Initial Moving-Boundary Value Problems Associated with the Wave Equation

by

Norman Christopher Corbett

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Abstract

The one-dimensional wave equation on a time-dependent domain is studied. Analytical solutions of an associated initial boundary value problem are sought. These solutions are derived by two distinct techniques, namely through an application of d'Alembert's solution and via the derivation of a series representation of these solutions.

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I am also indebted to the Department of Applied Mathematics for its financial support.

I dedicate this thesis to my parents, Brian and Betty, who have demonstrated by example the value of education, and to my wife Zoria whose patience and support have made this thesis possible.

Table of Contents

Introduction

Project Oedipus, undertaken by Bristol Aerospace Limited, culminated in the launching of a tethered satellite which was, among other things designed to investigate Earth's magnetic field. During its flight, the spin-stabilized Oedipus payload separated into two components, which remained connected by a thin flexible conducting tether throughout the deployment. Separation and tether deployment dynamics gave rise to a coning motion of both components of the payload. This coning motion acted to exite the ends of the tether as the components moved apart to a distance of nearly one kilometer. This deployment, and in particular the dynamics of the tether, as driven by the gross vehicle dynamics, is discussed by Berry in [Ber]. One of the models proposed in [Ber] is based upon a quasi-linear vector wave equation in one spatial variable whose domain is time-dependent. The result is an initial boundary value problem (IBVP) that models a vibrating string whose length grows with time, and whose ends are driven by the gross payload dynamics. Subsequent analysis of this model is accomplished chiefly through a numerical integration using the method of characteristics and does not address the question of analytical solutions.

The objective of this thesis is to derive analytical solutions of a simplified version of this model. However, this version does retain the complications associated with the time-dependent nature of the domain upon which the solution is sought. Two distinct approaches are taken in the derivation of these solutions. The first involves an application of the well-known d'Alembert solution of the wave equation to this IBVP. The second approach makes use of the theory of Lie groups and results in series solutions resembling those obtained by the usual separation of variables technique.

Chapter 1: Preliminaries

1.1: Introduction

As in [Ber], let us consider the vectorial initial moving-boundary value problem (IMBVP)

$$\frac{\partial^2 \vec{u}}{\partial t^2} = c^2 \frac{\partial^2 \vec{u}}{\partial x^2} + \vec{F}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \frac{\partial \vec{u}}{\partial t}\right), \qquad 0 < x < S(t), \quad t > 0, \quad |S'(t)| < 1, \tag{1.1.1a}$$

$$\vec{u}(x,0) = f_1(x)\hat{\imath} + f_2(x)\hat{\jmath}, \qquad \frac{\partial \vec{u}(x,0)}{\partial t} = g_1(x)\hat{\imath} + g_2(x)\hat{\jmath}, \qquad 0 \le x \le S(0), \tag{1.1.1b,c}$$

$$\vec{u}(0,t) = p_1(t)\hat{\imath} + p_2(t)\hat{\jmath}, \qquad \vec{u}(S(t),t) = q_1(t)\hat{\imath} + q_2(t)\hat{\jmath}, \qquad t \ge 0, \tag{1.1.1d,e}$$

where $\vec{u}(x,t) = u_1(x,t)\hat{\imath} + u_2(x,t)\hat{\jmath}$. In the above, $\vec{u}(x,t)$ represents the two-dimensional transverse displacement, at position x and time t, of the tether modelled by (1.1.1), where $\hat{\imath}$ and $\hat{\jmath}$ are orthogonal unit vectors lying along the u_1- and u_2-axes of the orthogonal coordinate system (u_1, u_2, x). We will assume that $S(T) > 0$ for any finite time $T > 0$ needed, we will restrict the variable t to a finite interval so that $S(t)$ remains non-negative. The functions $p_1(t)$, $p_2(t)$, $q_1(t)$ and $q_2(t)$ describe the displacement at the ends of the tether, and are determined by the motions of the fore and aft payloads. On the other hand, the functions $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ describe the initial displacements and velocities of points on the tether. In addition, the non-homogeneity $\vec{F}\left(x, t, \vec{u}, \frac{\partial \vec{u}}{\partial x}, \frac{\partial \vec{u}}{\partial t}\right)$ can be thought of as representing the net external force per unit length exerted on the tether at position x and time t. It should be noted that (1.1.1a) can be written as

$$\frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x^2} + F_1 \left(x, t, \vec{\mathrm{u}}, \frac{\partial \vec{\mathrm{u}}}{\partial x}, \frac{\partial \vec{\mathrm{u}}}{\partial t} \right), \tag{1.1.2a}$$

$$\frac{\partial^2 u_2}{\partial t^2} = c^2 \frac{\partial^2 u_2}{\partial x^2} + F_2 \left(x, t, \vec{\mathrm{u}}, \frac{\partial \vec{\mathrm{u}}}{\partial x}, \frac{\partial \vec{\mathrm{u}}}{\partial t} \right), \tag{1.1.2b}$$

which is a coupled quasi-linear system of wave equations. If the external forcing terms in (1.1.2a,b) are of the form $F_1\left(x, t, u_1, \frac{\partial u_1}{\partial x}, \frac{\partial u_1}{\partial t}\right)$ and $F_2\left(x, t, u_2, \frac{\partial u_2}{\partial x}, \frac{\partial u_2}{\partial t}\right)$ respectively, then system (1.1.2) becomes simply a pair of uncoupled quasi-linear wave equations. If we further assume that the forcing terms can be neglected altogether, then we can restrict our consideration to a single scalar IMBVP, based upon the usual (linear) wave equation. We note that this simplification is by no means physically reasonable, and the resulting system is not expected to accurately model the tether dynamics of the Oedipus deployment. However, the removal of the non-homogeneity allows us to concentrate our efforts on analytical techniques that will address the complication of the moving boundary $x = S(t)$.

For convenience, we introduce the dimensionless variables $x = \bar{x}$, $t = \frac{1}{c}\bar{t}$. Upon dropping the bar notation, we obtain an IMBVP of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < S(t), \qquad t > 0, \qquad |S'(t)| < 1, \tag{1.1.3a}$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad 0 \le x \le S(0), \tag{1.1.3b,c}$$

$$u(0,t) = p(t), \qquad u(S(t),t) = q(t), \qquad t \ge 0. \tag{1.1.3d,e}$$

More specifically, when we examine the case in which $S(t)$ is a linear function of the form $S(t) = bt + l_0$, we will employ an additional change of variables of the form $x = l_0 x^*$, $t = l_0 t^*$. In this case, upon dropping the $*$ notation, the function $S(t)$ becomes $S(t) = mt + 1$.

In the next section, we will begin our analysis of moving-boundary problems involving the wave equation by considering a problem, analogous to (1.1.3), on the semi-infinite domain $\mathbf{D} = \{(x,t) | S(t) < x < \infty, \quad t > 0\}$ with $S(t) = mt$. A solution of this problem is obtained by requiring that the usual d'Alembert solution of the wave equation satisfy the initial and boundary conditions. An extension of this analysis will result in a solution of the general problem (1.1.3) as demonstrated in chapter 2.

1.2: d'Alembert's Solution of a Semi-Infinite IMBVP

The general linear second-order homogenous partial differential equation, with constant coefficients, in two independent variables, has the form

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial u}{\partial t} + e\frac{\partial u}{\partial x} + fu = 0 \tag{1.2.1}$$

where a, b, c, d, e and f are constants and $a^2 + b^2 + c^2 \neq 0$. Further classification of (1.2.1) is made in terms of the sign of $b^2 - 4ac$. In particular, equation (1.2.1) is said to be hyperbolic if $b^2 - 4ac > 0$. In the sequel, we shall restrict consideration solely to the case when (1.2.1) is hyperbolic. The one-dimensional wave equation (often referred to as the "string equation") $u_{tt} = u_{xx}$, with $a = 1$, $c = -1$ and $b = d = e = f = 0$ belongs to this class of equations. The characteristics of equation (1.2.1) are the straight lines

$$\xi = x - \alpha t, \qquad \eta = x - \beta t, \tag{1.2.2}$$

where ξ and η are real constants and

$$\alpha = \frac{b - \sqrt{b^2 - 4ac}}{2a}, \qquad \beta = \frac{b + \sqrt{b^2 - 4ac}}{2a}. \tag{1.2.3}$$

Now suppose we think of (1.2.2) as defining a change of independent variables and that we want to express equation (1.2.1) in terms of the new variables (ξ, η). The Jacobian of the transformation is

$$J = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \alpha - \beta,$$

so that the coordinate transformation (1.2.2) is non-singular as long as $\alpha \neq \beta$, the latter being guaranteed by virtue of the assumption that (1.2.1) is hyperbolic.

Using the chain rule, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}, \\
\frac{\partial u}{\partial t} &= -\alpha \frac{\partial U}{\partial \xi} - \beta \frac{\partial U}{\partial \eta}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 U}{\partial \xi^2} + 2\frac{\partial^2 U}{\partial \xi \partial \eta} + \frac{\partial^2 U}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial x \partial t} &= -\alpha \frac{\partial^2 U}{\partial \xi^2} - (\alpha + \beta)\frac{\partial^2 U}{\partial \xi \partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial t^2} &= \alpha^2 \frac{\partial^2 U}{\partial \xi^2} + 2\alpha\beta \frac{\partial^2 U}{\partial \xi \partial \eta} + \beta^2 \frac{\partial^2 U}{\partial \eta^2},
\end{aligned}$$

where we have used the notation $u(x,t) = u(\frac{\beta\xi - \alpha\eta}{\beta - \alpha}, \frac{\xi - \eta}{\beta - \alpha}) \equiv U(\xi, \eta)$ and have assumed that $u(x,t)$ is twice continuously differentiable ($\mathbf{C}^2$) with respect to both x and t. Using (1.2.3), equation (1.2.1) can be written in the form

$$\begin{aligned}
&2(b^2 - 4ac)\frac{\partial^2 U}{\partial \xi \partial \eta} + (db - d\sqrt{b^2 - 4ac} - 2ae)\frac{\partial U}{\partial \xi} \\
&+(db + d\sqrt{b^2 - 4ac} - 2ae)\frac{\partial U}{\partial \eta} - 2afU = 0. \qquad (1.2.4)
\end{aligned}$$

Notice that, since $4(b^2 - 4ac)^2 > 0$, equation (1.2.4) is hyperbolic.

In the special case of the one-dimensional wave equation (1.1.3a), the characteristics are the straight lines

$$\xi = x + t, \qquad \eta = x - t. \qquad (1.2.5)$$

Moreover, in this case, equation (1.2.4) reduces to

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0,$$

which can be integrated to give

$$U = \varphi(\xi) + \psi(\eta),$$

or

$$u(x,t) = \varphi(x+t) + \psi(x-t), \tag{1.2.6}$$

where $\varphi(\xi)$ and $\psi(\eta)$ are arbitrary $\mathbf{C}^2$ functions of their respective single arguments. Equation (1.2.6) is said to define the general solution of the one-dimensional wave equation. Moreover, when (1.2.6) is used to solve an IBVP, the functions $\varphi(\xi)$ and $\psi(\eta)$ are to be chosen so that initial and/or boundary conditions are satisfied.

For example, consider the IVP problem for the infinite string:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad -\infty < x < \infty.$$

Using (1.2.6) (see [Tyn] pp. 47-48), it can be shown that the solution of the above problem is

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\tau)d\tau, \tag{1.2.7}$$

or equivalently,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[G(x+t) - G(x-t)],$$

where $G(\tau)$ is an antiderivative of $g(\tau)$ and

$$\varphi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2}\int_{\xi_0}^{\xi} g(\tau)d\tau + D, \tag{1.2.8a}$$

$$\psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2}\int_{\eta_0}^{\eta} g(\tau)d\tau - D, \tag{1.2.8b}$$

where D is any constant and $-\infty < \xi,\ \eta < \infty$, and $\xi_0 = \eta_0$ is some common fixed value of ξ and η. The solution (1.2.7) is known as the d'Alembert solution of the Cauchy problem for the one-dimensional wave equation. The general solution $u = \varphi(x+t) + \psi(x-t)$ can be thought of as the sum of two waves moving, with speed 1, in opposite directions. In turn, $\varphi(x+t)$ can be thought of as the amassed left-moving waves, while $\psi(x-t)$ represents the amassed right-moving waves.

Suppose we are interested in (1.2.7) at some particular point (x_0, t_0). In view of the form of (1.2.7), it is clear that this solution depends only on the initial data specified in the interval $[x_0 - t_0, x_0 + t_0]$. In particular, that part of the solution resulting from the initial displacement depends only on the end points of this interval, while that part due to the initial velocity depends on the whole interval. The interval $[x_0 - t_0, x_0 + t_0]$ is known as the interval of dependence of the solution, at the point (x_0, t_0) (see diagram 1.2.1).

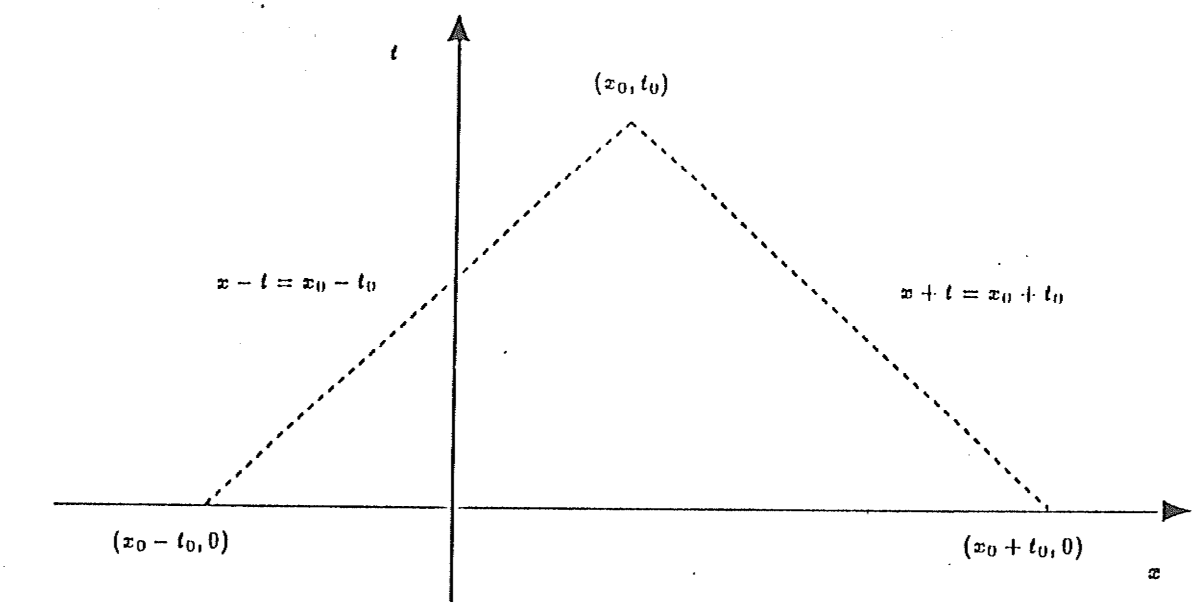

Diagram 1.2.1

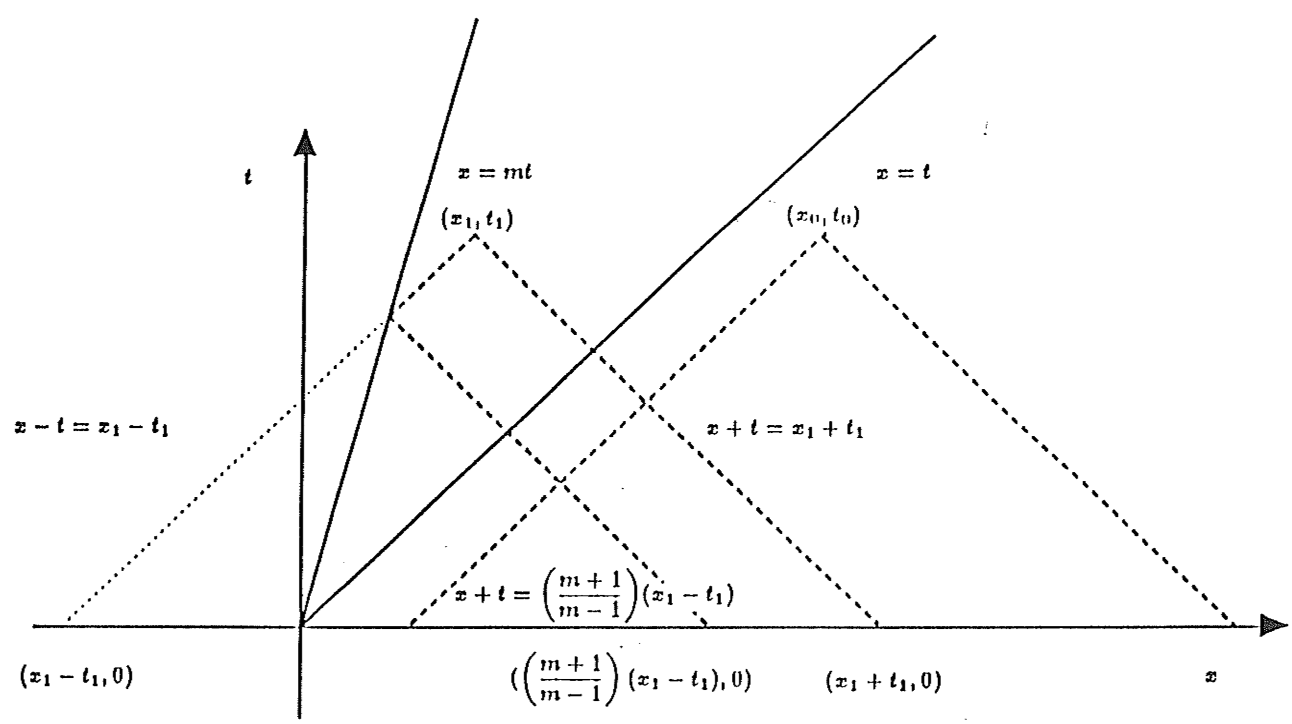

Diagram 1.2.2

As an additional example, consider the IMBVP (see [Zau] pp. 359-360):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad mt < x < \infty, \qquad t > 0, \qquad -1 < m < 1$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad 0 < x < \infty$$

$$u(mt,t) = 0, \qquad t > 0,$$

where the value of m has been restricted to guarantee that the boundary curve is time-like. For a discussion of the difficulties associated with space-like curves see [Zau] pp. 357-361.

Suppose we think of the string described by this problem as being infinite. Then one can interpret the moving boundary, $x = mt$, as defining the world-line of a restraint, that moves (without friction) along the x-axis and restricts the displacement of the string to zero.

For any point $P_0 = (x_0, t_0)$ to the right of the line $x = t$, the interval of dependence of the solution at P_0 is identical to that determined in the case of the infinite string of the previous example. That is, for $x > t$, the solution at P_0 depends only upon $\{f(x_0 - t_0), \quad f(x_0 + t_0), \quad g(x) \quad \text{for } x \in [x_0 - t_0, x_0 + t_0]\}$. Thus, throughout the region characterized by $x > t$, we have $u(x,t) = \varphi(x+t) + \psi(x-t)$, where $\varphi(\xi)$ and $\psi(\eta)$ are defined by (1.2.8), with $\xi_0 = \eta_0 = 0$ and $\xi, \eta > 0$.

For any point $P_1 = (x_1, t_1)$, lying in $mt < x < t$, the characteristic $x - t = x_1 - t_1$, through P_1, intersects the negative x-axis. Since the initial data is not specified for $x < 0$, the solution can not be written in the form (1.2.7) (see diagram 1.2.2). In particular, since $-t < mt < x < t$, we immediately have that $\xi = x + t > 0$ but $\eta = x - t < 0$, so that (1.2.8a) remains valid in this region even though (1.2.8b) does not. The problem presented by the deficiency in the initial data can be overcome by the requirement that the solution satisfy the specified boundary condition $u(mt, t) = 0$ along $x = mt$. Application of the boundary condition to the general solution (1.2.6) yields

$$\varphi([m+1]t) + \psi([m-1]t) = 0,$$

so that if we now let $\zeta = [m-1]t$, then we may write

$$\psi(\zeta) = -\varphi\left(\left[\frac{m+1}{m-1}\right]\zeta\right). \tag{1.2.9}$$

Upon setting $\zeta = x - t$ in (1.2.9), with (1.2.8a) being noted , we obtain

$$\psi(x-t) = -\frac{1}{2}f\left(\left[\frac{m+1}{m-1}\right](x-t)\right) - \frac{1}{2}\int_0^{\left[\frac{m+1}{m-1}\right](x-t)} g(\tau)d\tau - D, \tag{1.2.10}$$

so that the d'Alembert solution in this case is

$$u(x,t) = \frac{1}{2}[f(x+t) - f\left(\left[\frac{m+1}{m-1}\right](x-t)\right)] + \frac{1}{2}\int_{\left[\frac{m+1}{m-1}\right](x-t)}^{(x+t)} g(\tau)d\tau, \tag{1.2.11}$$

in the region given by $mt < x < t$.

Equation (1.2.9) can be regarded as an extension rule for the initial data. That is, the form of (1.2.9) tells us how the domains of the functions $f(x)$ and $g(x)$ must be extended so that these functions are defined on the negative x-axis. Once the data has been extended, solving the above IMBVP reduces to solving an "equivalent" infinite IVP. To illustrate the procedure for extension of the initial data, consider the following two special cases:

1. For $g(x) = 0$: Using the formula (1.2.8a) for $\varphi(\xi)$, where without loss of generality we may assume that D has been chosen to be zero, we have

$$\hat{f}(x) \equiv 2\psi(x) = -f\left(\left[\frac{m+1}{m-1}\right]x\right), \qquad -\infty < x < 0.$$

The function $\hat{f}(x)$ is the extension of $f(x)$, onto the negative x-axis. Indeed, $\hat{f}(x)$ is obtained by reflecting the graph of $f(x)$ through the origin and then "stretching" the result in the the x-direction by a factor of $\left|\frac{m-1}{m+1}\right|$ (see diagram 1.2.3). If $m = 0$, the function $f(x)$ is extended simply as an odd function.

2. For $f(x) = 0$: In this case, the resulting extension rule is

$$-\int_0^x \hat{g}(\tau)d\tau \equiv 2\psi(x) = -\int_0^{\left[\frac{m+1}{m-1}\right]x} g(\tau)d\tau.$$

Using Leibnitz' rule and differentiating with respect to x leads to

$$\hat{g}(x) = \frac{m+1}{m-1}g\left(\left[\frac{m+1}{m-1}\right]x\right), \qquad -\infty < x < 0.$$

Reflecting the graph of $g(x)$ through the origin and then "stretching" in both the vertical and horizontal directions by the factors $\left|\frac{m+1}{m-1}\right|$ and $\left|\frac{m-1}{m+1}\right|$ respectively produces the function $\hat{g}(x)$ (see diagram 1.2.4). Again, when $m = 0$, the function $g(x)$ is extended as an odd function.

