. "In solving problems students should be encouraged to look for patterns. When patterns are established, concepts are more easily understood and applied" (Manitoba Education, 2001, p. 13). Anna McMaken-Marsh (2007) explains the importance of searching for patterns as part of mathematical thinking. She writes, "the important thing here is to have students wrestle with trying to find patterns where there are not any. If we give students only problems whose solutions are neat and clear, we are not preparing them for the kind of mathematics that exists in life" (p. 386). Although the Pattern Math activities begin with an explicit pattern, the students are encouraged to look for other patterns as well. This often leads them to look for patterns in areas where there are not patterns. This process of looking for more patterns helps the students to better understand some of the characteristics of the pattern that they are studying.

In summary, patterns are an important part of what it means to study and understand mathematics. The generalization of patterns in arithmetic forms the basis for the study of algebra. Seeing the connection between the patterns of arithmetic helps students better understand algebra. This connection between algebra and arithmetic can also allow students to search and find new arithmetic patterns. Creating these connections helps students develop a deeper conceptual understanding of both arithmetic and algebra. Looking for patterns and explaining why patterns work is an example of mathematical thinking. Before being formally introduced to the students and the patterns they worked on in Chapter 4, the next chapter discusses the methodology of the research.

Chapter 3: Methodology and Method

Methodology: Design Experiment Research

Design experiment research (also called design-based research) is a methodology for developing or designing classroom materials while studying the effects of the materials on student cognition. Paul Cobb et al. (2003) write that “design experiments entail both ‘engineering’ particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them” (p. 9). For example, a unit may be designed first and then an experiment takes place to test the validity of the construct of the design. The design of a unit of instruction is based on theoretical considerations from research and the enactment of the unit of instruction tests the validity of theoretical constructs in a pragmatic way. The Design-Based Research Collective (2003) assert that “design-based research, which blends empirical educational research with the theory-driven design of learning environments, is an important methodology for understanding how, when, and why educational innovations work in practice” (p. 5).

Design experiment research acknowledges that classrooms are relational and that the relationship between the student, the teacher and the subject matter are always in negotiation. Thus, design experiment research allows for design to be developed and changed within the context of the unit and during the unit. A unit of instruction is designed and tested within the class allowing researchers to study both the validity of the unit of instruction while studying the learning in context. Because design experiment research examines both student cognition and instructional materials, it “can help create and extend knowledge about developing, enacting, and sustaining innovative learning environments” (p. 5). As a result, Cobb et al. (2003) write that “design experiments are pragmatic as well as theoretical in orientation in that the study of function – both of the design and of the resulting ecology of learning – is at the heart of the methodology” (p. 9).

The Design-Based Research Collective in their article “Design-based research: An emerging paradigm for educational inquiry” (2003) list five characteristics of design experiment research. First, the goals of designing the learning environment and the development of theories of learning are interwoven. Second, research and the development of materials are in a continuous cycle of design, experiment, analysis and revision. Third, research is conducted for the benefit of a larger community. Although research is conducted within the context of a single class, the results of the research are to be of benefit for other researchers or other practitioners. Fourth, the research is considerate of context. The "research must account for how designs function in authentic settings" (Design-Based Research Collective, 2003, p. 5). And finally, design experiment research aims at understanding learning issues, documenting how learning outcomes and student cognition relate to the enactment of the design.

Design-experiment research provides an opportunity to connect theory with practice. This is important as teachers will often say that theories lack practical implications and theorists argue that teachers do not consider theoretical insights that can enhance instruction. Theorists and teachers become collaborating researchers in the design and the experiment. The original design is based on theoretical ideas, but often the design is altered as practical issues based on context influence the design as well. As a result, design experiment research can "help researchers and designers understand the real-world demands placed on designs and on adopters of designs" (Design-Based Research Collective, 2003, p. 8). The design experiment research also connects the teacher to the theory and situates the teacher in "direct ownership of designs" (Design-Based Research Collective, 2003, p. 8). This connection between theorist and teacher helps break down some of the barriers formerly seen between theory and practice. Both the teacher and the theorist feel ownership in both the design and the enactment of the material. As a result, a unit of instruction that is strongly based on both theoretical and practical considerations can be developed.

Design experiment research can be seen as an extended form of lesson study (Stigler and Hiebert, 1999). James Stigler and James Hiebert (1999) describe lesson study in their book *The Teaching Gap*. Like design experiment research, lesson study consists of planning based on theoretical assumptions about learning. The planning is collaborative as it involves a group of teachers and or researchers. The goal of lesson study is to improve teaching and student learning. Stigler and Hiebert report that "lesson study is a process of improvement that is expected to produce small, incremental improvements in teaching over long periods of time" (p. 121). During the planning stages of lesson study, a constant focus is on student learning. After the planning of the lesson is completed, the lesson study group enacts the lesson. One person in the group teaches the lesson, while the others in the group observe students interacting with the lesson. Like design experiment research, a cyclical process of planning, enacting and revision takes place in lesson study. After the lesson study group presents their lesson for the first time, they go back and make revisions based on how students learned or interacted with the lesson that was created. Stigler and Hiebert state that "by attending to teaching as it occurs, lesson study represents teaching's complex and systemic nature, and so generates knowledge that is immediately usable" (p. 122). Design experiment research is also produced within the context of the classroom. This connection between the theoretical planning of the lesson and the pragmatic realities of the classroom make data and curriculum created from lesson study or design experiment of immediate use in the classroom.

Design experiment research is especially appealing for practitioners when considering that the design may be used to focus curriculum or enhance curriculum. Design experiment research formalizes the technique of design, reflection and revision that is used on a regular basis by practitioners more informally. For example, a practitioner designs a lesson to be taught to students. The lesson is enacted in the class and at times changed during the lesson to adapt to the needs of the students. Reflection upon the lesson will mean further adaptations to the design or enactment of the lesson. Comparatively, design experiment research follows the same format more formally. For example, a lesson is designed to test theory in a practical setting. During the lesson, practical consideration may alter the design. Data taken from the design is analyzed and further revisions to the design result from the data analysis. The results of cycles of data collection, analysis and revision mean that the design has been tested and altered to enhance student cognition. The result is that when the design is shared with other practitioners there is some validity to the design. Cobb (2000) explains that after a design experiment has taken place the "resulting theory can therefore be thought of as situated with respect to the activity of supporting students' learning in classrooms" (p. 59). A unit of instruction developed through design experiment research is grounded in theory, tested and adapted for practicality and presented in a way that allows for further adaptation and development. Design experiment research acknowledges that learning takes place in context. It asserts that although a design may have been developed successfully in one context, it may need adaptations to be successful in another context.

The Cyclical Nature of Data Collection and Data Analysis

The design experiment of Pattern Math naturally involved phases of data gathering and data analysis. In this section, I will outline how the cyclical nature of design, reflection and revision was done through the formal setting of design experiment research. As design experiment, Pattern Math had two complimentary research goals. The first was to examine student learning and cognition through inquiry and the second was to develop curricula that promotes student communication and conceptual learning. Data that was collected and analyzed from the first research goal became data to guide the development of the second research goal. As data was gathered and analyzed, it provided information to change, adapt or enhance the design. This cyclical nature is characteristic of design experiment research. Cobb et al (2003) explain that the "designed context is subject to test and revision, and the successive iterations that result play a role similar to that of systematic variation in experiment" (p. 9). The following represents the different phases of data collection and analysis which lead to further data collection, analysis and revision.

Phase 1: Interactive Writing: Data Collection and Preliminary Analysis

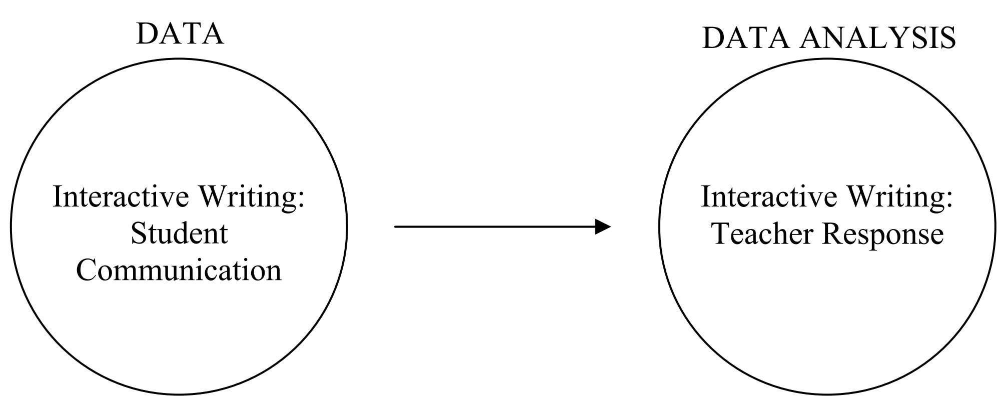


Figure 2: Transition from data to data analysis: interactive writing stage.

The first phase consists of interactive writing (Mason & McFeetors, 2002). Interactive writing was important both as a way to view student learning and as a way to promote student learning. By analyzing the students' communication of the pattern, I was able to see the extent of the students' thinking. By responding to the students, I could provide further instruction and push students to enter into the zone of proximal development. The importance of interactive writing to the development of learning and the analysis of data will be explicated through examples in subsequent chapters. Students explained their thoughts about different mathematical concepts in response to prompts provided in Pattern Math. This became the first set of data for the teacher/researcher. I read each student's writing and responded to each student individually. The response became the first phase of data analysis. At times, my response elicited more writing from students which resulted in more data.

Unlike some aspects of traditional research, design experiment research allows for changes and adaptations to occur during the research project. The early analysis and response to students' interactive writing allowed me to adapt and change Pattern Math activities to meet the needs of the students. This analysis helped inform me on how to improve and enhance the materials for further instruction.

Phase 2: Summative Responses to a Phase of Pattern Math: Secondary Analysis and Data Collection

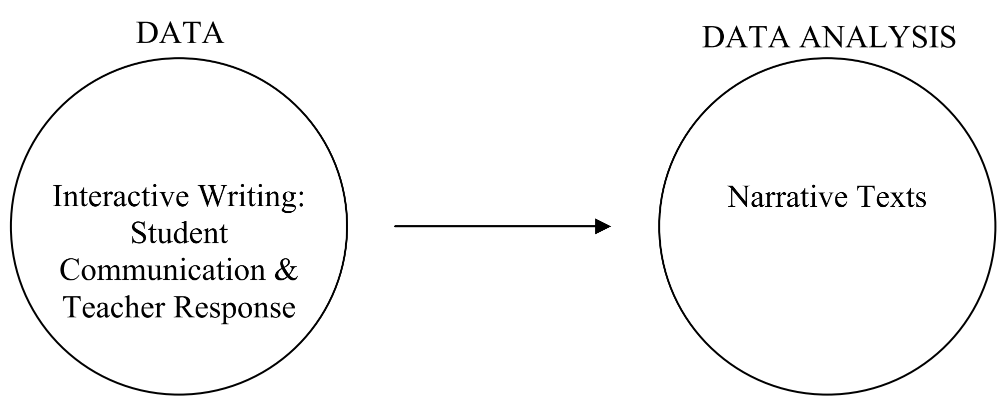


Figure 3: Transition from data to data analysis: narrative stage.

After a cycle of Pattern Math activities concluded, the students were provided with prompts to reflect upon the cycle (See Appendix C). I examined the data from the students' Pattern Math work and their summative responses to produce a narrative (Clandinin & Connelly, 2000) for each student. In the development of the narrative, I re-examined the data collected from the interactive writing and looked for themes that appeared throughout the cycle of Pattern Math. The themes were summarized in a narrative for each student. Students were given the opportunity to read the narratives that I wrote and had the opportunity to respond to the narratives in closure interviews.

Phase 3: Begin a New Cycle

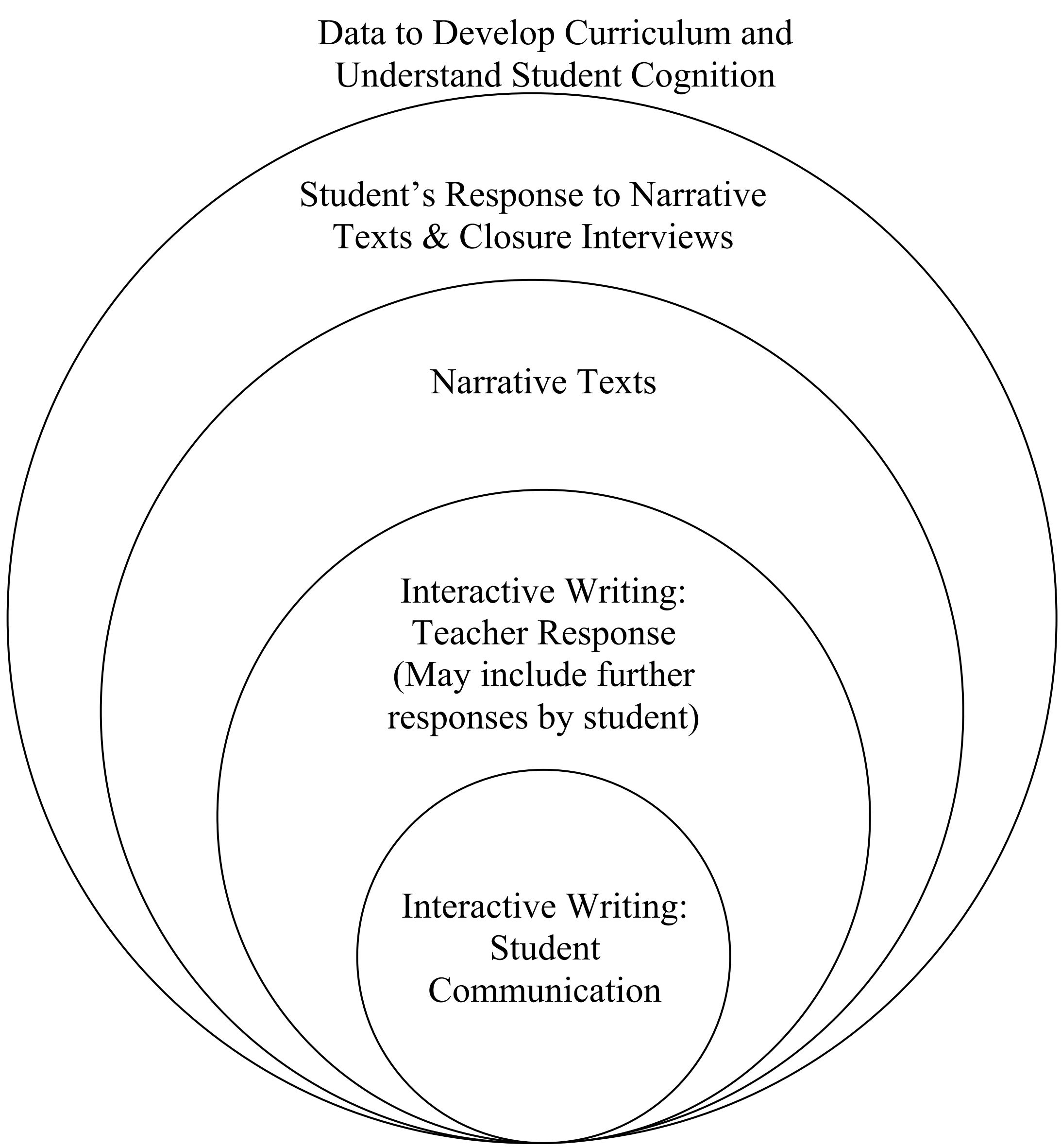


Figure 4: Data to develop curriculum and understand student cognition.

After summative comments have been made about a cycle of Pattern Math activities, the students began another cycle of Pattern Math activities. The second cycle was followed by reflections and the further development of the narrative of the student. The diagram in Figure 4 shows how the different levels of data gathering and analysis can help understand student cognition and be used to develop curriculum to meet the needs of the student.

Phase 4: Responses to Narrative Texts and Closure Interviews

After students completed the three cycles of Pattern Math, a narrative text was created for each student outlining the student's growth of mathematical understanding. Students were given time to read their own narratives and respond to the narratives through closure interviews. The closure interviews gave me summative views about the efficacy of the Pattern Math project. The questions that guided the closure interviews can be found in Appendix D.

Methodology: Pirie-Kieren Model

The Pirie-Kieren model of the growth of mathematical understanding served as a model to describe mathematical understanding in the research project. In Pattern Math, the students were given a pattern that invited them to think about and understand a mathematical concept. As students communicated and made connections as part of Pattern Math, the Pirie-Kieren model was used to describe a student's level of mathematical understanding and growth. The Pirie-Kieren theory provides a framework to understand the dynamic growth of mathematical understanding (Pirie and Kieren, 1994). The theory consists of eight nested circles with each circle representing a new level of growth and understanding. The nature of developing mathematical understanding is not linear and the model allows for the movement back and forth between levels. This movement back and forth between levels is called folding back, and each act of folding back creates a deeper or thicker understanding of a previous level. This section will describe the Pirie-Kieren theory and each of the eight rings involved in mathematical understanding. It will look at characteristics of the theory, such as 'don't need' boundaries, folding back, and the complementarities of acting and expressing.

Finally, the application of the theory in respect to the research project will be discussed as a way to categorize, describe and notice mathematical understanding as it occurred in Pattern Math.

Figure 5 shows the eight nested rings of Pirie-Kieren theory of the growth of mathematical understanding. Each ring represents a level in the growth of mathematical understanding. The first level is called *primitive knowing*. When a student begins to learn about a new mathematical concept they bring in previous knowledge. This previous

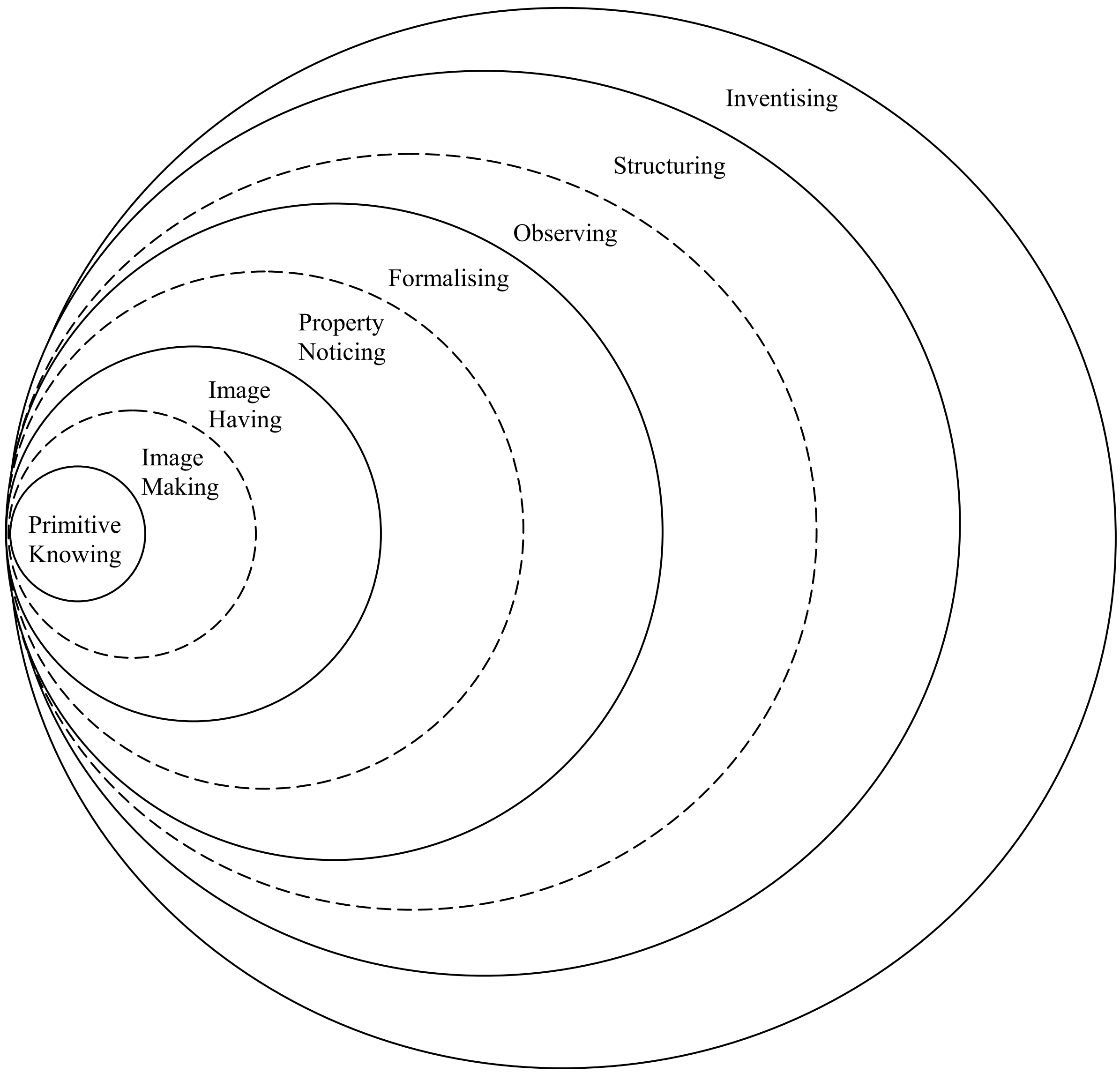


Figure 5: The Pirie-Kieren model of the growth of mathematical understanding.

knowledge is represented by the primitive knowing ring. It is the starting place where new growth and understanding about the mathematical topic will begin. This ring does not include any knowledge about the new topic to be learned as this will be shown in the outer rings of the model. Primitive knowing, from the perspective of the student, is everything that the student already knows and can do. Primitive knowing, from the perspective of the teacher or researcher, is everything that the teacher or research assumes that the student can already do. It is important to note that a teacher or researcher can never know the exact extent of a student's primitive knowledge, but only develop assumptions based on observations of the student.

The second level is called *image making*. During image making, the student is engaged in activities to help gain an understanding of the concept. These activities can create a wide variety of images. An image is not restricted to pictorial diagrams or visual symbols. An image could include a variety of constructs from thoughts, verbal descriptions, written symbols, diagrams or pictures to physical manipulation of objects related to the topic. The student who is engaged in image making develops a wide range of images about the topic. The purpose of image making is to gain an idea of what the topic is about.

When the images that are made during image making become part of the student's own mental image of the topic, the student enters the third level of mathematical understanding, *image having*. The student is moving away from the specific images that were made to a general image of the topic being learned. For example, if the image making process included the use of manipulatives, the image having process involves the student being able to work with the topic without needing to use the manipulatives. The student creates mental constructs of the topic. The learner is not confined to particular examples or activities when thinking about the concept. During this level, a mental image of the new topic is formed.

Having a mental image of a topic does not necessarily imply a deep level of understanding about the topic. More growth occurs as students enter the next level of understanding, *property noticing*. Pirie and Kieren (1994) write that property noticing "occurs when one can manipulate or combine aspects of one's images to construct context specific, relevant properties" (p. 170). During this level, the student can articulate patterns seen in images and make connections between properties of the images. Lyndon Martin (2008) writes, “Property noticing is an act of self-conscious reflection, of questioning one’s understandings and of looking for what can be said more generally about these” (p. 66). It is in this level that the student begins to notice and think about general characteristics of the topic being learned.

Property noticing naturally leads to a more formal description of the properties that have been observed. This is the fifth level of understanding called *formalising*. This is the stage where the student abstracts a common method or quality from the properties they noticed in the previous level. It is a stage where the student is able to think about the generalized properties of the topic. The student is able to articulate the properties of the topic in a general and abstract manner. Martin (2008) explains that at this stage the student is able to “work with the concept as a formal object, without specific reference to a particular action or image” (p. 66).

The next level of understanding, *observing*, is the act of taking formalizing one step further. The student reflects on the formal activity and tries to coordinate the formalizing as general theorems. The student who is developing a deeper understanding of a concept is continually moving from the specific to the general (and, as will be discussed later, moving back and forth between general and specific). Moving towards the general involves property noticing, formalising and reflecting on the formalising by observing.

This sets the stage for the second last ring, *structuring*. The process of observing creates theorems which describe the topic being learned. These theorems can be combined to form an overall theory through the act of structuring. Pirie and Kieren (1994) write, “this means that the person is aware of how a collection of theorems is inter-related and calls for justification or verification of statements through logical or meta-mathematical argument” (p. 171).

The last level of the Pirie-Kieren theory of mathematical understanding is called *inventising*. Prior to this level, the student has a complete and fully structured understanding of the topic that was being learned. It is at this stage that the student may have the ability to break away from the knowledge that created these structures and move towards new understandings. The student is able to invent new criteria that will lead to new image making. This in turn will lead towards new mathematical understanding.

The Pirie-Kieren model has three features that help explain the model. The first feature is that some of the boundaries between levels are called *'don't need' boundaries*. These boundaries are represented by the dotted lines in the diagram of the model. The first 'don't need' boundary occurs between the image making and image having levels. Once a person has acquired an image of a mathematical idea, they no longer need to make images of that mathematical idea. This does not mean that a person cannot make images, but the making of images becomes part of having an image. The second 'don't need' boundary occurs between property noticing and formalising. Once a person has developed a formal understanding of a mathematical concept, they have already noticed the patterns. The formal understanding can re-create the patterns that were noticed. The last 'don't need' boundary occurs between observing and structuring. The structuring level is where the person has a complete theory to describe the mathematical concept. Once a person has reached this level there is no longer a need for a boundary between previous levels of knowledge so the boundary between observing and structuring is a 'don't need' boundary.

The second key feature of the model is the idea of *folding back*. Although the levels have been presented as a linear string of different levels of understanding, the growth of mathematical understanding cannot be described simply as a linear progression. Pirie and Kieren (1994) describe folding back as the activity "which reveals the non-unidirectional nature of coming to understand mathematics" (p. 69). Folding back describes the action where a person is at one level in the model and is encountering problems understanding the mathematical concept. The person needs to fold back to a previous level. For example, a student may be at the pattern noticing stage and notice patterns that don't fit with the image they have of the mathematical concept. This student will need to fold back to the image making and image having stage, creating new images for the pattern they have noticed. The metaphor of folding back views folding back as creating a 'thicker' or deeper understanding at the previous level than existed before. Thus, folding back is essential to the growth of mathematical understanding.

Pirie and Kieren (1994) describe the third feature of the theory as the *complementarities of acting and expressing*. Acting and expressing is apparent at each of the levels beyond primitive knowing. Acting is what a student does at a level of understanding. It includes making images during image making or listing patterns during pattern noticing. Expressing is examining and articulating what was done during acting. It is more than just reflecting. A person must articulate an idea before it can become part of their understanding. A person who is growing in mathematical understanding at each level has to first act and then express before they can move beyond that level of understanding. Often, the complementarities of acting and expressing occur back and forth during a level of understanding as a person first acts, then expresses, the acts again and so on as the person grows in their mathematical understanding.

The patterns given to the students at the beginning of Pattern Math helped students in the stages of image-making and image having. After students created images of the pattern they went on to notice the pattern more fully and tried to formalise what they saw. Attempting to formalise or notice patterns often caused students to fold back to making new images so that they could create a deeper understanding of the new mathematical topic. Thus, the Pirie-Keiren model provided an excellent model of describing the growth of mathematical understanding as seen in my research project.

Timeline for Data Collection

Design experiment research consists of cycles of research. The purpose of this section is to provide a timeline for data collection. The timeline outlined below includes the three cycles of Pattern Math. During each of the cycles the students participated in three to five Pattern Math activities. All Pattern Math activities will be described fully in the next chapter. (Also, see Appendix B.) After each Pattern Math activity, students had time to compare their ideas with other students and discuss what they had found. Students were also given an opportunity to read the response from the teacher's interactive writing. After they had completed a cycle of Pattern Math activities, students reflected on the activities. These reflective questions can also be found in the appendix. (See Appendix C.) The students described what they felt was valuable from the activities.

Week 1:

- Monday: students were asked to write their preliminary thoughts about how they seek patterns while doing a typical homework assignment. (See Appendix A).
- Wednesday: students began the first activity of cycle 1 of Pattern Math.
- Friday: students read my response and completed the second activity of cycle 1.

Week 2:

- Monday: students read my response and completed the third activity of cycle 1.
- Wednesday: students read my response and completed the fourth activity of cycle 1.
- Friday: students read my response and completed the fifth activity of cycle 1.

Week 3:

- Monday: students read my response and reflected on the first cycle of Pattern Math.
- Wednesday: students began the first activity of cycle 2 of Pattern Math.
- Friday: students read my response and completed the second activity of cycle 2.

Week 4:

- Monday: students read my response and completed the third activity of cycle 2.
- Wednesday: students read my response and reflected on the second cycle of Pattern Math.
- Friday: students began the first activity of cycle 3 of Pattern Math.

Week 5:

- Monday: students read my response and completed the second activity of cycle 3.
- Wednesday: students read my response and completed the third activity of cycle 3.
- Friday: students read my response and completed the fourth activity of cycle 3.

Weeks 6 & 7

- Monday: students reflected on all three cycles of Pattern Math.
- Over these two weeks, the narratives for each student were developed.

Weeks 8

- Closure interviews were conducted with students to mark the end of data collection.

Chapter 4: Introduction of Pattern Math and Students

This section will describe the inquiry that took place in my classroom. It also describes the processes such as surveys, interactive writing, narratives and closure interviews that formed the data for this research. The purpose of this section is to describe the process used in the research, provide the rationale for this process, and introduce the students involved in the research.

The research began with a short homework survey. The survey consisted of four questions. Three of these questions were specifically about homework in a mathematics class while the last question examined students' views on the best way to learn mathematics. The survey questions are given below:

Homework Questionnaire

1. What is the purpose of a homework assignment in your view?
2. How do you learn while you are doing homework? Explain.
3. What do you think a teacher wants you to learn from doing a homework assignment?
4. In your opinion, what is the best way to learn mathematics?

The purpose of the homework survey was to get a sense of how students learn mathematics and their views on learning mathematics. In particular, I was curious to find out which students viewed learning mathematics conceptually and which students viewed learning mathematics procedurally. For example, with the first two questions, I was interested in whether or not students would describe their learning as learning how to do certain questions or learning why something works. This can be perceived as students' views towards mathematics as learning procedurally or learning conceptually. The third question examines the students' perceptions of their teacher. Do they view their teacher as one who wants them to conceptually understand concepts or do they view their teacher as one who wants them to be able to do the procedures involved in solving questions? The final question asked students to share what they thought was the best way to learn mathematics. Again, I was curious about whether or not this question would shed some light on students' views towards learning mathematics. I wondered if I should have asked this question first, before the homework questions, so that responses were not influenced by the previous questions. I had hoped that by prefacing the question with "In your opinion …”, that students would feel comfortable to answer this question without the worry of the previous questions biasing their thoughts.

The homework survey set the stage for the three cycles of Pattern Math activities that would follow. The survey provided some background on the students’ views towards learning mathematics. For those students who saw learning mathematics as understanding why a concept worked conceptually, the Pattern Math activities would give them an opportunity to explore and explain why things worked. For those students who saw learning mathematics as knowing how to do a procedure, the Pattern Math activities would stretch them to look at math differently by explaining why things worked.

The first cycle included five Pattern Math activities. Originally designed to have only four Pattern Math activities, an additional Pattern Math was introduced after the first activity to extend what the students had observed. The mathematical theme of the first cycle was patterns with perfect squares and differences of squares. The first four Pattern Math activities dealt with patterns with integral numbers. The last Pattern Math extended the theme of the patterns to algebraic expressions.

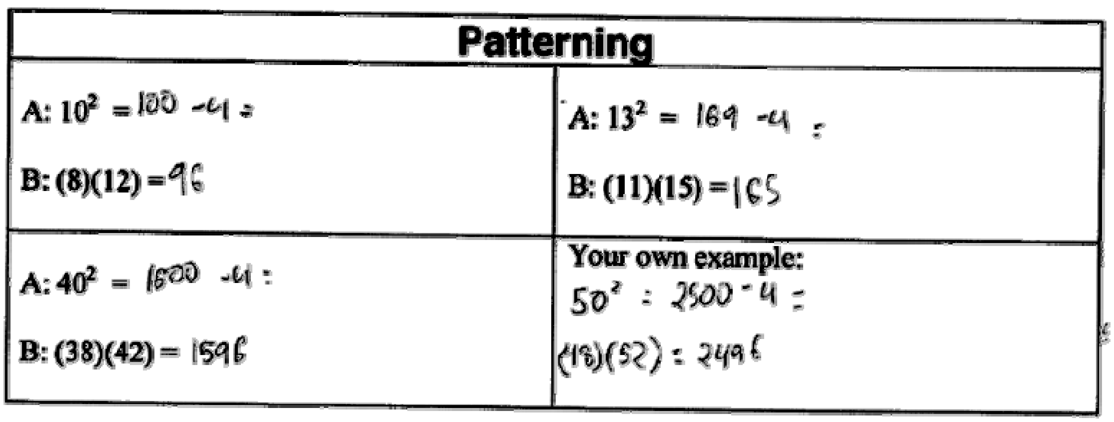
This section will describe the cycles of Pattern Math. To provide the reader with a more holistic view of the research and to allow the reader to become acquainted with the students who participated in the research, I have described the different Pattern Math activities by providing different students’ examples and a short commentary on these examples. I have also included the response that I provided as part of the interactive writing method. In order to get the most out of this section, I would encourage the reader to go through the each of the activities on their own first. They are all found in the appendix (Appendix B). I think that you will find that by doing the activities first and attempting to describe why they work that you will enjoy reading what the students have written more. You may also be surprised at some of the ideas the students have created and how these discoveries in turn help you to understand the patterns in ways that you may not have seen before.

In a subsequent chapter, I will provide the complete work of a specific student as she completed the three cycles of Pattern Math activities. It is my hope that this chapter and the subsequent chapter will give the reader a clear view of the research as well as become acquainted with various students who were involved in the research. This will set the stage for an analysis of the data that will be presented later.

Cycle 1

Cycle 1: Pattern 1: Two More Two Less

The first Pattern Math was titled "Two More Two Less" and can be seen in Justin's work below. The basic format of all the Pattern Math activities was to give three examples for students to try followed by a similar example that the student made. The patterns were designed so that they were relatively easy to identify so that the students could begin the process of trying to explain why the patterns worked. Justin has shown below in his work that he already has identified that the answer in part B is always four less than the answer in part A.



Patterning	
A: 10^2 = 100 -4 = B: (8)(12) = 96	A: 13^2 = 169 -4 = B: (11)(15) = 165
A: 40^2 = 1600 -4 = B: (38)(42) = 1596	Your own example: 50^2 = 2500 - 4 = (48)(52) = 2496

Figure 6: Justin's pattern work on "Two More Two Less"

Following the section titled "Patterning" was a section titled "Communicating". In this section, students attempted to describe the pattern and explain why the pattern worked. In the first Pattern Math, I included guiding instructions and questions to help the students write about what they saw. Asking students to "Talk about the pattern that you see," was intended to get students to communicate in words the pattern that they observed. The instruction to "Communicate as clearly as you can everything that you notice," was to encourage students not only to describe the main pattern that they saw but also to look for other patterns that they discovered while they were communicating and trying to explain why the pattern works. In the bottom right of Justin's communication below, you can see that he has discovered other patterns that he thought were significant. Justin noticed that the difference between the two products was equal to the difference between the two numbers in part B. His communication indicates that he wondered if this would work with other numbers, but his attempt to compare 13^2 and 10 times 16 showed that when the numbers were 6 apart, the product different by 9. This led him to conclude that the pattern that he noticed was specific to this example and could not be extended to other examples.

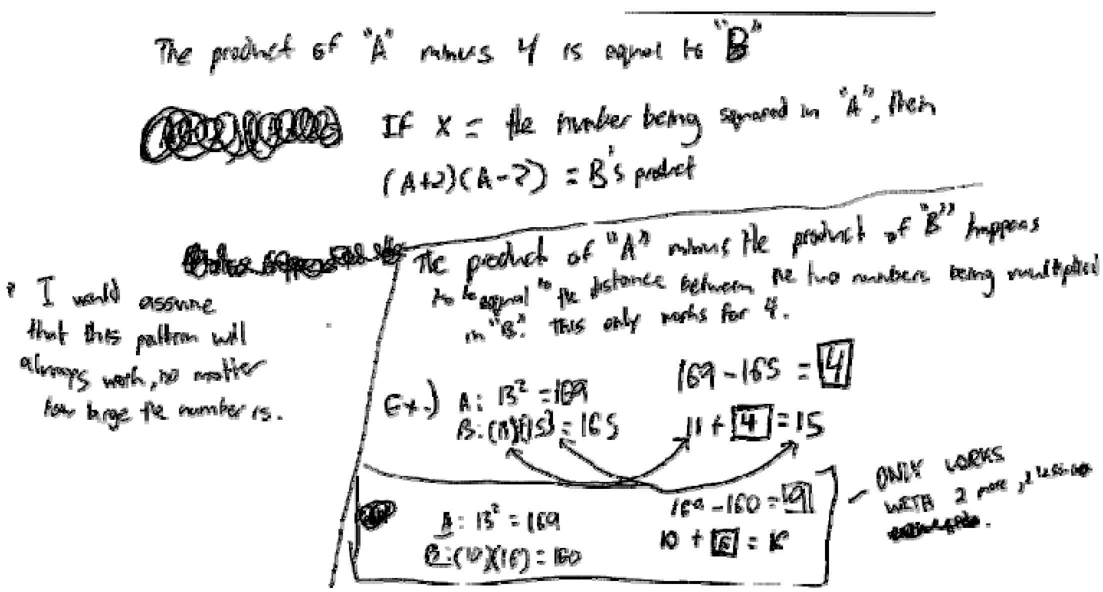


Figure 7: Justin's communication on "Two More Two Less"

Justin's work in the bottom right corner shows how powerful it can be to communicate clearly everything that you notice. Sometimes describing a pattern can cause students to see new patterns which extend the pattern to different contexts. Justin was really close to making a connection between "Two More Two Less" and what happens in general when you compare a number squared and the product of two numbers that are the same amount above and below that number. In fact, Justin's work inspired me to create a new Pattern Math for the students to explore this relationship closer.

