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COUPLING BETWEEN TWO COLLINEAR WAVEGUIDES

by

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TO MY PARENTS

AND

TO MY WIFE

ABSTRACT

The Wiener-Hopf technique is used to investigate the coupling between two collinear parallel-plate and circular waveguides, located in free space. Expressions for the reflected, transmitted and the radiated fields respectively in the exciting waveguide, coupled waveguide and free space are obtained and are presented graphically for some special cases. The exact solutions are then expanded to yield the solution of the ray theory of diffraction with modified diffraction coefficients. Also, for the case of circular waveguides, a spherical wavefactor is derived, which takes care of diffraction by small circular apertures in hard screens. It is shown that the results obtained by the Wiener-Hopf technique for the coupling between two collinear semi-infinite parallel-plate waveguides are more accurate than those obtained by Hu's transmission formula.

Numerical methods are also used. The expressions for the radiated fields due to the coupling between two collinear parallel-plate waveguides are derived in terms of the reflection and the transmission coefficients of the dominant mode and the evanescent currents on the waveguides walls. These coefficients and currents are then obtained by an application of the moment methods and are used to find the radiation fields. The results are shown to be in good agreement with those of the Wiener-Hopf technique.

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LIST OF SYMBOLS

Unless otherwise stated, the symbols most commonly used in this thesis have the following meaning.

Greek alphabet:

α Complex variable in the Fourier transform domain.

π^i Electric hertz vector.

$\psi_r(\rho,z)$ Total reflected field in the exciting waveguide as a function of the cylindrical coordinates ρ and z (circular waveguides case).

$\psi_r^{exc}(\rho,z)$ Reflected field in the exciting waveguide due to only its open end (circular waveguides case).

$\psi_r^{int(1)}(\rho,z)$, $\psi_r^{int,(2)}(\rho,z)$ Reflected field in the exciting waveguide due to the interactions between the open ends of the two waveguides with final diffraction at the open ends of the exciting and coupled waveguides, respectively, (circular waveguides case).

∇_t^2 Transverse part of the Laplacian operator.

$\delta_{n'n}$ Kronecker delta (=0 if $n' \neq n$; =1 if $n'=n$) .

$\prod_p$ Overall product.

ε Dielectric constant of free space .

ξ_{om} The m^{th} order zero of the zero order Bessel function.

ϕ, Φ Fourier transform pair, with Φ is the Fourier transform of ϕ.

$\phi^i(x,z)$ Incident electric field in the exciting waveguide as a function of the Cartesian coordinates x and z (parallel-plate waveguides case).

$\phi^{s}(x,z)$ Radiation electric field in the free space (parallel-plate waveguides case).

$\phi_{r}(x,z)$ Total reflected electric field in the exciting waveguide (parallel-plate waveguides case).

$\phi_{r}^{exc}(x,z)$ Reflected electric field in the exciting waveguide due to its open end (parallel-plate waveguides case).

$\phi_{r}^{int,(1)}(x,z)$, $\psi_{r}^{int,(2)}(x,z)$ Reflected electric field in the exciting waveguide due to the interactions between the open ends of the two waveguides with final diffraction at the open ends of the exciting and coupled waveguides, respectively, (parallel-plate waveguides case).

$\phi_{t}(x,z)$ Total transmitted electric field in the coupled waveguide (parallel-plate waveguides case).

$\phi_{t}^{exc}(x,z)$ Transmitted electric field in the coupled waveguide due to only the open end of the exciting waveguide (parallel-plate waveguides case).

$\phi_{t}^{int,(1)}(x,z)$, $\phi_{t}^{int,(2)}(x,z)$ Transmitted electric field in the coupled waveguide due to the two waveguides with final diffraction at the open ends of the exciting and coupled waveguides, respectively, (parallel-plate waveguides case).

γ_{ℓ} Propagation constant of mode ℓ in the waveguide .

$\Gamma(x)$ Gamma function.

γ Function of α and equal $(\alpha^2 - k^2)^{\frac{1}{2}}$ or $-i\ (k^2 - \alpha^2)^{\frac{1}{2}}$.

ρ, θ, z Polar coordinates.

σ, τ Real and imaginary parts of the complex variable α, respectively.

μ Permeability of free space .

ω Frequency of operation, in radians.

Latin alphabet (UPPER CASE):

$D(\theta,\theta_o)$ Diffraction coefficient with the incident and observation angles θ_o and θ , respectively.

$\bar{D}(\theta,\theta_o)$ Modified diffraction coefficient with the incident and observation angles θ_o and θ , respectively.

$E_y^i(x,z)$ Incident electric field in the parallel-plate exciting wave-guide.

$E_y^s(x,z)$ Scattered electric field in the free space .

E^*, H^* Complex conjugate of the electric and magnetic field components E and H , respectively.

$E_z^i(\bar{r})$, J_z^i Electric field in the far zone contributed by the current J_z^i induced on the walls of the exciting waveguide due to the incident electric field in it.

$E_z^r(\bar{r})$, J_z^r Electric field in the far zone contributed by the current J_z^r induced on the walls of the exciting waveguide due to the reflected electric field in it.

$E_z^t(\bar{r})$, J_z^t Electric field in the far zone contributed by the current J_z^t induced on the walls of the coupled waveguide due to the transmitted electric field in it.

$E_z^e(\bar{r})$, J_z^e Electric field in the far zone contributed by the evanescent current of J_z^e induced on the walls of the two waveguides.

$F(\theta)$ Radiation pattern.

G, G_+, G_- G is the transformed Green's function with G_+ and G_- as the plus and minus parts of G , respectively .

$\bar{G}$, $\bar{G}_+$, $\bar{G}_-$ $\bar{G}$ is the normalized transformed Green's function with $\bar{G}_+$ and $\bar{G}_-$ as the plus and minus parts of $\bar{G}$, respectively.

$H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ Hankel functions of first and second kind, respectively, having an order ν and argument x.

$H_\phi^s(r,\theta)$ Scattered magnetic field in the free space as a function of the spherical coordinates r and θ (circular waveguides case).

$H_\phi^r(\rho,z)$ Reflected magnetic field in the exciting waveguide (circular waveguides case).

$I_\nu(x)$, $K_\nu(x)$ Modified Bessel functions of first and second kind, respectively, having an order ν and x.

$J_\nu(x)$ Bessel function of order ν, having an argument x.

K Constant given in the figures of Chapter 3 and some figures of Chapter 4.

$P_o(\theta)$ Field radiated from the open end of the exciting waveguide alone as a function of the angle θ.

$P(\theta)$ Field radiated from the open end of the two collinear coupled waveguides.

$\frac{P_r}{P_t}$ Power received in the coupled waveguide / power transmitted by the exciting waveguide.

R_o, $R_{\ell,m}$ Reflection coefficient due to the open end of the exciting waveguide only.

R Total reflection coefficient, far from the open end of the exciting waveguide, due to the coupling.

$R_{\ell,m}^{(1)}$, $R_{\ell,m}^{(2)}$ Reflection coefficients due to interactions between the open ends of the two waveguides, with final diffraction at the open end of the exciting and coupled waveguides, respectively.

S_{ℓ}^{mn} Scattering matrix.

$T, T_{\ell,m}^{(2)}$ Transmission coefficient due to field transmitted in the coupled waveguide and far from its open end.

$T_{\ell,m}(z)$ and $T_{\ell,m}^{(1)}(z)$ Transmission coefficients as functions of the axial coordinate z .

$W_{u,\nu}(\xi)$ Whittaker's function.

Latin alphabet (lower case):

$f(u_1,u_2)$ Field distribution over the cross-sectional area of the waveguide with u_1,u_2 as the transverse coordinates.

$n!$ Factorial function.

r, θ, ϕ Spherical coordinates.

CHAPTER 1

INTRODUCTION

Scattering of electromagnetic and sound waves by parallel plate waveguides and cylindrical structures have recently received increasing attention due to their importance in radiation as transmitting or receiving antennas [1]-[3] and their application to sensor booms [4] and other microwave problems [5]. Optimization of the radiation characteristics of an open ended waveguide has also been investigated [6] and was achieved by varying the amplitude and phase of the exciting modes. This optimization may also be achieved by introducing another waveguide and producing mutual coupling effect. However, in large scale microwave arrays, the mutual coupling among various elements is a significant parameter and is generally used to control the radiation characteristics [7]-[10]. Previous analytical investigations of these structures are mostly based on the ray theory of diffraction [11]. This technique is limited to high frequency scattering and to certain geometries or orientations, due to difficulties in including all the rays.

Keller's theory of diffraction has been used extensively by many authors to find coupling between two antenna systems. Hamid [12] has used the ray theory to find the coupling between horn antennas under near field interactions. Dybdal et al [7] have applied the theory to obtain mutual coupling between TEM and $TE_{0,1}$ parallel-plate waveguide apertures.

There have been several studies on the coupled waveguides, mostly concerned with the waveguide apertures in one plane. The first was performed by Wheeler [13] who showed that a single-mode solution was adequate when the radiators are in the far field of one another. Later, Galejs [14] solved the problem of coupling between two parallel slots in a ground plane using a stationary formulation due to Richmond [15]. He avoided the solution of integral equations by assuming the tangential magnetic field at the coupled waveguide aperture to be the same as that on the ground plane in the absence of the coupled aperture. Similar work was done by Lyon et al [16] for the same problem as Galejs. The most recent work in this area has been handled by Mailleux [8] by solving an integral equation governing the coupling of two waveguides. For infinite arrays, the coupling effects are investigated in references [17] - [24].

To the best of the author's knowledge, no one has attempted to find the coupling between collinear waveguides either semi-infinite or finite waveguides, when one of the waveguides is an exciting (or transmitting) and the other is a coupled (or receiving) one. Kashyap and Hamid [25] have solved the problem of scattering of a plane wave by a slit in a thick conducting screen. This problem is closely related to the problem of coupling between collinear semi-infinite parallel plate waveguides.

The Wiener-Hopf technique was invented about 1931 to solve an integral equation of a special type and has been used to solve problems involving diffraction by geometries having discontinuities in the transverse or the longitudinal directions [26]. However, the technique usually involves laborious mathematics based on the complex variables theory. Moreover, some problems yield expressions which cannot be solved exactly by analytical techniques. As a result there have been some studies to overcome these problems. These studies are mainly due to Lee [27],[28] who investigated diffraction by two staggered plates. He used the Wiener-Hopf technique and then extracted its dominant asymptotic terms so that the result admits ray interpretation. He also introduced a modified diffraction coefficient which takes care of multiple reflections and diffractions along the shadow boundary only.

The modified diffraction coefficient of Lee is only applied where dimensions of the structures are very large compared to the wavelength. In other words, one needs to find a condition for its restriction, especially when introducing another waveguide in front of the exciting one.

Another active technique which is widely used in problems related to antennas and diffraction in unbounded space is the moment method [23]. Morita [30],[31] has investigated the scattering and diffraction by an arbitrary cross-sectional semi-infinite conductor, by introducing an evanescent current near the discontinuities which may be evaluated by the moment method. Morita's investigations have paved the way to solve problems of diffractions in bounded space [32], mainly waveguides. Wu and Chow [32] have obtained the reflection and transmission coefficients due to obstacles in a parallel-plate waveguide of infinite length. They also have obtained the reflection due to the open end of a semi-infinite parallel-plate waveguide.

To the best of the author's knowledge, the problem of coupling between waveguides, either semi-infinite or finite, has not been attacked by the above technique, especially for the radiation field. The aim of this thesis is to investigate the coupling between collinear parallel-plate and circular waveguides. The two waveguides may be of semi-infinite or finite length. The Wiener-Hopf technique is used to formulate the problem. The results are expanded and upon retaining the first term in the expansion, the final results are found to be the same as the ray theory results when using the modified diffraction coefficient of Lee [27],[28] in the case of parallel-plate waveguides. Getting ray theory results allowed us to know the condition for the applicability of the ray theory of diffraction in conjunction with the modified diffraction coefficient of Lee. The Wiener-Hopf solution is also restricted for large separations between two coupled waveguides. For small separation, or in fact any separation, the numerical technique is adopted and expressions are obtained for the radiation fields.

Chapter 2 represents the formulation and the solution for the problem of coupling between two collinear semi-infinite parallel-plate waveguides. One of the two waveguides is an exciting one (or transmitting antenna), while the other waveguide, which is separated from the exciting waveguide by a distance L, is the coupled waveguide (receiving one). $TE_{0,\ell}$ mode with ℓ odd is used as an excitation. ℓ may also be even, but this case is not included here, but is the same as ℓ odd. The TE case represents the soft boundary case in electromagnetics, where the electric field component is parallel to the edges. The formulation for the TM case, is the same as for TE and hence is not included. The Wiener-Hopf technique is used to formulate the problem and expressions are obtained for the reflected, transmitted and the radiated fields in the excited waveguide, coupled waveguide and free space, respectively. Some results are obtained and are discussed at the end of the chapter.

Chapter 3, treats the coupling between two collinear semi-infinite circular waveguides. The $TM_{0,m}$ mode is excited in one of the waveguides, while the other acts as a coupled one (or a receiving antenna). The case of $TM_{0,m}$ represents the hard boundary case in electromagnetics, where the magnetic field component is parallel to the rims of the waveguides. Again, the Wiener-Hopf technique is used in the formulation. Results for the reflected, transmitted and radiated fields are obtained and are presented at the end of the chapter.

In order to reduce the solutions to those of the ray theory of diffraction, the integral in the final expressions is approximated by expanding the transformed Green's function $G(\alpha)$ in a power series and retaining the first term only. Consequently, the results after integration are in terms of a series convergent under certain conditions. This is presented in Chapter 4, where for the case of circular waveguides, a modified diffraction coefficient and a spherical wavefactor are obtained. It is shown that the spherical wavefactor has to be introduced when treating problems of diffraction by a small aperture in hard screens. Results using the modified diffraction coefficients are obtained and are compared with the exact ones to show its validity. Since the first term of the convergent series yields the ray theory results in conjunction with a modified diffraction coefficient, the higher order terms of this series provide the correction when the separation distance is not large enough. This is discussed in detail in Chapter 4.

Chapter 5 deals with the coupling between waveguides of finite length. The scattering matrix approach is used in conjunction with the Wiener-Hopf results of coupling between two semi-infinite waveguides. The ray theory results are also given in order to examine its validity.

Unfortunately, as far as the author knows, no previous analytical or experimental results are available at the present time to compare the results obtained in Chapters 2, 3 and 5. This has encouraged the author to find other means for comparing these results. On this line, the investigations are concentrated on the case of parallel-plate waveguides, though the case of circular waveguides may be treated in the same manner. In Chapter 6, Hu's transmission formula [33] is used to examine the power received in the coupled waveguide using Kirchoff's approximations [34], the ray theory and the Wiener-Hopf technique. Some results are obtained and discussed at the end of this chapter.

Another way to compare the results is the numerical technique, since it can be used for any separation distance between the two parallel-plate waveguides. Formulas are derived for the radiation patterns in terms of the reflection and transmission coefficients and the physical explanations are given for those formulas. Some results are obtained for the semi-infinite and finite cases and are compared with those obtained in the previous chapters. Discussion of these results are shown in detail in Chapter 7.

The last chapter is concerned with the general discussion and the conclusion of these investigations. Some future research topics are also presented in this chapter.

CHAPTER 2

COUPLING BETWEEN TWO COLLINEAR SEMI-INFINITE PARALLEL-PLATE WAVEGUIDES

2.1 Introduction

As mentioned in the previous chapter, the Wiener-Hopf Technique is used to solve the problem of coupling between two waveguides, in other words, coupling between two antenna systems, where the exciting waveguide is used as a transmitting antenna and the coupled waveguide is used as a receiving antenna. In this chapter, the two antenna systems are two collinear semi-infinite parallel plate waveguides, as shown in figure 2-1. Jones' method of formulation is used, and a modified Wiener-Hopf equation of the second type [35]-[37] is obtained. Expressions for the reflected, transmitted and radiated fields are obtained respectively in the exciting waveguide, coupled waveguide and in the free space. The reflected and the transmitted waves are expressed in terms of the guided modes.

Also it has been shown in this chapter that results of the transmission (coupling between the two waveguides), reflection and radiation power can be obtained by *a priori* knowledge of the reflection coefficient of an exciting single semi-infinite waveguide, together with one value of the plus part of the Green's function associated with the obtained Wiener-Hopf equation. The last step is to evaluate numerically, an integral of the semi-infinite type [30] using the Gauss-Laguerre quadrature formula [38].

Some results for the reflected, transmitted and the radiated fields are obtained and are discussed at the end of this chapter.

2.2 Derivation of the modified Wiener-Hopf equation

Consider two infinitely thin and perfectly conducting parallel plate waveguides, having width 2a and separated by a distance L, located in free space as shown in figure 2-1. With a time factor $e^{-i\omega t}$, an incident field consisting of a $TE_{o,\ell}$ mode is assumed to be propagating in the exciting waveguide along the positive z-direction in the form:

$$E_y^i = \phi^i(x,z) = \cos(\frac{\ell\pi x}{2a})e^{-\gamma_\ell z} , \qquad \ell = 1,3,5,... \tag{2-1}$$

where

$$\gamma_\ell = [(\frac{\ell\pi}{2a})^2 - k^2]^{\frac{1}{2}} \tag{2-2}$$

and k is the free space propagation constant. Realizing that in any physical medium, there exists inevitably some loss no matter how slight it is, therefore, the idealized lossless medium (the free space in our case) should be regarded as a limiting case with vanishingly small loss. In the Wiener-Hopf technique, it is convenient to retain a small but non-zero loss, that is, writing $k = k_1 + ik_2$, $k_1 >> k_2 > 0$. By such an assumption, certain limiting processes can be avoided, as will be seen later. Hence for a lossless solution, we let $k_2 \rightarrow 0$ in the final solution of the lossy solution. Now, the resulting total electromagnetic fields may be found from $\phi^t = \phi + \phi^i$, where ϕ is the scattered electric field and satisfies a two-dimensional wave equation of the form:

$$(\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k^2)\phi = 0 \tag{2-3}$$

where ∇_t^2 is the transverse part of the Laplacian operator. Instead of of using the Wiener-Hopf integral method [39] and then the Fourier transform, the author uses Jones' method [40] which by-passes the integral equation. In the problem under consideration ∇_t^2 is $\frac{\partial^2}{\partial x^2}$. Fourier

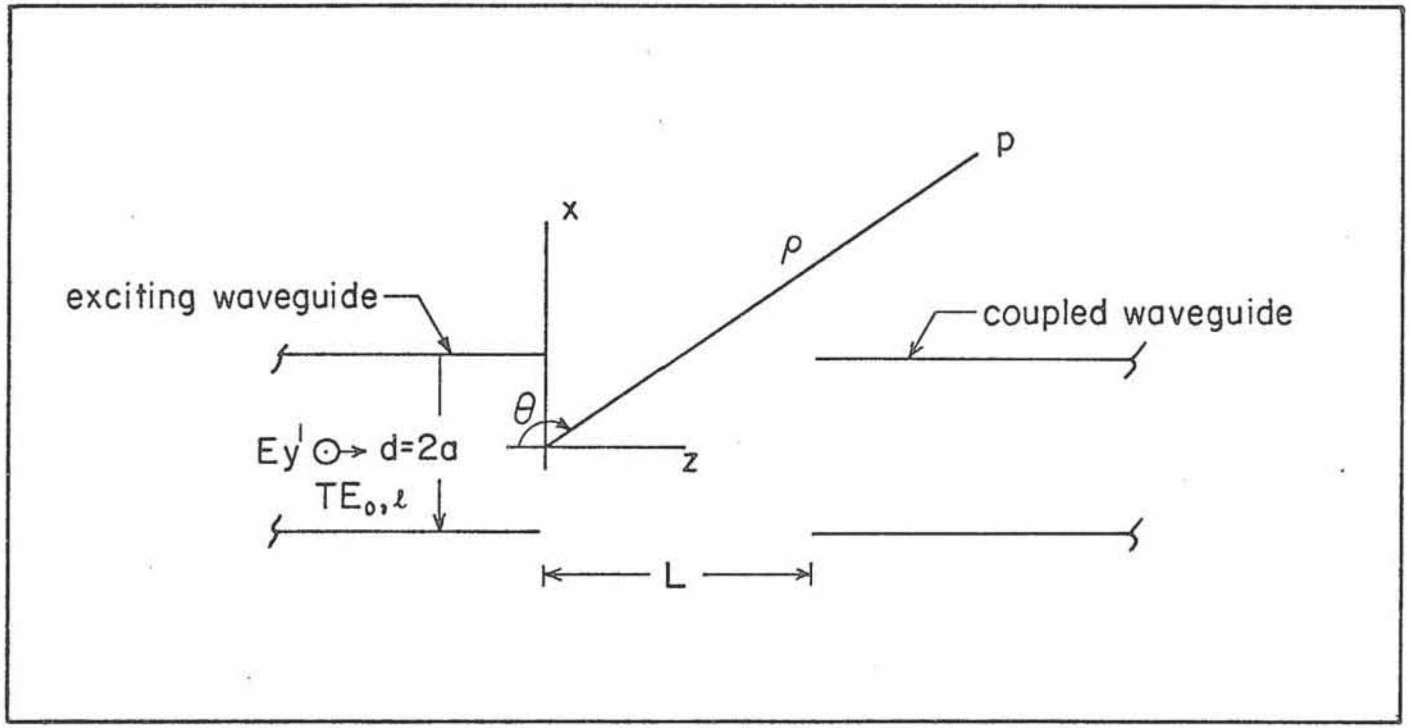

Fig. 2-1 Geometry of the problem for two collinear semi-infinite parallel-plate waveguides separated by L.

transforming of (2-3) with respect to z gives:

$$(\frac{d^2}{dx2} - \gamma^2)\Phi = 0 \qquad , \qquad |\tau| < k_2 \tag{2-4}$$

where ϕ and Φ are a Fourier transform pair given by

$$\Phi(x,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x,z)e^{i\alpha z}dz \qquad , \qquad \alpha = \sigma+i\tau \tag{2-5}$$

and

$$\phi(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x,\alpha)e^{-i\alpha z}d\alpha \tag{2-6}$$

In equation (2-4), γ is given by

$$\gamma = \sqrt{\alpha^2-k^2} = -i\sqrt{k^2-\alpha^2} \tag{2-7}$$

The branch cut of γ used in (2-7) has been chosen so that γ has a positive real part when $-k_2 < \tau < k_2$. Note that the scattered field ϕ attenuates at least as rapidly as $\exp(-k_2|z|)$ as $|z| \to \infty$ inside both waveguides. Outside the waveguides the scattered field is a radiated field, which is a cylindrical wave at large distances. It behaves asymptotically as

$$\phi = \rho^{-1/2}e^{i(k_1+ik_2)\rho} \qquad , \qquad \rho \to \infty \tag{2-8}$$

and attenuates at least as rapidly as $\exp(-k_2|z|)$ as $|z| \to \infty$. Therefore, the transformed wave equation in (2-4) holds at least in the strip $|\tau| < k_2$ in the complex α-plane, shown in figure 2-2.

The function $\Phi(x,\alpha)$ can be decomposed into three parts [26] to give

$$\Phi(x,\alpha) = \Phi_-(x,\alpha) + \Phi_1(x,\alpha) + e^{i\alpha L}\Phi_+(x,\alpha) \tag{2-9}$$

where

$$\Phi_-(x,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi(x,z)e^{i\alpha z}\, dz \tag{2-10a}$$

$$\Phi_1(x,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{L} \phi(x,z)e^{i\alpha z}\, dz \tag{2-10b}$$

$$\Phi_+(x,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{L}^{\infty} \phi(x,z)e^{i\alpha(z-L)}\, dz \tag{2-10c}$$

The functions $\Phi_-(x,\alpha)$ and $\Phi_+(x,\alpha)$ are analytic in the lower ($\tau < k_2$) and upper ($\tau > -k_2$) halves of the complex α-plane, respectively. As $L > z > o$, and $e^{i\alpha z}$ has an essential singularity as $|\alpha| \to \infty$ in the lower half of the α-plane, therefore the entire function $\Phi_1(\alpha)$ has an algebraic behavior at infinity only when $|\alpha| \to \infty$ in the upper half plane. Hence, $\Phi_1(\alpha)$ may be identified as a "plus" function.

Multiplying (2-10b) by $e^{-i\alpha L}$, we get

$$\begin{aligned} e^{-i\alpha L}\Phi_1(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{L} \phi e^{i\alpha(z-L)}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{0} \phi' e^{i\alpha z'} dz' \end{aligned} \tag{2-11}$$

By a similar argument, $e^{-i\alpha L}\Phi_1(\alpha)$ may be identified as a "minus" function.

Within the strip $|\tau| < k_2$, the solution of (2-4) can be expressed as

$$\Phi(x,\alpha) = \Phi_-(x,\alpha) + \Phi_1(x,\alpha) + e^{i\alpha L}\Phi_+(x,\alpha) = \begin{cases} A(\alpha)e^{-\gamma x} \quad , & x > a \\ -A(\alpha)e^{\gamma x} \quad , & x < -a \\ 2B(\alpha)\cosh(\gamma x) & |x| < a \end{cases} \tag{2-12}$$

Now, an application of the boundary conditions on the electric and magnetic fields at the plane $x = a$ gives

(i) $\Phi_-(a+,\alpha) = \Phi_-(a-,\alpha) = o$ (2-13a)

(ii) $e^{i\alpha L}\Phi_+(a+,\alpha) = e^{i\alpha L}\Phi_+(a-,\alpha) = o$ (2-13b)

(iii) $\Phi_1(a+,\alpha) = \Phi_1(a-,\alpha) = \Phi_1(a,\alpha)$ (2-13c)

(iv)
$$\Phi_1'(a+,\alpha) = \Phi_1'(a-,\alpha) + \frac{1}{\sqrt{2\pi}} \int_0^L \left.\frac{\partial\phi^i}{\partial x}\right|_{x=a} e^{i\alpha z}dz$$
$$= \Phi_1'(a-,\alpha) + \frac{1}{\sqrt{2\pi}} \frac{\ell\pi}{2a} (-1)^{\frac{\ell-1}{2}} \cdot \frac{e^{(i\alpha-\gamma_\ell)L} - 1}{\gamma_\ell - i\alpha} \tag{2-13d}$$

where the prime notation denotes differentiation with respect to x .

Differentiating (2-12) with respect to x , and setting $x = a$, we have

$$\Phi_-'(a+,\alpha) + \Phi_1'(a+,\alpha) + e^{i\alpha L}\Phi_+'(a+,\alpha) = -A(\alpha)\cdot\gamma e^{-\gamma a} \tag{2-14a}$$

$$\Phi_-'(a-,\alpha) + \Phi_1'(a-,\alpha) + e^{i\alpha L}\Phi_+'(a-,\alpha) = 2B(\alpha)\cdot\gamma\sinh(\gamma a) \tag{2-14b}$$

From (2-13a), (2-13b), (2-13c) and (2-12), we obtain

$$A(\alpha) = \Phi_1(a,\alpha)/\exp(-\gamma a) \tag{2-15}$$

$$2B(\alpha) = \Phi_1(a,\alpha)/\cosh(\gamma a) \tag{2-16}$$

Subtracting (2-14a) and (2-14b) and making use of (2-13d), (2-15) and (2-16), we obtain the following modified Wiener-Hopf equation of the second type [41] which is valid for $|\tau| < k_2$:

$$J_-(\alpha) + e^{i\alpha L}J_+(\alpha) + \Phi_1(a,\alpha)/G(\alpha) = \frac{i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{1-e^{(i\alpha-\gamma_\ell)L}}{\alpha + i\gamma_\ell} \quad , |\tau| < k_2 \tag{2-17}$$

where

$$J_-(\alpha) = \Phi_-'(a+,\alpha) - \Phi_-'(a-,\alpha) \tag{2-18}$$

$$, \quad J_+(\alpha) = \Phi_+'(a+,\alpha) - \Phi_+'(a-,\alpha) \tag{2-19}$$

and $$G(\alpha) = \cosh(\gamma a)/\gamma \exp(\gamma a) \tag{2-20}$$

The functions $J_-(\alpha)$ and $J_+(\alpha)$ are exactly proportional to the induced current, due to the scattered fields, on the wall $x = a$ of the exciting and the coupled waveguides, respectively. These two functions are analytic, respectively, in the lower $(\tau < k_2)$ and the upper $(\tau > -k_2)$ halves of the complex α-plane. $G(\alpha)$ is the transformed Green's function associated with the Wiener-Hopf technique, and can be factorized into a product $G_+(\alpha)G_-(\alpha)$, where $G_+(\alpha)$ and $G_-(\alpha)$ are analytic functions respectively in the upper $(\tau > -k_2)$ and the lower $(\tau < k_2)$ halves of the complex α-plane and are given by ([41], pp.154):

$$G_+(\alpha) = G_-(-\alpha) = \sqrt{\frac{\cos ka}{k+\alpha}}\, e^{i\frac{\pi}{4}}\, e^{i\frac{\alpha a}{\pi}[1-c+\ln\frac{\pi}{2ka}+i\frac{\pi}{2}]}$$
$$e^{i\frac{\gamma a}{\pi}\ln\frac{\alpha-\gamma}{k}} \prod_{n=1,3,5,\ldots}^{\infty} \left(1+\frac{\alpha}{i\gamma_n}\right)e^{i\frac{2\alpha a}{n\pi}} \tag{2-21}$$

where $c = 0.57721\ \ldots$ is Euler's constant, and

$$\gamma_n = \left[\left(\frac{n\pi}{2a}\right)^2 - k^2\right]^{1/2} \tag{2-22}$$

2.3 Solution of the modified Wiener-Hopf equation

Equation (2-17) can be modified to the form:

$$J_-(\alpha)G_-(\alpha) + \Phi_1(a,\alpha)/G_+(\alpha) = \frac{i\pi\ell}{2a\sqrt{2\pi}}\,(-1)^{\frac{\ell-1}{2}}\,\frac{G_-(\alpha)}{\alpha+i\gamma_\ell}$$
$$-e^{i\alpha L}G_-(\alpha)\left[\frac{i\pi\ell}{2a\sqrt{2\pi}}\,(-1)^{\frac{\ell-1}{2}}\,\frac{e^{-\gamma_\ell L}}{\alpha+i\gamma_\ell} + J_+(\alpha)\right] \tag{2-23}$$

The right hand side of (2-23) is then decomposed by isolating the pole in the first term and using decomposition formula for the second term. Thus, we have,

$$\frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_-(\alpha)}{\alpha+i\gamma_\ell} = \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}[G_-(\alpha) - G_+(i\gamma_\ell)]/(\alpha+i\gamma_\ell)$$

$$+ \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell} \qquad (2\text{-}24)$$

and

$$-e^{i\alpha L}G_-(\alpha)[\frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{e^{-\gamma_\ell L}}{\alpha+i\gamma_\ell} + J_+(\alpha)] = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{G_-(\beta)M(\beta)e^{i\beta L}}{\beta-\alpha}d\beta$$

$$+ \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_-(\beta)M(\beta)e^{i\beta L}}{\beta-\alpha}d\beta \ , \ -k_2<-d<\tau<d<k_2 \qquad (2\text{-}25)$$

where

$$M(\alpha) = \frac{-i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{e^{-\gamma_\ell L}}{\alpha+i\gamma_\ell} - J_+(\alpha) \qquad (2\text{-}26)$$

The first and the second terms in equations (2-24) and (2-25) are regular respectively in the lower $(\tau < k_2)$ and the upper $(\tau > -k_2)$ halves of the complex α-plane. Substituting (2-24) and (2-25) into (2-23), we have,

$$J_-(\alpha)G_-(\alpha) - \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{1}{\alpha+i\gamma_\ell}[G_-(\alpha) - G_+(i\gamma_\ell)] +$$

$$\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{G_-(\beta)M(\beta)e^{i\beta L}}{\beta-\alpha}d\beta = -\Phi_1(a,\alpha)/G_+(\alpha)$$

$$+ \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell} + \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_-(\beta)M(\beta)e^{i\beta L}}{\beta-\alpha}d\beta \qquad (2\text{-}27)$$

Note that the left hand side of (2-27) contains functions that are regular in the lower half of the α-plane defined by $\tau < k_2$, while the right-hand side contains functions that are regular in the upper half of the α-plane, defined by $\tau > -k_2$. Since these two half planes overlap, it follows from the analytic continuation that (2-27) is defined in the entire α-plane, and both sides are equal to an entire function $P(\alpha)$.

It can be shown with the help of Meixner's edge condition [42], that $P(\alpha)$ is bounded, and equals to zero as $|\alpha| \to \infty$. By Liouville's theorem [43], $P(\alpha)$ is identically zero everywhere. From (2-27) it follows that:

$$\frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell} + N(\alpha)G_-(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{G_-(\beta)M(\beta)e^{i\beta L}}{\beta-\alpha}\,d\beta$$

$$,\ -k_2<-d<\tau<d<k_2 \qquad (2\text{-}28)$$

for all α where $N(\alpha)$ is defined by:

$$N(\alpha) = J_-(\alpha) - \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{1}{\alpha+i\gamma_\ell} \qquad (2\text{-}29)$$

Equation (2-28) contains two unknown functions $N(\alpha)$ and $M(\alpha)$. Thus to find these functions, we need another equation which can be obtained in the following manner. Multiplying both sides of (2-17) by $e^{-i\alpha L}G_+(\alpha)$, we have:

$$J_+(\alpha)G_+(\alpha) + \frac{e^{-i\alpha L}\Phi_1(a,\alpha)}{G_-(\alpha)} = \frac{-i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{e^{-\gamma_\ell L}}{\alpha+i\gamma_\ell}G_+(\alpha)-e^{-i\alpha L}N(\alpha)G_+(\alpha)$$

$$(2\text{-}30)$$

In the last equation, the only term that has singularities in both halves of the α-plane is the second term in the right hand side, and must be decomposed in the same manner as equation (2-25), i.e.

$$e^{-i\alpha L}G_+(\alpha)N(\alpha) = \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)N(\beta)e^{-i\beta L}}{\beta-\alpha}\,d\beta - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{G_+(\beta)N(\beta)e^{-i\beta L}}{\beta-\alpha}\,d\beta$$

$$, \quad -k_2<-d<\tau<d<k_2 \qquad (2\text{-}31)$$

Substituting (2-31) into (2-30), after some rearrangement, we have,

$$\frac{e^{-i\alpha L}\Phi_1(a,\alpha)}{G_-(\alpha)} - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{G_+(\beta)N(\beta)e^{-i\beta L}}{\beta-\alpha}\,d\beta =$$

$$M(\alpha)G_+(\alpha) - \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)N(\beta)e^{-i\beta L}}{\beta-\alpha}\,d\beta \qquad (2\text{-}32)$$

Similar to the arguments after equation (2-27), both sides of (2-32) are zero, and hence we obtain from the right hand side,

$$M(\alpha)G_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)N(\beta)e^{-i\beta L}}{\beta-\alpha}\,d\beta \quad , \quad -k_2<-d<\tau<d<k_2 \qquad (2\text{-}33)$$

This is the other equation for the two unknowns $N(\alpha)$ and $M(\alpha)$. Equations (2-28) and (2-33) are two coupled integral equations in the two unknowns $N(\alpha)$ and $M(\alpha)$. These two equations can be decoupled [41] by changing β to $-\beta$ in (2-28) and α to $-\alpha$ in (2-33). Then the sum and difference of the obtained two equations, lead to

$$\frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell} + G_-(\alpha)S(\alpha) = \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)S(\beta)e^{-i\beta L}}{\beta+\alpha}\,d\beta$$

$$, \quad -k_2<-d<\tau<d<k_2 \qquad (2\text{-}34a)$$

and

$$\frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell}+G_-(\alpha)D(\alpha)=\frac{-1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)D(\beta)e^{-i\beta L}}{\beta+\alpha}\,d\beta$$

$$,\ -k_2<-d<\tau<d<k_2 \qquad (2\text{-}34b)$$

where

$$\begin{matrix} S(\alpha) \\ D(\alpha) \end{matrix} = N(\alpha)\pm M(-\alpha)$$

$$= [J_-(\alpha)\mp J_+(-\alpha)] - \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}}\left[\frac{1}{\alpha+i\gamma_\ell}\mp\frac{e^{-\gamma_\ell L}}{\alpha-i\gamma_\ell}\right] \qquad (2\text{-}35)$$

where the upper and lower signs correspond respectively to $S(\alpha)$ and $D(\alpha)$. Equations (2-34a) and (2-34b) are two decoupled integral equations and are of the same form. Solving these two equations for $S(\alpha)$ and $D(\alpha)$, we can obtain $J_+(\alpha)$ and $J_-(\alpha)$ using (2-35). Substitution of the results into the modified Wiener-Hopf equation (2-17) leads to $\Phi_1(a,\alpha)$. Consequently, using (2-15) and (2-16), $A(\alpha)$ and $B(\alpha)$ can be determined, and hence one can obtain the final solution through (2-12) and the inverse Fourier transform of $\Phi(a,\alpha)$, as will be shown later.

The integral in the right-hand side of (2-34) is of the form

$$I=\int_{-\infty-id}^{\infty-id}\frac{G_+(\beta)E(\beta)}{\beta+\alpha}e^{-i\beta L}\,d\beta$$

$$=\int_{-\infty-id}^{\infty-id}\frac{\cosh\gamma a\, e^{-\gamma a}E(\beta)}{\gamma(\beta+\alpha)G_-(\beta)}e^{-i\beta L}\,d\beta \qquad (2\text{-}36)$$

where $E(\alpha)$ is $S(\alpha)$ or $D(\alpha)$ and $G_+(\alpha)$ has been replaced by $G(\alpha)/G_-(\alpha)$ in equation (2-36) with $G(\alpha)$ given by (2-20).

For large L, the major contribution of I is from the integral over a small neighborhood around the branch point $\beta = -k$ [26]. The

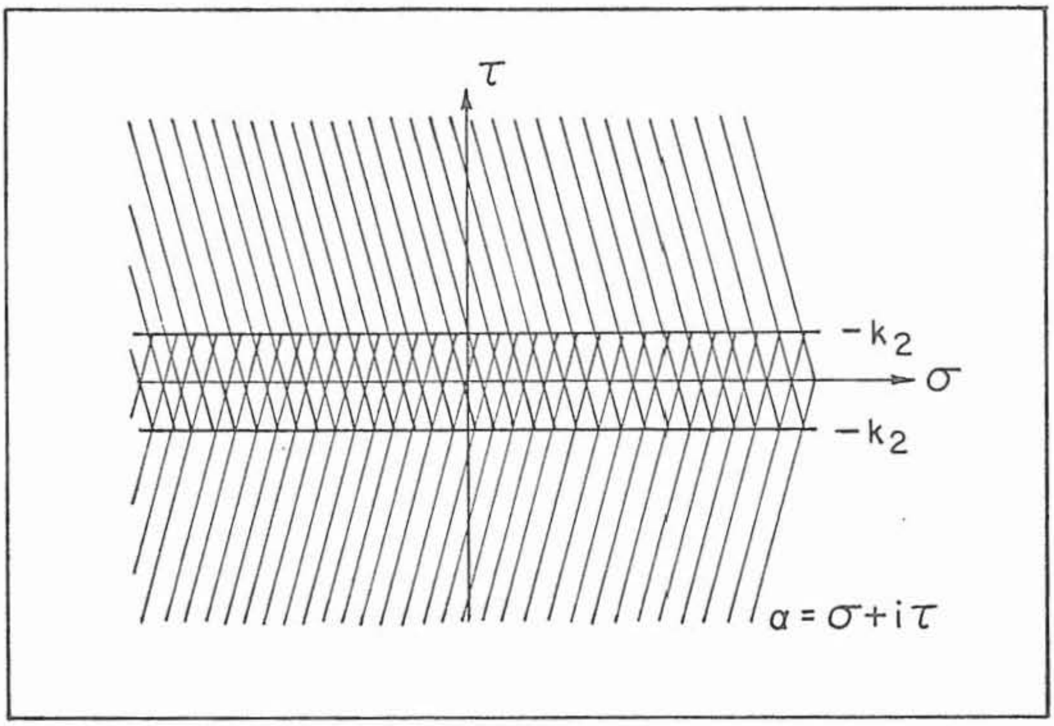

Figure 2-2 Complex α - plane

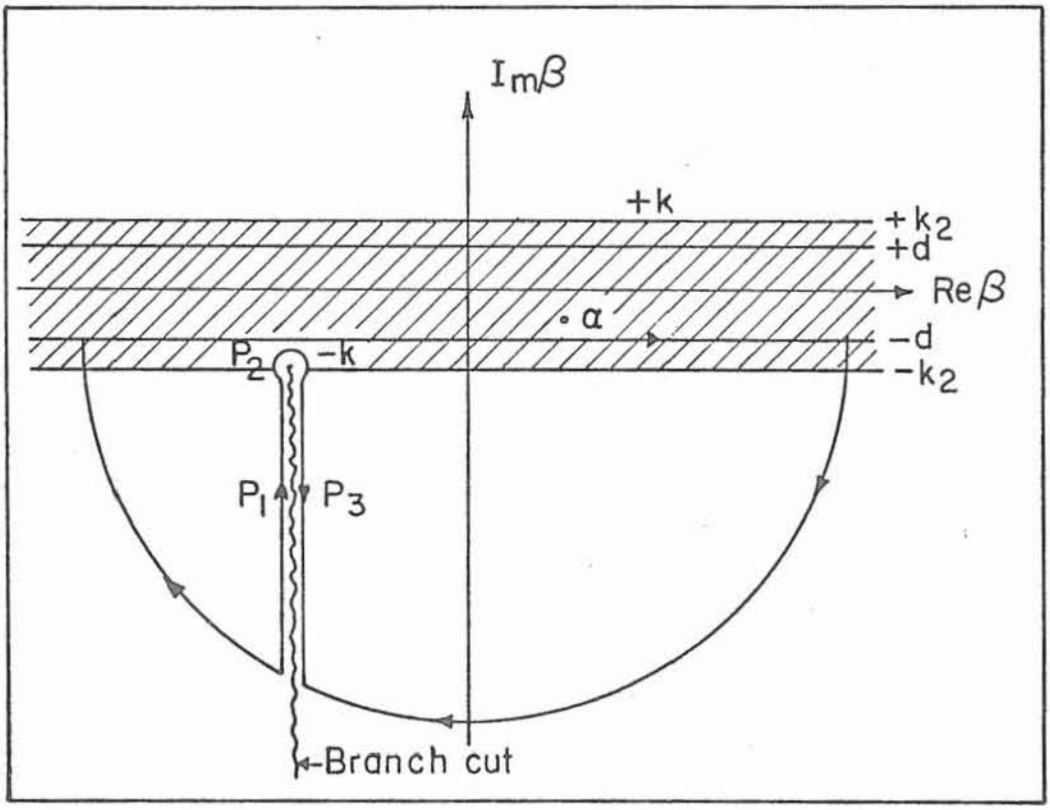

Figure 2-3 Contour of integral I in the β - plane

contour can then be deformed into the lower half of the β-plane, as shown in figure 2-3. The functions $G_-(\beta)$ and $E(\beta)$ are then expanded in a Taylor series about the branch point $\beta = -k$ and retaining the first term only one obtains

$$I = a\,\frac{E(-k)}{G_-(-k)} \int_p \frac{\cosh\gamma a}{\gamma a}\, e^{-\gamma a}\, \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta \tag{2-37}$$

where $p = p_1 + p_2 + p_3$. The integral over the small circle p_2 can be shown to be zero and hence (2-37) can be simplified to

$$I = a\,\frac{E(-k)}{G_+(k)} \left[\int_{-k}^{-k-i\infty} \frac{\cosh\gamma a}{\gamma a}\, e^{-\gamma a}\, \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta \right.$$

$$\left. - \int_{-k}^{-k-i\infty} \frac{\cosh\gamma a}{-\gamma a}\, e^{\gamma a}\, \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta \right]$$

$$= a\,\frac{E(-k)}{G_+(k)}\, T(\alpha) \tag{2-38}$$

where

$$T(\alpha) = 2 \int_{-k}^{-k-i\infty} \frac{\cosh^2\gamma a}{\gamma a(\beta+\alpha)}\, e^{-i\beta L}\, d\beta \tag{2-39}$$

Let $\beta = -k - \frac{iu}{L}$, where u is a new variable, then (2-39) becomes,

$$T(\alpha) = 2\,\frac{kL}{ka}\, e^{ikL} \int_0^\infty \frac{\cosh^2\left[\frac{ka}{kL}\sqrt{2ikL\,u-u^2}\right]}{\sqrt{2ikL\,u-u^2}\left[u+ikL\left(\frac{\alpha}{k}-1\right)\right]}\, e^{-u}\, du \tag{2-40}$$

This semi-infinite integral can be calculated numerically, for any value of α by the Gauss-Laguerre quadrature formula owing to the exponentially decreasing term $\exp(-u)$. Substituting (2-38) into (2-34) one obtains

$$E(\alpha) = \frac{-i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{G_-(\alpha)\cdot(\alpha+i\gamma_\ell)} \pm \frac{a}{2\pi i} \frac{E(-k)}{G_+(k)} \frac{T(\alpha)}{G_-(\alpha)} \tag{2-41}$$

where in the right hand side, the upper and lower signs correspond to $E(\alpha) = S(\alpha)$ or $D(\alpha)$, respectively. In (2-41), we have the unknown $E(-k)$, which can be obtained by letting $\alpha = -k$ in both sides of this equation, i.e.

$$E(-k) = \frac{-i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{G_+(k)(i\gamma_\ell-k)} \frac{1}{1\pm F} \tag{2-42}$$

where

$$F = \frac{-a}{2\pi i} \frac{T(-k)}{G_+^2(k)} \tag{2-43}$$

From (2-35) and (2-17), $\Phi_1(a,\alpha)$ can be written as a function of $S(\alpha)$ and $D(\alpha)$ in the form:

$$\Phi_1(a,\alpha) = \frac{1}{2}G(\alpha)[-\{S(\alpha) + D(\alpha)\} + e^{i\alpha L}\{S(-\alpha) - D(-\alpha)\}] \tag{2-44}$$

Using (2-41) in (2-44), we get:

$$\Phi_1(a,\alpha) = G_+(\alpha)[\frac{i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{\alpha+i\gamma_\ell} - \frac{a}{4\pi i} \frac{T(\alpha)}{G_+(k)}\{S(-k)-D(-k)\}]$$

$$+ \frac{a}{4\pi i} G_-(\alpha) \frac{T(-\alpha)}{G_+(k)} e^{i\alpha L}\{S(-k) + D(-k)\} \tag{2-45}$$

where $S(-k)$ and $D(-k)$ are given by (2-42) and are rewritten as

$$S(-k) = \frac{-i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{G_+(k)(i\gamma_\ell-k)} \frac{1}{1+F} \tag{2-46}$$

and

$$D(-k) = \frac{-i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{G_+(k)(i\gamma_\ell-k)} \frac{1}{1-F} \tag{2-47}$$

Substituting (2-46) and (2-47) into (2-45) and after some rearrangement, one has,

$$\Phi_1(a,\alpha) = \frac{i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} G_+(i\gamma_\ell) \frac{G_+(\alpha)}{\alpha+i\gamma_\ell}$$

$$- \frac{i\pi\ell}{2a\sqrt{2\pi}} (-1)^{\frac{\ell-1}{2}} G_+(i\gamma_\ell) \frac{1}{(i\gamma_\ell-k)} \frac{F}{1-F^2}(\frac{a}{2\pi i}) \frac{T(\alpha)G_+(\alpha)}{G_+^2(k)}$$

$$- \frac{i\pi\ell}{2a\sqrt{2\pi}}(-1)^{\frac{\ell-1}{2}} G_+(i\gamma_\ell) \frac{1}{(i\gamma_\ell-k)} \frac{1}{1-F^2}(\frac{a}{2\pi i}) \frac{T(-\alpha)G_+(-\alpha)}{G_+^2(k)} e^{i\alpha L} \quad (2\text{-}48)$$

where in (2-48), F is given by (2-43), and $T(-k)$ is obtained by calculating the integral in (2-40) for $\alpha = -k$. Once $\Phi_1(a,\alpha)$ is obtained, the constants $A(\alpha)$ and $B(\alpha)$ can be obtained from (2-15) and (2-16), which completes the solution of the modified Wiener-Hopf equation. The scattered field can then be determined by going back to (2-12) and taking the inverse Fourier transform which will be shown in the next section.

2.4 Evaluation of the scattered fields

2.4.1 Radiation field

In the region outside the two waveguides, the scattered electric field component E_y^s, for $x > a$, is given by,

$$E_y^s = \phi^s(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} A(\alpha)e^{-\gamma x} e^{-i\alpha z} d\alpha, \qquad |\tau| < k_2 \quad (2\text{-}49)$$

From (2-15) and (2-49) we have

$$\phi^s(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \Phi_1(a,\alpha)\, e^{\gamma(a-x)}\, e^{-i\alpha z} d\alpha \quad , \quad |\tau| < k_2 \quad (2\text{-}50)$$

with $\Phi_1(a,\alpha)$ given by (2-48).

In the far zone, i.e. $k\rho >> 1$, the saddle point method of integration can be applied to (2-50). Hence with the knowledge that $\Phi_1(a,\alpha)$ has no singularities in the two-sheeted α-plane except the branch singularities, it can be shown easily that

$$\phi^s(x,z) = \phi^s(\rho,\theta) = \frac{e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{k\rho}} k \sin\theta \, \Phi_1(a,k \cos\theta) e^{-ika \sin\theta} \quad (2\text{-}51)$$

where ρ and θ are polar coordinates defined in figure 2-1, substituting (2-48) into (2-51) with $\alpha = k \cos\theta$, we obtain

$$\phi^s(\rho,\theta) = \sqrt{\frac{2}{\pi k\rho}}\, e^{i(k\rho-\frac{\pi}{4})} F(\theta) \quad (2\text{-}52)$$

where

$$F(\theta) = \frac{i\pi\ell}{4a} (-1)^{\frac{\ell-1}{2}} G_+(i\gamma_\ell) k \sin\theta \, e^{-ika \sin\theta} \Big[\frac{G_+(k \cos\theta)}{k \cos\theta + i\gamma_\ell}$$

$$- \frac{a}{2\pi i} \frac{F\, T(k \cos\theta) G_+(k \cos\theta)}{G_+^2(k)(i\gamma_\ell - k)(1-F^2)} - \frac{a}{2\pi i} \frac{T(-k \cos\theta) G_+(-k \cos\theta) e^{ikL\cos\theta}}{G_+^2(k)(i\gamma_\ell - k)(1-F^2)} \Big]$$

$$(2\text{-}53)$$

Power radiated at point $p(\rho,\theta)$ is characterized by $|F(\theta)|^2$. The expression for the far field is originally obtained only for $x > a$, $o < \theta < \pi$, but it is easily seen that the formula for $F(\theta)$ will also hold for $x < -a$, $\pi < \theta < 2\pi$. Total power radiated per unit length of the y-axis is given by $\int_0^{2\pi} |F(\theta)|^2 d\theta$. The radiation field consists of three terms. The first term is the well known result of the radiation field from the open end of the exciting waveguide only (i.e. in the absence of the coupled waveguide). The second and third terms are due to interaction between the two waveguides. The second term represents the reradiation field from the open end of the exciting waveguide due to the interaction between the openings of the two waveguides, while the third term represents the radiation from the opening end of the coupled waveguide due to the previous interaction. These explanations will be more clear in Chapter 4.

