CLASSICAL MESON THEORY

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ABSTRACT

The subject of the thesis is classical meson theory. The purpose in writing this thesis was to present meson theory from an elementary point of view, a point of view which could be understood by students in physics not familiar with quantum field theory. Thus the development of the meson field equations is carried through by analogy with familiar classical fields, such as the electromagnetic field. The various formulations of meson field theory are considered: scalar, pseudo-scalar, vector, and pseudo-vector. The charged and symmetric meson theories are introduced by regarding the isotopic spin of a nucleon as a classical vector in an abstract charge space.

A sufficient background is developed to enable the reader to follow the analysis of several interesting problems in meson physics. The questions of nuclear forces, the structure of the deuteron, meson production in nucleon-nucleon and photon-nucleon collisions, meson scattering by nucleons, are treated by applications of the classical theory.

Finally the classical results are compared with quantum mechanics and with experiment.

TABLE OF CONTENTS

1. INTRODUCTION:

The mesons we shall be mainly concerned with are the π -mesons, believed to be responsible for the specific nuclear interactions. We assume the reader is familiar with the early historical development of meson theory. Hideki Yukawa[1] was the first to suggest that nuclear forces might be explained by a field theory similar to electromagnetic theory. Because of the short range nature of nuclear forces, Yukawa found it necessary to associate with his field, particles of finite mass intermediate between that of the proton and electron. He was thus able to predict the existence and mass of a hitherto unobserved elementary particle, the meson. Several years after the publication of Yukawa's first paper on meson theory, a particle of approximately the correct mass was discovered as a constituent of cosmic rays[2]. It was subsequently demonstrated that this particle, later called the μ-meson, interacted only weakly

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1. Hideki Yukawa: 'On the Interaction of Elementary Particles, I' Proceedings of the Physico-Mathematical Society of Japan (3), 17, 48 (1935). Reprinted in 'Foundations of Nuclear Physics', Dover, 1949.

2. First observed by Neddermeyer, Anderson (1939) and independently by Blackett, Wilson (1939).

with nucleons. However, further research[3] uncovered the presence of yet another meson in cosmic rays. This latter meson, the π -meson, did in fact seem to possess the properties of Yukawa's field particles.

Present day meson theory is based on the formalism of quantum field theory. It is felt that many physicists possess insufficient mathematical background to permit them to follow the rather complicated structure of meson theory, but are otherwise quite capable of grasping the physics involved. Perhaps, then, there is a need for an introductory, classical theory of mesons - a classical theory which could provide a basis for the reading of current literature on experimental meson physics. It is true that there are present in the literature many classical or semi-classical treatments of problems in meson physics. However, nowhere does there exist a unified account of meson theory from an elementary point of view.

It is our purpose to discuss the classical field theory for the π -meson. We begin by establishing the defining equations for the various types of meson fields: scalar, pseudo-scalar, vector, pseudo-vector. This is accomplished by considering well known classical fields as analogies, the source terms of the meson field being patterned

3. Powell et al, at Bristol, (1947).

after the source terms of the electrostatic, magnetostatic, and electromagnetic fields. Thus, just as one regards the electrostatic field as arising from a distribution of electric charge, so must one regard the meson field as arising from a distribution of nuclear matter.

By interpreting the meson as a field particle we find it possible to introduce electric charge into the meson field, and hence to set up the symmetric formulation of meson physics. It is in terms of this symmetric theory that we are able to discuss many interesting problems in meson physics. A solution of the static, symmetric, meson field equations makes possible a discussion of nuclear forces, and in addition yields a rather crude explanation of meson production in nucleon-nucleon collisions.

Within the classical framework, we also consider the scattering of mesons by nucleons and the production of mesons in photon-nucleon collisions.

Finally the results of the applications of our classical meson theory are compared with quantum theory and with experiment. The latter comparison is carried out to make the reader aware of the strengths and weaknesses of present day meson theory.

2. FORMALISM: FIELD EQUATIONS FOR MESON FIELD

2.1 Scalar Meson Field.

We have indicated that our purpose is to treat meson theory in a classical or semi-classical fashion. In doing so we will discuss two main formulations: first a scalar, and then a vector development of the meson field. We shall begin by considering the simpler of the two treatments, the scalar theory. One ~~would~~ says that a physical field is ~~was~~ scalar in nature, if it is possible to completely describe the field by specifying the value of a scalar $\phi(\vec{r},t)$ at an arbitrary space-time point $(\vec{r},t)$. Associated with this field ϕ are its sources, which may be given as definite functions $S(\vec{r},t)$ of time and position[1]. The manner in which the sources S give rise to the field ϕ is summed up in a field equation

$$D(\vec{r},t)\,\phi(\vec{r},t) = S(\vec{r},t) \tag{1}$$

where D is a multiplicative and differential operator.

Our problem with regard to the meson field ϕ is to choose appropriate functions characteristic of the nuclear sources for the field, and to relate these sources with ϕ through a set of field equations.

It will be instructive to consider a familiar scalar field, the classical electrostatic field arising from

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1. In such case one ignores the reaction of the field on its sources S.

a stationary distribution of electric charge. Let us suppose that in some region of space the field is described by the function $\psi(\vec{r})$ and the charge distribution by a density $\rho_Q(\vec{r})$. Then, it is well known that the behaviour of $\psi(\vec{r})$ will be governed by the Poisson equation

$$\nabla^2 \psi = -\rho_Q \tag{2}$$

(units chosen so dielectric constant = 4π)

At this point one often considers the force exerted on an elementary volume (dv) of charge, thereby introducing the notion of a field intensity. The force on dv will be

$$d\vec{F} = (\rho_Q \, dv) \vec{E} \tag{3}$$

where the field intensity is defined as

$$\vec{E} = -\nabla \psi \tag{4}$$

In terms of $\vec{E}$ it is possible to replace the single second order field equation (2) by a pair of first order equations. The first of these is the defining equation (4) for $\vec{E}$, and the second is obtained by substitution from (4) into (2).

The result is $$\nabla \cdot \vec{E} = \rho_Q \tag{5}$$

We are in particular interested in computing the field due to a single charged particle situated say at $\vec{r} = 0$. Such a charge distribution would be described by a density

$$\rho_Q = Q \, \delta(\vec{r}) \tag{6}$$

where Q is the charge on the particle and $\delta(\vec{r})$ is the Dirac delta function. Now a general solution of the Poisson equation (2) may be obtained by a Green's method. One uses the Green's function

$$\frac{1}{|\vec{r}-\vec{r}'|}$$

and obtains

$$\psi(\vec{r}) = \frac{1}{4\pi}\int \rho_Q(\vec{r}')\,\frac{dv'}{|\vec{r}-\vec{r}'|} \tag{7}$$

The field at $\vec{r}$ due to a single particle at the origin then is

$$\psi(\vec{r}) = \frac{Q}{4\pi r} \tag{8}$$

One notes that though the field $\psi(\vec{r})$ in (8) approaches zero as distance from the source increases, the fall off is rather gradual. For our present work, this is an important point. Experiments have shown that perhaps the main feature of specific nuclear forces is their very short range[1]. If then one is to develop a meson theory capable of explaining nuclear forces, one must incorporate in such a theory the short range nature of these forces.

This may be accomplished by selecting for the meson field a proper wave equation. One simple generalization of

~~1. One might say that nuclear forces determine the scale of the nucleus, coulomb electrostatic forces that of the atom.~~ Since the nuclear ~~radius $\sim 10^{-13}$ cm. and atomic radius $\sim 10^{-8}$ cm., one sees that the nuclear forces posses a~~ much ~~shorter range.~~

the Poisson equation (2) which will define a field, ϕ , of definite range is the following[1]:

$$(\nabla^2 - \kappa^2)\,\phi(\vec{r}) = \rho_n(\vec{r}) \tag{9}$$

(κ is some constant of dimensions (length)$^{-1}$)

For suppose one seeks the solution of (9) corresponding to a point source, i.e.

$$\rho_n(\vec{r}) = g\,\delta(\vec{r}) \tag{10a}$$

g being some constant whose nature we leave unspecified at the moment.

A general solution of (9) is obtained through use of a Green's Function

$$\frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

The result is

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int \rho_n(\vec{r}\,')\,\frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,dv' \tag{10b}$$

and the choice $\rho_n = g\,\delta(\vec{r})$ leads to a field

$$\phi(\vec{r}) = -\frac{g}{4\pi}\frac{e^{-\kappa r}}{r} \tag{11}$$

It is immediately evident that the field described by equation (11) will fall off far more rapidly for large r than will the coulomb field (8). One can in fact associate with the

1. The sign of ρ_n in the equation (9) is opposite to that of ρ_Q in the electrostatic field equation. The reason for this is that we wish to abide by standard convention for meson notation. The same question arises with reference to the meson field intensity.

field (11) a characteristic range $\frac{1}{K}$.

If one takes for $\frac{1}{K}$ a value in agreement with the experimentally determined range of nuclear forces, then perhaps equation (9) may be used as a basis for a scalar meson theory.

By analogy with the coulomb field, the source function $\rho_n(\vec{r})$ for the meson field will be taken as a density of nuclear source material. We shall in fact set $\rho_n(\vec{r}) = g\rho$, where $\rho(\vec{r})$ is to be the number of nuclear particles per unit volume. This would imply that we attribute to each nucleon, whether proton or neutron, a mesic charge g. In this respect the meson field would seem to parallel the gravitational field more closely than the electrostatic field - we have only one type of mesic charge.

Equation (11) will then give the field due to a single nucleon.

We may fix the units of g by considering the energy possessed by a nucleon placed in an external meson field. In electrostatics a charge Q placed in an external field ψ possesses energy $Q\psi$. Similarly, we shall take as the energy of a nuclear particle placed in a meson field ϕ, the quantity $g\phi$. Clearly then, $\frac{g^2 e^{-Kr}}{4\pi r}$ is to have the dimensions of energy, and g those of an electric charge.

It is also possible to introduce in meson field theory an intensity defined by

$$\vec{\chi} = \nabla \phi \tag{12}$$

Substitution into (9) then yields

$$\nabla \cdot \vec{\chi} = \kappa^2 \phi + g \rho(\vec{r}) \tag{13}$$

The equation (9) or the pair of equations (12) and (13) may be taken as defining equations for the scalar meson field. Actually such equations describe a rather limited field ϕ, in fact a static field. One would like to be able to discuss time varying fields arising from a moving distribution of nuclear matter. We shall demand that the structure of any time dependent meson theory be in conformity with the requirements of special relativity, i.e. we ask that the field equations be Lorentz invariant.[1]

We achieve a relativistic generalization of (9) or (12) and (13) by replacing all 3-dimensional quantities by corresponding 4-dimensional quantities. Such a transformation may be described by the following scheme:

$$\begin{aligned} \phi(\vec{r}) &\rightarrow \phi(\vec{r}, t) \\ \vec{\chi}(\vec{r}) &\rightarrow \chi_\nu \equiv (\vec{\chi}, \chi_4) \\ \nabla \cdot \vec{\chi} &\rightarrow \nabla \cdot \vec{\chi} + \frac{1}{ic} \frac{\partial \chi_4}{\partial t} \equiv \partial_\nu \chi_\nu \end{aligned}$$

Finally we must obtain a replacement for the nuclear density $\rho(\vec{r})$. This is most readily done by introducing a velocity field $\vec{v}(\vec{r}, t)$ to describe the flow of nuclear matter. We

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1. See Appendix A.

~~would~~ refer to $\vec{J} = \rho\vec{v}$ as a current density for this flow, and ~~would~~ then combine $\vec{J}$, ρ to form a 4-current $J_\nu \equiv (\vec{J}, ic\rho)$. The magnitude of this 4-current would be given by

$$J_\nu J_\nu = -c^2\rho^2(1 - v^2/c^2)$$

Hence it would follow

$$\rho\sqrt{1 - v^2/c^2} = \sqrt{-\frac{1}{c^2} J_\nu J_\nu}$$

is a 4-scalar or invariant.

For the static field $v \to 0$, and $\sqrt{-\frac{1}{c^2} J_\nu J_\nu} \longrightarrow \rho$. Clearly then $\sqrt{-\frac{1}{c^2} J_\nu J_\nu}$ provides us with a suitable generalization of ρ.

The time dependent meson field equations, then, may be written down

$$\vec{\chi} = \nabla\phi \qquad \text{(a)}$$

$$\chi_4 = \frac{1}{ic}\frac{\partial\phi}{\partial t} \qquad \text{(b)} \ldots\ldots \text{(14)}$$

$$\nabla\cdot\vec{\chi} + \frac{1}{ic}\frac{\partial\chi_4}{\partial t} = K^2\phi + \rho g\sqrt{1 - v^2/c^2} \qquad \text{(c)}$$

The Lorentz invariance of the field equations becomes apparent if one makes use of the 4-dimensional notation

$$\chi_\nu = \partial_\nu\phi \qquad \text{(a)}$$

$$\partial_\nu\chi_\nu = K^2\phi + g\sqrt{-\frac{1}{c^2} J_\nu J_\nu} \qquad \text{(b)} \ldots\ldots \text{(15)}$$

Either set, (14) or (15), is equivalent to the single wave equation

$$(\Box^2 - K^2)\phi = g\rho\sqrt{1 - v^2/c^2} = g\rho\beta \qquad (16)$$

$$\left(\begin{array}{l} \beta = \sqrt{1 - v^2/c^2} \\ \Box^2 \equiv \partial_\nu\partial_\nu \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \end{array}\right)$$

We see that the presence of nuclear matter makes itself felt only in the divergence equation (15b), through the scalar term $g\sqrt{-\frac{1}{c^2}J_\nu J_\nu}$.

We refer to such a situation as a scalar coupling of the meson field to its sources. A vector coupling may be introduced by adding to $\partial_\nu \phi$, in the definition of χ_ν, some 4-vector describing the nuclear sources. A rather obvious choice is the 4-current J_ν. We then might replace equation (15b) by

$$\chi_\nu = \partial_\nu \phi + g_2 J_\nu \tag{15b'}$$

where g_2 is a coupling constant. It will now be shown that the vector coupling contributes nothing to the field ϕ.

If we assume conservation of nuclear matter, then the flow of such matter must obey a continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{16}$$

Consider the wave equation resulting from both scalar and vector coupling, i.e. from equation (15a) and (15b'). This wave equation will be

$$(\Box^2 - \kappa^2)\phi = g\rho\sqrt{1 - v^2/c^2} - g_2 \partial_\nu J_\nu \tag{17}$$

The contribution from vector coupling vanishes by virtue of the continuity equation (16). Hence we will not lose any of the physical content of our theory if we ignore the vector coupling.

In later applications of meson theory we shall be interested in energy flow in the meson field. As a necessary part of our formalism, then, we must consider the possibility of energy retention in the field, and of energy contribution to the field by its source. We attempt to sum up such considerations in a continuity equation describing the transport of energy by the field.

To form this continuity equation we examine the vector

$$\vec{S} = \frac{c}{2} \chi_4 \vec{\chi} = -\dot{\phi} \nabla\phi , \quad \left(\frac{\partial\phi}{\partial t} \equiv \dot{\phi}\right) \tag{18}$$

We compute the divergence of this vector and substitute from the wave equation (17) to obtain

$$\nabla \cdot \vec{S} + \frac{\partial}{\partial t}(H_0) = -g\beta\dot{\phi}\rho \tag{19}$$

where $$H_0 = \frac{1}{2}\kappa^2\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2c^2}\dot{\phi}^2$$

This latter equation (19) is a continuity equation expressing the conservation of some physical entity characteristic of the field ϕ . To deduce the nature of this entity we examine the term $-g\rho\beta\dot{\phi}$ which is clearly a source for this field entity. By analogy with electrostatics we might attribute an energy density $g\rho\beta\phi$ to a distribution of nuclear matter, ρ, immersed in a meson field ϕ. It is then reasonable to suppose $-g\beta\rho\dot{\phi}$ represents the rate at which the nuclear sources contribute energy to the meson field[1]. This leads us to designate $\dot{H}_o$ as the energy density for the meson field, and $\vec{S}$ as the energy current density.

In quantum mechanics one refers to $g\beta\rho\phi$ as an interaction energy density peculiar to the nuclear sources, and writes the field energy density in the form

$$H = H_o + g\beta\rho\phi$$

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1. Let us suppose that the meson field is set up by a single nucleon. We expect that there is a true energy conservation in the system nucleon plus meson field. Thus, by integrating the right side of (19) over the nuclear volume we obtain as the time rate of change of the nucleon total energy $\mathcal{H}$

$$\frac{d\mathcal{H}}{dt} = g\beta\,\dot{\phi}(\vec{r}_n, t)$$

($\dot{\phi}$, β being evaluated at the nucleon position.)
Now one has as a theorem in particle mechanics $\frac{d\mathcal{H}}{dt} = -\frac{\partial L}{\partial t}$,

where $\mathcal{H}$ is the total energy or Hamiltonian for the particle and L is its Lagrangian. It is a simple matter to compute the Lagrangian for the nucleon - one adds to the free particle Lagrangian a contribution from the meson field, an interaction Lagrangian.

The relativistic free particle Lagrangian is

$$L_0 = -Mc^2\beta$$

M being the nucleon mass,

while the interaction Lagrangian is

$$L' = -\int_{\substack{\text{nucleon}\\ \text{volume}}} g\beta\rho\phi\, dv = g\beta\phi(\vec{r}_n,t)\ ,\ \rho = \delta(\vec{r}-\vec{r}_n)$$

Hence the total nucleon Lagrangian is

$$L = -(Mc^2 + g\phi)\beta$$

and one finds

$$\frac{d\mathcal{H}}{dt} = g\beta\dot{\phi}(\vec{r}_n,t)$$

Thus energy conservation in the system nucleon plus meson field is guaranteed.

The fact that one has labelled $g\beta\rho\phi(\vec{r},t)$ as an interaction energy density might suggest the potential energy of a nucleon in a meson field is $g\beta\phi(\vec{r}_n,t)$.

Actually the total energy for the nucleon is

$$\mathcal{H} = \frac{mc^2}{\beta} + \frac{g\phi(\vec{r}_n,t)}{\beta}$$

and hence the nucleon potential energy is $\frac{g\phi(\vec{r}_n,t)}{\beta}$.

In any case, the nucleon at rest in a scalar field ϕ possesses an energy $g\phi(\vec{r}_n)$.

2.2 Pseudo-Scalar Meson Field.

Suppose in place of electrostatics we had selected as a guiding analogy for meson theory the magnetic field set up by magnetized materials. One generally pictures magnetic material as consisting of elementary magnetic dipoles. The result is that on a macroscopic level the material possesses a net dipole moment per unit volume, $M(\vec{r})$ say.

The magnetic field intensity due to such a distribution of magnetic material is

$$\vec{B}(\vec{r}) = -\nabla\psi(\vec{r}) + \vec{M}(\vec{r}) \tag{1}$$

where ψ is to be considered as a potential for the magnetostatic field. As a consequence of the non-existence of magnetic monopoles one has

$$\nabla\cdot\vec{B} = 0 \tag{2}$$

From (1) and (2) one could deduce as the basic equation for the magnetic field ψ ,

$$\nabla^2\psi = \nabla\cdot\vec{M} \tag{3}$$

We note that $\vec{M}$ is not a true 3-vector. In fact, the magnetic moment $\vec{M}$ is very similar to an angular momentum, in that it possesses the properties of a cross product of two vectors. We suppose $\vec{a}$, $\vec{b}$ are a pair of 3-vectors, and we form the product $\vec{a} \times \vec{b}$. Then we consider a reflection of all three co-ordinate axes used in representation of $\vec{a}$ and $\vec{b}$. Under such a transformation we find $\vec{a}$ passes over into $-\vec{a}$, $\vec{b}$ into $-\vec{b}$, while $\vec{a} \times \vec{b}$ is unaffected.[1] It is this failure to transform properly under a reflection of axes that distinguishes pseudo-vectors such as $\vec{M}$, $\vec{a} \times \vec{b}$ from true vectors such as $\vec{a}$, $\vec{b}$.

In this sense a pseudo-scalar would be a scalar, which changes sign under a complete reflection of the co-ordinate axes.

We see then that for the field equation (1) to possess any meaning, $\vec{B}$ must be taken as a pseudo-vector and ψ as a pseudo scalar.

Why have we used magnetostatics as an analogy for the meson field? Principally because we are then able to introduce the effect of nuclear spin into our field equations. We generally associate with elementary particles an intrinsic angular momentum, which we may say is due to the spinning of the particle about some axis[2]. We might expect the nucleon spin to react on the meson field, i.e., act as a source for the meson field in the same fashion that elementary magnetic dipoles act as sources for the magnetic field.

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1. One is of course referring to component representation of $\vec{a}$, $\vec{b}$, i.e., cartesian components would change sign under reflection.

2. In other words we intend to treat spin as a classical vector.

The spin we attribute to a nucleon, being an angular momentum, must be described by a pseudo-vector. Hence if we are to include spin interaction inour scalar meson theory we must pass over to a pseudo-scalar formulation. For the source free pseudo-scalar theory, it is clear that the field equations are identical with those of scalar theory. In the static case then, the field $\phi(\vec{r})$ and the intensity $\vec{\chi}(\vec{r})$ are related by

$$\vec{\chi} = \nabla \phi \tag{4a}$$

$$\nabla \cdot \vec{\chi} = \kappa^2 \phi \tag{4b}$$

To couple the field ϕ with its sources one might introduce into (4a) a pseudo-vector and into (4b) a pseudo-scalar. In view of the non-existence of magnetic monopoles, however, we shall attempt to develop a pseudo-scalar theory ignoring the pseudo-scalar coupling[1]. The only pseudo-vector that can be associated with a nucleon is its intrinsic spin $\frac{1}{2}\vec{\sigma}$. We then make use of the nuclear density $\rho(\vec{r})$ to form a spin density $\frac{1}{2}\rho\vec{\sigma}$. $\vec{\sigma}$ is a unit vector since the nucleon has spin $\frac{1}{2}$.

The field equations for static pseudo-scalar theory then are

$$\vec{\chi} = \nabla \phi + \frac{f}{\kappa}\vec{\sigma}\rho \tag{5a}$$

$$\nabla \cdot \vec{\chi} = \kappa^2 \phi \tag{5b}$$

1. See Appendix B.

f being a coupling constant with the dimensions of an electric charge.

The next step in developing a pseudo-scalar theory is to generalize set (5) so as to encompass time-dependence. Once again we use special relativity as a guide for selecting source terms. We seek a time component for the spin density $\frac{1}{2}\rho\vec{\sigma}$, and find this to be $\frac{1}{2}\rho\frac{\vec{\sigma}\cdot\vec{v}}{c}$ [1].