The last guiding questions in the communicating section ask students to try to explain if the pattern always worked and how they would know that it always worked. This proved to be the most difficult task for many students. Justin was close to using algebra to explain the pattern, but his algebraic expressions only represent what he sees, they do not describe why it works. On the left, Justin writes, "I would assume that this will always work, no matter how large the number is." Justin is confident that the pattern he has found will always work, but he is unable to communicate exactly why it always works.

This first example shows a number of key elements that I encouraged students to do while they completed the communicating section of each Pattern Math. I encouraged students to start by describing the pattern that they saw in words. The description of the pattern may cause them to try more examples or see more patterns. At times, seeing more patterns may direct the student to explore other patterns that were similar to the original pattern. This is what Justin did when he looked at 10 times 16. I called this "extending the pattern" when students explored other patterns that were similar to the original pattern. Finally, I was always encouraging students to try to explain why a pattern always worked. Through explaining why the pattern always worked, I hoped that students would gain a deeper conceptual understanding about numbers, patterns, and algebra.

After each student completed their Pattern Math, I wrote back to them as part of the teaching/research technique called interactive writing (Mason & McFeetors, 2002). Responding to each student allowed me to communicate with each student individually in a way that would not have been possible using only the time allocated in class. It also gave me the opportunity to encourage each student in specific ways that would help their learning. Below is my response to Justin's first Pattern Math.

Justin,

Justin, I was really excited about the work that you were doing while you communicated the patterns that you saw. You correctly describe the pattern, but what was the most exciting thing for me was how you saw a secondary pattern and tried to explain if that pattern worked. You checked if three more and three less (a difference of 6) was 6 less than the number squared. You found out that it didn't work and concluded that it only worked for 2 more 2 less. This is a great example of mathematical thinking. There are two ways to look at proving if something always works or not. One way, what you did, was to look for an example that doesn't work. If you find an example that doesn't work then obviously it doesn't always work. The other way is to try to find a way to show that it always works. This looks like this was something that was difficult for you as you stated that you would assume that it always works and were unable to explain why it would always work. This is something I want you to think about as we do the next Pattern Math. What are the ways in which you would show something will always work? Think about this and try to communicate this in the next activities.

Your pattern that you noticed and checked led me to design another Pattern Math. Thank-you for your idea! I hope you continue to explore patterns as we look at three more and three less.

Feel free to respond to anything that I have written here.

Justin's work on the bottom right hand corner inspired me to create another Pattern Math before moving on to the next Pattern Math that I had originally planned. This is consistent with design experiment research as the products of the instruction may cause the researcher to adapt or modify the planned research to meet the needs of the design (Cobb et al., 2003; Design-Based Research Collective, 2003). This is consistent with what I saw from Justin's work as well as other students. Many students had difficulty explaining why the pattern worked. Few students made connections to broader understandings such as what would happen if you did three more and three less. As a result, I created another Pattern Math, with a little different format, to encourage students to explain why and make connections between different mathematical ideas.

Cycle 1: Pattern 2: One More One Less, Etc.

The second pattern of the first cycle can be seen below. This second Pattern Math served many purposes. I wanted students who had trouble explaining why the pattern worked in the first Pattern Math another chance with a similar pattern. I also wanted students who were able to explain the pattern to see how extending the pattern to similar examples can create a richer understanding of the concept that would lead them to a more generalized understanding of the pattern. I have included the work of Chase and my response to Chase to introduce the second Pattern Math. Chase was a student who had trouble explaining why the first pattern worked. You will also notice that I began this Pattern Math with an explanation to the students of why I was including it.

2^{nd} Pattern Math – Cycle 1

I was glad to see that everyone saw a pattern in the first activity. There were a few of you who saw a pattern that I wasn't expecting. It is exciting for me to see your thoughts, especially when you can describe patterns that I didn't see. Everyone noticed that the second answer was 4 less than the first answer. Some students also noticed that the difference between the two values in part B was 4 as well. One student asked the question whether or not this was a significant pattern and whether or not it worked for other numbers. For example, if we have (7)(13), which are 6 apart, is their product 6 less than (10)(10)? Sometimes when we see a pattern we asked if it will always work and we check other values to see if this works. Sometimes we extend a problem to see what happens when we change a few things to describe a new pattern. This is an example of great mathematical thinking. Because this was such a great example, I designed the second Pattern Math based on it. Sorry I took so much room at the top, you probably will have to write on the back to explore this pattern.

Pattern Math

Choose one or more of the following ideas to create a pattern. Once you have found a pattern try to explain what you have noticed and why it works. Try to make connections to other patterns or math you have learned.

One More One Less	Three More Three Less	Four More Four Less	Five more Five Less
A: $(10)^2$	A: $(10)^2$	A: $(10)^2$	A: $(10)^2$
B: (9)(11)	B: (7)(13)	B: (6)(14)	B: (5)(15)

What do you notice?

Figure 8: Second Pattern Math, "One More One Less, etc."

What do you notice?

A:100 $42^2 = 1764$
B:99 $(41)(43) = 1763$

$1^2 = 1$ $2^2 = 4$
$(0)(2) = 0$ $(1)(3) = 3$

$18^2 = 324$ $x^2 = y$
$(17)(19) = 323$ $(x-1)(x+1) = y-1$

The product of the number squared is always one more than the product of the two other numbers (x-1) and (x+1). I don't know why the pattern works, but it does. I see patterns, but I don't know why they work. I think that patterns work like the rest of math, you just have to believe that this is what the numbers equal. That's how math seems to be after adding, subtracting, multiplication and dividing. After that it just gets complicated.

Figure 9: Chase's communication on "One More One Less, etc."

Chase is having trouble explaining why the pattern works. He is able to describe the pattern and represent it with algebra, but he cannot see how the algebra might be used to explain why the pattern would always work. His comments about math being something you just have to believe indicates that Chase has been approaching math as sets of procedures and that he has not tried to conceptually understand math for quite some time. Here is my response to Chase.

Chase,

Thanks for your comments. I hope that you enjoyed the patterns we have seen so far. In some ways they seem like tricks to solve mathematical problems that you might not have been able to do mentally before. For example, 49*51 is 2499 because 25*25 is 2500.

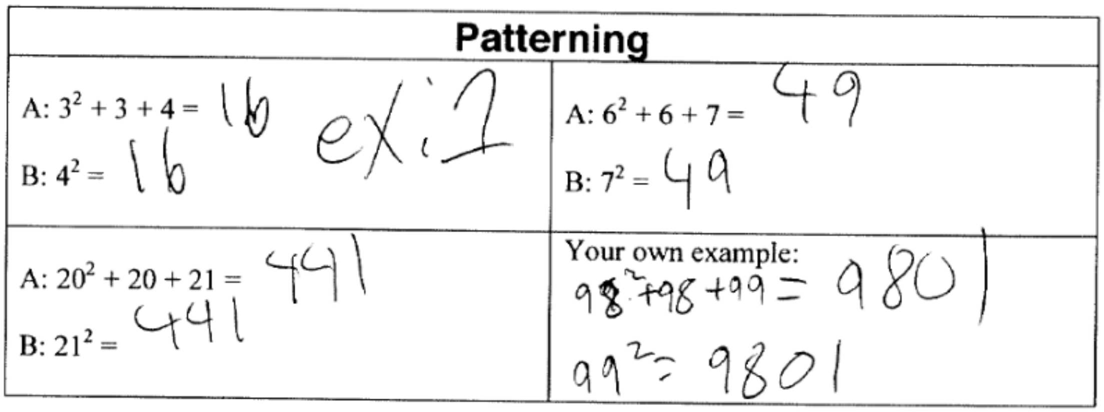
To understand why a pattern always works is a difficult thing to think about. It would be insane to try every number because that would take an infinite amount of time. However, that doesn't mean that it is impossible to show that something always works. You were, in fact, very close to showing that it always worked in the little bit of algebra that you had written down. You see, algebra is a way of looking at a series of patterns. When you start off with x, that means that x could be any number (whole numbers, negative numbers, rational numbers and even irrational numbers such as π.) Now if you think about what would happen if you took a number more than x and multiplied it by a number less than x. Algebraically that would mean that I have $(x + 1)(x - 1)$. You have learned some tools in algebra to expand this. What happens? Since we used x, this means that it works for any number. What are your thoughts on that?

I am glad that you believe what your math teachers are telling you is true. But I hope that you begin to discover that you can understand more about mathematics other than just adding, subtracting and multiplying are the things that make sense. The more complicated stuff can make sense as well, but you have to be willing to explore the math and ask why something works. The answers aren't easy, but I find that math is much more enjoyable if it makes sense and you understand it rather than just doing a bunch of steps to get the right answer.

Feel free to respond to anything that I have written here.

In my response to Chase, I am encouraging him to learn mathematics with more conceptual understanding rather than just memorizing the procedures needed to understand the math. I am also providing him with help in areas where I feel he may be stuck such as seeing how the algebra can help him explain the pattern. At the end of my response I encouraged Chase as well as other students to respond to what I had written. I found that students often did not respond directly to this invitation. Some students used their next Pattern Math to answer questions that I had asked while some students never responded. I used the closure interviews at the end of the research to ask the students to answer some of the questions that I had asked in my response.

Cycle 1: Pattern 3: Squared Plus Number Plus Next Number



Patterning	
A: $3^2 + 3 + 4 =$ 16 ex:1 B: $4^2 =$ 16	A: $6^2 + 6 + 7 =$ 49 B: $7^2 =$ 49
A: $20^2 + 20 + 21 =$ 441 B: $21^2 =$ 441	Your own example: $98^2 + 98 + 99 =$ 9801 $99^2 =$ 9801

Figure 10: Tom's pattern work on "Squared Plus Number Plus Next Number"

The third Pattern Math continued to look at squared numbers and their relationships. The example below is Tom's. Although he shows the relationship algebraically, he does not continue to show that the two algebraic expressions on either side of the equation are in fact equal. In order to test the scope of the pattern, Tom tries examples with zero, negative numbers as well as the irrational number π. I am not surprised that Tom believes that if the pattern works with a few numbers and it works with zero and π that it should work for all numbers. In fact, I think that is often how we teach math. We give students a few examples and then expect them to see from the few examples that the method we are using will always work. My response to Tom continues the conversation we had been having and how algebra can be used as a way to explain why a pattern always works. It also encourages him to look at patterns in a different way and how he could use different representations such as area to help him understand the pattern better.

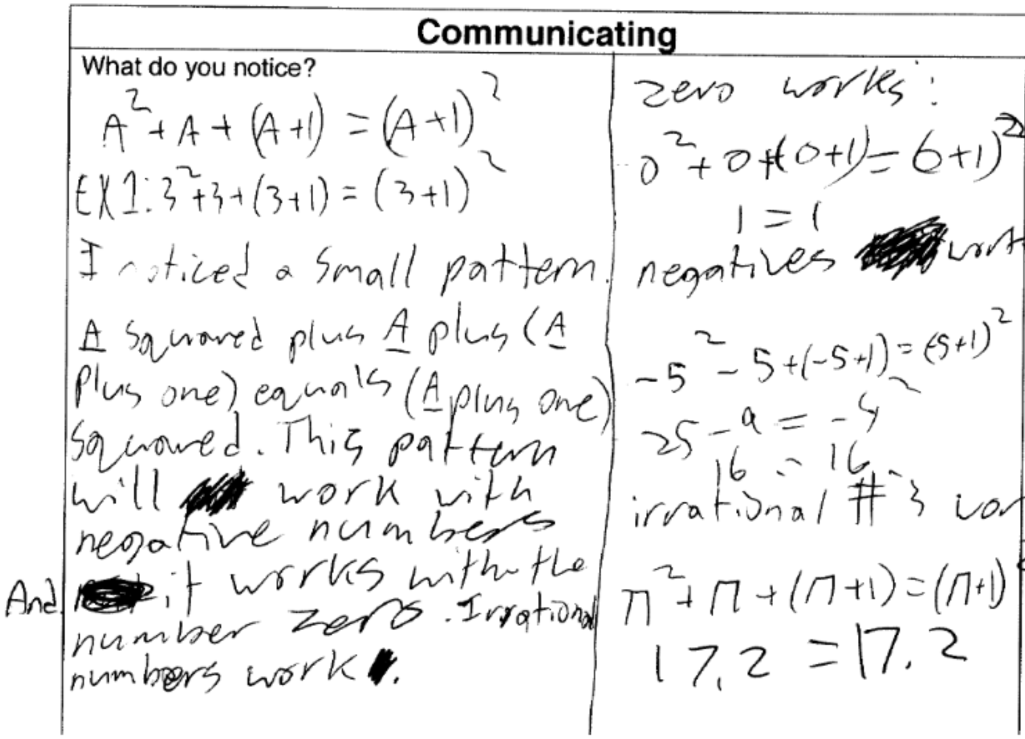


Figure 11: Tom's communication on "Squared Plus Number Plus Next Number"

Tom,

Last time I wrote you the following:

So how do we explain something works for every number? We often start with a bunch of examples as you did to see that it appears that it always works. Often diagrams help us see what is going on better than just the numbers. A diagram that would represent these questions well would be areas. Draw a 10 by 10 square. Draw a 9 by 11 rectangle. Why is there one less? Another idea that can really help is algebra. Let x be any number and see what happens when you represent the patterns with algebra. You have done areas before and you have done algebra before. This means that you can make connections to things that you have done in the past. In fact, I think that you have been using these patterns already in algebra, but never realized they were patterns.

I would like you to think about what you wrote as the pattern: $A^2 + A + (A + 1) = (A + 1)^2$. If you started with $A^2 + A + (A + 1)$ could you manipulate it so that it is equal to $(A + 1)^2$? Could you do it in the other direction? Start with $(A + 1)^2$ and get the other side? Do you think that this would explain why it works?

Or what would it mean to start with a square that is 3 by 3. Why does adding three and adding four make a square of 4 by 4? These are the kinds of things I would like you to explore while you are trying to explain why something works.

Cycle 1: Pattern 4: Difference Equals Sum

Patterning	
A: $6^2 - 5^2 = 11$ B: $6 + 5 = 11$	A: $10^2 - 9^2 = 19$ B: $10 + 9 = 19$
A: $50^2 - 49^2 = 99$ B: $50 + 49 = 99$	**Your own example:** $(-4)^2 - (-5)^2 = -9$ $(-4) + (-5) = -9$

Figure 12: Megan's pattern work on "Difference Equals Sum"

"Difference Equals Sum" continues the idea of arithmetic patterns involving square numbers. In the example, you can see how Megan has already developed communication and reasoning skills through the activities. In her own example, her use of negative numbers indicate that she is assuming that it will work with positive integers and is curious to see if it will work with negative numbers as well. Her communication begins with an algebraic explanation, but she continues to explore the pattern through geometric representations as well. In some of the earlier Pattern Math activities, she began to use areas to represent multiplications. She found this geometric representation to really help her understand why a pattern worked. Although she used a specific example in her diagram, a square with side lengths of 6 units, her diagram shows that she can see why it would work with other numbers as well. She writes in her concluding statement:

> "When making a square into another square whose sides are one unit shorter each, the number of square units removed is equal to the sum of the old and new side lengths."

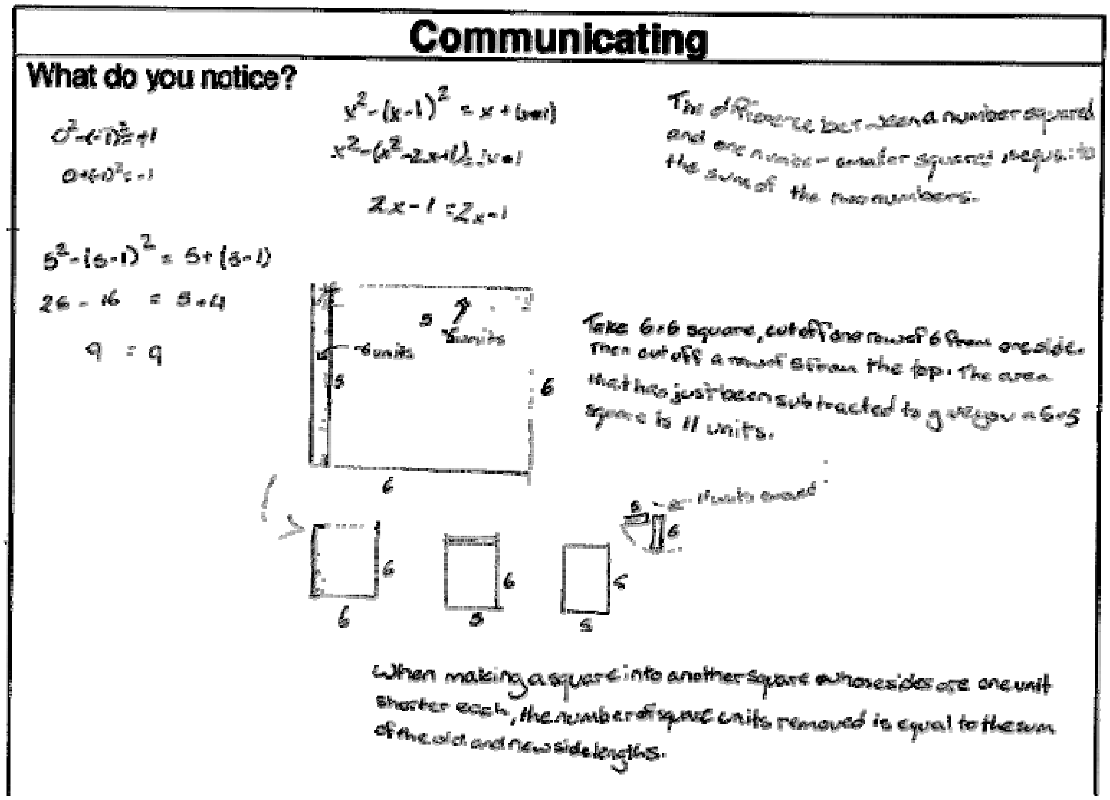


Figure 13: Megan's communication on "Difference Equals Sum"

She is able to take her specific example and generalize to include all examples. In a way her diagram which shows why it works is much more convincing than her algebraic proof that both expressions $x^2 - (x - 1)^2$ and $x + (x - 1)$ both reduce to $2x - 1$. What is powerful in Megan's work is that she can now explain the pattern in more than one way, both algebraically and geometrically. The result of this is a better understanding of the pattern than if she could explain it in only one way. My response below commends her on the excellent work she did.

Megan,

Excellent proofs with both the algebra and with the diagrams. I really liked how you generalized the area idea as well as provided a specific example. Your diagrams make it very clear why it would equal 6 + 5. This is a great example of great communication. Communicating in math is a mixture of words, numbers and diagrams and we use them back and forth so that a person who is reading your ideas can understand what you are saying. I just wanted to let you know that you did a great job communicating what you saw with the areas.

Also here is a technical thing that I also wrote to Nate. In your algebraic manipulations at the top, you start with an equation $(x)^2 - (x - 1)^2 = x + (x - 1)$. Mathematically this is true, but what you are wanting to do is to prove that it is true. So, the way you do this is to start with one expression, say the left side, and manipulate it on its own until you get the right side. This would prove that they are the same. By stating they are equal at the beginning technically you are assuming what is true before you have proven it (a mathematical no no). Does this make sense? If it doesn't, ask me and I will show you what I mean.

For you, which way of showing the pattern, the picture way or the algebra way, do you like better? Which do you think makes more sense to prove the pattern? Which do you think is a better proof of the pattern?

Feel free to respond to anything that I have written here.

Cycle 1: Pattern 5: Subtracting Binomial Squares

Patterning	
A: $(n+4)^2$ $(n+4)$ = $n^2+8n+16$ B: $(n+3)^2$ = n^2+6n+9 C: A – B = $2n+7$	A: $(2n+5)^2$ $(2n+5)$ = $4n^2+20n+25$ B: $(2n+4)^2$ $(2n+4)$ = $4n^2+16n+16$ C: A – B = $4n+9$
A: $(n-2)^2$ = n^2-4n+4 B: $(n-3)^2$ = n^2-6n+9 C: A – B $2n-5$	Your own example:

Figure 14: Hana's pattern work on "Subtracting Binomial Squares"

"Subtracting Binomial Squares" was the same pattern and same activity as "Difference Equals Sum" with the small difference of using algebraic expressions instead of using numbers. This proved to be extremely difficult for most students to see, with only a few students being able to recognize the same pattern. The work below is that of Hana. Hana was able to see that the pattern was the same as the one before. This was typical of Hana; she often saw the pattern quite quickly and came up with a way of explaining it. A characteristic of all of Hana's work was that she continued to play with the pattern and explore patterns even after she recognized and articulated a reason for the pattern. In this example you can see that she is extending the pattern again. She recognizes that the pattern works for consecutive numbers so she begins to look at numbers that are not consecutive to see if a similar pattern emerges. Her work with (x +3) and (x + 7) is intriguing because she notices a connection between the sum of the two numbers and the difference between the two expressions. Although this discovery was interesting, Hana decides not pursue more examples to see if it works for other differences. She also examines what happens if the coefficients of the expressions change. She hypothesizes that using (2x + 1) and (x – 1) should result in 3x if the pattern holds true. Her result of $3x^2 + 6x$ leads her to a tentative conclusion – that it only works with consecutive numbers. Hana's work was often characteristic of what is seen in this example. She would do lots of experiments with numbers and these experiments would lead her to new questions and more experiments.

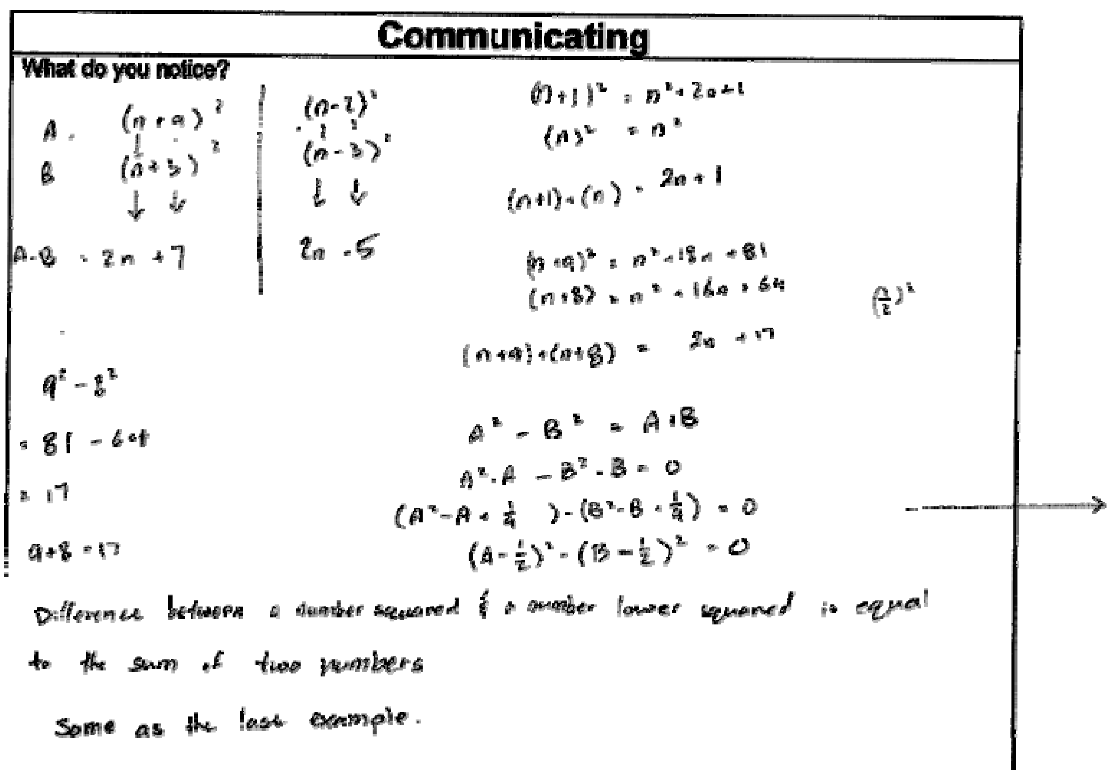


Figure 15: Hana's communication on "Subtracting Binomial Squares"

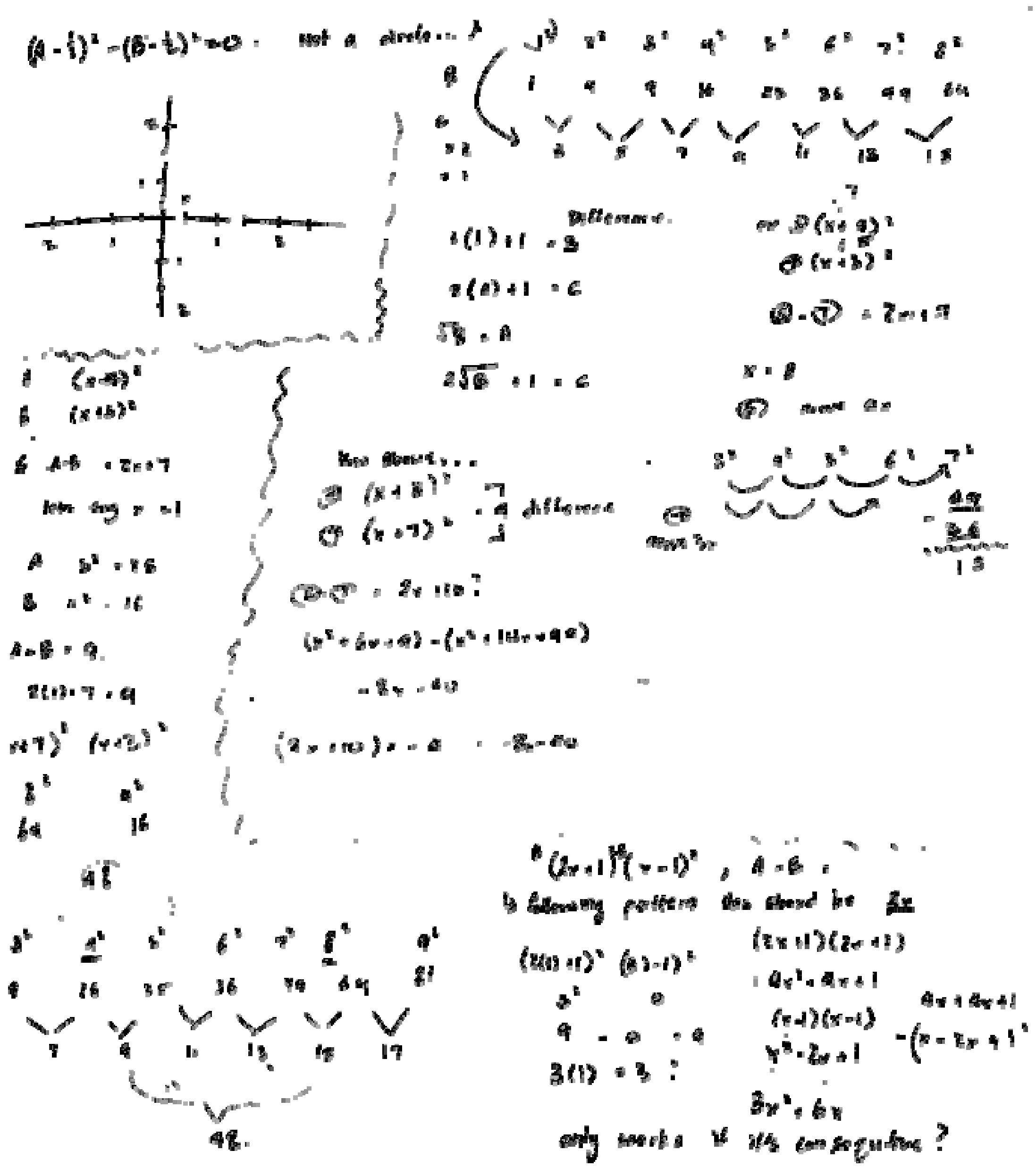


Figure 16: Hana's communication on "Subtracting Binomial Squares," continued.

In my response to Hana, I explained how I liked her extension of the pattern. I encouraged her to explore new patterns that she found with more examples and to try to explain these patterns as well. In subsequent activities, I would also encourage Hana to take the time to summarize what she had discovered in more words. I was encouraging her to use more communication within her experimenting.

Hana,

Looks like you did a lot of great experimenting! I also noticed that you made a connection between this example and the last one that we did. That is great!

I also liked how you extended the example to look at other examples like the one where there was a -4 difference. Did you find it interesting that the pattern didn't work, but that if you took the pattern and multiplied it by -4 then it did work? Why was that?

I also liked how you played around with the coefficients of the first variable and your results were quite different. This led you to state that you thought that it only worked with consecutive numbers.

One thing that I think you might find interesting is if you tried some of your other algebra skills when playing around with these patterns. For example, have you tried any factoring? You can factor expressions even if they have only numbers as in the previous examples. Let me know what you find.

Now that we have done 5 Pattern Maths, take a look at them all and see which Pattern Maths are connected to each other. I think this might help you explain why this last pattern works.

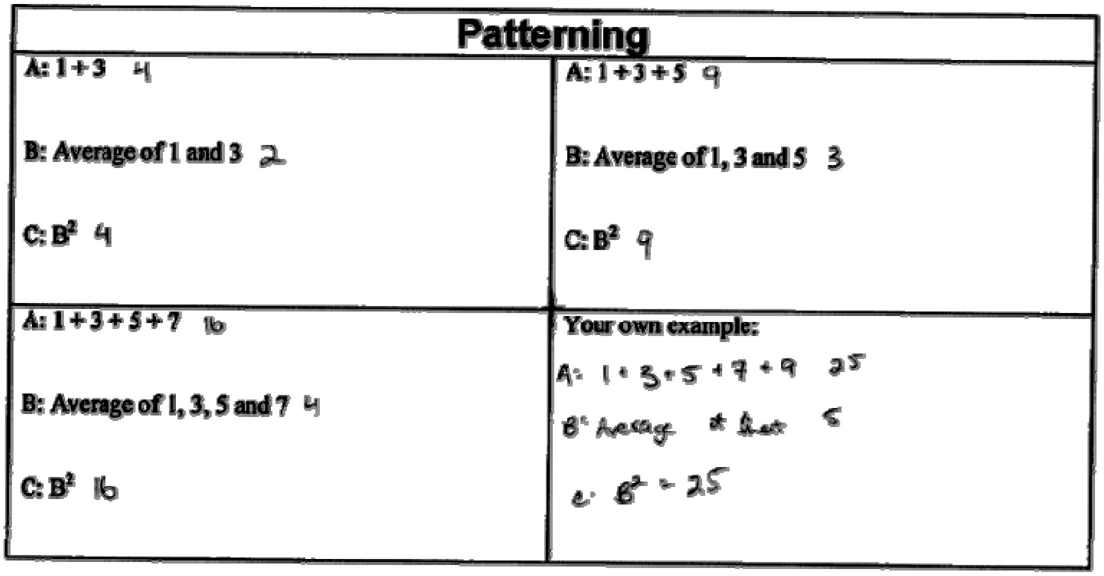
Feel free to respond to anything that I have written here.

At the end of each cycle I asked students to reflect on the cycle and what they have learned. Since the purpose of this chapter is to introduce the cycles and to introduce the students to the reader, I will share the results from student reflections in a later chapter. You will notice in my first response to Kayla in cycle 2 that I am referring to these reflections.

Cycle 2

Cycle 2: Pattern 1: Adding Odds

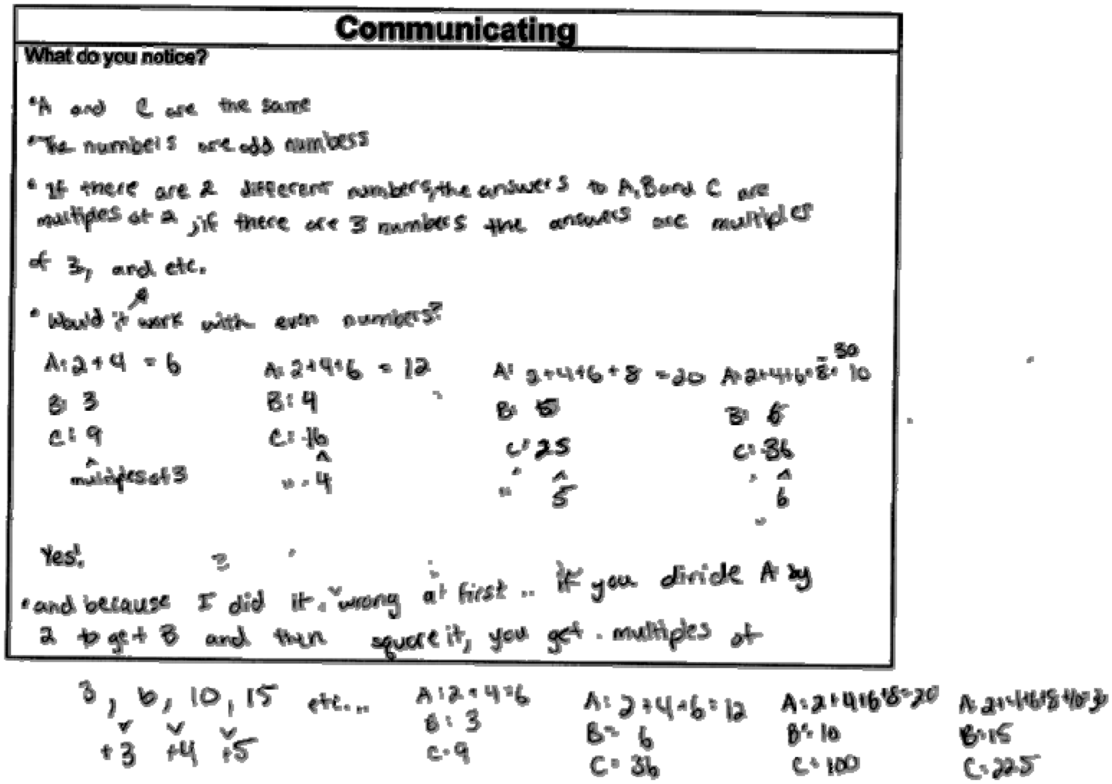
Whereas the first cycle dealt with square numbers and different patterns that emerged, the theme of the second cycle was patterns that emerge when we add a sequence of numbers. The first sequence was adding up odd numbers. Since the sum of a series of odd numbers is always a perfect square this connected to the previous cycle and provided a smooth transition from one cycle to the next.



Patterning	
A: 1 + 3 4 B: Average of 1 and 3 2 C: B^2 4	A: 1 + 3 + 5 9 B: Average of 1, 3 and 5 3 C: B^2 9
A: 1 + 3 + 5 + 7 16 B: Average of 1, 3, 5 and 7 4 C: B^2 16	Your own example: A: 1 + 3 + 5 + 7 + 9 25 B: Averag [illegible] 5 c: B^2 = 25

Figure 17: Kayla's pattern work on "Adding Odds"

The first Pattern Math was called "Adding Odds" and can be seen through Kayla's work below. Kayla was really good at describing the pattern that she saw and how it worked. In many of my responses to her I would often ask her to go beyond just describing the pattern and how it worked and attempt to explain why it worked. She found describing why a pattern worked to be very difficult and often felt stuck. As a result, when she was stuck trying to explain why something worked, she often changed the pattern and looked for new patterns. This is something that I encouraged students to do, to extend the pattern and to see when the pattern worked and when the pattern didn't work. The extension of the pattern to new situations often resulted in new patterns.


Communicating

What do you notice?

* A and C are the same
* The numbers are odd numbers
* If there are 2 different numbers, the answers to A, B and C are multiples of 2, if there are 3 numbers the answers are multiples of 3, and etc.
* Would it work with even numbers?

A: 2+4 = 6
B: 3
C: 9
multiples of 3

A: 2+4+6 = 12
B: 4
C: 16
4

A: 2+4+6+8 = 20
B: 5
C: 25
5

A: 2+4+6+8+10 = 30
B: 6
C: 36
6

Yes!

* and because I did it wrong at first .. if you divide A by 2 to get B and then square it, you get multiples of 3, 6, 10, 15 etc...

+3 +4 +5

A: 2+4 = 6
B: 3
C: 9

A: 2+4+6 = 12
B: 6
C: 36

A: 2+4+6+8 = 20
B: 10
C: 100

A: 2+4+6+8+10 = 30
B: 15
C: 225

Figure 18: Kayla's communication on "Adding Odds"

In Kayla's work, she is trying to see what happens when you add consecutive even numbers. Although the pattern is different with even numbers, the amount that Kayla explores the pattern shows that she is interested in finding patterns even though she does not take the time to explain why they work. Some interesting things that arise from Kayla's work is how the sum of consecutive even numbers is always double a triangle number and how the average of consecutive even numbers is always one more than the number of even numbers you started with.

Here is my response to Kayla.

