FREE LATTICES GENERATED BY

PARTIALLY ORDERED SETS

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ABSTRACT

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Let P be a partially ordered set. FL(P) is the free lattice generated by P preserving existing binary join and meet in P . A quasi-order is constructed on the lattice polynomials over P in terms of certain ideals and dual ideals of P ; under the induced partial order the quotient set is the lattice FL(P) . A characterization of FL(P) is derived and a generalization of a theorem of Sorkin on the extension of isotone maps is proved.

$\mathfrak{M}$ and $\mathfrak{N}$ are families of finite subsets of P such that every element of $\mathfrak{M}$ has a least upper bound in P and every element of $\mathfrak{N}$ has a greatest lower bound in P . FL(P ; $\mathfrak{M}$, $\mathfrak{N}$) , a generalization of FL(P) , is the free lattice generated by P preserving joins of elements of $\mathfrak{M}$ and meets of elements of $\mathfrak{N}$. The results on FL(P) are extended to FL(P ; $\mathfrak{M}$, $\mathfrak{N}$) . As an application certain results of R.A. Dean on completely free lattices are derived.

The results concerning FL(P) and FL(P ; $\mathfrak{M}$, $\mathfrak{N}$) are applied to solve the word problem for free products of lattices, partially ordered free products of lattices, and

amalgamated free products where the amalgamated sublattice is of finite length.

A lattice L generated by a partially ordered set P is said to admit canonical representations if every element of L can be represented by a polynomial over P of shortest length which is unique up to commutativity and associativity. Those free products and partially ordered free products of lattices that admit canonical representations are characterized.

PREFACE

The construction of free universal algebras was first accomplished by G. Birkhoff (see [7], Chapter 4). Birkhoff's construction essentially defines, in a highly non-effective manner, a congruence relation on the algebra of polynomials over the generating set. Whitman [10] analysed the structure of the free lattice generated by a totally unordered set; he defined a quasi-order on the lattice polynomials in an effective manner and thus was able to give an effective construction of the congruence relation yielding the free lattice. Among his conclusions was the result that every element of the free lattice can be represented by a polynomial of shortest length which is unique up to commutativity and associativity, that is, that the free lattice admits canonical representations.

In his analysis of the problem of embedding lattices in complemented lattices, Dilworth [4] had occasion to discuss the lattices $FL(P)$ and $CF(P)$ generated by a partially ordered set P. To construct $FL(P)$ he defined a quasi-order on the lattice polynomials in an inductive manner, which entailed knowing the quasi-order on a lower level on the polynomials and not just on P; thus effectiveness was lost. Of more import to our work, he had occasion to introduce certain elements of the generating set, the lower and upper covers, in a very special situation. These covers were further exploited by Chen and Grätzer [2] in their extension of Dilworth's results on embedding lattices in complemented lattices.

In [3] Dean completely analysed CF(P) and, generalizing Whitman's method, he showed that CF(P) admits canonical representations.

In [3a] Dean constructed FL(P) and a generalization $FL(P ; \mathcal{M}, \mathcal{N})$, the free lattice preserving more general sup and inf in P, in terms of certain ideals and dual ideals of P. Our construction of these lattices is essentially that of Dean; however, we use the results of Theorems 7 and 9 of Dean's paper to <u>define</u> the quasi-order that yields these lattices. This approach adheres more closely to the "cover" approach mentioned above; covers are now ideals and dual ideals of P, rather than elements.

For didactic reasons it was thought best to divide this work into two parts--one treating FL(P) and the other treating $FL(P ; \mathcal{M}, \mathcal{N})$. Consequently all sections relating to $(\mathcal{M}, \mathcal{N})$ concepts are marked with an asterisk (*); it is recommended that the reader omit these sections at first reading.

A special case of FL(P) is the free product of lattices. Sorkin [9] discussed free products of lattices and solved the problem of which are finite sets. One of the tools he used was the fact that isotone maps--and not necessarily lattice homomorphisms--from the factors to a lattice can be extended to an isotone map from the free product to the lattice. He also presented an example showing that the free product does not always admit canonical representations.

In Chapter I of this work we present the basic material needed in the analysis of FL(P) and FL(P ; $\mathcal{M}, \mathcal{N}$) ; most of these results are well-known, although assignment of specific references can be rather difficult.

In Chapter II we present the construction of FL(P) essentially due to Dean [3a]. We characterize FL(P) and, as an application of these methods, we present a generalization of Sorkin's theorem on the extension of isotone maps. In Chapter III these results are extended to FL(P ; $\mathcal{M}, \mathcal{N}$) . We apply these results to derive the results of Dean [3] concerning CF(P) .

In Chapter IV the results of Chapters II and III are specialized to free products, amalgamated free products, and the concept we chose to call partially ordered free products. In these cases the upper and lower covers reduce to <u>elements</u> of the factors.

Chapter VI summarizes the results of this work in the context of the "word problem".

And now a word on notation: The lattice-theoretic notation is explained in the text. Set-theoretic notation is standard; we need only mention that set union and intersection are denoted by $\bigcup$, $\bigcap$ while $\smile$, $\frown$ are used for "join" and "meet" of lattice polynomials. Set difference is denoted $A - B$. The symbol $\subseteq$ is used both for the concept "subset" and for the quasi-order defined on the lattice polynomials; there will be no danger of confusion. Maps are written on the left; thus fg is g followed by f.

