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DUALITY THEORY FOR QUASI-VARIETIES OF UNIVERSAL ALGEBRAS

by

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ABSTRACT

Many duality theorems for categories of universal algebras arise in the following manner. The category $\mathcal{A}$ is the quasi-variety generated by an algebra A which has a compact topology compatible with the operations (for example, if A is the circle group, the two-element Boolean algebra or the two-element semi-lattice with zero and unit, then $\mathcal{A}$ is the category of abelian groups, the category of Boolean algebras or the category of semi-lattices with zero and unit respectively). The category $\mathcal{X}$, which is dual to the category $\mathcal{A}$, is the category of compact topological spaces with some added structure (for the examples mentioned above, $\mathcal{X}$ is the category of compact topological abelian groups, the category of compact, totally disconnected spaces or the category of compact, totally disconnected topological semilattices respectively). In some sense the object A lies in both the category $\mathcal{A}$ and the category $\mathcal{X}$ and the duality between $\mathcal{A}$ and $\mathcal{X}$ is provided by the Hom-set functors $\mathcal{A}(-,A)\colon \mathcal{A} \to \mathcal{X}^{op}$ and $\mathcal{X}(-,A)\colon \mathcal{X}^{op} \to \mathcal{A}$. The study of dualities of this type is the central theme of this thesis.

In Chapter 1 the general theory is presented. Here we assume that $\mathcal{X}$ is a category with a faithful functor into the category of compact (Hausdorff) spaces. Under certain natural restrictions on $\mathcal{X}$, necessary and sufficient conditions are found for $\mathcal{X}$ to be

dual to A. The results of Chapter 1 are applied in Chapter 2 to the case where X is a category of compact topological partial algebras derived in a natural way from the category A. The approach developed in Chapter 2 is utilized in Chapter 3 to provide new proofs of Pontryagin's duality for abelian groups, Stone's duality for Boolean algebras, and the duality for semilattices with (zero and) unit most recently and thoroughly studied by K. H. Hofmann, M. Mislove and A. Stralka. New proofs are also obtained for two more recent duality theorems: the duality for an equational class generated by a primal algebra due to T. K. Hu, and the duality for the category of distributive lattices with (zero and) unit due to H. A. Priestley. The chapter closes with the presentation of a new duality theorem for the category of Stone algebras.

In Chapter 4 attention is focused on the equational class of Heyting algebras generated by an n-element chain and the equational class of Brouwerian algebras generated by an n-element chain. For these categories duality theorems are developed and then applied to describe the weak injective and injective algebras in each category. Finally we obtain a new proof for P. Köhlen's description of the finitely generated free algebras in these classes.

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I owe a great deal to my colleagues who have listened patiently to my ravings, but a particular note of thanks is due to Heini Werner who helped me steer a course through the initial ideas which eventually led to this thesis.

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0. PRELIMINARIES

Our standard references will be G. Grätzer [28] with respect to universal algebra, and S. MacLane [53] with respect to category theory. All notions from lattice theory may be found in G. Grätzer [30], H. Rasiowa and R. Sikorski [64], and G. Birkhoff [9]. The texts J. Kelley [48] and J. Dugundji [19] are more than adequate for our topological needs. Below we remark upon some specific points which require emphasis.

Since we often work with direct products and direct powers it will be convenient to use both the 'tuple' and the 'map' notations for elements of a direct product. If $(A_\gamma | \gamma < \kappa)$ is a family of sets, then an element of $\Pi(A_\gamma | \gamma < \kappa)$ will be denoted by $\langle a_\gamma \rangle_{\gamma<\kappa}$ and will be abbreviated to $\langle a_\gamma \rangle$ when no confusion arises. If A and B are sets, then the elements of A^B will be regarded as maps $g: B \to A$.

Universal Algebra

A universal algebra or partial algebra will always be denoted by its underlying set. If f is an m-ary operation on the partial algebra A, then the *domain of* f, that is $\{(a_0,\ldots,a_{m-1}) \in A^m | f(a_0,\ldots,a_{m-1})$ is defined$\}$, is denoted by D_f.

Let A be simultaneously a partial algebra and a Hausdorff topological space. Then A is called a *topological partial algebra* if each of the operations on A is continuous on its domain, that is, if f is an m-ary operation on A, $(a_0,\ldots,a_{m-1}) \in D_f$ and U is an open neighbourhood of $f(a_0,\ldots,a_{m-1})$, then there exist open neighbourhoods $U_0,\ldots,U_{m-1}$ of $a_0,\ldots,a_{m-1}$ respectively, such that $f(b_0,\ldots,b_{m-1}) \in U$ for all $(b_0,\ldots,b_{m-1}) \in U_0 \times \ldots \times U_{m-1} \cap D_f$.

A class $\mathcal{A}$ of similar algebras is called a *quasi-variety* if $\mathcal{A} = ISP(\mathcal{A})$. Clearly the quasi-variety generated by an algebra A is $\mathcal{A} = ISP(A)$. If $\mathcal{A}$ is a quasi-variety then, by [28, Corollary 1, p.167] all *free algebras* exist in $\mathcal{A}$ and we will denote the freely κ-generated algebra in $\mathcal{A}$ by $F_{\mathcal{A}}(\kappa)$, with free generators $\{x_\gamma | \gamma < \kappa\}$; in particular the free algebra in $\mathcal{A}$ on one generator is denoted by $F_{\mathcal{A}}(1)$, with free generator x_0.

Category Theory

With the single exception mentioned below for topological spaces, the set of arrows in a category $\mathcal{A}$ with domain B and codomain C will be denoted by $\mathcal{A}(B,C)$. If $\mathcal{A}$ is a class of similar algebras, then by *the category* $\mathcal{A}$ we will mean the category whose objects are the algebras in $\mathcal{A}$ and whose arrows are all homomorphisms between algebras in $\mathcal{A}$, that is $\mathcal{A}(B,C) = Hom(B,C)$ for all $B,C \in \mathcal{A}$.

If there is a faithful functor $|-|: \mathcal{A} \to \mathcal{C}$, then $\mathcal{A}$ is *grounded in* $\mathcal{C}$ and $|-|$ is called a *grounding*. $\mathcal{A}$ is *naturally grounded* if there exists $A_1 \in \mathcal{A}$ such that $\mathcal{A}(A_1,-): \mathcal{A} \to \mathcal{Set}$ is a grounding. If F: $\mathcal{C} \to \mathcal{A}$ is left adjoint to the grounding $|-|: \mathcal{A} \to \mathcal{C}$, then F is called a *$\mathcal{C}$-free functor*. If $\mathcal{C} = \mathcal{Set}$, then F is simply called a *free functor*.

If F: $\mathcal{Set} \to \mathcal{A}$ is a free functor for $\mathcal{A}$, then since

$A(F(\{0\}),A) \simeq Set(\{0\}, |A|) \simeq |A|$, $A_1 = F(\{0\})$ provides a natural grounding naturally isomorphic to $|-|$; in particular, if A is a quasi-variety then $A_1 = F_A(1)$ provides a natural grounding naturally isomorphic to the forgetful functor.

If A is grounded in *Set* by $|-|: A \to Set$, then an arrow g in A is called an A-*injection* (A-*surjection*) or simply an *injection* (*surjection*) if $|g|$ is one-one (onto). Clearly injections are monic and surjections are epic.

Let A be grounded in *Set*, let I be a class of injections in A and let S be a class of surjections in A. An object $A \in A$ is said to be I-*injective in* A if for all $g: B \to C \in I$ and $h: B \to A$, there exists $h': C \to A$ with $gh' = h$, and is said to be S-*projective in* A if for all $g: C \to B \in S$ and $h: A \to B$, there exists $h': A \to C$ with $h'g = h$. If A is S-projective with S the class of all A-surjections, then we say that A is *sur-projective*. If A is a category of universal algebras and A is I-injective in A with I the class of all injections (i.e., embeddings), then we drop the prefix and simply say that A is *injective in* A.

Topology

A topological space will always be denoted by its underlying set. The category of compact (Hansdorff) spaces and continuous maps will be denoted by *Comp*, and the full subcategory of zero dimensional compact spaces, otherwise known as *Boolean spaces*, will be denoted by *ZComp*. The Hom-sets *Comp*(X,Y) and *ZComp*(X,Y) will both be abbreviated to C(X,Y). As usual, the *Stone-Čech* *compactification functor* will be denoted by β: *Set* $\to$ *Comp*.

Since we often work with product spaces we introduce the following notation: if $U \subseteq X_\lambda$, then

$$(\lambda;U) = \{\langle x_\gamma\rangle \in \Pi(X_\gamma|\gamma < \kappa)|x_\lambda \in U\};$$

similarly, if $U \subseteq A$ and $b \in B$, then $(b;U) = \{g \in A^B|bg \in U\}$. Hence, if U is open in X_λ, then $(\lambda;U)$ is a sub-basic open set in $\Pi(X_\gamma|\gamma < \kappa)$, and if U is open in A, then (b;U) is a sub-basic open set in A^B.

Finally, for any space X, a continuous map $\phi \in C(X^\kappa,X)$ is said to have *finite support* if there exist $\gamma_0,\ldots,\gamma_{k-1} < \kappa$ such that, for all $\langle x_\gamma\rangle$, $\langle y_\gamma\rangle \in X^\kappa$, $x_{\gamma_j} = y_{\gamma_j}$ $(j < k)$ implies that $\langle x_\gamma\rangle\phi = \langle y_\gamma\rangle\phi$.

1. DUALITIES VIA STRUCTURED COMPACT SPACES

The concept of 'structure' referred to in the title of this chapter is an intuitive one. Formally, a category of structured compact spaces will be a category which is grounded in *Comp*.

DEFINITION (1.1). Let $\mathcal{A}$ and $\mathcal{X}$ be categories and assume that $E: \mathcal{X}^{op} \to \mathcal{A}$ is right adjoint to $D: \mathcal{A} \to \mathcal{X}^{op}$. Then (D,E) is called a *duality* (*between* $\mathcal{A}$ *and* $\mathcal{X}$) if the unit $\eta: Id_{\mathcal{A}} \to ED$ of the adjunction is a natural isomorphism, and is called a *full duality* if the counit $\varepsilon: Id_{\mathcal{X}} \to DE$ is also a natural isomorphism.

The following result, which is proved in [39], says in essence that when considering a duality (D,E) between 'concrete' categories $\mathcal{A}$ and $\mathcal{X}$ one necessarily has an object A lying in both categories such that D and E are given by Hom-functors in the contravariant argument with $D(-) = \mathcal{A}(-,A)$ and $E(-) = \mathcal{X}(-,A)$ inheriting their structure from an appropriate power of A.

PROPOSITION (1.2). *Let* $\mathcal{A}$ *and* $\mathcal{X}$ *be naturally grounded categories and suppose that* $E: \mathcal{X}^{op} \to \mathcal{A}$ *is right adjoint to* $D: \mathcal{A} \to \mathcal{X}^{op}$. *Let* $|-|_{\mathcal{A}} = \mathcal{A}(A_1,-)$ *and* $|-|_{\mathcal{X}} = \mathcal{X}(X_1,-)$ *be the respective grounding functors and set* $A = EX_1$ *and* $A^{\dagger} = DA_1$. *Then*

(i) $|A|_{\mathcal{A}} \simeq |A^{\dagger}|_{\mathcal{X}}$,

(ii) $|D(-)|_{\mathcal{X}} \simeq \mathcal{A}(-,A)$ *and* $|E(-)|_{\mathcal{A}} \simeq \mathcal{X}(-,A^{\dagger})$, *and*

(iii) *if* $\mathcal{X}$ *has products, then there is a monic natural transformation* $D \to (A^{\dagger})^{|-|_{\mathcal{A}}}: \mathcal{A} \to \mathcal{X}^{op}$. □

The development in the remainder of this chapter should be viewed in the light of this result.

LEMMA (1.3). *Let* A *be a compact topological algebra.*

(i) *For every topological space* X, C(X,A) *is closed under the pointwise operations.*

(ii) Hom(B,A) *is a closed subspace of* A^B *for every algebra* B *of the same type as* A.

Proof: (i) Let f be an m-ary operation, $\phi_0,\dots,\phi_{m-1} \in C(X,A)$ and assume that $xf(\phi_0,\dots,\phi_{m-1}) = f(x\phi_0,\dots,x\phi_{m-1}) \in U$ for some open subset U of A. Since f is a continuous operation there exist open subsets $U_0,\dots,U_{m-1}$ of A with $x\phi_j \in U_j$ $(j < m)$ and $f(U_0,\dots,U_{m-1}) \subseteq U$. Clearly $V = \bigcap(U_j\phi_j^{-1} | j < m)$ is an open neighbourhood of x with $Vf(\phi_0,\dots,\phi_{m-1}) \subseteq U$. Hence $f(\phi_0,\dots,\phi_{m-1}) \in C(X,A)$.
(ii) Let $g \in A^B$ and suppose that $g \notin$ Hom(B,A). We will construct an open neighbourhood W of g which is disjoint from Hom(B,A).

Since g is not a homomorphism there is an (m-ary) operation f and $b_0,\dots,b_{m-1} \in B$ such that $f(b_0,\dots,b_{m-1})g \neq f(b_0g,\dots,b_{m-1}g)$. Let U and V be disjoint open neighbourhoods of $f(b_0g,\dots,b_{m-1}g)$ and $f(b_0,\dots,b_{m-1})g$ respectively. By the continuity of f there exist open subsets $U_0,\dots,U_{m-1}$ of A with $b_jg \in U_j$ $(j < m)$ and $f(U_0,\dots,U_{m-1}) \subseteq U$. Let $W = \{h \in A^B | b_jg \in U_j \ (j < m); f(b_0,\dots,b_{m-1}) \in V\}$. W is a neighbourhood of g and clearly $f(b_0,\dots,b_{m-1})h \neq f(b_0h,\dots,b_{m-1}h)$ for all $h \in W$, whence W is disjoint from Hom(B,A). □

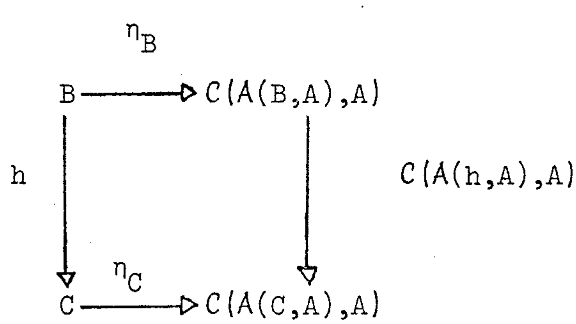
It follows from (ii) that if A is a compact topological algebra then $\mathcal{A}$(B,A) is compact, and if A is finite then $\mathcal{A}$(B,A) is a Boolean space.

For the remainder of this chapter we will assume that A is a compact topological algebra and that $\mathcal{A} = ISP(A)$ is the quasi-variety, qua category, generated by A. In the light of Lemma (1.3) it is easily seen that $\mathcal{A}(-,A)\colon \mathcal{A} \to \mathit{Comp}^{op}$ is a well-defined functor, where $g\mathcal{A}(h,A) = hg$, and that $C(-,A)\colon \mathit{Comp}^{op} \to \mathcal{A}$ is a well-defined functor, where $\phi C(\psi,A) = \psi\phi$.

PROPOSITION (1.4). *For all* $B \in \mathcal{A}$ *define* $\eta_B\colon B \to C(\mathcal{A}(B,A),A)$ *by* $b\eta_B = [b]$, *where* $g[b] = bg$, *and for all* $X \in \mathit{Comp}$ *define* $\varepsilon_X\colon X \to \mathcal{A}(C(X,A),A)$ *by* $x\varepsilon_X = [x]$, *where* $\phi[x] = x\phi$. *Then* $(\mathcal{A}(-,A),\ C(-,A);\ \eta,\ \varepsilon)$ *is an adjunction from* $\mathcal{A}$ *to* Comp^{op}.

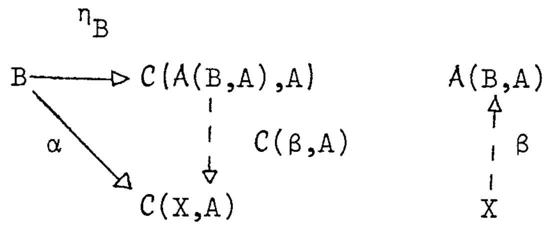
Proof. By [53, Theorem 2, p.81] it is sufficient to prove that η is a well defined natural transformation and that $\eta_B\colon B \to C(\mathcal{A}(B,A),A)$ is universal to $C(-,A)$ from B, for all $B \in \mathcal{A}$.

If U is open in A, then $U(b\eta_B)^{-1} = (b;U) \cap \mathcal{A}(B,A)$ and hence $b\eta_B$ is continuous for all $b \in B$. Furthermore, for each (m-ary) operation f, $g(f(b_0,\dots,b_{m-1})\eta_B) = f(b_0,\dots,b_{m-1})g$
$= f(b_0g,\dots,b_{m-1}g) = f(g(b_0\eta_B),\dots,g(b_{m-1}\eta_B)) = gf(b_0\eta_B,\dots,b_{m-1}\eta_B)$,
and so η_B is a homomorphism. To see that η is a natural transformation we must show that the following diagram commutes.



But if $b \in B$ and $g \in \mathcal{A}(C,A)$, then $g(b\eta_B \mathcal{C}(\mathcal{A}(h,A),A)) = g(\mathcal{A}(h,A)b\eta_B)$ $= (hg)b\eta_B = b(hg) = (bh)g = g(bh\eta_C)$, as required.

Now let $B \in \mathcal{A}$, $X \in$ *Comp* and let $\alpha: B \to \mathcal{C}(X,A)$ be a homomorphism. The unique fill-in map β for the diagram



if defined by $b(x\beta) = x(b\alpha)$. The map β is unique since it is clear that if $\gamma: X \to \mathcal{A}(B,A)$ is also a fill-in map, then $b(x\gamma) = x(b\alpha)$ by the commutativity of the diagram, and hence $\gamma = \beta$. It is easily checked that for all $x \in X$, $x\beta: B \to A$ is a homomorphism and thus β is well defined. It remains only to prove that β is continuous, but for any open subset U of A, $(b;U)\beta^{-1} = U(b\alpha)^{-1}$, which is open in X since $b\alpha$ is continuous.□

If A is finite we may restrict the codomain of $\mathcal{A}(-,A)$ and the domain of $\mathcal{C}(-,A)$ to the category *ZComp* of Boolean spaces.

COROLLARY (1.5). *If* A *is finite, then* $(\mathcal{A}(-,A), \mathcal{C}(-,A); \eta, \varepsilon)$ *is an adjunction from* $\mathcal{A}$ *to* $ZComp^{op}$.□

PROPOSITION (1.6). (i) *If* $h \in \mathcal{A}(B,C)$ *is a surjection, then* $\mathcal{A}(h,A): \mathcal{A}(C,A) \to \mathcal{A}(B,A)$ *is a homeomorphism onto a closed subspace.*

(ii) *For all* $\kappa > 0$ *the map* $\rho_\kappa: \mathcal{A}(F_{\mathcal{A}}(\kappa),A) \to A^\kappa$, *defined by* $g\rho_\kappa = \langle x_\gamma g \rangle_{\gamma<\kappa}$, *is a homeomorphism.*

(iii) *Assume that* $B \in \mathcal{A}$ *is generated by* $\{b_\gamma | \gamma < \kappa\}$ *and let*

$h\colon \mathcal{F}_A(\kappa) \to B$ *be determined by* $x_\gamma h = b_\gamma$ $(\gamma < \kappa)$. *Then* $A(h,A)\rho_\kappa\colon A(B,A) \to A^\kappa$ *is a homeomorphism onto a closed subspace.*

Proof. (i) Clearly $A(h,A)$ is one-one and since $A(C,A)$ is compact and $A(B,A)$ is Hausdorff we need only prove that $A(h,A)$ is continuous. But if U is open in A and $b \in B$, then $(b;U)A(h,A)^{-1} = (bh;U)$ which is open in $A(C,A)$.

(ii) By the definition of free algebra it follows that ρ_κ is a bijection. Again it is sufficient to prove that ρ_κ is continuous, but since $(\gamma;U)\rho_\kappa^{-1} = (x_\gamma;U)$ for every open set U in A, this is immediate.

(iii) This follows directly from (i) and (ii).□

REMARK (1.7). In the sequel we will abbreviate ρ_1 to ρ.

DEFINITION (1.8). Let $D\colon A \to \mathcal{X}^{op}$ be a functor into a category $\mathcal{X}$ with a grounding $|-|\colon \mathcal{X} \to \mathit{Comp}$ into the category of compact spaces. Then $(D,\mathcal{X},|-|)$ is a *proto-dual of* A if

(i) $|D(-)| = A(-,A)$, and

(ii) $E(X) = \{|\phi|\rho\colon \phi \in \mathcal{X}(X,D\mathcal{F}_A(1))\}$ is a subalgebra of $C(|X|,A)$ for all $X \in \mathcal{X}$.

REMARK (1.9). By analogy with Proposition (1.2) we will denote $D\mathcal{F}_A(1)$ by $A^\dagger$. This abuse of notation is justified by the fact that $|D\mathcal{F}_A(1)| = A(F_A(1),A)$ is homeomorphic to A by Proposition (1.6)(ii).

Let $(D,\mathcal{X},|-|)$ be a proto-dual of A. If f is an m-ary operation and $\phi_0,\ldots,\phi_{m-1} \in \mathcal{X}(X,A^\dagger)$, then by (1.8)(ii) there exists

$\phi \in \mathcal{X}(X,A^{\dagger})$ such that $|\phi|\rho = f(|\phi_0|\rho,\ldots,|\phi_{m-1}|\rho)$. Since $|-|$ is faithful, ϕ is unique and will be denoted by $f(\phi_0,\ldots,\phi_{m-1})$. Clearly this defines an algebraic structure on $\mathcal{X}(X,A^{\dagger})$ in such a way that the map $\phi \to |\phi|\rho$ is an isomorphism between $\mathcal{X}(X,A^{\dagger})$ and the subalgebra $E(X) = \{|\phi|\rho\colon \phi \in \mathcal{X}(X,A^{\dagger})\}$ of $C(|X|,A)$. If $\psi \in \mathcal{X}(X,Y)$, then $\mathcal{X}(\psi,A^{\dagger})\colon \mathcal{X}(Y,A^{\dagger}) \to \mathcal{X}(X,A^{\dagger})$ is defined by $\phi\mathcal{X}(\psi,A^{\dagger}) = \psi\phi$, and $E(\psi)\colon E(Y) \to E(X)$ is defined by $|\phi|\rho E(\psi) = |\psi||\phi|\rho$. The following lemma is obvious.

LEMMA (1.10). *If $(D,\mathcal{X},|-|)$ is a proto-dual of $\mathcal{A}$, then $\mathcal{X}(-,A^{\dagger})\colon \mathcal{X}^{op} \to \mathcal{A}$ and $E\colon \mathcal{X}^{op} \to \mathcal{A}$ are naturally isomorphic functors.* □

We will work with the functor E rather than $\mathcal{X}(-,A^{\dagger})$ since this makes some of the proofs a little less technical. Note that, like the functor D, E maps surjections to injections.

Our aim at the moment is to prove an analogue of Proposition (1.4) for the categories $\mathcal{A}$ and $\mathcal{X}$, and the next result shows that we may define the natural transformation $\eta\colon \mathrm{Id}_{\mathcal{A}} \to ED$ exactly as before.

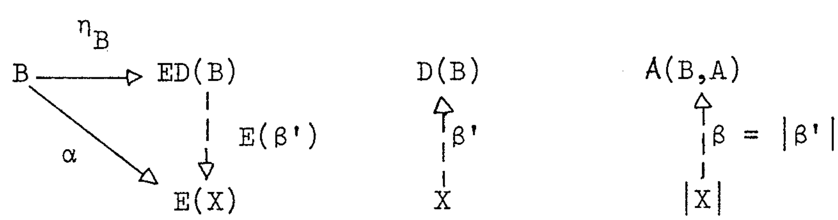
LEMMA (1.11). *Let $(D,\mathcal{X},|-|)$ be a proto-dual of $\mathcal{A}$. For all $B \in \mathcal{A}$ and each $b \in B$, the map $[b]\colon \mathcal{A}(B,A) \to A$, defined by $g[b] = bg$, is an element of $ED(B)$. In fact, $[b] = |D(h_b)|\rho$ where $h_b\colon F_{\mathcal{A}}(1) \to B$ is determined by $x_0h_b = b$.*

Proof. $g|D(h_b)|\rho = g\mathcal{A}(h_b,A)\rho = (h_bg)\rho = bg = g[b]$. □

DEFINITION (1.12). A continuous map $\phi\colon |X| \to |Y|$ *lifts to* $\mathcal{X}$ if there is an arrow $\phi' \in \mathcal{X}(X,Y)$ with $|\phi'| = \phi$.

PROPOSITION (1.13). *Let* $(D,\mathcal{X},|-|)$ *be a proto-dual of* $\mathcal{A}$ *and for all* $B \in \mathcal{A}$ *define* $\eta_B\colon B \to ED(B)$ *by* $b\eta_B = [b]$. *Then* $\eta\colon Id_{\mathcal{A}} \to ED$ *is a natural transformation and* E *is right adjoint to* D *with* η *as the unit of the adjunction if and only if, for all* $X \in \mathcal{X}$ *and every homomorphism* $\alpha\colon B \to E(X)$, *the continuous map* $\beta\colon |X| \to |D(B)| = \mathcal{A}(B,A)$, *defined by* $b(x\beta) = x(b\alpha)$, *lifts to* $\mathcal{X}$.

Proof. That η is a natural transformation follows as in the proof of Proposition (1.4). Let $\alpha\colon B \to E(X)$ be a homomorphism and consider the following diagram.



Clearly the commutativity of the diagram is equivalent to $x(b\alpha) = x(b\eta_B E(\beta')) = x(|\beta'|[b]) = x(\beta[b]) = b(x\beta)$, for all $b \in B$ and all $x \in |X|$. The result follows at once. □

REMARK (1.14). If E is right adjoint to D, then the counit $\varepsilon\colon Id_{\mathcal{X}} \to DE$ of the adjunction satisfies $x|\varepsilon_X| = [x]$, where $[x] \in |DE(X)| = \mathcal{A}(E(X),A)$ is defined by $\phi[x] = x\phi$ for all $\phi \in E(X)$.