The time dependent pseudo-scalar field equations have the form:

$$\vec{\chi} = \nabla\phi + \frac{f}{\kappa}\vec{\sigma}\rho \tag{6a}$$

$$\chi_4 = \frac{1}{ic}\frac{\partial\phi}{\partial t} + \frac{if}{\kappa}\rho\frac{\vec{\sigma}\cdot\vec{v}}{c} \tag{6b}$$

$$\nabla\cdot\vec{\chi} + \frac{1}{ic}\frac{\partial\chi_4}{\partial t} = \kappa^2\phi \tag{6c}$$

The wave equation equivalent to set (6) is easily shown to be

$$(\Box^2 - \kappa^2)\phi = -\frac{f}{\kappa}\nabla\cdot(\vec{\sigma}\rho) - \frac{f}{\kappa c^2}\frac{\partial}{\partial t}(\rho\vec{\sigma}\cdot\vec{v}) \tag{7}$$

One should point out that, although our treatment seems to indicate time varying fields can arise only from moving distributions of nuclear material, this need not be so. It is possible to have a time-varying field produced by precession of the spin vector of a nucleon at rest. With this in

1. See appendix A.

mind we now deduce a continuity equation for energy in the pseudo-scalar field, ignoring the velocity field $\vec{v}$. This will ~~shall~~ not prove to be any real restriction on our theory, since in later applications we generally assume the nucleon to possess infinite mass.

The energy continuity equation is again obtained by computing the divergence of the vector

$$\vec{S} = -\dot{\phi}\,\nabla\phi \tag{8}$$

The result is

$$\nabla\cdot\vec{S} + \frac{\partial H_0}{\partial t} = \frac{f}{k}(\vec{\sigma}\cdot\nabla\rho)\,\dot{\phi} \tag{9}$$

We interpret H_0 as an energy density, $\vec{S}$ as an energy flux, for the meson field. Once again one might introduce an interaction energy density $-\frac{f}{k}\phi\vec{\sigma}\cdot\nabla\rho$ and take the meson field energy density as

$$H = H_0 - \frac{f}{k}\phi\vec{\sigma}\cdot\nabla\rho = \frac{1}{2}k^2\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2c^2}\dot{\phi}^2 - \frac{f}{k}\phi\vec{\sigma}\cdot\nabla\rho \tag{10}$$

Since we have ignored the nucleon velocity we can deduce the potential energy of a nucleon immersed in a pseudo-scalar field ϕ , from

$$V = \int -\frac{f}{k}\phi(\vec{\sigma}\cdot\nabla\rho)\,dv \tag{11}$$

Partial integration yields

$$V = \int \frac{f}{k}\rho\,\vec{\sigma}\cdot\nabla\phi\,dv \tag{12}$$

(We suppose ϕ decreases sufficiently rapidly at large distances from the nucleon, so that the surface integral vanishes.)

i.e.,

$$V = \frac{f}{\kappa} \vec{\sigma} \cdot \nabla \phi (\vec{r}_n, t) \tag{13}$$

on setting $\rho = \delta(\vec{r} - \vec{r}_n)$

2.3 Vector Meson Field.

We wish now to study a meson theory of slightly greater complexity than that embodied in the scalar field. One might refer to the scalar case as a one component field. It should be possible to discuss fields in which the basic physical entities are expressed in terms of several components. If, however, we desire our theory to be Lorentz invariant we require that the set of components form, in general, a Lorentz tensor, i.e. possess certain well defined transformation properties. As a possible extension of the scalar theory we propose to establish a vector meson field[1]. We take as a basis for this field the 4-vector ϕ_ν, which we may refer to as the vector potential for the field.

Once again it will prove helpful to consider a well known classical vector field, viz. the electromagnetic field. One can define the electromagnetic field in terms of a 4-vector, A_ν, which has as its spatial component the magnetic vector potential $\vec{A}$ and as its time component

1. First investigated by Proca, J. Phys. Radium, 7, 347 (1936).

the scalar potential ϕ. The field intensities $\vec{E}$ and $\vec{B}$ are derivable from A_ν in the fashion,

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} \qquad \text{(a)}$$
$$\vec{B} = \nabla\times\vec{A} \qquad \text{(b)} \qquad (1)$$

This latter pair of equations is equivalent to the Maxwell equations

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \text{(a)}$$
$$\nabla\cdot\vec{B} = 0 \qquad \text{(b)} \qquad (2)$$

To take account of the presence of sources of the E.M. field one introduces the field vectors $\vec{H}$, $\vec{D}$ and the matter equations

$$\vec{B} = \vec{H} + 4\pi\vec{M} \qquad \text{(a)}$$
$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \text{(b)} \qquad (3)$$

The vectors $\vec{M}$ and $\vec{P}$ are descriptive of magnetic and electric materials. $\vec{M}$ is in fact the magnetic dipole moment per unit volume, $\vec{P}$ the polarization or electric dipole moment per unit volume.

Of course, free electric charge exists and gives rise to E.M. fields. The presence of charge is accounted for in the field equations

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \qquad \text{(a)}$$

$$\nabla \cdot \vec{D} = 4\pi\rho \qquad \text{(b)} \qquad (4)$$

A continuity equation for conservation of energy in the E.M. field follows from the field equations (2a) and (4a). This continuity equation takes on the form

$$\nabla \cdot \vec{S} + \frac{\partial}{\partial t}(W) = -\vec{E}\cdot\vec{J} \qquad \text{(a)} \qquad (5)$$

where

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H} \qquad \text{(b)}$$

and

$$W = \frac{1}{8\pi}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}) \qquad \text{(c)}$$

There is no difficulty in interpreting the term $-\vec{E}\cdot\vec{J}$ in (5a) as a loss of energy in the field due to Joule heating. Thus we would take W as the energy density and $\vec{S}$ as the energy flux in the E.M. field.

One might proceed to compute wave equations for the E.M. potentials. However, we have developed the E.M. field sufficiently far. We are now in a position to set down the field equations for a vector meson field. Such a field will be defined by a 4-potential $\phi_\nu \equiv (\vec{A}, i\phi)$ and two x sets of field vectors $(\vec{F}, \vec{G})$ and $(\vec{U}, \vec{V})$. These field vectors are related by matter equations for the meson field[1]

$$\vec{U} = \vec{F} + \vec{M} \qquad \text{(a)}$$
$$\vec{V} = \vec{G} + \vec{P} \qquad \text{(b)} \qquad (6)$$

$\vec{M}$, $\vec{P}$ are the sources of the field and are representative of the nuclear matter giving rise to the field. They are related to the dipole characteristics of the nucleons.

One has as the defining equations for the set $(\vec{F}, \vec{G})$

$$\vec{F} = \nabla \times \vec{A} \qquad \text{(a)}$$
$$\vec{G} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \text{(b)} \qquad (7)$$

The latter equations may be replaced by

$$\nabla \times \vec{G} + \frac{1}{c}\frac{\partial \vec{F}}{\partial t} = 0 \qquad \text{(a)} \qquad (8)$$
$$\nabla \cdot \vec{F} = 0 \qquad \text{(b)}$$

A last pair of field equations are obtained through introduction of the 'mesic charge' density ρ_n and 'mesic charge' current density $\vec{J_n} = \rho_n \vec{v}$. These equations are:

$$\nabla \times \vec{U} - \frac{1}{c}\frac{\partial \vec{V}}{\partial t} = -K^2\vec{A} + \frac{1}{c}\vec{J_n} \qquad \text{(a)}$$
$$\nabla \cdot \vec{V} = -K^2\phi + \rho_n \qquad \text{(b)} \qquad (9)$$

---

1. We use rationalized units for the meson fields.

The reason for including terms proportional to the potentials in the equations (9) is that we wish to obtain short range nuclear forces. This will become apparent when we deduce wave equations for the potentials. Set (9) constitutes a significant departure from the Maxwell equations of electromagnetic theory.

Let us compute $\nabla \cdot (9a) + \frac{1}{c}\frac{\partial}{\partial t}(9b)$. We find in this fashion

$$-\kappa^2\left(\frac{1}{c}\dot{\phi} + \nabla\cdot\vec{A}\right) + \left(\frac{1}{c}\dot{\rho}_n + \frac{1}{c}\nabla\cdot\vec{J}_n\right) = 0 \qquad (10)$$

Now if the flow of nuclear matter is to result neither in the creation nor destruction of nuclear mesic charge, we must have

$$\dot{\rho}_n + \nabla\cdot\vec{J}_n = 0 \qquad (11)$$

Whence it would follow

$$\frac{1}{c}\dot{\phi} + \nabla\cdot\vec{A} = 0 \qquad (12)$$

In discussing the electromagnetic field one could have deduced a continuity equation similar to (11), but the Lorentz condition (12) would have been imposed as a supplementary condition. In fact the Lorentz condition is introduced into electromagnetic theory in order to obtain wave equations for the potentials. In a vector meson theory (12) appears as a natural consequence of the conservation of nuclear matter, and of the existence of a characteristic range.

Moreover, the potentials $\vec{A}$, ϕ for the meson field automatically satisfy the wave equations

$$(\Box^2 - \kappa^2)\vec{A} = -\frac{1}{c}\vec{J}_n - \frac{1}{c}\frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M} \tag{13a}$$

$$(\Box^2 - \kappa^2)\phi = -\rho_n + \nabla \cdot \vec{P} \tag{13b}$$

One notes that though the Maxwell equations (2) and (4) for the electromagnetic field can be formulated independent of the electromagnetic potentials, the same is not true of the corresponding vector meson equations (8) and (9). This might suggest we ascribe to the meson potentials somewhat more physical reality than is ordinarily attributed to the electromagnetic potentials.

We would like to obtain explicit expressions for the vector coupling source quantities ($\vec{J}_n$, ρ_n), and the tensor coupling ($\vec{M}$, $\vec{P}$) in terms of nucleon properties. We have no trouble in seeing that reasonable choices for the vector coupling are

$$\vec{J}_n = g_v \rho \vec{v} \tag{14a}$$

$$\rho_n = g_v \rho \tag{14b}$$

where $\rho(\vec{r},t)$ is the familiar density for nuclear material, $\vec{v}(\vec{r},t)$ is the velocity field for the flow of nuclear matter, while g_v is a coupling constant with the dimensions of an electric charge.

Corresponding expressions for $\vec{M}$, $\vec{P}$ are obtained by appealing to relativity. One also makes use of the resemblance of $\vec{M}$ to a magnetic dipole moment density. We quote the results which are deduced in Appendix A.

$$\vec{M} = \left(\rho\, \vec{\sigma}_{\parallel} \sqrt{1 - v^2/c^2} + \frac{\rho\, \vec{\sigma}_{\perp}}{\sqrt{1 - v^2/c^2}}\right) g_T \tag{15a}$$

[1]

$$\vec{P} = g_T\, \rho \frac{\vec{\sigma} \times \vec{v}}{\sqrt{1 - v^2/c^2}} \tag{15b}$$

Finally we derive a continuity equation for energy in the vector meson field. If one considers $\vec{\mathcal{U}} \cdot (8a) - \vec{G} \cdot (9a)$ and then makes use of (6), (7b) and (12) one obtains

$$\nabla \cdot \vec{S} + \frac{\partial H}{\partial t} = - \vec{G} \cdot \vec{J}_n + \frac{\partial \vec{M}}{\partial t} \cdot \vec{F} - \frac{\partial \vec{P}}{\partial t} \cdot \vec{V} \tag{16}$$

where the energy flux is

$$\vec{S} = c\left(\vec{G} \times \vec{\mathcal{U}} + \vec{A}\, \phi\, \kappa^2\right) \tag{17a}$$

and the energy density

$$H = \frac{1}{2} \vec{V}^2 + \frac{1}{2} \vec{F}^2 + \frac{1}{2} \kappa^2 \vec{A}^2 + \frac{1}{2} \kappa^2 \phi^2 + \vec{M} \cdot \vec{F} + \vec{V} \cdot \vec{P} \tag{17b}$$

We note that the interaction energy density associated with the nucleon sources is

$$\vec{M} \cdot \vec{F} + \vec{P} \cdot \vec{V} \tag{18}$$

---

1. $\vec{\sigma}_{\parallel}$ is the component of $\vec{\sigma}$ parallel to $\vec{v}$; $\vec{\sigma}_{\perp}$ the component perpendicular to $\vec{v}$.

As a final possibility one might consider a pseudovector meson field, which differs from the vector field only in its source terms. We will, however, omit the pseudovector field from our discussions.

3. FORMALISM: PROPERTIES OF THE MESON FIELD.

3.1 The Meson as a Field Particle.

In the introduction we ~~had~~ indicated that the π-meson is a particle in some fashion related to nuclear forces. However, in our development of meson theory we have employed only the concept of a field without once implying that particles might be associated with this field. In a purely classical framework we could not bring in the notion of a field particle. Quantum mechanically, field particles enter through a so-called quantisation of the field. We intend to perform a crude quantisation of the meson field - which, despite its arbitrariness and inelegance allows us to obtain considerable insight into the nature of the field.

Before continuing we might point out that there exists a familiar example of the sort of thing we are discussing, viz. the electromagnetic theory of light. One generally discusses physical optics in terms of a field theory which has for its foundations the Maxwell equations. Yet one often introduces particles, called photons, which are supposed to be responsible for the propagation of energy in the light field.

As a basis for our quantisation of the meson field we shall use the De Broglie relations - with which we assume the reader to be acquainted. De Broglie postulated that with any free material particle it is possible to associate a plane wave; the energy-momentum 4 -vector, ($\vec{p}, \frac{iE}{c}$), for the particle being related to the propagation 4 -vector ($\vec{k}, \frac{i\omega}{c}$) for the wave in the fashion,

$$\begin{aligned} \vec{p} &= \hbar \vec{k} \\ E &= \hbar \omega \end{aligned} \tag{1}$$

where $\hbar$ is Planck's constant h , divided by 2π .

We shall assume the converse of De Broglie's hypothesis to be valid as well, i.e. with any wave motion we can associate material particles.

Consider now the wave equation for the scalar meson field in the absence of nuclear sources (or the equation for a single component of the vector field.) The particles we associate with such a meson field will correspond to free mesons. The equation for the field is:

$$(\Box^2 - \kappa^2)\phi = 0 \tag{2}$$

We shall seek plane wave solutions of this equation in the form

$$\phi = e^{i\vec{k}\cdot\vec{r} - i\omega t} + e^{-i(\vec{k}\cdot\vec{r} - \omega t)} \tag{3}$$

Such a choice for ϕ will be a valid solution provided

$$\omega^2 = c^2 \vec{k}^2 + c^2 \kappa^2 \tag{4}$$

If the field (3) is to represent a meson of energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$, then we will have

$$E^2 = c^2\vec{p}^{\,2} + c^4\left(\frac{\kappa\hbar}{c}\right)^2 \tag{5}$$

This latter equation is simply the relativistic relation between energy and momentum for a free particle of rest mass $m = \frac{\kappa\hbar}{c}$.

Our quantisation has not only introduced the meson as a field particle, but has also related the rest mass of the meson to the characteristic range of the meson field. It was roughly in the above fashion that Yukawa first introduced meson theory, and predicted the existence of the meson. He obtained an estimate for the meson mass by making use of the experimentally known range for nuclear forces.

By comparing the meson field with the electromagnetic field, one arrives at the conclusion that the particles associated with the field of short range are of non-zero rest mass. The photons of the electromagnetic field possess zero rest mass while the static electric field, which is proportional to $\frac{1}{r}$ for large distances from its sources, may be considered to be of infinite range.

We would expect disturbances in the meson field to be propagated with finite velocity less than that of light (c).

Thinking of these disturbances as streams of material particles one sees why their velocities should be less than c. In addition one would explain interactions between source particles (nucleons) in terms of exchanges of field particles (mesons). One nucleon may emit a meson which is subsequently absorbed by a second nucleon. This results in an exchange of momentum between the two nucleons - such an exchange would be observed as an interaction force[1].

3.2 Charge in the Meson Field.

Experimentally one observes positively and negatively charged as well as neutral mesons. Up till the present stage we have discussed only the neutral fields - it would, therefore, be impossible to attribute a charge to the field particles of the previous section. In this sense the meson field so far developed resembles the electromagnetic field. For photons are of course uncharged and the electromagnetic field is a neutral field. If our theory is to be realistic we must in some fashion extend it to allow for the retention and transfer of charge in the meson field.

~~1. We refer here to the usual force exerted by one nucleon on another. The exchange of momentum described above should not be confused with an exchange of charge; the latter resulting in a so-called exchange force.~~

We could, on a particle basis, explain the presence of charge by admitting the following reactions:

$$\text{neutron} \longrightarrow \text{proton} + \pi^-$$
$$\text{proton} \longrightarrow \text{neutron} + \pi^+$$

Thus the nucleons would act also as sources for the charge appearing in the meson field. It would then be natural to demand charge conservation for the entire system, nucleons plus meson field. We can incorporate such a conservation into our formalism through a continuity equation. One could not hope, however, to be able to define a charge density ρ_Q and a charge current density $\vec{I}_Q$ for so simple a field as (say) the scalar meson field we have so far developed.

The standard procedure for introducing charge into a field is to allow the field functions to be complex. (It has implicitly been assumed that we were dealing with real fields.) For the scalar field this is equivalent to considering a pair of field functions - we would then be able to associate positive and negative mesons with the field. Since, however, the meson field is to contain neutral particles as well, we shall consider a 3-component 'scalar' field.

We might write the three components simply as ϕ_ℓ $(\ell = 1, 2, 3)$. There would exist for each component a wave equation:

$$(\Box^2 - \kappa^2)\phi_\ell = \sum_n g^{(n)}{}_\ell \, \rho^{(n)} \tag{6}$$

where we distinguish between the components by labelling the coupling constants in the source terms. We have split the latter terms into a summation over the contributions of the separate nucleons.

We shall assume that ϕ_1, ϕ_2 describe the charged meson field and ϕ_3 the neutral field. Hence we shall attempt to deduce a continuity equation for charge in terms of ϕ_1 and ϕ_2. We multiply the first of the equations (6) by ϕ_2, the second by ϕ_1 and subtract to obtain:

$$\nabla \cdot [\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2] + \frac{\partial}{\partial t}\frac{1}{c^2}(\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1) = \sum_n \rho^{(n)}(g_1^{(n)}\phi_2 - g_2^{(n)}\phi_1) \tag{7}$$

This latter equation (7) is in the form of a continuity equation - we shall assume the entity which it defines is the charge in the scalar meson field. We shall in fact set

$$\rho_Q = \epsilon_0/c^2 \, (\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1) \tag{8a}$$

$$\vec{I}_Q = \epsilon_0 \, (\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2) \tag{8b}$$

The factor ϵ_0 is a constant whose only purpose at this point is to ensure that the physical units of ρ_Q and $\vec{I}_Q$ are correct.

The source term

$$\sum_n \epsilon_0 \rho^{(n)} \left(g_1^{(n)} \phi_2 - \phi_1 g_2^{(n)} \right) \tag{9}$$

must then represent the rate at which charge is contributed per unit volume to meson field by its nuclear sources. The contribution of a single nucleon (n) may be obtained by integrating (9) throughout the volume contained in any surface $\sigma_{(n)}$ enclosing this nucleon but excluding all others. We would then set $\rho^{(n)} = \delta(\vec{r} - \vec{r}_n)$, $\vec{r}_n$ being the position of (n) within $\sigma_{(n)}$, and would obtain as the rate at which (n) supplies charge to the field:

$$\frac{dq^{(n)}}{dt} = \epsilon_0 \left[g_1^{(n)} \phi_2 (\vec{r}_n, t) - g_2^{(n)} \phi_1 (\vec{r}_n, t) \right] \tag{10}$$

Let us suppose Q_n is the charge on (n) at any instant t . Then conservation of charge in system nucleons plus meson field is obtained by setting

$$\frac{dQ_n}{dt} + \frac{dq^{(n)}}{dt} = 0 \tag{11}$$

The charged and neutral meson field (ϕ_1, ϕ_2, ϕ_3) we have introduced might be referred to as a symmetric scalar field[1].

1. The symmetric formulation was introduced and developed by N. Kemner, Proc. Cambridge Phil. Soc. 34, 354, 1938.

It will be convenient to think of the three component field as constituting a vector in an abstract space - which we may call charge space.

We shall write

$$\underline{\phi} \equiv (\phi_1, \phi_2, \phi_3) \tag{12a}$$

and in addition[1]

$$\underline{g}^{(n)} \equiv (g_1^{(n)}, g_2^{(n)}, g_3^{(n)}) \tag{12b}$$

Then the wave equations will have the form

$$(\Box^2 - \kappa^2)\, \underline{\phi} = \sum_n \underline{g}^{(n)} \rho^{(n)} \tag{13}$$

It is interesting to compute $[\,\underline{\phi}, (13)\,]$.
One obtains in this fashion

$$\nabla \cdot \epsilon_0 [\nabla \underline{\phi}, \underline{\phi}] + \frac{\partial}{\partial t} \frac{\epsilon_0}{c^2} [\underline{\phi}, \dot{\underline{\phi}}] = \sum_n \epsilon_0 \rho^{(n)} [\underline{g}^{(n)}, \underline{\phi}] \tag{14}$$

---

1. We have had occasion to refer to three different types of space - ordinary 3-space, 4-dimensional space-time, and now 3-dimensional charge space. The notation employed in the first two of these has been obvious. To avoid confusion we shall introduce a somewhat different notation for charge space. In the latter we shall write vectors as $\underline{a}$, $\underline{b}$. The scalar product will be written as $(\underline{a}, \underline{b})$ and the vector product as $[\underline{a}, \underline{b}]$.

Clearly the 3-component of this latter equation is identical with the continuity equation (7). We require then for charge conservation

$$\frac{dQ_n}{dt} = \epsilon_0 [\underset{\sim}{\phi}(\vec{r}_n, t), \underset{\sim}{g}^{(n)}]_3 \qquad (14)$$

We will write for convenience

$$\underset{\sim}{g}^{(n)} = g\, \underset{\sim}{\tau}^{(n)}$$

interpreting g as a mesic charge common to all nucleons. It is because we associate the same mesic charge with each of the fields (ϕ_1, ϕ_2, ϕ_3), that we refer to our theory as a symmetric formulation.

One fashion in which charge conservation may be achieved, is to look on (14) as an equation of motion for the vector $\underset{\sim}{\tau}^{(n)}$.