2.4.2 Reflected field

In the region inside the exciting waveguide $(z < 0)$, the reflected electric field component may be given by,

$$\phi_r(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} 2B(\alpha)\cosh \gamma x \cdot e^{-i\alpha z}\, d\alpha \quad , \quad |\tau| < k_2 \qquad (2\text{-}54)$$

From (2-15) and (2-54), we have

$$\phi_r(x,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \Phi_1(a,\alpha)\, \frac{\cosh \gamma x}{\cosh \gamma a}\, e^{-i\alpha z}\, d\alpha \quad , \quad |\tau| < k_2 \qquad (2\text{-}55)$$

with $\Phi_1(a,\alpha)$ given by (2-48).

The integral (2-55) is evaluated for each term of $\Phi_1(a,\alpha)$. For the first and the second terms of $\Phi_1(a,\alpha)$, we close the contour of integration in the upper half of the complex α-plane, as shown in figure 2-4. The only singularities so enclosed are the poles at $\alpha = i\gamma_m$, where $\gamma_m = \sqrt{(\frac{m\pi}{2a})^2 - k^2}$ and $m = 1,3,5,\ldots$. Evaluating these residue contributions for each term, we have, due to first term,

$$\phi_r^{exc}(x,z) = \sum_{m=1,3,5,..}^{\infty} R_{\ell,m} \cos(\frac{m\pi}{2a}x) e^{\gamma_m z} \qquad (2\text{-}56)$$

where

$$R_{\ell,m} = -\frac{\ell\pi^2}{4a^3} (-1)^{\frac{\ell+m}{2}} \frac{m}{\gamma_m} \frac{G_+(i\gamma_\ell)G_+(i\gamma_m)}{(\gamma_m+\gamma_\ell)} \qquad (2\text{-}57)$$

While for the second term,

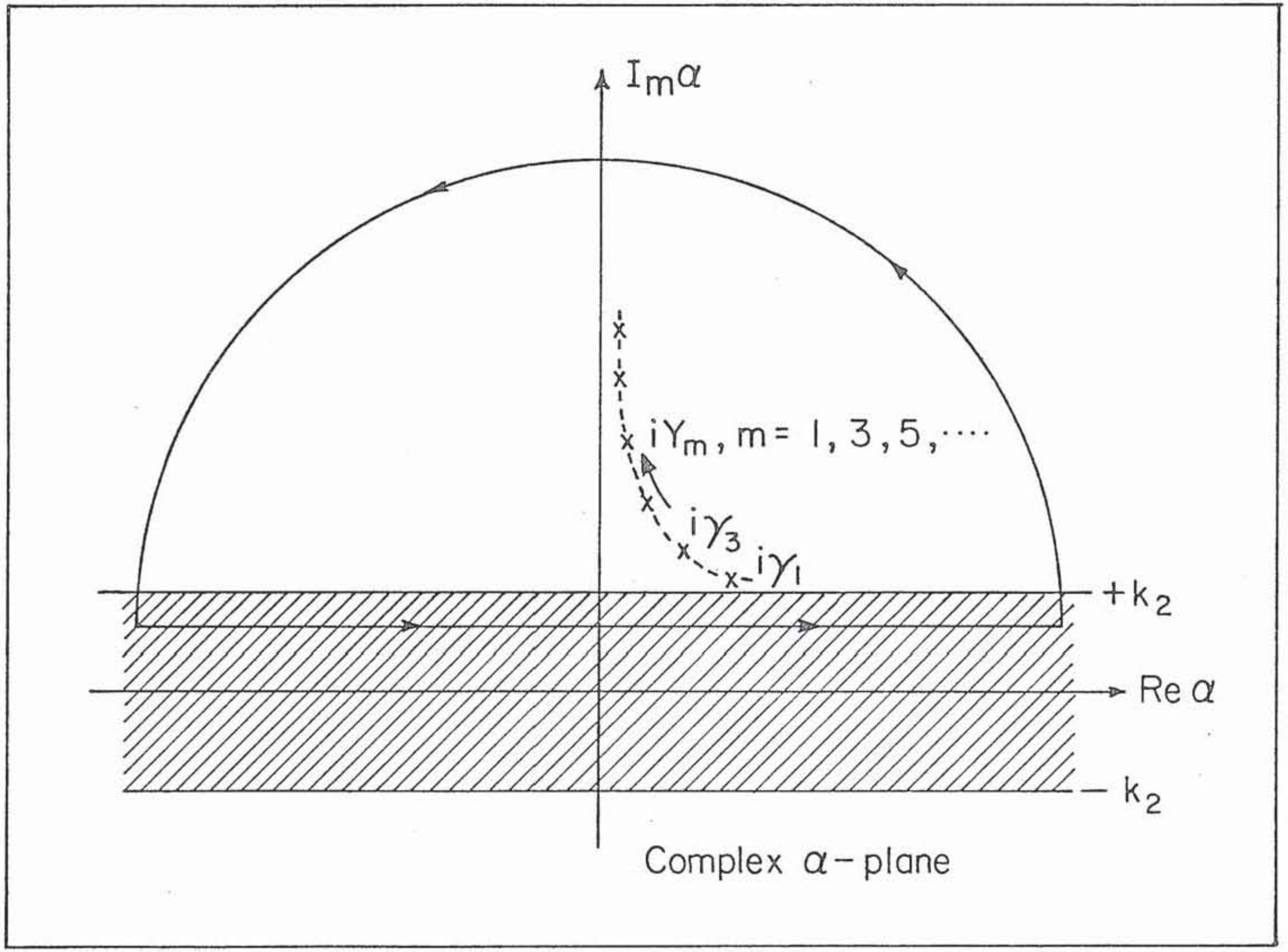

Fig. 2-4 Contour of integration for the first and second terms of $\Phi_1(a,\alpha)$ in equation (2-55)

$$\phi_r^{int,(1)}(x,z) = \sum_{m=1,3,5,..}^{\infty} R_{\ell,m}^{(1)} \cos(\frac{m\pi}{2a}x) e^{\gamma_m z} \tag{2-58}$$

where

$$R_{\ell,m}^{(1)} = \frac{i\ell\pi^2}{4a^3} (-1)^{\frac{\ell+m}{2}} \frac{m}{\gamma_m} \frac{G_+(i\gamma_\ell)G_+(i\gamma_m)}{(i\gamma_\ell-k)G_+^2(k)} T(i\gamma_m) \frac{a}{2\pi i} \frac{F}{1-F^2} \tag{2-59}$$

And for the third term in $\Phi_1(a,\alpha)$, (2-55) becomes,

$$\phi_r^{int,(2)}(x,z) = \frac{1}{\sqrt{2\pi}} (\frac{-i\pi\ell}{2a\sqrt{2\pi}})(-1)^{\frac{\ell-1}{2}} \frac{G_+(i\gamma_\ell)}{G_+^2(k)} \frac{1}{i\gamma_\ell-k} \frac{1}{1-F^2} (\frac{a}{2\pi i})$$

$$\int_{-\infty+i\tau}^{\infty+i\tau} T(-\alpha)G_+(-\alpha) \frac{\cosh \gamma x}{\cosh \gamma a} e^{i\alpha L} e^{-i\alpha z} d\alpha \quad , \quad |\tau| < k_2 \tag{2-60}$$

The superscripts exc and int refer to the scattered fields due to the exciting waveguide alone and to the interactions between the two waveguides, respectively. Assuming that $\phi_r^{int,(2)}$ is a modal series in the form

$$\phi_r^{int,(2)}(x,z) = \sum_{m=1,3,5,..}^{\infty} R_{\ell,m}^{(2)}(z) \cos(\frac{m\pi}{2a}x) \tag{2-61}$$

then with equation (2-60), and the orthogonality relation over cos x, one obtains

$$R_{\ell,m}^{(2)}(z) = \frac{-i\pi\ell m}{4a^3} \frac{G_+(i\gamma_\ell)}{G_+^2(k)} \frac{(-1)^{\frac{\ell+m}{2}}}{i\gamma_\ell-k} \frac{1}{1-F^2} (\frac{a}{2\pi i}) \int_{-\infty+i\tau}^{\infty+i\tau} \frac{T(-\alpha)\cosh \gamma a}{\gamma G_+(\alpha)(\alpha^2+\gamma_m{}^2)}$$

$$e^{-\gamma a+i\alpha(L-z)} d\alpha \quad , \quad |\tau| < k_2 \tag{2-62}$$

Closing the contour in the upper half of the complex α-plane, the only singularity so enclosed is the branch point at $\alpha = k$. It can be easily shown that

$$R^{(2)}_{\ell,m}(z) = \frac{-i\pi\ell m}{2a^2}\,\frac{G_+(i\gamma_\ell)}{G_+^2(k)}\,\frac{(-1)^{\frac{\ell+m}{2}}}{i\gamma_\ell-k}\,\frac{1}{1-F^2}\,\left(\frac{a}{2\pi i}\right)\int_k^{k+i\infty}\frac{\cosh^2\gamma a}{G_+(\alpha)\cdot\gamma a}\,\frac{T(-\alpha)}{\alpha^2+\gamma_m^2}\,e^{i\alpha(L-z)}\,d\alpha$$

(2-63)

The above integral cannot be evaluated analytically, but can be reduced to the following form which is more suitable for numerical integrations, by the Gauss-Laguerre quadrature formula, owing to an exponentially decreasing term:

$$R^{(2)}_{\ell,m}(z) = \frac{\pi\ell m}{2a^3}\,\frac{G_+(i\gamma_\ell)}{G_+^2(k)}\,\frac{e^{ik(L-z)}}{i\gamma_\ell-k}\,\frac{(-1)^{\frac{\ell+m}{2}}}{1-F^2}\,\left(\frac{a}{2\pi i}\right)$$

$$\int_0^\infty \frac{\cosh^2\left[\frac{a}{L}\sqrt{2ikLu-u^2}\right]T\left(-k-\frac{iu}{L}\right)}{\sqrt{2ikLu-u^2}\;G_+\left(k+\frac{iu}{L}\right)\;\left[\left(k+\frac{iu}{L}\right)^2+\gamma_m^2\right]}\,e^{-u\left(\frac{L-z}{L}\right)}\,du \qquad (2\text{-}64)$$

Hence, the reflected electric field is given by,

$$\phi_r(x,z) = \phi_r^{exc}(x,z) + \phi_r^{int,(1)}(x,z) + \phi_r^{int,(2)}(x,z)$$

$$= \sum_{m=1,3,5,..}^{\infty}\left[\left(R_{\ell,m}+R^{(1)}_{\ell,m}\right)e^{\gamma_m z} + R^{(2)}_{\ell,m}(z)\right]\cos\left(\frac{m\pi}{2a}x\right) \qquad (2\text{-}65)$$

where $R_{\ell,m}$, $R^{(1)}_{\ell,m}$ and $R^{(2)}_{\ell,m}$ are the reflection coefficients and are given respectively by (2-57), (2-59) and (2-64).

The reflected field is expressed by three terms. The first term $\phi_r^{exc}(x,z)$ is the reflected field due to the open end of the exciting waveguide, in the absence of the coupled waveguide, and the reflection coefficient is given by (2-57). The second and the third terms are due to the interaction between the two waveguides, with the second term $\phi_r^{int,(1)}(x,z)$ being the scattered field due to the open end of the exciting waveguide, when it is illuminated by the scattered field from the coupled waveguide, and the reflection coefficient of this component, $R^{(1)}_{\ell,m}$, is given by (2-59). The third term $\phi_r^{int,(2)}(x,z)$ is the scattered field from the coupled waveguide (in the absence of the exciting waveguide), and the reflection coefficient is given by (2-64). From (2-60), it is clear that this component is a continuous spectrum of inhomogeneous plane waves, and decays to zero at $z = -\infty$ (to satisfy the Sommerfeld radiation condition). Also, for large values of L or large values of z , the saddle point method of integration can be used to evaluate such an integral. The reflection coefficient $R^{(2)}_{\ell,m}$ has been expressed in this form, so that one can determine it at any values of z , expecially if one wants to find the aperture field at $z = 0$. It can be easily shown by saddle point method of integration that this component behaves as $\frac{e^{ik(L-z)}}{\sqrt{k(L-z)}}$ with z , and hence far from the opening, the only contributing coefficients are $R_{\ell,m}$ and $R^{(1)}_{\ell,m}$.

2.4.3 Transmitted field

In the region inside the exciting waveguide $(z > L)$, the transmitted electric field component may be given by,

$$\phi_t(x,z) = \phi^i(x,z) + \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\tau}^{\infty+i\tau} \Phi_1(a,\alpha)\,\frac{\cosh\,\gamma x}{\cosh\,\gamma a}e^{-i\alpha z}\,d\alpha, \quad |\tau| < k_2 \tag{2-66}$$

Closing the contour in the lower half of the complex α-plane, one can evaluate the integral separately for each term of $\Phi_1(a,\alpha)$. For the first term of $\Phi_1(a,\alpha)$ expressed in (2-48), the only singularities so enclosed are the pole at $\alpha = -i\gamma_\ell$ and the branch point at $\alpha = -k$. The contribution due to the pole cancels exactly the incident field component $\phi^i(x,z)$ in (2-66), and contributions due to the branch point can be obtained in the same manner as for $\phi_r^{int,(2)}(x,z)$ in (2-60). Hence, one may write

$$\phi_t^{exc}(x,z) = \sum_{m=1,3,5,..}^{\infty} T_{\ell,m}(z)\cos(\frac{m\pi}{2a}x) \tag{2-67}$$

where $T_{\ell,m}(z)$ is given by

$$T_{\ell,m}(z) = \frac{\pi\ell m}{2a^3}(-1)^{\frac{\ell+m}{2}} e^{ikz} G_+(i\gamma_\ell)\int_0^\infty \frac{\cosh^2[\frac{a}{z}\sqrt{2ikzu-u^2}]e^{-u}}{\sqrt{2ikzu-u^2}\; G_+(k+\frac{iu}{z})}$$

$$\frac{1}{[(k+\frac{iu}{z})^2+\gamma_m{}^2][-k-\frac{iu}{z}+i\gamma_\ell]}\; du \tag{2-68}$$

Similarly for the second term of $\Phi_1(a,\alpha)$, the only enclosed singularity is the branch point at $\alpha=-k$ and its contribution may be written in the form

$$\phi_t^{int,(1)}(x,z) = \sum_{m=1,3,5,..}^{\infty} T_{\ell,m}^{(1)}(z)\cos(\frac{m\pi}{2a}x) \tag{2-69}$$

where

$$T_{\ell,m}^{(1)}(z) = \frac{-\pi\ell m}{2a^3}(-1)^{\frac{\ell+m}{2}} e^{ikz}(\frac{a}{2\pi i})\frac{F\, G_+(i\gamma\ell)}{(1-F^2)G_+^2(k)(i\gamma_\ell-k)}$$

$$\int_0^\infty \frac{\cosh^2[\frac{a}{z}\sqrt{2ikzu-u^2}]T(-k-\frac{iu}{z})\; e^{-u}}{\sqrt{2ikzu-u^2}\;[(k+\frac{iu}{z})^2+\gamma_m^2]G_+(k+\frac{iu}{z})}\; du \tag{2-70}$$

For the third term of $\Phi_1(a,\alpha)$, the enclosed singularities are the poles at $\alpha = -i\gamma_m$, with $\gamma_m = \sqrt{(\frac{m\pi}{2a})^2-k^2}$, and $m = 1,3,5,..$. Evaluating these residue contributions, one obtains

$$\phi_t^{int,(2)}(x,z) = \sum_{m=1,3,5,..}^{\infty} T_{\ell,m}^{(2)}\cos(\frac{m\pi}{2a}x)\; e^{-\gamma_m z} \tag{2-71}$$

where

$$T_{\ell,m}^{(2)} = \frac{i\ell\pi^2}{4a^3}(-1)^{\frac{\ell+m}{2}}\frac{mG_+(i\gamma_\ell)G_+(i\gamma_m)T(i\gamma_m)}{\gamma_m(1-F^2)(i\gamma_\ell-k)G_+^2(k)}(\frac{a}{2\pi i})e^{\gamma_m L} \tag{2-72}$$

Therefore, transmitted electric field is given by,

$$\phi_t(x,z) = \phi_t^{exc}(x,z) + \phi_t^{int,(1)}(x,z) + \phi_t^{int,(2)}(x,z)$$

$$= \sum_{m=1,3,5,..}^{\infty} [T_{\ell,m}(z)+T_{\ell,m}^{(1)}(z)+T_{\ell,m}^{(2)}e^{-\gamma_m z}]\cos(\frac{m\pi}{2a}x) \qquad (2\text{-}73)$$

where $T_{\ell,m}(z)$, $T_{\ell,m}^{(1)}(z)$ and $T_{\ell,m}^{(2)}(z)$ are the transmission coefficients given respectively by (2-68), (2-70) and (2-72).

Again $T_{\ell,m}(z)$ and $T_{\ell,m}^{(1)}(z)$ are expressed in convenient forms for numerical integration and may be computed using a Gauss-Laguerre quadrature formula to determine the aperture field. Furthermore, these two components decay as $\frac{e^{ikz}}{\sqrt{kz}}$ in order to satisfy the radiation condition at infinity. At large distances from the opening, the only contributing coefficient to the transmission field is $T_{\ell,m}^{(2)}$ which is due to the interaction between the two waveguides. The transmission coefficient $T_{\ell,m}^{(2)}$ may be related to $R_{\ell,m}^{(1)}$ by

$$T_{\ell,m}^{(2)} = \frac{e^{\gamma_m L}}{F} R_{\ell,m}^{(1)} \qquad (2\text{-}74)$$

or, it can be related to the reflection coefficient of a single waveguide $R_{\ell,m}$ by the relation

$$T_{\ell,m}^{(2)} = \frac{e^{\gamma_m L}}{F} \cdot \frac{-a(\gamma_m+\gamma_\ell)T(i\gamma_m)}{2\pi G_+^2(k)(i\gamma_\ell-k)} \frac{F}{1-F^2} \quad R_{\ell,m} \qquad (2\text{-}75)$$

In other words, $T_{\ell,m}^{(2)}$ and $R_{\ell,m}^{(1)}$ can be evaluated by knowing the reflection coefficient of a single semi-infinite waveguide and $G_+(k)$ and then evaluating the function $T(\alpha)$ at $\alpha = -k$ and $i\gamma_m$. Similarly, the power radiated from the aperture between the two waveguides can be expressed in terms of $R_{\ell,m}$, which for the dominant propagating mode is given by

$$P_{rad} = P_i[1-|R_{\ell,\ell}|^2 - |R^{(1)}_{\ell,\ell}|^2 - |T^{(2)}_{\ell,\ell}|^2] \tag{2-76}$$

2.5 Results and discussion

Some results are obtained for a waveguide size $2a/\lambda = 0.6$ and $TE_{0,1}$ excitation. The resulting infinite integrals in the formulation of the problem are computed by the Gauss-Laguerre quadrature formula with 15 intervals. As there are no previous results available, no comparison is given. However, for comparison, numerical methods are also used to solve this problem, which are presented in Chapter 7. The results presented in this section are for the radiation pattern $|F(\theta)|^2$, and the reflection and the transmission coefficients.

Figure 2-5 shows the radiation pattern for $kL = \infty$, 50, 10 and 5. Since the radiation patterns are symmetric with respect to the waveguide geometry, only the patterns for $0 \leq \theta \leq 180°$ are presented. As expected, with decreasing kL, the direction of the radiation main lobe level moves progressively away from the forward direction. By increasing the separation distance kL, the radiated power oscillates with θ around the pattern corresponding to a single semi-infinite parallel plate waveguide. However, the amount of radiated power should be an oscillating function of kL similar to the reflection and the transmission fields discussed below.

The reflection coefficients for modes 1 and 3 are shown in figure 2-6. Since $R^{(2)}_{\ell,m}(z)$ decays with z as $\frac{1}{\sqrt{z}}$, shown in the formulation, its corresponding terms are not included in computation, the amplitude and phase of the reflection coefficients are oscillating functions of

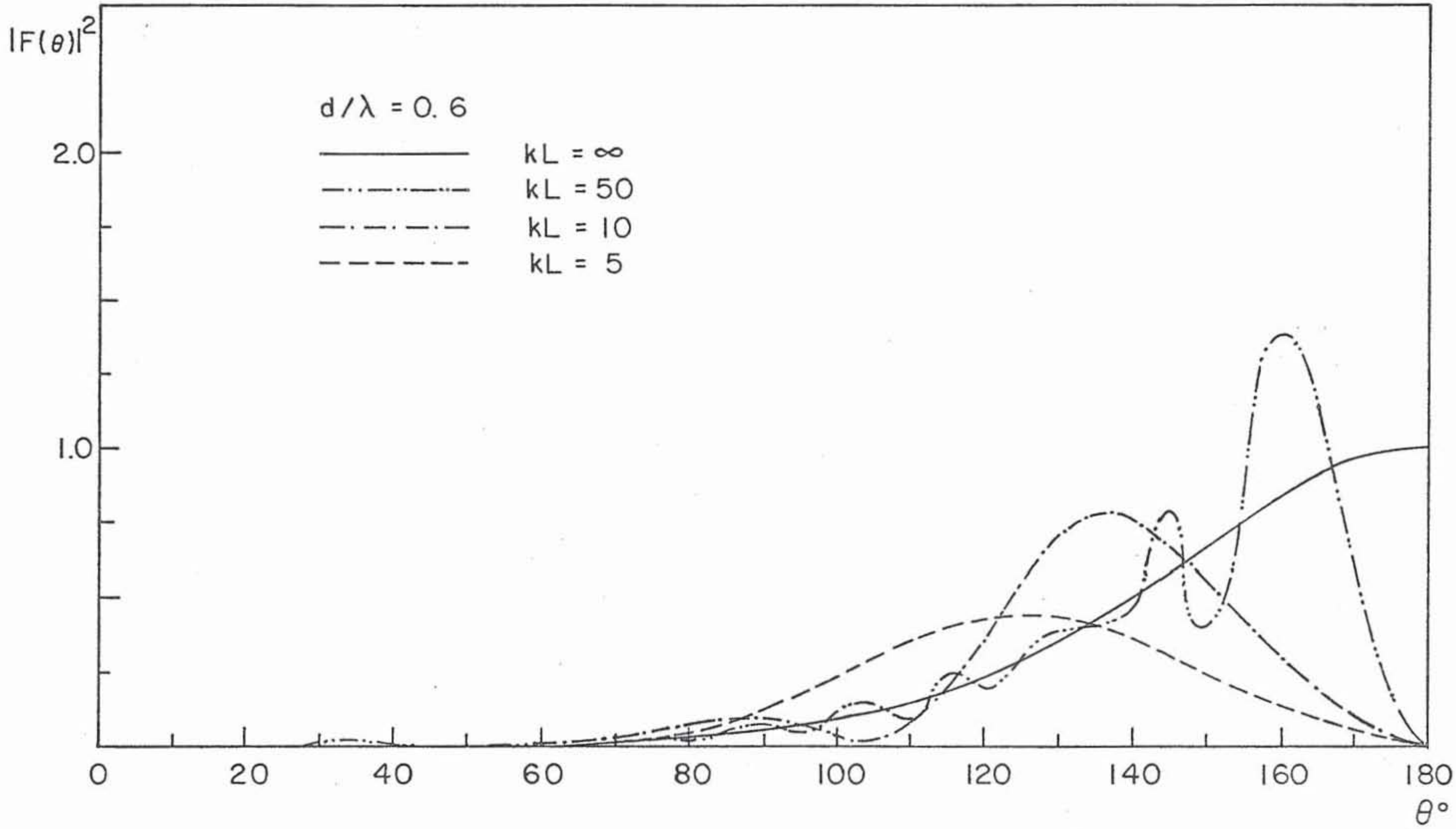

Figure 2-5 Radiation pattern of the $TE_{0,1}$ mode of Two collinear semi-infinite parallel-plate Waveguides.

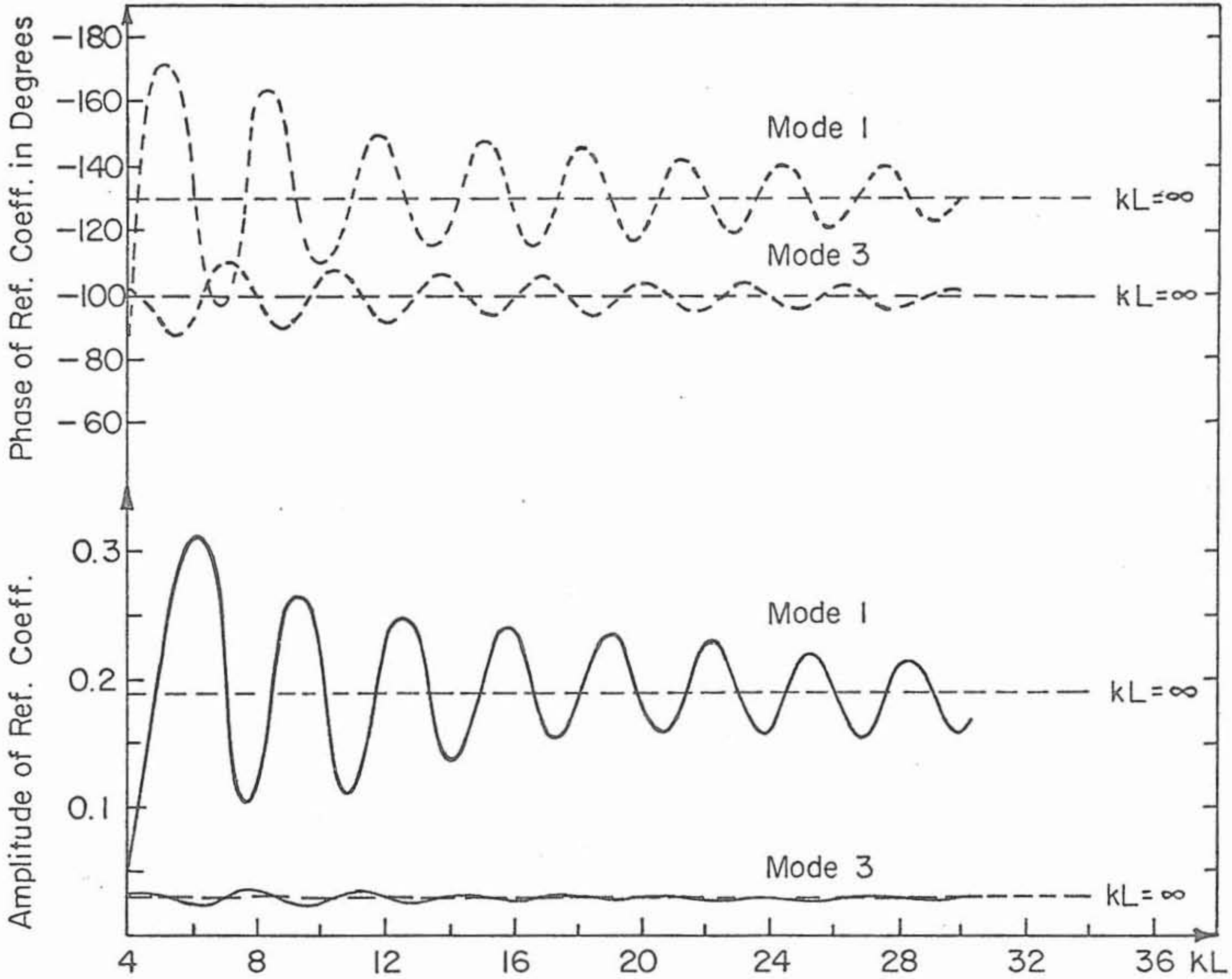

Figure 2-6 Reflection Coefficients of the $TE_{0,1}$ mode for $d/\lambda = 0.6$ of two collinear semi-infinite parallel-plate waveguides.

period kL equal π and decay continuously to reach the final values for $kL = \infty$, a single excited waveguide. Figure 2-7 represents the transmission coefficient (coupling to second waveguide) for the first dominant mode which is again an oscillating function, decaying to zero as kL approaches infinity. This transmission coefficient is again computed by neglecting the corresponding terms for the scattered fields which vanish at large distances from the opening.

In order to determine the aperture field distribution at the openings of both waveguides, we have to calculate the coefficients that are functions of z, i.e. $R^{(2)}_{\ell,m}(z)$, $T_{\ell,m}(z)$ and $T^{(1)}_{\ell,m}(z)$ which are given respectively by (2-64), (2-68) and (2-70). Although aperture field distribution can be obtained without expanding $\phi_r^{int,(2)}(x,z)$, $\phi_t^{exp}(x,z)$ and $\phi_t^{int,(1)}(x,z)$ in modal series, they have been expanded here to show the behaviour of field distribution with x. The functions $\phi_r^{int,(2)}(x,z)$, $\phi_t^{exp}(x,z)$ and $\phi_t^{int,(1)}(x,z)$ are continuous spectrums of inhomogeneous plane waves, and may be evaluated using the saddle point method of integration. However, the mode coefficients that are functions of z are computed again using the Gauss-Laguerre quadrature formula, and the total reflected and transmitted electric fields at the centre of the open end of the exciting waveguide and coupled waveguide are shown in figures 2-8 and 2-9, respectively, for modes 1, 3 and 5. These results can be used to find the resulting aperture fields.

The analysis in this chapter was carried out for $TE_{o,\ell}$ excitation with ℓ odd. The extension to $TE_{o,\ell}$ with ℓ even and $TM_{o,\ell}$ with even or odd ℓ is trivial and can be carried out with the proper Green's functions. In this chapter, the problem of coupling between two semi-infinite parallel plate waveguides is solved using the Wiener-Hopf

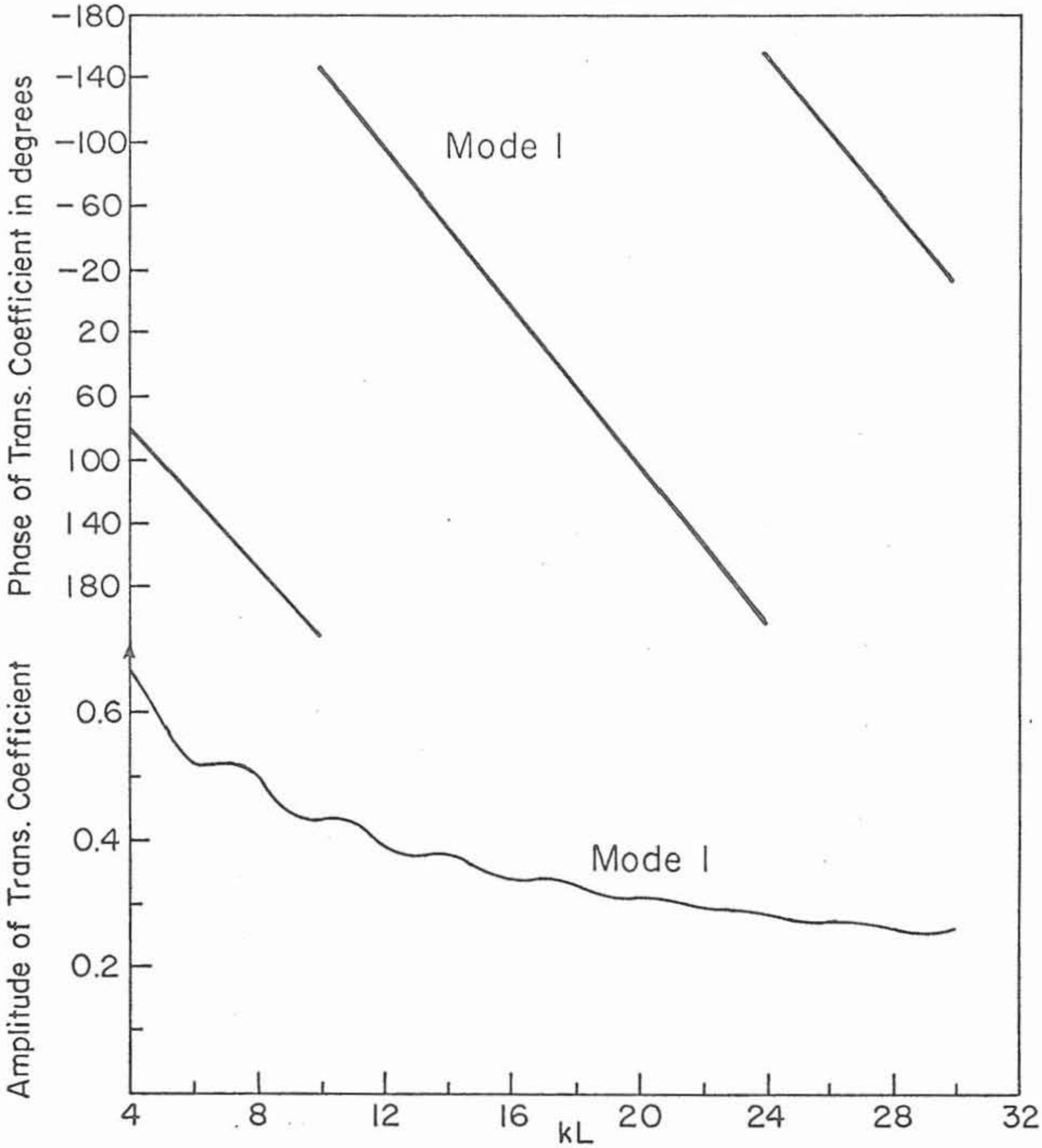

Figure 2-7 Transmission Coefficient of the $TE_{0,1}$ mode for $d/\lambda=0.6$ of two collinear semi-infinite parallel-plate Waveguides.

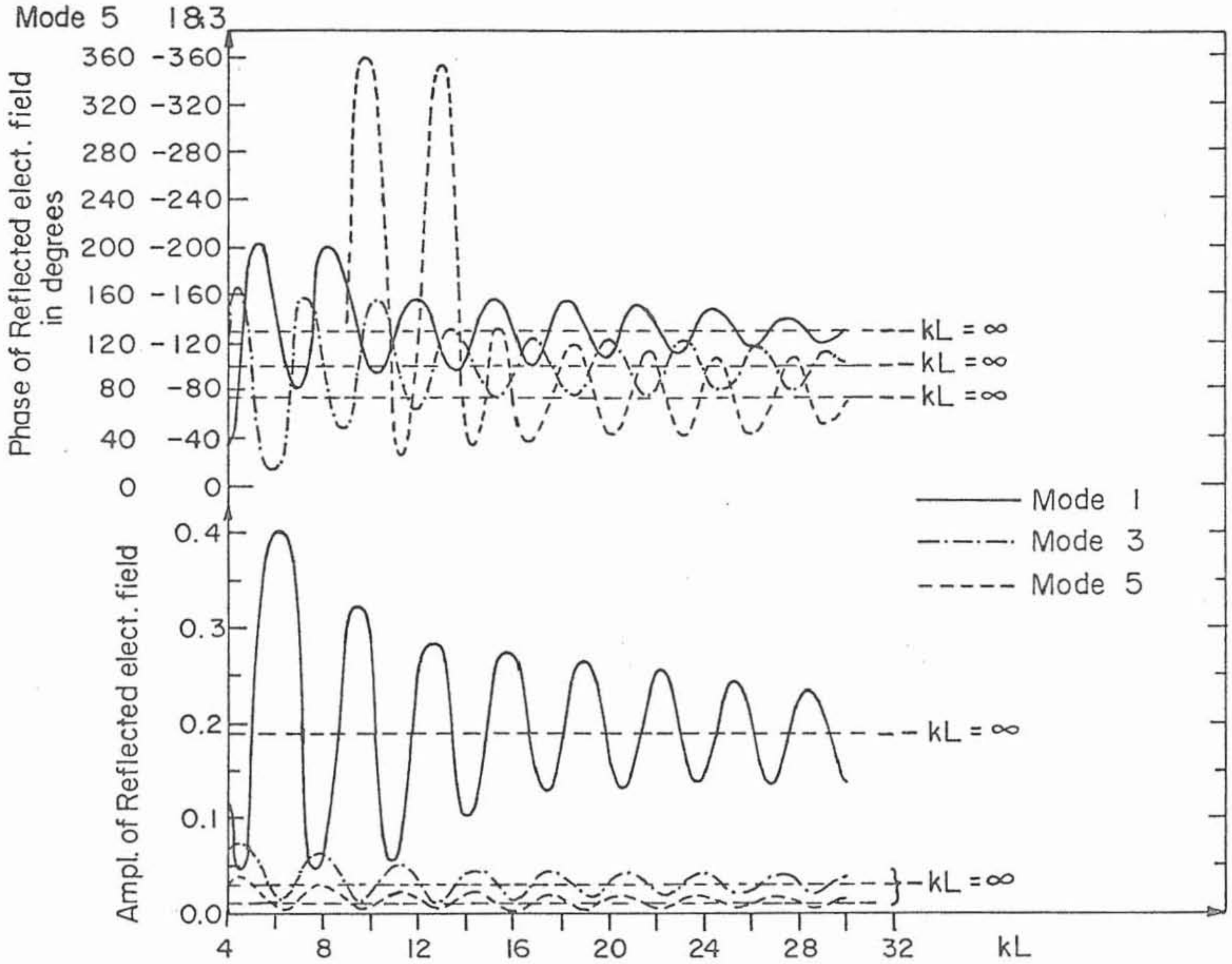

Figure 2-8 Reflected electric field at the center of the opening end of the exciting parallel-plate waveguide for an exciting $TE_{0,1}$ mode with $d/\lambda = 0.6$.

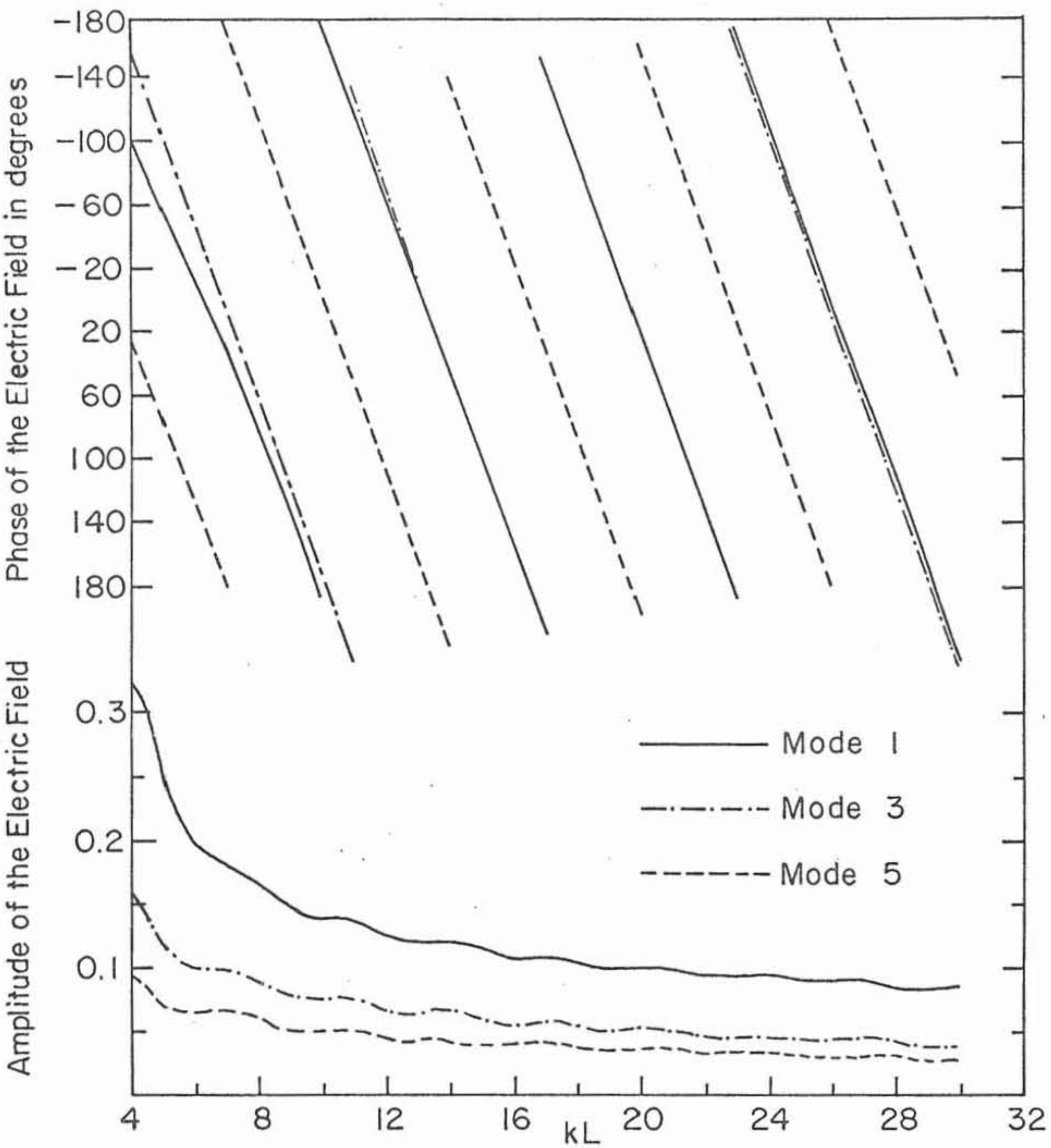

Figure 2-9 Electric field at the center of the opening of the coupled parallel-plate waveguide for an exciting $TE_{0,1}$ mode with $d/\lambda = 0.6$

technique. The analysis is being limited by the separation distance between the two waveguides. To complement the problem for any separation, numerical methods are adopted as are discussed in Chapter 7. Also, as the problem becomes more complicated for different waveguide widths, the results in this chapter are used to obtain the solution using the ray theory of diffraction and hence they can be modified to get an approximate solution for any waveguide width with any orientation. This is investigated in one of the sections of Chapter 4. However the results of coupling between two semi-infinite waveguides obtained in this chapter are useful to find an approximate solution for the coupling between waveguides of finite length. This is investigated in Chapter 5.

CHAPTER 3

COUPLING BETWEEN TWO COLLINEAR SEMI-INFINITE CIRCULAR WAVEGUIDES

3.1 Introduction

Scattering of electromagnetic and sound waves by cylindrical structures have recently received increasing attention due to their importance in radiation or other microwave problems [1],[2],[4],[6], [44]-[48]. Coupling between two semi-infinite circular waveguides provide useful information for optimizing the radiation characteristics of an open ended waveguide [6], since the two waveguides may act as a two-antennas system, with the exciting waveguide as the transmitting antenna and the coupled waveguide as the receiving antenna. This system can be used for near field measurements for a variety of applications such as sensor booms [4].

In this chapter the problem of coupling between two collinear semi-infinite circular waveguides is investigated for the symmetrical excitation of $TM_{o,m}$ mode. The Wiener-Hopf technique is used and a modified Wiener-Hopf equation is obtained which is then solved with a similar approach to that of Chapter 2. Solutions for the radiated, reflected and transmitted fields are obtained in terms of a semi-infinite integral which can be evaluated numerically by the Gauss-Laguerre formula [30],[38]. Solutions obtained here are similar to those of Chapter 2, and are used in Chapter 4 to get an approximate solution, which can be represented in terms of a new diffraction coefficient and a spherical wavefactor.

From the exact solution of $TM_{o,m}$ excitation, some results are obtained and shown in the last section.

3.2 Formulation of the problem

Consider two collinear infinitely thin and perfectly conducting semi-infinite circular waveguides of diameter 2a, separated by a distance L and located in free space, as shown in figure 3-1. With a time factor $e^{-i\omega t}$, an incident $TM_{o,m}$ mode is assumed to be propagating in the left (exciting) waveguide along the positive z-axis, with an electric hertz vector given by

$$\pi^i = \psi^i \hat{a}_z \tag{3-1}$$

where

$$\psi^i = J_o\left(\frac{\xi_{om}\rho}{a}\right) e^{-\gamma_{om} z} \tag{3-2}$$

with

$$\gamma_{om} = \left[\left(\frac{\xi_{om}}{a}\right)^2 - k^2\right]^{\frac{1}{2}} \qquad , \qquad m = 1,2,3,\ldots \tag{3-3}$$

In this equation ξ_{om} is the mth order zero of J_o, the zero order Bessel function, and $k = k_1 + ik_2$ as defined in Chapter 2. Similar to the analysis given in Chapter 2, Jones' method of formulation gives the following modified Wiener-Hopf equation of the second type (see Appendix A).

$$J_-(\alpha) + e^{i\alpha L} J_+(\alpha) - \frac{i\omega\varepsilon\Phi_1(a,\alpha)}{\gamma^2 a\cdot G(\alpha)} = \frac{-i}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{1-e^{(i\alpha-\gamma_{om})L}}{\alpha+i\gamma_{om}}$$

$$, \qquad |\tau| < k_2 \tag{3-4}$$

where α, τ and γ are defined in Chapter 2. $G(\alpha)$ is the Fourier transform Green's function associated with the Wiener-Hopf equation and given by

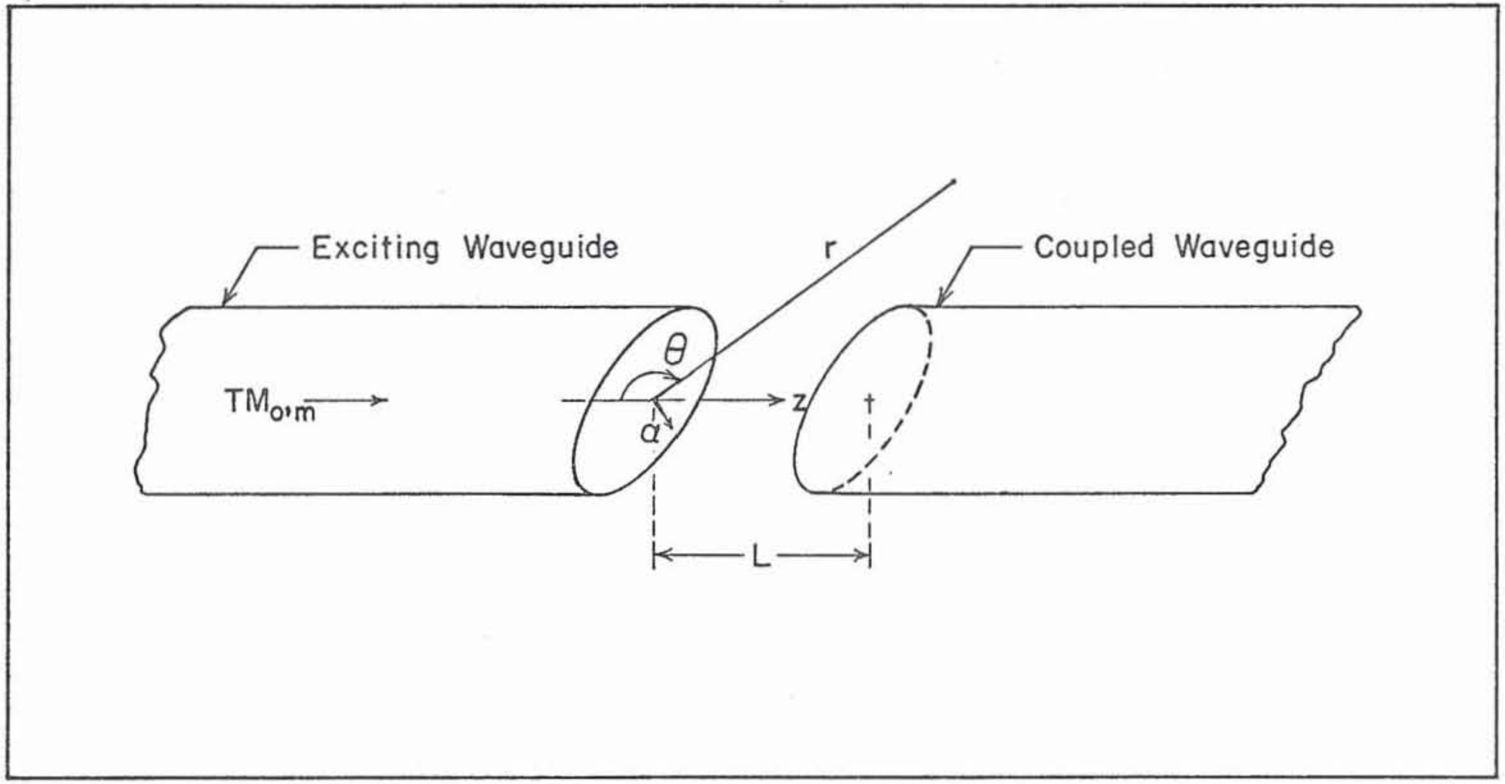

Fig. 3-1 Coupling between two collinear semi-infinite circular waveguides

$G(\alpha) = I_o(\gamma a)K_o(\gamma a)$ where I_o and K_o are the zero order modified Bessel functions of the first and second kind, respectively. J_1 is the Bessel function of first order. $\Phi_1(a,\alpha)$ is the Fourier transform of the scattered electric field component E_z in the aperture between the rims of the two waveguides. The functions $J_+(\alpha)$ and $J_-(\alpha)$ are unknowns and are exactly proportional to the induced current (due to only the scattered fields) on the walls of the coupled and excited waveguides, respectively. $J_+(\alpha)$ and $J_-(\alpha)$ are analytic respectively in the upper ($\tau > -k_2$) and lower ($\tau < k_2$) halves of the complex α-plane. It can be shown that $\Phi_1(a,\alpha)$ satisfies the following equation (see Appendix A):

$$\Phi_1(a,\alpha) = \frac{\gamma^2 a}{i\omega\varepsilon} G(\alpha)[e^{i\alpha L}\{S(-\alpha)-D(-\alpha)\} - \{S(\alpha)+D(\alpha)\}] \tag{3-5}$$

where

$$\left.\begin{matrix} S(\alpha) \\ \\ D(\alpha) \end{matrix}\right\} = J_-(\alpha) \mp J_+(-\alpha) + \frac{i}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om})\left[\frac{1}{\alpha+i\gamma_{om}} \mp \frac{e^{-\gamma_{om}L}}{\alpha-i\gamma_{om}}\right] \tag{3-6}$$

The upper and lower signs belong respectively to $S(\alpha)$ and $D(\alpha)$. These functions satisfy the following integral equation

$$\frac{-i\xi_{om}}{a\sqrt{2\pi}} J_1(\xi_{om}) \frac{k+i\gamma_{om}}{\alpha+i\gamma_{om}} G_+(i\gamma_{om}) + (k-\alpha)G_-(\alpha)E(\alpha) =$$

$$\frac{\lambda}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{(\beta+k)G_+(\beta)E(\beta)}{(\beta+\alpha)} e^{-i\beta L}\, d\beta\ , \quad -k_2 < -d < \tau < d < k_2 \tag{3-7}$$

where

$$E(\alpha) = \begin{cases} S(\alpha) & , \quad \lambda = 1 \\ D(\alpha) & , \quad \lambda = -1 \end{cases} \tag{3-8}$$

and $G_+(\alpha)$ is the "plus part" of $G(\alpha)$, $(G(\alpha) = G_+(\alpha)G_-(\alpha))$, and is given by ([41], pp. 238):

$$G_+(\alpha) = G_-(\alpha) = \sqrt{\frac{\pi i}{2} H_o^{(1)}(ka) J_o(ka)}\; e^{\frac{i\alpha a}{\pi}[1-c+\ln\frac{2\pi}{ka}+i\frac{\pi}{2}]}$$

$$e^{-i\frac{ka}{2}+i\frac{\gamma a}{\pi}\ln(\frac{\alpha-\gamma}{k})+q(\alpha)} \prod_{m=1,2,3,..}^{\infty} (1+\frac{\alpha}{i\gamma_m}) e^{\frac{i\alpha a}{m\pi}} \tag{3-9}$$

where, $\gamma_m = \sqrt{(\frac{\xi_m}{a})^2-k^2}$, ξ_m is the mth ordered zero of $J_o(x)$, $q(\alpha)$ is given by:

$$q(\alpha) = \frac{a}{\pi} \int_0^{\infty} [1 - \frac{2}{\pi\omega a} \frac{1}{J_o^2(\omega a)+Y_o^2(\omega a)}] \ln [1 + \frac{\alpha}{\sqrt{k^2-\omega^2}}] \, d\omega \tag{3-10}$$

and c is the Euler's constant = 0.57721... .