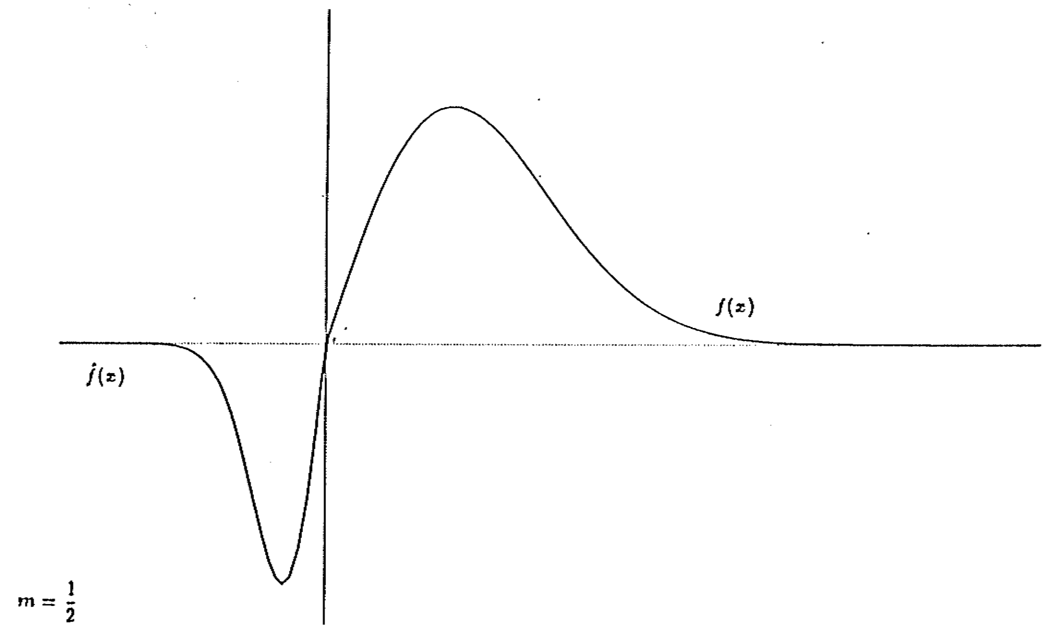

Diagram 1.2.3a

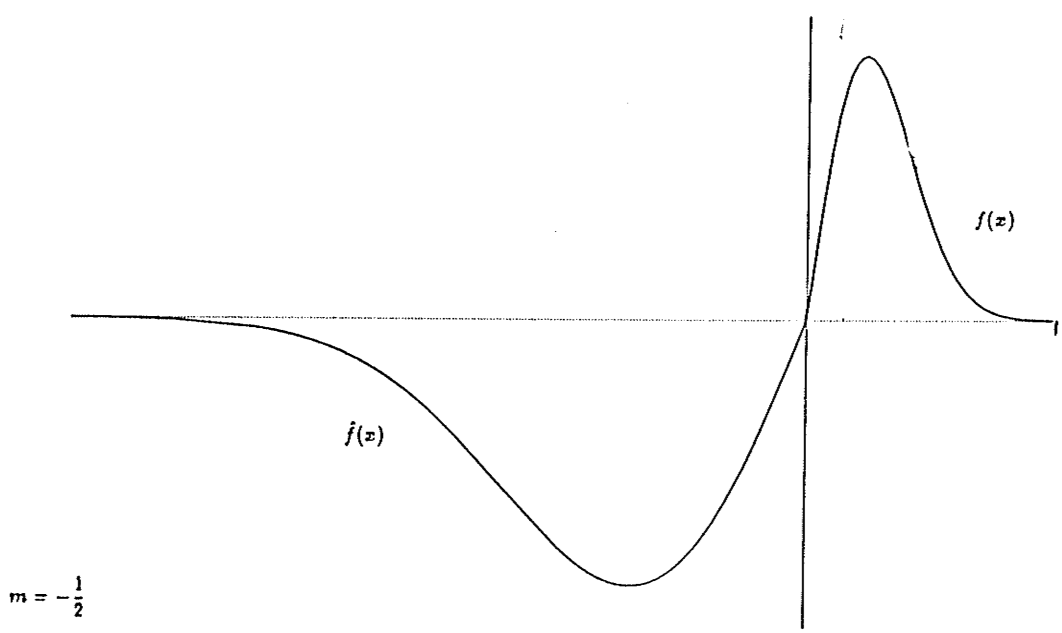

Diagram 1.2.3b

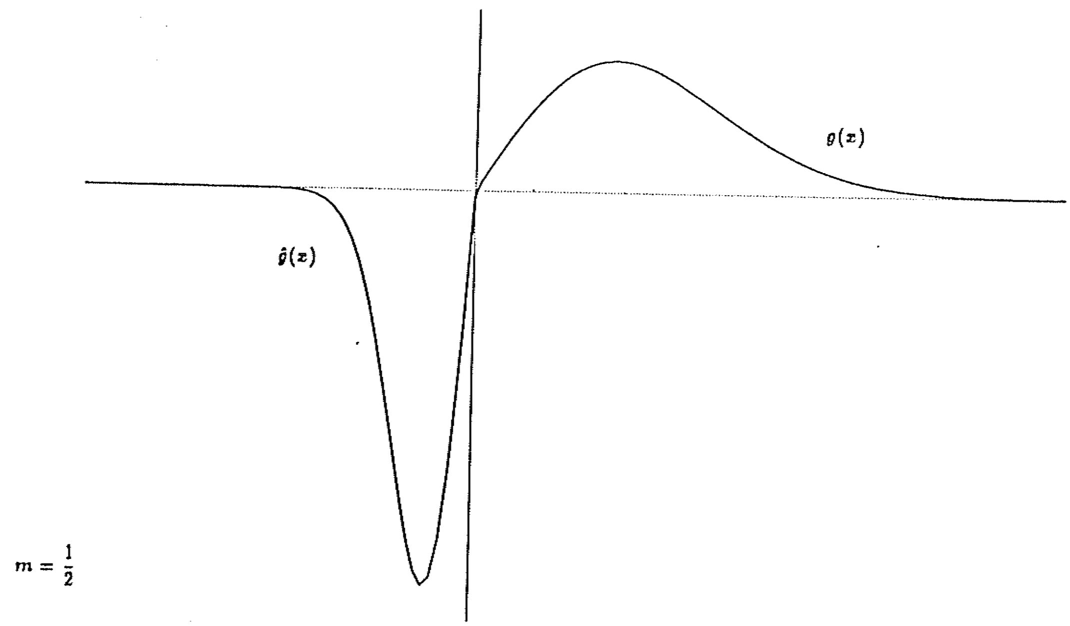

Diagram 1.2.4a

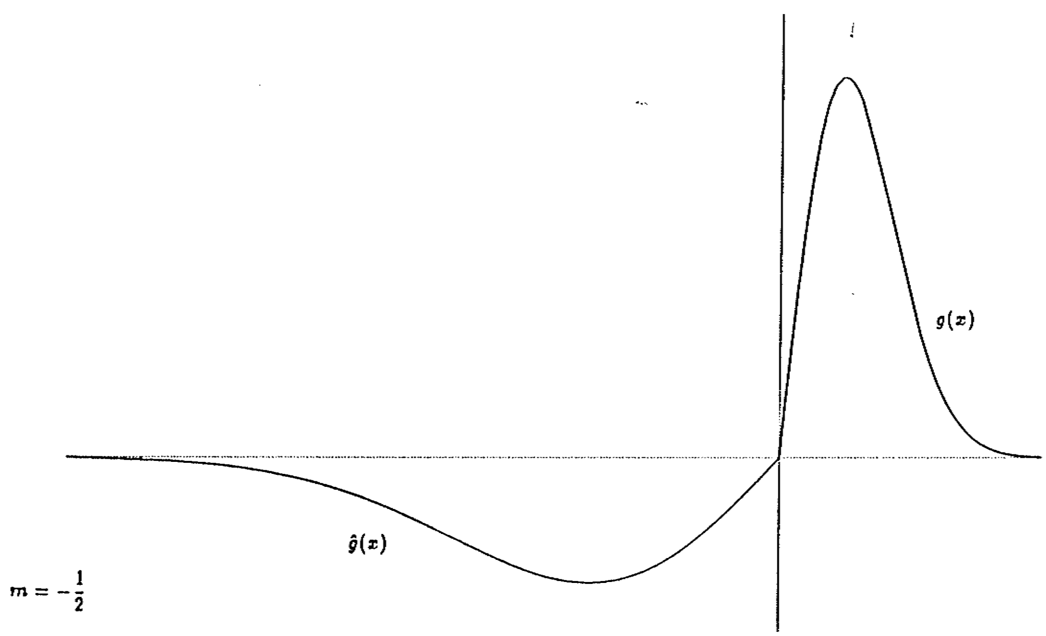

Diagram 1.2.4b

Note: In view of diagram 1.2.2, and (1.2.11), the solution at $P_1 = (x_1, t_1)$ for $P_1 \in \{(x,t)|mt < x < t\}$ depends upon the initial data specified in the interval $\left[\left[\frac{m+1}{m-1}\right](x_1 - t_1), (x_1 + t_1)\right]$. The process of extending the initial data to the negative x-axis can therefore be thought of as finding the appropriate interval of dependence (on the x-axis) for the solution to the left of $x = t$.

The differentiability of the solution is determined by the differentiability of the initial and/or boundary data. Indeed if:

1. $f(x) \in \mathbf{C}^0$, $g(x)$ is integrable and $f(0) = 0$ then $u(x,t) \in \mathbf{C}^0$.

2. $f(x) \in \mathbf{C}^1$, $g(x) \in \mathbf{C}^0$, $f(0) = 0$ and $mf'(0) + g(0) = 0$ then $u(x,t) \in \mathbf{C}^1$.

3. $f(x) \in \mathbf{C}^2$, $g(x) \in \mathbf{C}^1$, $f(0) = 0$, $mf'(0) + g(0) = 0$ and $(m^2+1)f''(0) + 2mg'(0) = 0$ then $u(x,t) \in \mathbf{C}^2$.

On the other hand, if $f(x)$ has a jump discontinuity at some point $x_0 > 0$ in its domain, then by (1.2.7) and (1.2.11), there will be a jump discontinuity in $u(x,t)$ across each of the characteristics $x + t = x_0$, $x - t = x_0$ and $x - t = \left[\frac{m-1}{m+1}\right] x_0$. In general, discontinuities in the data (initial and boundary) and their derivatives will be propagated along the characteristics. In what follows, we will not make any assumptions about the differentiability of $u(x,t)$ unless it is required by the analysis.

We conclude the section with a special case of the preceding IMBVP:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad mt < x < \infty, \qquad t > 0, \qquad -1 < m < 1$$

$$u(x,0) = \sin(\omega x), \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad 0 < x < \infty$$

$$u(mt,t) = 0, \qquad t > 0.$$

The solution of the above, as given by (1.2.7) and (1.2.11), is

$$u(x,t) = \frac{1}{2}[\sin(\omega(x+t)) + \sin(\omega(x-t))], \qquad x > t,$$

$$u(x,t) = \frac{1}{2}[\sin(\omega(x+t)) - \sin\left(\omega\left[\frac{m+1}{m-1}\right](x-t)\right)], \qquad mt < x < t.$$

The right moving wave $-\frac{1}{2}\sin\left(\omega\left[\frac{m+1}{m-1}\right](x-t)\right)$ can be interpreted as the reflection of the incident wave $\frac{1}{2}\sin(\omega(x+t))$. The frequency of the reflected wave is $\Omega = \left|\frac{m+1}{m-1}\right|\omega$. If $0 < m < 1$ then $\Omega > \omega$ and if $-1 < m < 0$ then $\Omega < \omega$ (see diagrams 1.2.5). That is, if the directions of motion of the moving endpoint and the incident wave are opposite, the frequency of the reflected wave is increased. Conversely, when the moving endpoint and incident wave move in the same direction, the frequency of the reflected wave is decreased. The phenomenon of a change in frequency, resulting from either the reflection of a wave off a moving obstacle (boundary), or the emission of a wave from a moving source, is known as the Doppler effect.

The profile of the string, for a few different values of t, is shown in diagrams 1.2.6. We note that the graphs depicted in 1.2.6 are not smooth (there are discontinuities in u_x and u_t across the characteristic $x = t$). This is due to the fact that $f'(0) \neq 0$. It is be worthwhile to to note that, in this example, if the endpoint were not moving (i.e., if $m = 0$), one would expect to see simply standing waves. Thus, we would expect, under the current circumstances, to see essentially standing waves with a disturbance caused by reflection off the "fixed" (zero amplitude) moving endpoint.

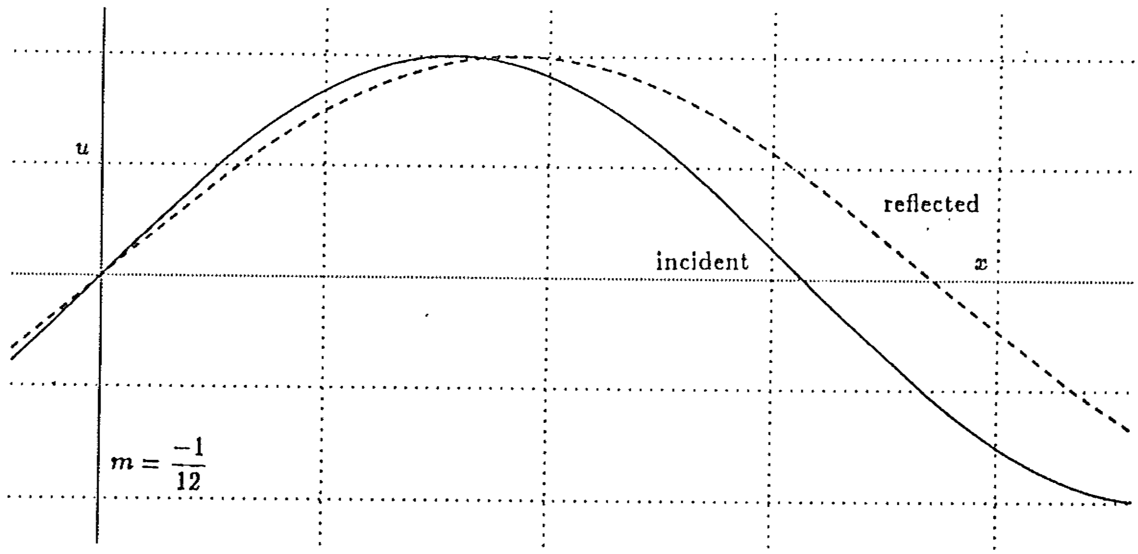

Diagram 1.2.5a

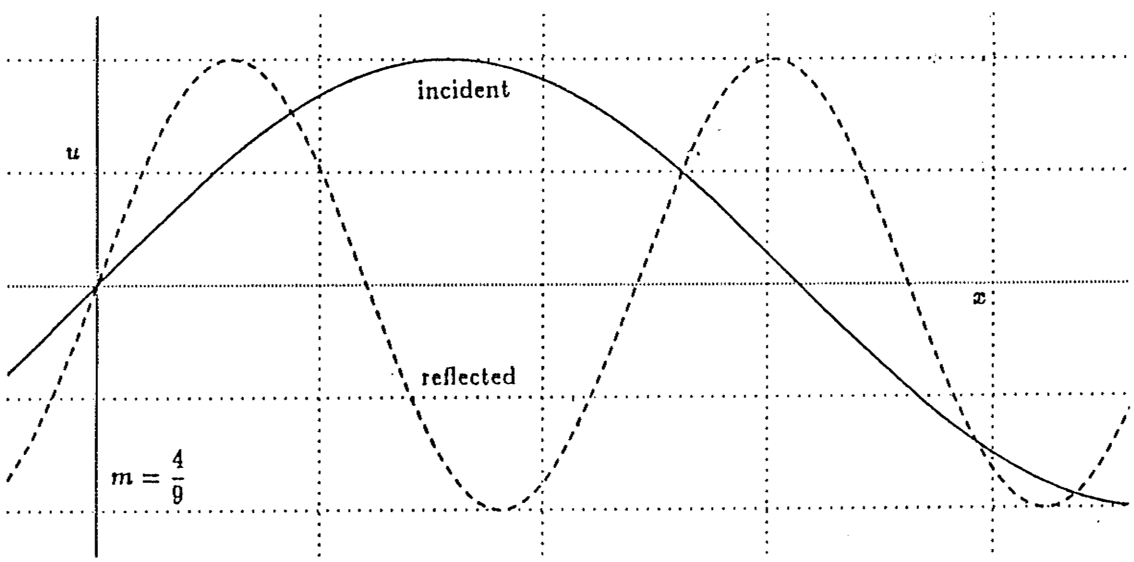

Diagram 1.2.5b

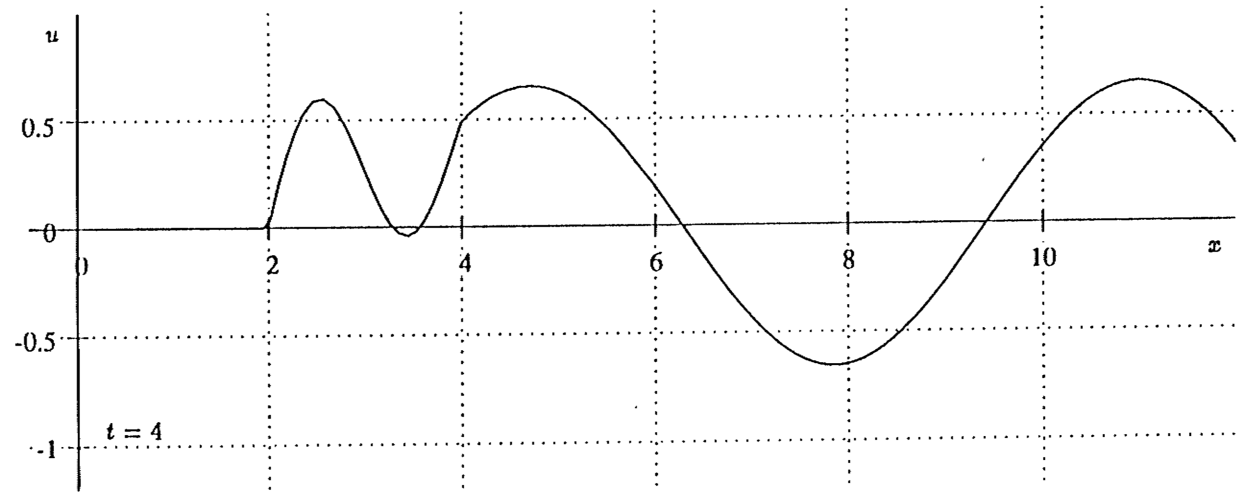

Diagram 1.2.6a

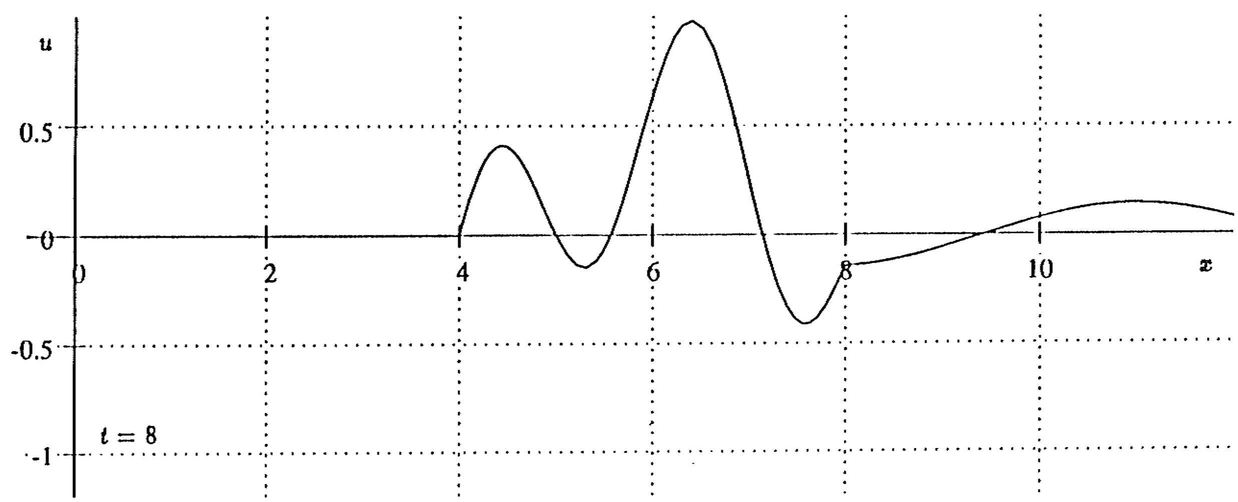

Diagram 1.2.6b

Chapter 2: d'Alembert's Solution

2.1: Introduction

In this chapter, we will solve the IMBVP (1.1.3) using d'Alembert's form of the general solution of the wave equation. The derivation of this solution will establish both the existence and uniqueness of solutions for IMBVP (1.1.3). In addition, we provide a less general uniqueness argument which is based upon a modification of an energy argument presented by Kevorkian (see [Kev] pp. 166-167). This energy argument is useful, since it provides us with some insight into the long term behavior of solutions to the string retrieval problem.

2.2: An IMBVP with Linearly Moving Endpoint

Before analyzing the general case, it is beneficial to inspect a simpler example. Consider the IMBVP

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < mt + 1, \qquad t > 0, \qquad |m| < 1 \tag{2.2.1a}$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad 0 \le x \le 1 \tag{2.2.1b, c}$$

$$u(0,t) = p(t), \qquad u(mt+1,t) = q(t), \qquad t \ge 0, \tag{2.2.1d, e}$$

where we note that if $-1 < m < 0$ then we must require that $0 < t < \frac{1}{m}$. As in section 1.2, we must enforce certain additional requirements on the functions $f(x)$, $g(x)$, $p(t)$ and $q(t)$ in order that the solution $u(x,t)$ will be in the desired differentiability class (i.e., $\mathbf{C}^0$, $\mathbf{C}^1$, etc.). We will delay listing these requirements until the next section when we solve the general MIBVP (1.1.3).

Due to repeated reflections off of the two boundaries, the solution of (2.2.1) assumes distinct functional forms in different subregions denoted $R_{(i,j)}$ of the domain $\mathbf{D} = \{(x,t) | 0 < x < mt + 1, \quad t > 0\}$ as depicted in diagram 2.2.1. The particular form of the solution in any one region depends upon the number of times the waves, generated by the initial and boundary data, reflect off of a given boundary before they reach the subregion in question. In analogy with the usual d'Alembert form of the solution of the wave equation, we denote the solution in region $R_{(i,j)}$ by

$$u(x,t) = \varphi_i(x+t) + \psi_j(x-t). \tag{2.2.2}$$

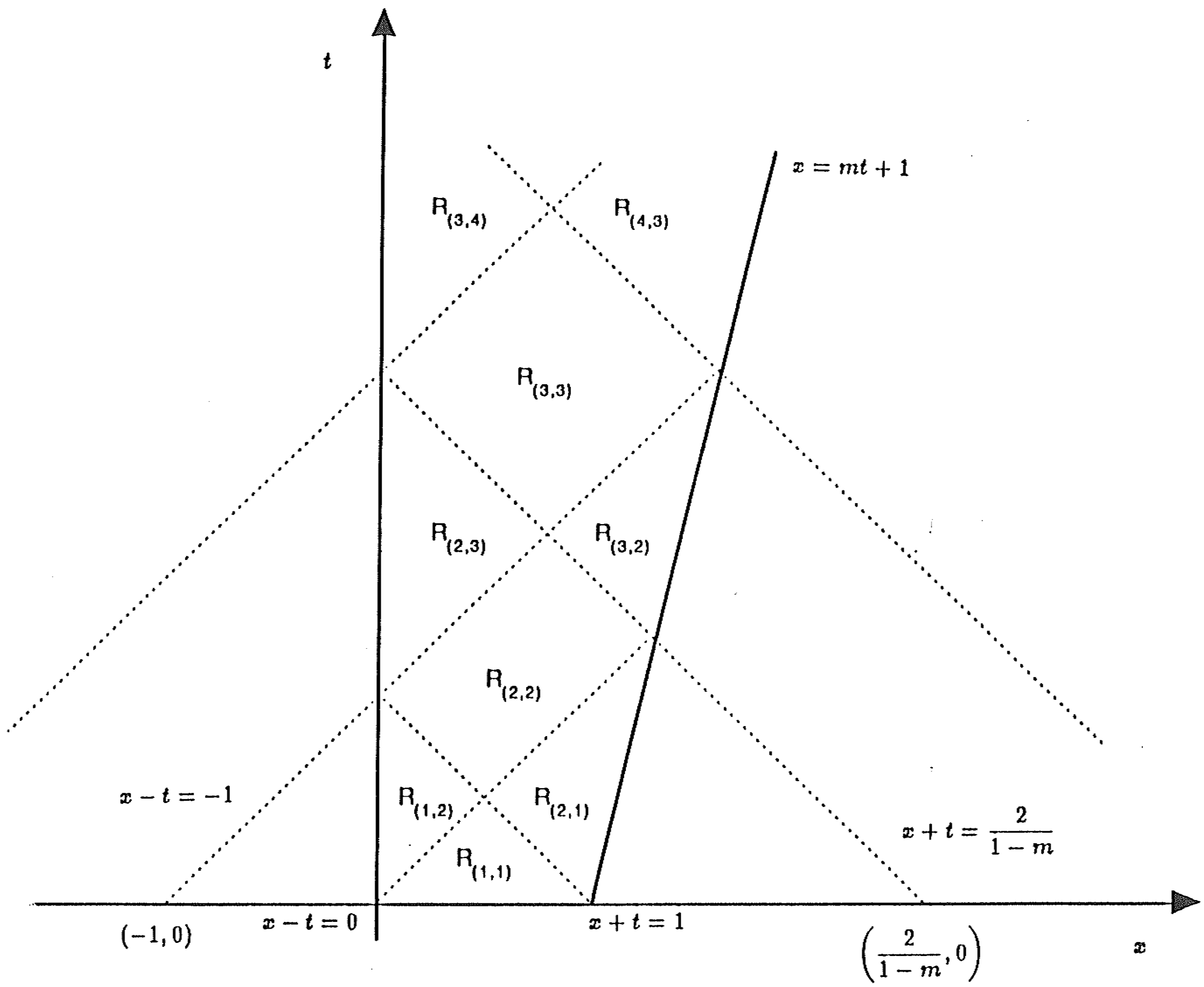
t
$x = mt + 1$
$R_{(3,4)}$
$R_{(4,3)}$
$R_{(3,3)}$
$R_{(2,3)}$
$R_{(3,2)}$
$R_{(2,2)}$
$x - t = -1$
$R_{(1,2)}$
$R_{(2,1)}$
$x + t = \frac{2}{1-m}$
$R_{(1,1)}$
$(-1, 0)$
$x - t = 0$
$x + t = 1$
$\left(\frac{2}{1-m}, 0\right)$
x

Diagram 2.2.1

In $R_{(1,1)}$, we have the formulae

$$\varphi_1(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2}\int_0^{\xi} g(\tau)d\tau + D, \qquad 0 \le \xi \le 1 \tag{2.2.3a}$$

and

$$\psi_1(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2}\int_0^{\eta} g(\tau)d\tau - D, \qquad 0 \le \eta \le 1, \tag{2.2.3b}$$

where D is some arbitrary constant. The solution in $R_{(1,2)}$ is composed of a left-moving wave emerging from $R_{(1,1)}$ (as described by (2.2.3a)) and a right-moving wave emanating from the boundary $x = 0$. The latter right-moving wave is a combination of transmitted boundary disturbances and reflections of a left-moving wave. Applying (2.2.1d) to (2.2.2) with $i = 1$ and $j = 2$ results in

$$\psi_2(\eta) = p(-\eta) - \varphi_1(-\eta),$$

or using (2.2.3a)

$$\psi_2(\eta) = p(-\eta) - \frac{1}{2}f(-\eta) - \frac{1}{2}\int_0^{-\eta} g(\tau)d\tau - D.$$