We shall in fact set $\dot{Q}_n \sim \dot{\tau}_3^{(n)}$

or in effect

$$Q_n = a\tau_3^{(n)} + b \qquad a, b \text{ constants} \qquad (15)$$

We ~~would take~~ set $Q_n = 0$ if (n) is a neutron, $Q_n = \epsilon$ if (n) is a proton. (ϵ is the magnitude of the electronic charge.) One possibility would be to set

$\tau_3^{(n)} = 1$ for (n) a proton

$\tau_3^{(n)} = -1$ for (n) a neutron

and then to have

$$Q_n = \frac{\epsilon}{2} (1 + \tau_3^{(n)}) \qquad (16)$$

Whence the equation (14) becomes:

$$\dot{\tau}_3^{(n)} = \frac{2\epsilon_0}{\epsilon} g \, [\underline{\phi}, \underline{\tau}^{(n)}]_3 \tag{17}$$

One might then postulate that the motion of $\underline{\tau}^{(n)}$ in charge space is governed by the equation

$$\dot{\underline{\tau}}^{(n)} = \frac{2\epsilon_0}{\epsilon} g \, [\underline{\phi}, \underline{\tau}^{(n)}] \tag{18}$$

It is clear that (18) predicts the magnitude of $\underline{\tau}^{(n)}$ to be invariant. We may then suppose $\underline{\tau}^{(n)}$ is a unit vector thus guaranteeing that the charge on a nucleon is never outside the range $0 \longrightarrow \epsilon$. The motion of $\underline{\tau}^{(n)}$ may be likened to a precession or spinning in charge space. One refers to $\underline{\tau}^{(n)}$ as the isotopic spin vector (i -spin) of the nucleon (n). Classically, $\underline{\tau}^{(n)}$ must be allowed to take on all orientations in charge space - implying that $\tau_3^{(n)}$ assumes all values between -1 and 1. Hence our classical picture would allow a nucleon to exist in a state intermediate between proton and neutron. One might then picture the proton and neutron as different states of a single particle, the 'nucleon'.

We shall be interested in the energy density and flux for the symmetric meson field. These quantities are obtained simply by summation over the component fields ϕ_1, ϕ_2, ϕ_3. For example, the energy flux density may be written

$$\vec{S} = \sum_{\alpha=1}^{3} -\dot{\phi}_\alpha \nabla \phi_\alpha \tag{18}$$

$$= -(\dot{\underline{\phi}}, \nabla \underline{\phi})$$

In a similar fashion one may obtain the potential energy possessed by a nucleon in a symmetric scalar field from the result deduced in 2.1 for the neutral field.

One finds this potential energy is

$$V = \int (\underline{\tau}^{(n)}, g \underline{\phi} \rho)\, dv \tag{20a}$$

$$\text{i.e. } V = g\,(\underline{\tau}^{(n)}, \underline{\phi}(\vec{r}_n, t)) \tag{20b}$$

3.3 Plane Waves in the Symmetric Meson Field.

Let us now examine the structure of plane meson waves in a symmetric theory. How is one to specify the charge on the mesons associated with such waves? We shall find it helpful to make use of the notation

$$\phi_+ = \phi_1 + i\,\phi_2 \tag{1a}$$

$$\phi_- = \phi_1 - i\,\phi_2 \tag{1b}$$

Then one ~~would~~ finds from 3.2(8)

$$\rho_Q = \frac{i\epsilon_0}{2c^2}(\phi_+ \dot{\phi}_- - \phi_- \dot{\phi}_+) \tag{2a}$$

$$\vec{I}_Q = \frac{i\epsilon_0}{2}(\phi_- \nabla\phi_+ - \phi_+ \nabla\phi_-) \tag{2b}$$

Similarly the energy flux for charged mesons may be expressed as

$$\vec{S} = -\tfrac{1}{2}(\dot{\phi}_+ \nabla\phi_- + \dot{\phi}_- \nabla\phi_+) \tag{3}$$

A plane meson wave will result when one considers the meson field at great distances from the nuclear sources. As we saw ~~had~~ ~~seen~~ earlier a meson field ϕ_+ representing a plane wave will be required to satisfy the source free equation

$$(\Box^2 - \kappa^2)\phi_+ = 0 \tag{4}$$

One might then take

$$\phi_+ = a e^{i\omega t - i\vec{\mu}\cdot\vec{r}} \qquad \text{(a some constant)} \tag{5}$$

with the proviso

$$\omega^2/c^2 = \mu^2 + \kappa^2$$

It is important to note that the disturbance represented by ϕ_+ will be damped out unless $\omega > \kappa c$. This simply means that the field particles associated with ϕ_+ must possess an energy, $\hbar\omega$, greater than the rest energy of the meson.

The charge density and charge current density for the plane wave (5) are from (2)

$$\rho_q = \frac{\epsilon_0 |a|^2 \omega}{c^2} \tag{6a}$$

$$\vec{I}_q = \epsilon_0 |a|^2 \vec{\mu} \tag{6b}$$

The important point to note here is that both ρ_q and $\vec{I}_q$ are independent of time and positive. One might say ϕ_+ is a plane wave of positive mesons travelling in the direction of the vector $\vec{\mu}$. Had we taken

$$\phi_+ = a e^{-i\omega t + i\vec{\mu}\cdot\vec{r}} \tag{7}$$

we would have obtained a wave of negative mesons travelling in the $\vec{\mu}$ direction.

We have in 3.1 related to the field (5) particles of energy $\hbar\omega$ and momentum $\hbar\vec{\mu}$. In light of the results in (6) we should wish to attribute to these particles a charge $+\epsilon$. Similarly the particles of the field (7) would possess a charge $-\epsilon$. These demands are not independent and as we shall show lead to a specification of the constant ϵ_0.

We compute first the energy flux in the wave (5) from (3). This will be

$$\vec{S} = |a|^2 \omega \vec{\mu} \tag{8}$$

Since the energy of each meson in the wave (5) is $\hbar\omega$, it follows that the particle flux is

$$\vec{P} = \frac{|a|^2 \omega \vec{\mu}}{\hbar\omega} = \frac{|a|^2 \vec{\mu}}{\hbar} \tag{9}$$

But then if the charge per particle is $+\epsilon$, the charge flux must be

$$\vec{I_Q} = \frac{\epsilon |a|^2 \vec{\mu}}{\hbar} \tag{10}$$

Equating this latter result to that in (6b) we will have $\epsilon_0 = \epsilon/\hbar$. This specification of ϵ_0 followed from our wish to conform with the observed fact that a positively charged π-meson bears the charge $+\epsilon$. The z-spin equation of motion 3.2(18) now becomes

$$\dot{\underline{\tau}} = \frac{2g}{\hbar} [\underline{\phi}, \underline{\tau}] \tag{11}$$

3.4 Motion of Isotopic Spin - Specification of the Neutron and Proton States.

As a second example which ~~shall~~ will later prove useful we ~~will~~ shall consider the meson field generated by a certain prescribed motion of the $\dot{\tau}$ -spin of a nucleon. We shall assume the nucleon is at rest. The $\dot{\tau}$ -spin will be given by

$$\begin{aligned} \tau_3 &= \text{constant} \\ \tau_+ &= \tau_1 + i\tau_2 = \tau_{+0}\, e^{i\omega t} \end{aligned} \tag{1}$$

Then the field ϕ_+ generated by τ_+ satisfies:

$$(\Box^2 - \kappa^2)\, \phi_+ = g\, \tau_+\, \delta(\vec{r}) \tag{2}$$

and is easily found to be

$$\phi_+ = - \frac{g\, \tau_{+0}}{4\pi} \frac{e^{i\omega t - i\mu r}}{r} \tag{3}$$

with $\mu^2 = \omega^2/c^2 - \kappa^2$

This field will represent a travelling wave provided $\omega > \kappa c$. We have deliberately chosen the solution of (2) in the form of a wave propagated in a direction outwards from the nucleon source, since we are interested in the energy and charge appearing in the far field. The energy (and similarly the charge) is obtained by integrating ($r^2 \vec{i_r} \cdot \vec{S}$) over the surface of a large sphere, hence it is necessary to compute $\vec{S}$ only to terms of first order in $\frac{1}{r^2}$. One obtains

$$\vec{S} = \left(\frac{g|\tau_{+0}|}{4\pi}\right)^2 \frac{\vec{\iota_r}}{r^2}\,\omega\,\mu \qquad (4)$$

$\vec{\iota_r}$ representing a unit vector in the radiated direction. The radiation from the $\dot{z}$ -spin is clearly isotropic in ordinary space. It follows that the energy radiated per unit time by the nucleon is

$$\frac{g^2|\tau_{+0}|^2\omega\mu}{4\pi} \qquad (5)$$

The charge radiated in a similar time is

$$\frac{g^2|\tau_{+0}|^2\epsilon\mu}{4\pi\hbar} \qquad (6)$$

Clearly both the radiated charge and energy lead to the same number of particles emitted per unit time, i.e.

$$\frac{g^2|\tau_{+0}|^2\mu}{4\pi\hbar} \qquad (7)$$

We conclude the meson field (3) represents a spherical wave of positive mesons travelling outwards from the nucleon source. In an analagous fashion the choice

$$\tau_+ = \tau_{+0}e^{-i\omega t} \qquad (8)$$

would lead to a radiated field of negative mesons.

Previously we stated that a given nucleon (n) was a proton if $\tau_3^{(n)} = +1$, a neutron if $\tau_3^{(n)} = -1$. We ~~will~~ shall later find it necessary to relax this restriction somewhat, and permit the τ -spin vectors of neutrons and protons to possess 1, 2 components. Such a picture of the nucleon would in fact be closer to the quantum mechanical situation. We have demanded $\underline{\tau}^{(n)}$ possess an invariant magnitude equal to unity in order that the charge be conserved in the system nucleon plus meson field, and also in order that the nucleon never possess a charge greater than the electronic charge ϵ in magnitude. Quantum mechanically, the magnitude of $\underline{\tau}^{(n)}$ ~~might be said~~ is taken to be $\sqrt{3}$. The charge on the nucleon is still determined by

$$Q^{(n)} = \frac{\epsilon}{2}\left(1 + \tau_3^{(n)}\right)$$

but the inclination of the τ -spin vector to the 3-axis in charge space can never be such as to make $|\tau_3^{(n)}| > 1$. In fact, if the nucleon is in a quantum state definitely identifiable as a proton or neutron state, then $\tau_3^{(n)} = \pm 1$.

The same situation exists quantum mechanically with regard to ordinary or mechanical spin, $\frac{1}{2}\vec{\sigma}$. We have implied $\vec{\sigma}^2 = 1$, while quantum mechanically the ordinary spin operator has a magnitude $\frac{1}{4}\vec{\sigma}^2 = \frac{3}{4}$. In a few isolated cases we will find that agreement is obtained between classical and quantum mechanical results by taking $\vec{\sigma}^2 = 3$ in the classical formulation.

3.5 Symmetric Pseudo-Scalar and Vector Theories.

A symmetric scalar meson theory has been established. It is possible to carry out a similar program for the pseudo-scalar and vector fields. Most of the formalism involved will be identical with that for the scalar field. The only differences will arise when one considers the motion of τ-spin of nucleons.

(a) Pseudo-Scalar:

For reasons indicated in 2.2 we ignore all but the pseudo-vector coupling in the pseudo-scalar theory. We have then as the wave equation for the neutral pseudo-scalar field

$$(\Box^2-\kappa^2)\,\phi = \sum_n -\frac{f}{\kappa}\,\vec{\sigma}^{(n)}\cdot\nabla\rho^{(n)} \tag{1}$$

A symmetric formulation is obtained by introducing the τ-spin vector $\underline{\tau}^{(n)}$ for each nucleon, and by writing the field as a vector $\underline{\phi} \equiv (\phi_1, \phi_2, \phi_3)$ in charge space. One obtains ~~would have~~ as wave equation for the symmetric pseudo-scalar theory:

$$(\Box^2-\kappa^2)\,\underline{\phi} = \sum_n -\underline{\tau}^{(n)}\frac{f}{\kappa}\,\vec{\sigma}^{(n)}\cdot\nabla\rho^{(n)} \tag{2}$$

The charge continuity equation will be

$$\nabla\cdot\epsilon_0\left[\nabla\underline{\phi},\underline{\phi}\right]_3 + \frac{\partial}{\partial t}\frac{\epsilon_0}{c^2}\left[\underline{\phi},\dot{\underline{\phi}}\right]_3 = \sum_n \epsilon_0\frac{f}{\kappa}\,\vec{\sigma}^{(n)}\cdot\nabla\rho^{(n)}\left[\underline{\phi},\underline{\tau}^{(n)}\right]_3 \tag{3}$$

We once again take the charge on the nucleon to be

$$Q_n = \frac{e}{2}\left(1+\tau_3^{(n)}\right) \tag{4}$$

To obtain charge conservation we set

$$\frac{dQ_n}{dt} + \int_{V_n} \epsilon_0 \left[\underset{\sim}{\phi}, \underset{\sim}{\tau}^{(n)} \right]_3 \frac{f}{\kappa} \vec{\sigma}^{(n)} \cdot \nabla \rho^{(n)} \, dv = 0 \tag{5}$$

where V_n is a region surrounding the nucleon (n) but excluding all other nucleons. Partial integration yields the following equation of motion for $\underset{\sim}{\tau}^{(n)}$:

$$\dot{\underset{\sim}{\tau}}^{(n)} = \frac{2f}{\hbar \kappa} \left[\vec{\sigma}^{(n)} \cdot \nabla \underset{\sim}{\phi}(\vec{r}_n, t) , \underset{\sim}{\tau}^{(n)} \right] \tag{6}$$

We might also list the energy flux density for $\underset{\sim}{\phi}$

$$\vec{S} = - (\dot{\underset{\sim}{\phi}}, \nabla \underset{\sim}{\phi})$$

Finally we wish to make note of the potential energy possessed by a nucleon in a static, symmetric pseudo-scalar field. This energy is obtained by a generalization of the result expressed in 2.2(13) and is

$$V = \frac{f}{\kappa} \left(\underset{\sim}{\tau}^{(n)}, \vec{\sigma}^{(n)} \cdot \nabla \underset{\sim}{\phi}(\vec{r}_n, t) \right) \tag{7}$$

(b) Vector:

For each of the field vector components $\phi_\nu \equiv (\vec{A}, i\phi)$ we introduce a vector $\underset{\sim}{\phi}_\nu$ in charge space. It will also prove helpful to write the spatial part of the field potential as a vector in charge space in the form $\vec{A}_\alpha$ ($\alpha = 1, 2, 3$), the time part as ϕ_α ($\alpha = 1, 2, 3$).

In 2.3 we had for the neutral vector field the wave equations

$$(\square^2 - \kappa^2) \vec{A} = \sum_n \nabla \times g_T \, \rho^{(n)} \vec{\sigma}^{(n)} \tag{8a}$$

$$(\square^2 - \kappa^2) \phi = - \sum_n g_v \, \rho^{(n)} \tag{8b}$$

We have again neglected any possible motion of the nucleons. The transition to a symmetric theory is easily obtained in the fashion

$$(\Box^2-\kappa^2)\vec{A}_\alpha = \sum_n \tau_\alpha^{(n)} g_T \nabla\times\vec{\sigma}^{(n)}\rho^{(n)} \tag{9a}$$

$$(\Box^2-\kappa^2)\phi_\alpha = -\sum_n \tau_\alpha^{(n)} g_V \rho^{(n)} \tag{9b}$$

One computes a charge density and flux for each of the components of the field potential ($\vec{A}_\alpha$, $i\phi_\alpha$) and then obtains the corresponding quantities for the entire vector field by summation. The results are

$$\vec{I}_Q = \epsilon_0 \left[\nabla\underline{\phi}_\nu , \underline{\phi}_\nu\right]_3 \tag{10a}$$

$$\rho_Q = \frac{\epsilon_0}{c^2}\left[\underline{\phi}_\nu , \dot{\underline{\phi}}_\nu\right]_3 \tag{10b}$$

summation over the repeated index ν is implied. Consideration of the source term in the charge continuity equation would yield an equation of motion for the $\dot{z}$ -spin.

We may write this equation in the somewhat awkward form:

$$\dot{\underline{\tau}}^{(n)} = \frac{2g_T}{\hbar}\left[\vec{\sigma}^{(n)}\cdot\nabla\times\vec{\underline{A}} , \underline{\tau}^{(n)}\right] - \frac{2g_V}{\hbar}\left[\underline{\phi} , \underline{\tau}^{(n)}\right] \tag{11}$$

i.e.,

$$\dot{\underline{\tau}}^{(n)} = \frac{2g_T}{\hbar}\left[\vec{\sigma}^{(n)}\cdot\vec{\underline{F}} , \underline{\tau}^{(n)}\right] - \frac{2g_V}{\hbar}\left[\underline{\phi} , \underline{\tau}^{(n)}\right]$$

3.6 Equations of Motion for the Ordinary Spin.

Equations of motion of the $\dot{z}$ -spin of a nucleon have been developed in terms of charge conservation. We would like also to consider the precession of the ordinary spin when the nucleon is placed in various types of fields. Ordinary spin enters into our theories only in the pseudo-scalar and vector formulations, and we shall ignore the latter.

We first study the effect of an external meson field on the spin vector $\frac{1}{2}\vec{\sigma}$ of a nucleon. We expect the meson field to exert a torque on $\vec{\sigma}$, resulting in a precession of the form

$$\frac{d}{dt}\vec{\sigma} = \vec{\Omega} \times \vec{\sigma} \tag{1}$$

where $\vec{\Omega} \times \vec{\sigma}$ is proportional to the generating torque. One might recall the analogy, previously developed, between the pseudo-scalar field and the magnetic field arising from a distribution of magnetic material. Ordinary spin had its analogue in the elementary magnetic dipoles of the material. Since the torque on a magnetic dipole $\vec{m}$ in a field of intensity $\vec{B}$ is $\vec{m} \times \vec{B}$, we would take as the torque on a spin $\frac{1}{2}\vec{\sigma}$ in a pseudo-scalar field of intensity $\nabla\phi$, $-\frac{f}{\kappa}\vec{\sigma} \times \nabla\phi$. We have taken $\frac{f}{\kappa}\vec{\sigma}$ as the meson moment of the nucleon. The equation of motion is then obtained by equating the torque to the time rate of change of nucleon angular momentum. We attribute to a nucleon with mechanical spin vector $\vec{\sigma}$, an angular momentum $\frac{1}{2}\hbar\vec{\sigma}$. Thus the equation of motion for ordinary spin in a pseudo-scalar field ϕ is

$$\frac{d\vec{\sigma}}{dt} = \frac{2f}{\hbar\kappa}\nabla\phi\times\vec{\sigma} \tag{2}$$ [1]

The equation (2) refers to a neutral field. The generalization to a symmetrical field is easily seen to be

$$\frac{d\vec{\sigma}}{dt} = \frac{2f}{\hbar\kappa}\nabla(\underline{\phi},\underline{\tau})\times\vec{\sigma} \tag{3}$$

---

1. The reader who is familiar with Hamiltonian mechanics will be able to follow an alternative method for arriving at the equation of motion for $\vec{\sigma}$. If we suppose the nuclear sources to be at rest, the interaction Hamiltonian for a nucleon in a neutral pseudo-scalar field will be from 2.2(13)

$$H' = \frac{f}{\kappa}\vec{\sigma}\cdot\nabla\phi(\vec{r}_n,t)$$

One supposes the intrinsic angular momentum $\vec{s} = \frac{1}{2}\hbar\vec{\sigma}$ associated with the nucleon spin satisfies the classical Poisson bracket ($P.B.$)relations $\{s_i, s_j\} = s_k$ taken cyclically. It follows

$$\{\sigma_i,\sigma_j\} = \frac{2\sigma_k}{\hbar}$$

The time rate of change of a component σ_i is obtained by computing the $P.B.$ of σ_i with the interaction Hamiltonian. The result is

$$\frac{d\vec{\sigma}}{dt} = \frac{2f}{\hbar\kappa}\nabla\phi(\vec{r}_n,t)\times\vec{\sigma}$$

It will also be necessary to consider the motion of ordinary spin when a nucleon is immersed in an electromagnetic field. We obtain the equation of motion in this case by supposing there is associated with the spin of the nucleon a magnetic moment

$$\vec{m} = \mu_n \left(\frac{\epsilon\hbar}{2Mc}\right) \vec{\sigma} \tag{4}$$

where μ_n is to be the value of this moment measured in units of the Bohr magneton $\frac{\epsilon\hbar}{2Mc}$, M is the nuclear mass. μ_n would have different values for a proton and a neutron.

The torque on a nucleon when it is placed in a magnetic field of intensity $\vec{B}$ is $\vec{m} \times \vec{B}$.

Also the angular momentum associated with the intrinsic spin $\frac{1}{2}\vec{\sigma}$ is $\frac{1}{2}\hbar\vec{\sigma}$. Hence the equation of motion for the ordinary spin vector in the magnetic field $\vec{B}$ will be

$$\frac{d\vec{\sigma}}{dt} = \frac{2}{\hbar}\, \vec{m} \times \vec{B} = \frac{\mu_n \epsilon}{Mc}\, \vec{\sigma} \times \vec{B} \tag{4}$$

3.7 Interaction between Electromagnetic and Meson Fields.

We shall consider an interaction between an electromagnetic field specified by the potential ($\vec{A}, \psi$) and a scalar meson field in the absence of its sources. By regarding the meson field as an assemblage of particles we shall be able to treat the present problem in terms of the simple interaction between the electromagnetic field and a charged particle. The key to a solution of our problem lies in observing how a free particle, with charge e, behaves in an electromagnetic field. The particle seems to take on in addition to its kinetic momentum $\vec{p}$ and kinetic energy E (both relativistic), a potential momentum $\frac{e\vec{A}}{c}$ and a potential energy $e\psi$.

Suppose we then consider a free positive meson which is represented by the plane wave

$$\phi_+ = e^{-i\vec{\mu}\cdot\vec{r} + i\omega t} \tag{1}$$

In the absence of an electromagnetic field the positive meson (1) possesses a total momentum $\hbar\vec{\mu}$ and a total energy $\hbar\omega$ [1], which may be evaluated in the fashion:

$$\hbar\vec{\mu}\phi_+ = i\hbar\nabla\phi_+ \tag{2a}$$

$$\hbar\omega\phi_+ = -i\hbar\frac{\partial\phi_+}{\partial t} \tag{2b}$$

We now postulate

(a) The field (1) represents a positive meson even in the presence of an electromagnetic field.

(b) The total momentum and energy of the positive meson may be evaluated from $\left(i\hbar\nabla\phi_+, -i\hbar\frac{\partial\phi_+}{\partial t}\right)$ even in the

---

1. For a completely free particle:

total momentum = kinetic momentum.

presence of an electromagnetic field. If a positive meson in an electromagnetic field ($\vec{A}, \psi$) possesses kinetic momentum $\vec{p}$, kinetic energy E , then it will possess total momentum $\vec{p} + \frac{\epsilon \vec{A}}{c}$, and total energy $E + \epsilon\psi$. Hence we should set

$$i\hbar \nabla \phi_+ = \left(\vec{p} + \frac{\epsilon \vec{A}}{c}\right)\phi_+ \tag{3a}$$

$$-i\hbar \frac{\partial \phi_+}{\partial t} = (E + \epsilon\psi)\phi_+ \tag{3b}$$

The latter two equations (3a), (3b) imply

$$\left(\nabla + \frac{i\epsilon\vec{A}}{\hbar c}\right)^2 \phi_+ = -\vec{p}^{\,2}/\hbar^2 \, \phi_+ \tag{4a}$$

$$\left(\frac{\partial}{\partial t} - \frac{i\epsilon\psi}{\hbar}\right)\phi_+ = -E^2/\hbar^2 \, \phi_+ \tag{4b}$$

Now the kinetic energy and momentum of a relativistic particle always satisfy

$$E^2/c^2 = \vec{p}^{\,2} + m^2c^2 \tag{5}$$

where m is the rest mass of the particle.

