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SOME NEW ASPECTS OF CONGRUENCE TRANSFORMATION

AS APPLIED TO RC LADDER NETWORKS

by

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A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

with love

To my parents

my wife Nivin

and my son Tarek

ABSTRACT

In a canonic RC ladder network realized from specified two-port parameters a proper transformation is applied to generate a family of equivalent networks in which the total resistance, total capacitance and the open-circuit voltage gain change continuously with respect to a single variable. A straight forward method is then developed to minimize cost function in ladder networks. The cost function is formed by weighting each of the three components - total resistance, total capacitance and the gain - and combining them.

The thesis also develops a systematic method of realizing RC low-pass ladder networks with nonuniformly lossy capacitors. The method is based on applying a proper congruence transformation to the capacitance and conductance matrices of the canonic network. The transformation matrix is constructed in such a way that the capacitors of the transformed network have shunt conductances while the driving-point impedance is kept invariant and the transmission characteristics of the two-port are preserved. Application of the results to other types of ladder structures is also investigated. In addition, the synthesis of RC low-pass ladder networks with equal-valued capacitors is studied. Realizability conditions are determined such that the transformed network includes as many equal capacitors as possible without increasing the total number of capacitors. Explicit formulas for the element values of the equivalent network are given throughout the thesis.

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Adel Ahmed Ali

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SYMBOLS AND ABBREVIATIONS

A	Transformation matrix
A^t	Transpose of matrix A
a_i	The diagonal elements of A
a_{ij}	The (i, j)th element of A
a_i, b_i	Polynomial coefficients
B	Transformation matrix
C	Capacitance matrix
C'	Capacitance matrix of the transformed network
C_T	Total capacitance
c_{ij}	The (i, j)th element of C
c'_{ij}	The (i, j)th of C'
c_k	The k-th capacitor of the network
c'_k	The k-th capacitor of the transformed network
diag( )	Diagonal matrix
E	Vector of the node voltages
E'	Vector of node voltage in the transformed network
$F(\cdot)$	Cost function
G	Conductance matrix
G'	Conductance matrix of the transformed network
G_T	Total conductance
G_{ij}	The (i, j)th element of G
G'_{ij}	The (i, j)th of G'
G_k	The k-th conductor of the network
G'_k	The k-th conductor of the transformed network
g'_k	The k-th shunt conductor of the transformed network

H	Open-circuit voltage gain
I	Vector of equivalent source currents
I'	Vector of equivalent source current in the transformed network
I_n	Unit matrix of order n
K, $K(\cdot)$	Relative gain
k	Positive constant
N	Canonical network
N'	Transformed network
R	Resistance matrix
R'	Resistance matrix of the transformed network
R_k, r_k	k-th resistor
S	Elastance matrix
S_k	The k-th elastance
s	Complex frequency variable
Y	Node admittance matrix
Y'	Node admittance matrix of the transformed network
y_{ij}	The (i, j)th element of Y
y'_{ij}	The (i, j)th element of Y'
Z	Mesh impedance matrix
Z'	Mech impedance matrix of the transformed network
z_{ij}	The (i, j)th element of Z
z'_{ij}	The (i, j)the element of Z'
β, β_i	Normalized variables; $0 \leq \beta \leq 1$
γ_k	$\gamma_k = \sum_{i-k}^{n} \frac{1}{C_i}$
δ_i	Loss factor
θ_k	$\theta_k = \sum_{i=k}^{n} G_i$
σ_k	$\sigma_k = \sum_{i=k}^{n} R_i$

CHAPTER I

INTRODUCTION

In circuits designed for low frequency applications the use of inductors is undesirable mainly for the fact that they are bulky, nonlinear dissipative and uneconomical. There are other disadvantages: where specifications are precise, the usual synthesis techniques for lossless networks cannot be used because of the inherent resistance of the inductors. There is, however, a great need for precise performance at low frequency, especially in the control field where accurate compensations are required. Thus, the use of a network containing only resistors and capacitors is very important.

This thesis deals with three important problems in RC synthesis: i) precorrecting for capacitor dissipations, ii) minimizing duly defined cost function, and iii) designing RC ladder networks with equal capacitors.

With the advent of thin film and integrated circuit techniques, design criteria have changed significantly. In particular, the new techniques have led to less emphasis on the number of elements in the network and to more interest in design methods which take into account the higher losses that occur with thin-film capacitors at the present state of development [1]. Another important problem that faces the designer of RC circuits fabricated by integrated-circuit techniques is the size of the circuit chip. As the chip size increases, yield decreases, and cost rises. In some manufacturing processes, the area required by a resistor or a capacitor is proportional to its value [1]. Hence, for a small chip size, it is necessary to minimize the total resistance and capacitance in the circuit. The use of equal capacitors in integrated circuit design is advantageous in other ways as well: such networks are more easily made and tested; they also minimize mask layout errors, and errors in the network functions caused by temperature variations [1].

The following chapters treats these design problems. The basic approach to them is the application of the matrix transformation technique. The results of these studies reveal some new implications of the equivalent network theory.

Generating equivalent networks is one of the central concerns in network theory. Methods of realizing equivalent networks are quite important, both from theoretical and practical reasons.

It is well known that two networks differing in their topology and elements may be completely equivalent if the relations between variables at a selected subset of their terminals are concerned.

Cauer [2] showed in 1929 that congruence transformation applied to the matrix representing a realizable network can generate another realizable network in such a manner that a specified set of input and transfer admittances will be invariant. He also pointed out that these transformations would preserve the positive semidefinite character of the parameter matrices. Since this latter property is both necessary and sufficient for realization of networks containing ideal transformers, he therefore established a basic method for generating equivalent networks.

If ideal transformers are not used, however, the positive semidefinite character of the parameter matrices is no longer sufficient to guarantee realizability. Unfortunately, the general conditions that are both necessary and sufficient for this transformation procedure to yield realizable parameter matrices are not known. This lack of information has therefore limited the practical applications of Cauer's method.

Guillemin [3], [4], was the first to recognize the great importance of the normal form for Cauer's approach in generating equivalent two-element-kind networks. The normal form ties together the so-called methods of direct transformation and matrix synthesis. The former refers to Cauer's method of transforming a given network directly into an equivalent one, while the latter refers to Guillemin's procedure of generating networks from the normal form by reversing the normal coordinate transformation procedure. Since direct transformation can always be viewed as a conversion from the network to the normal form, followed by a conversion from the normal form to an equivalent network, it is clear that the normal form introduced by Guillemin is the common feature of equivalent networks produced by linear transformations.

Duda [5], and Schneider [6] applied Guillemin's procedure to generate equivalent realizations of RC driving-point impedances.

The theory of continuously equivalent networks, which was introduced by Schoeffler [7], [8], is an extension of the Howitt transformation [9], for deriving equivalent networks; however, it allows all elements in a network to vary continuously as functions of single parameter. The method is quite general, flexible, and easily programmed. Schoeffler and others [10]-[16], have applied the method to generate equivalent networks for the purpose of minimizing sensitivity.

The main objective of this thesis is the investigation of a special class of congruence transformations for novel features which facilitate the solution of the stated problems. The application of the transformation technique for the stated objectives is new, and to the best knowledge of the author, the same problems has not been previously treated in the literature.

The thesis consists of six chapters, including the introduction. In Chapter II, the general linear transformation is discussed and then congruence transformation is presented in detail. The equilibrium equations for the low-pass ladder networks are set out and the transformation matrix is constructed to generate equivalent networks within the ladder topology. Element values are given explicitly in terms of those of the canonic network. The results are then directly extended to high-pass and other types of ladder networks using the topological duality and the GC:CG:GC transformation.

The minimization of cost functions is dealt with in Chapter III. The relevant quantities - total resistance, total capacitance, and the gain - are combined to form a penalty function which has to be minimized to produce the optimum network. In this chapter a family of n-canonic ladder networks is shown to have some interesting properties. We then generate a one-parameter family of continuously equivalent networks having useful characteristics. In the two preceding cases, the optimum network will be found uniquely. Explicit formulas for the element values of the transformed networks are derived and illustrative examples are provided.

Chapter IV develops a systematic method for the compensation of lossy capacitors. Uniform and nonuniform loss are treated and explicit formulas for the element values of the precompensated networks are given.

Chapter V discusses the possiblity of realizing ladder networks with equal capacitors. It shows that a canonic network can be transformed into an equivalent network of equal-valued capacitors under certain specified conditions. This chapter provides element values and examples to illustrate the synthesis procedure.

Chapter VI then outlines some concluding remarks. Finally, six appendices set out the mathematical proofs of realizability conditions and the construction of transformation matrices.

CHAPTER II

EQUIVALENT TRANSFORMATION OF RC LADDER NETWORKS

2.1 Introduction

Ladder networks are one of the simplest and most important network structures. Because of their versatile properties, they are the most widely used configurations. The transmission zeros, for example, can easily be realized in such networks by creating poles of series impedances and shunt admittances. Zero shifting and tuning are accomplished by adjusting a readily accessible element in a simple way. They are also characterized by their high reliability and low sensitivity to element variations [17].

The main objective of this chapter is to develop a general transformation technique that can be used in devleoping a continuously equivalent family of RC ladder networks. Network matrices are subjected to congruence transformation in such a way that one of the driving point functions remains unchanged, while the transmission characteristics of the two-port are preserved. But first it is useful to introduce the general theory of linear transformations and discuss the physical interpretation of congruence transformation. It is essential to identify the proper transformation matrices for the two-element-kind ladder networks. Then, as a result of the transformation of the parameter matrices, it is possible to explicitly establish the formulas for element values.

2.2 Equivalent Network Theory

The earliest theory of equivalent networks was based upon congruence transformations of loop impedance matrices. The basic idea of the theory is to keep the port variables fixed while modifying the internal loop variables to obtain a desired structure. Since constant-congruence transformations on impedance matrices keep the internal energy invariant, they can be realized with transformers. Using such transformations in the manner developed by Howitt, Cauer and others [18]-[21], we will generate equivalent ladder networks. However, as pointed out by Brune [22], not all equivalent realizations can be determined by the method.

Theorem

Given an $n + p$-port network N, described by the admittance matrix Y, consider an $n + p$-port N' described by Y' such that

$$Y' = B\,Y\,A, \tag{2.1}$$

where

$$B = \begin{array}{cc} \left[\begin{array}{cc} I_n & B_{12} \\ 0 & B_{22} \end{array}\right] & \begin{array}{l} \}n \\ \}p \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{p} \end{array} & \end{array},$$

$$A = \begin{array}{cc} \left[\begin{array}{cc} I_n & 0 \\ A_{21} & A_{22} \end{array}\right] & \begin{array}{l} \}n \\ \}p \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{p} \end{array} & \end{array} \tag{2.2}$$

and

$$Y = \begin{array}{cc} \left[\begin{array}{cc} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{array}\right] & \begin{array}{l} \}n \\ \}p \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{p} \end{array} & \end{array} \tag{2.3}$$

For nonsingular B and A, Y and Y' have the same nxn admittance matic Y_n

$$Y_n = Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}. \tag{2.4}$$

which is obtained by opening the last p-ports of the network. The theorem can be readily proved by using (2.1)-(2.4).

Corollary (Howitt)

If A is a constant matrix and B is the transpose of A then

$$Y' = A^t Y A \tag{2.5}$$

is called a congruence transformation.

It should be noted that B and A of (2.1) may actually be rational or irrational functions of the complex frequency variable s [23].

Since the congruence transformation is applied to generate equivalent ladder networks, it is useful to physically interpret the transformation. Consider a network with n independent node pairs. Such a network is described by a set of n equations of the form

$$I = Y E \tag{2.6}$$

where Y is the nxn admittance matrix of the network, E is the nxl column matrix of node voltages and I is the nxl column matrix of equivalent source currents. If this network is imbedded in a 2n-port network of ideal transformers, another n-port results in which the variables of the original and the new network are related by a linear transformation:

$$E = A E'$$

$$I' = A^t I \tag{2.7}$$

Where A is an nxn nonsingular constant matrix and E' and I' are the new voltage and current variables [24]. The last equation follows from the energy conservation of the transformer network. The new variables are related by the equation

$$I' = Y' E' \tag{2.8}$$

where

$$Y' = A^t Y A \tag{2.9}$$

From the method of construction it is clear that the resulting admittance matrix is physically realizable. The usefulness of the theory stems from the possibility of interpreting (2.9) as the equilibrium equations of an equivalent transformerless network. In this case, not all of the elements will be positive in general, a fact that has limited the applications of the theory [25]. By proper choice of the transformation matrix A, certain driving-point and/or transfer functions can be kept invariant. For example, if the k-th row of A is the k-th unit vector (all zeros except for the k-th entry which is unity), the driving point impedance at the k-th terminal pair is invariant to the transformation. If two rows are so chosen, the driving point impedance at each port is invariant, as is the transfer impedance between the two ports. Thus, it is possible to maintain desired transfer functions invariant to the transformation.

2.3 Low-Pass RC Ladder Networks

2.3.1 z_{11}, z_{12} are specified

Let us consider an RC two-port with two of the open-circuit impedance parameters:

$$z_{11} = \frac{\Sigma_o^m a_i s^i}{\Sigma_o^p b_i s^i}, \qquad z_{12} = \frac{k}{\Sigma_o^p b_i s^i} \tag{2.10}$$

Since the transmission zeros are all at infinity, the synthesis of z_{11} in the first Cauer form, which will be called the reference network, gives the low-pass ladder realization shown in Figure 1.a, where $C_1 = o$ for p=m and $C_1 \neq o$ for p=m+1, with n=m+1 in either case, $b_o \neq o$ is assumed throughout.

The node admittance matrix Y of N is given by

$$Y = G + sC \tag{2.11}$$

where G and C are the nxn conductance and capacitance matrices, respectively, and s is the complex frequency variable.

$$G = \begin{bmatrix} G_1 & -G_1 & 0 & \cdot & \cdot & \cdot & 0 \\ -G_1 & G_1+G_2 & -G_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & & & \\ & & & & & & \\ 0 & 0 & \cdot & \cdot & G_{n-2}+G_{n-1} & & -G_{n-1} \\ 0 & 0 & & & -G_{n-1} & & G_{n-1}+G_n \end{bmatrix} \tag{2.12}$$

and

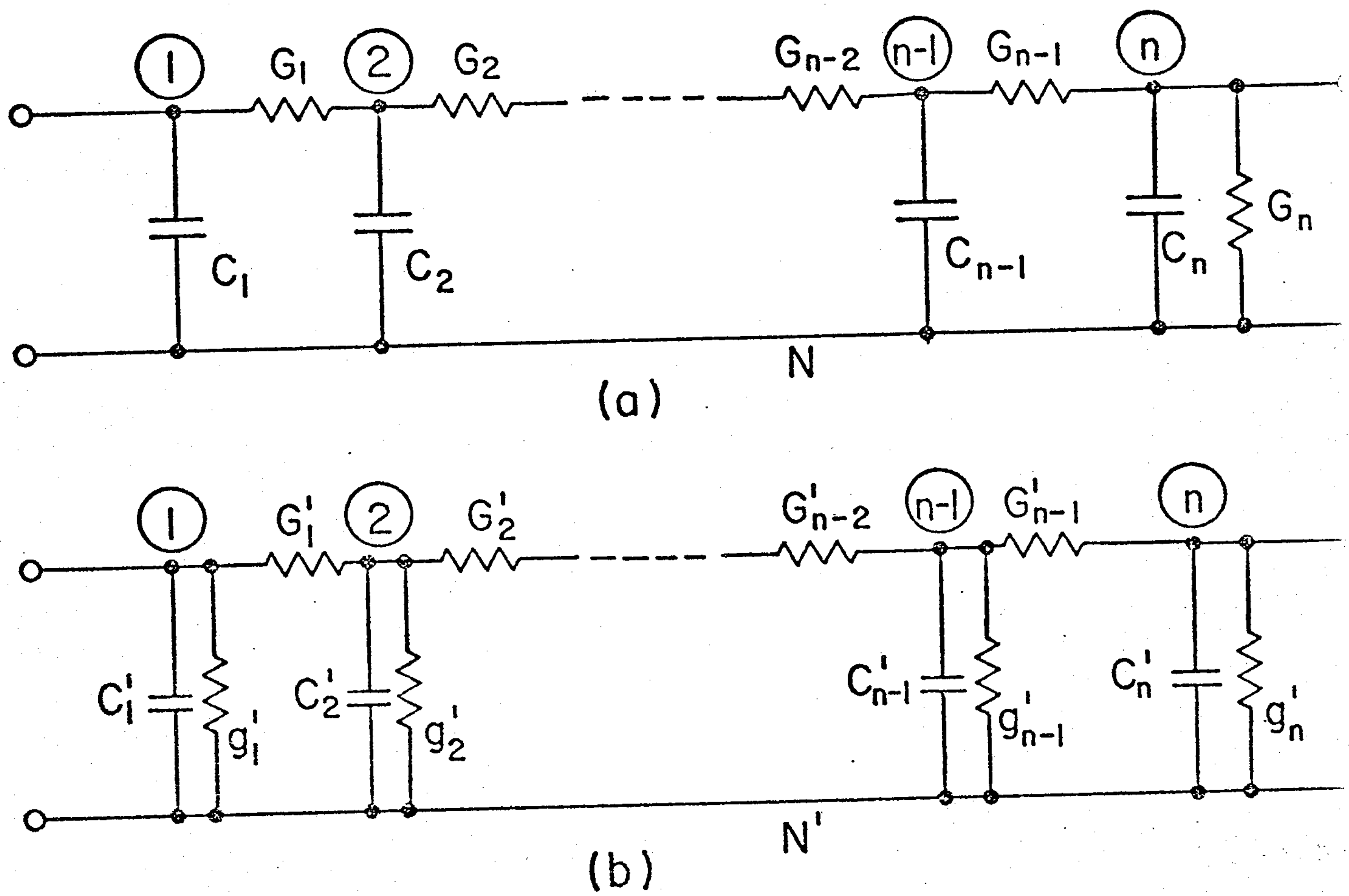


Figure 1 Low-pass realization of specified (z_{11}, z_{12})

(a) Canonic network.
(b) Equivalent network.

$$C = \text{diag } (C_1, C_2, \ldots, C_n) \tag{2.13}$$

In standard matrix notation

$$Y = [y_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{2.14}$$

$$G = [G_{ij}] \qquad i, j = 1, 2, \ldots, n$$

$$C = [C_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{2.15}$$

We now apply the congruence transformation on Y to obtain a new network N' corresponding to Y' as given by (2.9). The transformed network N' of Figure 1.b has $z'_{11} = z_{11}$; $z'_{12} = kz_{12}$ and each capacitor in N' has a shunt conductance. To keep z_{11} invariant, and preserve the ladder topology of N' characterized by Y' as

$$Y' = G' + sC' \tag{2.16}$$

it is required that the matrix A has the form (Appendix A)

$$A = \text{diag } (1, a_2, a_3, \ldots, a_n) \tag{2.17}$$

Substituting (2.14) in (2.9) we get

$$Y' = [a_i a_j y_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{2.18}$$

$$G' = [a_i a_j G_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{2.19}$$

$$C' = [a_i a_j C_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{2.20}$$

In order to determine the element values in N' consider the node admittance matrix Y' of the equivalent ladder shown in Figure 1.b.