For large L, the major contribution to the integral in the right hand side of equation (3-7) is from the integral over a small neighborhood around the branch point $\beta = -k$. Hence, similar to Chapter 2, it can be shown that (see Appendix B)

$$\text{R.H.S. of (3-7)} = \frac{\lambda E(-k)}{G_+(k)} T(\alpha) , \tag{3-11}$$

where

$$T(\alpha) = \frac{\pi}{L} e^{iKL} \int_0^{\infty} \frac{u \; I_o^2[\frac{a}{L}\sqrt{2ikLu-u^2}\,]}{u+ikL(\frac{\alpha}{k}-1)} e^{-u} \, du \tag{3-12}$$

Equation (3-12) cannot be evaluated analytically, but a numerical evaluation using the Gauss-Laguarre formula is feasible. Substituting equation (3-11) into (3-7), one obtains expressions for $S(\alpha)$ and $D(\alpha)$, which together with equation (3-5) lead to

$$\Phi_1(a,\alpha) = \frac{\gamma^2}{i\omega\varepsilon} \frac{i\xi_{om}}{\sqrt{2\pi}} J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om})\{ \frac{G_+(\alpha)}{(k-\alpha)(\alpha+i\gamma_{om})}$$

$$- \frac{1/2\pi i}{2k\; G_+^2(k)(1-F^2)(i\gamma_{om}-k)} [F\frac{T(\alpha)G_+(\alpha)}{k-\alpha} + \frac{T(-\alpha)G_+(-\alpha)e^{i\alpha L}}{k+\alpha}]\} \tag{3-13}$$

where

$$F = \frac{-1}{2\pi i} \frac{T(-k)}{2kG_+^2(k)} \tag{3-14}$$

$T(-k)$ is the value of the semi-infinite integral given by (3-12) with $\alpha = -k$. It is clear that (3-13) is of the same form as the one given in Chapter 2, by equation (2-48). Once $\Phi_1(a,\alpha)$ is obtained, the scattered fields in free space or in either of the two waveguides can be determined. The field components of the $TM_{o,m}$ mode are H_ϕ, E_z and E_ρ which can be determined from the electric hertz vector. In the following section, the component of the scattered magnetic field in different regions is determined, but the details of the analysis are omitted.

3.3 Evaluation of the scattered fields

3.3.1 Radiation field

In the region outside the two waveguides, the scattered magnetic field H_ϕ^s is given by:

$$H_\phi^s(\rho,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \gamma \frac{K_1(\gamma\rho)}{K_o(\gamma a)} \overline{\Phi}_1(a,\alpha) e^{-i\alpha z}\, d\alpha \quad , \ |\tau| < k_2 \tag{3-15}$$

where

$$\overline{\Phi}_1(a,\alpha) = \frac{-i\omega\varepsilon}{\gamma^2} \Phi_1(a,\alpha) \tag{3-16}$$

$K_1(x)$ is the first order modified Bessel function of the second kind. In the far zone (i.e. $kr >> 1$), the saddle point method of integration can be applied to (3-15). Hence with the knowledge that $\overline{\Phi}_1(a,\alpha)$ has no singularities in the two-sheeted α-plane, except the branch singularities of $\gamma = \sqrt{\alpha^2 - k^2}$, we obtain

$$H_\phi^s(r,\theta) = -\sqrt{\frac{2}{\pi}} \frac{ke^{ikr}}{kr} \frac{k\sin\theta}{H_o^{(1)}(ka\sin\theta)} \overline{\Phi}_1(a,k\cos\theta) \tag{3-17}$$

the spherical coordinates r and θ are shown in figure 3-1. Equation

(3-17) together with equations (3-13) and (3-16) give

$$H_{\phi}^{s}(r,\theta) = \frac{e^{i(kr-\frac{\pi}{2})}}{kr} F(\theta) \tag{3-18}$$

where

$$F(\theta) = -\frac{\xi_{om}}{\pi} J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om})\frac{k^2\sin\theta}{H_o^{(1)}(ka\sin\theta)}$$

$$\left\{ \frac{G_+(k\cos\theta)}{(k-k\cos\theta)(k\cos\theta+i\gamma_{om})} - \frac{1}{2\pi i}\frac{F\,T(k\cos\theta)G_+(k\cos\theta)}{2k(1-F^2)G_+^2(k)(i\gamma_{om}-k)(k-k\cos\theta)}\right.$$

$$\left. - \frac{1}{2\pi i}\frac{T(-k\cos\theta)G_+(-k\cos\theta)e^{ikL\cos\theta}}{2k(1-F^2)G_+^2(k)(i\gamma_{om}-k)(k+k\cos\theta)} \right\} \tag{3-19}$$

The radiation pattern is given by $|F(\theta)|^2$. It is clear that $F(\theta)$ is exactly of the same form as thatfor parallel plate waveguides. This radiation field has been expressed by three terms. The first term is the well known radiation field from the open end of a single semi-infinite circular waveguide for $TM_{o,m}$ excitation, while the second and third terms are due to interactions between the two waveguides. Equation (3-19) can be rewritten in the form

$$F(\theta) = \frac{A(\theta)}{\tan\frac{\theta}{2}H_o^{(1)}(ka\sin\theta)} + \frac{B(\theta)}{\cot\frac{\theta}{2}\,H_o^{(1)}(ka\sin\theta)} \tag{3-20}$$

where $A(\theta)$ and $B(\theta)$ are finite functions with respect to θ. It is interesting to note that $F(0) = \infty$ and $F(\pi) = \infty$.

In the direction of waveguide walls and with the conditions that $ka\sin\theta << 1$ and $|\cos\theta| \simeq 1, F(\theta)$ may be expressed by a very simple formula given by

$$F(\theta) = \frac{-i\pi A(\theta)}{\sin\theta\,\ln(ka\sin\theta)}\ , \quad \theta \simeq 0 \tag{3-21}$$

and

$$F(\theta) = \frac{-i\pi B(\pi)}{\sin\theta\,\ln(ka\sin\theta)}\ , \quad \theta \simeq \pi \tag{3-22}$$

The strong radiations of the $TM_{o,m}$ modes in the forward and backward directions are caused by the directive effect of the outer surface of the waveguides. This phenomena was observed by Weinstein [2] for a single semi-infinite circular waveguide excited by $TM_{o,m}$ mode.

Equation (3-18) gives the magnetic field component $H_\phi^S(r,\theta)$, which is the dominant component of H. This is related to the dominant component of E, E_θ^S, by the free space wave impedance $\eta_0 = 120\pi$ ohm (i.e. $E_\theta^S = \eta_0 H_\phi^S$).

3.3.2 Reflected field

In the region inside the exciting waveguide $(z < o)$, the reflected magnetic field is given by:

$$H_\phi^r(\rho,z) = -\frac{\partial\psi_r(\rho,z)}{\partial\rho} \tag{3-23}$$

where

$$\psi_r(\rho,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{I_o(\gamma\rho)}{I_o(\gamma a)} \overline{\Phi}_1(a,\alpha) e^{-i\alpha z}\, d\alpha \quad , \quad |\tau| < k_2 \tag{3-24}$$

The contour of integration for the first and the second terms of $\overline{\Phi}_1(a,\alpha)$ may be closed in the upper half of the complex α-plane. The enclosed singularities are the poles at $\alpha = i\gamma_{on}$ which are the zeros of $I_o(\gamma a)$, when $\gamma_{on} = \sqrt{(\xi_{on}/a)^2 - k^2}$ and $n = 1,2,3,\ldots$. An application of the residue theorem gives the following contributions for the reflected field. The first term of $\overline{\Phi}_1(a,\alpha)$ gives

$$\psi_r^{exc}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} R_{m,n} J_o\left(\frac{\xi_{on}}{a}\rho\right) e^{\gamma_{on} z} \tag{3-25}$$

where

$$R_{m,n} = -\frac{\xi_{om}}{a}\frac{\xi_{on}}{a}\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})}{(k-i\gamma_{on})(\gamma_{on}+\gamma_{om})\gamma_{on}} \tag{3-26}$$

While the second term of $\bar{\Phi}_1(a,\alpha)$ yields

$$\psi_r^{int,(1)}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} R_{m,n}^{(1)} J_o(\frac{\xi_{on}}{a}\rho)e^{\gamma_{on}z} \tag{3-27}$$

where

$$R_{m,n}^{(1)} = \frac{-1}{2\pi}\frac{F}{1-F^2}\frac{\xi_{om}}{a}\frac{\xi_{on}}{a}\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})T(i\gamma_{on})}{2k\gamma_{on}(k-i\gamma_{om})(k-i\gamma_{on})G_+^2(k)} \tag{3-28}$$

and the third term of $\bar{\Phi}_1(a,\alpha)$ yields

$$\psi_r^{int,(2)}(\rho,z) = \frac{-i\xi_{om}}{2\pi} J_1(\xi_{om}) \frac{(k+i\gamma_{om})G_+(i\gamma_{om})}{2k\; G_+^2(k)(k-i\gamma_{om})}\frac{1}{2\pi i}\frac{1}{1-F^2}\, II \tag{3-29}$$

where

$$II = \int_{-\infty+i\tau}^{\infty+i\tau} \frac{I_o(\gamma\rho)}{I_o(\gamma a)}\frac{T(-\alpha)G_+(-\alpha)}{k+\alpha} e^{i\alpha(L-z)}\, d\alpha \quad , \quad |\tau| < k_2 \tag{3-30}$$

The superscripts exc and int refer to the scattered fields due to the exciting waveguide alone and to the interactions between the two waveguides, respectively. Equation (3-29) may be assumed in the following form:

$$\psi_r^{int,(2)}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} R_{m,n}^{(2)}(z) J_o(\frac{\xi_{on}}{a}\rho) \tag{3-31}$$

Equating equations (3-29) and (3-31) and multiplying both sides by $\rho J_o(\frac{\xi_{on'}}{a}\rho)$, an integration over ρ between 0 and a and using the following identity [49]

$$\int_o^a \rho J_\nu(x_{\nu n'}\frac{\rho}{a})J_\nu(x_{\nu n}\frac{\rho}{a})d\rho = \frac{a^2}{2}\,[J_{\nu+1}(x_{\nu n})]^2\,\delta_{n'n} \tag{3-32}$$

gives

$$R_{m,n}^{(2)}(z) = i\frac{\xi_{om}}{2\pi}\frac{2J_1(\xi_{om})}{a^2J_1^2(\xi_{on})}\frac{(k+i\gamma_{om})G_+(i\gamma_{om})}{2kG_+^2(k)(k-i\gamma_{om})}\frac{1}{2\pi i}\frac{1}{1-F^2}\, III \tag{3-33}$$

where

$$III = \int_{-\infty+i\tau}^{\infty+i\tau} [\int_o^a \rho J_o(i\gamma\rho)J_o(\frac{\xi_{on}}{a}\rho)d\rho]\, \frac{T(-\alpha)K_o(\gamma a)e^{i\alpha(L-z)}}{(k+\alpha)G_+(\alpha)}\, d\alpha \quad , \quad |\tau| < k_2 \tag{3-34}$$

using the Lommel's integral:

$$\int_0^1 x\, J_o(ux) J_o(vx) dx = \frac{u\, J_o'(u) J_o(v) - v\, J_o'(v) J_o(u)}{v^2 - u^2} \quad (3\text{-}35)$$

Equation (3-34) reduces to

$$III = \xi_{on} J_1(\xi_{on}) \int_{-\infty+i\tau}^{\infty+i\tau} \frac{T(-\alpha) K_o(\gamma a) I_o(\gamma a)}{G_+(\alpha)(k+\alpha)(\alpha^2+\gamma_{on}^2)}\, e^{i\alpha(L-z)} d\alpha \quad , \quad |\tau| < k_2 \quad (3\text{-}36)$$

Closing the contour in the upper half of the complex α-plane, the contour of the integration is $p = p_1 + p_2 + p_3$ as shown in figure 2-3, and the only enclosed singularity is the branch point at $\alpha = k$. The integral over the small circle p_2 can be shown to be zero and equation (3-36) reduces to

$$III = \pi i\, \xi_{on} J_1(\xi_{on}) \int_k^{k+i\infty} \frac{T(-\alpha) I_o^2(\gamma a)}{G_+(\alpha)(k+\alpha)(\alpha^2+\gamma_{on}^2)}\, e^{i\alpha(L-z)}\, d\alpha \quad (3\text{-}37)$$

Since the integral cannot be evaluated analytically, it may be converted to a more suitable form for numerical integration by letting $\alpha = k + \frac{iu}{L}$. The result is a semi-infinite type integral which can be evaluated again using the Gauss-Laguerre quadrature formula. Thus equation (3-33) becomes

$$R_{m,n}^{(2)}(z) = -i\, \frac{\xi_{om}}{a} \frac{\xi_{on}}{a} \frac{J_1(\xi_{om})}{J_1(\xi_{on})} \frac{e^{ik(L-z)}}{kL} \frac{(k+i\gamma_{om}) G_+(i\gamma_{om})}{(k-i\gamma_{om}) 2kG_+^2(k)} \frac{1}{2\pi i} \frac{1}{1-F^2}$$

$$\int_0^{\infty} \frac{T[-k-\frac{iu}{L}]\, I_o^2[\frac{a}{L}\sqrt{2ikLu-u^2}]}{G_+(k+\frac{iu}{L})(2+\frac{iu}{kL})[(k+\frac{iu}{L})^2+\gamma_{on}^2]}\, e^{-u(L-z)/L}\, du \quad (3\text{-}38)$$

Hence the total reflected magnetic field is:

$$H_\phi^r(\rho,z) = -\frac{\partial}{\partial\rho}[\psi_r^{exc}(\rho,z) + \psi_r^{int,(1)}(\rho,z) + \psi_r^{int,(2)}(\rho,z)]$$

$$= -\frac{\partial}{\partial\rho} \sum_{n=1,2,3,..}^{\infty} [(R_{m,n} + R_{m,n}^{(1)}) e^{\gamma_{on} z} + R_{m,n}^{(2)}(z)] J_o(\frac{\xi_{on}}{a}\rho) \quad (3\text{-}39)$$

where the coefficients $R_{m,n}$, $R^{(1)}_{m,n}$ and $R^{(2)}_{m,n}(z)$ are given respectively by equations (3-26), (3-28) and (3-38). It can be shown easily that $R^{(2)}_{m,n}(z)$ decays with z as $\dfrac{e^{ik(L-z)}}{k(L-z)}$ which represents radiation from an origin located at the centre of the open end of the coupled waveguide.

The reflected field has been expressed by three terms. The first term, $\psi_r^{exc}(\rho,z)$ is the reflection due to the open end of the exciting waveguide, in the absence of the coupled waveguide, while the other two, $\psi_r^{int,(1)}(\rho,z)$ and $\psi_r^{int,(2)}(\rho,z)$ are due to the interactions between the two waveguides. This is exactly the same as the case of parallel plate waveguides. Far from the open end of the exciting waveguide, the main contribution for the reflected field is due to $R_{m,n}$ and $R^{(1)}_{m,n}$ only. Hence the total reflection coefficient equals $R_{m,n} + R^{(1)}_{m,n}$. The third term in (3-39) is a continuous spectrum of inhomogeneous plane waves and decays to zero as $z \to -\infty$ to satisfy the Sommerfeld radiation condition. The contribution of $\psi_r^{int,(2)}$ to the reflected field is significant only at points near the open end of the exciting waveguide, and in particular the aperture field. For large L or z, $\psi_r^{int,(2)}$ can be computed analytically using the saddle point method of integration.

3.3.3 Transmitted field

Inside the coupled waveguide, where $z > L$, transmitted magnetic field may be expressed as:

$$H_\phi^t(\rho,z) = -\frac{\partial}{\partial\rho}\,\psi_t(\rho,z) \tag{3-40}$$

where

$$\psi_t(\rho,z) = \psi^i(\rho,z) + \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{I_o(\gamma\rho)}{I_o(\gamma a)}\,\overline{\Phi}_1(a,\alpha)e^{-i\alpha z}\,d\alpha \quad , \quad |\tau| < k_2 \tag{3-41}$$

The integral again can be evaluated by closing the contour in the lower half of the complex α-plane. The first term in $\overline{\Phi}_1(a,\alpha)$ has a pole at $\alpha = -i\gamma_{om}$ and a branch point at $\alpha = -k$. The contribution due to the pole cancels exactly the incident field and the branch point contribution can be evaluated similar to $\psi_r^{int,(2)}(\rho,z)$. The result may be shown to be (see Appendix C):

$$\psi_t^{exc}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} T_{m,n}(z) J_o\left(\frac{\xi_{on}}{a}\rho\right) \tag{3-42}$$

where

$$T_{m,n}(z) = -i\,\frac{\xi_{om}}{a}\,\frac{\xi_{on}}{a}\,\frac{e^{ikz}}{kz}\,\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\,(k+i\gamma_{om})G_+(i\gamma_{om})\int_0^{\infty}$$

$$\frac{I_o^2[\frac{a}{z}\sqrt{2ikzu-u^2}]}{(2+\frac{iu}{kz})G_+(k+\frac{iu}{z})}\;\frac{e^{-u}}{(-k-\frac{iu}{z}+i\gamma_{om})[(k+\frac{iu}{L})^2+\gamma_{on}^2]}\,du \tag{3-43}$$

Similarly for the second term of $\overline{\Phi}_1(a,\alpha)$, the only enclosed singularity is the branch point at $\alpha = -k$, and hence

$$\psi_t^{int,(1)}(\rho,z) = \sum_{n=1,2,3,...}^{\infty} T_{m,n}^{(1)}(z) J_o\left(\frac{\xi_{on}}{a}\rho\right) \tag{3-44}$$

where $T_{m,n}^{(1)}(z)$ is given by:

$$T_{m,n}^{(1)}(z) = -i\,\frac{\xi_{om}}{a}\,\frac{\xi_{on}}{a}\,\frac{e^{ikz}}{kz}\,\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\,\frac{(k+i\gamma_{om})G_+(i\gamma_{om})}{(k-i\gamma_{om})2k\,G_+^2(k)}\,\frac{1}{2\pi i}\,\frac{F}{1-F^2}$$

$$\int_0^{\infty}\frac{I_o^2[\frac{a}{z}\sqrt{2ikzu-u^2}]T(-k-\frac{iu}{z})e^{-u}}{(2+\frac{iu}{kz})G_+(k+\frac{iu}{z})[(k+\frac{iu}{L})^2+\gamma_{on}^2]}\,du \tag{3-45}$$

It can be shown easily that $T_{m,n}(z)$ and $T_{m,n}^{(1)}(z)$ decay by the factor $\frac{e^{ikz}}{kz}$. This is true as $\psi_t^{exc}(\rho,z)$ and $\psi_t^{int,(1)}(\rho,z)$ represent radiation terms from the open end of the exciting waveguide.

For the third term of $\overline{\Phi}_1(a,\alpha)$, the enclosed singularities are the poles at $\alpha = -i\gamma_{on}$ which are the zeros of $I_o(\gamma a)$, with $\gamma_{on} = \sqrt{(\xi_{on}/a)^2-k^2}$, and $n = 1,2,3,\ldots$. Evaluating these residue contributions, one obtains

$$\psi_t^{int,(2)}(\rho,z) = \sum_{n=1,2,3,\ldots}^{\infty} T_{m,n}^{(2)} J_o(\frac{\xi_{on}}{a}\rho)e^{-\gamma_{on}z} \tag{3-46}$$

where

$$T_{m,n}^{(2)} = \frac{-e^{\gamma_{on}L}}{2\pi(1-F^2)} \frac{\xi_{om}}{a} \frac{\xi_{on}}{a} \frac{J_1(\xi_{om})}{J_1(\xi_{on})} \frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})T(i\gamma_{on})}{\gamma_{on}\cdot 2k(k-i\gamma_{om})(k-i\gamma_{on})G_+^2(k)} \tag{3-47}$$

Hence the transmitted magnetic field will be given by:

$$\begin{aligned} H_\phi^t(\rho,z) &= -\frac{\partial}{\partial\rho}[\psi_t^{exc}(\rho,z) + \psi_t^{int,(1)}(\rho,z) + \psi_t^{int,(2)}(\rho,z)] \\ &= -\frac{\partial}{\partial\rho}\sum_{n=1,2,3,\ldots}^{\infty}[T_{m,n}(z)+T_{m,n}^{(1)}(z)+T_{m,n}^{(2)}e^{-\gamma_{on}z}]J_o(\frac{\xi_{on}}{a}\rho) \end{aligned} \tag{3-48}$$

where $T_{m,n}(z)$, $T_{m,n}^{(1)}(z)$ and $T_{m,n}^{(2)}$ are the transmission coefficients given by equations (3-43), (3-45) and (3-47). Again the transmitted field is expressed by three terms. The coefficients $T_{m,n}(z)$ and $T_{m,n}^{(1)}(z)$ are expressed in a convenient form for the numerical integration, especially for evaluation of the aperture field at the open end of the coupled waveguide. These two terms represent fields scattered by the rim of the exciting waveguide. They decay to zero, as mentioned before, to satisfy the Sommerfeld radiation condition, and do not contribute to the transmission coefficient at large distances from the open end of the coupled waveguide. The only contribution to the transmitted field in the coupled waveguide at large distances comes from $T_{m,n}^{(2)}$ which is due to the interaction between the rims of the two waveguides.

Similar to the case of coupling between parallel plate waveguides, the coupling coefficient (or transmission coefficient) $T_{m,n}^{(2)}$ can be related to the reflection coefficient $R^{(1)}_{m,n}$ by

$$T^{(2)}_{m,n} = \frac{e^{\gamma_{on}L}}{F} R^{(1)}_{m,n} \tag{3-49}$$

which is exactly the same as equation (2-74). Furthermore, this transmission coefficient could be related to the total reflection coefficient by the equation

$$\begin{aligned} T^{(2)}_{m,n} &= f(a,L)\, R^{total}_{m,n} \\ &= f(a,L)\left[1 + A\frac{F}{1-F^2}\, T(i\gamma_{on})\right] R_{m,n} \end{aligned} \tag{3-50}$$

where

$$f(a,L) = \frac{A\,\dfrac{e^{\gamma_{on}L}}{1-F^2}\, T(i\gamma_{on})}{1+A\,\dfrac{F}{1-F^2}\, T(i\gamma_{on})} \tag{3-51}$$

and A is a constant given by

$$A = \frac{1}{2\pi}\, \frac{\gamma_{on} + \gamma_{om}}{2kG^2_+(k)(k-i\gamma_{om})} \tag{3-52}$$

Hence the radiation power can be related to the incident power by the equation

$$P_{rad} = P_i\left[1 - \sum_{n=1,2,3,..}^{\infty} |R^{total}_{m,n}|^2\{1 + |f(a,L)|^2\}\right] \tag{3-53}$$

It is clear that as L approaches infinity, $f(a,L)$ tends to zero and $R^{total}_{m,n}$ reduces to $R_{m,n}$, the reflection coefficient due to the exciting waveguide alone. The previous arguments hold also for a parallel plate waveguide and it may be noted that all previous interesting results can be obtained by an evaluation of the integral in equation (3-12) and using the known values of $G_+(k)$ and $R_{m,n}$, the reflection coefficient of mode n due to the open end of the exciting waveguide.

3.4 Results and discussion

For a $TM_{0,1}$ excitation and $ka = 2.5$ and 5, some numerical results are obtained and are shown in figures 3-2 to 3-4. The infinite integrals in the field equations are evaluated numerically by the Gauss-Laguerre quadrature formula. However, the integral in the expression for $G_+(\alpha)$ is found to be more complex and, for its evaluation, a combination of the Gauss and the Gauss-Laguerre quadrature formulas are used. This integral is a function of ka and its contribution is negligible for $ka << 1$. To examine the accuracy of the computation, the computed values of $G_+(\alpha)$ using the expression given by equation (3-9) are compared with those of Weinstein [2] where $G_+(\alpha)$ has a different form although Green's functions are the same. It is found that for $ka << 1$, the agreement is excellent, but deteriorates as ka increases. For numerical results of $TM_{0,1}$ excitation, the discrepancy is less than 5 percent. Figure 3-2 shows the radiation patterns for $kL = 50$, 10 and 5. The results here are given by

$$|F(\theta)^2 \;/\; |-\frac{\xi_{om}}{\pi} J_1(\xi_{om})(k + i\gamma_{om})G_+(i\gamma_{om})|^2$$

As indicated by equations (3-20), (3-21) and (3-23), figure 3-2 shows a strong radiation field in the forward and backward directions along the waveguide walls. This strong radiation is due to the directive effect of the outer surface of the waveguide walls for $TM_{0,\ell}$ modes of excitation. Figure 3-2a is for the radiation pattern of the circular waveguide with radius $ka = 2.5$. A significant change in the radiation pattern occurs only for $\theta > 160°$. For $ka = 5.0$, the radiation patterns are shown in figure 3-2b, which shows stronger variation in the radiated power for $\theta > 100°$. For two cases of $ka = 2.5$ and 5, the radiation pattern for $kL = 50$ is the same as for single semi-infinite circular waveguide [2].

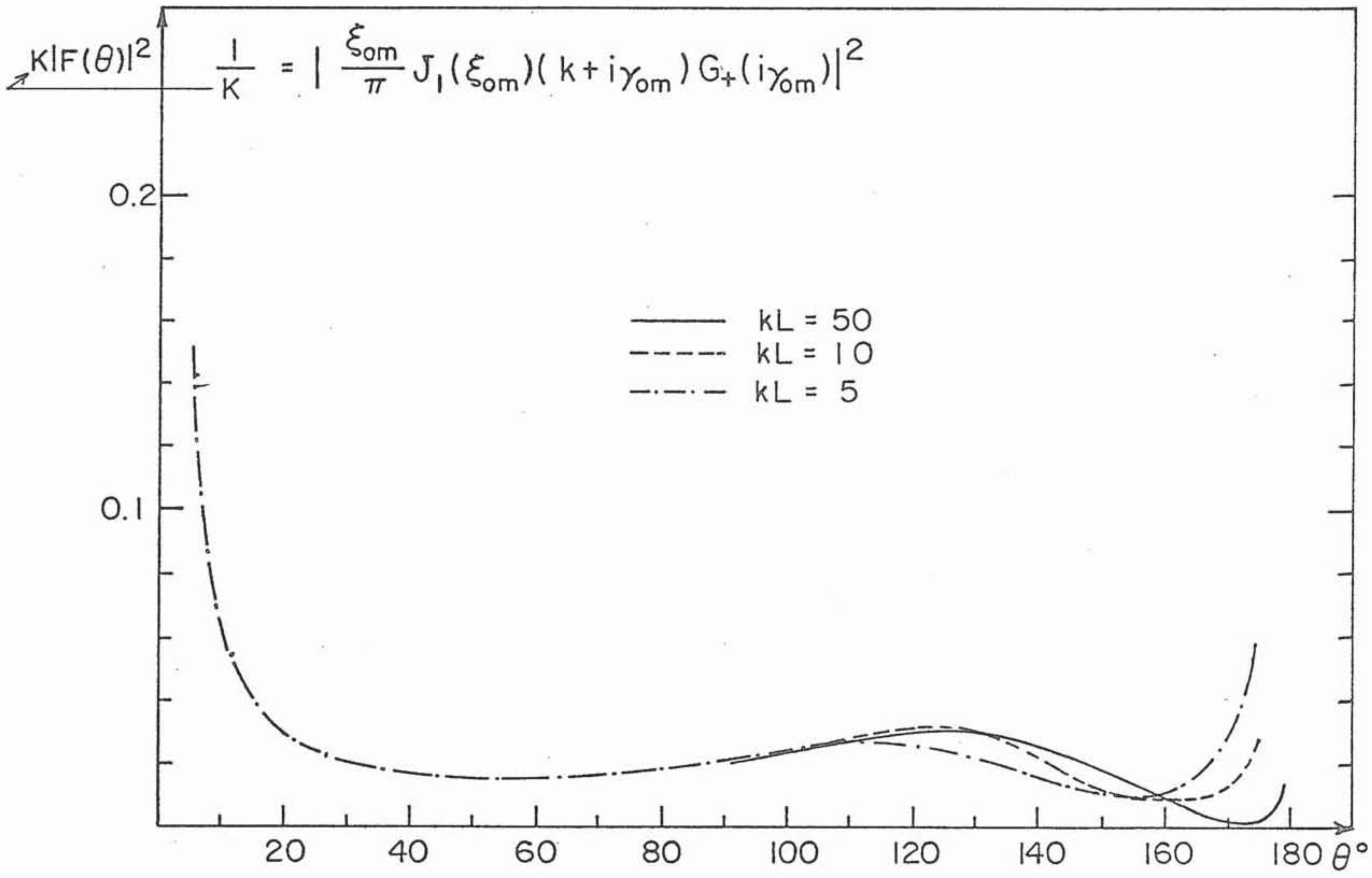

Figure 3-2a Radiation pattern of two semi-infinite circular waveguides separated by L and with $TM_{0,1}$ mode of excitation and radius ka = 2.5

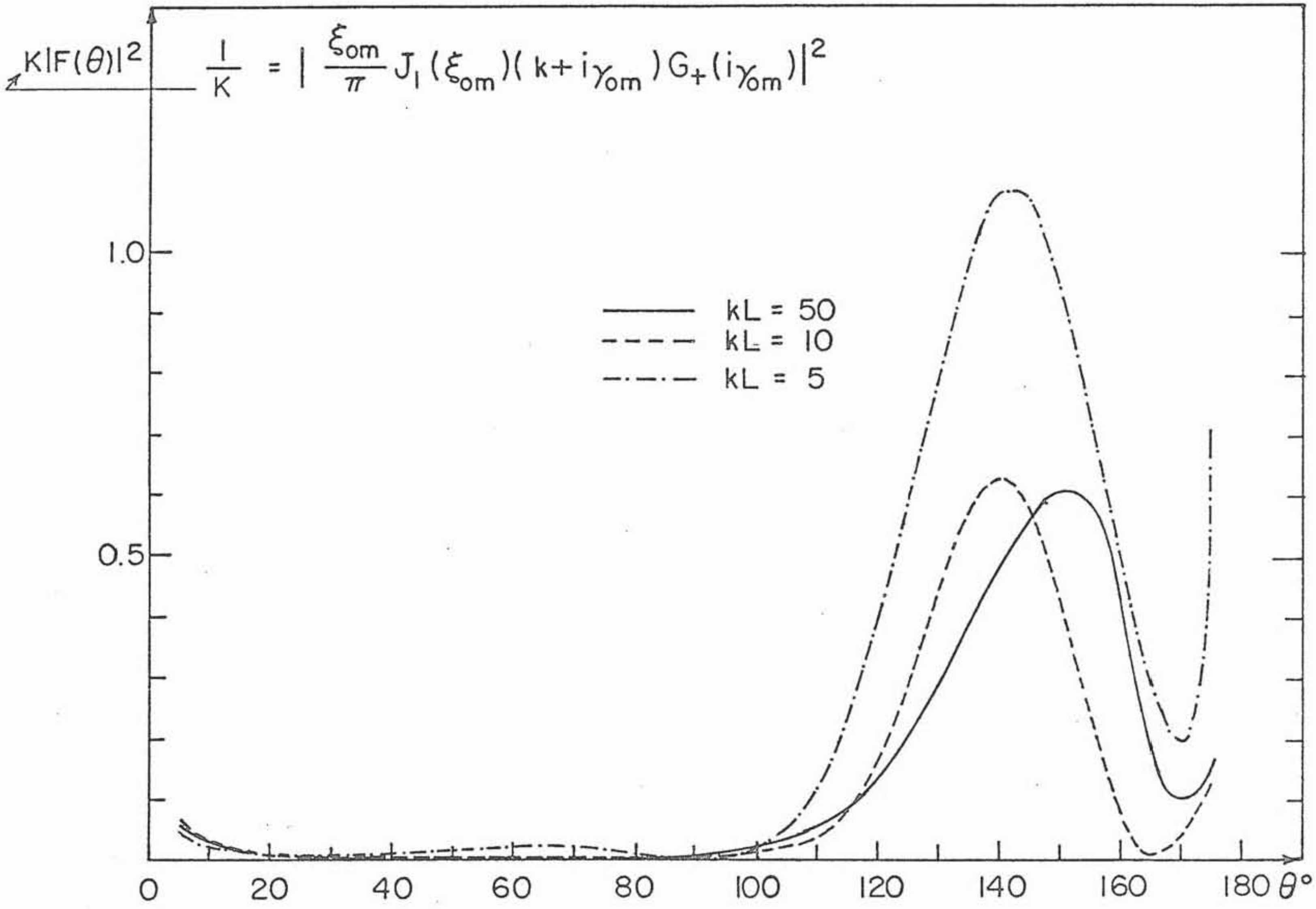

Figure 3-2b Radiation pattern of two semi-infinite circular waveguides separated by L and with $TM_{0,1}$ mode of excitation and radius ka = 5.0

Some results for the reflection and transmission coefficients are also obtained. Since $R^{(2)}_{m,n}(z)$, $T_{m,n}(z)$ and $T^{(1)}_{m,n}(z)$ decay with the factor $\frac{e^{ikz}}{kz}$ and do not contribute to the reflected and the transmitted fields at large distances, their corresponding terms are not included in the computations. Also, their effect will be the same as for parallel-plate waveguides investigated in Chapter 2. However, they should be considered for evaluation of the aperture fields, or field distributions near an open end. Figure 3-3 shows the magnitude and phase of the re-reflection coefficient for the dominant propagation mode with $ka = 5$. As shown, the magnitude is an oscillating function of kL with a period of oscillation approaching kL equal π as kL increases. In addition, the variation in the results decreases with kL and approaches gradually the reflection coefficient of an open ended single circular waveguide. Closer to the lower limit of ka for the dominant mode to propagate, the reflection coefficient has negligible variation as a function of kL. The results for such cases are not shown. In general, it was found that the effect of kL on the reflection coefficient decreases with ka and the coupled waveguide has negligible effect on the reflection coefficient. Figure 3-4 shows the coupling (transmission) coefficient of the dominant mode for $ka = 2.5$ and 5. The curves decay continuously with negligible oscillation as kL approaches infinity. The radiated power for $ka = 2.5$ and 5.0 are also computed and are shown in figure 3-5.

It should be noted that equations (3-49) to (3-53) give the reflection coefficient, the coupling coefficient and the radiated power of the geometry in terms of the reflection coefficient of an open ended waveguide with $G_+(\alpha)$ evaluated at $\alpha = k$. The final step is the evaluation of $T(\alpha)$ from (3-12) for a given α and L .

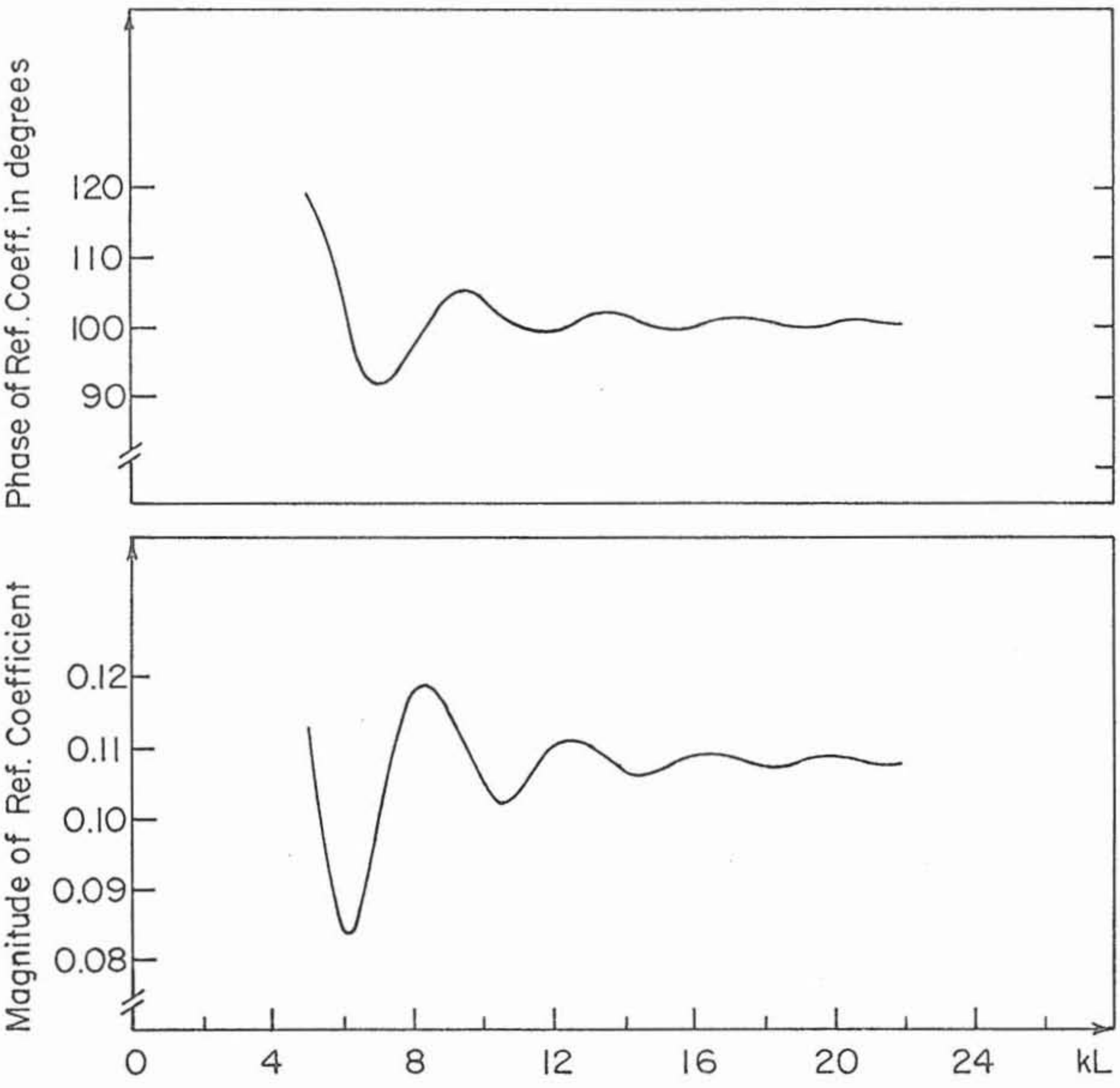

Figure 3-3 Reflection coefficient for $ka = 5$ and $TM_{0,1}$ excitation of two collinear semi-infinite circular waveguides.

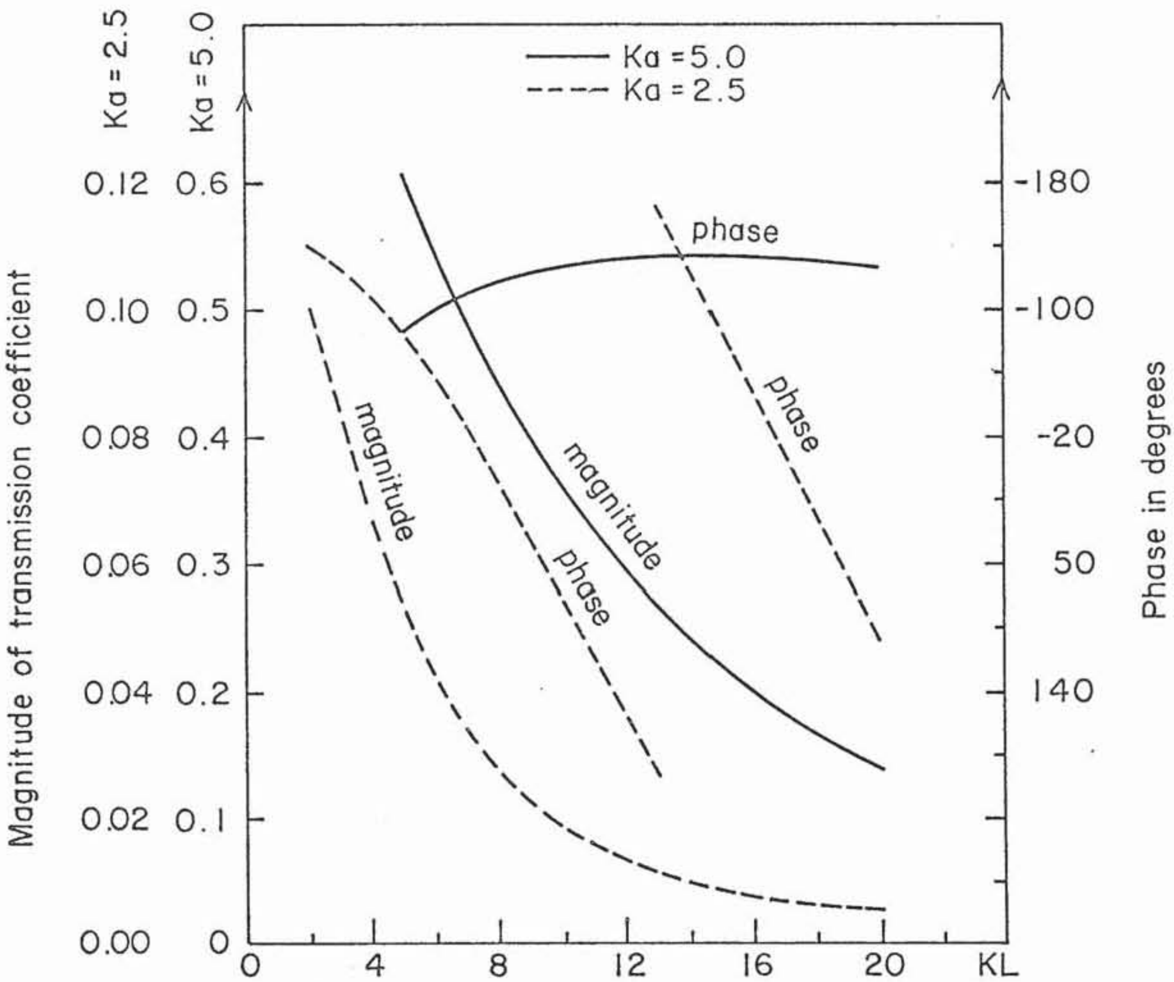

Fig. 3-4 Transmission coefficient for $TM_{0,1}$ mode of two collinear semi-infinite circular waveguides.

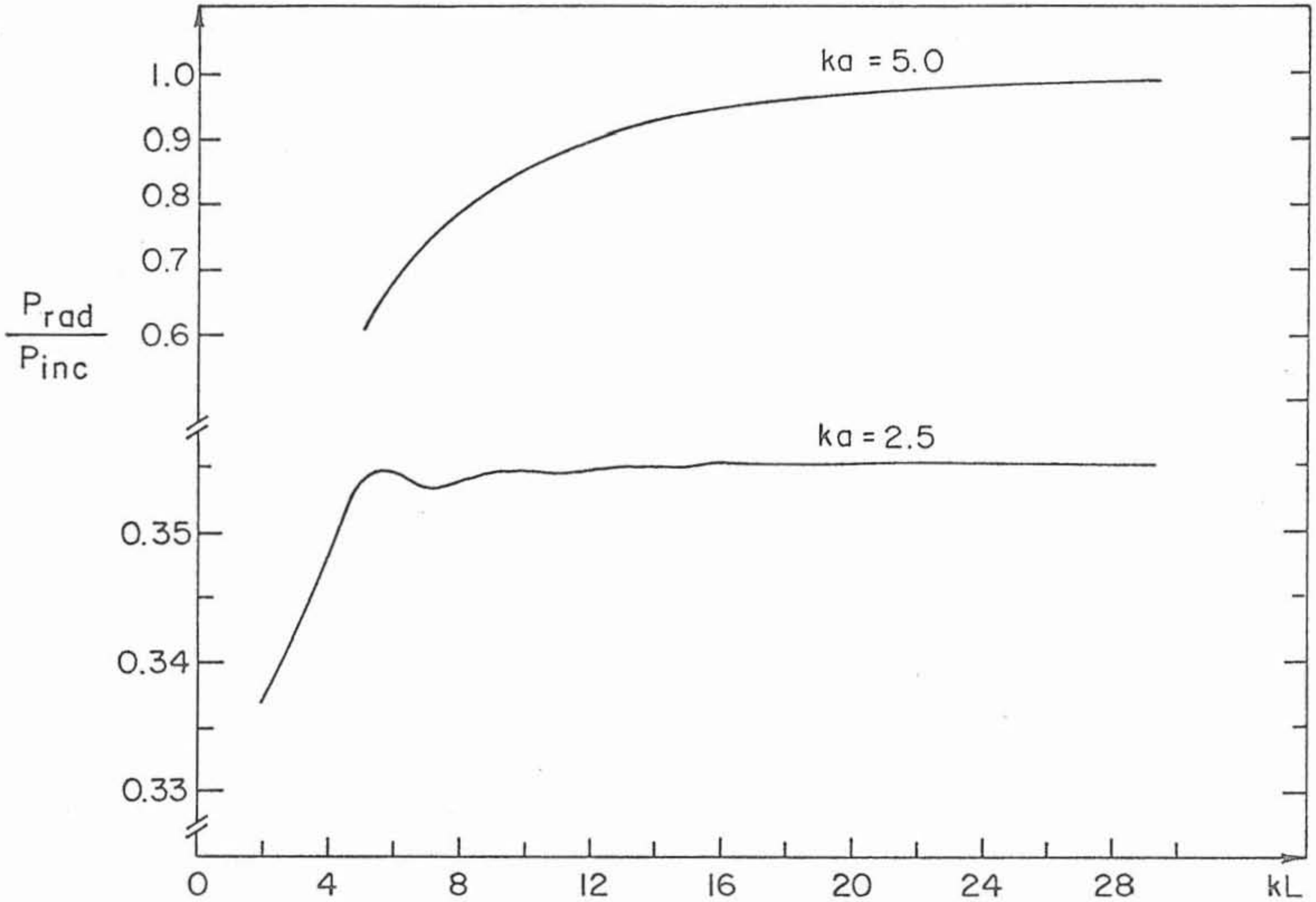

Figure 3-5 Power radiation / power incident, for two collinear semi-infinite circular waveguides separated by L and $TM_{0,1}$ excitation.

However, the analysis in this chapter will be used later to get a modified diffraction coefficient for a circular waveguide structure and a spherical wavefactor which is related to the scattered field from the rim of a circular waveguide. This is presented in the next chapter, using the results of this chapter.

CHAPTER 4

EXTENSION TO RAY THEORY OF DIFFRACTION

4.1 Introduction

Keller's Geometrical theory of diffraction [11], which is an extension of geometrical optics has been used extensively in diffraction and antenna problems. Diffracted rays are produced when incident rays hit edges, corners or vertices of boundary surfaces. A field is associated with each ray and the total field at a point is the sum of the fields on all rays passing through that point. The phase of the field on a ray is proportional to the optical length of the ray from some reference point. The amplitude varies in accordance with the principle of conservation of energy in a narrow tube of rays. The initial value of the field on a diffracted ray is determined from the incident field with the aid of an appropriate diffraction coefficient which is determined from the solution of certain canonical problems and they all vanish as the wavelength approaches zero. In the latter case, the remaining field is only the geometrical optics field, since the diffraction term is usually attributed to the fact that λ is not zero. In a homogeneous medium, the diffracted rays are complex straight lines, while in an inhomogeneous medium, they are complex-valued solutions of the differential equations for rays [50]. This theory has paved the way to get an approximate solution for complicated structures [51],[52]. Diffraction by an aperture in soft and hard screens has been investigated by Keller [53] and Karp and Keller [54], respectively. Also, Jull [55] has treated the problem of diffraction by a wide aperture in an anisotropic medium. Several examples of edge diffraction theory are given in references [7],[12],[56]-[59]. The edge diffraction coefficients normally used in Keller's geometrical theory of diffraction are based on plane wave assumption for the edge field. As a result of the distance limitation on these coefficients, the diffracted field due to multiple edge interactions lead to some deviations from the exact solutions. Hamid [12] has presented amplitude and phase correction factors in the diffraction coefficient by comparison with the exact solution. The divergence phenomena caused by the use of the plane wave diffraction coefficients was overcome for H-polarization by Yu and Rudduck [60] by the use of appropriate cylindrical wave diffraction functions. Their formulations were basically the same as the formulations by Karp and Russek [61] and by Ufimtsev [62],[63]. Recently, Mohsen and Hamid [64] has improved the accuracy of the results by including higher order terms in Keller's diffraction coefficient.

Lee [27],[28], has extended Keller's diffraction ray method to problems involving two or more parallel plates by introducing a modified diffraction coefficient which automatically takes care of the coupling along a shadow boundary.