(5) coupled with (4a) and (4b) leads to

$$\left(\nabla + \frac{i\epsilon\vec{A}}{\hbar c}\right)^2 \phi_+ - \frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{i\epsilon\psi}{\hbar}\right)^2 \phi_+ - \kappa^2 \phi_+ = 0 \tag{6}$$

where for the meson we set $\kappa = \frac{mc}{\hbar}$.

Equation (6) is the wave equation to be associated with a positive meson immersed in an electromagnetic field.

Had we considered a negative meson using the field

$$\phi_+ = e^{i\vec{\mu}\cdot\vec{r} - i\omega t}$$

we would have obtained an <u>identical</u> wave equation. We need not, then, treat positive and negative mesons separately.

The results of this section apply equally well to the scalar and pseudo-scalar fields. The generalization to interaction between electromagnetic fields and meson fields with sources is obtained by placing appropriate source terms on the right side of (6). If a gradient operator appears in these source terms it must be replaced by ($\nabla + \frac{i\epsilon\vec{A}}{\hbar c}$).

In the presence of an electromagnetic field the wave equations for meson theory are altered in form; hence the continuity equations for energy and charge are no longer valid. For the scalar meson field arising from a single nucleon, the wave equation has the form

$$\left(\nabla + \frac{i\epsilon\vec{A}}{\hbar c}\right)^2 \phi_+ - \left(\frac{\partial}{\partial t} - \frac{i\epsilon\psi}{\hbar}\right)^2 \phi_+ - \kappa^2 \phi_+ = g\tau_+ \rho \qquad (7)$$

The complex conjugate of equation (7) may be written as

$$\left(\nabla - i\epsilon\frac{\vec{A}}{\hbar c}\right)^2 \phi_- - \left(\frac{\partial}{\partial t} + \frac{i\epsilon\psi}{\hbar}\right)^2 \phi_- - \kappa^2 \phi_- = g\tau_- \rho \qquad (8)$$

One may then compute $\epsilon_0 \phi_- (7) - \epsilon_0 \phi_+ (8)$, and obtain the continuity equation for charge,

$$\nabla \cdot \left[\frac{c\epsilon}{2\hbar}\left(\phi_- \nabla\phi_+ - \phi_+ \nabla\phi_-\right) - \frac{\epsilon^2}{\hbar^2 c}\vec{A}\phi_+\phi_-\right]$$
$$+ \frac{\partial}{\partial t}\left[\frac{c\epsilon}{2\hbar c^2}\left(\phi_+ \dot{\phi}_- - \phi_- \dot{\phi}_+\right) - \frac{\epsilon^2}{\hbar^2 c^2}\psi\phi_+\phi_-\right]$$
$$= \frac{\epsilon g \rho}{2\hbar} i\left(\tau_+ \phi_- - \tau_- \phi_+\right) \qquad (9)$$

We have made use of the Lorentz condition

$$\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0$$

The charge density and current density have been altered by the addition of terms linear in the electromagnetic field. For a meson field perturbed by low intensity electromagnetic field such terms are small and one is often justified in retaining the definitions

$$\vec{I}_Q = \frac{i\epsilon}{2\hbar}\left(\phi_- \nabla\phi_+ - \phi_+ \nabla\phi_-\right)$$

$$\rho_Q = \frac{i\epsilon}{2\hbar c^2}\left(\phi_+ \dot{\phi}_- - \phi_- \dot{\phi}_+\right)$$

In a similar fashion the energy continuity equation for a meson field in interaction with an electromagnetic field differs from the equation 2.1(19) by the addition of terms proportional to the electromagnetic fields. For most applications of the formalism in this section one may ignore the flow of energy represented by the extra terms.

4. APPLICATIONS.

4.1 General Solution for the Meson Wave Equation.

One might say the scalar meson field is defined by the equation

$$(\Box^2 - \kappa^2)\underline{\phi} = \sum_n g\, \underline{\tau}^{(n)} \rho^{(n)} \tag{1}$$

If one ignores the reaction of the meson field on its nuclear sources, then one sets a problem in scalar meson theory by specification of the values of $\rho^{(n)}(\vec{r},t)$ and $\underline{\tau}^{(n)}(t)$. We have previously indicated that a Green's method could be applied in ordinary 3-space to obtain a solution of the static equation 2.1(9). The equivalent method for (1) is somewhat more complicated; our treatment will be in terms of the Lorentz space-time which is to be regarded formally as a Euclidean 4-space.

Let us take x_μ as the field point in space-time at which $\underline{\phi}$ in (1) is to be evaluated, and x'_μ as a variable source point. We introduce into the Lorentz 4-space a set of spherical co-ordinates ($R, \theta_1, \theta_2, \phi$) with origin at x_μ [1]; so that we have

$$R^2 = (x_\mu' - x_\mu)^2 \tag{2}$$

The Green's Function to be associated with (1) is then the spherically symmetric solution $u(R)$ of the

1. For further discussion of this co-ordinate system and related details, see Appendix A.

equation

$$(\Box^2 - \kappa^2)\, u(R) = 0 \tag{3}$$

i.e., of

$$\frac{1}{R^3}\frac{d}{dR}\left(R^3 \frac{du}{dR}\right) - \kappa^2 u = 0$$

We must choose the solution of (3) which behaves properly at $R = \infty$. One finds an appropriately normalized choice is

$$u(R) = \frac{\kappa K_1(\kappa R)}{R} \tag{4}$$

$K_1(\kappa R)$ is a Bessel function of imaginary argument and may be written in the form

$$K_1(z) = \tfrac{1}{2}\pi i e^{\frac{1}{2}\pi i} H_1^{(1)}(iz) \tag{5}$$

where $H_1^{(1)}(x)$ is the Hankel Function of the first kind. It is of interest to us to note

$$K_1(z) \xrightarrow[|z| \to +\infty]{} \sqrt{\frac{\pi}{2z}}\, e^{-z} \tag{6a}$$

and also

$$K_1(z) \xrightarrow[z \to 0]{} \frac{1}{z} \tag{6b}$$ [1]

It is evident that $u(R)$ will have a singularity at the source point $x_\mu' = x_\mu$. Since we shall perform an

1. See e.g. G. N. Watson: 'Bessel Functions'

integration over the source material present in space-time, we enclose the singularity $x_\mu' = x_\mu$ by a small hypersphere S_0 of radius R_0. We apply a 4-dimensional Green's Theorem[1] to the region V_4 interior to a large hypersphere S_{R_1} and exterior to S_0, obtaining

$$\int_{V_4} (u \Box^2 \phi' - \phi' \Box^2 u)\, dv_4' = -\int_{S_0} \left(u \frac{\partial \phi'}{\partial R_0} - \phi' \frac{\partial u}{\partial R_0}\right) ds' + \int_{S_{R_1}} \left[u \frac{\partial \phi'}{\partial R_1} - \phi' \frac{\partial u}{\partial R_1}\right] ds' \tag{7}$$

The primes are to remind us that we are to take $\phi' = \phi(x_\mu')$ in the integration.

It will be assumed that ϕ is sufficiently well behaved at $R_1 = \infty$ to allow the integral over S_{R_1} to be neglected.

---

1. In 3-space one has for ψ_1, ψ_2 a pair of scalars regular in a region V_3 enclosed by a surface S_3,

$$\int_{V_3} (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1)\, dv_3 = \int_{S_3} \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n}\right) ds$$

where n is the normal direction outwards from V_3.

In 4-space one has an analagous result for χ_1, χ_2, i.e.

$$\int_{V_4} (\chi_1 \Box^2 \chi_2 - \chi_2 \Box^2 \chi_1)\, dv_4 = \int_{S_4} \left(\chi_1 \frac{\partial \chi_2}{\partial n} - \chi_2 \frac{\partial \chi_1}{\partial n}\right) dS$$

We note in (7) that the outwards normal from V_4 is radially inwards over S_0, hence the negative sign.

Over S_o one has:

$$\frac{\partial u}{\partial R_0} = \frac{\kappa\, dK_1(\kappa R_0)}{R_0\, d R_0} - \frac{\kappa K_1(\kappa R_0)}{R_0^2} \tag{8}$$

Then in view of (6b) we will find

$$-\int_{S_0} \left(u \frac{\partial \phi'}{\partial R_0} - \phi' \frac{\partial u}{\partial R_0} \right) ds' \longrightarrow \int_{S_0} \left[-\frac{1}{R_0^2} \frac{\partial \phi'}{\partial R_0} + \left(\frac{-2}{R_0^3} \right) \phi' \right] R_0^3\, d\Omega \tag{9}$$

We have written an element of area on S_o in form

$$ds' = R_0^3\, d\Omega \quad 1 \tag{10}$$

Clearly then as $R_0 \longrightarrow 0$

$$-\int_{S_0} \left(u \frac{\partial \phi'}{\partial R_0} - \phi' \frac{\partial u}{\partial R_0} \right) ds' \longrightarrow -2\phi(x_\mu) \int d\Omega = -4\pi^2 \phi(x_\mu) \tag{11}$$

Making use of (7), (11) and the wave equations (1), (3) one obtains as the formal solution of (1)

$$\phi(x_\mu) = -\frac{\kappa}{4\pi^2} \int \frac{K_1(\kappa R)\, s(x_\mu')}{R}\, dv_4' \tag{12a}$$

where $\quad s(x_\mu') = \sum_n g\, \tau^{(n)} \rho^{(n)} \quad$ (12B)

The integration in (12a) is over all of 4-space and may be written as a double integral over 3-space and time $\left(t' = \frac{x_4'}{ic} \right)$

$$\phi(\vec{r}, t) = \frac{-\kappa}{4\pi^2} \int_{x_4'} \int_{V_3'} \frac{K_1(\kappa R)}{R}\, s(x_\mu')\, dV_3'\, dx_4' \tag{13}$$

---

1. See appendix A for discussion of surface area and solid angle in 4-space.

We shall apply (13) to obtain the field of an arbitrary moving nucleon source, for which we take

$$S(x_\mu') = g\,\underline{\tau}(t')\beta(t')\,\delta(\vec{r}\,'-\vec{r}_n(t')) \tag{14}$$

Performing the volume integration indicated in (13) we get

$$\phi(\vec{r},t) = -\frac{g\kappa}{4\pi^2}\int_{-\infty}^{\infty}\underline{\tau}(t')\frac{K_1(\kappa R)}{R}\,\beta(t')\,dx_4' \tag{15a}$$

where now

$$R^2 = |\vec{r}-\vec{r}_n(t')|^2 + x_4'^2 \tag{15b}$$

For convenience we measure the nucleon field at the time $t=0$. An explicit evaluation of ϕ is possible when the nucleon is at rest (at $\vec{r}_n = 0$ say) and its $\dot{\tau}$ -spin is not varying in time.

In such a case

$$\beta(t') \equiv 1 \quad , \quad R = \sqrt{x_4'^2 + r^2} \tag{16}$$

and we must compute the value of the integral

$$I = \int_{-\infty}^{\infty}\frac{K_1\left(\kappa\sqrt{x_4'^2+r^2}\right)}{\sqrt{x_4'^2+r^2}}\,dx_4' \tag{17}$$

This is most readily done by allowing x_4' to take on complex values, and regarding I as a contour integration in the x_4' -plane.

We observe R is multivalued with branch points at $x_4' = \pm i r$, and in addition the integrand in I

has simple poles at $x_4' = \pm i r$. We then will take as branch lines for the integrand the imaginary axis from $-i\infty$ to $-ir$ and $i\infty$ to ir . This is accomplished by setting

$$x_4' - ir = a_1 e^{i\alpha_1} \tag{18a}$$

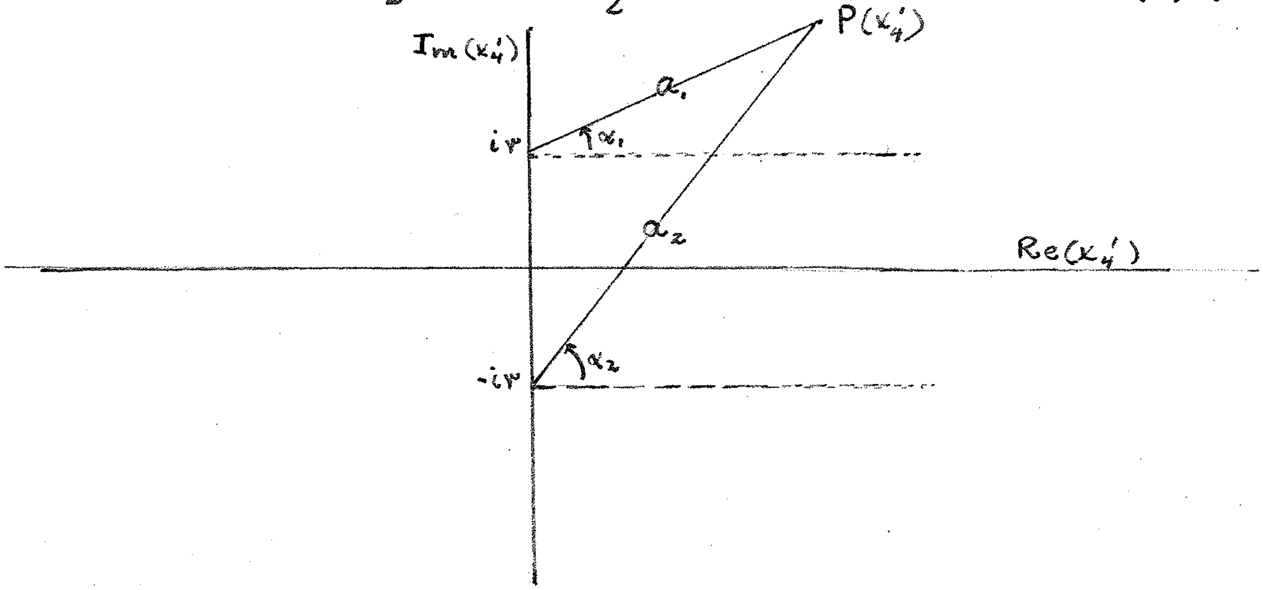
$$x_4' + ir = a_2 e^{i\alpha_2} \tag{18b}$$

(See Fig. 1. below)

and then limiting the ranges of α_1 and α_2 in the fashion

$$-\frac{3\pi}{2} < \alpha_1 < \frac{\pi}{2} \tag{19a}$$

$$-\frac{\pi}{2} < \alpha_2 < \frac{3\pi}{2} \tag{19b}$$



<u>Figure 1</u>.

We have then:

$$R = \sqrt{a_1 a_2}\; e^{\frac{1}{2} i(\alpha_1 + \alpha_2)} \tag{20}$$

and it is clear R is discontinuous across the branch lines we have introduced.

It is important to note that in the negative imaginary half of the x_4' -plane $\frac{\alpha_1 + \alpha_2}{2}$ is limited to the range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. It follows that all integrals involved are convergent and we may transform the contour of integration from along the real axis to one along the path indicated in Figure 2. below:

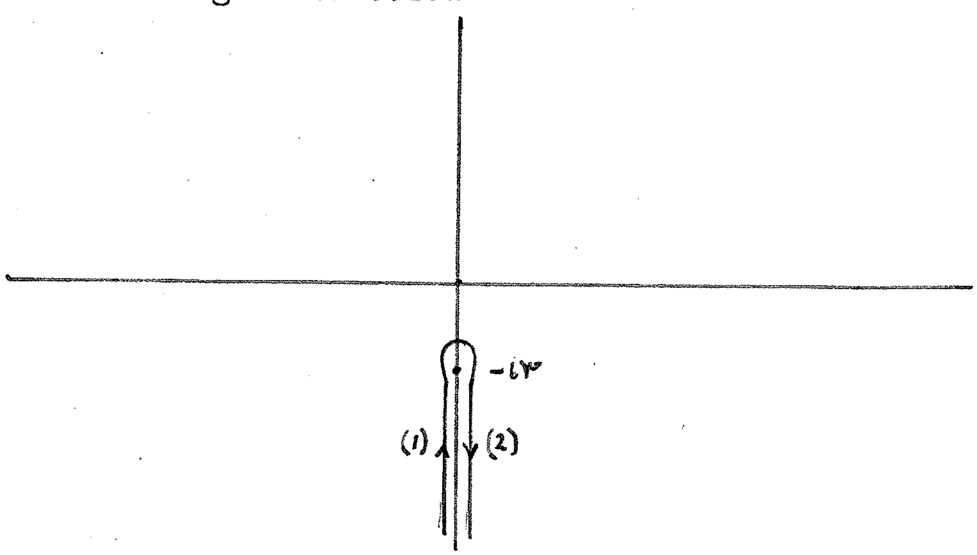


Figure 2.

We might then write

$$I = I_1 + I_2 + P_1 \tag{21}$$

where I_1 and I_2 are integrals taken along the paths (1), (2) in Figure 2., while P_1 is a contribution at the pole $x_4' = -ir$

Since the details of the calculation are not of any interest, we simply quote the results,

$$I_1 + I_2 = -\frac{\pi}{\kappa r}\left(1 - e^{-\kappa r}\right) \tag{22a}$$

$$P_1 = \frac{\pi}{\kappa r} \tag{22b}$$

There results the familiar static meson field

$$\phi(\vec{r}) = -g\,\tau\,\frac{e^{-\kappa r}}{4\pi r} \tag{23}$$

It is evident that the field of an arbitrarily moving nucleon might be obtained, in theory, by an analagous integration[1].

---

1. The field of a uniformly moving nucleon could be evaluated by a Lorentz transformation between the rest frame of the nucleon and the lab frame. One simply recalls that the meson field ϕ is a 4-scalar, and obtains

$$\phi = -\frac{g\tau}{4\pi}\,\frac{e^{-\kappa\sqrt{x_1^2 + x_2^2 + (x_3 + vt)^2 \frac{1}{\beta^2}}}}{\sqrt{x_1^2 + x_2^2 + (x_3 + vt)^2 \frac{1}{\beta^2}}}$$

for the field of a nucleon moving with speed v in the x_3 -direction.

One would obtain the field in such a case by an integration over the past motion of the nucleon up to the time $\bar{t}$, which is a solution of

$$\bar{t} = \frac{|\vec{r} - \vec{r}_n(\bar{t})|}{c} \tag{24}$$

In electromagnetic theory the field of an arbitrarily moving charge involves only the position of the charge at the retarded time (24). It is possible to consider only a single retarded time because all photons travel with a unique speed c. The field particles of meson theory, however, may possess any speed less than c. Hence the field, at the space time point ($\vec{r}$, $t=0$), of a moving nucleon is due to the mesons emitted by the nucleon throughout its past history previous to the retarded time (24).

4.2 (a) Nuclear Forces.

We have completed a formalism sufficient for handling many interesting problems in meson physics. Perhaps the first we should wish to tackle is the question of nuclear interactions. On the basis of our relativistic formulation it would appear that a complete treatment of internucleon forces must include velocity dependent forces. However, we shall consider only the static nuclear interactions.

We will discuss the interaction between a pair of nucleons n_1, n_2 separated by a distance r_{12}. It will be sufficient for our purposes to compute the potential energy of such a nucleon pair. Our static approximation amounts to taking the τ -spins $\underline{\tau}^{(1)}$, $\underline{\tau}^{(2)}$ of the nucleons to be independent of time.

The potential energy V_{12} is arrived at by picturing the nucleon n_1, say, to be immersed in the field of the nucleon n_2. On the basis of a scalar theory the meson field arising from n_2 is deduced by solving the equation

$$(\nabla^2 - \kappa^2)\underline{\phi} = g\,\underline{\tau}^{(2)}\,\delta(\vec{r}) \tag{1a}$$

i.e.,

$$\underline{\phi} = -\frac{g}{4\pi}\,\underline{\tau}^{(2)}\,\frac{e^{-\kappa r}}{r} \tag{1b}$$

We have placed n_2 at the origin. The energy of a distribution of nuclear matter, with τ -spin density $g\,\underline{\tau}\,\rho(\vec{r})$, immersed in a field $\underline{\phi}(\vec{r})$ is, from 3.2(20a):

$$V = \int \left(g\,\underline{\tau}\,\rho(\vec{r}),\ \underline{\phi}\right) dv \tag{2}$$

Hence the interaction energy between the pair n_1, n_2 in a symmetric scalar theory is

$$V_{12} = \int \left(g\,\underline{\tau}^{(1)}\,\delta(\vec{r}-\vec{r}_1),\ \underline{\phi}\right) dv \tag{3a}$$

$$V_{12} = -\frac{g^2}{4\pi}\left(\underline{\tau}^{(1)}, \underline{\tau}^{(2)}\right)\frac{e^{-\kappa r_{12}}}{r_{12}} \tag{3b}$$

Let us consider next the pseudo-scalar theory. The static meson field due to a nucleon n_2 with ordinary spin $\vec{\sigma}^{(2)}$ is determined from 3.5(2):

$$(\nabla^2-\kappa^2)\,\underline{\phi} = -\frac{f}{\kappa}\,\underline{\tau}^{(2)}\,\vec{\sigma}^{(2)}\cdot\nabla\,\delta(\vec{r}) \tag{4}$$

and is found by a Green's method to be

$$\underline{\phi} = \frac{f\,\underline{\tau}^{(2)}}{4\pi\kappa}\,\vec{\sigma}^{(2)}\cdot\nabla\frac{e^{-\kappa r}}{r} \tag{5}$$

From 3.5(7) we have that a nucleon with τ-spin $\underline{\tau}$, will, in a pseudo-scalar field $\underline{\phi}$, possess energy

$$V = \frac{f}{\kappa}\,(\underline{\tau},\vec{\sigma}\cdot\nabla\underline{\phi}) \tag{6}$$

Hence the potential energy of a nucleon pair n_1, n_2 is, on the basis of the symmetric pseudo-scalar theory,

$$V_{12} = \frac{f^2}{4\pi\kappa^2}\,(\underline{\tau}^{(1)},\underline{\tau}^{(2)})(\vec{\sigma}^{(1)}\cdot\nabla)(\vec{\sigma}^{(2)}\cdot\nabla)\frac{e^{-\kappa r_{12}}}{r_{12}} \tag{7a}$$

Expansion of the operator $(\vec{\sigma}^{(1)}\cdot\nabla)(\vec{\sigma}^{(2)}\cdot\nabla)$ yields

$$V_{12} = \frac{f^2}{4\pi\kappa^2}(\underline{\tau}^{(1)},\underline{\tau}^{(2)})\left[(\vec{\sigma}^{(1)}\cdot\vec{\iota}_{r_{12}})(\vec{\sigma}^{(2)}\cdot\vec{\iota}_{r_{12}})\left(\frac{3}{r_{12}^3}+\frac{3\kappa}{r_{12}^2}+\frac{\kappa^2}{r_{12}}\right)-(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\left(\frac{1}{r_{12}^3}+\frac{\kappa}{r_{12}^2}\right)\right]e^{-\kappa r_{12}} \tag{7b}$$

Finally we consider the vector meson theory. The field equations for the vector field are listed in 3.5. The static equations for the field ($\vec{A}_\alpha, \phi_\alpha$) are obtained from those in 2.3 by dropping terms proportional to the velocity of the nucleons. We ~~had seen~~ saw reasonable choices for $\vec{P}_\alpha$ and $\vec{J}_\alpha$ indicated these terms were of first order in the nucleon velocity.