$$Y' = G' + sC' \tag{2.21}$$

where

$$G' = \begin{bmatrix} g_1'+G_1' & -G_1' & 0 & \cdot & \cdot & 0 \\ -G_1' & G_1'+G_2'+g_2' & -G_2' & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot & \\ & & & & & \\ 0 & 0 & 0 & \cdot\ \cdot & G_{n-2}'+G_{n-1}'+g_{n-1}' & -G_{n-1}' \\ 0 & 0 & 0 & \cdot\ \cdot & -G_{n-1}' & G_{n-1}'+g_n' \end{bmatrix} \tag{2.22}$$

and

$$C' = \text{diag}\ (C_1', C_2', \ldots, C_n') \tag{2.23}$$

Comparing the main diagonal elements of (2.19) and (2.22), we write

$$G_{k-1}'+G_k' + g_k' = a_k^2\ (G_{k-1} + G_k) \qquad k = 1, 2, \ldots, n \tag{2.24}$$

where

$$G_o = G_o' = 0$$

Since

$$G_k' = a_k\ a_{k+1}\ G_k \quad \text{and} \quad G_{k-1}' = a_{k-1}\ a_k\ G_{k-1} \tag{2.25}$$

we identify g_k' from (2.24) as

$$g_k' = a_k^2\ \{G_k(1- \frac{a_{k+1}}{a_k}) - G_{k-1}\ (\frac{a_{k-1}}{a_k} - 1)\} \qquad k = 1, 2, \ldots, n \tag{2.26}$$

where

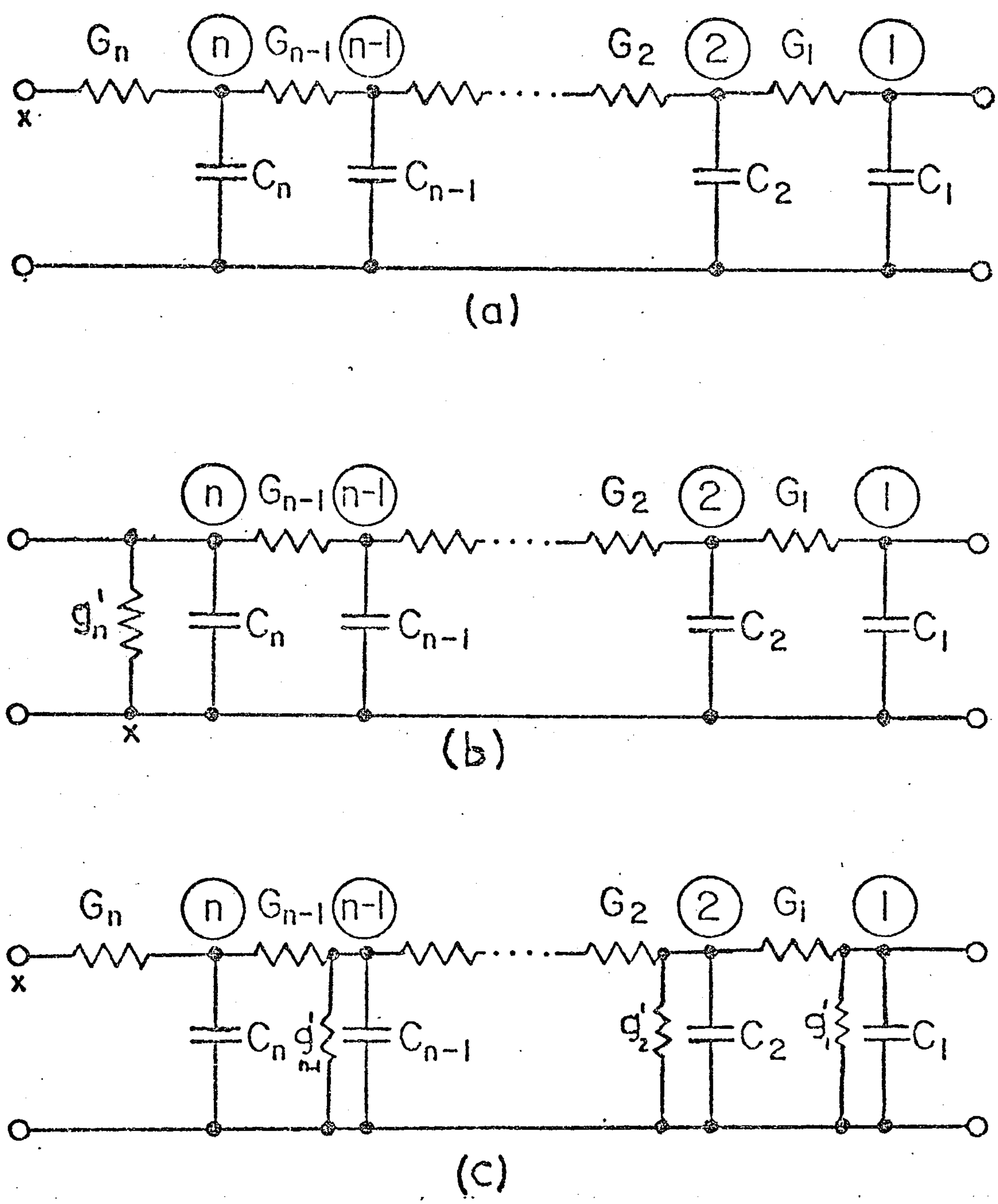


Figure 2 Low-pass realization of specified (y_{22}, y_{12}).

(a) Canonic network.
(b) Intermediate stage network.
(c) Equivalent network.

$$a_o = a_{n+1} = G_o = G_{n+1} = 0$$

Similarly, from (2.20) and (2.23) we write

$$C_k' = a_k^2 \, C_k \qquad\qquad k = 1, 2, \ldots, n \tag{2.27}$$

Equations (2.25)-(2.27) give explicit formulas for the element values of the transformed network in terms of the element values of the reference network and the transformation parameters a_k. The transformation matrix A is chosen to meet the design criteria.

2.3.2 y_{12}, y_{22} are specified

The network shown in Figure 2.a is the canonic realization of

$$y_{22} = \frac{\sum_{o}^{m} a_i s^i}{\sum_{o}^{q} b_i s^i}, \qquad y_{12} = \frac{-k}{\sum_{o}^{q} b_i s^i} \tag{2.28}$$

where $C_1 = o$, for $m=q$ and $n=q+1$. Since y_{22} is the driving-point admittance with terminal x grounded, we have the same situation as in Figure 1.a except that the input and output ports are now interchanged. Thus, we can directly apply the transformation method developed in the preceeding section to obtain the element values of the transformed network. Those values are given by (2.25)-(2.27) and the equivalent network is shown in Figure 2.c.

2.4 High-Pass Ladder Networks

2.4.1 y_{11}, y_{12} are specified

When the RC two-port short circuit admittance parameters y_{11} and y_{12} are given as

$$y_{11} = \frac{\Sigma_o^m a_i s^i}{\Sigma_o^p b_i s^i} , \qquad y_{12} = - \frac{k s^m}{\Sigma_o^p b_i s^i} \tag{2.29}$$

we can realize y_{11} in the Cauer second form as shown in Figure 3.a. The network can be described by the mesh impedance matrix of

$$Z = R + \frac{1}{s} S \tag{2.30}$$

where R and S are the nxn resistance and elastance matrices, respectively.

$$R = \begin{bmatrix} R_1 & -R_1 & 0 & \cdot & \cdot & & 0 \\ -R_1 & R_1+R_2 & -R_2 & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdot & R_{n-2}+R_{n-1} & & -R_{n-1} \\ 0 & 0 & 0 & \cdot & -R_{n-1} & & R_{n-1}+R_n \end{bmatrix} \tag{2.31}$$

and

$$S = \text{diag}\ (\frac{1}{C_1}, \frac{1}{C_2}, \ldots, \frac{1}{C_n}) = \text{diag}\ (S_1, S_2, \ldots, S_n) \tag{2.32}$$

As a result of the topological duality existing between the low-pass ladder of Figure 1.a and the high-pass ladder of Figure 3.a, we observe the similarity between resistance and conductance matrices given by (2.12) and (2.31), and between capacitance and elastance matrices, given by equations (2.13) and (2.32), respectively. Thus, the congruence transformation

$$Z' = A^t Z A \tag{2.33}$$

can be applied to give the network of Figure 3.b with impedance matrix Z'.

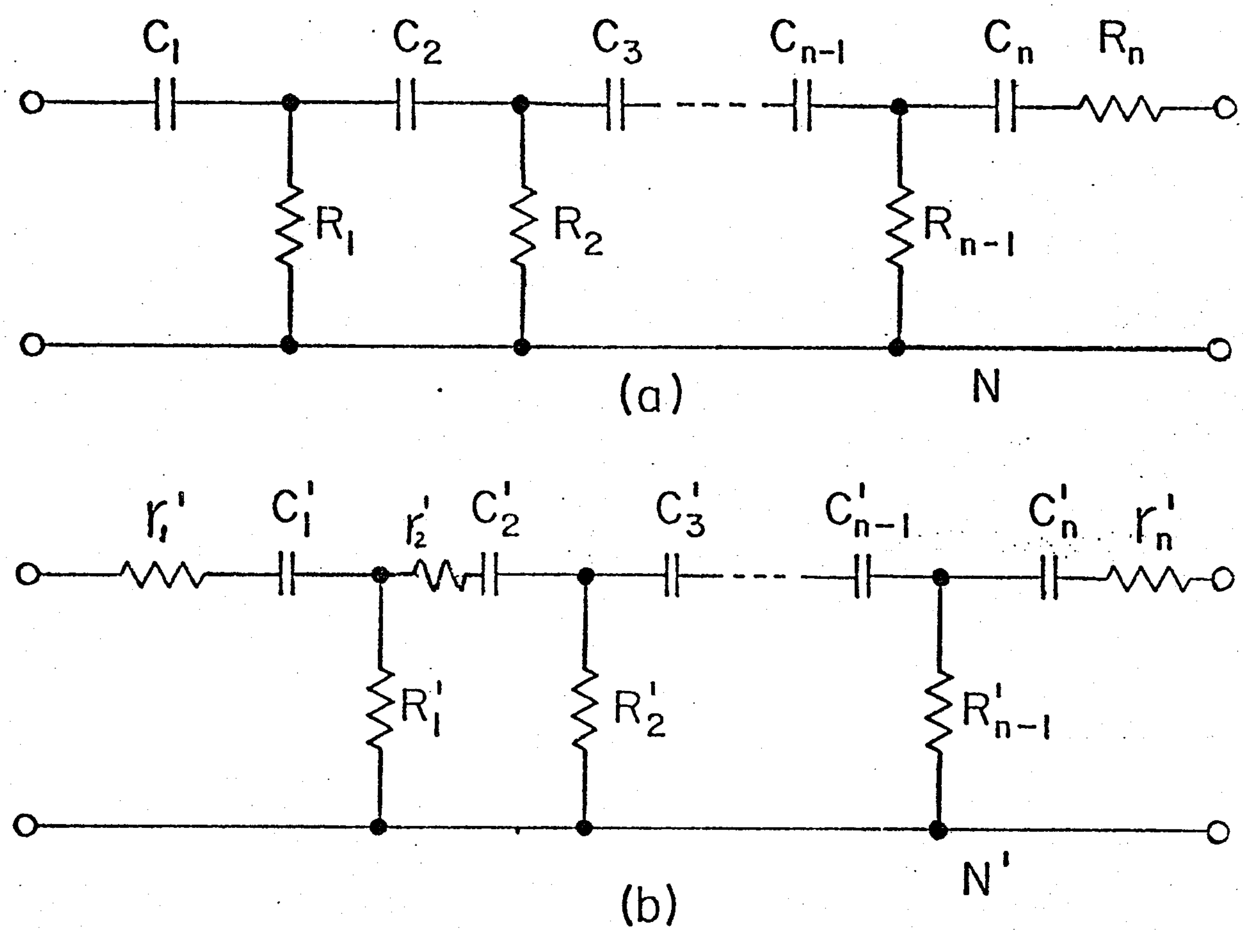


Figure 3 High-pass realization of specified (y_{11}, y_{12}).

(a) Canonic network.
(b) Equivalent network.

The element values can be obtained by using the results of the low-pass ladder of Figure 1, with conductances replaced by resistances according to (2.25) and (2.26) and capacitances replaced by elastances according to (2.27).

The element values of the equivalent high-pass ladder of Figure 3 are given by

$$R_k' = a_k\, a_{k+1}\, R_k \qquad k = 1, 2, \ldots, n-1 \qquad (2.34)$$

$$r_k' = a_k^2\, \{R_k\, (1 - \frac{a_{k+1}}{a_k}) - R_{k-1}(\frac{a_{k-1}}{a_k} - 1)\} \qquad k = 1, 2, \ldots, n \qquad (2.35)$$

$$C_k' = \frac{1}{a_k^2}\, C_k \qquad k = 1, 2, \ldots, n \qquad (2.36)$$

and

$$a_o = a_{n+1} = R_o = R_{n+1} = 0$$

where R_k and C_k are the elements of the Cauer second form shown in Figure 3.a and the transformation parameters a_k are to be chosen. It should be noted that throughout the transformation of (2.33) y_{11} remains unchanged; y_{12} is maintained within a multiplicative constant.

2.4.2 z_{11}, z_{12} are specified

In the previous section the topological duality existing between low and high pass ladders was used to find the element values of the transformed high-pass ladder. The GC:CG transformation [26] can be used to derive equivalent networks for the high-pass ladder shown in Figure 4.a, while preserving the open-circuit impedance parameters z_{11}, z_{12}. The procedure can be viewed as an GC:CG transformation followed by a CG:GC transformation.

Let the specifications for the two-port be given in the form of the open-circuit impedance parameters as

$$z_{11a} = \frac{\Sigma_o^m a_i \; s^i}{\Sigma_o^p b_i \; s^i} \quad , \qquad z_{12a} = \frac{k \; s^m}{\Sigma_o^p b_i \; s^i} \tag{2.37}$$

The canonic realization of z_{11a} in the second Cauer form yields the high-pass ladder network of Figure 4.a. If we replace each conductance of G_i mhos with a capacitance of G_i farads, and each capacitance of C_i farads with a conductance of C_i mhos (GC:CG transformation [26]), we obtain the low-pass network of Figure 4.b. Correspondingly, we now have,

$$z_{11b} = \frac{1}{s} \; z_{11a} \; (\frac{1}{s}) \quad , \qquad z_{12b} = \frac{1}{s} \; z_{12a} \; (\frac{1}{s}) \tag{2.38}$$

Since Figure 4.a has the same structure as the low-pass ladder of Figure 1.a, we can apply the congruence transformation to obtain the network of Figure 4.c, which is characterized by

$$z_{11c} = \frac{1}{s} \; z_{11a} \; (\frac{1}{s}) \quad , \qquad z_{12c} = \frac{k}{s} \; z_{12a} \; (\frac{1}{s}) \tag{2.39}$$

Then, reapplying the CG:GC transformation to the network of Figure 4.c, we find the high-pass network depicted in Figure 4.d. That network has the same driving-point impedance and transmission characteristics as the original network

$$z_{11d} = z_{11a} \quad , \qquad z_{12s} = k \; z_{12a} \tag{2.40}$$

The element values of the equivalent network are found from (2.25)-(2.27) by replacing conductances with capacitances and capacitances with conductances as explained. The element values are then given by

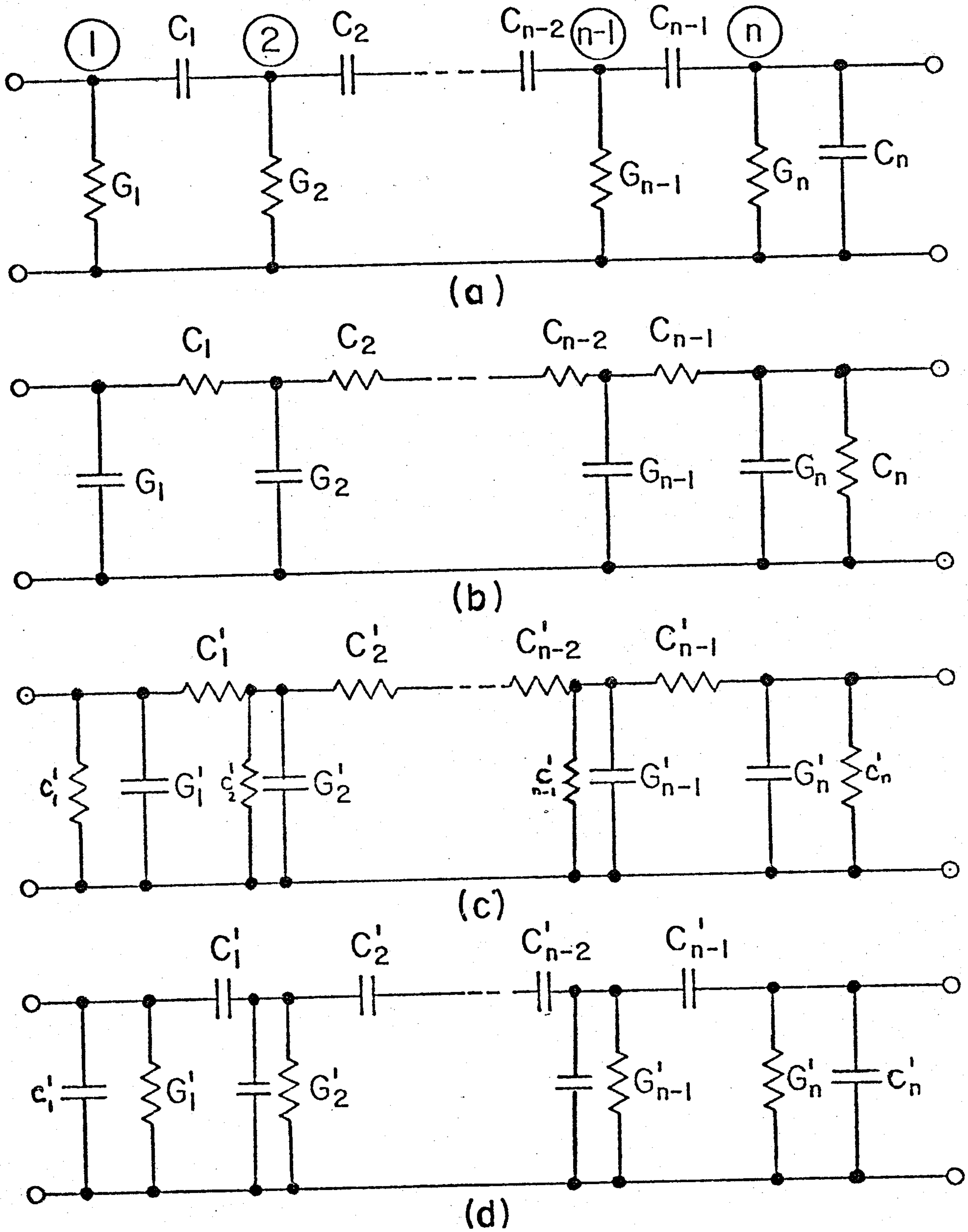


Figure 4 High-pass realization of specified (z_{11}, z_{12}).

(a) Canonic network

(b) Intermediate network obtained by GC:CG transformation.

(c) Equivalent network of (b).

(d) Equivalent network obtained by CG:GC Transformation.

$$G_k' = a_k^2 G_k \qquad k = 1, 2, \ldots, n \quad (2.41)$$

$$C_k' = a_k a_{k+1} C_k \qquad k = 1, 2, \ldots, n-1 \quad (2.42)$$

$$c_k' = a_k^2 \{C_k (1 - \frac{a_{k+1}}{a_k}) - C_{k-1} (\frac{a_{k-1}}{a_k} - 1)\} \qquad k = 1, 2, \ldots, n \quad (2.43)$$

where

$$a_o = a_{n+1} = C_o = C_{n+1} = 0.$$

It should be noted that the above results can be obtained by applying the congruence transformation to the admittance matrix of the original high-pass ladder and identifying the element values, as in the low-pass case.

2.5 Ladder Networks with Finite Zeros of Transmission

The method of generating equivalent networks described in the previous sections can also be applied to low-pass and high-pass ladders with finite zeros of transmission.

2.5.1 Mid-Series Networks

The node admittance matrix Y of the mid-series ladder network of Figure 5.a can be written as

$$Y = \begin{bmatrix} G+G_{12} & -G_{12} \\ -G_{12} & sC+G_{12} \end{bmatrix} \quad (2.44)$$

where

$$G_{12} = \text{diag}\ (G_{n+1}, G_{n+2}, \ldots, G_{2n}) \quad (2.45)$$

and G and C are the conductance and capacitance matrices given by (2.12) and (2.13), respectively.