The main theorem of this chapter may now be stated; its proof will consist of a series of lemmas.

THEOREM (1.15). *Let* $(D,\mathcal{X},|-|)$ *be a proto-dual of* $\mathcal{A}$. *Then* (D,E) *is a duality between* $\mathcal{A}$ *and* $\mathcal{X}$ *if and only if*

(D_0) *for all* $B \in \mathcal{A}$, *all* $X \in \mathcal{X}$ *and every* $\alpha \in \mathcal{A}(B,E(X))$, *the map* $\beta \in C(|X|,\mathcal{A}(B,A))$, *defined by* $b(x\beta) = x(b\alpha)$, *lifts to* $\mathcal{X}$,

(D_1) *there is a class* I *of* X*-injections, containing the image under* D *of the class of all* A*-surjections, such that* $A^\dagger$ *is* I*-injective in* X,

(D_2) *for all* $1 \leq n < \omega$ *and each* $\phi \in X(D\,F_A(n), A^\dagger)$, *the map* $\rho_n^{-1}|\phi|\rho\colon A^n \to A$ *is a polynomial function, and*

(D_3) *for all* $\kappa \geq \omega$ *and each* $\phi \in X(D\,F_A(\kappa), A^\dagger)$, *the map* $\rho_\kappa^{-1}|\phi|\rho\colon A^\kappa \to A$ *has finite support.*

Furthermore, if A *is finite, then* (D,E) *is a duality if and only if* (D_0), (D_1) *and* (D_2) *hold.*

The necessity of (D_0) follows from Proposition (1.13), (D_2) is clearly necessary for η_B to be an isomorphism whenever B is a finitely generated free algebra, and (D_3) is necessary for η_B to be an isomorphism whenever B is an infinitely generated free algebra since κ-ary polynomial functions clearly have finite support. Finally, the necessity of (D_1) follows from the next lemma and its corollary.

LEMMA (1.16). *Let* $(D, X, |-|)$ *be a proto-dual of* A. *If* (D,E) *is a duality, then* D *is full and faithful.*

Proof. For any $g \in A(B,C)$ we have $\eta_B ED(g) = g\eta_C$. Thus if $g, h \in A(B,C)$ satisfy $D(g) = D(h)$, then $g\eta_C = \eta_B ED(g) = \eta_B ED(h) = h\eta_C$ and hence $g = h$ since η_C is monic.

Let $\phi \in X(D(C), D(B))$. We will construct $g \in A(B,C)$ satisfying $|D(g)| = A(g,A) = |\phi|$, from which it follows that $D(g) = \phi$

since $|-|$ is faithful. Let $g = \eta_B E(\phi)\eta_C^{-1}$ and note that for all $b \in B$, $bg = (|\phi|[b])\eta_C^{-1}$. Now for any $\psi \in ED(C)$ we have $h\psi = h(\psi\eta_C^{-1}\eta_C) = h[\psi\eta_C^{-1}] = (\psi\eta_C^{-1})h$. Hence, setting $\psi = |\phi|[b]$, we have $b(gh) = (bg)h = ((|\phi|[b])\eta_C^{-1})h = h(|\phi|[b]) = (h|\phi|)[b]$ $= b(h|\phi|)$. This gives $h|\phi| = gh = h\mathcal{A}(g,A)$ for all $h \in \mathcal{A}(C,A)$, whence $|\phi| = \mathcal{A}(g,A) = |D(g)|$.□

COROLLARY (1.17). *Let $(D,\mathcal{X},|-|)$ be a proto-dual of* A. *If* (D,E) *is a duality, then $A^\dagger$ is $\mathcal{I}$-injective where $\mathcal{I} = \{D(g) | g$ is an $\mathcal{A}$-surjection$\}$.*

Proof. It follows from Proposition (1.6)(i) that $\mathcal{I}$ is a class of $\mathcal{X}$-injections and hence $\mathcal{I}$-injectivity is a well-defined concept. The $\mathcal{I}$-injectivity of $A^\dagger = D\,F_{\mathcal{A}}(1)$ follows immediately from the sur-projectivity of $F_{\mathcal{A}}(1)$ and the fullness of D.□

We will now show the sufficiency of the conditions (D_0), (D_1), (D_2) and (D_3). Note that for all $B \in \mathcal{A}$, $\mathcal{A}(B,A)$ separates the points of B since $\mathcal{A} = ISP(A)$ and hence η_B is an embedding. Thus it remains to show that η_B is a surjection for all $B \in \mathcal{A}$.

LEMMA (1.18). *Let $(D,\mathcal{X},|-|)$ be a proto-dual of* A. *If (D_2) and (D_3) hold, then for all κ, η_B is an isomorphism for $B = F_{\mathcal{A}}(\kappa)$.*

Proof. Clearly (D_2) implies that for all $1 \leq n < \omega$, η_B is a surjection for $B = F_{\mathcal{A}}(n)$. Let $\kappa \geq \omega$ and let $\phi \in \mathcal{X}(D\,F_{\mathcal{A}}(\kappa),A^\dagger)$. Then $\sigma = \rho_\kappa^{-1}|\phi|\rho\colon A^\kappa \to A$ has a finite support, say $\{\gamma_0,\ldots,\gamma_{n-1}\}$. Define $h\colon F_{\mathcal{A}}(\kappa) \to F_{\mathcal{A}}(n)$ by $x_{\gamma_j}h = x_j$ $(j < n)$ and h arbitrary on all other generators. Then $D(h)\phi \in \mathcal{X}(D\,F_{\mathcal{A}}(n),A^\dagger)$ and hence

$\lambda = \rho_n^{-1}|D(h)\phi|\rho\colon A^n \to A$ is a polynomial function. Let $p(x_0,\ldots,x_{n-1})$ be an n-ary polynomial with $p(a_0,\ldots,a_{n-1}) = (a_0,\ldots,a_{n-1})\lambda$ for all $a_0,\ldots,a_{n-1} \in A$. It is easily seen that the κ-ary polynomial $q(\langle x_\gamma\rangle_{\gamma<\kappa}) = p(x_{\gamma_0},\ldots,x_{\gamma_{n-1}})$ satisfies $q(\langle a_\gamma\rangle) = \langle a_\gamma\rangle\sigma$ for all $\langle a_\gamma\rangle \in A^\kappa$ and hence η_B is a surjection for $B = F_A(\kappa)$.□

Condition (D_1) now allows us to extend to arbitrary algebras in A by taking homomorphic images.

LEMMA (1.19). *Let* $(D, X, |-|)$ *be a proto-dual of* A. *If* η_B *is an isomorphism for* $B = F_A(\kappa)$ *and* (D_1) *holds, then* η_B *is an isomorphism for every* κ*-generated algebra* $B \in A$.

Proof. Suppose that B is generated by $\{b_\gamma | \gamma < \kappa\}$ and let $h\colon F_A(\kappa) \to B$ be determined by $x_\gamma h = b_\gamma$ $(\gamma < \kappa)$. Since h is a surjection, for every arrow $\phi \in X(D(B),A^\dagger)$ there is an arrow $\psi \in X(DF_A(\kappa),A^\dagger)$ with $\phi = D(h)\psi$, by (D_1). Now there is a κ-ary polynomial, say q, with $q = \rho_\kappa^{-1}|\psi|\rho$. For all $g \in A(B,A)$ we have $g(|\phi|\rho) = g(|D(h)||\psi|\rho) = g(A(h,A)|\psi|\rho) = hg(|\psi|\rho) = (hg)\rho_\kappa\rho_\kappa^{-1}|\psi|\rho$ $= q(\langle b_\gamma g\rangle_{\gamma<\kappa}) = q(\langle b_\gamma\rangle_{\gamma<\kappa})g = g[q(\langle b_\gamma\rangle_{\gamma<\kappa})]$. Hence $|\phi|\rho = [b] = b\eta_B$, where $b = q(\langle b_\gamma\rangle_{\gamma<\kappa})$, and thus η_B is a surjection.□

The sufficiency of (D_0), (D_1), (D_2) and (D_3) now follows immediately from Proposition (1.13), Lemma (1.18) and Lemma (1.19). That D_3 is superfluous when A is finite follows from the following observation.

LEMMA (1.20). *Let* A *be a finite, discrete topological space. Then every continuous map* $\phi\colon A^\kappa \to A$ *has finite support.*

Proof. It is not difficult to see that U is clopen in A^κ if and only if there exist $\gamma_{ij} < \kappa$ and $a_{ij} \in A$ such that $U = \bigcup(\bigcap((\gamma_{ij};\ \{a_{ij}\})|j < n_i)|i < n)$; we will say that $\{\gamma_{ij}|j < n_i;\ i < n\}$ *fixes* U.

Now $\{a\phi^{-1}|a \in \mathrm{Im}(\phi)\}$ is a finite partition of A^κ into clopen sets. For each $a \in \mathrm{Im}(\phi)$ let Γ_a be a finite set of indices which fixes the clopen set $a\phi^{-1}$. Clearly $\bigcup(\Gamma_a|a \in \mathrm{Im}(\phi))$ is a finite support for ϕ. □

This completes the proof of Theorem (1.15). □

It follows from Proposition (1.6) that for all $B \in \mathcal{A}$ there exists κ and an injection $\tau\colon D(B) \to DF_{\mathcal{A}}(\kappa) \in \mathcal{I}$ since $\mathcal{I}$ contains the image under D of the class of all $\mathcal{A}$-surjections. If for every $X \in \mathcal{X}$ there exists κ and an injection $\tau\colon X \to DF_{\mathcal{A}}(\kappa)$ (which must happen if (D,E) is a full duality), then in Theorem (1.15) we can slightly weaken the assumption that $(D,\mathcal{X},|-|)$ should be a proto-dual of $\mathcal{A}$. Namely, the assumption that E(X) is a subalgebra of $C(|X|,A)$ can be dropped.

THEOREM (1.21). *Let* $D\colon \mathcal{A} \to \mathcal{X}^{op}$ *be a functor into a category* $\mathcal{X}$ *with a grounding* $|-|\colon \mathcal{X} \to$ *Comp such that* $|D(-)| = \mathcal{A}(-,A)$. *Assume* (D_0), (D_1), (D_2) *(and* D_3*, if* A *is infinite) and assume that for every* $X \in \mathcal{X}$ *there exists* κ *and an injection* $\tau\colon X \to DF_{\mathcal{A}}(\kappa) \in \mathcal{I}$. *Then* $(D,\mathcal{X},|-|)$ *is a proto-dual of* $\mathcal{A}$ *and* (D,E) *is a duality.*

Proof. We must show that for all $X \in \mathcal{X}$, $E(X) = \{|\phi|\rho\colon \phi \in \mathcal{X}(X,A^\dagger)\}$ is a subalgebra of $C(|X|,A)$. Let $\phi_0,\ldots,\phi_{m-1} \in \mathcal{X}(X,A^\dagger)$ and let

f be an m-ary operation. Since $A^\dagger$ is $\mathcal{I}$-injective there exist $\psi_0,\ldots,\psi_{m-1} \in \mathcal{X}(D\,\mathcal{F}_A(\kappa),A^\dagger)$ with $\phi_j = \tau\psi_j$ $(j < m)$. Now (D_2) and (D_3) imply that $ED\,\mathcal{F}_A(\kappa)$ is a subalgebra of $C(A(\mathcal{F}_A(\kappa),A),A)$ isomorphic to $\mathcal{F}_A(\kappa)$, as in the proof of Lemma (1.18). Thus $f(|\psi_0|\rho,\ldots,|\psi_{m-1}|\rho) \in ED\,\mathcal{F}_A(\kappa)$ and so there exists $\psi \in \mathcal{X}(D\mathcal{F}_A(\kappa),A^\dagger)$ with $|\psi|\rho = f(|\psi_0|\rho,\ldots,|\psi_{m-1}|\rho)$. Hence $f(|\phi_0|\rho,\ldots,|\phi_{m-1}|\rho)$ $= f(|\tau||\psi_0|\rho,\ldots,|\tau||\psi_{m-1}|\rho) = |\tau|f(|\psi_0|\rho,\ldots,|\psi_{m-1}|\rho) = |\tau||\psi|\rho$ $= |\tau\psi|\rho \in E(X)$. □

We will now give a brief discussion of conditions under which a duality (D,E) between $\mathcal{A}$ and $\mathcal{X}$ will be a full duality. Since so little has been assumed about $\mathcal{X}$ it is not surprising that such conditions are little more than tautological.

LEMMA (1.22). *Let $(D,\mathcal{X},|-|)$ be a proto-dual of $\mathcal{A}$ and assume that (D,E) is a duality. Then (D,E) is a full duality if and only if for each $X \in \mathcal{X}$, $|\varepsilon_X|$ is a homeomorphism and there exists $\phi \in \mathcal{X}(DE(X),X)$ with $|\phi| = |\varepsilon_X|^{-1}$, that is $|\varepsilon_X|^{-1}$ lifts to $\mathcal{X}$.*

Proof. Only the sufficiency requires proof. Clearly we must show that $\phi = \varepsilon_X^{-1}$. But $|Id_X| = Id_{|X|} = |\varepsilon_X||\varepsilon_X|^{-1} = |\varepsilon_X||\phi| = |\varepsilon_X\phi|$ and hence $Id_X = \varepsilon_X\phi$ since $|-|$ is faithful. Similarly, $Id_{DE(X)} = \phi\varepsilon_X$. □

LEMMA (1.23). *Let $(D,\mathcal{X},|-|)$ be a proto-dual of $\mathcal{A}$ and assume that (D,E) is a duality. Then the following are equivalent for each $X \in \mathcal{X}$:*

(i) *$|\varepsilon_X|$ is a homeomorphism onto a closed subspace;*

(ii) *$\mathcal{X}(X,A^\dagger)$ separates the points of $|X|$;*

(iii) *there is a* κ *and an arrow* $\tau \in X(X, DF_A(\kappa))$ *such that* $|\tau|\rho_\kappa\colon |X| \to A^\kappa$ *is a homeomorphism onto a closed subspace.*

Proof. (i) <=> (ii). This follows immediately from Remark (1.14).

(i) => (iii). Assume that E(X) is generated by $\{b_\gamma | \gamma < \kappa\}$ and let $h\colon F_A(\kappa) \to E(X)$ be determined by $x_\gamma h = b_\gamma$ $(\gamma < \kappa)$. Then D(h) is an injection by Proposition (1.6)(i) and $\tau = \varepsilon_X D(h)$ is the required injection.

(iii) => (ii). Let $x, y \in |X|$ be distinct. Since $\langle a_\gamma \rangle_{\gamma<\kappa} = x|\tau|\rho_\kappa \neq y|\tau|\rho_\kappa = \langle b_\gamma \rangle_{\gamma<\kappa}$, there exists $\lambda < \kappa$ such that $a_\lambda \neq b_\lambda$. Let $h_\lambda\colon F_A(1) \to F_A(\kappa)$ be determined by $x_0 h_\lambda = x_\lambda$. Then $\phi = \tau D(h_\lambda) \in X(X, A^\dagger)$ and $x|\phi| = x|\tau|A(h_\lambda, A) = h_\lambda(x|\tau|) = a_\lambda \neq b_\lambda = h_\lambda(y|\tau|) = y|\tau|A(h_\lambda, A) = y|\phi|$, as required. □

If (D,E) is a full duality between A and X, we may describe the left adjoint to the grounding $|-|\colon X \to Comp$, that is the *Comp*-free functor for X.

PROPOSITION (1.24). *Let* $(D, X, |-|)$ *be a proto-dual of* A. *Then the following statements are related by* (i) => (ii) => (iii).

(i) (D,E) *is a full duality.*

(ii) DC(-,A): *Comp* $\to X$ *is left adjoint to* $|-|\colon X \to$ *Comp.*

(iii) $DA^{(-)}$: *Set* $\to X$ *is left adjoint to the forgetful functor* $|-|\colon X \to$ *Set.*

Proof. (i) => (ii). C(-,A): *Comp* $\to A^{op}$ is left adjoint to $A(-,A)$: $A^{op} \to$ *Comp* by Proposition (1.4), and if (D,E) is a full duality then D: $A^{op} \to X$ is left adjoint to E: $X \to A^{op}$. Since left adjoints compose, (ii) follows.

(ii) => (iii). Since left adjoints compose it follows that $DC(\beta(-),A)$: *Set* $\to \mathcal{X}$ is left adjoint to $|-|: \mathcal{X} \to$ *Set*, where β: *Set* $\to$ *Comp* is the Stone-Čech compactification functor. Since $C(\beta S,A) \simeq C(S,A) \simeq A^S$, (iii) follows. □

REMARK (1.25). If A is finite, then we may replace (ii) in Proposition (1.24) by

(ii)' $DC(-,A)$: *ZComp* $\to \mathcal{X}$ *is left adjoint to* $|-|: \mathcal{X} \to$ *ZComp*.

In Chapter 4 we will see examples where $DC(-,A)$: *ZComp* $\to \mathcal{X}$ is left adjoint to $|-|: \mathcal{X} \to$ *ZComp* while (D,E) is not a full duality. If A is injective in $\mathcal{A}$, then we can obtain a partial converse to Proposition (1.24).

PROPOSITION (1.26). *Let* $(D,\mathcal{X},|-|)$ *be a proto-dual of* A *and assume that* (D,E) *is a duality. If* A *is injective in* $\mathcal{A}$, $\mathcal{X}(X,A^\dagger)$ *separates the points of* $|X|$ *for each* $X \in \mathcal{X}$, *and* $DC(-,A)$ *is left ajoint to* $|-|$, *then* $|\varepsilon_X|: |X| \to |DE(X)|$ *is a homeomorphism for each* $X \in \mathcal{X}$.

Proof. Since $\mathcal{X}(X,A^\dagger)$ separates the points of $|X|$, by Lemma (1.23) it is sufficient to show that $|\varepsilon_X|$ is a surjection for each $X \in \mathcal{X}$. Firstly we will show that $|\varepsilon_{D(B)}|$ is a surjection for each $B \in \mathcal{A}$; in fact, we will show that $\varepsilon_{D(B)}$ is an isomorphism in $\mathcal{X}$. Since D and E are adjoint functors it follows that $\varepsilon_{D(B)}D(\eta_B) = Id_{D(B)}$ (see [53, Theorem 2, p.81]). But since η_B is invertible, $D(\eta_B)$ is invertible and consequently $\varepsilon_{D(B)} = Id_{D(B)}D(\eta_B)^{-1}$ is invertible.

Denote the unit and counit of the adjoint pair $(|-|,DC(-,A))$

by ζ: $Id_{Comp} \to |DC(-,A)|$ and ξ: $DC(|-|,A) \to Id_{X}$. Since $\zeta_{|X|}|\xi_X| = Id_{|X|}$ (see [53, Theorem 2, p.81]) it follows that $|\xi_X|$ is a surjection and hence $E(\xi_X)$: $E(X) \to EDC(|X|,A)$ is an A-injection.

We will now show that $|\varepsilon_X|$: $|X| \to |DE(X)|$ is a surjection. Let $g \in A(E(X),A)$. Since $E(\xi_X)$ is an injection and A is injective in A, there exists a homomorphism g': $EDC(|X|,A) \to A$ with $g = E(\xi_X)g'$. Let $B = C(|X|,A)$. Since $|\varepsilon_{D(B)}|$ is a surjection it follows that there exists $\phi \in |D(B)| = A(C(|X|,A),A)$ with $[\phi] = \phi|\varepsilon_{D(B)}| = g'$. Set $x = \phi|\xi_X|$ and let $\psi \in E(X) \subseteq C(|X|,A)$. Then $\psi(x|\varepsilon_X|) = \psi[x] = x\psi = (\phi|\xi_X|)\psi = \phi(|\xi_X|\psi) = (|\xi_X|\psi)[\phi]$ $= (|\xi_X|\psi)g' = (\psi E(\xi_X))g' = \psi(E(\xi_X)g') = \psi g$, and hence $x|\varepsilon_X| = g$, as required. □

By collecting together (1.22) - (1.26), observing that the proof of Proposition (1.26) only requires that A be injective with respect to the images under E of the X-surjections, and noting that if (D,E) is a full duality, then E is full (and faithful) we obtain the following characterization of full dualities.

THEOREM (1.27). *Let* $(D,X,|-|)$ *be a proto-dual of* A *and assume that* (D,E) *is a duality. Then* (D,E) *is a full duality if and only if*

(E_1) $X(X,A^{\dagger})$ *separates the points of* $|X|$ *for all* $X \in X$,

(E_2) *there is a class* I *of* A*-injections, containing the image under* E *of the class of* X*-surjections, such that* A *is* I*-injective in* A,

(E_3) $DC(-,A)$ *is left adjoint to* $|-|$, *and*

(E_4) *the homeomorphism* $|\varepsilon_X|^{-1}: |DE(X)| \to |X|$ *lifts to* X *for each* $X \in \mathcal{X}$.

Proof. Note that (E_1), (E_2) and (E_3) guarantee that (E_4) is meaningful.

Only the necessity of (E_2) has not already been established. (E_3) implies that $DA^{(-)}$ is the free functor for $\mathcal{X}$ and hence $D(A)$ is sur-projective in $\mathcal{X}$ (see Proposition (1.29)). Let $\mathcal{I} = \{E(\phi) | \phi$ is an $\mathcal{X}$-surjection$\}$. Then the $\mathcal{I}$-injectivity of A follows from the fact that $D(A)$ is sur-projective and E is full. □

We close the chapter with some remarks on injectives and projectives.

PROPOSITION (1.28). *Let* $(D,\mathcal{X},|-|)$ *be a proto-dual of* $\mathcal{A}$ *with* (D,E) *a full duality, and assume that* A *is injective in* $\mathcal{A}$. *Then the following are equivalent:*

(i) I *is injective in* $\mathcal{A}$;

(ii) $D(I)$ *is sur-projective in* $\mathcal{X}$;

(iii) $I \simeq E(P)$ *for some* P, *sur-projective in* $\mathcal{X}$.

Proof. (i) => (ii). Let $\phi \in \mathcal{X}(X,Y)$ be a surjection and let $\psi \in \mathcal{X}(D(I),Y)$. Then $E(\phi)$ is an injection and hence, since $ED(I) \simeq I$ is injective, there exists $g \in \mathcal{A}(E(X),ED(I))$ with $E(\phi)g = E(\psi)$. Since the duality is full, $\psi' = \varepsilon_{D(I)}D(g)\varepsilon_X^{-1} \in \mathcal{X}(D(I),X)$ satisfies $\psi'\phi = \psi$, as required.

(ii) => (iii). This is trivial.

(iii) => (i). Let $g \in \mathcal{A}(B,C)$ be an injection and let $h \in \mathcal{A}(B,I)$. Since $\mathcal{A}$ is injective, $D(g)$ is a surjection and since $D(I) \simeq DE(P) \simeq P$ is sur-projective there exists $\phi \in \mathcal{X}(D(I),D(C))$ with $\phi D(g) = D(h)$. Thus $h' = \eta_C E(\phi)\eta_I^{-1} \in \mathcal{X}(C,I)$ satisfies $gh' = h$, as required. □

The following useful result is proved in [40].

PROPOSITION (1.29). *If* $\mathcal{X}$ *is a category grounded in Comp (ZComp) and the grounding has a left adjoint* F: *Comp* $\to \mathcal{X}$ (F: *ZComp* $\to \mathcal{X}$), *then the following are equivalent:*

(i) P *is sur-projective in* $\mathcal{X}$;

(ii) P *is a retract of* $F(\beta S)$ *for some set* S;

(iii) P *is a retract of* $F(X)$ *for some compact, extremally disconnected space* X. □

2. DUALITIES VIA COMPACT TOPOLOGICAL PARTIAL ALGEBRAS: GENERAL THEORY

As in Chapter 1, throughout this chapter we will assume that A is a compact topological algebra and that $\mathcal{A}$ = ISP(A) is the quasi-variety generated by A.

For any B ε $\mathcal{A}$ and any n-ary polynomial p we may regard p, defined pointwise, as an operation on the set A^B. By relativization p becomes a partial operation on the subset $\mathcal{A}(B,A)$. Hence for all $g_0,\ldots,g_{n-1} \in \mathcal{A}(B,A)$, $(g_0,\ldots,g_{n-1}) \in D_p$ if and only if $p(g_0,\ldots,g_{n-1})$, defined pointwise, is a homomorphism.

REMARK (2.1). (i) Clearly $(g_0,\ldots,g_{n-1}) \in D_p$ if and only if for every m-ary operation f and all $b_0,\ldots,b_{m-1} \in B$,

$$p(f(b_0g_0,\ldots,b_{m-1}g_0),\ldots,f(b_0g_{n-1},\ldots,b_{m-1}g_{n-1}))$$
$$= f(p(b_0g_0,\ldots,b_0g_{n-1}),\ldots,p(b_{m-1}g_0,\ldots,b_{m-1}g_{n-1})).$$

(ii) Note that if D_p is non-empty, then $p(0,\ldots,0) = 0$ for every nullary operation 0. Hence if 0 and 1 are distinct nullary operations on A, then D_0 and D_1 are both empty.

Throughout the remainder of this chapter $\mathcal{Q}$ will denote a fixed set of finitary polynomials (of the type of the algebra A) and whenever we refer to the partial algebra $\mathcal{A}(B,A)$ it will be implied that the operations on $\mathcal{A}(B,A)$ are exactly those which arise as described above from polynomials in $\mathcal{Q}$.

LEMMA (2.2). *For each* B ε $\mathcal{A}$, $\mathcal{A}(B,A)$ *is a compact topological partial algebra.*

Proof. $\mathcal{A}(B,A)$ is compact by Lemma (1.3)(ii). For each p ε $\mathcal{Q}$, p is a continuous (full) operation on A^B and hence the relativization of p to $A(B,A)$ is also continuous. □

DEFINITION (2.3). A map $\phi: X \to Y$ between two similar topological partial algebras will be called an *iseomorphism* if it is both an isomorphism and a homeomorphism.