As Keller's geometrical theory of diffraction cannot be applied to complicated problems that involve interacting and bouncing rays, Dybdal, R.B. et al [7] has applied this theory to obtain mutual coupling between TEM and $TE_{0,1}$ parallel-plate waveguide apertures, with special geometries and oritentations in which $\theta_o > \theta_g$, as shown in figure 4-1. They formulated the solution of the coupling problem as follows: A unit-amplitude wave with its associated model voltage or current is incident in the exciting waveguide. An equivalent line source, having an omnidirectional

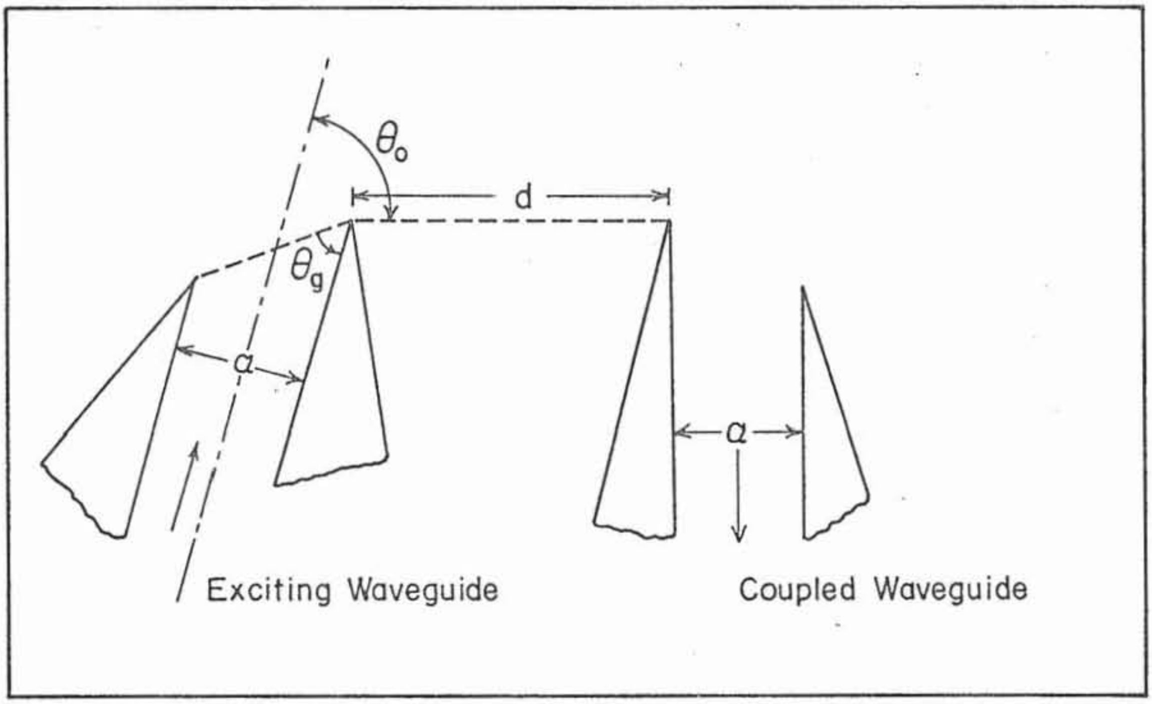

Fig. 4-1 Coupling between TEM and $TE_{0,1}$ parallel-plate waveguide apertures (after Dybdal et al [7])

pattern and a field matching that of the guide in the direction of θ_o, is substituted for the exciting waveguide. A coupled wave with its associated model voltage or current is induced in the coupled waveguide. The mutual coupling is then defined as the ratio of the model quantity in the coupled guide to that of the exciting guide.

This chapter represents extension of the Wiener-Hopf technique which has been used in the second and third chapters to obtain the coupling between parallel-plate and circular waveguides respectively. The Wiener-Hopf results are reduced to those corresponding to ray theory of diffraction in conjunction with the modified diffraction coefficient of Lee [27], [28]. To obtain the solution using ray theory of diffraction, the Green's function associated with the modified Wiener-Hopf equation is expanded in a power series and truncated after the first term. This is shown in detail in the following sections. For parallel plate waveguides, the steps for obtaining the ray theory results from the Wiener-Hopf technique are given and the limitations of the method are discussed. For the case of circular waveguides, a modified diffraction coefficient is obtained and a spherical wavefactor is defined and is shown how it is related to Keller's results [53] for circular apertures. With these definitions, results for the reflected, transmitted and radiated fields are obtained and the limitation of the method is derived in the same manner as for parallel plate waveguides.

The last section deals with some results obtained using the ray theory and the Wiener-Hopf technique with particular attention to the case of parallel plate waveguides. A discussion of the results is also presented.

4.2 Coupling between two collinear semi-infinite parallel-plate waveguides

4.2.1 Expansion of the transformed Green's function

If the transformed Green's function $G(\alpha)$ given by equation (2-20), is expanded in a power series, then the function $T(\alpha)$ given by equation (2-39) can be written as:

$$T(\alpha) = \sum_{n=0,1,2,..}^{\infty} T_n(\alpha) \tag{4-1}$$

where

$$T_n(\alpha) = \frac{1}{\varepsilon_n} \int_{-\infty-id}^{\infty-id} \frac{(-2\gamma a)^n}{n!\ \gamma a} \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta \tag{4-2}$$

with $\varepsilon_n = \begin{cases} 1 & , \text{ for } n = 0 \\ 2 & , \text{ for } n \neq 0 \end{cases}$

In the neighborhood of $\beta = k$, the function $(\beta-k)^{\frac{n-1}{2}}$ is regular and smooth, and can be replaced by $(-2k)^{\frac{n-1}{2}}$. Therefore, equation (4-2), after deforming the contour, becomes (see Appendix D):

$$T_n(\alpha) = \begin{cases} \dfrac{(-1)^n 2^{n+1} a^{n-1} (2k)^{\frac{n-1}{2}} e^{-i\frac{\pi}{2}(n-1)}}{n!\ \varepsilon_n} \displaystyle\int_{-k}^{-k-i\infty} (\beta+k)^{\frac{n-1}{2}} \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta & , \ n = 0,2,4,6,\ldots \quad (4\text{-}3a) \\ 0 & , \ n = 1,3,5,\ldots \quad (4\text{-}3b) \end{cases}$$

A change of variable via $\beta = -k - \frac{iu}{L}$ gives

$$T_n(\alpha) = \frac{(-1)^{n+1} 2^{\frac{3n+1}{2}} (ka)^{n-1} e^{i\frac{\pi}{4}(n-1)} e^{ikL}}{n!\ \varepsilon_n\ (kL)^{\frac{n-1}{2}}} W_{\frac{n-2}{2}}(\xi) \quad , \ n = 0,2,4,6,\ldots \tag{4-4}$$

where

$$\xi = -ikL(1-\frac{\alpha}{k}) \tag{4-5}$$

and

$$W_{j-\frac{1}{2}}(\xi) = \int_0^\infty \frac{u^j e^{-u}}{u+\xi}\, du \tag{4-6}$$

The above function is related to the Whittaker function $W_{u,v}(\xi)$ by the relation

$$W_{j-\frac{1}{2}}(\xi) = \Gamma(j+1)e^{\frac{\xi}{2}}\, \xi^{\frac{j-1}{2}}\, W_{-\frac{1}{2}(j+1),\frac{j}{2}}(\xi) \tag{4-7}$$

Using the asymptotic expansion of $W_{u,v}(\xi)$, [65], in (4-4), one obtains

$$T_n(\alpha) = \frac{(-1)^{n+1} 2^{\frac{3n+1}{2}} (ka)^{n-1} e^{i\frac{\pi}{4}(n-1)} e^{ikL} \Gamma(\frac{n+1}{2})}{n!\ \varepsilon_n\ (kL)^{\frac{n-1}{2}}}\ \frac{1}{\xi}\ \{ 1 + \sum_{s=1}^{\infty}$$

$$\frac{1}{s!\ \xi^s}\ [(\tfrac{n-1}{4})^2-(\tfrac{n+3}{4})^2][(\tfrac{n-1}{4})^2-(\tfrac{n+7}{4})^2]---[(\tfrac{n-1}{4})^2-(\tfrac{n+4s-1}{4})^2]\} \tag{4-8}$$

Retaining the first term in (4-8), its substitution into (4-1) gives

$$T(\alpha) = \sum_{n=0,2,4,..}^{\infty} T_n(\alpha)$$

$$= \frac{-i\sqrt{2}\ e^{i(kL-\frac{\pi}{4})}}{a\sqrt{kL}(k-\alpha)} \sum_{n=0,2,4,..}^{\infty} \Gamma(\tfrac{n+1}{2})\ \frac{(-1)^n 2^{\frac{3n}{2}} (ka)^n e^{i\frac{n\pi}{4}}}{n!\ \varepsilon_n\ (kL)^{n/2}} \tag{4-9}$$

which can be rewritten as

$$T(\alpha) = \frac{-2\pi i e^{i(kL-\frac{\pi}{4})}}{a\ \sqrt{2\pi kL}}\ \frac{1}{(k-\alpha)}\ [1 + i\nu - \nu^2 - \tfrac{2}{3}i\nu^3 + \ldots.] \tag{4-10}$$

where $\nu - (ka)^2/kL$

Equation (4-10) is an asymptotic expansion of $T(\alpha)$. It is clear that for fast convergence of $T(\alpha)$, ν must be less than unity, i.e. $(ka)^2 << kL$. However, if $(ka)^2 << kL$, the first term in (4-10) can be retained, and equation (4-10) reduces to

$$T(\alpha) = T_o(\alpha) = \left(\frac{-2\pi i}{a}\right) \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k-\alpha} \tag{4-11}$$

Finally, this equation together with equation (2-43) gives

$$F = \overline{F} = \frac{e^{i(kL-\frac{\pi}{4})}}{2k\sqrt{2\pi kL}} \frac{1}{G_+^2(k)} \tag{4-12}$$

This function will be shown to be the same as that obtained in the next section using the ray theory approach. A substitution of (4-11), (4-12) into the expressions of the radiated, reflected and transmitted fields, obtained by the Wiener-Hopf technique, gives

i - For the radiation pattern given by (2-52) and (2-53) we obtain

$$\phi^s(\rho,\theta) = \sqrt{\frac{2}{\pi k\rho}}\, e^{i(k\rho-\frac{\pi}{4})} F(\theta) \tag{4-13}$$

where

$$F(\theta) = \frac{i\pi\ell}{4a} (-1)^{\frac{\ell-1}{2}} G_+(i\gamma_\ell)k\sin\theta\, e^{-ika\sin\theta}\Big[\frac{G_+(k\cos\theta)}{k\cos\theta+i\gamma_\ell}$$

$$+ \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k(1-\cos\theta)} \frac{\overline{F}}{1-\overline{F}^2} \frac{G_+(k\cos\theta)}{G_+^2(k)(i\gamma_\ell-k)}$$

$$+ \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k(1+\cos\theta)} \frac{1}{1-\overline{F}^2} \frac{G_+(-k\cos\theta)}{G_+^2(k)(i\gamma_\ell-k)} e^{ikL\cos\theta} \Big] \tag{4-14}$$

ii - For the reflected field given by (2-65), (2-57), (2-59) and (2-64) we obtain

$$\phi_r(x,z) = \phi_r^{exc}(x,z) + \phi_r^{int,(1)}(x,z) + \phi_r^{int,(2)}(x,z)$$

$$= \sum_{n=1,3,5,..}^{\infty} [(R_{\ell,m}+R_{\ell,m}^{(1)})e^{\gamma_m z} + R_{\ell,m}^{(2)}(z)]\cos(\frac{m\pi}{2a}x) \tag{4-15}$$

where $R_{\ell,m}$ is the same as that of (2-57) and $R_{\ell,m}^{(1)}$ and $R_{\ell,m}^{(2)}$ are given by

$$R_{\ell,m}^{(1)} = \frac{-i\ell\pi^2}{4a^3}(-1)^{\frac{\ell+m}{2}} \frac{m}{\gamma_m} \frac{G_+(i\gamma_\ell)G_+(i\gamma_m)}{(i\gamma_\ell-k)G_+^2(k)} \frac{\overline{F}}{1-\overline{F}^2} \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k-i\gamma_m} \tag{4-16}$$

and

$$R^{(2)}_{\ell,m}(z) = \frac{-\pi\ell m}{2a^3}\,\frac{G_+(i\gamma_\ell)}{G_+^2(k)}\,\frac{e^{ik(L-z)}}{(i\gamma_\ell-k)}\,\frac{(-1)^{\frac{\ell+m}{2}}}{1-\overline{F}^2}\,\frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}}$$

$$\int_0^\infty \frac{\cosh^2[\frac{a}{L}\sqrt{2ikLu-u^2}]\qquad e^{-u(\frac{L-z}{L})}}{\sqrt{2ikLu-u^2}\;G_+(k+\frac{iu}{L})[(k+\frac{iu}{L})^2+\gamma_m^2][2k+\frac{iu}{L}]}\,du \tag{4-17}$$

iii - For the transmitted field given by (2-73), (2-68), (2-70) and (2-72) we obtain

$$\begin{aligned}\phi_t(x,z) &= \phi_t^{exc}(x,z) + \phi_t^{int,(1)}(x,z) + \phi_t^{int,(2)}(x,z)\\ &= \sum_{m=1,3,5,..}^{\infty} [T_{\ell,m}(z) + T^{(1)}_{\ell,m}(z) + T^{(2)}_{\ell,m}e^{-\gamma_m z}]\cos(\tfrac{m\pi}{2a}x)\end{aligned} \tag{4-18}$$

where $T_{\ell,m}(z)$ is the same as that of (2-68) and $T^{(1)}_{\ell,m}(z)$ and $T^{(2)}_{\ell,m}$ are given by

$$T^{(1)}_{\ell,m}(z) = \frac{\pi\ell m}{2a^3}(-1)^{\frac{\ell+m}{2}}e^{ikz}\,\frac{\overline{F}}{1-\overline{F}^2}\,\frac{G_+(i\gamma_\ell)e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}\;G_+^2(k)(i\gamma_\ell-k)}$$

$$\int_0^\infty \frac{\cosh^2[\frac{a}{z}\sqrt{2ikzu-u^2}]\;e^{-u}}{\sqrt{2ikzu-u^2}[(k+\frac{iu}{z})^2+\gamma_m^2][2k+\frac{iu}{z}]G_+^2(k+\frac{iu}{z})}\,du \tag{4-19}$$

and

$$T^{(2)}_{\ell,m} = \frac{-i\ell\pi^2}{4a^3}(-1)^{\frac{\ell+m}{2}}\,\frac{mG_+(i\gamma_\ell)G_+(i\gamma_m)\;e^{i(kL-\frac{\pi}{4})}\;e^{\gamma_m L}}{\gamma_m(1-\overline{F}^2)(i\gamma_\ell-k)(k-i\gamma_m)G_+^2(k)\sqrt{2\pi kL}} \tag{4-20}$$

Previous expressions for the radiated, reflected and transmitted fields are obtained in the next section using ray theory of diffraction with the modified diffraction coefficient of Lee [27], [28].

4.2.2 Application of the ray theory of diffraction

S.W. Lee [27],[28] has introduced a modified diffraction coefficient for problems involving two or more parallel plates, which takes care of

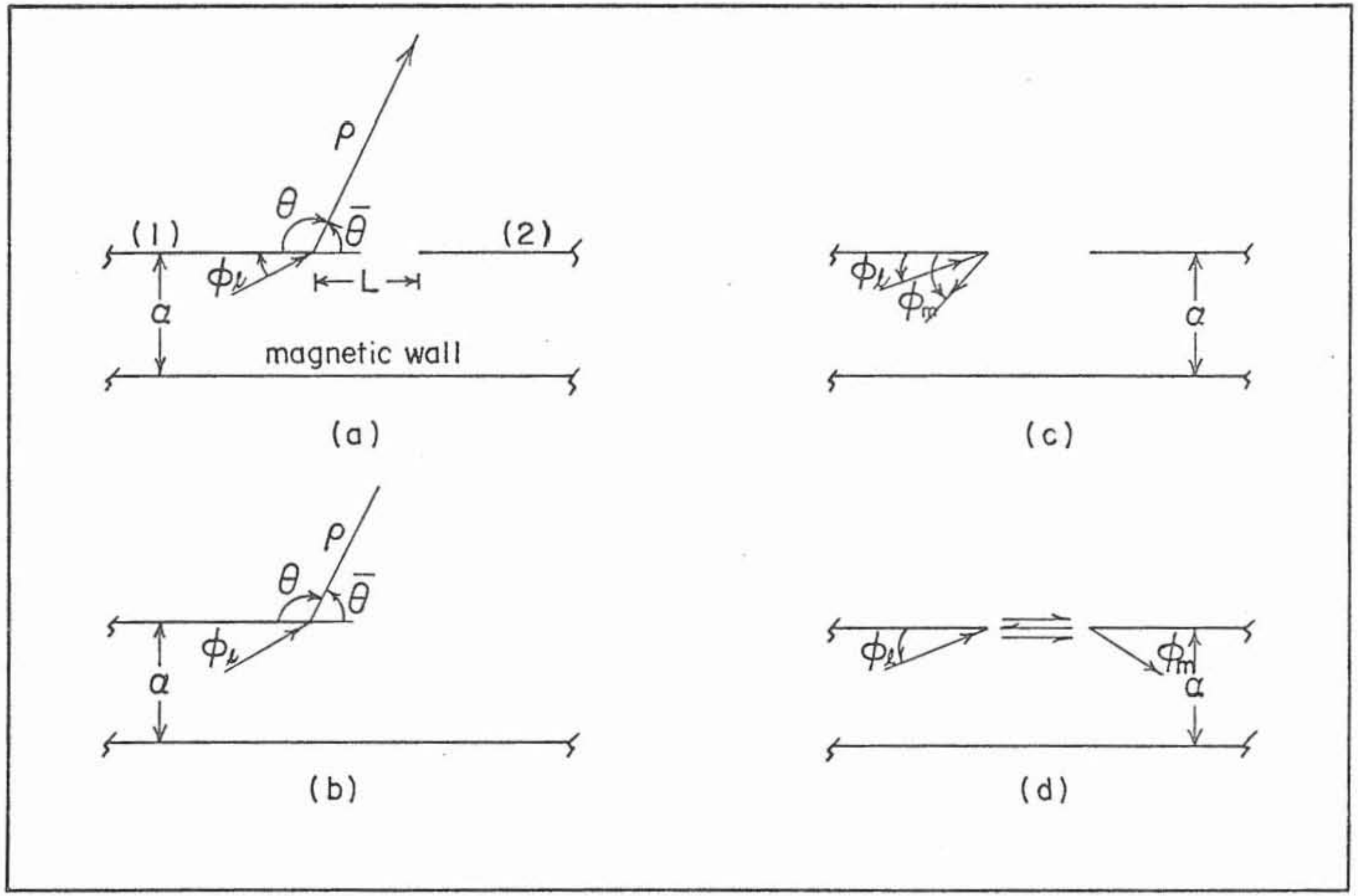

Fig. 4-2 Geometrics for the application of the ray theory of diffraction

coupling along a shadow boundary. To apply it to the present problem with an excitation of $TE_{o,\ell}$ mode and ℓ odd, one utilizes the symmetry of the geometry with respect to z-axis and introduces an infinitely large magnetic wall at the centre of the waveguides, as shown in figure 4-2a. The incident field is then a plane wave illuminating the upper edge of the exciting waveguide at an angle ϕ_ℓ, where $\sin\phi_\ell = \frac{\ell\pi}{2ka}$. The resulting diffracted, reflected and transmitted waves can then be found by an application of the above modified diffraction coefficient.

i - Diffraction patterns:

The diffraction patterns consist of the diffraction due to the exciting waveguide alone and the multiple diffractions between plates 1 and 2, which may be considered separately as follows:

A - Diffraction due to the open end of the exciting waveguide, figure 4-2b:

The field $\phi_1(\rho,\theta)$ on the ray diffracted at the edge of upper plate is given by

$$\phi_1(\rho,\theta) = \frac{e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{2\pi k\rho}} \overline{D}(\overline{\theta},\theta_o)E_y^i \quad , \quad \overline{\theta} = \pi-\theta \tag{4-21}$$

where $$E_y^i = \frac{i}{2}(-1)^{\frac{\ell-1}{2}} \quad , \text{ at the upper edge} \tag{4-22}$$

and θ_o is the direction of the incident plane wave. ρ and θ are the coordinates of the observation point with respect to the upper edge and the factor $\overline{D}(\overline{\theta},\theta_o)$ is the modified diffraction coefficient in the form

$$\overline{D}(\overline{\theta},\theta_o) = \frac{-2i\cos\frac{\theta_o}{2}\cos\frac{\overline{\theta}}{2}}{\cos\theta_o+\cos\overline{\theta}} \; f(\overline{\theta})f(\theta_o) \tag{4-23}$$

with

$$f(\theta) = \begin{cases} \overline{G}_+(-k\cos\theta) & \frac{\pi}{2} < |\theta| < \pi \\ [\overline{G}_+(k\cos\theta)]^{-1} & |\theta| < \frac{\pi}{2} \end{cases} \tag{4-24}$$

The function $\overline{G}_+(\alpha)$ used in [27] and [28], is related to G_+ of the present work by

$$\overline{G}_+(\alpha) = \sqrt{2(\alpha+k)}\, e^{-i\frac{\pi}{4}}\, G_+(\alpha) \tag{4-25}$$

Thus, from equation (4-21) and equations (4-22) to (4-25), we obtain

$$\phi_1(\rho,\theta) = \frac{e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{2\pi k\rho}}\,[\frac{i}{2}(-1)^{\frac{\ell-1}{2}}][\frac{2k\sin\phi_\ell \sin\theta}{\cos\phi_\ell+\cos\theta}\, G_+(k\cos\phi_\ell)G_+(k\cos\theta)] \tag{4-26}$$

Replacing $k\,\sin\phi_\ell$ by $\frac{\ell\pi}{2a}$ and $k\,\cos\phi_\ell$ by $i\gamma_\ell$ in the above expression, it becomes:

$$\phi_i(\rho,\theta) = \frac{i\pi\ell e^{i(k\rho-\frac{\pi}{4})}}{2a\sqrt{2\pi k\rho}}(-1)^{\frac{\ell-1}{2}}\, G_+(i\gamma_\ell)\, \frac{k\sin\theta G_+(k\cos\theta)}{k\cos\theta + i\gamma_\ell} \tag{4-27}$$

Note that for $\overline{\theta} < \frac{\pi}{2}$, the specular reflection at the magnetic wall requires the multiplication of the results by a factor of $[1 + e^{2ika\sin\overline{\theta}}]$, which when combined by $[\overline{G}_+(k\cos\overline{\theta})]^{-1}$ gives $\overline{G}_+(-k\cos\theta)$. Thus, for the range $0 < \overline{\theta} < \pi$, one can use a single expression $f(\overline{\theta}) = \overline{G}_+(-k\cos\overline{\theta})$, which gives the above equation (4-27) for $\phi_1(\rho,\theta)$ valid for $0 < \theta < \pi$.

B. Multiple diffraction between two waveguides:

There are two kinds of multiple diffracted rays as shown in table 4-1, with the integer n being at least unity.

TABLE 4-1

Types of multiple diffracted rays between edges of plates 1 & 2

Type	Initial Diffraction at the edge of plate	Number of diffracted rays at the edge of plate		Final Diffraction at the edge of plate
		(1)	(2)	
(A)	(1)	n + 1	n	(1)
(B)	(1)	n	n	(2)

The first column designates the type of ray. The second column shows which edge is first hit by the incident ray that gives rise to multiply diffracted rays. The last column shows the edge at which the multiply diffracted ray is finally diffracted. The other columns give the number of times the ray is diffracted at each edge. Now consider the fields due to rays of the type (A):

$$\phi_{2,A}(\rho,\theta) = \sum_{n=1}^{\infty} \bar{F}_i \bar{F}^{(2n-1)} \bar{F}_{f_1} \tag{4-28}$$

where $\bar{F}_i$ is the diffraction field at the edge of plate (2) due to the initial diffraction of the incident plane wave at the edge of plate (1). $\bar{F}$ is the diffraction field at the edge of plate (1) or (2) with a plane wave of unit amplitude incident with angle zero at the edge of plate (2) or (1), respectively. $\bar{F}_{f_1}$ is the diffraction field, at the observation point, due to the final diffraction at the edge of plate (1) of an incident plane wave having a unit amplitude. Equation (4-28) can be written in the form

$$\phi_{2,A}(\rho,\theta) = \bar{F}_i \bar{F}_{f_1} \sum_{n=1}^{\infty} \bar{F}^{(2n-1)} = \bar{F}_i \bar{F}_{f_1} \frac{\bar{F}}{1-\bar{F}^2} \tag{4-29}$$

where $\bar{F}_i$, $\bar{F}$ and $\bar{F}_{f_1}$, with the aid of equation (4-21), are given by

$$\begin{aligned} \bar{F}_i &= \frac{e^{i(kL - \frac{\pi}{4})}}{\sqrt{2\pi kL}} \bar{D}(0,-(\pi-\phi_\ell))E_y^i \\ &= \frac{-\pi\ell}{2a} (-1)^{\frac{\ell-1}{2}} \frac{e^{i(kL - \frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{G_+(i\gamma_\ell)}{G_+(k)(i\gamma_\ell - k)} \end{aligned} \tag{4-30}$$

$$\bar{F} = \frac{e^{i(kL - \frac{\pi}{4})}}{\sqrt{2\pi kL}} \bar{D}(0,0) = \frac{e^{i(kL - \frac{\pi}{4})}}{2k\sqrt{2\pi kL}} \frac{1}{G_+^2(k)} \tag{4-31}$$

and $$\bar{F}_{f_1} = \frac{e^{i(k\rho - \frac{\pi}{4})}}{\sqrt{2\pi k\rho}} \bar{D}(\bar{\theta},0) = \frac{-i\, e^{i(k\rho-\pi/4)}}{\sqrt{2\pi k\rho}} \frac{\sin\theta}{1-\cos\theta} \frac{G_+(k\cos\theta)}{G_+(k)} \tag{4-32}$$

It is clear that $\overline{F}$ given by (4-31) is the same as that given by equation (4-12). Or, in other words, the function $T_o(\alpha)$, which is the first term of $T(\alpha)$ given by (4-10) gives the result if we use the ray theory of diffraction.

A substitution of (4-30) and (4-32) into (4-29), gives

$$\phi_{2,A}(\rho,\theta) = \frac{i\pi\ell(-1)^{\frac{\ell-1}{2}}}{2a\sqrt{2\pi k\rho}} e^{i(k\rho-\frac{\pi}{4})} \frac{\overline{F}}{1-\overline{F}^2} \frac{G_+(i\gamma_\ell)\cdot k\sin\theta}{(i\gamma_\ell-k)G_+^2(k)} G_+(k\cos\theta)$$

$$\left[\frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k(1-\cos\theta)}\right] \tag{4-33}$$

$$\left.\begin{array}{l}\text{Similarly fields due to rays of type (B) with}\\ \text{distances measured from edge of plate (1)}\end{array}\right\} = \sum_{n=1}^{\infty} \overline{F}_i \overline{F}^{(2n-2)} \overline{F}_{f_2} e^{ikL\cos\theta} \tag{4-34}$$

$$\text{or} \quad \phi_{2,B}(\rho,\theta) = \overline{F}_i\, \overline{F}_{f_2} \frac{1}{1-\overline{F}^2} e^{ikL\cos\theta} \tag{4-35}$$

where $\overline{F}_i$ and $\overline{F}$ are given by (4-30) and (4-31) and $\overline{F}_{f_2}$ is

$$\overline{F}_{f_2} = \frac{-i\, e^{i(k\rho-\frac{\pi}{4})}}{\sqrt{2\pi k\rho}} \frac{\sin\theta}{1+\cos\theta} \frac{G_+(-k\cos\theta)}{G_+(k)} \tag{4-36}$$

Again a substitution of (4-30) and (4-36) into (4-35), gives

$$\phi_{2,B}(\rho,\theta) = \frac{i\pi\ell(-1)^{\frac{\ell-1}{2}}}{2a\sqrt{2\pi kL}} e^{i(k\rho-\frac{\pi}{4})} \frac{1}{1-\overline{F}^2} \frac{G_+(i\gamma_\ell)\cdot k\sin\theta}{(i\gamma_\ell-k)G_+^2(k)} G_+(-k\cos\theta)$$

$$\left[\frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}} \frac{1}{k(1+\cos\theta)}\right] e^{ikL\cos\theta} \tag{4-37}$$

Thus, the total diffracted field with distances measured from origin at $x=0$ and $z=0$, is given by

$$\phi^s(\rho,\theta) = [\phi_1(\rho,\theta) + \phi_{2,A}(\rho,\theta) + \phi_{2,B}(\rho,\theta)]e^{-ika\sin\theta} \tag{4-38}$$

Upon combining (4-26), (4-33) and (4-37) with (4-38), the diffracted field $\phi^s(\rho,\theta)$ can be shown to be the same as (4-13). Or, in other words, the results obtained by the ray theory of diffraction are the same as that obtained by the Wiener-Hopf technique when the first term in the asymptotic expansion of $T(\alpha)$ is retained. Hence, $T_o(\alpha)$ yields the ray theory results.

ii - Fields inside the exciting waveguide:

Again the reflected field consists of the diffraction due to the exciting waveguide and the multiple diffraction between the two waveguides. However, the diffracted rays are now converted into modes inside the exciting waveguide. The reflection due to the open end of the exciting waveguide is,figure 4-2c:

$$\phi_{r,1}(x,z) = \sum_{m=1,3,5,\ldots}^{\infty} [2i(-1)^{\frac{m-1}{2}}]\cos(\frac{m\pi}{2a}x)e^{\gamma_m z}\,\overline{D}[-(\pi-\phi_m),-(\pi-\phi_\ell)]$$

$$[\text{Ray to mode conversion factor}]\; E_y^i \tag{4-39}$$

where the first bracket is to normalize the amplitude of the rays travelling in the $-(\pi-\phi_m)$ direction at $x=a$, $z=0$, for an incident plane wave of unit amplitude at the above point. The second bracket is the ray to mode conversion factor given by [27].

$$\text{conversion factor} = \left\{ [\frac{d}{d\alpha}(\gamma\overline{G}(\alpha))]^{-1} \right\}_{\alpha=k\cos\phi_m} = \frac{1}{2ka\cos\phi_m} \tag{4-40}$$

and E_y^i is given by equation (4-22). Equation (4-30), after some manipulation becomes:

$$\phi_{r,1}(x,z) = \sum_{m=1,3,5,..}^{\infty} R_{\ell,m}\cos(\frac{m\pi}{2a}x)e^{\gamma_m z} \tag{4-41}$$

with

$$R_{\ell,m} = \frac{-\ell\pi^2}{4a^3} (-1)^{\frac{\ell+m}{2}} \frac{mG_+(i\gamma_m)G_+(i\gamma_\ell)}{\gamma_m(\gamma_m+\gamma_\ell)} \tag{4-42}$$

Similarly, the reflection due to rays of type (A) can be shown to be:

$$\phi_{r,2,A}(x,z) = \sum_{m=1,3,5,..}^{\infty} [2i(-1)^{\frac{m-1}{2}}]\cos(\frac{m\pi}{2a}x)e^{\gamma_m z}[\frac{\overline{F}}{1-\overline{F}^2}\overline{F}_i\overline{F}_{f_1}]$$

$$[\text{Ray to mode conversion factor}] \tag{4-43}$$

where $\overline{F}$, $\overline{F}_i$ and $\overline{F}_{f_1}$ are again given respectively by (4-31), (4-30) and (4-32), with θ replaced by ϕ_m and $\frac{\theta^{i(k\rho-\pi/4)}}{\sqrt{2\pi k\rho}}$ dropped in (4-32). Hence one finds

$$\phi_{r,2,A}(x,z) = \sum_{m=1,3,5,..}^{\infty} R^{(A)}_{\ell,m} \cos(\frac{m\pi}{2a}x)\, e^{\gamma_m z} \tag{4-44}$$

with

$$R^{(A)}_{\ell,m} = \frac{-i\ell\pi^2}{4a^3} (-1)^{\frac{\ell+m}{2}} \frac{mG_+(i\gamma_\ell)G_+(i\gamma_m)}{\gamma_m(i\gamma_\ell-k)G_+^2(k)} \frac{\overline{F}}{1-\overline{F}^2} \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}\,(k-i\gamma_m)} \tag{4-45}$$

The field, due to rays of type (B), is a radiated field and has the same form as equation (4-37) with $\cos\theta \simeq 1$. This field can be converted into a modal series to give $R^{(B)}_{\ell,m}(z)$ similar to $R^{(2)}_{\ell,m}(z)$, given by equation (4-17). Based on this discussion and upon combining (4-41) and (4-44), the total reflected field using the ray theory of diffraction is the same as (4-15). Or, in other words, $T_o(\alpha)$, which is the first term of the asymptotic expansion of $T(\alpha)$, yields ray theory results.

iii - Fields inside the coupled waveguide

The transmitted fields in the coupled waveguide also consist of the diffracted fields due to the exciting waveguide and multiple diffraction between the two waveguides, figure 4-2d. Here the diffraction due to the exciting waveguide and rays of type (A) are of the scattering type and give transmission coefficients which are functions of z and can be treated similar to the rays of type (B) in part ii. The remaining contribution comes from rays of type (B) which may be shown to be

$$\phi_{t,2,B}(x,z) = \sum_{m=1,3,5,..}^{\infty} [2i(-1)^{\frac{m-1}{2}}]\cos(\frac{m\pi}{2a}x)e^{-\gamma_m(z-L)}\frac{\overline{F}_i\overline{F}_{f_2}}{1-\overline{F}^2}$$

$$[\text{Ray to mode conversion factor}] \qquad (4\text{-}46)$$

where again $\overline{F}$, $\overline{F}_i$ and $\overline{F}_{f_2}$ are given respectively by (4-31), (4-30) and (4-36), with θ replaced by $\pi - \phi_m$ and $\frac{e^{i(k\rho-\pi/4)}}{\sqrt{2\pi k\rho}}$ dropped in (4-36). Hence one finds

$$\phi_{t,2,B}(x,z) = \sum_{m=1,3,5,..}^{\infty} T^{(B)}_{\ell,m} \cos(\frac{m\pi}{2a}x)\ e^{-\gamma_m z} \qquad (4\text{-}47)$$

with

$$T^{(B)}_{\ell,m} = \frac{-i\ell\pi^2}{4a^3}(-1)^{\frac{\ell+m}{2}} \frac{mG_+(i\gamma_\ell)G_+(i\gamma_m)e^{\gamma_m L}}{\gamma_m(i\gamma_\ell-k)G_+(k)(1-\overline{F}^2)} \frac{e^{i(kL-\frac{\pi}{4})}}{\sqrt{2\pi kL}\ (k-i\gamma_m)} \qquad (4\text{-}48)$$

Again, it is clear that $T^{(B)}_{\ell,m}$ is the same as $T^{(2)}_{\ell,m}$ given by (4-20). As a result $T_o(\alpha)$, given by equation (4-11), yields ray theory results.

Some results for a $TE_{o,1}$ mode excitation will be shown in the last section of this chapter, to verify the validity of the ray theory for solving these kinds of problems.

4.3 Coupling between two collinear circular waveguides

4.3.1 Expansion of the transformed Green's function

If the transformed Green's function $G(\alpha)$ given in appendix A by equation (A-12), is expanded in a power series, then the function $T(\alpha)$, given by equation (3-12) can be written as

$$T(\alpha) = \pi i \int_{-k}^{-k-i\infty} \frac{(\beta+k)}{\beta+\alpha} [\sum_{n=o,1,2,..}^{\infty} \frac{(\gamma a)^{2n}}{4^n(n!)^2}]^2\ e^{-i\beta L}\ d\beta \qquad (4\text{-}49)$$

In the neighborhood of $\beta = -k$, the function $\beta-k$ is regular and smooth and can be replaced by $-2k$, and after some manipulations (4-49) gives

$$T(\alpha) = \pi i e^{ikL} \{ (\frac{-i}{L}) W_{\frac{1}{2}}(\xi) - ka^2 (\frac{-i}{L})^2 W_{\frac{3}{2}}(\xi) + \frac{3}{8} k^2 a^4 (\frac{-i}{L})^3 W_{\frac{5}{2}}(\xi) + \ldots\} \tag{4-50}$$

where ξ and $W_{j-\frac{1}{2}}(\xi)$ are defined by equations (4-5) and (4-6), respectively. The function $W_{j-\frac{1}{2}}(\xi)$ is related to the Whittaker function $W_{u,v}(\xi)$ by equation (4-7). Upon using the asymptotic expansion of the Whittaker functions and retaining only the first term, equation (4-50) reduces to

$$T(\alpha) = \frac{ie^{ikL}}{L^2(k-\alpha)} [1 + i\mu - \frac{9}{16}\mu^2 + \ldots] \tag{4-51}$$

where, $\mu = 2(ka)^2/kL$

Equation (4-51) is an asymptotic expansion of $T(\alpha)$, which is convergent for $\mu << 1$. In other words, kL must be much greater than $2(ka)^2$. Thus, retaining only the first term in equation (4-51), it reduces to

$$T(\alpha) = T_o(\alpha) = \frac{\pi i e^{ikL}}{L^2(k-\alpha)}$$

Combining this equation with equation (3-14), one obtains

$$F = \overline{F} = \frac{-e^{ikL}}{8(kL)\ G_+^2(k)} \tag{4-53}$$

This function will be shown later to be the same as that obtained by ray theory. A substitution of (4-52) and (4-53) into the expressions of the radiated, reflected and transmitted fields, obtained by the Wiener-Hopf technique, gives

i - For the radiated field given by (3-18) and (3-19):

$$H_\phi^s(r,\theta) = \frac{e^{i(kr-\frac{\pi}{2})}}{kr}\ F(\theta) \tag{4-54}$$

where

$$F(\theta) = -\frac{\xi_{om}}{\pi} J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om}) \frac{k^2\sin\theta}{H_o^{(1)}(ka\sin\theta)}$$

$$\left\{\frac{G_+(k\cos\theta)}{(k-k\cos\theta)(k\cos\theta+i\gamma_{om})} - \frac{\overline{F}}{1-\overline{F}^2} \frac{e^{ikL}}{(2kL)^2(1-\cos\theta)} \frac{G_+(k\cos\theta)}{G_+^2(k)(i\gamma_{om}-k)(k-k\cos\theta)}\right.$$

$$\left. - \frac{1}{1-\overline{F}^2} \frac{e^{ikL}}{(2kL)^2(1+\cos\theta)} \frac{G_+(-k\cos\theta)}{G_+^2(k)(i\gamma_{om}-k)(k+k\cos\theta)} e^{ikL\cos\theta}\right\} \qquad (4\text{-}55)$$

ii - For the reflected field given by (3-39), (3-26), (3-28) and (3-38):

$R^{(2)}_{m,n}(z)$ is the same as that given by (3-38) with $T(\alpha)$ inside the integral being replaced by equation (4-52). Also, $R_{m,n}$ is the same as (3-26). For the third component of the reflected field, the reflection coefficient $R^{(1)}_{m,n}$ is given by

$$R^{(1)}_{m,n} = -\frac{i}{2} \frac{\overline{F}}{1-\overline{F}^2} \frac{\xi_{om}}{a} \frac{\xi_{on}}{a} \frac{J_1(\xi_{om})}{J_1(\xi_{on})} \frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})e^{ikL}}{2kL^2\gamma_{on}(k-i\gamma_{om})(k-i\gamma_{on})^2G_+^2(k)} \qquad (4\text{-}56)$$

iii - For the transmitted field given by (3-48), (3-43), (3-45) and (3-47).

$T_{m,n}(z)$ and $T^{(1)}_{m,n}(z)$ are the same as (3-43) and (3-45) with $T(\alpha)$ in the integrand of equation (3-45), being replaced by equation (4-52). However, the main contributing term to the transmitted field is modified to

$$T^{(2)}_{m,n} = -\frac{i\, e^{\gamma_{on}L}}{2(1-\overline{F}^2)} \frac{\xi_{om}}{a} \frac{\xi_{on}}{a} \frac{J_1(\xi_{om})}{J_1(\xi_{on})} \frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})e^{ikL}}{2kL^2\gamma_{on}(k-i\gamma_{om})(k-i\gamma_{on})^2G_+(k)} \qquad (4\text{-}57)$$

The above expressions for the radiated, reflected and transmitted fields are determined later using the ray theory of diffraction, after introducing a modified diffraction coefficient, a spherical wavefactor and a ray to mode conversion factor.

4.3.2 Determination of the modified diffraction coefficient, conversion factor and the spherical wavefactor

In equation (3-18), the first term corresponds to the radiation from the open end of the exciting waveguide only. Rewriting this term with θ replaced by $\pi-\overline{\theta}$, where $\overline{\theta}$ is measured from the positive z-axis, one has

$$H_{\phi}^{s,exc}(r,\overline{\theta}) = \frac{-e^{i(kr-\frac{\pi}{2})}}{\pi kr}\,\xi_{om}J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om})\,\frac{k^2\sin\overline{\theta}}{H_o^{(1)}(ka\sin\overline{\theta})}$$

$$\frac{G_+(-k\cos\overline{\theta})}{(k+k\cos\overline{\theta})(i\gamma_{om}-k\cos\overline{\theta})} \tag{4-58}$$

Introducing the Green's function obtained by following the procedure of Lee [28], one has

$$\overline{G}(\alpha) = 2\gamma a\ I_o(\gamma a)K_o(\gamma a) = 2\gamma a\ G(\alpha) \tag{4-59}$$

Hence

$$\overline{G}_+(\alpha) = e^{-i\frac{\pi}{4}}\sqrt{2a(k+\alpha)}\ \ G_+(\alpha) \tag{4-60}$$

From equations (4-58) and (4-60), with the relations $k\sin\overline{\theta}_{om} = \frac{\xi_{om}}{a}$ and $k\cos\overline{\theta}_{om} = -i\gamma_{om}$, one obtains

$$H_{\phi}^{s,exc}(r,\overline{\theta}) = [\frac{\xi_{om}}{a}\,J_1(\xi_{om})]\,\frac{e^{i(kr-\frac{\pi}{2})}}{2\pi kr}\,\left(\frac{2i\sin\frac{\overline{\theta}_{om}}{2}\sin\frac{\overline{\theta}}{2}}{\cos\overline{\theta}_{om}+\cos\overline{\theta}}\right)$$

$$\frac{\overline{G}_+(-k\cos\overline{\theta}_{om})\overline{G}_+(-k\cos\overline{\theta})}{\sin\overline{\theta}\ H_o^{(1)}(ka\sin\theta)} \tag{4-61}$$

It is clear that the first bracket is the incident magnetic field, H_{ϕ}^{i}, at the rim of the exciting waveguide. By analogy with the case of parallel-plate waveguides, which has been investigated by Lee [27], the modified diffraction coefficient can be written as

$$\overline{D}(\overline{\theta},\overline{\theta}_{om}) = D(\overline{\theta},\overline{\theta}_{om})f(\overline{\theta})f(\overline{\theta}_{om}) \tag{4-62}$$

where

$$D(\overline{\theta},\overline{\theta}_{om}) = \frac{2i \sin\frac{\overline{\theta}_{om}}{2} \sin\frac{\overline{\theta}}{2}}{\cos\overline{\theta}_{om}+\cos\overline{\theta}} \tag{4-63}$$

and

$$f(\theta) = \begin{cases} \overline{G}_{+}(-k\cos\theta) & \frac{\pi}{2} < |\theta| < \pi \\ [\overline{G}_{+}(k\cos\theta)]^{-1} & |\theta| < \frac{\pi}{2} \end{cases} \tag{4-64}$$

$D(\overline{\theta},\overline{\theta}_{om})$ is the well known diffraction coefficient for a half plane illuminated by a TM plane wave in the direction of $\overline{\theta}_{om}$. The angles $\overline{\theta}$ and $\overline{\theta}_{om}$ take values between $-\pi$ and π and are measured from the positive z-axis.

The factor $\frac{1}{\sin\overline{\theta}\, H_o^{(1)}(ka \sin\overline{\theta})}$ in equation (4-61) corresponds to the spherical waves of the diffracted rays and reduces to unity for the cylindrical waves, in the case of parallel plate waveguide. Hence equation (4-61) can be rewritten as

$$H_\phi^{s,exc}(r,\overline{\theta}) = H_\phi^{i} \text{ at the rim of the exciting waveguide}$$
$$\times \frac{e^{i(kr-\frac{\pi}{2})}}{2\pi kr} \times \overline{D}(\overline{\theta},\overline{\theta}_{om}) \times \text{spherical wavefactor} \tag{4-65}$$

It is quite interesting to note that the spherical wave variation $\frac{e^{i(kr-\frac{\pi}{2})}}{2\pi kr}$ and the spherical wavefactor are related to the curvature of the rim [53]. If $ka \sin\overline{\theta} \gg 1$, i.e. in the entire rear half space, or if the radius of the circular waveguide is large, then by using the asymptotic expression for $H_o^{(1)}(ka \sin\theta)$, we can write

$$\frac{e^{i(kr-\frac{\pi}{2})}}{2\pi kr} \times \text{spherical wavefactor} \simeq \frac{e^{i(kr-ka \sin\overline{\theta}-\frac{\pi}{4})}}{2\sqrt{r}\,(2\pi k)^{1/2}} \sqrt{\frac{a}{r \sin\overline{\theta}}}$$

$$= \frac{e^{i(kr-ka\sin\overline{\theta}-\frac{\pi}{4})}}{2(2\pi k)^{1/2}} \frac{1}{[r(1-\frac{r_1}{a}\sin\theta)]^{1/2}}$$

$$\simeq \frac{e^{i(kr_1-\frac{\pi}{4})}}{2(2\pi k)^{1/2}} \frac{1}{[r_1(1-\frac{r_1}{a}\sin\theta]^{1/2}} \qquad (4\text{-}66)$$

Where r_1 is measured from the diffracting point to the observation point, as shown in figure 4-3. However, equation (4-66), is the same equation that has been obtained by Keller [53] in treating the problem of diffraction by a circular aperture. In other words, the spherical wavefactor $\frac{1}{\sin\overline{\theta}\, H_o^{(1)}(ka\sin\overline{\theta})}$ may be used together with (4-63), to obtain the diffraction field by a circular aperture of small radius in hard screens.

Inside the exciting circular waveguide, the reflected magnetic field, due to only its open end, can be put in a form similar to that of the parallel-plate waveguide [27]:

$$H_\phi^r(\rho,z) = \sum_{n=1,2,..}^{\infty} R_{m,n} J_1(\frac{\xi_{on}}{a}\rho)e^{\gamma_{on}z} \qquad (4\text{-}67)$$

where

$$R_{m,n} = H_\phi^i \text{ at the rim of the exciting waveguide x } \overline{D}(\overline{\theta}_{on},\overline{\theta}_{om}) \text{ x}$$
$$\text{Ray to mode conversion factor} \qquad (4\text{-}68)$$

with $\overline{\theta}_{on} = \pi-\theta_{on}$, $k\cos\theta_{on} = i\gamma_{on}$ and $k\sin\theta_{on} = \frac{\xi_{on}}{a}$, and the conversion factor is given by

$$\text{conversion factor} = \frac{-1}{2ka\cos\theta_{on}\, J_1(\xi_{on})}$$

$$= \left[\frac{d}{d\alpha}\left(\frac{-i\,\overline{G}(\alpha)}{a\,K_o(\gamma a)}\right]^{-1}\right|_{\alpha=k\cos\theta_{on}} \qquad (4\text{-}69)$$

For the parallel plate waveguide of width 2a, and $TE_{o,\ell}$ excitation,

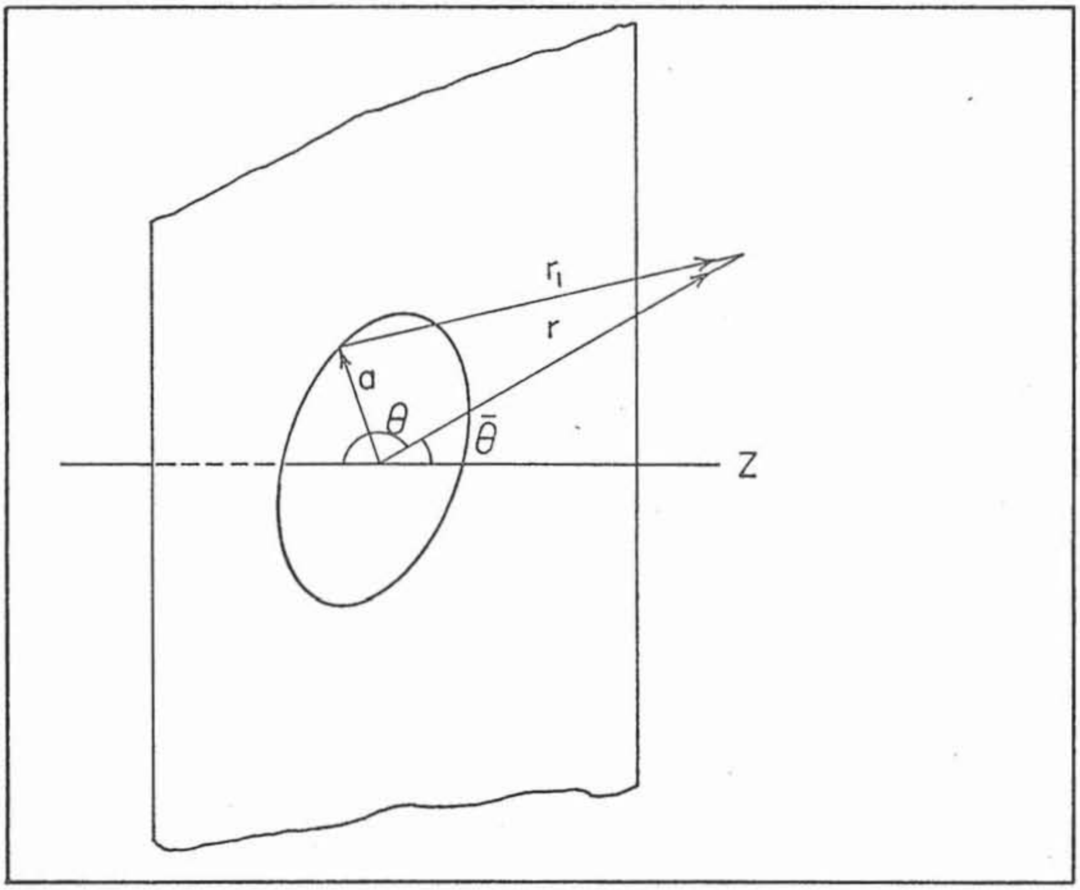

Fig. 4-3 A plane screen with a circular aperture of radius a.

this factor reduces to $\frac{1}{2ka \cos\theta_{on}}$, [27].

In the following section, the above results for the modified diffraction coefficient, the spherical wavefactor and the ray to mode conversion factor, are used to obtain the radiated, reflected and coupled (transmitted) fields of two axially coupled circular waveguides.