If the design specifications are given as the open-circuit impedance parameters (z_{11}, z_{12}), the transformation of (2.9) can be applied with

$$A = \text{diag}\ (1, a_2, \ldots, a_n, 1, a_2, \ldots, a_n) \tag{2.46}$$

or in more compact notation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} \tag{2.47}$$

where

$$A_1 = \text{diag}\ (1, a_2, \ldots, a_n) \tag{2.48}$$

It should be noted that the symmetrical form of A given by (2.47) is chosen so that the transformed network of Figure 5.b has only n resistors connected across the shunt arms. More general structures can be found by using diagonal transformation matrices which are not symmetrical. (If A is not symmetrical, the transformed network will have n resistors across the capacitors and n resistors across the shunt arms of the ladder). The node admittance matrix Y of the equivalent network shown in Figure 5.b is given by

$$Y' = \begin{bmatrix} G'+G'_{12} & -G'_{12} \\ -G'_{12} & G'_{12}+sC' \end{bmatrix} \tag{2.49}$$

and from (2.9) and (2.47) the congruence transformation yields

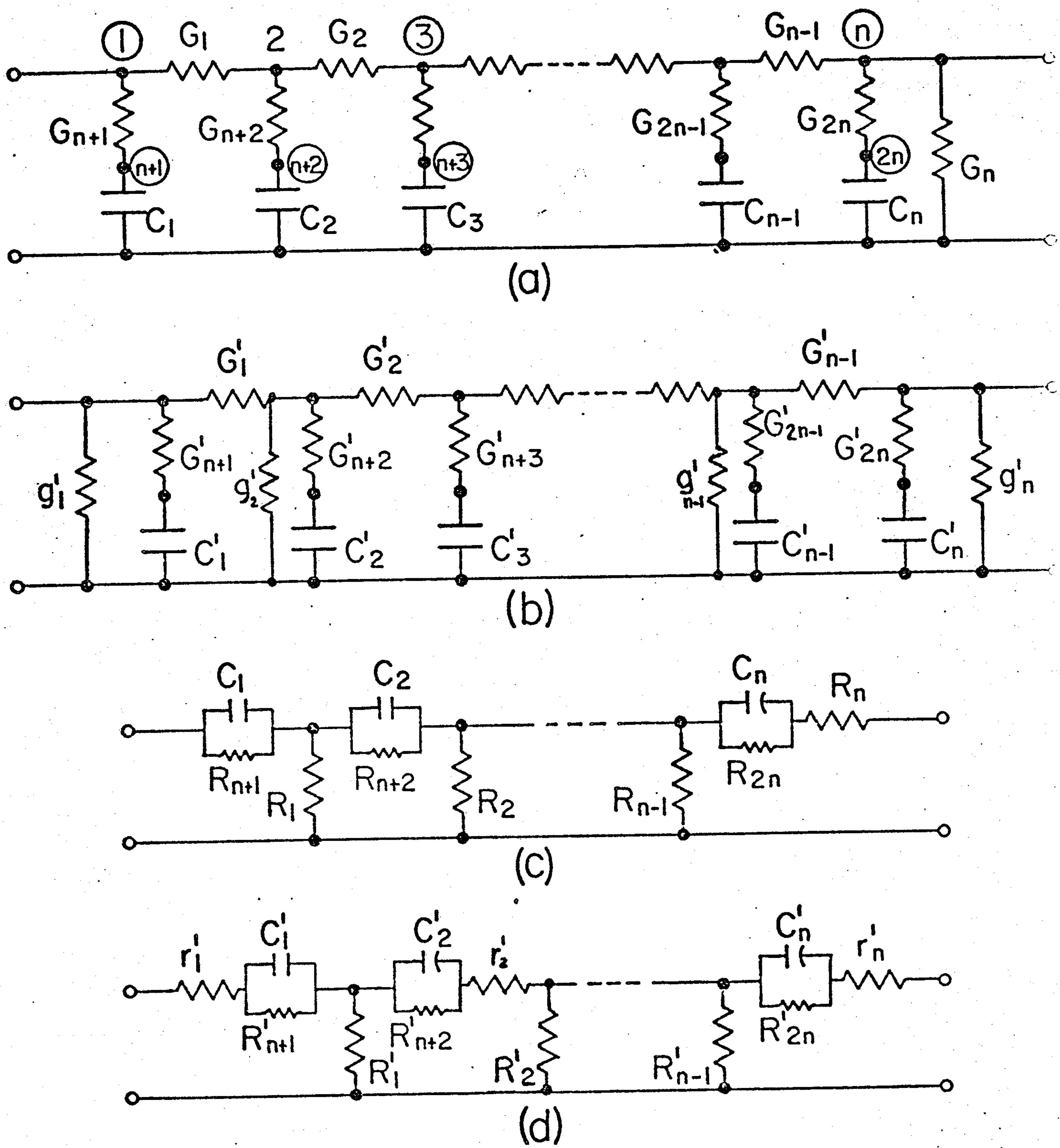


Figure 5 Ladder networks with finite zeros of transmission.

(a) A mid-series ladder network.
(b) Transformed equivalent network of (a).
(c) A mid-shunt ladder network.
(d) Transformed equivalent network of (c).

$$Y' = \begin{bmatrix} A_1^t GA_1 + A_1^t G_{12} A_1 & -A_1^t G_{12} A_1 \\ \\ -A_1^t G_{12} A_1 & A_1^t G_{12} A_1 + sA_1^t CA_1 \end{bmatrix} \tag{2.50}$$

from which we can identify

$$G' = A_1^t GA_1 \tag{2.51}$$

$$C' = A_1^t CA_1 \tag{2.52}$$

$$G'_{12} = A_1^t G_{12} A_1 \tag{2.53}$$

Since G' and C' are similar to the conductance and the capacitance matrices of the low-pass ladder, the new element values are given by (2.25)-(2.27) together with

$$G'_{n+k} = a_k^2 \, G_{n+k} \qquad k = 1, 2, \ldots, n \tag{2.54}$$

If the design specifications are given in terms of the short-circuit admittance parameters (y_{22}, y_{12}), a procedure similar to that used in section 2.3.2 can be followed to find equivalent networks.

2.5.2 Mid-Shunt Networks

The mid-shunt ladder network shown in Figure 5.c is the topological dual of the mid-series ladder network of Figure 5.a. This relationship can be used to find element values of the equivalent ladder of Figure 5.d for specified admittance parameters (y_{11}, y_{12}). Element values can also be derived by applying the transformation method used in section 2.5.1, but in this case it is based on impedance analysis. A similar development can be made for the case where (z_{11}, z_{12}) are specified.

2.6 RL, LC Ladders

By using RC:RL and RC:LC transformations [26] the techniques developed in this chapter can easily be implemented to RL and LC ladder networks.

CHAPTER III

COST FUNCTION MINIMIZATION IN RC LADDER NETWORKS

3.1 Introduction

In the realization of a certain transfer function $T(s)$, it is often the case that a pair of two-port parameters, viz., (z_{11}, z_{12}) or (y_{22}, y_{12}) or both are specified. If the poles of $T(s)$ are not located on the negative real axis, decomposition and partitioning of $T(s)$ is necessary for RC realization, and the resulting network is the interconnection of RC subnetworks with active elements properly embedded. There are two distinct cases: (a) decomposition arbitrary within RC realizability, and (b) decomposition under a constraint due to a design criteria such as sensitivity minimization [27]-[29]. In case (a) there is a degree of freedom, viz., the choice of the denominator polynomials, in identifying the two-port parameters, while in case (b) the parameters are specified as a result of the constraint.

The primary objective of this chapter is to propose a systematic method of minimizing the cost function in the RC ladder realization of case (b) where the two-port parameters are given as specified. The important components in the cost function are considered to be the total resistance R_T, total capacitance C_T, and the gain constant K. These factors are particularly relevant in integrated-circuit design. In certain fabrications, an important design specification is the minimization of the chip area which is proportional to the total resistance and the total capacitance. Accordingly, if a farad of capacitance requires W times as much area to realize as an ohm of resistance then the objective would be to realize a network with resistors R_i for $i = 1, 2, \ldots, \ell$ and capacitors C_j for $j = 1, 2, \ldots, p$ such that the weighted sum $\sum_{1}^{\ell} R_i + W \sum_{1}^{p} C_j$ is minimum. As an example, for silicon integrated circuits, typical values of resistance and capacitance per unit area are 800 Ω/mil^2 and 0.4 pF/mil^2, respectively [30], [31], yielding a W of 2 $\text{K}\Omega/\text{pF}$.

Resistance and capacitance minimization in RC circuits has been studied by Hagopian and Frisch [32], [33], who identified the network structures yielding minimum R_T or minimum C_T in RC driving-point function realizations. Minimization of the product R_TC_T was investigated by Protonatarios [34]. He has given necessary and sufficient conditions for the minimum value of R_TC_T in the ladder realization of an open-circuit low-pass voltage transfer function.

Recently, through the optimal choice of denominator polynomial, Stein and Salama [35] have given the conditions for low-pass RC ladder networks that realize a voltage transfer function with minimum weighted sum of resistance and capacitance. In this chapter we formulate the following problem of practical use: given the RC two-port parameters, specified in accordance with certain design criteria, such as sensitivity minimization, realize the two-port RC ladder network which also minimizes the duly defined cost function. For the solution, we first realize the two-port in a canonic form, and use it as the reference network, which will be subjected to a proper transformation to generate equivalent networks. Two distinct cases will be treated; first, the transformed network is canonic and second, the transformed network has one more element than the canonic realization. It will be shown that for a reference network with n independent nodes, there exists n canonical realizationsfor which the total resistance, total capacitance and the open circuit voltage gain monotonically vary between two limits. If the equivalent network is allowed to have one more element than the canonic realization, a one parameter family of equivalent ladder networks can be generated such that the cost function continuously vary with respect to a single parameter and the optimum network can be uniquely found. The analysis will be carried out for low-pass ladder networks and results will then be extended to other ladder structures using the techniques of the previous chapter.

3.2 Low-Pass Ladder Networks

3.2.1 Equivalent Canonic Networks

The low-pass ladder of Figure 1.a is the Cauer canonic realization of the open-circuit impedance parameters given by (2.10). Figure 1.b, shows the equivalent realization found from the canonic realization by application of the congruence transformation of (2.9) and (2.17).

Let us assume that the conductances satisfy the condition:

$$g_k = 0 \qquad k = 1, 2, \ldots, n; \quad k \neq i \tag{3.1}$$

then as a consequence (Appendix B), we have

$$\begin{aligned} a_k &= 1 & k &= 1, 2, \ldots, i \\ a_k &= \frac{\sigma_k}{\sigma_i} & k &= i + 1, \ldots, n \end{aligned} \tag{3.2}$$

where

$$\sigma_k = \sum_{j=k}^{n} R_j \tag{3.3}$$

and

$$R_j = \frac{1}{G_j}$$

The canonic network N_i with a shunt conductance at the i-th node is dipicted in Figure 6.a. The element values of N_i can be found by substituting (3.2) in (2.25)-(2.27) and we obtain

$$G_k' = G_k \qquad k = 1, 2, \ldots, i-1$$

$$G_k' = \frac{\sigma_k \sigma_{k+1}}{\sigma_i^2} G_k \qquad k = i, \ldots, n-1 \tag{3.4}$$

$$g_k' = \frac{1}{\sigma_i}$$

$$C_k' = C_k \qquad k = 1, 2, \ldots, i$$

$$C_k' = (\frac{\sigma_k}{\sigma_i})^2 C_k \qquad k = i+1, \ldots, n \tag{3.5}$$

It is interesting to note that in terms of the driving-point impedance the n-canonic networks N_i, i=1, 2, ...,n constitute a family of equivalent ladder networks where the first and the last networks N_1 and N_n are the networks of minimum total capacitance and minimum total resistance, respectively [32]. Through a straightforward algebraic manipulations, the total $R_T(i)$ and the total capacitance $C_T(i)$ of N_i are given by

$$R_T(i) = \sum_{k=1}^{n-1} (\frac{1}{G_k'} + \frac{1}{g_k'})$$

$$= \sum_{k=1}^{i-1} R_k + \sum_{k=i}^{n-1} \frac{R_i \sigma_i^2}{\sigma_k \sigma_{k+1}} + \sigma_i \tag{3.6}$$

which can further be arranged as

$$R_T(i) = R_T\min + (\frac{\sigma_i}{\sigma_n} - 1) \sigma_i \qquad i=1, 2, \ldots, n \tag{3.7}$$

and $R_T(i)$ assumes its minimum value $R_T\min$ for $i = n$ and

$$C_T(i) = \sum_{k=1}^{i} C_k + \sum_{k=i+1}^{n} a_k^2 C_k$$

$$= \sum_{k=1}^{i} C_k + \sum_{k=i+1}^{n} \left(\frac{\sigma_k}{\sigma_i}\right)^2 C_k \qquad i=1, 2, \ldots, n \tag{3.8}$$

The open circuit voltage gain is given by [36] as

$$H = \frac{V_o'}{V_{in}'} = \frac{z_{12}'}{z_{11}'} \tag{3.9}$$

$$H = \frac{z_{12}}{z_{11}} \cdot \frac{1}{a_n} \tag{3.10}$$

Since the gain of the reference network is given by $\frac{z_{12}}{z_{11}}$, the relative gain of the transformed network can be defined as

$$K = \frac{1}{a_n} \tag{3.11}$$

and with (3.2) the relative gain is given by

$$K(i) = \frac{\sigma_i}{\sigma_n} \tag{3.12}$$

From (3.7), (3.8) and (3.12) it is clear that the total resistance and the gain monotonically decrease as the integer i increases while the total capacitance increases with i as shown in Figure 6. The cost function $F(i)$, corresponding to the i-th network, can be formed to include the three components as

$$F(i) = R_T(i) + W_1 C_T(i) + \frac{W_2}{K(i)} \tag{3.14}$$

where W_1 and W_2 are weighing factors to be properly chosen to meet

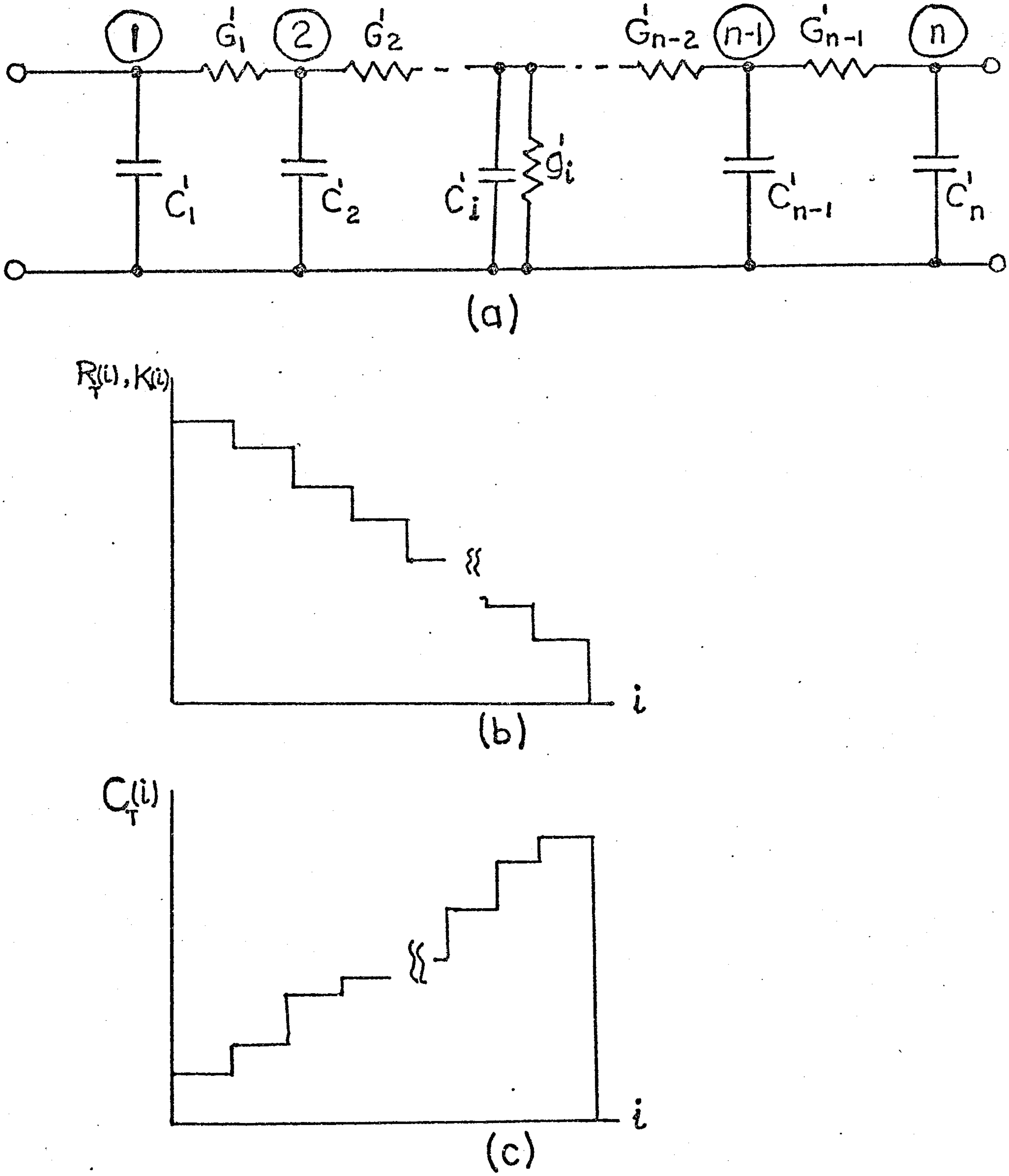


Figure 6 Canonic low-pass realizations.

(a) The i-th canonic network N_i.
(b) Total resistance and gain of N_i.
(c) Total capacitance of N_i.

design specifications and application. It should be noted that the term $\frac{W_2}{K(i)}$ is incorporated with a view that it is desirable to have higher gains in the passive subnetworks. However, the choice W_2=o reduces the cost function of (3.14) to the weighted sume of total resistance and total capacitance and the problem of minimizing cost function is reduced to finding the equivalent network of the minimum weighted sum.

The optimum canonic network can be readily obtained by finding the integer i such that

$$F_{min.} = \text{Minimum of } \{F(1), F(2), \ldots, F(n)\} \qquad (3.15)$$

which requires n evaluations. The element values of the optimum network can be found from (3.4) and (3.5) as illustrated in Example 3.6.1.