Define a partial algebra structure on the compact space A^κ as follows: if $p \in \mathcal{Q}$ is n-ary, then for all $\underline{a}^0,\ldots,\underline{a}^{n-1} \in A^\kappa$, $(\underline{a}^0,\ldots,\underline{a}^{n-1}) \in D_p$ if and only if

$$p(q(\underline{a}^0),\ldots,q(\underline{a}^{n-1})) = q(p(\underline{a}^0,\ldots,\underline{a}^{n-1}))$$

for every κ-ary polynomial q. If $(\underline{a}^0,\ldots,\underline{a}^{n-1}) \in D_p$, then $p(\underline{a}^0,\ldots,\underline{a}^{n-1})$ is defined pointwise and hence the partial algebra A^κ is a weak subalgebra of the full algebra A^κ.

We may now prove an analogue of Proposition (1.6).

PROPOSITION (2.4). (i) *If* $h \in A(B,C)$ *is a surjection, then* $A(h,A): A(C,A) \to A(B,A)$ *is an iseomorphism onto a closed subalgebra.*

(ii) *For all* $\kappa > 0$ *the map* $\rho_\kappa: A(F_A(\kappa),A) \to A^\kappa$, *defined by* $g\rho_\kappa = \langle x_\gamma g\rangle_{\gamma<\kappa}$, *is an iseomorphism.*

(iii) *Assume that* $B \in \mathcal{A}$ *is generated by* $\{b_\gamma | \gamma < \kappa\}$ *and let* $h: F_A(\kappa) \to B$ *be determined by* $x_\gamma h = b_\gamma$ $(\gamma < \kappa)$. *Then* $A(h,A)\rho_\kappa: A(B,A) \to A^\kappa$ *is an iseomorphism onto a closed subalgebra.*

Proof. (i) By Proposition (1.6) and the definition of $A(h,A)$ it remains to prove that for all $p \in \mathcal{Q}$, $p(g_0,\ldots,g_{n-1})$ is defined in $A(C,A)$ if and only if $p(hg_0,\ldots,hg_{n-1})$ is defined in $A(B,A)$. But if $p(g_0,\ldots,g_{n-1})$ is a homomorphism, then $hp(g_0,\ldots,g_{n-1})$

$= p(hg_0,\ldots,hg_{n-1})$ is a homomorphism, and conversely, if $p(hg_0,\ldots,hg_{n-1}) = hp(g_0,\ldots,g_{n-1})$ is a homomorphism then, since h is onto, $p(g_0,\ldots,g_{n-1})$ is a homomorphism.

(ii) Let $p \in \mathcal{Q}$ be n-ary, let $g_0,\ldots,g_{n-1} \in \mathcal{A}(\mathcal{F}_A(\kappa),A)$, and define $\underline{a}^0,\ldots,\underline{a}^{n-1} \in A^\kappa$ by $a^j_\gamma = x_\gamma g_j$ for all $\gamma < \kappa$ and $j < n$. Then $g_j\rho_\kappa = \underline{a}^j$ for all $j < n$.

Now $p(g_0,\ldots,g_{n-1})$ is a homomorphism if and only if it is determined by its values on the generators, that is if and only if for each κ-ary polynomial q,

$$q(\langle x_\gamma\rangle_{\gamma<\kappa})p(g_0,\ldots,g_{n-1}) = q(\langle x_\gamma p(g_0,\ldots,g_{n-1})\rangle_{\gamma<\kappa}),$$

i.e., $p(q(\langle x_\gamma\rangle)g_0,\ldots,q(\langle x_\gamma\rangle)g_{n-1}) = q(\langle p(x_\gamma g_0,\ldots,x_\gamma g_{n-1})\rangle)$,

i.e., $p(q(\underline{a}^0),\ldots,q(\underline{a}^{n-1})) = q(p(\underline{a}^0,\ldots,\underline{a}^{n-1}))$.

Thus $p(g_0,\ldots,g_{n-1})$ is defined in $\mathcal{A}(\mathcal{F}_A(\kappa),A)$ if and only if $p(g_0\rho_\kappa,\ldots,g_{n-1}\rho_\kappa)$ is defined in A^κ. Since ρ_κ is a homeomorphism by Proposition (1.6)(ii), the result follows.

(iii) This follows directly from (i) and (ii).□

If $h \in \mathcal{A}(B,C)$, then it is easily verified that $\mathcal{A}(h,A)\colon \mathcal{A}(C,A) \to \mathcal{A}(B,A)$ is a continuous homomorphism. Hence it is natural to define a category $\mathcal{X} = \mathcal{X}_{\mathcal{Q}}$ by declaring that X is an object of $\mathcal{X}$ if and only if X is iseomorphic to a closed subalgebra of the compact partial algebra A^κ for some κ, and $\phi \in \mathcal{X}(X,Y)$ if and only if ϕ is a continuous homomorphism.

It is important to note that $\mathcal{X} = \mathcal{X}_{\mathcal{Q}}$ depends upon the choice of the set $\mathcal{Q}$. Since no confusion will arise, no notational distinction will be drawn between an object $X \in \mathcal{X}$ and its underlying compact space, nor between an arrow $\phi \in \mathcal{X}$ and its underlying continuous map, that is we will suppress the grounding $|-|: \mathcal{X} \to Comp$. Hence $D = \mathcal{A}(-,A): \mathcal{A} \to \mathcal{X}^{op}$ is a well-defined functor.

Clearly when applying the results of Chapter 1 to this situation we may replace $A^{\dagger} = DF_A(1)$ by its iseomorphic copy A (regarded as a compact partial algebra - let $\kappa = 1$ in the discussion preceeding Proposition (2.4)). This abuse of notation should cause no problems since in any given situation it will be clear whether A is acting as an object of $\mathcal{A}$ or as an object of $\mathcal{X}$.

In general $\mathcal{X}$ need not be a proto-dual of $\mathcal{A}$ for we do not know that $\mathcal{X}(X,A)$ is a subalgebra of $C(X,A)$. If $\mathcal{X}(X,A)$ is a subalgebra of $C(X,A)$ for all $X \in \mathcal{X}$, then $E = \mathcal{X}(-,A): \mathcal{X}^{op} \to \mathcal{A}$ is a well-defined functor by Lemma (1.10).

PROPOSITION (2.5). *If* $\mathcal{X}(X,A)$ *is a subalgebra of* $C(X,A)$ *for all* $X \in \mathcal{X}$, *then* $(D,E;\eta,\varepsilon)$ *is an adjunction from* $\mathcal{A}$ *to* $\mathcal{X}^{op}$, *where* $\eta_B: B \to ED(B)$ *is defined by* $b\eta_B = [b]$ $(g[b] = bg$ *for all* $g \in \mathcal{A}(B,A))$ *and* $\varepsilon_X: X \to DE(X)$ *is defined by* $x\varepsilon_X = [x]$ $(\phi[x] = x\phi$ *for all* $\phi \in \mathcal{X}(X,A))$.

Proof. By Proposition (1.13) it remains to be shown that for all $B \in \mathcal{A}$, all $X \in \mathcal{X}$ and every $\alpha \in \mathcal{A}(B,\mathcal{X}(X,A))$ the map $\beta: X \to \mathcal{A}(B,A)$, defined by $b(x\beta) = x(b\alpha)$, is a homomorphism.

If $p \in \mathcal{Q}$ is n-ary, $x^0,\dots,x^{n-1} \in X$ and $p(x^0,\dots,x^{n-1})$ is defined in X, then $p(x^0,\dots,x^{n-1})\beta\colon B \to A$ is a homomorphism (see Proposition (1.4)). But

$$b(p(x^0,\dots,x^{n-1})\beta) = p(x^0,\dots,x^{n-1})(b\alpha)$$
$$= p(x^0(b\alpha),\dots,x^{n-1}(b\alpha)), \text{ since } b\alpha \in \mathcal{X}(X,A)$$
$$= p(b(x^0\beta),\dots,b(x^{n-1}\beta)) = bp(x^0\beta,\dots,x^{n-1}\beta),$$

and hence $p(x^0\beta,\dots,x^{n-1}\beta)$ is a homomorphism, since it is pointwise equal to the homomorphism $p(x^0,\dots,x^{n-1})\beta$. Thus $p(x^0\beta,\dots,x^{n-1}\beta)$ is defined in $\mathcal{A}(B,A)$ and equals $p(x^0,\dots,x^{n-1})\beta$. Hence β is a homomorphism. □

Theorem (1.15) may now be applied.

THEOREM (2.6). *If* $\mathcal{X}(X,A)$ *is a subalgebra of* $C(X,A)$ *for all* $X \in \mathcal{X}$, *then* (D,E) *is a duality between* A *and* $\mathcal{X}$ *if and only if*

(D_1) *there is a class* $\mathcal{I}$ *of* $\mathcal{X}$*-injections, containing the image under* D *of the class of all* A*-surjections, such that* A *is* $\mathcal{I}$*-injective in* $\mathcal{X}$,

(D_2) *for all* $1 \leq n < \omega$, *every* $\phi \in \mathcal{X}(A^n,A)$ *is a polynomial function, and*

(D_3) *for all* $\kappa \geq \omega$, *every* $\phi \in \mathcal{X}(A^\kappa,A)$ *has finite support.*

Furthermore, if A *is finite, then* (D,E) *is a duality if and only if* (D_1) *and* (D_2) *hold.* □

DEFINITION (2.7). If I is $\mathcal{I}$-injective in $\mathcal{X}$ for the class $\mathcal{I}$ of all iseomorphisms onto closed subalgebras, we will simply say that I is *injective in* $\mathcal{X}$.

If A is injective in $\mathcal{X}$ then Theorem (1.21) is immediately applicable.

THEOREM (2.8). *E: $\mathcal{X}^{op} \to \mathcal{A}$ is a well-defined functor and (D,E) is a duality between $\mathcal{A}$ and $\mathcal{X}$ whenever*

$(D_1)'$ A is injective in $\mathcal{X}$,

(D_2) for all $1 \leq n < \omega$, every $\phi \in \mathcal{X}(A^n,A)$ is a polynomial function, and

(D_3) for all $\kappa \geq \omega$, every $\phi \in \mathcal{X}(A^\kappa,A)$ has finite support.

Furthermore, if A is finite, then $(D_1)'$ and (D_2) will suffice. □

In the present situation Lemma (1.22) becomes somewhat more satisfactory.

THEOREM (2.9). *Assume that $\mathcal{X}(X,A)$ is a subalgebra of $C(X,A)$ for all $X \in \mathcal{X}$ and that (D,E) is a duality between $\mathcal{A}$ and $\mathcal{X}$. Then (D,E) is a full duality if and only if ε_X: X → DE(X) is a surjection for all $X \in \mathcal{X}$.*

Proof. It is sufficient to prove that if (D,E) is a duality, then ε_X is an iseomorphism onto a closed subalgebra for all $X \in \mathcal{X}$. But ε_X is a homeomorphism onto a closed subspace by Lemma (1.23) and is a homomorphism since $\varepsilon_X \in \mathcal{X}$. Thus we must show that for all $p \in \mathcal{Q}$ and all $x^0,\ldots,x^{n-1} \in X$, $p(x^0,\ldots,x^{n-1})$ is defined in X whenever $p(x^0\varepsilon_X,\ldots,x^{n-1}\varepsilon_X) = p([x^0],\ldots,[x^{n-1}])$ is defined in DE(X).

Without loss of generality we may assume that X is a closed subalgebra of A^κ for some κ. Note that the γ-th projection π_γ: X → A is a continuous homomorphism since it is the restriction to X of the continuous homomorphism $\rho_\kappa^{-1}D(h_\gamma)\rho$: $A^\kappa \to A$, where h_γ: $F_A(1) \to F_A(\kappa)$ is determined by $x_0 h_\gamma = x_\gamma$.

If $p([x^0],\ldots,[x^{n-1}])$ is defined in DE(X), then $p([x^0],\ldots,[x^{n-1}])$: $X(X,A) \to A$ is a homomorphism. Hence if q is ν-ary polynomial and $\phi_\lambda \in X(X,A)$ for all $\lambda < \nu$, then

$$q(<\phi_\lambda>)p([x^0],\ldots,[x^{n-1}]) = q(<\phi_\lambda p([x^0],\ldots,[x^{n-1}])>).$$

In particular, let q be any κ-ary polynomial, let $\phi_\lambda = \pi_\lambda$ $(\lambda < \kappa)$ and assume that $x^j = <a_\gamma^j>_{\gamma<\kappa}$ $(j < n)$. Then

$$\begin{aligned} & p(q(<a_\gamma^0>),\ldots,q(<a_\gamma^{n-1}>)) \\ &= p(q(<<a_\lambda^0>\pi_\gamma>),\ldots,q(<<a_\lambda^{n-1}>\pi_\gamma>)) \\ &= p(<a_\lambda^0>q(<\pi_\gamma>),\ldots,<a_\lambda^{n-1}>q(<\pi_\gamma>)) \\ &= p(q(<\pi_\gamma>)[x^0],\ldots,q(<\pi_\gamma>)[x^{n-1}]) \\ &= q(<\pi_\gamma>)p([x^0],\ldots,[x^{n-1}]) \\ &= q(<\pi_\gamma p([x^0],\ldots,[x^{n-1}])>) \\ &= q(<p(\pi_\gamma[x^0],\ldots,\pi_\gamma[x^{n-1}])>) \\ &= q(<p(x^0\pi_\gamma,\ldots,x^{n-1}\pi_\gamma)>) \\ &= q(<p(a_\gamma^0,\ldots,a_\gamma^{n-1})>). \end{aligned}$$

It follows that $p(x^0,\ldots,x^{n-1}) = p(<a_\gamma^0>,\ldots,<a_\gamma^{n-1}>)$ is defined in A^κ and hence since X is a subalgebra of A^κ, $p(x^0,\ldots,x^{n-1})$ is defined in X.□

We may now combine this result with Proposition (1.26).

COROLLARY (2.10). *Assume that* $X(X,A)$ *is a subalgebra of* $C(X,A)$ *for all* $X \in X$ *and that* (D,E) *is a duality between* A *and* X. *If* DC(-,A): *Comp* (*ZComp*) $\to X$ *is left adjoint to the forgetful* *functor from* $\mathcal{X}$ *into Comp (ZComp) and* A *is injective in* $\mathcal{A}$, *then* (D,E) *is a full duality.* □

REMARK (2.11). S. Fatjlowicz ([22], [23], [24]) has also considered duality theory for quasi-varieties. Rather than consider compact topological algebras he assumes that both the category $\mathcal{A}$ and the category $\mathcal{X}$ are quasi-varieties, but a proper class of possibly infinitary operations is allowed. The results presented here and his results have non-trivial symmetric difference and intersect in the case where $\mathcal{X}$ is a category of compact topological algebras of the same type as A.

If each operation in the type of A gives rise to a full operation on $\mathcal{A}(B,A)$ for all $B \in \mathcal{A}$, then, by choosing $\mathcal{Q}$ to be the set of all operations on A, the category $\mathcal{X} = \mathcal{X}_{\mathcal{Q}}$ becomes a category of compact topological algebras of the same type as A.

PROPOSITION (2.12). (T. Evans [21]). *Let* f_1 *be an* n*-ary operation in the type of* A. *Then the following are equivalent:*

(i) f_1 *induces a full operation on* $\mathcal{A}(B,A)$ *for all* $B \in \mathcal{A}$*;*

(ii) f_1 *induces a full operation on* $\mathcal{A}(F_{\mathcal{A}}(\omega),A)$*;*

(iii) f_1 *induces a full operation on* $\mathcal{A}(F_{\mathcal{A}}(m),A)$ *for all* $m < \omega$

(iv) *if* $m < \omega$, f_2 *is an* m*-ary operation in the type of* A, *and* $\underline{a}^0,\dots,\underline{a}^{m-1} \in A^n$, *then*

$$f_1(f_2(\underline{a}^0,\dots,\underline{a}^{m-1})) = f_2(f_1(\underline{a}^0),\dots,f_1(\underline{a}^{m-1})).$$

Proof. (i) => (ii) is trivial.

(ii) => (iii). $F_{\mathcal{A}}(m)$ is (isomorphic to) a subalgebra of $F_{\mathcal{A}}(\omega)$

and any homomorphism $g\colon \mathcal{F}_A(m) \to A$ extends to a homomorphism $g'\colon \mathcal{F}_A(\omega) \to A$. If $g_0,\ldots,g_{n-1} \in \mathcal{A}(\mathcal{F}_A(m),A)$, then $f_1(g_0,\ldots,g_{n-1})$ is the restriction of $f_1(g'_0,\ldots,g'_{n-1})$ to $\mathcal{F}_A(m)$. Since $f_1(g'_0,\ldots,g'_{n-1})$ is a homomorphism, so is $f_1(g_0,\ldots,g_{n-1})$.

(iii) => (iv). This follows from Proposition (2.4)(ii) with $\kappa = m$.

(iv) => (i). Let $g_0,\ldots,g_{n-1} \in \mathcal{A}(B,A)$. Since $f_1(g_0,\ldots,g_{n-1})$ is defined pointwise, the equations of (iv) imply that $f_1(g_0,\ldots,g_{n-1})$ is a homomorphism.□

REMARK (2.13). The operation f_1 may be defined on Hom(B,C) for every algebra C of the same type as A. Since condition (iv) is a set of equations, (i) - (iv) are equivalent to

(v) f_1 *induces a full operation on* Hom(B,C) *for all* B,C $\in$ HSP(A).

If $\mathcal{Q}$ is the set of all operations in the type of A and $\mathcal{A}(B,A)$ is a full algebra for all $B \in \mathcal{A}$, then $\mathcal{X}(X,A)$ is a subalgebra of $\mathcal{C}(X,A)$ for all $X \in \mathcal{X}_{\mathcal{Q}}$, as the following slightly more general lemma implies.

LEMMA (2.14). *Assume that each operation in the type of* A *induces a full operation on* $\mathcal{A}(B,A)$ *for all* $B \in \mathcal{A}$, *and let* $\mathcal{X}$ *be any full subcategory of the category of all compact topological algebras of the same type as* A *with* $A \in \mathcal{X}$. *Then* $\mathcal{X}(X,A)$ *is a subalgebra of* $\mathcal{C}(X,A)$ *for all* $X \in \mathcal{X}$ *and hence* $E = \mathcal{X}(-,A)\colon \mathcal{X}^{op} \to \mathcal{A}$ *is a well-defined functor.*

Proof. Let f be an m-ary operation and let $\phi_0,\ldots,\phi_{m-1} \in \mathcal{X}(X,A)$.

Then $f(\phi_0,\ldots,\phi_{m-1})$ is a homomorphism since $\mathcal{A}(G(X),A)$ is a full algebra, where $G\colon \mathcal{X} \to \mathcal{A}$ is the forgetful functor, and $\mathcal{X}(X,A) \subseteq \mathcal{A}(G(X),A)$. Similarly, $f(\phi_0,\ldots,\phi_{m-1})$ is continuous since $\mathcal{X}(X,A) \subseteq C(X,A)$. Hence $f(\phi_0,\ldots,\phi_{m-1}) \in \mathcal{X}(X,A)$ since $\mathcal{X}$ is a full subcategory of the category of all compact topological algebras of the same type as A. □

The following result describes an analogue of the *Bohr compactification functor* from the theory of compact abelian groups (see [37]).

THEOREM (2.15). *Let* $\mathcal{Q}$ *be the set of operations in the type of* A *and assume that* $\mathcal{A}(B,A)$ *is a full algebra for all* $B \in \mathcal{A}$. *If* (D,E) *is a full duality between* $\mathcal{A}$ *and* $\mathcal{X} = \mathcal{X}_{\mathcal{Q}}$, *then* $DGD\colon \mathcal{A} \to \mathcal{X}$ *is left adjoint to the forgetful functor* $G\colon \mathcal{X} \to \mathcal{A}$.

Proof. (Sketch). The unit $\mu\colon \mathrm{Id}_{\mathcal{A}} \to GDGD$ of the adjunction is given by $g(b\mu_B) = bg$ for all $b \in B$ and all $g \in GD(B) = G\mathcal{A}(B,A)$. Since every $X \in \mathcal{X}$ is iseomorphic to $D(C)$ for some $C \in \mathcal{A}$, it is sufficient to prove that for each homomorphism $\alpha\colon B \to GD(C)$, there is a unique arrow $\beta\colon DGD(B) \to D(C) \in \mathcal{X}$ with $\mu_B G(\beta) = \alpha$. Define $\nu\colon C \to G\mathcal{A}(B,A)$ by $b(c\beta) = c(b\alpha)$. Then $\beta = D(\nu)$ is the required arrow. □

We close by noting an important restriction on the applicability of the theory expounded in this chapter.

PROPOSITION (2.17). *If* (D,E) *is a duality between* $\mathcal{A}$ *and* $\mathcal{X} = \mathcal{X}_{\mathcal{Q}}$, *then the algebra* A *has at most one one-element subalgebra.*

Proof. If (D,E) is a duality, then $\eta_A: A \to X(A(A,A),A)$ is an isomorphism. If {b} and {c} are one-element subalgebras of A, then the constant endomorphisms $\bar{b}$, $\bar{c}$: $A \to A$ onto {b} and {c} respectively, are elements of $A(A,A)$.

If $0 \in Q$ is nullary, then b = c since any one-element subalgebra of A must equal {0}. Thus we may assume that every $p \in Q$ has arity greater than zero. For any $p \in Q$, $p(b,\ldots,b)$ is defined in the partial algebra A since for any unary polynomial q, $p(q(b),\ldots,q(b)) = p(b,\ldots,b) = b = q(b) = q(p(b,\ldots,b))$. Thus the constant map $\hat{b}$: $A(A,A) \to A$ onto {b} is an element of $X(A(A,A),A)$ and hence there exists $a \in A$ with $[a] = \hat{b}$. Thus $b = \bar{c}\hat{b} = \bar{c}[a] = a\bar{c} = c$. □

3. DUALITIES VIA COMPACT TOPOLOGICAL PARTIAL ALGEBRAS: EXAMPLES

3.1 *Pontryagin's Duality for Abelian Groups*

This famous duality theorem is due to L. S. Pontryagin ([57], [58]). The main distinguishing feature of a proof based on Theorem (2.8) and Theorem (2.9) is that we need only prove the duality for $F_A(1)$ (i.e., Z) rather than for all elementary groups, which is the usual technique of proof.

Let *Ab* be the category of abelian groups and let *K* be the category of compact abelian groups. Denote by T the compact group R/Z.

THEOREM (3.1.1). $(Ab(-,T), K(-,T))$ *is a full duality between Ab and K.*

Proof. Since {0} is a one-element subalgebra, $-(a+b) = (-a) + (-b)$ and $(a+b) + (c+d) = (a+c) + (b+d)$, $Ab(B,T)$ is an abelian group for all $B \in Ab$ (Proposition (2.12)) and hence both $Ab(-,T)$ and $K(-,T)$ are well-defined functors (Lemma (2.14)).

Since all subdirectly irreducible abelian groups, the cyclic groups of prime-power order and the Prüfer groups, are isomorphic to subgroups of T it follows that $Ab = \mathrm{ISPS(T)} = \mathrm{ISP(T)}$. That every compact abelian group is iseomorphic to a closed subalgebra of a power of T follows from the fact that $K(X,T)$ separates the points of X, but unfortunately the only proofs of this result known to the author require some high-powered representation theory for (locally) compact abelian groups (see [59, C, p.241] or [37, Theorem (22.17), p.345]). The author also knows of no direct proof of the fact that T is injective in K so the following indirect proof will have to suffice. By [37, Theorem (23.20), p.364], if Y is a compact abelian group and B is a subgroup of $K(Y,T)$ which separates the points of Y, then $B = K(Y,T)$. Now let Y be a olosed subgroup of a compact abelian group X and let $B = \{\phi|Y : \phi \in K(X,T)\}$. Since B separates the points of Y it follows that $B = K(Y,T)$ and hence every character $\phi \in K(Y,T)$ is the restriction to Y of a character $\phi' \in K(X,T)$, as required.

It is clear that $K(T^n,T) \simeq \sum_{j<n} K(T,T)$ for all $1 \leq n < \omega$. Hence to show that each $\phi \in K(T^n,T)$ is a polynomial function it is sufficient to show that each $\phi \in K(T,T)$ is a polynomial function, that is since $F_{Ab}(1) \simeq Z$ it is sufficient to show that $\eta_Z : Z \to K(T,T)$, defined by $a(n\eta_Z) = na$ for all $a \in T$, is an isomorphism. For this we refer to [59, C, p.247].

We will now prove that for all $\kappa \geq \omega$, every $\phi \in K(T^\kappa,T)$ has finite support. Note that $\{\Lambda_n | 1 \leq n < \omega\}$ is a neighbourhood basis of zero in T, where $\Lambda_n = \{ x/Z \in T: |x| < 1/3n\}$, and if $a \in T$ satisfies $na \in \Lambda_1$ for all $1 \leq n < \omega$, then $a = 0$. Let $\phi \in K(T^\kappa, T)$. Then $\Lambda_1\phi^{-1}$ is open in T^κ and hence there exist $\gamma_0,\ldots,\gamma_{k-1} < \kappa$ and Λ_m such that $\bigcap((\gamma_j;\Lambda_m)|j < k) \subseteq \Lambda_1\phi^{-1}$. If $<a_\gamma>$, $<b_\gamma> \in T^\kappa$ with $a_{\gamma_j} = b_{\gamma_j}$ $(j < k)$, then $a_{\gamma_j} - b_{\gamma_j} = 0$ $(j < k)$ and hence $n(<a_\gamma> - <b_\gamma>) \in \bigcap((\gamma_j;\Lambda_m)|j < k)$ for all $1 \leq n < \omega$. Thus $n(<a_\gamma>\phi - <b_\gamma>\phi) \in \Lambda_1$ for all $1 \leq n < \omega$ and so $<a_\gamma>\phi - <b_\gamma>\phi = 0$. Consequently $\{\gamma_j | j < k\}$ is a finite support for ϕ.

Hence by Theorem (2.8), $(Ab(-,T),K(-,T))$ is a duality and by Theorem (2.9) it remains only to prove that $\varepsilon_X\colon X \to Ab(K(X,T),T)$ is a surjection for all $X \varepsilon K$. If ε_X is not onto, then there is a non-zero character of the topological factor group $Ab(K(X,T),T)/X\varepsilon_X$ and hence there is a non-zero character of $Ab(K(X,T),T)$ whose restriction to $X\varepsilon_X$ is the zero character. Thus it is sufficient to prove that any character $\phi\colon Ab(K(X,T),T) \to T$ which is zero on $X\varepsilon_X$ is identically zero.