Using (2.2.1c) and (2.2.2), we obtain a similar result in $R_{(2,1)}$, with

$$\varphi_2(\xi) = q\left(\frac{\xi - 1}{m+1}\right) - \psi_1\left(\left[\frac{m-1}{m+1}\right](\xi - 1) + 1\right),$$

or equivalently, by virtue of (2.2.3b),

$$\varphi_2(\xi) = q\left(\frac{\xi - 1}{m+1}\right) - \frac{1}{2}f\left(\left[\frac{m-1}{m+1}\right](\xi - 1) + 1\right) - \frac{1}{2}\int_0^{\left[\frac{m-1}{m+1}\right](\xi-1)+1} g(\tau)d\tau + D.$$

In general, for any $R_{(j,j+1)}$ along $x = 0$, the solution (2.2.2) must satisfy

$$\psi_{j+1}(\eta) = p(-\eta) - \varphi_j(-\eta), \tag{2.2.4}$$

where $j = 1, 2, 3, \cdots$. Similarly, in $R_{(i+1,i)}$, for each $i = 1, 2, 3, \ldots$, we must have

$$\varphi_{i+1}(\xi) = q\left(\frac{\xi - 1}{m + 1}\right) - \psi_i\left(\left[\frac{m - 1}{m + 1}\right](\xi - 1) + 1\right),$$

or

$$\varphi_{i+1}(\xi) = q(\sigma(\xi)) - \psi_i(-\nu(\xi)), \qquad i = 1, 2, 3, \ldots, \tag{2.2.5a}$$

where

$$\sigma(\xi) = \frac{\xi - 1}{m + 1}, \tag{2.2.5b}$$

and

$$-\nu(\xi) = \left[\frac{m - 1}{m + 1}\right](\xi - 1) + 1. \tag{2.2.5c}$$

Suppose we are interested in the solution $u(x, t_1)$, for some fixed time $t_1 > 0$. We must first determine what particular regions $R_{(i,j)}$ the line $t = t_1$ intersects and then, using (2.2.4) and (2.2.5), we can recursively generate the appropriate functions, $\varphi_i(\xi)$ and $\psi_j(\eta)$, that will allow the determination of $u(x, t_1)$. To illustrate this procedure, let

$$h^{(n)}(\alpha) \equiv \begin{cases} \alpha, & n = 0 \\ h(h^{(n-1)}(\alpha)), & n = 1, 2, \ldots \end{cases}$$

denote the n^{th} composition of the function $h(\alpha)$ with itself. Let $\hat{q}(\xi) = q(\sigma(\xi))$ in

(2.2.5). Then, by induction, we have

$$\varphi_2(\xi) =\hat{q}(\xi) - \psi_1(-\nu(\xi)),$$

$$\varphi_3(\xi) =\hat{q}(\xi) - p(\nu(\xi)) + \varphi_1(\nu(\xi)),$$

$$\varphi_4(\xi) =\hat{q}(\xi) - p(\nu(\xi)) + \hat{q}(\nu(\xi)) - \psi_1\left(-\nu^{(2)}(\xi)\right),$$

$$\varphi_5(\xi) =\hat{q}(\xi) - p(\nu(\xi)) - \hat{q}(\nu(\xi)) - p\left(\nu^{(2)}(\xi)\right) + \varphi_1\left(\nu^{(2)}(-\xi)\right),$$

$$\ldots$$

and

$$\psi_2(\eta) =p(-\eta) - \varphi_1(-\eta),$$

$$\psi_3(\eta) =p(-\eta) - \hat{q}(-\eta) + \psi_1(-\nu(-\eta)),$$

$$\psi_4(\eta) =p(-\eta) - \hat{q}(-\eta) + p(\nu(-\eta)) - \varphi_1(\nu(-\eta)),$$

$$\psi_5(\eta) =p(-\eta) - \hat{q}(-\eta) + p(\nu(-\eta)) - \hat{q}(\nu(-\eta)) + \psi_1\left(-\nu^{(2)}(-\eta)\right),$$

$$\ldots$$

from which we deduce the following formulae:

1. If $i, j = 2, 4, 6, \ldots$ then:

$$\varphi_i(\xi) = \sum_{k=1}^{\frac{i}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{i}{2}-1} p\left(\nu^{(k)}(\xi)\right) - \psi_1\left(-\nu^{(\frac{i}{2})}(\xi)\right),$$

$$0 \leq -\nu^{(\frac{i}{2})}(\xi) \leq 1, \tag{2.2.6}$$

$$\psi_j(\eta) = \sum_{k=1}^{\frac{j}{2}} p\left(\nu^{(k-1)}(-\eta)\right) - \sum_{k=1}^{\frac{j}{2}-1} \hat{q}\left(\nu^{(k-1)}(-\eta)\right) - \varphi_1\left(\nu^{(\frac{j}{2}-1)}(-\eta)\right),$$

$$0 \leq \nu^{(\frac{j}{2}-1)}(-\eta) \leq 1. \tag{2.2.7}$$

2. If $i, j = 3, 5, 7, \ldots$ then:

$$\varphi_i(\xi) = \sum_{k=1}^{\frac{i-1}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{i-1}{2}} p\left(\nu^{(k)}(\xi)\right) + \varphi_1\left(\nu^{(\frac{i-1}{2})}(\xi)\right),$$

$$0 \leq \nu^{(\frac{i-1}{2})}(\xi) \leq 1, \tag{2.2.8}$$

$$\psi_j(\eta) = \sum_{k=1}^{\frac{j-1}{2}} p\left(\nu^{(k-1)}(-\eta)\right) - \sum_{k=1}^{\frac{j-1}{2}} \hat{q}\left(\nu^{(k-1)}(-\eta)\right) + \psi_1\left(-\nu^{(\frac{j-1}{2})}(-\eta)\right),$$

$$0 \leq -\nu^{(\frac{j-1}{2})}(-\eta) \leq 1. \tag{2.2.9}$$

In reference to the above equations, we note the following:

1. Whenever the upper limit of summation is smaller than the lower limit, the summation is to be ignored.

2. The quantity $\nu^{(n)}(\alpha)$, for a positive integer n, is given by

$$\nu^{(n)}(\alpha) = (-1)^{n+1}\left(\frac{m-1}{m+1}\right)^n (1-\alpha) + 2\sum_{k=1}^{n-1}(-1)^{k+1}\left(\frac{m-1}{m+1}\right)^k - 1,$$

$$n = 1, 2, 3, \ldots,$$

with, as usual, $\nu^{(0)}(\alpha) = \alpha$.

3. The inequalities, given with the formulae (2.2.6)-(2.2.9), define regions in which the formulae are valid. The equations $\nu^{(n)}(\alpha) = 0$ and $\nu^{(n)}(\alpha) = \pm 1$, (where

$+1$ or -1, n and α are chosen in accordance with the particular region $R_{(i,j)}$ of interest) define two pairs of parallel characteristics. The intersection of the area bounded by these four characteristics and the domain of the IMBVP is $R_{(i,j)}$.

A justfication of the above formulae will be postponed until section 2.2.3, when we consider the general case.

As in the previous section, (2.2.4) and (2.2.5), with $p(t) = q(t) = 0$, can be thought of as extension rules for the initial data. Application of (2.2.4) and (2.2.5) allows for the recursive extension of the initial data to the successive intervals along the x-axis, as depicted in diagram 2.2.1, which lie to the left and right of $[0,1]$ respectively. Alternatively, the formulae (2.2.6)-(2.2.9) can be used. For instance, suppose we want to extend $f(x)$ and $g(x)$ to the $(i-1)^{\text{th}}$ interval lying to the right of $[0,1]$. Setting $p(t) = q(t) = 0$, the following are found:

1. If $i = 2, 4, 6, \ldots$, then:

   a) for $g(x) = 0$:

$$\hat{f}(x) = 2\varphi_i(x) = -f\left(-\nu^{(\frac{i}{2})}(x)\right), \qquad 0 \leq -\nu^{(\frac{i}{2})}(x) \leq 1$$

b) for $f(x) = 0$:

$$\hat{g}(x) = -\frac{d}{dx}\left[\nu^{(\frac{i}{2})}(x)\right] g\left(-\nu^{(\frac{i}{2})}(x)\right), \qquad 0 \le -\nu^{(\frac{i}{2})}(x) \le 1$$

2. If $i = 3, 5, 7, \ldots$, then:

a) for $g(x) = 0$:

$$\hat{f}(x) = 2\varphi_i(x) = f\left(\nu^{(\frac{i-1}{2})}(x)\right), \qquad 0 \le \nu^{(\frac{i-1}{2})}(x) \le 1$$

b) for $f(x) = 0$:

$$\hat{g}(x) = \frac{d}{dx}\left[\nu^{(\frac{i-1}{2})}(x)\right] g\left(\nu^{(\frac{i-1}{2})}(x)\right), \qquad 0 \le \nu^{(\frac{i-1}{2})}(x) \le 1$$

In this instance, the attached inequalities determine intervals on the x-axis which are the domains of the extended functions $\hat{f}(x)$ and $\hat{g}(x)$. The endpoints of these intervals are the x-intercepts of the corresponding pairs of parallel characteristics previously discussed. Suppose we want to extend $f(x)$ to the first interval on the right of $[0,1]$. We have

$$\hat{f}(x) = -f\left(-\nu^{(1)}(x)\right), \qquad 0 \le -\nu^{(1)}(x) \le 1,$$

where the inequality defines the interval

$$0 \le \left(\frac{m-1}{m+1}\right)(x-1) + 1 \le 1,$$

which can be written as

$$0 \leq x - 1 \leq \frac{1+m}{1-m},$$

or

$$1 \leq x \leq \frac{2}{1-m}.$$

Similar extension rules can be developed using (2.2.7), (2.2.8) and (2.2.9).

In their present forms, (2.2.6)-(2.2.9) can be difficult to use. Suppose that we want to graph $u(x, t_1)$ for some fixed time $t_1 > 0$. We must first determine which particular regions $R_{(i,j)}$ the line $t = t_1$ intersects and then, using (2.2.6)-(2.2.9), piece together the function $u(x, t_1)$. This difficulty can be overcome through a slight modification of (2.2.6)-(2.2.9).

The line $t = t_1$ intersects the boundaries $x = 0$ and $x = mt + 1$ at the points $(0, t_1)$ and $(mt_1 + 1, t_1)$ respectively. On the other, the characteristics $x - t = -t_1$ and $x + t = (m+1)t_1 + 1$ intersect the x-axis at the points $P_1 = (-t_1, 0)$ and $P_2 = ((m+1)t_1 + 1, 0)$ respectively. The points P_1 and P_2 tell us how far along the x-axis the initial data must be extended so that the desired solution $u(x, t_1)$, for $0 < x < mt_1 + 1$, can be obtained from an "equivalent" infinite IVP.

Now suppose we find the smallest positive integers i and j such that either

$$0 \leq -\nu^{(\frac{i}{2})}((m+1)t_1 + 1) \leq 1 \quad \text{if } i \text{ is even},$$

or

$$0 \leq \nu^{(\frac{i-1}{2})}((m+1)t_1+1) \leq 1 \quad \text{if } i \text{ is odd}$$

and either

$$0 \leq \nu^{(\frac{j}{2}-1)}(t_1) \leq 1 \quad \text{if } j \text{ is even,}$$

or

$$0 \leq -\nu^{(\frac{j-1}{2})}(t_1) \leq 1 \quad \text{if } j \text{ is odd,}$$

then the solution is given by

$$u(x,t_1) = \tilde{\varphi}_i(x+t_1) + \tilde{\psi}_{i-1}(x-t_1) + \tilde{\psi}_j(x-t_1) + \tilde{\varphi}_{j-1}(x+t_1),$$

where the functions $\tilde{\varphi}_i(\xi)$ and $\tilde{\psi}_j(\eta)$ are defined as follows:

1. If $i, j = 2, 4, 6, \ldots$, then:

$$\tilde{\varphi}_i(\xi) = \varphi_i(\xi)\left[H\left(-\nu^{(\frac{i}{2})}(\xi)\right) - H\left(-\nu^{(\frac{i}{2})}(\xi) - 1\right)\right],$$

$$\tilde{\psi}_j(\eta) = \psi_j(\eta)\left[H\left(\nu^{(\frac{j}{2}-1)}(-\eta)\right) - H\left(\nu^{(\frac{j}{2}-1)}(-\eta) - 1\right)\right].$$

2. If $i, j = 3, 5, 7, \ldots$, then:

$$\tilde{\varphi}_i(\xi) = \varphi_i(\xi)\left[H\left(\nu^{(\frac{i-1}{2})}(\xi)\right) - H\left(\nu^{(\frac{i-1}{2})}(\xi) - 1\right)\right],$$

$$\tilde{\psi}_j(\eta) = \psi_j(\eta)\left[H\left(-\nu^{(\frac{i-1}{2})}(-\eta)\right) - H\left(-\nu^{(\frac{i-1}{2})}(-\eta) - 1\right)\right],$$

where

$$H(\zeta) = \begin{cases} 1 & \zeta \geq 0 \\ 0 & \zeta < 0 \end{cases}$$

is the unit Heaviside funtion and $\varphi_i(\xi)$ and $\psi_j(\eta)$ are defined by (2.2.6)-(2.2.9).

Finally, as an example of the preceding discussion, consider (2.2.1) with $g(x) = f(x) = q(t) = 0$, $p(t) = \sin(6t)$ and $m = \frac{1}{2}$. Suppose we want to find the solution at time $t = 3$. It can be shown that $i = j = 3$ so

$$u(x,3) = \tilde{\varphi}_3(x+3) + \tilde{\psi}_2(x-3) + \tilde{\psi}_3(x-3) + \tilde{\varphi}_2(x+3),$$

where

$$\varphi_3(\xi) = -\sin(6\nu(\xi))$$

$$\varphi_2(\xi) = 0$$

$$\psi_3(\eta) = \sin(-6\eta)$$

$$\psi_2(\eta) = \sin(-6\eta)$$

The graph of $u(x,3)$ is given in diagram 2.2.2.

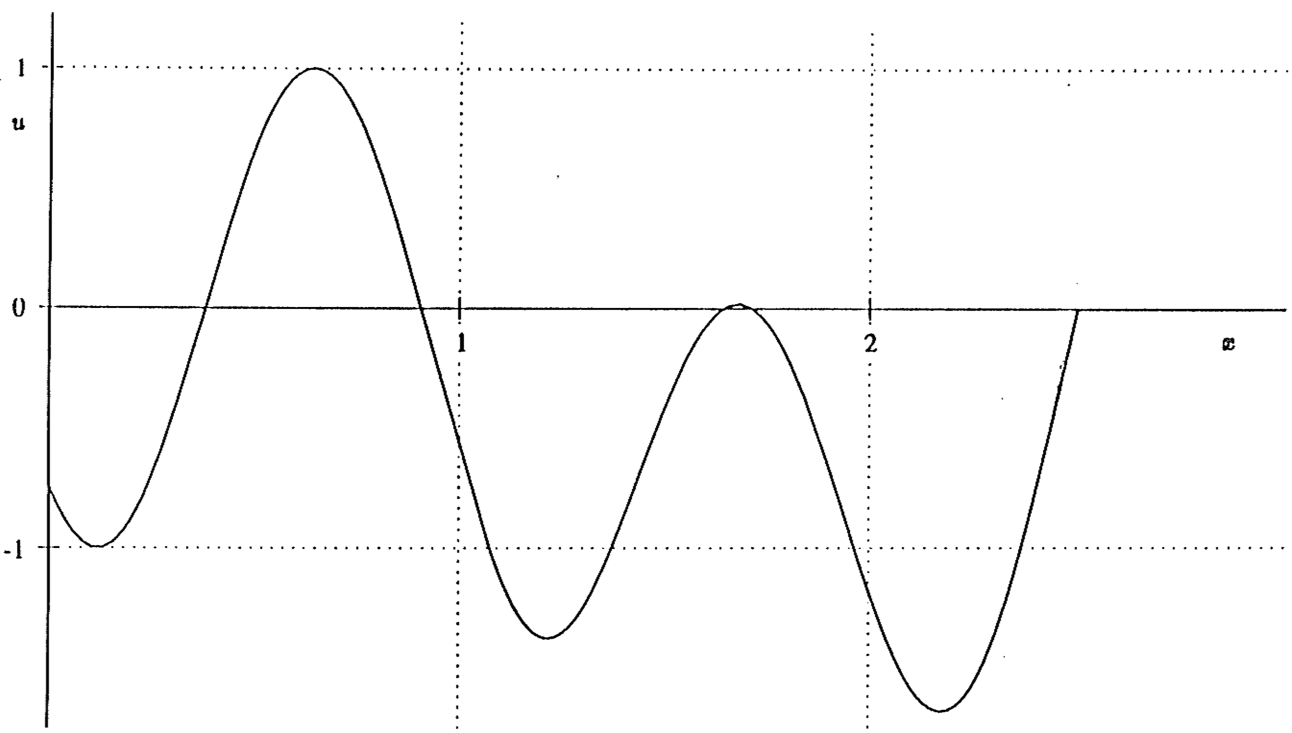

Diagram 2.2.2

2.3: An IMBVP with General Moving Endpoint

We now consider the IMBVP (1.1.3). Using an approach that is similar to that of the previous section, the general solution of (1.1.3) can be constructed. For convenience, we restate (1.1.3).

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < S(t), \qquad t > 0, \qquad |S'(t)| < 1$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad 0 \leq x \leq S(0)$$

$$u(0,t) = p(t), \qquad u(S(t),t) = q(t), \qquad t \geq 0.$$

The order of differentiability of solutions of (1.1.3) is determined through the following conditions:

1. If $f(x)$, $p(t)$, $q(t) \in \mathbf{C}^0$, $g(x)$ is integrable, $f(0) = p(0)$ and $f(S(0)) = q(0)$ then $u(x,t) \in \mathbf{C}^0$.

2. If $f(x)$, $p(t)$, $q(t) \in \mathbf{C}^1$, $g(x) \in \mathbf{C}^0$, $f(0) = p(0)$, $f(S(0)) = q(0)$, $g(0) = p'(0)$ and $S'(0)f'(S(0)) + g(S(0)) = q'(0)$ then $u(x,t) \in \mathbf{C}^1$.

3. If $f(x)$, $p(t)$, $q(t) \in \mathbf{C}^2$, $g(x) \in \mathbf{C}^1$, $f(0) = p(0)$, $f(S(0)) = q(0)$, $g(0) = p'(0)$,

$S'(0)f'(S(0)) + g(S(0)) = q'(0)$, $f''(0) = p''(0)$ and $([S'(0)]^2 + 1) f''(S(0)) + 2S'(0)g'(S(0)) + S''(0)f'(S(0)) = q''(0)$ then $u(x,t) \in \mathbf{C}^2$.

As before, the domain $\mathbf{D}= \{(x,t)|0 \leq x \leq S(t), \quad t \geq 0\}$ must be subdivided into regions $R_{(i,j)}$ using the characteristics $x = t$, $x + t = s(0)$ and their respective reflections at the boundary curves $x = 0$ and $x = S(t)$ (see diagram 2.3.1). Solving (1.1.3) requires that the appropriate form of the solution be found for each $R_{(i,j)}$.

As in the previous section, let

$$u(x,t) = \varphi_i(x+t) + \psi_j(x-t) \tag{2.3.1}$$

denote the solution of (1.1.3) in region $R_{(i,j)}$. In analogy with the results of the previous section, it can be shown that the solution in $R_{(i,j)}$ is given by (2.3.1) with $i = 1$ and $j = 1$, with

$$\varphi_1(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2}\int_0^\xi g(\tau)d\tau + D, \qquad 0 \leq \xi \leq S(0)$$

and

$$\psi_1(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2}\int_0^\eta g(\tau)d\tau - D, \qquad 0 \leq \eta \leq S(0).$$

Similarly, using (2.3.1) and (1.1.3b) we have, for each of the regions $R_{(j,j+1)}$,

$$\psi_{j+1}(\eta) = p(-\eta) - \varphi_j(-\eta), \tag{2.3.2}$$

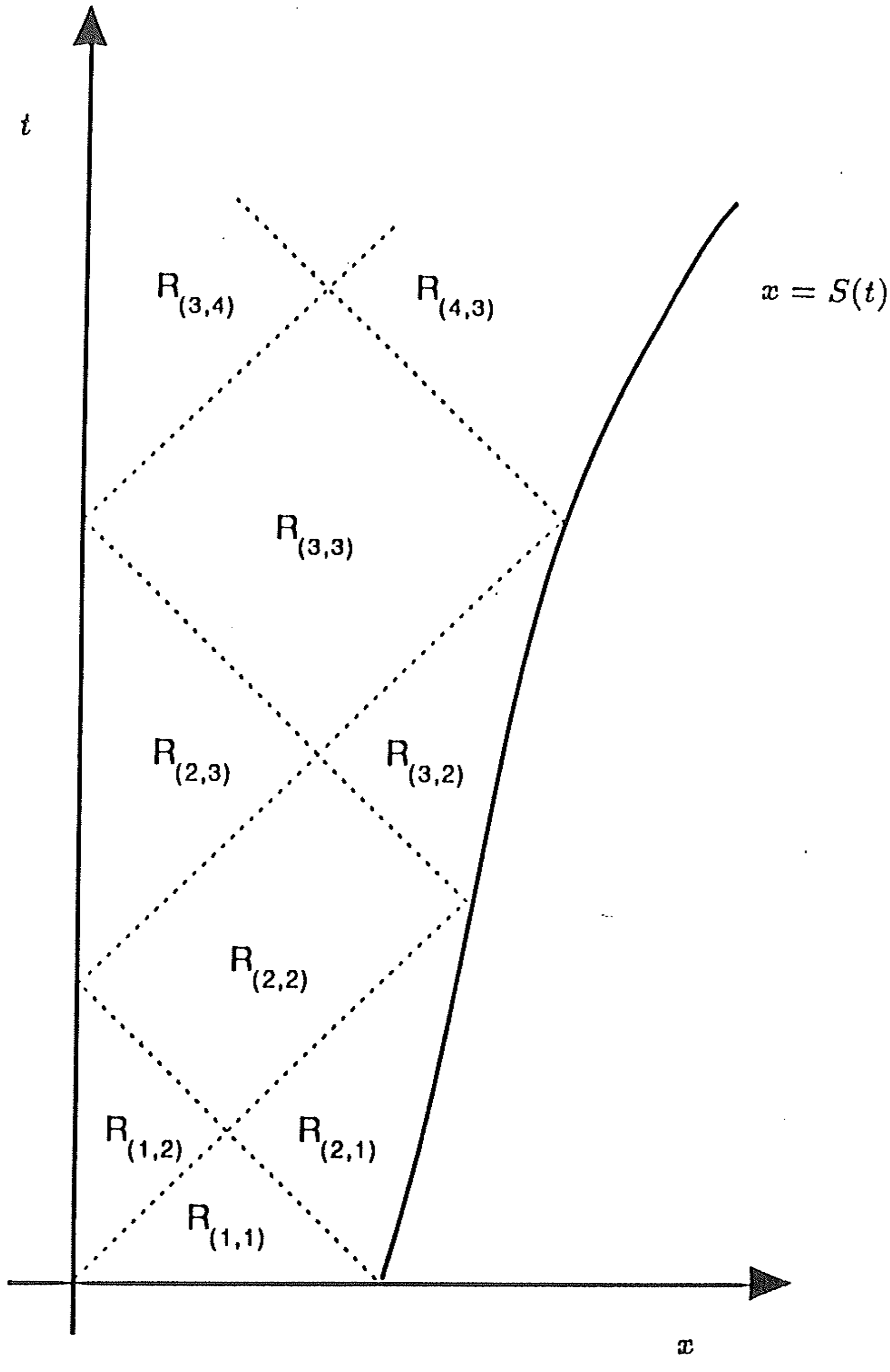

Diagram 2.3.1

while on the other hand, for each of the regions $R_{(i+1,i)}$, the boundary condition (1.1.3c) provides

$$\varphi_{i+1}(S(t)+t)+\psi_i(S(t)-t)=q(t).$$

Let $\xi=\theta(t)=S(t)+t$. Since $|S'(t)|<1$, the function $\theta(t)$ is invertible and the above may be written as

$$\varphi_{i+1}(\xi)=q(\theta^{-1}(\xi))-\psi_i(S(\theta^{-1}(\xi))-\theta^{-1}(\xi)). \tag{2.3.3}$$

Putting $t=\theta^{-1}(\xi)$ in $S(t)=\theta(t)-t$ yields

$$S(\theta^{-1}(\xi))=\xi-\theta^{-1}(\xi). \tag{2.3.4}$$

Substitution of (2.3.4) into (2.3.3) produces the result,

$$\varphi_{i+1}(\xi)=q(\theta^{-1}(\xi))-\psi_i(\xi-2\theta^{-1}(\xi))$$

or

$$\varphi_{i+1}(\xi)=\hat{q}(\xi)-\psi_i(-\nu(\xi)) \tag{2.3.5a}$$

where

$$\hat{q}(\xi)=q(\theta^{-1}(\xi)) \tag{2.3.5b}$$

and

$$\nu(\xi)=2\theta^{-1}(\xi)-\xi. \tag{2.3.5c}$$

In analogy with the procedures developed in the previous section, it can be shown that the functions $\varphi_i(\xi)$ and $\psi_j(\eta)$, generated recursively by (2.3.5) and (2.3.2), are given as follows:

1. If $i, j = 2, 4, 6, \ldots$ then:

$$\varphi_i(\xi) = \sum_{k=1}^{\frac{i}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{i}{2}-1} p\left(\nu^{(k)}(\xi)\right) - \psi_1\left(-\nu^{(\frac{i}{2})}(\xi)\right),$$

$$0 \leq -\nu^{(\frac{i}{2})}(\xi) \leq S(0), \tag{2.3.6}$$

$$\psi_j(\eta) = \sum_{k=1}^{\frac{j}{2}} p\left(\nu^{(k-1)}(-\eta)\right) - \sum_{k=1}^{\frac{j}{2}-1} \hat{q}\left(\nu^{(k-1)}(-\eta)\right) - \varphi_1\left(\nu^{(\frac{j}{2}-1)}(-\eta)\right),$$

$$0 \leq \nu^{(\frac{j}{2}-1)}(-\eta) \leq S(0). \tag{2.3.7}$$

2. If $i, j = 3, 5, 7, \ldots$ then:

$$\varphi_i(\xi) = \sum_{k=1}^{\frac{i-1}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{i-1}{2}} p\left(\nu^{(k)}(\xi)\right) + \varphi_1\left(\nu^{(\frac{i-1}{2})}(\xi)\right),$$

$$0 \leq \nu^{(\frac{i-1}{2})}(\xi) \leq S(0), \tag{2.3.8}$$

$$\psi_j(\eta) = \sum_{k=1}^{\frac{j-1}{2}} p\left(\nu^{(k-1)}(-\eta)\right) - \sum_{k=1}^{\frac{j-1}{2}} \hat{q}\left(\nu^{(k-1)}(-\eta)\right) + \psi_1\left(-\nu^{(\frac{j-1}{2})}(-\eta)\right),$$

$$0 \leq -\nu^{(\frac{j-1}{2})}(-\eta) \leq S(0). \tag{2.3.9}$$

where

$$\nu^n(\alpha) = \begin{cases} \alpha, & n = 0 \\ \nu(\nu^{(n-1)}(\alpha)), & n = 1, 2, \ldots \end{cases}.$$

We illustrate a proof of (2.3.6) as follows:

For $i = 2$, (2.3.6) becomes

$$\varphi_2(\xi) = \hat{q}(\xi) - \psi_1(-\nu(\xi)),$$

which is simply (2.3.5a). Suppose that (2.3.6) holds for $i = n$ (n even), so that

$$\varphi_n(\xi) = \sum_{k=1}^{\frac{n}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{n}{2}-1} p\left(\nu^{(k)}(\xi)\right) - \psi_1\left(-\nu^{(\frac{n}{2})}(\xi)\right),$$

$$0 \leq -\nu^{(\frac{n}{2})}(\xi) \leq S(0).$$

Equation (2.3.2) provides

$$\psi_{n+1}(\xi) = p(-\xi) - \varphi_n(-\xi),$$

so that (2.3.5a) becomes

$$\begin{aligned}
\varphi_{n+2}(\xi) =& \hat{q}(\xi) - p(\nu(\xi)) + \varphi_n(\nu(\xi)) \\
=& \hat{q}(\xi) - p(\nu(\xi)) - \sum_{k=1}^{\frac{n}{2}} \hat{q}\left(\nu^{(k)}(\xi)\right) - \sum_{k=1}^{\frac{n}{2}-1} p\left(\nu^{(k)1)}(\xi)\right) \\
& - \psi_1\left(-\nu^{(\frac{n+2}{2})}(\xi)\right).
\end{aligned}$$

Re-indexing the sums leads to

$$\varphi_{n+2}(\xi) = \sum_{k=1}^{\frac{n+2}{2}} \hat{q}\left(\nu^{(k-1)}(\xi)\right) - \sum_{k=1}^{\frac{n+2}{2}-1} p\left(\nu^{(k)}(\xi)\right) - \psi_1\left(-\nu^{(\frac{n+2}{2})}(\xi)\right),$$

which is exactly (2.3.6) in the case when $i = n+2$ (even). Thus by the Principle of Mathematical Induction, (2.3.6) holds for all $i = 2, 4, 6, \cdots$. The proof of (2.3.7) follows in an analogous manner, and the formulae (2.3.8) and (2.3.9) follow respectively from (2.3.7) and (2.3.6) by using (2.3.5) and (2.3.2).

We note that the existence of formulae (2.3.6)-(2.3.9) establish the existence of a solution to IMBVP (1.1.3). Furthermore, by virtue of (2.3.1) and (2.3.6)-(2.3.9) we see that the solution is determined uniquely, by the functions $f(x)$, $g(x)$, $p(t)$, $q(t)$ and $S(t)$, for each $R_{(i,j)}$. That is, all solutions of the wave equation must be of the form (2.3.1), and requiring that (2.3.1) satisfy the initial and boundary conditions determines (2.3.1) uniquely in each $R_{(i,j)}$. More spcifically, formulae (2.3.6)-(2.3.9), along with the formulae for $\varphi_1(\xi)$ and $\psi_1(\eta)$, determine the sequences $\{\varphi_i(\xi)\}$ and $\{\psi_j(\eta)\}$ to within an arbitrary constant D. When terms from these sequences are added (according to (2.3.1)) D cancels and (2.3.1) depends only upon the data specified in (1.1.3).

As an example, consider (1.1.3) with $S(t) = \sqrt{(t^2+1)}$, as discussed by Balazs in [Bal]. The domain of this problem is subdivided into a finite number of regions $R_{(i,j)}$, as shown in diagram 2.3.2. Equation (2.3.6) gives

$$\varphi_2(\xi) = \hat{q}(\xi) - \psi_1(-\nu(\xi))$$

while (2.3.7) and (2.3.9) provide

$$\psi_2(\eta) = p(-\eta) - \varphi_1(-\eta)$$

and

$$\psi_3(\eta) = p(-\eta) - \hat{q}(-\eta) + \psi_1(-\nu(-\eta)).$$

The function $\theta(t)$ is given by

$$\zeta = \theta(t) = \sqrt{t^2+1} + t,$$

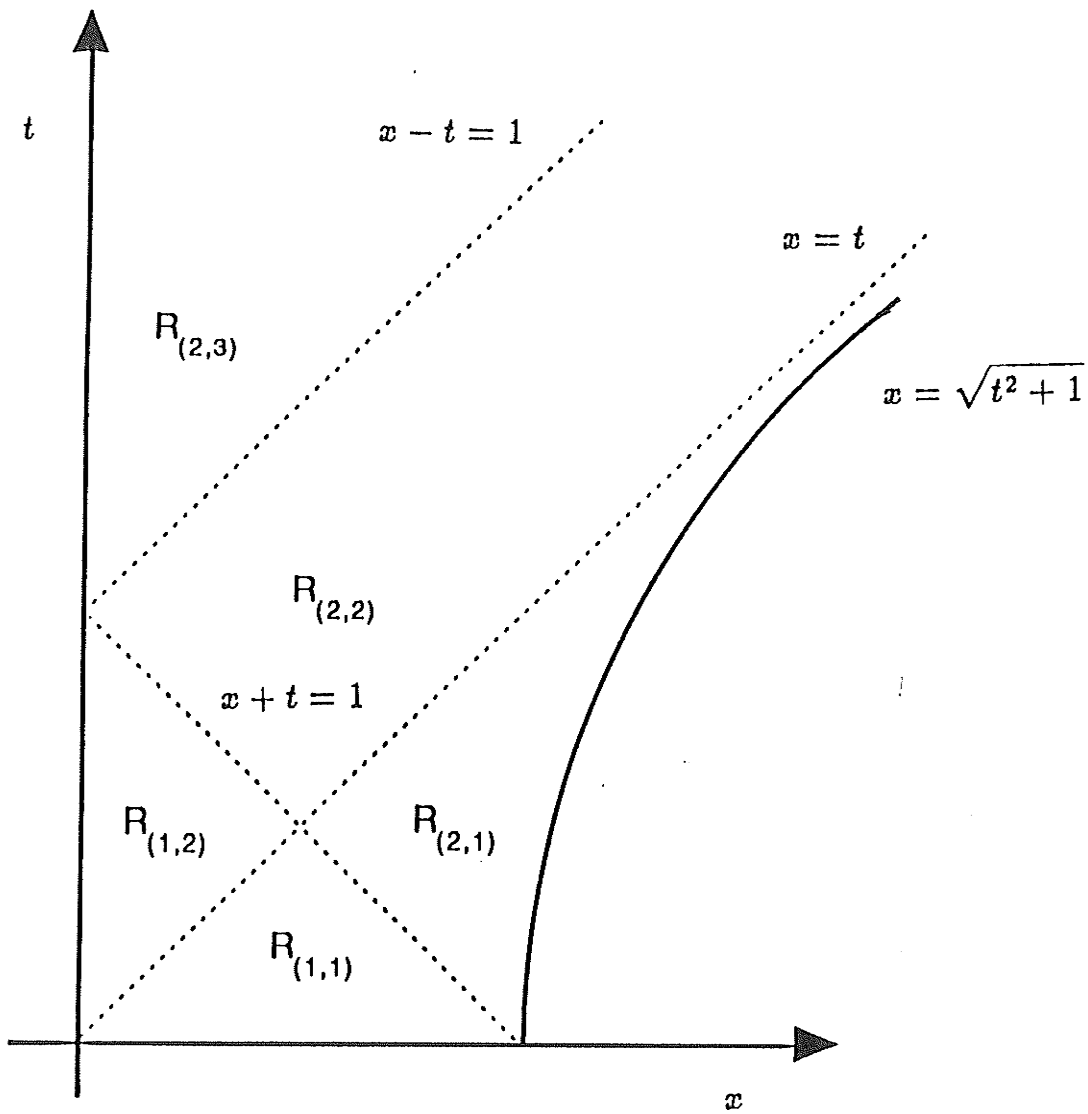

Diagram 2.3.2

from which we deduce that

$$\zeta^2 - 2\zeta t + t^2 = t^2 + 1,$$

which may be solved for t as a function of ζ to give

$$t = \theta^{-1}(\zeta) = \frac{\zeta^2 - 1}{2\zeta}.$$

The above leads us to

$$\hat{q}(\zeta) = q\left(\frac{\zeta^2 - 1}{2\zeta}\right)$$

and

$$\nu(\zeta) = 2\theta^{-1}(\zeta) - \zeta = \frac{-1}{\zeta}.$$

Using the preceding analysis, the complete solution is given as follows:

1. In $R_{(1,1)}$,

$$u(x,t) = \varphi_1(x+t) + \psi_1(x-t)$$

2. In $R_{(1,2)}$,

$$u(x,t) = \varphi_1(x+t) - \varphi_1(t-x) + p(t-x)$$

3. In $R_{(2,1)}$,

$$u(x,t) = -\psi_1\left(\frac{1}{x+t}\right) + \psi_1(x-t) + q\left(\frac{(x+t)^2 - 1}{2(x+t)}\right)$$

4. In $R_{(2,2)}$,

$$u(x,t) = -\psi_1\left(\frac{1}{x+t}\right) - \varphi_1(t-x) + q\left(\frac{(x+t)^2-1}{2(x+t)}\right) + p(t-x)$$

5. In $R_{(2,3)}$,

$$u(x,t) = -\psi_1\left(\frac{1}{x+t}\right) + \psi_1\left(\frac{1}{t-x}\right) + q\left(\frac{(x+t)^2-1}{2(x+t)}\right) + p(t-x)$$
$$-q\left(\frac{(t-x)^2-1}{2(t-x)}\right)$$

where the functions $\varphi_1(\xi)$ and $\psi_1(\eta)$ have been previously defined. The above solution, derived via the formulae (2.3.6)-(2.3.9), is consistent with the results presented by Balazs. However, it should be noted that finding an explicit solution for a given boundary curve $x = S(t)$ depends directly upon finding the function $t = \theta^{-1}(\zeta)$ explicitly which, in general, may be difficult to do.

Graphs of Balazs' solution $u(x,t)$ for several values of t, with $p(t) = f(x) = g(x) = 0$ and $q(t) = \sin(6t)$, appear in diagram 2.3.3. It should be noted that the motion of the endpoint of the string at which the driver is located effectively modifies the "wavelength" of the waves generated by the driving force $q(t)$. Like the Doppler effect (caused by a linearly moving endpoint), the motion of the endpoint induces an increase in the "wavelength" which results in a horizontal stretching of the graph of $u(x,t)$. However, the stretching in this case is not uniform, since the

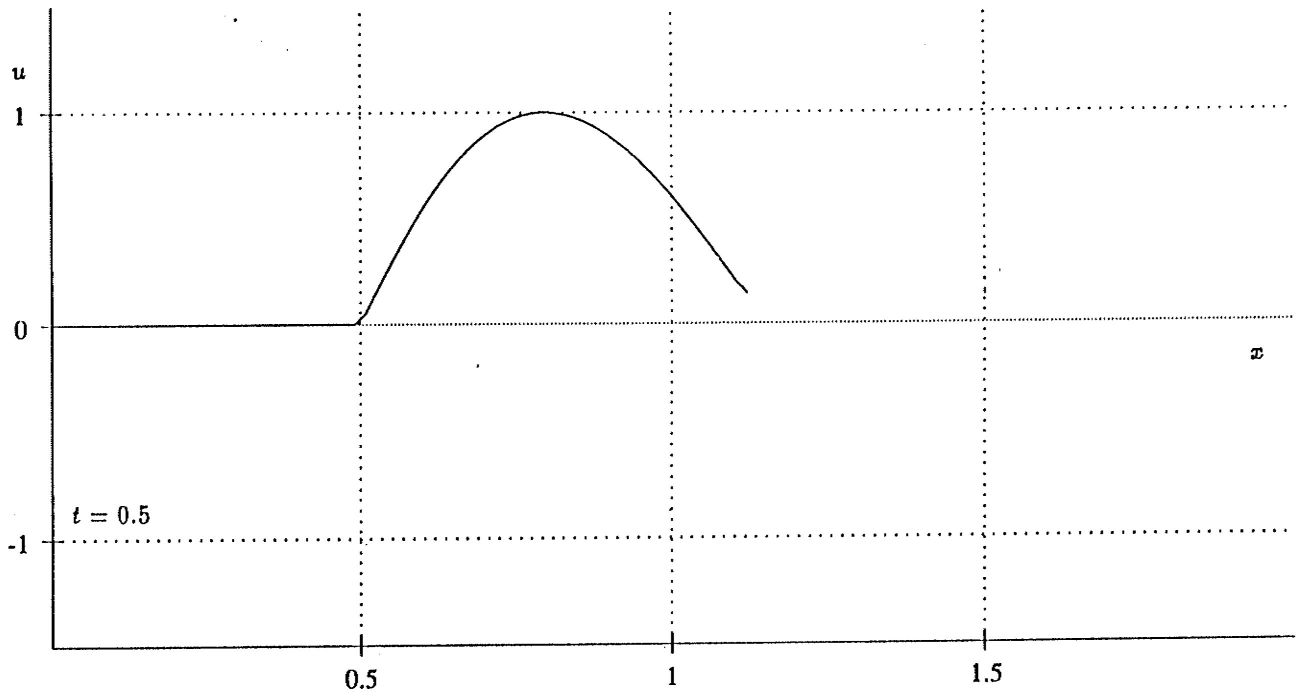

Diagram 2.3.3a

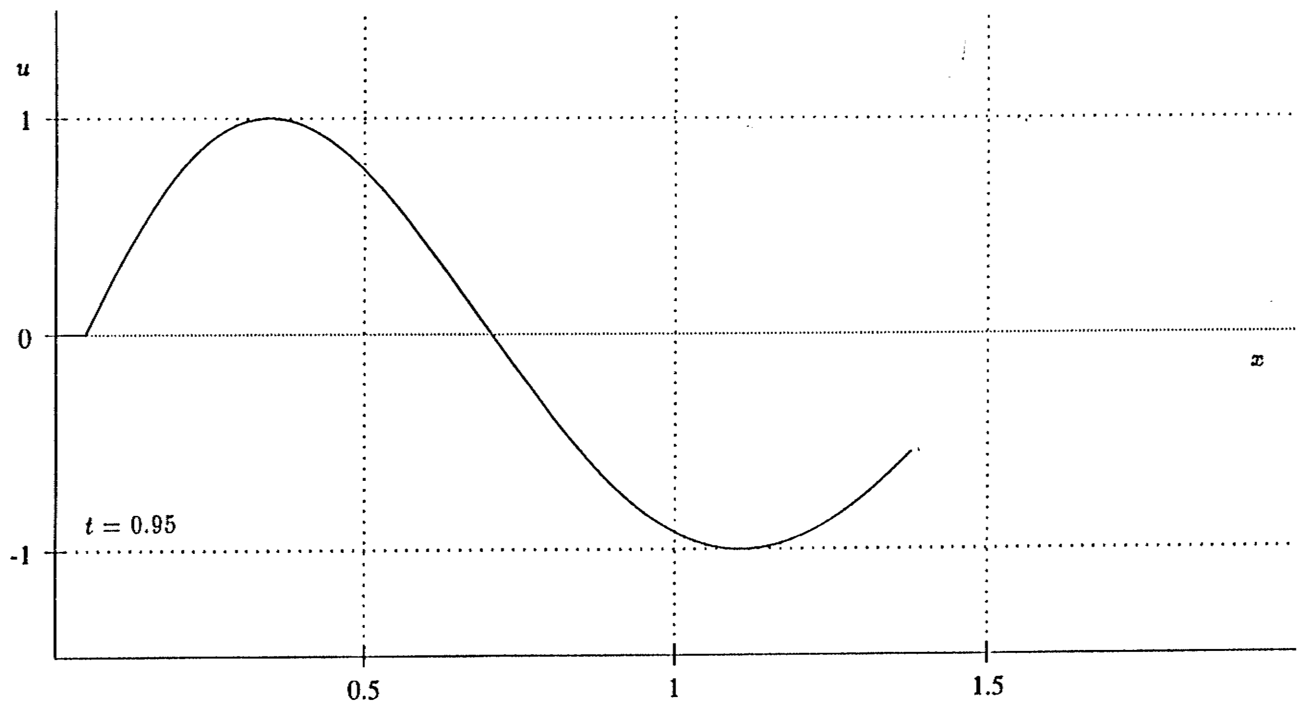

Diagram 2.3.3b

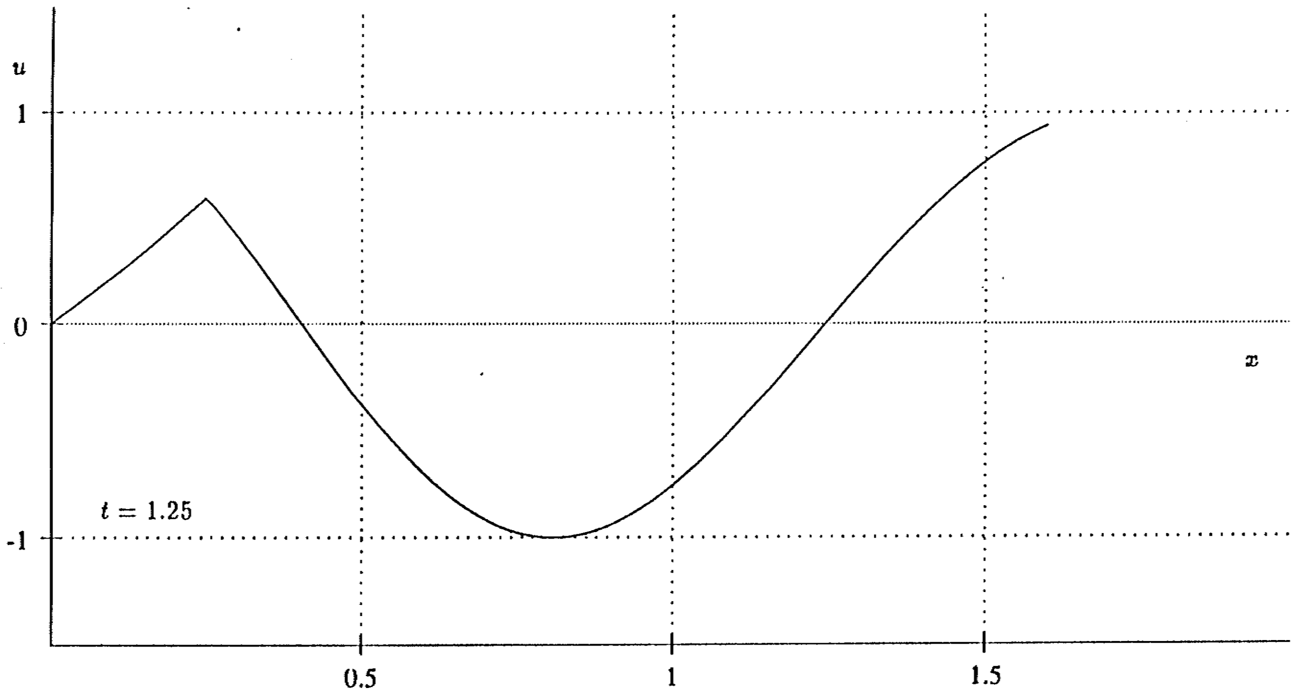

Diagram 2.3.3c

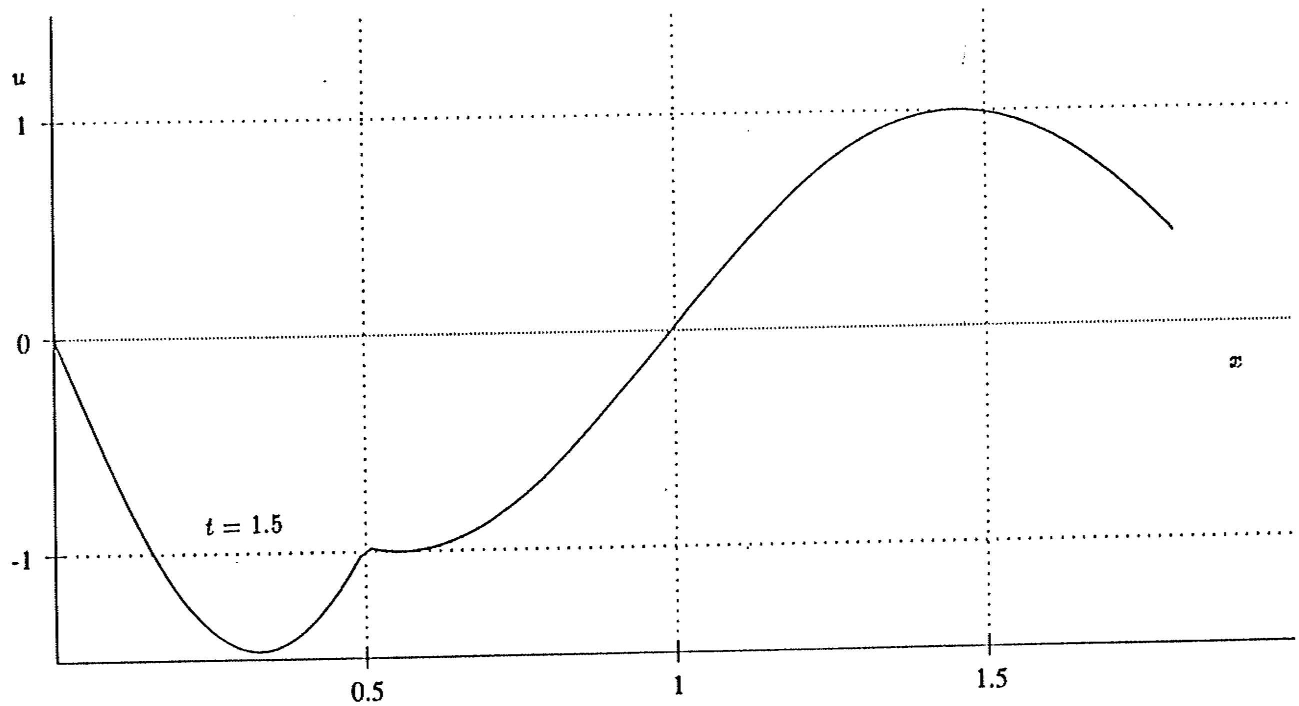

Diagram 2.3.3d

speed of the endpoint at the driver is not constant. The sharp corners appearing in the graphs are caused by discontinuities in the first derivative of $u(x,t)$. These dicontinuities are a result of the fact that $q'(0) \neq 0$.