3.2.2 One-Parameter Family of Equivalent Networks

It has been shown, in the last section, that the canonic network N_1 is the minimum total capacitance realization. The element values of N_1 are found by subjecting the reference network to the transformation (2.9) with $a_k = a_k$ min. where

$$a_k\text{min.} = \frac{\sigma_k}{\sigma_1} \qquad (3.16)$$

In fact, $a_k$min. is the minimum value that can be assigned to the transformation parameters while maintaining the elements of the transformed network non-negative (Appendix C). In order to continuously generate the equivalent networks we introduce a variable β such that

$$A(\beta) = (1 - \beta) A_{min} + \beta I_n \qquad (3.17)$$

where I_n is the unity matrix of order n and

$$A_{min} = \text{diag } (1, a_2\text{min}, \ldots, a_n\text{min}) \qquad (3.18)$$

and

$$0 \leq \beta \leq 1 \tag{3.19}$$

The entries to the transformation matrix $A(\beta)$ are then written as

$$a_k(\beta) = \frac{\sigma_k}{\sigma_1} + \beta(1 - \frac{\sigma_k}{\sigma_1}) \tag{3.20}$$

The congruence transformation (2.9), with the transformation matrix $A(\beta)$ given by (3.17), continuously generates a one-parameter family of equivalent ladder networks which have an additional resistor across the input terminals. The substitution of (3.20) into (2.25) - (2.27) gives

$$\begin{aligned}
C_k'(\beta) &= [\frac{\sigma_k}{\sigma_1} + \beta(1 - \frac{\sigma_k}{\sigma_1})]^2 C_k \qquad k = 1, 2, \ldots, n \\
G_k'(\beta) &= [\frac{\sigma_k}{\sigma_1} + \beta(1 - \frac{\sigma_k}{\sigma_1})][\frac{\sigma_{k+1}}{\sigma_1} + \beta(1 - \frac{\sigma_{k+1}}{\sigma_1})]G_k \qquad k = 1, 2, \ldots, n-1 \\
g_1'(\beta) &= \frac{1 - \beta}{\sigma_1} \\
g_n'(\beta) &= \beta[\frac{1}{\sigma_1} + \beta(\frac{1}{\sigma_n} - \frac{1}{\sigma_1})]
\end{aligned} \tag{3.21}$$

It is seen that a family of continuously equivalent networks in the form of Fig. 7C can be generated within (3.19) and subsequently, the total resistance, total capacitance, and the relative gain are expressed explicity as functions of the variable β as

$$\begin{aligned}
R_T(\beta) &= \sum_{k=1}^{n-1} \frac{1}{G_k'(\beta)} + \frac{1}{g_1'(\beta) + g_n'(\beta)} \\
&= \frac{\sigma_1 (\sigma_1 - \sigma_n)}{\sigma_n + \beta(\sigma_1 - \sigma_n)} + \frac{\sigma_1 \quad \sigma_n}{\sigma_n + \beta^2(\sigma_1 - \sigma_n)}
\end{aligned} \tag{3.22}$$

$$C_T(\beta) = \sum_{k=1}^{n} C_k'(\beta)$$

$$= (1-\beta)^2 \sum_{k=1}^{n} \left(\frac{\sigma_k}{\sigma_1}\right)^2 C_k + 2\beta(1-\beta) \sum_{k=1}^{n} \left(\frac{\sigma_k}{\sigma_1}\right) C_k + \beta^2 \sum_{k=1}^{n} C_k \tag{3.23}$$

$$K(\beta) = \frac{\sigma_1}{\sigma_n + \beta(\sigma_1 - \sigma_n)} \tag{3.24}$$

It is noted that $\beta = 0$ and $\beta = 1$ correspond to the minimum total capacitance and minimum total resistance networks respectively. As β increases from zero to one, $C_T(\beta)$ increases monotonically while $R_T(\beta)$ and $K(\beta)$ decrease as depicted in Figure 7b. Since the network under consideration is of low-pass structure, the cost function $F(\beta)$ may be constructed as

$$F(\beta) = R_T(\beta) + w_1 C_T(\beta) + \frac{w_2}{K(\beta)} \tag{3.25}$$

where w_1 and w_2 are weighting factors the optimum network can be obtained by finding the value of β which minimizes $F(\beta)$. The condition

$$\frac{dF(\beta)}{d\beta} = 0 \tag{3.26}$$

results in an algebraic equation in β of order 7. It can be shown with (3.22) - (3.24) that

$$\frac{d^2F(\beta)}{d\beta^2} \quad 0 \tag{3.27}$$

for all β of $0 \leq \beta \leq 1$, and accordingly $F(\beta)$ has only one minimum at $\beta = \beta_o$. Three cases may be considered as shown in Figure 7.c.:

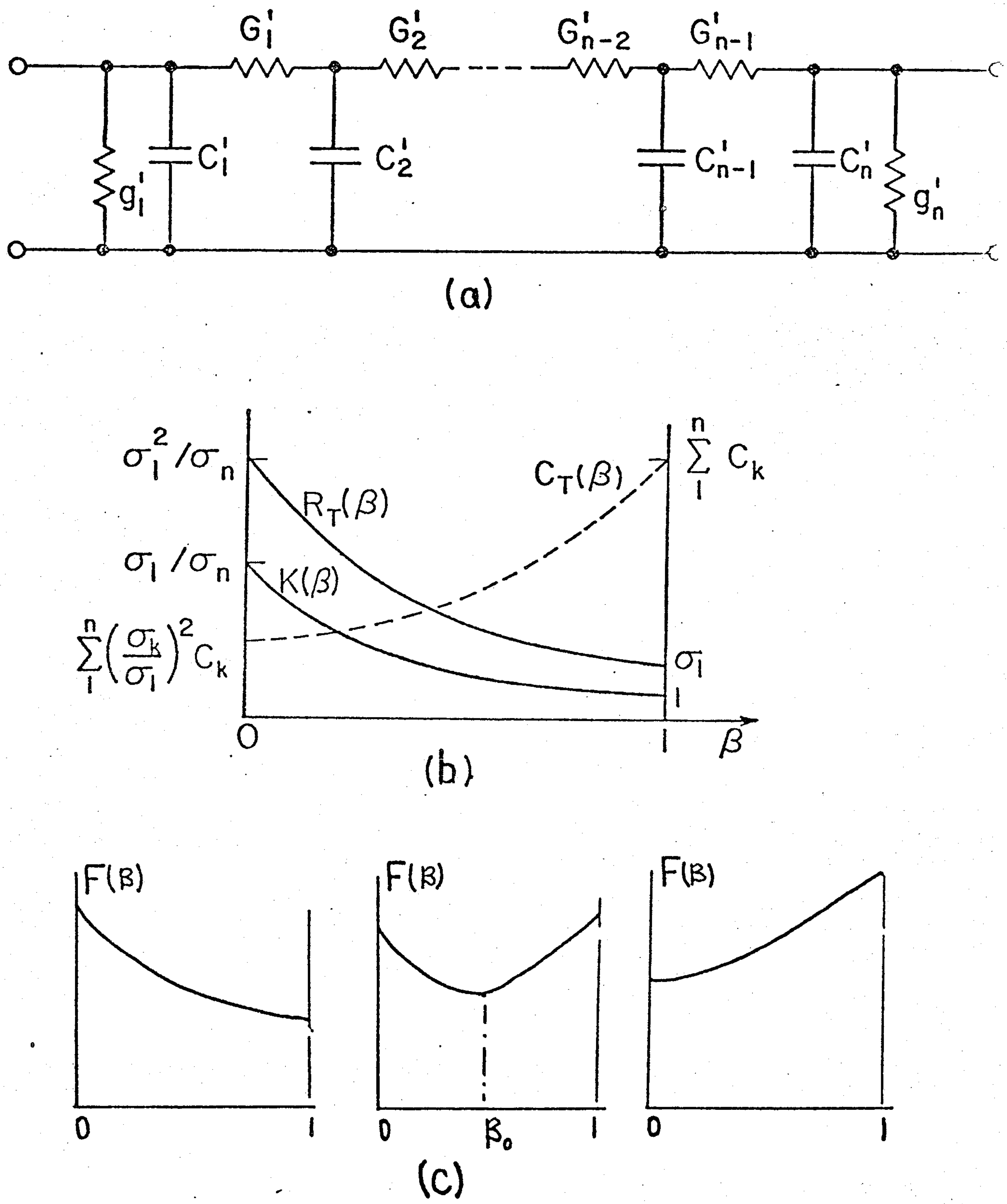


Figure 7 One-parameter family of low-pass networks.
(a) Equivalent network for cost function minimization.
(b) A typical plot of R_T, C_T and K as function of β.
(c) Cost function plot: the three different cases.

(i) $\beta_o \leq 0$; the optimum network corresponds to $\beta = 0$.

(ii) $0 < \beta_o < 1$; the optimum network corresponds to $\beta = \beta_o$.

(iii) $\beta_o \geq 1$; the optimum network is the reference network which corresponds to $\beta = 1$.

It should be noted that the derivation of explicit formulas for the element values of the optimum network is beyond easy access.
However, the optimum network can be readily obtained by finding the root of the 7-th order polynomial, resulting from (3.26), which lies in the finite interval $0 \leq \beta \leq 1$. Example 3.6.2 is provided to illustrate the proposed method.

Although the analysis in this section dealt with the realization of the impedance parameters, the outlined minimization technique, together with the results of section (2.3.2), can be applied to obtain the equivalent network of minimum cost when the specifications are given in terms of admittance parameters.

3.3 High-Pass Ladder Networks

When the RC two-port is specificed by the short-circuit admittance parameters of (2.29), the network is realized in the Cauer second form of Figure 8.a to yield the reference network. If the congruence transformation is applied to the Z-matrix, we then obtain analogous results to those of the low-pass case. A family of n-canonic networks and a one-parameter family of equivalent realizations can be found as follows.

3.3.1 Equivalent Canonic Networks

The canonic network N_i; $i = 1, 2, \ldots, n$ which differs from the reference network of Figure 3.a in having a resistance in series with the i-th capacitor, is shown in Figure 8.a. The entries to the transformation

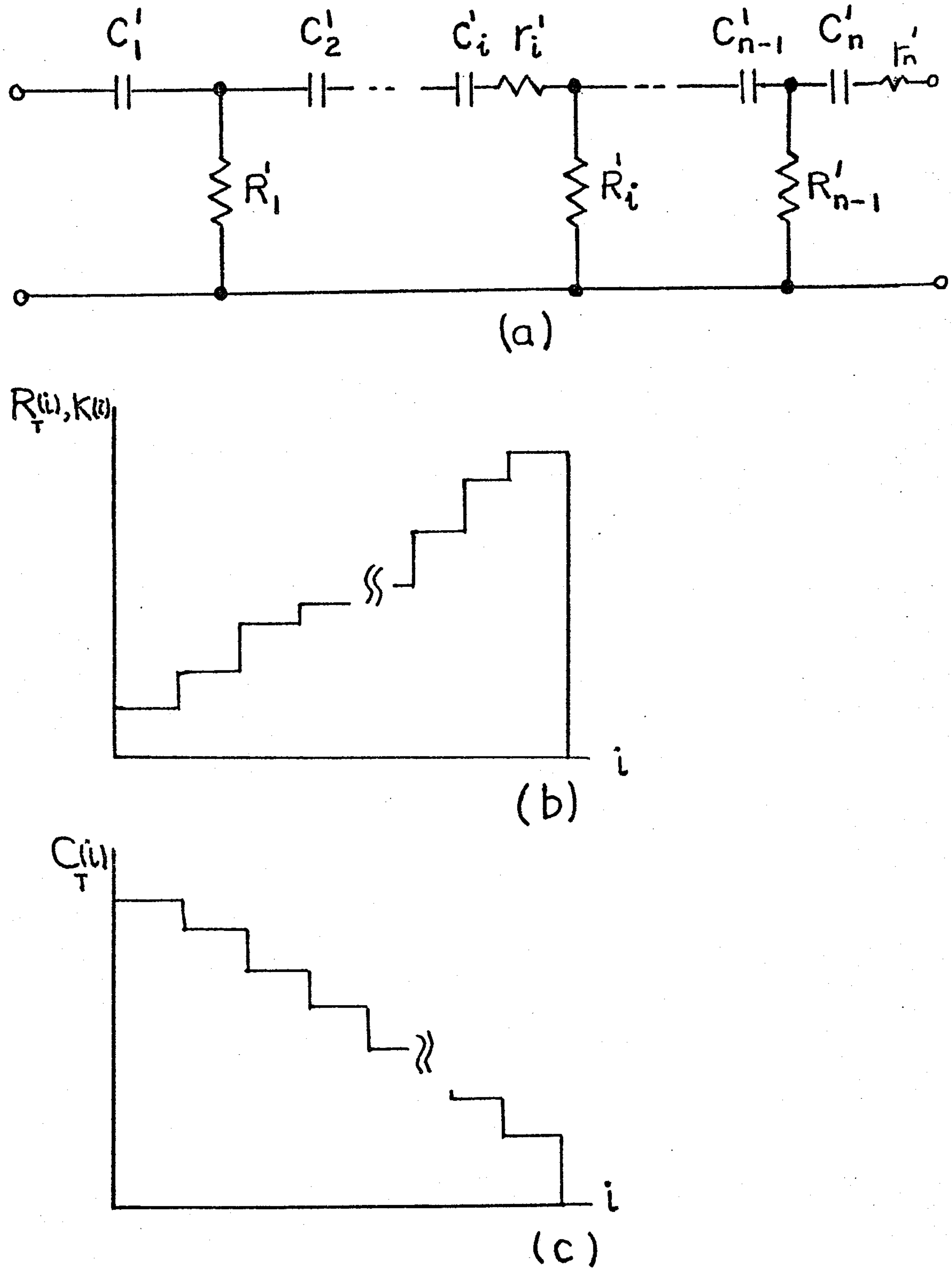


Figure 8 Canonic high-pass realizations.
(a) The i-th canonic network N_i.
(b) Total resistance and gain of N_i.
(c) Total capacitance of N_i.

matrix which yields the proposed family of networks can be found in an analogous way to the low-pass case and we have

$$a_k = 1 \qquad k = 1, 2, \ldots, i$$
$$= \frac{\theta_k}{\theta_i} \qquad k = i+1, \ldots, n \tag{3.28}$$

where

$$\theta_k = \sum_{j=k}^{n} G_j \tag{3.29}$$

$$G_j = \frac{1}{R_j}$$

Accordingly, the element values can be identified as

$$R_k' = R_k \qquad k = 1, 2, \ldots, i-1$$
$$R_k' = \frac{\theta_k \, \theta_{k+1}}{\theta_i^2} R_k \qquad k = i, \ldots, n-1 \tag{3.30}$$
$$r_i' = \frac{1}{\theta_i}$$

$$C_k' = C_k \qquad k = 1, 2, \ldots, i$$
$$C_k' = \left(\frac{\theta_i}{\theta_k}\right)^2 C_k \qquad k = i+1, \ldots, n \tag{3.31}$$

and the relative open circuit gain of the i-th canonic network is given by

$$K(i) = \frac{\theta_i}{\theta_n} \qquad i = 1, 2, \ldots, n \tag{3.32}$$

From the above equations, it is clear that R_k' increases with an increase of i, while C_k' decreases as i increases. The total resistance, total capacitance and the gain vary with i as shown in Figure 8. If a cost function is defined for the specific high-pass ladder network, the optimum network can then be found as in section 3.2.1.

3.3.2 One-Parameter Family of Equivalent Networks

When the two-port is specified by the short-circuit-admittance parameters of (2.29), we synthesize y_{11} in the second Cauer form as shown in Figure 3.a. As a consequence of the topological duality existing between the networks of Figure 1.a and Figure 3.a, the following expressions are directly derived from the results of section 3.2.2 for possible minimization of a cost function.

$$R_k' = [\frac{\theta_k}{\theta_1} + \beta(1 - \frac{\theta_k}{\theta_1})][\frac{\theta_{k+1}}{\theta_1} + \beta(1 - \frac{\theta_{k+1}}{\theta_1})]R_k \qquad k=1, 2, \ldots, n-1$$

$$r_1' = \frac{1 - \beta}{\theta_1}$$

$$r_n' = \beta[\frac{1}{\theta_1} + \beta(\frac{1}{\theta_n} - \frac{1}{\theta_1})] \tag{3.33}$$

$$C_k' = \frac{C_k}{[\frac{\theta_k}{\theta_1} + \beta(1 - \frac{\theta_k}{\theta_1})]^2} \qquad k = 1, 2, \ldots, n$$

$$G_T(\beta) = \frac{\theta_1(\theta_1 - \theta_n)}{\theta_n + \beta(\theta_1 - \theta_n)} + \frac{\theta_1 \theta_n}{\theta_n + \beta^2(\theta_1 - \theta_n)} \tag{3.34}$$

$$S_T(\beta) = (1 - \beta)^2 \sum_1^n (\frac{\theta_k}{\theta_1})^2 S_k + 2\beta(1 - \beta) \sum_1^n (\frac{\theta_k}{\theta_1}) S_k + \beta^2 \sum_1^n S_k \tag{3.35}$$

$$K(\beta) = \frac{\theta_1}{\theta_n + \beta(\theta_1 - \theta_n)} \tag{3.35}$$

where

$$S_k = \frac{1}{C_k}$$

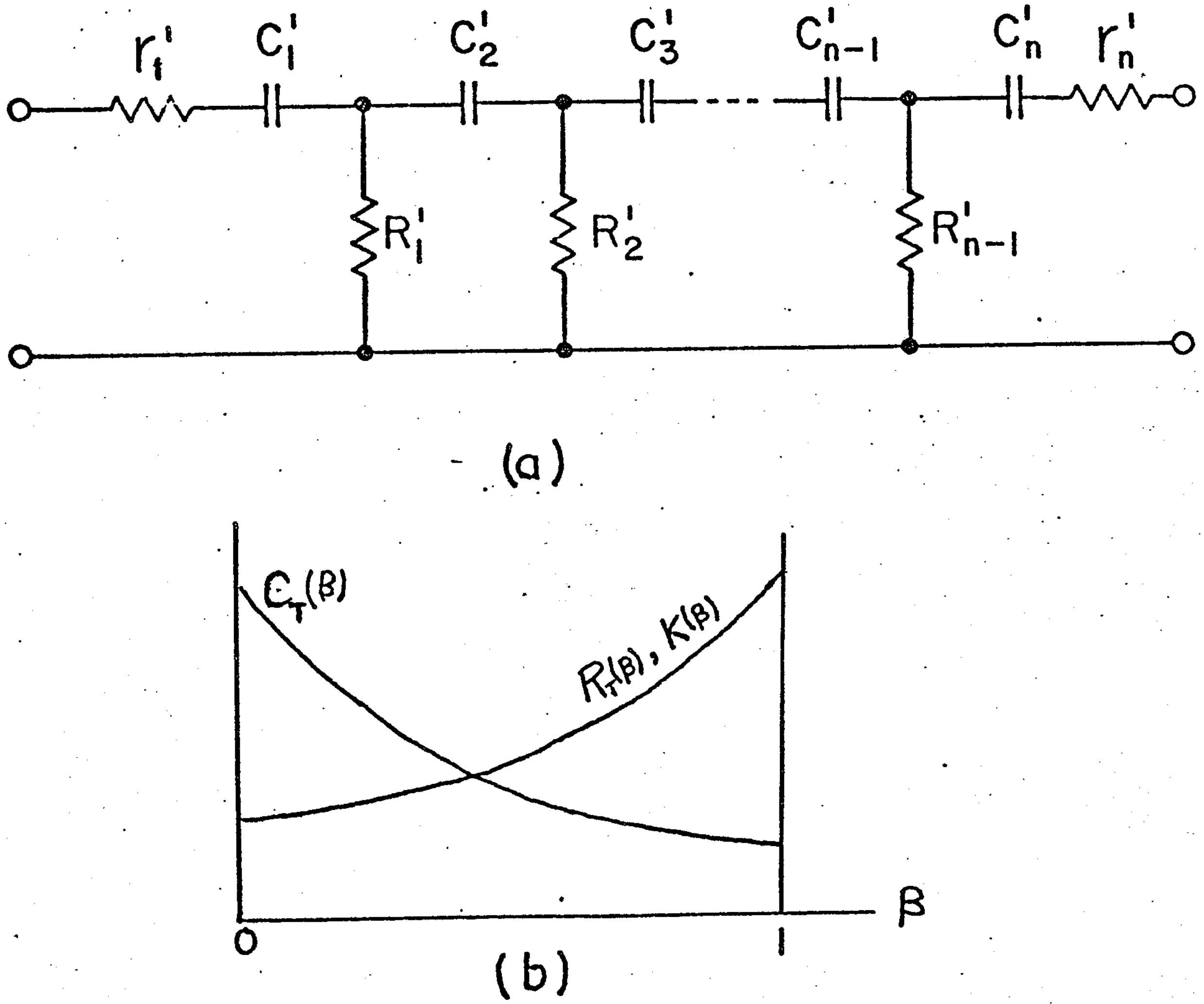


Figure 9 One-parameter family of high-pass networks.

(a) Equivalent network for cost function minimization.
(b) A typical plot of R_T, C_T and K as functions of β.

The equivalent network amenable to cost function minimization for the high-pass structure is shown in Figure 9.a, while the total resistance and total capacitance are plotted as functions of β in Figure 9.b.

If the specifications for the RC two-port are given in the form of the open-circuit impedance parameters of (2.37), the GC:CG:GC transformation, outlined in section 2.4.2, can be applied to obtain element values of the equivalent network and we then have,

$$R_k'(\beta) = \frac{R_k}{\left[\frac{\gamma_k}{\gamma_1} + \beta\left(1 - \frac{\gamma_k}{\gamma_1}\right)\right]^2} \qquad k = 1, 2, \ldots, n \tag{3.37}$$

$$C_k'(\beta) = \left[\frac{\gamma_k}{\gamma_1} + \beta\left(1 - \frac{\gamma_k}{\gamma_1}\right)\right]\left[\frac{\gamma_{k+1}}{\gamma_1} + \beta\left(1 - \frac{\gamma_{k+1}}{\gamma_1}\right)\right]C_k \qquad k=1,2,\ldots,n-1$$

$$c_1'(\beta) = \frac{1-\beta}{\gamma_1} \tag{3.38}$$

$$c_n'(\beta) = \beta\left[\frac{1}{\gamma_1} + \beta\left(\frac{1}{\gamma_n} - \frac{1}{\gamma_1}\right)\right]$$

where

$$\gamma_k = \sum_{j=k}^{n} \frac{1}{C_j} \tag{3.39}$$

From the above expressions it can be shown that the total resistance and total capacitance vary with β as depicted in Figure 10.b. Accordingly, a minimization technique similar to the one presented earlier in this chapter can be applied to a cost function properly formed for the specific high-pass ladder network.