Since $(Ab(-,T),K(-,T))$ is a duality there exists $\psi \varepsilon K(X,T)$ with $\psi\eta_{K(X,T)} = [\psi] = \phi$. Now, denoting the constant map onto $\{0\}$ by $\hat{0}$, we have $\phi|X\varepsilon_X = \hat{0} \iff [\psi]|X\varepsilon_X = \hat{0} \iff x\varepsilon_X[\psi] = 0$ for all $x \varepsilon X \iff \psi(x\varepsilon_X) = 0$ for all $x \varepsilon X \iff x\psi = 0$ for all $x \varepsilon X \iff \psi = \hat{0} \iff \phi = [\psi] = \hat{0}$, as required. □

An application of Proposition (1.24) yields the *Comp*-free functor for the category K of compact abelian groups and an application of Theorem (2.15) yields the Bohr compactification functor.

PROPOSITION (3.1.2). (i) $Ab(C(-,T),T)\colon$ *Comp* $\to K$ *is left adjoint to the forgetful functor from* K *into Comp.*

(ii) $Ab(T^{(-)},T)\colon$ *Set* $\to K$ *is left adjoint to the forgetful functor from* K *into Set.*

(iii) $Ab(GAb(-,T),T)\colon Ab \to K$ *is left adjoint to the forgetful functor* $G\colon K \to Ab$. □

Little need be said about the applications of this duality; one need only refer to the texts L. S. Pontryagin [59], E. Hewitt and K. A. Ross [37] and K. H. Hofmann [38].

3.2 *Duality for $\wedge$-Semilattices*

Although this duality dates back to C. W. Austin [4], the most recent and thorough exposition is most certainly the monograph [40] by K. H. Hofmann, M. Mislove and A. Stralka. The proof given here is quite different from the proof given in [40], which uses the concepts of density and codensity in categories.

Let $\mathcal{S\ell}$ be the category of $\wedge$-semilattices with zero and unit, let Z be the category of compact, totally disconnected topological $\wedge$-semilattices, and let $\underset{\sim}{2}$ denote the two-element chain.

THEOREM (3.2.1). *$(\mathcal{S\ell}(-,\underset{\sim}{2}),Z(-,\underset{\sim}{2}))$ is a full duality between $\mathcal{S\ell}$ and Z.*

To prove this result we require two elementary facts about Z.

LEMMA (3.2.2). *For all $X \in Z$, each $x \in X$ has a neighbourhood basis of clopen subsemilattices.*

Proof. (L. B. Schneperman [65]). Let U be a clopen neighbourhood of x and by an application of Zorn's lemma let S be a subsemilattice of X which contains x, is a subset of U, and is maximal with respect to these properties. Since the closure of a subsemilattice is a subsemilattice and since U is closed, it follows that S is closed.

Let $y \in S$. Then $y \wedge S \subseteq S \subseteq U$ and hence by a simple compactness argument using the continuity of the meet operation there exists an open neighbourhood V of y with $V \subseteq U$ and $V \wedge S \subseteq U$. We claim that $V \subseteq S$, whence S is open. If $z \in V - S$, then

$T = S \cup (z \wedge S) \cup \{z\}$ is a subsemilattice of X. But $S \subseteq U$, $z \wedge S \subseteq V \wedge S \subseteq U$ and $z \in V \subseteq U$, and so $T \subseteq U$, contradicting the maximality of S. $\square$

Although every $X \in Z$ is necessarily a meet-complete lattice, we need only the fact that X has a zero.

For each $x \in X$ define the translation $t_x: X \to X$ by $yt_x = y \wedge x$. Clearly t_x is continuous and hence $(x] = Im(t_x)$ is closed in X.

LEMMA (3.2.3). *Every* $X \in Z$ *has a zero.*

Proof. Consider X as a downward directed net on itself. Since X is compact, X has an accumulation point, say x. If x is not the zero of X, then there exists $y \in X$ with $y < x$. But then $U = X - (y]$ is an open neighbourhood of x such that for all $z \leq y$, $z \notin U$, contradicting the fact that x is an accumulation point of X. $\square$

Clearly $S\ell(B,\underset{\sim}{2})$ separates the points of B for all $B \in S\ell$, and hence $S\ell = ISP(\underset{\sim}{2})$. If we let $Q = \{\wedge\}$, then since $(a\wedge b) \wedge (c\wedge d) = (a\wedge c) \wedge (b\wedge d)$, $0 \wedge 0 = 0$ and $1\wedge 1 = 1$, it follows that $\wedge$ is a full operation on $S\ell(B,\underset{\sim}{2})$ for all $B \in S\ell$ (Proposition (2.12)), and hence X_Q is a subcategory of Z and $S\ell(-,\underset{\sim}{2}): S\ell \to Z$ is well defined. We will now show that $X_Q = Z$, that is every compact, totally disconnected topological $\wedge$-semilattice is iseomorphic to a closed subsemilattice of a power of $\underset{\sim}{2}$.

Clearly it is sufficient to show that $Z(X,\underset{\sim}{2})$ separates pairs of comparable elements of X. Let $x < y$. Since $U = X - (x]$ is an open neighbourhood of y, by Lemma (3.2.2) there is a clopen subsemilattice S of X with $x \in S \subseteq U$. Since S is closed in X it follows that $S \in Z$ and hence S has a zero, say θ, by Lemma (3.2.3). Now $[\theta) = \{z \in X | z \wedge \theta \in S\} = St_\theta^{-1}$ is a clopen filter of X containing y but not containing x and hence the characteristic function of $[\theta)$ is an element of $Z(X,\underset{\sim}{2})$ which separates x from y.

REMARK (3.2.4). An alternative, though not particularly intuitive proof, may be obtained by showing that if $x < y$ and U is a clopen neighbourhood of y with $x \notin U$, then $F = \{z \in X | z \wedge w \wedge y \in U \iff w \wedge y \in U, \text{ for all } w \in X\}$ is a clopen filter with $y \in F$ and $x \notin F$ — this can be proved without resource to Lemma (3.2.2) and Lemma (3.2.3).

To prove that $\underset{\sim}{2}$ is injective in Z it is clearly sufficient to prove that if Y is a closed subsemilattice of X and U is a clopen filter of Y, then there exists a clopen filter V of X with $V \cap Y = U$. Let $x \in U$ and for all $y \in Y-U$ let $V_{x,y}$ be a clopen filter of X with $x \in V_{x,y}$ and $y \notin V_{x,y}$. Then $\{X - V_{x,y} | y \in Y - U\}$ is an open cover of $Y-U$ and hence has a finite subcover, say $\{X - V_{x,y_0}, \ldots, X - V_{x,y_{n-1}}\}$. Let $V_x = \bigcap(V_{x,y_j} | j < n)$. Then V_x is a clopen filter containing x and satisfies $V_x \cap Y \subseteq U$. Since U is closed in Y it follows that $U \in Z$ and hence U has a zero, say θ, by Lemma (3.2.3). Clearly $V = V_\theta$ is the required clopen filter.

That each $\phi \in Z(\underset{\sim}{2}^n,\underset{\sim}{2})$ is a polynomial function is easily seen. If $1\phi^{-1}$ is empty, then $\phi = 0$ and if $1\phi^{-1} = \underset{\sim}{2}^n$, then

$\phi = 1$. Otherwise there exists $\underline{a} \in \underset{\sim}{2}^n$ with $1\phi^{-1} = [\underline{a})$ and it is clear that the polynomial $p(x_0,\ldots,x_{n-1}) = \bigwedge(x_j \mid a_j = 1)$ satisfies $\underline{b}\phi = p(\underline{b})$ for all $\underline{b} \in \underset{\sim}{2}^n$.

Thus by Theorem (2.8), $(S\ell(-,\underset{\sim}{2}),Z(-,\underset{\sim}{2}))$ is a duality and by Theorem (2.9) it remains only to prove that $\varepsilon_X\colon X \to S\ell(Z(X,\underset{\sim}{2}),\underset{\sim}{2})$ is a surjection for all $X \in Z$.

Let $g \in S\ell(Z(X,\underset{\sim}{2}),\underset{\sim}{2})$. Then $\{1\phi^{-1} \mid \phi \in 1g^{-1}\}$ has the finite intersection property and consequently $\bigcap(1\phi^{-1} \mid \phi \in 1g^{-1})$ is a non-empty closed subsemilattice which by Lemma (3.2.3) has a zero, say x. We will prove that $x\varepsilon_X = [x] = g$. Clearly $g \leq [x]$ since $\psi g = 1 \Rightarrow x\psi = 1 \Rightarrow \psi[x] = 1$. Now assume that $\psi[x] = x\psi = 1$. Then $1\psi^{-1} \supseteq [x) = \bigcap(1\phi^{-1} \mid \phi \in 1g^{-1})$. By compactness there exist $\phi_0,\ldots,\phi_{n-1} \in 1g^{-1}$ with $1\psi^{-1} \supseteq \bigcap(1\phi^{-1} \mid j < n)$. Hence $\psi \geq \bigwedge(\phi_j \mid j < n) \in 1g^{-1}$ and so $\psi g = 1$. Thus $[x] \leq g$.

This completes the proof of Theorem (3.2.1).□

Applying Proposition (1.24) we obtain the $ZComp$-free functor for Z. It is easily proved that for all $X \in ZComp$, $S\ell(C(X,\underset{\sim}{2}),\underset{\sim}{2})$ is iseomorphic to the hyperspace of X regarded as a $\wedge$-semilattice with set union as the operation (see [40, Proposition 2.5 and Lemma 2.14]).

PROPOSITION (3.2.5). (i) $S\ell(C(-,\underset{\sim}{2}),\underset{\sim}{2})\colon ZComp \to Z$ *is left adjoint to the forgetful functor from* Z *into* $ZComp$.

(ii) $S\ell(\underset{\sim}{2}^{(-)},\underset{\sim}{2})\colon Set \to Z$ *is left adjoint to the forgetful functor from* Z *into* Set.□

A similar duality theorem may be proved for the category $\mathcal{S}\ell_1$ of $\wedge$-semilattices with unit and the category Z_1 of compact, totally disconnected topological $\wedge$-semilattices with unit. In this case we could let $\mathcal{Q} = \{\wedge,1\}$, prove that $Z_1 = X_{\mathcal{Q}}$ and then apply Theorem (2.8) and Theorem (2.9), but there is no need since the duality for $\mathcal{S}\ell_1$ and Z_1 can be obtained as a corollary of Theorem (3.2.1).

THEOREM (3.2.6). *$(\mathcal{S}\ell_1(-,\underset{\sim}{2}),Z_1(-,\underset{\sim}{2}))$ is a full duality between $\mathcal{S}\ell_1$ and Z_1.*

Proof. For any $B \in \mathcal{S}\ell_1$ let ${}_0B$ denote the bounded semilattice obtained by adjoining a new zero to B. If $g \in \mathcal{S}\ell_1(B,\underset{\sim}{2})$, then ${}_0g \in \mathcal{S}\ell({}_0B,\underset{\sim}{2})$ will denote the obvious extension of g. Clearly $g \to {}_0g$ is an iseomorphism of $\mathcal{S}\ell_1(B,\underset{\sim}{2})$ onto $\mathcal{S}\ell({}_0B,\underset{\sim}{2})$. It is also clear that for all $X \in Z_1$, $Z(X,\underset{\sim}{2}) = Z_1(X,\underset{\sim}{2}) \cup \{\hat{0}\} = {}_0Z_1(X,\underset{\sim}{2})$. It follows at once that the functors $\mathcal{S}\ell_1(-,\underset{\sim}{2})$ and $Z_1(-,\underset{\sim}{2})$ are well defined and furthermore, $Z_1(\mathcal{S}\ell_1(B,\underset{\sim}{2}),\underset{\sim}{2}) \simeq Z(\mathcal{S}\ell({}_0B,\underset{\sim}{2}),\underset{\sim}{2}) - \{\hat{0}\} \simeq {}_0B - \{0\} = B$ and $\mathcal{S}\ell_1(Z_1(X,\underset{\sim}{2}),\underset{\sim}{2}) \simeq \mathcal{S}\ell({}_0Z_1(X,\underset{\sim}{2}),\underset{\sim}{2}) = \mathcal{S}\ell(Z(X,\underset{\sim}{2}),\underset{\sim}{2}) \simeq X.$ □

There is an obvious analogue of Proposition (3.2.5) for $\mathcal{S}\ell_1$ and Z_1. Furthermore, since the algebras in both categories are of the same type, Theorem (2.15) describes the left adjoint to the forgetful functor from Z_1 to $\mathcal{S}\ell_1$.

PROPOSITION (3.2.7). (i) *$\mathcal{S}\ell_1(C(-,\underset{\sim}{2}),\underset{\sim}{2})$: ZComp $\to Z_1$ is left adjoint to the forgetful functor from Z_1 into ZComp.*

(ii) *$\mathcal{S}\ell_1(\underset{\sim}{2}^{(-)},\underset{\sim}{2})$: Set $\to Z_1$ is left adjoint to the forgetful functor* *from* Z_1 *into Set.*

(iii) $S\ell_1(GS\ell_1(-,\underset{\sim}{2}),\underset{\sim}{2})$: $S\ell_1 \to Z_1$ *is left adjoint to the forgetful functor* G: $Z_1 \to S\ell_1$.□

For applications of these dualities one need not look beyond the monograph [40]. Proposition (2.16) shows that the approach given here cannot be extended to develop a duality theory for $\wedge$-semilattices in general.

3.3 *Duality for Equational Classes Generated by Primal Algebras*

In his paper [43], T. K. Hu develops a duality theory for class of algebras 'determined' by locally primal algebras. The methods developed in Chapter 2 are well suited to proving the duality in the particular case where the locally primal algebra is finite and therefore primal. Since the two-element Boolean algebra is primal, Stone's duality for Boolean algebras arises as a particular case.

DEFINITION (3.3.1). A finite, non-trivial algebra A is said to be *primal* if for all $1 \leq n < \omega$, every map ϕ: $A^n \to A$ is a polynomial function.

A discussion of primal algebras more than adequate for our purposes may be found in [56].

Let A be a primal algebra and let $\mathcal{A}$ = HSP(A) be the equational class generated by A. Let $\mathcal{Q}$ be the empty set of polynomials. Then $X_{\mathcal{Q}}$ is simply the category *ZComp* of Boolean spaces.

THEOREM (3.3.2). *$(A(-,A),C(-,A))$ is a full duality between $\mathcal{A}$ and ZComp.*

Proof. We have already seen that the functors are well defined (Lemma (1.3) and Proposition (1.4)). Since a primal algebra is simple, has no proper subalgebras and every algebra in $\mathcal{A}$ has distributive congruences, it follows by Jónsson's lemma (see [45, Corollary 3.4, p.115] that $\mathcal{A} = ISPHS(A) = ISP(A)$. It is well known that any finite discrete space is injective in ZComp, and every $\phi \in C(A^n,A)$ is a polynomial function as this is just the definition of primality. Hence by Theorem (2.8), $(A(-,A),C(-,A))$ is a duality.

We could now prove that the duality is full by appealing to Jónsson's lemma again. Instead we will prove a more general result which will imply the fullness of this duality and will have several other applications in the future. As usual, if B is a subalgebra of A^X, then for all $x \in X$, $[x]: B \to A$ is defined by $\phi[x] = x\phi$ for all $\phi \in B$. We will denote the monoid of endomorphisms of A by End(A).

PROPOSITION (3.3.3). *Let A be a non-trivial finite algebra all of whose non-trivial subalgebras are subdirectly irreducible and assume that every algebra in $\mathcal{A} = ISP(A)$ has distributive congruences. If X is a Boolean space and B is a subalgebra of $C(X,A)$ containing the constant maps, then every homomorphism $g \in A(B,A)$ is of the form $[x]e$ for some $x \in X$ and some $e \in End(A)$.*

Proof. If $Im(g) = \{a\}$, then choose $x \in X$ arbitrarily and let $\bar{a}$

be the constant endomorphism onto $\{a\}$; clearly $g = [x]\bar{a}$. If $\mathrm{Im}(g)$ is non-trivial then it is subdirectly irreducible and by Jónsson's lemma [45, Lemma 3.1, p.114] there is an ultrafilter U on X with $\Theta_U|B \leq \mathrm{Ker}(g)$, where Θ_U is the congruence on A^X given by $(\phi,\psi) \in \Theta_U \Leftrightarrow \mathrm{Eq}(\phi,\psi) = \{x \in X | x\phi = x\psi\} \in U$. Let $F = \{Y \in U | Y \text{ is clopen in } X\}$. Then F is an ultrafilter of the Boolean algebra of clopen subsets of X and hence since X is a Boolean space, there exists (a unique) $x \in X$ with $F = \{Y \subseteq X | Y \text{ is clopen and } x \in Y\}$.

Now $\mathrm{Eq}(\phi,\psi) = \bigcup(a\phi^{-1} \cap a\psi^{-1} | a \in A)$ and thus if $\phi, \psi \in C(X,A)$, then $\mathrm{Eq}(\phi,\psi)$ is clopen in X. Hence $(\phi,\psi) \in \Theta_U|B$ if and only if $x\phi = x\psi$.

Define $e: A \to A$ by $ae = \hat{a}g$, where $\hat{a}: X \to A$ is the constant map onto $\{a\}$. Since $A \simeq \{\hat{a} | a \in A\}$, e is an endomorphism. We claim that $g = [x]e$. If $\phi \in B$, then $\phi([x]e) = (x\phi)e = \widehat{(x\phi)}g$. But $(\phi,\widehat{(x\phi)}) \in \Theta_U|B$ since $x\phi = x\widehat{(x\phi)}$ and hence $(\phi,\widehat{(x\phi)}) \in \mathrm{Ker}(g)$ since $\Theta_U|B \leq \mathrm{Ker}(g)$. Thus $\phi g = \widehat{(x\phi)}g$, and so $\phi([x]e) = \phi g$, as required. □

COROLLARY (3.3.4). *Assume that the conditions of the proposition hold and that* $\{a_0,\ldots,a_{n-1}\}$ *is the set of all mutually distinct elements which form one-element subalgebras of* A. *Let*

$$F(X) = (X \times (\mathrm{End}(A) - \{\bar{a}_0,\ldots,\bar{a}_{n-1}\})) \cup \{a_0,\ldots,a_{n-1}\}$$

and define $\Gamma_X: F(X) \to A(C(X,A),A)$ *by* $(x,e)\Gamma_X = [x]e$ *and* $a_j\Gamma_X = \hat{a}_j$ $(j < n)$. *Then* Γ_X *is a homeomorphism of* $F(X)$ *onto* $A(C(X,A),A)$.

Proof. The proposition guarantees that Γ_X is onto. We now show that Γ_X is one-one. Let (x,e), $(y,f) \in F(X)$. If $e \neq f$, then there exists $a \in A$ with $ae \neq af$ and consequently $\hat{a}((x,e)\Gamma_X) = (\hat{a}[x])e = ae \neq af = (\hat{a}[y])f = \hat{a}((y,f)\Gamma_X)$. If $e = f$ and $x \neq y$, then let U be a clopen set with $x \in U$ and $y \notin U$. Since $\mathrm{Im}(e)$ is non-trivial there exist $a, b \in A$ with $ae \neq be$. Thus, defining $\phi\colon X \to A$ by $z\phi = a$ $(z \in U)$ and $z\phi = b$ $(z \notin U)$, we have $\phi((x,e)\Gamma_X) = (\phi[x])e = (x\phi)e = ae \neq be = (y\phi)e = (\phi[y])e = \phi((y,e)\Gamma_X)$. It follows at once that Γ_X is one-one.

Let $\phi \in C(X,A)$, let $a \in A$ and set $U = \{(x,e) \in F(X) \mid (x\phi)e = a\}$. If $\{a\}$ is not a subalgebra of A, then $(\phi;\{a\})\Gamma_X^{-1} = U$ and if $\{a\}$ is a subalgebra of A, then $(\phi;\{a\})\Gamma_X^{-1} = U \cup \{a\}$. Hence to prove that Γ_X is continuous it is sufficient to prove that U is open. But for every $(x,e) \in U$, $(ae^{-1})\phi^{-1} \times \{e\}$ is an open neighbourhood of (x,e) contained in U. Thus Γ_X is continuous and since it is a bijection, it is a homeomorphism. □

Since a primal algebra A has no proper subalgebras and no proper endomorphisms it follows that for all $X \in \mathcal{Z}Comp$, $X \simeq F(X)$ and $\varepsilon_X \simeq \Gamma_X$, whence Corollary (3.3.4) implies that ε_X is a homeomorphism. Thus the duality $(\mathcal{A}(-,A), \mathcal{C}(-,A))$ between $\mathcal{A}$ and $\mathcal{Z}Comp$ is full. □

M. H. Stone's duality [66] for the category $\mathcal{B}$ of Boolean algebras follows at once.

THEOREM (3.3.5). $(\mathcal{B}(-,\underset{\sim}{2}), \mathcal{C}(-,\underset{\sim}{2}))$ *is a full duality between* $\mathcal{B}$ *and* $\mathcal{Z}Comp$.

Proof. It is well known that the two-element Boolean algebra is primal — if $\phi: \underset{\sim}{2}^n \to \underset{\sim}{2}$, then $\underline{a}\phi = p(\underline{a})$ for all $\underline{a} \in \underset{\sim}{2}^n$, where

$$p(x_0,\ldots,x_{n-1}) = \bigvee(\bigwedge(x_j \mid a_j = 1) \wedge \bigwedge(x_j' \mid a_j = 0) \mid \underline{a} \in 1\phi^{-1}).\square$$

REMARK (3.3.6). K. Keimel and H. Werner [47] have generalized Theorem (3.3.2) to equational classes generated by quasi-primal algebras. Since a quasi-primal algebra may have many one-element subalgebras, in fact a quasi-primal algebra can be idempotent, Proposition (2.16) shows that a duality via compact partial algebras cannot in general be obtained for an equational class $\mathcal{A}$ generated by a quasi-primal algebra A. In Keimel and Werner's duality the dual $\mathcal{A}(B,A)$ of an algebra $B \in \mathcal{A}$ is a Boolean space endowed with an action, by partial homeomorphisms, of the semigroup H of inner isomorphisms of A. If one attempts to apply Theorem (1.15) to prove this duality one finds that condition (D_0) is easily checked and that condition (D_2) is just the definition of quasi-primality. Unfortunately the only proof the author knows for the injectivity condition (D_1) amounts to compiling the necessary facts from the proof of the duality given in [47] and there is no significant saving.

Applications of Stone's duality abound in both algebra and topology. Some applications of Hu's duality for equational classes generated by primal algebras may be found in [43] and [44].

3.4 *Duality for Distributive Lattices*

The most famous duality for distributive lattices is certainly that of M. H. Stone [67]. His duality, which utilizes the concept of a spectral space, is purely topological.

That a finite distributive lattice is completely determined by its poset of join-irreducibles (and therefore by its poset of prime filters) is part of the folklore of the theory. In retrospect, it is surprising that a general duality theory for distributive lattices utilizing the natural partial order of the prime filters was not developed until the late 1960's.

Let $\underset{\sim}{2}$ denote the two-element distributive lattice with the zero and unit as nullary operations. Then by the prime ideal theorem, $\mathcal{D} = \mathrm{ISP}(\underset{\sim}{2})$ is the category of bounded distributive lattices. If $\mathcal{Q} = \{\wedge\}$, then the objects of the category $\mathcal{P} = X_{\mathcal{Q}}$ are (at least) compact, totally disconnected topological partial $\wedge$-semilattices.

Recall that a subset U of the poset X is said to be *increasing* (*decreasing*) if $x \in U$ and $y \geq x$ ($y \leq x$) imply that $y \in U$.

DEFINITION (3.4.1). A partially ordered topological space X is said to be *totally order-disconnected* if for all $x, y \in X$ with $x \nleq y$, there exists a clopen, increasing subset U of X such that $x \in U$ and $y \notin U$.

PROPOSITION (3.4.2). (i) *Let* X *be a compact, totally disconnected topological partial* $\wedge$*-semilattice. Then* $X \in \mathcal{P}$ *if and only if it is totally order-disconnected when partially ordered by* '$x \leq y \iff x \wedge y$ *exists and equals* x'.

(ii) *Let* X *be a compact partially ordered space. Then* X *is totally order-disconnected if and only if* $X \in P$ *when a partial* $\wedge$ *operation is defined on* X *by* '$x \wedge y$ *exists and equals* $x \iff x \leq y$'.

(iii) *P is isomorphic to the category whose objects are compact totally order-disconnected spaces and whose arrows are continuous, order-preserving maps.*

Proof. Both (i) and (ii) will hold provided we can establish the necessity of the condition stated in each. Clearly (iii) is a corollary of (i) and (ii).

Firstly we describe the structure of the partial algebra $\underset{\sim}{2}^{\kappa}$; namely, for all $\underline{a}, \underline{b} \in \underset{\sim}{2}^{\kappa}$, $\underline{a} \wedge \underline{b}$ is defined if and only if $\underline{a} \leq \underline{b}$ or $\underline{b} \leq \underline{a}$ (pointwise). For $\kappa = 1$ this is trivial, hence assume that $\kappa > 1$. If $\underline{a} \leq \underline{b}$, then for any κ-ary polynomial q, $q(\underline{a}) \wedge q(\underline{b}) = q(\underline{a}) = q(\underline{a} \wedge \underline{b})$ since polynomial functions are order-preserving. Conversely, assume that $\underline{a}$ and $\underline{b}$ are incomparable. Then there exist $\gamma, \lambda < \kappa$ such that $0 = a_{\gamma} < b_{\gamma} = 1$ and $1 = a_{\lambda} > b_{\lambda} = 0$. Letting $q(\underline{x}) = x_{\gamma} \vee x_{\lambda}$, we obtain $q(\underline{a}) \wedge q(\underline{b}) = (0 \vee 1) \wedge (1 \vee 0) = 1 \neq 0 = (0 \wedge 1) \vee (1 \wedge 0) = q(\underline{a} \wedge \underline{b})$ whence $\underline{a} \wedge \underline{b}$ is undefined.