4.3.3 Application of the ray theory of diffraction

i - Diffraction pattern:

The diffraction pattern consists of the diffracted rays due to the exciting waveguide alone and the multiple diffractions between the rims of the two cylindrical waveguides. If $H^1_\phi(r,\theta)$ is the contribution of the diffracted rays due to the exciting waveguide alone, equation (4-58) gives the contribution of these diffracted rays. The contribution of the multiple diffracted rays comes from the two type of rays explained in table 4-1. Type (A), with initial and final diffractions at the rim of the exciting waveguide and type (B) with the initial and final diffractions at the rim of the exciting (rim 1) and the coupled waveguides (rim 2), respectively. Similar to the case of parallel plate waveguides, the contribution of rays of type (A) and (B) are respectively given by,

$$H^{2,A}(r,\theta) = \frac{\overline{F}}{1-\overline{F}^2}\, \overline{F}_i \overline{F}_{f_A} \tag{4-70}$$

and

$$H^{3,B}(r,\theta) = \frac{1}{1-\overline{F}^2}\, \overline{F}_i \overline{F}_{f_B} \; e^{ikL\cos\theta} \tag{4-71}$$

where $\overline{F}_i$ is the diffracted field at rim (2) due to the intial diffraction of the incident wave H^i_ϕ at rim (1) and $\overline{F}$ is the diffraction field at rim (1) or (2) of an incident wave of unit amplitude and incidence angle zero at rims (2) or (1), respectively. Also, $\overline{F}_{f_A}$ and $\overline{F}_{f_B}$ are respectively the diffracted fields, due to the final diffraction at rim (1) and (2) for a unit amplitude wave of angle zero at rim (1) or (2). Now, equation (4-63) shows that if the incident field vanishes at an edge, the diffracted field also vanishes and following Karp and Keller [54], the diffracted field is proportional to the normal derivative of the incident field at the edge. Thus it may be shown that:

$$\overline{F}_i = \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{e^{i(kL-\frac{\pi}{2})}}{2\pi kL} \frac{2}{L}\{\frac{\partial}{\partial\overline{\theta}}[\overline{D}(\overline{\theta},\overline{\theta}_{om}) \text{x spherical wavefactor}]\}_{\overline{\theta}=o}$$

$$= \xi_{om} J_1(\xi_{om}) \frac{e^{ikL}}{2L^2} \frac{k+i\gamma_{om}}{k-i\gamma_{om}} \frac{G_+(i\gamma_{om})}{G_+(k)} \quad , \tag{4-72}$$

$$\overline{F}_{f_A} = \frac{e^{i(kr-\frac{\pi}{2})}}{2\pi kr} (\frac{-2}{ik})\{ \frac{\partial}{\partial\overline{\theta}_{om}} [\overline{D}(\overline{\theta},\overline{\theta}_{om})]\}_{\overline{\theta}_{om}=o} \text{ x spherical wavefactor}$$

$$= \frac{-e^{i(kr-\frac{\pi}{2})}}{2\pi kr} \frac{k \sin\theta\, G_+(k\cos\theta)}{G_+(k) H_o^{(1)}(ka \sin\theta)(k-k\cos\theta)^2} \quad , \tag{4-73}$$

$$\overline{F}_{f_B} = \frac{-e^{i(kr-\frac{\pi}{2})}}{2\pi kr} \frac{k \sin\theta\, G_+(-k\cos\theta)}{G_+(k) H_o^{(1)}(ka \sin\theta)(k+k\cos\theta)^2} \tag{4-74}$$

and

$$\overline{F} = \frac{e^{i(kL-\frac{\pi}{2})}}{2\pi kL} (\frac{2}{L})(\frac{-2}{ik})\{ \frac{\partial^2}{\partial\overline{\theta}\,\partial\overline{\theta}_{om}} \overline{D}(\overline{\theta},\overline{\theta}_{om})\}_{\overline{\theta} = \overline{\theta}_{om} = 0}$$

$$= \frac{-e^{ikL}}{8(kL)^2 G_+^2(k)} \tag{4-75}$$

It is clear that $\overline{F}$ given by (4-75) is the same as that given by equation (4-53). In other words, the function $T_o(\alpha)$, which is the first term of $T(\alpha)$, given by (4-51), gives the results of the ray theory of diffraction. A substitution (4-72), (4-73) and (4-74) into (4-70) and (4-71) gives:

$$H_\phi^{2,A}(r,\theta) = \frac{e^{i(kr-\frac{\pi}{2})}}{\pi kr} \frac{\xi_{om}\overline{F}}{1-\overline{F}^2} \frac{J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om})\cdot k\sin\theta\cdot G_+(k\cos\theta)e^{ikL}}{(2kL)^2\cdot(1-\cos\theta)^2\cdot(i\gamma_{om}-k)G_+^2(k)H_o^{(1)}(ka\sin\theta)}$$

(4-76)

and

$$H_\phi^{3,B}(r,\theta) = \frac{e^{i(kr-\frac{\pi}{2})}}{\pi kr} \frac{\xi_{om}}{1-\overline{F}^2} \frac{J_1(\xi_{om})(k+i\gamma_{om})G_+(i\gamma_{om})\ k\sin\theta\cdot G_+(k\cos\theta)e^{ikL(1+\cos\theta)}}{(2kL)^2(1+\cos\theta)^2(i\gamma_{om}-k)G_+^2(k)H_o^{(1)}(ka\sin\theta)}$$

(4-77)

The total radiated field is then the sum of $H_\phi^1(r,\theta)$, $H_\phi^{2,A}(r,\theta)$ and $H^{3,B}(r,\theta)$. It is clear that this result is exactly the same as that given by (4-55). In other words, when retaining only the first term of the asymptotic expansion of $T(\alpha)$, the results yield the results of the ray theory of diffraction with the defined modified diffraction coefficient in conjunction with the spherical wavefactor. Consequently, since $T_o(\alpha)$ corresponds to the solution of the ray theory of diffraction, the higher order terms of $T(\alpha)$ given by (4-51) provide the correction when $2(ka)^2/kL$ is not small enough.

ii - Fields inside the exciting waveguide:

The reflected field consists of the diffraction field at the open end of the exciting waveguide and the multiple diffraction between the two waveguides. To find this reflected field, the diffracted fields are, of course, converted into waveguide modes, inside the exciting waveguide. The reflected field due to the exciting waveguide only, is given by equation (4-67), and that of the multiple diffracted fields is due to rays of type (A) only. The rays of type (B) yield the radiation field inside the exciting waveguide and, at large distance, they do not contribute to the reflected fields. Hence, the reflected field due to rays of type (A) may be shown to be:

$$H_{\phi}^{r,A}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} J_1(\frac{\xi_{on}}{a}\rho)e^{\gamma_{on}z}[\frac{\overline{F}}{1-\overline{F}^2}\overline{F}_i\overline{F}_{f_A}][\text{Ray to mode conversion factor}] \quad (4\text{-}78)$$

where $\overline{F}$, $\overline{F}_i$ and $\overline{F}_{f_A}$ are given, respectively by (4-75), (4-72) and (4-73), by replacing θ by θ_{on} and dropping $e^{i(kr-\frac{\pi}{2})}/2\pi kr$ and the spherical wavefactor in (4-73). Thus one finds

$$H_{\phi}^{r,A}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} R_{m,n}^{(A)} J_1(\frac{\xi_{on}}{a}\rho)e^{\gamma_{on}z} \quad (4\text{-}79)$$

where

$$R_{m,n}^{(A)} = -\frac{i}{2}\frac{\overline{F}}{1-\overline{F}^2}\frac{\xi_{om}}{a}(\frac{\xi_{on}}{a})^2\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})e^{ikL}}{2kL^2\gamma_{on}(k-i\gamma_{om})(k-i\gamma_{on})^2G_+^2(k)} \quad (4\text{-}80)$$

It is clear, also, that $R_{m,n}^{(A)}$ is the same as $(\frac{\xi_{on}}{a}R_{m,n}^{(1)})$, with $R_{m,n}^{(1)}$ given by (4-56). Hence, once again $T_o(\alpha)$ gives the solution of the ray theory of diffraction.

iii - Fields inside the coupled waveguide:

Far from the open end of the coupled waveguide, the transmitted field mainly comes from the rays of type (B). The contributions of these rays after conversion into waveguide modes is given by:

$$H_{\phi}^{t,(B)}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} J_1(\frac{\xi_{on}}{a}\rho)e^{-\gamma_{on}(z-L)}[\frac{1}{1-\overline{F}^2}\overline{F}_i\overline{F}_{f_B}][\text{Ray to mode conversion factor}] \quad (4\text{-}81)$$

where $\overline{F}$, $\overline{F}_i$ and $\overline{F}_{f_B}$ are given respectively by (4-75), (4-72) and (4-74), by replacing θ by $\pi-\theta_{on}$ and dropping $e^{i(kr-\frac{\pi}{2})}/2\pi kr$ and the spherical wavefactor in (4-74). Hence one obtains

$$H_{\phi}^{t,(B)}(\rho,z) = \sum_{n=1,2,3,..}^{\infty} T_{m,n}^{(B)} J_1(\frac{\xi_{on}}{a}\rho)e^{-\gamma_{on}z} \quad (4\text{-}82)$$

where

$$T^{(B)}_{m,n} = \frac{-i}{2}\,\frac{e^{\gamma_{on}L}}{1-\overline{F}}\,\frac{\xi_{om}}{a}\,\left(\frac{\xi_{on}}{a}\right)^2 \frac{J_1(\xi_{om})}{J_1(\xi_{on})}\,\frac{(k+i\gamma_{om})G_+(i\gamma_{om})G_+(i\gamma_{on})e^{ikL}}{2kL^2\gamma_{on}(k-i\gamma_{om})(k-i\gamma_{on})^2G_+^2(k)} \quad (4\text{-}83)$$

Equation (4-83) is identical to $(\frac{\xi_{on}}{a}\,T^{(2)}_{m,n})$, with $T^{(2)}_{m,n}$ given by (4-57), which was obtained from (3-47) by replacing $T(i\gamma_{on})$ from equation (4-52).

4.4 Results and discussion

The asymptotic expansions for $T(\alpha)$, for the two cases of parallel-plate and circular waveguides, areobtained and are given by (4-10) and (4-51), respectively. It has been shown that the first term in the asymptotic expansion of $T(\alpha)$, which is $T_o(\alpha)$, yields the solution using ray theory of diffraction. In other words, higher order terms of $T(\alpha)$ provide corrections when $(ka)^2/kL$ is not small enough in the case of parallel-plate waveguides, and when $2(ka)^2/kL$ is not small enough in the case of circular waveguides. Kashyap and Hamid [25] have investigated the problem of diffraction by a slit in a thick screen and have obtained the same condition $(ka)^2/kL \ll 1$, such that first term of their solution leads to ray theory results.

Besides treating the circular waveguide by ray theory of diffraction and deriving the condition of its validity, a spherical wavefactor has been obtained and was shown to be necessary for treating problems of diffraction by a small aperture in a hard screen.

To study the effect of including higher terms of $T(\alpha)$ on the results, $T(\alpha)$ was investigated in detail for the case of parallel-plate waveguides. This is shown in figures 4-4, 4-5 and 4-6, for different values of α. It is clear from these figures that for α close but not equal to k, the results deviate from the exact form of $T(\alpha)$. However,

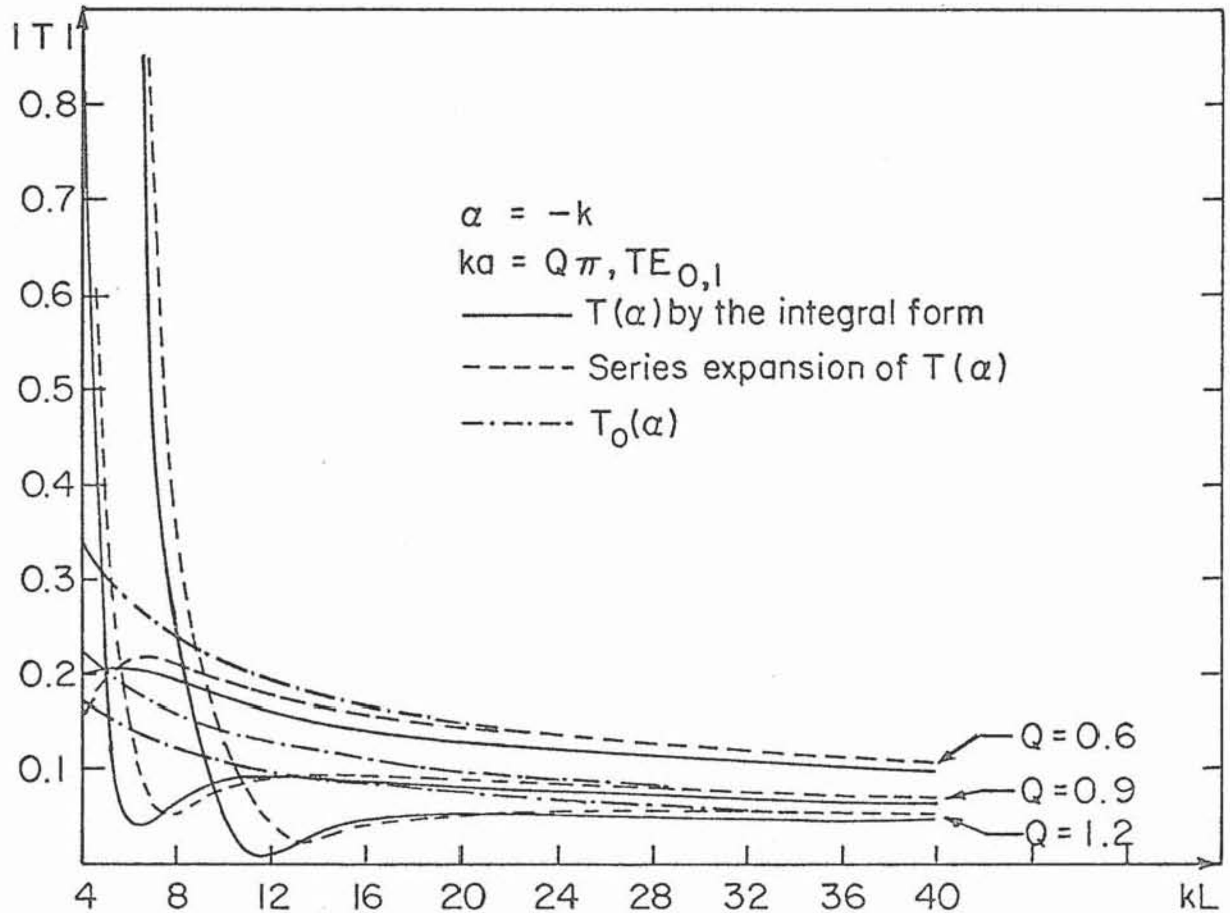

Figure 4-4 Evaluation of $T(\alpha)$ for parallel plate waveguides case using : 1 - The semi - infinite integral , equation (2-40)
2- The asymptotic expansion , equation (4-10)
3- First term in the asymptotic expansion of $T(\alpha)$, equation (4-11)

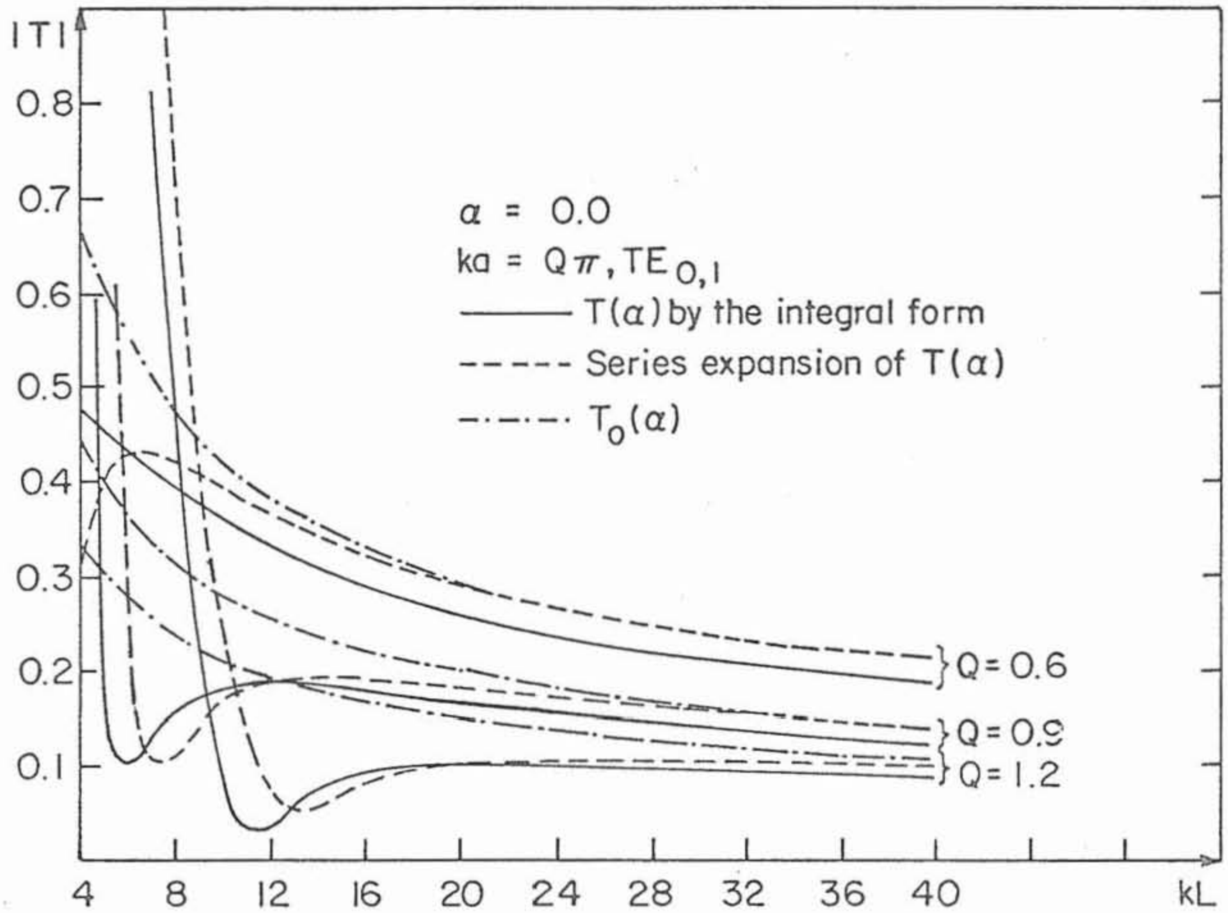

Figure 4-5 Evaluation of T(α) for parallel plate waveguides case using : 1 - The semi-infinite integral, equation (2-40)
2 - The asymptotic expansion, equation (4-10)
3 - First term in the asymptotic expansion of T(α), equation (4-11)

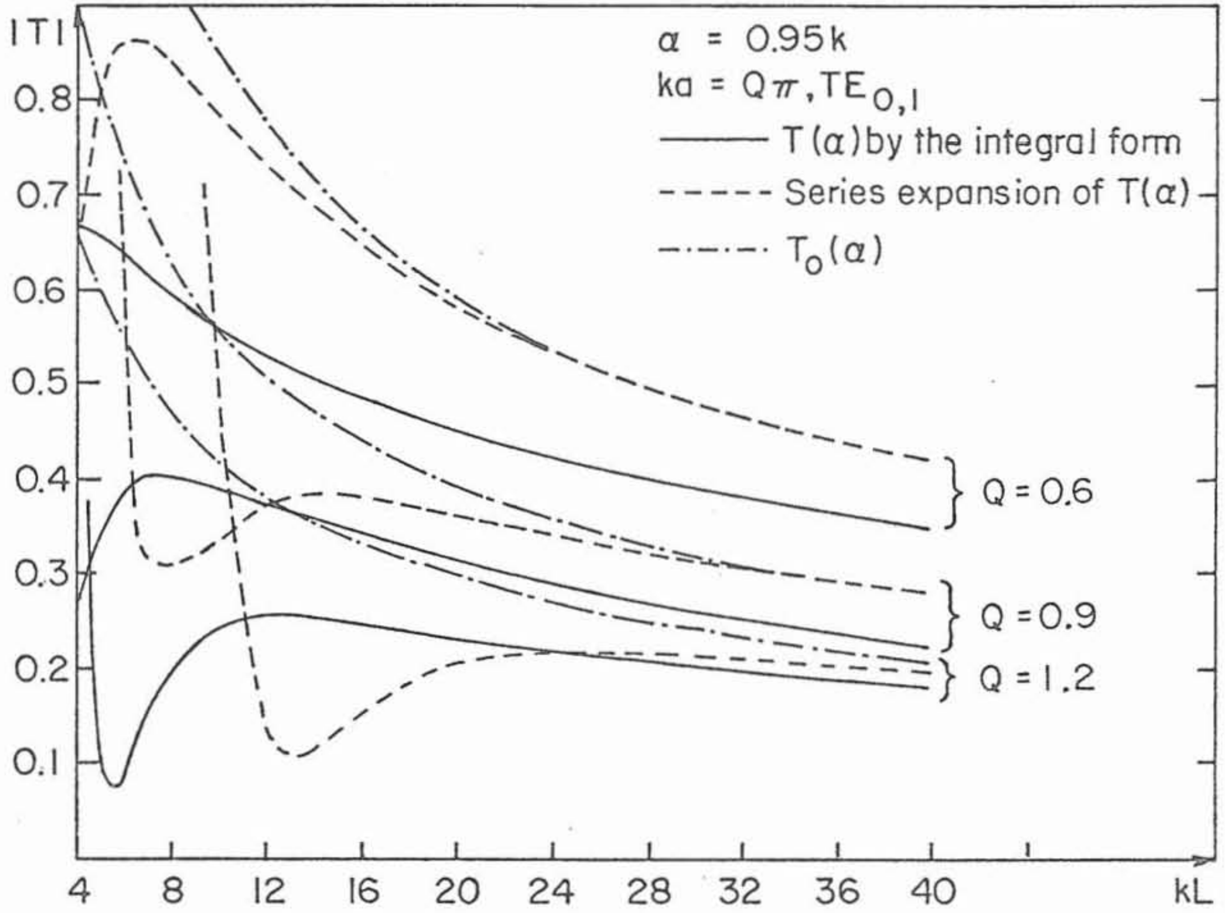

Figure 4-6 Evaluation of $T(\alpha)$ for parallel plate waveguides case using: 1- The semi-infinite integral, equation (2-40)
2- The asymptotic expansion, equation (4-10)
3- First term in the asymptotic expansion of $T(\alpha)$, equation (4-11)

for $\alpha = -k$, the results are in good agreement, even for smaller values of kL. Although $(ka)^2/kL$ must be much less than kL for good approximation for the ray theory results, α must be far from $-k$, $(-k < \alpha \leq k)$. This can be seen by noticing the effect on the radiation pattern in the forward direction, where $T(-\alpha) = T(-k \cos\theta) \simeq T(k)$. On the other hand, the results using the Wiener-Hopf technique do not blow up at $\theta = 180°$ even when the separation between the waveguides is relatively small. Similarresults and arguments can be mentioned for the case of circular waveguides. Figures 4-7 and 4-8, show the radiation pattern for $kL = 5$ and $kL = 50$, respectively, for $TE_{o,1}$ excitation of two parallel-plate waveguides. The results are obtained using the Wiener-Hopf technique with the integral form of $T(\alpha)$ and with the asymptotic form of $T(\alpha)$, and using the ray theory of diffraction. As mentioned previously, the results are in good agreement except in the forward direction, especially when kL is relatively small. Again, figures 4-7 and 4-8 show the validity of the ray theory of diffraction. For circular waveguides with $TM_{o,1}$ excitation, figures 4-9 and 4-10 show the radiation pattern using the Wiener-Hopf and the ray theory of diffraction, for the two cases of $KL = 10$ and 50, respectively. Once again, results are in good agreement except in the forward direction, especially when kL is relatively small.

Some results are also obtained for the reflection and transmission coefficients for two parallel-plate waveguides and are shown in table 4-2. It is clear from this table that the reflection and the transmission coefficients are in good agreement when kL is large.

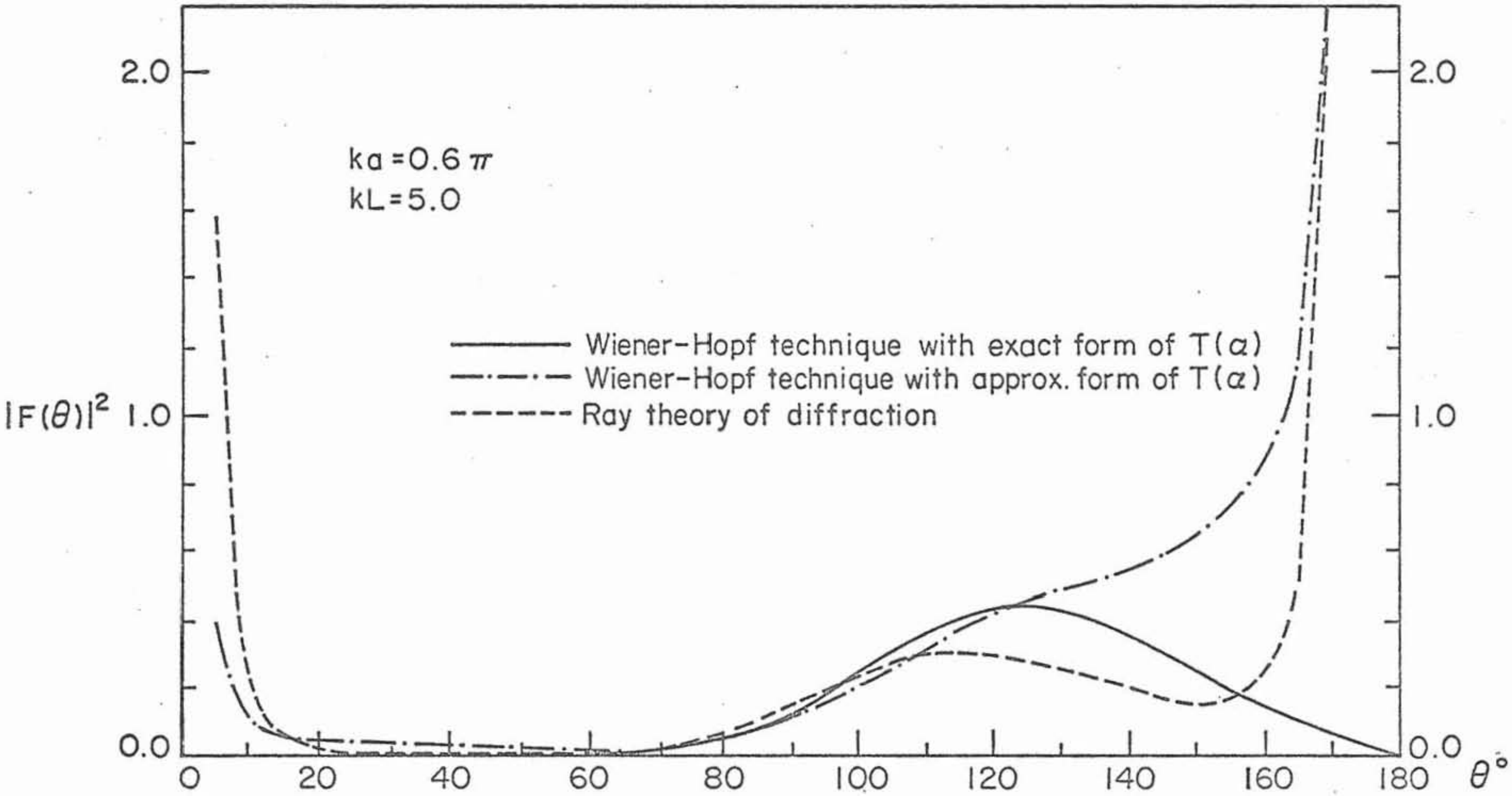

Fig. 4-7 Radiation pattern of two parallel-plate waveguides separated by L with $TE_{0,1}$ excitation

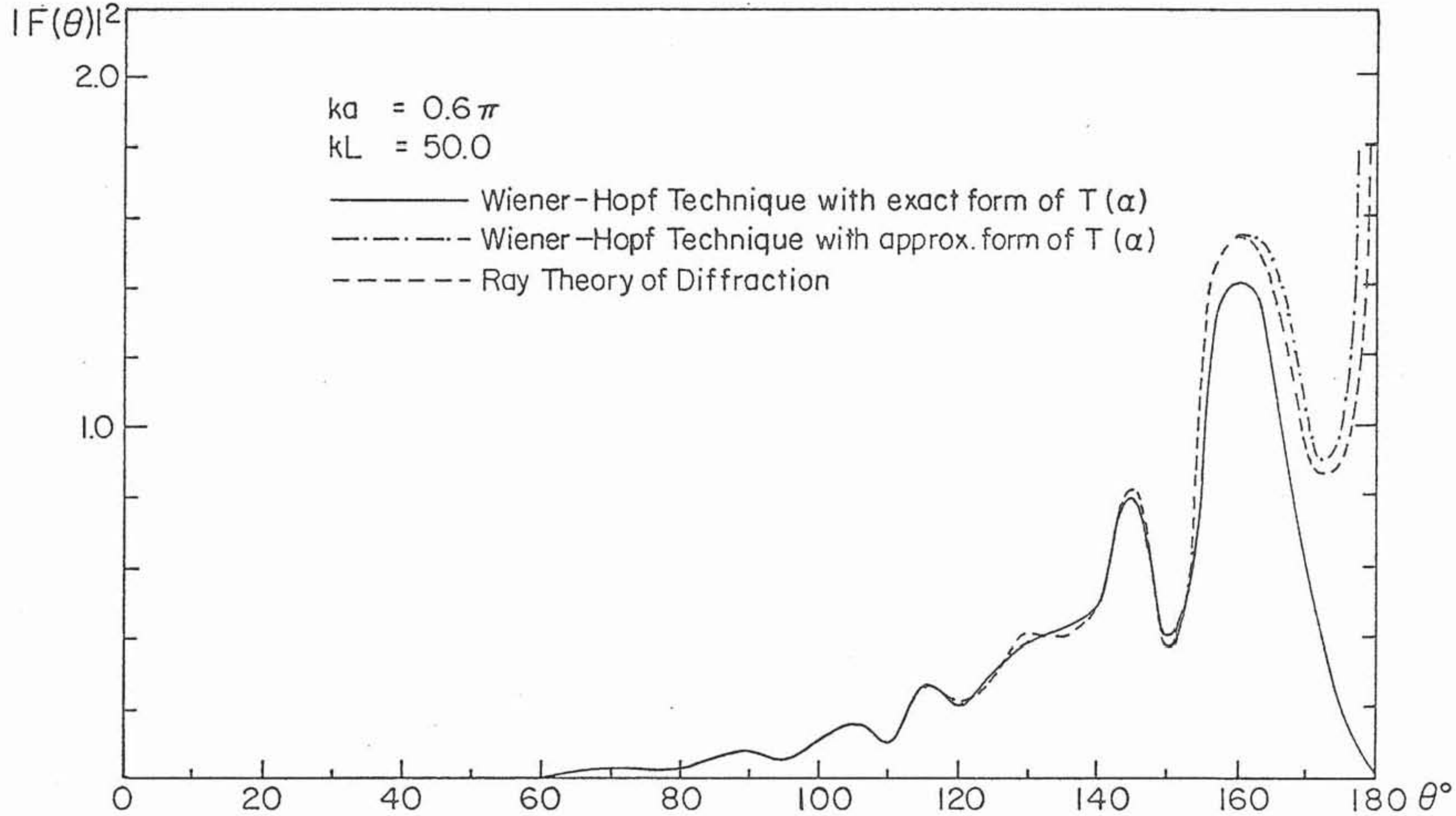

Figure 4-8 Radiation pattern of two parallel-plate waveguides separated by L with $TE_{0,1}$ excitation.

TABLE 4-2

Values of R, T using the different forms of T(α) including that corresponding to ray theory solution

kL	Reflection Coefficient using T(α) given by			Transmission Coefficient using T(α) given by		
	equation (2-40)	equation (4-10)	equation (4-11) Ray theory	equation (2-40)	equation (4-10)	equation (4-11) Ray theory
5	.215/-171	.218/-162	.083/ 44	.586/105	.486/116	.801/ 70
50	.200/-135	.203/-136	.201/-137	.195/147	.222/147	.223/143

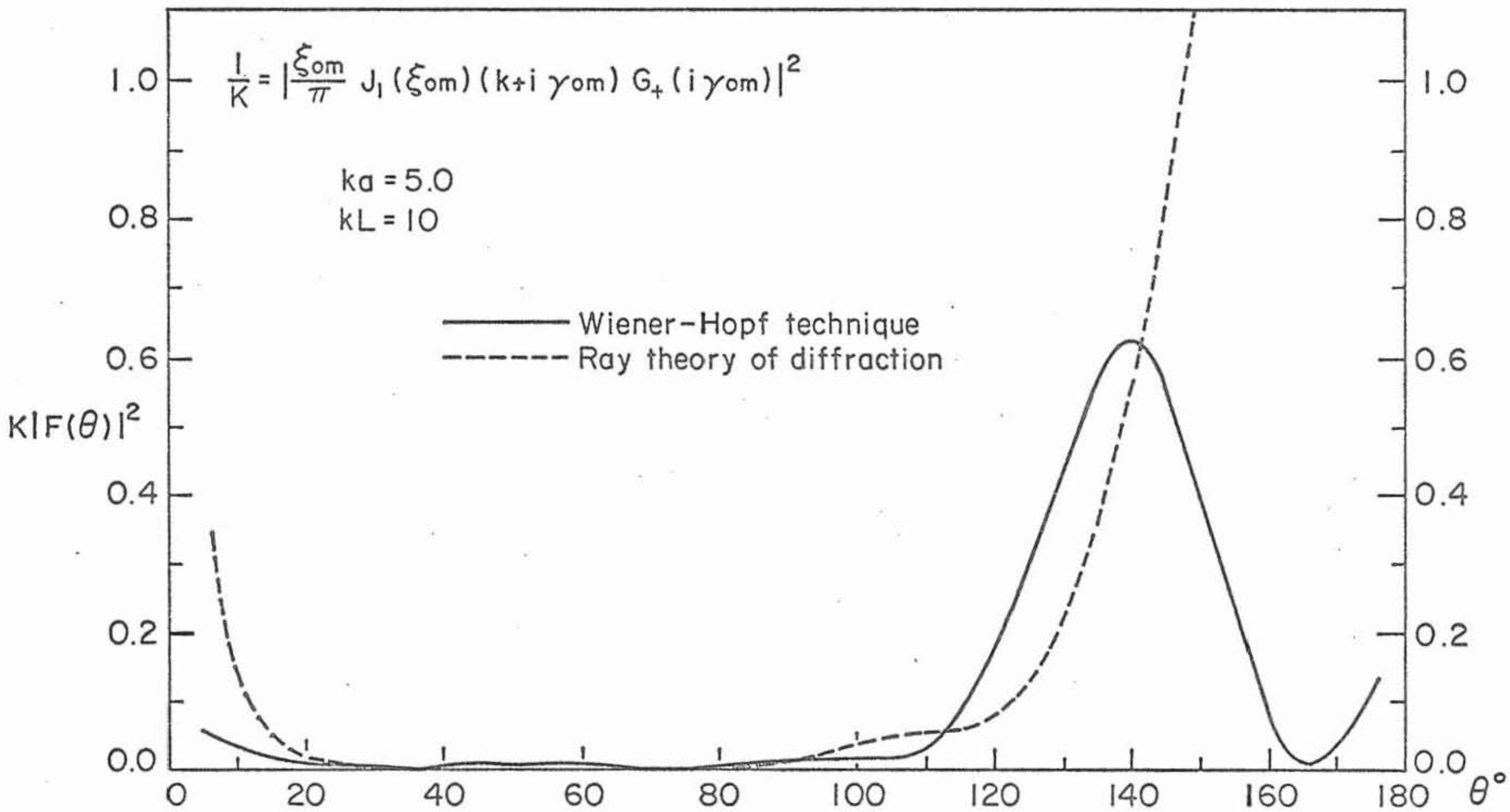

Fig. 4-9 Radiation pattern of two circular waveguides separated by L with $TM_{0,1}$ excitation

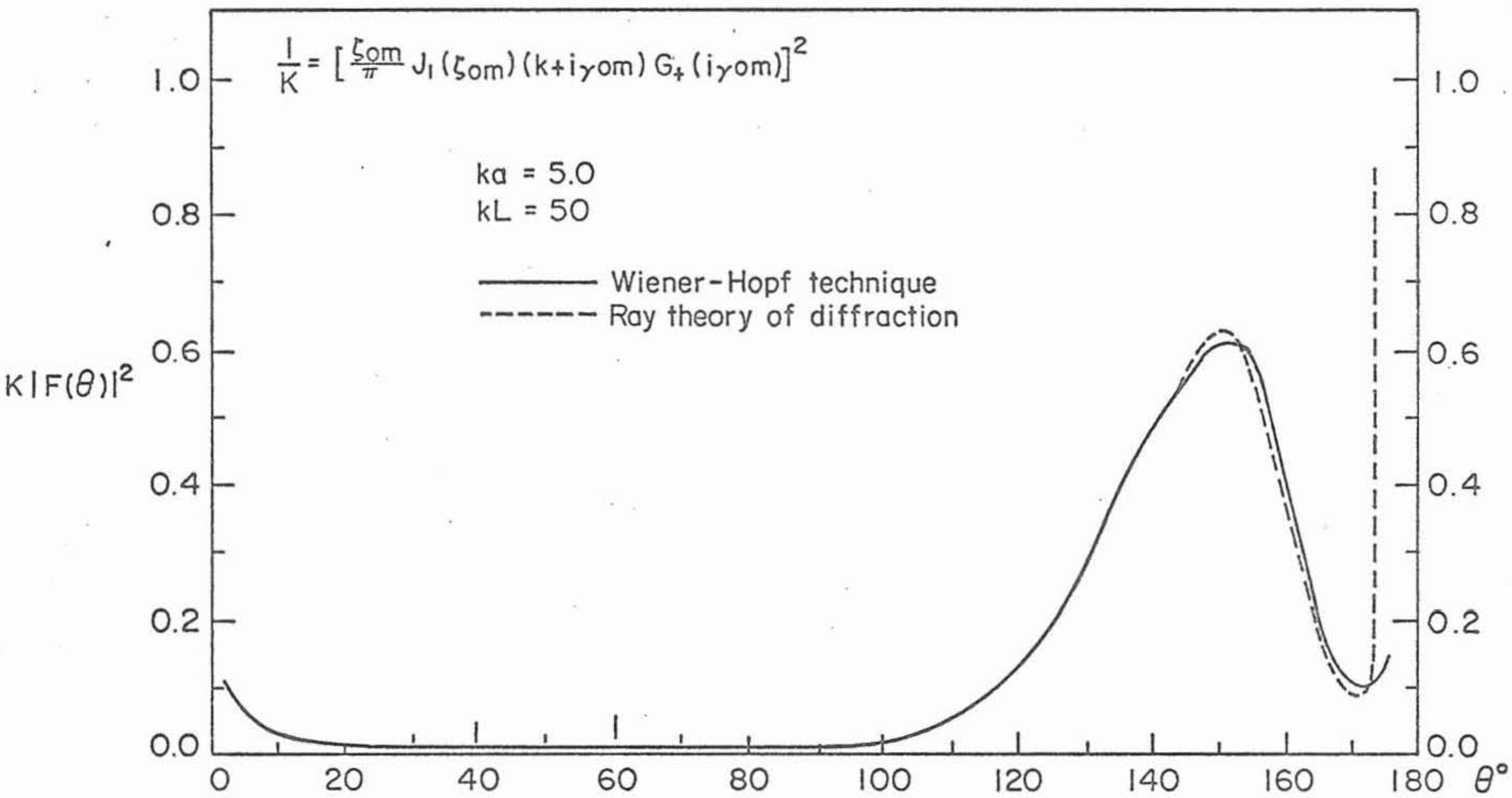

Fig. 4-10 Radiation pattern of two circular waveguides separated by L with $TM_{0,1}$ excitation

CHAPTER 5

COUPLING BETWEEN TWO COLLINEAR WAVEGUIDES OF FINITE LENGTH

5.1 Introduction

As waveguides may have finite length in practice, their corresponding problems have been investigated by many authors. Jones [35] has studied the diffraction by a parallel-plate waveguide of finite length, for an incident plane electromagnetic wave polarized parallel to the edges of the guide. While for a plane wave polarized perpendicular to the edges of the guide, the problem has been treated by Williams [35]. Williams [37] has also studied the diffraction of a plane harmonic sound wave by a hollow circular cylinder of finite length. Jones and Williams have used the Wiener-Hopf technique for solving these problems. Ufimtsev [66] has obtained the scattered field in the far zone due to the diffraction of a general plane wave by a thin, ideally conducting cylinder of finite length, using the multiple diffraction of the fringe waves of the induced currents on the conductor. Each fringe wave of the current, reaching the opposite end of the conductor is assumed to be reflected as from the end of a semi-infinite conductor, giving rise to a new wave. Fialkovskii [67] has solved Ufimtsev's problem [66] by successive approximations of the integral equations [68],[69] and has obtained results identical to those of Ufimtsev [66] based on the boundary-value technique. Kao [70] has investigated the scattering of E and H polarized plane waves incident normally, on a circular tube of any radius and length. This was achieved by transforming the problem into determining an infinite set of Fourier components in various integral forms, which were then solved numerically.

In all above cases, scattering by only a single scatterer were investigated. Recently, Hurd [71] has investigated the mutual coupling of two tubular collinear antennas of unequal lengths and with arbitrary feedpoint locations. The results however are restricted to antennas of small radii (thin dipoles).

From the practical point of view, it is interesting and valuable to utilize the results of previous chapters for obtaining the coupling between two waveguides of finite length. This is demonstrated in this chapter by two examples. The first is the coupling between two collinear parallel-plate waveguides of finite length, while the other is the coupling between two collinear circular waveguides of finite length. This chapter shows in detail how the results of a finite case can be obtained from those of an infinite one. Some graphical results are given at the end of this chapter, and, for the case of parallel-plate waveguides, they are compared with those of the ray theory of diffraction. Examination of the results for accuracy of the solution is given in Chapter 7.

5.2 Formulation of the problem

Consider two collinear perfectly conducting waveguides of finite length, separated by a distance L. Both waveguides have the same transverse dimension, but the first waveguide (exciting waveguide) having a length ℓ_1, while the second waveguide (coupled waveguide) has a length ℓ_2, as shown in figure 5-1. Before going through the analysis, we may assume the following conditions:

1. The exciting waveguide is matched at the far end and the effect of

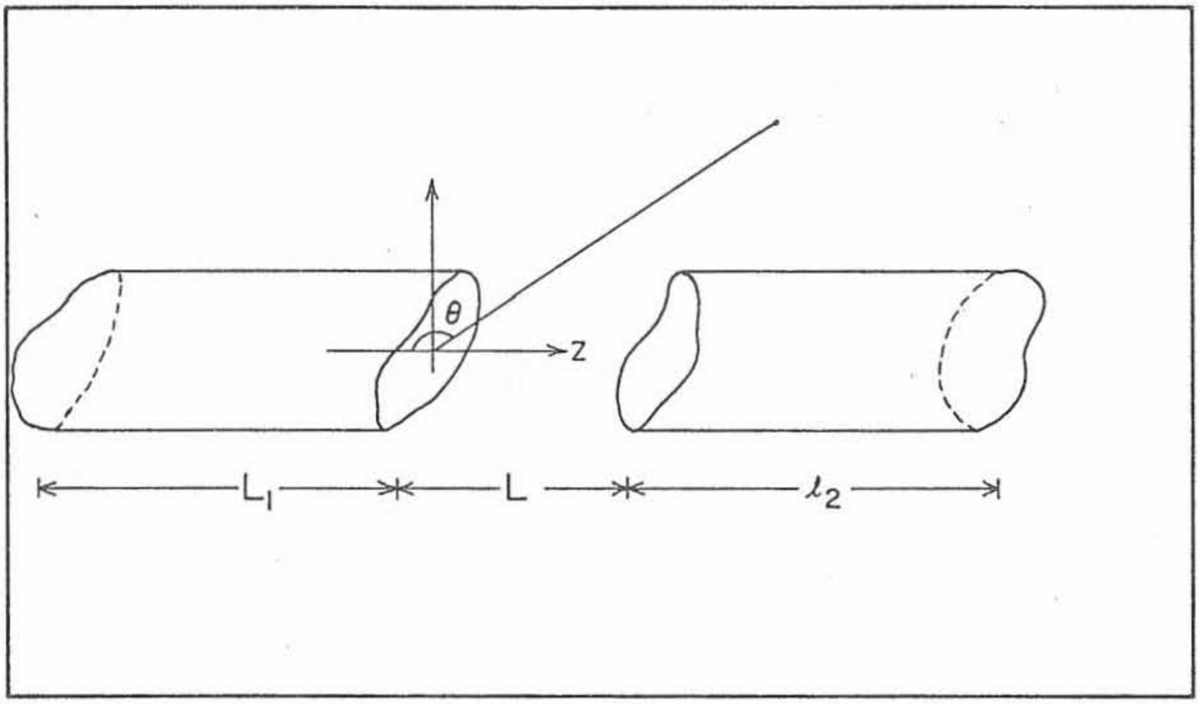

Fig. 5-1 Coupling between waveguides of finite lengths separated by a distance L.

the feed system is neglected.

2. The coupled waveguide is open ended at both terminals.

3. Coupling between the two terminals of the exciting waveguide is neglected, i.e., $K\ell_1 >> 1$.

4. Coupling between the two terminals of the coupled waveguide is neglected, i.e., $K\ell_2 >> 1$.

Due to the assumption (1), no reflections occur at the far end of the exciting waveguide. Combining this with assumption (3), the exciting waveguide acts as a semi-infinite one. Due to the assumptions (2) and (4), the fields inside the coupled waveguide are only due to the multiple reflections at both ends.

The generalized scattering-matrix technique [72] is used to solve this problem with the above assumptions taken into consideration . This technique is very closely related to the scattering matrix of circuit theory [73], or that of microwave network theory [74]. The only difference, however, is that it is extended to include evanescent as well as propagating modes in waveguides.

Let a field be excited in the exciting waveguide and be defined by the scalar quantity ϕ^i. The two dimensional problem can then be described in terms of three apertures 1, 2 and 3, and the solution may be expressed in terms of the multiple scattering phenomena at these apertures.

First, consider the incident wave arriving at the aperture 1, as represented symbolically by 1' in figure 5-2. Let the wave generated at the aperture 1 and reflected into region A (inside the exciting waveguide) be denoted symoblically by $S_1^{AA}\,\phi^i$. Also let the wave generated, due to ϕ^i and the multiple scattering between

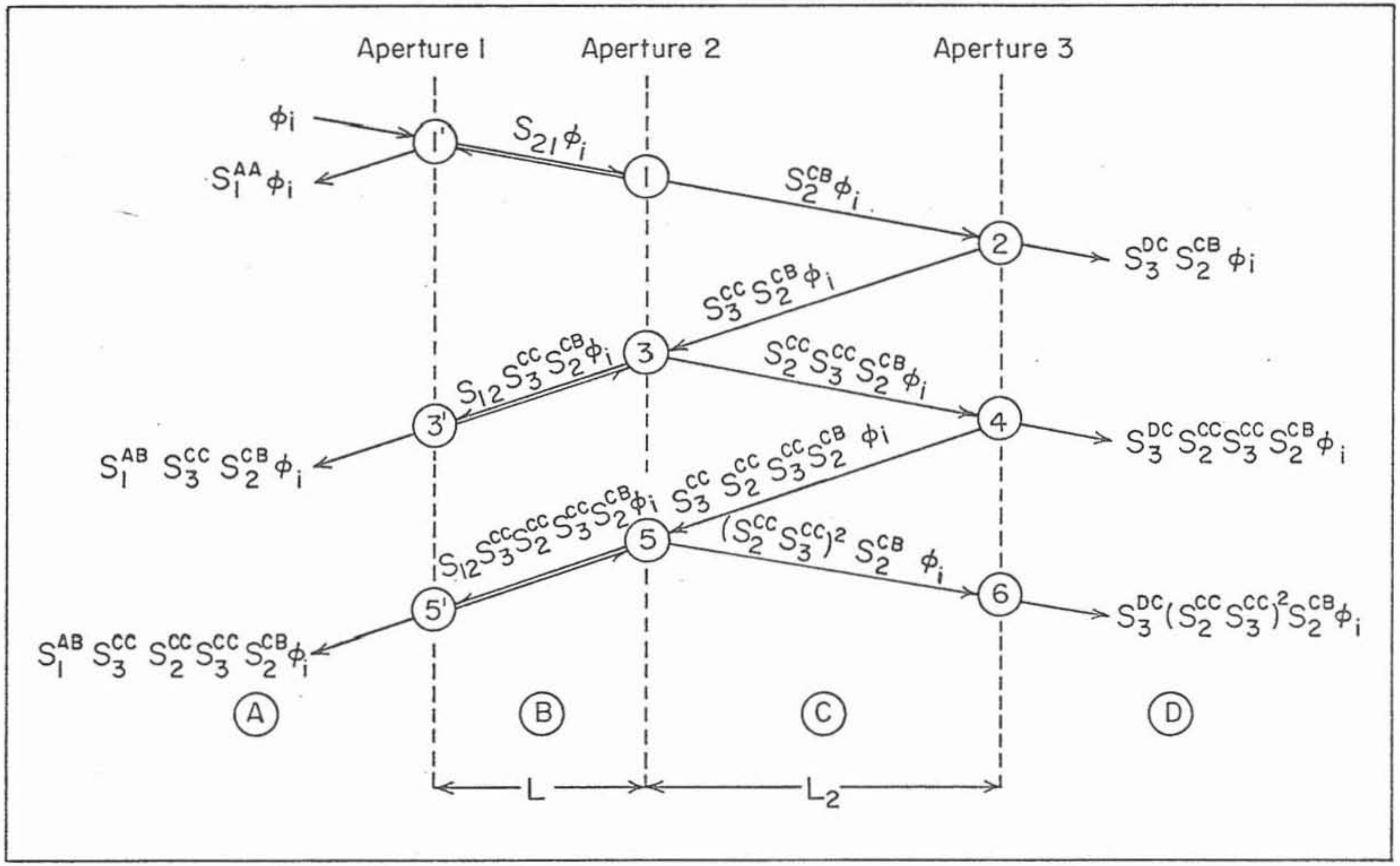

Fig. 5-2 Multiple scattering between apertures 1, 2 and 3 for the problem of figure 5-1.

apertures 1 and 2, in the region B (free space) be denoted symbolically by $S_{21}\ \phi^i$. Simultaneously, let the wave transmitted into the region C (inside the coupled waveguide) be indicated by $S_2^{CB}\ \phi^i$. The subscript 1 of S_1^{AA} is to be associated with aperture 1; the first A in the superscript indicates that the reflected wave is in region A, and the second A signifies that the incident wave is also from region A. Similar interpretation can be readily associated with S_2^{CB}. The subscript of S_{21} is to be associated with aperture 1 and 2, with the first aperture (number 2) indicates interactions due to the incident wave from the second aperture (number 1).