3.4 Ladder Networks with Finite Zeros of Transmission

3.4.1 Mid-Series Networks

(a) Equivalent Canonic Networks

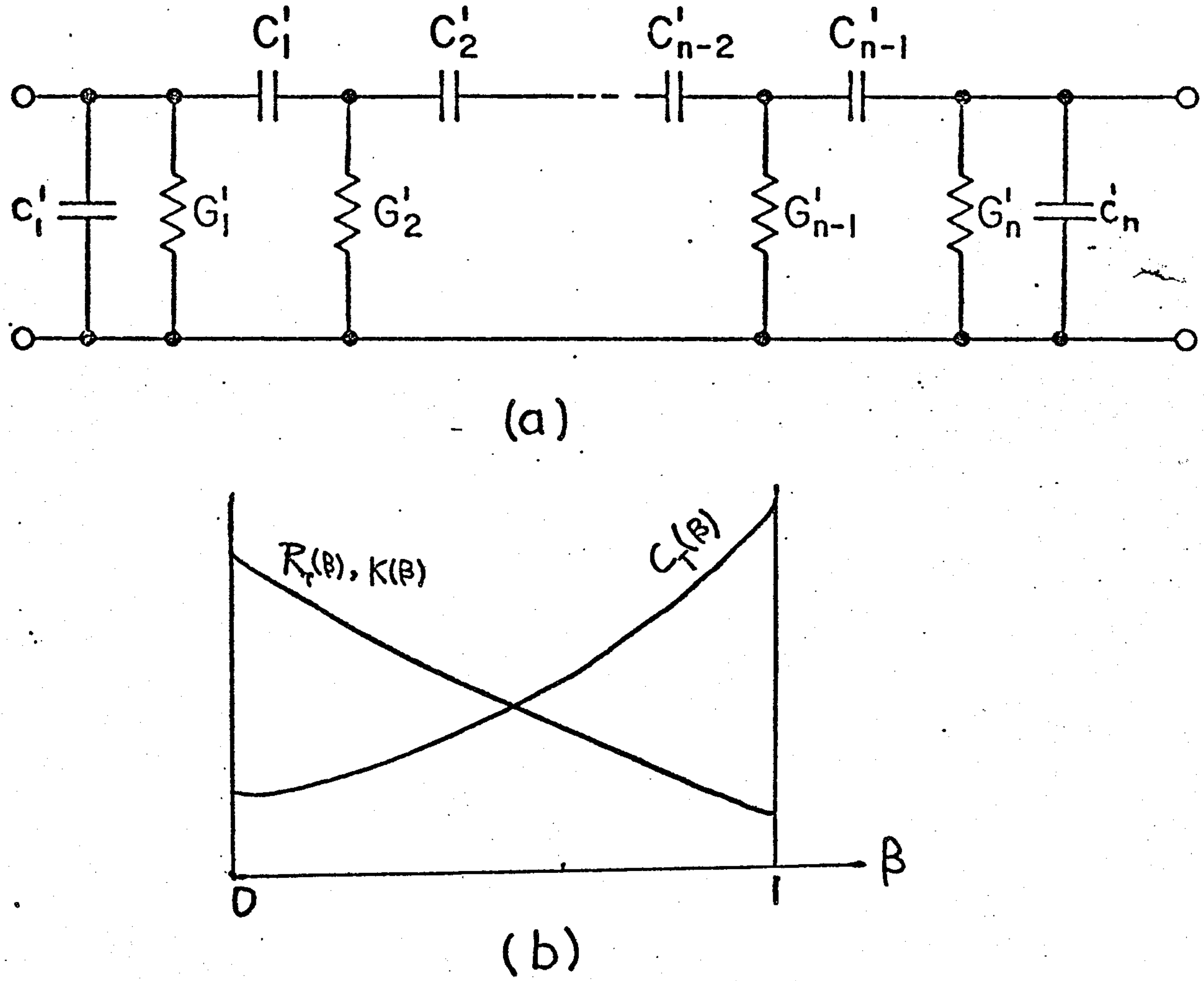


Figure 10 Equivalent high-pass realizations of (z_{11}, z_{12}).

(a) Equivalent network for cost function minimization.

(b) A typical plot of R_T, C_T and K.

The canonic mid-series ladder network is shown in Figure 5.a. If the design specifications are given in the form of the impedance parameters (z_{11}, z_{12}), the nodal admittance matrix Y of (2.49) is subjected to the congruence transformation of (2.47) in order to generate equivalent network. The matrix A is constructed in such a way that $g_k' = 0$ for all k except k=i. This transformation yields a family of equivalent canonic networks with shunt resistor at the i-th node as shown in Figure 11.a, and the element values are readily given by (3.4), (3.5) together with

$$G_{n+k}' = G_{n+k} \qquad k = 1, 2, \ldots, i$$

$$G_{n+k}' = \left(\frac{\theta_k}{\theta_i}\right)^2 G_{n+k} \qquad k = i+1, \ldots, n \qquad (3.40)$$

The total resistance, total capacitance and the gain of the i-th canonic network are plotted in Figure 6.b and Figure 6.c, respectively.

(b) <u>One-Parameter Family of Equivalent Networks</u>

Continuously equivalent networks can be found in the same manners as in low-pass networks. For transformed network shown in Figure 11.b we have

$$G_{n+k}' = \left[\frac{\sigma_k}{\sigma_1} + \beta\left(1 - \frac{\sigma_k}{\sigma_1}\right)\right]^2 G_{n+k} \qquad k = 1, 2, \ldots, n \qquad (3.41)$$

The element values are obtained from (3.16) and (3.41).

It should be noted that the total resistance, total capacitance and the gain of the mid-series network vary with β in a similar way as in the low-pass networks. Hence, cost function can be minimized accordingly.

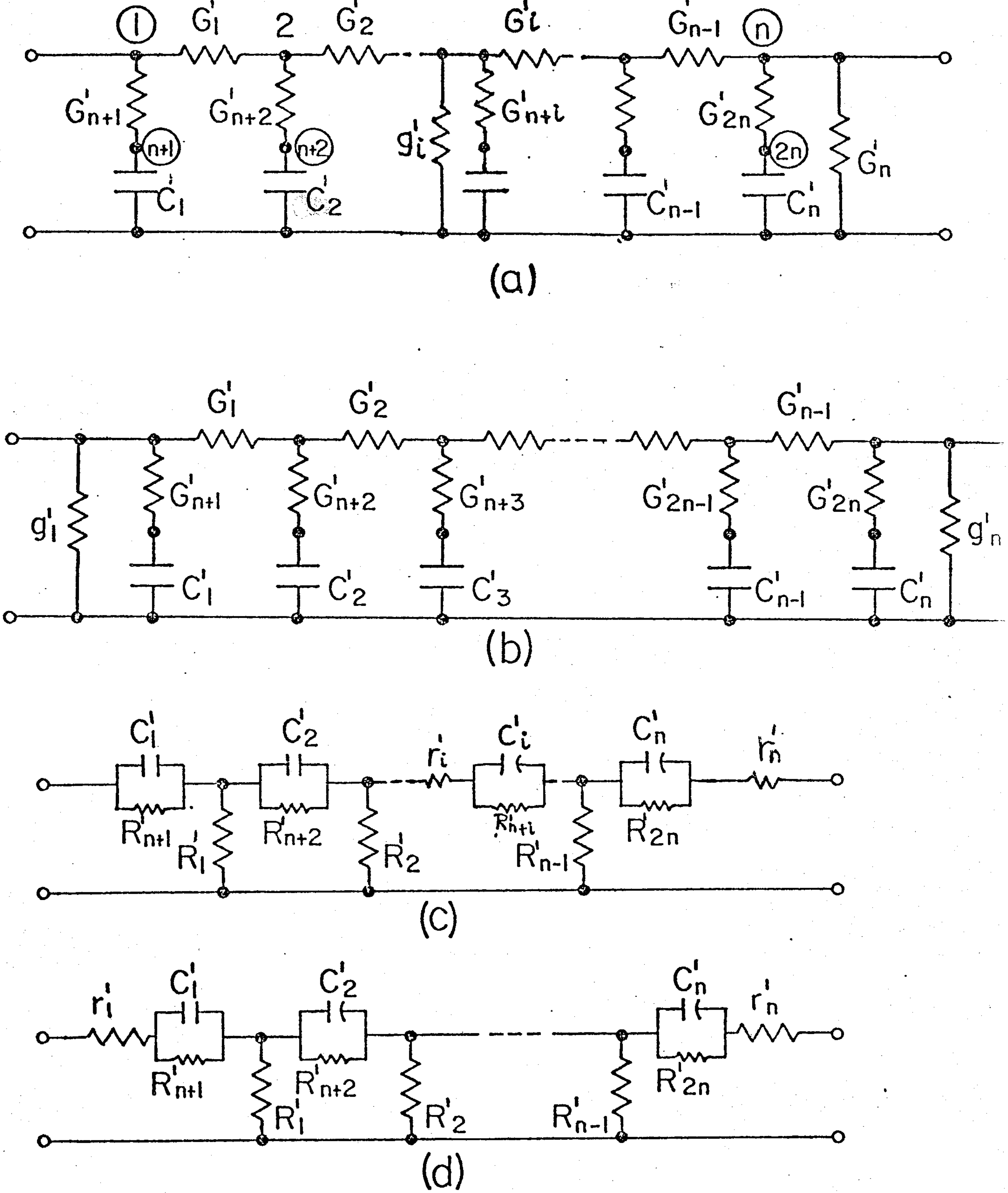


Figure 11 (a) The i-th canonic mid-series network.
(b) One-parameter family of equivalent mid-series networks.
(c) The i-th canonic mid-shunt network.
(d) One-parameter family of equivalent mid-shunt networks.

3.4.2 Mid-Shunt Networks

The mid-shunt ladder network is the topological dual of the mid-series ladder network. This relationship can be used, together with the results obtained for the high-pass structure to obtain n-canonic realizations and a one-parameter family of equivalent networks as follows.

(a) Equivalent Canonic Networks

The element values for the i-th canonic network of Figure 11.c are given by (3.30), (3.31) and

$$R'_{n+k} = R_{n+k} \qquad k = 1, 2, \ldots, i$$
$$R'_{n+k} = \left(\frac{\theta_k}{\theta_i}\right) R_{n+k} \qquad k = i+1, \ldots, n \tag{3.42}$$

The total resistance and total capacitance of the i-th network are shown in Figure 8.b and Figure 8.c, respectively.

(b) One-Parameter Family of Equivalent Networks

For the mid-shunt ladder of Figure 5.c, an equivalent one-parameter family of realizations are shown in Figure 11.d. It should be noted that the driving-point admittance y_{11} is kept invariant in the transformed network where

$$R'_{n+k}(\beta) = \frac{R_{n+k}}{\left[\frac{\theta_k}{\theta_1} + \beta\left(1 - \frac{\theta_k}{\theta_1}\right)\right]^2} \qquad k = 1, 2, \ldots, n \tag{3.43}$$

Element values are given by (3.33) and (3.43), while total resistance and total capacitance vary with β as in Figure 9.b.

3.5 Other Two-Element Kind Networks

It is noteworthy to observe that the interesting results obtained

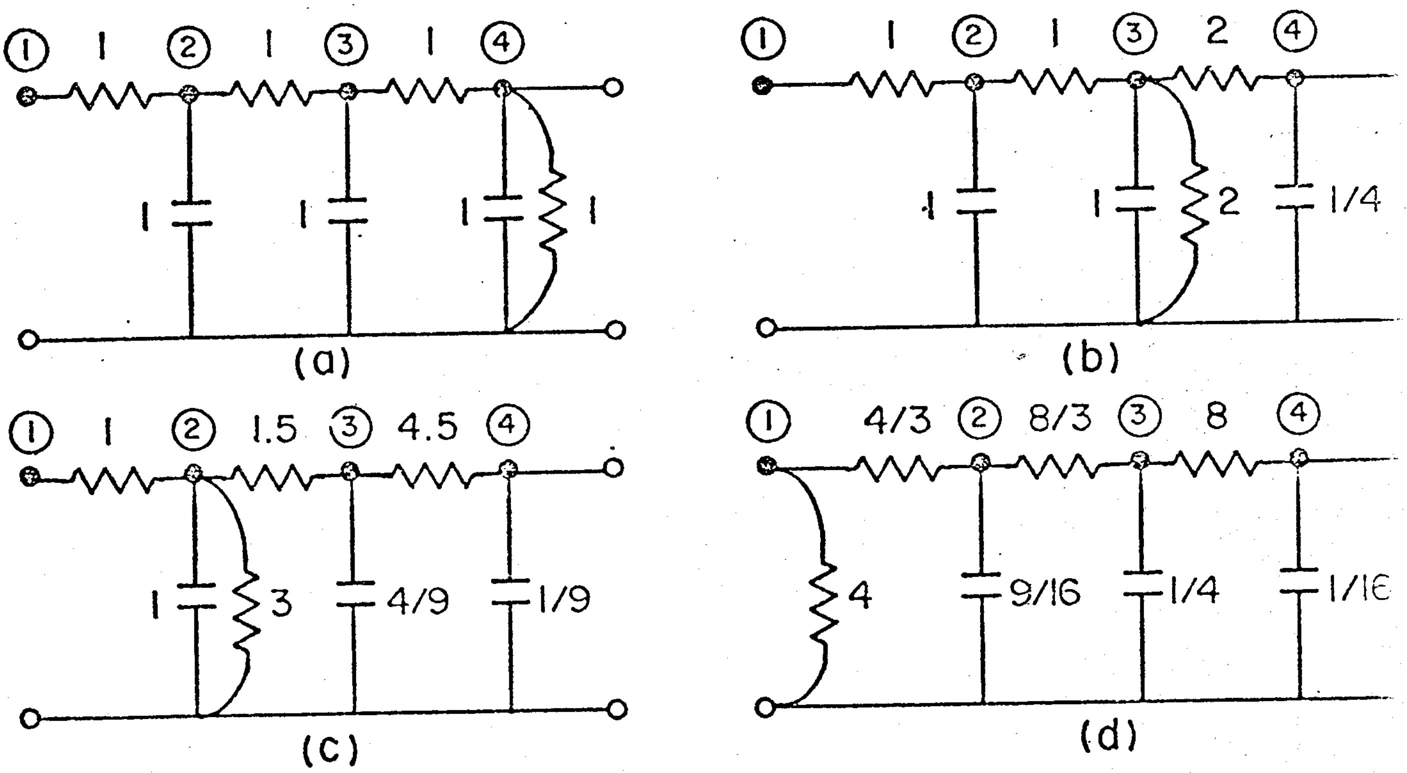


Figure 12 Networks in example 3.6.1

(a) The canonic network N_4.
(b) The canonic network N_3.
(c) The canonic network N_2.
(d) The canonic network N_1.

	$R_T(i)$	$C_T(i)$	$K(i)$
(a) i=4	4.000	3.000	1.000
(b) i=3	6.000	2.250	2.000
(c) i=2	10.000	1.556	3.000
(d) i=1	16.000	0.875	4.000

Table 1 Comparison of four canonic realizations of example 3.6.1

for RC ladder networks hitherto can easily be applied to RL and LC ladder networks by using the RC:RL and RC:LC transformations [26]. Consequently, the proposed method of cost function minimization is amenable to all two-element-kind networks, provided that a cost function suitable for the specific case is properly defined.

3.6 Examples

3.6.1 Example

Given

$$z_{11} = \frac{s^3 + 6s^2 + 10s + 4}{s^3 + 5s^2 + 6s + 1}, \qquad z_{12} = \frac{k}{s^3 + 5s^2 + 6s + 1},$$

four different realizations using the minimum number of elements are shown in Figure 12, and the total resistance, total capacitance and the relative gain for each network is tabulated in Table 1 for a direct comparison.

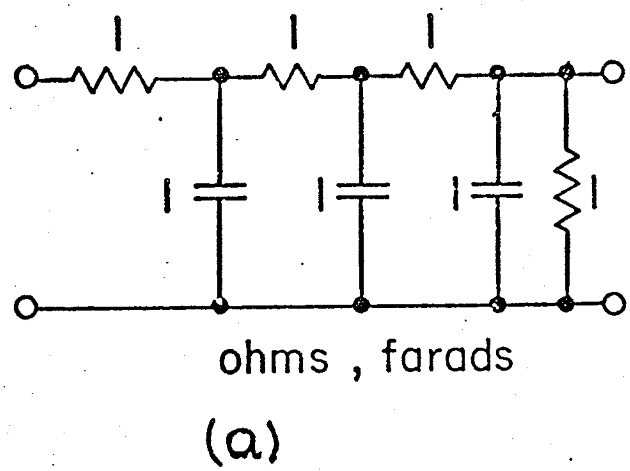
With the weighting factors of $W_1 = 5$ and $W_2 = 2$, for example, the network of Figure 12.b has the minimum cost function.

3.6.2 Example

If the open circuit impedance parameters of example 1 are specificed, we realize z_{11} in the first Cauer form as shown in Figure 13.a which is the reference network consisting of $R_1 = R_2 = R_3 = R_4 = 1.$, $C_1 = 0$, $C_2 = C_3 = C_4 = 1$. From (3.3) we calculate $\sigma_1 = 4$, $\sigma_2 = 3$, $\sigma_3 = 2$, $\sigma_n = \sigma_4 = 1$ and using (3.22)-(3.25) the cost function is expressed as

$$F(\beta) = \frac{12}{1 + 3\beta} + \frac{4}{1 + 3\beta^2} + W_1 \{\frac{7}{8}(1 - \beta)^2 + 3\beta(1 - \beta) + 3\beta^2\}$$

$$+ W_2 \{\frac{1 + 3\beta}{4}\}.$$



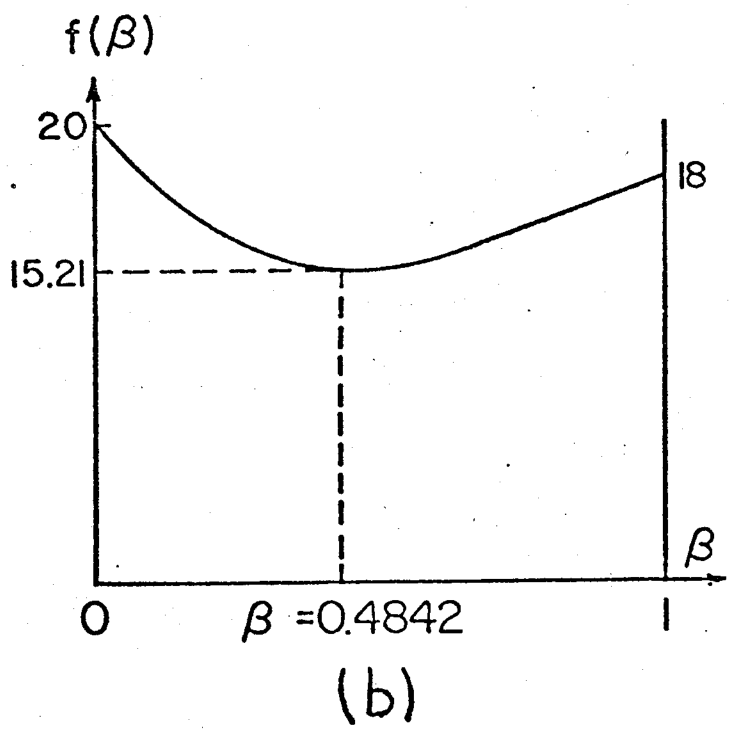


Figure 13 Network in example 3.6.2

(a) The canonic realization.
(b) The plot of F(β).

For weighting factors of, say, $W_1 = 4$, $W_2 = 2$, the minimum of $F(\beta)$ occurs at $\beta_o = 0.4842$. The curve of $F(\beta)$ is shown in Figure 13.b and element values are then computed by (3.21) to be $\{c_2', c_3', c_4'\} = \{.76, .55, .38\}$, $\{R_1', R_2', R_3'\} = \{1.15, 1.55, 2.20\}$ and $\{g_1, g_4\} = \{.13, .30\}$, respectively.