Now let X be a closed subalgebra of $\underset{\sim}{2}^{\kappa}$ and assume that $\underline{a}, \underline{b} \in X$ satisfy $\underline{a} \nleq \underline{b}$. Then there exists $\gamma < \kappa$ such that $a_{\gamma} = 1$ and $b_{\gamma} = 0$, and thus $U = \{\underline{c} \in \underset{\sim}{2}^{\kappa} \mid c_{\gamma} = 1\}$ is a clopen, increasing subset of X with $\underline{a} \in U$ and $\underline{b} \notin U$. This shows that X is totally order-disconnected and establishes necessity in (i).

Let X be a compact, totally order-disconnected space and endow X with the partial $\wedge$-semilattice structure described in (ii). Since X is totally order-disconnected, the set $\mathcal{C}_{\leq}(X,\underset{\sim}{2})$ of continuous order-preserving maps into $\underset{\sim}{2}$ separates the points of X, whence X is homeomorphic and order-isomorphic to a closed subspace of the compact totally order-disconnected space $\underset{\sim}{2}^{\kappa}$, where κ is the cardinality of $\mathcal{C}_{\leq}(X,\underset{\sim}{2})$. But any closed subspace of $\underset{\sim}{2}^{\kappa}$ is a closed subalgebra (by the characterization of $\wedge$ in $\underset{\sim}{2}^{\kappa}$ given above) and hence by the definition of $\wedge$ in X, any continuous order-isomorphism of X into $\underset{\sim}{2}^{\kappa}$ is an iseomorphism of the partial algebra X onto a closed subalgebra. Thus X ε $\mathcal{P}$, establishing necessity in (ii). $\square$

REMARK (3.4.3). We will identify $\mathcal{P}$ with the category of compact, totally order-disconnected spaces. Note that if we were to choose $\mathcal{Q}$ to be the set of all operations in the type of $\underset{\sim}{2}$, that is $\mathcal{Q} = \{\wedge,\vee,0,1\}$, then Proposition (3.4.2)(iii) would still hold since $D_{\vee} = D_{\wedge}$, and hence $\vee$ induces the same partial order as $\wedge$, and both D_0 and D_1 are empty.

THEOREM (3.4.4). (H. A. Priestley [60], [61]). $(\mathcal{D}(-,\underset{\sim}{2}),\mathcal{P}(-,\underset{\sim}{2}))$ *is a full duality between* $\mathcal{D}$ *and* $\mathcal{P}$.

Proof. Let Y be closed in X and let U be a clopen increasing subset of Y. As in the proof of the injectivity of $\underset{\sim}{2}$ in $\mathcal{Z}$ (p.38), for x ε U let V_x be clopen increasing in X with $x \in V_x \cap Y \subseteq U$. Let $\{V_{x_0},\ldots,V_{x_{m-1}}\}$ be a finite subcover of the cover $\{V_x | x \in U\}$ of U. Then $V = \bigcup(V_{x_j} | j < m)$ satisfies $V \cap Y = U$ and hence $\underset{\sim}{2}$ is injective in $\mathcal{P}$.

$\phi \in P(\underset{\sim}{2}^n, \underset{\sim}{2})$ if and only if ϕ is order-preserving and hence $F = 1\phi^{-1}$ is increasing. If F is empty, then $\phi = 0$ and if $F = \underset{\sim}{2}^n$, then $\phi = 1$. Otherwise there exist distinct, non-zero elements $\underline{a}^0, \ldots, \underline{a}^{m-1} \in \underset{\sim}{2}^n$ with $F = \bigcup([\underline{a}^j) \mid j < m)$. Define

$$p(x_0, \ldots, x_{n-1}) = \bigvee(\bigwedge(x_i \mid a_i^j = 1) \mid j < m).$$

Clearly $\underline{b}\phi = p(\underline{b})$ for all $\underline{b} \in \underset{\sim}{2}^n$.

Observe that $\underset{\sim}{2}$ satisfies the conditions of Proposition (3.3.3) and that $\underset{\sim}{2}$ has no proper endomorphisms. Hence for all $X \in P$, $\varepsilon_X \colon X \to D(P(X, \underset{\sim}{2}), \underset{\sim}{2})$ is onto by Proposition (3.3.3) with $B = P(X, \underset{\sim}{2})$.

Theorem (2.8) and Theorem (2.9) now combine to yield the result. $\square$

REMARK (3.4.5). Let B be a bounded distributive lattice and let X_B be the set of prime filters of B. Order X_B by inclusion and let $\{X_b \mid b \in B\} \bigcup \{X_B - X_b \mid b \in B\}$, where $X_b = \{x \in X_B \mid b \in x\}$, be a basis for a topology on X_B. Define $\phi \colon X_B \to D(B, \underset{\sim}{2})$ by $b(x\phi) = 1 \iff b \in x$. Then ϕ is an order-isomorphism and a homeomorphism and thus Theorem (3.4.4) implies that B is isomorphic to the lattice of clopen increasing subsets of X_B. If B is finite then X_B is discretely topologized and hence B is isomorphic to the lattice of increasing subsets of the poset X_B of prime filters of B.

REMARK (3.4.6). Products in the category P are a little unorthodox since the ordered space structure on $X \times Y$ is just the pointwise structure while the partial algebra structure on $X \times Y$ need not be the pointwise one: if $x_1 < x_2$ in X and $y_1 > y_2$ in Y, then in the pointwise structure, $(x_1,y_1) \wedge (x_2,y_2)$ is defined, but $(x_1,y_1) \wedge (x_2,y_2)$ is not defined in the product of X and Y in P since (x_1,y_1) and (x_2,y_2) are incomparable.

There is a natural embedding of *ZComp* into P which endows a Boolean space with the discrete partial order '$x \leq y \iff x = y$'. The Stone-Čech compactification functor β: *Set* $\to$ *ZComp* may be lifted to a functor β: *Set* $\to P$ in the same way. Proposition (1.24) becomes rather trivial in the present setting.

PROPOSITION (3.4.7). (i) *The natural embedding of ZComp into P is left adjoint to the forgetful functor from P into ZComp.* (ii) β: *Set* $\to P$ *is left adjoint to the forgetful functor from P into Set.* □

A duality for distributive lattices with unit may now be obtained very simply. Since the proof is almost identical to the proof of the duality between $S\ell_1$ and Z_1 (Theorem (3.2.6)), it will be omitted.

Let $\underset{\sim}{2}$ now denote the two-element distributive lattice with the unit as a nullary operation. By the prime ideal theorem, $\mathcal{D}_1 = \mathrm{ISP}(\underset{\sim}{2})$ is the category of distributive lattices with unit. If $\mathcal{Q} = \{\wedge, 1\}$, then the objects of the category $P_1 = X_{\mathcal{Q}}$ are compact totally order-disconnected spaces with unit and $\phi \in P_1(X,Y)$ if and only if ϕ is continuous, order-preserving and unit-preserving.

THEOREM (3.4.8). $(\mathcal{D}_1(-,\underset{\sim}{2}), P_1(-,\underset{\sim}{2}))$ *is a full duality between* $\mathcal{D}_1$ *and* P_1. □

Again Proposition (2.16) shows that the natural extension of this approach will not yield a duality theory for distributive lattices in general.

Many applications have been found for these dualities. See for example [1], [2], [3], [13], [14], [61], [62], [63], and Chapter 4 of this work.

3.5 *Duality for Stone Algebras*

The duality presented here is new but is very similar to the duality for Stone algebras developed by H. A. Priestley in [62] as an application of her duality for bounded distributive lattices. The main difference between the two approaches lies in the fact that here the category $\mathcal{St}$ of Stone algebras is treated independently of the category $\mathcal{D}$ of bounded distributive lattices, and hence the three-element chain plays an important role, while in [62] the category $\mathcal{St}$ is treated as a subcategory of $\mathcal{D}$ and hence the two-element chain has the upper hand. (See Remark (3.5.4)). The proof presented here and the proof in [62] bear no relation to one another.

Let $\underset{\sim}{3}$ denote the three-element Stone algebra, $0 < a < 1$. Then $\underset{\sim}{3}$ is an algebra of type $(2,2,1,0,0)$ with operations $(\wedge,\vee,{}^{*},0,1)$. Since $\underset{\sim}{2}$ and $\underset{\sim}{3}$ are the only subdirectly irreducible Stone algebras (see [29], [51] or [52]) it follows that $\mathcal{St} = \mathrm{ISP}(\underset{\sim}{3})$ is the category of Stone algebras. Let $\mathcal{W} = \mathcal{X}_{\mathcal{Q}}$ where $\mathcal{Q} = \{\wedge, {}^{**}\}$. Recall that whenever the operations $\wedge$ and ** are defined in $\underset{\sim}{3}^{K}$, they are defined pointwise.

LEMMA (3.5.1). (i) *Define a partial order* $\leq^*$ *on the set* $\underset{\sim}{3}^\kappa$ *as follows:* '$\underline{b} \leq^* \underline{c} \iff$ *for all* $\gamma < \kappa$, *either* $b_\gamma = c_\gamma$, *or* $b_\gamma = a$ *and* $c_\gamma = 1$', *or equivalently* '$\underline{b} \leq^* \underline{c} \iff \underline{b} \leq \underline{c}$ *in* $\underset{\sim}{3}^\kappa$, *and* $b_\gamma = 0$ *implies* $c_\gamma = 0$'. *Then* $\underline{b} \wedge \underline{c}$ *is defined in* $\underset{\sim}{3}^\kappa$ *if and only if* $\underline{b} \leq^* \underline{c}$ *or* $\underline{c} \leq^* \underline{b}$.

(ii) $\underline{b}^{**}$ *is defined for all* $\underline{b} \in \underset{\sim}{3}^\kappa$ *and* $\underline{b}^{**}$ *is the unique element in* $\underset{\sim}{3}^\kappa$ *which majorizes* $\underline{b}$ *and is maximal with respect to* $\leq^*$.

Proof. Throughout the following, $\leq$ will denote the partial order on $\underset{\sim}{3}$ and its pointwise extension to $\underset{\sim}{3}^\kappa$, and $\leq^*$ will denote the partial order defined above.

(i) Assume that $\underline{b} \wedge \underline{c}$ is defined. Since lattice polynomials are a fortiori Stone algebra polynomials, it follows that $\underline{b} \leq \underline{c}$ or $\underline{c} \leq \underline{b}$ as in Proposition (3.4.2). Without loss of generality assume that $\underline{b} \leq \underline{c}$. If $b_\gamma < c_\gamma$, then set $q(\underline{x}) = x_\gamma^*$. Since $\underline{b} \wedge \underline{c}$ is defined we have $b_\gamma^* = q(\underline{b}) = q(\underline{b} \wedge \underline{c}) = q(\underline{b}) \wedge q(\underline{c}) = b_\gamma^* \wedge c_\gamma^* = c_\gamma^*$, whence $b_\gamma = a$ and $c_\gamma = 1$.

Conversely, assume that $\underline{b} \leq^* \underline{c}$. We claim that for every κ-ary polynomial q, $q(\underline{b} \wedge \underline{c}) = q(\underline{b}) \wedge q(\underline{c})$, that is $q(\underline{b}) \leq q(\underline{c})$. If $q(\underline{x}) \in \{x_\gamma, x_\gamma \wedge x_\lambda, x_\gamma \vee x_\lambda, x_\gamma^*\}$, then it is clear that $q(\underline{b}) \leq q(\underline{c})$. A simple induction on the rank of q now shows that $q(\underline{b}) \leq q(\underline{c})$ for every κ-ary polynomial q.

(ii) The identities $0^{**} = 0$, $1^{**} = 1$, $(x_\gamma \wedge x_\lambda)^{**} = x_\gamma^{**} \wedge x_\lambda^{**}$ and $(x_\gamma^*)^{**} = (x_\gamma^{**})^*$ hold in any pseudocomplemented lattice. Since the identity $(x_\gamma \vee x_\lambda)^{**} = x_\gamma^{**} \vee x_\lambda^{**}$ is characteristic of Stone algebras (see [26]) it follows by Proposition (2.12) that ** is a full operation on $\underset{\sim}{3}^{\kappa}$. Since $\underline{c} \in \underset{\sim}{3}^{\kappa}$ is maximal with respect to $\leq^{*}$ if and only if $c_{\gamma} \in \{0,1\}$ for all $\gamma < \kappa$, and $0^{**} = 0$ and $a^{**} = 1^{**} = 1$, it is obvious that $\underline{b}^{**}$ is the unique maximal element majorizing $\underline{b}$.□

Let V be the category defined by: X is an object of V if and only if X is a compact, totally order-disconnected space, every $x \in X$ is majorized by a unique maximal element x^{+}, and the map $^{+}: X \to X$ is continuous; ϕ is an arrow of V if and only if ϕ is continuous, order-preserving and $^{+}$-preserving. The partial order on an object of V will be denoted by $\leq^{*}$. The partial order on $\underset{\sim}{3}$, as an object of V, is determined by '$x <^{*} y \iff x = a$ and $y = 1$' and $^{+}$ is given by $0^{+} = 0$, $a^{+} = 1^{+} = 1$.

PROPOSITION (3.5.2). (i) *Every object* X *of* W *becomes an object of* V *when partially ordered by* '$x \leq^{*} y \iff x \wedge y$ *exists and equals* x'. *The map* $^{+}: X \to X$ *is then given by* $x^{+} = x^{**}$.

(ii) *Every object* X *of* V *becomes an object of* W *when a partial* $\wedge$ *operation is defined on* X *by* '$x \wedge y$ *exists and equals* $x \iff x \leq^{*} y$', *and a unary operation* $^{**}: X \to X$ *is defined by* $x^{**} = x^{+}$.

(iii) W *is isomorphic to the category* V.

Proof. (i) Since every object in W is iseomorphic to a subalgebra of $\underset{\sim}{3}^{\kappa}$ for some κ, this follows from Lemma (3.5.1).

(ii) Let $X \in V$ and define $\wedge$ and ** on X as indicated. We will show that $\mathcal{V}(X,\underset{\sim}{3})$ separates the points of X. Since a map ϕ: $X \to \underset{\sim}{3}$ is order-preserving if and only if it is meet-preserving, it then follows that X is iseomorphic to a closed subalgebra of $\underset{\sim}{3}^{\kappa}$, where κ is the cardinality of $\mathcal{V}(X,\underset{\sim}{3})$. If x and y are distinct elements of X, then either $x \nleq^* y$ or $y \nleq^* x$, say $x \nleq^* y$. Thus there exists a clopen increasing subset U of X with $x \in U$ and $y \notin U$. Let $U' = \{z \in X | z^+ \in U\}$. Since $^+$ is continuous and order-preserving, U' is clopen and increasing, and since $U' = (U]$, U' is also decreasing. Define ϕ: $X \to \underset{\sim}{3}$ by

$$z\phi = \begin{cases} 1 & \text{if } z \in U, \\ a & \text{if } z \in U' - U, \\ 0 & \text{if } z \notin U'. \end{cases}$$

Clearly $x\phi = 1$, $y\phi \in \{a,0\}$ and ϕ is continuous. Since the partial order on $\underset{\sim}{3}$ is determined by '$x <^* y \iff x = a$ and $y = 1$', it is also clear that ϕ is order-preserving and $^+$-preserving.

(iii) This follows from (i) and (ii). □

REMARK (3.5.3). If we had chosen $\mathcal{Q} = \{\wedge,\vee,0,1,^*,^{**}\}$, then $\mathcal{W}$ and $\mathcal{V}$ would still be isomorphic since $D_\vee = D_\wedge$, and hence $\vee$ induces the same partial order as $\wedge$, and D_0, D_1 and D_* are all empty. The polynomial $p(x_0) = x_0^{**}$ is an example of a polynomial which gives rise to a full operation while all its non-trivial, proper sub-polynomials (namely x_0^*) give rise to partial operations with empty domain.

REMARK (3.5.4). We will identify $\mathcal{W}$ with the category $\mathcal{V}$. Note that for all $B \in St$, the partial order $\leq^*$ on $St(B,\underset{\sim}{3})$ is given by $g \leq^* h \Leftrightarrow bg \leq bh$ for all $b \in B$, and for all $g \in St(B,\underset{\sim}{3})$, $g^+ \in St(B,\underset{\sim}{3})$ is defined by $bg^+ = (bg)^{**}$. It is well known (see [30] or [34]) that in a Stone algebra B every prime filter is contained in a unique maximal filter, and it is easily shown (see [5] or [30]) that $g \in St(B,\underset{\sim}{3})$ if and only if $1g^{-1}$ is a prime filter and $\{a,1\}g^{-1}$ is the unique maximal filter containing $1g^{-1}$. Define $T: St(B,\underset{\sim}{3}) \to \mathcal{D}(B,\underset{\sim}{2})$ by $gT = g_1$, where g_1 is the characteristic function of $1g^{-1}$. It is easily seen that T is a homeomorphism and an order-isomorphism. This remark provides the link between our approach and the approach in [62].

THEOREM (3.5.5). *$(St(-,\underset{\sim}{3}),\mathcal{W}(-,\underset{\sim}{3}))$ is a full duality between St and $\mathcal{W}$.*

Proof. By Theorem (2.8), to show that we have a duality it is sufficient to prove that $\underset{\sim}{3}$ is injective in $\mathcal{W}$ and that for all $1 \leq n < \omega$, every $\phi \in \mathcal{W}(\underset{\sim}{3}^n,\underset{\sim}{3})$ is a polynomial function. But this is precisely the content of the next two lemmas. Since $\underset{\sim}{3}$ is injective in St (see [5] or [30]), to show that the duality is full it is sufficient to prove that $St(C(-,\underset{\sim}{3}),\underset{\sim}{3})$: $ZComp \to \mathcal{W}$ is the $ZComp$-free functor for $\mathcal{W}$ (see Corollary (2.10)). This is proved in Proposition (3.5.8).□

LEMMA (3.5.6). *$\underset{\sim}{3}$ is injective in $\mathcal{W}$.*

Proof. Let X be a closed subalgebra of Y, let $\phi \in \mathcal{W}(X,\underset{\sim}{3})$ and let $J_0 = 0\phi^{-1}$, $J_a = a\phi^{-1}$ and $J_1 = 1\phi^{-1}$. Since $\{a\}$ decreasing in $\underset{\sim}{3}$ and

$\{1\}$ is increasing in $\underset{\sim}{3}$, J_a is clopen decreasing in X and J_1 is clopen increasing in X. Let U_a be clopen decreasing in Y with $U_a \cap X = J_a$, and let U_1 be clopen increasing in Y with $U_1 \cap X = J_1$ — applying the injectivity of $\underset{\sim}{2}$ in $\mathcal{P}$. Set $V_1 = \{y \in Y | y^+ \in U_1\}$ and $V_a = U_a - U_1$. The following three observations are easily checked: (i) $J_0 \subseteq Y - V_1$, (ii) $J_a \subseteq V_1 \cap V_a$, and (iii) $J_1 \subseteq V_1 - V_a$. Define $\phi' : Y \to \underset{\sim}{3}$ by

$$y\phi' = \begin{cases} 1 & \text{if } y \in V_1 - V_a, \\ a & \text{if } y \in V_1 \cap V_a, \\ 0 & \text{if } y \in Y - V_1. \end{cases}$$

ϕ' is continuous since V_1 and V_a are clopen. The set V_1 is both increasing and decreasing while V_a is decreasing. Hence $V_1 - V_a$ is increasing, $V_1 \cap V_a$ is decreasing and $Y - V_1$ is both increasing and decreasing, whence ϕ' is order-preserving. Since ϕ' is order-preserving, to prove that ϕ' is $^+$-preserving it is sufficient to prove that if $y\phi' = a$, then $y^+\phi' = 1$. If $y\phi' = a$, then $y \in V_1 \cap V_a$ and so $y^+ \in U_1$. Thus $(y^+)^+ = y^+ \in U_1$ and so $y^+ \in V_1$. Also $y^+ \in U_1$ implies that $y^+ \notin V_a = U_a - U_1$. Hence $y^+ \in V_1 - V_a$, giving $y^+\phi' = 1$ as required. Finally, $\phi' | X = \phi$ by (i), (ii) and (iii). □

LEMMA (3.5.7). *For all* $1 \leq n < \omega$, *every* $\phi \in W(\underset{\sim}{3}^n, \underset{\sim}{3})$ *is a polynomial function.*

Proof. Throughout the following discussion we make the convention that the meet of an empty set of variables is the nullary polynomial 1 and the join of an empty set of variables is the nullary polynomial 0.

Set $X = \underset{\sim}{3}^n$, $X^+ = \{\underline{b} \in X | b_i \in \{0,1\}$ for all $i < n\}$, $X_0^+ = 0\phi^{-1} \cap X^+$ and $X_1^+ = 1\phi^{-1} \cap X^+$. Clearly $X^+ = X_0^+ \cup X_1^+$. Define $p^0(\underline{x})$ by

$$p^0(\underline{x}) = \bigwedge(\bigvee(x_i^{**} | b_i = 0) \vee \bigvee(x_i^* | b_i = 1) | \underline{b} \in X_0^+).$$

Since $\underline{c}\phi = 0 \iff \underline{c}^+ \in X_0^+$ and $\underline{c}\phi \neq 0 \iff \underline{c}^+ \in X_1^+$, it follows that $\underline{c}\phi = 0 \iff p^0(\underline{c}) = 0$ and $\underline{c}\phi \neq 0 \iff p^0(\underline{c}) = 1$.

Let $(z]_* = \{y \in X | y \overset{*}{\leq} z\}$. If X_1^+ is non-empty, then for all $z \in X_1^+$, $1\phi^{-1} \cap (z]_*$ is a non-empty increasing subset of $(z]_*$. Let M_z be the set of minimal elements of $1\phi^{-1} \cap (z]_*$ and define $p^z(\underline{x})$ by

$$p^z(\underline{x}) = \bigvee(\bigwedge(x_i | b_i = 1) \wedge \bigwedge(x_i^{**} | b_i = a) \wedge \bigwedge(x_i^* | b_i = 0) | b \in M_z).$$

We claim that $p(\underline{x}) = p^0(\underline{x}) \wedge \bigvee(p^z(\underline{x}) | z \in X_1^+)$ is the required polynomial. Clearly is is sufficient to prove that $\underline{c}\phi \neq 0$ and $\underline{c}^+ = z$ imply that $p^z(\underline{c}) = \underline{c}\phi$ and $p^w(\underline{c}) = 0$ for $z \neq w \in X_1^+$.

We will constantly, and without specific reference, use the fact that $\underline{b} \overset{*}{\leq} \underline{c}$ and $b_i = 0$ imply that $c_i = 0$.

If $\underline{c}\phi = 1$, then $\underline{b} \overset{*}{\leq} \underline{c}$ for some $\underline{b} \in M_z$. Thus $\bigwedge(c_i | b_i = 1) = 1$, $\bigwedge(c_i^{**} | b_i = a) = 1$ and $\bigwedge(c_i^* | b_i = 0) = 1$, and so $p^z(\underline{c}) = 1$.

If $\underline{c}\phi = a$, then for all $b \in M_z$, $\underline{b} \overset{*}{\nleq} \underline{c}$. Thus for all $b \in M_z$, there exists $j < n$ such that $c_j = a$ and $b_j = 1$. It follows that for all $\underline{b} \in M_z$, $\bigwedge(c_i | b_i = 1) = a$, $\bigwedge(c_i^{**} | b_i = a) = 1$ and $\bigwedge(c_i^* | b_i = 0) = 1$, and so $p^z(\underline{c}) = a$.

If $z \neq w \in X_1^+$, then there exists $j < n$ such that either

$z_j = 0$ and $w_j = 1$, or $z_j = 1$ and $w_j = 0$. If $z_j = 0$ and $w_j = 1$, then for all $\underline{b} \in M_w$ either $b_j = a$ or $b_j = 1$. Thus for all $\underline{b} \in M_w$ either $\bigwedge(c_i | b_i = 1) = 0$ or $\bigwedge(c_i | b_i = a) = 0$, giving $p^w(\underline{c}) = 0$. If $z_j = 1$ and $w_j = 0$, then for all $\underline{b} \in M_w$, $b_j = 0$ and hence $\bigwedge(c_i^* | b_i = 0) = 0$, giving $p^w(\underline{c}) = 0$. □

If $X \in ZComp$, then let $F(X) = X \times \underset{\sim}{2} \in \mathcal{W}$ be determined by $(x,0)^+ = (x,1)^+ = (x,1)$ and $(x,a) <^* (y,b) \iff x = y$, $a = 0$ and $b = 1$. If $\phi \in C(X,Y)$, then define $F(\phi) \in \mathcal{W}(F(X),F(Y))$ by $(x,a)F(\phi) = (x\phi,a)$. Clearly $F: ZComp \to \mathcal{W}$ is a well-defined functor.

PROPOSITION (3.5.8). (i) $F: ZComp \to \mathcal{W}$ *is naturally isomorphic to* $St(C(-,\underset{\sim}{3}),\underset{\sim}{3}): ZComp \to \mathcal{W}$ *and is left adjoint to the forgetful functor from* $\mathcal{W}$ *into* $ZComp$.

(ii) $F(\beta(-)): Set \to \mathcal{W}$ *is left adjoint to the forgetful functor from* $\mathcal{W}$ *into* Set.

Proof. Let $End(\underset{\sim}{3}) = \{e_0,e_1\}$, where for all $c \in \underset{\sim}{3}$, $ce_0 = c$ and $ce_1 = c^{**}$. Define $\Gamma_X: F(X) \to St(C(X,\underset{\sim}{3}),\underset{\sim}{3})$ by $(x,d)\Gamma_X = [x]e_d$. Since $\underset{\sim}{3}$ satisfies the conditions of Proposition (3.3.3), Corollary (3.3.4) implies that Γ_X is a homeomorphism. Since $e_0 \leq e_1$ (pointwise), Γ_X is an order-isomorphism, and since $e_0^+ = e_1^+ = e_1$, Γ_X is $^+$-preserving. Thus Γ_X is an iseomorphism.

We now show that Γ is a natural transformation. Let $\psi \in C(X,Y)$, let $\phi \in C(Y,\underset{\sim}{3})$ and let $(x,d) \in F_X$. Then $\phi((x,d)\Gamma_X St(C(\psi,\underset{\sim}{3}),\underset{\sim}{3})) = \phi(C(\psi,\underset{\sim}{3})[x]e_d) = (\psi\phi)[x]e_d = (x\psi\phi)e_d$ $= \phi[x\psi]e_d = \phi((x\psi,d)\Gamma_Y) = \phi((x,d)F(\psi)\Gamma_Y)$, and hence $\Gamma_X St(C(\psi,\underset{\sim}{3}),\underset{\sim}{3})$ $= F(\psi)\Gamma_Y$ as required. Thus F and $St(C(-,\underset{\sim}{3}),\underset{\sim}{3})$ are naturally isomorphic.