As a final example of the applications of (2.3.6)-(2.3.10), we consider the question of continuous dependence of the solution $u(x,t)$ on the initial and boundary data. In other words, we want to verify that the solution $u(x,t)$ remains bounded when small changes in the initial and boundary data are introduced. Since the functions $f(x)$, $g(x)$, $p(t)$ and $q(t)$ are to be chosen to represent complicated physical quantities whose exact nature may not be understood, it is not unreasonable to assume that some error may be made during their determination. If this is indeed the case, then it is desirable that the solution, resulting from the erroneous data, be in some sense close to the solution that would result from perfect data. Suppose, for the moment, that $p(t) = q(t) = 0$ and consider the two sets of initial data

$$u_1(x,0) = f_1(x), \qquad \frac{\partial u_1(x,0)}{\partial t} = g_1(x), \qquad 0 \leq x \leq S(0) \qquad (2.3.11)$$

and

$$u_2(x,0) = f_2(x), \qquad \frac{\partial u_2(x,0)}{\partial t} = g_2(x), \qquad 0 \leq x \leq S(0) \qquad (2.3.12)$$

such that

$$|f_1(x) - f_2(x)| < \delta, \qquad |g_1(x) - g_2(x)| < \epsilon, \qquad \text{for all } x \in [0, S(0)],$$

where δ and ϵ are small positive constants. Consider the solutions resulting from (2.3.11) and (2.3.12) in a regions $R_{(2k,2k)}$ for $k = 1, 2, \cdots$. Using (2.3.6) and (2.3.7) we obtain

$$u_n(x,t) = -\frac{1}{2}[f_n\left(-\nu^{(k)}(x+t)\right) + f_n\left(\nu^{(k-1)}(t-x)\right)] + \frac{1}{2}\int_{\nu^{(k-1)}(t-x)}^{-\nu^{(k)}(x+t)} g_n(\tau)d\tau,$$

for $n = 1, 2$ and where $0 \leq -\nu^{(k)}(x+t) \leq S(0)$ and $0 \leq \nu^{(k-1)}(t-x) \leq S(0)$. Let $\zeta_1 = -\nu^{(k)}(x+t)$ and $\zeta_2 = \nu^{(k-1)}(t-x)$. Then,

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| &\leq \frac{1}{2}|f_1(\zeta_1) - f_2(\zeta_1)| + \frac{1}{2}|f_1(\zeta_2) - f_2(\zeta_2)| + \int_{\zeta_2}^{\zeta_1} |g_1(\tau) - g_2(\tau)|d\tau \\ &\leq \delta + \frac{1}{2}\epsilon(\zeta_1 - \zeta_2) \leq \delta + \frac{1}{2}\epsilon S(0). \end{aligned}$$

For the remaining five "types" of regions $R_{(2k+1,2k+1)}$, $R_{(2k+1,2k)}$, $R_{(2k+1,2(k+1))}$, $R_{(2k,2k+1)}$ and $R_{(2(k+1),2k+1)}$, it can similarly be shown that

$$|u_1(x,t) - u_2(x,t)| \leq \delta + \frac{1}{2}\epsilon S(0),$$

from which we can conclude that the solution $u(x,t)$ depends continuously on the initial data. Now suppose that $f(x) = g(x) = 0$ and that we have two sets of boundary data

$$u_1(0,t) = p_1(t), \qquad u_1(S(t),t) = q_1(t), \qquad t \geq 0$$

and

$$u_2(0,t) = p_2(t), \qquad u_2(S(t),t) = q_2(t), \qquad t \geq 0,$$

such that

$$|p_1(t) - p_2(t)| < \delta, \qquad |q_1(t) - q_2(t)| < \epsilon, \qquad \text{for all } t \geq 0.$$

In an analogous manner, one can show that for $(x,t) \in R_{(i,j)}$, where $i,j = 0,1,2,\ldots$

$$|u_1(x,t) - u_2(x,t)| \leq (i-1)\delta + (j-1)\epsilon.$$

The above bound depends on i and j, and as $t \to \infty$, $i,j \to \infty$, so that the above bound is no longer finite. However, for $0 \leq t \leq t_1$, with t_1 finite, the solution depends continuously on the boundary data. Finally, using the principle of superposition, we can write the solution of (1.1.3) as

$$u(x,t) = \tilde{u}(x,t) + \hat{u}(x,t),$$

where $\tilde{u}(x,t)$ satisfies (1.1.3) with $f(x) = g(x) = 0$ and $\hat{u}(x,t)$ satisfies (1.1.3) with $p(t) = q(t) = 0$, and therefore the solution of (1.1.3) depends continuously on specified initial and boundary data for all finite times.

2.4: Uniqueness of Solutions

In the preceeding section the existence of solutions of IMBVP (1.1.3) was established by construction, and the continuous dependence of solutions on the initial and boundary data was demonstated. In this section we present a uniqueness argument that, while not quite as general as one would like, allows us to deduce an important conclusion concerning the long term stability of the case of (1.1.3) when $-1 < S'(t) < 0$.

Suppose there exists two solutions $u_1(x,t)$, and $u_2(x,t)$ of (1.1.3). If $w(x,t) = u_1(x,t) - u_2(x,t)$, then it must satisfy the IMBVP

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0, \qquad 0 < x < S(t), \qquad t > 0, \qquad |S'(t)| < 1 \qquad (2.4.1a)$$

$$w(x,0) = 0, \qquad \frac{\partial w(x,0)}{\partial t} = 0, \qquad 0 \leq x \leq S(0) \qquad (2.4.1b,c)$$

$$w(0,t) = 0, \qquad w(S(t),t) = 0, \qquad t \geq 0. \qquad (2.4.1d,e)$$

Multiplying (2.4.1a) by $\frac{\partial w}{\partial t}$ yields

$$\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2} - \frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2} = 0$$

or equivalently

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\left[\frac{\partial w}{\partial t}\right]^2 + \left[\frac{\partial w}{\partial x}\right]^2\right) - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right) = 0,$$

where we have assumed that the second order mixed partial derivative commutes (i.e., $\frac{\partial^2 w}{\partial x \partial t} = \frac{\partial^2 w}{\partial t \partial x}$). A sufficient condition for this is $u(x,t) \in \mathbf{C}^2$. Integration of the latter result with respect to x leads to

$$\int_0^{S(t)} \frac{\partial}{\partial t} \frac{1}{2} \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) dx = \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial t} \right) \Bigg|_{x=0}^{x=S(t)} . \tag{2.4.2}$$

On the other hand, by Leibnitz' rule, it can be shown that

$$\frac{d}{dt} \int_0^{S(t)} \frac{1}{2} \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) dx = \int_0^{S(t)} \frac{\partial}{\partial t} \frac{1}{2} \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) dx$$
$$+ \frac{1}{2} S'(t) \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) \Bigg|_{x=S(t)} . \tag{2.4.3}$$

Substitution of (2.4.2) into (2.4.3) produces

$$\frac{d}{dt} \int_0^{S(t)} \frac{1}{2} \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) dx = \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial t} \right) \Bigg|_{x=0}^{x=S(t)}$$
$$+ \frac{1}{2} S'(t) \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) \Bigg|_{x=S(t)} . \tag{2.4.4}$$

The quantity

$$E(t) = \int_0^{S(t)} \frac{1}{2} \left(\left[\frac{\partial w}{\partial t} \right]^2 + \left[\frac{\partial w}{\partial x} \right]^2 \right) dx$$

represents the total energy in the string at any time t, while along the boundary curve $x = S(t)$, we have

$$\frac{dw}{dt} \Bigg|_{x=S(t)} = \left(S'(t) \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \right) \Bigg|_{x=S(t)} = 0, \tag{2.4.5}$$

by virtue of (2.4.1e). Using (2.4.5), we can rewrite (2.4.4) as

$$E'(t) = -S'(t) \left[\frac{\partial w}{\partial x} \right]^2 \Bigg|_{x=S(t)} - \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial t} \right) \Bigg|_{x=0}$$

$$+\frac{1}{2}S'(t)\left(\left[\frac{\partial w}{\partial x}\right]^2+\left[S'(t)\frac{\partial w}{\partial x}\right]^2\right)\Bigg|_{x=S(t)}.$$

However, from (2.4.1d), we see that $w_t(0,t)=0$, for all times $t \geq 0$. Therefore, the preceding equation becomes

$$E'(t)=\frac{1}{2}S'(t)\left([S'(t)]^2-1\right)\left[\frac{\partial w}{\partial x}\right]^2\Bigg|_{x=S(t)}.$$

If we now restrict $S'(t)$ to satisfy $0 \leq S'(t) < 1$, then $E'(t) \leq 0$, so that $E(t)$ is a non-increasing function of t. However by definition, $E(t) \geq 0$ and in particular, by (2.4.1b,c), $E(0)=0$. This leads to the conclusion that $E(t) \equiv 0$ for all t. Finally, since the integrand in $E(t)$ is a sum of squares, we may conclude that $w_t(x,t) \equiv 0$ and $w_x(x,t) \equiv 0$, so that $w(x,t) \equiv$ constant, and therefore $w \equiv 0$, in view of (2.4.1b). Thus, $u_1(x,t) \equiv u_2(x,t)$ in contradiction to our original assumption, so that the solution of (1.1.3) must be unique. It should be noted that if $S(t)$ satisfies $-1 < S'(t) < 0$ or $S'(t) > 1$, then the above uniqueness proof becomes invalid since $E'(t) > 0$. The condition $S'(t) \leq -1$ leads to $E'(t) \leq 0$, but is precluded from consideration due to the condition $|S'(t)| < 1$.

The failure of the above argument when $-1 < S'(t) < 0$ illustrates that difficulties may be encountered when one considers the string retrieval problem. In particular, the process of winding up the string, modelled by (1.1.3), is intrinsically unstable since the energy density (i.e., the energy per unit length of string) in a system, free from external excitation, could increase without bound as the string is wound up.

Finally, we note that if (1.1.3e) is replaced with the condition

$$\frac{\partial u(S(t),t)}{\partial x} = 0$$

then $E'(t) \equiv 0$ independent of the function $S(t)$.

Chapter 3: The Solutions of Balazs and Greenspan

3.1: Introduction

In this chapter, we will consider methods presented by Greenspan [Gre] and Balazs [Bal] for the solution of IMBVP (2.2.1). The resulting series expansions, as formulated by Greenspan and Balazs, appear to be quite different from each other. However, our derivation will show that the representations of the solution are consequences of each other, as one should expect in view of the fact that uniqueness of the solution of this problem has been established in section 2.4.

As in the original articles, the presentation here is strictly formal and hence a discussion of the problem of convergence of the resulting series is ignored.

3.2: Series Solution for an IMBVP with a Linearly Moving Endpoint

As in [Gre], consider the coordinate transformation

$$\zeta = \frac{mx}{mt+1}, \qquad \tau = \ln(mt+1), \tag{3.2.1}$$

which is non-singular on the domain $\mathbf{D} = \{(x,t)|0 \leq x \leq mt+1, \quad t \geq 0\}$. Under (3.2.1) the IMBVP (2.2.1), with $0 < m < 1$ and $p(t) = q(t) = 0$, becomes

$$(1-\zeta^2)\frac{\partial^2 U}{\partial \zeta^2} + 2\zeta\frac{\partial^2 U}{\partial \zeta \partial \tau} - \frac{\partial^2 U}{\partial \tau^2} - 2\zeta\frac{\partial U}{\partial \zeta} + \frac{\partial U}{\partial \tau} = 0, \qquad 0 < \zeta < m, \qquad \tau > 0, \tag{3.2.2a}$$

$$U(\zeta,0) = f\left(\frac{\zeta}{m}\right), \qquad \frac{\partial U(\zeta,0)}{\partial \tau} = \frac{1}{m}\left(g\left(\frac{\zeta}{m}\right) + \zeta f'\left(\frac{\zeta}{m}\right)\right), \qquad 0 \leq \zeta \leq m, \tag{3.2.2b, c}$$

$$U(0,\tau) = 0, \qquad U(m,\tau) = 0, \qquad \tau \geq 0, \tag{3.2.2d, e}$$

where $U(\zeta,\tau) \equiv u\left(\frac{1}{m}\zeta e^{\tau}, \frac{1}{m}(e^{\tau}-1)\right) = u(x,t)$. We observe that the effect of transformation (3.2.1) is to map $\mathbf{D}$ to a rectangular strip $\mathbf{D}' = \{(\zeta,\tau)|0 \leq \zeta \leq m, \quad \tau \geq 0\}$. That is, the time dependent domain of IMBVP (2.2.1) has been mapped to a domain which allows the application of standard solution techniques. Transformation (3.2.1), originally quoted by Carrier [Car], is employed by Greenspan without any mention as to how one might deduce such a simplifying transformation. In the next chapter we will see that (3.2.1) arises as a set of canonical coordinates of a Lie group.

To find solutions of (3.2.2a) we employ a separation of variables technique by assuming that there exist solutions of the form

$$U(\zeta,\tau) = e^{i\lambda_n\tau}P_n(\zeta), \tag{3.2.3}$$

in which λ_n is some constant and $P_n(\zeta)$ is a function of ζ only. Substitution of (3.2.3) into (3.2.2a) yields the result

$$(1-\zeta^2)P_n''(\zeta) + 2(i\lambda_n - 1)\zeta P_n'(\zeta) + (i\lambda_n + \lambda_n^2)P_n(\zeta) = 0. \tag{3.2.4}$$

Carrier [Car] points out that equation (3.2.4) can be put in hypergeometric form. With this in mind, we momentarily digress and present some basic results about Gauss' hypergeometric equation

$$t(1-t)y''(t) + [\gamma - (\alpha+\beta+1)t]y'(t) - \alpha\beta y(t) = 0,$$

where α, β, and γ are constants, and $\gamma - 1 \neq 1, 2, 3, \ldots$ (see [Arf] pp. 497). Under the transformation

$$t = \frac{1}{2}(1-z),$$

the above differential equation takes the form

$$(1-z^2)Y''(z) + [\alpha+\beta+1-2\gamma-(\alpha+\beta+1)z]Y'(z) - \alpha\beta Y(z) = 0, \tag{3.2.5}$$

where $Y(z) \equiv y\left(\frac{1}{2}(1-z)\right)$. Using the method of Frobenius, one can show that the solution of (3.2.5) is

$$Y(z) = \hat{A}F\left[\alpha,\beta;\gamma;\frac{1-z}{2}\right] + \hat{B}\left(\frac{1-z}{2}\right)^{1-\gamma} F\left[1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; \frac{1-z}{2}\right],$$

where $\hat{A}$ and $\hat{B}$ are arbitrary constants and

$$F[\alpha,\beta;\gamma;t] = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} t^k, \qquad |t| < 1, \tag{3.2.6}$$

which is known as Gauss' hypergeometric series, with the Pochammer symbol $(\alpha)_k$ being defined by

$$(\alpha)_k = \begin{cases} 1, & k = 0 \\ \alpha(\alpha+1)\cdots(\alpha+k-1), & k = 1,2,3,\ldots \end{cases}. \tag{3.2.7}$$

To see the relevance of the above to the current discussion, put $\sigma_n = i\lambda_n - 1$ in (3.2.4) to obtain

$$(1-\zeta^2)P_n''(\zeta) + 2\sigma_n\zeta P_n'(\zeta) - \sigma_n(\sigma_n+1)P_n(\zeta) = 0. \tag{3.2.8}$$

Upon comparison of (3.2.5) and (3.2.8), we deduce the system of equations

$$\alpha+\beta+1-2\gamma = 0, \qquad \alpha+\beta+1 = -2\sigma_n, \qquad \alpha\beta = \sigma_n(\sigma_n+1),$$

which has the solutions

$$(\alpha,\beta,\gamma) \in \{(-\sigma_n, -\sigma_n - 1, -\sigma_n), \quad (-\sigma_n - 1, -\sigma_n, -\sigma_n)\}.$$

Notice that since $F[\alpha,\beta;\gamma;t]$ is unchanged with respect to an interchange of α and β, both solutions of the above system will give rise to the same series. Formally, the general solution of (3.2.4) is

$$P_n(\zeta) = \hat{A}F\left[-\sigma_n, -\sigma_n - 1; -\sigma_n; \frac{1-\zeta}{2}\right] + \hat{B}\left(\frac{1-\zeta}{2}\right)^{1+\sigma_n} F\left[1, 0; 2+\sigma_n; \frac{1-\zeta}{2}\right],$$

from which we obtain, by virtue of (3.2.6),

$$P_n(\zeta) = \hat{A} \sum_{k=0}^{\infty} \frac{(-\sigma_n)_k(-\sigma_n - 1)_k}{k!(-\sigma_n)_k} \left(\frac{1-\zeta}{2}\right)^k$$

$$+\hat{B} \left(\frac{1-\zeta}{2}\right)^{1+\sigma_n} \sum_{k=0}^{\infty} \frac{(0)_k(1)_k}{k!(2+\sigma_n)_k} \left(\frac{1-\zeta}{2}\right)^k$$

and hence, by (3.2.7),

$$P_n(\zeta) = \hat{A} \sum_{k=0}^{\infty} \frac{(-\sigma_n - 1)_k}{k!} \left(\frac{1-\zeta}{2}\right)^k + \hat{B} \left(\frac{1-\zeta}{2}\right)^{1+\sigma_n},$$

where, due to the restriction imposed on γ, we require $1 + \sigma_n \notin \{-1, -2, -3, \cdots\}$. Indeed, it is easily verified by direct calculation that $v(\zeta) = (1-\zeta)^{i\lambda_n}$ is a solution of (3.2.4). Now suppose that we seek a second solution of (3.2.4) of the form

$$P_n(\zeta) = v(\zeta)w(\zeta) = (1-\zeta)^{i\lambda_n} w(\zeta), \tag{3.2.9}$$

as is customary in the so-called reduction of order technique. Substitution of (3.2.9) into (3.2.4) yields

$$(1-\zeta^2)v(\zeta)w''(\zeta) + [2(1-\zeta^2)v'(\zeta) + 2(i\lambda_n - 1)\zeta w(\zeta)]w'(\zeta)$$

$$+[(1-\zeta^2)v''(\zeta) + 2(i\lambda_n - 1)\zeta v'(\zeta) + \lambda_n(\lambda_n + i)v(\zeta)]w(\zeta) = 0.$$

Since $v(\zeta)$ is a solution of (3.2.4), the above can be written as

$$(1-\zeta^2)v(\zeta)w''(\zeta) + [2(1-\zeta^2)v'(\zeta) + 2(i\lambda_n - 1)\zeta v(\zeta)]w'(\zeta) = 0$$

or

$$\hat{w}'(\zeta) - 2\left(\frac{i\lambda_n + \zeta}{1-\zeta^2}\right)\hat{w}(\zeta) = 0, \tag{3.2.10}$$

where $\hat{w}(\zeta) = w'(\zeta)$. Equation (3.2.10) is a separable first order differential equation whose solution is given by

$$\hat{w}(\zeta) = \hat{K} \exp\left[2\int \frac{i\lambda_n + \zeta}{1-\zeta^2} d\zeta\right], \tag{3.2.11}$$

where $\hat{K}$ is an arbitrary constant. Integration by partial fractions provides

$$2\int \frac{i\lambda_n + \zeta}{1-\zeta^2} d\zeta = (i\lambda_n - 1)\ln(1+\zeta) - (i\lambda_n + 1)\ln(1-\zeta),$$

so that (3.2.11) becomes

$$\hat{w}(\zeta) = \hat{K} \exp\left[i\lambda_n \ln\left(\frac{1+\zeta}{1-\zeta}\right) - \ln(1-\zeta^2)\right].$$

Thus the desired function $w(\zeta)$ must satisfy the first order differential equation

$$w'(\zeta) = \frac{\hat{K}}{1-\zeta^2} \exp\left[i\lambda_n \ln\left(\frac{1+\zeta}{1-\zeta}\right)\right],$$

upon which integration gives us

$$w(\zeta) = K \exp\left[i\lambda_n \ln\left(\frac{1+\zeta}{1-\zeta}\right)\right],$$

where we have assumed, without any loss in generality, that the constant of integration is zero. Hence, we may conclude that the solution of (3.2.4) takes the form

$$\begin{aligned} P_n(\zeta) &= A(1-\zeta)^{i\lambda_n} \exp\left(i\lambda_n \ln\left(\frac{1+\zeta}{1-\zeta}\right)\right) + B(1-\zeta)^{i\lambda_n} \\ &= Ae^{i\lambda_n \ln(1+\zeta)} + Be^{i\lambda_n \ln(1-\zeta)}, \end{aligned} \tag{3.2.12}$$

where A and B are arbitrary constants.

Boundary conditions (3.2.2d,e) imply that

$$P_n(0) = 0, \qquad P_n(m) = 0. \tag{3.2.13a, b}$$

Application of (3.2.13a) to (3.2.12) yields $B = -A$, whereas condition (3.2.13b) requires

$$A\left(e^{i\lambda_n \ln(1+m)} - e^{i\lambda_n \ln(1-m)}\right) = 0$$

or

$$Ae^{i\lambda_n \ln(1-m)}\left(e^{i\lambda_n L} - 1\right) = 0,$$

where $L = \ln\left(\frac{1+m}{1-m}\right)$. The above is satisfied if we take

$$\lambda_n = \frac{2n\pi}{L}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{3.2.14}$$

which are admissible values for the constant λ_n since the restriction $1 + \sigma_n \notin \{-1, -2, -3, \ldots\}$ implies $i\lambda_n \notin \{-1, -2, -3, \ldots\}$, which is not in conflict with (3.2.14), since clearly $i\lambda_n$ is imaginary. Since (3.2.2a) is linear, the solutions

$$U_n(\zeta, \tau) = A_n\left(e^{i\lambda_n \ln(1+\zeta)} - e^{i\lambda_n \ln(1-\zeta)}\right)e^{i\lambda_n \tau}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

can be superposed to obtain a more general solution of the form

$$U(\zeta, \tau) = \sum_{n=-\infty}^{\infty} A_n\left(e^{i\lambda_n \ln(1+\zeta)} - e^{i\lambda_n \ln(1-\zeta)}\right)e^{i\lambda_n \tau} \tag{3.2.15}$$

where the A_n are to be chosen so that the initial conditions will be satisfied. Determination of the A_n requires us to expand a linear combination of the "initial" functions (3.2.2b,c) in a series based upon expressions of the form

$$A_n\left(e^{i\lambda_n \ln(1+\zeta)} - e^{i\lambda_n \ln(1-\zeta)}\right).$$

This incorporation of the initial conditions is more readily accomplished when we work in terms of the original variables (x, t). To this end, we write (3.2.15) as

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n \left(e^{i\lambda_n \ln(m[t+x]+1)} - e^{i\lambda_n \ln(m[t-x]+1)} \right)$$

and apply (2.2.1b,c) to obtain the equations

$$\sum_{n=-\infty}^{\infty} A_n \left(e^{i\lambda_n \ln(1+mx)} - e^{i\lambda_n \ln(1-mx)} \right) = f(x) \tag{3.2.16}$$

and

$$\sum_{n=-\infty}^{\infty} A_n \left(\frac{i\lambda_n m}{1+mx} e^{i\lambda_n \ln(1+mx)} - \frac{i\lambda_n m}{1-mx} e^{i\lambda_n \ln(1-mx)} \right) = g(x).$$

Moreover, integration of the second equation gives

$$\sum_{n=-\infty}^{\infty} A_n \left(e^{i\lambda_n \ln(1+mx)} + e^{i\lambda_n \ln(1-mx)} \right) = G(x), \tag{3.2.17}$$

whereupon addition and subtraction of (3.2.16) and (3.2.17) leads to the equations

$$\sum_{n=-\infty}^{\infty} A_n e^{i\lambda_n \ln(1+mx)} = \frac{1}{2} \left(G(x) + f(x) \right) \tag{3.2.18}$$

and

$$\sum_{n=-\infty}^{\infty} A_n e^{i\lambda_n \ln(1-mx)} = \frac{1}{2} \left(G(x) - f(x) \right), \tag{3.2.19}$$

where, for an arbitrary constant D, the function $G(x)$ is given by

$$G(x) = \int_0^x g(\tau) d\tau + D.$$

Due to the form the series solution takes, the constant D always cancels out and so there is no loss in generality in assuming that $D = 0$. In view of (2.2.4), the functions $f(x)$ and $g(x)$ are to be extended from $0 \leq x \leq 1$ to $-1 \leq x < 0$ as odd functions. With this in mind, we see that (3.2.19) is a consequence of (3.2.18). Thus, we may restrict our attention to equation (3.2.18). In particular, we note that for $n \neq k$

$$\int_{x=-1}^{x=1} e^{i\lambda_n \ln(1+mx)} e^{-i\lambda_k \ln(1+mx)} d(\ln(1+mx))$$
$$= \frac{-i}{\lambda_n - \lambda_k} \left[e^{i(\lambda_n-\lambda_k)\ln(1+m)} - e^{i(\lambda_n-\lambda_k)\ln(1-m)} \right]$$
$$= \frac{-i}{\lambda_n - \lambda_k} e^{i(\lambda_n-\lambda_k)\ln(1-m)} \left[e^{i(\lambda_n-\lambda_k)L} - 1 \right] = 0,$$

while, for $n = k$, the corresponding result is

$$\int_{x=-1}^{x=1} d(\ln(1+mx)) = \ln(1+m) - \ln(1-m) = L.$$

Using (3.2.18) and the above, it can be shown that

$$A_n = \frac{1}{L} \int_{x=-1}^{x=1} \frac{1}{2} (G(x) + f(x))\, e^{-i\lambda_n \ln(1+mx)} d(\ln(1+mx)), \tag{3.2.20}$$

so that IMBVP (2.2.1) is formally solved by

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n \left(e^{i\lambda_n \ln(m[t+x]+1)} - e^{i\lambda_n \ln(m[t-x]+1)} \right). \tag{3.2.21}$$

We note that we can produce analogous results by differentiating (3.2.16) and proceeding in a manner similar to that of the foregoing analysis. In such a case, one will obtain an integral (for determination of the A_n) that is based upon $f'(x)$ and $g(x)$ as opposed $f(x)$ and $G(x)$. In fact, this integral can be obtained directly from (3.2.20) via integration by parts. As it turn out, the integral will not be defined for $n = 0$. However, from (3.2.21) it is apparent that we can set $A_0 = 0$, without any loss in generality. In [Bal], Balazs follows this approach, basing his analysis on the series

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n \left(e^{i\lambda_n \ln\left(\frac{t+x}{t_0}\right)} - e^{i\lambda_n \ln\left(\frac{t-x}{t_0}\right)} \right),$$

where t_0 is a positive constant. To within a scaling and translation in $t+x$ and $t-x$, the above series is identical to (3.2.21). It should be noted that (3.2.20) implies $A_{-n} = \bar{A}_n$ for $n = 1, 2, 3, \ldots$ so that (3.2.21) may be written in the form of a real Fourier-like series involving $\sin(\lambda_n \ln(m[t \pm x] + 1)$ and $\cos(\lambda_n \ln(m[t \pm x] + 1)$, with a summation over positive integer values of n.