3.7 Sensitivity Considerations

Once a network satisfying a given two post parameters has been realized in a canonical form, a family of equivalent newtworks can be generated as in the preceeding sections. The network with minimum cost function can also be found. It should be noted that the cost function can be formed to include as many design parameters as desired. Further more, if the transformed network is allowed to have more shunt resistors at the different nodes, a more general transformation that has (n-1) independant parameters β_i, i = 2, ..K can be constructed as in Chapter V. It can be easily shown that the one parameter family of equivalent networks presented in this chapter is a special case of the transformation presented in Chapter V, and hence, if more shunt resistors are allowed, a network with smaller cost function can be obtained. In this section the dirving point function sensitivity will be found explicitly as a function of β, no partial differentiation is required. A sensitivity index can be chosen arbitrarily for the specific application and then, a cost function can be formed to include the sensitivity index. The driving point admittance sensitivity is defined as

$$S_{e_i}^{Yin} = \frac{\partial Yin}{\partial e_i} \frac{e_i}{Yin} \tag{3.44}$$

for the network element e_i and a sensitivity index can be defined as

$$\Phi = \sum_i \omega_i \, S^{Yin}_{ei} \, S^{*Yin}_{ei} \tag{3.45}$$

with the summation taken over all the network elements and S^{*Yin}_{ei} denotes the complex conjugate of S^{Yin}_{ei} and W_i the weight that represents the degree of importance of the i-th element.

3.7.1. Low Pass Networks

Let us consider the low Pan ledder of Fig. 1 (a). The node admittance matrix is given by

$$\begin{aligned} Y &= G + SC \\ &= \begin{array}{cc} \begin{array}{cc} (1) & (n-1) \end{array} & \\ \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} & \begin{array}{c} (n-1) \\ \\ \end{array} \end{array} \end{aligned} \tag{3.47}$$

and the driving point admittance is given by

$$Yin = Y_{11} - Y_{12}\, Y_{22}^{-1}\, Y_{21} \tag{3.48}$$

where Y_{11} is a single element, Y_{21} is (n-1)x1 vector, Y_{12} is the transpose of Y_{21} and Y_{22} and Y_{22} is an (n-1) x (n-1) matrix. From (2.12) and (2.13) we write

$$\begin{aligned} Y_{11} &= G_1 + SG \\ Y_{12} &= (-G_1 \quad 0 \quad 0 \; . \; . \; . \quad 0) \end{aligned} \tag{3.49}$$

$$Y_{22} = \begin{bmatrix} G_1 + G_2 + SC_2 & -G_2 & 0 \; . \; . \; . & 0 \\ -G_2 & G_2 + G_3 + SC_3 & -G_3 \; . \; . \; . & 0 \\ 0 & & & \\ \vdots & & \vdots & \\ 0 & & & G_{n-1} + G_n + SC_n \end{bmatrix}$$

From (3.44) - (3.49) the sensitivity can be obtained as

$$S_{ei}^{Yin} = \frac{ei}{Yin}\left(\frac{\partial Y_{11}}{\partial e_i} - \frac{\partial}{\partial e_i} Y_{21} Y_{22}^{-1} Y_{21}\right) \quad (3.50)$$

and hence,

$$S_{c_1}^{Yin} = \frac{c_1}{Yin} \frac{\partial Yin}{\partial c_1} \quad (3.51)$$

$$= \frac{c_1}{Yin} \frac{\partial Y_{11}}{\partial c_1}$$

$$S_{c_1}^{Yin} = \frac{sc_1}{Yin} \quad (3.52)$$

$$S_{G_1}^{Yin} = \frac{G_1}{Yin} \left\{\frac{\partial Y_{11}}{\partial G_1} - \frac{\partial Y_{12}}{\partial G_1} Y_{22}^{-1} Y_{21} + Y_{12} Y_{22}^{-1} \frac{\partial Y_{22}}{\partial G_1} Y_{22}^{-1} Y_{21} - Y_{12} Y_{22}^{-1} \frac{\partial Y_{21}}{\partial G_1}\right\} \quad (3.53)$$

and from (3.48) we have

$$\frac{\partial Y_{22}}{\partial G_i} = \begin{bmatrix} 1 & 0 \ . \ . \ . & 0 \\ 0 & & . \\ . & & . \\ . & & . \\ 0 & . \ . \ . & 0 \end{bmatrix} \quad (3.54)$$

where as

$$Y_{22}^{-1} = [\Delta ij] \qquad i,j = 1, 2, \ . \ . \ n-1 \quad (3.55)$$

Manipulating (3.49), (3.54) into and (3.55)

Yeilds

$$\frac{\partial Y_{11}}{\partial G_1} = 1$$

$$\frac{\partial Y_{12}}{\partial G_1} Y_{22}^{-1} Y_{21} = G_1 D_{11} \tag{3.56}$$

$$Y_{12} Y_{22}^{-1} \frac{\partial Y_{22}}{\partial G_1} Y_{22}^{-1} Y_{21} = G_1^{2} \Delta_{11}^{2}$$

substituting from (3.56) into (3.53) and arranging we get

$$S_{G_1}^{Y_1} = \frac{G_1}{Y_1} (1 - \Delta_{11} G_1)^2 \tag{3.57}$$

Similarly, for C_k we obtain

$$S_{c_k}^{Yin} = \frac{c_k}{Yin} (Y_{12} Y_{22}^{-1} \frac{\partial Y_{22}}{\partial c_k} Y_{22}^{-1} Y_{21})$$

where

$$\frac{\partial Y_{22}}{\partial c_k} = \begin{array}{cl} \begin{array}{cccccccc} & & & k-1 & & & & \end{array} & \\ \left[\begin{array}{cccccccc} 0 & \cdots & & & & & & 0 \\ 0 & & & \cdot & & & & 0 \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & & \cdot \\ 0 & 0 & \cdots & S & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & & & & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ k-1 \\ \\ \end{array} \end{array}$$

and simple manipulation gives

$$S_{c_k}^{Yin} = \frac{sc_k}{Yin} (G_1^{2} \Delta_{1\,k-1}^{2}) \qquad k = 2, 3, \ldots n-1 \tag{3.58}$$

Similar manipulations yeilds

$$S_{G_k}^{Yin} = \frac{G_k}{Yin} G_1^2 (\Delta_{1\,k-1} - \Delta_{1k})^2 \quad k = 2, 3, \ldots n-1 \tag{3.59}$$

and

$$S_{G_n}^{Yin} = \frac{G_n}{Yin} G_1^2 \Delta_{1n-1}^2 \tag{3.60}$$

Equations (3.52) and (3.57) - (3.60) give the different driving point function sensitivities for the canonic realization. Let us now find the sensitivities of the equivalent network in Figure 1 (b). From (3.52) - (3.60) we write

$$S^{Yin}_{c'_1} = \frac{sC_1}{Yin} \tag{3.61}$$

$$S^{Yin}_{G'_1} = \frac{G'_1}{Yin} - (1 - \Delta'_{11}\, G'_1)^2 \tag{3.62}$$

where

$$G'_k = \alpha_k\, \alpha_{k+1}\; G_k \qquad k = 1, 2, . . n-1$$

$$\Delta'_{jk} = \frac{\Delta jk}{\alpha_{j+1}\; \alpha_{k+1}} \tag{3.63}$$

and (3.62) can be expressed in terms of the elements of the canonic realization as

$$S^{Yin}_{Gi} = \frac{\alpha_2 G_1}{Yin}\; (1 - \frac{\Delta_{11} G_1}{\alpha_2})^{\;2} \tag{3.64}$$

similarly for C'_k and G'_k we have

$$S^{Yin}_{c'_k} = \frac{sC'_k}{Yin}\; (G'^{2}_1\; \Delta'^{2}_{1k-1}) \qquad k = 2, 3, . . n$$

$$S^{Y}_{c'_k} = \frac{sC_k}{Yin}\; .\; G^2_1\; \Delta^2_{1\;k-1} \qquad k = 2, 3, . . n \tag{3.65}$$

and

$$S^{Y}_{G'_k} = \frac{G_k G_1^2}{Yin} \alpha_2^2 \alpha_k \alpha_{k+1} \left(\frac{\Delta_{1k-1}}{\alpha_2 \alpha_k} - \frac{\Delta_{1k}}{\alpha_2 \alpha_{k+1}}\right)^2 \qquad (3.66)$$

$$= \frac{G_1^2 G_k}{Yin} \alpha_k \alpha_{k+1} \left(\frac{\Delta_{1k-1}}{\alpha_k} - \frac{\Delta_{1k}}{\alpha_{k+1}}\right)^2 \qquad k = 2, 3, \ldots n-1$$

$$S^{Y}_{g'_1} = \frac{g'_1}{Yin}$$

$$= \frac{(1 - \alpha_2)\, G_1}{Yin}$$

$$S^{Y}_{g'_k} = \frac{g_k}{Yin} (G_1^{12} \quad \Delta_{1k-1}^{12}) \qquad k = 2, 3, \ldots n-1$$

$$S^{Y}_{g'_k} = \frac{G_1^2 \Delta_{1\cdot k-1}^2}{Yin} \left[G_k \left(1 - \frac{\alpha_{k+1}}{\alpha_k}\right) - G_{k-1}\left(\frac{G_{k-1}}{\alpha_k} - 1\right)\right] \qquad (3.67)$$

$$k = 2, 3, \ldots n-1$$

$$S^{Y}_{g'_n} = \frac{G_1^2 \Delta_{1\; n-1}^2}{Yin} \left[G_n - G_{n-1} \left(\frac{\alpha_{n-1}}{\alpha_n} - 1\right)\right] \qquad (3.68)$$

It is interesting to note that the above sensitivity expressions are given for the general case with (n-1) transformation parameter α_k; α_k = 2, n. All the special cases can be easily drivied by substituting the corresponding values of α_k.

3.7.2 Example

Consider the reactive friving point admittance

$$Y(s) = \frac{6s^4 + 22s^2 + 18}{5s^3 + 12s}$$

which can be realized in Cauer's first form as shown in figure example 3.7.2 (a). The equivalent network of Figure 3.7.2 (b) was considered by Schoeffler [7] and Leon & Yokomoto [12].

For comparison, we also calculate sensitivities of the equivalent networks at $\omega = 3/2$. It should be noted that the analysis made for RC networks is applicable for LC networks simply replacing each conductance G_k by $\frac{\mathcal{L}_k}{s}$ where $\mathcal{L}_k$ is the inverse inductance and s is the complex frequency variable.

$$S^Y_{c_1'} = \frac{sC_1}{Y}$$

$$S^Y_{\mathcal{L}_1'} = \frac{a_2\,1}{sY}\left[1 - \frac{\mathcal{L}_1}{a_2(\mathcal{L}_1 + \mathcal{L}_2 + s^2c_2)}\right]^2$$

$$S^Y_{c_2^1} = \frac{s\;c_2\mathcal{L}_1^2}{Y(\mathcal{L}_1 + \mathcal{L}_2 + s^2C_2)^2}$$

$$S^Y_{\mathcal{L}_2^1} = \frac{\mathcal{L}_1^2\left[\mathcal{L}_2 - \mathcal{L}_1\left(\frac{1}{a_2} - 1\right)\right]}{sY(\mathcal{L}_1 + \mathcal{L}_2 + sC_2)^2}$$

$$S^Y_{\ell_1^1} = \frac{(1 - a_2)\mathcal{L}_1}{sY}$$

$$\phi = \Sigma\, S\overset{*}{S} = 13.48 + \frac{1.0258444}{a_2^2}(a_2 - .2105263)^4 +$$

$$.000837\ (115.52 - \frac{1.52}{a_2})^2 + 1.0268\ (1 - a_2)^2$$

which has a minimum at $a = .0235$

If we realize Zin in Cauer's second form we get the network shown in Figure Example 3.7.(d) and the equivalent network of Figure Example 3.7.(e) has the lower sensitivity index. It should be noted the optimum network obtained by Yokomoto [ ] has a sensitivity index of 18.34 while the network of Figure 3.7.(e) has a lower sensitivity index of 16.579. Table 2 summarizes the results in a matrix format.

$$Z_{in} = \frac{5S^3 + 12S}{6S^4 + 22S^2 + 18}$$

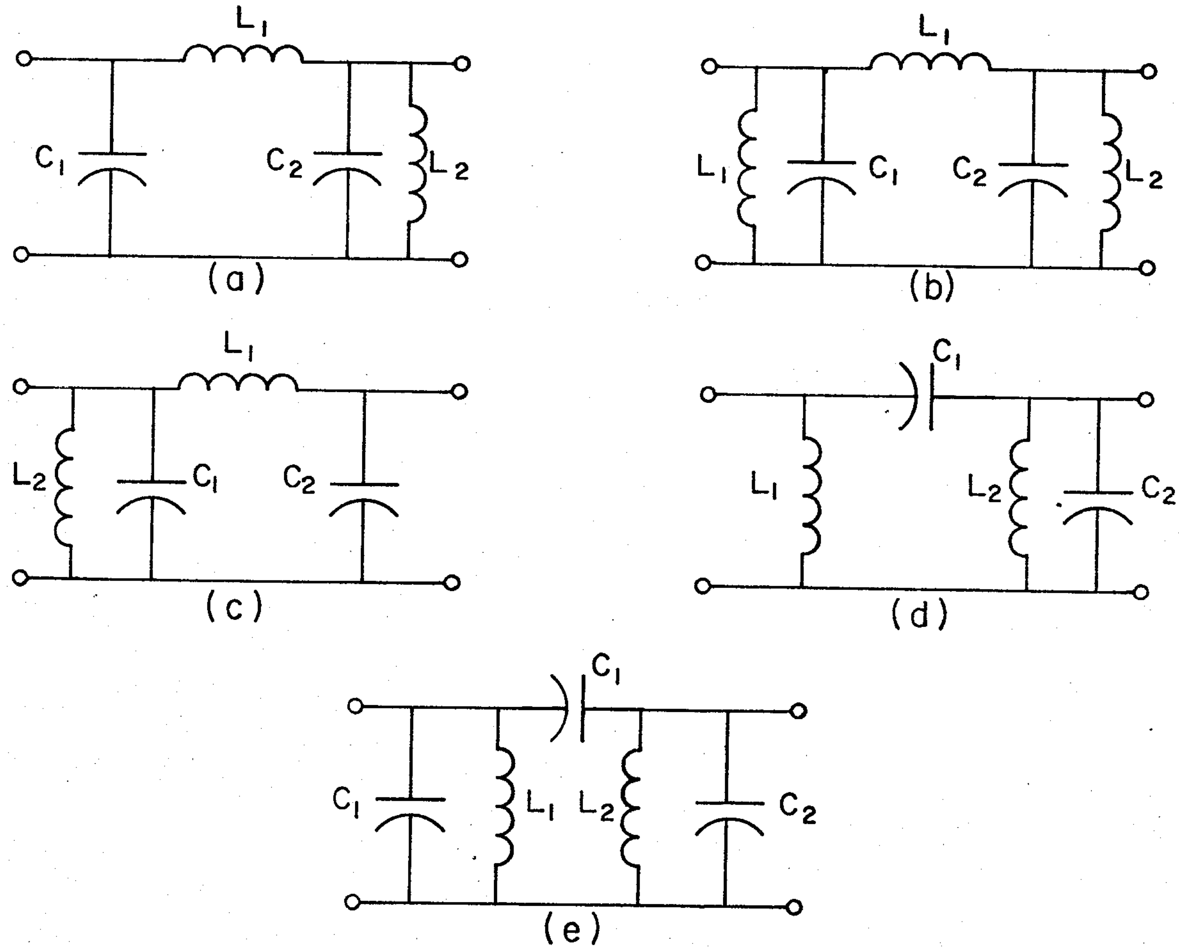


Figure example 3.7 , elements in farads & inverse henery.

(a) Cauer's first form

(b) Equivalent realization with min.Φ

(c) Scheffler's network

(d) Cauer's second form

(e) Equivalent realization with min.Φ

Table 2.

NETWORK	L_1	L_2	l_1	C_1	C_2	c_1	a	ϕ
(a)	1.52	114	—	1.2	48.1333	—	1	24.7576
(b)	.03572	.028	1.48213	1.2	.02658	—	.0235	18.99
(c)	.02	—	1.5	1.2	.00833	—	$\frac{1}{76}$	23.48
(d)	1.5	420.5	—	1.2083	174	—	1	19.924
(e)	1.5	.118357	—	.020283	.02909	2.191513	.016787	16.57911

CHAPTER IV

RC LADDER REALIZATION WITH LOSSY CAPACITORS

4.1 Introduction

Networks realized by ordinary synthesis methods do not perform exactly as intended due to the nonidealness of the elements. To circumvent the problem, several methods [37]-[39] were developed to estimate and compensate for dissipation in reactive elements. These methods developed the well known predistortion technique, under the assumption that all reactive elements have a uniform dissipation. Predistortion has been used in these methods as a means for associating resistance with each reactive element during a synthesis procedure. In brief, predistortion is accomplished by replacing the complex frequency variable s in a driving point or transfer function by a new variable $(\lambda = s + \delta_o)$, where δ_o is a positive real constant which can be measured along the real axis of the s-plane. When $(\lambda = s + \delta_o)$ is substituted for s, the poles and zeros of the function are moved to the left by a distance δ_o. Consequently, it is possible to realize a resistance δ_o with each reactive element. The criterion upon which predistortion is based is that all poles and zeros of a function must lie at least δ_o units to the left of the jw-axis.

More realistic methods [40]-[43], which assume semi-uniform loss, viz., all inductors have the same dissipation factor $d\ell$ while all capacitors have dissipation factor dc, was then developed.

In an interesting paper [44], Desoer proposed the first feasible method of precompensation for arbitrarily distributed losses. His method is basically a first-order perturbation technique. The precorrection is carried out through the derivation of a predistorted transmittance with critical fequencies displaced by increments equal in magnitude but opposite in direction to the shifts due to dissipation, and the realization of a network exhibiting the predistorted transmittance and having the same configuration as the lossless circuit. Desoer's method was investigated and developed by many researchers [45]-[49], who applied the method of first-order perturbation.

Although this approach is conceptually simple and attractive, its application is laborious. Tedious calculations are necessary to find the critical frequencies, the lossy transmittance and the derivation of the ideal response, all of which are required for the derivation of zero and pole shifts.

The main objective of this chapter is to develop an exact method of precompensating uniform and non-uniform losses in capacitors. The method is based on subjecting the conductance and capacitance matrices of the ideal-capacitor network to a proper congruence transformation. The transformation matrix is constructed in such a way that it associates a resistor with every capacitor, in the transformed network, while preserving the driving-point impedance and ladder topology.

4.2 Low-pass Ladder Networks

4.2.1 Uniform Loss

Consider the network N shown in Figure 1.a which is the canonic realization of the RC impedance parameters (z_{11}, z_{12}) of (2.10). It has been shown, in Chapter II, that the equivalent network N' of Figure 1.b can be obtained from N by subjecting the nodal admittance matrix Y of N to the congruence transformation given by (2.9). Since all capacitors in N' have shunt conductances, precompensation for capacitor loss can be carried out. Let us assume that the dissipations to be compensated for are uniformly distributed. Define the dissipation factor as

$$\delta_o = \frac{g_k'}{C_k'} \qquad k = 1, 2, \ldots, n \tag{4.1}$$

where C_k' is the k-th capacitor of the transformed network and g_k' is the associated shunt conductance.

The element values of the precompensated network N' are given by (2.25) - (2.27), while the last shunt conductance g_n' is written as

$$g_n' = g_n'' - \delta_o C_n' \tag{4.2}$$

The maximum value for the uniform loss, that can be compensated for, can be measured as the distance along the real axis of the complex plane between the first pole p_o and the origin and hence, we can write

$$\delta_o \max = p_o \tag{4.3}$$

If δ_o is smaller than p_o, the ladder will be terminated in a conductance g_n'' such that

$$g_n'' = g_n' - \delta_o C_n' \tag{4.4}$$

Furthermore, we note that g_n'' will be zero if δ_o assumes its upper limit given by (4.3), however, if δ_o exceeds p_o, g'' becomes accordingly negative.

4.2.2 Non-uniform Loss

Dissipation in capacitors does not increase linearly with size. This suggests a synthesis problem of practical interest in which capacitor loss of nonuniform nature can be precompensated for. In some situations, when the network function of interest is more sensitive to the distortion caused by the loss in some capacitors, it is then desirable to precorrect especially for these losses, which in turn, implies the need for nonuniform loss precorrection.