One has, then, for the field set up by n_2 :

$$(\nabla^2-\kappa^2)\vec{A}_\alpha = \nabla\times g_T\,\delta(\vec{r})\,\vec{\sigma}^{(2)}\tau_\alpha^{(2)} \tag{8a}$$

$$(\alpha = 1,2,3)$$

$$(\nabla^2-\kappa^2)\,\phi_\alpha = -g_V\,\delta(\vec{r})\,\tau_\alpha^{(2)} \tag{8b}$$

In analogy with electromagnetic theory, the ordinary spin is to be regarded as a quasi-magnetic dipole, and the nuclear density $g_V\,\delta(\vec{r})$ as a quasi-electric charge density. One would then take as the mutual energy of the nucleon pair n_1 , n_2 , on the basis of a symmetric vector theory

$$V_{12} = \sum_{\alpha=1}^{3}\left[g_V\,\tau_\alpha^{(1)}\phi_\alpha^{(12)} + g_T\,\tau_\alpha^{(1)}\vec{\sigma}^{(1)}\cdot\vec{F}_\alpha^{(12)}\right] \tag{9}$$

where $\vec{F}_\alpha^{(12)}$ is the quasi-magnetic field intensity at n_1 , due to n_2 and $\phi_\alpha^{(12)}$ the quasi-electric potential at n_1 due to n_2 .

Solving (8) and performing the indicated calculations one obtains

$$V_{12} = (\tau^{(1)}\cdot\tau^{(2)})\left[\frac{g_V^2}{4\pi}\frac{e^{-\kappa r_{12}}}{r_{12}} + \frac{g_T^2}{4\pi}\left(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\nabla^2\frac{e^{-\kappa r_{12}}}{r_{12}} - (\vec{\sigma}^{(1)}\cdot\nabla)(\vec{\sigma}^{(2)}\cdot\nabla)\frac{e^{-\kappa r_{12}}}{r_{12}}\right)\right] \tag{10}$$

(b): The Deuteron:

The potentials 4.2(3,7,10) provide us with a means for investigating the nature of a two-nucleon system. We are, in particular, interested in the possibility of bound states for such a system. We would like to see if meson theory can predict the existence of the deuteron and also account for the experimentally observed electric quadrupole moment of the deuteron ground state.

The symmetric scalar forces between a pair of nucleons are central - attractive for a like pair, repulsive for an unlike pair. Thus scalar meson theory allows for the occurrence in nature of a di-proton or di-neutron, while denying that of the deuteron. This is of course contrary to experience. A neutral scalar theory would lead to a potential differing from 4.2(3) by the factor ($\underline{\tau}^{(1)}, \underline{\tau}^{(2)}$) and hence would permit all three two-nucleon systems to appear in nature.

Even a neutral scalar theory, however, could not explain the finite electric quadrupole moment of the deuteron. If we suppose an electrical charge distribution $\rho_Q(\vec{r})$ to possess an axis of symmetry (the z -axis, say), then it is possible to speak of a single quadrupole moment for the system, which may be taken as

$$M_Q = \int \rho_Q \, (3z^2 - r^2) \, dv \tag{11}$$

Classically, the charge distribution of the deuteron is set up by the rotation of the positively charged proton relative to the neutron. Quantum mechanically one may think of the proton as being smeared throughout space, thus creating a diffuse charge cloud. Since the scalar forces are isotropic the quantum mechanical charge distribution associated with the ground state of the deuteron is spherically symmetric. This would imply $M_Q = 0$ on the basis of a scalar theory.

In vector and pseudo-scalar theories the ordinary spin vector defines a preferred direction in space for each nucleon. The fashion in which the ordinary spin vectors of the neutron and proton line up, when these particles form a deuteron, will determine the existence or non-existence of a deuteron quadrupole moment. We ~~shall~~ assume the ordinary spins of the neutron and proton shall combine so as to yield a minimum potential energy in the deuteron ground state.

Let us choose the line joining the neutron and proton as polar axis for a set of spherical co-ordinates, specifying the spin orientations for the two nucleons in the fashion:

$$\vec{\sigma}^{(p)} \equiv (\sin\theta_p \cos\phi_p, \sin\theta_p \sin\phi_p, \cos\theta_p) \qquad (12a)$$

$$\vec{\sigma}^{(n)} \equiv (\sin\theta_n \cos\phi_n, \sin\theta_n \sin\phi_n, \cos\theta_n) \qquad (12b)$$

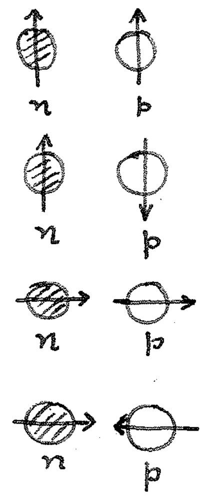
Then for the potential energy of the deuteron in pseudo-scalar theory we have from 4.2(7):

$$V_{12} = -\frac{f^2}{4\pi\kappa^2}\Big[\Big(\frac{3}{r_{12}^3} + \frac{3\kappa}{r_{12}^2} + \frac{\kappa^2}{r_{12}}\Big)e^{-\kappa r_{12}}\cos\theta_p\cos\theta_n \qquad (13)$$

$$- e^{-\kappa r_{12}}\Big(\frac{1}{r_{12}^3} + \frac{\kappa}{r_{12}^2}\Big)\big(\cos\theta_p\cos\theta_n + \sin\theta_p\sin\theta_n\cos(\phi_p - \phi_n)\big)\Big]$$

It is easily shown that the orientations which lead to stationary values of V_{12} are those pictured below:

$$\begin{aligned}
&(a) \quad \theta_p = \theta_n = \frac{\pi}{2} \\
&(b) \quad \theta_n = +\frac{\pi}{2}, \ \theta_p = -\frac{\pi}{2} \\
&(c) \quad \theta_n = \theta_p = 0 \\
&(d) \quad \theta_n = 0, \ \theta_p = \pi
\end{aligned} \tag{14}$$



One may compute the potential energies associated with each of these cases

$$\begin{aligned}
&(a) \quad V_{12} = \frac{f^2}{4\pi K^2} e^{-K r_{12}} \left(\frac{1}{r_{12}^3} + \frac{K}{r_{12}^2} \right) \\
&(b) \quad V_{12} = -\frac{f^2}{4\pi K^2} e^{-K r_{12}} \left(\frac{1}{r_{12}^3} + \frac{K}{r_{12}^2} \right) \\
&(c) \quad V_{12} = -\frac{f^2}{4\pi K^2} e^{-K r_{12}} \left(\frac{2}{r_{12}^3} + \frac{2K}{r_{12}^2} + \frac{K^2}{r_{12}} \right) \\
&(d) \quad V_{12} = \frac{f^2}{4\pi K^2} e^{-K r_{12}} \left(\frac{2}{r_{12}^3} + \frac{2K}{r_{12}^2} + \frac{K^2}{r_{12}} \right)
\end{aligned} \tag{15}$$

It would appear that the pseudo-scalar deuteron is described by a spin-orientation (c), suggesting the charge distribution for the deuteron is elongated in the direction of its spin axis. This implies:

$$\int z^2 \rho_Q \, dv > \int x^2 \rho_Q \, dv$$

and

$$\int z^2 \rho_Q \, dv > \int y^2 \rho_Q \, dv$$

hence $M_Q > 0$ for the pseudo-scalar deuteron.

Furthermore one might say the deuteron (c) possesses a spin angular momentum $\frac{1}{2}\hbar + \frac{1}{2}\hbar = \hbar$, the spin vectors of the neutron and proton being parallel in such a case. Experimentally one finds the deuteron possesses a spin angular momentum $\hbar$ and a positive electric quadrupole moment. Qualitatively, then, pseudo-scalar theory agrees with experiment.

One may separate the vector deuteron potential energy into an isotropic part

$$V_{12}^{s} = -\frac{g_v^2}{4\pi} \frac{e^{-\kappa r_{12}}}{r_{12}} \tag{16a}$$

and a spin dependent part

$$V_{12}^{a} = -\frac{g_T^2}{4\pi}\left[\vec{\sigma}^{(n)}\cdot\vec{\sigma}^{(p)}\kappa^2 \frac{e^{-\kappa r_{12}}}{r_{12}} - (\vec{\sigma}^{(n)}\cdot\nabla)(\vec{\sigma}^{(p)}\cdot\nabla)\frac{e^{-\kappa r_{12}}}{r_{12}}\right] \tag{16b}$$

Only V_{12}^{a} can give rise to a quadrupole moment. We note V_{12}^{a} differs from the pseudo-scalar potential energy only in sign and in the addition of the term

$$-\frac{g_T^2}{4\pi}\, \vec{\sigma}^{(n)}\cdot\vec{\sigma}^{(p)}\, \kappa^2 \frac{e^{-\kappa r_{12}}}{r_{12}} .$$

Clearly this term cannot affect the choices of (a), (b), (c), (d) as orientations yielding stationary values for V_{12}^a.

It follows that V_{12}^a assumes a minimum value for one of the orientations (a) or (d). A deuteron ground state described by (a) would be associated with a negative quadrupole moment and a spin angular momentum $\hbar$, while (d) leads to a positive quadrupole moment and a spin angular momentum zero. Thus the ~~vector~~ deuteron will not at all resemble the natural particle it is supposed to describe.

4.3 Meson Production in Nucleon-Nucleon collisions.

A collision between nucleons is brought about by using one nucleon as a projectile and firing it at a stationary nucleon which acts as a target. If the energy of the incident nucleon is sufficiently high one or more mesons may be produced in the collision[1]. We shall show how the scalar and pseudo-scalar formulations might predict such a reaction.

First let us consider a scalar collision of a nucleon pair n_1, n_2. If one assumes that the time during which the two nucleons are within the range of nuclear forces is small, then it should be possible to

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1. See Appendix A for a discussion of energy thresholds in meson production.

describe the collision in terms of the interaction of the nucleons at a single, representative, instant of time. We shall in fact suppose the nucleons interact only at the instant of closest approach, taking the interaction at this instant to be static. If the bremsstrahlung radiation, arising from the acceleration suffered by the nucleons, is ignored, then only an induced motion of the nucleon τ -spins can give rise to meson production.

The equations of motion for the τ -spins are from 3.3(11),

$$\dot{\underline{\tau}}^{(1)} = \frac{2g}{\hbar}\left[\underline{\phi}_{12}, \underline{\tau}^{(1)}\right] \tag{1a}$$

$$\dot{\underline{\tau}}^{(2)} = \frac{2g}{\hbar}\left[\underline{\phi}_{21}, \underline{\tau}^{(2)}\right] \tag{1b}$$

where $\underline{\phi}_{12}$ is the field at n_1 due to n_2 when the two nucleons are at the minimum separation r_{12} ; $\underline{\phi}_{21}$ is a similar field at n_2 due to n_1 .

We shall in fact have on the basis of a static approximation:

$$\underline{\phi}_{12} = \underline{\tau}_0^{(2)}\left(\frac{-g}{4\pi}\frac{e^{-\kappa r_{12}}}{r_{12}}\right) \tag{2a}$$

$$\underline{\phi}_{21} = \underline{\tau}_0^{(1)}\left(\frac{-g}{4\pi}\frac{e^{-\kappa r_{12}}}{r_{12}}\right) \tag{2b}$$

Previously we have mentioned that difficulties would be encountered if we insisted on labelling the proton and neutron states for the nucleon with $\tau_3 = \pm 1$. In the present problem, if at the start of the collision ($t=0$) we take

$$\underline{\tau}_0^{(1)} = \tau_0^{(1)} \underline{e}_3$$
$$\underline{\tau}_0^{(2)} = \tau_0^{(2)} \underline{e}_3$$

$\underline{e}_3$ being a unit vector in the 3-direction in charge space, then we would have

$$\underline{\Phi}_{12} \parallel \underline{\tau}_0^{(1)}$$

and

$$\underline{\Phi}_{21} \parallel \underline{\tau}_0^{(2)}$$

This would lead to $\dot{\underline{\tau}}^{(1)} = \dot{\underline{\tau}}^{(2)} = 0$ initially, and the $\dot{\imath}$ -spins would remain directed along the 3-axis in charge space. If, however, the $\dot{\imath}$ -spins are allowed to possess small 1, 2 components, there will result a precession of the $\dot{\imath}$ -spins in charge space.

Consider first the motion of $\underline{\tau}^{(1)}$. We may set

$$\Omega = \frac{2g^2}{4\pi} \frac{e^{-\kappa r_{12}}}{r_{12}} \tag{3}$$

and have

$$\dot{\underline{\tau}}^{(1)} = [\underline{\tau}^{(1)}, \underline{\tau}_0^{(2)} \Omega] \tag{4}$$

The motion of $\underline{\tau}^{(1)}$ is clearly a precession, described by an angular velocity $\underline{\Omega} = -\Omega \underline{\tau}_0^{(2)}$. An explicit solution of (4) may be obtained by setting

$$\underline{\tau}^{(1)} = \underline{\alpha}^{(1)} \cos \omega t + \underline{\beta}^{(1)} \sin \omega t + \underline{\gamma}^{(1)} \tag{5}$$

where $\underline{\alpha}^{(1)}$, $\underline{\beta}^{(1)}$, $\underline{\gamma}^{(1)}$ are constant vectors in charge space.

Substitution into (4) then yields:

$$\omega \underline{\beta}^{(1)} = [\underline{\Omega}, \underline{\alpha}^{(1)}] \tag{5a}$$

$$-\omega \underline{\alpha}^{(1)} = [\underline{\Omega}, \underline{\beta}^{(1)}] \tag{5b}$$

$$[\underline{\Omega}, \underline{\gamma}^{(1)}] = 0 \tag{5c}$$

Whence it follows

$$(-\omega^2)\,\underline{\alpha}^{(1)} = [\underline{\Omega}, [\underline{\Omega}, \underline{\alpha}^{(1)}]] \tag{6}$$

The latter equation 4.3(6) constitutes a set of linear homogeneous equations in the three components of $\underline{\alpha}^{(1)}$. For there to exist a non-null solution the determinant of the equations must vanish. This yields $\omega = \pm\Omega, 0$, and we can take $\omega = \Omega$ (We reject $\omega = 0$, and $\omega = -\Omega$ simply changes the sign of $\underline{\beta}^{(1)}$.)

The vectors $\underline{\alpha}^{(1)}$, $\underline{\beta}^{(1)}$, $\underline{\gamma}^{(1)}$ may be deduced through use of the initial conditions and the fact that $\underline{\tau}^{(1)}$ is of invariant magnitude equal to unity. The results are

$$\underline{\alpha}^{(1)} = [\underline{\tau}_0^{(2)}, [\underline{\tau}_0^{(1)}, \underline{\tau}_0^{(2)}]] \tag{7a}$$

$$\underline{\beta}^{(1)} = [\underline{\tau}_0^{(1)}, \underline{\tau}_0^{(2)}] \tag{7b}$$

$$\underline{\gamma}^{(1)} = (\underline{\tau}_0^{(1)}, \underline{\tau}_0^{(2)})\,\underline{\tau}_0^{(2)} \tag{7c}$$

It is then possible to compute the partial meson field due to n_1. One seeks the solution of the meson field equation 3.2(13) corresponding to waves travelling outwards from the centre of the collision. (For definiteness we take an origin at the position of n_1 .)

One finds

$$\underline{\phi}^{(1)} = \frac{-g}{4\pi}\,\underline{\alpha}^{(1)}\frac{\cos(\omega t-\mu r)}{r} - \frac{g}{4\pi}\,\underline{\beta}^{(1)}\frac{\sin(\omega t-\mu r)}{r} \tag{8}$$

where $\mu^2 = \omega^2/c^2 - \kappa^2$.

The static field arising from $\underline{\gamma}^{(1)}$ has been discarded.

In a similar fashion the partial field due to the motion of $\underline{\tau}^{(2)}$ is found to be

$$\underline{\phi}^{(2)} = -\frac{g}{4\pi}\left[\underline{\alpha}^{(2)}\frac{\cos(\omega t-\mu[r-r_{12}\cos\theta])}{r-r_{12}\cos\theta} + \underline{\beta}^{(2)}\frac{\sin(\omega t-\mu[r-r_{12}\cos\theta])}{r-r_{12}\cos\theta}\right] \tag{9}$$

$\underline{\alpha}^{(2)}$, $\underline{\beta}^{(2)}$ are obtained from $\underline{\alpha}^{(1)}$, $\underline{\beta}^{(1)}$ by an interchange of the superscripts (1), (2). We have set down the asymptotic field $\underline{\phi}^{(2)}$, for $r_{12} \ll$ the distance from the collision centre to the point at which the meson field is evaluated. The angle between the vectors $\vec{r}_{12}$, $\vec{r}$ is taken to be θ .

The total meson field is simply

$$\underline{\phi} = \underline{\phi}^{(1)} + \underline{\phi}^{(2)} \tag{10}$$

From Φ we may compute the energy and charge current densities in the far field. When averaged over a period of precession of $\tau^{(1)}$, $\tau^{(2)}$ these densities are

$$\vec{S}_{,t} = \left(\frac{g}{4\pi}\right)^2 \omega\mu \frac{\vec{l_r}}{r^2}\left[1-(\tau_0^{(1)},\tau_0^{(2)})^2\right]\left[2-(1+(\tau_0^{(1)},\tau_0^{(2)}))\cos(\mu r_{12}\cos\theta)\right] \quad (11)$$

$$\vec{I}_{,t} = \frac{\epsilon}{\hbar}\left(\frac{g}{4\pi}\right)^2 \mu \frac{\vec{l_r}}{r^2}\left[1-(\tau_0^{(1)},\tau_0^{(2)})^2\right]\Big[\left(\tau_{30}^{(1)}+\tau_{30}^{(2)}\left(\cos(\mu r_{12}\cos\theta)-1\right)\right) + \left(\tau_{10}^{(2)}\tau_{20}^{(1)}-\tau_{20}^{(2)}\tau_{10}^{(1)}\right)\sin(\mu r_{12}\cos\theta)\Big] \quad (12)$$

Integrating over all radiation directions $\vec{l_r}$, one obtains for the energy and charge radiated per unit time by the 2-nucleon system, just after collision:

$$E = \frac{g^2\omega\mu}{4\pi}\left[1-(\tau_0^{(1)},\tau_0^{(2)})^2\right]\left[2-(1+(\tau_0^{(1)},\tau_0^{(2)}))\frac{\sin\mu r_{12}}{\mu r_{12}}\right] \quad (13a)$$

$$q = \frac{g^2\epsilon\mu}{4\pi\hbar}\left[1-(\tau_0^{(1)},\tau_0^{(2)})^2\right]\left[(\tau_{30}^{(1)}+\tau_{30}^{(2)})\left(\frac{\sin\mu r_{12}}{\mu r_{12}}-1\right)\right] \quad (13b)$$

An examination of (5) and (7) leads us to conclude that $\tau_3^{(1)}$, $\tau_3^{(2)}$ oscillate with small amplitude about their initial values $\tau_{30}^{(1)}$, $\tau_{30}^{(2)}$. Thus it would seem that the system is not losing any charge, in contradiction to the result in (13). Actually the fashion in which (13) are deduced is only approximate. One would, in a second approximation, introduce the reaction of the meson field on the nucleon system - this reaction compensating for the energy and charge loss.

We note that the interaction energy of the two nucleon system is given at any instant by

$$g(\underline{\tau}^{(1)}, \underline{\Phi}_{12}) = g(\underline{\tau}^{(2)}, \underline{\Phi}_{21}) \tag{14}$$

Hence the rate at which the system loses energy due to the precession of the $\dot{z}$ -spin vectors is

$$W = -g(\dot{\underline{\tau}}_R^{(1)}, \underline{\Phi}_{12}) = -g(\dot{\underline{\tau}}_R^{(2)}, \underline{\Phi}_{21}) \tag{15}$$

where $\dot{\underline{\tau}}_R^{(1)}$ and $\dot{\underline{\tau}}_R^{(2)}$ are the reaction changes produced in $\underline{\tau}^{(1)}$ and $\underline{\tau}^{(2)}$. We would obtain energy conservation by setting

$$W = E \tag{16}$$

and get then

$$(\dot{\underline{\tau}}_R^{(2)}, \underline{\tau}_0^{(1)}) = (\dot{\underline{\tau}}_R^{(1)}, \underline{\tau}_0^{(2)}) = \frac{E}{\left(\frac{g^2 e^{-\kappa r_{12}}}{4\pi r_{12}}\right)} \tag{17}$$

To a good approximation (we are assuming $|\tau_{30}^{(1)}|$ and $|\tau_{30}^{(2)}|$ are almost unity) one finds

$$\dot{\tau}_{R3}^{(2)} = \frac{E}{\tau_{30}^{(1)}\left(\frac{g^2 e^{-\kappa r_{12}}}{4\pi r_{12}}\right)} \tag{18a}$$

$$\dot{\tau}_{R3}^{(1)} = \frac{E}{\tau_{30}^{(2)}\left(\frac{g^2 e^{-\kappa r_{12}}}{4\pi r_{12}}\right)} \tag{18b}$$

In a collision of two protons both $\dot{\tau}_{R3}^{(1)}$ and $\dot{\tau}_{R3}^{(2)}$ would be positive. Hence the i -spins $\tau^{(1)}$ and $\tau^{(2)}$ would tend toward states in which $\tau_3^{(1)} = \tau_3^{(2)} = 1$. We expect, in this case, that the initial motion produced in $\tau^{(1)}, \tau^{(2)}$ would be damped out - no radiation appearing as a result of the collision. Similarly, one would not observe any meson production in a scalar neutron-neutron collision.

However, in a neutron-proton collision, with n_1 a neutron and n_2 a proton we find

$$\dot{\tau}_{R3}^{(2)} < 0 \qquad , \dot{\tau}_{R3}^{(1)} > 0$$

This would result in a decrease in $\tau_3^{(2)}$ from its initial value $\tau_{30}^{(2)} \gtrsim 1$ and an increase in $\tau_3^{(1)}$ from its initial value $\tau_{30}^{(1)} \gtrsim -1$. Thus in a collision of unlike nucleons the perturbation in $\tau_3^{(1)}, \tau_3^{(2)}$ would grow - with perhaps the i -spin vectors $\tau^{(1)}$, $\tau^{(2)}$ 'flipping over.' Mesons might then be produced in a scalar n-p collision.