In order that a connection between Greenspan's and Balazs' solutions be established, let us return to (3.2.15) which with the aid of Euler's formula can be written as

$$U(\zeta,\tau) = \sum_{n=-\infty}^{\infty} A_n \{\cos[\lambda_n \ln(1+\zeta)] - \cos[\lambda_n \ln(1-\zeta)]$$

$$+ i \sin[\lambda_n \ln(1+\zeta)] - i \sin[\lambda_n \ln(1-\zeta)]\} \, e^{i\lambda_n \tau}.$$

Moreover, the identities

$$\sin(\theta) - \sin(\phi) = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

and

$$\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

allow us to write the last result as

$$U(\zeta,\tau) = \sum_{n=-\infty}^{\infty} 2iA_n \sin\left(\lambda_n \ln\sqrt{\frac{1+\zeta}{1-\zeta}}\right) e^{i\lambda_n (\ln\sqrt{1-\zeta^2}+\tau)}$$

which is the form of the series solution given by Greenspan. The change of variable $\zeta = mx$ (obtained from (3.2.1) with $t = 0$) can be applied to (3.2.20) to obtain an expression for the A_n in terms of ζ.

It should be pointed out that, due to the complicated form of the integrand in (3.2.20), finding explicit anti-derivatives to use in the evaluation of the coefficients A_n will be in general impossible. In such cases, numerical integration will be required. We observe that when m is near 1, the function

$$e^{-i\lambda_n \ln(1+mx)} \tag{3.2.22}$$

oscillates wildly as $x \to -1$ (in fact, when $m = 1$ the variation of (3.2.22) becomes unbounded as $x \to -1$). When using numerical integration in conjunction with highly oscillatory integrands, one must ensure that the size of the mesh is chosen with the frequency of the oscillations in mind. If this is not done, the accuracy of integration procedure will be reduced. This oscillatory behaviour persists in the series representations of the solution and can reduce the accuracy of partial sum approximations

$$u_N(x,t) = \sum_{n=-N}^{N} A_n \left(e^{i\lambda_n \ln(m[t+x]+1)} - e^{i\lambda_n \ln(m[t-x]+1)} \right),$$

obtained by truncating the infinite series. As an example, consider IMBVP (2.2.1) with $f(x) = 1$ and $g(x) = p(t) = q(t) = 0$. Since $f(x)$ does not satisfy the required end-point conditions (given in chapter 2), the solution will be discontinuous across the characteristics $x = t$ and $x + t = 1$. We will express the solution using (3.2.21) and check the accuracy of a partial sum $u_N(x,t)$ at $t = 0$ for various values of m and N. From (3.2.20), we have

$$A_n = \frac{1}{2n\pi i}\left[1 - e^{-i\lambda_n \ln(1-m)}\right].$$

From diagrams 3.2.1 we see that for $N = 5$ and $N = 10$ an increase in the value of m reduces the accuracy of $u_N(x,t)$ in a neighborhood of the moving end-point.

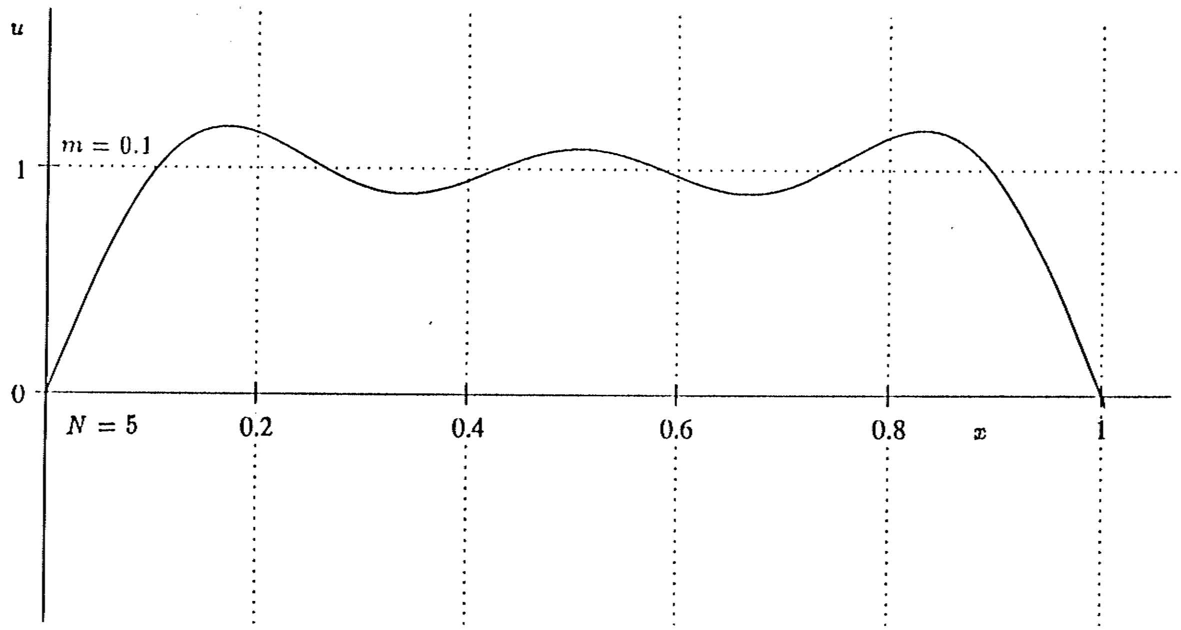

Diagram 3.2.1a

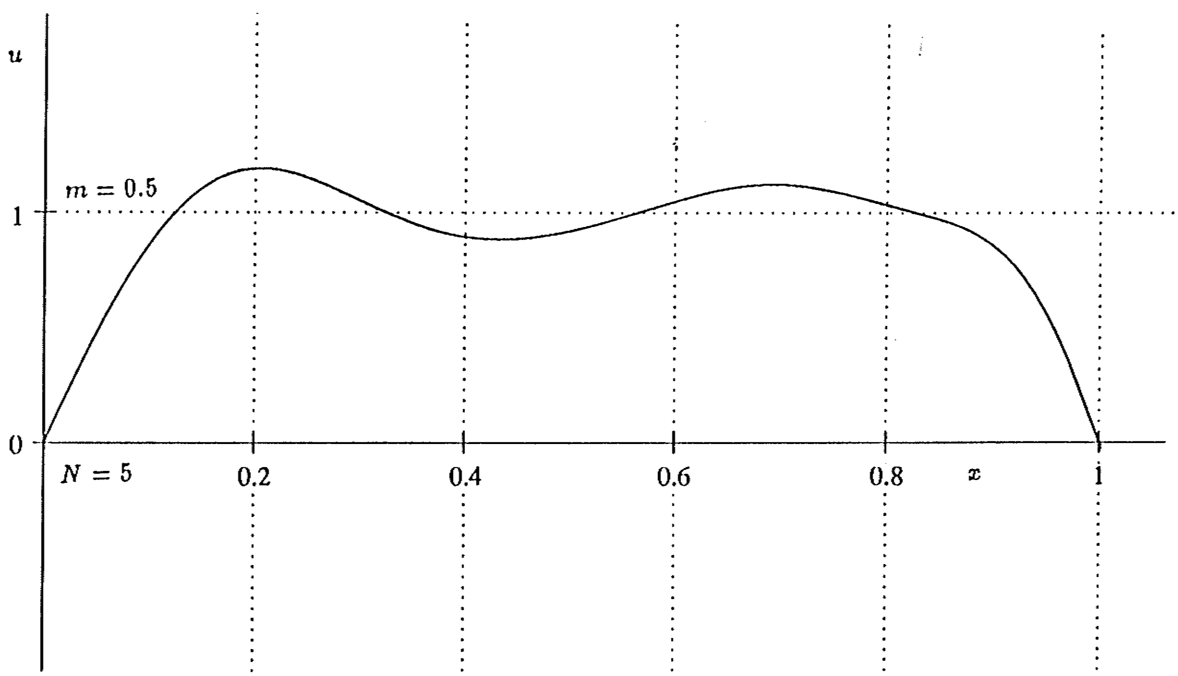

Diagram 3.2.1b

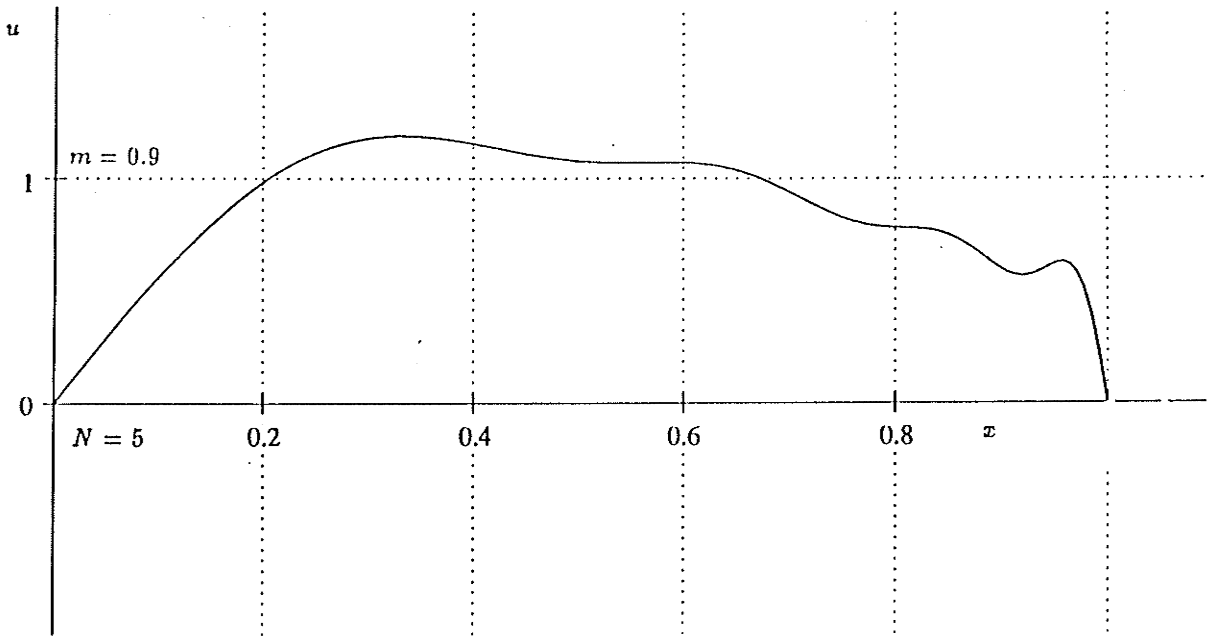

Diagram 3.2.1c

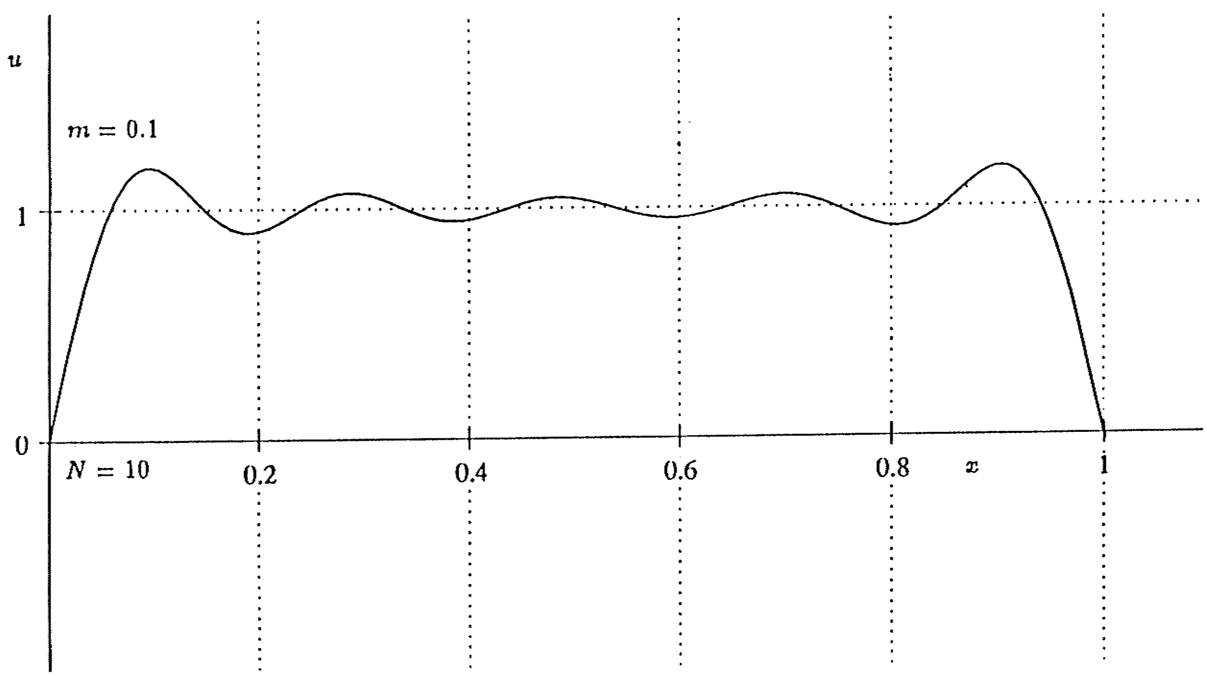

Diagram 3.2.1d

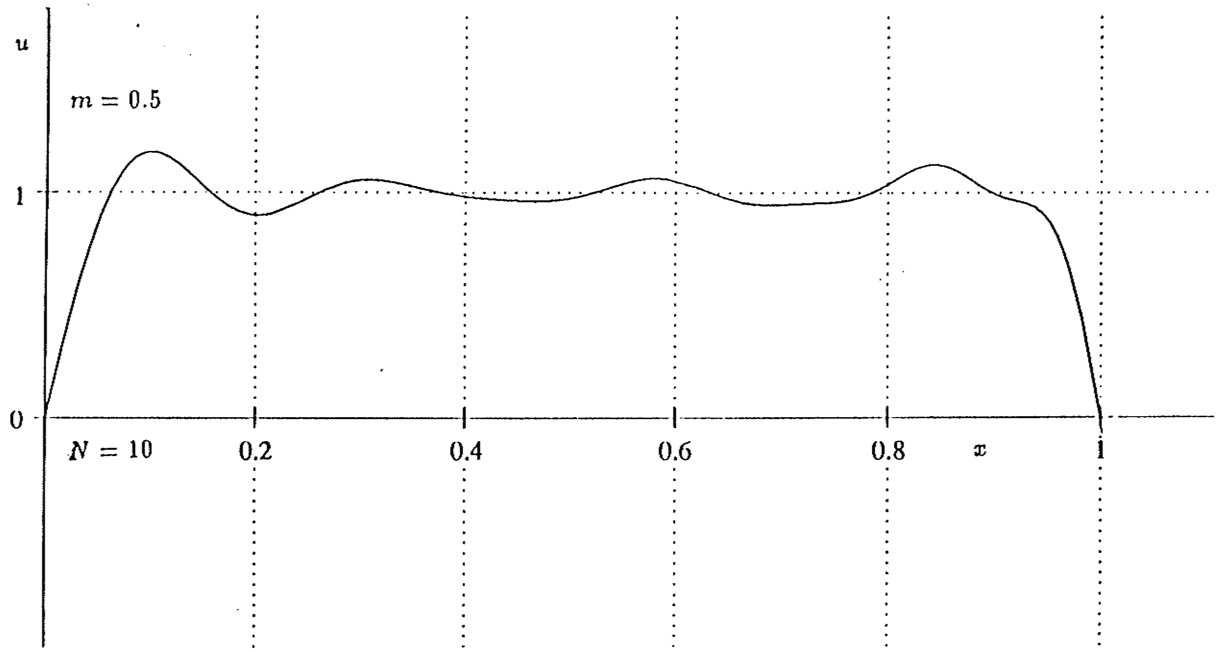

Diagram 3.2.1e

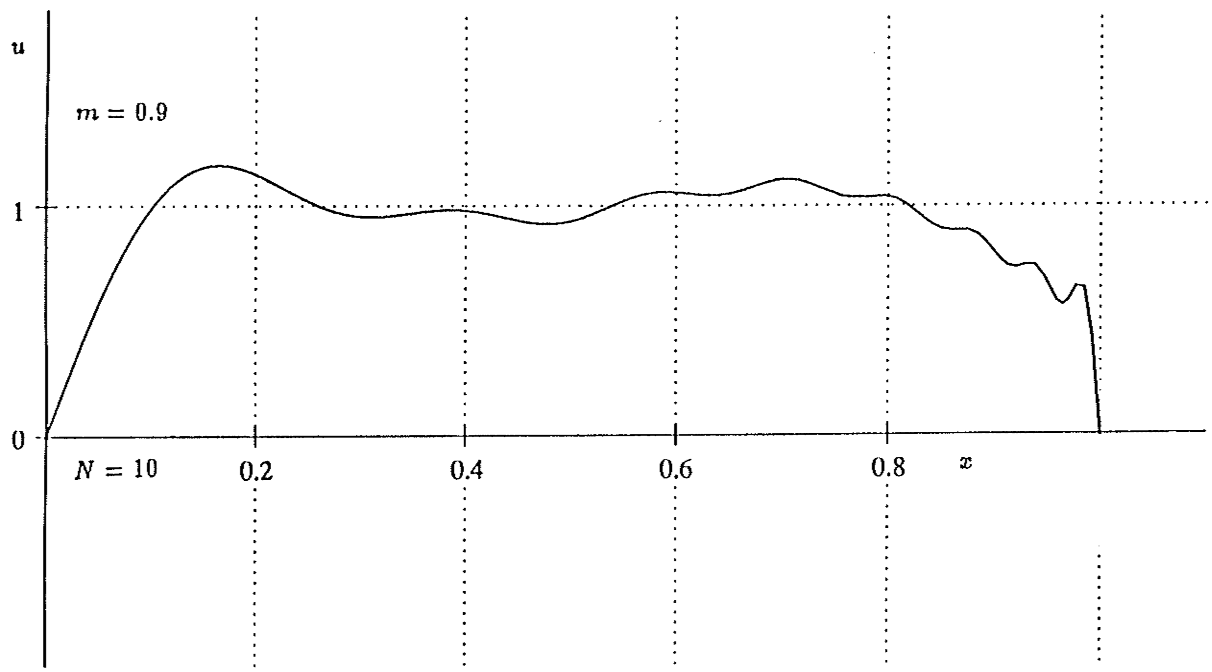

Diagram 3.2.1f

Chapter 4: Series Solutions via Lie Group Theory

4.1: Introduction

The purpose throughout the remainder of this thesis is to extend the analysis of the previous chapter in the sense that we seek a series representation of the solution of the IMBVP (1.1.3) with general moving boundary given by $x = S(t)$ where $|S'(t)| < 1$. The derivation of this solution, which is postponed until Section 4.3 makes use of the unifying techniques of Lie group theory. In the next section, we shall make use of the material presented in chapters 2 and 4 of Bluman and Kumei [Blu] to derive the infinitesimals and the associated infinitesimal generators of the group of tranformations that leaves the wave equation invariant. It is these infinitesimal generators that allow us to derive the series representation of Section 4.3 in a step by step manner. This analysis will be accomplished by finding the invariant surfaces, that satisfy the wave equation, of an infinitesimal generator that leaves the boundary curves $x = S(t)$ and $x = 0$ as well as the homogeneous boundary condition $u = 0$ invariant.

Solutions developed in this manner resemble Fourier series and well-established theorems concerning the convergence of Fourier series are used to provide some justification of the results to be presented.

Lastly, since it is not our purpose to expound upon Lie group theory, the pertinent material in [Blu] is assumed, so that the continuity of this thesis can be preserved.

4.2: Infinitesimals of the Wave Equation

We now make use of the material presented in chapters 2 and 4 of Bluman and Kumei [Blu] to derive the infinitesimals arising from the groups of transformations that leave the wave equation invariant. It will be shown that the wave equation admits an infinite number of one-parameter Lie groups, or more succinctly, that the wave equation admits a non-trivial infinite parameter Lie group.

Suppose the wave equation admits an infinitesimal generator of the form

$$V = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{4.2.1}$$

Let $u_{(1)} = (u_x, u_t)$ and $u_{(2)} = (u_{xx}, u_{xt}, u_{tt})$, then the first and second extensions of V are

$$V^{(1)} = V + \eta_1^{(1)}(x,t,u,u_{(1)})\frac{\partial}{\partial u_x} + \eta_2^{(1)}(x,t,u,u_{(1)})\frac{\partial}{\partial u_t}$$

and

$$V^{(2)} = V^{(1)} + \eta_{11}^{(2)}(x,t,u,u_{(1)},u_{(2)})\frac{\partial}{\partial u_{xx}} + \eta_{12}^{(2)}(x,t,u,u_{(1)},u_{(2)})\frac{\partial}{\partial u_{xt}}$$

$$+\eta_{22}^{(2)}(x,t,u,u_{(1)},u_{(2)})\frac{\partial}{\partial u_{tt}}. \tag{4.2.2}$$

By Theorem 4.1.1-1 in chapter 4 of [Blu] the wave equation admits the group determined by (4.2.1) if and only if

$$V^{(2)}(u_{tt} - u_{xx}) = 0$$

whenever $u_{tt} = u_{xx}$. Application of (4.2.2) results in the following invariance condition for the wave equation

$$\eta_{22}^{(2)} = \eta_{11}^{(2)} \tag{4.2.3}$$

Using Theorem 2.3.4-1 from chapter 2 of [Blu], (4.2.3) can be written as

$$\begin{aligned}
&\frac{\partial^2\eta}{\partial t^2} + \left(2\frac{\partial^2\eta}{\partial t\partial u} - \frac{\partial^2\xi_2}{\partial t^2}\right)u_t - \frac{\partial^2\xi_1}{\partial t^2}u_x + \left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_2}{\partial t}\right)u_{tt}\\
&-2\frac{\partial\xi_1}{\partial t}u_{xt} + \left(\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_2}{\partial t\partial u}\right)(u_t)^2 - 2\frac{\partial^2\xi_1}{\partial t\partial u}u_xu_t - \frac{\partial^2\xi_2}{\partial u^2}(u_t)^3\\
&-\frac{\partial^2\xi_1}{\partial u^2}u_x(u_t)^2 - 3\frac{\partial\xi_2}{\partial u}u_tu_{tt} - \frac{\partial\xi_1}{\partial u}u_xu_{tt} - 2\frac{\partial\xi_1}{\partial u}u_tu_{xt}\\
&=\frac{\partial^2\eta}{\partial x^2} + \left(2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi_1}{\partial x^2}\right)u_x - \frac{\partial^2\xi_2}{\partial x^2}u_t + \left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_1}{\partial x}\right)u_{xx}\\
&-2\frac{\partial\xi_2}{\partial x}u_{xt} + \left(\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_1}{\partial x\partial u}\right)(u_x)^2 - 2\frac{\partial^2\xi_2}{\partial x\partial u}u_xu_t - \frac{\partial^2\xi_1}{\partial u^2}(u_x)^3\\
&-\frac{\partial^2\xi_2}{\partial u^2}(u_x)^2u_t - 3\frac{\partial\xi_1}{\partial u}u_xu_{xx} - \frac{\partial\xi_2}{\partial u}u_tu_{xx} - 2\frac{\partial\xi_2}{\partial u}u_xu_{xt}.
\end{aligned}$$

Collection of common powers of $u_{(1)}$ and $u_{(2)}$ and replacement of u_{tt} by u_{xx} leads to the equation

$$\begin{aligned}
&\left(\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2}\right) + \left(2\frac{\partial^2\eta}{\partial t\partial u} - \frac{\partial^2\xi_2}{\partial t^2} + \frac{\partial^2\xi_2}{\partial x^2}\right)u_t + \left(-2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi_1}{\partial t^2} + \frac{\partial^2\xi_1}{\partial x^2}\right)u_x\\
&+\left(-2\frac{\partial\xi_2}{\partial t} + 2\frac{\partial\xi_1}{\partial x}\right)u_{xx} + \left(2\frac{\partial\xi_2}{\partial x} - 2\frac{\partial\xi_1}{\partial t}\right)u_{xt} + \left(\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_2}{\partial t\partial u}\right)(u_t)^2\\
&+\left(2\frac{\partial^2\xi_2}{\partial x\partial u} - 2\frac{\partial^2\xi_1}{\partial t\partial u}\right)u_xu_t - \frac{\partial^2\xi_2}{\partial u^2}(u_t)^3 - \frac{\partial^2\xi_1}{\partial u^2}u_x(u_t)^2 - 2\frac{\partial\xi_1}{\partial u}u_tu_{xt}\\
&+\left(2\frac{\partial^2\xi_1}{\partial x\partial u} - \frac{\partial^2\eta}{\partial u^2}\right)(u_x)^2 + \frac{\partial^2\xi_1}{\partial u^2}(u_x)^3 + \frac{\partial^2\xi_2}{\partial u^2}(u_x)^2u_t + 2\frac{\partial\xi_1}{\partial u}u_xu_{xx} - 2\frac{\partial\xi_2}{\partial u}u_tu_{xx}\\
&+2\frac{\partial\xi_2}{\partial u}u_xu_{xt} = 0,
\end{aligned}$$

whereupon setting the coefficients of the above to zero, we obtain the determining equations