Let us assume that the network to be precompensated is described by the impedance parameter of (2.10). The canonic and the transformed networks are shown in Figure 1.a and Figure 1.b, respectively. For the elements in the transformed network to be positive valued, a_k must be properly chosen as follows: we first write (2.25) with k=1 as

$$G_1' = a_2 G_1$$

from which we note that a_2 must be nonnegative. By similar reasoning we can show that

$$a_k > o \qquad k = 2, 3, \ldots, n \tag{4.5}$$

is a necessary condition for non-negative elements. Furthermore, from (2.26) with k=1, the condition $a_2 \leq 1$ must be satisfied. Similarly, for k=2, 3, ..., n we derive

$$1 \geq a_2 \geq a_3 \geq \ldots \geq a_{n-1} \geq a_n > o \tag{4.6}$$

If we now define the dissipation factors as

$$\delta_k = \frac{g_k'}{C_k'} \qquad k = 1, 2, \ldots, n \tag{4.7}$$

then, from (2.25) and (2.26) it can easily be shown that the condition for non-negative elements is given by

$$\delta_k C_k + G_{k-1} \left(\frac{a_{k-1}}{a_k} - 1\right) \leq G_k \qquad k = 1, 2, \ldots, n \tag{4.8}$$

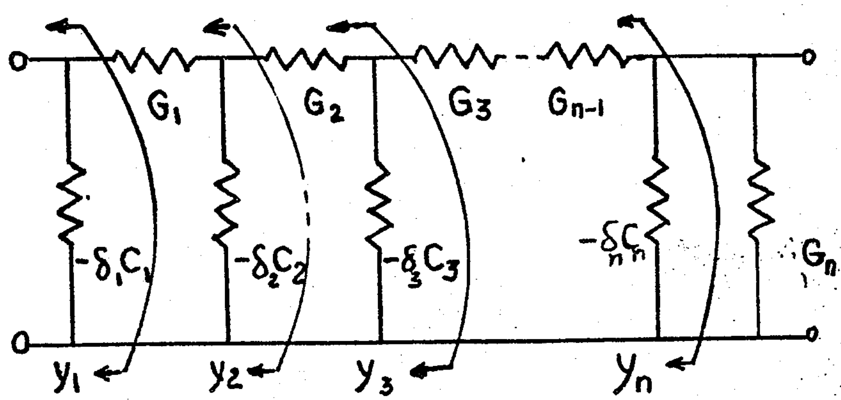


Figure 14 Definition of y_k.

It is useful now to eliminate a_k from (4.8) such that the realizability condition is expressed explicitly in terms of δ_k. If we introduce y_k as,

$$\frac{a_{k+1}}{a_k} = 1 + \frac{y_k}{G_k} \qquad k = 1, 2, \ldots, n \qquad (4.9)$$

then from (2.25) we can derive the recurrence formula

$$y_k = -\delta_k C_k + \frac{y_{k-1}}{1 + y_{k-1}/G_{k-1}} \qquad k = 1, 2, \ldots, n \qquad (4.10)$$

The realizability condition (4.8) is then equivalent to

$$y_k + G_k \geq o \qquad k = 1, 2, \ldots, n \qquad (4.11)$$

It is interesting to note that the parameter y_k in (4.10) can be computed by replacing each capacitor of C_i farads in the original network N with a conductance of $-\delta_i C_i$ mhos and computing the admittance at C_k as depicted in Figure 14.

To find the upper limit on the dissipation factors, we manipulate (4.8) - (4.10) to obtain

$$\delta_k(\text{max.}) = \frac{1}{C_k\ \sigma_k} \qquad k = 1, 2, \ldots, n \qquad (4.12)$$

The detailed proof of (4.12) is given in appendix D. It should be noted that the absolute maximum value for δ_k given by (4.12) can be realized only under the assumption that all capacitors C_i, $i \neq k$ are lossless. In fact, the maximum value for δ_k depends on the values of preceeding loss factors. Subsequently, if $\delta_1, \delta_2, \ldots, \delta_i$ are chosen, within realizability, the maximum value for δ_k, $k > i$ is given by (Appendix E).

$$\delta_k(\text{max.}) = \frac{1}{C_k(1/y_i + \sigma_i - \sigma_k)} + \frac{1}{C_k\,\sigma_k} \qquad k = i+1, \ldots, n \quad (4.13)$$

This upper limit, together with the formulas for the element values given in (2.25) - (2.27) can be used to generate design tables for practical use. Furthermore, a given set of lossy capacitor can be tested for realizability as shown in example 4.4.1.

4.3 Ladder Networks of Other Structures

The proposed method for precompensation can be extended to other ladder structures such as high-pass, mid-series and mid-shunt networks by using the results of Chapter II.

4.4 Examples

4.4.1 Example

$$z_{11} = \frac{s + 2}{s^2 + 4s + 3}, \qquad z_{12} = \frac{k}{s^2 + 4s + 3}.$$

We realize z_{11} in a canonic form as shown in Figure 15.a. Using (4.12) the maximum loss factors are obtained as

$$(\delta_1 \text{ max.}, \delta_2 \text{ max.}) = (1.5, 1.5)$$

Thus we know that a set of $(\delta_1, \delta_2) = (0.1, 1.5)$ is not realizable in this particular case since δ_k max. is calculated under the assumption that $\delta_i = o$ for $i \neq k$. For a new set $(\delta_1, \delta_2) = (1.25, .66)$ where $(\delta_1, \delta_2) < (\delta_1 \text{ max.}, \delta_2 \text{ max.})$, we take the following simple steps. From (4.10) and (4.13) we find $y_1 = -1.25$, $\delta_2 \leq .66$, hence, the set is realizable. To obtain the element values of the precorrected network we use (4.9) to find $(a_1, a_2) = (1, .375)$ which in turn, given Figure 15.b. Now let us consider another set (0.1, 0.3). Obviously it is realizable

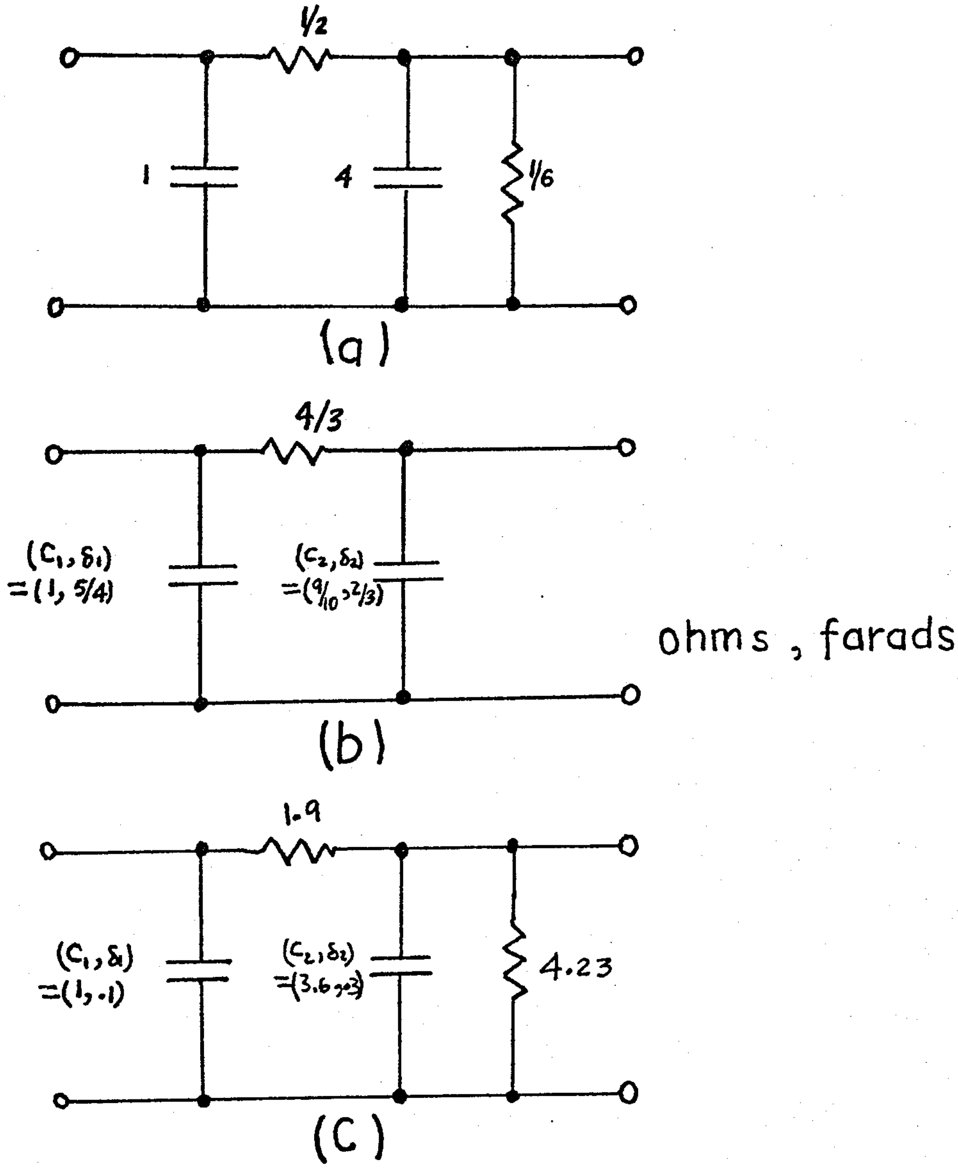


Figure 15 Networks in example 4.4.1

since each element is smaller than the corresponding element in the previous set. Here, $y_1 = -0.1$, $(a_1, a_2) = (1, .95)$ and we have the network of Figure 15.c.

CHAPTER V

RC LADDER REALIZATIONS WITH EQUAL-VALUED CAPACITORS

5.1 Introduction

In an integrated circuit fabrication as well as a conventional lumped network design it is advantageous to produce a network in which the number of capacitors is minimum and most of them are equal-valued. This is particularly true when the cost and size minimization is the design specification of prime importance [32], [50].

In this chapter, a systematic method is developed to generate an equivalent two-port in which as many capacitors are equal-valued as the given conditions permit. This is achieved by applying congruence transformation on the admittance matrix of the canonic ladder form realization of the given two-port parameters. The resulting network which keeps a driving point function and the transmission characteristics invariant, also retain the topology of the canonic ladder structure with additional conductances across the capacitors. These shunt conductances render the possibility of realizing real capacitors that are dissipative in nature as shown in Chapter IV. Although computer-aided techniques for the minimization of element-value spread exist [8], [51], no explicit conditions for the equal-capacitor realization appear in the literature. The method developed in this chapter deals with the low-pass networks, however, it can easily be extended to other types of ladder networks.

5.2 Transformation of Low-Pass Ladder Networks

Let us consider that the RC two-port to be realized is specified by the two of the open-circuit impedance parameters given by (2.10). Since the transmission zeros are all at infinity, the synthesis of z_{11}, in the first Cauer form gives the low-pass ladder network N as shown in Figure 1.a. If we now apply the congruence transformation of (2.9) and (2.11) to the nodal admittance matrix Y of N we get the equivalent network N' shown in Figure 1.b, and the element values of N' are given by (2.25)-(2.27).

A necessary condition for non-negative element is

$$a_k \geq a_{k+1} \tag{5.1}$$

and the minimum value of a_k given by (3.16) follows

$$1 \geq a_k \geq a_k \text{ min.} \qquad k = 2, 3, \ldots, n \tag{5.2}$$

Introducing a set of (n-1) parameters

$$\beta = \{\beta_2, \beta_3, \ldots, \beta_n\} \tag{5.3}$$

in the transformation matrix such that

$$0 \leq \beta_k \leq 1 \qquad k = 2, 3, \ldots, n \tag{5.4}$$

we obtain the recurrence formula (Appendix F) for $a_k (a_1 = 1)$ as

$$a_k(\beta) = \frac{\sigma_k}{\sigma_{k-1}} \quad a_{k-1}(\beta) + \frac{R_{k-1}}{\sigma_{k-1}} \prod_2^k \beta_i \qquad k = 2, 3, \ldots, n \tag{5.5}$$

with (2.11) and (5.5) we can generate continuously equivalent networks of n-1 parameters in which element values are obtained by

$$C_k' = a_k^2 (\beta) \; C_k \qquad k = 1, 2, \ldots, n \tag{5.6}$$

$$G_k' = a_k (\beta) \; a_{k+1}(\beta) \; G_K \qquad k = 1, 2, \ldots, n-1 \tag{5.7}$$

$$g_k' = \frac{a_k(\beta)}{\sigma_k} \quad (1 - \beta_{k+1}) \prod_2^k \beta_i \qquad k = 1, 2, \ldots, n \tag{5.8}$$

5.3 Realizability Conditions

5.3.1 Equal-valued Capacitors

For the invariance of the driving-point impedance z_{11}, it is required that $a_1=1$, and hence the first capacitor $C_1' = a_1^2 C_1 = C_1$ is also invariant under the transformation. To make the rest of the capacitors equal-valued, to say, C_{01}, we write

$$C_k' = a_k^2(\beta)\ C_k = C_{01} \tag{5.9}$$

or

$$a_k^2 = \frac{C_{01}}{C_k} \qquad k = 2, 3, \ldots, n \tag{5.10}$$

Substituting a_k by the recurrence formula of (5.5) and solving for β we obtain

$$\beta_2 = \frac{\sqrt{C_{01}}}{R_1}\left(\frac{\sigma_1}{\sqrt{C_2}} - \frac{\sigma_2}{\sqrt{C_{01}}}\right) \tag{5.11a}$$

$$\beta_3 = \frac{R_1\left(\dfrac{\sigma_2}{\sqrt{C_3}} - \dfrac{\sigma_3}{\sqrt{C_2}}\right)}{R_2\left(\dfrac{\sigma_1}{\sqrt{C_2}} - \dfrac{\sigma_2}{\sqrt{C_{01}}}\right)} \tag{5.11b}$$

$$\beta_k = \frac{R_{k-2}\left(\dfrac{\sigma_{k-1}}{\sqrt{C_k}} - \dfrac{\sigma_k}{\sqrt{C_{k-1}}}\right)}{R_{k-1}\left(\dfrac{\sigma_{k-2}}{\sqrt{C_{k-1}}} - \dfrac{\sigma_{k-1}}{\sqrt{C_{k-2}}}\right)} \qquad k = 4, 5, \ldots, n \tag{5.11c}$$

For $C_{k-1}' = C_k'$ within (5.4), we use (5.5) and (5.6) which, in turn, necessitates that

$$\frac{\sigma_{k-1}}{\sigma_k} \geq \sqrt{\frac{C_k}{C_{k-1}}} \geq 1 \tag{5.12}$$

To derive the necessary and sufficient conditions for the equal-valued capacitors realization, we first note that all β's must satisfy (5.4). Then from (5.11), it is also noted that only β_2 and β_3 are functions of C_{01}. Now, from (5.6) with k=2 and k=3 we obtain

$$\frac{1}{1 + \frac{R_1}{R_2}\left(1 - \sqrt{\frac{C_2}{C_3}}\right)} \leq \sqrt{\frac{C_{01}}{C_2}} \leq 1 \tag{5.13}$$

For all values of C_{01} satisfying (5.13) we have $C_2' = C_3'$, and if C_{01} can assume the value of C_1 we can make $C_1' = C_2' = C_3' = C_{01}'$.

To make all the capacitors equal-valued it is further required that (5.4) holds for $k = 4, 5, \ldots, n$, independent of C_{01}.

5.3.2 Locally equal-valued Capacitors

We have illustrated the case where an original network was transformed to have all capacitors equal-valued. If, (5.4) holds only for $k = 2, 3, \ldots, n_1$ where $n_1 < n$, we make the first set of n_1 capacitors equal-valued and then try for a second set of n_2 $(n_2 \leq n - n_1)$ equal-valued capacitors by applying a similar procedure as developed in the preceeding section. Successive applications of this procedure result in a cascade of ladder sections of locally equal-valued capacitors.

For m number of local sections we write

$$\sum_{i=1}^{m} n_i = n \tag{5.14}$$

where n_i is the number of capacitors of value C_{oi} in the i-th section. Expressions of β_k are now obtained as follows:

$$\beta_j = \frac{R_{j-2}\left(\frac{\sigma_{j-1}}{\sqrt{C_j}}\sqrt{\frac{C_{oi}}{C_{oi-1}}} - \frac{\sigma_j}{\sqrt{C_{j-1}}}\right)}{R_{j-1}\left(\frac{\sigma_{j-2}}{\sqrt{C_{j-1}}} - \frac{\sigma_j}{\sqrt{C_{j-1}}}\right)} \qquad j=1+n_1+\ldots+n_i=1 \tag{5.15a}$$

$$\beta_j = \frac{R_{j-2}\left(\frac{\sigma_{j-1}}{\sqrt{C_j}} - \frac{\sigma_j}{\sqrt{C_{j-1}}}\right)}{R_{j-1}\left(\frac{\sigma_{j-2}}{\sqrt{C_{j-1}}} - \frac{\sigma_{j-1}}{\sqrt{C_{j-2}}}\sqrt{\frac{C_{oi-1}}{C_{oi}}}\right)} \qquad j=2+n_1+\ldots+n_i-1 \tag{5.15b}$$

$$\beta_k = \frac{R_{k-2}\left(\frac{\sigma_{k-1}}{\sqrt{C_k}} - \frac{\sigma_k}{\sqrt{C_{k-1}}}\right)}{R_{k-1}\left(\frac{\sigma_{k-2}}{\sqrt{C_{k-1}}} - \frac{\sigma_{k-1}}{\sqrt{C_{k-2}}}\right)} \qquad k \neq j \tag{5.15c}$$

It should be noted that β_j is a function of $\frac{C_{oi}}{C_{oi-1}}$, and hence, (5.6) requires

$$\frac{1}{\sqrt{\frac{C_{j-1}}{C_j}}\left(1 + \frac{R_{j-1}}{R_j}\right) - \frac{R_{j-1}}{R_j}\sqrt{\frac{C_{j-1}}{C_{j+1}}}} \leq \sqrt{\frac{C_{oi}}{C_{oi-1}}} \leq \frac{\sqrt{C_j}}{\sigma_{j-1}}\left\{\frac{R_{j-1}}{R_{j-2}}\left(\frac{\sigma_{j-2}}{\sqrt{C_{j-1}}} - \frac{\sigma_{j-1}}{\sqrt{C_{j-2}}}\right) + \frac{\sigma_j}{\sqrt{C_{j-1}}}\right\}$$

$$i = 2, 3, \ldots, m \tag{5.16}$$

It is interesting to note that the upper and lower values of $\frac{C_{oi}}{C_{oi-1}}$ given by (5.16), correspond to the minimum total resistance and minimum total capacitance, respectively.

5.4 Examples

5.4.1 Example

The low-pass ladder of Figure 16.a, is the canonical realization of the open-circuit impedance parameters

$$z_{11} = \frac{(s+1)\ (s+3)}{(s+2)\ (s^2+4s+2)} \quad , \quad z_{12} = \frac{k}{(s+2)\ (s^2+4s+2)} \quad .$$

For the second capacitor in the transformed network to assume the required value $C_2' = 1$, it is required that β_2 satisfies (5.4). From (5.11a) we get $\beta_2 = \frac{1}{4}$. Similarily for $C_3' = 1$, β_3 must satisfy (5.4) thus from (5.11b) we have $\beta_3 = 1$. From (5.5) we get $\{a\}=\{1, \frac{1}{2}, \frac{1}{4}\}$ and subsequently, element values are computed by (5.6)-(5.8). The resulting network with all equal-valued capacitors is shown in Figure 16.b.