Kayla,

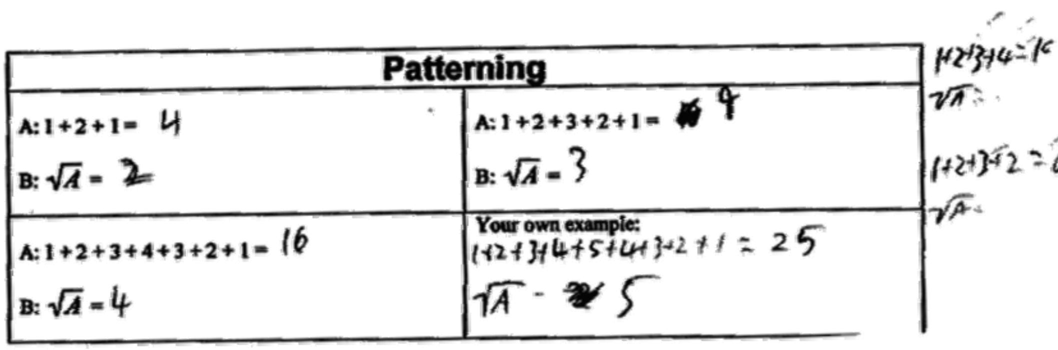
First of all I would like to respond to your reflections on the first cycle. I am glad that you found looking for other patterns in the difference equals sums to be so interesting. There is something about finding patterns in mathematics that is interesting. It is especially fun when you find the patterns yourself and take the time to explore them. I hope you take the time over the next couple of Pattern Maths to not only try to explain why things work, but to also take the time to explore other patterns as well.

Now, back to adding odds. I noticed that you found the pattern right away and that you were able to describe it. I also noticed that you asked whether or not it would work with even numbers and this intrigued you to try some examples. That was great! I wish you wouldn't have erased what you said you did wrong. I don't mind if you make mistakes and sometimes we learn a lot by making mistakes and sometimes we find new patterns through our mistakes.

Both the odd numbers and the even numbers make some interesting patterns. I am glad that you are taking the time to find these patterns. I would also like you to try to explain why these patterns work. You may not come up with a complete explanation, but I would like you to try by using algebra or graphs or pictures or whatever you think might work. Finding patterns is exciting, but it is also interesting to try to explain why they work.

Feel free to respond to anything that I have written here.

Cycle 2: Pattern 2: Add Up Add Down

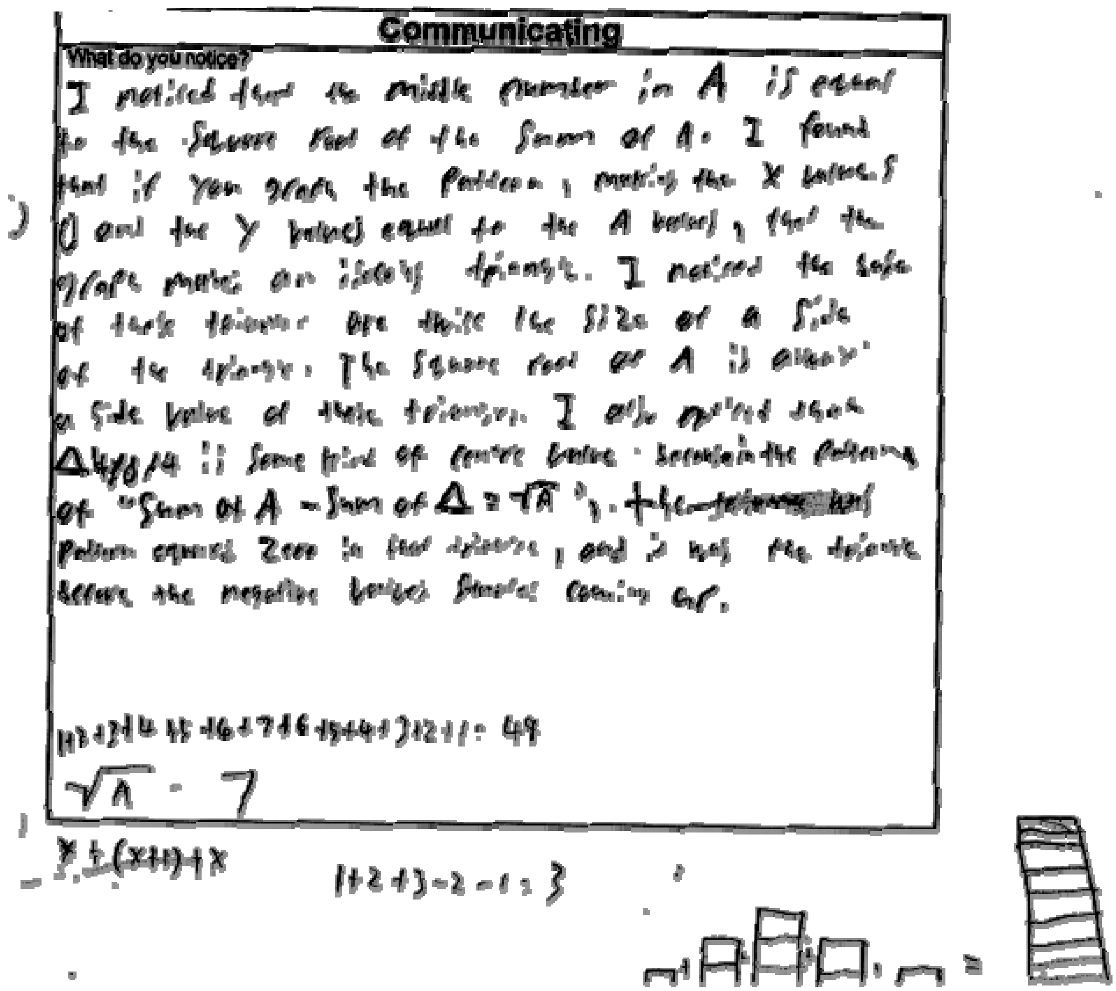


Patterning	
A: 1+2+1= 4 B: $\sqrt{A}$ = 2	A: 1+2+3+2+1= 9 B: $\sqrt{A}$ = 3
A: 1+2+3+4+3+2+1= 16 B: $\sqrt{A}$ = 4	Your own example: 1+2+3+4+5+4+3+2+1 = 25 $\sqrt{A}$ - 5

1+2+3+4=16
$\sqrt{A}$
1+2+3+2=8
$\sqrt{A}$

Figure 19: Bryson's pattern work on "Add Up Add Down"

The second Pattern Math in this cycle had the same results as the first Pattern Math. The pattern was that if you add up from one to a certain number and then back down to one then the result is always that number squared. The following is Bryson's work. I really liked interacting with Bryson on the Pattern Math activities. I never knew what to expect from Bryson. In the first couple of Pattern Math examples, Bryson didn't know where to start. He had trouble seeing a pattern and had trouble describing and explaining the pattern. He would often hand in a Pattern Math later than the other students because he had been working on it a long time and just couldn't put into words what he was seeing. Then, every once in a while, Bryson would explode with a huge explanation of why something worked and he would do lots of experiments into the pattern. Some of the things that he found during these extended explorations were wonderful. In "Add Up Add Down", Bryson decided to look at graphing the numbers in the sequence. His results made a triangle and his explorations into these triangles gave him a new way of seeing and being able to explain the pattern.


Communicating

What do you notice?

I noticed that the middle number in A is equal to the square root of the Sum of A. I found that if you graph the pattern, making the X values () and the Y values equal to the A values, that the graph makes an isoceles triangle. I noticed the sides of these triangles are twice the size of a side of the triangle. The square root of A is always a side value of these triangles. I also noticed that Δ4/8/14 is some kind of center value because in the pattern of "Sum of A − Sum of Δ = √A", the [illegible] pattern equals zero in that triangle, and it was the triangle before the negative values started coming up.

1+2+3+4+5+6+7+6+5+4+3+2+1 = 49

√A = 7

x + (x+1) + x

1+2+3−2−1 = 3

Figure 20: Bryson's communication on "Add Up Add Down"

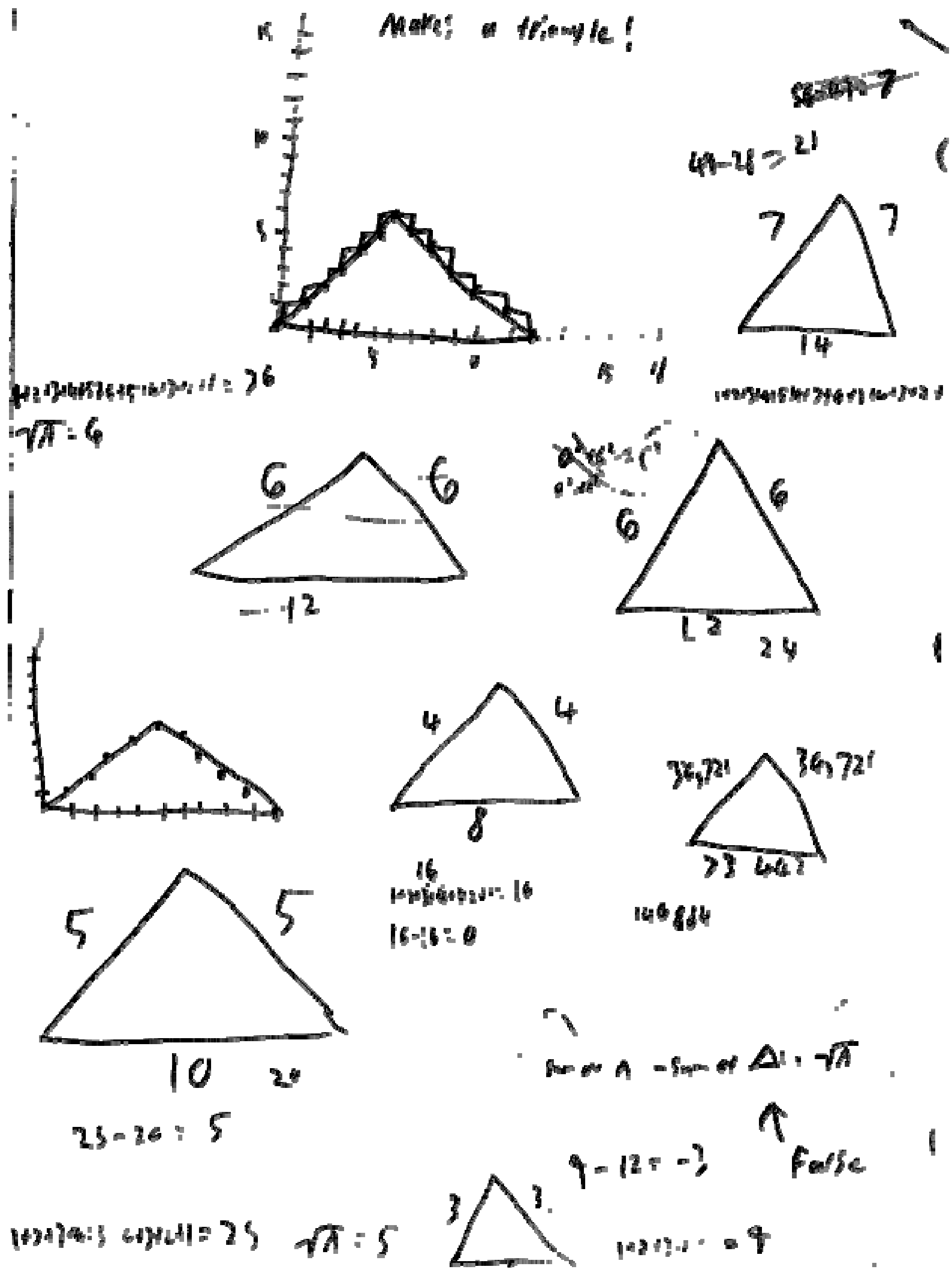


Figure 21: Bryson's communication on "Add Up Add Down," continued.

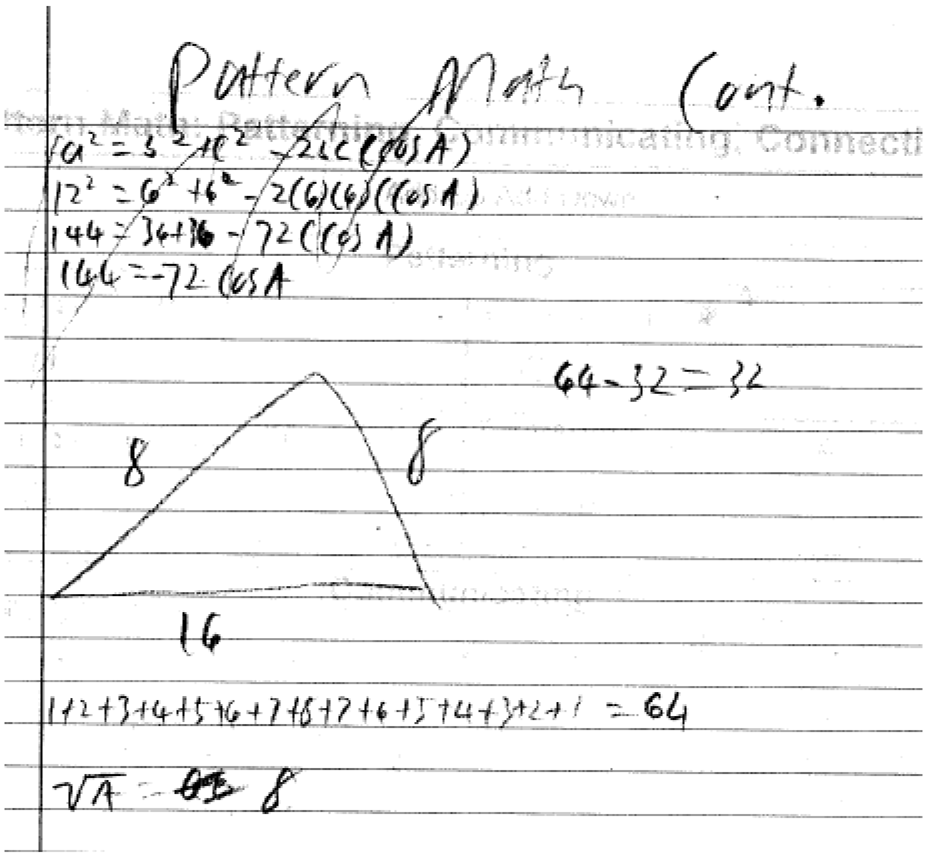


Figure 22: Bryson's communication on "Add Up Add Down," continued

My response to Bryson is below:

Bryson,

Cool ideas with the triangles! Your ideas gave me a couple of ideas that I wanted to try as well. I will show you them here at the bottom. When I was looking at your graph which made a triangle you mentioned that the base was twice the length of the sides. You have to be careful when you are counting the length of the sides since it is on a diagonal, the length from one corner to another corner is actually more than one (it is the square root of 2). But if you look at the height of your triangle, the height is actually half of the base. For example, look at your sum up to 6 then back down that you showed on the back of your page. Notice that the base is 12 and the height is 6. What is crazy is that if you use the area formula what do you get??? 36!! I wanted to do something with areas of triangles, but my ideas were more like your ideas on the first page. A first idea I had to count the areas was to draw a triangle and count the area. The height in figure one is 3 and the base is 5, but base times height divided by two doesn't actually give the correct number because it looks like there is more area above than below. The area of the triangle is 7.5 while the actual area is 9. So then I wondered how I could make the area of the triangles cut off on top equal the amount missed underneath. In figure 2, I decided to draw lines going through the midpoints of the sides. Can you see that in figure 2 that all the little triangles above the line could fit into little spaces under the line? This would mean that the area of the triangle would be exactly the same as the sum. The bottom of the triangle is now 1 more than the number of numbers you add and the height is still the bigger number. So this triangle has a base of 6 and height of 3 making an area of 9. I also noticed that if you put points on the top of each square where line crosses you get the same diagram you made. Cool.

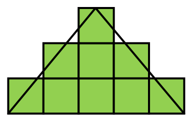


Figure 1

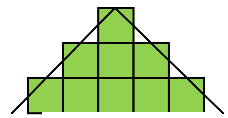
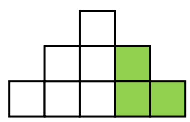


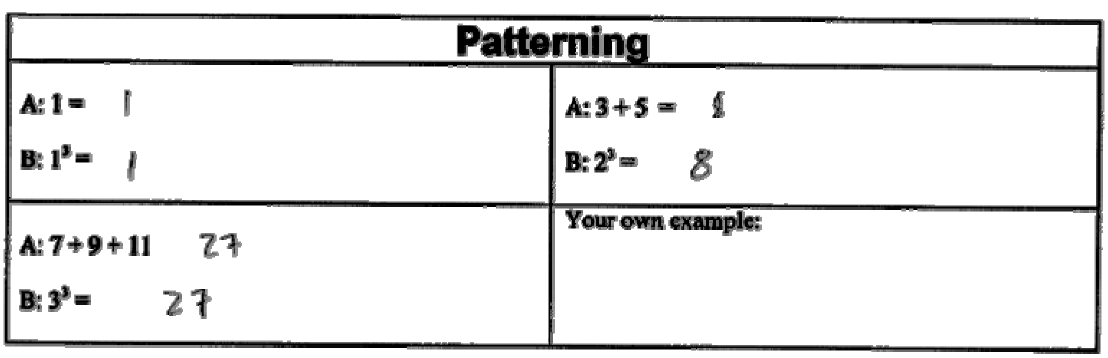
Figure 2

I also thought about what would happen if you move part of the diagram over to make a square. Can you see if you moved the smaller shaded region and rotated it you could make a 3 by 3 square?



Thanks for all your effort on this one. I hope you learned something that you find valuable. Feel free to respond to anything that I have written here.

Cycle 2: Pattern 3: Adding Up Odds

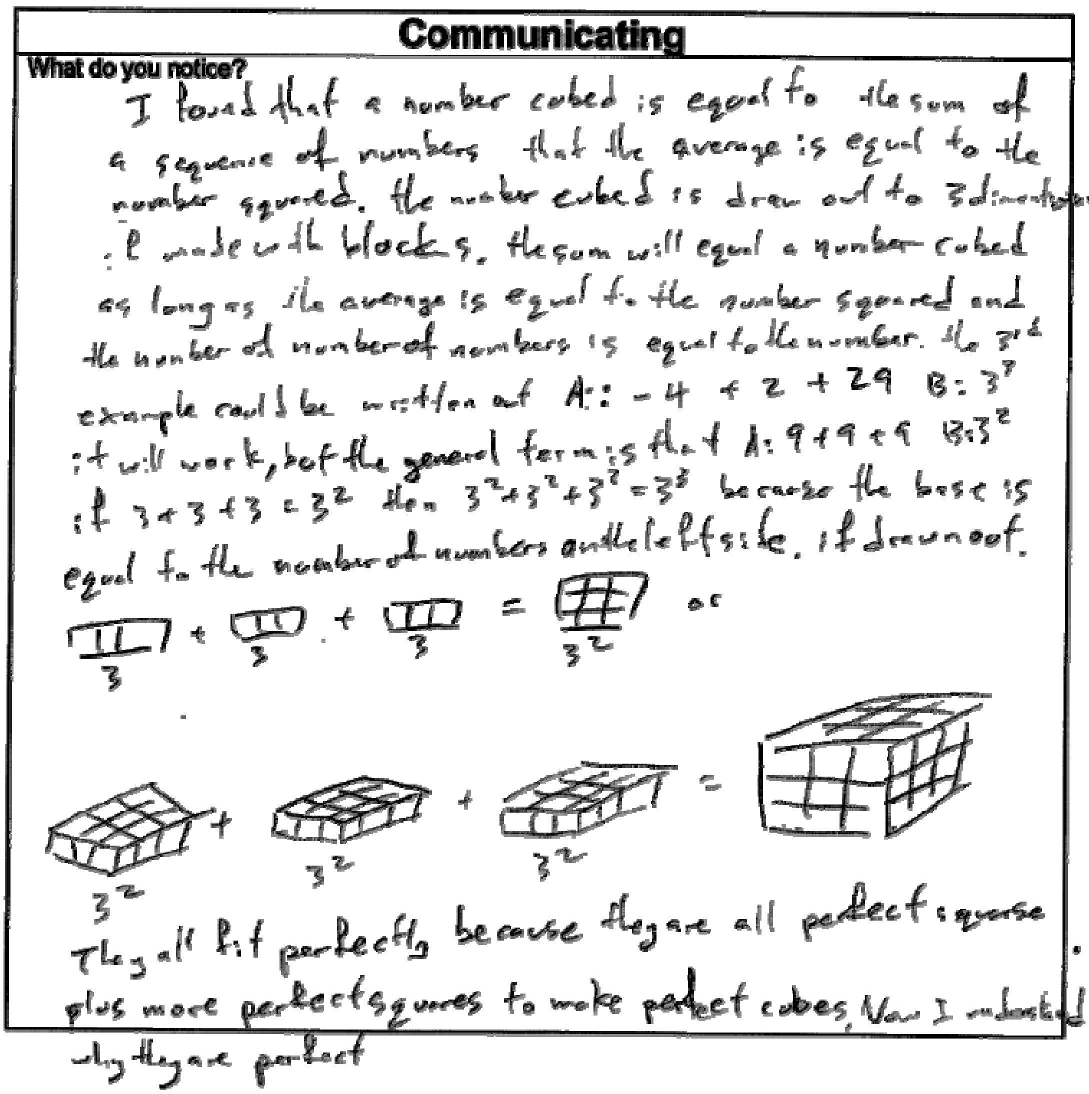


Patterning

A: 1 = 1 B: 1^3 = 1	A: 3 + 5 = 8 B: 2^3 = 8
A: 7 + 9 + 11 27 B: 3^3 = 27	Your own example:

Figure 23: Ivan's pattern work on "Adding Up Odds"

The following is Ivan's work. Ivan quickly described the pattern that worked and then immediately went into an explanation of why it worked in most of his Pattern Math examples. His explanations were clear, but often only used algebra as a way to describe what he saw. The following example is one of Ivan's first attempts to use diagrams in addition to algebra to explain what he sees. Ivan's explanation shows that he understands why it is important that the average of the three terms must be 32. Ivan's work seems to show that he has a new appreciation for the perfect squares and perfect cubes when he says, "Now I understand why they are perfect."


Communicating

What do you notice?

I found that a number cubed is equal to the sum of a sequence of numbers that the average is equal to the number squared. the number cubed is drawn out to 3 dimentions & made with blocks. the sum will equal a number cubed as long as the average is equal to the number squared and the number of number of numbers is equal to the number. the 3rd example could be written as A: $-4+2+29$ B: 3^3 it will work, but the general term is that A: $9+9+9$ B: $3 \cdot 3^2$ if $3+3+3=3^2$ then $3^2+3^2+3^2=3^3$ because the base is equal to the number of numbers and the left side. if drawn out.

They all fit perfectly because they are all perfect squares plus more perfect squares to make perfect cubes. Now I understand why they are perfect

Figure 24: Ivan's communication on "Adding Up Odds"

Here is my response to Ivan.

Ivan,

I like the explanation with the diagrams. I tried in my head to think about how I would draw the next one. How would you draw $3^3 + 3^3 + 3^3 = 3^4$? I thought about it for a while until my head hurt and then I stopped trying to think about it. It would make a hypercube but I can't quite imagine that very well. The problem with thinking about it is adding the other dimension. I notice that your first examples were two dimensional which make sense since you were using areas. Then your nine from the first example changed into volumes when you add the 9 cubes three times to get the 3 by 3 by 3 cube. I just can't imagine adding another dimension very well.

I really like how you are trying more ways of describing things such as using diagrams and such. Your last sentence says that you now understand why they are perfect. Can you explain that a little bit more? What didn't you understand about it before that you have a better understanding of it now?

Feel free to respond to anything that I have written here.

Cycle 3

Cycle 3: Pattern 1: Ancient Chinese Mathematics

The third cycle of mathematics looked at different techniques to multiply numbers together. It began with an ancient Chinese method where lines were drawn and the intersection of those lines created a way to calculate the product. The work below is Dave's.

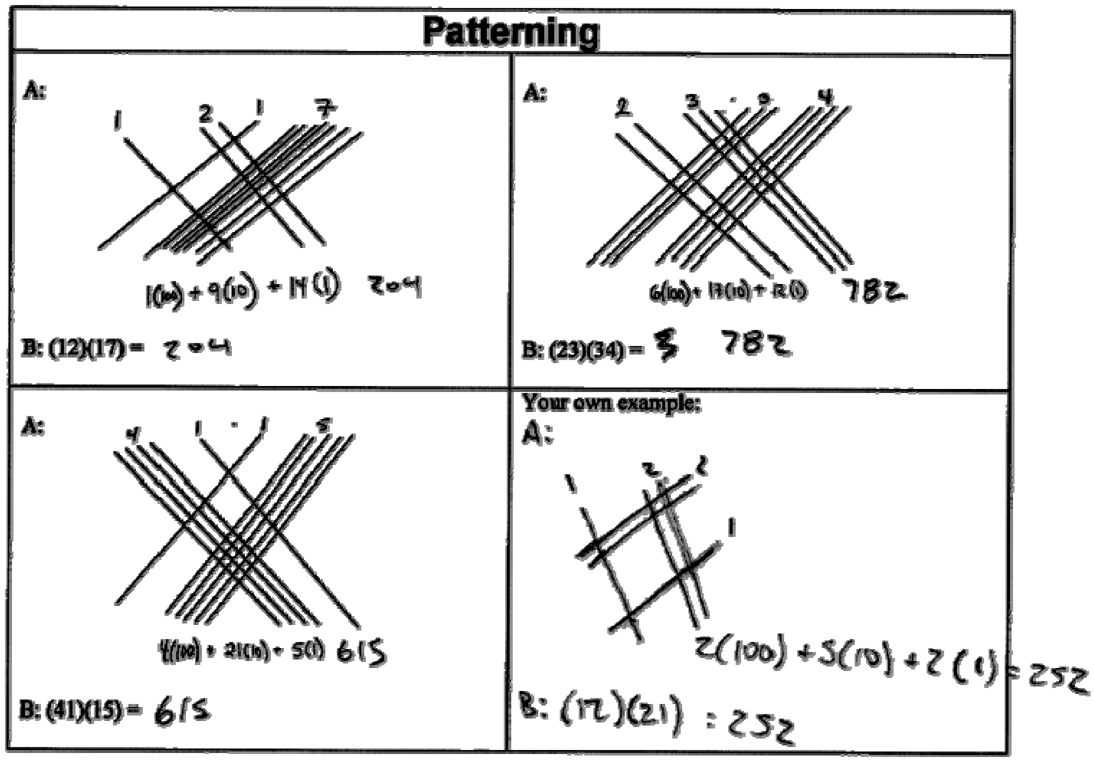


Figure 25: Dave's pattern work on "Ancient Chinese Mathematics"

Dave does a very good job at describing the pattern he sees. A theme that has emerged from Dave's work is that he did not explain why things work and he was usually quite concise in his description of patterns as well. In this Pattern Math, he provided more description than he usually did, but again you will notice that he does not attempt to explain why the pattern works. His description however is excellent and he creates new terminology (such as using the "diamond" that appears) to help describe what he notices.

Communicating

What do you notice?

For each number you draw lines. The first number you split up into the first digit and second digit. The first digit you draw a diagonal line, then for the second digit you draw a diagonal line above the first line and it has to be parralel to the first line. Then you do the same thing for the second number except the opposite. The first digit will be a diagonal line through the second digit of the first number and the second digit of the second number will go through the first digit of the first number. Whatever the digit is, that is how many lines there will be.

Then look at the drawings and try to see a diamond. First count the intersections on the left side of the diamond than multiply that number by 100. Next count the intersections at the top and bottom and multiply that number by 10. Than count the intersections on the right of the diamond than multiply that by one. To get your answer add up all the numbers.

Figure 26: Dave's communication on "Ancient Chinese Mathematics"

Dave was not alone in having difficulty in both explaining why something works or even attempting to explain why a pattern works. Most of the Pattern Maths were designed in a way that finding and describing the pattern was easy to see, but the actual explanation took some work. In my response to Dave, you will notice that I included a comment I made to Chase as well. This was a comment that I included in my response to most of the students who were having difficulty explaining why things worked. I was trying to provide students some scaffolding so that they could attempt to explain something that at first they felt they were unable to explain.

Dave,

Great description of what is happening. You correctly noticed which values get multiplied by 100, which by 10 and which ones by 1. One thing that I really liked in your description was the fact that you noticed the diamond shape in the middle and how you used the diamond to figure out which dots belong to which sets. Did you find it interesting that when you added up these numbers that it equaled the multiplication? Why do you think that works? Will it always work? After you finished describing what you see as the pattern, I want you to look at the pattern in more detail to try to see if you can explain why it works.

Here is something that I wrote to Chase to help him think about explaining why the pattern might work and I thought you might find it useful as well:

Sometimes the process of finding out why it works is the process of describing all of the little details and being observant while you describe the little details. For example, a little detail to look at is how we found the number of 100s that would be added. In the first example, there was only 1 100 because it was the intersection of the 1 from the 12 and the 1 from the 17. In the second example there were 6 100s because it was the intersection of the 2 from the 23 and the 3 from the 34. And in the third example there were 4 100s because it was the intersection of the 4 from the 41 and the 1 from the 15. Now, I just took the time to describe part of the pattern. After describing it I can now look back and try to figure out why this works. By writing it down, I can perhaps see patterns that I might miss if I didn't write it down as I don't have to mentally think about so many things at once. Now I can ask why the hundreds work. Perhaps you want to look over these numbers again and think about why they work. Try some ideas and if you still are stuck come and ask me some questions. Understanding why this pattern works may help you see why the next couple of Pattern Math ideas work.

Cycle 3: Pattern 2: Napier's Bones

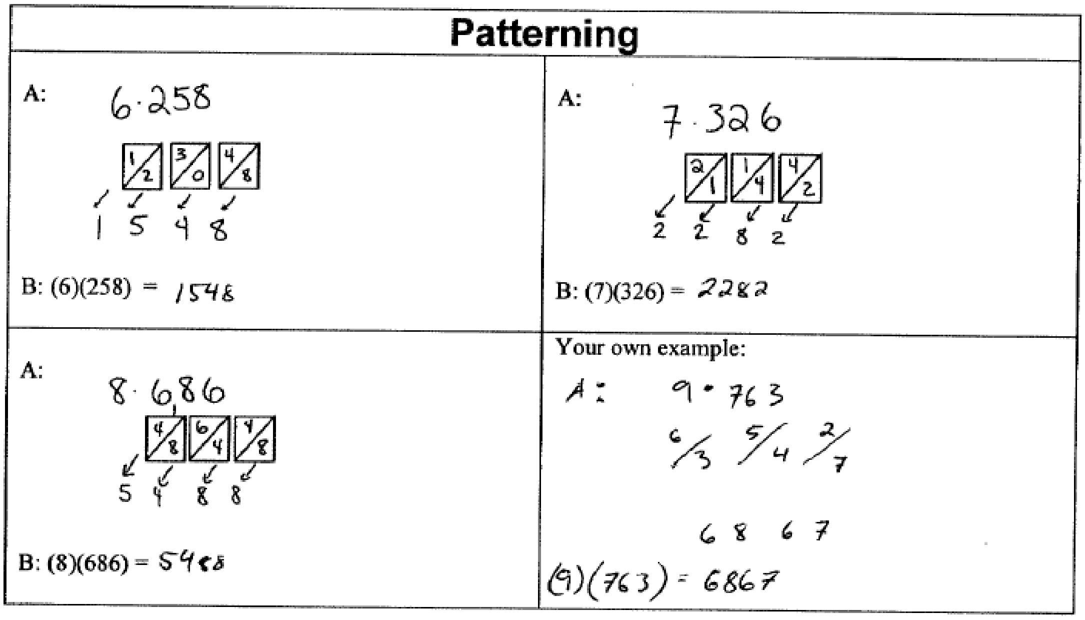


Figure 27: Nate's pattern work on "Napier's Bones"

The second Pattern Math also looked at a historical method for multiplying numbers, called Napier's Bones. Instead of drawing lines, students filled in boxes based on multiplying single digit numbers and then added along the diagonal to find the final answer. I included Nate's work below because my initial response to Nate was that he did not try hard enough to describe why the pattern worked, he just described what the pattern was. In my response to him I asked him to try to explain why it worked and to resubmit it. In my response, I often asked students questions and asked them to look at something again or to write back to me if they wanted to. Generally, students rarely responded directly to my questions. Often they responded to my comments by making some changes in their next Pattern Math or commenting in their next Pattern Math. You will see this quite a bit in the next chapter where I follow Zoe's work throughout the entire Pattern Math cycle. Below is Nate's first response and my response followed by his second response and my response.

Communicating

What do you notice?

In this pattern, when you add up the product of one number and each of the digits of another number, you get the same result as if you just multiply the 2 numbers together

Figure 28: Nate's initial communication on "Napier's Bones"

Nate,

Nate, I would like you to take the time and add more to this Pattern Math. I want to see you explain why it works. You described the pattern well, but you didn't take the time to explain why it works. Please resubmit it once you have attempted an explanation.

Thank-you

Also, I know that you responded to my questions on the last Pattern Math. Could you hand that in as well. Thanks.

Communicating

What do you notice?

In this pattern, when you add up the product of one number and each of the digits of another number, you get the same result as if you just multiply the 2 numbers together.

This works because the single digit number is multiplying the second number, just in steps, and each of the triangles represents a multiple of 10.

An advantage of this pattern is that the largest number that you need to multiply is 9×9, which is quite simple. Also, the addition required is easy, making the math required for this pattern a lot easier than multiplying the 2 numbers at once.

Figure 29: Nate's second communication on "Napier's Bones"

This pattern can be seen as a simpler version of the charts multiplying just set up differently. The methods are basically the same

Comparison

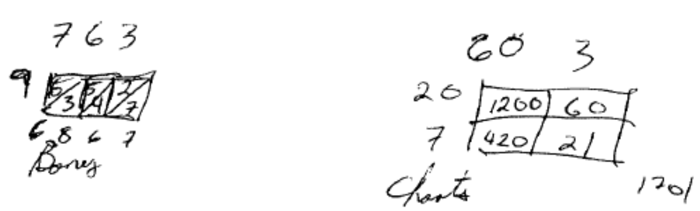


Figure 30: Nate's second communication on "Napier's Bones," continued.

Nate,

Thanks for taking the time to explain your ideas more fully. I wanted to see that you understood the pattern in the same way as you understood the last pattern. I also liked that you took the time to compare this method to other methods and to describe its advantages. You mentioned that this is a simpler version of the charts method. I would agree that they are very similar. What makes this method simpler?

Cycle 3: Pattern 3: Charts

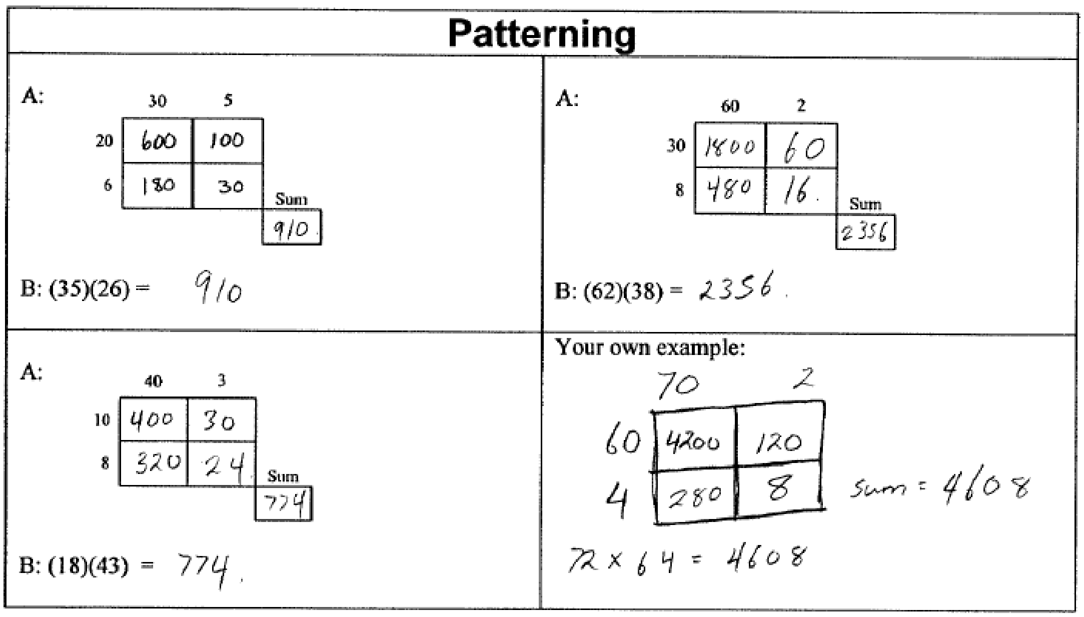


Figure 31: Sun's pattern work on "Charts"

The third Pattern Math in the cycle used a chart to multiply numbers and add them together. Many students had made connections in the previous cycles that place value was an important reason why both the "Ancient Chinese Mathematics" and "Napier's Bones" worked. I wanted students to think beyond place value in the "Charts'" method so I provided them with an additional example which can be seen at the beginning of Sun's communication section. I then challenged students to make connections and explain why the "Charts'" method works even when you don't split numbers up with place values. Sun's work shows how she worked with the idea of splitting the numbers up with different values. She tried a couple of more examples to make sure that it worked in more than one case. She then made the connection to seeing multiplication as areas, something that was discussed in the first cycle of Pattern Math. She even made connections to percentages as well.