The unit $\zeta\colon \mathrm{Id}_{\mathcal{ZComp}} \to \mathcal{W}$ of the adjunction from $\mathcal{ZComp}$ to $\mathcal{W}$ is defined by $x\zeta_X = (x,0)$. If $Y \in \mathcal{W}$ and $\alpha\colon X \to Y$ is continuous, then define $\beta\colon F(X) \to Y$ by $(x,0)\beta = x\alpha$ and $(x,1)\beta = (x\alpha)^+$. Clearly $\beta \in \mathcal{W}(F(X),Y)$. Since $x\zeta_X\beta = (x,0)\beta = x\alpha$ we have $\zeta_X\beta = \alpha$ and the uniqueness of β is immediate. This establishes (i) and (ii) follows as an obvious corollary.□

The duality can be applied to show that if $(B_\delta | \delta \in \Delta)$ is a family of Stone algebras, then their free product in $\mathcal{D}$ is a Stone algebra and is in fact their free product in $\mathcal{St}$. The description of the finitely generated free Stone algebras also follows easily (see [30] and [31]). For other applications of this duality one may turn to [62]. In that work, Priestley applies (her version of) this duality to carry out a thorough analysis of the triple construction for Stone algebras (see [10] and [11]) and then applies the dual triple to obtain information on free Stone algebras and to obtain a new proof of the characterization of the injectives in $\mathcal{St}$ (see [5], [30], [33] and [51]). We now give yet another proof of the characterization of the injectives in $\mathcal{St}$. Our proof is completely different from the proof in [62] and illustrates well the use of the $\mathcal{ZComp}$-free functor.

THEOREM (3.5.9). *The following are equivalent:*

(i) P *is sur-projective in* $\mathcal{W}$*;*

(ii) P *is a retract of* $F(X) = X \times \underset{\sim}{2}$ *for some compact, extremally* *disconnected space* X;

(iii) *there are compact, extremally disconnected spaces* X_0 *and* X_1 *such that* P *is iseomorphic to* $F(X_0)\dot{\cup}X_1$, *where* X_1 *is endowed with the discrete structure* ($x \leq^* y \iff x = y$, *and* $x^+ = x$).

Proof. By Proposition (1.29), only the equivalence of (ii) and (iii) remains to be proved.

(ii) => (iii). For convenience we will assume that P is a subalgebra of $X \times \underset{\sim}{2}$ and that $\tau = X \times \underset{\sim}{2} \to P$ is a retraction onto P. If $(x,0) \in P$, then $(x,1) = (x,0)^+ \in P$, but it may happen that $(x,1) \in P$ while $(x,0) \notin P$. Hence setting $X_0 = \{x \in X | (x,0) \in P\}$ and $X_1 = \{x \in X | (x,1) \in P; x \notin X_0\}$, we have $P = X_0 \times \underset{\sim}{2} \dot{\cup} X_1 \times \{1\}$. We claim that (a) $X_0 \cup X_1$ is a retract of X, and (b) X_0 and X_1 are clopen in $X_0 \cup X_1$. The result then follows since (a) and (b) imply that X_0 and X_1 are retracts of X and it is well known that a retract of a compact, extremally disconnected space is itself compact and extremally disconnected (see [35]).

(a) Define $\rho: X \to X_0 \cup X_1$ by $x\rho = (x,1)\tau\pi$, where $\pi: X \times \underset{\sim}{2} \to X$ is the natural projection. Let U be open in X. Then

$$U\rho^{-1} = \{x \in X | (x,1)\tau\pi \in U\}$$
$$= [(X \times \{1\}) \cap (U \times \underset{\sim}{2})\tau^{-1}]\pi,$$

which is open since τ is continuous and π is open.

(b) We will construct a set V which is clopen in X and satisfies $V \cap (X_0 \cup X_1) = X_0$. In fact, let $V = [(X \times \{0\}) \cap (X \times \{0\})\tau^{-1}]\pi$. Then V is clopen since τ is continuous and π, being a projection parallel to a compact factor, is both open and closed. It is easily verified that $X_0 \subseteq V$ and $X_1 \cap V$ is empty, whence $V \cap (X_0 \cup X_1) = X_0$.

(iii) => (ii). If P is iseomorphic to $X_0 \times \underset{\sim}{2} \dot{\cup} X_1$, then it is also iseomorphic to the retract $X_0 \times \underset{\sim}{2} \dot{\cup} X_1 \times \{1\}$ of $(X_0 \dot{\cup} X_1) \times \underset{\sim}{2}$. Since X_0 and X_1 are both compact and extremaly disconnected, so is $X_0 \dot{\cup} X_1$.□

If B is a Boolean algebra, then let $B^{[2]} = \{(b_0,b_1) \in B^2 | b_0 \leq b_1\}$. It is easily seen that $B^{[2]}$ is a Stone algebra in which $(b_0,b_1)^* = (b_1',b_1')$. Observe that $B^{[2]} \simeq C(X,\underset{\sim}{3})$, where X is the Stone space of B.

THEOREM (3.5.10). (R. Balbes and G. Gratzer [5]). *The following are equivalent:*

(i) I *is injective in St;*

(ii) $I \simeq C(X_0,\underset{\sim}{3}) \times C(X_1,\underset{\sim}{2})$ *for some compact, extremally disconnected spaces* X_0 *and* X_1*;*

(iii) $I \simeq B_0^{[2]} \times B_1$ *for some complete Boolean algebras* B_0 *and* B_1.

Proof. (i) <=> (ii). It is easily seen that $\underset{\sim}{3}$ is injective in *St* (see [5] or [30]). Hence by Proposition (1.28), I is injective in *St* if and only if there is a sur-projective P in $\mathcal{W}$ with $I \simeq \mathcal{W}(P,\underset{\sim}{3})$. Hence I is injective in *St* if and only if there are compact, extremally disconnected spaces X_0 and X_1 with $I \simeq \mathcal{W}(F(X_0) \dot{\cup} X_1,\underset{\sim}{3}) \simeq \mathcal{W}(F(X_0),\underset{\sim}{3}) \times \mathcal{W}(X_1,\underset{\sim}{3})$. By the duality, $\mathcal{W}(F(X_0),\underset{\sim}{3}) \simeq \mathcal{W}(St(C(X_0,\underset{\sim}{3}),\underset{\sim}{3}),\underset{\sim}{3}) \simeq C(X_0,\underset{\sim}{3})$ and since $x^+ = x$ for all $x \in X_1$, $\mathcal{W}(X_1,\underset{\sim}{3}) \simeq C(X_1,\underset{\sim}{2})$ as required.

(ii) <=> (iii). This is immediate since compact, extremally disconnected spaces are precisely the Stone spaces of complete Boolean algebras (see [35]). □

4. DUALITIES FOR EQUATIONAL CLASSES OF RELATIVE STONE ALGEBRAS

A *Brouwerian algèbra* B is a (necessarily distributive) lattice in which for all $a, b \in B$, the relative annihilator $(a;b) = \{x \in B \mid x \wedge a \leq b\}$ is a principal ideal; the generator of the ideal $(a;b)$ is denoted by $a*b$. Since B necessarily has a unit ($1 = a*a$ for all $a \in B$), we regard Brouwerian algebras as algebras of type (2,2,2,0) with operations $(\wedge, \vee, *, 1)$. A *Heyting algebra* is a Brouwerian algebra with zero, whence it is an algebra of type (2,2,2,0,0) with operations $(\wedge, \vee, *, 0, 1)$. The standard results on Brouwerian and Heyting algebras may be found in [64] where they are referred to as relatively pseudo-complemented lattices and pseudo-Boolean algebras respectively. In particular, recall that the classes of Brouwerian algebras and Heyting algebras are equational and that the lattice of congruences on a Brouwerian or Heyting algebra is isomorphic to its lattice of filters. It follows immediately from the latter fact that each Brouwerian or Heyting algebra has distributive congruences and that every equational class of Brouwerian or Heyting algebras has the congruence extension property (see Definition (4.2.1)).

We will denote the n-element chain $0 = a_0 < a_1 < \cdots < a_{n-2} < a_{n-1} = 1$ as a Brouwerian algebra by C_n^1 and as a Heyting algebra by C_n.

A Brouwerian algebra satisfying the identity $(x*y) \vee (y*x) = 1$

is known as a *relative Stove algebra*. We will denote the equational class of all relative Stone algebras by R_ω and for $1 \leq n < \omega$, R_n will denote the equational subclass generated by C_n^1. A Heyting algebra satisfying the identity $(x*y) \vee (y*x) = 1$ is known as an *L-algebra*. We will denote the equational class of all *L-algebra* by L_ω and for $1 \leq n < \omega$, L_n will denote the equational subclass generated by C_n. It is well known (see [6], [12], [15] or [54]) that every interval in a relative Stone algebra is a Stone algebra, whence the name. Relative Stone algebras date back to [34] and L-algebras arise naturally in the study of intermediate logics (see [41] and [42]). T. Hecht and T. Katriňák [36] have shown that the lattices of equational subclasses of R_ω and L_ω are given by the $(\omega+1)$-chains $R_1 \subset R_2 \subset \ldots \subset R_\omega$ and $L_1 \subset L_2 \subset \ldots \subset L_\omega$. Identities characterizing the classes R_n and L_n may also be found in [36].

In this chapter dualities are established for each of the classes R_n and L_n $(2 \leq n < \omega)$. Then the dualities are applied to describe the weak injectives, injectives, free products and finite generated free algebras in each of the classes.

Throughout this chapter n will be fixed with $2 \leq n < \omega$.

4.1 *The Dualities*

We will establish the duality for L_n, from which the duality for R_{n-1} will follow easily. The following result, which is also valid for Brouwerian algebras, is crucial.

PROPOSITION (4.1.1). *Let* B *be a Heyting algebra.*

(i) $B \in L_\omega$ *if and only if for any prime filter* F *of* B, *the set of prime filters containing* F *forms a chain.*

(ii) $B \in L_n$ *if and only if for any prime filter* F *of* B, *the set of prime filters containing* F *forms a chain with at most* n - 1 *elements.*

(iii) *Let* $g: B \to C_n$ *be an onto map and for* $1 \leq j < n$ *let* $F_j = [a_j)g^{-1}$. *Then* g *is a homomorphism if and only if* F_j *is a prime filter for all* $1 \leq j < n$ *and the chain* $F_{n-1} \subset F_{n-2} \subset \cdots \subset F_1$ *is the set of all prime filters containing* $F_{n-1} = 1g^{-1}$.

Proof. (i) See [12], [15], [34], [54] or [68].

(ii) See [36].

(iii) If F is a filter in a distributive lattice D, then the smallest congruence Θ_F on D with F as a congruence class is given by $(a,b) \in \Theta_F \iff a \wedge f = b \wedge f$ for some $f \in F$ (see [9]). Let H be the set of all prime filters containing F. It is easily verified that $(a,b) \in \Theta_F \iff (a \in P \iff b \in P$, for all $P \in H)$. Thus it is sufficient to show that the unique Heyting algebra congruence on B determined by the filter $F_{n-1} = 1g^{-1}$ coincides with the lattice congruence $\Theta_{F_{n-1}}$. But this is proved in [55]. $\square$

We can use (4.1.1)(iii) to describe the endomorphisms of C_n. Let $1 \leq k < n$ and define a map $e: C_n \to C_n$ by declaring that

$0e = 0$, $[a_k)e = \{1\}$ and for $0 < i < j < k$ choosing $0 < a_i e < a_j e < 1$. Then e is an endomorphism of C_n and every endomorphism of C_n is of this form. It follows that

$$|\mathrm{End}(C_n)| = \binom{n-2}{0} + \binom{n-2}{1} + \ldots + \binom{n-2}{n-2} = 2^{n-2}.$$

Note that $\mathrm{End}(C_n)$ is a monoid with unit $\iota = \mathrm{Id}_{C_n}$.

DEFINITION (4.1.2). Let $B \in L_n$ and let $F = F_k \subset F_{k-1} \subset \cdots \subset F_1$ be the chain of all prime filters containing the prime filter F. The *homomorphism* $h_F \in L_n(B,C_n)$ *determined by* F is defined by

$$bh_F = \begin{cases} 1 & \text{if } b \in F = F_k, \\ a_j & \text{if } b \in F_j - F_{j+1} \ (1 \leq j < k), \\ 0 & \text{if } b \in B - F_1. \end{cases}$$

Proposition (4.1.1) guarantees that h_F is well defined. Essentially, h_F maps all the elements of F to 1 and maps all the other elements of B as low as possible in the chain C_n.

We can now prove a useful factorization lemma.

LEMMA (4.1.3). *Let* $g \in L_n(B,C_n)$ *and let* g^0 *be the homomorphism determined by* $F = 1g^{-1}$. *Then there exists an endomorphism* $e \in \mathrm{End}(C_n)$ *with* $g = g^0 e$.

Proof. Let $F = F_k \subset F_{k-1} \subset \cdots \subset F_1$ be the chain of all prime filters containing F. For all $1 \leq j < k$ choose $b_j \in F_j - F_{j+1}$ and define $e: C_n \to C_n$ by $0e = 0$, $a_j e = 1$ for $k \leq j < n$ and $a_j e = b_j g$ for $1 \leq j < k$. By Proposition (4.1.1)(iii) g is constant on $B - F_1$, $F_j - F_{j+1}$ $(1 \leq j < k)$ and F_k. Thus $g = g^0 e$, and $e \in \mathrm{End}(C_n)$ since $0 < i < j < k$ implies that $0 < a_i e < a_j e < 1$. □

Let X be a Boolean space. Then the set $E(X) = C(X,X)$ is a monoid with Id_X as unit. Let $\mathcal{X}_n$ be the category of Boolean spaces which have a continuous action of the monoid $End(C_n)$, that is a semigroup homomorphism $e \to \tilde{e}$ from $End(C_n)$ into $E(X)$ such that $\tilde{\imath} = Id_X$. A map $\phi \in C(X,Y)$ is an arrow of $\mathcal{X}_n$ if and only if ϕ preserves the action of $End(C_n)$, that is $x\tilde{e}\phi = x\phi\tilde{e}$ for all $x \in X$ and all $e \in End(C_n)$. Observe that $C_n \in \mathcal{X}_n$: for all $e \in End(C_n)$, $\tilde{e} = e$.

For any $B \in \mathcal{L}_n$, the Boolean space $\mathcal{L}_n(B,C_n)$ can be lifted to an object of $\mathcal{X}_n$: for all $e \in End(C_n)$, $\tilde{e} \in E(\mathcal{L}_n(B,C_n))$ is defined by $g\tilde{e} = ge$. If $h \in \mathcal{L}_n(B,D)$, then it is clear that $\mathcal{L}_n(h,C_n) \in \mathcal{X}_n$, whence $\mathcal{L}_n(-,C_n)\colon \mathcal{L}_n \to \mathcal{X}_n^{op}$ is a well-defined functor. It is also easy to verify that for all $X \in \mathcal{X}_n$, $\mathcal{X}_n(X,C_n)$ is a subalgebra of $C(X,C_n)$ and that for all $\phi \in \mathcal{X}_n(X,Y)$, $\mathcal{X}_n(\phi,C_n) \in \mathcal{L}_n$, whence $\mathcal{X}_n(-,C_n)\colon \mathcal{X}_n^{op} \to \mathcal{L}_n$ is a well-defined functor.

PROPOSITION (4.1.4). *For all* $B \in \mathcal{L}_n$ *define* $\eta_B\colon B \to \mathcal{X}_n(\mathcal{L}_n(B,C_n),C_n)$ *by* $b\eta_B = [b]$, *where* $g[b] = bg$, *and for all* $X \in \mathcal{X}_n$ *define* $\varepsilon_X\colon X \to \mathcal{L}_n(\mathcal{X}_n(X,C_n),C_n)$ *by* $x\varepsilon_X = [x]$, *where* $\phi[x] = x\phi$. *Then* $(\mathcal{L}_n(-,C_n),\mathcal{X}_n(-,C_n);\ \eta,\varepsilon)$ *is an adjunction from* $\mathcal{L}_n$ *to* $\mathcal{X}_n^{op}$.

Proof. By Proposition (1.13) it is sufficient to show that for all $X \in \mathcal{X}_n$ and every homomorphism $\alpha\colon B \to \mathcal{X}_n(X,C_n)$, the continuous map $\beta\colon X \to \mathcal{L}_n(B,C_n)$, defined by $b(x\beta) = x(b\alpha)$, lifts to $\mathcal{X}_n$. But if $e \in End(C_n)$, then since $b\alpha$ preserves $\tilde{e}$, $b(x\tilde{e}\beta) = x\tilde{e}(b\alpha)$ $= (x(b\alpha))e = (b(x\beta))e = b(x\beta\tilde{e})$ as required. □

LEMMA (4.1.5). *If* $\phi \in \mathcal{X}_n(\mathcal{L}_n(B,C_n),C_n)$, *then* $g\phi \in Im(g)$ *for all*

$g \in L_n(B,C_n)$.

Proof. Since ϕ preserves the action of $End(C_n)$, by Lemma (4.1.3) it is sufficient to show that $g^0\phi \in Im(g^0)$ for all $g \in L_n(B,C_n)$. Let $Im(g^0) = (a_{k-1}] \dot{\bigcup} \{1\}$ and let e_k be the endomorphism of C_n determined by the prime filter $[a_k)$. Clearly $g^0 = g^0e_k = g^0\tilde{e}_k$ and hence $g^0\phi = (g^0\tilde{e}_k)\phi = (g^0\phi)\tilde{e}_k = (g^0\phi)e_k$, whence $g^0\phi \in Im(e_k)$ $= Im(g^0)$. $\square$

THEOREM (4.1.6). *$(L_n(-,C_n), X_n(-,C_n))$ is a duality between L_n and X_n.*

Proof. Since the only subdirectly irreducible algebras in L_n are the chains $C_2,\ldots,C_n$ (see [36] or apply Jónsson's lemma [45]), it follows that $L_n = ISP(C_n)$ and hence for all $B \in L_n$, $\eta_B: B \to X_n(L_n(B,C_n),C_n)$ is an embedding. We will now apply the duality for bounded distributive lattices (Theorem (3.4.4)) to show that η_B is always a surjection.

Define an equivalence relation R on $L_n(B,C_n)$ by $(g,h) \in R$ $\Leftrightarrow 1g^{-1} = 1h^{-1}$ and note that $g/R = g^0/R$, where g/R denotes the equivalence class of g in $L_n(B,C_n)/R$. Define a partial order $\leq$ on the quotient space $L_n(B,C_n)/R$ by $g/R \leq h/R \Leftrightarrow 1g^{-1} \subseteq 1h^{-1}$. Observe that $g/R \leq h/R \Leftrightarrow g^0 \leq h^0$ (pointwise) $\Leftrightarrow g^0e = h^0$ for some $e \in End(C_n)$.

We claim that $L_n(B,C_n)/R$ is iseomorphic to $\mathcal{D}(B,\underset{\sim}{2})$. Define $\lambda: L_n(B,C_n) \to \mathcal{D}(B,\underset{\sim}{2})$ by $g\lambda = g\sigma$, where $\sigma \in \mathcal{D}(C_n,\underset{\sim}{2})$ is determined by $1\sigma^{-1} = \{1\}$. λ is continuous since for all $b \in B$,

$(b;\{1\})\lambda^{-1} = \{g \in L_n(B,C_n) | bg\sigma = 1\} = \{g \in L_n(B,C_n) | bg = 1\}$ $= (b;\{1\})$, and $(b;\{0\})\lambda^{-1} = \{g \in L_n(B,C_n) | bg\sigma = 0\} = (b;(a_{n-2}])$. Since λ is constant on the equivalence classes of R, it induces a homeomorphism $\bar{\lambda}$ between $L_n(B,C_n)/R$ and $\mathcal{D}(B,\underset{\sim}{2})$ (see [19, Corollary 2.2, p.227]). Clearly $g/R \leq h/R \iff 1g^{-1} \subseteq 1h^{-1} \iff g\lambda \leq h\lambda$ $\iff g/R\bar{\lambda} \leq h/R\bar{\lambda}$ and hence $\bar{\lambda}$ is an iseomorphism.

If $\phi \in X_n(L_n(B,C_n),C_n)$, then $\phi\sigma\colon L_n(B,C_n) \to \underset{\sim}{2}$ is continuous. Let $(g,h) \in R$ and let $g = g^0e$ and $h = h^0f$ be factorizations of g and h via Lemma (4.1.3). Then $g^0 = h^0$ and hence $g^0\phi = h^0\phi$. If $g^0\phi = h^0\phi = 1$, then $g\phi = (g^0\phi)e = 1 = (h^0\phi)e = h\phi$. If $g^0\phi = h^0\phi \neq 1$, then $(g^0\phi)e \neq 1$ and $(h^0\phi)f \neq 1$ by Lemma (4.1.5), and thus $g\phi \neq 1$ and $h\phi \neq 1$. In either case it follows that $g\phi\sigma = h\phi\sigma$. Thus $\phi\sigma$ is a continuous map into $\underset{\sim}{2}$ which is constant on the equivalence classes of R, and hence it induces a continuous map $\overline{\phi\sigma}\colon L_n(B,C_n)/R \to \underset{\sim}{2}$.

If $g/R \leq h/R$, then there is an endomorphism $e \in \mathrm{End}(C_n)$ with $g^0e = h^0$. Thus, if $g^0\phi\sigma = 1$, then $g^0\phi = 1$ and so $h^0\phi = (g^0e)\phi = (g^0\phi)e = 1e = 1$, which implies that $h^0\phi\sigma = 1$. Consequently $g^0\phi\sigma \leq h^0\phi\sigma$, whence $g/R\overline{\phi\sigma} = g^0/R\overline{\phi\sigma} \leq h^0/R\overline{\phi\sigma} = h/R\overline{\phi\sigma}$. Hence $\overline{\phi\sigma}$ is a continuous, order-preserving map, that is $\overline{\phi\sigma} \in P(L_n(B,C_n)/R,\underset{\sim}{2})$.

Since $L_n(B,C_n)/R$ is iseomorphic to $\mathcal{D}(B,2)$, Theorem (3.4.4) implies that there exists $b \in B$ such that $g/R\overline{\phi\sigma} = b(g\sigma)$ for all $g \in L_n(B,C_n)$. We claim that $b\eta_B = \phi$.

Note that for all $h \in L_n(B,C_n)$, $h\phi = 1 \Leftrightarrow h\phi\sigma = 1 \Leftrightarrow h/R\overline{\phi\sigma}$ $= 1 \Leftrightarrow bh\sigma = 1 \Leftrightarrow bh = 1$, that is

(*) for all $h \in L_n(B,C_n)$, $h\phi = 1$ if and only if $bh = 1$.

By Lemma (4.1.3) it is sufficient to prove that for all $g \in L_n(B,C_n)$, $g^0\phi = bg^0$. Let $1g^{-1} = F_k \subset F_{k-1} \subset \ldots \subset F_1$ be the chain of all prime filters containing $1g^{-1}$. For $1 \leq j \leq k$ let $g_j: B \to C_n$ be the homomorphism determined by the prime filter F_j, let $e_j \in End(C_n)$ be the endomorphism determined by the prime filter $[a_j)$ and observe that $g_j = g^0 e_j$.

If $b \in F_k = 1g^{-1}$, then $g^0\phi = 1 = bg^0$ by (*). If $b \in B - F_1$, then $bg_1 \neq 1$ and hence $g_1\phi \neq 1$ by (*). But $g_1\phi \in Im(g_1)$ by Lemma (4.1.5) and hence $g_1\phi = 0$. Consequently $g^0\phi e_1 = g^0 e_1 \phi$ $= g_1\phi = 0$, but since $b \in B - F_1$ we also have $bg^0 = 0$, whence $g^0\phi = bg^0$. Finally, assume that $b \in F_\ell - F_{\ell+1}$ with $1 \leq \ell < k$. Clearly $bg_\ell = 1$ and $bg_{\ell+1} \neq 1$. Thus by (*), $g_\ell\phi = 1$ and $g_{\ell+1}\phi \neq 1$. Hence $g^0\phi e_\ell = g^0 e_\ell \phi = g_\ell\phi = 1$ and $g^0\phi e_{\ell+1} = g^0 e_{\ell+1}\phi$ $= g_{\ell+1}\phi \neq 1$, that is $g^0\phi \in [a_\ell) - [a_{\ell+1}) = \{a_\ell\}$. Hence $g^0\phi = a_\ell$, but since $b \in F_\ell - F_{\ell+1}$ we also have $bg^0 = a_\ell$, whence $g^0\phi = a_\ell = bg^0$. □

For n = 2 the duality reduces to Stone's duality for Boolean algebras and hence is full.

For n = 3 the duality is also full. Any algebra $B \in L_3$ is a fortiori a Stone algebra and since any chain of prime filters in B has at most two elements it follows that $\mathcal{L}_3(B,C_3) = \mathcal{St}(B,\underset{\sim}{3})$. The operation $^+$ on $\mathcal{St}(B,\underset{\sim}{3})$ corresponds to the action of the proper endomorphism $e_1 \in End(C_3)$ on $\mathcal{L}_3(B,C_3)$; in fact, $g^+ = g^{**} = ge_1 = g\tilde{e}_1$. Similarly, for any $X \in \mathcal{X}_3$ we may define an operation $^+$ on X by $x^+ = x\tilde{e}_1$ and a partial order $\leq$ on X by $x \leq y \iff x = y$ or $x\tilde{e}_1 = y$. It is easily seen that X is totally order-disconnected with respect to $\leq$ and hence $X \in \mathcal{W}$. Since both $^+$ and $\leq$ are defined in terms of the endomorphisms of C_3 it follows that $\mathcal{X}_3(X,C_3) = \mathcal{W}(X,\underset{\sim}{3})$. Since the duality for Stone algebras is full (Theorem (3.5.5)) it follows that the duality between $\mathcal{L}_3$ and $\mathcal{X}_3$ is full.