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2} = 0 \tag{4.2.4a}$$

$$2\frac{\partial^2 \eta}{\partial t \partial u} - \frac{\partial^2 \xi_2}{\partial t^2} + \frac{\partial^2 \xi_2}{\partial x^2} = 0 \tag{4.2.4b}$$

$$2\frac{\partial^2 \eta}{\partial x \partial u} + \frac{\partial^2 \xi_1}{\partial t^2} - \frac{\partial^2 \xi_1}{\partial x^2} = 0 \tag{4.2.4c}$$

$$\frac{\partial \xi_2}{\partial t} - \frac{\partial \xi_1}{\partial x} = 0 \tag{4.2.4d}$$

$$\frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_2}{\partial x} = 0 \tag{4.2.4e}$$

$$\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_2}{\partial t \partial u} = 0 \tag{4.2.4f}$$

$$\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_1}{\partial x \partial u} = 0 \tag{4.2.4g}$$

$$\frac{\partial \xi_1}{\partial u} = \frac{\partial \xi_2}{\partial u} = 0. \tag{4.2.4h}$$

From (4.2.4h) we see that the infinitesimals ξ_1 and ξ_2 are independent of u. Differentiating (4.2.4d) and (4.2.4e) with respect to x and t, one can easily show that both ξ_1 and ξ_2 satisfy the wave equation. Incorporation of the two preceding consequences into (4.2.4) leads to the reduced system

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = 0 \tag{4.2.5a}$$

$$\frac{\partial^2 \eta}{\partial x \partial u} = \frac{\partial^2 \eta}{\partial t \partial u} = 0 \tag{4.2.5b}$$

$$\frac{\partial \xi_2}{\partial t} - \frac{\partial \xi_1}{\partial x} = 0 \tag{4.2.5c}$$

$$\frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_2}{\partial x} = 0 \tag{4.2.5d}$$

$$\frac{\partial^2 \eta}{\partial u^2} = 0 \tag{4.2.5f}$$

Equation (4.2.5f) implies that η is linear in u. Now, using equations (4.2.5b) and (4.2.5a), we obtain

$$\eta = \lambda u + w(x,t), \tag{4.3.6}$$

where λ is an arbitrary constant and $w(x,t)$ is a solution of the wave equation. Since ξ_1 must satisfy the wave equation, let

$$\xi_1(x,t) = \varphi(x+t) + \psi(x-t), \tag{4.2.7}$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are arbitrary functions. Equations (4.2.5c,d) will be satisfied if

$$\xi_2(x,t) = \varphi(x+t) - \psi(x-t). \tag{4.2.8}$$

Thus, the wave equation admits the infinitesimal generator

$$V = (\varphi(x+t) + \psi(x-t))\frac{\partial}{\partial x} + (\varphi(x+t) - \psi(x-t))\frac{\partial}{\partial t} + (\lambda u + w(x,t))\frac{\partial}{\partial u}.$$

The infinitesimal generators arising from a multi-parameter Lie group (infinite or finite) form a Lie algebra, that is a vector space $\mathcal{L}$ over some field $\mathcal{F}$ which is closed under an additional operation known as commutation. The commutator of two infinitesimal generators V_1, $V_2 \in \mathcal{L}$ is the infinitesimal generator $V_c \in \mathcal{L}$ defined by

$$V_c = [V_1, V_2] \equiv V_1V_2 - V_2V_1.$$

Since generators form Lie algebras, we can form arbitrary linear combinations that produce new generators. It can be shown that the wave equation admits

$$V_\xi = (\varphi(x+t) + \psi(x-t))\frac{\partial}{\partial x} + (\varphi(x+t) - \psi(x-t))\frac{\partial}{\partial t}, \tag{4.2.9a}$$

$$V_\lambda = \lambda u\frac{\partial}{\partial u}, \tag{4.2.9b}$$

and

$$V_w = w(x,t)\frac{\partial}{\partial u} \tag{4.2.9c}$$

from which we can construct new generators.

From Lie's first fundamental theorem (Theorem 2.2.1-1 in [Blu]), it can be shown that the Lie groups giving rise to (4.2.9b) and (4.3.9c) are

$$\bar{x} = x, \qquad \bar{t} = t, \qquad \bar{u} = e^{\varepsilon}u \tag{4.2.10}$$

and

$$\bar{x} = x, \qquad \bar{t} = t, \qquad \bar{u} = u + \varepsilon w \tag{4.2.11}$$

respectively, where $\varepsilon \in \Re$. The group (4.2.11) arises as a consequence of the linearity of the wave equation and tells us that if u and w are solutions, then so is $u + \varepsilon w$. The group (4.2.10) tells us that solutions of the wave equation are invariant under arbitrary scaling that is, if u is a solution then so is $e^{\varepsilon}u$. More generally, groups of similar type appear for any linear homogenous partial differential equation. The group (4.2.10) has a second important interpretation which we will make use of in the next section. As an illustration of the interpretation, consider the infinitesimal generator $V_t = \frac{\partial}{\partial t}$ which arises from the group

$$\bar{x} = x, \qquad \bar{t} = t + \varepsilon, \qquad \bar{u} = u, \tag{4.2.12}$$

which by Definition 2.2.5-1 in [Blu] is expressed in terms of canonical coordinates. It is a simple exercise to show that the wave equation is invariant under (4.2.12).

Now, suppose that we want solutions of the wave equation that are also invariant surfaces of the generator

$$V_t + \lambda V_\lambda = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial u}. \qquad (4.2.13)$$

Using Theorem 2.2.7-1 from [Blu], $u = \theta(x,t)$ is an invariant surface of (4.2.13) if and only if

$$(V_t + \lambda V_\lambda)(u - \theta(x,t)) = 0, \qquad (4.2.14)$$

whenever $u = \theta(x,t)$. The last equation leads to the first order partial differential equation

$$\frac{\partial \theta}{\partial t} = \lambda \theta,$$

which has characteristic equations

$$\frac{dx}{0} = \frac{dt}{1} = \frac{d\theta}{\lambda \theta},$$

from which we are able to construct the general solution

$$u = \theta(x,t) = F(x)e^{\lambda t},$$

in which $F(x)$ is some arbitrary function of x. The function $F(x)$ is to be chosen so that $u = \theta(x,t)$ is a solution of the wave equation. Substitution of the above into $u_{tt} = u_{xx}$ leads to the ordinary differential equation

$$F''(x) - \lambda^2 F(x) = 0.$$

The preceding analysis produces results analogous to those obtained by the use of the method of separation of variables, and in general the invariant solutions of

$$\hat{V} + \lambda V_\lambda \qquad (4.2.14)$$

will be the "separated" solutions

$$u = \theta(x,t) = F(r)e^{\lambda s}, \tag{4.3.15}$$

where $r = r(x,t)$ and $s = s(x,t)$ are the canonical coordinates of the Lie group giving rise to the infinitesimal generator $\hat{V}$ admitted by the wave equation. Again, we must ensure that $F(r)$ is chosen so that the wave equation is satisfied.

Suppose that we want to use $\hat{V}$ to solve the wave equation. According to Bluman and Kumei (see section 4.4 in [Blu]), if $\hat{V}$ leaves the boundary curves and the corresponding boundary conditions invariant, then the invariant solution corresponding to $\hat{V}$ satisfies the boundary value problem. It is important to note that if the conditions specified on the boundary curves involve derivatives of u, then the appropriate extension of $\hat{V}$ must be used when testing the invariance of said conditions. The above requirements are quite restrictive and in practice one will not often be able to find an appropriate $\hat{V}$.

In the next section we will utilize a weaker form of invariance that will allow us to derive series solutions for a wide variety of IMBVPs. For example, consider IMBVP (1.1.3) in the case when $p(t) = q(t) = 0$. Suppose that $\hat{V}$ leaves the boundary curves and the corresponding homogenous boundary condition invariant. It will be shown that the boundary conditions (1.1.3b,c) correspond to the boundary conditions $u(0,s) = 0$ and $u(L,s) = 0$, for constant L, in the rs-plane. As pointed out in section 3.2, the canonical coordinates (r, s) of $\hat{V}$ map a time dependent domain to a domain in which the boundary curves are given by $r = 0$ and $r = L$. This critical fact sets the stage for the application of the method of separation of variables to the wave equation, once it is expressed in terms of the canonical coordinates of $\hat{V}$. No attempt will be made to find an infinitesimal generator that leaves the initial conditions and/or initial curve invariant (the preceding fact makes the invariance weak). These additional conditions must be addressed by other means.

4.3: Series Solution for a General IMBVP

The infinitesimal generator

$$V = (\varphi(x+t) + \psi(x-t))\frac{\partial}{\partial x} + (\varphi(x+t) - \psi(x-t))\frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u} \tag{4.3.1}$$

is admitted by the wave equation and leaves the boundary condition $u = 0$ invariant, since $Vu = 0$ whenever $u = 0$. In order that (4.3.1) leave the boundary $x = 0$ invariant, we must have

$$\psi(\zeta) = -\varphi(-\zeta).$$

Thus, if the boundary curve $x = 0$ is to be invariant, the infinitesimal generator may be written in the form

$$V = (\varphi(t+x) - \varphi(t-x))\frac{\partial}{\partial x} + (\varphi(t+x) + \varphi(t-x))\frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}. \tag{4.3.2}$$

Similarly, the invariance of the moving boundary curve $x = S(t)$ under (4.3.2) demands that

$$\varphi(t+S(t)) - \varphi(t-S(t)) - [\varphi(t+S(t)) + \varphi(t-S(t))]\, S'(t) = 0,$$

which leads to the condition

$$\varphi(t+S(t))(1 - S'(t)) = \varphi(t-S(t))(1+S'(t))$$

or equivalently

$$\frac{d(t-S(t))}{\varphi(t-S(t))} = \frac{d(t+S(t))}{\varphi(t+S(t))}. \tag{4.3.3}$$

Let us assume that for any given $S(t)$ there exists a function $\varphi(\zeta)$, defined on $[-S(0),\infty)$, which satisfies (4.3.3), and that $\frac{1}{\varphi(\zeta)}$ has an anti-derivative $\Phi(\zeta)$ which is also defined on $[-S(0),\infty)$. Then (4.3.3) can be written as

$$\Phi\left(t+S(t)\right)-\Phi\left(t-S(t)\right)=L, \tag{4.3.4}$$

where L is a constant of integration.

The invariant surfaces $u=\theta(x,t)$ of (4.3.2) must satisfy the first order partial differential equation

$$\left(\varphi(t+x)-\varphi(t-x)\right)\frac{\partial\theta}{\partial x}+\left(\varphi(t+x)+\varphi(t-x)\right)\frac{\partial\theta}{\partial t}=\lambda\theta,$$

which has characteristic equations

$$\frac{dx}{\varphi(t+x)-\varphi(t-x)}=\frac{dt}{\varphi(t+x)+\varphi(t-x)}=\frac{d\theta}{\lambda\theta}$$

or equivalently

$$\frac{d(t+x)}{\varphi(t+x)}=\frac{d(t-x)}{\varphi(t-x)}=\frac{2d\theta}{\lambda\theta}.$$

In view of our previous assumptions, these characteristic equations lead to the general solution

$$\theta(x,t)=H\left(\Phi(t+x)-\Phi(t-x)\right)e^{\frac{\lambda}{2}\Phi(t-x)} \tag{4.3.5}$$

or equivalently

$$\theta(x,t)=G\left(\Phi(t+x)-\Phi(t-x)\right)e^{\frac{\lambda}{2}\Phi(t+x)}, \tag{4.3.6}$$

where $H(\zeta)$ and $G(\zeta)$ are arbitrary functions. If we let $G(\zeta) = \hat{G}(\zeta)\exp\left(\frac{-\lambda}{2}\zeta\right)$, where $\hat{G}(\zeta)$ is an arbitrary function, then one can see that (4.3.6) is equivalent to (4.3.5), so that we may restrict our attention to (4.3.5).

It should be noted that

$$r(x,t) = \Phi(t+x) - \Phi(t-x), \qquad s(x,t) = \frac{1}{2}\Phi(t-x), \tag{4.3.7}$$

define a set of canonical coordinates for the group generated by

$$V_\xi = (\varphi(t+x) - \varphi(t-x))\frac{\partial}{\partial x} + (\varphi(t+x) + \varphi(t-x))\frac{\partial}{\partial t}.$$

Furthermore, it can be shown that $s(x,t)$ is a particular solution of the first order partial differential equation $V_\xi w(x,t) = 0$, while $r(x,t)$ is a particular solution of $V_\xi v(x,t) = 1$. The coordinate transformation (4.3.7) is non-singular as long as

$$\begin{vmatrix} r_x & r_t \\ s_x & s_t \end{vmatrix} = \Phi'(t+x)\Phi'(t-x) \neq 0,$$

which is satisfied if there is a constant M such that

$$|\varphi(\zeta)| < M$$

for all $\zeta \in [-S(0), \infty)$. If we further require that $\frac{1}{\varphi(\zeta)} \in \mathrm{C}^1$, then $\Phi(\zeta) \in \mathrm{C}^2$ and IMBVP (1.1.3) can be expressed in terms of the canonical coordinates (4.3.7). In particular, the boundary curves $x = 0$ and $x = S(t)$ are respectively mapped to the boundary curves $r = 0$ and $r = L$ in the rs-plane, so that (4.3.7) is a change of the independent variables that "fixes" the moving endpoint. Unfortunately, the image of the initial line $t = 0$ under (4.3.7) will not usually be of the form s =constant, but rather will be some curve $h(r, s) = 0$. Although we should be able to solve (1.1.3) when it is expressed in terms of the canonical coordinates (4.3.7), the form of the "initial" curve in the rs-plane may make this difficult to do.

Note: It should be pointed out that, for a particular $\varphi(\zeta)$ satisfying (4.3.3), it is possible that there will be alternative choices for the canonical coordiantes under which (1.1.3) will take a more convenient form. For instance, consider the infinitesimal generator

$$\hat{V} = mx\frac{\partial}{\partial x} + (mt + 1)\frac{\partial}{\partial t}.$$

The canonical coordinates of $\hat{V}$ are particular solutions of the partial differential equations

$$mx\frac{\partial w}{\partial x} + (mt + 1)\frac{\partial w}{\partial t} = 0$$

and

$$mx\frac{\partial v}{\partial x} + (mt + 1)\frac{\partial v}{\partial t} = 1,$$

which have characteristic equations

$$\frac{dx}{mx} = \frac{dt}{mt + 1} = \frac{dw}{0}$$

$$\frac{dx}{mx} = \frac{dt}{mt + 1} = \frac{dv}{1} \qquad (4.3.8)$$

or

$$\frac{d(t+x)}{m(t+x)+1} = \frac{d(t-x)}{m(t-x)+1} = \frac{dw}{0}$$

$$\frac{d(t+x)}{m(t+x)+1} = \frac{d(t-x)}{m(t-x)+1} = \frac{dv}{1}, \tag{4.3.9}$$

respectively. From (4.3.8), we obtain the canonical coordinates

$$w = r(x,t) = \frac{mx}{mt+1}, \qquad v = s(x,t) = \frac{1}{m}\ln(mt+1),$$

which are analogous to (3.2.1), the coordinates employed by Greenspan, as demonstrated in chapter 3. In this case, the transformed equation will have separable solutions of the form $P_n(r)\exp(im\lambda_n s)$. On the other hand, from (4.3.9) we obtain

$$w = r(x,t) = \ln\left(\frac{m(t+x)+1}{m(t-x)+1}\right), \quad v = s(x,t) = \frac{1}{m}\ln(m(t-x)+1),$$

which are in keeping with (4.3.7).

Expressing (1.1.3) in terms of the canonical coordinates (4.3.7) can be avoided altogether by working directly with the general solution (4.3.5). The arbitrary function $H(r)$ must be chosen so that $u = \theta(x,t)$ is a solution of the wave equation. Upon differentiation of (4.3.5), we find

$$\frac{\partial^2\theta}{\partial x^2} = e^{\lambda s}\left[\left(\frac{\partial r}{\partial x}\right)^2 H''(r) + \left(\frac{\partial^2 r}{\partial x^2} + 2\lambda\frac{\partial r}{\partial x}\frac{\partial s}{\partial x}\right)H'(r)\right.$$
$$\left. + \lambda\left(\frac{\partial^2 s}{\partial x^2} + \lambda\left(\frac{\partial s}{\partial x}\right)^2\right)H(r)\right]$$

and

$$\frac{\partial^2 \theta}{\partial t^2} = e^{\lambda s}\left[\left(\frac{\partial r}{\partial t}\right)^2 H''(r) + \left(\frac{\partial^2 r}{\partial t^2} + 2\lambda \frac{\partial r}{\partial t}\frac{\partial s}{\partial t}\right) H'(r) + \lambda\left(\frac{\partial^2 s}{\partial t^2} + \lambda\left(\frac{\partial s}{\partial t}\right)^2\right) H(r)\right].$$

Substitution of the above into the wave equation, with (4.3.7) being noted, results in

$$-2\Phi'(t+x)\Phi'(t-x)H''(r) + \lambda\Phi'(t+x)\Phi'(t-x)H'(r) = 0$$

or

$$H''(r) - \frac{\lambda}{2}H'(r) = 0. \tag{4.3.10}$$

From (4.3.5) we see that boundary conditions (1.1.3b,c) (with $p(t) = q(t) = 0$) will be satisfied , for an arbitrary $\Phi(\zeta)$, if and only if

$$H(0) = 0 \tag{4.3.11}$$

and

$$H(L) = 0, \tag{4.3.12}$$

the last two conditions serving as boundary conditions for the differential equation (4.3.10). Integration of (4.3.10) leads to

$$H'(r) - \frac{\lambda}{2}H(r) = \hat{A},$$

where $\hat{A}$ is a constant. The last equation is a non-homogenous first order linear differential equation, with integrating factor $\exp\left(\frac{-\lambda}{2}r\right)$, which may be rewritten in the form

$$\frac{d}{dr}\left(e^{\frac{-\lambda}{2}r}H(r)\right) = \hat{A}e^{\frac{-\lambda}{2}r},$$

whereupon a second integration yields

$$e^{\frac{-\lambda}{2}r}H(r) = \frac{-2}{\lambda}\hat{A}e^{\frac{-\lambda}{2}r} + B$$

or

$$H(r) = A + Be^{\frac{\lambda}{2}r},$$

for some arbitrary constants A and B. Condition (4.3.11) implies that $A = -B$, after which (4.3.12) requires that

$$B\left(e^{\frac{\lambda}{2}L} - 1\right) = 0, \tag{4.3.13}$$

which is satisfied trivially when $B = 0$. Alternatively, if we set $\frac{\lambda}{2}L = 2n\pi i$ for some integer n, then (4.3.13) holds for any B. Thus we set

$$\lambda_n = \frac{2n\pi}{L}$$

and write the solution $H(r)$ of (4.3.10), (4.3.11) and (4.3.12) in the form

$$H(r) = B\left(e^{i\lambda_n r} - 1\right),$$

from which we deduce, by virtue of (4.3.5), that

$$\theta(x,t) = B\left(e^{i\lambda_n(\Phi(t+x)-\Phi(t-x))} - 1\right)e^{i\lambda_n\Phi(t-x)}$$

or

$$\theta(x,t) = B\left(e^{i\lambda_n\Phi(t+x)} - e^{i\lambda_n\Phi(t-x)}\right),$$

in analogy with (3.2.21). Since the above satisfies the homogenous wave equation with homogenous boundary conditions, we may superimpose solutions to obtain the solution

$$u(x,t) = \sum_{n=-\infty}^{\infty} B_n \left(e^{i\lambda_n \Phi(t+x)} - e^{i\lambda_n \Phi(t-x)} \right), \tag{4.3.14}$$

where once again the coefficients B_n are to be determined by the initial conditions. These initial conditions, when applied to (4.3.14), produce the equations

$$f(x) = \sum_{n=-\infty}^{\infty} B_n \left(e^{i\lambda_n \Phi(x)} - e^{i\lambda_n \Phi(-x)} \right) \tag{4.3.15}$$

and

$$g(x) = \sum_{n=-\infty}^{\infty} B_n \left(i\lambda_n \Phi'(x) e^{i\lambda_n \Phi(x)} - i\lambda_n \Phi'(-x) e^{i\lambda_n \Phi(-x)} \right). \tag{4.3.16}$$

Integrating (4.3.16) with respect to x, we obtain

$$G(x) = \sum_{n=-\infty}^{\infty} B_n \left(e^{i\lambda_n \Phi(x)} + e^{i\lambda_n \Phi(-x)} \right), \tag{4.3.17}$$

where, as in chapter 3, there is no loss in generality in defining the anti-derivative $G(x)$ by

$$G(x) = \int_0^x g(\tau)d\tau.$$

Addition and subtraction of equations (4.3.15) and (4.3.17) produces the equations

$$\frac{1}{2}\left(G(x) + f(x)\right) = \sum_{n=-\infty}^{\infty} B_n e^{i\lambda_n \Phi(x)} \tag{4.3.18}$$

and

$$\frac{1}{2}\left(G(x) - f(x)\right) = \sum_{n=-\infty}^{\infty} B_n e^{i\lambda_n \Phi(-x)}. \tag{4.3.19}$$

If the equations (4.3.15) and (4.3.16) are to be satisfied, it is apparent that $f(x)$ and $g(x)$ must be odd functions on the interval $[-S(0), S(0)]$, which is in keeping with the extension rule motivated by (2.3.2) with $p(t) = 0$. With this in mind, we see that (4.3.19) is merely a consequence of (4.3.18).

In an effort to determine the coefficients B_n appearing in (4.3.18), consider the integral

$$I = \int_{-S(0)}^{S(0)} e^{i\lambda_n \Phi(x)} e^{-i\lambda_k \Phi(x)} \Phi'(x) dx,$$

for any integer k. If $n = k$, then by (4.3.4), we have

$$I = \int_{-S(0)}^{S(0)} \Phi'(x) dx = \Phi(S(0)) - \Phi(-S(0)) = L,$$

while if $n \neq k$, then

$$\begin{aligned} I &= \int_{-S(0)}^{S(0)} e^{i(\lambda_n - \lambda_k)\Phi(x)} \Phi'(x) dx \\ &= \frac{-i}{\lambda_n - \lambda_k} \left[e^{i(\lambda_n - \lambda_k)\Phi(S(0))} - e^{i(\lambda_n - \lambda_k)\Phi(-S(0))} \right] \\ &= \frac{-i}{\lambda_n - \lambda_k} e^{i(\lambda_n - \lambda_k)\Phi(-S(0))} \left[e^{i(\lambda_n - \lambda_k)L} - 1 \right] = 0. \end{aligned}$$

Thus, upon multiplying (4.3.18) by $\exp\left(-i\lambda_k \Phi(x)\right) d\Phi(x)$ and integrating with respect to x, we obtain

$$B_n = \frac{1}{L} \int_{x=-S(0)}^{x=S(0)} \frac{1}{2} \left(G(x) + f(x) \right) e^{-i\lambda_n \Phi(x)} d\Phi(x), \tag{4.3.20}$$

and therefore IMBVP (1.1.3) is formally solved by (4.3.14), where the B_n are determined by (4.3.20). The validity of the above is based on a number of assumptions which are summarized below:

1. For any $S(t)$ satisfying $|S'(t)| < 1$, there exists a $\varphi(\zeta)$ defined on $[-S(0), \infty)$ such that

   a) $\frac{1}{\varphi(\zeta)} \in \mathbf{C}^1$.

   b) there exists a constant M such that $|\varphi(\zeta)| < M$.

2. The series defined by (4.3.14) satisfies the following conditions:

   a) (4.3.14) and all series obtained from it by termwise differentiation, up to and including second order derivatives, converge uniformly for all (x, t) satisfying the inequalities $-S(t) < x < S(t), \quad 0 < t < T, T < \infty$.

   b) (4.3.14), with t=0, converges uniformly to $f(x)$ on $-S(0) < x < S(0)$ and the series obtained from (4.3.14) by termwise differentiation with respect to t converges uniformly to $g(x)$ when $t = 0$ on $-S(0) < x < S(0)$.

It is also noted that, under the above assumptions, the series (4.3.14) will also solve IMBVP (1.1.3) on $D_1 = \{(x,t)| - S(t) < x < S(t), \quad t > 0\}$ as long as $f(x)$ and $g(x)$ are odd functions on the interval $[-S(0), S(0)]$.

The series solution (4.3.14) can be thought of as a generalization of the results presented by Balazs. Indeed, if we set $\Phi(\zeta) = \ln(m\zeta + 1)$ and $L = \ln\left(\frac{1+m}{1-m}\right)$ then we obtain series (3.2.21). Furthermore, if we let $\Phi(\zeta) = \zeta$ and $L = 2$ in (4.3.14), it can be shown that (4.3.14) is equivalent to the series obtained when one solves the fixed endpoint problem on $0 < x < 1$.