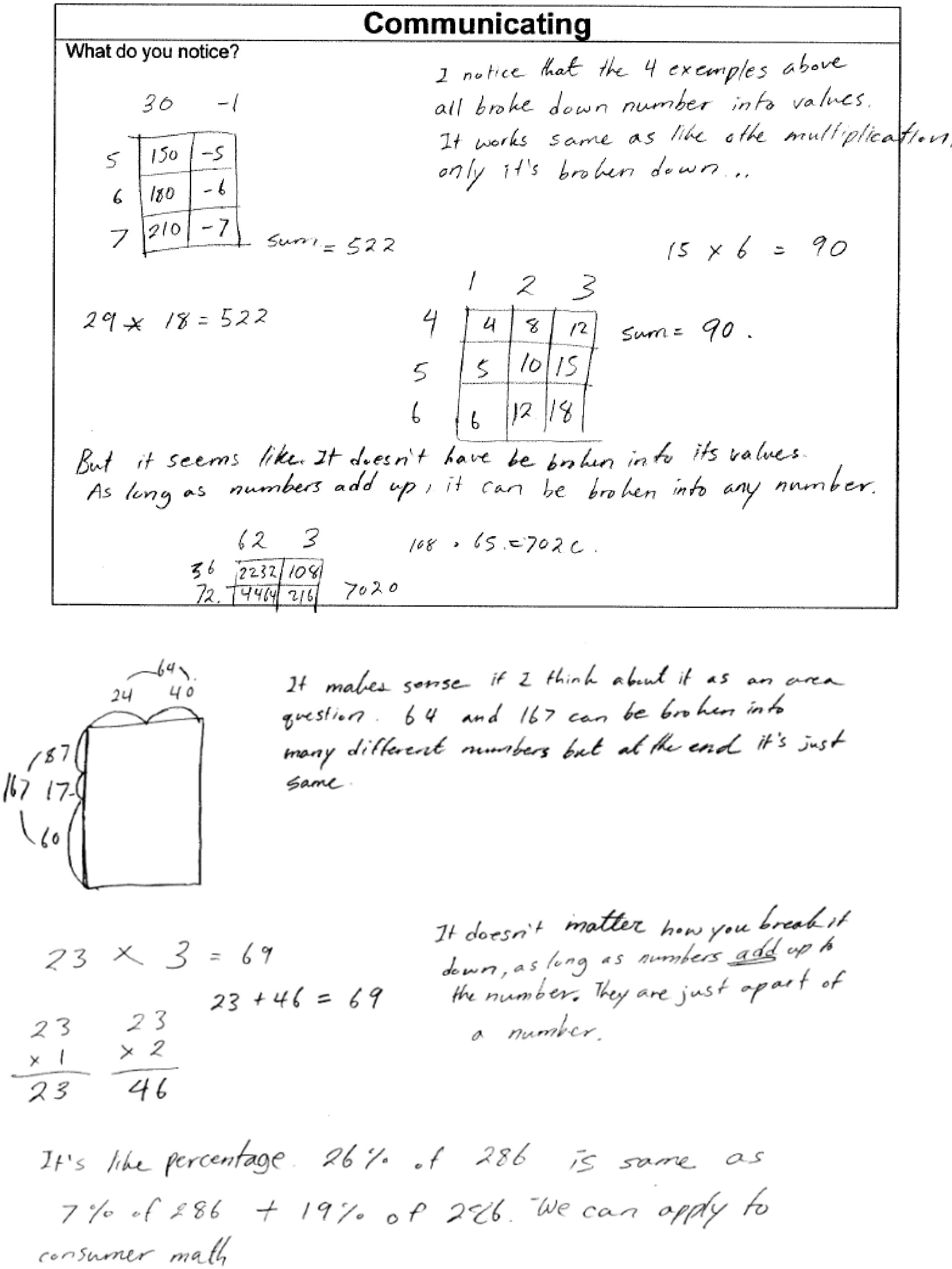


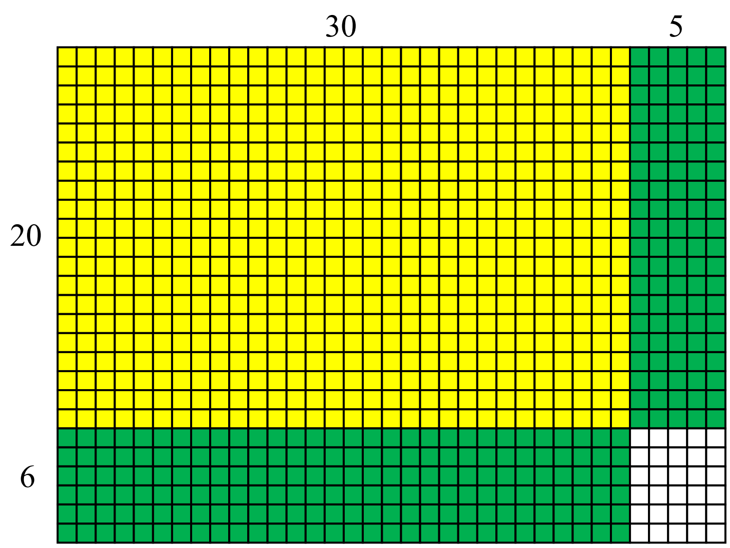
Figure 32: Sun's communication on "Charts"

Here is my response to Sun.

Sun,

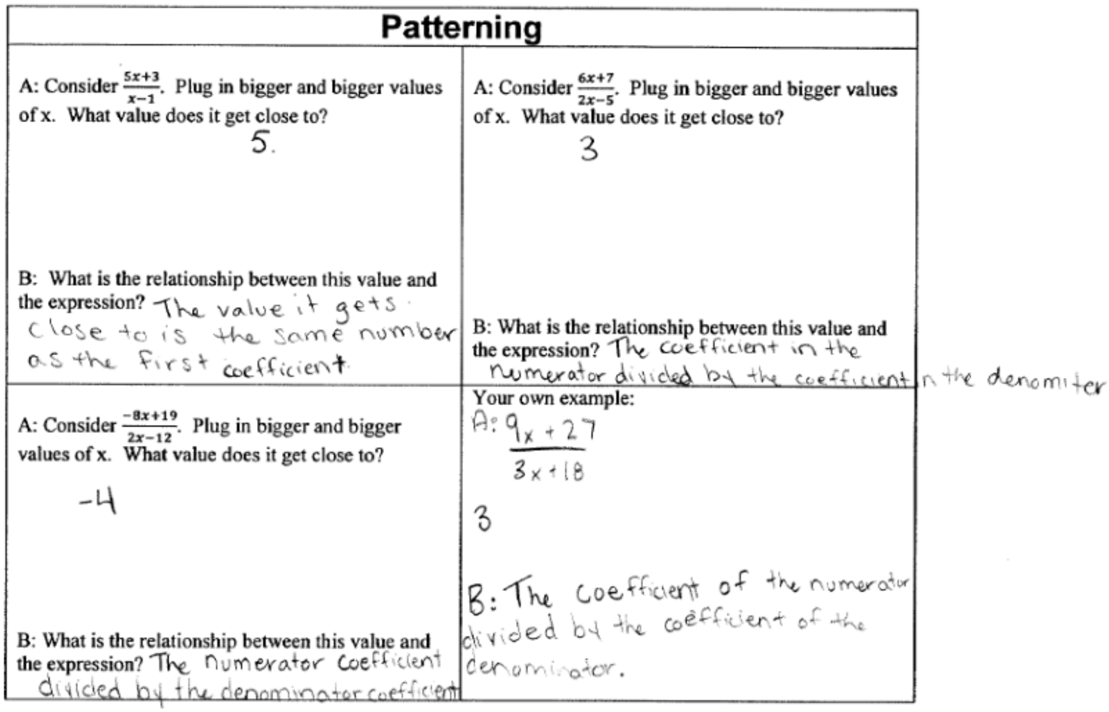
I really liked how you showed that it works with percentages as well. This can help explain why 5% GST plus 7% PST is the same as taking 12% tax once. In some provinces they are changing the taxes so that there will only be one tax and they are going to call the one tax a Harmonized Tax.

I also liked that you thought about areas when you went to explain it. I thought of this as well. The following is my own work at trying to explain the chart method with areas. See if it makes sense to you. It uses area to explain why it works. Take our first example, 35 by 26 and make a rectangle with those dimensions. You can see in my example below that I can break the area up into smaller rectangles and the number of boxes will be the same for the total. Can you see how the number of boxes in each of the different shaded areas is the same as the chart method? Can you see why it wouldn't matter what numbers I use to show the rectangles as long as the length added to 35 and the width added to 26?



Cycle 3: Pattern 4: Horizontal Asymptotes

The last pattern in this cycle did not match with the theme of the other three. In a sense it could have been in a cycle on its own. I wanted to see if I could create a Pattern Math that would help introduce a topic that we were currently learning. We were about to learn how to graph different rational functions and I wanted students to be able to understand how to find the horizontal asymptotes in the graphs.



Patterning

A: Consider $\frac{5x+3}{x-1}$. Plug in bigger and bigger values of x. What value does it get close to?
5.

B: What is the relationship between this value and the expression? The value it gets close to is the same number as the first coefficient

A: Consider $\frac{6x+7}{2x-5}$. Plug in bigger and bigger values of x. What value does it get close to?
3

B: What is the relationship between this value and the expression? The coefficient in the numerator divided by the coefficient in the denomiter

A: Consider $\frac{-8x+19}{2x-12}$. Plug in bigger and bigger values of x. What value does it get close to?
-4

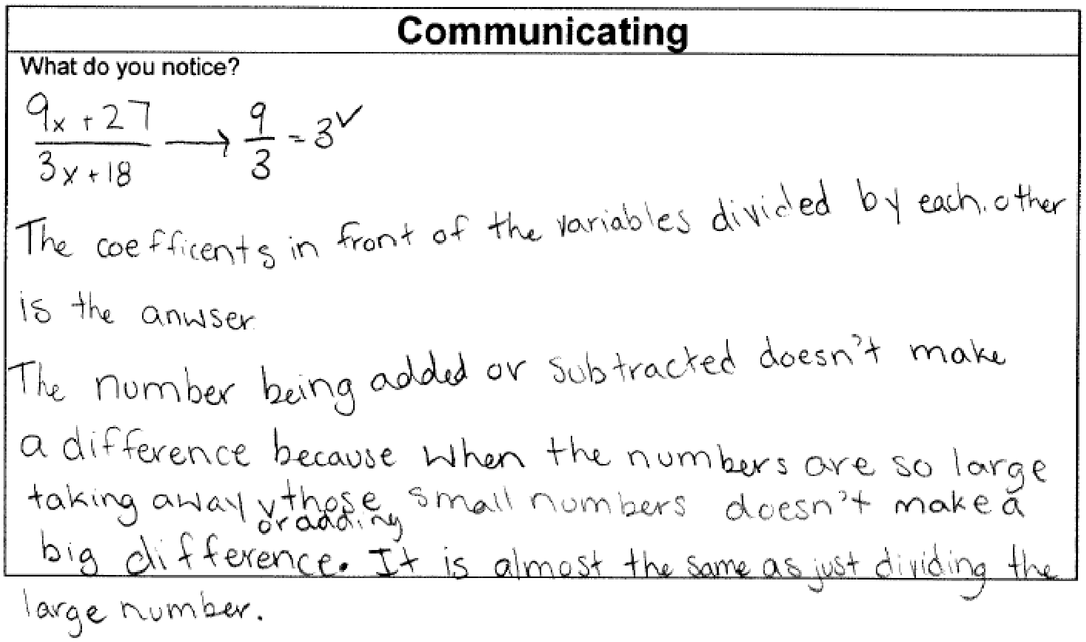
B: What is the relationship between this value and the expression? The numerator coefficient divided by the denominator coefficient

Your own example:
A: $\frac{9x+27}{3x+18}$
3

B: The coefficient of the numerator divided by the coefficient of the denominator.

Figure 33: Carly's pattern work on "Horizontal Asymptotes"

The Pattern Math asked students to consider a rational expression and to calculate the value of the rational expression for larger and larger values of x. In Carly's work you can see how she refined what she noticed in the first box to what she described in the final box. She also did a great job using proper terminology in her description. Her work below also describes why the values that are added or subtracted do not factor into the answer when large numbers are used for x.


Communicating

What do you notice?

$\frac{9x+27}{3x+18} \longrightarrow \frac{9}{3} = 3$ ✓

The coefficents in front of the variables divided by each other is the anwser

The number being added or subtracted doesn't make a difference because when the numbers are so large taking away these or adding small numbers doesn't make a big difference. It is almost the same as just dividing the large number.

Figure 34: Carly's communication on "Horizontal Asymptotes"

I did not respond directly to students on this last Pattern Math as we used a class discussion to look at the pattern and relate it to how horizontal asymptotes are apparent in rational graphs.

The purpose of this chapter was to introduce the reader to both the activity and the students involved in the activity. Through the examples given, 12 of the 14 students who participated in this study were introduced. In the sixth chapter you will get to meet Zoe as I share her journey through the Pattern Math activities. In the analysis of the data, you will reconnect with some of these students and see data from Jeanette as well, the only student whom has yet to be mentioned through the data.

Chapter 5: Mathematical Discoveries

Throughout the Pattern Math activities there were many patterns that were explored and explained. These patterns in turn led to other discoveries, like Sun's discovery that consecutive numbers always add to an odd number. This discovery led her to make a connection between the first and second Pattern Math. She showed that $1 + 2 + 3 + 4 + 3 + 2 + 1 = 1 + 1 + 2 + 2 + 3 + 3 + 4 = 1 + 3 + 5 + 7$. Each discovery that students made allowed them to take ownership of their learning and create an environment where they could think mathematically. In this chapter, I explore two mathematical discoveries that were made during the Pattern Math activities. The first, the "Sum of Cubes", is an example of the power of many minds coming together for a mathematical discovery. The second, "Bryson's Bones", is an example of extending an idea successfully to create a new discovery.

Part 1: Sum of Cubes

In the Pattern Math activities the teacher has the benefit to read and respond to all students who participated. It is in reading these responses that the teacher can see one pattern from many different perspectives. This was the case during the second cycle, where I was able to read the responses of many different students and see their interpretations of the pattern. Often students would come up with patterns or ideas that I had not seen before. Often their ideas led me to make new connections to the material as well. A number of these responses led me to new ways of looking for a formula for the sum of cubes.

The last Pattern Math of the second cycle was "Adding Up Odds". After reading all the responses together I decided to share my own communication with "Adding Up Odds" in an effort to show students some of my thinking. I wanted to credit the students because it was the ideas of Hana, Nate and Ivan that led me to look at this pattern in a different way and to explain the pattern in new way. Before presenting the work that I shared with students, it is important to highlight and describe fully some of the patterns that the students saw that led me to make my connections.

Nate: Average is the number squared

In Nate's work on "Adding Up Odds" he made the discovery that the average of the sequence with 4 odd numbers is 4^2, the average of the sequence with 5 odd numbers is 5^2 and so on. Since there are 4 numbers with an average of 4^2, it is easy to see that the sum would be 4^3. The following is Nate's work and my response to Nate showing how his work made me make some connections I hadn't seen before.

By ~~squaring~~ cubing a number, you get the same result as if you add up that many consecutive odd numbers, with the average of them being the original number squared.

one number

$1 = 1$ ← one squared = one

$1^3 = 1$

one

two numbers

average $(4) = 2^2$

$3+5 = 8$

$2^3 = 8$

two

average $= x^2$

x number of numbers

x

x^3

This works because A is just the number squared, times the original number. However the squared numbers are ~~arranged~~ changed, but if the average is still the original number squared, it works

Figure 35: Nate's communication on "Adding Up Odds"

Nate,

Your comments helped me think about how I could create this pattern backwards. Let's say I wanted to show 7^3. Well, 7^2 is 49, so if I wanted 7 odd numbers whose average was 49 I would need 43 + 45 + 47 + 49 + 51 + 53 + 55. If you did an even number like 12^3, then $12^2 = 144$, so you have to have 6 numbers on one side of 144 and 6 on the other giving you 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 151 + 153 + 155 + 157. Your comments about average gave me a place to start to think about how I could start if I didn't have the previous ones. Since I have 12 numbers whose average is 144 it is the same as 12 times 144 which is the same as 12^3.

Ivan: Triangle numbers.

In "Add Up Add Down", Ivan noticed the triangle numbers. He separated the sum of the sequence into two parts. Adding up, he noticed one triangle number and adding down he noticed the previous triangle number. His worked showed that when you add any two consecutive triangle numbers that you get a perfect square.

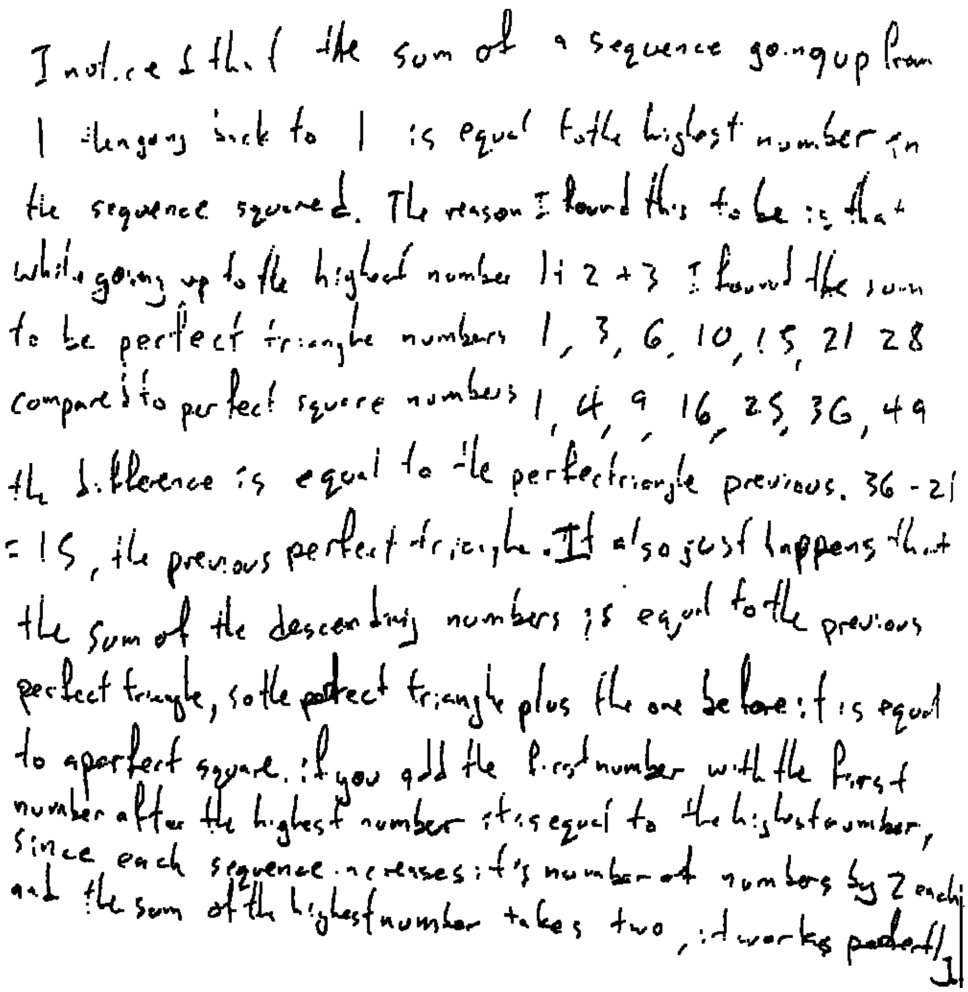

I noticed that the sum of a sequence going up from
1 then going back to 1 is equal to the highest number in
the sequence squared. The reason I found this to be is that
while going up to the highest number 1 + 2 + 3 I found the sum
to be perfect triangle numbers 1, 3, 6, 10, 15, 21 28
compared to perfect square numbers 1, 4, 9, 16, 25, 36, 49
the difference is equal to the perfect triangle previous. 36 - 21
= 15, the previous perfect triangle. It also just happens that
the sum of the descending numbers is equal to the previous
perfect triangle, so the perfect triangle plus the one before it is equal
to a perfect square. If you add the first number with the first
number after the highest number it is equal to the highest number,
since each sequence increases it's number of numbers by 2 each
and the sum of the highest number takes two, it works perfectly.

Figure 36: Ivan's communication on "Add Up Add Down"

Hana: Every Cube is equal to a Difference of Squares

Hana made an amazing discovery by connecting the work in the first Pattern Math, "Adding Odds", to the third Pattern Math, "Adding Up Odds". In "Adding Odds", adding up any sequence of consecutive odd numbers will result in a number squared. In "Adding Up Odds", adding up different sequences of odd numbers will result in a number cubed. Hana connected these two ideas to show that any cube can be represented by a difference of two squares. Interestingly, the two squares always happen to be triangle numbers. Here is part of Hana's work.

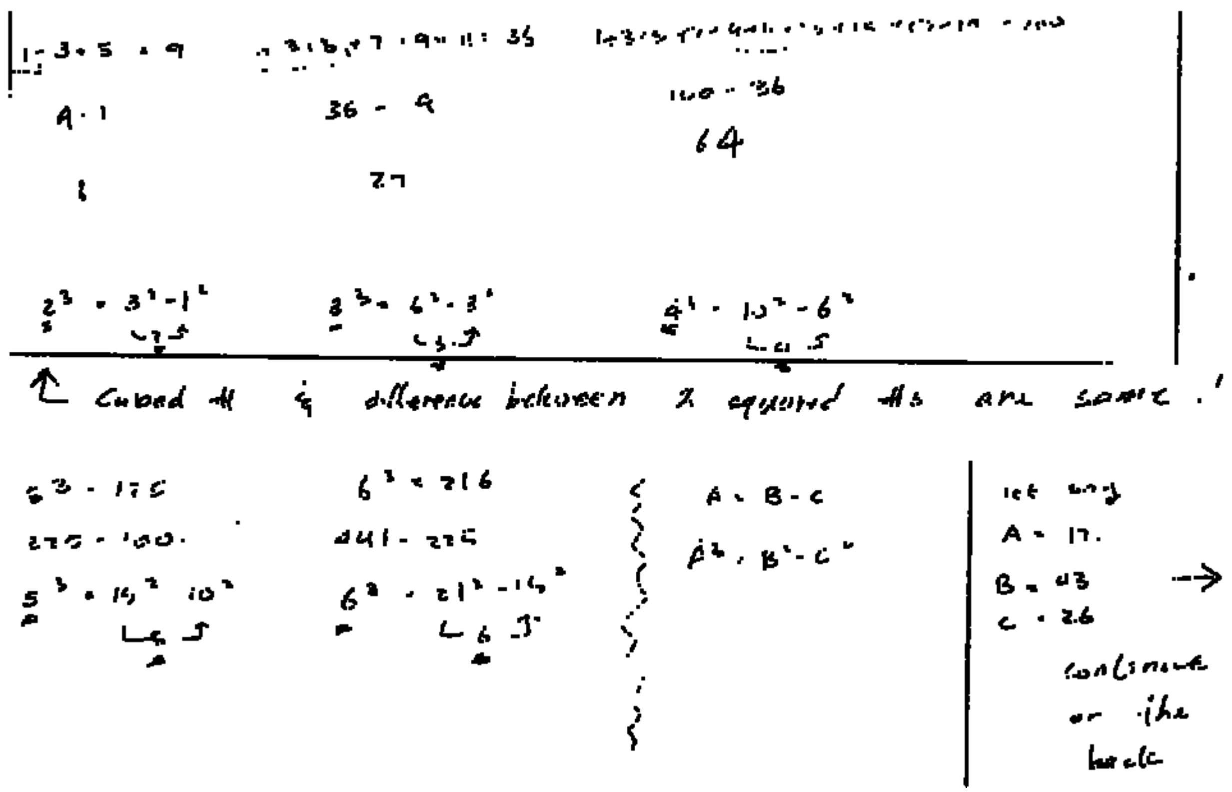


Figure 37: Hana's communication on "Adding Up Odds"

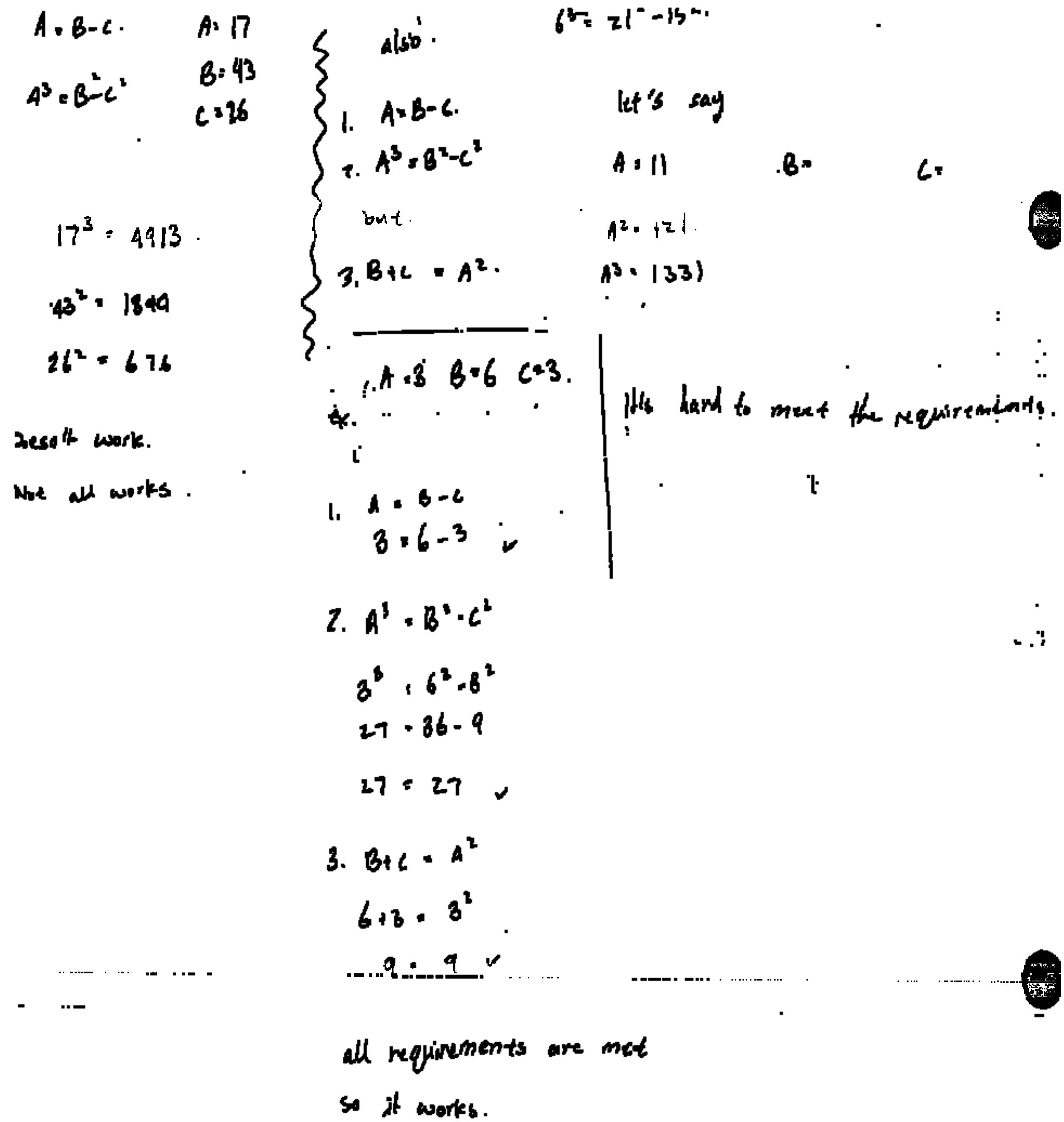


Figure 38: Hana's communication on "Adding Up Odds," continued.

My response to Hana was intended to commend her on her discovery and to help her see how she could use the math that she knows to meet all her requirements she mentioned above.

Hana,

I really liked the two squares minus each other equals the cubes. The first time I read it I thought, wow, this is amazing, how did she come up with this! You took the idea from the last one where we added up odds and noticed that you get a number squared and then subtracted the other number squared. Very cool and very neat! I also liked how you generalized it on the back to describe when it works and when it doesn't work. Now you mentioned on the back that 17, 43 and 26 don't work. I think that is because although 43 – 26 = 17, 43 + 26 doesn't equal 17^2 = 289. But if you had used 153 and 136 it does work. Very cool. You should share this with the people who sit next to you because it is very neat.

Do you see how your requirements in part 1 and part 3 are related to your requirements in part 2? Think difference of squares!

The following is my work that I gave to the students. I think I spent too much time coming up with a formal proof so that the students could see what that looked like. I also included all of my thoughts and math even though I made mistakes while I was figuring out things as well. I wanted students to see that when you are exploring mathematical concepts that it is okay if you make mistakes as long as you use those mistakes to keep thinking and exploring the idea.

Pattern Math: Patterning, Communicating, Connecting

Adding Up Odds

Patterning	
A: 1 = B: 1^3 =	A: 3 + 5 = B: 2^3 =
A: 7 + 9 + 11 B: 3^3 =	Your own example:

Communicating
What do you notice? Okay, so I will let you know some of the patterns that I see and I will try to prove the patterns as well. But the main reason that I have decided to look at this pattern more in depth is because there are a lot of formulas for adding up a bunch of numbers. For example, if you wanted to add 1 + 2 + 3 + ... + n it would equal $\frac{n(n+1)}{2}$. Similarly, there is a formula for $1^2 + 2^2 + 3^2 + ... + n^2$ which equals $\frac{n(n+1)(2n+1)}{6}$ (This formula I always

have to look up). So I want to make a formula using the math that you guys know and this pattern for $1^3 + 2^3 + 3^3 + ... + n^3$. It seems that this would be possible since the sum of n cubes is just the sum of a bunch of odd numbers. All I need to do is find out how to represent the last odd number in terms of the number being cubed.

But first, I will attempt to prove why the pattern that you see works. The first thing to notice is that the number of odd numbers being added is the base of the number cubed. Also, the average of the odd numbers is always the base squared. For example, with 13 + 15 + 17 + 19 there are 4 numbers, the sum is 4^3 and the average is 4^2. So if the average is x^2 and there are x numbers then it makes sense that the sum would be x^3. This is something that Nate and Ivan showed in their work and I think that it was great. Hana did some amazing work with showing how two cubes is the same as subtracting two squares. If you have the time, ask them to explain to you what they did. What I am about to do is to show you some formal mathematics that I wouldn't expect you to be able to do. I think that you would be able to understand it though once you see it and I think if you had the time you could do it as well. I will try to communicate clearly. Sometimes typing makes it more difficult because it is harder to show math, but I will try my best.

So now I want to think about the first number in the odd number sequence with n terms and the last number. If I can find a formula for the first number in the sequence, then the last number will be easy to find since you just add 2 (n – 1 times) to the first number. So the first number ... For 2 it is 3, for 3 it is 7 and for 4 it is 13. The pattern that I see is 4*3 + 1 = 13 and 3*2 + 1 = 7 so the pattern that I see is $n(n-1) + 1$. So that is the same as $n^2 - n + 1$. This would mean that the last number is $n^2 - n + 1 + 2(n-1) = n^2 + n - 1 = n(n+1) - 1$. Now does that always work??

For this I think I will try a mathematical proof idea called mathematical induction. The way it works is first you show that the formula you have works for the first couple of examples. I already know that $n^2 - n + 1$ works to represent the first odd number for 1, 2, 3, and 4 odd numbers!! Then you assume that it works for some arbitrary number, say k, and show that if that is true then it works for the next number, k + 1. What this means is that if it works for 4, you proved it works for 5, but then if it works for 5, it works for 6 and so on and so on and so on forever so then it has to work for every number. This is what it means to prove something by mathematical induction.

So we have shown it works for 1, 2, 3, and 4. Now I assume that it works for a number k. That is, the first number in the sequence of odd numbers is $k^2 - k + 1$, there are k numbers in the sequence and the last number in the sequence is $k^2 + k - 1$. Now we want to show that if this is true, we can prove that it works when we have k + 1 terms in the sequence. So we want to show that with k + 1 terms in the sequence, the first term is $(k+1)^2 - (k+1) + 1$ and that the last term is $(k+1)^2 + (k+1) - 1$.
The first term of the sequence is 2 more than the last term of the previous sequence. So the first term when there are k + 1 terms will be

$= (k^2 + k - 1) + 2$

$= k^2 + k + 1$ (now I will add and subtract the same values to get $(k+1)^2$)

$= k^2 + k + 1 + k - k$
$= k^2 + 2k + 1 - k$
$= (k + 1)^2 - k$ (now I will add and subtract the same values to get $-(k + 1)$)
$= (k + 1)^2 - k - 1 + 1$
$= (k + 1)^2 - (k + 1) + 1$ (This is what I wanted to show!!)

Now I want to show that the last term will be $(k + 1)^2 + (k + 1) - 1$ and I can do this by taking the first term in the sequence and add 2, k times.

$= (k + 1)^2 - (k + 1) + 1 + 2k$
$= (k + 1)^2 - k - 1 + 1 + 2k$
$= (k + 1)^2 + k$ (Now I add and subtract 1 to get what I want)
$= (k + 1)^2 + k + 1 - 1$
$= (k + 1)^2 + (k + 1) - 1$ (This is what I wanted to show!!)

So I showed that if I assumed that it worked for k terms, that it will also work for k + 1 terms. This means that it will always work. In other words, this proves that if it works for 4 terms then it works for 5 and then for 6 and then 7 and so on and so on forever. Okay, so now I will show why if you start with k consecutive odd numbers starting with $k^2 - k + 1$ and ending with $k^2 + k - 1$ that the sum will equal k^3. Perhaps it is easiest to see by using the average. And since the numbers are distributed evenly (they are all two apart) the average will be the same as the average of the first and the last. This is true if you have an even or an odd number of consecutive odds. The average of the first and last will be:

$$= \frac{(k^2-k-1)+(k^2+k-1)}{2}$$

$$= \frac{2k^2}{2}$$

$$= k^2$$

So, since there are k numbers and the average is k^2, the sum will be k^3.

Okay, so all of the work before was just a formal way to prove something that seems to always work.

Now back to what I originally was interested in finding. A formula for the sum of up to n cubes! What I am going to do is use the idea that the sum of n cubes is equal to:

$= 1 + 3 + 5 + ... + (n^2 + n - 1)$ (since $n^2 + n - 1$ was proven to be the last number)

Now, this is an arithmetic sequence which you have already learned that the sum of an arithmetic sequence is the first term plus the last term divided by two times the number of terms.

This means that the sum of the first n cubes is:

$$= \frac{1+(n^2 + n - 1)}{2}$$

$=\frac{n^2+n}{2}$

In mathematical terms we would write

$$\sum_{i=1}^{n} i^3 = \frac{n^2+n}{2}$$

So I tried this and ... it didn't work. Part of me is glad because I thought the formula looked way too simple. Well, I should have written out the formula for the sum of an arithmetic sequence before I wrote this out. Oops! Sometimes we make mistakes and that is okay. That is why I always do little checks along the way just to make sure things still make sense. Instead of erasing what I did, I will just correct it here. I forgot to multiply by the number of terms. The formula for an arithmetic sequence is:

$$S_n = \frac{n(t_1 + t_n)}{2}$$

So how many terms are there?? Well the first cube had 1 term, the second cube had 2 terms, the third had 3 terms, up to the n^{th} which had n terms. So if I am adding up to the n^{th} cube the number of terms would be 1 + 2 + 3 + ... + n = $\frac{n(1+n)}{2}$.

Now I just need to add this to the work I did before:

$=\frac{1+(n^2+n-1)}{2}\cdot\frac{n(1+n)}{2}$

$=\frac{n^2+n}{2}\cdot\frac{n(1+n)}{2}$

$=\frac{n(n+1)}{2}\cdot\frac{n(n+1)}{2}$

$=\frac{n^2(n+1)^2}{4}$

In mathematical terms we would write

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Now this is a formula that I have seen before, but never been able to understand where it came from. I didn't know that the Pattern Maths I was giving you would show me how to find it.

Figure 39: My response to "Adding Up Odds"

Now after I wrote this and after the Pattern Math cycles were done, I continued to think about this pattern and continued to think about what my students had written. In particular, I was intrigued by the fact that Hana had discovered that every cube was just the difference of two triangle numbers. I used this fact to come up with the proof of the sum of cubes in an even easier way.

First, the formula for the triangle numbers is : $t_n = \frac{n(n+1)}{2}$

Second, subtracting the squares of two consecutive triangle numbers is equal to a number cubed. For example,

$$1^3 = 1^2 - 0^2$$

$$2^3 = 3^2 - 1^2$$

$$3^3 = 6^2 - 3^2$$

$$4^3 = 10^2 - 6^2$$

In terms of algebra, this can be proved as follows:

Subtract the squares of any two consecutive triangle numbers, $(t_n)^2$ and $(t_{n-1})^2$:

$$\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2$$

Simplifying, by expanding and factoring out a common factor gives: (Note: factoring it as a difference of squares will work as well – as learned by observations in the first cycle of Pattern Math)

$$= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2n^2}{4}$$

$$= \frac{n^2}{4}\left(\frac{(n+1)^2 - (n-1)^2}{1}\right)$$

$$= \frac{n^2}{4}\left(\frac{4n}{1}\right)$$

$$= n^3$$

With this in mind we can now approach the sum of the first n cubes as a telescoping sum:

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

$$\sum_{i=1}^{n} i^3 = (1^2 - 0^2) + (3^2 - 1^2) + (6^2 - 3^2) + \cdots + \left(\frac{n^2(n+1)^2}{4} - \frac{(n-1)^2 n^2}{4}\right)$$

Since all the sums will simplify to zero except for the last triangle number squared we have:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

That was fun!