For $n \geq 4$ the duality is not full. Let $X = \{0,1\}$, let $\tilde{\iota} = Id_X$ and for all $e \neq \iota$ let $\tilde{e}$ be the retraction onto the point 1. It is easily checked that the action of $End(C_n)$ is well defined and that $\mathcal{X}_n(X,C_n) = \{\phi_0,\phi_1,\phi_2\}$ where $0\phi_0 = 1\phi_0 = 0$, $0\phi_1 = 1$ and $1\phi_1 = a_{n-2}$, and $0\phi_2 = 1\phi_2 = 1$. Hence $\mathcal{X}_n(X,C_n) \simeq C_3$ which gives $|\mathcal{L}_n(\mathcal{X}_n(X,C_n),C_n)| = |\mathcal{L}_n(C_3,C_n)| = n - 1 \neq 2$, whence ε_X is not a surjection.

We turn now to the category $\mathcal{R}_n$. The endomorphisms of C_n^1 are determined just as the endomorphisms of C_n were, except that the restriction $0e = 0$ is dropped. By identifying C_n^1 with the filter $[a_1)$ of C_{n+1} we obtain a one-one correspondence, in fact a semigroup isomorphism, between $End(C_n^1)$ and $End(C_{n+1})$. Hence $|End(C_n^1)| = 2^{n-1}$. The retraction onto 1 acts as a zero of the monoid $End(C_n^1)$ and we will denote it by θ.

Let X be a pointed Boolean space. Then the set $E^1(X)$ of point-preserving, continuous maps $\phi: X \to X$ is a monoid with Id_X as unit and the retraction onto the distinguished point as a zero. Let $\mathcal{Y}_n$ be the category of pointed Boolean spaces which have a continuous action, $e \to \tilde{e}$, of the monoid $End(C_n^1)$ such that $\tilde{\theta}$ is the retraction onto the distinguished point. The arrows of $\mathcal{Y}_n$ are the point-preserving, continuous maps which also preserve the action of $End(C_n^1)$. Observe that $C_n^1 \varepsilon \mathcal{Y}_n$: 1 is the distinguished point and for all $e \varepsilon End(C_n^1)$, $\tilde{e} = e$. The Hom-functors $\mathcal{R}_n(-,C_n^1): \mathcal{R}_n \to \mathcal{Y}_n^{op}$ and $\mathcal{Y}_n(-,C_n^1): \mathcal{Y}_n^{op} \to \mathcal{R}_n$ are defined as they were for $\mathcal{L}_n$ and $\mathcal{X}_n$. Note that for all $B \varepsilon \mathcal{R}_n$, the constant map $\hat{1}$ onto $\{1\}$ acts as the distinguished point of $\mathcal{R}_n(B,C_n^1)$.

THEOREM (4.1.7). *$(\mathcal{R}_n(-,C_n^1),\mathcal{Y}_n(-,C_n^1))$ is a duality between $\mathcal{R}_n$ and $\mathcal{Y}_n$.*

Proof. That the functors are adjoint follows as in the proof of Proposition (4.1.4). If $B \varepsilon \mathcal{R}_n$, then ${}_0B$, the Heyting algebra obtained by adjoining a new zero to B, is an object of $\mathcal{L}_{n+1}$ by Proposition (4.1.1)(ii). If $g \varepsilon \mathcal{R}_n(B,C_n^1)$, then, identifying C_n^1 with the filter $[a_1)$ of C_{n+1} as mentioned above, we obtain ${}_0g \varepsilon \mathcal{L}_{n+1}({}_0B,C_{n+1})$ by extending g in the obvious way. Since $End(C_n^1) \simeq End(C_{n+1})$ it follows that $\mathcal{R}_n(B,C_n^1) \simeq \mathcal{L}_{n+1}({}_0B,C_{n+1})$ where the distinguished point of $\mathcal{L}_{n+1}({}_0B,C_{n+1})$ is the homomorphism $h_1: {}_0B \to C_{n+1}$ determined by the prime filter B. Note that for all $g \in \mathcal{L}_{n+1}({}_0B,C_{n+1})$, $ge_1 = h_1$. If $\phi \in \mathcal{X}_{n+1}(\mathcal{L}_{n+1}({}_0B,C_{n+1}),C_{n+1})$, then $h_1\phi \in Im(h_1) = \{0,1\}$ by Lemma (4.1.5). If $h_1\phi = 0$, then for all g we have $g\phi e_1 = ge_1\phi = h_1\phi = 0$ and so $g\phi = 0$, that is $\phi = \hat{0}$. Similarly, if $h_1\phi = 1$, then for all g, $g\phi \in [a_1)$. It follows readily that

$$\mathcal{X}_{n+1}(\mathcal{L}_{n+1}({}_0B,C_{n+1}),C_{n+1}) \simeq \mathcal{Y}_n(\mathcal{L}_{n+1}({}_0B,C_{n+1}),C_n^1)\bigcup\{0\}.$$

Thus $\mathcal{Y}_n(\mathcal{R}_n(B,C_n^1),C_n^1) \simeq \mathcal{Y}_n(\mathcal{L}_{n+1}({}_0B,C_{n+1}),C_n^1) \simeq \mathcal{X}_{n+1}(\mathcal{L}_{n+1}({}_0B,C_{n+1}),$ $C_{n+1}) - \{0\} \simeq {}_0B - \{0\} = B.$ □

For n = 2 the duality is full. A *dual generalized Boolean algebra* (DGBA) is a distributive lattice with unit in which each principal filter is a Boolean algebra. It is well known that a DGBA is a Brouwerian algebra ($a*b$ is the complement of a in the principal filter $[a\wedge b)$) and that $\mathcal{R}_2$ is the class of all DGBA's (see [46]). Since $End(C_2^1) = \{\iota,\theta\}$ the action of $End(C_2^1)$ is trivial and hence $\mathcal{Y}_2$ is isomorphic to the category of pointed Boolean spaces. Since the duality for distribution lattices with unit is full (Theorem (3.4.7)) it follows that the duality between $\mathcal{R}_2$ and $\mathcal{Y}_2$ is full.

For $n \geq 3$ the duality is not full. Let $X = \{0,1\}$, let $\tilde{\iota} = Id_X$ and for $e \neq \iota$ let $\tilde{e}$ be the retraction onto the distinguished point 1. It is easily checked that the action of $End(C_n^1)$ is well defined and that $\mathcal{Y}_n(X,C_n^1) = \{\phi_0,\phi_1\}$ where $0\phi_0 = a_{n-2}$ and $1\phi_0 = 1$, and $0\phi_1 = 1\phi_1 = 1$. Hence $|\mathcal{R}_n(\mathcal{Y}_n(X,C_n^1),C_n^1)|$ $= |\mathcal{R}_n(C_2^1,C_n^1)| = n \neq 2$, and thus ε_X is not a surjection.

Even though the dualities we have established are almost never full the functors $L_n(C(-,C_n),C_n)$: $ZComp \to X_n$ and $R_n(C(-,C_n^1),C_n^1)$: $ZComp \to Y_n$ are $ZComp$-free for the categories X_n and Y_n respectively (see Proposition (1.24)). In fact by utilizing Corollary (3.3.4) we can give explicit constructions for these functors.

For $X \in ZComp$ let $F(X) = X \times End(C_n)$ and define the action of $End(C_n)$ on $F(X)$ by $(x,e)\tilde{f} = (x,ef)$. If $\phi \in C(X,Y)$, then define $F(\phi) \in X_n(F(X),F(Y))$ by $(x,e)F(\phi) = (x\phi,e)$. Clearly F: $ZComp \to X_n$ is a well-defined functor.

For $X \in ZComp$ let $F^1(X) = X \times (End(C_n^1) - \{\theta\})\dot{\cup}\{\theta\}$, let θ be the distinguished point of $F^1(X)$ and define the action of $End(C_n^1)$ on $F^1(X)$ by

$$(x,e)\tilde{f} = \begin{cases} (x,ef) & \text{if} \quad ef \neq \theta, \\ \theta & \text{if} \quad ef = \theta, \end{cases} \quad \text{and } \theta\tilde{f} = \theta.$$

If $\phi \in C(X,Y)$, then define $F^1(\phi) \in Y_n(F^1(X),F^1(Y))$ by $(x,e)F^1(\phi) = (x\phi,e)$ and $\theta F^1(\phi) = \theta$. Clearly F^1: $ZComp \to Y_n$ is a well-defined functor.

Since the proof of the following result is just a slight generalization of the proof of Proposition (3.5.8) it will be omitted. Note that Brouwerian and Heyting algebras have distributive congruences and that for $2 \leq n < \omega$, C_n and C_n^1 are subdirectly irreducible and hence Corollary (3.3.4) is applicable.

PROPOSITION (4.1.8). (i) F: $ZComp \to X_n$ *is naturally isomorphic* *to* $L_n(C(-,C_n),C_n)$: *ZComp* $\to X_n$ *and is left adjoint to the forgetful functor from* X_n *into* ZComp.

(ii) F^1: *ZComp* $\to Y_n$ *is naturally isomorphic to* $R_n(C(-,C_n^1),C_n^1)$: *ZComp* $\to Y_n$ *and is left adjoint to the forgetful functor from* Y_n *into* ZComp. □

As usual the free functors from *Set* into X_n and Y_n may be obtained by composing F and F^1 respectively, with β, the Stone-Čech compactification functor.

4.2 *Weak Injectives and Injectives*

We recall some definitions. Throughout this preamble *A* will denote an equational class.

DEFINITION (4.2.1). (i) An algebra I ε *A* is a *weak injective in A* if, given algebras B,C ε *A*, B a subalgebra of C, and given a surjection g: B → I there is a homomorphism g': C → I with g'|B = g.

(ii) An algebra I ε *A* is an *absolute subretract in A* if it is retract of each of its extensions in *A*.

(iii) *A* is said to have *enough injectives* if every algebra in *A* has an injective extension in *A*.

(iv) An algebra I ε *A* is *self-injective* if every homomorphism of a subalgebra of I into I extends to an endomorphism of I.

(v) *A* satisfies the *congruence extension property* if, given B,C ε *A*, B a subalgebra of C, and given any congruence Θ on B there is a congruence Θ' on C with Θ'|B = Θ.

(vi) A *maximal subdirectly irreducible* algebra in A is a subdirectly irreducible algebra with no subdirectly irreducible, proper extension in A.

(vii) Let $(B_\delta | \delta \in \Delta)$ be a family of algebras and let $g: B \to \Pi(B_\delta | \delta \in \Delta)$ be an embedding of B as a subdirect product. If g also embeds B as a retract of $\Pi(B_\delta | \delta \in \Delta)$ we say that B is a *subdirect retract* of the family $(B_\delta | \delta \in \Delta)$.

These concepts are related as follows (see [8], [16], [32] and [33]).

PROPOSITION (4.2.2). (i) *If* I *is an injective in* A, *then* I *is a weak injective. If* I *is a weak injective in* A, *then* I *is an absolute subretract.*

(ii) *If* A *satisfies the congruence extension property, then* I *is a weak injective in* A *if and only if* I *is an absolute subretract in* A.

(iii) *If* A *has enough injectives, then (in* A*) the concepts of injective, weak injective and absolute subretract are equivalent.*

(iv) *Let* A *be a congruence distributive equational class and assume that* A = ISP(A), *where* A *is a finite subdirectly irreducible algebra whose subalgebras are either injective or subdirectly irreducible. Then* A *has enough injectives if and only if* A *is self-injective.*

(v) *Any maximal subdirectly irreducible algebra in* A *is an absolute subretract.*

(vi) *A subdirect retract of a family of weak injectives is itself a weak injective.* □

If $B,C \in \mathcal{A}$ and Θ and Φ are congruences on B and C respectively, then we may define a congruence (Θ,Φ) on $B \times C$ by $((b_1,c_1),(b_2,c_2)) \in (\Theta,\Phi) \Leftrightarrow (b_1,b_2) \in \Theta$ and $(c_1,c_2) \in \Phi$. If for all algebras $B,C \in \mathcal{A}$, every congruence on $B \times C$ can be factored in this way we say that $\mathcal{A}$ has the *product property on congruences*. Note that if Θ is a congruence on $B \times C$ which factors as $\Theta = (\Theta_B,\Theta_C)$, then $(b_1,b_2) \in \Theta_B \Leftrightarrow ((b_1,c),(b_2,c)) \in \Theta$ for some $c \in C \Leftrightarrow ((b_1,c),(b_2,c)) \in \Theta$ for all $c \in C$, and similarly for Θ_C. It is well known (see [25]) that if every algebra in $\mathcal{A}$ has distributive congruences then $\mathcal{A}$ has the product property on congruences.

PROPOSITION (4.2.3). *If* $\mathcal{A}$ *has the product property on congruences and* $\Pi(B_\delta|\delta \in \Delta)$ *is an absolute subretract in* $\mathcal{A}$*, then each* B_δ *is also an absolute subretract in* $\mathcal{A}$.

Proof. Let $\sigma: B_\gamma \to D$ be an injection and define $\bar{\sigma}: \Pi B_\delta \to D \times \Pi(B_\delta|\delta \neq \gamma)$ by $\underline{b}\bar{\sigma} = (b_\gamma\sigma,\underline{c})$ where $\underline{c} \in \Pi(B_\delta|\delta \neq \gamma)$ is given by $c_\delta = b_\delta$; clearly $\bar{\sigma}$ is an injection. Let $\tau: D \times \Pi(B_\delta|\delta \neq \gamma) \to \Pi B_\delta$ be a retraction of $\bar{\sigma}$. Let $\underline{c} \in \Pi(B_\delta|\delta \neq \gamma)$ and define $\lambda: D \to B_\gamma$ by $d\lambda = (d,\underline{c})\tau\pi_\gamma$, where $\pi_\gamma: \Pi B_\delta \to B_\gamma$ is the natural projection.

We claim that λ is independent of the choice of $\underline{c}$. Since $\mathcal{A}$ has the product property on congruences there exist congruences Θ and Φ on D and $\Pi(B_\delta|\delta \neq \gamma)$ respectively, such that $\mathrm{Ker}(\tau\pi_\gamma) = (\Theta,\Phi)$. Clearly it is sufficient to prove that Φ is the improper congruence, that is for all $\underline{b}$, $\underline{c} \in \Pi(B_\delta | \delta \neq \gamma)$ there exists $d' \in D$ with $(d',\underline{b})\tau\pi_\gamma = (d',\underline{c})\tau\pi_\gamma$. Let $a \in B_\gamma$. Then $d' = a\sigma$ will suffice since $(d',\underline{b})\tau\pi_\gamma = (a\sigma,\underline{b})\tau\pi_\gamma$ $= (a,\underline{b})\bar{\sigma}\tau\pi_\gamma = (a,\underline{b})\pi_\gamma = a$ and similarly $(d',\underline{c})\tau\pi_\gamma = a$.

It follows immediately that λ is a homomorphism and since $\sigma\lambda = \mathrm{Id}_{B_\gamma}$ we are through. □

The following result was proved in [33] for the case where A is an equational class of distributive pseudocomplemented lattices.

PROPOSITION (4.2.4). *If* I *is a finite, weak injective algebra in* A*, then endowing* I *with the discrete topology,* $C(X,I)$ *is a weak injective in* A *for every compact, extremally disconnected space* X.

Proof. It is easy to verify that the functor $C(-,I)\colon \mathit{ZComp}^{op} \to A$ has the following properties (since I is finite): (i) if ϕ is a surjection, then $C(\phi,I)$ is an injection, (ii) if ϕ is an injection, then $C(\phi,I)$ is a surjection, and (iii) if $(X_\delta | \delta \in \Delta)$ is a family of Boolean spaces and $X = \beta(\dot{\bigcup}(X_\delta | \delta \in \Delta))$ is their coproduct in *ZComp*, then $C(X,I) \simeq \Pi(C(X_\delta,I) | \delta \in \Delta)$.

Any Boolean space X is the surjective image of βS for some set S (for example $S = |X|$), from which it follows, by (i), (ii) and (iii), that $C(X,I)$ may be embedded in $C(\beta S,I) \simeq I^S$ as a subdirect product. Since X is extremally disconnected it is sur-projective (see [27] or [35]) and hence is a retract of βS. It follows that $C(X,I)$ is a retract of $C(\beta S,I) \simeq I^S$, whence it is a subdirect retract of copies of I. Hence $C(X,I)$ is a weak injective in $\mathcal{A}$ by Proposition (4.2.2)(vi). □

REMARK (4.2.5). It is easily seen that for any Boolean algebra B and any finite algebra A, the Boolean extension A[B] of A by B (see [28]) is isomorphic to $C(X,A)$, where X is the Stone space of B. Since compact, extremally disconnected spaces are precisely the Stone spaces of complete Boolean algebras, Proposition (4.2.4) may be restated as:

'If I *is a finite, weak injective algebra in* $\mathcal{A}$, *then for every complete Boolean algebra* B, *the Boolean extension* I[B] *of* I *by* B *is a weak injective in* $\mathcal{A}$.'

We will now characterize the sur-projectives in $\mathcal{X}_n$ and $\mathcal{Y}_n$ and then apply the dualities to characterize the weak injectives and injectives in $\mathcal{L}_n$ and $\mathcal{R}_n$. Proofs will only be presented for the case of $\mathcal{R}_n$ and $\mathcal{Y}_n$, which is the more technical of the two cases.

LEMMA (4.2.6). *Let* $X \in$ *ZComp and assume that* $P \in \mathcal{Y}_n$ *is a retract of* $F^1(X) = X \times (\mathrm{End}(C_n^1) - \{\theta\}) \dot{\bigcup} \{\theta\}$. *Then* $(x,e) \in P$ *implies that* $(x,e^0) \in P$ *for all* $e \in \mathrm{End}(C_n^1) - \{\theta\}$.

Proof. Let $\tau\colon F^1(X) \to P$ be a retraction and let $(x,e) \in P$. If $(x,\iota)\tau = \theta$, then $(x,e) = (x,e)\tau = (x,\iota)\tau\tilde{e} = \theta\tilde{e} = \theta$, a contradiction. Hence $(x,\iota)\tau = (y,f) \in P$.

Now $(x,e) = (x,e)\tau = (x,\iota)\tau\tilde{e} = (y,f)\tilde{e} = (y,fe)$ and thus $x = y$ and $e = fe$. But $e = fe$ implies that $af = a$ for all

$a \in C_n^1$ such that $ae \neq 1$. Hence $fe^0 = e^0$, and since $(x,f) \in \tilde{P}$ it follows that $(x,e^0) = (x,fe^0) = (x,f)e^0 \in P$. □

Recall that for $1 \leq k < n$, $e_k \in \mathrm{End}(C_n^1)$ denotes the endomorphism determined by the prime filter $[a_k)$. Note that $e_{n-1} = \iota$. Let $E_k = e_k\mathrm{End}(C_n^1) - \{\theta\}$ be the deleted right ideal generated by e_k. If $X_1,\ldots,X_{n-1}$ are (possibly empty) disjoint Boolean spaces then $X = \dot{\bigcup}(X_k \times E_k | 1 \leq k < n) \dot{\bigcup} \{\theta\}$ is a subobject of $F^1(\bigcup(X_k|1 \leq k < n)$, in fact is is easily verified that $\tau\colon F^1(\bigcup(X_k|1 \leq k < n) \to X$, defined by

$$(x,e)\tau = \begin{cases} (x,e_k e) & \text{if } x \in X_k \text{ and } e_k e \neq \theta, \\ \theta & \text{if } x \in X_k \text{ and } e_k e = \theta, \end{cases} \quad \text{and } \theta\tau = \theta,$$

is a retraction onto X.

THEOREM (4.2.7). *The following are equivalent:*

(i) P *is sur-projective in* $\mathcal{Y}_n$;

(ii) P *is a retract of* $F^1(X)$ *for some compact, extremally disconnected space* X;

(iii) *there are disjoint, compact, extremally disconnected spaces* $X_1,\ldots,X_{n-1}$ *such that* P *is isomorphic (in* $\mathcal{Y}_n$*) to* $\dot{\bigcup}(X_k \times E_k|1 \leq k < n) \dot{\bigcup} \{\theta\}$.

Proof. By Proposition (1.29), (i) is equivalent to (ii). If (iii) holds, then by the discussion above, P is a retract of $F^1(\bigcup(X_k|1 \leq k < n))$. Since each X_k is compact and extremally disconnected, so is $\bigcup(X_k|1 \leq k < n)$, and hence (ii) holds. It remains only to prove that (ii) implies (iii).

Without loss of generality assume that P is a subobject of $F^1(X)$ and that $\tau: F^1(X) \to P$ is a retraction onto P. Let $X_{n-1} = \{x \in X | (x,\iota) \in P\}$ and for $1 \leq k < n - 1$ let $X_k = \{x \in X | (x,e_k) \in P;\ x \notin X_{k+1}\}$. Since $e^0 \in \{e_k | 1 \leq k < n\}$ for all $e \in \mathrm{End}(C_n^1) - \{\theta\}$, it follows by Lemma (4.2.6) that $P = \dot{\bigcup}(X_k \times E_k | 1 \leq k < n) \dot{\bigcup} \{\theta\}$.

We will show that $\bigcup(X_k | 1 \leq k < n) \bigcup \{\theta\}$ is a retract of $X \bigcup \{\theta\}$ and that each X_k is clopen in $\bigcup(X_k | 1 \leq k < n)$. Since θ is an isolated point of $F^1(X)$ and since clopen subsets of retracts of compact, extremally disconnected spaces are compact and extremally disconnected (see [35]), the result will then follow.

Define $\rho: X \bigcup \{\theta\} \to \bigcup(X_k | 1 \leq k < n) \bigcup \{\theta\}$ by $\theta\rho = \theta$ and $x\rho = (x,e_1)\tau\pi$, where $\pi: F^1(X) \to X \bigcup \{\theta\}$ is the obvious projection. If $(x,e) \in P$, then $(x,e^0) \in P$ and hence $(x,e_1) = (x,e^0 e_1)$ $= (x,e^0)\tilde{e}_1 \in P$. Thus for all $x \in \bigcup(X_k | 1 \leq k < n)$, $x\rho = (x,e_1)\tau\pi$ $= (x,e_1)\pi = x$.

If U is open in X, then

$$\begin{aligned} U\rho^{-1} &= \{x \in X | (x,e_1)\tau\pi \in U\} \\ &= \{x \in X | (x,e_1)\tau \in U \times (\mathrm{End}(C_n^1) - \{\theta\})\} \\ &= [(X \times \{e_1\}) \bigcap (U \times (\mathrm{End}(C_n^1) - \{\theta\}))\tau^{-1}]\pi, \end{aligned}$$

which is open in X since τ is continuous and π is open.

Similarly,
$$\begin{aligned} \{\theta\}\rho^{-1} &= \{x \in X | (x,e_1)\tau = \theta\} \bigcup \{\theta\} \\ &= [(X \times \{e_1\}) \bigcap \{\theta\}\tau^{-1}]\pi \bigcup \{\theta\}, \end{aligned}$$

which is open in $X \bigcup \{\theta\}$ since θ is an isolated point of $F^1(X)$, τ is continuous and π is open.

To show that each X_k is clopen in $\bigcup(X_k | 1 \leq k < n)$ it is sufficient to show that for all $1 \leq k < n$, the set $U_k = \{x \in X | (x,e_k) \in P\}$ is clopen in $\bigcup(X_k | 1 \leq k < n)$. Since τ is continuous and π, being a projection parallel to a compact factor, is both open and closed, it follows that

$$V_k = [(X \times \{e_k\}) \bigcap (X \times \{e_k\})\tau^{-1}]\pi$$

is clopen in X. We claim that $U_k = V_k \bigcap (\bigcup(X_k | 1 \leq k < n))$. If $x \in U_k$, then $(x,e_k) \in P$ and hence $x \in V_k \bigcap (\bigcup(X_k | 1 \leq k < n))$ since $(x,e_k)\tau = (x,e_k)$. Conversely, assume that $x \in V_k \bigcap (\bigcup(X_k | 1 \leq k < n))$. Then $(x,e_k)\tau = (y,e_k) \in P$ for some $y \in X$, and $(x,e) \in P$ for some $e \in End(C_n^1) - \{\theta\}$. By Lemma (4.2.6), $(x,e^0) \in P$, but $e^0 = e_j$ for some $1 \leq j < n$, whence $(x,e_j) \in P$. If $j \leq k$, then $e_k e_j = e_j$ and hence $(x,e_j) = (x,e_j)\tau = (x,e_k)\tilde{e}_j\tau = (x,e_k)\tau\tilde{e}_j = (y,e_k)\tilde{e}_j = (y,e_j)$, which gives $x = y$. Thus $(x,e_k) = (y,e_k) \in P$. If $j > k$, then $e_j e_k = e_k$ and hence $(x,e_k) = (x,e_j)\tilde{e}_k \in P$ since $(x,e_j) \in P$. In either case $(x,e_k) \in P$, giving $x \in U_k$ as required. □

LEMMA (4.2.8). *Let $3 \leq n < \omega$. Then for any non-empty Boolean space X, $C(X,C_n^1)$ is not self-injective, and for $2 \leq k < n$, $C(X,C_k^1)$ is not an absolute subretract in R_n.*

Proof. Let $D = \{a_{n-2},1\} \subset C_n^1$. Then $C(X,D)$ is a subalgebra of $C(X,C_n^1)$. Define $\lambda: D \to C_n^1$ by $1\lambda = 1$ and $a_{n-2}\lambda = 0$, and define

$\overline{\lambda}$: $C(X,D) \to C(X,C_n^1)$ by $\phi\overline{\lambda} = \phi\lambda$. If $C(X,C_n^1)$ were self-injective there would be a homomorphism γ: $C(X,C_n^1) \to C(X,C_n^1)$ satisfying $\phi\gamma = \phi\overline{\lambda}$ for all $\phi \in C(X,D)$. For $a \in C_n^1$, let $\hat{a} \in C(X,C_n^1)$ be the corresponding constant map. Since $\hat{0} = \hat{a}_{n-2}\overline{\lambda} = \hat{a}_{n-2}\gamma$ and $\hat{a}_{n-2}*\hat{0} = \hat{0}$, we obtain $\hat{0}*\hat{0}\gamma = \hat{a}_{n-2}\gamma*\hat{0}\gamma = (\hat{a}_{n-2}*\hat{0})\gamma = \hat{0}\gamma$, which gives the contradiction $\hat{0} > \hat{0}\gamma$. Hence $C(X,C_n^1)$ is not self-injective.