Now consider the scattering phenomena at aperture 3, the wave $S_2^{CB}\ \phi^i$ progresses toward this aperture and is scattered there at 2. The result is a transmitted wave $S_3^{DC}\ S_2^{CB}\ \phi^i$ and a reflected wave $S_3^{CC}\ S_2^{CB}\ \phi^i$ in regions D (free space) and C, respectively. The reflected wave travels in the negative z-direction towards aperture 2, where it is again scattered at 3. This process of multiple scattering continues for an infinite number of times. Now all contributions in region A, C, B and D due to the multiple scattering process can be added up to yield:

(i) Reflected field in the exciting waveguide (region A):

$$\phi_A(u_1,u_2,z) = S_1^{AA}\phi^i + S_1^{AB}\ S_3^{CC}\ S_2^{CB}\phi^i + S_1^{AB}\ S_3^{CC}\ S_2^{CC}\ S_3^{CC}\ S_2^{CB}\phi^i$$
$$+ S_1^{AB}\ S_3^{CC}\ S_2^{CC}\ S_3^{CC}\ S_2^{CC}\ S_3^{CC}\ S_2^{CB}\ \phi^i + \ldots \qquad (5\text{-}1)$$

The Series of equation (5-1) is known as the Neumann Series [72] and is convergent [72]. It can be written in a compact form as

$$\phi_A(u_1,u_2,z) = S_1^{AA}\phi^i + S_1^{AB}\ S_3^{CC}\ (I - S_2^{CC}\ S_3^{CC})^{-1}\ S_2^{CB}\ \phi^i \qquad (5\text{-}2)$$

where I is an identity matrix.

(ii) Field inside the coupled waveguide (region C):

$$\phi_C(u_1,u_2,z) = S_2^{CB}\phi^i + S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i + S_2^{CC}\,S_3^{CC}\,S_2^{CC}\,S_3^{CC}\phi^i$$

$$+ \ldots\ldots\ldots + S_3^{CC}\,S_2^{CB}\phi^i + S_3^{CC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i$$

$$+ S_3^{CC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i + \ldots.$$

$$= (I - S_2^{CC}\,S_3^{CC})^{-1}\,S_2^{CB}\phi^i + S_3^{CC}(I - S_2^{CC}\,S_3^{CC})^{-1}\,S_2^{CB}\phi^i \tag{5-3}$$

(iii) Field in free space (region B + D)

$$\phi_{B+D} = S_{21}\phi^i + S_{12}\,S_3^{CC}\,S_2^{CB}\phi^i + S_{12}\,S_3^{CC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i + \ldots\ldots.$$

$$+ S_3^{DC}\,S_2^{CB}\phi^i + S_3^{DC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i + S_3^{DC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CC}\,S_3^{CC}\,S_2^{CB}\phi^i$$

$$= S_{21}\phi^i + S_{12}\,S_{33}^{CC}\,(I - S_2^{CC}\,S_3^{CC})^{-1}\,S_2^{CB}\phi^i$$

$$+ S_3^{DC}\,(I - S_2^{CC}\,S_3^{CC})^{-1}\,S_2^{CB}\phi^i \tag{5-4}$$

the scattering matrices S_1^{AA}, S_1^{AB}, S_{12}, S_{21}, S_2^{CB}, S_2^{CC}, S_3^{CC} and S_3^{DC} are determined in the same manner as those of Lee and Mittra [75] who solved the problem of diffraction by a thick conducting half-plane and a dielectric loaded waveguide. For most practical cases, the evanescent modes are not taken into account, and consequently all previous matrices are unit matrices. Hence equations (5-2), (5-3) and (5-4) reduce respectively to (see Appendix E)

Reflected field inside the exciting waveguide $= f(u_1,u_2)e^{\gamma_m z}$

$$\left[R + \frac{T^2\,R_o\,e^{-2\gamma_m\,(L + \ell_2)}}{1 - R\,R_o\,e^{-2\gamma_m\,\ell_2}}\right] \tag{5-5}$$

$$\text{Field inside the coupled waveguide} = f(u_1,u_2) \frac{T}{1 - R\,R_o\,e^{-2\gamma_m \ell_2}}$$

$$[e^{-\gamma_m z} + R_o\,e^{-2\gamma_m(L + \ell_2)}\,e^{\gamma_m z}] \tag{5-6}$$

$$\text{Radiation field:}\quad \text{Rad}(\theta) = P(\theta) + \frac{T\,e^{-\gamma_m(L + \ell_2)}\,e^{ikL\cos\theta}}{1 - R\,R_o\,e^{-2\gamma_m \ell_2}}$$

$$[P_o(\theta)\,e^{ik\ell_2\cos\theta} + P(\pi - \theta)\,R_o\,e^{-\gamma_m \ell_2}] \tag{5-7}$$

where: $f(u_1,u_2)$ = field distribution over the cross-section of a two dimensional waveguide with transverse coordinates u_1 and u_2.

γ_m = propagation constant inside the waveguide.

R_o = reflection coefficient due the open-end of a semi-infinite waveguide.

R = reflection coefficient of two semi-infinite waveguides separated by a distance L.

T = transmission coefficient of two semi-infinite waveguides separated by a distance L.

$P_o(\theta)$ = radiation field of a single exciting semi-infinite waveguide, with its open-end at $z = o$.

$P(\theta)$ = radiation field of two semi-infinite waveguides separated by a distance L, with the open-end of the exciting waveguide at $z = o$.

For the case of parallel-plate waveguides with $TE_{o,\ell}$ excitation, $P_o(\theta)$ or $P(\theta)$ is the electric field component E_y, given in Chapter 2 for the case of ℓ odd. While for the case of circular waveguides with $TM_{o,m}$ excitation, $P_o(\theta)$ and $P(\theta)$ are the magnetic field component H_ϕ, given in Chapter 3. The radiated power as a function of θ for the whole system is then given by $|Rad(\theta)|^2$, for which some results are shown in the next section for both previous cases.

In the case of two parallel-plate waveguides with $TE_{o,\ell}$ excitation (ℓ odd), $P_o(0) = P(0) = P(\pi) = 0$, giving $Rad(0) = 0$ and

$$Rad(\pi) = \frac{T\, e^{-\gamma_m(L + \ell_2)}}{1 - R\, R_o\, e^{-2\gamma_m \ell_2}}\, e^{-ik(L + \ell_2)}\, P_o(\pi) \tag{5-8}$$

which means that, $Rad(\pi)$ is the radiated field from an open-end of a semi-infinite waveguide, $P_o(\pi)$, multiplied by the constant factor dependent on ℓ_2 and L. When the separation distance between two col-linear semi-infinite waveguides tends to zero, the reflection and trans-mission coefficients, R and T, go to zero and unity, respectively, and the radiated field $P(\theta)$ becomes very small. Hence (5-7) reduces to

$$\lim_{L\to 0} Rad(\theta) = P_o(\theta).\, e^{-\gamma_m(L + \ell_2)} \,.\, e^{ik(L + \ell_2)\cos\theta} \tag{5-9}$$

which represents the radiation from an open end (at $z = L + \ell_2$) of a semi-infinite waveguide excited by a wave $\phi^i = f(u_1,u_2)e^{-\gamma_m z}$.

5.3 Results and discussion

Some results are obtained for two cases of parallel-plate and circular waveguides. Figure 5-3 shows the radiation of two parallel-plate waveguides of finite length with $ka = 0.6\pi$ and $TE_{o,1}$ excitation, and for different lengths of the coupled waveguide. It is clear that the main change in the magnitude of the radiated field is in the forward direction with small variations in the backward direction. For this case, figure 5-4 shows the radiation pattern for different values of kL, with $k\ell_2$ fixed. This figure shows the behaviour of the radiated

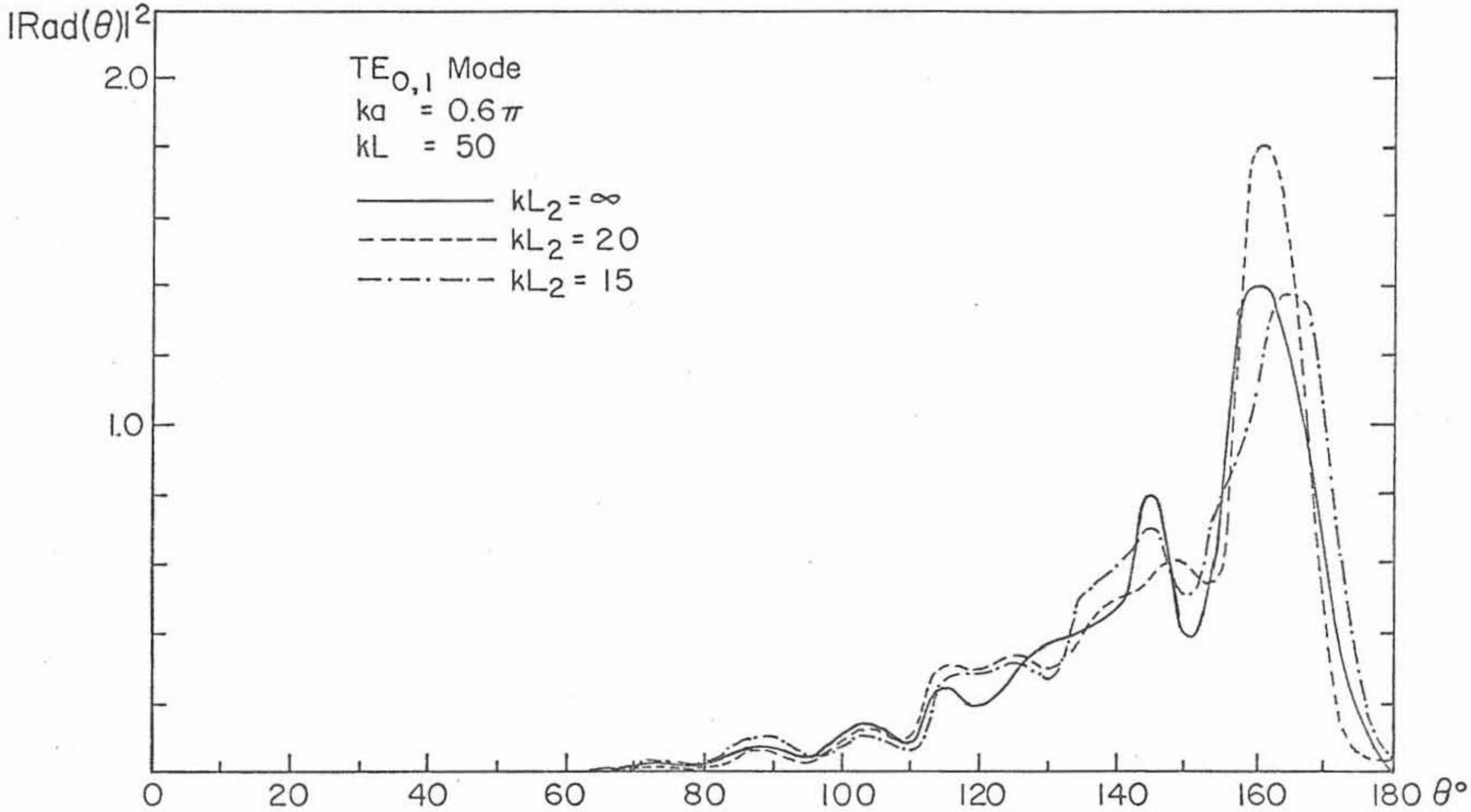

Figure 5-3 Radiation pattern of two parallel-plate waveguides of finite length using Wiener-Hopf Results.

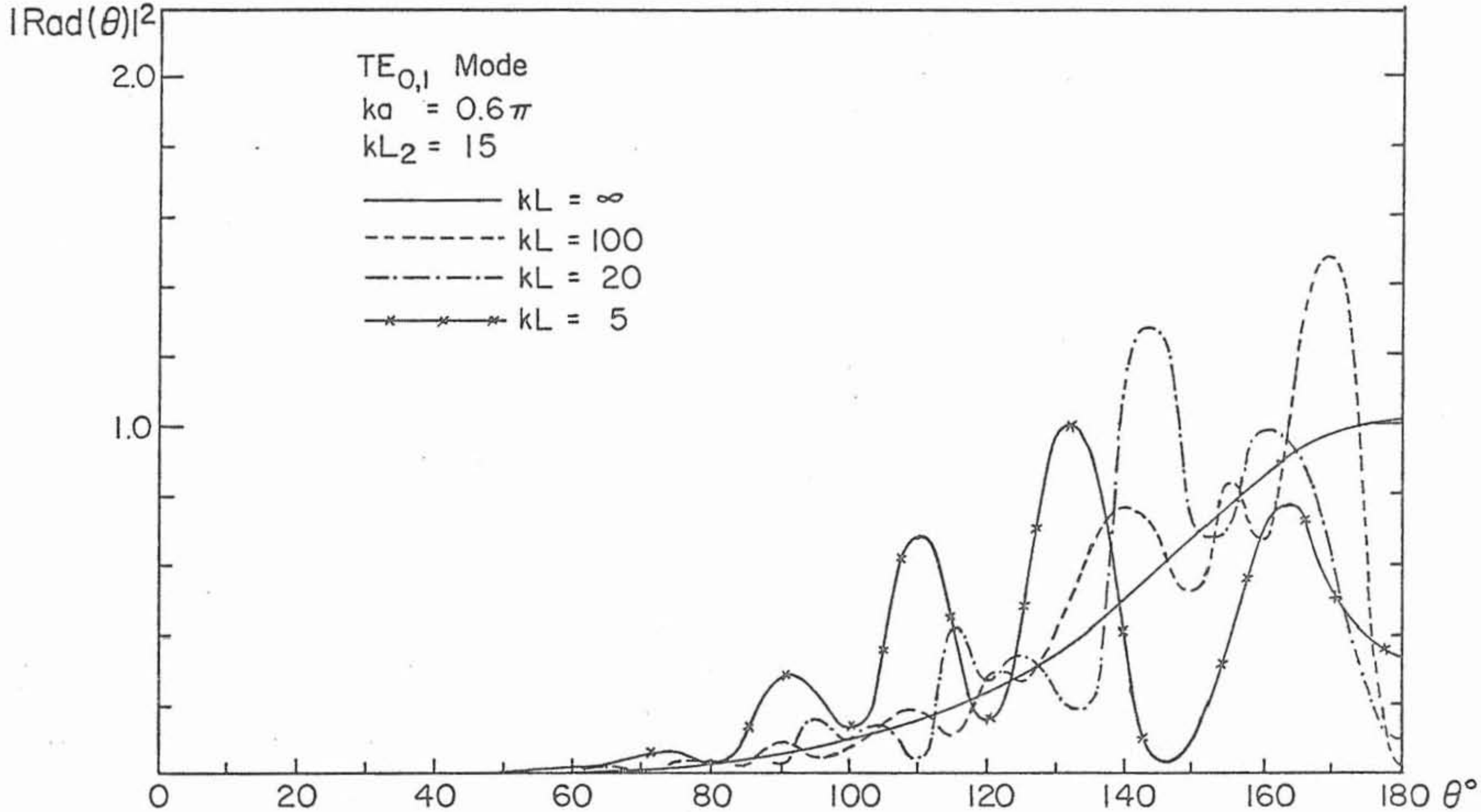

Figure 5-4 Radiation pattern of two colliner parallel-plate waveguides of finite length using the Wiener – Hopf technique

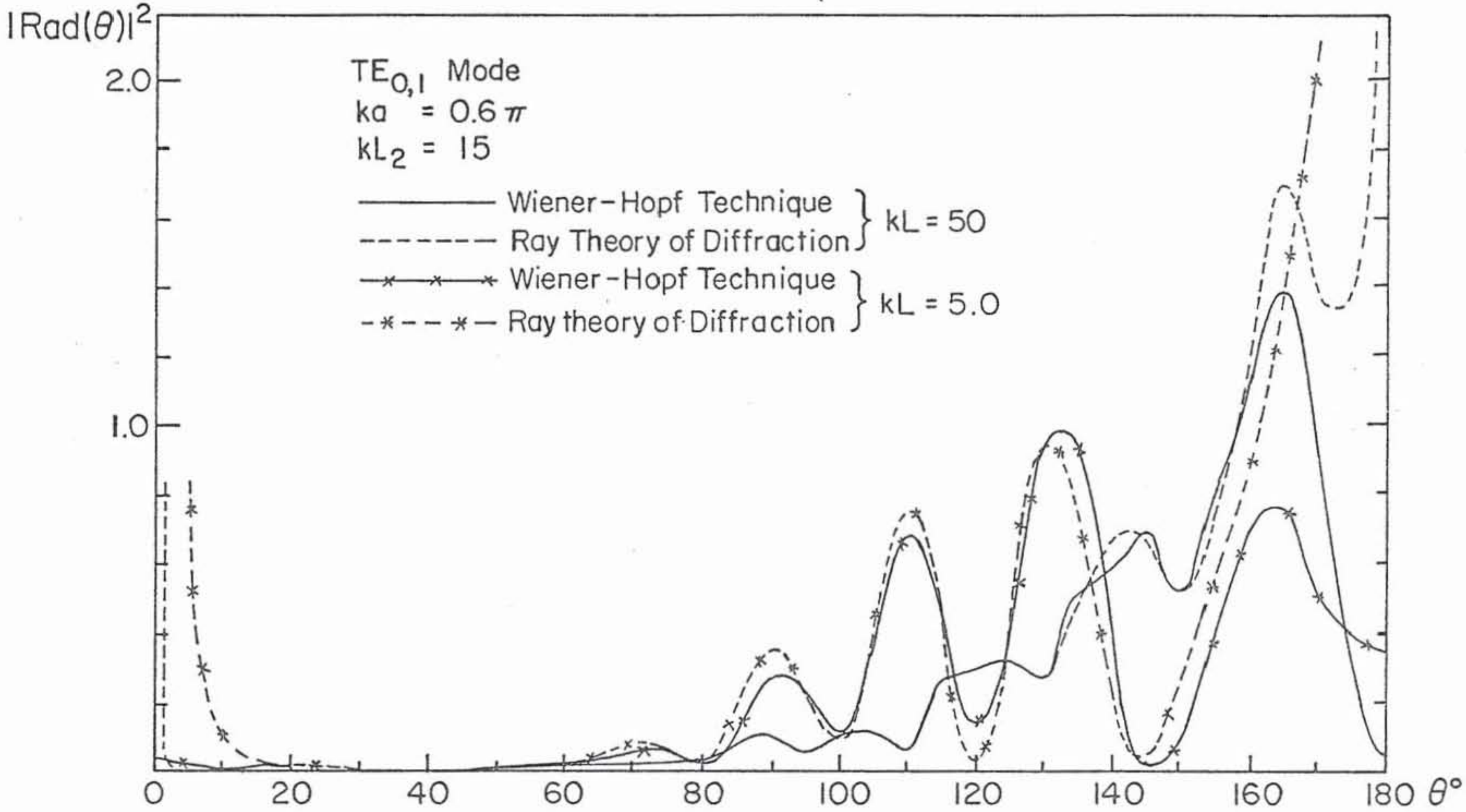

Figure 5-5 Radiation pattern of two collinear parallel-plate waveguides of finite length.

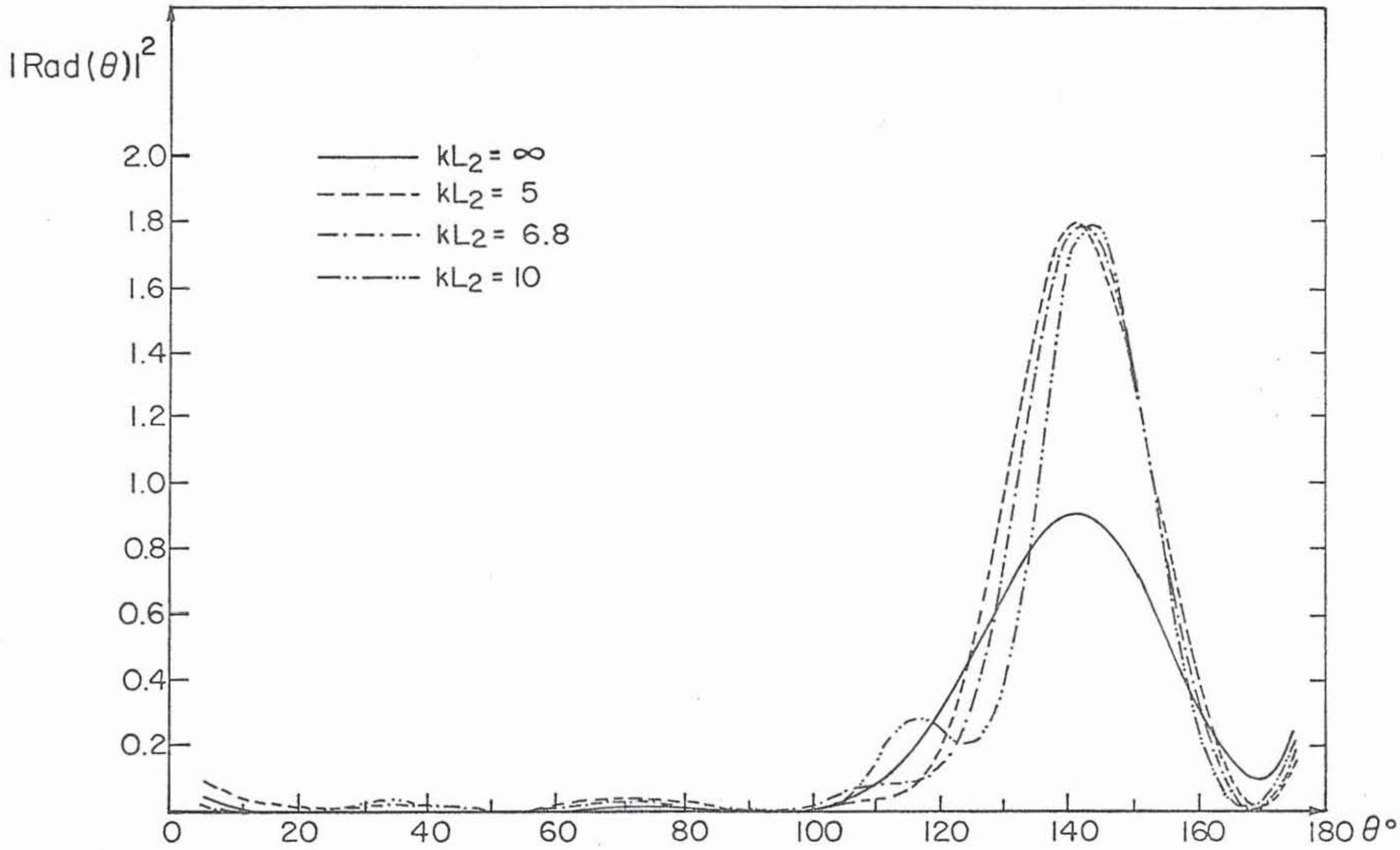

Figure 5-6 Radiation pattern for a two coupled circular waveguides of radius $ka = 5$ with a $TM_{0,1}$ excitation and separation distance $kL = 6$.

power with KL in the forward direction of $\theta = \pi$.

To show the accuracy of the ray theory of diffraction some results are obtained for the radiated power in the case of parallel-plate waveguides. This is shown in figure 5-5, where results are compared with those of the Wiener-Hopf technique. It is clear that the results of the Wiener-Hopf and ray theory deviate especially in the forward and backward directions, especially near the walls of the two waveguides.

Similar results are also obtained for circular waveguides with $TM_{0,1}$ mode excitation and are shown in figure 5-6. Due to the truncation of the coupled waveguide the main lobe is strongly affected.

Further examination of the results for the case of parallel-plate waveguides will be given in Chapter 7 using numerical methods.

CHAPTER 6

HU'S TRANSMISSION FORMULA AND THE WIENER-HOPF TECHNIQUE

6.1 Introduction

The Gain of a standard horn can be determined by measuring the transmission loss versus separation between two identical standard horns. Friis' transmission formula [76] is only valid when the separation distance between the two identical horns is large enough compared to the wavelength. Therefore the Gain formula

$$G = \frac{4\pi r}{\lambda} \left(\frac{P_r}{P_t}\right)^{\frac{1}{2}} \tag{6-1}$$

may introduce considerable error when the far-zone gain of electromagnetic horns is measured at relatively short distances [77].

In 1958, M.K. Hu [33] introduced a general power transmission formula for a matched lossless two antennas system, using the Lorentz reciprocity theorem in combination with Maxwell's equations. Hu's transmission formula may be used as a near zone power transmission formula when written as

$$\frac{P_r}{P_t} = \frac{1}{4} \frac{\left| \int_s (\bar{H}_2 \times \bar{E}_1 + \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds \right|^2}{\{R_e \int_{s_1} (\bar{E}_1 \times \bar{H}_1^*) \cdot \hat{n}_1\, ds\}\{R_e \int_{s_2} (\bar{E}_2 \times \bar{H}_2^*) \cdot \hat{n}_2\, ds\}} \tag{6-2}$$

where $\frac{P_r}{P_t}$ is the ratio of the received to the transmitted powers between two antennas at any separation distance . $\bar{E}_1$ and $\bar{H}_1$ are the fields when antenna 1 is transmitting, $\bar{E}_2$, $\bar{H}_2$ are the fields when antenna 2 is transmitting, and $\hat{n}$, $\hat{n}$ and $\hat{n}_2$ are the unit normals to the surfaces. The surface s may be either one of the two antenna apertures. Hu's transmission formula given by (6-2) is an exact formula if all the field quantities are evaluated with both antennas in place and under matched conditions. Neglecting the reflections between the two antenna systems and mismatch due to their feeds; and assuming that the tangential fields E_t and H_t are related to each other by the free space impedance at each point, equation (6-2) may be reduced to the more suitable form [33] used by Chu and Semplak [77] to calculate the ratio between the Fraunhofer and Fresnel gain of a pyramidal electromagnetic horn as a function of horn dimensions and separation distance. Jull [78]-[81] has investigated the gain of parallel-plate waveguides and errors in the predicted gain of sectorial and pyramidal horns, while Hamid [12] has studied the near field coupling between horn antennas.

In the following section, the author wishes to compare his results with those obtained using Hu's transmission formula. Simple formulas are obtained using Kirchhoff's approximation, modified Kirchhoff's approximation, ray theory of diffraction and the Wiener-Hopf technique. In the comparison, it should be noted that in equation (6-2), P_r represents power received in the aperture of the receiving antenna. When the receiving antenna is a semi-infinite waveguide, P_r must be multiplied by the ratio of the waveguide and the free space wave impedances in order to get the power far from the aperture. One should note that $\frac{P_r}{P_t}$ using the Wiener-Hopf technique equals $|T|^2$, where T is the transmission coefficient. Hence $|T|^2$ will be the value that will be compared with $\frac{P_r}{P_t}$ obtained using Hu's transmission formula. Some results and discussion are given in another section.

6.2 Formulation of the problem using Hu's transmission formula

Rewriting equation (6-2) in the form

$$\frac{P_r}{P_t} = \frac{1}{4} \frac{\left| \int_{s_2} (\overline{H}_2 \times \overline{E}_1 + \overline{E}_2 \times \overline{H}_1) \cdot \hat{n}_2 \, ds \right|^2}{\{R_e \int_{s_1} (\overline{E}_1 \times \overline{H}_1^*) \cdot \hat{n}_1 \, ds\}\{R_e \int_{s_2} (\overline{E}_2 \times \overline{H}_2^*) \cdot \hat{n}_2 \, ds\}} \tag{6-3}$$

The surface s in (6-2) is chosen to be the aperture of the receiving antenna (antenna 2). Neglecting the reflections between the antennas (i.e. in evaluating $\overline{E}_1$ and $\overline{H}_1$, antenna 2 will be removed and in evaluating $\overline{E}_2$ and $\overline{H}_2$, antenna 1 will be removed) and considering only linearly polarized uniform phase plane aperture antennas [33], equation (6-3) for parallel plate waveguides case, can be written, after some manipulations, in the form:

$$\frac{P_r}{P_t} = \frac{\left| \int_{s_2} \overline{E}_1 \cdot \overline{E}_2 \, ds \right|^2}{\{\int_{s_1} |\overline{E}_1|^2 \, ds\}\{\int_{s_2} |\overline{E}_2|^2 \, ds\}} \tag{6-4}$$

Here it should be noted that $\overline{E}_1$ in the numerator is the electric field due to antenna 1 in the aperture of antenna 2 in the absence of antenna 2, while $\overline{E}_1$ in the denominator is the electric field of antenna 1 in its own aperture. Far from antenna 2, $\overline{E}_1$ in the numerator can be assumed fairly constant for the case of $TE_{o,1}$ mode, and (6-4) may hence be written in the form:

$$\frac{P_r}{P_t} = |\overline{E}_1|^2 \frac{\left| \int_{s_2} \overline{E}_2 \, ds \right|^2}{\{\int_{s_1} |\overline{E}_1|^2 \, ds\}\{\int_{s_2} |\overline{E}_2|^2 \, ds\}} \tag{6-5}$$

In the denominator, the contribution of the first integral is the same as the contribution of the second integral and hence (6-5) reduces to

$$\frac{P_r}{P_t} = |\overline{E}_1|^2 \frac{\left| \int_{s_2} \overline{E}_2 \, ds \right|^2}{[\int_{s_2} |\overline{E}_2|^2 \, ds]^2} \tag{6-6}$$

which for $TE_{0,1}$ mode excitation in the parallel-plate waveguides and aperture fields of unit amplitudes reduces to

$$\frac{P_r}{P_t} = \frac{16}{\pi^2} |\bar{E}_1|^2 \tag{6-7}$$

Again $\bar{E}_1$ is the electric field due to antenna 1 in the aperture of antenna 2 (due to a unit amplitude in the aperture of antenna 1), in the absence of antenna 2 and is constant over aperture of antenna 2 and equal its value at the centre of this aperture. The aperture of the coupled waveguide is acting like a transformer and hence P_r may be multiplied by the waveguide impedance. $1/\sqrt{1 - (\pi/kd)^2}$, in order to get the power far from the aperture, i.e.

$$\frac{P_r}{P_t} = \frac{16}{\pi^2} \quad \frac{1.0}{\sqrt{1 - (\pi/kd)^2}} \quad |\bar{E}_1|^2 \tag{6-8}$$

The radiated field of the transmitting antenna (the exciting waveguide) is evaluated using Kirchhoff's approximation, modified Kirchhoff's approximation, the ray theory of diffraction and the Wiener-Hopf technique (exact solution). Table 6-1 shows the evaluation of (6-8) for different formulas of $|\bar{E}_1|$ [79]. Second column states $|\bar{E}_1|$ for the different methods of formulations listed in the first column, while the third column represents $\frac{16}{\pi^2} |\bar{E}_1|^2$.

In comparison, the Wiener-Hopf technique applied to two collinear semi-infinite parallel plate waveguides gives $\frac{P_r}{P_t} = |T|^2$, where T represents the transmission coefficient in the coupled waveguide far from the open end as derived in Chapter 2.

6.3 Results and discussion

Some results have been obtained for the solutions using Hu's transmission formula and are compared with the exact solution using the Wiener-Hopf technique. They are shown in figures 6-1 and 6-2. Figure 6-1 shows $\frac{P_r}{P_t}$ for different separation distances $K L$ and for a waveguide width

TABLE 6-1

Evaluation of (6-8) for the different formulas of $|\bar{E}_1|$

Formulas Using	$\lvert\bar{E}_1\rvert = \lvert E_y\rvert$	$(P_r/P_t)\ \frac{\beta}{k}^{*}$
Kirchhoff's approximation	$\frac{2kd}{\pi(2\pi kL)^{\frac{1}{2}}}$	$\frac{32}{\pi^5}\ \frac{(kd)^2}{kL}$
Modified Kirchhoff's approximation	$\frac{kd(1+\beta/k)}{\pi(2\pi kL)^{\frac{1}{2}}}$ when β is replaced by k,	$\frac{8}{\pi^5}\ \frac{(kd)^2}{kL}\ (1+\beta/k)^2$ they reduce to those corresponding to Kirchhoff's approximation
Ray theory of diffraction	$\frac{kd}{\pi(\pi kL)^{\frac{1}{2}}}\ (1+\beta/k)^{\frac{1}{2}}$	$\frac{16}{\pi^5}\ \frac{(kd)^2}{kL}\ (1+\beta/k)$
Wiener-Hopf technique	$\frac{kd}{\pi}\left[\frac{\beta/k}{(1+k/\beta)kL}\right]^{\frac{1}{2}}\ e^{\frac{kd}{4}(1-\beta/k)}$	$\frac{16}{\pi^4}\ \frac{(kd)^2}{kL}\ \frac{\beta/k}{(1+k/\beta)}\ e^{\frac{kd}{2}(1-\beta/k)}$

* $\beta = k\sqrt{1-(\frac{\pi}{kd})^2}$

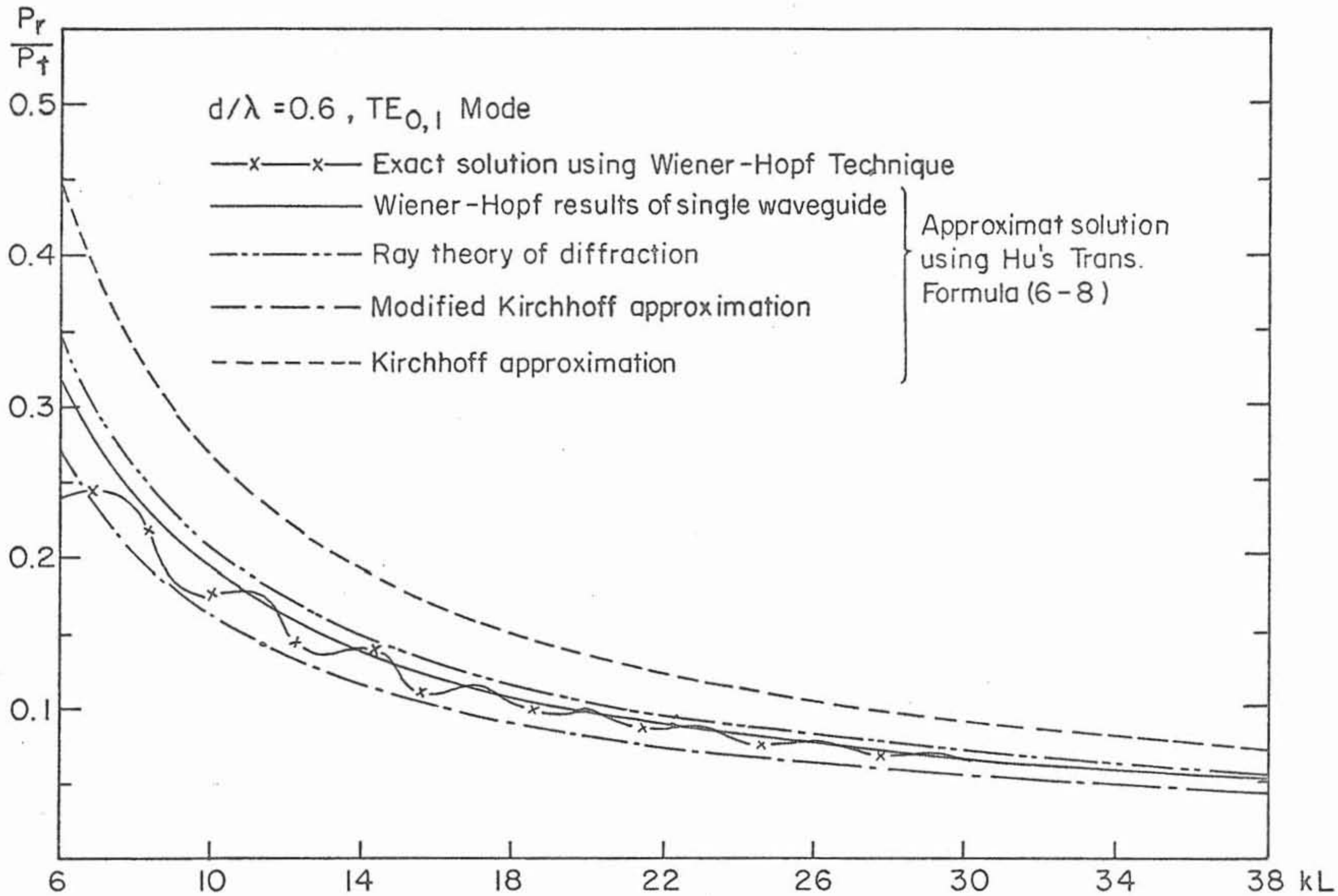

Figure 6-1 Comparison of results for two antenna systems composed of two collinear semi-infinite parallel-plate waveguides separated by a distance kL.

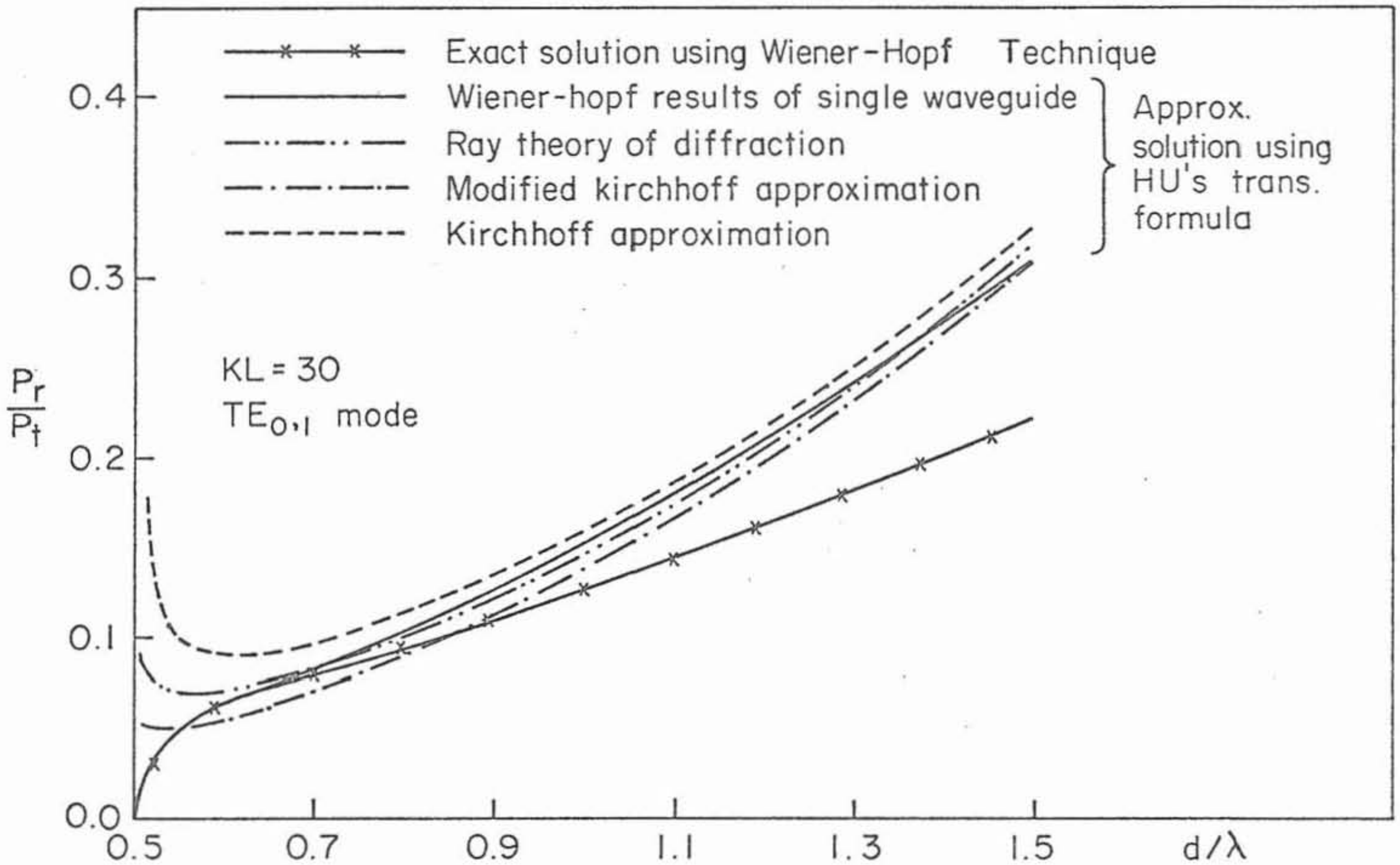

Figure 6-2 Comparison of results for two antennae systems composed of two collinear semi-infinate parallel plate waveguides of width d and separation distance KL=30

$d/\lambda = 0.6$. It is clear that, except for $KL < 10$, the exact solution oscillates around that obtained by Hu's formula using Wiener-Hopf results for a single waveguide. These oscillations are with a period π. This phenomenon is due to the fact that in equation (6-8), $\bar{E}_1$ is evaluated far from the open end of the exciting waveguide. This is similar to the condition on Friis' transmission formula [82], for $L \geq \frac{2d^2}{\lambda}$. Also, it is clear that $|\bar{E}_1|$ approximated by ray theory of diffraction or by the modified Kirchhoff's method gives better results. Figure 6-2 shows the results for a separation distance $KL = 30$ and for different waveguide widths. It is clear that the exact solution coincides with that of Hu's formula, using the Wiener-Hopf results of single waveguide, when $0.5 < d/\lambda < 0.65$, and deviates for larger values of d/λ, for the same reason mentioned previously. In other words, for reasonably large values of L, Hu's formula gives very good results, since they approach zero when d/λ goes to .5, while the results of the other approaches blow up at this value of d/λ. Hu's formula given by (6-2) cannot be solved exactly. Neglecting multiple reflections, (6-2) reduces to (6-4) which is used in this chapter to get the previous results. Using expression (6-4) to get initial diffraction, multiple reflections may be used to improve the results.

CHAPTER 7

NUMERICAL TECHNIQUES FOR COUPLING BETWEEN WAVEGUIDES

7.1 Introduction

Among the methods used for investigating scattering and radiation problems is the numerical technique. This technique is widely used for problems that do not have exact analytical or even approximate solutions. Numerical solutions for scattering by perfectly conducting rectangular cylinders, both for parallel and perpendicularly polarized incident waves, have been obtained by Mei and Van Bladel [83]. Similarly Andreasen [84] has investigated the scattering by parallel metallic cylinders with arbitrary cross sections. For metallic cylinders, the problem is reduced to the numerical evaluation of integral equations for the surface currents. For dielectric cylinders, Richmond [85],[86] has obtained solutions by numerical evaluation of the integral equations for the polarization currents in the dielectric material. The numerical evaluation of the integral equations is usually carried out by using a moment or a point matching method [29] to convert the integral equations to a set of simultaneous linear equations. The number of matching points depends on the length of the contour of the cross section for metallic cylinders and on the size of the cross sectional area for dielectric cylinders. For problems involving discontinuities in the contour of the cross section, a higher number of matching points is needed. Abdelmessih and Sinclair [87] have used Meixner's edge condition for treatment of the singularities of the surface current at the discontinuities. An alternative method for treatment of the singularities is the method of coordinate transformation which has been used successfully by Shafai [88].

In all cases, the rate of convergence of the solution depends on the size of the scatterer and may be improved by utilizing the symmetry arguments, increasing the efficiency of numerical evaluation of the matrix elements or by optimizing the basic set used for the expansion. A technique for improving the convergence of the moment method has been studied by Tew and Tsai [89] through the use of *a priori* knowledge of the solution. Their idea was that a known good approximation, such as the physical-optics current, is subtracted from the unknown total current, with the result that the residual difference current, which is now the quantity to be determined, will converge more rapidly.

So far the moment method has been applied in open space to solve scattering problems of obstacles of finite size. The extension of the method to diffraction by arbitrary cross sectional semi-infinite conductors, that has recently been investigated by Morita [30],[31], has paved the way for use of the moment method to many new problems. Wu and Chow [32] have utilized Morita's investigation and extended the direct moment method to the closed space inside a waveguide which has infinitely long walls along the propagation direction, taking an advantage from the localized nature of the evanescent waves to assure the convergence of the solution. A similar approach was also used by Burnside et al [90], Thiele and Newhouse [91] and Chow and Seth [92].

In this chapter the author formulates the problem of two parallel plate waveguides using the direct moment method and the modified moment method [89]. The direct moment method is used when the separation distance between the two waveguides is large ($KL >> 1$), while the modified moment method is used when the separation distance is small. In the direct moment method, as the separation distance between the two waveguides is large, coupled waveguide can be assumed to be illuminated by the field radiated from the open end of the exciting waveguide. This field has an exact value and is given by many authors and is included in the previous chapters. This illuminating field is scattered by the coupled waveguide which may have any length and any width, i.e. the coupled waveguide may support other modes and not necessarily the dominant modes in the exciting waveguide. The method of coordinate transformation, which has been used by Shafai [88], is used here and the walls of the coupled waveguide is conformally mapped on to two circular cylinders and hence overcoming the singularities in the transformed domain. The surface current is evaluated from which the scattered field is determined. Consequently, radiation patterns due to coupling between two waveguides is obtained by adding the radiated field from the open end of the exciting waveguide to the scattered field from the coupled waveguide. This will be shown in detail in the next section.

In the modified moment method, where the separation distance between the two waveguides is small, the propagating wave in the exciting waveguide is considered to be a plane wave bouncing off the walls, in the case of parallel plate waveguides. When this bouncing wave meets the discontinuity, a scattered field results. This field is then decomposed into reflected, transmitted and evanscent waves. The reflected and transmitted waves are represented by plane waves in the exciting and coupled waveguides, respectively, having reflection and transmission coefficients R and T, respectively. The evanescent waves are represented by extra induced current densities on the conducting walls of two waveguides. As the evanescent waves decay exponentially from the discontinuities, the basic functions are required to cover only a finite space, i.e. reducing the infinitely long walls to finite ones. Once the reflection and transmission coefficients and the evanescent currents are determined, the radiation field can be readily obtained. This will be shown in detail in this chapter.

In the following sections, the case of parallel plate waveguide is treated with $TE_{o,1}$ mode incident in the exciting waveguide. Numerical results are given for both methods of formulations and a comparison between Wiener-Hopf results and those of numerical methods are discussed.

7.2 Direct moment method (DMM)

The geometry of the problem is again shown in figure 7-1, where KL is the length of the coupled waveguide. As indicated in the introduction when the separation distance between the two waveguides is large, the waveguide can be assumed to be illuminated by the radiated field from the open end of the exciting waveguide in the absence of the coupled waveguide. For $TE_{o,1}$ incident wave in the exciting waveguide, the radiation field is given by:

$$E_y = \frac{i\pi}{2a\sqrt{2\pi K\rho}}\, e^{i(K\rho - Ka|\sin\theta| - \frac{\pi}{4})}\, G_+(i\gamma_\ell)\, \frac{K|\sin\theta|\cdot G_+(K\cos\theta)}{K\cos\theta + i\gamma_\ell} \qquad (7\text{-}1)$$

As the coupled waveguide is illuminated normally by the above electromagnetic wave which is polarized parallel to the edges, a solution for the total electromagnetic field may be obtained by an application of Green's identity and the result is [29].

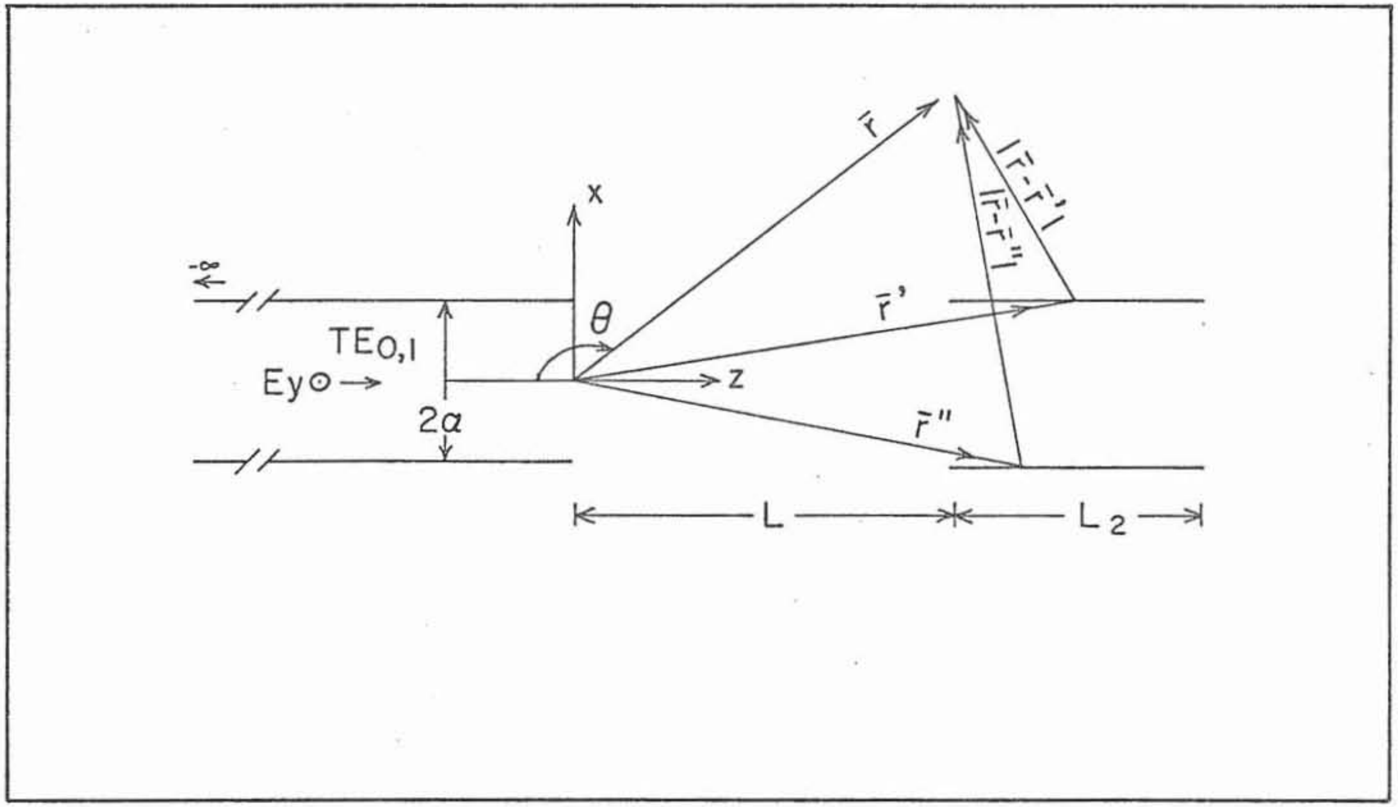

Fig. 7-1 Coupling between two waveguides using DMM.

$$E_y^{total} = E_y^{inc.} - \frac{\eta}{4} \int_W H_o^{(1)}(K|\bar{r}-\bar{r}'|) I_y(\bar{r}')\, d(K\bar{r}') \tag{7-2}$$

where $E_y^{inc.}$ is given by (7-1) , $\eta = (\mu_o/\varepsilon_o)^{1/2} = 120\pi$ is the intrinsic impedance of free space, and I_y is the induced current on the walls of the coupled waveguide. The integral path W is along two walls of the coupled waveguide and r and r' are the coordinates of the field points and of the source points on the walls.