Part 2: Bryson's Bones

I had been showing students "Napier's Bones" in class for many years. I had found it to be an excellent way to quickly do multiplications of any number by a single digit number. Although I had wondered for years if there would be a way to multiply two multi-digit numbers together, I had never been able to come up with a method I was happy with. During the Pattern Math, Bryson took time to explore multi-digit numbers and came up with an ingenious method to do it. I have since been proud to call this Bryson's method.

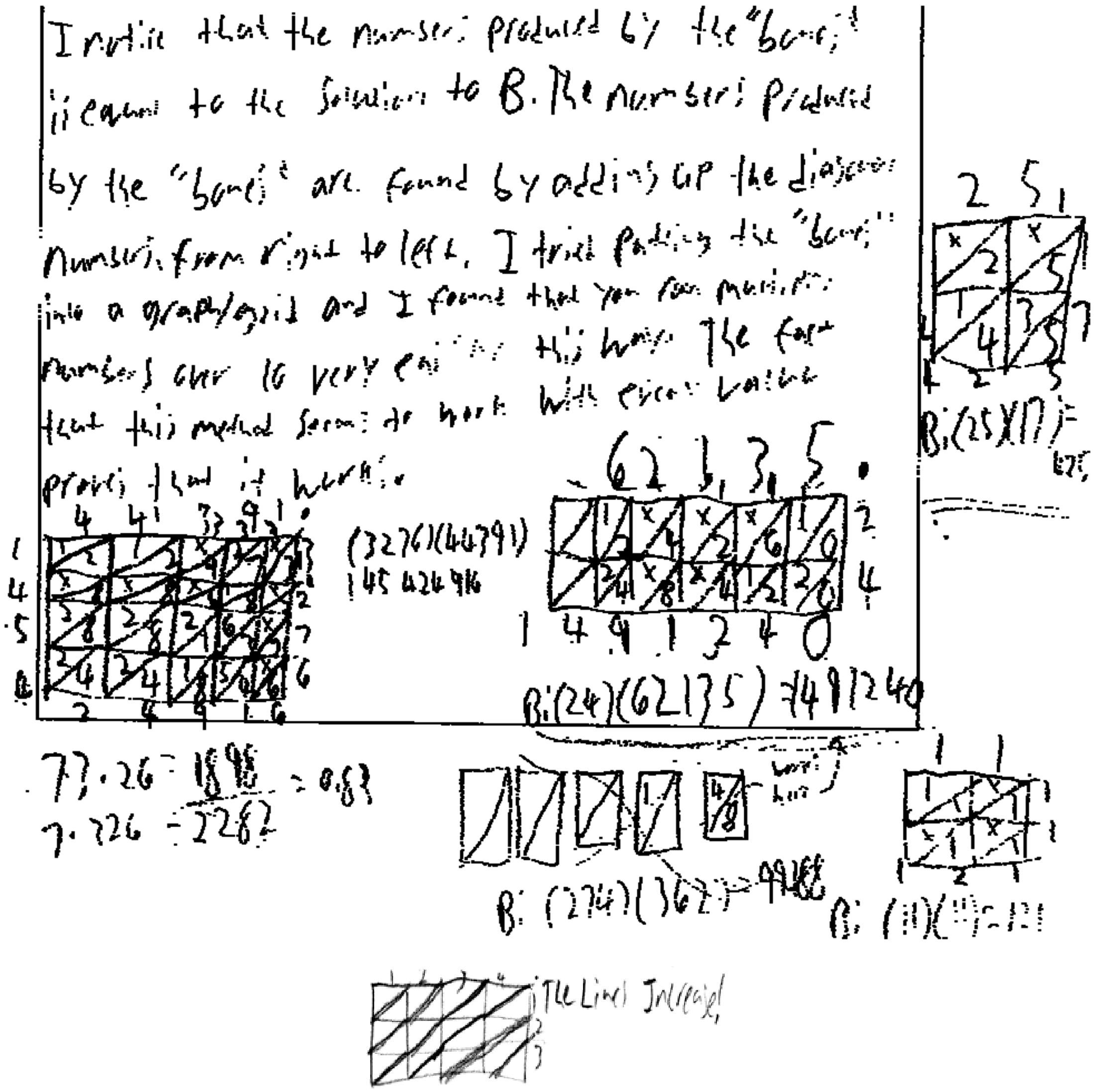
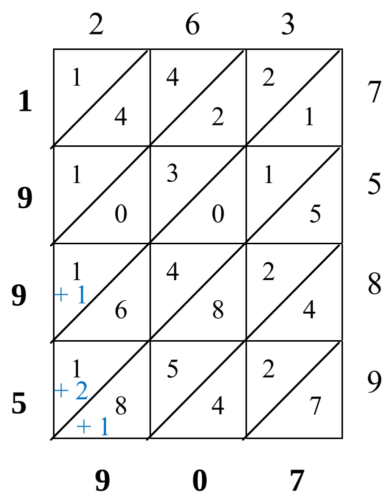


Figure 37: Bryson's communication on "Napier's Bones"

Part of what I really like about "Bryson's Bones" is that he puts the one number he is multiplying down the right side of the grid. I don't know if I would have ever thought of that because I have also used the Charts method so many times and this method would have influenced me to try to put the number down the left side. Putting the number on the right side allows the solution to wrap around the corner. Here is another example so that you can follow what Bryson did. I am going to multiply 263 by 7589. I write 263 across the top and 7589 down the right side. Multiply to fill in the chart and add along the diagonals. Any sum greater than 10 gets carried over to the next diagonal (indicated by the + 1 and +2 in the chart). The answer then appears wrapped around the box and I have made it bold below.



263 x 7589 = 1995907

Figure 38: Bryson's Bones to calculate 263 x 7589

The beautiful thing about the method is that you never have to multiply large numbers nor add large numbers. Here is my response to Bryson:

Bryson,

Wow Bryson!! Did you ever think that you could multiply 3276 by 44391 that fast? That was amazing. To tell you the truth, I had never found a way as good as yours for using Napier's bones to multiply when both numbers have more than 1 digit. I have many ways that I like to multiply and I always used another method instead. I think that I will start using your method as a preferred method. There are a couple of things that I really like. One, is how you put your numbers down the right side. That way when you add things up you can just go around the corner. Very cool! I will be telling others in class to come and look at yours after class because it is a really neat method. Keep up the good work!

When students are given the time to explore mathematics they will make great discoveries. Students will also feel pride and ownership of these discoveries. When teachers take the time to read and respond to students' explorations, they will also have the opportunity to make discoveries. This is what happened over the second cycle of the Pattern Math. Combining the discoveries of Nate, Ivan and Hana led me to make my own discoveries with the formula for the sum of cubes. Allowing students to explore when they are stuck trying to explain ideas led Bryson to a great method for multiplying two multi-digit numbers.

Chapter 6: Zoe's Story

The following is a look at one student's journey throughout the entire Pattern Math project. The purpose of this chapter is to give the reader a sense for the flow of the project; how one student engages with an activity and engages with the interactive writing. By exploring a single student in particular, one can see how the student grew and changed through the activity. I chose Zoe in particular because her work very clearly demonstrates a reaction to my responses in the interactive writing process and how she learned as a result of that process. Examining a single student in particular also allows a demonstration of the Pirie-Kieren model to be done within context. This should help the reader make sense of the analysis of the data in subsequent chapters.

When Zoe completed the survey that preceded the Pattern Math activities, she stated that the purpose of homework was to practice. She writes, "The purpose of a homework assignment is to practice the stuff we learn at school and try it on our own. Homework may contain questions that we did not go over in class so it helps us be ready for any kind of questions, and, if we didn't understand it we can ask the teacher." Although Zoe refers to understanding as important, it is unclear what she means by understanding math. Does she mean that it is important to only understand why a concept works or does she mean that it is important to understand both how and why a concept works? She comments that teachers want students to be prepared for tests and exams. Although most teachers would say that they would love to have students understand why concepts work, traditional tests generally only test whether a student knows how to do certain questions.

I will introduce each Pattern Math that Zoe did with her own work on the patterning section. This will re-introduce the pattern to the reader and show some of Zoe's early work on the pattern. For the most part, I will type Zoe's responses, but I may include images where typing her work would not suffice.

Pattern Math: Cycle 1 Two More, Two Less

Patterning	
A: 10^2 = 100 B: (8)(12) = 96	A: 13^2 = 169 B: (11)(15) = 165
A: 40^2 = 1600 B: (38)(42) = 1596	Your own example: 20^2 = 400 (18)(22) = 396

Figure 39: Zoe's pattern work on "Two More, Two Less"

I gave students some guidelines for the first Pattern Math to help them get started. I asked them to describe the pattern that they saw and anything else that they noticed. The description would help them improve their communication skills as well as give them an easy way to begin writing and thinking about the problem. I then challenged students to question whether or not the pattern would always work and how they would come up with an explanation that described why it always worked. Here is Zoe's response.

$(1000)^2 = 1000000$
$(998)(1002) = 999996$

I see a pattern in these examples. The pattern is the fact that when a number is squared and it equals a certain number, when you multiply a number which is two more than the original number by a number that is two less than the original number, the answer will be four less than the original answer. This pattern will always work because if you try it with a variable it totally works. When you multiply two "xs" it will equal x^2 and if you FOIL $(x + 2)(x - 2)$, it equals $x^2 - 4$. $(x + 2)$ is two more than "x" and $(x - 2)$ is two less than "x" and the answer being "$x^2 - 4$" shows that it is four less than the original answer which is "x^2".

$(x)(x) = x^2$
$(x + 2)(x - 2) = x^2 - 4$

Zoe's response shows that she has algebraic skills and understands that algebra can be used to generalize a pattern that she sees. Although her explanation completely described the pattern and why it worked, I wanted to push her to see other things and to think about the pattern more deeply. In particular, I wanted her to continue to think about a pattern in different ways even though you might have been able to explain it. Although she was able to explain the pattern with algebra, I also wanted her to think about how a geometric representation, using areas, would also explain why the pattern worked. My response to students was a place where I could challenge them to make more connections and learn things more deeply. The following is my response to Zoe.

Zoe,

Great job describing the pattern. I liked how you looked at more than one example to check if the pattern works for other examples. To describe whether or not the pattern always works you went to algebra and used x to represent the number. Why is it that this proves that it will work for every number? Thinking about this will help you understand the power of algebra.

Now when you went to algebra you noticed the part B was $(x - 2)(x + 2)$ and that part A was $x^2 - 4$. I am wondering if you have ever thought of foiling or factoring a difference of squares as something that could help you with mental math? Have you ever made this connection before or are there other things that this makes you think of? Another thing you might want to look at is how multiplications are related to areas of rectangles. Whenever you multiply 2 numbers you can imagine a rectangle where one number is the length and the other the width. How does the pattern that we have seen here relate to areas of rectangles? Do the rectangles also help explain why the pattern works? Something to think about.

Feel free to respond to anything that I have written here.

Pattern Math: Cycle 1

Pattern Math: Cycle 1 One More One Less, Etc.

Pattern Math

Choose one or more of the following ideas to create a pattern. Once you have found a pattern try and explain what you have noticed and why it works. Try to make connections to other patterns or math you have learned.

One More One Less	Three More Three Less	Four More Four Less	Five more Five Less
A: $(10)^2$ = 100	A: $(10)^2$ = 100	A: $(10)^2$ = 100	A: $(10)^2$ = 100
$-(1)^2$	$-(3)^2$	$-(4)^2$	$-(5)^2$
B: (9)(11) 99	B: (7)(13) = 91	B: (6)(14) = 84	B: (5)(15) = 75

I noticed that when a number a is squared and we multiply x less than that number and x more than that, the answer will be x^2 less than the original answer. So, for example, if our original number was A and we'll do x more and x less the formula will look like this:

$(A)(A) = A^2$
$(A + X)(A - X) = A^2 - X^2$

$(12)^2 = 144$ $(24)^2 = 576$
$(1)(23) = 23$ $(14)(34) = 476$

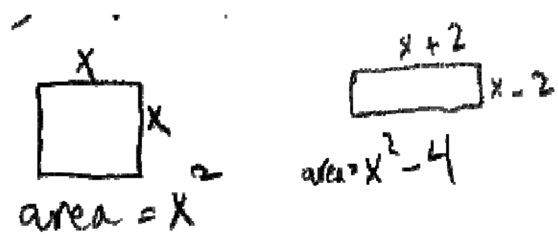


Figure 40: Zoe's pattern work and communication on "One More, One Less, etc."

In the second Pattern Math, you can see that Zoe is using her algebra skills again to describe the pattern and explain why it works. She not only uses one variable, but introduces a second variable as well to show how one variable (A) could be the number you started with and the other variable (X) is how much you add or subtract to the variable. At the end, you can see that she is attempting to respond to my comment about areas. Although she correctly starts with a square, her use of algebra at this point skews how an observation about areas would help her understand why the pattern works through a different medium. In my response to her, I tried to explain how areas could help her understand it better. In reading my response, it is quite wordy and I think it is difficult to understand. The thing that is missing in my response is diagrams. I suggest that Zoe try to fill in the gap that I was missing by drawing the diagrams herself.

Zoe,

Great answer. I liked how you generalized the statement to work with any number. I also liked how you took my suggestion to think about multiplication as an area question. When I think about area this is how I think about it. Suppose you had a square with sides of x (I will also use 10 in brackets to think about a specific number). Then if you want to make a rectangle with dimensions 2 more and 2 less than x (8 by 12) you could do this by imagining the square separated into squares that were 1 unit by 1 unit. Then if you took away two rows of such squares your one side would be (x – 2) (or 8) and the other side would still be x (10). Now if you rotated the row you took away and added it onto the x side it would now be (x + 2) (or 12). Thinking about this row that you first took away and then added on, it was originally on the square so its length is x (10). Now when you rotate this and add it on, you only need (x – 2) to complete the rectangle which will leave a square with sides of 2 by 2 left over. This would explain why the square x by x has an area four larger than the (x – 2) by (x + 2) rectangle. This is hard to explain using only words. I am wondering how well I communicated. Could you draw diagrams that show what I have written? Thanks,

Pattern Math: Cycle 1 Squared plus number plus next number

Patterning	
A: $3^2 + 3 + 4 = 16$ B: $4^2 = 16$	A: $6^2 + 6 + 7 = 49$ B: $7^2 = 49$
A: $20^2 + 20 + 21 =$ B: $21^2 = 441$	Your own example: $10^2 + 10 + 11 = 12$ $11^2 = 121$

I notice that a number squared plus that number and plus one more than that number is equal to one more than the original number squared.

$x^2 + x + x + 1 = x^2 + 2x + 1$
$(x + 1)^2 = x^2 + 2x + 1$
$1^2 + 1 + 2 = 4$
$2^2 = 4$

$\pi^2 + \pi + (\pi + 1) = 17.152789...$
$(\pi + 1)^2 = 17.152789...$

$0^2 + 0 + 1 = 1$
$1^2 = 1$
$(-2)^2 + (-2) + (-1) = 1$ It even works with negatives and zero.
$(-1)^2 = 1$

Figure 41: Zoe's pattern work and communication on "Squared Plus Number Plus Next Number"

The interesting thing in this response is that although Zoe has already shown that the pattern works with algebra, she still wants to verify specific cases where patterns normally break down, negatives and zero. This further investigation that she does at this point may indicate that she has thought of all the examples up to this point with only positive whole numbers so her algebra might represent only positive whole numbers. Perhaps her exploration into negative numbers and zero may be expanding her view of what it means that x could be any number. Now x, or an algebraic representation, may mean that x can be any number; positive, negative, zero or even π.

My response to Zoe indicated that I wanted Zoe to spend more time communicating why the pattern worked. Perhaps she was getting tired of explaining why it worked because her algebraic manipulations were a sufficient explanation. Whatever the reason, I encouraged her to communicate what she saw so that if someone else were to read her work they would understand why it worked. I also encouraged her to use proper mathematical terminology in an effort to improve her communication skills.

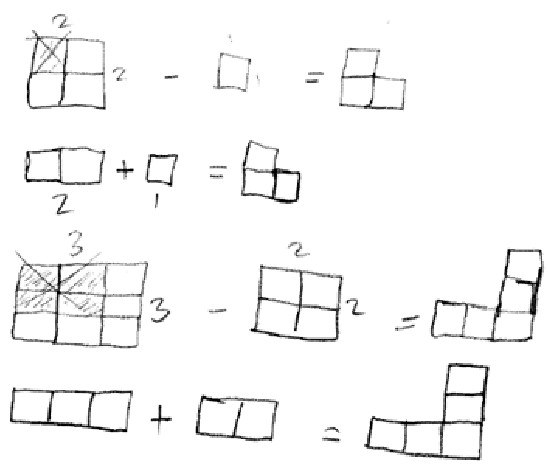
Zoe,

Zoe, when I read your answers from last time I was impressed with how you described the pattern and explained why it worked. This last time, I saw that you were able to describe the pattern, but I was not convinced by your explanation that you understood why it always worked. When you explain what you notice, I also want you to explain the reason it works. I want you to explore the details and see if you can convince anyone who reads this why the pattern works. When you are explaining your ideas, also try to use as much proper mathematical terminology as possible.

Pattern Math: Cycle 1 Difference Equals Sum

Patterning	
A: $6^2 - 5^2 = 11$ B: $6 + 5 = 11$	A: $10^2 - 9^2 = 19$ B: $10 + 9 = 19$
A: $50^2 - 49^2 = 99$ B: $50 + 49 = 99$	Your own example: $39^2 - 38^2 = 77$ $39+38 = 77$

I notice that a number squared minus one less than that number squared is the same as the original number + one less than it.



It works because the two squares cancel each other out and the signs cancel each other out. So after the squares get cancelled out we are left with the numbers who get added and give us the same result as the addition.

$$x^2 - (x - 1)^2 = x^2 - (x^2 - 2x + 1) = x^2 - x^2 + 2x - 1 = 2x - 1$$

$$x + (x - 1) = 2x - 1$$

Figure 42: Zoe's pattern work and communication on "Difference Equals Sum"

What is interesting in this response was that Zoe decided to use diagrams to explain why it worked first, rather than algebra. It appears that she was responding to my earlier comments. Perhaps she was trying to show the idea with areas as another way to represent the problem. Perhaps she was responding to my request that she explain the idea so that anyone could understand and she felt that geometric representations would be easier to understand rather than the algebra. Whatever the case, this is the first explanation that Zoe gives that provides two different representations of the problem and two different explanations as to why the pattern works. She has extended her ability to explain why something works by explaining it in more than one way.

In my response, I thank Zoe for taking my advice about looking at these patterns as areas as well. I invite her to share opinions in her communications as well. I want her to explain to me which proof she liked better and why. I also continue to encourage her to use proper terminology in her explanations, but that she should still explain things in her own words as well.

Zoe,

I am glad that you took my advice to think about the questions as area questions. I think your diagrams provide a very visual reason why the pattern works. You were able to show why the pattern works with algebra as well. As you are writing on these Pattern Math activities you can share your opinions as well. For example, I am curious what you think about the two proofs. Which do you like better? The proof involving algebra or the proof involving the pictures? Which proof do you think is better? Which proof do think is better mathematically?

In your explanation you said that the signs cancel each other out. I am not sure what you mean by this. Did you mean that when you distribute the negative that the signs inside the brackets change to the opposite sign? Try to use proper mathematical terminology when you are explaining things. If you don't know how to say it mathematically you can try first in your own words and then ask how you would say it mathematically.

Pattern Math: Cycle 1 Subtracting Binomial Squares

Patterning	
A: $(n+4)^2 = n^2+8n+16$ B: $(n+3)^2 = n^2+6n+9$ C: A−B $= 2n+7$	A: $(2n+5)^2 = 4n^2+10n+10n+25$ $4n^2+20n+25$ B: $(2n+4)^2$ $4n^2+16n+16$ C: A−B $= 4n+9$
A: $(n-2)^2 = n^2-4n+4$ B: $(n-3)^2 = n^2-6n+9$ C: A−B $2n-5$	Your own example: A: $(n-5)^2 = n^2-10n+25$ B: $(n-6)^2 = n^2-12n+36$ C: A-B $= 2n-11$

I notice that a polynomial squared minus a polynomial that is one more or less than the former polynomial will result in addition of the numbers and variables in both polynomials without squaring them in the first place. I think it only works if the polynomials have the same signs. This pattern is <u>very</u> hard to explain.

Figure 43: Zoe's pattern work and communication on "Subtracting Binomial Squares"

Zoe states at the end of her writing that this pattern was very hard to explain. She does a great job describing the pattern that she sees. Her description uses proper terminology and is easy to understand. In my response to Zoe, I suggest that she look for connections between this pattern and the previous patterns we had been doing to help her explain this pattern.

Zoe,

Great job describing the pattern that you saw. I am curious for you to explain to me why you stated that the second polynomial can be one more or one less than the first polynomial. All of the examples given have the first polynomial one more than the second one. Was that a little confusing because of the negative signs?

Look at the Pattern Math that we did just before this one. You told me in this Pattern Math that the pattern was very hard to explain. Can you describe to me what were the differences and similarities between these two Pattern Maths? Can you tell me why one is harder to explain over the other one?

Now that we have done 5 Pattern Maths, take a look at them all and see which Pattern Maths are connected to each other. I think this might help you explain why this last pattern works.

Feel free to respond to anything that I have written here.

Pattern Math: Cycle 1 Reflective Questions after the First Cycle

After completing the first cycle, students were asked to respond to three questions to reflect on the first cycle. The following is a list of the questions and Zoe's responses.

1. Which Pattern Math activity did you find most interesting and why?

The pattern that was most interesting to me was the x more and x less one because it was easy to understand and more useful and easy to use. You can always use it in mental math.

2. Which activity did you find you explained the best and why?

I understood the pattern "Difference Equals Sum" the best because I tried to explain it with drawings and it helped me understand why it works.

3. Describe one thing that you learned that you felt was valuable.

I learned that there always are patterns in math that can make doing math much easier or much harder. ☹ But the good thing is that it makes you think.

In the first question, Zoe explains how she sees value in both understanding why a pattern works and the usefulness of the pattern. She liked the Pattern Math that dealt with x more and x less because she can use the pattern to solve mental math problems. In the second question, we see that when Zoe used drawings as well as algebra to describe the pattern she felt that she was able to explain and understand the pattern better. Although many of her other explanations used only algebra and were sufficient in describing why a pattern worked, the use of diagrams enhanced her understanding and provided a deeper level of meaning for her. In the third question, Zoe mentions that the Pattern Math activities have had a positive impact on her because they have made her think. The activities have caused her to think about how and why mathematical patterns work. The activities have not been problems where a precise method of doing the question will produce a correct answer. Her comment about making math easier or harder is intriguing. Perhaps when she was unable to explain why a pattern worked this bothered her and caused her to think that the pattern was hard. When she was able to describe and explain a pattern, the pattern made mathematics easier. I responded to Zoe about her reflective comments on the same response to the first pattern of the second cycle of Pattern Maths. You will notice my response after I introduce Adding Odds.

Pattern Math: Cycle 2

Pattern Math: Cycle 2 Adding Odds

Patterning	
A: 1 + 3 = 4 B: Average of 1 and 3 = 2 C: B^2 = 4	A: 1 + 3 + 5 = 9 B: Average of 1, 3 and 5 = 3 C: B^2 = 9
A: 1 + 3 + 5 + 7 = 16 B: Average of 1, 3, 5 and 7 = 4 C: B^2 = 16	Your own example: 1+3+5+7+9+11+13+15+17 = 81 B: average = 9 C: B^2 = 81

I notice that if you add consecutive odd numbers together it equals a perfect square number. The average of the numbers added is the square root of the answer. And the average squared is the original answer. If there are an odd number of "odd numbers" being added the average is odd. The Average is the number that is exactly in the middle so if you square the number that is exactly in the middle then you get the sum. If the numbers of the odd numbers is even and there is no exact number in the middle, you choose an even number between the two middle odds. I hope it makes sense.

P.S. I haven't read the comment for the previous one so I can't really answer the questions … sorry ☹

Figure 44: Zoe's pattern work and communication on "Adding Odds"

Zoe's response in this first Pattern Math of the second cycle shows that she sees and can describe a number of patterns. What is missing from her response is an explanation or an attempt to explain why those patterns happen. In my response, I wanted to move beyond just describing the patterns but also to explain why the patterns work. I also encouraged her to extend the pattern or make connections to other mathematics she had learned in the past.

Zoe,

First I will respond to your reflective questions. I am glad that you find some of these patterns useful for mental math. That is the reason that I really like them too. I am also happy that using the drawings helped make the patterns make sense. Sometimes a drawing can be a much more powerful tool in understanding something rather than just looking at numbers or the algebra behind the pattern. I am also glad that these patterns have made you think a lot. It is by thinking that we really learn about math. Always look for patterns in whatever math that you are doing and try to explain patterns that you see. This will cause you to have a deeper understanding of mathematics.

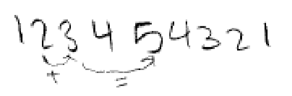
Now I will talk about adding odds. Yes your comments make sense. You described a lot of patterns that appeared. That is great. The next step would be to attempt to explain the patterns that you see. You could have also extended the pattern and seen where that would have gone or tried to make connections to math that you have done in the past. Do you remember a unit where you added up a bunch of numbers? Maybe this will help you explore ideas in these Pattern Maths.

Pattern Math: Cycle 2 Add Up Add Down

Patterning	
A: $1+2+1 = 4$ B: $\sqrt{A} = 2$	A: $1+2+3+2+1 = 9$ B: $\sqrt{A} = 3$
A: $1+2+3+4+3+2+1 = 16$ B: $\sqrt{A} = 4$ $1+1+2+2+3+3+4$	Your own example: A: $1+2+3+4+5+6+7+6+5+4+3+2+1 = 49$ B. $\sqrt{A} = 7$

I notice that when you add consecutive numbers and then stop at a specific number and go backwards repeating every other number except the specific number you stopped at, the answer will be the specific number you stopped at. If you add the first number with the number beside the middle number it adds up to the middle number.

If you add the second number with the second closest it also adds up to the middle number.



This pattern doesn't work if you don't start from 1.

Figure 45: Zoe's pattern work and communication on "Add Up Add Down"

In this Pattern Math, Zoe is attempting to explain why the pattern emerges, but her attempt is not very clear or easy to follow. She sets some constraints on the problem by noticing that the pattern only works if you start from one. She also notices that certain numbers in the pattern will add to the middle number. Despite these observations, her overall explanation is still unclear. In my response, I provided Zoe with some communication strategies to help her when she gets stuck explaining a pattern. The strategy was to actually write out questions as they arise and attempt to answer these questions instead of just leaving a page blank without writing anything. Here is my response.

Zoe,

I noticed in class that you were stuck on this one while you were trying to explain the pattern. I helped you a bit by leading you to think about adding up the numbers in a different way. This helped you see why it worked in a specific example, but your communication didn't explain it completely. You were able to explain it with one example. So how does it show it works for other examples? Why will that always happen? Why is the final number a perfect square? Perhaps one thing that you could do to help you explain ideas and look for patterns is to actually write out the questions that arise. For example you might say, *I noticed that the sum was always a perfect square* and then ask, *I wonder why the number is a perfect square*. By writing out these questions they will guide you to communicate more because you will have to try to answer your own questions. It is alright if you can't answer all of your questions. Sometimes asking questions only leads to more questions which is okay.

Pattern Math: Cycle 2 Adding Up Odds

Patterning	
A: $1 = 1$ B: $1^3 = 1$	A: $3 + 5 = 8$ B: $2^3 = 8$
A: $7 + 9 + 11 = 27$ B: $3^3 = 27$	Your own example: $13 + 15 + 17 + 19 = 64$ $4^3 = 64$

I notice this weird pattern that I don't know how to explain. You know what? I am going to treat this thing as a math journal kind of thing. So I'll write down whatever I think. How's that? I notice that one odd number equals one^3 and two odd numbers added equals two^3 and three odd numbers equals three^3 and so on. I wonder why it works. The thing is that the odd numbers have to be consecutive, like if you want to get 2^3 you've got to add the consecutive numbers that come after one. Hey!! I just did 9 + 7 to test my previous strategy and saw that it equals 16 which is 2^4. I guess that was a total accident cause I did 11 + 13 and it equals 24 which is kind of off topic. Well that was disappointing. I hope you're not getting bored of this journal thing Mr. JR cause it's totally boring. Okay back to the topic, before I start thinking about how cute my new sandals are ... well ... I don't get why the pattern works ☹ I guess the journal thing was useless.

Figure 46: Zoe's pattern work and communication on "Adding Up Odds"

In her response to the second Pattern Math of this cycle, Zoe attempted a new strategy to try to figure out the pattern. She decided to think of her response as a journal entry where she wrote down whatever she thinks. This was in part a response to my previous comments where I suggested that she write down questions that she thinks of and then tries to solve the questions. Her response shows more of her thinking than the previous responses. For example, I had hoped that the patterns that I had chosen would elicit a response such as "I wonder why it works." Zoe's exclamation of "Hey!!" shows the excitement of making a connection with another mathematical concept only to see a disappointment when her new found pattern was proven to be false. It is in this statement that we still see that Zoe is not writing down everything that she is thinking, but her comments give us a window to see what she may have been thinking. For example, since $3 + 5$ was 2^3 and $7 + 9$ was 2^4 perhaps she was checking to see if $11 + 13$ would be 2^5. Even though the pattern did not work, the process of looking for patterns, making hypothesis and checking her hypothesis is very apparent. The initial pattern caused Zoe to look for new patterns and to make new hypothesis about the patterns she was seeing.

In the end, Zoe concludes that the journal idea was a waste of time since she was unable to explain why the pattern worked. I wanted to encourage her that the idea of journaling would be very helpful, but she needs to experiment with the numbers as she is journaling. This mixture of asking questions and making observations coupled with mathematical experiments could really help her understand the mathematical concepts. My response also attempts to discuss her use of the word boring and exactly what she meant by this. This Pattern Math would set the stage for Zoe's future responses and great mathematical discoveries to come.

Zoe,

Neat that $9 + 7 = 2^4$. I played around with that for a bit and found $25 + 27 + 29$ is 3^4 and that $61 + 63 + 65 + 67$ is 4^4. I am not sure what kind of pattern that makes but I thought you might find it interesting.

I liked your idea of thinking of this as a journal and writing down what you think. However, communicating in math is more than just writing your thoughts. You also have to experiment using some numbers. You did experiment a little with 7 + 9, but you didn't really experiment with the numbers in the examples. I think that that is part of the reason you were unable to explain why the pattern works. If you had tried to play around with the pattern and tried to examine the numbers you might have been surprised with what you could find. Take a look at what Kayla did with looking for patterns. She found a method of explaining the idea. In fact, I used her method to come up with the patterns for the cubes which I showed you in the first paragraph.

Boring? I would be interested in what you mean by boring. For me, I would say that sometimes writing back to all of you guys is overwhelming and tiring. It takes a while, but I wouldn't say that it is boring. In fact, it is quite interesting to see what you guys are thinking and learning. I find that writing back to you is a way that I can communicate with each student in a way that there isn't time for in class. I hope that by communicating with each of you that I can encourage you to think in new ways and to learn more than you have in the past.

Now when you say boring it makes me think that you don't find these patterns interesting. Is that true? A person is bored when they are not interested at all in doing something. So are you not interested in the patterns or are you not interested in explaining why it works? Or are you not interested because explaining the pattern is too hard and too frustrating? I would really like to know. Could you either write back to me or come and talk some time to explain to me more?

Thanks.

Pattern Math: Cycle 2 Reflective Questions after the Second Cycle

After completing the second cycle, students were again asked to respond to three questions to reflect on the cycle. The following is a list of the questions and Zoe's responses.

1. Which Pattern Math activity did you find most interesting and why?

I found the Pattern Math "Add Up, Add Down" because it was very interesting and useful I think. We've been working with squares all semester so it would be good to see different ways of doing it.

2. Which activity did you find you explained the best and why?

I think I explained the last Pattern Math best because I explained it the journal way and it helped me see where I'm wrong or right and what I wasn't doing right.

3. Describe one thing that you learned that you felt was valuable.

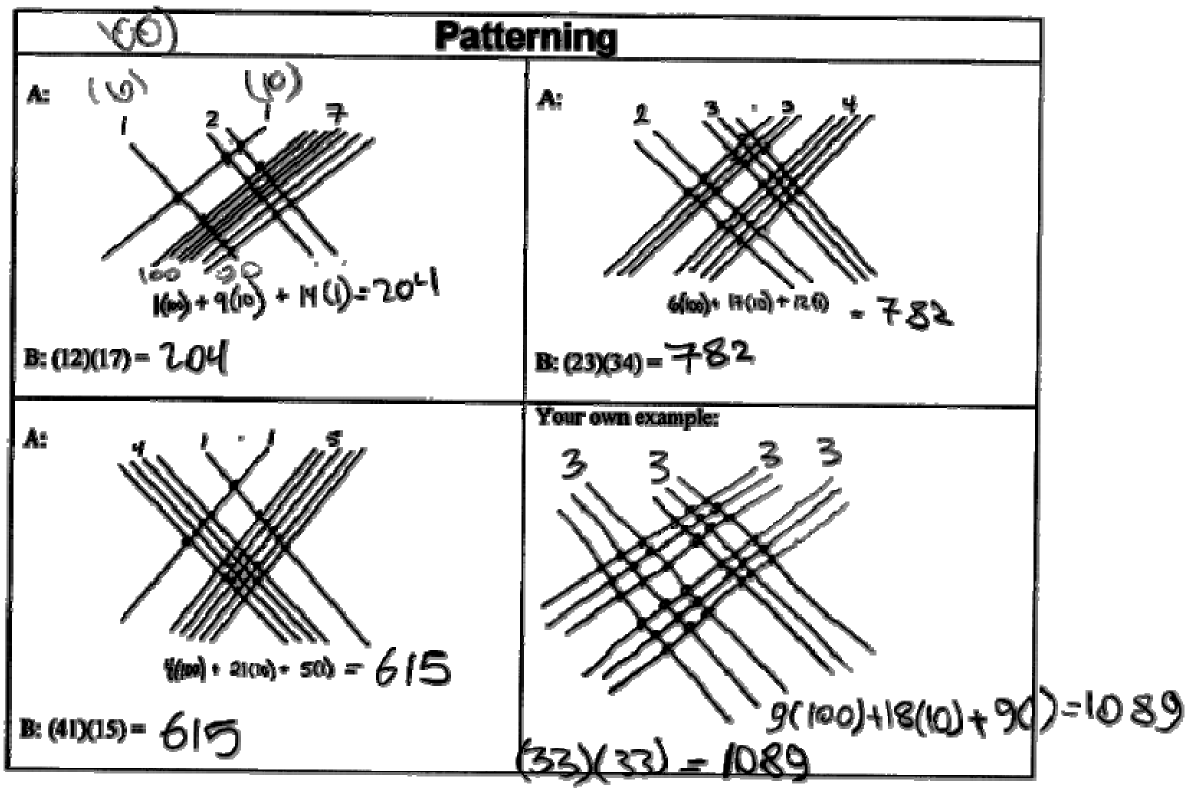
The pattern I learned the most was "Adding Odds." It was the easiest one and easy to understand. I also understood how it works which means I'll remember it and use sometimes. Maybe.

I would like to make a few comments about Zoe's reflections. In the first comment she refers to the mathematics that we were learning over the semester (quadratics) and was making connections to the patterns we were doing and the mathematics she was learning. In the second comment she refers to her journaling as method that helped explain her ideas best. This method would help her in the next cycle of Pattern Maths. Her third response shows her idea that math can be memorized more easily if it is based on understanding how and why it works.

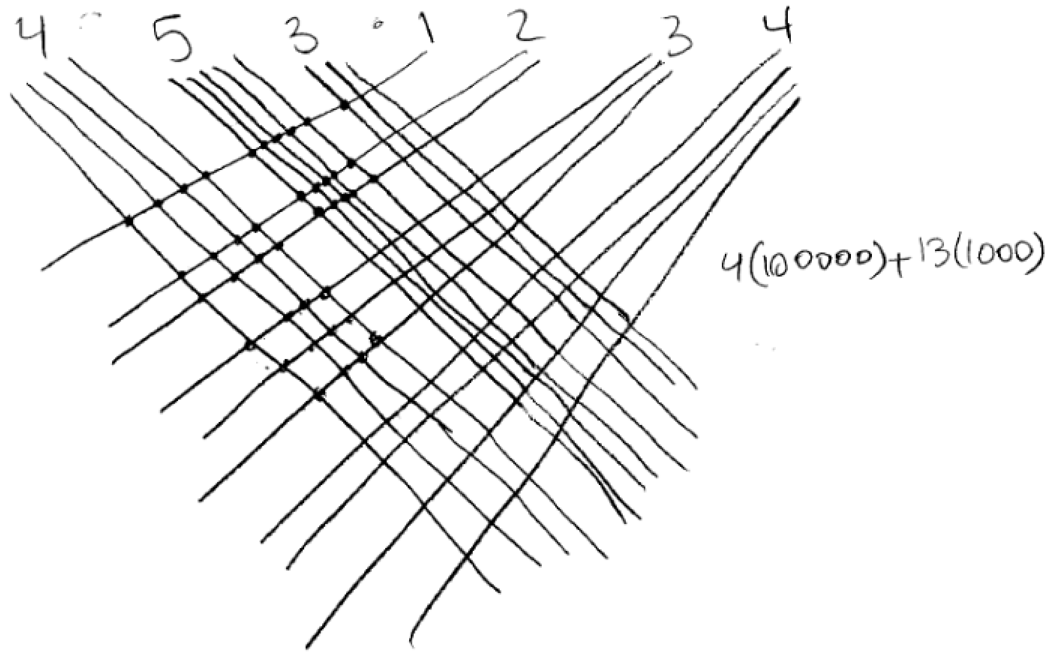
Pattern Math: Cycle 3

Pattern Math: Cycle 3 Ancient Chinese Multiplication

The last series of Pattern Math examples dealt with different ideas of multiplication.