Let σ: $C_k^1 \to C_n^1$ embed C_k^1 as a filter of C_n^1 and define $\overline{\sigma}$: $C(X,C_k^1) \to C(X,C_n^1)$ by $\phi\overline{\sigma} = \phi\sigma$. If $C(X,C_k^1)$ is an absolute subretract then there exists a retraction τ: $C(X,C_n^1) \to C(X,C_k^1)$ of $\overline{\sigma}$. Since $0\sigma > 0$ it follows that $\widehat{(0\sigma)}*\hat{0} = \hat{0}$, and hence $\hat{0}*\hat{0}\tau = \hat{0}\overline{\sigma}\tau*\hat{0}\tau = (\hat{0}\overline{\sigma}*\hat{0})\tau = (\widehat{(0\sigma)}*\hat{0})\tau = \hat{0}\tau$; again we have the contradiction $\hat{0} > \hat{0}\tau$. Hence $C(X,C_k^1)$ is not an absolute subretract in $\mathcal{R}_n$. □

If B is a Boolean algebra, then let $B^{<n>} = \{(b_0,\ldots,b_{n-1}) \in B^n \mid b_0 \le b_1 \le \ldots \le b_{n-1}\}$. It is easily seen that $B^{<n>}$ is a Brouwerian (in fact Heyting) algebra in which

$$(b_0,\ldots,b_{n-1})*(c_0,\ldots,c_{n-1}) = \left(\bigwedge_{i=0}^{n-1} b_i'\vee c_i,\ \bigwedge_{i=1}^{n-1} b_i'\vee c_i,\ldots,b_{n-1}'\vee c_{n-1}\right).$$

Furthermore, it is clear that $B^{<n>} \cong C(X,C_{n+1}^1)$ where X is the Stone space of B. Hence $B^{<n-1>} \in \mathcal{R}_n$.

For convenience we will regard the one-element algebra as an n-valued Post algebra (see [20]).

THEOREM (4.2.9). *The following are equivalent:*

(i) I *is a weak injective in* R_n;

(ii) I *is an absolute subretract in* R_n;

(iii) $I \simeq C(X,C_n^1)$ *for some compact, extremally disconnected space* X;

(iv) I *is a complete* n-*valued Post algebra;*

(v) $I \simeq B^{<n-1>}$ *for some complete Boolean algebra* B.

Proof. (i) <=> (ii). Since Brouwerian algebras have the congruence extension property this follows from Proposition (4.2.2)(ii).

(ii) => (iii). Since $R_n = ISP(C_n^1)$, I is isomorphic to a subalgebra of $(C_n^1)^S \simeq C(\beta S,C_n^1)$ and hence I is a retract of $C(\beta S,C_n^1)$. It follows that $R_n(I,C_n^1)$ is a retract of $R_n(C(\beta S,C_n^1),C_n^1)$, which by Proposition (4.1.8)(ii) is isomorphic to $F^1(\beta S)$. Hence by Theorem (4.2.7) there are (possibly empty) disjoint, compact, extremally disconnected spaces $X_1,\ldots,X_{n-1}$ with $R_n(I,C_n^1)$ isomorphic in Y_n to $X = \dot{\bigcup}(X_k \times E_k | 1 \le k < n) \dot{\bigcup} \{\theta\}$. For all $1 \le k < n$, $Y_k = (X_k \times E_k) \bigcup \{\theta\}$ is a subobject of X and if $\phi_k \in Y_n(Y_k,C_n^1)$ for all $1 \le k < n$, then $\phi \in Y_n(X,C_n^1)$ may be defined by $\phi|Y_k = \phi_k$; in fact X is the Y_n-coproduct of the family $(Y_k | 1 \le k < n)$. Hence $I \simeq Y_n(R_n(I,C_n^1),C_n^1) \simeq Y_n(X,C_n^1) \simeq \Pi(Y_n(Y_k,C_n^1)|1 \le k < n)$.

We claim that $Y_n(Y_k,C_n^1) \simeq C(X_k,C_{k+1}^1)$ and hence $I \simeq \Pi(C(X_k,C_{k+1}^1)|1 \le k < n)$. If $\phi \in Y_n(Y_k,C_n^1)$, then $(x,e_k)\phi = (x,e_k)\tilde{e}_k\phi = (x,e_k)\phi e_k$ and thus $(x,e_k)\phi \in Im(e_k) = \{0,a_1,\ldots,a_{k-1},1\}$. Hence ϕ induces a continuous map $\hat{\phi} \in C(X_k,C_{k+1}^1)$ defined by $x\hat{\phi} = (x,e_k)\phi$. Conversely, any continuous map $\psi \in C(X_k,C_{k+1}^1)$ induces an arrow $\hat{\psi} \in Y_n(Y_k,C_n^1)$ defined by $(x,e)\hat{\psi} = x\psi e$ and

$\theta\hat{\psi} = \theta$. The map $\phi \to \hat{\phi}$ is clearly a homomorphism and since $\hat{\hat{\phi}} = \phi$ and $\hat{\hat{\psi}} = \psi$ it follows that $\mathcal{Y}_n(Y_k, C_n^1) \simeq C(X_k, C_{k+1}^1)$.

Lemma (4.2.3) and Lemma (4.2.8) imply that X_k is empty for $1 \leq k < n - 1$ and hence (iii) follows.

(iii) => (i). Since C_n^1 is a maximal subdirectly irreducible in $\mathcal{R}_n$ it follows that C_n^1 is a weak injective in $\mathcal{R}_n$ by Proposition (4.2.2)(ii)(v), since Brouwerian algebras have the congruence extension property. Hence (iii) implies (i) by Proposition (4.2.4).

(iii) <=> (iv). For any non-empty Boolean space X, $C(X, C_n^1)$ is an n-valued Post algebra and conversely any n-valued Post algebra I is isomorphic to $C(X, C_n)$, where X is the Stone space of the centre of I (see [20]). Finally, an n-valued Post algebra is complete if and only if its centre is complete (see [20]).

(iii) <=> (v). This follows immediately since we have already noted that $B^{<n-1>} \simeq C(X, C_n^1)$, where X is the Stone space of B. □

THEOREM (4.2.10). *$\mathcal{R}_n$ has enough injectives if and only if* $n = 2$. *An algebra in $\mathcal{R}_2$ is injective if and only if it is a complete Boolean algebra, and for* $3 \leq n < \omega$, *$\mathcal{R}_n$ has only trivial injectives.*

Proof. C_2^1 is trivially self-injective and by Lemma (4.2.8) with $|X| = 1$, C_n^1 is not self-injective for all $n \geq 3$. Hence only $\mathcal{R}_2$ has enough injectives by Proposition (4.2.2)(iv). By Proposition (4.2.2)(iii) and Theorem (4.2.9), an algebra in $\mathcal{R}_2$ is injective if and only if it is a complete Boolean algebra since the 2-valued Post algebras are exactly the Boolean algebras.

Since injective algebras are necessarily self-injective and weak injective, for $n \geq 3$, Lemma (4.2.8) and Theorem (4.2.9) imply that $\mathcal{R}_n$ has only trivial injectives. □

We now turn our attention to L_n and X_n.

Let $E_k = e_k End(C_n)$ be the right ideal generated by e_k. If $X_1, \ldots, X_{n-1}$ are (possibly empty) disjoint Boolean spaces, then $X = \dot{\bigcup}(X_k \times E_k | 1 \leq k < n)$ is a subobject of $F(\bigcup(X_k | 1 \leq k < n))$ and τ: $F(\bigcup(X_k | 1 \leq k < n)) \to X$, defined by $(x,e)\tau = (x, e_k e)$ for all $x \in X_k$, is a retraction onto X.

THEOREM (4.2.11). *The following are equivalent:*

(i) P *is sur-projective in* X_n;

(ii) P *is a retract of* F(X) *for some compact, extremally disconnected space* X;

(iii) *there are disjoint, compact, extremally disconnected spaces* $X_1, \ldots, X_{n-1}$ *such that* P *is isomorphic (in* X_n*) to* $\dot{\bigcup}(X_k \times E_k | 1 \leq k < n)$. □

LEMMA (4.2.12). *Let* $4 \leq n < \omega$. *Then for any non-empty Boolean space* X, $C(X,C_n)$ *is not self-injective and for* $3 \leq k < n$, $C(X,C_k)$ *is not an absolute subretract in* L_n.

Proof. Mimic the proof of Lemma (4.2.8). Let $D = \{0, a_{n-2}, 1\}$, define λ: $D \to C_n$ by $0\lambda = 0$, $1\lambda = 1$ and $a_{n-2}\lambda = a_1$, and obtain $\hat{a}_1\gamma = \hat{0}$ and the contradiction $\hat{0} = \hat{0}\gamma = (\hat{a}_1 * \hat{0})\gamma = \hat{a}_1\gamma * \hat{0}\gamma = \hat{0} * \hat{0} = \hat{1}$.

Let σ: $C_k \to C_n$ be the embedding characterized by $0\sigma = 0$ and $[a_1)\sigma$ is a filter of C_n. Show that $\hat{a}_1\tau = \hat{0}$, and obtain the contradiction $\hat{0} = \hat{1}$, as above. □

THEOREM (4.2.13). *The following are equivalent:*

(i) I *is a weak injective in* L_n;

(ii) I *is an absolute subretract in* L_n;

(iii) $I \approx C(X_0,C_2) \times C(X_1,C_n)$ *for some compact, extremally disconnected spaces* X_0 *and* X_1;

(iv) $I \approx B_0 \times P$ *for some complete Boolean algebra* B_0 *and some complete* n-*valued Post algebra* P;

(v) $I \approx B_0 \times B_1^{<n-1>}$ *for some complete Boolean algebras* B_0 *and* B_1.

Proof. A proof may be obtained by making the obvious minor changes in the proof of Theorem (4.2.9). In particular, note that for every compact, extremally disconnected space X_0, $C(X_0,C_2)$ is a weak injective in L_n by Proposition (4.2.4), since it is easily seen that C_2 is injective in L_ω — the set of complemented elements in an L-algebra is a retract and hence we can apply the fact that C_2 is an injective Boolean algebra; in fact, this approach has been generalized in [7] to show that every complete Boolean algebra is an injective Heyting algebra. Clearly where Lemma (4.2.8) was used in the proof of Theorem (4.2.9) we now call on Lemma (4.2.12). □

Except for the characterization of the injectives in L_3, the following result is due to A. Day [17], [18].

THEOREM (4.2.14). L_n *has enough injectives if and only if* n = 2 *or* n = 3. *An algebra in* L_2 *is injective if and only if it is a complete Boolean algebra. An algebra in* L_3 *is injective if and only if it is isomorphic to the direct product of a complete Boolean algebra with a complete* 3-*valued Post algebra. For* $n \geq 4$

an algebra in L_n is injective if and only if it is a complete Boolean algebra.

Proof. Clearly C_2 and C_3 are self-injective, and by Lemma (4.2.12) with $|X| = 1$, C_n is not self-injective for all $n \geq 4$. Hence by Proposition (4.2.2)(iv), L_n has enough injectives if and only if $n = 2$ or $n = 3$. The characterization of the injectives in L_2 and L_3 follows from Proposition (4.2.2)(iii) and Theorem (4.2.13). It is very easily shown that in any category A with products, if $A(B,C)$ is non-empty for all $B, C \in A$, then $I_1 \times I_2$ is injective in A if and only if both I_1 and I_2 are injective in A (see [5]). Hence if I is injective in L_n $(n \geq 4)$, then I is a complete Boolean algebra by Theorem (4.2.13) and Lemma (4.2.12). Since complete Boolean algebras are injective in the class of all Heyting algebras ([7]), the result follows.□

4.3 *Free Products and Free Algebras*

Free products in L_n and R_n are readily described by using the dualities. The free product of the family $(B_\delta | \delta \in \Delta)$ will be denoted by ${}^*\Pi(B_\delta | \delta \in \Delta)$.

THEOREM (4.3.1). (i) *Let $(B_\delta | \delta \in \Delta)$ be a family of non-trivial L_n-algebras. Then ${}^*\Pi(B_\delta | \delta \in \Delta)$ exists in L_n and is isomorphic to $X_n(\Pi(X_\delta | \delta \in \Delta), C_n)$, where $X_\delta = L_n(B_\delta, C_n)$.*

(ii) *Let $(B_\delta | \delta \in \Delta)$ be a family of R_n-algebras. Then ${}^*\Pi(B_\delta | \delta \in \Delta)$ exists in R_n and is isomorphic to $Y_n(\Pi(X_\delta | \delta \in \Delta), C_n^1)$, where $X_\delta = R_n(B_\delta, C_n^1)$.*

Proof. (i) Since L_n is equivalent to (the dual of) its image X_n' in X_n, it follows that the image of a direct product in X_n' under the functor $X_n(-,C_n)$ is a coproduct in L_n. Free products are distinguished from coproducts only by the requirement that the natural homomorphism g_δ: $B_\delta \to {}^*\Pi(B_\delta | \delta \in \Delta)$ be an embedding. But $g_\delta = \eta_{B_\delta} X_n(\pi_\delta, C_n)$, where π_δ: $\Pi(X_\delta | \delta \in \Delta) \to X_\delta$ is the natural projection, and thus g_δ is an embedding since π_δ is a surjection. (ii) This follows exactly as in (i).□

In (i) the algebra B_δ was assumed to be non-trivial so that X_δ would be non-empty. Note that the free product of C_1 and C_2 does not exist in L_n and hence this requirement is necessary.

For all $B \in L_n$ let $E_n(B)$ be the subset of $L_n(B,C_n)$ consisting of those homomorphism which are determined by some prime filter of B (see Definition (4.1.2)). If $E_n(B)$ is ordered pointwise, then the correspondence $F \to h_F$ is an order-isomorphism between the poset of prime filters of B and the poset $E_n(B)$. Observe that if g, h $\in E_n(B)$, then $g \leq h$ if and only if $ge = h$ for some $e \in \mathrm{End}(C_n)$. Since a finite distributive lattice is determined by its poset of prime filters (see Remark (3.4.5)), it follows that a finite algebra $B \in L_n$ is determined by the poset $E_n(B)$. For an algebra $B \in R_n$, $E_n^1(B)$ is defined similarly, and again if B is finite, then the poset $E_n^1(B)$ determines B.

Using this observation we can describe the finitely generated free algebras in L_n and R_n. Since the description in L_n is dependent upon the description in $\mathcal{R}_n$ we commence with the latter.

Define the action of $\mathrm{End}(C_n^1)$ on $(C_n^1)^m$ pointwise, that is $(c_0,\dots,c_{m-1})\tilde{e} = (c_0e,\dots,c_{m-1}e)$. Then it is clear that the map $\rho_m\colon \mathcal{R}_n(F_{\mathcal{R}_n}(m),C_n^1) \to (C_n^1)^m$ is an isomorphism in $\mathcal{Y}_n$ (see Proposition (1.6)). Let $E_n^1(m)$ be the image of $E_n^1(F_{\mathcal{R}_n}(m))$ under ρ_m and define a partial order on $E_n^1(m)$ by '$\underline{c} \leq \underline{d} \iff \underline{c}\tilde{e} = \underline{d}$ for some $e \in \mathrm{End}(C_n^1)$'. Obviously $E_n^1(F_{\mathcal{R}_n}(m))$ and $E_n^1(m)$ are order-isomorphic.

PROPOSITION (4.3.2). (i) $F_{\mathcal{R}_n}(m) \simeq \mathcal{Y}_n((C_n^1)^m, C_n^1)$ *for all* $m < \omega$.
(ii) $\underline{c} \in E_n^1(m)$ *if and only if* $\{c_j \mid j < m\} \cup \{1\} = (a_{\ell-1}] \cup \{1\}$ *for some* $1 \leq \ell < n$.

Proof. (i) Apply the duality.
(ii) If $g \in \mathcal{R}_n(F_{\mathcal{R}_n}(m), C_n^1)$, then $\mathrm{Im}(g)$ is the subalgebra of C_n^1 generated by $\{x_jg \mid j < m\}$ and hence $\mathrm{Im}(g) = \{x_jg \mid j < m\} \cup \{1\}$, since for all $c, d \in C_n^1$, $c{*}d \in \{c,d,1\}$. Thus g is determined by a prime filter of $F_{\mathcal{R}_n}(m)$ if and only if $\{x_jg \mid j < m\} \cup \{1\} = (a_{\ell-1}] \cup \{1\}$ for some $1 \leq \ell < n$. After translating from $\mathcal{R}_n(F_{\mathcal{R}_n}(m), C_n^1)$ to $(C_n^1)^m$ via ρ_m, the result follows. □

With the exception of Proposition (4.3.4), the remaining results in this section are due to P. Köhler [49] (see also [42]), but the proofs of Theorem (4.3.3) and Theorem (4.3.5) are new.

Recall that ${}_0B$ is obtained from B by adjoining a new zero.

THEOREM (4.3.3). (i) $F_{\mathcal{R}_2}(m) \simeq (C_2^1)^{2^m-1}$ *for all* $m < \omega$.
(ii) *For* $n \geq 3$, $F_{\mathcal{R}_n}(0) \simeq C_1^1$ *and for* $1 \leq m < \omega$,

$$F_{\mathcal{R}_n}(m) \simeq \prod_{k=0}^{m-1} [{}_0(F_{\mathcal{R}_{n-1}}(k))]^{\binom{m}{k}}.$$

Proof. It is clear that $F_{R_n}(0) \simeq C_1^1$ for all $n \geq 2$ and hence we will assume that $m \geq 1$.

Applying Proposition (4.3.2)(i) we find that $F_{R_2}(m) \simeq (C_2^1)^{2^m-1}$ since $Y_2((C_2^1)^m, C_2^1)$ is the set of all arrows which map $(1,\ldots,1)$ to 1 and map each of the other 2^m-1 elements of $(C_2^1)^m$ arbitrarily into C_2^1.

Now consider $n \geq 3$. If $\underline{c} \in E_n^1(m)$, then there exists $j < m$ such that $c_j = 0$. Thus if $\underline{c}, \underline{d} \in E_n^1(m)$ and $\underline{c} \leq \underline{d}$, then $c_j = 0 \iff d_j = 0$ (if $\underline{c}\tilde{e} = \underline{d}$, then $0e = 0$ for otherwise $\underline{c}\tilde{e} \notin E_n^1(m)$). Hence $M^1 = (C_2^1)^m - \{(1,\ldots,1)\}$ is the set of maximal elements of $E_n^1(m)$, and in fact $E_n^1(m)$ is the disjoint union of the family $((\underline{c}] \mid \underline{c} \in M^1)$ of principal ideals. (Note that since $E_n^1(m)$ is isomorphic to the poset of prime filters of an algebra in R_n, $E_n^1(m)$ must be a disjoint union of principal ideals by Proposition (4.1.1)(ii).) By Remark (3.4.5), $F_{R_n}(m)$ is isomorphic to the lattice of increasing subsets of $E_n^1(m)$ and so is isomorphic to $\Pi(B_{\underline{c}} \mid c \in M^1)$, where $B_{\underline{c}}$ is the lattice of increasing subsets of $(\underline{c}]$. But it is easily verified that if $\underline{c} \in M^1$ has exactly $k < m$ coordinates equal to 1 (and hence m-k equal to 0), then $(\underline{c}]$ is order-isomorphic to $E_{n-1}^1(k) \cup \{(1,\ldots,1)\}$. Since there are $\binom{m}{k}$ elements of M^1 with exactly k coordinates equal to 1, and since the lattice of increasing subsets of $E_{n-1}^1(k) \cup \{(1,\ldots,1)\}$ is isomorphic to ${}_0(F_{R_{n-1}}(k))$, the result follows. □

The proof of the following result is similar to the proof of Proposition (4.3.2) and will be omitted. As before, the action of

$\mathrm{End}(C_n)$ on $(C_n)^m$ is defined pointwise, and $\mathcal{E}_n(m)$, the image of $\mathcal{E}_n(\mathcal{F}_{L_n}(m))$ under ρ_m, is order-isomorphic to $\mathcal{E}_n(\mathcal{F}_{L_n}(m))$ when ordered by '$\underline{c} \leq \underline{d} \iff \underline{c}\tilde{e} = \underline{d}$ for some $e \in \mathrm{End}(C_n)$'.

PROPOSITION (4.3.4). (i) $\mathcal{F}_{L_n}(m) \simeq \mathcal{X}_n((C_n)^m, C_n)$ *for all* $m < \omega$.
(ii) $\underline{c} \in \mathcal{E}_n(m)$ *if and only if* $\{0\} \cup \{c_j \mid j < m\} \cup \{1\}$
$= \{0\} \cup [a_1, a_{\ell-1}] \cup \{1\}$ *for some* $1 \leq \ell < n$. □

THEOREM (4.3.5). (i) $\mathcal{F}_{L_2}(m) \simeq (C_2)^{2^m}$ *for all* $m < \omega$.
(ii) *For* $n \geq 3$ *and for all* $m < \omega$,

$$\mathcal{F}_{L_n}(m) \simeq \prod_{k=0}^{m} [{}_0(\mathcal{F}_{R_{n-1}}(k))]^{\binom{m}{k}}.$$

Proof. Again it is clear that $\mathcal{F}_{L_n}(0) \simeq C_2$ for all $n \geq 2$ and hence we will assume that $m \geq 1$.

That $\mathcal{F}_{L_2}(m) \simeq (C_2)^{2^m}$ follows immediately from Proposition (4.3.4)(i) since C_2 has no proper endomorphisms.

Now consider $n \geq 3$. It follows as in the proof of Theorem (4.3.3) that $M = (C_2)^m$ is the set of maximal elements in the poset $\mathcal{E}_n(m)$ and that $\mathcal{E}_n(m)$ is the disjoint union of the family $((\underline{c}] \mid \underline{c} \in M)$ of principal ideals. Consequently $\mathcal{F}_{L_n}(m)$ is isomorphic to $\Pi(B_{\underline{c}} \mid \underline{c} \in M)$ where $B_{\underline{c}}$ is the lattice of increasing subsets of $(\underline{c}]$. In this case if $\underline{c} \in M$ has exactly $k \leq m$ coordinates equal to 1, then $(\underline{c}]$ is order-isomorphic to $\mathcal{E}^1_{n-1}(k) \cup \{(1,\ldots,1)\}$ since
(a) if $\underline{d}^1, \underline{d}^2 \in \mathcal{E}_n(m)$ and $\underline{d}^1 \leq \underline{d}^2$, then $d^1_j = a_1 \iff d^2_j = a_1$, and
(b) by identifying C^1_{n-1} with the filter $[a_1)$ of C_n, we see that $\mathrm{End}(C_n) \simeq \mathrm{End}(C^1_{n-1})$ (as was observed in the proof of Theorem (4.1.7)).

Since there are $\binom{m}{k}$ elements of M with exactly k coordinates equal to 1, and since the lattice of increasing subsets of $E^1_{n-1}(k) \cup \{(1,\ldots,1)\}$ is isomorphic to $_0(\mathcal{F}_{R_{n-1}}(k))$, the result follows. □

The following simple result indicates the isomorphisms existing between the finitely generated algebras in various classes and also allows us to describe the finitely generated free algebras in L_ω and R_ω.

LEMMA (4.3.6). (i) *Let* B ε L_ω. *If* B *is* n-*generated, then* B ε L_{n+2}. (ii) *Let* B ε R_ω. *If* B *is* n-*generated, then* B ε R_{n+1}.

Proof. (i) Let B ε L_ω be n-generated and let F be a prime filter of B. We will prove that the chain of prime filters containing F has at most n+1 elements, whence B ε L_{n+2} by Proposition (4.1.1)(ii).

Let Θ be the unique congruence on B with F as a congruence class ($(a,b) \,\varepsilon\, \Theta \iff a*b \wedge b*a \,\varepsilon\, F$). For all a, b ε B, $a*b \vee b*a = 1 \,\varepsilon\, F$ and hence either $a*b \,\varepsilon\, F$ or $b*a \,\varepsilon\, F$, that is $[a]\Theta * [b]\Theta = [1]\Theta$ or $[b]\Theta * [a]\Theta = [1]\Theta$. But a Heyting (or Brouwerian) algebra C is a chain if and only if for all a, b ε C, $a*b = 1$ or $b*a = 1$; thus B/Θ is a chain. Since B/Θ is generated by the images of the n generators of B, it follows that $|B/\Theta| \leq n+2$, the 2 being added for the nullary operations 0 and 1. Hence the chain of all prime filters containing F has at most n+1 elements by Proposition (4.1.1)(iii).

(ii) If B is an n-generated in $\mathcal{R}_\omega$, then ${}_0B$ is n-generated in $\mathcal{L}_\omega$ and hence ${}_0B \in \mathcal{L}_{n+2}$. It follows that ${}_0B \in \mathcal{R}_{n+2}$ and thus $B \in \mathcal{R}_{n+1}$. □

Our final result follows easily since if $\mathcal{B}$ is an equational subclass of an equational class $\mathcal{A}$ and every n-generated algebra in $\mathcal{A}$ is an algebra in $\mathcal{B}$, then $F_{\mathcal{A}}(n) \simeq F_{\mathcal{B}}(n)$.

PROPOSITION (4.3.7). (i) $F_{\mathcal{R}_\omega}(m) \simeq F_{\mathcal{R}_{m+1}}(m)$.

(ii) $F_{\mathcal{L}_\omega}(m) \simeq F_{\mathcal{L}_{m+2}}(m)$.

(iii) *If* $n \geq m+1$, *then* $F_{\mathcal{R}_n}(m) \simeq F_{\mathcal{R}_{m+1}}(m)$.

(iv) *If* $n \geq m+2$, *then* $F_{\mathcal{L}_n}(m) \simeq F_{\mathcal{L}_{n+2}}(m)$.

(v) $F_{\mathcal{R}_\omega}(m) \simeq \prod_{k=0}^{m-1} [{}_0(F_{\mathcal{R}_\omega}(k))]^{\binom{m}{k}}$.

(vi) $F_{\mathcal{L}_\omega}(m) \simeq \prod_{k=0}^{m} [{}_0(F_{\mathcal{R}_\omega}(k))]^{\binom{m}{k}} \simeq F_{\mathcal{R}_\omega}(m) \times {}_0(F_{\mathcal{R}_\omega}(m))$. □

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