5.4.2 Example

The network shown in Figure 17.a is the canonic realization of the open-circuit impedance parameters

$$z_{11} = \frac{1080\ s^4 + 2442\ s^3 + 1374\ s^2 + 2474\ s + 12}{540\ s^4 + 1086\ s^3 + 438\ s^2 + 535\ s + 1} \quad ;$$

$$z_{12} = \frac{k}{540\ s^4 + 1086\ s^3 + 438\ s^2 + 535\ s + 1}$$

in which k is a constant. Since $C_1 = 0$, then the capacitance C_{01} of the first section is chosen by (5.13) so that

$$2 \geq C_{01} \geq 1.58$$

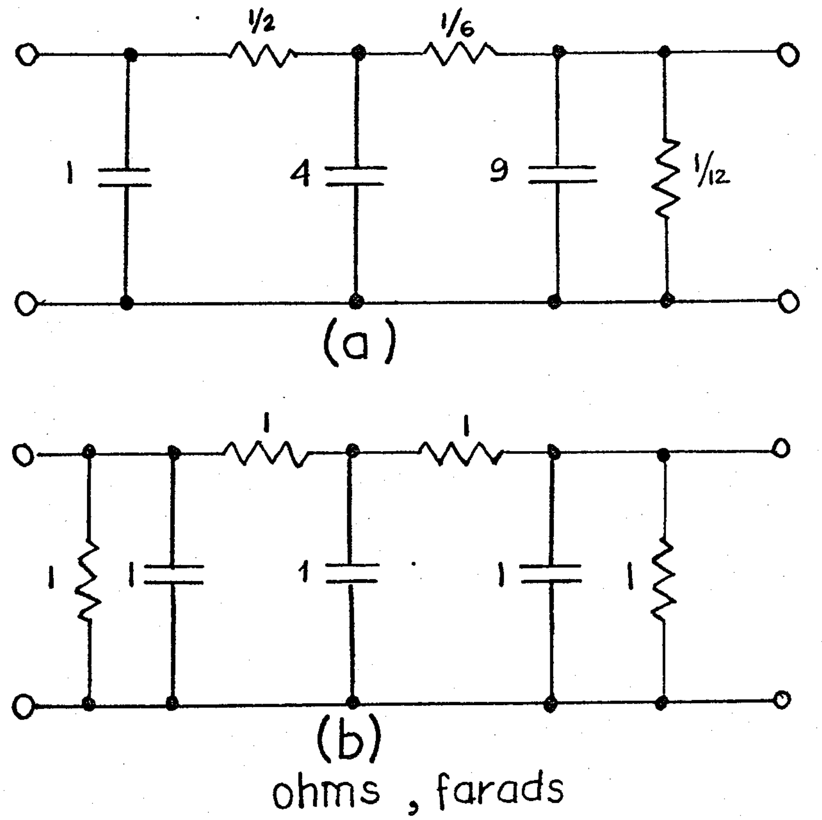


Figure 16 Networks in example 5.4.1

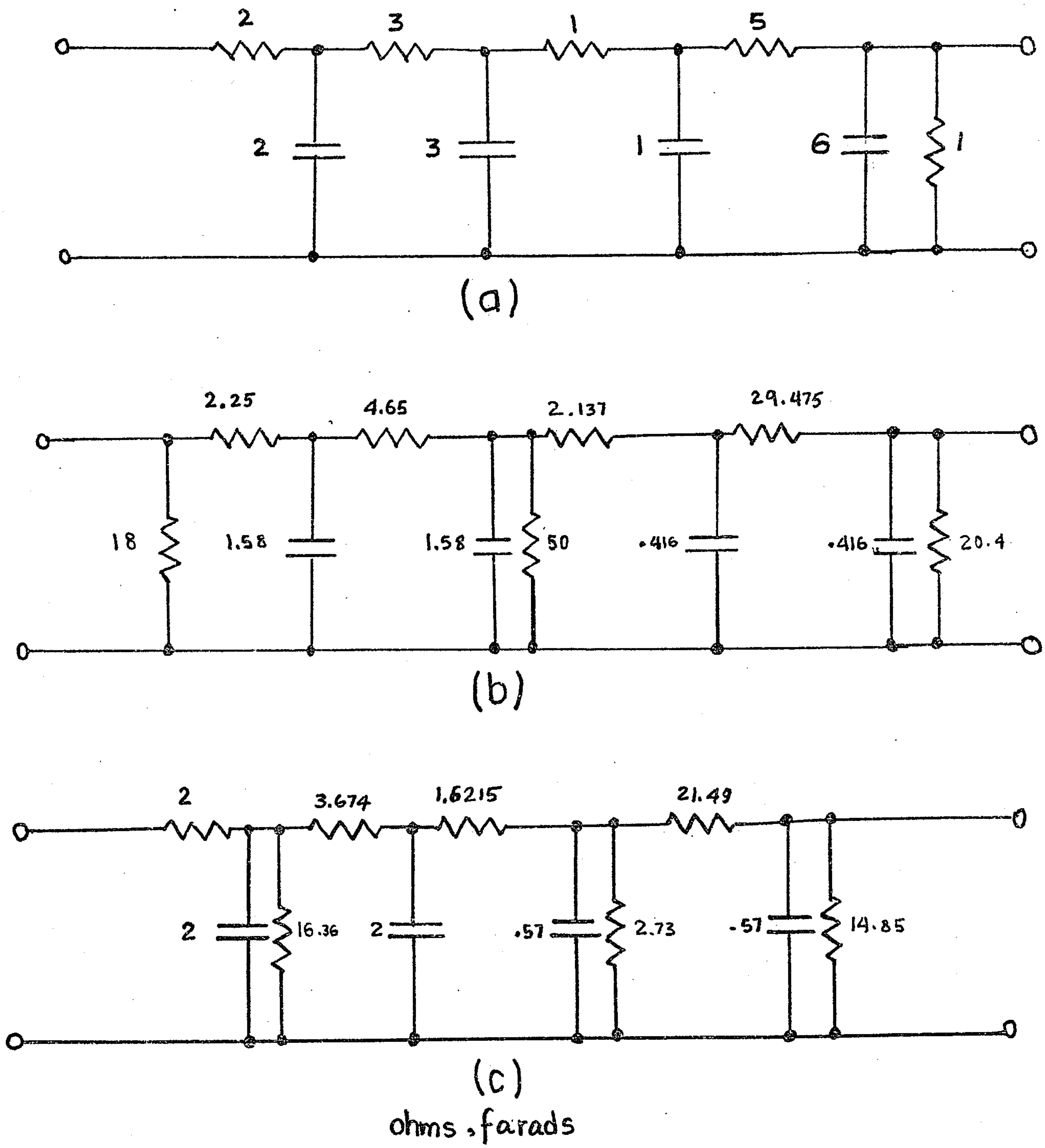


Figure 17 Networks in example 5.4.2

which in turn yields $C_2' = C_3' = C_{o1}$. Now, since β_4 does not satisfy (5.4) another cycle of equalization takes place. First from (5.16) we write

$$0.5341 \geq \frac{C_{o2}}{C_{o1}} \geq 0.513$$

which assures that $C_4' = C_5' = C_{o2}$.

In determining C_{o1} and C_{o2} we can consider some other design criteria such as minimization of total capacitance. For this case we choose $C_{o1} = 1.58$, $\frac{C_{o2}}{C_{o1}} = 0.513$, which gives $C_{o2} = 0.416$ and $\beta = \{0.3328, 1, 0.56, 1\}$, $a = \{1, 0.8888, 0.7255, 0.645, 0.263\}$. The resulting network is shown in Figure 17.b.

If minimum total resistance is desired, we choose $C_{o1} = 2$, $\frac{C_{o2}}{C_{o1}} = 0.5341$ and subsequently we obtain $C_{o2} = 0.57$, $\beta = \{1, 0.339, 1, 0.5611\}$ and $a = \{1, 1, 0.8165, 0.7553, 0.308\}$ which results in the network shown in Figure 17.c.

CHAPTER VI

CONCLUSIONS

1. It has been shown that for an RC low-pass ladder network synthesized from a specified two-port parameters, a proper transformation is possible to generate a class of equivalent networks which can be divided into two kinds. The first kind includes n-canonical realizations while the second produces is a one-parameter family of equivalent networks. In either case, the total resistance, total capacitance and the open-circuit voltage gain vary monotonically with a single variable. As a consequence a straightforward method has been developed for the minimization of the cost function duly constructed for the low-pass network. It has also been demonstrated by means of GC:CG:GC transformation and topological duality that a similar technique is applicable to other types of ladder structures.

2. The transformation technique has been applied to develop a method of realizing RC ladder networks of nonideal capacitors. Unlike existing methods which assume small loss-factors to accomodate approximations and first-order perturbations, present method develops exact realizations using capacitors of uniform and nonuniform loss. Explicit formulas derived are amenable to the machine computation.

3. RC ladder realization with equal-valued capacitors has been studied within the proposed transformation methods. A systematic procedure has been advanced to generate an equivalent two-port in which capacitor values are equalized as many as possible. The proposed structure renders the possibility of realizing real capacitors that are dissipative in nature, and the flexibility of the method provides a direct control over the total

resistance and total capacitance of the network.

4. Although the techniques developed in the dissertation dealt exclusively with RC ladder networks, analogous results can be obtained for LC and RL ladders through RC:LC and RC:RL transformations, respectively.

5. Finally, it is hoped that the novel aspects of the transformation theory presented in the thesis, may open up a new vista for future developments of optimization procedures.

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APPENDIX A

DERIVATION OF EQUATION (2.17)

From (2.11) and (2.16) we have

$$A^t CA = C' \tag{A.1}$$

$$A^t GA = G'. \tag{A.2}$$

Since C and C' are both diagonal matrices of the positive elements, $C^{\frac{1}{2}}$ and $C'^{\frac{1}{2}}$ exist, and from (A.1) we write

$$C'^{-\frac{1}{2}} A^t C^{\frac{1}{2}} C^{\frac{1}{2}} AC'^{-\frac{1}{2}} = I$$

or

$$Q^t Q = I$$

Now let A be a square matrix of

$$A = [a_{ij}] \qquad i,j = 1, 2, \ldots, n. \tag{A.3}$$

then

$$Q = [q_{ij}] = [\frac{C_i}{C'_j} a_{ij}]. \qquad i,j = 1.2, \ldots, n. \tag{A.4}$$

For the invariance of z_{11} it is required that $a_{1j} = 0$ for $j \neq 1$. Moreover, the orthogonality of Q imposes that $a_{i1} = 0$ for $i \neq 1$.

Let us now introduce a unit vector u_k whose components are zero except at the kth entry. With (A.2) we write

$$A^t GAu_1 = G'u_1 \tag{A.5}$$

or

$$G_1 \begin{bmatrix} 1 \\ -a_{22} \\ -a_{23} \\ \circ \\ \circ \\ -a_{2n} \end{bmatrix} = \begin{bmatrix} g_1' + G_1' \\ -G_1' \\ 0 \\ \circ \\ \circ \\ 0 \end{bmatrix} \tag{A.5}$$

where we get $a_{2j} = 0;\ j = 3, 4, \ldots, n$, while orthogonality of Q gives $a_{i2} = 0$ for $i = 3, 4, \ldots, n$. Repeating the similar operation with vectors $u_2, u_3, \ldots, u_n$, we conclude that

$$a_{ij} = 0 \qquad \text{for } i \neq j, \tag{A.6}$$

and the square matrix A is reduced to a diagonal matrix of the form

$$A = \mathrm{diag}\,\{1, a_2, a_3, \ldots, a_n\} \ . \tag{A.7}$$

APPENDIX B

DERVIATION OF EQUATION (3.2)

To find a_k which yields the proposed structure of Figure 6.a we first write from (2.26) with $k=1$

$$g_1' = G_1 (1 - a_2) \tag{B.1}$$

From (3.1) and (B.1) we obtain

$$a_2 = 1 \tag{B.2}$$

Repeating the above steps for $k = 2, 3, \ldots, i-1$ we have

$$a_k = 1 \qquad k = 2, 3, \ldots, i \tag{B.3}$$

Now let $k=n$ in (2.26) and apply the condition (3.1) to get

$$a_n = \frac{R_n}{R_{n-1} + R_n} a_{n-1} \tag{B.4}$$

Similarly with $k=n-1$, we write

$$a_{n-1} = \frac{R_{n-1} + R_n}{R_{n-2} + R_{n-1} + R_n} a_{n-2}$$

and by induction we derive the recurrence formula

$$a_{k+1} = \frac{\sigma_{k+1}}{\sigma_k} a_k \qquad k = i, i+1, \ldots, n-1 \tag{B.5}$$

Referring to (B.3) we rewrite (B.5) as

$$a_k = \frac{\sigma_k}{\sigma_i} \qquad k = i+1, \ldots, n \tag{B.6}$$

APPENDIX C

DERIVATION OF EQUATION (3.16)

From (2.26) we write

$$g_n' = a_n \{a_n (G_n + G_{n-1}) - a_{n-1} G_{n-1}\}$$

where a_n assumes its minimum value when $g_n' = 0$, i.e.,

$$a_{n(\min)} = \frac{R_n}{R_n + R_{n-1}} a_{n-1} \quad . \tag{C.1}$$

Similarly from

$$g_{n-1}' = a_{n-1} \{a_{n-1} (G_{n-1} + G_{n-2}) - (a_n G_{n-1} + a_{n-2} G_{n-2})\}$$

we write

$$a_{n-1(\min)} = \frac{R_n + R_{n-1}}{R_n + R_{n-1} + R_{n-2}} a_{n-2} \tag{C.2}$$

For kth entry, it gives

$$a_{k(\min)} = \frac{\sigma_k}{\sigma_{k-1}} a_{k-1} \tag{C.3}$$

Since we let $a_1 = 1$ for the invariance of z_{11}, we have

$$a_{2(\min)} = \frac{\sigma_2}{\sigma_1}$$

$$a_{3(\min)} = \frac{\sigma_3}{\sigma_2} a_2 = \frac{\sigma_3}{\sigma_2} \cdot \frac{\sigma_2}{\sigma_1} = \frac{\sigma_3}{\sigma_1}$$

and

$$a_{k(\min)} = \frac{\sigma_k}{\sigma_1} \tag{C.4}$$

APPENDIX D

DERIVATION OF EQUATION (4.12)

From (4.10) and (4.11) we can write

$$-y_k \leq G_k \tag{D.1}$$

and for $k=1, 2, 3$

$$-y_1 = \delta_1 C_1 \leq G_1 \tag{D.2}$$

$$-y_2 = \delta_2 C_2 + \frac{\delta_1 C_1 G_1}{G_1 - \delta_1 C_1} \leq G_2 \tag{D.3}$$

$$-y_3 = \delta_3 C_3 + \frac{\delta_2 C_2 + \dfrac{\delta_1 C_1 G_1}{G_1 - \delta_1 C_1}}{1-(\delta_2 C_2 + \dfrac{\delta_1 C_1 G_1}{G_1 - \delta_1 C_1})/G_2} \leq G_3 \tag{D.4}$$

If we solve for δ_1 with $\delta_k = 0$ $(k \neq 1)$

$$\delta_1 \leq \frac{1}{C_1 R_1}$$

$$\delta_1 \leq \frac{1}{C_1 (R_1 + R_2)}$$

$$\delta_1 \leq \frac{1}{C_1 (R_1 + R_2 + R_3)}$$

Thus

$$\delta_1 \leq \frac{1}{C_1 \sum_{i=1}^{n} R_i} \tag{D.5}$$

and, in general

$$\delta_k \leq \frac{1}{C_k \sum_{i=k}^{n} R_i} \quad \text{with } \delta_j = 0 \text{ for } j \neq k \tag{D.6}$$

APPENDIX E

DERIVATION OF EQUATION (4.13)

Using $-y_k \leq G_k$ we write

$$-y_{r+1} = \delta_{r+1} C_{r+1} - \frac{y_r}{1 + y_r/G_r} \leq G_{r+1} \tag{E.1}$$

$$-y_{r+2} = \delta_{r+2} C_{r+2} - \frac{y_{r+1}}{1 + y_{r+1}/G_{r+1}} \leq G_{r+2} \tag{E.2}$$

$$-y_{r+3} = \delta_{r+3} C_{r+3} - \frac{y_{r+2}}{1 + y_{r+2}/G_{r+2}} \leq G_{r+3} \tag{E.3}$$

The maximum value for δ_{r+1} which satisfies

(E.1) is $\delta_{r+1} \leq \dfrac{1}{C_{r+1}\ (\frac{1}{y_r} + R_r)} + \dfrac{1}{C_{r+1}\ R_{r+1}}$

(E.2) is $\delta_{r+1} \leq \dfrac{1}{C_{r+1}\ (\frac{1}{y_r} + R_r)} + \dfrac{1}{C_{r+1}\ (R_{r+1} + R_{r+2})}$

(E.3) is $\delta_{r+1} \leq \dfrac{1}{C_{r+1}\ (\frac{1}{y_r} + R_r)} + \dfrac{1}{C_r (R_{r+1} + R_{r+2} + R_{r+3})}$

And in general, the maximum value for δ_{r+1} is

$$\delta_{r+1} \leq \frac{1}{C_{r+1}\ (\frac{1}{y_r} + R_r)} + \frac{1}{C_{r+1} \sum_{i=x+1}^{n} R_i} \tag{E.4}$$

To find the maximum value for δ_{r+2} we start from (E.2) and we have

$$\delta_{r+2} \leq \frac{1}{C_{r+2}\left(\frac{1}{y_r} + R_r + R_{r+1}\right)} + \frac{1}{C_{r+2} \sum_{i=r+2}^{n} R_i} \tag{E.5}$$

repeating, we have

$$\delta_k \leq \frac{1}{C_k\left(\frac{1}{y_r} + \sum_{i=r}^{k-1} R_i\right)} + \frac{1}{C_k \sum_{i=k}^{n} R_i} \qquad k = r+1, \ldots, n \tag{E.6}$$

APPENDIX F

DERVIATION OF EQUATION (5.4)

The realizability condition (5.2), with $K=2$ can be written in terms of the parameter β as

$$a_2 = a_{2\ min} + \beta_2 \, (a_{2\ max} - a_{2\ min}) \tag{F.1}$$

where

$$0 \leq \beta_2 \leq 1 \tag{F.2}$$

Substituting for $a_{2\ max} = 1$ and $a_{2\ min} = \frac{\sigma_2}{\sigma_1}$ in (F.1) we get

$$a_2(\beta) = \frac{\sigma_2 + \beta_2 \, R_1}{\sigma_1} \tag{F.3}$$

Now from (2.26) with $k=2$, it is clear that a_3 assumes its upper limit when $g_2' = 0$ and we have

$$a_{3\ max} = a_2 + \frac{R_2}{R_1} \, (1 - a_2) \tag{F.4}$$

while a_3 assumes the lower limit, $a_{2\ min}$, when $g_k' = 0$ for $k = 3, 4, \ldots, n$ as it can be seen from (2.26). Substituting $g_k' = 0; \ k = 3, 4, \ldots, n$ in (2.26) and manipulating we obtain

$$a_{3\ min} = \frac{\sigma_3}{\sigma_2} \, a_2(\beta) \tag{F.5}$$

Introducing the normalized parameter β_3; $0 \leq \beta, \leq 1$, a_3 can be expressed as

$$a_3 = a_{3\ min} + \beta_3 \, (a_{3\ max} - a_{3\ min}) \tag{F.6}$$

Substituting from (F.4) and (F.5) in (F.6) we write

$$a_3(\beta) = \frac{\sigma_3}{\sigma_2} a_2 + \frac{R_2}{\sigma_2} \beta_2 \beta_3 \tag{F.7}$$

Repeating for $k = 4, 5, \ldots, n$ the maximum value for a_k is obtained when $g'_{k-1} = 0$, while a_k is minimum when $g'_i = 0$; $i = k, \ldots, n$ and we get

$$a_{k\,min} = \frac{\sigma_k}{\sigma_{k-1}} a_{k-1}(\beta) \tag{F.8}$$

$$k = 3, 4, \ldots, n$$

$$a_{k\,max} = a_{k-1} + \frac{R_{k-1}}{R_{k-2}} (a_{k-2} - a_{k-1}) \tag{F.9}$$

If we introduce the normalized parameters β_k; $k = 2, \ldots, n$ such that $0 \leq \beta_k \leq 1$, the transformation parameters a_k are written as

$$a_k(\beta) = a_{kmin} + \beta_k (a_{kmax} - a_{kmin}) \qquad k = 2, 3, \ldots, n \tag{F.10}$$

Substituting for a_{kmin} and a_{kmax}, and manipulating, (F.10) gives

$$a_k = \frac{\sigma_k}{\sigma_{k-1}} a_{k-1} + \frac{R_{k-1}}{\sigma_{k-1}} \prod_{2}^{k} \beta_i \qquad k = 2, 3, \ldots, n \tag{F.11}$$