I notice that Chinese people had lots of spare time to draw lines and then count where they intersect. I also notice that they like making things complicate(r) than it should be, no wonder their writing is sooooo complicated. Aside from my new foundations about Chinese People I also notice an interesting pattern. The pattern is that the intersections on the left is (100's) and the intersections in the middle are (10's) and the intersections of the lines on the right is (1's). These lines are based on the numbers that make the number, for example for (52) (23) we would draw 5 lines for five, 2 lines for 2, 2 lines for the other two and three lines for the three. The lines of the first # goes in the right direction and the lines for the second # goes in the left direction. I wonder if it works for numbers that have more than two digits.



(453)(1234)

Oh!! I got why it works! Well I guess not why it works but why the farthest left is the "100's" and so on, because in (52) (23) the "5" and the "2" are "10's" of the number so (10) (10) is 100 so that's why their intersection is the "100's" and the intersection between the "2" of "52" and "2" of "23" is the "10's" because the first "2" is in the ones and the second in the 10's so them multiplied equals "10", and that's why the middle intersection is "10's". I guess it's something really obvious, and probably everyone knew it just by looking at it but it took me a while. Well, not a while, but I got it when I was trying to explain to my sister what the pattern was. I guess I learned an important thing from these Pattern Maths. I learned that by explaining something to others, it'll be easier for you to understand the concept of your explanation. I hope that makes sense.

Figure 47: Zoe's pattern work and communication on "Ancient Chinese Multiplication"

Zoe's response shows the power of communicating what you see and trying new examples. It leads to seeing why the pattern works. It also shows the power of trying to explain an idea to another person and how you can learn from this. You can see the moment when Zoe realized how the pattern worked. She was about to do an extended example when the idea came to her. You can see in her example of (453)(1234) that she has abandoned her example to explain why the idea worked. You can tell that she really understood the idea because once she finished explaining the mathematics behind Ancient Chinese Mathematics she referred to it as obvious. In my response to her I wanted to commend her on her mathematical findings and to point out that what she had discovered was not obvious. Her comments about learning from explaining to others made me question how the Pattern Math activities were set up. Perhaps students would gain more from their work if they shared it with each other instead of just submitting their work to their teacher. I asked Zoe her opinion on this as well. At this point, the interactive writing between Zoe and me can be seen like an ongoing conversation. I ask questions or pose questions and Zoe responds to them later in her Pattern Maths. I had originally expected some students to comment back on my responses. Most however seemed to comment directly on their next Pattern Math either through comments or through their methods of solving new problems.

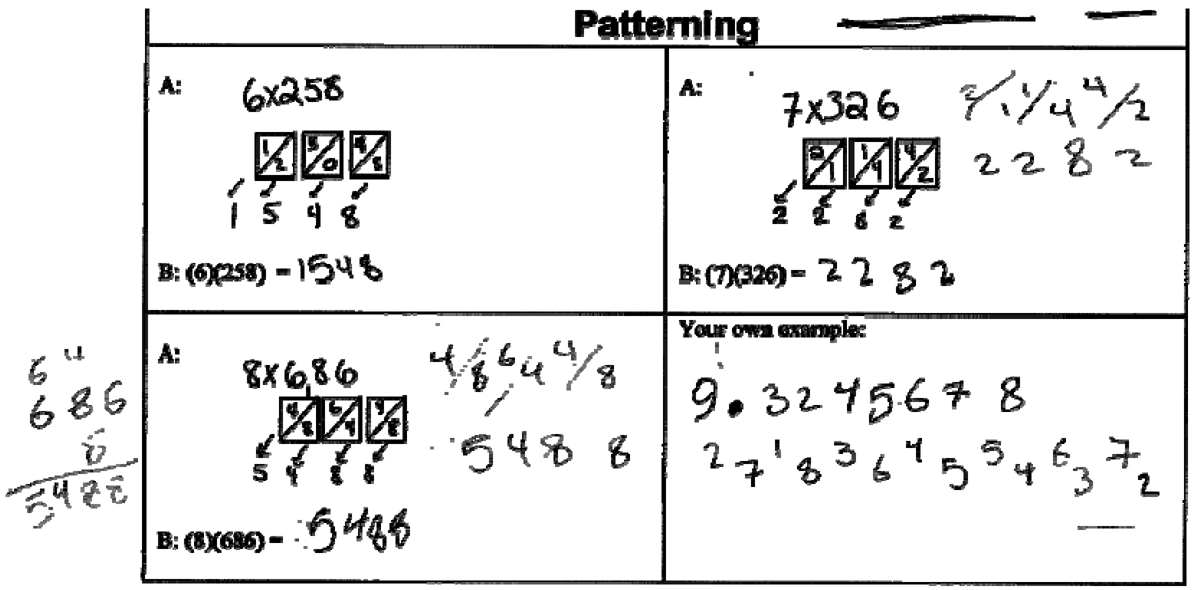
Zoe,

Wow!! That was my first response after reading your Pattern Math. First of all, what you noticed (with the two tens intersecting making the hundreds, the ones and the tens making the tens and so on) is not obvious, but once you see how it works it seems obvious. Many people who see these lines and see that the multiplication works think that it is just some sort of crazy magic, but you were able to see past that and break down the pattern and explain it. Excellent!

You said something in the Pattern Math that I was hoping that people would realize. By explaining something to someone else you can understand it better. That is what I am trying to get you to do when you have to explain these ideas to me. By getting you guys to explain the ideas to me I hope that you will come to a better understanding of why it works. This definitely worked for you in this case, but it seemed to work better as you were trying to explain it to your sister rather than explaining it to me. I wonder if that is because your sister didn't see the reason why the pattern worked just as you initially didn't see the pattern. Is it easier to try to describe it to someone who doesn't know rather than to the teacher (whom you may assume knows why these patterns work)? I wonder if these Pattern Maths would be even more effective in helping you understand ideas if you had to explain them to someone other than the teacher. Perhaps you should write to each other and then I could also just read what you found? Now I am just talking to myself and writing just like you were just writing whatever came to your mind (like the observations about the Chinese or perhaps your nice sandals from the last Pattern Math).

Sorry about getting off topic! Another thing that I really liked was your wondering. Like when you wondered whether the pattern would work beyond just two digits. It gets a little messy but it does. Now that you know why it works you can figure out why the pattern works in other sections. The next Pattern Maths are other ways of doing complicated multiplication with big numbers. I think you will be able to explain them well because you did such an amazing job with this one. Once you have finished explaining the next ones, I hope that you have fun playing around with them and wondering how they would work in different situations.

Pattern Math: Cycle 3 Napier's Bones



Thanks Mr. JR for being honest about the fact that "I am a genius". Just kidding! Wait, hope I spelled "genius" right, did I? Sorry about bugging you so much in class. About the idea of us students writing to each other, I don't think it would be too awesome ☹ because then we'd be afraid that people would make fun of our observations and think us stupid so then we wouldn't write our observations. Some people might not take it seriously. I don't know, I just wanted to tell you what I thought. Another thing I wanted to say is about the "boring" thing. By saying its boring, I didn't mean that the Pattern Maths are boring. What I meant was that my explanations might be boring for you. Well now that I've cleared out certain "things" I shall talk about the new <u>square</u> pattern. I notice that when you want to multiply big numbers by a number, you can do so through this specific pattern. When you want to multiply (8) by (3465), first you multiply the first # by the first digit of the 2^{nd} # and write it down 8 x 3 = 24. You write it one up, one down like ${}^{2}{}_{4}$ and go on until you get to ${}^{2}{}_{4}{}^{3}{}_{2}{}^{4}{}_{8}{}^{4}{}_{0}$ then you add the numbers that are in the bottom row with the number on the top right side of it and it'll be 27720. I think this pattern works like basic Grade 3r multiplication. In the basic multiplication you do it this way

$$\begin{array}{r} {}^{3} \\ 27 \\ 5 \\ \hline 135 \end{array}$$

You carry the tens over and add it to the tens of the bigger number and go on like that. It's multiplying and adding consecutively step by step. But in the pattern you do all of the multiplication first then do the addition which makes it easier because you don't have to switch back and forth between methods. I'm not 100000% sure this is how it works but I <u>think</u> it is how it works.

Figure 48: Zoe's pattern work and communication on "Napier's Bones"

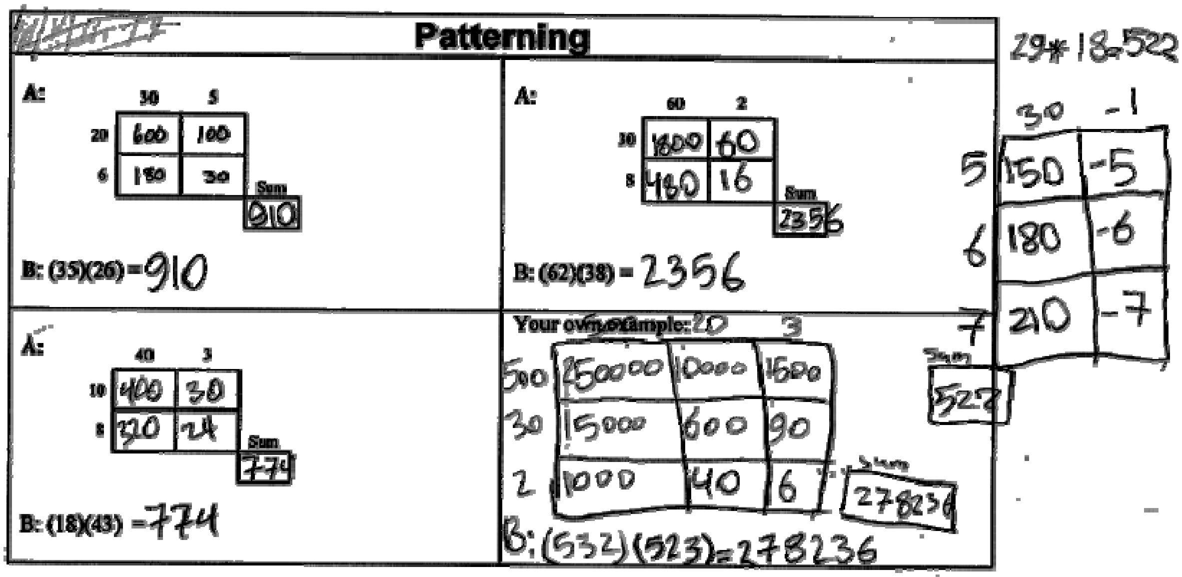
When I first read Zoe's response, I was a bit surprised that she did not use more of her ideas from the previous pattern to explain this pattern. In some ways this is good as it causes her to look at this pattern without presupposing ideas she had learned from the previous pattern. In some ways I wanted her to see the connections between the two and how the two ideas were based on the same principle. I mentioned this to her in my response.

Zoe,

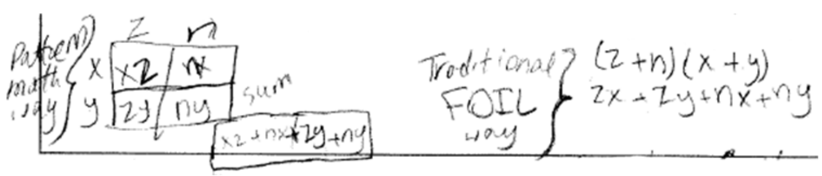
Yes, genius was spelled correctly. Thanks for being honest about the idea of writing to other students. I appreciated that. Thanks for also clearing up the boring thing. I don't find anyone's responses boring because, funny as it might seem, as a Math teacher I don't get to see how you guys think mathematically very often and these Pattern Maths help me see you guys think.

Very cool connection between what you did in Grade 3 and this method. I had never thought about the old method where you are always switching back and forth between multiplication and adding. I can definitely see how doing all the multiplication first and then all the adding at the end would be easier. Did you make any connections between this method for multiplying and the ancient Chinese method? Do you see how Napier's bones uses 10s, 100s and 1s?? It maintains place value (that's the word for 10s 100s 1s etc.) in a similar way that the ancient Chinese method maintains place value.

Pattern Math: Cycle 3 Charts



I notice that this Pattern Math is really close to the other Pattern Maths. I think this Pattern Math also depends on the place value of the numbers. It is very similar to the traditional way of multiplying. It's like breaking down the steps and putting them into the boxes. Although I wonder why it works with (30 – 1) (5 + 6 + 7). Well I guess it's because it's basically the same thing because if you do (x^2) (x) it equals x^3 and if you do (x) (x) (x) it still equals to x^3. I wonder if it would work if you put variables instead of #'s. I'll try one, I hope I won't end up confusing myself like I did with the Chinese thing.



Well I guess this Pattern Math is like <u>FOILing</u>. I <u>guess</u> it works the same way FOILing does. Let me try it with numbers.

(30 – 1) (5 + 6) → 29 11 = 319. Hey!! I guess it is FOILing except

150 + 180 – 5 – 6 = 319 with #'s and $\square$'s! Cool!!

Figure 49: Zoe's pattern work and communication on "Charts"

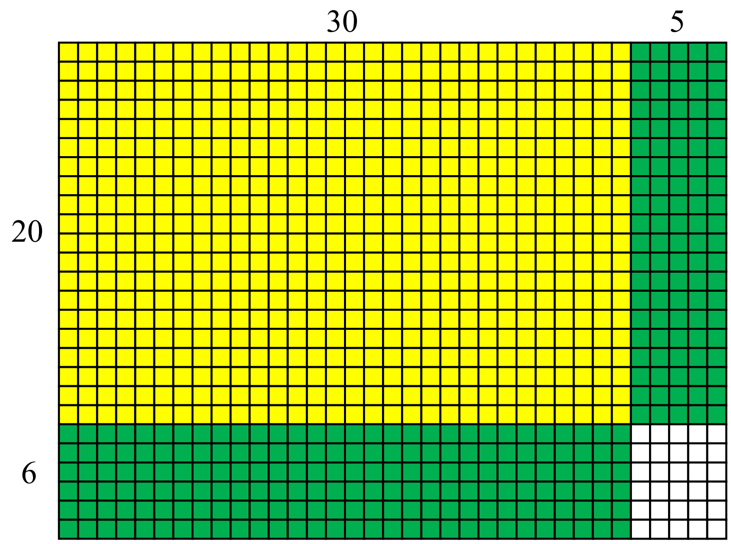
In my previous response to Zoe, I had asked if she had made the connection between the Ancient Chinese Multiplication method and the method in Napier's Bones.

It is not surprising that she starts off this Pattern Math by looking for and stating a comparison between place values in this Pattern Math. Because I wanted students to think beyond place values in this last example, I introduced the additional pattern on the side where numbers other that the place values were used. This intrigued students and also meant that an explanation based solely on place values would not suffice. Again you can really see how Zoe's thoughts led her to her conclusions. She saw a pattern and wondered if it would work with variables as well as numbers. This led her to see a connection between what was done in Charts and what is normally taught in mathematics, FOILing. It also led her to see that the techniques used in algebra could also apply to numbers. To summarize, a pattern in numbers led her to see that the pattern worked with variables, which in turn led her to see that an algebraic operation, normally not used with numbers, also worked with numbers. In my response to Zoe, I summarized all the connections she made and how I believed that she didn't write down anything that she hadn't already known, but that the connections she just made may have caused her to understand the things that she knew in a deeper way. I also tried to expand her knowledge even further by reintroducing the ideas of areas that we had used in the first cycle of Pattern Maths.

Zoe,

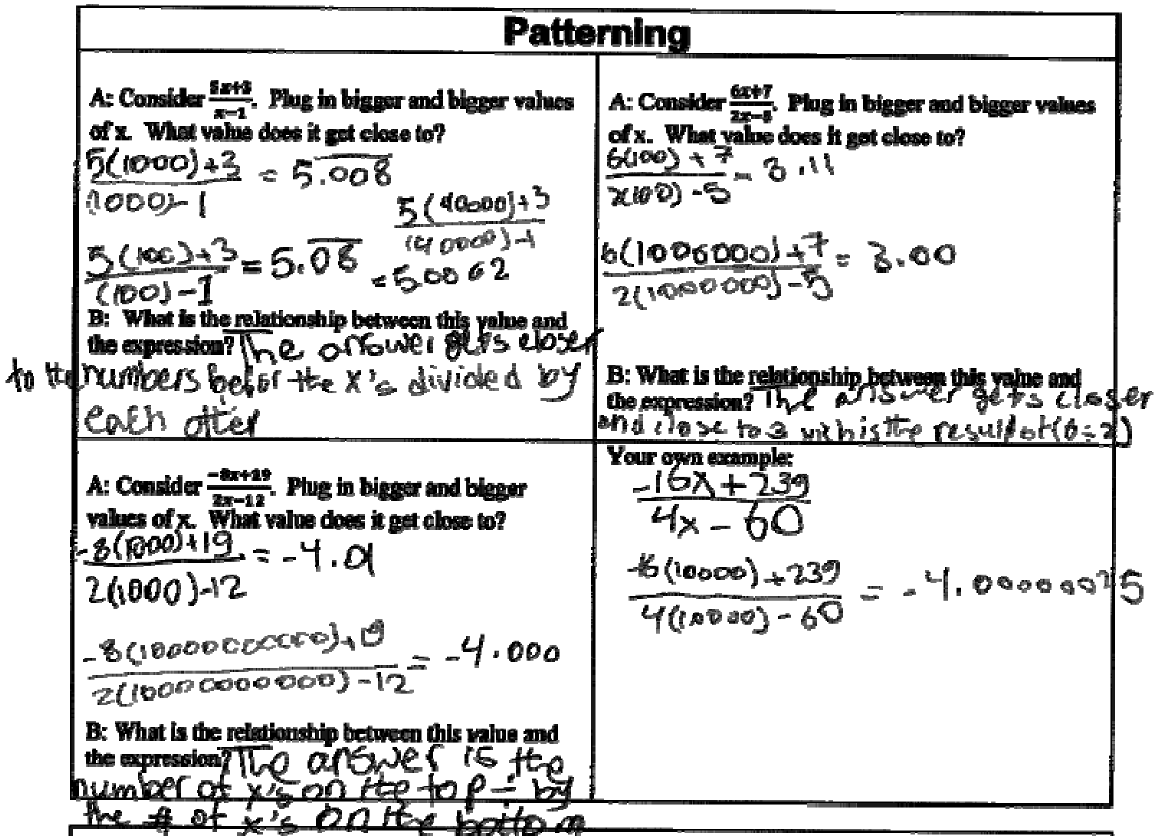
Great! I don't think you wrote down anything that you didn't already know, but I think you made some connections between things that you hadn't seen before. Here are some connections that I think you made that were amazing: 1) You noticed how the chart method was the same idea as FOILing. You might not have seen this if you hadn't wondered if the chart method also worked with variables. When you went to check the chart's results you FOILed and only with FOILing did you realize it was the same thing. 2) You noticed that you could FOIL regular numbers the same way that you FOIL variables. After you saw that it was the same as FOILing you tested your observation to see if it was the same with regular numbers. It was! Great work and great observations! Perhaps you can see how all of these things put together help you understand math better. Thank-you for all your effort!

The following is my own work at trying to explain the chart method with areas. See if it makes sense to you. It uses area to explain why it works. Take our first example, 35 by 26 and make a rectangle with those dimensions. You can see in my example below that I can break the area up into smaller rectangles and the number of boxes will be the same for the total. Can you see how the number of boxes in each of the different shaded areas is the same as the chart method? Can you see why it wouldn't matter what numbers I use to show the rectangles as long as the length added to 35 and the width added to 26?



Pattern Math: Cycle 3 Horizontal Asymptotes

This Pattern Math really doesn't belong to Cycle 3, but since it was the last Pattern Math it also didn't make sense to call it a cycle by itself. The purpose of this last Pattern Math was to design a Pattern Math that introduced a new topic in the curriculum. Our class was about to learn about rational graphs and I wanted to design a Pattern Math that got them to think about horizontal asymptotes and why they appeared in the graphs. Compared to the other Pattern Maths, I had to do a lot more assisting during this Pattern Math to explain what I wanted the students to see during the Patterning section.



Patterning

A: Consider $\frac{5x+3}{x-1}$. Plug in bigger and bigger values of x. What value does it get close to?

$\frac{5(1000)+3}{1000)-1} = 5.008$

$\frac{5(100)+3}{(100)-1} = 5.08$

$\frac{5(40000)+3}{(40000)-1} = 5.0062$

B: What is the relationship between this value and the expression? The answer gets closer to the numbers befor the x's divided by each other

A: Consider $\frac{6x+7}{2x-5}$. Plug in bigger and bigger values of x. What value does it get close to?

$\frac{6(100)+7}{2(100)-5} = 3.11$

$\frac{6(1000000)+7}{2(1000000)-5} = 3.00$

B: What is the relationship between this value and the expression? The answer gets closer and close to 3 which is the result of $(6\div2)$

A: Consider $\frac{-8x+19}{2x-12}$. Plug in bigger and bigger values of x. What value does it get close to?

$\frac{-8(1000)+19}{2(1000)-12} = -4.01$

$\frac{-8(1000000000)+19}{2(1000000000)-12} = -4.000$

B: What is the relationship between this value and the expression? The answer is the number of x's on the top ÷ by the # of x's on the bottom

Your own example:

$\frac{-16x+239}{4x-60}$

$\frac{-16(10000)+239}{4(10000)-60} = -4.00000075$

I notice that the answer is the coefficients of x divided by each other and as the value of x increases the answer gets closer and closer to the answer.

Figure 50: Zoe's pattern work and communication on "Horizontal Asymptotes"

This exercise took longer than most of the other Pattern Maths to complete the Patterning section. Because of this length, it is not surprising that the Communicating section was less developed. The patterns were also introducing new concepts instead of teaching old concepts. Nevertheless, I was curious as to how the Pattern Math would help students understand the graphing concepts that I was about to teach in that same class. I did not respond directly to this Pattern Math, but decided to have the students let me know how the Pattern Math helped them in the reflective questions at the end of this final cycle.

Pattern Math: Cycle 3 Reflective Questions

1. What was the purpose of doing the Pattern Math activities?

The purpose of doing Pattern Math activities is to see new patterns and think about it. It'll help us see things from another perspective and see new ways of doing math.

2. Describe your abilities to communicate mathematically.

My ability to communicate mathematically wasn't very good and isn't very good. It's easy to write things but it's hard to explain. The Pattern Maths helped me develop my skills in explaining mathematically. It was a way of practicing.

3. Describe what you learned from the Pattern Math activities.

I learned that you should always look beyond what you see in the first impression. When I looked at the Pattern Maths for the first time, I thought it was something you learn and remember and is always the same but once I saw the different cool patterns it made me think of math as a whole new world ready to be discovered and explored.

4. What does it mean to understand something mathematically?

I think "understanding" math means that you see what is going on and what is the purpose of it. If you just do $2 + 2 = 4$ and memorize it, it wouldn't be very useful because you don't see the purpose or the usefulness. By understanding math you know how to use it and when to use it.

5. How did the last Pattern Math help you understand the horizontal asymptotes in rational graphs?

The last Pattern Math helped me a lot in understanding the y-asymptotes. The good thing about the Pattern Maths is that once you understand, you also remember them easily.

6. Describe how you could use what you have learned from the Pattern Math activities while you learn the content in a math class.

I would try to find patterns when I learn new concepts because there always are patterns that help you remember them easily.

7. Describe how you could use what you have learned from the Pattern Math activities while you do your homework.

Try to use the patterns to find the answers in a way that would help me understand it better.

Pattern Math: Zoe's Narrative

Prior to the closure interview, I created a narrative for each of the students that described their learning throughout the three cycles of the Pattern Math. In this narrative, I often included questions that I felt I still had of the student and used these questions as part of the closure interview. The following is Zoe's narrative.

Zoe's Story

Zoe, the following is a summary of what I have read in your Pattern Maths. It creates a story of how you learned and thought about things as we did the Pattern Math activities. While I was writing this story and reading over the communications we had during the activities I still had some questions. These questions will be asked as part of the interview to help clarify the questions that I still have.

First, I would like to summarize some of the big things that I saw during these activities. Before we started the Pattern Math activities, I asked your opinion about the purpose of homework. You stated that homework was for practice. Through practice you would learn how the concept worked.

Since the very first Pattern Math you showed that you had some great algebra skills. You stated that "This pattern will always work because if you try it with a variable it totally works." I was really happy that you knew that you could show that something always worked by using a variable to represent any number. When I wrote back to you I wondered if you had ever thought of using algebra as something to help you with mental math.

Question: Have you ever thought about using your algebra skills as something that could help you with mental math?

Another thing that was very apparent in all of your work is that you always considered what I said when I wrote back to you and tried to incorporate the ideas into your work. In my first response, I was glad that you were able to show the algebra, but challenged you to explain the idea using areas as well. In the second Pattern Math, you played around with the area idea a little bit, but your work didn't really lead you to any *discoveries. It wasn't until Difference Equals Sum that you really made a breakthrough using areas. In this Pattern Math you were able to not only describe the pattern using algebra, but also with diagrams as well. In your reflections about this cycle of math you stated that you had explained this Pattern Math the best because you were able to explain it with drawings and that helped you understand why it worked.*

Question: I wonder if you could explain to me more how the drawings helped you understand why it works.

Question: In Difference Equals Sum, which of the two ways of explaining, the picture way or the algebra way, do you like better? Why? Which proof do you think is better?

One of the things that you appreciated in the Pattern Maths was when you found them to be useful. For example, you liked x-more and x-less because you found it useful for mental math.

Question: Can you describe how you would use some of the things that you have learned?

My last question from the first cycle has to do with your comment from what you felt was valuable. You wrote, "But the good thing is that it makes you think."

Question: What did you mean by this statement?

An idea that you developed in the second cycle was to respond to the Pattern Math questions as if they were a math journal. I think this was in part because of what I had suggested to you that you could write down what you notice and then ask questions like "I wonder why that works?" At first you didn't have a lot of success. In Adding Up Odds, you described the pattern wonderfully then went on to wonder why it worked. You looked for other patterns and noticed that $7 + 9 = 2^4$*. You thought you had found something interesting but 11 + 13 was not interesting at all. You thought that what you wrote was boring and felt that the journal thing was useless.*

I however, really liked your journal idea and encouraged you to keep with it. An important thing that you were doing I would describe as noticing, wondering and then checking. This strategy you created by first noticing and describing the pattern, then wondering why it worked and finally checking your questions with examples. This ended up being the key to some of your major discoveries in later Pattern Math questions.

After Adding Up Odds, I responded to you and asked you to look at Kayla's Pattern Math.

Question: How much did you look at other people's Pattern Math responses while we did the activities?

At the end of the second cycle of Pattern Math activities, I asked you which Pattern Math you explained the best. You said, "I explained the last Pattern Math best because I explained it the journal way and it helped me see where I'm wrong or right and what I wasn't doing right."

Question: Could you expand on this for me. How did the journal way help you see where you were wrong or right? How did this do it more than what you were doing before?

Responding to what you felt was valuable, you stated that Adding Odds was valuable because you understood how it works which means that you would remember it and maybe use it.

Question: How does understanding how it works help you remember it better? Can you explain this to me more?

It was in the third cycle where your journal idea really helped you see the reason why a Pattern Math worked. In Ancient Chinese Multiplication, you were thorough in noticing the pattern and describing the pattern. Then you wondered a little bit. You wondered if it worked with 3 digit numbers and checked an example. It was while you checked this example that you came to an understanding of why it worked in general. (Or *as you explained it, why certain numbers were 100s, certain ones tens and so on.) At the end of your journal you wrote, "I guess I learned an important thing from these Pattern Maths. I learned that by explaining something to others, it'll be easier for you to understand the concepts of your explanations."*

Question: Can you describe to me how you will use what you have learned for learning math in the future?

Just like the first Pattern Math, I was always very impressed how you read what I had written and incorporated it into your writing. I was appreciative of you explaining what you meant by "boring" and how writing to other students in the class may not be the best idea. After Napier's Bones, I asked if you had made any connections between it and the last Pattern Math. In the next Pattern Math, Charts, you made sure to tell me that you saw that it was similar to the other Pattern Maths. I think you made some great connections to ideas that you had learned in the past and this was very apparent in the last Pattern Math where you wondered and check if it worked with algebra. If you hadn't wondered and checked, I don't think you would have made the connection with the idea of FOILing. I think you would have been able to explain the Charts Pattern Math in the same way you explained Napier's Bones and Ancient Chinese Math. That is why I purposefully added the (30 – 1) times (5 + 6 + 7) to make you think even more. It wasn't until you made the connection to FOIL that you were able to figure this out as well. I hope this is something that you learn from the Pattern Maths: You know a lot of things about a lot of different areas of math, but when you can make connections between what you already know, you will have a deeper understanding of the math. For example, I think that you knew how to FOIL and know how to multiply using charts, but you didn't know that they were connected. Perhaps you could use charts to multiply now or use FOIL to multiply numbers. It just allows you to do more with what you already know.

Question: Did you read what I wrote about using areas to explain the chart method? What did you think about that?

Finally I have some questions about your final statements. You mentioned that the purpose of the Pattern Maths was to "help us see things from another perspective and see new ways of doing math."

Question: Can you expand on this? Is it important to see things from another perspective? Explain. Is it important to see new ways of doing math? Explain.

Here is another statement you made: "When I looked at the Pattern Maths for the first time, I thought it was something you learn and remember and is always the same but once I saw the different cool patterns it made me think of math as a whole new world ready to be discovered and explored."

Question: Can you expand on what you meant by this? How did your view of math change?

Question: How does understanding something help you remember it?

Question: Are you ever satisfied with memorizing how to do something in math even though you don't understand it? Explain.

Interview with Zoe

The following is the closure interview with Zoe. The narrative provided a framework for some of the questions that were asked in the interview. In addition to the questions from the narrative, I asked some summative questions as well. I have chosen to represent Zoe with the same font that I used throughout this chapter for her communications.

Have you ever thought about using your algebra skills as something that could help you with mental math?

No, I only do algebra when I have to do algebra.

So when we did the Pattern Maths, was this something new that you saw?

Yeah, because I used algebra with not variables, numbers sometimes, for the FOILing thing.

You used some diagrams as well as algebra. I'm wondering if you could explain to me more how the drawings help you understand why it works.

Because it's a visual, it's easier to understand visual stuff, like you understand movies better than books ... no, I actually understand books better than movies.

But you can read the words ...

Yeah, if you know the words that's good, but if you don't know the words you're screwed.

So, in the Differences Equals Sums, which way of explaining, the picture way or the algebra way, did you like better?

I liked the picture way because then I could just see how it worked, but the algebra way I just knew that it works but I don't see how.

Excellent. One thing that you wrote at the end of the first cycle was "but the good thing is that it makes you think." What did you mean by that?

Because math, if you're good at math, you just do math you don't care, you don't think but then if it's something tricky you have to think about it and when you think about it, you see patterns. That's what I learned.

And that's a good thing?

Yeah, it made me think, not just do it. Think before doing stuff.

So you enjoyed thinking about it?

I guess, not really when it was difficult.

Not when it was difficult but when you figured it out you really liked it?

Yup.

How much did you look at other people's Pattern Math responses while we were doing the activities?

Not much.

Not much?

Yeah, I just did my own. I looked at Sun's sometimes.

You wrote, "I explained the last Pattern Math best because I explained it in the journal way and it helped me see where I'm wrong or right and what I was doing right." Can you expand on this for me, how did the journal way help you see where you were wrong or right and how did this do it more than what you were doing before?

Because if I was doing it the thinking way, I was just thinking, and then it was just in my mind and then if I think of something else, then it would go away. But if I write it down then it's right there. If I think something else then I can see it written down then I can think of that instead of something else. You get it?

Yeah, … How does understanding how it works help you remember it better? Can you explain this to me more?

Because if you don't understand it, you're just like OK it's just math stuff and you forget it next year and then you remember it OK but if you understand it you know how it works and then if you see something like that you just remember OK that's how it works and everything. It's just not memorized.

You mention that the purpose of the Pattern Maths is to help us see from another perspective and see new ways of doing math. Can you expand on why it is important to see things from another perspective? Explain. Is it important to see new ways of doing math? Explain.

Lots of explaining.

Well, I want to know.

OK, I think, because, … OK, it's just math we just thought it's something straightforward you just do it and there's always a solution and everything just one solution that's how it works but then Pattern Maths did something else I don't know something all the same thing in a different way so I thought it was like, oh there's other ways to solve it and there is different ways to do it and that's why maybe the way I'm solving it is not the way everyone is solving it. OK.

And is that important to see new ways of doing it?

Yeah, because when you forget the actual way then you can remember other ways to solve it.

OK, you also wrote "when I looked at the Pattern Maths for the first time I thought it was something you learn and remember and it's always the same but once I saw the different cool patterns it made me think of math as a whole new world ready to be discovered and explored. Can you explain what you meant by the whole new world ready to be discovered and explored more?

Because you love math and you're still here and you've been doing it like for so many years probably it is a big world to be discovered. I don't know, oh oh oh I know, because, oh, because oh oh, this stuff that you learn when you were in grades whatever it's just nothing compared to the stuff we learn now so probably there is even more and more and more and more and people go crazy over stuff to infinity.

I always thought it interesting that even in the stuff that you would probably think of as adding up numbers like 1+2+3+4+3+2+1 that's not that exciting, like, isn't this a grade two assignment, and yet all the patterns that you see with that... I find that very interesting.

And that guy is making everything 13 in the video thing.

... Oh right.

Yeah that guy.

Are you ever satisfied with memorizing how to do something in math even though you don't understand it?

Not really because if I don't understand it then I will forget it then I will have to do it again because I think I memorize stuff when it's just there but then later on I don't get them and then when I do math homework and everything and I actually understand how works and I remember it and I do good on tests.

Excellent. Describe your ability to explain why things work.

Not very good at explaining because I always confuse people. Yeah, I do, because I have to explain lots of stuff to lots of people because I'm a smart person and then I always confuse them out there like oh you are just confusing me. So, but right now I think my explanations are getting better because at least I made you understand stuff in Pattern Math.

OK. Describe what it means to think mathematically.

Well I guess to think mathematically is thinking mathematically. Oh, use numbers to think.

OK. How did the Pattern Math activities make you think?

Well they were different, and they didn't look like the stuff we used to do in math. They were just like something new and you are like hey, if you understand then you were like hey, I didn't know that's how it works. That's interesting. And then it makes you think because it's interesting. Not boring like that.

Describe your ability to communicate math.

My ability to communicate math is OK. I don't know. You have to explain stuff and prove it. Not just by yelling and stuff.

How did the Pattern Math activities affect your ability to communicate mathematically?

My communicating mathematically and do I need to improve it? Can I at all communicate mathematically? So it helped me see where I am.

Did the Pattern Math activities help you to see connections between different mathematical concepts?

Yes, because everything is connected in math. If you do something wrong you will do everything wrong. It is because if you put one sign wrong then the whole thing is wrong. You do one page of math and if you did something wrong at the beginning it's wrong to the end and then everything is connected because you need to do, ... you need to add stuff to multiply and you need to multiply stuff to add stuff.

What is the importance of making connections between different mathematical ideas or concepts?

Well if you see that everything is connected then you don't think there's lots to understand. You understand one and you can make connections with everything else that you know then it's easier to remember that one because you're like oh it's just the same thing that I learned before it's just more advanced.

In high school math, how important is it to know why a mathematical concept works?

Because in high school if you don't understand things you get bored and fall asleep but if you understand it you don't fall asleep and that's good.

Did you ever fall asleep in my class?

Yes. That has a special explanation you weren't in the room and you weren't teaching so I had to fall asleep there was nothing else...

Now that we've finished doing Pattern Maths, describe all the things that you've learned that you found valuable.

I learned that... That you should always have a proof and always try to prove it to yourself before proving it to others and if you are actually satisfied then you should tell others because sometimes I thought I was right and I was wrong and that was not a good feeling.

How will you approach a learning math next year?

I'll try to see patterns in it and make my mark better than it is right now and improve by more than 0.5 on the exam.

Chapter 7 – Using the Pirie-Kieren model – Case Study of Zoe

The purpose of this chapter is to give the reader an in-depth look at how the Pirie-Kieren model can be used to represent a student's growth of mathematical understanding. The first cycle of Zoe's Pattern Math responses will be analyzed because the reader is familiar with her example from the previous chapter. An examination of the first cycle will provide the reader with sufficient information about how the model was applied to other students. The following chapter will discuss some common themes the analysis of the Pirie-Kieren elicited with regards to all students.

Below is my mapping of the growth of understanding of the first cycle for Zoe. There are five numbers listed in the Primitive Knowing circle and these five numbers represent the 5 different Pattern Maths of the first cycle in the order that they were completed. I have chosen not to connect the cycles physically by drawing a line. One could imagine a line connecting the end of the first Pattern Math to the beginning of the second Pattern Math and so on, as what has been learned in the first Pattern Math may become part of the student's *primitive knowing* for the second Pattern Math. However, I was not always positive that students made connections from one Pattern Math to the next so I did not presume a connectedness between the different Pattern Math activities. In some cases, it is clear that students made connections between Pattern Math activities. In these cases, I would make a comment that the connection was made and that the student had created a "thicker" or deeper level of understanding.