On the walls of the coupled waveguide, the boundary condition $E_y^{total} = 0$ reduces (7-2) to

$$E_y^{inc.}(\bar{r}_o) = \frac{\eta}{4} \int_W H_o^{(1)}(K|\bar{r}_o-\bar{r}'|) I_y(\bar{r}')\, d(K\bar{r}') \tag{7-3}$$

This is an integral equation for the current distribution I_y and is solved by the direct moment method, in which the integral equation is converted to a set of simultaneous linear equations. In the most common method, the path of integration w is divided into N segments Δw and a step or a linear approximation to I_y is used [29]. As mentioned in the introduction, sharp edges create singularities of the induced currents and for accurate results, a transformation is needed to map the cross sectional contour of the scatterers onto circles on which induced currents are finite. After conformal mapping of the region outside the walls of the coupled waveguide to the region outside two circles in the transformed domain, (7-3) reduces to

$$E_y^{inc.}(\bar{r}_o) = \frac{\eta}{4} \int_0^{2\pi} H_o^{(1)}(K|\bar{r}_o-\bar{r}'|)\, J_{y_1}(\theta')\, d\theta' + \frac{\eta}{4} \int_0^{2\pi} H_o^{(1)}(K|\bar{r}_o-\bar{r}''|) J_{y_2}(\theta')\, d\theta' \tag{7-4}$$

where $\bar{r}'$ and $\bar{r}''$ are functions of θ' and are given by equation (F-5a) and (F-5b) in Appendix F. J_{y_1} and J_{y_2} are two unknown induced currents on the walls of coupled waveguide in the transform domain. A detailed analysis of the transformation and the regularity of the current

J_y is given in Appendix F. For the solution of J_y, a series of trigonometric functions with unknown coefficients are assumed [88] and are found by an application of the point matching technique. The reason for choosing such a solution was pointed out by Shafai [88] which is to provide an approximate lower limit of the number of terms in the series, for the desired degree of accuracy and by comparing the behavior of J_y with the current distribution on a circular cylinder. For a general illumination, J_y may be assumed in the form

$$J_y(\theta') = \sum_{n=-\infty}^{\infty} a_n e^{in\theta'} \tag{7-5}$$

From equation (7-1) it is clear that the illumination is symmetrical and the above form can be reduced to

$$J_y(\theta') = \sum_{n=o}^{\infty} a_n \cos n\theta' \tag{7-6}$$

For $TE_{o,1}$ mode the induced current on the walls of the waveguide is the same. Hence, if the coupled waveguide has the same width as the exciting waveguide, then the induced currents on the walls of the coupled waveguide are equal. Therefore equation (7-4) reduces to

$$E_y^{inc.}(\overline{r}_o) = \frac{\eta}{4} \sum_{n=o}^{\infty} a_n \int_0^{2\pi} [H_o^{(1)}(K|\overline{r}_o-\overline{r}'|)+H_o^{(1)}(K|\overline{r}_o-\overline{r}''|)]\cos n\theta' d\theta' \tag{7-7}$$

On truncating the series to N terms, one obtains an equation in N unknowns. Dividing the circumference of any one of the circles into N segments and using the point matching, (7-7) can be written in the following matrix form [29]:

$$[\ell_{m,n}][f_m] = [g_m] \tag{7-8}$$

where

$$f_m = \frac{\eta}{4} a_m \quad , \quad m = 0,1,2,\ldots N-1 \tag{7-9}$$

$$\ell_{m,n} = \int_0^{2\pi} [H_o^{(1)}(K|\overline{r}_m-\overline{r}'|)+H_o^{(1)}(K|\overline{r}_m-\overline{r}''|)]\cos n\theta' d\theta'$$

$$, \quad m= o,1,2,.. N-1 \; , \; n=o,1,2,.. N-1 \qquad (7\text{-}10)$$

and

$$g_m = E_y^{inc}(\overline{r_m})$$

$$= \frac{i\pi}{2a\sqrt{2\pi Kr_m}} e^{i(Kr_m-Ka|\sin\theta_m|-\frac{\pi}{4})} G_+(i\gamma_\ell) \frac{K|\sin\theta_m|\cdot G_+(K\cos\theta_m)}{K\cos\theta_m + i\gamma_\ell} \qquad (7\text{-}11)$$

where $\theta_m = \tan^{-1}\dfrac{x_m}{y_m}$

when matching points are on the upper wall, the element $\ell_{m,n}$ may have singular points at $\overline{r}_m = \overline{r}'$. For matching points on the lower wall, these singularities will be at $\overline{r}_m = \overline{r}''$. These singularities can be treated analytically, but in numerical integration they can be avoided by choosing proper integration points. In (7-10), the integrand is an even function. Hence $\ell_{m,n}$ can be reduced to

$$\ell_{m,n} = 2 \int_0^{\pi} [H_o^{(1)}(K|\overline{r}_m-\overline{r}'|)+H_o^{(1)}(K|\overline{r}_m-\overline{r}''|)]\cos n\theta' d\theta'$$

$$, \quad m=o,1,2,...N-1 \; , \; n=o,1,2,...N-1 \qquad (7\text{-}12)$$

Equation (7-8) is the matrix form of an N simultaneous linear equations in N unknowns and can be solved numerically by known method. Once the coefficients are known, induced currents can be determined and consequently the scattered field from coupled waveguide can be obtained as

$$E_y^{sc.}(\overline{r}) = -\frac{\eta}{2}\int_0^{\pi} [H_o^{(1)}(K|\overline{r}-\overline{r}'|)+H_o^{(1)}(K|\overline{r}-\overline{r}''|)]J_y(\theta')d\theta' \qquad (7\text{-}13)$$

At a large distance from the scatterer, (7-13) can be written in the form

$$E_y^{sc.}(\overline{r}) = -\frac{\eta}{2}\sqrt{\frac{2}{\pi K\rho}}\, e^{i(K\rho-\frac{\pi}{4})}\int_0^{\pi} J_y(\theta')[e^{iK\rho'\cos(\theta+\phi')}+e^{iK\rho''\cos(\theta+\phi'')}]d\theta' \qquad (7\text{-}14)$$

where

$$\left.\begin{aligned} \rho' &= [x'(\theta') + z'(\theta')]^{1/2} \\ \rho'' &= [x''(\theta') + z''(\theta')]^{1/2} = \rho', \\ \phi' &= \tan^{-1}[x'(\theta')/z'(\theta')] \quad ; \\ \text{and} \quad \phi'' &= \tan^{-1}[x''(\theta')/z''(\theta)] = -\phi' \end{aligned}\right\} \tag{7-15}$$

Hence, the total radiation field may be expressed as

$$E_y^{total} = E_y^{inc.} + E_y^{sc.}$$

$$= \sqrt{\frac{2}{\pi K\rho}}\, e^{i(K\rho - \frac{\pi}{4})} F(\theta) \tag{7-16}$$

where

$$F(\theta) = \frac{i\pi}{4a} G_+(i\gamma_\ell) \frac{K|\sin\theta| e^{-iKa|\sin\theta|} G_+(K\cos\theta)}{K\cos\theta + i\gamma_\ell}$$

$$- \frac{\eta}{2} \int_0^{\pi} J_y(\theta')[e^{iK\rho'\cos(\theta+\phi')} + e^{iK\rho'\cos(\theta-\phi')}]d\theta' \tag{7-17}$$

7.3 Modified Moment Method (MMM)

The direct moment method has been used in the previous section to find the radiation field when the separation distance between two waveguides was large. When KL_2 becomes very large, the required computer time for achieving a reasonable convergence of the series of the induced currents on the scatterer becomes larger. Consequently, an investigation of the scattering by a semi-infinite waveguide becomes formidable. On the other hand, the separation distance in MMM is arbitrary, which enables us to obtain the reflection and transmission coefficients and the radiation pattern for any arbitrary value of KL_2. The procedure of the solution is as follows [32]: The propagating wave in the exciting waveguide is considered to be a plane wave bouncing off the walls. When this wave meets the discontinuity, scattering occurs. The scattered field is then decomposed into reflected, transmitted and the evanescent waves. The reflected and transmitted waves are represented by plane waves with a reflection and transmission coefficients R and T, respectively, whereas the evanescent waves are represented by an extra induced current density on the conducting walls of the two waveguides. As the evanescent waves decay exponentially from the discontinuity, the base functions are required to cover only a finite space, i.e. reducing the infinitely long walls to finite ones.

For the case of two semi-infinite waveguides, let a $TE_{o,1}$ mode propagate in the exciting waveguide. The field is considered to be a plane wave bouncing off the upper and the lower walls, with an angle $\theta_o = \sin^{-1}\frac{\lambda}{2d}$ as shown in figure 7-2. With the time factor $e^{i\omega t}$ being suppressed, the incident plane wave can be written in the form:

$$E_z^i = \hat{z}\,\frac{1}{2}\,e^{i(Kx\cos\theta_o + Ky\sin\theta_o)} \tag{7-18}$$

The induced current density on the lower or the upper walls are equal for $TE_{o,1}$ mode and are given by

$$J_z^i = 2\hat{n}xH^i\Big|_{y=o,\text{or } y=d} = \hat{z}\,\frac{\sin\theta_o}{\eta}\,e^{iKx\cos\theta_o} \tag{7-19}$$

where $\hat{n}$ is the unit normal to the wall inside the waveguide, and η is the intrinsic impedance of the medium inside the waveguide ($\eta = 120\pi$, if the medium is free space).

Due to the discontinuity at $x=0$, part of the incident field is reflected back into the exciting waveguide and another part is diffracted at the edges. These diffracted waves are diffracted again at the opening

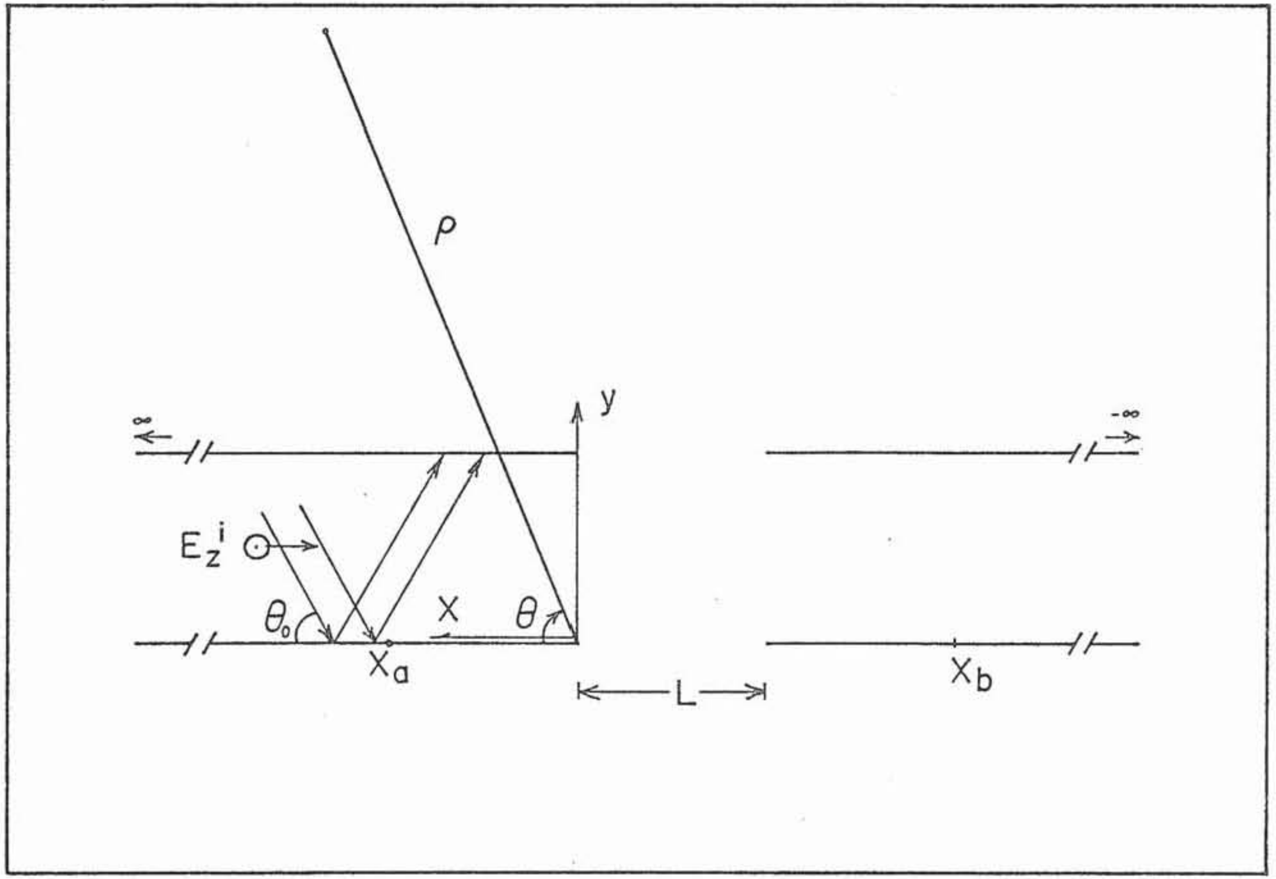

Fig. 7-2 Coupling between two waveguides using MMM

of the coupled waveguide, which gives rise to transmitting fields. The current densities due to the reflected and transmitted waves can be written in the form:

$$J_z^r = \hat{z}\ R\ \frac{\sin\theta_o}{\eta}\ e^{-iKx\cos\theta_o} \qquad \text{for } x > 0\ ,\ y = 0 \text{ and } y = d \qquad (7\text{-}20)$$

and

$$J_z^t = \hat{z}\ T\ \frac{\sin\theta_o}{\eta}\ e^{iKx\cos\theta_o} \qquad \text{for } x < -L\ ,\ y = 0 \text{ and } y = d \qquad (7\text{-}21)$$

An evanescent current $J^e = \hat{z}\ J_z^e$ exists on the walls near the discontinuities, i.e. between $x=0$ and $x=x_a$ and between $x=-L$ and $x=x_b$, where x_a and x_b are the values of x after which J_z^e may be assumed zero. From the boundary condition $E_z=0$ on the walls, the following integral equation is satisfied [29]:

$$o = \frac{\eta}{4}\int_w [J^e(\bar{r}')+J^i(\bar{r}')+J^r(\bar{r}')+J^t(\bar{r}')]H_o^{(2)}(K|\bar{r}-\bar{r}'|)d(K\bar{r}') \qquad (7\text{-}22)$$

where $\bar{r}$ and $\bar{r}'$ are the coordinates of the field and source points on the walls. The integral path w is along all relevant waveguide walls, i.e., on the walls of the exciting waveguide for J^i, J^r and J^e and on the walls of the coupled waveguide for J^t and J^e . Equation (7-22) cannot be solved exactly and approximate results may be obtained numerically by using a point matching technique with base functions in the form of unit pulses. Dividing the integral path w (for the integral over J^e) into N segments with a current J_n^e at the centre of each segment, equation (7-22) will contain $N+2$ unknowns, J_1^e , J_2^e , ..., J_N^e , R and T. The N segments are between $x=0$ and $x=x_a$ and between $x=-L$ and $x=x_b$. Another two test points corresponding to R and T should be chosen on the walls of the waveguide far from two open ends of the two waveguides. In this manner equation (7-22) combined with (7-19), (7-20) and (7-21) may be written in the following matrix form:

$$[\ell_{m,n}][f_m] = [g_m] \tag{7-23}$$

where

$$f_m = \begin{cases} \frac{\eta}{4} J^e_m & , \quad \text{for } m = 1,2,\ldots, N \\ R & , \quad \text{for } m = N+1 \\ T & , \quad \text{for } m = N+2 \end{cases} \tag{7-24}$$

$$g_m = -\frac{\sin\theta_o}{4} [I_+(0,\infty,0) + I_+(0,\infty,Kd)] \tag{7-25}$$

And for the linear operator ℓ, the elements $\ell_{m,n}$ will be given by:

(i) For $m = 1,2,\ldots N+2$, and $n = 1,2,\ldots N$

$$\ell_{m,n} = \begin{cases} [H_o^{(2)}(|Kx_m - Kx_n'|) + H_o^{(2)}(((Kx_m - Kx_n')^2 + Kd^2)^{1/2})]\Delta(Kx_n') & , \text{ for } m \neq n \\ [1 - i\frac{2}{\pi}(\ln\frac{\Delta(Kx_n')}{4} + \gamma - 1) + H_o^{(2)}(Kd)]\Delta(Kx_n') & , \text{ for } m = n \end{cases} \tag{7-26}$$

where $\gamma = 0.577215665 \ldots$

(ii) For $m = 1,2,\ldots N+2$ and $n = N+1$ which correspond to R

$$\ell_{m,N+1} = \frac{\sin\theta_o}{4} [I_-(0,\infty,0) + I_-(0,\infty,Kd)] \tag{7-27}$$

(iii) For $m = 1,2,\ldots, N+2$ and $n = N+2$ which correspond to T

$$\ell_{m,N+2} = \frac{\sin\theta_o}{4} [I_+(-\infty,-L,0) + I_+(-\infty,-L,Kd)] \tag{7-28}$$

In equations (7-25), (7-27) and (7-28), I is the integral defined by:

$$I_\pm(A,B,y) = \int_A^B e^{\pm iKx'\cos\theta_o} H_o^{(2)}[((Kx' - Kx_m)^2 + y^2)^{1/2}]d(Kx') \tag{7-29}$$

The semi-infinite integrals shown above should be converted into a more convenient form. For the source elements g_m can be written as

$$g_m = -\frac{\sin\theta_o}{4} [\int_0^\infty e^{iKx'\cos\theta_o} H_o^{(2)}(|Kx' - Kx_m|)dKx' + \int_0^\infty e^{iKx'\cos\theta_o} H_o^{(2)}\{[(Kx' - Kx_m)^2 + Kd^2]^{1/2}\}dKx' \tag{7-30}$$

changing the variable of integration and making use of the following relations [32]:

$$\int_0^\infty e^{iKx\cos\theta_o} H_o^{(2)}(Kx)\,dKx = \frac{2(\pi-\theta_o)}{\pi\sin\theta_o} \quad , \tag{7-31}$$

$$\int_0^\infty e^{iKx\cos\theta_o} H_o^{(2)}[(Kx^2+Kd^2)^{1/2}]\,dKx = \frac{2}{\sin\theta_o} e^{-iKd\sin\theta_o} - \int_0^\infty e^{-iKx\cos\theta_o} H_o^{(2)}[(Kx^2+Kd^2)^{1/2}]\,dKx \tag{7-32}$$

then (7-30) reduces to:

$$g_m = \frac{\theta_o}{2\pi} e^{iKx_m\cos\theta_o} - \frac{\sin\theta_o}{4} e^{iKx_m\cos\theta_o} \Big[\int_0^{Kx_m} e^{-iKx\cos\theta_o} H_o^{(2)}(|Kx|)\,dKx + \int_0^{Kx_m} e^{-iKx\cos\theta_o} H_o^{(2)}\{(Kx^2+Kd^2)^{1/2}\}\,dKx - J\Big] \tag{7-33}$$

where J is the integral given by:

$$J = \int_0^\infty e^{-iKx\cos\theta_o} H_o^{(2)}[(Kx^2+Kd^2)^{1/2}]\,dKx \tag{7-34}$$

Similarly, the elements $\ell_{m,N+1}$ and $\ell_{m,N+2}$ can be expressed in the more convenient forms:

$$\ell_{m,N+1} = \frac{\theta_o}{2\pi} e^{-iKx_m\cos\theta_o} + \frac{\sin\theta_o}{4} e^{-iKx_m\cos\theta_o} \Big[\int_0^{Kx_m} e^{iKx\cos\theta_o} H_o^{(2)}(|Kx|)\,dKx + \int_0^{Kx_m} e^{iKx\cos\theta_o} H_o^{(2)}\{(Kx^2+Kd^2)^{1/2}\}\,dKx + J\Big] \tag{7-35}$$

$$\ell_{m,N+2} = \frac{\theta_o}{2\pi} e^{i(Kx_m+KL)\cos\theta_o} - \frac{\sin\theta_o}{4} e^{i(Kx_m+KL)\cos\theta_o} \Big[\int_0^{Kx_m+KL} e^{-iKx\cos\theta_o} H_o^{(2)}(|Kx|)\,dKx + \int_0^{Kx_m+KL} e^{-iKx\cos\theta_o} H_o^{(2)}\{(Kx^2+Kd^2)^{1/2}\}\,dKx - J\Big] \tag{7-36}$$

In deriving (7-35), the following relation has been utilized [32]:

$$\int_0^\infty e^{-iKx\cos\theta_o} H_o^{(2)}(Kx)\,dKx = \frac{2\theta_o}{\pi\sin\theta_o} \tag{7-37}$$

It is now clear that in all elements g_m, $\ell_{m,N+1}$ and $\ell_{m,N+2}$, expressed by (7-33), (7-35) and (7-36), all integrals except J can be evaluated numerically by convenient known methods, since the limit of integrations in these integrals are of finite extent. The integral J is again of a semi-infinite type with very low convergence and must be converted to a suitable form. This integral was obtained by Morita [30] and was modified such that it can be evaluated numerically, i.e. J can be written in the form:

$$J = -i\int_0^{Kd} e^{-iKx\cos\theta_o} H_o^{(2)}[(Kd^2-Kx^2)^{1/2}]dKx + \frac{2}{\pi} e^{-Kd\cos\theta_o}$$

$$[\frac{1}{2\cos\theta_o}(-\gamma+\ln\frac{2\cos\theta_o}{Kd} - \frac{1}{2Kd\cos\theta_o})+\int_0^{\infty}[K_o\{(Kx^2+2Kd\cdot Kx)^{1/2}\}$$

$$+\gamma+\frac{1}{2}\ln\frac{Kd\cdot Kx}{2} + \frac{Kx}{4Kd}]e^{-Kx\cos\theta_o}\, dKx] \tag{7-38}$$

where $K_o(z)$ is the modified Bessel function of the second kind. The first integral in (7-38) can be evaluated numerically due to the finite integration range, while the second integral can be evaluated numerically [30] by the Gauss-Laguerre quadrature formula [93] owing to the exponentially decreasing factor $\exp(-Kx\cos\theta_o)$.

Once the base functions $\ell_{m,n}$ and the incident field elements g_m are determined, the system of N+2 linear equations given by (7-23) can be solved for the unknowns R, T and the evanescent current J^e, by the known methods. Once all unknowns are determined, radiation field can be obtained through the integral equation

$$E_z^{total}(\bar{r}) = \frac{\eta}{4}\int_W [J^e(\bar{r}')+J^i(\bar{r}')+J^r(\bar{r}')+J^t(\bar{r}')]H_o^{(2)}(K|\bar{r}-\bar{r}'|)d(K\bar{r}') \tag{7-39}$$

which can be written as:

$$E_z^{total}(\bar{r}) = E_z^e(\bar{r}) + E_z^i(\bar{r}) + E_z^r(\bar{r}) + E_z^t(\bar{r}) \tag{7-40}$$

where E_z^e , E_z^i , E_z^r and E_z^t represent respectively the radiation field, contributed by the evanescent, incident, reflected and the transmitted currents. These radiation fields can be evaluated in the following ways.

7.3.1 Contribution of the induced evanescent currents

$$E_z^e(\bar{r}) = \frac{\eta}{4}\int_{\Delta c} J^e(\bar{r}')H_o^{(2)}(|K\bar{r}-K\bar{r}'|)d(K\bar{r}') \tag{7-41}$$

where Δc is the path of the integral between $x=0$ and $x=x_a$ and between $x=-L$ and $x=x_b$, and J^e is evaluated in these regions. Far from the waveguides and using the asymptotic expansion of the Hankel function, (7-41) can be expressed as

$$\begin{aligned} E_z^e(\bar{r}) &= \frac{\eta}{4}\sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})}\int_{\Delta c} J^e(Kx)e^{iKx\cos\theta}[1+e^{iKd\sin\theta}]d(Kx) \\ &= \frac{\eta}{4}\sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})}[1+e^{iKd\sin\theta}]\sum_{n=1,2,..}^{N} J_n^e(Kx_n)e^{iKx_n\cos\theta}\Delta(Kx_n) \\ &= \sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})}F^e(\theta) \end{aligned} \tag{7-42}$$

where

$$F^e(\theta) = [1+e^{iKd\sin\theta}]\sum_{n=1,2,..}^{N}\frac{\eta}{4}J_n^e(Kx_n)e^{iKx_n\cos\theta}\Delta(Kx_n) \tag{7-43}$$

7.3.2 Contribution of the induced incident current

The contribution of the incident current induced on the walls of the exciting waveguide may be represented by

$$E_z^i(\bar{r}) = \frac{\eta}{4}\int_{\omega} J^i(\bar{r}')H_o^{(2)}(K|\bar{r}-\bar{r}'|)dK\bar{r}' \tag{7-44}$$

From (7-19) and (7-44) one obtains

$$E_z^i(\bar{r}) = \frac{\sin\theta_o}{4} [\int_0^\infty e^{iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 + Ky^2\}^{1/2}]dKx'$$

$$+ \int_0^\infty e^{iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 + (Ky-Kd)^2\}^{1/2}]dKx'] \tag{7-45}$$

changing θ_o to $\pi-\theta'$ in the above equation yields

$$E_z^i(\bar{r}) = \frac{\sin\theta'}{4} [\int_0^\infty e^{-iKx'\cos\theta'} H_o^{(2)}[\{(Kx-Kx')^2 + Ky^2\}^{1/2}]dKx'$$

$$+ \int_0^\infty e^{-Kx'\cos\theta'} H_o^{(2)}[\{(Kx-Kx')^2 + (Ky-Kd)^2\}^{1/2}]dKx'] \tag{7-46}$$

The integrals in the above equation are similar to that obtained by Morita [31] and for a region far from origin, equation (7-46) reduces to

$$E_z^i(\bar{r}) = \frac{i}{4} \frac{\sin\theta'}{\cos\theta-\cos\theta'} [H_o^{(2)}(K\rho) + H_o^{(2)}(K\rho_1)] \tag{7-47}$$

where $\rho = \{x^2+y^2\}^{1/2}$, $\cos\theta = x/\rho$

and $\rho_1 = \{x^2+(y-d)^2\}^{1/2}$

Using the asymptotic expansion of $H_o^{(2)}$, (7-47) reduces to

$$E_z^i(\bar{r}) = \sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})} F^i(\theta) \tag{7-48}$$

where

$$F^i(\theta) = [1+e^{iKd\sin\theta}] \frac{i}{4} \frac{\sin\theta_o}{\cos\theta+\cos\theta_o} \tag{7-49}$$

It should be noted that this result gives the radiation field from the open end of a semi-infinite waveguide when Kirchhoff's method (Huygens' Principle) is used. In other words this is another approach to obtain the radiation pattern of a semi-infinite waveguide using Kirchhoff's approximations [2], which sets the field in the aperture (the open end of the waveguide) equal to the field of the incident mode in an infinite waveguide and the field on the outer walls equal to zero.

7.3.3 Contribution of the induced reflected current

The reflected current induced on the walls of the exciting waveguide may contribute to the radiation field by an expression of the form:

$$E_z^r(\bar{r}) = \frac{\eta}{4}\int_W J^r(\bar{r}')H_o^{(2)}(K|\bar{r}-\bar{r}'|)dK\bar{r}' \tag{7-50}$$

From (7-20), equation (7-50) reduces to

$$E_z^r(\bar{r}) = R\,\frac{\sin\theta_o}{4}\Big[\int_0^\infty e^{-iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 + Ky^2\}^{1/2}]dKx' + \int_0^\infty e^{-iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 + (Ky-Kd)^2\}^{1/2}]dKx'\Big] \tag{7-51}$$

This equation is of the same form as (7-46) and hence reduces to

$$E_z^r(\bar{r}) = \sqrt{\frac{2}{\pi K\rho}}\; e^{-i(K\rho-\frac{\pi}{4})} F^r(\theta) \tag{7-52}$$

where

$$F^r(\theta) = [1+e^{iKd\sin\theta}]\,\frac{iR}{4}\,\frac{\sin\theta_o}{\cos\theta-\cos\theta_o} \tag{7-53}$$

7.3.4 Contribution of the induced transmitted current

The contribution of the transmitted current induced on the walls of the coupled waveguide may be represented by

$$E_z^t(\bar{r}) = \frac{\eta}{4}\int_W J^t(\bar{r}')H_o^{(2)}(K|\bar{r}-\bar{r}'|)dK\bar{r}' \tag{7-54}$$

From (7-21), equation (7-54) reduces to

$$E_z^t(\bar{r}) = T\,\frac{\sin\theta_o}{4}\Big[\int_{-\infty}^{-L} e^{iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 + Ky^2\}^{1/2}]dKx' + \int_{-\infty}^{-L} e^{iKx'\cos\theta_o} H_o^{(2)}[\{(Kx-Kx')^2 +(Ky-Kd)^2\}^{1/2}]dKx'\Big] \tag{7-55}$$

changing the variable of integration and after some manipulation, (7-55) becomes

$$E_z^t(\bar{r}) = T\,\frac{\sin\theta_o}{4}\; e^{-iKL\cos\theta_o}\Big[\int_0^\infty e^{-iKx'\cos\theta_o} H_o^{(2)}[\{(KX-Kx')^2+Ky^2\}^{1/2}]dKx'$$

$$+\int_0^\infty e^{-iKx'\cos\theta_o} H_o^{(2)}[\{(KX-Kx')^2 + (Ky-Kd)^2\}^{1/2}]dKx' \qquad (7\text{-}56)$$

where $KX = -(Kx+KL)$.

This equation is of the same form as (7-46) and hence reduces to

$$E_z^t(\bar{r}) = \frac{iT}{4}\frac{\sin\theta_o}{\cos\bar{\theta}-\cos\theta_o}[H_o^{(2)}(K\rho_2) + H_o^{(2)}(K\rho_3)] \qquad (7\text{-}57)$$

where

$$\rho_2 = \{X^2 + y^2\}^{1/2} ,$$

$$\rho_3 = \{X^2 + (y-d)^2\}^{1/2}$$

and $\cos\bar{\theta} = X/\rho_2 \simeq -x/\rho = -\cos\theta$

using the asymptotic expansion of $H_o^{(2)}(K\rho)$, (7-57) becomes

$$E_z^t(\bar{r}) = \sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})} F^t(\theta) \qquad (7\text{-}58)$$

where

$$F^t(\theta) = -[1+e^{iKd\sin\theta}]\frac{iT}{4}\frac{\sin\theta_o}{\cos\theta+\cos\theta_o}\, e^{-iKL(\cos\theta+\cos\theta_o)} \qquad (7\text{-}59)$$

which can be related to $F^i(\theta)$ by the relation

$$F^t(\theta) = -T\, e^{-iKL(\cos\theta+\cos\theta_o)}\, F^i(\theta) \qquad (7\text{-}60)$$

Combining (7-40), (7-42), (7-48), (7-52) and (7-58), one obtains

$$E_z^{total}(\bar{r}) = \sqrt{\frac{2}{\pi K\rho}}\, e^{-i(K\rho-\frac{\pi}{4})} F(\theta)$$

where

$$F(\theta) = F^e(\theta) + F^i(\theta) + F^r(\theta) + F^t(\theta)$$

$$= [1+e^{iKd\sin\theta}][\frac{i}{4}\frac{\sin\theta_o}{\cos\theta+\cos\theta_o}(1-T\, e^{-iKL(\cos\theta+\cos\theta_o)})$$

$$+\frac{iR}{4}\frac{\sin\theta_o}{\cos\theta-\cos\theta_o} + \sum_{n=1,2,..}^{N}\frac{\eta}{4}J_n^e(Kx_n)e^{iKx_n\cos\theta}\Delta(Kx_n)] \qquad (7\text{-}61)$$

The above equation represents the radiation pattern due to coupling between two semi-infinite waveguides, as a function of the reflection and transmission coefficients and the evanescent currents. In the absence of the coupled waveguide, T=0 and radiation pattern is given by:

$$F(\theta) = [1+e^{iKd\sin\theta}][\frac{i}{4}\frac{\sin\theta_o}{\cos\theta+\cos\theta_o} + \frac{iR}{4}\frac{\sin\theta_o}{\cos\theta-\cos\theta_o}$$

$$+ \sum_{n=1,2,..}^{N} \frac{\eta}{4} J_n^e(Kx_n)e^{iKx_n\cos\theta} \Delta(Kx_n)] \qquad (7\text{-}62)$$

where R represents the reflection coefficient of a semi-infinite waveguide, and evanescent currents are along the exciting waveguide walls betwen $x=0$ and $x=x_a$. The system of equations, in this case, is for N+1 unknowns, J_1^e, J_2^e, ..., J_N^e and R, and thus the elements $\ell_{m,N+2}$, which correspond to T, i.e., existence of the coupled waveguide should be removed from the system of equations represented by (7-23).

When the coupled waveguide is not of the semi-infinite type but has a finite length L_2, then equation (7-62) can be used. In this case R will represent the reflection coefficient in the presence of the coupled waveguide and the evanescent currents will be along the exciting waveguide walls between $x=0$ and $x=x_a$ and along the whole length of the coupled waveguide. Again, the system of equations is for N+1 unknowns.

7.4 Results and discussion

Some results are obtained using direct moment method (DMM) for the radiation pattern of two collinear parallel plate waveguides of width $d = 0.6\lambda$ and a separation distance $KL = 50$. These results are shown in figure 7-3 for different lengths of the coupled waveguide. The solid

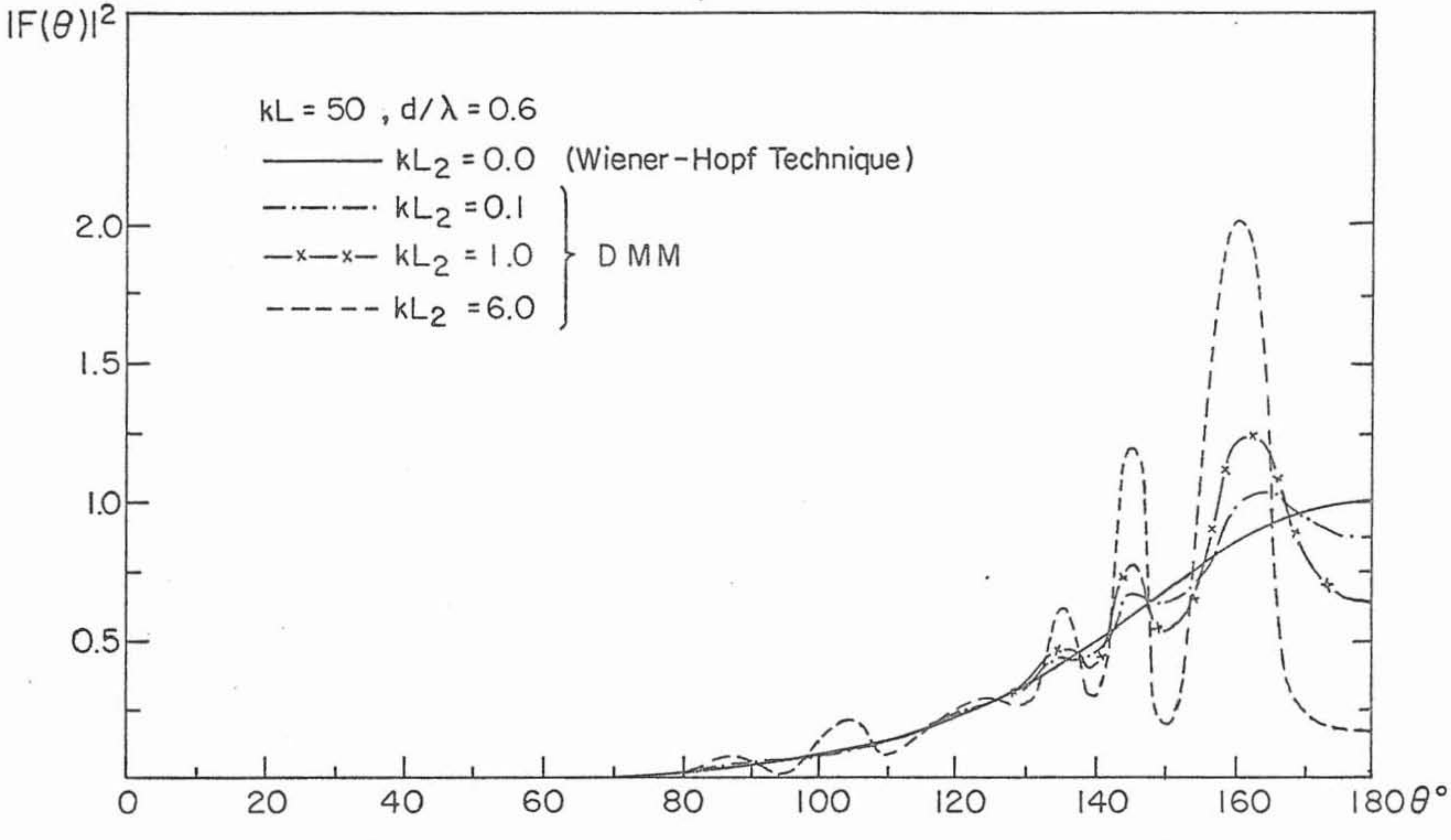

Figure 7-3 Radiation pattern of two collinear parallel-plate waveguides with a separation kL and $TE_{0,1}$ excitation.

curve shows the radiation pattern for $KL = 0.0$, the case of a single waveguide, obtained using the Wiener-Hopf technique, while the broken line curves show the patterns by numerical methods, for $KL = 0.1$, 1.0 and 6.0. It is clear from this figure that for non zero values of KL_2 the radiated power oscillates around the value corresponding to $KL_2 = 0.0$. DMM have been used to check the results of MMM and the results of the Wiener-Hopf technique. Also, it has the advantage that the coupled waveguide may have any width and orientation. The disadvantage of DMM is that the coupled waveguide cannot have large length, i.e. the case of two semi-infinite waveguides cannot be treated because of the need for a large number of matching points and consequently large computation time. This is overcome by MMM as shown in previous sections. As an application of MMM, radiation patterns and the reflection coefficients are obtained and shown in figures 7-4 and 7-5 and table 7-1. It is clear that the radiation pattern of a single waveguide is exactly the same as the one obtained by the Wiener-Hopf technique which is shown by the solid line in figures 7-4 and 7-5. Also it is interesting to note that the contribution of the incident current gives the results obtained by the Kirchhoff approximation. The contributions of the reflected and evanescent currents are shown separately and then together. The evanescent current has the main contribution in the forward direction ($\theta = 180°$) while the reflected current gives the main contribution in the forward and backward directions. Two examples for $d = 0.51\lambda$ and 0.6λ are shown in figures 7-4 and 7-5. As the reflection coefficient decreases rapidly by increasing the waveguide width, it will not contribute significantly to the radiated power. This is clear by two examples of $d = 0.51\lambda$ and 0.6λ where the reflection coefficients

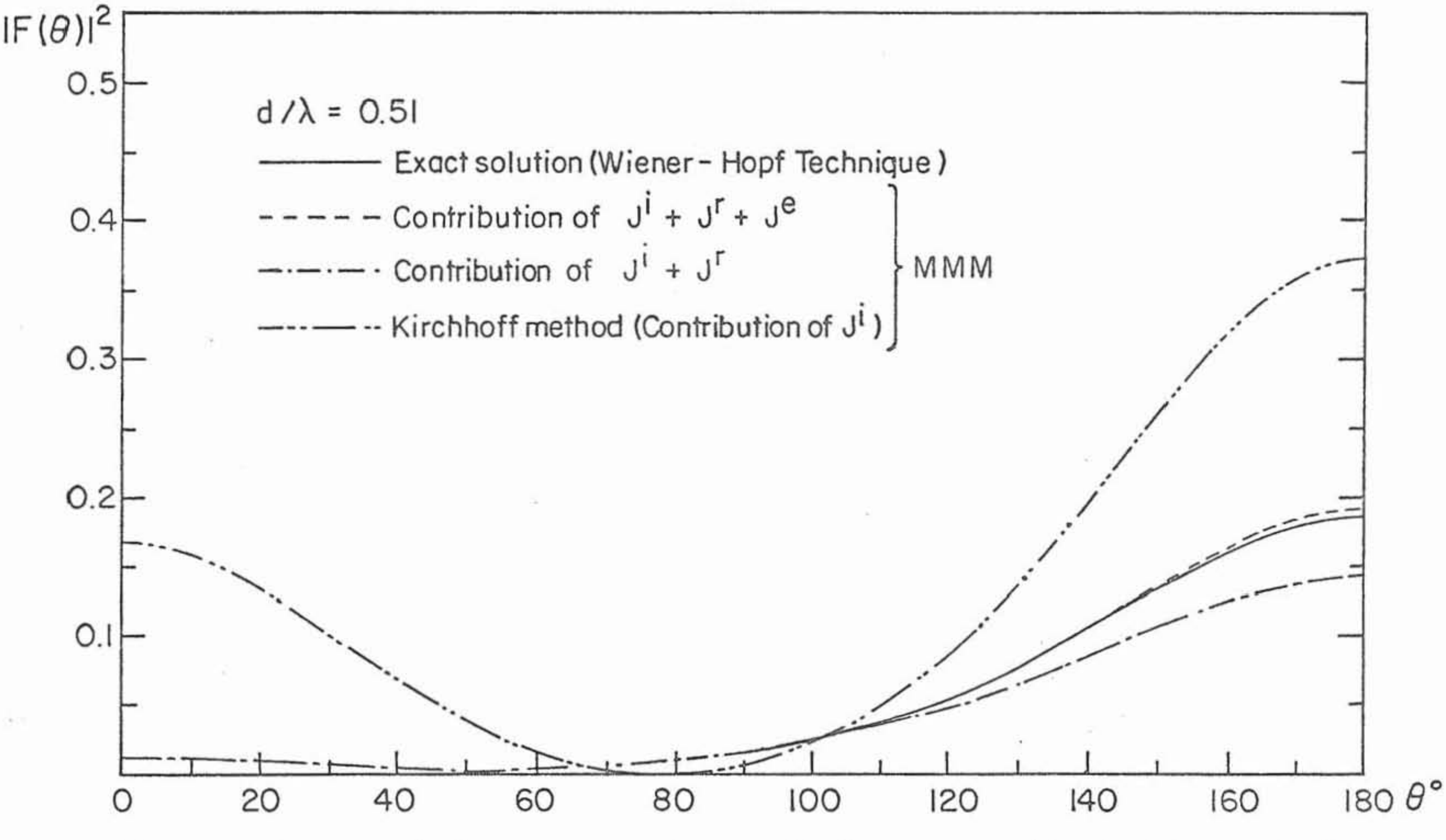

Figure 7-4 Radiation pattern of a semi-infinite waveguide with $TE_{0,1}$ excitation

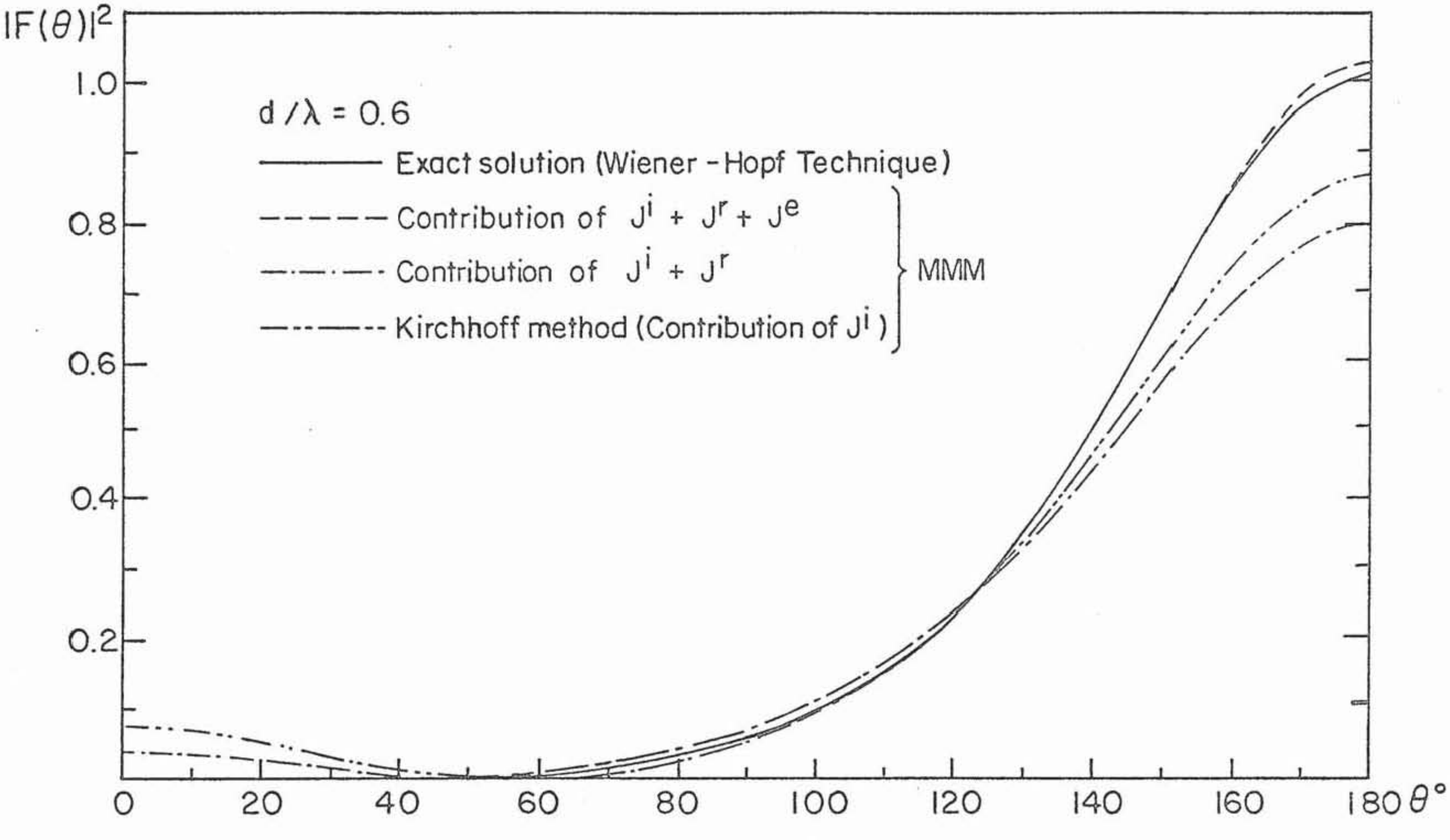

Figure 7-5 Radiation pattern of a semi-infinite waveguide with $TE_{0,1}$ excitation.

TABLE 7-1

Reflection Coefficient of a
semi-infinite waveguide with $TE_{o,1}$ mode of excitation

Waveguide width in wavelength	Using Wiener-Hopf Technique		Using MMM	
	Magnitude	phase in degrees	Magnitude	phase in degrees
.51	.5971	-164.3	.6026	-165.7
.60	.1891	-130.9	.1911	-134.6
1.00	.0176	- 80.8	.0194	- 95.1

are $0.6 \underline{/-164}$ and $0.19 \underline{/-131}$, respectively. Also, table 7-1 shows the reflection coefficient of a single waveguide using the Wiener-Hopf technique and MMM for different waveguide widths and indicates very good agreement. By increasing the waveguide width, the reflection coefficients become small and, as a result, the phase errors in the computations become large. But, as mentioned before, the contribution of the small reflection coefficient to the radiated power is very small and hence one expects to get very good results for the radiation pattern even with a large waveguide width ($0.5\lambda < d < 1.5\lambda$).

Some results have also been obtained for MMM applied to two collinear parallel plate waveguides with the coupled waveguide of finite length. Figure 7-6 shows the radiation pattern for $d/\lambda = 0.6$ and $KL_2 = 15.0$ with different values of the separation distance KL. It is clear that for $KL = 0.1$, i.e. small separation distance, the radiation pattern is the same as the one corresponding to the radiation pattern from the open end of a single semi-infinite waveguide. This is true because the main radiation comes from the far end of the coupled waveguide. Some results are also shown for $KL = 1$ and 10. For the previously mentioned case, the reflection coefficients are shown in table 7-2. Also, it is clear from this table that for $KL = 0.1$, the magnitude of the reflection coefficient is the same as that of a single semi-infinite waveguide, while the phase is different, since it represents approximately the reflection from the far end of the coupled waveguide. A comparison of the reflection coefficient for the finite lengths of the coupled waveguide using the Wiener-Hopf technique and MMM is shown in table 7-3, which shows a fairly close agreement. Radiation patterns for the finite lengths of the coupled waveguide and using the Wiener-Hopf technique, MMM and DMM are shown in figure 7-7.

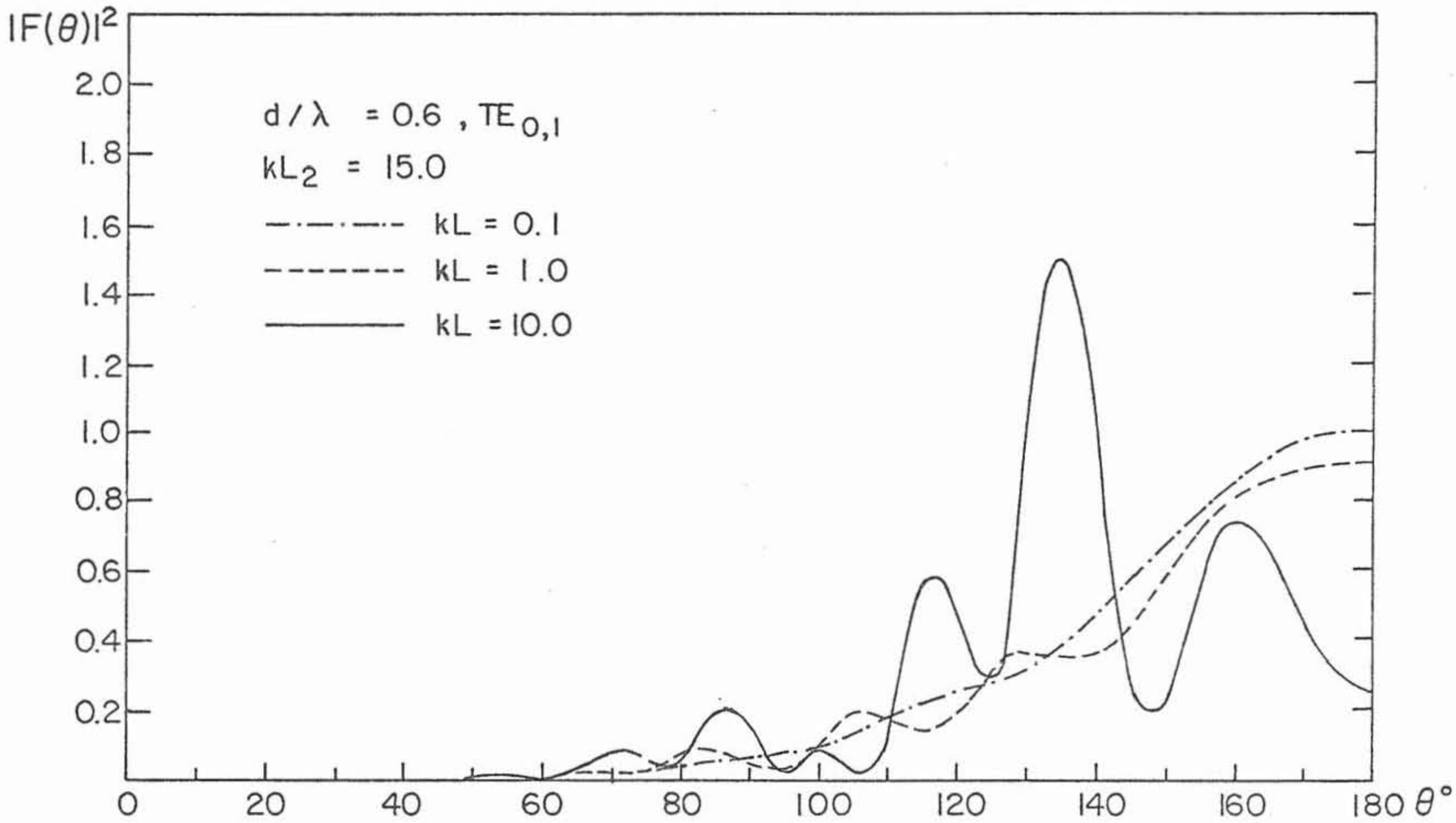

Figure 7-6 Radiation pattern of two collinear finite waveguides by MMM.