Before leaving the subject of scalar nucleon collision we should like to make note of a point which we have ignored. For the fields $\phi^{(1)}$, $\phi^{(2)}$ to represent actual travelling waves one requires

$$\mu^2 \geqq 0$$
$$\text{i.e. } \omega \geqq Kc \tag{19}$$

This condition yields an estimate of the cross-section for the reaction.

For (19) implies

$$\frac{2g^2}{4\pi\hbar}\ \frac{e^{-\kappa r_{12}}}{r_{12}} \geqq \kappa c \tag{20}$$

$$\text{i.e.}\quad r_{12}\, e^{\kappa r_{12}} \leqq \left(\frac{g^2}{4\pi\hbar c}\right)\frac{2}{\kappa}$$

A rough approximation to the solution of the inequality (20) is

$$r_{12} \leqq \left(\frac{g^2}{4\pi\hbar c}\right)\frac{2}{\kappa} \tag{21}$$

One sees thatthe two nucleons must approach within a distance

$$d = \left(\frac{g^2}{4\pi\hbar c}\right)\frac{2}{\kappa}$$

before any meson production can take place. The cross section for the production of mesons in a collision of unlike nucleons is then $\sim d^2$.

The analysis of a pseudo-scalar nucleon-nucleon collision follows an almost identical pattern to that outlined for the scalar theory. The equation of motion for the τ -spin vector of n_1 is now, from 3.5(6)

$$\dot{\underline{\tau}}^{(1)} = \frac{2f}{\hbar\kappa}\left[\vec{\sigma}^{(1)}\cdot\nabla\underline{\phi}_{12},\ \underline{\tau}^{(1)}\right] \tag{22a}$$

where

$$\underline{\phi}_{12} = \frac{f}{4\pi\kappa}\ \underline{\tau}_0^{(2)}\ \vec{\sigma}^{(2)}\cdot\nabla\frac{e^{-\kappa r_{12}}}{r_{12}} \tag{22b}$$

A formal identification with scalar theory is obtained by setting

$$\Omega = \frac{2f^2}{4\pi\hbar\kappa^2}(\vec{\sigma}^{(1)}\cdot\nabla)(\vec{\sigma}^{(2)}\cdot\nabla)\frac{e^{-\kappa r_{12}}}{r_{12}} \tag{23a}$$

and writing

$$\dot{\underline{\tau}}^{(1)} = [\Omega\, \underline{\tau}_0^{(2)}, \underline{\tau}^{(1)}] \tag{23b}$$

We will not analyze the $\dot{z}$ -spin motion completely; but shall simply compute the reaction force exerted on the $\dot{z}$ -spin motion and decide in this fashion whether mesons can be produced in a pseudo-scalar nucleon-nucleon collision.

The initial-spin motion produced by the collision of nucleons n_1 , n_2 will result in a rate of radiation of energy which we denote as E . We must equate E to the initial rate at which the nucleon system supplies energy to the meson field. The interaction energy possessed by a nucleon pair in a pseudo-scalar field is from 2.2(10). (Altered so as to include $\dot{z}$ -spin.)

$$(\underline{\tau}^{(1)}, \underline{\tau}^{(2)})\frac{f^2}{4\pi\kappa^2}(\vec{\sigma}^{(1)}\cdot\nabla)(\vec{\sigma}^{(2)}\cdot\nabla)\frac{e^{-\kappa r_{12}}}{r_{12}} \tag{24}$$

We may then set

$$(\dot{\underline{\tau}}_R^{(1)}, \underline{\tau}_0^{(2)}) \times \frac{\Omega}{2} = (\dot{\underline{\tau}}_R^{(2)}, \underline{\tau}_0^{(1)})\frac{\Omega}{2} = -E \tag{25}$$

Assuming the greatest contribution to the reaction comes from the 3-components of $\dot{\underline{\tau}}_R^{(1)}$, $\dot{\underline{\tau}}_R^{(2)}$, we get

$$\dot{\tau}_{R3}^{(1)} = -\frac{2E}{\Omega\,\tau_{30}^{(2)}} \qquad (26a)$$

$$\dot{\tau}_{R3}^{(2)} = \frac{-2E}{\Omega\,\tau_{30}^{(1)}} \qquad (26b)$$

We cannot decide in which direction the z -spin will start to turn until we know the algebraic sign of Ω . Clearly this sign will vary with the spin orientations $\vec{\sigma}^{(1)}$, $\vec{\sigma}^{(2)}$.

Let us first consider the case $\Omega > 0$. Then we have

$$\dot{\tau}_{R3}^{(1)} \sim -\frac{1}{\tau_{30}^{(2)}} \qquad (27a)$$

$$\dot{\tau}_{R3}^{(2)} \sim -\frac{1}{\tau_{30}^{(1)}} \qquad (27b)$$

If now n_1 , n_2 are a proton pair, both $\dot{\tau}_{R3}^{(1)}$ and $\dot{\tau}_{R3}^{(2)}$ will be negative; the z -spins $\underline{\tau}^{(1)}$, $\underline{\tau}^{(2)}$ would tend to flip over and radiated mesons will be observed. Moreover, an analysis of the z -spin motion would show that at least initially the charge radiated per unit time is proportional to ($\tau_{30}^{(1)} + \tau_{30}^{(2)}$), i.e., is positive. Thus one would expect positive mesons to be produced in the proton-proton collision. Similarly, for $\Omega > 0$, the neutron-neutron collision results in the production of negative mesons.

However, if n_1 is a neutron and n_2 a proton, we see for $\Omega > 0$

$$\tau_{R3}^{(1)} < 0$$
$$\tau_{R3}^{(2)} > 0$$

Thus the i -spins would tend toward states in which $\tau_3^{(1)} = -1$, $\tau_3^{(2)} = +1$. No meson production is expected for this collision.

With $\Omega < 0$, the situation is reversed. The unlike nucleon collision may yield positive and negative mesons, the like collision cannot yield any mesons.

In all cases where meson production is possible one finds the rate of radiation of neutral mesons is initially negligible. This does not, necessarily imply neutral mesons will not appear in the collision.

One notes that the ordinary spin will also be set in motion by the static interaction (22b); from 3.6(3) we see the equation of motion for the ordinary spin of n, is

$$\frac{d\vec{\sigma}^{(1)}}{dt} = \frac{2f}{\hbar\kappa} \nabla(\underline{\tau}_0^{(1)}, \underline{\phi}_{12}) \times \vec{\sigma}^{(1)} \qquad (28)$$

We may set

$$\vec{\Omega} = \frac{2f}{\hbar\kappa} \nabla(\underline{\tau}_0^{(1)}, \underline{\phi}_{12}) \qquad (29)$$

and have

$$\frac{d\vec{\sigma}^{(1)}}{dt} = \vec{\Omega} \times \vec{\sigma}^{(1)} \qquad (30)$$

Then in analogy with the treatment of i -spin motion we obtain a solution of (29) in the fashion

$$\vec{\sigma}^{(1)} = \vec{\alpha}^{(1)}\cos\omega t + \vec{\beta}^{(1)}\sin\omega t + \vec{\gamma}^{(1)} \tag{31}$$

where $\omega = |\vec{\Omega}|$

and

$$\vec{\alpha}^{(1)} = \frac{1}{\omega^2}\,\vec{\Omega}\times(\vec{\sigma}_0^{(1)}\times\vec{\Omega}) \tag{32a}$$

$$\vec{\beta}^{(1)} = \frac{1}{\omega}\,\vec{\Omega}\times\vec{\sigma}_0^{(1)} \tag{32b}$$

$$\vec{\gamma}^{(1)} = \frac{1}{\omega^2}\vec{\Omega}\,(\vec{\sigma}_0^{(1)}\cdot\vec{\Omega}) \tag{32c}$$

Similarly

$$\vec{\sigma}^{(2)} = \vec{\alpha}^{(2)}\cos\omega t + \vec{\beta}^{(2)}\sin\omega t + \vec{\gamma}^{(2)} \tag{33}$$

where $\vec{\alpha}^{(2)}, \vec{\beta}^{(2)}, \vec{\gamma}^{(2)}$ are obtained from $\vec{\alpha}^{(1)}, \vec{\beta}^{(1)}, \vec{\gamma}^{(1)}$ in an obvious fashion. (33) and (34) will give rise to a radiation field

$$\Phi(\vec{r},t) = \Phi^{(1)} + \Phi^{(2)} \tag{34}$$

with

$$\Phi^{(1)} = \frac{f}{4\pi K}\,\Sigma_0^{(1)}\left[\vec{\beta}^{(1)}\cdot\nabla\frac{\sin(\omega t-\mu r)}{r} + \vec{\alpha}^{(1)}\cdot\nabla\frac{\cos(\omega t-\mu r)}{r}\right] \tag{35}$$

and

$$\Phi^{(2)} = \frac{f}{4\pi K}\,\Sigma_0^{(2)}\left[\vec{\beta}^{(2)}\cdot\nabla\frac{\sin(\omega t-\mu r')}{r'} + \vec{\alpha}^{(2)}\cdot\nabla\frac{\cos(\omega t-\mu r')}{r'}\right] \tag{36}$$

where r is the distance from n_1 to the field point $\vec{r}$ and r' is the distance from n_2 to $\vec{r}$.

Without computing the energy flux associated with 34, we see that this field will give rise to an appreciable radiated neutral meson field. Since the 1,2-components of $\underline{\Sigma}_0^{(1)}, \underline{\Sigma}_0^{(2)}$ are small, only a low intensity charged radiation will appear. Thus we can conclude in a pseudo-scalar collision of nucleons, neutral mesons are produced by the gyration of ordinary spin.

It is interesting to compare the results of our rather crude picture of nucleon-nucleon production of mesons with the corresponding quantum theory results. The following table[1] contains such a comparison:

Meson Field	Collision Particles	Mesons Produced: Quantum Theory	Mesons Produced: Classical Theory
Symmetric Scalar with scalar coupling	n-n	none	none
	p-p	none	none
	n-p	none	π^+, π^-
Symmetric pseudo-scalar with pseudo-vector coupling	n-n	π^-	π^-, π^0
	p-p	π^+	π^+, π^0
	n-p	π^+, π^-, π^0	π^+, π^-, π^0

1. This table and the following brief discussion on the experimental evidence for meson production are quoted from R. E. Marshak: 'Meson Physics' (see in particular Chap.2, Table 7). One should consult current literature since publication of this book for further experimental results.

Clearly, there is considerable disagreement between quantum mechanics and our classical theory on the question of meson production in nucleon-nucleon collisions. ~~One recalls the classical theory predicted meson production in the scalar n-p collision and neutral meson production in all three pseudo-scalar collisions.~~

Experimentally, only π^+ and π^0 production in p-p collisions have been examined in any detail. The results suggest there is π^+ production but no π^- or π^0 production. One would on the basis of this meagre evidence tend to favour the pseudo-scalar (pseudo-vector) quantum theory over the scalar theory. One finds even the pseudo-scalar (pseudo-vector) theory cannot give an accurate description of the dependence of the cross-section for meson production on the energy of the incident nucleon.

4.4 Scattering of Mesons by Nucleons.

A scattering experiment consists of the firing of a beam of projectiles at a stationary target. Interaction between the particles in the beam and those in the target results in a change in direction of the paths followed by the projectiles. To obtain a measure of the strength of this interaction one introduces a differential scattering cross-section. If one supposes the target to contain n particles, and the projectile beam to be described by a particle flux P_0 then the differential cross section, $\frac{d\sigma}{d\eta}$, is specified by the requirement that the number of particles scattered per unit time into an element of solid angle $d\eta$ in the direction (θ, ϕ) with respect to the direction of incident beam, be

$$dN = n P_0 \frac{d\sigma}{d\eta}(\theta, \phi)\, d\eta \tag{1}$$

From dN one could compute the scattered particle flux at a distance r from the scattering centre; this would be

$$P = \frac{1}{r^2} \frac{dN}{d\eta} \tag{2}$$

Thus

$$\frac{d\sigma}{d\eta} = \frac{r^2 P}{n P_0} \tag{3}$$

If the target contains a single particle

$$\frac{d\sigma}{d\eta} = \frac{r^2 P}{P_0} \tag{4}$$

In terms of energy fluxes S_0 and S one would have

$$\frac{d\sigma}{d\eta} = \frac{r^2 S}{S_0} \tag{5}$$

A total cross section for the scattering is computed in the fashion

$$\Sigma = \int \frac{d\sigma}{d\eta}\, d\eta \tag{6}$$

It is our present purpose to deduce values of the total and differential cross sections when the projectiles are mesons and the targets nucleons.

In general one would have to treat the scattering or recoil of the target particles as well as that of the projectiles. The analysis is considerably simplified if the target particle mass is much greater than that of the projectile. One then ignores any motion of the target. Our discussion of meson scattering by nucleons will make use of the latter simplification.

If one takes $\frac{1}{\kappa} = 2 \times 10^{-13}$ cm as a reasonable experimental value for the range of nuclear forces, then one finds the ratio of meson mass to nucleon mass is of the order of one-sixth. It is then possible to see to what extent the 'infinite target mass' approximation is justified for meson-nucleon scattering.

The scattering of scalar mesons is relatively straightforward. Since one ignores any possible motion of the nucleon target, the incident mesons can interact with the nucleon only through the latter's isotopic spin.

The total meson field during the scattering would have the form:

$$\underline{\Phi} = \underline{\Phi}^{(i)} + \underline{\Phi}^{(s)} \tag{7}$$

The field $\underline{\Phi}^{(i)}$ is associated with the incident beam of mesons and hence will be taken as a plane wave.

The term $\underline{\Phi}^{(s)}$ is a distortion in the meson field produced by the perturbation of the nucleon $\dot{\imath}$ -spin, hence

$\underline{\phi}^{(s)}$ represents the scattered mesons. At large distances from the nucleon we must take $\underline{\phi}^{(s)}$ to be an outward travelling, spherical wave.

The $\dot{z}$ -spin motion is determined from 3.3(11)

$$\dot{\underline{\tau}} = \frac{2g}{\hbar}\left[\underline{\phi}^{(i)}(0), \underline{\tau}\right] \tag{8}$$

where the incident field $\underline{\phi}^{(i)}(0)$ is evaluated at the position of the nucleon. A first approximation to the solution of (8) is obtained by setting

$$\underline{\tau} = \underline{\tau}_0 + \underline{\tau}^{(s)} \tag{9}$$

$\underline{\tau}_0$ being the initial $\dot{z}$ -spin of the nucleon and $\underline{\tau}^{(s)}$ the perturbation, resulting from the weak meson field $\underline{\phi}^{(i)}$.

One might then write

$$\dot{\underline{\tau}}^{(s)} \approx \frac{2g}{\hbar}\left[\underline{\phi}^{(i)}(0), \underline{\tau}_0\right] \tag{10}$$

and could solve explicitly for $\underline{\tau}^{(s)}$ in order to deduce the scattered field from

$$(\Box^2 - \kappa^2)\,\underline{\phi}^{(s)} = g\,\underline{\tau}^{(s)}\,\delta(\vec{r}) \tag{11}$$

Finally by computing the energy fluxes

$$\vec{S}_{(i)} = -\left(\dot{\underline{\phi}}^{(i)}, \nabla\underline{\phi}^{(i)}\right) \tag{12a}$$

$$\vec{S}_{(s)} = -\left(\dot{\underline{\phi}}^{(s)}, \nabla\underline{\phi}^{(s)}\right) \tag{12b}$$

one could arrive at a value for the differential cross-section

$$\frac{d\sigma}{d\eta} = r^2 \frac{\overrightarrow{S_{(s)}} \cdot \overrightarrow{e_r}}{|\overrightarrow{S_{(i)}}|} \tag{13}$$

In the present discussion a nucleon in the proton or neutron states is to possess the initial i -spin

$$\underline{\tau}_0 = \tau_0 \underline{i}_3 \quad , \quad \tau_0 = \pm 1 \tag{14}$$

It is then fairly evident that neutral scalar mesons are not scattered by either a neutron or a proton. For one would take for an incident beam of neutral mesons

$$\underline{\phi}^{(i)} = \underline{i}_3 \, a \cos(\omega t - \mu z) \; , \; \mu^2 = \omega^2/c^2 - \kappa^2 \tag{15}$$

then $\underline{\phi}^{(i)}$ is parallel to $\underline{\tau}_0$ and from (10) $\dot{\underline{\tau}}_s = 0$.

We obtain finite cross-sections, however, for the scattering of charged scalar mesons by the nucleon (14).

It is convenient to describe a beam of charged mesons by the field

$$\phi_+^{(i)} = \phi_1^{(i)} + i \, \phi_2^{(i)} \tag{16}$$

For a beam composed only of positive mesons one has

$$\phi_+^{(i)} = a \, e^{i\omega t - i\mu z} \tag{17}$$

One attributes to the field (17) an energy flux

$$\vec{S}_{(i)} = |a|^2 \omega\mu \vec{i_z} \tag{18}$$

The equation of motion (10) then leads to

$$\tau_1^{(s)} = \frac{2ag\tau_0}{\hbar\omega}(1-\cos\omega t) \tag{19a}$$

$$\tau_2^{(s)} = -\frac{2ag\tau_0}{\hbar\omega}\sin\omega t \tag{19b}$$

Since the constant part of $\tau_1^{(s)}$ cannot give rise to any scattered energy, we may compute the scattered field from

$$(\Box^2 - \kappa^2)\,\phi_+^{(s)} = g\,\tau_+^{(s)}\,\delta(\vec{r}) \tag{20a}$$

where

$$\tau_+^{(s)} = -\frac{2ag\tau_0}{\hbar\omega}e^{i\omega t} \tag{20b}$$

Since $\tau_+^{(s)} \sim e^{i\omega t}$ one concludes that only positive mesons are present in the scattered field.

One ~~would~~ gets

$$\phi_+^{(s)} = \frac{2g^2 a\tau_0}{4\pi\hbar\omega}\,\frac{e^{i\omega t - i\mu r}}{r} \tag{21}$$

with $\mu = \sqrt{\omega^2/c^2 - \kappa^2}$

and hence one ~~would~~ deduces

$$\vec{S}_{(s)} = \left(\frac{2g^2|a|\tau_0}{4\pi\hbar\omega}\right)^2 \frac{\omega\mu}{r^2}\,\vec{i_r} \tag{22}$$

Clearly the scattered energy would be the same whether the target is a neutron or a proton[1]. It follows that the differential cross-section for the scattering of scalar positive mesons by a neutron or proton is

$$\left(\frac{d\sigma}{d\eta}\right) = \frac{g^4}{4\pi^2\hbar^2\omega^2} \tag{23}$$

An identical result is obtained for the scattering of negative mesons by neutron or proton. The scattering of mesons in a scalar field is isotropic with respect to ordinary space, as one might expect. Finally the total cross-section is seen to be

$$\Sigma^{\pm} = \frac{g^4}{\pi\hbar^2\omega^2} \tag{24}$$

The dependence of the scalar scattering on the energy of the incident mesons is particularly simple - there being a fairly rapid decrease in cross-section for increase in energy.

The scattering of pseudo-scalar mesons is somewhat more complex. The incident field may produce a motion of both the isotopic and ordinary spins of the nucleon target. The $\dot{\tau}$ -spin motion is approximated by

$$\dot{\underline{\tau}}^{(5)} \approx \frac{2f}{\hbar K}\left[\vec{\sigma}_o\cdot\nabla\underline{\phi}^{(4)}(0)\, ,\ \tau_o\underline{\zeta}_3\right] \tag{25}$$

$\vec{\sigma}_0$ being the initial ordinary spin.

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1. We ignore a possible coulomb interaction between proton and π^+ or π^-.

Since $\dot{\underline{\tau}}^{(s)}$ does not possess a 3-component, the perturbed $\dot{\imath}$ -spin cannot give rise to neutral mesons and the $\dot{\imath}$ -spin will scatter only charged mesons.

We might just as well set

$$\phi_{+}^{(i)} = a e^{i\omega t - i\mu z} \quad , \quad \phi_3^{(i)} = 0 \tag{26}$$

and then deduce

$$\tau_{+}^{(s)} = \frac{i 2 f \tau_0 \mu a}{\hbar \omega K} (\vec{\sigma}_0 \cdot \vec{i}_z) e^{i\omega t} \tag{27}$$

The scattered field arising from (27) is

$$\phi_{+}^{(s)} = \frac{f}{4\pi K} \tau_{+}^{(s)} \vec{\sigma}_0 \cdot \nabla \frac{e^{-i\mu r}}{r} \tag{28}$$

There is no difficulty in arriving at the differential cross-sections

$$\left(\frac{d\sigma}{d\eta}\right)^{\pm} = \frac{f^4 \mu^4}{(2\pi\hbar\omega)^2 K^4} (\vec{\sigma}_0 \cdot \vec{i}_z)^2 (\vec{\sigma}_0 \cdot \vec{i}_r)^2 \tag{29}$$

Averaging over all possible initial spin directions one finds[1]

$$\left(\frac{d\sigma}{d\eta}\right)^{\pm}_{,\vec{\sigma}_0} = \frac{f^4 \mu^4}{4\pi^2 \hbar^2 \omega^2 K^4} \frac{1}{15} (1 + 4\cos^2\theta) \tag{30}$$

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1. The average of a function $T(\vec{\sigma}_0)$ for all possible orientations of $\vec{\sigma}_0$ is obtained in the fashion

$$(T)_{,\vec{\sigma}_0} = \frac{1}{4\pi} \int T(\vec{\sigma}_0)\, d\eta$$

where $d\eta$ is an element of solid angle. One evaluates $(T)_{,\vec{\sigma}_0}$ by specifying the orientation of $\vec{\sigma}_0$ with spherical polar angles relative to some polar axis. For averaging $\left(\frac{d\sigma}{d\eta}\right)^{\pm}$ in 4.3(29) one would take the polar axis in the direction of $\vec{i}_z$.

where θ is the angle between the direction $\vec{l_z}$ of the incident beam and the radiation direction $\vec{l_\mu}$.

The pseudo-scalar scattering (30) of charged mesons differs from the scalar result (23) in dependence on both energy and direction. Since $\mu^2 = \omega^2/c^2 - \kappa^2$, the pseudo-scalar theory predicts an increase in the scattering cross-section with increase in meson energy, for high meson energies. Moreover, the pseudo-scalar scattering is anisotropic, even though there exists a symmetry about the direction of the incident meson beam.