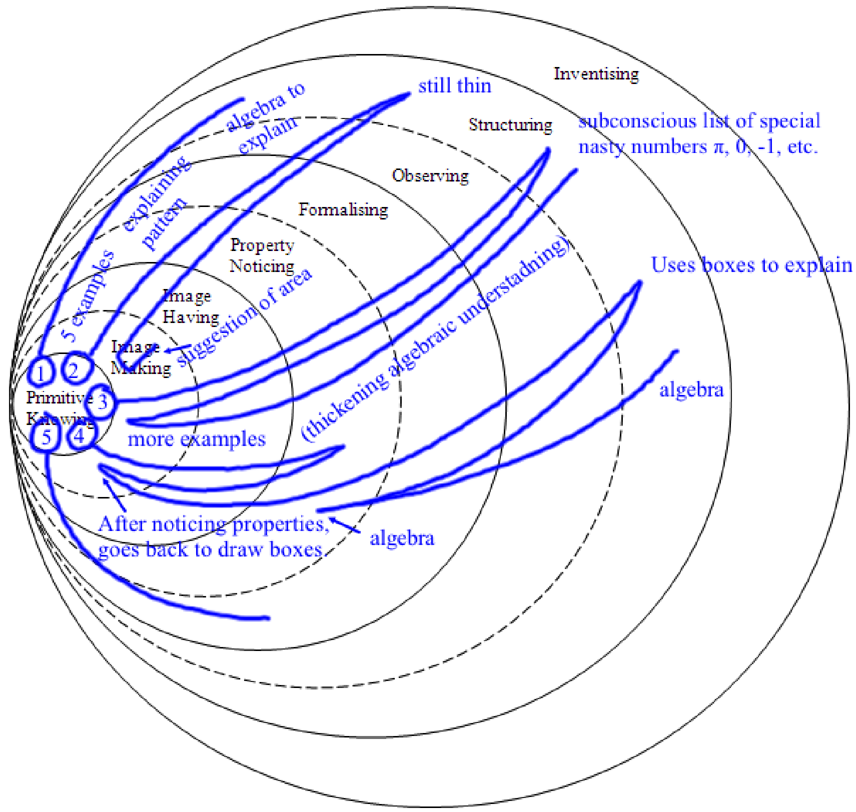


Figure 51: Pirie-Kieren Model for Zoe's first cycle of Pattern Math.

Before discussing Zoe's understanding as shown through the Pirie-Kieren model, it is important to re-examine the idea of developing a "thicker" level of understanding as shown through the idea of "folding back." When Pirie-Kieren use the idea of folding back, they describe the action of being at one level, say *structuring*, and then moving back to a previous level. Take Zoe's actions in the third Pattern Math as a reference. A student may be in the process of *structuring*, and this may cause them to question whether or not the theory they have works in a new situation. The student then tries some more examples that will help clarify these questions. The process of trying more examples moves the student back to the stage of *image making*. This does not mean that the student has regressed or has lost understanding gained at a higher level. Instead, the student has created more examples to deepen their understanding of the concept so that when they return to the *structuring* stage they have a deeper or "thicker" understanding than they did before. The process of "folding back" allows the student to examine more examples which leads to a better understanding of the concept.

In the Pirie-Kieren model, the act of folding back creates a line from an advance stage back to a previous stage (such as *image making*). If the process of folding back enhances a student's later stage (for example *structuring*) then a line back to the later stage can be drawn. The result is a series of lines three thick to represent the student's growth of understanding from *primitive knowing* to *structuring*. The existence of the three lines indicates that coming to understand the material was not a linear process, but in the process of going back and forth the student develops a "thicker" and therefore deeper level of understanding. With these ideas in mind, I will now examine and analyze the five Pattern Math activities of Zoe in the first cycle.

Using the Pirie-Kieren model for the first Pattern Math, Zoe was able to move to a high level of mathematical understanding, all the way to the *structuring* stage. Although she was able to reach this high level, you will notice that there was no folding back and I have thus categorized her understanding of the first stage as thin, or at least, I am unable to fully comprehend the depth of her understanding. I have included her writing below to help in the analysis.

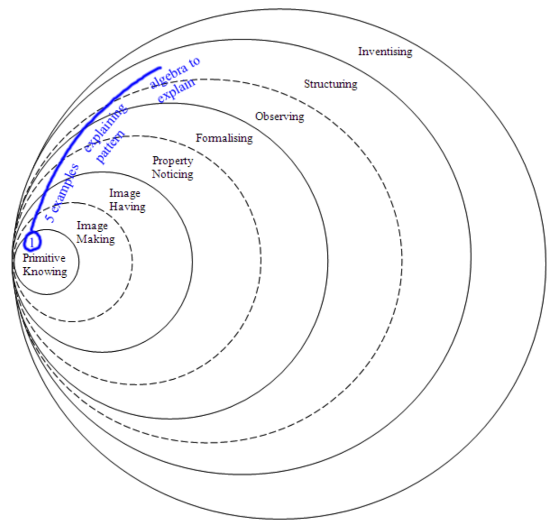


Figure 52: Pirie-Kieren Model for Zoe's first Pattern Math.

I see a pattern in these examples. The pattern is the fact that when a number is squared and it equals a certain number, when you multiply a number which is two more than the original number by a number that is two less than the original number, the answer will be four less than the original answer. This pattern will always work because if you try it with a variable it totally works. When you multiply two "xs" it will equal x^2 and if you FOIL $(x + 2)(x - 2)$, it equals $x^2 - 4$. $(x + 2)$ is two more than "x" and $(x - 2)$ is two less than "x" and the answer being "$x^2 - 4$" shows that it is four less than the original answer which is "x^2".

$(x)(x) = x^2$
$(x + 2)(x - 2) = x^2 - 4$

Zoe moved through the *image making* and *image having* stages by creating five examples. The structure of the Pattern Math activities themselves created an environment where all students were given the opportunity to see the pattern and to indicate that they had an image of the pattern by creating their own example showing the pattern. By describing the pattern that she sees in words, Zoe moves into the *property noticing* stage. Her description of the pattern already uses generalizations such as "a number" rather than a specific number which indicates that she is already at the more advanced level of *formalising*.

Zoe's writing above is a great example of what Pirie-Kieren call 'don't need' boundaries. Once Zoe has developed a formal description of the patterns that she has noticed, there no longer needs to be a boundary between *property noticing* and *formalising*. Similarly, once Zoe had an image of the pattern, she is able to make images of the pattern as often as she wants so there is no longer a need for a boundary between the *image making* level and the *image having* level.

After *formalising* by describing the pattern, Zoe continues to explain why the pattern works with algebra. She is taking the *formalising* one step further by reflecting upon it and organizing it as a general theorem. The process of reflecting and organizing the pattern as a general theorem is part of the level of *observing*. To move to the level of *structuring*, Zoe must show that she understands that the theorems created and explained in the *observing* stage are related and can be logically defended. I believe that her use of algebra shows that she does have the structure to warrant putting her at the *structuring* level of understanding. I can see how someone may argue that she has only exhibited work at the *observing* stage. Although her level of understanding has reached the *structuring* level, I am not able to make comments about the depth of her understanding. In this regard, I can only consider her understanding to be "thin" and not deep. Although I did not have the technicalities of the Pirie-Kieren model in my mind when I responded to Zoe, I did want students to develop a deep understanding of the pattern. My response to Zoe shows that I wanted her to explain more and to think about the pattern in more ways than just the algebra.

Great job describing the pattern. I liked how you looked at more than one example to check if the pattern works for other examples. To describe whether or not the pattern always works you went to algebra and used x to represent the number. Why is it that this proves that it will work for every number? Thinking about this will help you understand the power of algebra.

Now when you went to algebra you noticed the part B was $(x - 2)(x + 2)$ and that part A was $x^2 - 4$. I am wondering if you have ever thought of foiling or factoring a difference of squares as something that could help you with mental math? Have you ever made this connection before or are there other things that this makes you think of? Another thing you might want to look at is how multiplications are related to areas of rectangles. Whenever you multiply 2 numbers you can imagine a rectangle where one number is the length and the other the width. How does the pattern that we have seen here relate to areas of rectangles? Do the rectangles also help explain why the pattern works? Something to think about.

My response pushes Zoe to examine the pattern through other representations besides algebra. Although her algebra allowed her to explain why the pattern worked, I wanted her to have an even deeper understanding of the pattern and to see how it connected not only arithmetic to algebra, but geometric relationships such as area to arithmetic and algebra as well. If she were able to make these connections, I felt that her level of understanding would be even deeper than she currently portrayed.

The last level of the Pirie-Kieren model is called *inventising*. Although Zoe did not reach this level, I feel that this example provided an excellent opportunity for students to reach this level had they wanted to. At the *inventising* level, students take the theorems they have learned from the structuring level and break away from them to invent or examine new ideas. This Pattern Math compared the products of a number squared (x^2) and the product of two above and two below this number $(x + 2)(x - 2)$. A student could have entered into this level by creating new criteria such as looking at what would happen with three more and three less or four more and four less and so on. I was surprised at how few students went beyond and looked at other examples so I encouraged students in my responses to explore and look for new patterns. Often students would look for new patterns when they were stuck trying to explain the original pattern. I felt that these explorations may help them develop a strong image of the pattern and may help them come back and create a deeper understanding of the pattern.

I created the second Pattern Math to help students see how they could connect the previous Pattern Math to a more general idea. In a way, I was showing them what it meant to see a pattern and to extend the pattern. For Zoe, it was a case where she could again show the same structuring skills that she showed in the previous example since her algebraic technique would once again work to explain the mathematics.

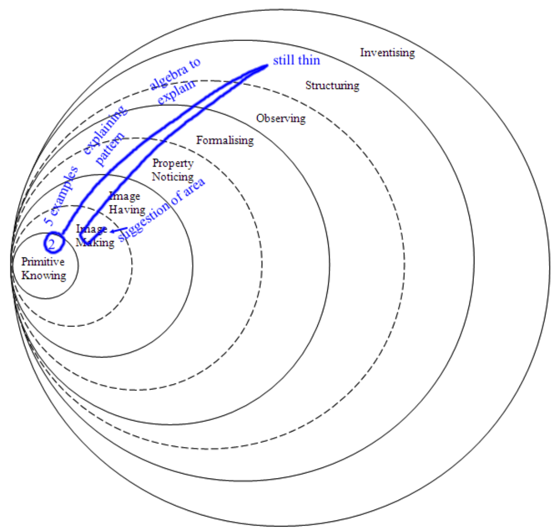


Figure 53: Pirie-Kieren Model for Zoe's second Pattern Math.

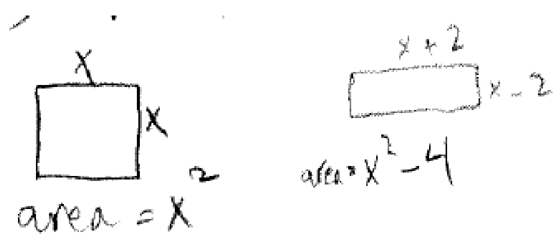
I noticed that when a number a is squared and we multiply x less than that number and x more than that, the answer will be x^2 less than the original answer. So, for example, if our original number was A and we'll do x more and x less the formula will look like this:

$(A)(A) = A^2$

$(A + X)(A - X) = A^2 - X^2$

$(12)^2 = 144 \quad (24)^2 = 576$

$(1)(23) = 23 \quad (14)(34) = 476$



Notice at the end of her writing that she includes the area idea that I had suggested through my response after the first Pattern Math. Using the Pirie-Kieren model, I noted once again that her understanding on the topic was still thin because she was able to explain it only with algebra. Her attempt to use areas is indicated by a folding back to *image making*. Despite making these images, Zoe did not show that she understood how the idea of areas was linked to the pattern. For this reason, the line ends in the *image making* level. She still has the *structuring* level as far as an algebraic understanding is concerned, but her overall understanding of the concept could be deeper. In my response to her I once again tried to connect the ideas of areas to how it could help her explain the pattern in a different way.

There are some very interesting things that happened during the third Pattern Math for Zoe. One question that I had asked Zoe in the first Pattern Math was for her to explain why algebra showed that it worked for every number. I often wondered when my students wrote algebra whether they understood that using x to represent a number meant that the pattern worked for all numbers. Looking at students' work, I often was led to believe that although students were using algebra to represent the pattern; their view of algebra was that a pattern only worked with natural numbers since those were the ones used in the examples. For example, in the first Pattern Math, I would love to be able to ask students who used algebra to explain the pattern if they had started with 9.3 as a the number, would (7.3)(11.3) still be 4 less than 9.3^2?

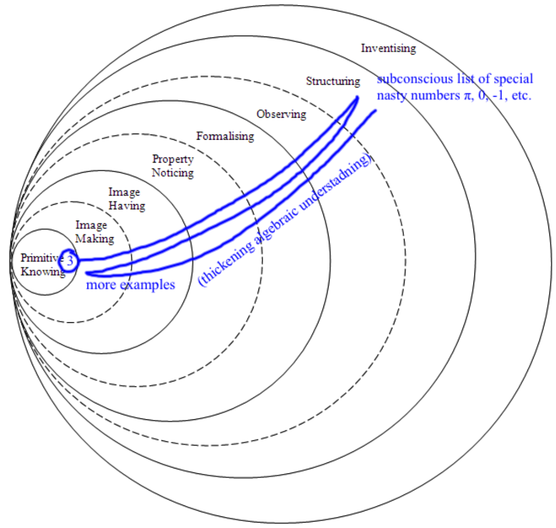


Figure 54: Pirie-Kieren Model for Zoe's third Pattern Math.

Zoe's response in the third Pattern Math seems to answer this question. After proving the pattern algebraically she continues to show examples of numbers other than natural numbers that also fit the pattern. She uses the numbers -1, 0 and π to show that the algebra does not only satisfy the natural numbers that appear in the examples, but also other numbers as well. It seems like Zoe has used this special list of 'nasty' numbers before as numbers that don't always seem to fit the pattern and therefore are great to use to check to see if the pattern still works.

I notice that a number squared plus that number and plus one more than that number is equal to one more than the original number squared.

$x^2 + x + x + 1 = x^2 + 2x + 1$
$(x + 1)^2 = x^2 + 2x + 1$
$1^2 + 1 + 2 = 4$
$2^2 = 4$
$0^2 + 0 + 1 = 1$
$1^2 = 1$
$(-2)^2 + (-2) + (-1) = 1$
$(-1)^2 = 1$

$\pi^2 + \pi + (\pi + 1) = 17.152789...$
$(\pi + 1)^2 = 17.152789...$

It even works with negatives and zero.

There can be two interpretations of this data. One interpretation is that Zoe did not initially see the algebra as extending to numbers beyond the natural numbers that were given in the first example. If this was the case, when she tried the numbers -1, 0 and π, she was folding back to *image making* and redefining what she understood as algebra. Although she may have understood that algebra meant that the pattern works for all numbers, she has now seen examples that verify this statement to be true. A second interpretation is that Zoe is directly responding to my comments from the first Pattern Math. She is telling me that she does understand that algebra means that it works for all numbers and she is showing me these results by using a specific list of numbers that would indicate numbers of all types work. She has chosen a negative number, zero and an irrational number to show this. Regardless of which interpretation is used, what is known at the end of this Pattern Math is that Zoe does fully understand that using algebra is representative of all numbers. After the first two Pattern Maths, I was not sure that this was true and for this reason characterized her understanding as thin. After the third Pattern Math, I was aware of this deeper understanding. This is also reflected in the Pirie-Kieren model where the folding back and the returning to the *structuring* level shows a 'thicker' level of understanding.

During the fourth Pattern Math, Zoe does not immediately turn to algebra to explain why the pattern works. Although she has noticed the pattern and can describe it in words, she turns to diagrams and areas to explain the meaning of the pattern. Using the Pirie-Kieren model, I mapped this as movement towards *formalising* and then the act of drawing the diagrams was a movement back towards *image making* and *image having*. She generalizes from specific examples she has drawn that when you subtract a square from a larger square you are left with the sum of the numbers.

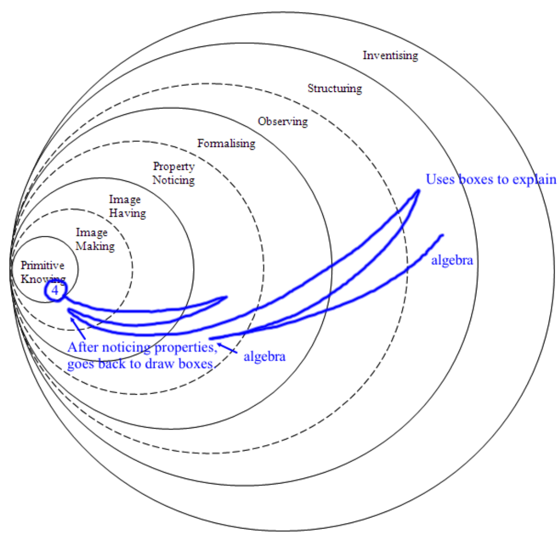
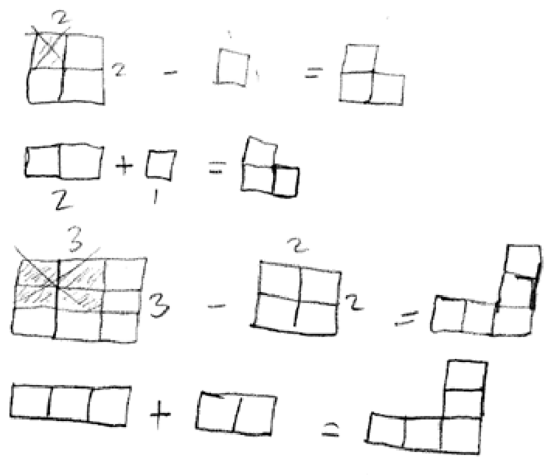


Figure 55: Pirie-Kieren Model for Zoe's fourth Pattern Math.

She writes, "So after the squares get cancelled out we are left with the numbers who get added and give us the same result as the addition."



]

I notice that a number squared minus one less than that number squared is the same as the original number + one less than it.

It works because the two squares cancel each other out and the signs cancel each other out. So after the squares get cancelled out we are left with the numbers who get added and give us the same result as the addition.

$$x^2 - (x-1)^2 = x^2 - (x^2 - 2x + 1)$$
$$= x^2 - x^2 + 2x - 1$$
$$= 2x - 1$$

$$x + (x - 1)$$
$$= 2x - 1$$

This process of generalizing from the specific examples she has drawn is an act of *observing* and *structuring*. A general theorem is emerging from the diagrams she has drawn. This theorem is strengthened by her return to algebra. This is noted in the Pirie-Kieren as a folding back to the *formalising* stage. I have indicated that she returns to the same stage as she was before her work with diagrams. Her algebraic explanation further explains why it is that her geometric explanation makes sense. This explanation helps her understand the theorems she was creating in a richer and deeper way as she has seen the explanations through two different representations. She has explained the pattern both algebraically and geometrically. At this point, I can see that she is making the connection between multiplication and areas that I suggested in my first response back to her.

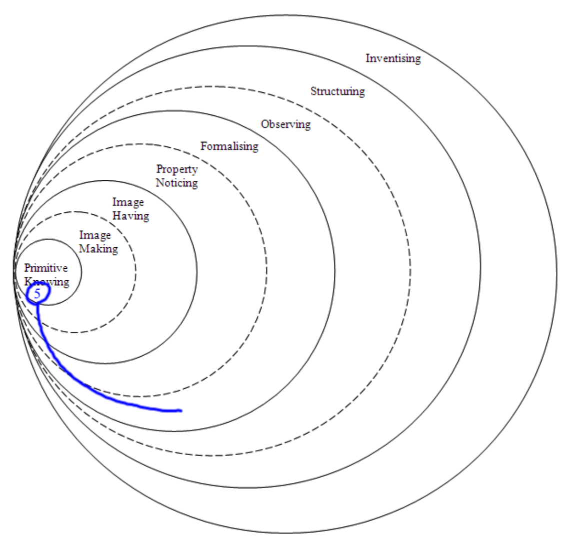


Figure 56: Pirie-Kieren Model for Zoe's first Pattern Math.

The fifth Pattern Math asked students to think about patterns in a different way than the previous four Pattern Math activities. During the first four Pattern Math activities, Zoe was showing how she could understand patterns and explain patterns with algebra and with diagrams. I challenged her in the first Pattern Math to be more rigorous in explaining why algebra meant that the pattern worked for all numbers. During the third Pattern Math, Zoe showed that she understood that algebra meant that it worked beyond just 'nice' natural numbers and that the pattern worked for zero, negative numbers, and irrational numbers such as π. The fifth Pattern Math challenged students to see if they could make the connection that algebra also works for algebra. In other words, the fifth Pattern Math was just an extension of the previous Pattern Math. Instead of using two numbers whose difference was one, the Pattern Math always used two expressions whose difference was one unit. Therefore, if the algebraic expression worked for any numerical value in the fourth Pattern Math activity, then the algebraic expression would also work for any algebraic expression as well. Thus, the fifth Pattern Math was examining the idea that algebra works for algebra as well.

As could be expected, this fifth Pattern Math proved to be difficult for Zoe and the other students. Her response is short and is limited to doing some examples and describing the pattern she notices. Although she notices the pattern, she finds that it is very difficult to explain. She notices that the two polynomials differ only by one, but she is unable to make the connection to the previous Pattern Math or the algebra that she has already developed. As a result, for the fifth Pattern Math she is able to work at the *pattern noticing* and *formalising* level, but unable to advance from there.

I notice that a polynomial squared minus a polynomial that is one more or less than the former polynomial will result in addition of the numbers and variables in both polynomials without squaring them in the first place. I think it only works if the polynomials have the same signs. This pattern is very hard to explain.

The Pirie-Kieren model was used to document the growth of understanding for all students in the study. As can be seen in Zoe's example, the model can be used to look at how a student moves back and forth between various stages. This movement indicates a growth in understanding and documents how a student comes to understand mathematics. The following chapter will look at some of the themes that emerged from the development of the Pirie-Kieren for all students.

Chapter 8 – Themes from Analysis of Pirie-Kieren Mappings

The purpose of this chapter is to summarize the findings from the development of the Pirie-Kieren model for all students and to give the reader a sense of some of the common themes that emerged. I will explore each of the three cycles separately and describe some of the themes that I noticed during these cycles.

During the first cycle, I noticed three important themes. The first theme was that many students were able to reach the *property noticing* and *formalising* levels, but were not able to move beyond these levels. Since algebra provided one way for students to explain why a pattern worked, a second theme emerged that while some students were able to use algebra to explain a pattern, other students were only using algebra to represent a pattern. A third characteristic of the first cycle was that students began to do more folding back as the activity progressed. I will now explore each of these three themes in more detail.

The first theme from the first cycle was that students were unable to move beyond the *property noticing* and *formalising* levels of the Pirie-Keiren model. I was surprised at how few students were able to reach the *observing* and *structuring* levels of the model. During the first two Pattern Math activities only 6 of the 14 students were able to use algebra as a way to generalize and explain why the pattern worked. Even within those six students, I was sometimes unsure from the explanation whether or not they understood why the algebra explained the pattern.

I used my response in the interactive writing process to help students communicate and explain the patterns that they saw. To many students I suggested using algebra as a way to explain the pattern. I also suggested the use of areas as a way to see the pattern from a geometric perspective. Below is an excerpt from a response that I wrote to more than one student. The purpose was to give students more options to explore and to explain when they felt they were stuck or felt that they had finished explaining their ideas fully.

If you feel like you were finished, which I think that you did since you described the pattern well with algebra, you can take the time to extend the problem or connect the problem to things that we have already done. This will develop a deeper understanding for you. For example, once you “finished” your explanation, try to play with it, extend it or connect it to other ideas. By playing with the pattern, I mean try a bunch of numbers to see what happens. I saw that you tried negative numbers to see if you could find one that didn't work. That was good. Did you try a bunch in your head? Like 48 x 52, 68 x 72? Did you try any backwards? Like what is 22^2? Well you know 20 x 24. Could you do 18 squared, or 102 squared? By extending the pattern I mean that you could play around with the pattern to see if other patterns exist. For example what about the relationship when it is one more and one less such as 19 x 21 and 20 squared? What about 3 more 3 less such as 7 x 13 and 10 squared? This is actually something we are going to do in the second Pattern Math so you will be able to extend the pattern and play with the pattern more. By connecting the pattern I mean that you can take the pattern and see what it means if you looked at it in different ways. You looked at it algebraically and it explained why it worked. It is a parabola, you could have made this connection and explained it graphically. You could also have thought about the multiplication as an area. Any time you have the product of two numbers you can think of it as the length and width of a rectangle and the answer is the area of the rectangle. How do the rectangles also explain why the patterns work?

As seen above my response suggested a number of things that could help students communicate what they saw and to explain the patterns they saw. I suggested the use of algebra. I also suggested examining different numbers to help clarify where the pattern worked and where it did not. These suggestions helped students look at patterns in more ways, but I was still surprised at how few were able to move to a level where they had a full understanding of the pattern and how it worked.

In order to answer why it was that so many students were unable to move beyond the *property noticing* and *formalising* stages I noticed an important theme relating to the use of algebra. Although initially only 6 of the 14 students moved on towards the *observing* and *structuring* level using either algebra or areas, all students had used algebra in some way in their Pattern Math explanations. The main difference was that some students were using algebra as a way to *explain* the pattern that they saw while some students only used algebra to *represent* the pattern that they saw. Those students who used *algebra to explain* were able to see how the pattern was generalized and how the algebra provided a way to explain the pattern and why it always worked. In Carly's work below you can see that she is using algebra to explain why the pattern works. Even though the communication is somewhat difficult to follow, a careful reading shows that Carly has made the connection that when you take $(x – 2)$ and $(x + 2)$ and multiply them you get an answer that is always four less than x^2.

The pattern is the answer of A subtract 4 is the
Answer of B. Also, the numbers of B is A subtract
two and A plus two. The pattern always works.

$y=(x-2)(x+2)$

By subtracting 2 and adding 2 it is like finding a number in
the middle. Once you square that number you have to
Subtract 4 So you can get the answer to the other two
numbers. If number ~~A~~ is x then the answer to B is
y it is the same thing as Squaring x and subtracting
4. When you factor the equation it becomes 36
$y=x^2-4$. Which is the same as Answer A subtract
4 to get answer B.

Figure 57: Carly's communication using algebra to explain why the pattern works.

For Sun, the algebra shows that it works for every number. She describes that the use of a variable implies that the pattern works for any number. She understands the power of algebra to explain why the pattern worked.

This pattern always work because
when we use variable it works.
we can plug in any number.

$$(x)(x) = (x+2)(x-2)\ (+4)$$
$$x^2 = x^2 \cancel{-4}\ (+4)$$
$$x^2 = x^2$$

Figure 58: Sun's communication using algebra to explain why the pattern works.

Other students were able to use *algebra to represent* the situation, but were unable to use the algebra to show why this worked. In Kayla's work below, algebra is used to represent the values and to describe the patterns that are apparent in the question, but the algebra does not explain why the pattern works nor why the pattern would work with any number.

1) The two numbers that are used in B are two less and two more than the number squared in A. Then what B equals is 4 less than what A equals. You would be able to do mental math easier, because maybe you'd know 13^2 but not (11)(15) and you could use this pattern to help you solve things mentally that would be harder otherwise.

$x^2 = y$

$(x-2)(x-2) = y-4$

Figure 59: Kayla's communication using algebra to represent the pattern.

In my response to Kayla, I tried to encourage her to look beyond just using algebra as a way to represent the pattern, but as a way to explain the pattern.

Kayla,

I liked your comment about how the pattern was a useful way to do mental math. This is one of the reasons that I like the pattern so much. If you know 50x50 = 2500 then you also know 48x52=2496. Now, why does that work? How do you show that a pattern that seems to work for some numbers works for other numbers without trying all the other numbers? This is one of the things that I hope you explore more as we look at more patterns similar to this one.

Kayla, at the end of your writing it looks like you tried to write out the pattern using algebraic symbols. This is another way to represent the pattern. You might want to explore what your algebraic symbols mean and maybe that will help explain why the pattern works.

My response to Kayla was typical of my responses to students who used algebra to represent. I wanted her to look at the ideas further and explain why the pattern worked with algebra. Although I encouraged all of the students who used algebra to represent to explore the idea of using algebra to explain, in subsequent Pattern Math activities, only two students made the shift. Kayla was one of the students and you can see in her work below that she is starting to use algebra to explain.

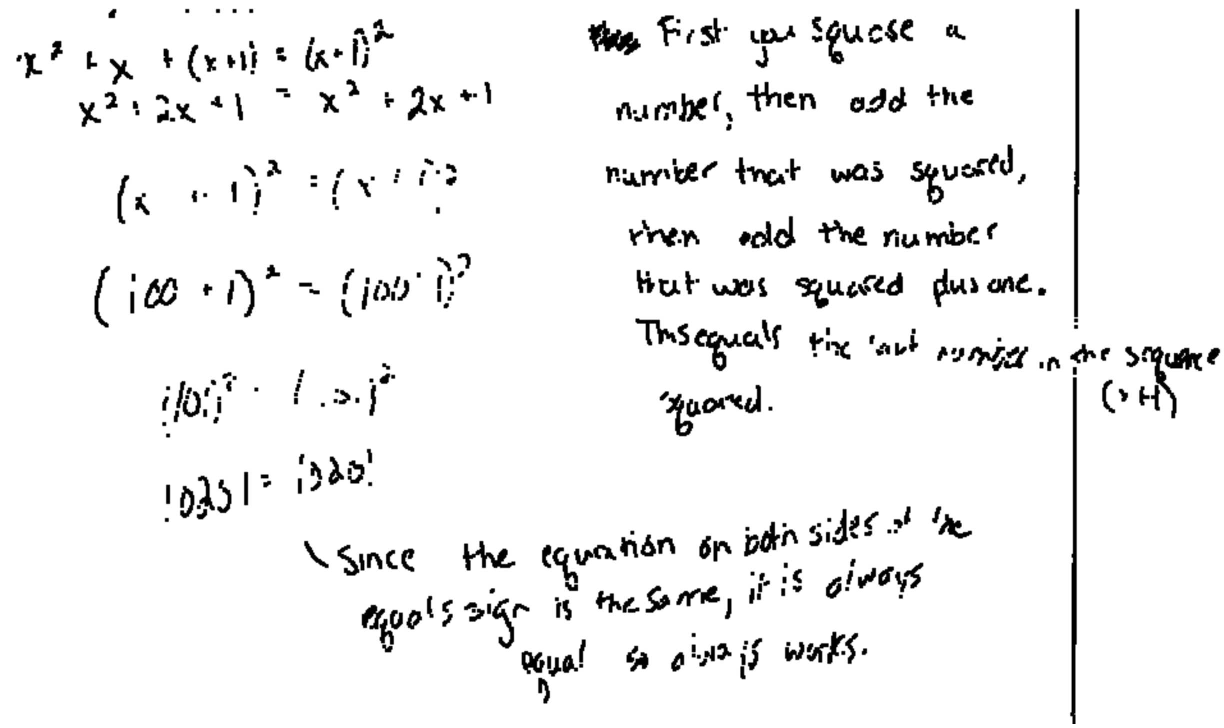


Figure 60: Kayla’s communication using algebra to explain why the pattern works.

While only 6 of 14 students were using algebra to explain at the beginning of cycle 1, only 8 of 14 students were using algebra to explain by the end of cycle. This helps explain why many students were not able to pass the *pattern noticing* stage of the Pirie-Kieren Model. Those students who were stuck and unable to use algebra to explain were continually encouraged to explain why the pattern they saw worked and given suggestions such as algebra and areas to do so. Those students who did use algebra to explain were encouraged to explore other methods of explaining the pattern (such as the use of areas) or to extend the pattern to see what else they could discover.

It is not surprising that as a result of my responses, students began to do more folding back as the activities progressed. My responses provided students with suggestions when they felt that they were either done or stuck trying to explain the pattern. Earlier I mentioned in one of my responses that when students felt that they were stuck or done explaining that they could play with the pattern, extend the pattern, or make connections to other mathematical concepts that they knew. With many of the students, this meant that when they were stuck that they tried more examples to see if they saw other patterns or tried similar patterns with different restraints. The results of these explorations were that students were often able to more fully describe the pattern and some of the restrictions of the pattern. The folding back did not necessarily result in the student moving beyond the *property noticing* and *formalising* stage, but it did mean that they had a more developed view of the pattern as compared to those students who did not fold back to more examples. Below is Dave's work on Difference Equals Sum. Although he is stuck in trying to explain why the initial pattern works, he does fold back to do more examples and notices a new pattern that all of the sums happen to be odd numbers.

I noticed the sum of the two numbers that aren't squared equal the difference of the same two numbers squared. I find this to be an easier way to find the answer, if you find "B" first then "A" is the same answer. I have tried examples with five decimal places and this pattern still works.

$2+1=3$
$3+2=5$
$4+3=7$
$5+4=9$
$6+5=11$
$7+6=13$
$8+7=15$
$9+8=17$
$10+9=19$

All of the answers are odd numbers.

A number squared minus one less than a number squared will always equal an odd number.

$$x^2-(x-1)^2 = \text{odd number}$$

Figure 61: Dave's communication showing folding back.

The second cycle of Pattern Math activities focused on the sums of different sequences of numbers. The patterns that emerged were less likely to be explained by algebra and more likely to be explained through an understanding of concepts such as averages and rearranging the sums in different ways. The students noticed this shift in the second Pattern Math and very few (only 3 of the 14) students tried to use algebra as a way to represent and explain the patterns that they saw. An analysis of the Pirie-Kieren models that I created led to the following two themes that I would like to discuss. First, all of the students on all of the activities reached the stages of *property noticing*, but only 8 students moved on to *formalising* and one student to the *observing* stage in only one activity. Second, students spent more time extending and modifying the patterns compared to the previous cycle. Following a discussion of these two themes, I will explore an interesting observation that emerged from a comment Tom made about explaining mathematical ideas.

The first theme that emerged from the data was that the majority of students settled at the *property noticing* or *formalising* stage on the Pirie-Kieren model. This contrasted the first cycle where many students were able to generalize and structure the patterns they saw at an *observing* or *structuring* stage. In the first cycle, students generalized using algebra as a way of explaining why the patterns always worked. The sequences in the second cycle were not conducive to using algebra and very few students attempted this. When students entered the formalising stage by explaining the pattern they saw, they often did it through specific examples as opposed to generalized ideas.

The sequences proved to be more difficult for students to explain the patterns that they saw. Of all the students, only one student was able to explain why the pattern worked with specific examples (thus reaching the *formalising* stage) in all three Pattern Math activities. Another student explained why the pattern worked in two of the three activities and 6 out of the 14 students were able to explain one of the three Pattern Math activities in this cycle.

These results indicate that the level of difficulty of this cycle was in the students' zone of proximal development. Students were in the process of developing a mathematical understanding of the concepts. They were unable to generalize the concepts and continued to fold back to *image making* and back to *property noticing* to see if they could formalise what they saw.

The second theme of the cycle, that students spent more time extending and modifying the pattern, is a direct result of the fact that students were attempting to formalise the pattern. Since students were attempting to explain the pattern and getting stuck, they followed a suggestion that I gave them in the first cycle through my responses. I suggested to students that when they get stuck that they could extend the pattern or modify it to see what other patterns they saw. When students modified the pattern to look for other patterns they often described the scope and limitations of the pattern. In Jeanette's work below in Add Up Add Down, she has reached the formalising stage with a specific example, but is unable to generalize the pattern. As a result, she goes back to image making and modifies the pattern. Her modification leads her to a stronger image and she notices certain limitations to the pattern as a result.

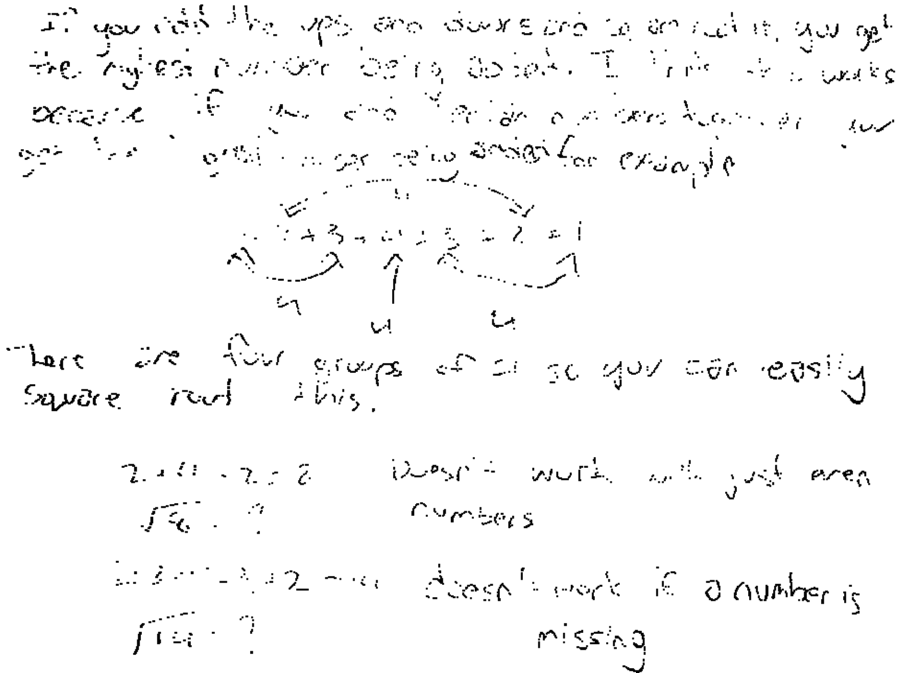


Figure 62: Jeanette's communication showing extensions and modifications to the pattern.

Jeanette's work above was typical of many students when they were stuck trying to explain why the pattern worked. Students would try even numbers or starting at different numbers to see if these properties were important to the pattern.