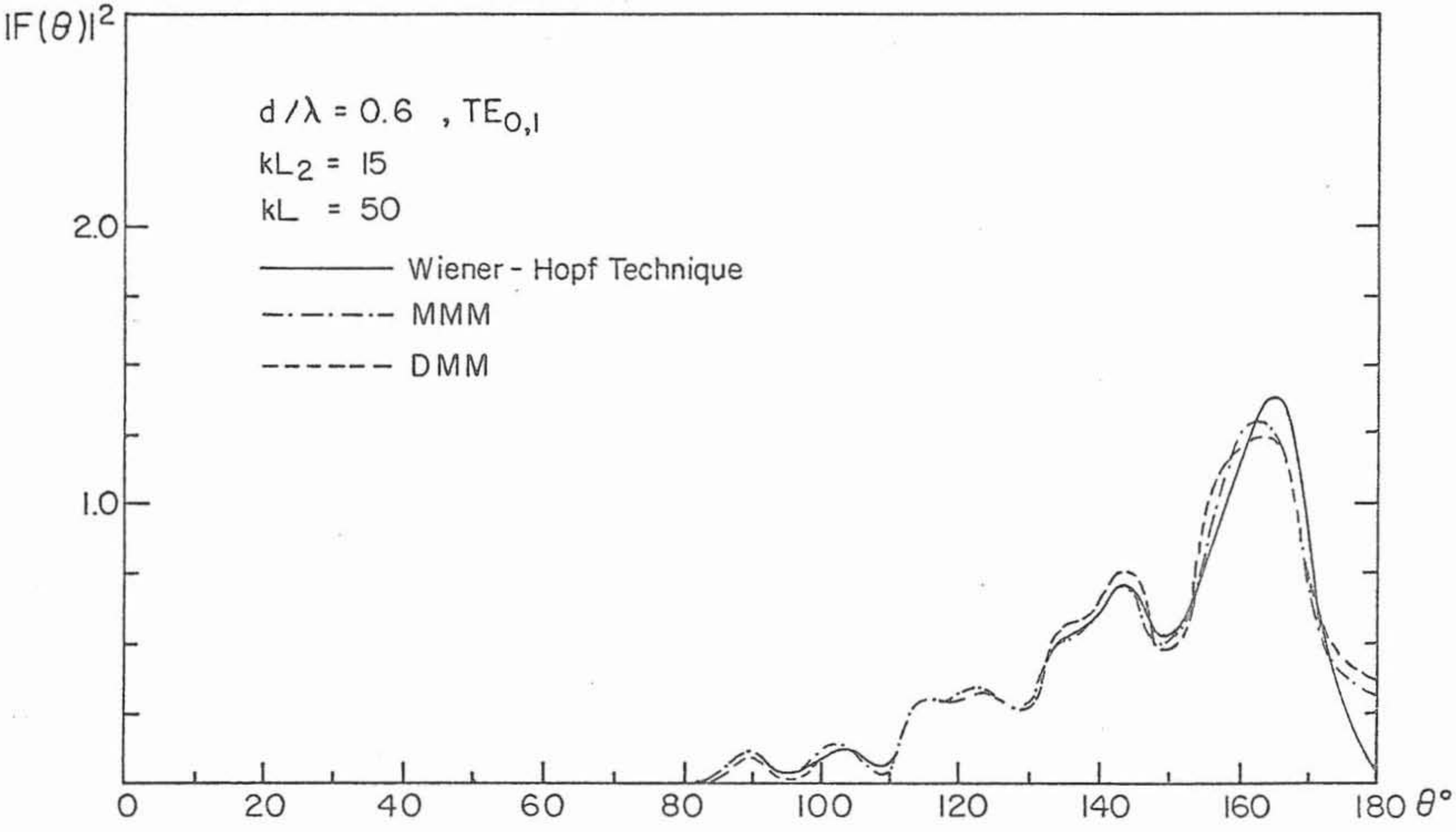

Figure 7-7 Comparison of radiation pattern by different methods of formulation.

TABLE 7-2

Reflection Coefficient by MMM

for two collinear parallel-plate waveguides of finite length

$d/\lambda = 0.6$, $KL_2 = 15$

KL	Magnitude	phase in degrees
10	0.233	-103.6
1.0	0.325	148.1
0.1	0.198	85.1

TABLE 7-3

Reflection Coefficient

for two collinear parallel-plate waveguides of finite length

$d/\lambda = 0.6$, $KL_2 = 15$

KL	Using Wiener-Hopf Technique		Using MMM	
	Magnitude	phase in degrees	Magnitude	phase in degrees
10	0.200	-113	0.233	-103.6
20	0.168	-123	0.151	-113
50	0.199	-133	0.194	-142

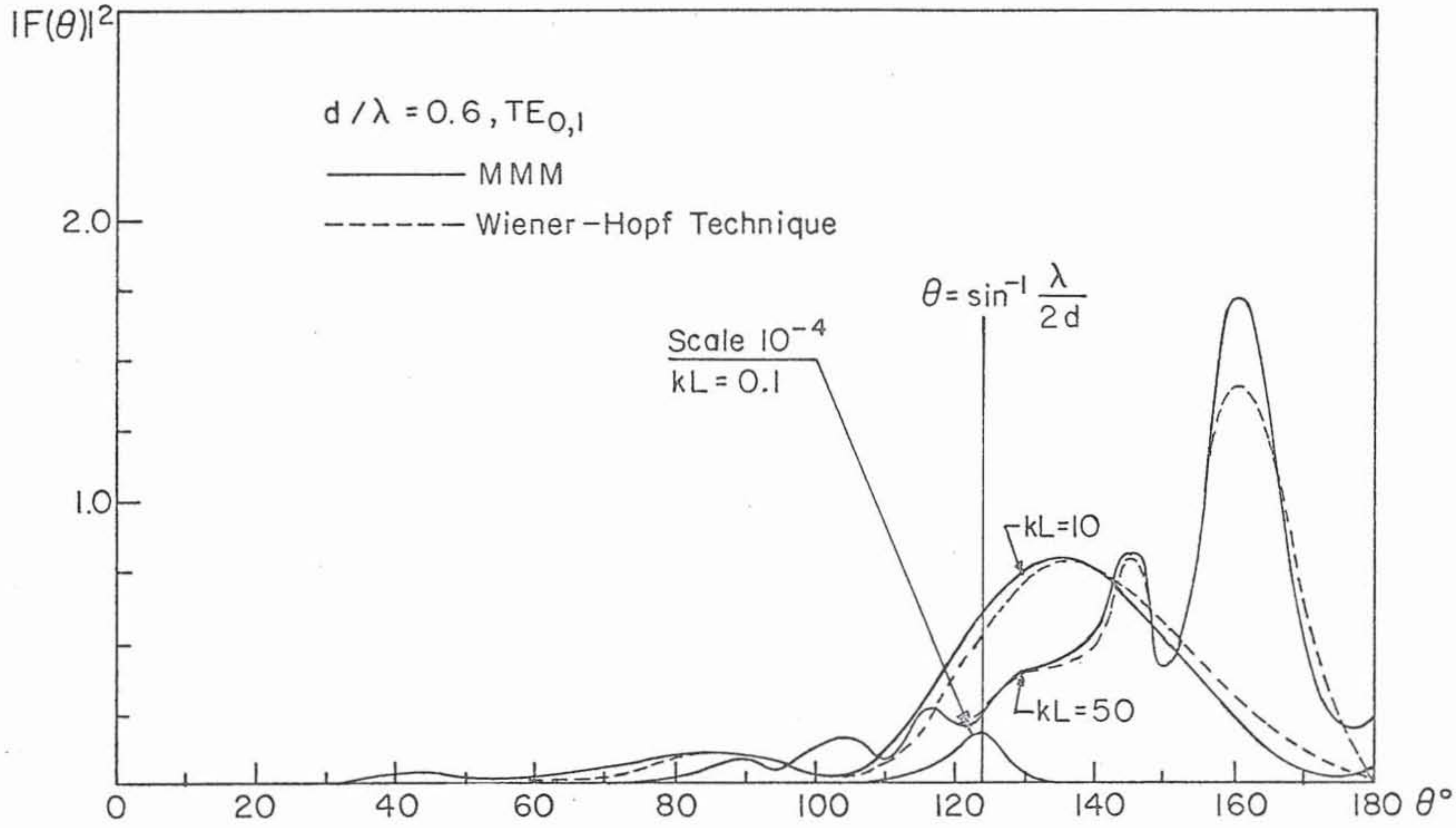

Figure 7-8 Radiation pattern for two collinear semi-infinite waveguides.

It is clear that the results using DMM and MMM are approximately the same. The results using the Wiener-Hopf technique are slightly different, especially in the forward direction, and this is because of the fact that some interacting rays are not included in the derivation.

For the case of two semi-infinite waveguides, some results are obtained using MMM and the Wiener-Hopf technique and are shown in figure 7-8 and table 7-4. Figure 7-8 shows the radiation patterns for different separation distances. For $KL = 0.1$ and by MMM, the radiation pattern is like a beam at an angle given by $\theta = -\sin^{-1}\frac{\lambda}{2d} + \pi$. It is clear also that the results by MMM are slightly different from those by the Wiener-Hopf technique in the forward direction. This may be due to the fact that in the MMM, the far testing points, corresponding to the reflection and transmission coefficients, are arbitrary and, as a result the results of MMM are too sensitive at these two points. This point will be clarified later. Table 7-4 shows some results for R and T using MMM and the Wiener-Hopf technique. It is noticed that values corresponding to $|T|$ do not agree because of the phenomena mentioned before. In the case of single semi-infinite parallel-plate waveguide, the sensitivity of the results to location of the far testing point was studied. by Wu, S.C. and was shown by unpublished results. The same phenomenon can be said about two semi-infinite waveguides and cannot be studied here because of another important phenomenon, which is the dependence of R and T or the evanescent currents on the number of matching points. This has been shown in figures 7-9 and 7-10. Figure 7-9 shows R and T versus the number of matching points N_2 (corresponding to the evanescent current) on the coupled waveguide for $N_1 = 20$ (N_1 = number of matching points corresponding to evanescent current on the exciting waveguide). Similar results are also obtained for R and T versus N_1 for $N_2 = 20$ and are shown in figure 7-10. Figures 7-9 and 7-10 show sensitivity of R and T to the number of matching points corresponding to the evanescent currents. For very small separation distances, R and T do not change significantly, which means that the MMM is very reliable when the separation distance is small.

The numerical results obtained here were for waveguides of equal width. The method, however, can readily be extended to waveguides of different widths. It can also be used for waveguides with a flare angle and coupling between adjacent horn antennas.

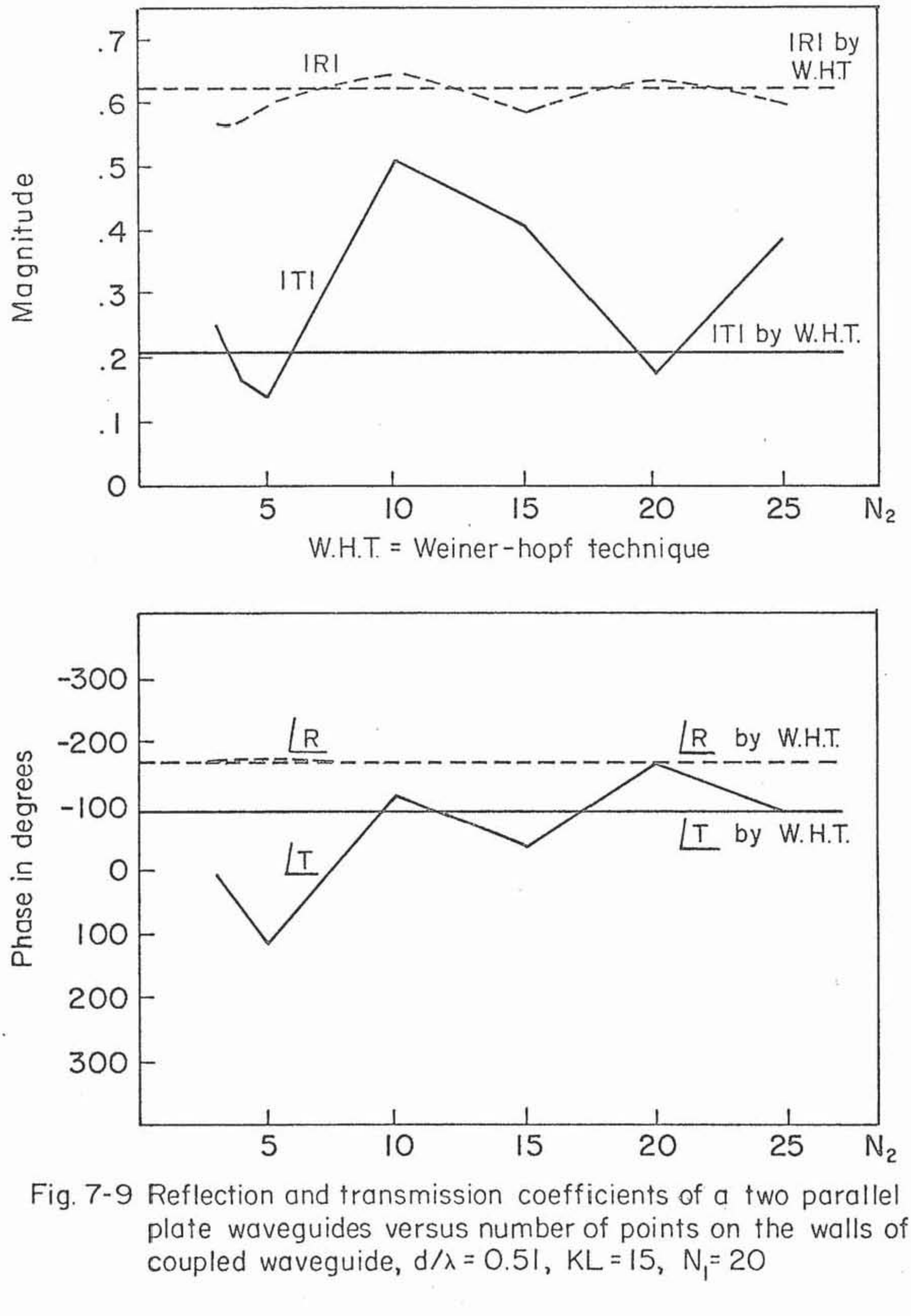

Fig. 7-9 Reflection and transmission coefficients of a two parallel plate waveguides versus number of points on the walls of coupled waveguide, $d/\lambda = 0.51$, $KL = 15$, $N_1 = 20$

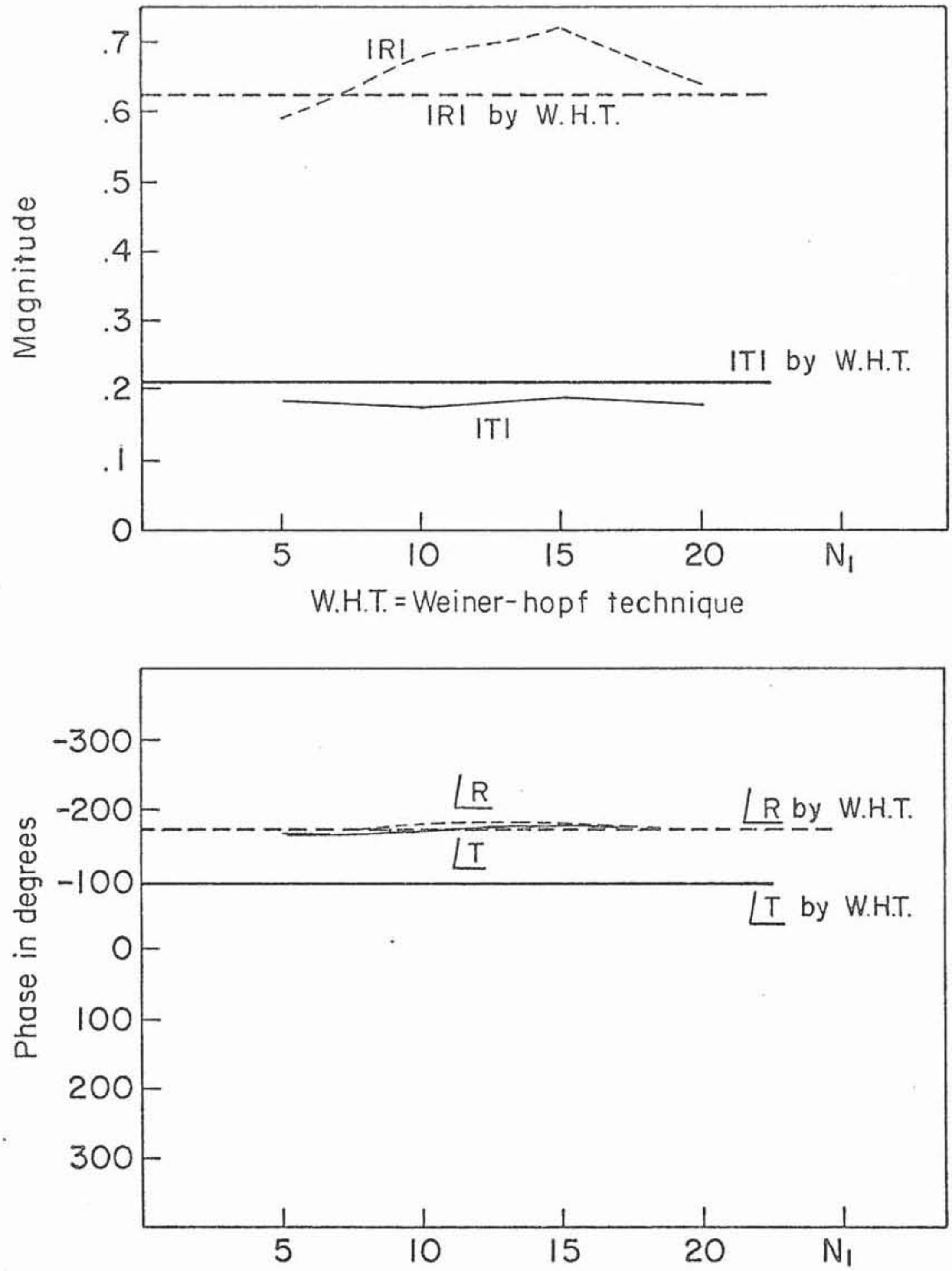

Fig.7-10 Reflection and transmission coefficients of a two parallel plate waveguides versus number of points on the walls of exciting waveguide, $d/\lambda = 0.51$, $KL=15$, $N_2 = 20$

TABLE 7-4

Comparison between Wiener-Hopf technique and MMM
for reflection and transmission coefficients of $TE_{o,1}$ mode

KL	Reflection Coefficient		Transmission Coefficient	
	Wiener-Hopf Tech.	MMM	Wiener-Hopf Tech.	MMM
0.1	-	0.0015/-103	-	0.999/ .5
1.0	-	0.116/131	-	0.991/7.5
10.0	0.236/-111	0.285/-113	0.429/-148	0.501/-159
50.0	0.199/-135	0.192/-135	0.195/147	0.261/140

CHAPTER 8

DISCUSSION AND CONCLUSION

8.1 Discussion

The Wiener-Hopf technique was used to investigate the coupling between two collinear semi-infinite parallel-plate and circular waveguides. Obtained results for the reflection, transmission and radiation fields in the exciting waveguide, coupled waveguide and free space were expressed by three terms. The first term was due to the open end of the exciting waveguide alone (i.e. in the absence of coupled waveguide). The second and third terms were due to the interactions between the two opening ends of the waveguides. The third component of the reflection coefficients and the first and second components of the transmission coefficients were functions of axial distance z and they represent radiation of either waveguide inside the other. These functions when represented as inhomogeneous waves in the direction of the axis would decay to zero at the far end, in order to satisfy the Sommerfeld radiation condition. Far from the open ends, the main contribution to the reflected field was due to the first and second terms (equation 2-65 for the parallel-plate waveguides and equation 3-39 for the circular waveguides), while the main contribution for the transmitted field was due to third term (equation 2-73 for the parallel-plate waveguides and equation 3-48 for the circular waveguides). The amplitude and phase of the reflection and transmission coefficients were oscillating functions of period $kL = \pi$. The reflection coefficient decays continuously with kL to reach its final value for $kL = \pi$, a single excited semi-infinite waveguide (e.g. see figures 2-6 and 3-3), while the transmission coefficient would decay to zero as kL approaches infinity (e.g. see figures 2-7 and 3-4).

Equation (4-66) shows that the spherical wavefactor $\dfrac{1}{\sin\bar{\theta}\ H_o^{(1)}(ka\ \sin\bar{\theta})}$ combined with the spherical wave variation $\dfrac{e^{i(kr-\pi/2)}}{2\pi\ kr}$ is related to the curvature of the rim of the circular waveguide. It was shown that this factor was necessary for treating problems of diffraction by small apertures in hard screens.

Results using the ray theory of diffraction in conjunction with the modified diffraction coefficients, were in good agreement with the rigorous solution, especially for very large values of kL. However, the ray theory results showed that, for any separation distance, the radiation patterns would blow up in the front direction (see e.g. figures 4-8 and 4-9).

Using the results of the coupling between semi-infinite waveguides and scattering matrix technique, the coupling between waveguides of finite length was obtained. This system may act as an open type resonator with its frequency determined from the equation $R\ R_o\ e^{-\gamma_m \ell_2} = 1$, where γ_m is a function of $k = k_1 + ik_2$ where k_1 and k_2 correspond respectively to the frequency of oscillation and loss factor [41].

From figure 6-1, it is clear that, except for $kL < 10$, the exact solution, based on the Wiener-Hopf technique, oscillates around that obtained by Hu's formula using the Wiener-Hopf results of a single waveguide. These oscillations have a period of $kL = \pi$. This phenomenon is because of the fact that in equation (6-8), $\bar{E}$, is evaluated far from the open end of the exciting waveguide. This is similar to the condition of Friis' transmission formula [82] for $L \geq \dfrac{2d^2}{\lambda}$.

Results obtained by the modified moment method for the radiation from an open end of a semi-infinite parallel-plate waveguide are in very good agreement with the Wiener-Hopf results. The contribution of J^i (see equation 7-49) to the radiation field gives results of the Huygen's principle, as shown by figures 7-4 and 7-5. The contribution of R to the radiation field is significant when the waveguide width is slightly larger than 0.5λ ($0.5 < d/\lambda < 1.5$ for $TE_{0,1}$ mode).

As the Wiener-Hopf technique cannot be applied when the separation distance between the waveguides is small, the modified moment method is another way to solve this problem. Some results were shown in figure 7-8. The results of this method for small separation distance were not sensitive to the testing and matching points.

Analysis of the parallel-plate and circular waveguides (symmetrical modes) for other modes of excitations can also be treated in the same way as shown in this thesis. For asymmetrical modes in circular waveguides, the problem will be different as the diffracted fields (reflected, transmitted and radiated) will be combinations of TE and TM modes.

8.2 Conclusion

The problem of coupling between two collinear parallel-plate and circular waveguides located in free space has been solved. Expressions were obtained for the reflected, transmitted and the radiated fields and were expressed by three terms. The first term was due to the open end of the exciting waveguide, while the other two terms were due to interactions. The rigorous solution was expanded to obtain the ray theory results with the help of a modified diffraction coefficient. The modified diffraction coefficient for the circular waveguide was found to be in the same form as that of Lee for parallel-plate waveguides, but another factor called the spherical wavefactor, has to be introduced. This factor was shown to be necessary in treating problems of diffraction by small apertures in hard screens.

Results using the Wiener-Hopf technique for parallel-plate waveguides, was found to be more accurate than those obtained using Hu's transmission formula.

The modified moment method was applied to two separated structures and very accurate results in good agreement with the other available results, were obtained. The radiation fields obtained in terms of the evanescent currents, reflection and transmission coefficients gave another explanation for the physical meaning of the ray theory of diffraction.

Finally, the Wiener-Hopf technique together with the modified moment method gave a complete analysis of the coupling between parallel-plate waveguides.

8.3 Suggestions for future research

During the formulation of the problem by the Wiener-Hopf technique, an expansion of the function $E(\alpha)$ and $G_-(\alpha)$, in a Taylor series, about the branch point $\beta = -k$ was needed (see, e.g. equation (2-36)). An improved expansion may be obtained by expanding $E(\alpha)$ and $G_-(\alpha)$ about another point p chosen such that the second term in the Taylor series vanishes [26] on page 200. This requires further investigations to examine the possible improvements on results presented in this thesis.

As the insertion of dielectrics in waveguides reduces the loss and may give an optimum radiation for certain dielectric constants, it is interesting to investigate the case of coupling between dielectric loaded waveguides. The only problem here is the difficulty in factorizing the new Green's function and finding the roots of a characteristic equation [26], [94]. The author suggests using scattering matrix technique with the elements of scattering matrices to be determined by the use of the results presented here for the coupling between waveguides.

It is rather interesting to see the significance of the spherical wavefactor and its application to any convex aperture in hard screens. This may be an introduction to solving problems of radiation from semi-infinite circular waveguides with oblique openings of elliptical shape. In this case, the excitation of either $TE_{o,m}$ or $TM_{o,m}$ will produce diffracted waves of both types of asymmetrical modes i.e. $TE_{m,n}$ and $TM_{m,n}$ and $n \neq 0$. Consequently two modified Wiener-Hopf equations will appear. For $TE_{o,m}$ excitation in circular waveguides, the attenuation theoretically decreases indefinitely with increasing frequency [95]. Therefore, it is interesting to solve this problem in the same way as presented in Chapter 3.

The modified moment method is only adopted in this thesis for $TE_{o,\ell}$ parallel-plate waveguides. In order to apply it to $TM_{o,\ell}$ one needs first to investigate the diffraction of an H-polarized plane wave by a semi-infinite conductor of arbitrary cross-section. Also, one may try to apply this method to circular waveguides. Investigation of waveguides arrays is also interesting, especially if the waveguides are loaded.

APPENDIX A

FORMULATION OF EQUATIONS (3-4) AND (3-5)

The total electromagnetic fields may be found from $\psi^t = \psi + \psi^i$, where ψ is a scalar potential associated with the scattered fields. The scattered field components are

$$\left.\begin{aligned} H_\phi &= -\frac{\partial\psi}{\partial\rho} \\ E_\rho &= \frac{i}{\omega\varepsilon}\,\frac{\partial^2\psi}{\partial\rho\partial z} \\ E_z &= \frac{i}{\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi \end{aligned}\right\} \tag{A-1}$$

Since the problem in this chapter is for symmetrical modes, it is a two dimensional one and the associated wave equation for ψ must be solved subject to the boundary and edge conditions.

A Fourier Transform Ψ of the scattered field ψ may be assumed in the form

$$\Psi(\rho,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(\rho,z)\, e^{i\alpha z}\, dz \quad , \quad \alpha = \sigma + i\tau \tag{A-2}$$

Let $\Phi(\rho,\alpha)$ be the Fourier Transform of E_z. Hence the relation between $\Phi(\rho,\alpha)$ and $\Psi(\rho,\alpha)$ is

$$\Phi(\rho,\alpha) = \frac{\gamma^2}{i\omega\varepsilon}\,\Psi(\rho,\alpha) \tag{A-3}$$

The function $\Phi(\rho,\alpha)$ can be decomposed into three parts as shown in Chapter 2, to give

$$\Phi(\rho,\alpha) = \Phi_-(\rho,\alpha) + \Phi_1(\rho,\alpha) + e^{i\alpha L}\,\Phi_+(\rho,\alpha) \tag{A-4}$$

where Φ_-, Φ_1 and Φ_+ are as defined by equations(2-10a), (2-10b) and (2-10c). The transform function Ψ which satisfies the transformed wave equation is analytic in the strip $|\tau| < k_2$ and has a solution of the form:

$$\Psi(\rho,\alpha) = \begin{cases} A(\alpha)\ K_0(\gamma\rho)/K_0(\gamma a) & , \quad \rho \geq a \\ \\ B(\alpha)\ I_0(\gamma\rho)/I_0(\gamma a) & , \quad 0 \leq \rho \leq a \end{cases} \tag{A-5}$$

or

$$\Phi(\rho,\alpha) = \Phi_-(\rho,\alpha) + \Phi_1(\rho,\alpha) + e^{i\alpha L}\Phi_+(\rho,\alpha) = \begin{cases} \dfrac{\gamma^2}{i\omega\varepsilon}\ A(\alpha)\dfrac{K_0(\gamma\rho)}{K_0(\gamma a)} , & \rho \geq a \\ \dfrac{\gamma^2}{i\omega\varepsilon}\ B(\alpha)\dfrac{I_0(\gamma\rho)}{I_0(\gamma a)} , & 0 \leq \rho \leq a \end{cases} \tag{A-6}$$

where $\gamma^2 = \alpha^2 - k^2$

Now, an application of the boundary conditions on electric and magnetic fields at the surface $\rho = \alpha$ give

$$\Phi_-(a+,\ \alpha) = \Phi_-(a-,\ \alpha) = 0\ , \tag{A-7a}$$

$$e^{i\alpha L}\ \Phi_+(a+,\ \alpha) = e^{i\alpha L}\ \Phi_+(a-,\ \alpha) = 0\ , \tag{A-7b}$$

$$\Phi_1(a+,\ \alpha) = \Phi_1(a-,\ \alpha) = \Phi_1(a,\alpha) \tag{A-7c}$$

and $$\Phi_1'(a+,\ \alpha) = \Phi_1'(a-,\ \alpha) + \frac{1}{\sqrt{2\pi}}\ \frac{\xi_{om}}{a}\ J_1(\xi_{om})\ \frac{e^{(i\alpha - \gamma_{om})L} - 1}{i\alpha - \gamma_{om}}\ \frac{\gamma^2}{i\omega\varepsilon} \tag{A-7d}$$

where the prime denotes differentiation with respect to ρ, and ξ_{om} is the m^{th} order zero of J_o, the zero Bessel function. Equation (A-6) together with equation (A-7a) to (A-7c) give

$$A(\alpha) = B(\alpha) = \frac{i\omega\varepsilon}{\gamma^2}\ \Phi_1(a,\alpha) \tag{A-8}$$

Differentiating (A-6) with respect to ρ and letting $\rho = a$, then after using (A-7d), one obtains equation (3-4), i.e. ,

$$J_-(\alpha) + e^{i\alpha L} J_+(\alpha) - \frac{i\omega\varepsilon\ \Phi_1(a,\alpha)}{\gamma^2 a\ G(\alpha)} = \frac{-i}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{1 - e^{(i\alpha - \gamma_{om})L}}{\alpha + i\gamma_{om}}$$

$$, \quad |\tau| < k_2 \qquad \text{(A-9)}$$

and is a modified Wiener-Hopf equation of the second kind where

$$J_-(\alpha) = \Psi_-'(a+,\ \alpha) - \Psi_-'(a-,\ \alpha) \qquad \text{(A-10)}$$

$$J_+(\alpha) = \Psi_+'(a+,\ \alpha) - \Psi_+'(a-,\ \alpha) \qquad \text{(A-11)}$$

and $G(\alpha) = I_o(\gamma a)\ K_o(\gamma a)$

$$= G_+(\alpha)\ G_-(\alpha) \qquad \text{(A-12)}$$

Equation (A-9) can be modified to the form:

$$-iJ_-(\alpha)(k-\alpha)G_-(\alpha) + \frac{(\omega\varepsilon/a)\ \Phi_1(a,\alpha)}{(k+\alpha)\ G_+(\alpha)} = -\frac{\xi_{om}}{\sqrt{2\pi}\ a} J_1(\xi_{om}) \frac{(k-\alpha)}{\alpha + i\gamma_{om}} G_-(\alpha)$$

$$+ i(k-\alpha)\ G_-(\alpha)e^{i\alpha L}\left[\frac{-i}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{e^{-\gamma_{om}L}}{\alpha + i\gamma_{om}} + J_+(\alpha)\right]$$

$$, \quad |\tau| < k_2 \qquad \text{(A-13)}$$

Decomposing the right hand side of (A-13), by isolating the pole in the first term and using a decomposition formula for the second term, one has

$$-\frac{1}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{(k-\alpha)}{\alpha + i\gamma_{om}} G_-(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{1}{\alpha+i\gamma_{om}}[-(k-\alpha)G_-(\alpha)$$

$$+ (k+i\gamma_{om})G_+(i\gamma_{om})] - \frac{1}{\sqrt{2\pi}} \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{(k + i\gamma_{om})}{\alpha + i\gamma_{om}} G_+(i\gamma_{om})$$

$$\text{(A-14)}$$

and

$$i(k-\alpha)G_-(\alpha)e^{i\alpha L}\ [J_+(\alpha) - \frac{i}{\sqrt{2\pi}}\frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{e^{-\gamma_{om}L}}{\alpha + i\gamma_{om}}] =$$

$$\frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{-i(k-\beta)G_-(\beta)\ M(\beta)\ e^{i\beta L}}{\beta - \alpha} + \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{-i(k-\beta)G_-(\beta)\ M(\beta)}{\beta - \alpha}\ e^{i\beta L}\ d\beta$$

$$, \quad -k_2 < -d < \tau < d < k_2 \qquad \text{(A-15)}$$

where

$$M(\alpha) = \frac{i}{\sqrt{2\pi}}\ \frac{\xi_{om}}{a} J_1(\xi_{om}) \frac{e^{-\gamma_{om}L}}{\alpha + i\gamma_{om}} - J_+(\alpha) \qquad \text{(A-16)}$$

The first and second terms in equations (A-14) and (A-15) are regular respectively in the lower $(\tau < k_2)$ and the upper $(\tau > -k_2)$ halves of the complex α-plane. Substituting (A-14) and (A-15) into (A-13), one obtains

$$-i\ J_-(\alpha)(k-\alpha)G_-(\alpha) + \frac{1}{\sqrt{2\pi}}\frac{\xi_{om}}{a}\frac{J_1(\xi_{om})}{\alpha + i\gamma_{om}}\ [(k-\alpha)G_-(\alpha) - (k+i\gamma_{om})G_+(i\gamma_{om})]$$

$$+ \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{-i(k-\beta)G_-(\beta)\ M(\beta)}{\beta - \alpha}\ e^{i\beta L}\ d\beta = \frac{(\omega\varepsilon/a)\ \Phi_1(a,\alpha)}{(k+\alpha)\ G_+(\alpha)} - \frac{i\xi_{om}}{a\sqrt{2\pi}} J_1(\gamma_{om})$$

$$- \frac{i(k+i\gamma_{om})}{\alpha + i\gamma_{om}} G_+(i\gamma_{om}) + \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{-i(k-\beta)G_-(\beta)\ M(\beta)\ e^{i\beta L}}{\beta - \alpha}\ d\beta \qquad \text{(A-17)}$$

Since H_ϕ and E_z satisfy the edge condition, it can be shown that both sides of (A-17) are zero and the equation is valid for all values of α. Equating the left-hand side of (A-17) to zero gives

$$\frac{-\xi_{om}}{a\sqrt{2\pi}} \frac{k+i\gamma_{om}}{\alpha+ i\gamma_{om}} J_1(\xi_{om})\ G_+(i\gamma_{om}) - i(k-\alpha)G_-(\alpha)\ N(\alpha)$$

$$= \frac{-1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{-i(k-\beta)\ G_-(\beta)\ M(\beta)\ e^{i\beta L}}{\beta - \alpha}\ d\beta$$

$$, -k_2 < -d < \tau < d < k_2$$

(A-18)

where

$$N(\alpha) = J_-(\alpha) + \frac{i\xi_{om}}{a\sqrt{2\pi}} \frac{J_1(\xi_{om})}{\alpha + i\gamma_{om}} \tag{A-19}$$

Multiplying both sides of (A-9) by $-i(k+\alpha)\ e^{-i\alpha L}\ G_+(\alpha)$ one obtains

$$-i(k+\alpha)\ J_+(\alpha)\ G_+(\alpha) - \frac{(\omega\varepsilon/a)e^{-i\alpha L}\ \Phi_1(a,\alpha)}{(k-\alpha)\ G_-(\alpha)} = \frac{\xi_{om}}{a\sqrt{2\pi}} J_1(\xi_{om}) \frac{(k+\alpha)e^{-\gamma_{om}L}}{\alpha+ i\gamma_{om}}\ G_+(\alpha)$$

$$+ ie^{-i\alpha L}\ N(\alpha)\ (k + \alpha)\ G_+(\alpha) \tag{A-20}$$

In this equation the only term that has singularities in both halves of the α-plane is the second term in the right-hand side which can be decomposed in the same manner as equation (A-15), i.e.,

$$-i\ e^{-i\alpha L}\ N(\alpha)(k + \alpha)G_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty-id}^{\infty-id} \frac{-i(k+\beta)G_+(\beta)\ N(\beta)}{\beta - \alpha}\ e^{-i\beta L}\ d\beta$$

$$- \frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{-i(k+\beta)G_+(\beta)\ N(\beta)\ e^{-i\beta L}}{\beta - \alpha}\ d\beta$$

$$, -k_2 < -d < \tau < d < k_2 \tag{A-21}$$

Substituting (A-21) into (A-20) one obtains

$$- \frac{(\omega\varepsilon/a)e^{-i\alpha L}\ \Phi_1(a,\alpha)}{(k-\alpha)G_-(\alpha)} - \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{-i(k+\beta)G_+(\beta)\ N(\beta)}{\beta-\alpha}\ e^{-i\beta L} = -i(k+\alpha)$$

$$G_+(\alpha)M(\alpha) \quad - \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{-i(k+\beta)\ G_+(\beta)\ N(\beta)}{\beta-\alpha}\ e^{-i\beta L}\ d\beta \qquad \text{(A-22)}$$

From the edge condition, both sides of (A-22) are zero and the right hand side gives

$$-i(k+\alpha)G_+(\alpha)M(\alpha) = \frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{-i(k+\beta)G_+(\beta)\ N(\beta)}{\beta-\alpha} e^{-i\beta L}\ d\beta$$

$$, \ -k_2 < -d < \tau < d < k_2 \qquad \text{(A-23)}$$

Equations (A-18) and (A-23) are two coupled integral equations for the two unknowns $N(\alpha)$ and $M(\alpha)$. These two equations can be decoupled by changing β to $-\beta$ in (A-18) and α to $-\alpha$ in (A-23). The sum and difference of these equations lead to the integral equation (3-7) where $S(\alpha) = N(\alpha) + M(-\alpha)$ and $D(\alpha) = N(\alpha) - M(-\alpha)$ which are given by equation (3-6). Finally equations (3-6) and (3-4) give $\Phi_1(a,\alpha)$ as shown in equation (3-5).

APPENDIX B

DERIVATION OF THE FUNCTION $T(\alpha)$ GIVEN BY (3-12)

The integral in the R.H.S. of (3-7) is of the form

$$I = \frac{\lambda}{2\pi i} \int_{-\infty-id}^{\infty-id} \frac{(\beta+k)\ I_o(\gamma a)\ K_o(\gamma a)\ E(\beta)}{G_-(\beta)(\beta+\alpha)}\ e^{-i\beta L}\ d\beta \tag{B-1}$$

where $G_+(\alpha)$ has been replaced by $G(\alpha)/G_-(\alpha)$ in equation (3-7) with $G(\alpha) = I_o(\gamma a)\ K_o(\gamma a)$.

For large L, the major contribution of I is from the integral over a small neighborhood around the branch point $\beta = -k$ [26]. The contour can be deformed into the lower half of the β-plane, as shown in figure 2-3. The functions $G_-(\beta)$ and $E(\beta)$ are then expanded in a Taylor series about the branch point $\beta = -k$ and retaining the first term only, this leads to

$$I = \frac{\lambda}{2\pi i} \frac{E(-k)}{G_-(-k)} \int_p \frac{(\beta+k)\ I_o(\gamma a)\ K_o(\gamma a)\ e^{-i\beta L}}{(\beta+\alpha)}\ d\beta \tag{B-2}$$

where $p = p_1 + p_2 + p_3$. The integral over the small circle p_2 can besshown to be zero and hence (B-2) can be simplified to

$$I = \frac{\lambda}{2\pi i} \frac{E(-k)}{G_+(k)} \int_{-k}^{-k-i\infty} \frac{(\beta+k)}{(\beta+\alpha)}\ e^{-i\beta L}\ [I_o(\gamma a)K_o(\gamma a) - I_o(-\gamma a)\ K_o(-\gamma a)]\ d\beta$$

$$= \frac{\lambda}{2\pi i} \frac{E(-k)}{G_+(k)}\ T(\alpha) \tag{B-3}$$

where

$$T(\alpha) = \int_{-k}^{-k-i\infty} \frac{(\beta+k)}{\beta+\alpha} e^{-i\beta L} [I_o(\gamma a)K_o(\gamma a) - I_o(-\gamma a)K_o(-\gamma a)] \, d\beta$$

$$= \pi i \int_{-k}^{-k-i\infty} \frac{(\beta+k) \; I_o^2(\gamma a)}{\beta+\alpha} e^{-i\beta L} \; d\beta \qquad \text{(B-4)}$$

Letting $\beta = -k - \frac{iu}{L}$, where u is a new variable, then (B-4) reduces to equation (3-12).

APPENDIX C

DERIVATION OF THE EQUATION (3-42)

When closing the contour of integration in equation (3-41), in the lower half of the α-plane, the first term in $\overline{\Phi}_1(a,\alpha)$ has a pole at $\alpha = -i\gamma_{om}$ and a branch point at $\alpha = -k$. The contribution due to the pole cancels exactly the incident field and the branch point contribution can be evaluated similar to $\psi_r^{int,(2)}(\rho,z)$.

$$\psi_t^{exc}(\rho,z) = \frac{-i\xi_{om}}{2\pi} J_1(\xi_{om})(k+i\gamma_{om}) G_+(i\gamma_{om}) \int_p \frac{I_o(\gamma\rho)}{I_o(\gamma a)} \frac{G_+(\alpha)}{(k-\alpha)(\alpha+i\gamma_{om})} e^{-i\alpha z} \, d\alpha \tag{C-1}$$

replacing $G_+(\alpha)$ by $G(\alpha)/G_-(\alpha)$ in (C-1), we obtain

$$\psi_t^{exc}(\rho,z) = \int_p \frac{I_o(\gamma\rho) K_o(\gamma a)}{G_-(\alpha)(k-\alpha)(\alpha+i\gamma_{om})} e^{-i\alpha z} \, d\alpha \tag{C-2}$$

Assuming

$$\psi_t^{exc}(\rho,z) = \sum_{n=1,2,3,\ldots}^{\infty} T_{m,n}(z) \, J_o\left(\frac{\xi_{on}}{a}\rho\right) \tag{C-3}$$

and using the orthagonality of the Bessel function in both sides of (C-2) and (C-3), we obtain

$$T_{m,n}(z) = \frac{2}{a} \frac{J_1(\xi_{om})}{J_1^2(\xi_{on})} \frac{-i\xi_{om}}{2\pi} (k+i\gamma_{om}) G_+(i\gamma_{om})$$

$$\int_p \frac{K_o(\gamma a) \, e^{-i\alpha z}}{(k-\alpha) \, G_-(\alpha)(\alpha+i\gamma_{om})} \int_o^a J_o(i\gamma a) \, \rho \, J_o\left(\frac{\xi_{on}}{\alpha}\rho\right) d\rho d\alpha$$

$$= \frac{-i}{\pi a^2} \xi_{on} \, \xi_{om} \frac{J_1(\xi_{om})}{J_1(\xi_{on})} (k+i\gamma_{om}) \, G_+(i\gamma_{om})$$

$$\int_{p} \frac{K_o(\gamma a)\ I_o(\gamma a) \quad e^{-i\alpha z}}{G_-(\alpha)(k-\alpha)(\alpha+i\gamma_{om})(\alpha^2+\gamma_{on}^2)}\ d\alpha \tag{C-4}$$

where $p = p_1 + p_2 + p_3$. The integral over p_2 can be shown to be zero and hence (C-4) reduces to

$$T_{m,n}(z) = \frac{\xi_{on}}{a}\frac{\xi_{om}}{a}\frac{J_1(\xi_{om})}{J_1(\xi_{on})}\ (k+i\gamma_{om})\ G_+(i\gamma_{om})$$

$$\int_{-k}^{-k-i\infty} \frac{I_o^2(\gamma a)}{G_-(\alpha)(k-\alpha)(\alpha+i\gamma_{om})(\alpha^2+\gamma_{on}^2)}\ e^{-i\alpha z}\ d\alpha \tag{C-5}$$

Letting $\alpha = -k - \frac{iu}{z}$ where u is a new variable of integration, then equation (C-5) reduces to (3-43).

APPENDIX D

DERIVATION OF THE EQUATION (4-3)

After deforming the contour of the integral in equation (4-2), we obtain

$$T_n(\alpha) = \frac{1}{\varepsilon_n} \int_p \frac{(-2\gamma a)^n}{n!\,\gamma a} \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta \tag{D-1}$$

where $p = p_1 + p_2 + p_3$. The integral over the small circle p_2 can be shown to be zero and hence (D-1) can be simplified to

$$T_n(\alpha) = \frac{(-2)^n\, a^{n-1}}{n!\,\varepsilon_n} \int_{-k}^{-k-i\infty} \frac{e^{-i\beta L}}{\beta+\alpha}\, \gamma^{n-1}\, [1 - (-1)^{n-1}]\, d\beta \tag{D-2}$$

for odd values of n $(n = 1,3,5,\ldots)$, $T(\alpha)$ is zero and for other values of n , equation (D-2) becomes

$$T_n(\alpha) = \frac{(-1)^n\, 2^{n+1}\, a^{n-1}}{n!\,\varepsilon_n} \int_{-k}^{-k-i\infty} \gamma^{n-1}\, \frac{e^{-i\beta L}}{\beta+\alpha}\, d\beta\ ,\ n=0,2,4,\ldots \tag{D-3}$$

In the neighbourhood of $\beta = -k$, the function $(\beta-k)^{n-1/2}$ is regular and smooth, and can be replaced by $(-2k)^{n-1/2}$ and equation (D-3) leads to (4-3a).

APPENDIX E

DERIVATION OF EQUATIONS (5-5), (5-6) AND (5-7)

Since for many practical purposes, the evanescent modes are not taken into account, all scattering matrices reduce to a unity matrix. Hence, from figure 5-2, $S_1^{AA}\,\phi^i$ is approximated by

$$S_1^{AA}\,\phi^i = f(u_1,u_2)\,e^{\gamma_m z}\,R \tag{E-1}$$

Also, S_3^{CC} will represent the reflection coefficient $R_o\,e^{-\gamma_m \ell_2}$ of a single semi-infinite waveguide, while S_2^{CC} will represent the reflection coefficient $R\,e^{-\gamma_m \ell_2}$ of the two semi-infinite waveguides separated by the distance L. Also, S_2^{CB} will correspond to the transmission coefficient $T\,e^{-\gamma_m L}$ of two semi-infinite waveguides separated by the distance L. Combining these with equation (5-2) we obtain

Reflected field inside the exciting waveguide =

$$f(u_1,u_2)\,e^{\gamma_m z}\,R + \frac{T^2 R_o\,e^{-2\gamma_m (L+\ell_2)}}{1 - R\,R_o\,e^{-2\gamma_m \ell_2}}\,f(u_1,u_2)\,e^{\gamma_m z}$$

$$= f(u_1,u_2)\,e^{\gamma_m z}\left[R + \frac{T^2\,R_o\,e^{-2\gamma_m (L+\ell_2)}}{1 - R\,R_o\,e^{-2\gamma_m \ell_2}}\right] \tag{E-2}$$

which is the same as (5-5). However, with similar arguments, (5-6) and (5-7) may be obtained.

APPENDIX F

CONFORMAL TRANSFORMATION AND REGULARITY OF J_y in DMM

In the conformal transformation of the region outside the scatterers, in the $W = Kz + iKx$ plane to the region outside two circles in the $t = \theta + i\beta$ plane, the coordinates β = constant and θ = constant constitute an orthogonal coordinate system with a metric coefficient

$$h = |dW/dt| \tag{F-1}$$

and the coordinate $\beta = 0$ is the cross-sectional contour of the scatterers [96]. The transformed geometry, the two circles, in the t-plane has a uniform curvature. The behaviour of the metric coefficient h is directly related to the curvature of the original geometry in the W-plane. Thus the behaviour of the current distribution I_y being dependent on the surface curvature is related to the behaviour of the metric coefficient h. Shafai [88] has shown that the behaviour of the singular components of the field is described by the reciprocal of h. Hence, the induced current in the transform domain $J_y = hI_y$ is regular and independent of the curvature of the scattering surface. From the relation

$$d(Kr') = \frac{(dKx^2 + dKz^2)^{1/2}}{d\theta} d\theta = h(\theta)\ d\theta, \tag{F-2}$$

equation (7-3) leads to (7-4). The transformation which maps the region outside the upper or lower wall to the region outside a circular cylinder is given by [88]

$$W = \frac{1}{2} KL_2 \cos(t) + (KL + \frac{1}{2} KL_2) \pm i\ Ka \tag{F-3}$$

where the positive and negative signs correspond respectively to the upper and lower plate. Hence the coordinates z and x of the upper and lower walls on the cross-sectional contours are related to θ through the relations

$$z = \frac{1}{2} KL_2 \cos\theta + (KL + \frac{1}{2} KL_2), \quad x = Ka \quad \text{for the upper wall} \tag{F-4a}$$

$$z = \frac{1}{2} KL_2 \cos\theta + (KL + \frac{1}{2} KL_2), \quad x = -Ka \quad \text{for the lower wall} \tag{F-4b}$$

i.e $$K\bar{r}' = \frac{1}{2} KL_2 \cos\theta + (KL + \frac{1}{2} KL_2) + iKa \tag{F-5a}$$

$$K\bar{r}'' = \frac{1}{2} KL_2 \cos\theta + (KL + \frac{1}{2} KL_2) - iKa \tag{F-5b}$$

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