The total cross-section for charged pseudo-scalar mesons is

$$\Sigma^{\pm} = \left(\frac{1}{45}\right) \frac{f^4 \mu^4}{\pi \hbar^2 \omega^2 \kappa^4} \tag{31}$$

The ordinary spin motion is determined from 3.6(3),

$$\frac{d\vec{\sigma}^{(s)}}{dt} \approx \frac{2f}{\hbar \kappa} \nabla\left(\underline{\Phi}^{(\omega)}(0), \underline{\tau}_0 \right) \times \vec{\sigma}_0 \tag{32}$$

Clearly if we take $\underline{\tau}_0 = \tau_0 \underline{l}_3$ for the target nucleon, the ordinary spin will scatter only neutral mesons. The method for computing the cross-sections is by now quite familiar and we just quote the results:

$$\left(\frac{d\sigma}{d\eta}\right)^0 = \frac{1}{2} \frac{f^4 \mu^4}{(2\pi\hbar\omega)^2 \kappa^4} \left(\vec{l_z} \times \vec{\sigma_0} \cdot \vec{l_\mu}\right)^2 \tag{33}$$

Averaging over initial spin directions yields

$$\left(\frac{d\sigma}{d\eta}\right)^{0}_{,\vec{\sigma}_0} = \frac{1}{6}\,\frac{f^4\mu^4}{(2\pi\hbar\omega)^2 K^4}\,\sin^2\theta \tag{34}$$

The total cross-section for neutral pseudo-scalar mesons is

$$\Sigma^{0} = \frac{1}{9\pi}\cdot\frac{f^4\mu^4}{\hbar^2\omega^2 K^4} \tag{35}$$

One might compare the results (30)and (34). Whereas charged mesons are scattered most strongly in the forward or backward directions, the maximum scattering of neutral mesons occurs in a direction perpendicular to that of the incident meson beam. The magnitudes of the total cross-sections (31) and (35) are roughly comparable. We would like to point out that our classical treatment of meson scattering is by no means complete, and must be considered as a rough approximation to the true picture. For further discussion of classical meson scattering by nucleons one might refer to W. Pauli 'Meson Theory of Nuclear Forces' and W. W. Wada, 'Scattering of Charge Symmetric Pseudoscalar Mesons by Nucleons' Phys. Rev. 88, 5 (1952).

We wish to complete this discussion of meson scattering by setting down the quantum mechanical results and contrasting these with the corresponding classical results. The following table lists the quantum mechanical cross-sections for scalar theory with scalar coupling and pseudo-scalar theory with pseudo-vector coupling.

	Scattered Particle	Target Particle	Differential Cross Section	
Scalar (S)	π^+ or π^-	n or p	$\frac{g^4}{16\pi^2(\hbar\omega)^2}$	1
Pseudo-Scalar (PV)	π^+ or π^-	n or p	$\frac{f^4\mu^4}{4\pi^2 K^4\hbar^2\omega^2}$	2

As in classical theory, the preceding table is computed for nucleons of infinite mass and for mesons which are weakly coupled to their nucleon sources. The quantum mechanical and classical results differ in the scalar theory only by a multiplicative factor, but disagree in pseudo-scalar theory on directional dependence.

The one definite thing that can be said experimentally is that the cross-section for, say, π^+'s scattered on hydrogen increases fairly rapidly with the energy of the incident meson. It would seem, then, that scalar theory is incapable of explaining meson nucleon scattering. Marshak[3] is of the opinion that the experimental evidence for meson scattering favours the pseudo-scalar (PV) theory.

1. G. Wentzel 'Quantum Theory of Fields' p. 59, Eq. (9.18)
2. R. E. Marshak 'Meson Physics' p. 247, Eq. (3a',b'). Marshak uses a different set of units, which we have converted into our own. In addition to differences in choice for g, one notes Marshak has taken $\hbar = c = 1$.
3. R. E. Marshak 'Meson Physics' p. 249.

4.5 Photomesic Production.

An electromagnetic photon of sufficient energy when in collision with a nucleon is capable of producing mesons. To treat this problem classically we must allow the nucleon τ -spin to possess finite 1,2-components (see 3.4). Then the nucleon will be surrounded by a charged field which we imagine to contain equal numbers of positive and negative mesons. The electromagnetic field will interact with the charged meson field surrounding the nucleon; photons of energy $\hbar\omega > \kappa c$ may eject mesons from the charged meson field.

Before an electromagnetic field is introduced the ~~scalar~~ nucleon may be described by an τ -spin

$$\tau_+ = \tau_{+0} \tag{1a}$$

$$\tau_3 = \tau_{30} \tag{1b}$$

The charged field surrounding the nucleon satisfies, from 3.4(2)

$$(\Box^2 - \kappa^2)\,\phi_+^{(0)} = g\,\tau_{+0}\,\delta(\vec{r}) \tag{2}$$

It follows

$$\phi_+^{(0)} = -\frac{g\tau_{+0}}{4\pi}\,\frac{e^{-\kappa r}}{r} \tag{3}$$

The electromagnetic field incident on the nucleon is specified by a vector potential

$$\vec{A} = A\vec{a}\cos(\omega t - kz) \tag{4}$$

where $\vec{a}$ is a unit vector in the direction of polarization and $k = \omega/c$.

The wave equation for the charged meson field when the electromagnetic field is 'turned on' becomes

$$\left(\nabla + \frac{ie\vec{A}}{\hbar c}\right)^2 \phi_+ - \frac{1}{c^2}\ddot{\phi}_+ - \kappa^2 \phi_+ = g\tau_+ \delta(\vec{r}) \tag{5}$$

We attempt to write

$$\phi_+ = \phi_+^{(0)} + \phi_+^{(1)} \tag{6}$$

where the field $\phi_+^{(1)}$ is a small perturbation in the meson field produced by a low intensity photon beam. Substituting from 4.5(3) for $\phi_+^{(0)}$ and retaining only those terms which are of first order in the small quantities A, $\phi_+^{(1)}$, we get as an approximate field equation for $\phi_+^{(1)}$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2\right)\phi_+^{(1)} = \frac{2ieg\tau_{+0}}{4\pi\hbar c}\vec{A}\cdot\nabla\frac{e^{-\kappa r}}{r} \tag{7}$$

We set

$$\phi_+^{(1)} = \phi_a + \phi_b \tag{8}$$

where

$$(\Box^2 - \kappa^2)\phi_a = \frac{ieg\tau_{+0}A}{4\pi\hbar c}e^{i\omega t}e^{-ikz}\,\vec{a}\cdot\nabla\frac{e^{-\kappa r}}{r} \tag{9a}$$

and

$$(\Box^2-\kappa^2)\,\phi_b = \frac{i\epsilon g\tau_{+0}A}{4\pi\hbar c}\, e^{-i\omega t} e^{ikz}\,\vec{a}\cdot\nabla\frac{e^{-\kappa r}}{r} \tag{9b}$$

One seeks solutions for ϕ_a , ϕ_b which represent waves travelling outward from the nucleon. We get

$$\phi_a = \frac{-i\epsilon g\tau_{+0}A e^{i\omega t}}{(4\pi)^2\hbar c}\int e^{-ikz'}\frac{e^{-i\mu|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,\vec{a}\cdot\nabla'\frac{e^{-\kappa r'}}{r'}\,dv' \tag{10a}$$

$$\phi_b = \frac{-Ai\epsilon g\tau_{+0}e^{-i\omega t}}{(4\pi)^2\hbar c}\int e^{ikz'}\frac{e^{i\mu|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,\vec{a}\cdot\nabla'\frac{e^{-\kappa r'}}{r'}\,dv' \tag{10b}$$

with $\mu^2 = \frac{1}{c^2}\omega^2 - \kappa^2$.

It is clear ϕ_a , ϕ_b will be undamped travelling waves and hence will represent true radiation only if $\omega > \kappa c$. This would imply the photon energy must be at least as great as the rest energy of a meson if photomesic production is to occur.

To deduce the energy cross-section for the photo-production in scalar theory one must compute the asymptotic values of ϕ_a , ϕ_b for large r . These areobtained by expanding the Green's functions in the integrands of (10a), (10b) in the fashion

$$\frac{e^{-i\mu|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \xrightarrow[r\to\infty]{} \frac{e^{-i\mu(r-r'\cos\gamma)}}{r} \tag{11}$$

whence γ is the angle between the vectors $\vec{r}$, $\vec{r}'$. The subsequent integrations, although quite lengthy, are

are elementary and yield

$$\phi_a = \frac{g\epsilon}{8\pi\hbar c}\,\frac{\mu}{k^2}\,\frac{(\vec{a}\cdot\vec{i_r})A_{\tau 0}}{[1-\mu/_k \cos\theta]}\,\frac{e^{-i\mu r + i\omega t}}{r} \qquad (12a)$$

$$\phi_b = \frac{-g\epsilon}{8\pi\hbar c}\,\frac{\mu}{k^2}\,\frac{(\vec{a}\cdot\vec{i_r})A_{\tau 0}}{[1-\mu/_k \cos\theta]}\,\frac{e^{i\mu r - i\omega t}}{r} \qquad (12b)$$

where θ is the angle between the radiation direction $\vec{i_r}$, and the direction of the incident photon beam $\vec{i_z}$.

The total perturbation in the meson field is then

$$\phi_+^{(1)} = \phi_a + \phi_b \qquad (13)$$

As we have seen in 3.4 ϕ_a represents a wave of positive mesons and ϕ_b a wave of negative mesons. It is puzzling that both π^+'s and π^-'s should be produced by a collision of photons with the nucleon 4.5(1). If $\tau_{30} > 0$, the nucleon is presumably a proton and one would expect only positive mesons to be torn loose from the charged field surrounding the nucleon. On the other hand if $\tau_{30} < 0$, the nucleon is a neutron and one expects only negative mesons to be produced in the photon-nucleon collision.

One finds, in fact, that quantum theory permits only π^+'s to be produced in a photon-proton collision and only π^-'s in a photon-neutron collision, in agreement with experiment. Our classical theory, however, attributes to the nucleon 4.5(1) and its associated charged field 4.5(3) a zero charge density coupled with a non-zero energy density. Hence one concludes an arbitrary region of the field 4.5(1) contains equal numbers of π^+'s and π^-'s. It is not then surprising that the interaction of the electromagnetic field with the meson field 4.5(3) should produce both types of charged mesons.

The energy cross-section for the scalar photomesic reaction is evaluated by first computing the energy flux in the field $\phi_+^{(1)}$

$$\vec{S} = 2\left(\frac{g\epsilon}{8\pi\hbar c}\right)^2 \frac{\vec{i}_r}{r^2} \frac{\mu^3\omega}{k^4} \frac{(\vec{a}\cdot\vec{i}_r)^2 |A|^2 |\tau_{+0}|^2}{[1-\mu/k\cos\theta]^2} \tag{14}$$

and the energy flux in the electromagnetic field

$$\vec{S}_{E.M} = \vec{i}_z \frac{\omega^2 |A|^2}{8\pi c} \tag{15}$$

Whence the energy cross section is

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{g^2\epsilon^2\mu^3 |\tau_{+0}|^2}{8\pi\hbar^2 c\omega k^4} \frac{(\vec{a}\cdot\vec{i}_r)^2}{[1-\mu/k\cos\theta]^2} \tag{16}$$

Finally, averaging over all possible polarization states for the electromagnetic field, one gets

$$\left(\frac{d\sigma}{d\Omega}\right)_{\vec{a}} = \frac{1}{2} \frac{g^2\epsilon^2\mu^3}{4\pi\hbar^2 c\omega k^4} \frac{\sin^2\theta}{[1-\mu/k\cos\theta]^2} |\tau_{+0}|^2 \tag{17}$$

This cross-section would be obtained for a photon-neutron or photon-proton collision.

The pseudo-scalar photon-nucleon collision results in both neutral and charged mesons. For, in addition to the direct interaction between photons and the charged cloud surrounding the nucleon, one must consider the effect of the electromagnetic field on the ordinary spin of the nucleon. With this latter spin we have associated an intrinsic magnetic moment, capable of being set in motion by an impinging electromagnetic field.

We can take for the electromagnetic field incident on the nucleon, a magnetic intensity

$$\vec{B} = B\vec{i_B} \sin(\omega t - kz) \tag{18}$$

$\vec{i_B}$ being a unit vector.

The equation of motion for the ordinary spin is from 3.6(4)

$$\frac{d\vec{\sigma}}{dt} = \frac{2\bar{\mu}}{\hbar} \vec{B} \times \vec{\sigma} \tag{19}$$

where

$$\bar{\mu} = \frac{\mu_n \epsilon \hbar}{2MC}$$

Writing

$$\vec{\sigma} = \vec{\sigma_0} + \vec{\sigma_1} \tag{20}$$

where $\vec{\sigma_1}$ is a small perturbation produced by the electromagnetic field, we can approximate (19) by

$$\frac{d\vec{\sigma}_1}{dt} \approx \frac{2\bar{\mu}}{\hbar}\vec{B}(0)\times\vec{\sigma}_0 = \frac{2\bar{\mu}}{\hbar}B(\vec{\iota}_B\times\vec{\sigma}_0)\sin\omega t \qquad (21)$$

The magnetic intensity is evaluated at the position of the nucleon.

We would get

$$\vec{\sigma}_1 = \frac{2\bar{\mu}B}{\hbar\omega}(\vec{\sigma}_0\times\vec{\iota}_B)\cos\omega t \qquad (22)$$

The meson field radiated by (22) is determined from

$$(\Box^2-\kappa^2)\underline{\phi}^{(1)} = -\frac{f}{\kappa}\,\underline{\tau}_0\,\vec{\sigma}_1\cdot\nabla\delta(\vec{r}) \qquad (23)$$

and is

$$\underline{\phi}^{(1)} = \frac{2f\bar{\mu}B\underline{\tau}_0}{4\pi\hbar\kappa\omega}(\vec{\sigma}_0\times\vec{\iota}_B)\cdot\nabla\frac{\cos(\omega t-\mu r)}{r} \qquad (24)$$

If we take $\underline{\tau}_0 = \tau_0\,\underline{\zeta}_3$, then the ordinary spin will give rise only to neutral photo-mesons. The energy cross-section for the reaction will be

$$\left(\frac{d\sigma}{d\eta}\right)^0 = \frac{\omega\mu^3}{2}\left(\frac{2f\bar{\mu}B}{4\pi\hbar\kappa\omega}\right)^2\frac{(\vec{\sigma}_0\times\vec{\iota}_B\cdot\vec{\iota}_r)^2}{c/8\pi\,B^2_{\rightarrow}} \qquad (25)$$

Averaging over all initial spin directions $\vec{\sigma}_0$ and all polarization states $\vec{\iota}_B$ one gets

$$\left(\frac{d\sigma}{d\eta}\right)^0_{,\vec{\sigma}_0,\vec{\iota}_B} = \frac{1}{2}\left(\frac{\mu^3f^2e^2\mu_n^2}{4\pi M^2c^3\kappa^2\omega}\right)\frac{\sigma_0^2}{3}(1+\cos^2\theta) \qquad (26)$$

The reason for retaining σ_0^2 in the above expression (we generally set $\sigma_0^2=1$) will become apparent when we compare the result in (26) with the corresponding quantum mechanical calculation.

The pseudo-scalar theory for charged photomesic production is almost identical with the scalar computation. Hence we simply quote the result in terms of an energy cross-section

$$\left(\frac{d\sigma}{d\eta}\right)_{\vec{v}_0, \vec{u}} = \frac{\epsilon^2 f^2 \mu |\tau_{+0}|^2}{4\pi\hbar^2 c \kappa^2 \omega} \frac{\vec{v}_0^{\,2}}{3}\left[1 - \frac{\mu^2 \kappa^2 \sin^2\theta}{2k^4(1-\mu/k \cos\theta)^2}\right] \qquad (27)$$

Quantum mechanically one finds that only π^+'s can be produced in a photon-proton collision, only π^-'s in a photon-neutron collision. Moreover the cross-sections for these two reactions are equal. We have listed these cross-sections below, for scalar theory with scalar coupling and pseudo-scalar theory with pseudo-vector coupling[1].

(Scalar) $$\left(\frac{d\sigma^\pm}{d\eta}\right) = \frac{g^2 \epsilon^2 \mu^3}{4\pi\hbar^2 c \omega k^4} \frac{\sin^2\theta}{[1-\mu/k \cos\theta]^2} \qquad (29a)$$

(Pseudo-Scalar) $$\left(\frac{d\sigma^\pm}{d\eta}\right) = \frac{\epsilon^2 f^2 \mu}{4\pi\hbar^2 c \kappa^2 \omega}\left[1 - \frac{\mu^2 \kappa^2 \sin^2\theta}{2k^4(1-\mu/k \cos\theta)^2}\right] \qquad (29b)$$

In the pseudoscalar case one also finds in quantum theory that neutral mesons are produced in either the photon-proton or photon-neutron collisions. The cross-sections for the two collisions are similar and are given by[2]

1. R. E. Marshak 'Meson Physics' p. 13, eqns, (20), (21)
2. R. E. Marshak (Meson Physics' p. 18, eqn. (28).

$$\left(\frac{d\sigma^{00}}{d\eta}\right) = \frac{1}{2}\left(\frac{e^2 g^2 \mu_n^2 \mu^3}{4\pi \kappa^2 M^2 c^3 \omega}\right)(1+\cos^2\theta) \qquad (30)$$

Our results would agree with the quantum theory results if we ignored the factor $|\tau_{+0}|^2$, in (17), (26), and (27), while taking $\sigma_0^2 = 3$ in (26) and (27). In quantum mechanics $\vec{\sigma}_0$ is an operator possessing the constant square $\vec{\sigma}_0^{\,2} = 3$. In addition $|\tau_{+0}|^2$ could be looked on as a matrix element to be evaluated between the initial and final states of the reaction being considered.

Marshak has deduced the cross-sections (17), (26), (27) classically; his results agree with ours only in the case of the neutral meson production. For the charged cross-sections his values differ from ours by a factor of 2. This discrepancy arises because he has taken a complex representation for the electromagnetic field without accounting for this at a later stage. It would seem that he obtains (say in scalar theory) a perturbed meson field proportional to $\frac{e^{i\mu r - i\omega t}}{r}$, whereas he should have this field proportional to $\frac{\sin(\omega t - \mu r)}{r}$.

4.6 Comparison of Meson Theory with Experiment.

One might ask which of the four possible formulations - scalar, pseudo-scalar, vector, pseudo-vector - yields the best description of the π-meson. By 'best' we of course mean a description in closest agreement with nature. Hence we must appeal to physical experience before deciding in favour of one or other of the four theories.

One point which we have avoided in this work is the question of meson spin. The spin of the π -meson is, indeed, one of the crucial criteria to be used in determining the nature of the meson field. For one finds, in crude terms, that the higher is the numerical value assigned to the meson spin, the greater is the complexity of the meson field. Thus a spin 0 is associated with a scalar or pseudo-scalar formulation, while a spin 1 is associated with a vector or pseudo-vector formulation.

As R. E. Marshak[1] points out, experiments performed by Clark et al at Rochester[2] and Durbin et al at Columbia[3] establish beyond doubt that the spin of the charged π -meson is 0. Marshak states further that related experiments with the neutral meson suggest this meson possesses spin 0 as well.

We then consider the results of sections 4.2 to 4.5, assuming the π -meson is either scalar or pseudo-scalar. The discussion of nuclear forces in 4.2 seemed to

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1. R. E. Marshak, 'Meson Physics'
2. Clark, Roberts, and Wilson, Phys. Rev. <u>83</u>: 649 (1951)
3. Durbin, Loar, Steinberger, Phys. Rev. <u>83</u>: 646 (1951).

favour the pseudo-scalar field. The symmetric scalar theory denied the possible existence of the deuteron, while any scalar theory was too simple to allow for a finite electric quadrupole moment for the deuteron. On the surface, the pseudo-scalar theory of nuclear forces permitted the existence of a bound state for the neutron-proton system, and predicted the correct sign for the neutron-proton quadrupole moment. Actually the strong singularity exhibited by the pseudo-scalar potential 4.2(7) at $r_{12}=0$ makes it impossible to use this potential for a direct analysis of the n-p system (using a Schrödinger equation, say.) Bethe[1] suggested a 'cutting off', of r_{12} at small values. However, this procedure would seem to be unsatisfactory since it is non-relativistic[2].

We have previously stated in 4.3 that the experimental results for meson production in nucleon-nucleon collisions favour the pseudo-scalar theory (with pseudo-vector coupling) over the scalar theory. Briefly, this followed from the actual observation of π^+ -mesons in proton-proton collisions - the scalar theory (quantum mechanically) predicts no meson production in such a collision while the pseudo-scalar theory

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1. H. A. Bethe, Phys. Rev. 57, 260, 390 (1940).
2. In effect one is attributing a finite size and structure to the elementary particles, the proton and neutron, without explanation. The size and structure would not then be relativistically covariant.

yields a finite π^+ production.

The results of charged meson scattering by nucleons pointed towards a pseudo-scalar theory. The scalar theory scattering cross-section decreased with an increase in the energy of the incident meson, the pseudo-scalar cross-section increased with energy, while experimentally the cross-section increased with energy.

Finally in photomesic production one notes neutral mesons can result from a pseudo-scalar collision of photon and nucleon, but not from a scalar collision. The appearance of π^0 -mesons in such collisions have been observed by more than one experimenter.[1]

On the basis of present experimental evidence, incomplete as it may be, one would surely conclude that the π -meson field is pseudo-scalar. However, even the pseudo-scalar field (with pseudo-vector coupling) is unable to give more than qualitative agreement with experiment. Aside from this quantitative disagreement with practice, there are many outstanding difficulties in meson theory (such as the poorly behaved nuclear potentials) which make it impossible to obtain an entirely consistent theory. It is hoped, nevertheless, that the present meson theory of nuclear forces will point out a path along which we may proceed to a correct theory.

1. e.g., Silverman and Stearns (at Cornell), Phys. Rev. 83 : 853 (1951).

5. APPENDICES.

5.1 Appendix A.

We have used special relativity as a guide in the selection of source terms for the various meson fields. In this appendix we would like to elaborate on the specific choices for these source terms.

The central concept of special relativity is the covariance of the laws of physics in transformations between frames of reference in uniform relative motion. Suppose we consider two such frames F, F' in which position and time are specified by

$$(x_1, x_2, x_3, x_4 = ict) \tag{1a}$$

and

$$(x_1', x_2', x_3', x_4' = ict') \tag{1b}$$

respectively.

It may be shown that the transition $F \to F'$ is accomplished through use of the Lorentz transformation equations[1]

$$\begin{aligned} x_1' &= (x_1 - i \tfrac{v}{c} x_4) \tfrac{1}{\beta} \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= (x_4 - i \tfrac{v}{c} x_1) \tfrac{1}{\beta} \end{aligned} \tag{2}$$

1. See for example H. G. Goldstein 'Classical Mechanics' or P. G. Bergmann 'Introduction to the Theory of Relativity.'

where $\beta = \sqrt{1 - v^2/c^2}$.

The form of the set (2) implies that the spatial axes of F and F' are parallel, that the origin of F' travels along the positive 1-axis of F with speed v , and finally that the origins of F , F' are coincident at the time $t = 0$.