Although the majority of students attempted to explain why the patterns worked, there were some students who only described the pattern that they saw. Most of these students I gave encouragement and suggestions after the first Pattern Math of the cycle and they attempted to explain the patterns that they saw in the second and third Pattern Math examples. There was one student, Tom, who did not attempt to explain the pattern in all three examples and I noticed something in his writing that explained the situation. Since the explanation has significant implications, I thought that it would be important to share as a result from the second cycle.

Throughout the first and second cycle I was constantly responding to Tom by encouraging him to explain why the pattern worked. After the first Pattern Math of the second cycle I wrote, "Also, while you are exploring patterns always take the time to attempt to explain why the pattern works." After the second Pattern Math I reiterated this request, stating, "Another thing that I am going to keep pushing you to do more of is trying to explain why the pattern works." Below is Tom's work on the third Pattern Math of the second cycle.

I can see that this pattern isn't completely
making scence to me.
So after a cauple days I still couldn't get
this pattern. I think I was thinking too far outside
of the box it's way simpler. But hard to
explain. I will try to explain it like so...
the first odd # is one cubed, the next 2 are 2 cubed,
the next 3 are 3 cubed and so on. A pattern
similar to this won't work you would have to
radicaly change it or something. This pattern
is very interesting but it seems pretty useless
other than to admire it. Like to try to count
to the hundreds just to figure out the very
very very next would be redicalous.

Figure 63: Tom's communication where a description is seen as an explanation.

In Tom's writing, he communicates that the pattern was difficult for him and hard to explain. He then states, "I will try to explain it like so …" Following this statement is a description, not an explanation of the pattern. It appears that for Tom, the idea of explaining is limited to only describing the pattern not showing how or why the pattern always works. Perhaps he felt he was responding to my comments to explain the patterns when he was describing patterns. I wonder how many students felt that their description of a pattern was the same as explaining why the pattern worked. This would be consistent of different types of understanding as presented by Skemp (1978). Skemp differentiates between relational understanding and instrumental understanding. Perhaps the students who continue to describe the pattern they see feel like they are explaining the pattern. They don't understand that describing "how it works" is different from explaining "why it works." Before Tom is able to explain why a pattern works, he first must come to understand what it means to explain something mathematically.

The third cycle of the Pattern Math focused on different methods of multiplication. The level of difficulty was less than the second cycle as many students made connections to previous mathematics that they had learned to explain the patterns that they saw. The purpose of the cycle was to allow students to see multiple methods of multiplication in the hopes that understanding multiple methods would give them a stronger understanding of the methods they already knew and a stronger sense of multiplication in general. Two themes emerged from the last cycle. The first theme was the use of place values in the explanations of why the patterns worked. The second theme was the ability of students to compare and connect the different multiplication techniques with methods they had used in the past.

An understanding of place values in multiplication provided students with a way of explaining and understanding the Pattern Math activities of the third cycle. Eight out of the fourteen students were able to make this observation and use it to explain the pattern in Ancient Chinese Multiplication. For these eight students, I thought that the explanations for the next two Pattern Math activities might become monotonous because the same pattern could be explained in the same way. Surprisingly, in the Napier's Bones, only 4 of these students continued with the idea of place values to explain and only 6 students used the idea to explain Charts. In Charts, I also created a question that was not separated by place values to challenge these eights students to explain the pattern in a different way. Of these eight students, three of them came up with clear explanations why it worked in a different context. Four students made attempts that showed some understanding but their communication was unclear.

The six students who did not discuss place values only provided a description of the pattern and did not attempt any form of explanation. Perhaps they felt that their description of how it worked was an explanation of why it worked as discussed earlier with Tom. In order to provide scaffolding for these students I suggested the following to each of the students in my response to them. I wrote the following to Chase and copied it in my responses to other students who did not see how they could use place values to explain.

Sometimes the process of finding out how it works is the process of describing all of the little details and being observant while you describe the little details. For example, a little detail to look at is how we found the number of 100s that would be added. In the first example there was only 1 100 because it was the intersection of the 1 from the 12 and the 1 from the 17. In the second example there were 6 100s because it was the intersection of the 2 from the 23 and the 3 from the 34. And in the third example there were 4 100s because it was the intersection of the 4 from the 41 and the 1 from the 15. Now, I just took the time to describe part of the pattern. After describing it I can now look back and try to figure out why this works. By writing it down, I can perhaps see patterns that I might miss if I didn't write it down as I don't have to mentally think about so many things at once. Now I can ask why the hundreds work. Perhaps you want to look over these numbers again and think about why they work. Try some ideas and if you still are stuck come and ask me some questions. Understanding why this pattern works may help you see why the next couple of Pattern Math ideas work.

The purpose of my writing was not to show them the correct way of explaining the pattern. You will see in my writing that I was presenting the students with a way of organizing the data and a way of providing a venue where the students can ask themselves why the pattern works. My response is leading, but not telling. Despite this response to each of the six students, only Chase, and only in the third Pattern Math, Charts, considered place values and was able to explain why the pattern worked.

The second theme that was apparent was that students made connections to other forms of multiplication that they had learned in the past. I was pleased to see this because I think it helped students not only understand the pattern they were studying better, but it also helped them understand their 'old' techniques better as well. These connections came in various forms. Some students connected this to a traditional method of multiplying as shown in Zoe's example below.

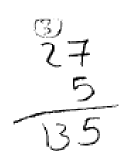


Figure 64: Zoe demonstrating a method of multiplication.

Hana connected the mathematics of Ancient Chinese math to a method she had learned earlier. I really liked this method because it was one that I had never seen before. Her work is shown below.

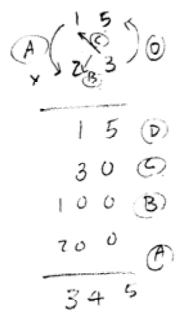


Figure 65: Hana demonstrating a method of multiplication.

Other students made connections to the multiplication of two binomials, commonly called FOILing, to help them see why the pattern in Charts worked. What fascinated me about this connection is that students were using something that they had learned almost exclusively for multiplying with variables to multiplying with regular numbers. Students who made this connection were able to see again that the use of variables is to represent any number. If a process works with variables then it works with numbers. It is interesting to note that using this technique of FOILing with regular numbers does go against procedures (such as brackets first before multiplying) that the students had learned. I think this shows that the students are developing their conceptual understanding as opposed to their procedural knowledge. Below is Megan's work as she connects the Charts method to the method of FOILing.

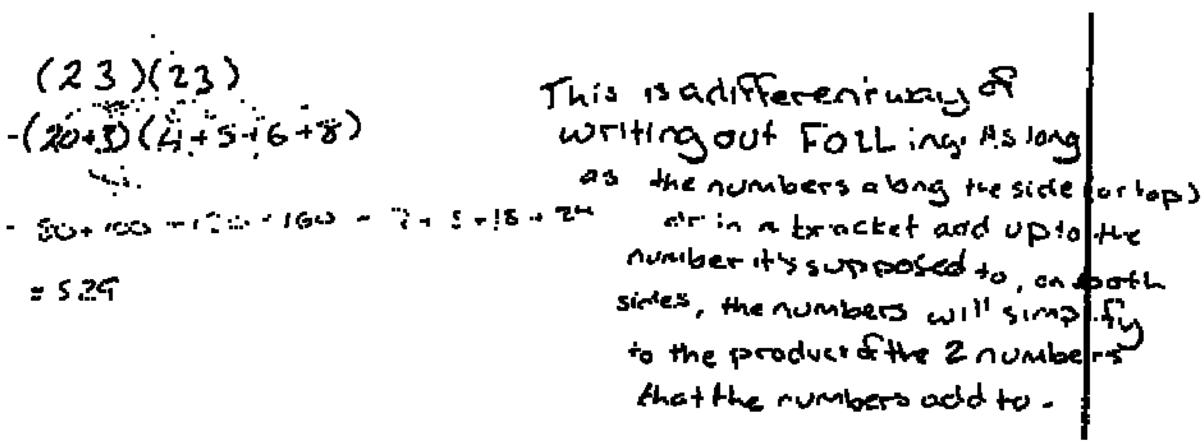


Figure 66: Megan connecting "Charts" to FOILing.

This chapter discussed the different themes that emerged from the analysis of the Pattern Math activities. The next chapter looks at student reflections from the end of each cycle and from the closure interviews.

Chapter 9: Student Reflections

At the end of each cycle students were given questions to reflect on what they had learned and what they had found valuable. These reflections along with the closure interviews provided the data that will be used to answer two of the three research questions of this study. In particular, the data will shed light on how the Pattern Math activities contributed to the development of students' abilities to think mathematically, to communicate mathematically and to connect mathematical concepts. The data also describes how students interpreted their experiences with the Pattern Math activities with respect to their view of learning mathematics. The Pattern Math activities were set up to provide students with an opportunity to think mathematically, communicate mathematically and make mathematical connections. The extent to which the students did these things is reflected in their comments.

This chapter will focus on the students' reflections of the Pattern Math activities. Students' comments have been organized around the research questions of thinking mathematically, communicating mathematically and making mathematical connections. The comments will also examine how students interpreted their experience with respect to learning mathematics.

Thinking Mathematically

"I learned that there always are patterns in math that can make doing math much easier or harder. But the good thing is that it makes you think." – Zoe

Zoe's comment after the first cycle of Pattern Math activities expresses a positive attitude toward thinking in a math class. The Pattern Math activities gave students the opportunity to think about mathematics in ways that was different to their other mathematical experiences. For most students, many mathematical problems can be solved using methods that were presented during class. The Pattern Math activities did not present students with a method for solving problems, but invited the students to notice patterns and explain these patterns.

Many students valued the process of finding and noticing new patterns. Justin writes, "After spending a lot of time working on it, I found that there were a lot of other patterns and rules that I hadn't noticed before." Justin's comment shows that thinking about a concept leads to new mathematical understandings that he had not noticed before. There is also a sense of ownership when students are thinking mathematically and noticing patterns themselves.

After the first cycle, Bryson explains that the most interesting Pattern Math activity for him was the "one that got me thinking the most." Bryson writes, "I felt that by the end of this cycle I was really able to think about patterns in ways I couldn't before. The Pattern Maths really got me thinking of clever ways to explain the pattern. I also found that having me explain the patterns got me to think about the patterns more than I would have if I was just told to find the pattern." The last statement that Bryson writes summarizes how the Pattern Math activities contributed to the development of students' abilities to think mathematically. The Pattern Math activities asked students to not only find patterns but to explain the meaning behind those patterns. Communicating the meaning of the pattern led students to think about the pattern more than if they were just asked to describe the pattern or show another example that followed the same pattern. As part of the interactive writing process, the response that I gave to students often encouraged students to extend their thinking in new ways or to encourage them to think further. Explaining the patterns provided a venue where students could develop their abilities to think mathematically.

Communicating Mathematically

"I have also developed the ability to describe mathematical ideas using words, due to all my experience with doing so on the Pattern Maths." – Nate

"When we first started the Pattern Math activities, I found that what I was trying to say was hard to put in words. Over time, I noticed that communicating my own ideas and noticing other patterns became easier." – Justin

"I think that my communication skills have greatly improved since the beginning of the Pattern Math activity. I am definitely better at communicating mathematically thanks to the Pattern Maths." – Chase

Just as the Pattern Math activities provided opportunities for students to develop their abilities to think mathematically, they also provided students with the chance to enhance their communication skills. It may be argued that any activity that practices communication skills will enhance communication skills. It is not my purpose to point out the obvious that since students had little practice communicating mathematical concepts prior to the research that the act of communicating mathematics over a period of 13 Pattern Math activities increased the students' abilities. I believe that the Pattern Math activities provided a rich environment where the development of communication skills was not only the result of practicing, but seen as something that was valuable to mathematical understanding.

Mathematical communication is seen in curriculum documents as a "critical aspect of learning, doing and understanding mathematics" (Manitoba Education, 2009, p. 6). The document stresses that "students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas" (p. 6). The Pattern Math activities provide students with the opportunity to communicate in mathematics especially in the areas of writing about, viewing, representing and reading about mathematical ideas. The response from the teacher also provides a forum for further reading about, listening to and discussing mathematical ideas.

The interactive writing process was essential to the development of students' abilities to communicate mathematically. The process began with students writing their explanations for why the pattern worked. I wrote back to each student individually commenting on their mathematical explanations and encouraging them to explore new ideas or explain things further. This meant that the Pattern Math activities were more than just practicing mathematical communication. Students knew that the teacher was reading and responding to each of their communications. This created an audience for their communication and gave them a reason to communicate their ideas. The responses also encouraged them to improve their communications. It gave them opportunities to think about the patterns in new ways which they could use in subsequent Pattern Math activities. I believe that the structure of the interactive writing provided students with the opportunity to greatly improve their communication skills.

After the first and second Pattern Math cycles, I asked students what they found was the most valuable thing that they had learned. After the first cycle, Jeanette wrote that the most valuable thing she learned was "learning how to communicate." She reiterated this after the second cycle by writing, "I felt that learning how to communicate helped me a lot and is valuable." What Jeanette and others discovered was that they were able to communicate best when they understood the mathematics. Being unable to fully describe why a pattern worked meant that they did not have a full understanding of the mathematical ideas. Jeanette writes, "I feel that adding odds was my best explanation because I explained how it worked and how it doesn't work." Bryson also articulates that the process of communication helps understanding. He writes, "From this set of Pattern Maths, I really learned that explaining new ideas and checking them is the best way to find the pattern." Bryson continues further to state, "The Pattern Maths have really pushed me to describe the math in words, and I feel that's helped me immensely."

Connecting Mathematical Ideas

"Everything has some connection with each other through patterns, except that they are written differently." – Ivan

"One thing I learned that I think is valuable is seeing similarities between patterns, and how patterns work, and why they work the way they do. Also, all the patterns themselves seemed quite valuable." – Nate

"I think doing the Pattern Math helped me make connections in math to better understand concepts better." – Hana

In addition to thinking mathematically and communicating mathematically, the Pattern Math activities also allowed students to make connections between different mathematical concepts. Each cycle had certain mathematical themes. Because of these common themes, students were able to make connections between the different Pattern Math activities in the cycle. The first cycle looked at how algebraic and geometric concepts could be seen through arithmetic examples. In particular, topics such as differences of squares occurred over many of the activities in the first cycle. Ivan commented that he made the connection between using algebra to find new arithmetic formulas. He writes, "When patterns are written out algebraically, new ones can be formed."

The second cycle looked at the sums of different sequences and the patterns that emerged. Megan noticed how adding up a sequence of numbers was connected to the idea of averages. She writes, “Adding odd numbers can produce averages, which can make the question easier to solve. If there are averages, you might be able to multiply, which will simplify the question even more. The more you manipulate the numbers, the more things you find that you wouldn’t have thought of otherwise, leading to more patterns, and there is no one right answer.”

The third cycle examined different techniques of multiplication. Hana commented that learning different ways to do the same thing was valuable. She writes, “I think learning that there are many methods to learn one concept was valuable. For example, for multiplying there was Chinese Multiplication, Napier’s Bones and my elementary school methods. I think this will help me find other methods to learn if I don’t quite understand any other math I learn in the future.”

The Pattern Math activities were designed to give students the opportunity not only to see connections between the different activities, but to also make connections to other things they have learned in mathematics. A great example is that of Megan. In the third cycle of Pattern Math she made the connection that FOILing (multiplying two binomials) also works with regular numbers. Although Megan had multiplied two algebraic equations numerous times before, she had not made this connection. In her closure interview she explained, “without doing these Pattern Maths I would have not made connections like the FOILing with regular numbers or just making connections with different things that wouldn’t have occurred to me otherwise.”

In another example, Ivan explained how he made a connection between a problem he had learned in the past, but saw no use for the numbers. In Adding Up Adding Down, Ivan discovered the triangle numbers appeared as he added the numbers up. This connection intrigued him and made him feel like he explained the pattern really well. He writes, “I think I explained the adding up adding down Pattern Math the best because I finally found a use for perfect triangle numbers that I researched upon many years ago because a riddle asked me, ‘What does 1, 3, 6, 10, 15, 21, and 28, have in common?’”

Making connections between different mathematical concepts helps develop mathematical understanding. The Manitoba Curriculum Framework (2009) of outcomes lists connections as one of seven critical mathematical processes. “Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant, and integrated" (p. 9). Making connections between concepts creates a better mathematical understanding. Responding to a question about the importance of making connections between mathematical concepts, Zoe reflects, "Well if you see that everything is connected then you don't think there's lots to understand. You understand one and you can make connections with everything else that you know then it's easier to remember that one because you're like, oh, it's just the same thing that I learned before it's just more advanced." This statement highlights the power of making mathematical connections. If you take the time to see how a concept is connected to other concepts it makes understanding the concept easier.

Students' Views of Learning Mathematics

"The value of what is learned from these patterns is much greater than the value of knowing the pattern itself. I have learned to look at math from a different point of view and to explore math problems because they are not always as they seem. I learned that when you really look into a problem, you begin to find all sorts of patterns, rules and exceptions." – Justin

"Doing Pattern Math helps me discover simple patterns in math which can be used in more complicated and difficult problems to reduce the number of steps used and make things easier. It also helps open the mind to new and creative methods and ideas of doing math." – Nate

"I learned that you should always look beyond what you see in the first impression. When I looked at the Pattern Maths for the first time, I thought it was something you learn and remember and is always the same but once I saw the different cool patterns it made me think of math as a whole new world ready to be discovered and explored." – Zoe

At the end of each cycle as well as in the closure interviews, I asked students what they learned that they felt was valuable. The purpose of this question was to see how students' interpreted their experience with the Pattern Math activities with respect to their view of learning mathematics. From an analysis of these statements I noticed two main themes. The first theme was that having ownership in the discovery of mathematical ideas was valuable to students. The second theme was that students came to view understanding mathematics as both understanding how and why something worked. I ended my closure interviews with students by asking how they will approach learning math next year. After discussing each of these themes, I will explore this last question.

Students commented that making mathematical discoveries was fun. I was actually surprised at how often students used the word fun when describing what valuable things they had learned. Here are some examples from the students' reflections.
Hana writes, "I found 'Adding Up Odds' the most interesting probably because I found something in the pattern. I don't know about the other ones, but for this one, I was very excited through the whole exercise that I may have found something interesting. The more I played with and made connections, it narrowed down the possibilities of the answers. Very fun." Kayla writes, "That if you try and just experiment with everything and then you can find different patterns and new things. Anything you can do could end up being a new pattern. Also the more you say and the more patterns you find the more fun it is. Also even if what you try doesn't work, it's still communicating." Nate writes, "I found the Adding Up Odds pattern to be the most interesting because I was able to discover and explain how it works. Also, someone was able to discover and explain something from mine, which was fun as well."

The students found making their own mathematical discoveries very valuable. Notice the language used by the students. Kayla and Hana both claim they 'found' something, while Nate states he was able 'to discover' something. These words show that the students feel ownership in their learning and feel proud of what they have accomplished. These feelings helped make the process more enjoyable for the students. In the closure interview with Bryson, I asked him how it felt to make a mathematical discovery. His response, "It made me feel awesome. That really got my math interests, what's the word, higher."

One thing that I hoped to communicate to students during the Pattern Math activities was that understanding math was more than just knowing how to do a problem, but also why it worked. I asked students to reflect on what it meant to understand something mathematically at the end of the third cycle. The students' responses show that they have come to know what it means to understand mathematics. Here are some of the students' responses.

“To ‘understand’ math you have to know what you are doing and why it works for every number. You also have to know how it works.” – Carly

“I think understanding math involves not just knowing how to get the answer, but also why that method works.” – Nate

“What it means to ‘understand’ mathematics is to not only be able to go through the steps of getting to the solution but understand why those steps work to get to the solution, which then helps you to remember the steps anyway.” – Kayla

“I think that ‘understanding’ mathematics is when you understand how something works and why something works.” – Bryson

“Understanding math, in my opinion, is the point in time where you are able to take a problem, solve it, realize why it works and then follow up by realizing when it wouldn’t work.” – Justin

“To understand is to know step by step how the math works in an equation. I can just solve a question by just memorizing plugging in the numbers. That would be understanding the ‘method’ not the math of the equation.” – Hana

“To understand math is to have an understanding of why it works. Through the Pattern Math, I realized that getting an answer is the easy part, the hard part is why.” – Sun

These comments were expected since the Pattern Math activities stressed the process of trying to not only explain how the pattern worked but why it worked as well. The Pattern Math activities provided an opportunity for students to explain things and understand them deeper. In addition to describing what it meant to understand math, many students made comments that they valued this process. Justin writes, “Being able to understand why and how a pattern works was the most valuable part of these activities. They also helped me think about math in a different way which made it seem less scary/stressful.” Chase also felt that it was valuable to understand how and why a pattern works. He writes, “Something valuable that I learned from Pattern Maths was how to understand how and why patterns work.” Sun comments that the Pattern Math activities helped her develop a deeper understanding of mathematics by exploring patterns and making mathematical connections. She writes, “The Pattern Math allowed me to go deeper into a simple pattern and made me realize that each pattern is connected to something else.” Tom comments that he has realized that there is more to equations that just how to do them. Tom writes, “I learned that there is a deeper understanding on how all equations work. They don't just work for no reasons, they work because math is all interconnected."

The last question I asked students was how they were going to approach learning math next year. I also asked them how they may approach their math homework in the future. I wondered what other affects the Pattern Math would have on their views of learning math. Their previous comments showed that they valued making their own mathematical discoveries and understanding both how and why a pattern worked. Carly commented that she would like to take more time on her homework "so I can really understand how and why it works." This means that she is moving away from just completing steps but interested in noticing why the mathematics works. Hana reiterates this idea by stating that she "can try to figure out how a new concept works and understand why rather than just memorizing the method. I think it will help me remember the concept (But it's gonna take a lot of time)". Dave comments on his approach to learning next year, "Learning that next year, I would probably, I would want to understand why it works and how it works, not just being able to answer the question, maybe, sometimes explain and write it out how it works, that would probably help." The Pattern Math activities gave students the opportunity to communicate mathematics, connect mathematical concepts and explain mathematical concepts. This opportunity was valuable and affected the students in their views of learning math. Students stated that they will approach learning math in the future with a focus on both how and why a mathematical concept works. I will end with Nate's thoughts towards approaching math next year. He writes, "I think I will find it more interesting because I think all these patterns help me approach it in new ways, knowing that there's just different methods and that there are patterns everywhere that, if I just think about it, I could probably find like a new way or an easier way of doing things of I mean breaking down numbers or something, so I think I'll enjoy it."

Chapter 10: Conclusion

The Pattern Math activities were great opportunities for students to develop an understanding of mathematics. The inquiry was designed to allow all students to enter into the process of thinking and communicating mathematically. The interactive writing component was an opportunity for each student to engage within their zone of proximal development. The students were aware that I would be reading their mathematical communication and felt they had an audience to share their mathematical thinking and discoveries. Reading the students' writings allowed me to view the students thinking and cognition. My responses to students encouraged them to explore and explain the patterns in new ways.

The purpose of this final chapter is to reflect on my professional goals and the research questions that were presented at the beginning of this thesis. I will also examine ways in which this research could be altered or enhanced to be used in different situations.

Professional Goals

In the first chapter I mentioned four professional goals that I wished to accomplish with this research. I have listed these four goals again below.

1. To formally incorporate inquiry as a way of learning and understanding mathematics.
2. To become aware of student cognition or mathematical thinking in my classroom.
3. To give students the opportunity to learn math with a deeper conceptual understanding and to develop an appreciation for mathematics.
4. To create materials to share with other teachers.

I believe that I was able to accomplish each of these goals.

The first goal was to incorporate inquiry as a way of learning and understanding mathematics. The Pattern Math activities were open-ended activities which allowed students to explore and think about mathematics. They formally introduced inquiry into the mathematics classroom. By having students describe what they saw and attempt to explain why that happened, students were given an opportunity to learn mathematics with a conceptual understanding. The interactive writing process allowed me to respond to each student individually. This encouraged students to inquire more into how the patterns worked and what they could discover.

The second goal was to become aware of student cognition and mathematical thinking in my classroom. I was quite surprised at how effective the Pattern Math activities were at making me aware of how the students thought. I can remember on numerous times that I was surprised at some students' abilities to think mathematically. For example, students who often did not perform well on traditional assessments showed great abilities to think mathematically. At other times, I was surprised by the relatively weak performance to explain mathematical concepts by students who generally did quite well on other assessments. I also felt that this awareness made me feel more confident in assessing their mathematical abilities. In particular, I felt I had valuable information about students' abilities to conceptually understand mathematics.

The third goal was to give students an opportunity to develop a deeper conceptual understanding and an appreciation for mathematics. I have already discussed how the Pattern Math activities provided students an opportunity to develop a conceptual understanding of mathematics. Having to explain why the patterns worked and encouraging students through my responses provided this opportunity. The last chapter also highlighted some of the ways in which students developed an appreciation for mathematics. Student made comments that they enjoyed the patterns that they saw, and they found explaining the patterns and finding new patterns to be 'fun.'

The last goal was to create activities that I could share with other teachers. I am quite excited to share my ideas with other teachers. I think that the Pattern Math activities are accessible and easy to use. I also think that the format of the activity invites teachers to make modifications or to create their own similar activities. The Pattern Math activities provide an easy way for teachers to enhance the communication skills of their students.

Revisiting the Research Questions

In the first chapter I had listed my research questions. I have included them here again, below.

- How did Pattern Math activities contribute to the development of students' abilities to think mathematically, communicate mathematically and connect mathematical concepts?
- How did Pattern Math activities contribute to students' conceptual understanding of the mathematics?
- How did students interpret their experience with the Pattern Math activities with respect to their view of learning mathematics?

Each of these questions was addressed directly through chapters 7-9 of the thesis; however evidence was also seen in chapters 4-6. Chapter 4 introduced the research project and the students involved. This chapter gave many examples showing students thinking mathematically, communicating mathematically and making mathematical connections. Chapter 5 focused on particular discoveries that students made and provided further examples of students thinking mathematically. In chapter 6, the entire work of Zoe displayed the flow of learning throughout the three cycles of Pattern Math. This chapter provided the reader with an opportunity to see how the Pattern Math activities contributed to a student's conceptual understanding. The chapter also provided the background needed to understand how the Pirie-Kieren model could be applied to the data as a way of mapping the growth of mathematical understanding.

Chapter 7 illustrated the Pirie-Kieren model with respect to Zoe's first cycle of Pattern Math. Chapter 8 analyzed the data from all of the students using the Pirie-Kieren model and summarized different themes that appeared. This analysis described the extent to which the Pattern Math activities contributed to students' conceptual understanding of the mathematics. Finally, chapter 9 examined the reflections of students. These reflections described the influences that their engagement with the Pattern Math activities had on the students' mathematical abilities and attitudes. Students described how the activities made them think, communicate and connect mathematical ideas. Students also described how they saw learning mathematics as not only seeking procedures but trying to understand how it worked.

The design of the Pattern Math activities enabled students to think mathematically, communicate mathematically and make mathematical connections. The simplicity of the patterns invited each student to notice the patterns and to think about the mathematics behind the pattern. Describing the pattern provided students with the opportunity to begin to communicate mathematically and an opportunity to express mathematics with proper terminology. As students attempted to explain why the patterns worked they made many connections to other mathematical ideas and displayed many examples of thinking mathematically. Communicating these observations and discoveries further enhanced the students' communication skills and their ability to think mathematically.

It was through this process of thinking mathematically and communicating mathematically that the Pattern Math activities contributed to students' conceptual understanding of mathematics. Students were able to generalize the specialized patterns that they saw. The results of specific arithmetic patterns could be described through algebraic generalizations. In the process of describing the patterns algebraically, students were given an opportunity to understand how algebra was connected to arithmetic and how they could use algebra to find new arithmetic patterns. Some students rediscovered the meaning of algebraic operations as they noticed how they worked for numbers as well as variables. It was through these experiences that the Pattern Math activities provided an excellent means for students to develop their conceptual understanding of mathematics.

The Pattern Math activities presented students with patterns and asked students to explain why the patterns worked to the best of their abilities. My response to the students as part of the interactive writing process continually encouraged the students and pushed the students to explain why the patterns worked. As a result of this process, students experienced success and fulfillment when they discovered a pattern on their own or were able to explain why a pattern worked. After explaining why a pattern worked, students felt that they understood the mathematics better and had a feeling of ownership because they were able to figure it out on their own. They interpreted this experience positively and expressed that in the future they will attempt to look for patterns and try to explain why things work rather than just doing math without conceptually understanding it. The Pattern Math activities not only provided the students with an opportunity to learn mathematics with a conceptual understanding, it also provided them with an opportunity to view mathematics as something that they can learn and can understand.

Limitations and Possibilities

The Pattern Math activities were an example of design experiment research. In design experiment research, materials and instruction are modified as needed depending on how well the design works. The materials themselves are open to modification and alteration. It must be realized that although the Pattern Math activities worked very well in the context in which they were delivered; this may not be the case in other contexts. Certain aspects of the process may need to be changed depending on the context. For example, the research was conducted with students who were taking a grade 11 pre-calculus class while they were in grade 10. Perhaps these students would be more willing to think mathematically and explain mathematical concepts. However, it is important that throughout this research, the Pattern Math activities have been suggested as a way to provide students with the opportunity to think and communicate mathematically. I also believe that the simplicity of the original patterns along with the complexity of their explanation suggests that other teachers could use this activity to engage students of all abilities.

Another contextual factor was the size of the class. A class of 14 students allowed for communication to happen individually with students through the interactive writing process. With larger classes, time restraints may make this part of the project less possible. This factor would need to be considered when implementing a similar project. The benefits of the interactive writing have been seen throughout this project. Implementing the project over a greater period of time may help alleviate some of the time restraints a teacher may feel.

An exciting aspect of the Pattern Math activities is the prospect of teachers creating their own Pattern Math activities. Mathematics is full of patterns and simple patterns provide the opportunity for great mathematical explorations and discoveries. In chapter 5, I discussed some of the discoveries that the students made and how these discoveries helped me make my own discoveries.

Final Thoughts

I thoroughly enjoyed sharing the Pattern Math activities with my students. I liked to hear them get the same excitement that I had the first time that I saw these patterns. I enjoyed seeing their natural curiosities emerge as they wondered why the patterns worked. I appreciated the opportunity to communicate directly with the students and provide them with an opportunity to extend their abilities in both thinking mathematically and communicating mathematically. I was surprised to see the numerous discoveries that the students made over the three cycles and I was also pleasantly surprised that I made some mathematical discoveries of my own. The students were able to feel ownership in their discoveries and I could see this reflected in their attitudes towards mathematics. I will end with a quote from the closure interview with Nate. Reflecting on his discoveries during the Pattern Math activities, he comments, "They made me feel smart, like I was discovering new things and, like, I was just really amazed that I could discover so much from just one little Pattern Math, and all the places I could go with it."

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Appendices

Appendix A: Noticing Patterns in Homework

This first appendix provides the questions that students will answer about how they work at homework to understand the preconceptions students have about homework and the extent to which students look for patterns while doing homework assignments.

Homework Questionnaire

1. What is the purpose of a homework assignment in your view?

2. How do you learn while you are doing homework? Explain.

3. What do you think a teacher wants you to learn from doing a homework assignment?

4. In your opinion, what is the best way to learn mathematics?

Appendix B: Three Cycles of Pattern Math

The following are three cycles of Pattern Math. The first cycle will introduce students to the idea of seeing algebra in arithmetic. The idea behind each of the activities is to present intriguing patterns that occur in arithmetic. Each of these patterns will illicit a natural response of "why does that work?"

Cycle 1

Pattern Math: Patterning, Communicating, Connecting

Two More, Two Less

Patterning	
A: 10^2 = B: (8)(12) =	A: 13^2 = B: (11)(15) =
A: 40^2 = B: (38)(42) =	Your own example:

Communicating
What do you notice? • Talk about the pattern that you see. • Communicate as clearly as you can everything you notice. • Does the pattern you see always work? How would you know?

Pattern Math: Patterning, Communicating, Connecting

Squared plus number plus next number

Patterning	
A: $3^2 + 3 + 4 =$ B: $4^2 =$	A: $6^2 + 6 + 7 =$ B: $7^2 =$
A: $20^2 + 20 + 21 =$ B: $21^2 =$	Your own example:

Communicating
What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Difference Equals Sum

Patterning	
A: $6^2 - 5^2 =$ B: $6 + 5 =$	A: $10^2 - 9^2 =$ B: $10 + 9 =$
A: $50^2 - 49^2 =$ B: $50 + 49 =$	Your own example:

Communicating
What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Subtracting Binomial Squares

Patterning	
A: $(2n + 4)^2$ B: $(2n + 3)^2$ C: A – B	A: $(2n + 5)^2$ B: $(2n + 4)^2$ C: A – B
A: $(2n + 9)^2$ B: $(2n + 8)^2$ C: A – B	Your own example:

Communicating
What do you notice?

Cycle 2

Pattern Math: Patterning, Communicating, Connecting

Adding Odds

Patterning	
A: 1 + 3 B: Average of 1 and 3 C: B^2	A: 1 + 3 + 5 B: Average of 1, 3 and 5 C: B^2
A: 1 + 3 + 5 + 7 B: Average of 1, 3, 5 and 7 C: B^2	Your own example:

Communicating
What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Add Up Add Down

Patterning	
A: $1 + 2 + 1 =$ B: $\sqrt{A} =$	A: $1 + 2 + 3 + 2 + 1 =$ B: $\sqrt{A} =$
A: $1 + 2 + 3 + 4 + 3 + 2 + 1 =$ B: $\sqrt{A} =$	Your own example:

Communicating
What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Adding Up Odds

Patterning	
A: 1 = B: 1^3 =	A: 3 + 5 = B: 2^3 =
A: 7 + 9 + 11 B: 3^3 =	Your own example:

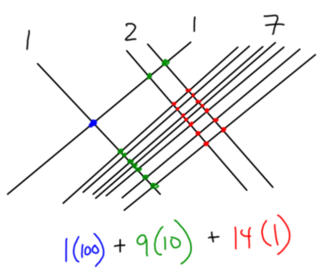
Communicating
What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Ancient Chinese Multiplication

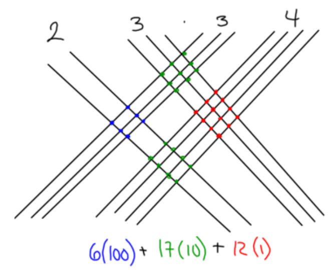
Patterning

A:



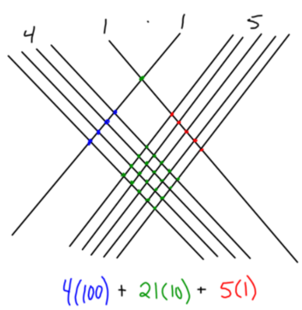
B: (12)(17) =

A:



B: (23)(34) =

A:



B: (41)(15) =

Your own example:

Communicating

What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Napier's Bones

Patterning

A: $6 \cdot 258$

1/2 3/0 4/8

1 5 4 8

B: (6)(258) =

A: $7 \cdot 326$

2/1 1/4 4/2

2 2 8 2

B: (7)(326) =

A: $8 \cdot 686$

4/8 6/4 4/8

5 4 8 8

B: (8)(686) =

Your own example:

Communicating

What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Charts

Patterning

A:

	30	5	
20	600	100	
6	180	30	Sum

B: (35)(26) =

A:

	60	2	
30			
8			Sum

B: (62)(38) =

A:

	40	3	
10			
8			Sum

B: (18)(43) =

Your own example:

Communicating

What do you notice?

Pattern Math: Patterning, Communicating, Connecting

Horizontal Asymptotes

Patterning

A: Consider $\frac{5x+3}{x-1}$. Plug in bigger and bigger values of x. What value does it get close to? B: What is the relationship between this value and the expression?	A: Consider $\frac{6x+7}{2x-5}$. Plug in bigger and bigger values of x. What value does it get close to? B: What is the relationship between this value and the expression?
A: Consider $\frac{-8x+19}{2x-12}$. Plug in bigger and bigger values of x. What value does it get close to? B: What is the relationship between this value and the expression?	Your own example:

Communicating

What do you notice?

Appendix C: Reflective Questions After a Cycle of Pattern Math

Cycle 1:

1. Which Pattern Math activity did you find most interesting and why?

2. Which activity did you find you explained the best and why?

3. Describe one thing that you learned that you felt was valuable.

Cycle 2:

1. Which Pattern Math activity did you find most interesting and why?

2. Which activity did you find you explained the best and why?

3. Describe one thing that you learned that you felt was valuable.

4. This cycle only had 3 Pattern Math activities. Design a fourth Pattern Math activity that you feel would fit with this idea and provide your answer to this problem.

Cycle 3:

1. What is the purpose of doing Pattern Math activities?

2. Describe your ability to communicate mathematically.

3. Describe one thing that you learned that you felt was valuable.

4. What does it mean to “understand” mathematics?

Appendix D: Guiding Questions for Closure Interviews

1. The narrative that you have read describes your learning throughout three cycles of Pattern Math. Describe your learning over the 3 cycles in your own words. You may highlight or dispute items in the narrative that you agree with or disagree with.

2. Describe what it means to "understand" something in mathematics.

3. Describe the ways that you have found that you learn and understand mathematics best.

4. In your words, describe the importance of patterns in mathematics.

5. One of the things we worked hard on during mathematics was communicating our ideas. Describe the role communication had in your learning over the three cycles of Pattern Math.

6. Describe what you have felt is the most valuable thing you have learned during the Pattern Math cycles.

7. How will you apply what you have learned in other areas?