Before studying the convergence properties of (4.3.14), let us consider some examples of boundary curves $x = S(t)$ that can arise from choosing specific functions $\Phi(\zeta)$. For example, let

$$\Phi(\zeta) = \sqrt{\zeta + b}, \tag{4.3.21}$$

where b is some constant. Then, by (4.3.4), we have

$$\sqrt{t + S(t) + b} - \sqrt{t - S(t) + b} = L,$$

which leads to

$$t + S(t) + b = t - S(t) + b + 2L\sqrt{t - S(t) + b} + L^2$$

or

$$\left(2S(t) - L^2\right)^2 = 4L^2\left(t - S(t) + b\right).$$

This is simply a quadratic equation in $S(t)$, which may be solved to give

$$S(t) = L\sqrt{t + b - \frac{L^2}{4}}. \tag{4.3.22}$$

Unless otherwise stated, we will henceforth assume that $L > 0$. From (4.3.21), we must have $\zeta \geq -b$ which leads to

$$-(t+b) \leq x \leq (t+b)$$

or, by (4.3.22),

$$-(t+b) \leq L\sqrt{t+b-\frac{L^2}{4}} \leq (t+b),$$

which will be satisfied as long as

$$L^2\left(t+b-\frac{L^2}{4}\right) \leq (t+b)^2$$

or equivalently

$$\left(t+b-\frac{L^2}{2}\right)^2 \geq 0. \tag{4.3.23}$$

Inequality (4.3.23) is satisfied for all times $t \geq 0$, and thus the requirement $\zeta \geq -b$ does not impose any restrictions on the constants L and b. Now, from (4.3.22), we must require that $b \geq \frac{L^2}{4}$, so that $S(t)$ will be defined for all times $t \geq 0$. Graphs of the (symmetric) fifth and tenth partial sums u_5, u_{10}, for various values of t, in the case when $f(x) = 1$, $g(x) = p(t) = q(t) = 0$, $L = 2$ and $b = 5$ appear in diagrams 4.3.1.

Note: In the above example $\Phi(\zeta) \to 0$ as $\zeta \to \infty$, so that the canonical coordinates (4.3.7) become singular (for any fixed x) as $t \to \infty$. Accordingly, we cannot necessarily expect a solution expressed in the form (4.3.14) to exhibit the correct limiting behaviour as $t \to \infty$. Let us restrict our attention to the case

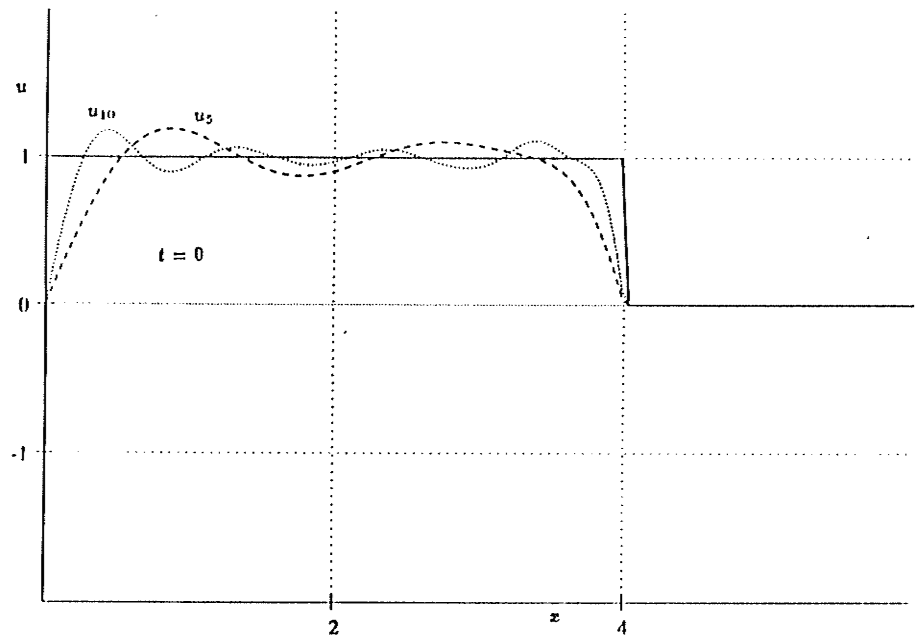

Diagram 4.3.1a

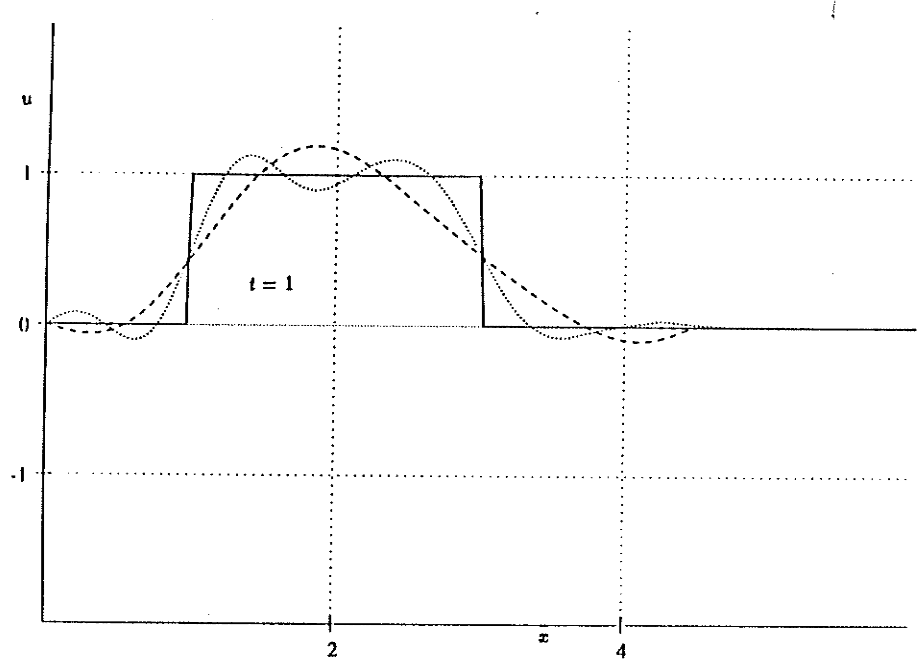

Diagram 4.3.1b

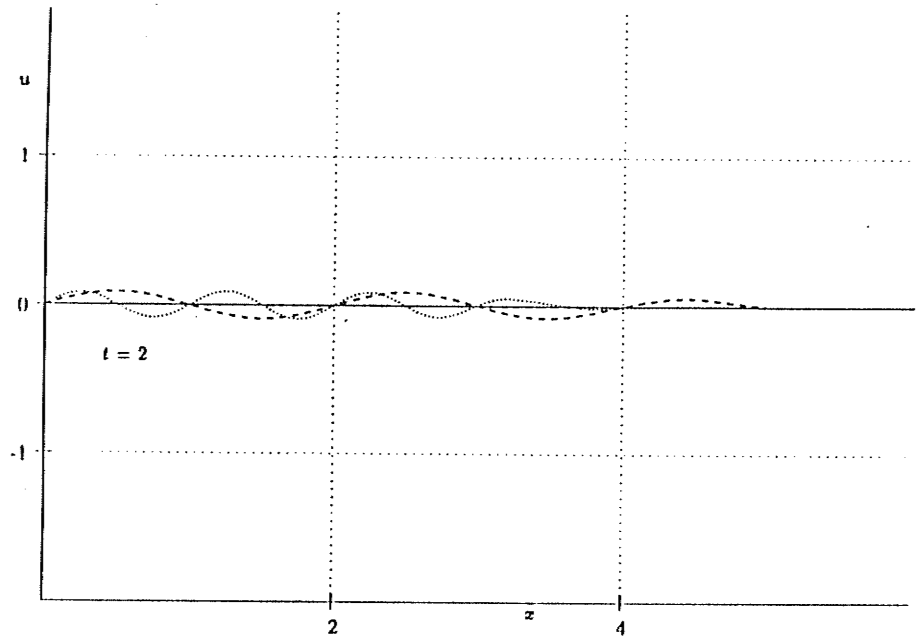

Diagram 4.3.1c

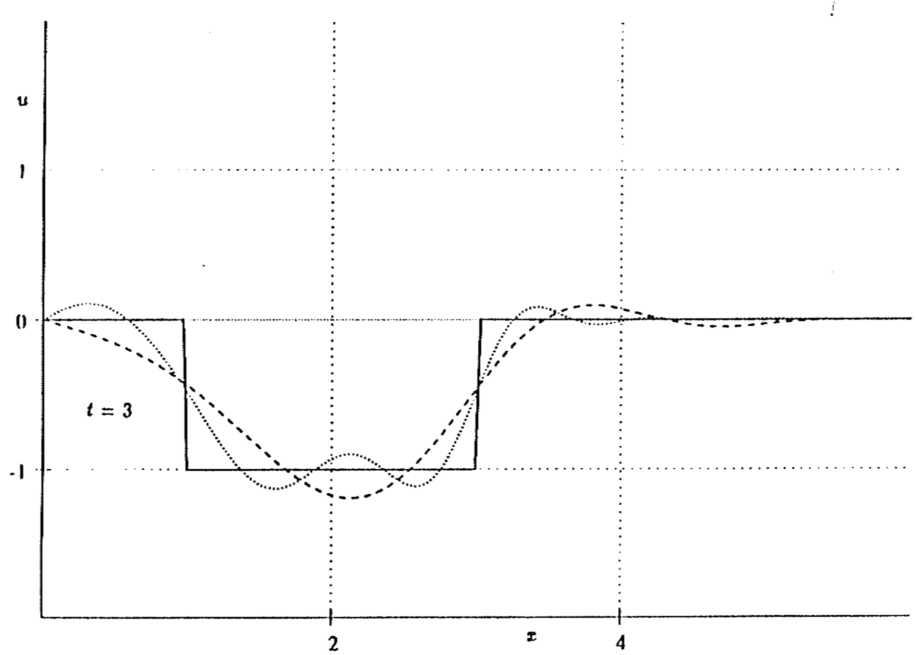

Diagram 4.3.1d

$0 < S'(t) < 1$ as is required if (4.3.14) is to be unique. Now, equation (4.3.14) can be written in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} 2iB_n \sin\left[\frac{\lambda_n}{2}\left(\Phi(t+x) - \Phi(t-x)\right)\right] e^{\frac{i\lambda_n}{2}(\Phi(t+x)+\Phi(t-x))}.$$

Moreover, for each n

$$\begin{aligned} &\lim_{t\to\infty} \sin\left[\frac{\lambda_n}{2}\left(\Phi(t+x) - \Phi(t-x)\right)\right] = \\ &\lim_{t\to\infty} \sin\left[\frac{\lambda_n}{2}\left(\sqrt{t+x+b} - \sqrt{t-x+b}\right)\right] = 0, \end{aligned} \tag{4.3.24}$$

which leads us to the conclusion $\lim_{t\to\infty} u(x,t) = 0$ for each fixed x. In addition, from section 2.4, we have the equation

$$E'(t) = \frac{1}{2}S'(t)\left([S'(t)]^2 - 1\right)\left[\frac{\partial u}{\partial x}\right]\Bigg|_{x=S(t)}, \tag{4.3.25}$$

whose validity is dependent only upon the fact that the boundary conditions of (1.1.3b,c) are assumed to homogenous. Recall that the energy of the string modelled by (1.1.3) is given by

$$E(t) = \int_0^{S(t)} \frac{1}{2}\left(\left[\frac{\partial u}{\partial t}\right]^2 + \left[\frac{\partial u}{\partial x}\right]^2\right) dx$$

for any time $t \geq 0$. From (4.3.22) and (4.3.25), with $L > 0$, we see that $E(t) \leq E(0)$ where

$$E(0) = \int_0^{S(0)} \frac{1}{2}\left([g(x)]^2 + [f'(x)]^2\right) dx < \infty.$$

In particular,

$$\lim_{t\to\infty} E(t) < \infty,$$

from which we deduce that $u_t \longrightarrow 0$ and $u_x \longrightarrow 0$, as $t \longrightarrow \infty$, which leads to the conclusion $u \longrightarrow$ constant as $t \longrightarrow \infty$. If the solution is continuous, then we have $u \longrightarrow 0$, which is in agreement with the limit (4.3.24). Unfortunately, the current example does not possess a continuous solution (as seen in diagrams 4.3.1) and thus we can not comment on its long term behaviour.

More generally, we note that, by varying our choice of the function $\Phi(\zeta)$, we can solve (1.1.3) for a variety of different boundary curves $x = S(t)$. As additional examples, we observe that if $\Phi(\zeta) = \frac{1}{\zeta+b}$ then

$$S(t) = \frac{1}{L}\left(1 \pm \sqrt{L^2(t+b)^2+1}\right),$$

while, if $\Phi(\zeta) = (\zeta + b)^2$ then

$$S(t) = \frac{L}{4(t+b)}.$$

4.4: Convergence of Series Solutions

In this section we show how a simple change of variable in (4.3.16) and (4.3.20) will allow for the application of standard convergence theorems concerning Fourier series to the investigation of the convergence properties of (4.3.14). Let us restrict our attention to

$$F(w) = \sum_{n=-\infty}^{\infty} B_n e^{i\lambda_n \Phi(w)}, \tag{4.4.1}$$

where

$$B_n = \frac{1}{L}\int_{-S(0)}^{S(0)} F(w)e^{-i\lambda_n \Phi(w)}\Phi'(w)dw. \tag{4.4.2}$$

Let $\zeta = \Phi(w)$ then since $\Phi'(w) \neq 0$, (4.4.1) and (4.4.2) become

$$H(\zeta) = \sum_{n=-\infty}^{\infty} B_n e^{i\lambda_n \zeta} \tag{4.4.3}$$

and

$$B_n = \frac{1}{L}\int_{\Phi(-S(0))}^{\Phi(S(0))} H(\zeta)e^{-i\lambda_n \zeta}d\zeta, \tag{4.4.4}$$

where $H(\zeta) = F\left(\Phi^{-1}(\zeta)\right)$. Formally, (4.4.3) is the Fourier series of $H(\zeta)$ where the Fourier coefficients B_n are determined by (4.4.4). Series (4.4.3) can be written in the equivalent form

$$H(\zeta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(\lambda_n\zeta) + b_n \sin(\lambda_n\zeta)\right], \tag{4.4.5}$$

where

$$a_n = \frac{2}{L}\int_0^{\Phi(S(0))} H(\zeta)\cos(\lambda_n\zeta)d\zeta, \qquad n = 0, 1, 2, \ldots,$$

and

$$b_n = \frac{2}{L}\int_0^{\Phi(S(0))} H(\zeta)\sin(\lambda_n\zeta)d\zeta, \qquad n = 1, 2, 3, \ldots.$$

The coefficients B_n are related to a_n and b_n by the equations

$$a_0 = 2B_0, \qquad a_n = B_n + B_{-n}, \qquad b_n = i\,(B_n - B_{-n})\,.$$

The following theorems refer to (4.4.5) and can be found in texts like [Chu] and [Tyn].

Theorem 1. *Let $H(\zeta)$ be a continuous L-periodic function, with $H'(\zeta)$ piecewise continuous. Then (4.4.5) converges uniformly to $H(\zeta)$.*

Theorem 2. *Let $H(\zeta)$ be a continuous L-periodic function and let $H'(\zeta)$ be piecewise smooth. Then the Fourier expansion of $H'(\zeta)$ can be obtained from termwise differentiation of (4.4.5) and converges to $H'(\zeta)$ at all points of continuity and to the average of the jump at all points of dicontinuity of $H'(\zeta)$.*

Suppose we require $H(\zeta)$ to be twice continuously differentiable and $H'''(\zeta)$ to be piecewise continuous for all ζ then by Theorem 1, the Fourier expansions of $H(\zeta)$, $H'(\zeta)$ and $H''(\zeta)$ are uniformly convergent. Furthermore by Theorem 2, the expansions of $H'(\zeta)$ and $H''(\zeta)$ can be obtained via termwise differentiation of the expansions of $H(\zeta)$ and $H'(\zeta)$ respectively. Differentiating $H(\zeta)$, we obtain

$$\begin{aligned} H'(\zeta) &= \frac{d}{d\zeta}\left(\Phi^{-1}(\zeta)\right) F'\left(\Phi^{-1}(\zeta)\right) \\ &= \frac{1}{\Phi'(w)} F'(w), \end{aligned} \tag{4.4.6}$$

$$\begin{aligned} H''(\zeta) &= \left[\frac{d}{d\zeta}\left(\Phi^{-1}(\zeta)\right)\right]^2 F''\left(\Phi^{-1}(\zeta)\right) + \frac{d^2}{d\zeta^2}\left(\Phi^{-1}(\zeta)\right) F'\left(\Phi^{-1}(\zeta)\right) \\ &= \frac{1}{\left[\Phi'(w)\right]^2} F''(w) - \frac{\Phi''(w)}{\left[\Phi'(w)\right]^3} F'(w) \end{aligned} \tag{4.4.7}$$

$$\begin{aligned} H'''(\zeta) &= \left[\frac{d}{d\zeta}\left(\Phi^{-1}(\zeta)\right)\right]^3 F'''\left(\Phi^{-1}(\zeta)\right) \\ &+ 3\frac{d}{d\zeta}\left(\Phi^{-1}(\zeta)\right) \frac{d^2}{d\zeta^2}\left(\Phi^{-1}(\zeta)\right) F''\left(\Phi^{-1}(\zeta)\right) + \frac{d^3}{d\zeta^3}\left(\Phi^{-1}(\zeta)\right) F'\left(\Phi^{-1}(\zeta)\right) \\ &= \frac{1}{\left[\Phi'(w)\right]^2} F'''(w) - \frac{3\Phi''(w)}{\left[\Phi'(w)\right]^3} F''(w) + \frac{3\Phi''(w) - \Phi'''(w)\Phi'(w)}{\left[\Phi'(w)\right]^5} F'(w) \end{aligned} \tag{4.4.8}$$

from which we see that if:

1. $F(w) \in \mathbf{C}^2$ and $F'''(w)$ is piecewise continuous, and

2. $\Phi(w) \in \mathbf{C}^2$ $\Phi'''(w)$ is piecewise continuous and $\frac{1}{\Phi'(w)}$ is continuously differentiable,

then the conditions imposed on $H(\zeta)$ are satisfied. Furthermore since (4.4.3) is uniformly convergent, (4.4.1) is uniformly convergent. From (4.4.6) and (4.4.7) the series

$$\frac{1}{\Phi'(w)} F'(w) = \sum_{n=-\infty}^{\infty} i\lambda_n B_n e^{i\lambda_n \Phi(w)}$$

and

$$\frac{1}{[\Phi'(w)]^2}F''(w) - \frac{\Phi''(w)}{[\Phi'(w)]^3}F'(w) = \sum_{n=-\infty}^{\infty} (i\lambda_n)^2 B_n e^{i\lambda_n \Phi(w)}$$

are both uniformly convergent. Solving the above for $F'(w)$ and $F''(w)$, we obtain the following convergent series

$$F'(w) = \Phi'(w) \sum_{n=-\infty}^{\infty} i\lambda_n B_n e^{i\lambda_n \Phi(w)}$$

and

$$F''(w) = \Phi''(w) \sum_{n=-\infty}^{\infty} i\lambda_n B_n e^{i\lambda_n \Phi(w)} + [\Phi'(w)]^2 \sum_{n=-\infty}^{\infty} (i\lambda_n)^2 B_n e^{i\lambda_n \Phi(w)},$$

which can be obtained by termwise differentiation of (4.4.1). Since $H(\zeta)$ is assumed to be L-periodic, the function $F(w)$ must satisfy $F\left(\Phi^{-1}(\zeta)\right) = F\left(\Phi^{-1}(\zeta - L)\right)$ or

$$F(w) = F\left(\Phi^{-1}\left[\Phi(w) - L\right]\right). \tag{4.4.9}$$

Now, suppose that in addition to our standing assumptions, we require $f'''(x)$ and $g''(x)$ to be piecewise continuous, then (4.3.18) and (4.3.19) converge uniformly to $\frac{1}{2}\left(G(x) + f(x)\right)$ and $\frac{1}{2}\left(G(x) - f(x)\right)$ respectively (on $(-S(0), S(0))$). Furthermore the series obtained by termwise differentiation with respect to x converges uniformly to $\frac{1}{2}\left(g(x) - f'(x)\right)$ and $\frac{1}{2}\left(g(x) - f'(x)\right)$ and therefore series (4.3.14) converges uniformly to $f(x)$ and

$$\sum_{n=-\infty}^{\infty} \left(\Phi'(x)e^{i\lambda_n \Phi(x)} - \Phi'(-x)e^{i\lambda_n \Phi(-x)}\right)$$

converges uniformly to $g(x)$ when $t = 0$ and $-S(0) < x < S(0)$. In view of (4.4.9) we must have

$$f(x) + G(x) = f\left(\Phi^{-1}\left[\Phi(w) - L\right]\right) + G\left(\Phi^{-1}\left[\Phi(w) - L\right]\right)$$

and

$$f(x) - G(x) = f\left(\Phi^{-1}\left[\Phi(w) - L\right]\right) - G\left(\Phi^{-1}\left[\Phi(w) - L\right]\right),$$

which lead to

$$f(x) = f\left(\Phi^{-1}\left[\Phi(w) - L\right]\right) \tag{4.4.10}$$

and

$$G(x) = G\left(\Phi^{-1}\left[\Phi(w) - L\right]\right). \tag{4.4.11}$$

In analogy with section 2.2, (4.4.10) and (4.4.11) can be thought of as extension rules for the functions $f(x)$ and $G(x)$. For example, let $\Phi(\zeta) = \ln(m\zeta + 1)$ and $L = \ln\left(\frac{1+m}{1-m}\right)$ then (4.4.10) becomes

$$\begin{aligned} f(x) =& f\left(\frac{1}{m}\left[\exp\left(\ln(mx+1) - \ln\left(\frac{1+m}{1-m}\right)\right) - 1\right]\right) \\ =& -f\left(\frac{(m-1)x+2}{m+1}\right), \end{aligned}$$

which is analogous with results presented in section 2.2 for the extension of $f(x)$ from $[0,1]$ to $[1, \frac{2}{1-m}]$.

Finally, since we can construct (4.3.14) from (4.4.1) using the substitutions $w = t + x$ and $w = t - x$, (4.3.14) is uniformly convergent and furthermore the expansions of u_x, u_t, u_{xx}, u_{xt} and u_{tt} converge uniformly and can be obtained from (4.3.14).

Summary

Using a constructive procedure, it was shown (in section 2.3) that a solution of IMBVP (1.1.3) exists as long as $|S'(t)| < 1$. It was then demonstrated how this solution could be used to verify certain important theoretical properties, that in part, let us verify that (1.1.3) is well posed. Furthermore, the form of this solution allows us to conclude that the fundamental effect of a moving endpoint is simply an elongation/contraction (albeit non-linear in most cases) of the waves that are reflected from this endpoint. In fact, we have seen that every time a wave interacts with the moving endpoint, this elongation/contraction appears in the form of the function $\nu(\xi)$. One of the most important properties of the d'Alembert solution is that the solution of (1.1.3) can be computed exactly for any finite time t. Furthermore, formulae (2.3.6)-(2.3.9) show us exactly how the solution evolves as time progresses, by keeping track of individual disturbances (waves), the sources of these diturbances and the effect of the moving endpoint.

The series solutions, derived in section 4.3, are an extension of Balazs' work and arise in a unified manner through the theory of Lie groups. These solutions are dependent upon the the discovery of a suitable function $\Phi(\zeta)$, satisfying (4.3.4), for each particular function $S(t)$, identifying the moving boudary. Once an appropriate function $\Phi(\zeta)$ has been obtained, we can construct a set of canonical coordinates that map a time dependent domain to a semi-infinite "strip". When (1.1.3) is expressed in terms of the canonical coordinates, we are free to apply solution techniques that are designed to handle the "fixed" end case. The function $\Phi(\zeta)$ is like the function $\nu(\xi)$ in the respect that $\Phi(\zeta)$ can be thought of as inducing a stretch of standard eigenfunctions to obtain new ones which allow us to solve (1.1.3). However, unlike $\nu(\xi)$, we do not have a sufficient conditon that will guarantee the existence of a suitable choice of $\Phi(\zeta)$. A possible avenue of exploration presents itself in the form of equation (4.4.9). It was verified for the case $S(t) = mt + 1$ that

$$\Phi^{-1}\left(\Phi(\zeta) - L\right) = \nu(\zeta),$$

which can be shown to be a restatement of (4.3.4). Using this relationship, it is possible that one could establish the existence of a suitable $\Phi(\zeta)$ for a given function $S(t)$ and at the same time provide a unifying link between the solutions of sections 2.3 and 4.3.

Unfortunately, the existence of the function $\Phi(\zeta)$ is not the only question left unanswered in regards to (4.3.14). Section 4.4 simply indicates how rigorous convergence statements might be made and is by no means an exhaustive study of the convergence properties of (4.3.14). Work in this area could build directly upon the material we have presented, or on the other hand a more abstract approach may prove beneficial.

Returning to the IMBVP (1.1.3), or more precisely the model from which it was obtained, there is no reason to suspect that (1.1.3) retains all of the important physical behaviour described by the more general vectorial problem (1.1.1). Therefore, any subsequent analysis would not be complete without an attempt to address the possibility of including non-homogeneities in both the differential equation and the boundary conditions of (1.1.3). Furthermore, such analysis should have as its ultimate objective the solution of the vectorial IMBVP, as stated in the introduction.

Finally, recent discussion about the initial Oedipus flight has introduced a need to generalize the boundary conditions of this model. This in turn could motivate further study of (1.1.3) where Neumann and/or Robin conditons are given in place of the existing Dirichlet conditions.

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