We have guaranteed the covariance of the meson field theory by writing the equations for the field in tensor form. We have stated that the various coupling terms possessed certain tensor transformation properties. In scalar theory we showed an appropriate generalization of the nuclear density $\rho(\vec{r})$, was the 4-current magnitude $\sqrt{-\frac{1}{c^2} J_\nu J_\nu}$ where

$$J_\nu \equiv (\rho \vec{v}, ic\rho) \tag{3}$$

That J_ν is a 4-vector, and hence its magnitude a 4-scalar, we deduce from the continuity equation

$$\partial_\nu J_\nu = 0 \quad ^1 \tag{4}$$

We simply note the 4-gradient ∂_ν transforms like a 4-vector and apply the well known tensor product rule[2].

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1. We abide by the Einstein summation convention, whereby a repeated index is summed. Greek indices take on values 1 to 4, Latin indices values 1 to 3.

2. For our purposes this rule may be stated in the following manner. If the set of quantities $A(\nu)$ $(\nu = 1, 2, 3, 4)$ are defined for all frames, and combine with an arbitrary 4-vector B_ν so as to render $A(\nu) B_\nu$ a 4-scalar, then this set $A(\nu)$ forms a 4-vector.

The choice made in 2.2 for the pseudo-vector coupling in the pseudo-scalar theory will be valid provided we assign definite transformation properties to the ordinary spin. With a single nucleon in its rest frame, we associate a density ρ_0 and an ordinary spin $\frac{1}{2}\vec{\sigma}_0$. We ask that the pseudo-vector coupling in the rest frame be

$$S_\mu^0 \equiv (\rho_0 \vec{\sigma}_0 , 0) \tag{5}$$

An arbitrary lab frame may be taken as one moving relative to the nucleon rest frame with velocity $-v$ in the direction of their common 1-axes. The pseudo-vector coupling in the lab frame will clearly have the form

$$S_\mu \equiv (\rho \vec{\sigma} , S_4) \tag{6}$$

Moreover, from (2) we must have

$$S_1 = \rho_0 \sigma_{01} \frac{1}{\beta} \tag{7a}$$

$$S_2 = \rho_0 \sigma_{02} \tag{7b}$$

$$S_3 = \rho_0 \sigma_{03} \tag{7c}$$

$$S_4 = i \rho_0 \sigma_{01} v/c \frac{1}{\beta} \tag{7d}$$

Since $c\rho$ is the 4-component of the current vector J_ν it follows

$$\rho = \rho_0 \frac{1}{\beta} \qquad (\;\;)$$

Whence we get

$$\sigma_1 = \sigma_{01} \tag{8a}$$

$$\sigma_2 = \sigma_{02}\,\beta \tag{8b}$$

$$\sigma_3 = \sigma_{03}\,\beta \tag{8c}$$

while

$$S_4 = i\rho \, v/c \; \sigma_1 \tag{8d}$$

In three dimensional vector form the equations are

$$\vec{\sigma}_{\parallel} = \vec{\sigma}_{0\parallel} \tag{9a}$$

$$\vec{\sigma}_{\perp} = \beta\,\vec{\sigma}_{0\perp} \tag{9b}$$

$$S_4 = i\rho\,\frac{\vec{v}\cdot\vec{\sigma}}{c} \tag{9c}$$

where $\vec{\sigma}_{\parallel}$ is the component of $\vec{\sigma}$ parallel to $\vec{v}$, $\vec{\sigma}_{\perp}$ the component perpendicular to $\vec{v}$.

It is also possible to demonstrate the consistency of the choice of the tensor coupling in the vector meson theory. In 2.3 (15) we took this coupling to be specified by

$$\vec{M} = g_T\left(\rho\,\vec{\sigma}_{\parallel}\sqrt{1-v^2/c^2} + \frac{\rho\,\vec{\sigma}_{\perp}}{\sqrt{1-v^2/c^2}}\right) \tag{10a}$$

$$\vec{P} = g_T\,\rho\,\frac{\vec{\sigma}\times\vec{v}}{\sqrt{1-v^2/c^2}} \tag{10b}$$

where

$$p_{\nu\mu} \equiv [\vec{M}, \vec{P}] \tag{11}$$

is a second rank Lorentz tensor.[1]

If one examines the transformation equations (2) one couldshow that the set $(\vec{M}^{(1)}, \vec{P}^{(1)})$ in $F^{(1)}$ is related to the set $(\vec{M}^{(2)}, \vec{P}^{(2)})$ in $F^{(2)}$ by

$$\vec{M}^{(2)}_{\parallel} = \vec{M}^{(1)}_{\parallel} \tag{12a}$$

$$\vec{M}^{(2)}_{\perp} = (\vec{M}^{(1)}_{\perp} - \frac{1}{c}\vec{u} \times \vec{P}^{(1)})\frac{1}{\beta} \tag{12b}$$

$$\vec{P}^{(2)}_{\parallel} = \vec{P}^{(1)}_{\parallel} \tag{12c}$$

$$\vec{P}^{(2)}_{\perp} = (\vec{P}^{(1)}_{\perp} + \frac{1}{c}\vec{u} \times \vec{M}^{(1)})\frac{1}{\beta} \tag{12d}$$

where $\vec{u}$ is the velocity of $F^{(2)}$ relative to $F^{(1)}$.

We fix on $F^{(1)}$ as the rest frame of the nucleon and $F^{(2)}$ as an arbitrary lab frame. Then $\vec{v} = -\vec{u}$ is the velocity of the nucleon as observed in the lab frame. We would wish to have in the rest frame

$$\vec{M}^{(0)} = \vec{M}^{(1)} = 4\rho_0 \vec{\sigma}_s \tag{13a}$$

$$\vec{P}^{(0)} = \vec{P}^{(1)} = 0 \tag{13b}$$

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1. We ask that $p_{\mu\nu}$ be antisymmetric in its two indices, i.e. $p_{\mu\nu} = -p_{\nu\mu}$. Then clearly $p_{\mu\nu}$ has only six independent components and we set:

$$p_{12} = M_3 \;,\quad p_{13} = -M_2 \;,\quad p_{23} = M_1$$
$$p_{14} = -iP_1 \;,\quad p_{24} = -iP_2 \;,\quad p_{34} = -iP_3$$

Whence it would follow in the lab frame

$$\vec{M} = \vec{M}^{(2)} = \left(\beta \int \rho \vec{\sigma}_{\parallel} + \frac{\int \rho \vec{\sigma}_{\perp}}{\beta}\right) g_{T}$$

$$\vec{P} = \vec{P}^{(2)} = g_{T} \frac{\int \rho \vec{\sigma} \times \vec{v}}{\beta}$$

In terms of the momentum-energy 4-vector it is possible to deduce the energy threshold for meson production in, say, a nucleon-nucleon collision; i.e. the minimum energy the nucleon pair must possess before any mesons can appear in the collision. If in the collision one nucleon is indident on another at rest, the energy available for meson production is not the excess of the energy of the incident nucleon over its rest mass. Rather it is the sum of such excesses for both nucleons, when these excesses are computed in the centre of mass system. The kinetic energy associated with the motion of the centre of mass of the two-nucleon system cannot contribute to meson production, since this energy will remain constant during a collision involving only internal forces in the system.

In the lab frame the incident nucleon n_1 and target nucleon n_2 possess momentum energy $(\vec{p}, i\frac{E}{c})$ and $(0; i\frac{M}{c}c^2)$ respectively[1]. We ignore the proton-neutron mass difference,

1. The momentum-energy we refer to are the relativistic quantities

$$\vec{p} = \frac{M\vec{v}}{\sqrt{1-v^2/c^2}}$$

$$E = \frac{Mc^2}{\sqrt{1-v^2/c^2}}$$

where $\vec{v}$ is the nucleon velocity. Clearly $\vec{p}$ and E are related by $\left(\frac{E}{c}\right)^2 = \vec{p}^{\,2} + M^2c^2$

taking M as a common nucleon mass. Since the centre of mass frame moves with uniform velocity with respect to the lab frame we may perform a Lorentz transformation between the frames to obtain $(\vec{p}_1, \frac{E_1 i}{c})$ and $(\vec{p}_2, \frac{iE_2}{c})$ as the momentum-energy of n_1 and n_2 respectively. The centre of mass frame is in fact defined by

$$\vec{p}_1 = -\vec{p}_2 \tag{15}$$

Now the momentum-energy of each particle is a 4-vector and hence the total momentum-energy of the system is also a 4-vector. The magnitude of this latter 4-vector is the same in the lab and centre of mass frames; hence we get

$$\left(\frac{E_1+E_2}{c}\right)^2 - (\vec{p}_1+\vec{p}_2)^2 = \left(\frac{E}{c} + Mc\right)^2 - \vec{p}^{\,2} \tag{16}$$

In addition we have

$$\left(E/c\right)^2 = \vec{p}^{\,2} + M^2c^2$$

whence it follows

$$\left(\frac{E_1+E_2}{c}\right)^2 = 2ME + 2M^2c^2 \tag{17}$$

The energy available for meson production is seen to be

$$(E_1+E_2) - 2Mc^2 = \sqrt{2MEc^2+2M^2c^4} - 2Mc^2 \tag{18}$$

The threshold energy $E = E_t$ is obtained when a single meson appears with zero momentum, i.e. when the right side of (18) is equated to the rest energy of the meson.

One finds

$$E_t = mc^2\left(2 + \frac{m}{2M}\right) + Mc^2 \tag{19}$$

where m is the meson rest mass.

In a similar fashion onemay compute the threshold photon energy for single meson production in a photon-nucleon collision. One finds this threshold photon energy to be

$$\hbar\omega_t = mc^2\left[1 + \frac{m}{2M}\right] \tag{20}$$

We had deduced in 4.5 that the threshold photon energy for collision with a nucleon of infinite mass was $\hbar\omega_t = mc^2$ in agreement with (20).

Finally we would like to consider the question of spherical polar co-ordinates in a 4-dimensional Euclidean space with reference to the discussion in 4.1.[1] If we look on Lorentz space-time as a Euclidean space we may set up the polar co-ordinates $R, \theta_1, \theta_2, \phi$ defined by

$$x_1 = R\sin\theta_1\sin\theta_2\cos\phi \qquad x_2 = R\sin\theta_1\sin\theta_2\sin\phi \tag{21a}$$

$$x_3 = R\sin\theta_1\cos\theta_2 \qquad x_4 = R\cos\theta_1 \tag{21b}$$

The ranges for the co-latitude angles θ_1, θ_2 are 0 to π, that for the azimuth ϕ is 0 to 2π. An element of volume in the 4-space is given by

1. See J. A. Stratton 'Electromagnetic Theory.'

$$dv_4 = dx_1 dx_2 dx_3 dx_4 = R^3 \sin^2\theta_1 \sin\theta_2 \, dR \, d\theta_1 \, d\theta_2 \, d\phi \quad (22)$$

It follows that an element of surface area for a hypersphere of radius R is

$$dS_4 = R^3 \sin^2\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi \quad (23)$$

and hence an element of solid angle by

$$d\eta_4 = \sin^2\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi \quad (24)$$

Finally, the total solid angle subtended by a hypersphere at its centre is

$$\eta_4 = \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \sin^2\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\phi$$

$$\text{i.e. } \eta_4 = 2\pi^2 \quad (25)$$

5.2 Appendix B. Lagrange Formulation of Meson Field Theory.

An alternative and perhaps more sophisticated fashion of obtaining the wave equations for the meson field is through use of a variational principle. One defines a Lagrangian density $\mathcal{L}$ for the meson field and attempts to obtain the field equations by demanding that

$$\delta \int_{V_4} \mathcal{L} \, d v_4 = 0 \tag{1}$$

the integration is over an arbitrary region V_4 in Lorentz space time. In order to explain the nature of the variation implied by the δ -symbol we consider a general field defined by a set of functions $\phi_\nu(x_\mu)$. We suppose

$$\mathcal{L} = \mathcal{L}(\phi_\nu, \partial_\mu \phi_\nu, x_\lambda) \tag{2}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$

Then the variation in (1) is with regard to the ϕ_ν. The variations $\delta\phi_\nu$ are to be arbitrary except on the surface S_4 of V_4, where we set

$$\delta \phi_\nu = 0 \tag{3}$$

Moreover the position-time co-ordinates x_μ are to be treated as constant parameters in the variations

$$i.e. \ \delta x_\mu = 0 \tag{4}$$

It follows the δ -symbol commutes with all space-time operations, and (1) may be written

$$\int_{V_4} \delta \mathcal{L} \, dv_4 = 0 \tag{5}$$

Now

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_\nu} \delta \phi_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \delta(\partial_\mu \phi_\nu)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_\nu} \delta \phi_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \partial_\mu(\delta \phi_\nu)$$

$$\text{i.e.} \quad \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_\nu} \delta \phi_\nu + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \delta \phi_\nu \right] - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \right) \delta \phi_\nu \tag{6}$$

One notes the second term in the last line of (6) is a 4-divergence and yields on integration

$$\int_{V_4} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \delta \phi_\nu \right] dv_4 = \int_{S_4} n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \delta \phi_\nu \, dS_4 \tag{7}$$

where n_μ is a unit vector normal to S_4 . However, on S_4

$$\delta \phi_\nu = 0$$

hence (5) is equivalent to

$$\int_{V_4} \left[\frac{\partial \mathcal{L}}{\partial \phi_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \right) \right] \delta \phi_\nu \, dv_4 = 0 \tag{8}$$

Since V_4 is arbitrary the integrand in (8) must vanish, and then since $\delta\phi_\nu$ is arbitrary in V_4 we are led to

$$\frac{\partial \mathcal{L}}{\partial \phi_\nu} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)} \right] = 0 \tag{9}$$

for all time and position.

The equation (9) constitutes an Euler-Lagrange equation for the field ϕ_ν ; there is one such equation for each of the components ϕ_ν .

We may obtain a continuity equation for the energy in the field simply by computing

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{L}}{\partial \phi_\nu}\frac{\partial \phi_\nu}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)}\frac{\partial}{\partial t}(\partial_\mu \phi_\nu) + \left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\phi_\nu, \partial_\mu \phi_\nu} \tag{10}$$

where the term $\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\phi_\nu, \partial_\mu \phi_\nu}$ is the time rate of change of $\mathcal{L}$ computed at a given position for constant ϕ_ν and $\partial_\mu \phi_\nu$.

Proper manipulation of (10) together with a use of the field equations (9) yields

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_\nu}{\partial t}\right)}\frac{\partial \phi_\nu}{\partial t} - \mathcal{L}\right] + \partial_k\left[\frac{\partial \mathcal{L}}{\partial(\partial_k \phi_\nu)}\dot{\phi}_\nu\right] = -\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\phi_\nu, \partial_\mu \phi_\nu} \tag{11}$$

(Summation on k is from 1 to 3).

We interpret

$$H = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_\nu}{\partial t}\right)}\frac{\partial \phi_\nu}{\partial t} - \mathcal{L} \tag{12}$$

as an energy density for the field, while

$$S_k = -\frac{\partial \mathcal{L}}{\partial(\partial_k \phi_\nu)} \frac{\partial \phi_\nu}{\partial t}, \quad k = 1, 2, 3 \tag{13}$$

is the energy flux for the field. The source term for field energy is

$$-\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\phi_\nu, \partial_\mu \phi_\nu}$$

We note relativistic invariance for our field theory is obtained by demanding $\mathcal{L}$ be a 4-scalar. Then because the volume element

$$dv_4 = dx_1 dx_2 dx_3 dx_4$$

is also an invariant[1], the equation (1) is an invariant equation.

For the scalar meson field one sets

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}' \tag{14}$$

where $\mathcal{L}_0$ is a source free Lagrangian density which yields the meson field equations in the absence of nuclear matter; while $\mathcal{L}'$ is an interaction Lagrangian density representing the effect of nuclear sources.

---

1. Actually the volume element dv_4 is a scalar density, i.e. a pseudo-scalar. Under reflection of an odd number of the co-ordinates x_1, x_2, x_3, x_4 the volume element changes in sign. However, it is clear whether we equate

$$\delta \int_{V_4} \mathcal{L}\, dv_4 \quad \text{or} \quad -\delta \int_{V_4} \mathcal{L}\, dv_4$$

to zero is immaterial.

One can take for a neutral scalar field

$$\mathcal{L}_0 = -\frac{1}{2}\kappa^2\phi^2 - \frac{1}{2}(\partial_\mu\phi)^2 \tag{15}$$

evidently $\mathcal{L}_0$ is a Lorentz invariant. We might include in $\mathcal{L}'$ a scalar coupling proportional to ϕ and a vector coupling proportional to $\partial_\mu\phi$; this latter is, however, ignored for reasons indicated in 2.1. We then get $\mathcal{L}'$ by multiplying ϕ by an appropriate 4-scalar characterising the nuclear sources. Such a scalar is $\rho\beta$ and we might have

$$\mathcal{L}' = -g\rho\beta\phi \tag{16}$$

where g is a coupling constant.

There is no trouble in seeing that (15), (16), coupled with (9) yield the scalar meson wave equation 2.1(16).

Also the energy flux which we used in 2.1 to deduce the energy continuity equation for the scalar field is deduced from

$$S_k = -\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)}\dot{\phi} = -\dot{\phi}\,\partial_k\phi$$

i.e.,

$$\vec{S} = -\dot{\phi}\,\nabla\phi \tag{17}$$

For the symmetric scalar field one treats ϕ_ℓ $(\ell = 1, 2, 3)$ as independent field functions and takes

$$\mathcal{L} = -\tfrac{1}{2}\kappa^2(\underline{\phi},\underline{\phi}) - \tfrac{1}{2}(\partial_\mu\underline{\phi},\partial_\mu\underline{\phi}) - g\rho\beta(\underline{\tau},\underline{\phi}) \qquad (18)$$

It is possible to proceed in an identical fashion for the pseudo-scalar and vector meson fields. We simply quote the Lagrangian densities for the symmetrized versions of these fields:

Pseudo-scalar (pseudo-vector coupling)

$$\mathcal{L}_0 = -\tfrac{1}{2}\kappa^2(\underline{\phi},\underline{\phi}) - \tfrac{1}{2}(\partial_\mu\underline{\phi},\partial_\mu\underline{\phi}) \qquad (19a)$$

$$\mathcal{L}' = -\frac{f}{\kappa} S_\mu(\underline{\tau},\partial_\mu\underline{\phi}) \qquad (19b)$$

with $S_\mu \equiv (\rho\vec{\sigma}, i\rho\frac{\vec{\sigma}\cdot\vec{v}}{c})$

Vector (vector and tensor coupling)

$$\mathcal{L}_0 = -\tfrac{1}{2}\kappa^2(\underline{\phi}_\mu,\underline{\phi}_\mu) - \tfrac{1}{2}(\partial_\mu\underline{\phi}_\nu - \partial_\nu\underline{\phi}_\mu, \partial_\mu\underline{\phi}_\nu - \partial_\nu\underline{\phi}_\mu) \qquad (20a)$$

$$\mathcal{L}' = -g_v J_\mu(\underline{\phi}_\mu,\underline{\tau}) - p_{\mu\nu}(\underline{\tau},\partial_\nu\underline{\phi}_\mu) \qquad (20b)$$

where $J_\mu \equiv (\rho\vec{v}, ic\rho)$

and $p_{\mu\nu} \equiv (\vec{M},\vec{P})$

The interaction densities (16), (19b), (20b) were actually selected using the Dirac particle as a model of the nucleon. For example, the classical interaction Lagrangian

~~1. See Bibliography for references.~~

density (16) for the scalar meson field was patterned after the corresponding Quantum Mechanical interaction density

$$\mathcal{L}'_{Q.M.} = -g(\psi^* \beta \psi) \qquad (21)$$

where ψ and its 'complex conjugate' ψ^* are Dirac[1] wave functions forthe nucleon and β is the usual Dirac operator. Roughly speaking one would say $\psi^* \psi$ passes over, in the classical limit, into a nuclear particle density; while β goes over into $\sqrt{1-v^2/c^2}$.

We would like to point out that the classical limit of the Quantum Mechanical pseudo-scalar coupling in the pseudo-scalar theory is in a first order of approximation, zero. In further orders the pseudo-scalar coupling introduces into the Lagrangian density a term proportional to ϕ^2 and a term equivalent to the pseudo-vector coupling.

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1. For discussion of Dirac wave equation, see for example: 'Principles of Quantum Mechanics' - P.A.M. Dirac, 3rd edition - Oxford University Press, 1947.
'Quantum Mechanics' - L. I. Schiff, McGraw Hill, 1949.
For explicit expressions for possible coupling terms, see 'Meson Physics', R. E. Marshak - McGraw Hill, 1952.

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(2) G. Wentzel, 'Quantum Theory of Fields' - Interscience (1949) Though this treatise is devoted principally to developing the methods of quantum field theory, it spends a certain amount of time on the general classical formulation of field theory. It deals with both scalar and vector meson theories as examples of quantum fields.

(3) W. Pauli, 'Meson Theory of Nuclear Forces' - Interscience (1946). An essentially classical treatment of various aspects of meson theory. In particular the questions of nuclear forces, meson scattering by nucleons, are discussed. The possibility of a non-relativistic meson theory using extended nuclear sources is considered, and also the classical strong coupling theory is introduced.

(4) L. Rosenfeld, 'Nuclear Forces I and II' - Interscience (1948)
These volumes include an excellent discussion of the importance of meson theory in the study of the nucleus. The sections on meson theory are concerned mainly with establishing the nucleon-nucleon potentials.

(5) L. Rosenfeld and C. Møller:

(i) 'On the Field Theory of Nuclear Forces' - Kgl. Danske Vid Sels. XVII, No. 8 (1940.) The symmetric field theories are discussed mainly in relation to nuclear forces. Attempts are made to subtract off the bothersome terms in the nucleon-nucleon potentials.

(ii) 'Electromagnetic Properties of Nucleon Systems on Meson Theory' Kgl. Danske Vid. Sels. XX, No. 12 (1943).

(iii) 'Non-Central Coupling in the Mixed Meson Theory of Nuclear Forces' - Kgl. Danske Vid. Sels. XXIII, No. 13 (1945).

(6) W. W. Wada, 'Scattering of Charge Symmetric Pseudo-scalar Mesons by Nucleons' - Phys. Rev. 88, 5, (1952).
A classical discussion of meson-nucleon scattering, patterned after Pauli's treatment.

(7) N. Kemmer, Proc. Cambridge Phil. Soc., 34, 354 (1938).

B. Reference Material for Electromagnetic Theory and Relativity:

(1) J. A. Stratton, 'Electromagnetic Theory' - McGraw Hill (1941)
Contains a lucid treatment of the 4-dimensional integration of the electromagnetic wave equations. We have patterned the discussion in 4.1 after Stratton's treatment - the method is apparently due to Sommerfeld. This book may also be used to provide the background in electromagnetic theory required for the present work on meson field theory.

(2) H. Goldstein, 'Classical Mechanics' - Addison-Wesley (1950).

(3) P. G. Bergmann, 'Introduction to the Theory of Relativity' - Prentice-Hall (1942).
The latter two books might be used as sources for special relativity.