The theorems and definitions are numbered consecutively in each chapter. In referring to a theorem, definition, or section the chapter number is given only if that theorem, definition, or section is in a chapter other than the one in which the reference is made. Thus, for example, "Lemma 5" refers to Lemma 5 of that chapter while "Definition 3.2" refers to Definition 2 of Chapter III.

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CHAPTER I

INTRODUCTORY CONCEPTS

1. Posets and lattices.

A relational system $\langle P ; \leqslant \rangle$ with a binary relation $\leqslant$ is said to be a partially ordered set (poset) if the following three properties hold:

1) Reflexive, for all $x \in P$, $x \leqslant x$;

2) Antisymmetric, for all $x, y \in P$, if $x \leqslant y$ and $y \leqslant x$ then $x = y$;

3) Transitive, for all $x, y, z \in P$, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

The relation $\leqslant$ is said to be a partial order. The statement $x \leqslant y$ is often also written $y \geqslant x$. If $x \leqslant y$ and $x \neq y$ we write $x < y$.

If, in addition, the property

4) for all $x, y \in P$ either $x \leqslant y$ or $y \leqslant x$;

holds then the order $\leqslant$ is said to be a total order and P is said to be a chain.

A binary relation $\subseteq$ that is reflexive and transitive, but not necessarily antisymmetric, is said to be a quasi-order. Lemma 1 on p. 21 of [1] states:

1. Lemma. In any quasi-ordered set $\langle S ; \subseteq \rangle$ define $x \sim y$ when $x \subseteq y$ and $y \subseteq x$. Then:

(i) $\sim$ is an equivalence relation on S ;

(ii) if E and F are two equivalence classes for $\sim$, then $x \subseteq y$ either for no $x \in E$, $y \in F$ or for all $x \in E$, $y \in F$;

(iii) the quotient set $S/\sim$ is a poset if $E \leqslant F$ is defined to mean that $x \subseteq y$ for some $x \in E$, $y \in F$.

If $\langle P ; \leqslant \rangle$ is a poset, A a subset of P, and $x \in P$, then x is said to be an <u>upper bound</u> of A if $x \geqslant a$ for all $a \in A$. x is said to be the <u>least upper bound</u> of A, denoted by $\sup A$ (or $\sup_P A$ if the poset P is to be stressed) if

(i) x is an upper bound of A ;

(ii) if y is an upper bound of A then $x \leqslant y$.

The concepts of <u>lower bound</u> and <u>greatest lower bound</u>, denoted $\inf A$, are dual to the above. It is clear from the definition that if $\sup A$ (and dually $\inf A$) exists it is unique.

A <u>lattice</u> L is a poset in which every set consisting of a pair of elements has a least upper bound and a greatest lower bound. We denote $\sup \{x, y\}$ by $x \vee y$, "join", and $\inf \{x, y\}$ by $x \wedge y$, "meet". It is to be stressed that in this work $\vee$ and $\wedge$ are used only in a lattice, i.e. when all pairs have a sup and an inf.

A lattice L can also be thought of as an algebra with two binary operations, $\vee$ and $\wedge$, satisfying:

1) for all $x \in L$, $x \vee x = x \wedge x = x$;

2) for all $x, y \in L$, $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;

3) for all $x, y, z \in L$, $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

4) for all $x, y \in L$, $x \vee (x \wedge y) = x \wedge (x \vee y) = x$.

The connection between these two approaches to a lattice is provided by:

$x \leqslant y$ is equivalent to $x \vee y = y$ which is equivalent to $x \wedge y = x$ ([1] p. 8).

A lattice L may or may not have a greatest element and a least element; if L has both a greatest and a least element it is said to be <u>bounded</u>. Any lattice L may be embedded in a bounded lattice L^b. To construct L^b from L we adjoin two symbols 0 and 1 to L ; $L^b = L \cup \{0, 1\}$. The partial order on L^b is that defined on L along with the requirement that $1 \geqslant x$ for all $x \in L^b$ and $0 \leqslant x$ for all $x \in L^b$. Thus, for all $x \in L^b$, $1 \vee x = 1$, $1 \wedge x = x$, $0 \vee x = x$, $0 \wedge x = 0$.
We should like to point out that even if L is bounded L^b consists of two more elements than L ; the reason for this approach is to preserve a degree of effectiveness in certain constructions, because for an arbitrary lattice L there is no algorithm to decide whether or

not L is bounded.

If L is a poset such that <u>all</u> subsets have a sup and an inf then L is said to be a <u>complete lattice</u>. By the method of "completion by cuts", [1] p. 126:

2. <u>Lemma</u>. Any lattice L can be embedded as a sublattice of a complete lattice L*.

Finally we mention a metatheorem, the principle of duality:

<u>Principle of duality</u>. Any theorem about a poset remains true if $\leqslant$ is replaced by $\geqslant$ and sup and inf are interchanged.

The principle of duality will be referred to rather frequently in the sequel in order to reduce the length of proofs.

2. <u>Ideals, homomorphisms, hereditary sets, and isotone maps</u>.

Let $\langle P ; \leqslant \rangle$ be a poset.

3. <u>Definition</u>. a) A subset I of P is said to be an <u>ideal</u> of P if:

(i) $x \in I$ and $y \leqslant x$ imply $y \in I$;

(ii) $x, y \in I$ and $\sup \{x, y\}$ exists imply $\sup \{x, y\} \in I$.

b) A subset D of P is said to be a <u>dual ideal</u> of P if:

(i) $x \in D$ and $y \geqslant x$ imply $y \in D$;

(ii) $x, y \in D$ and $\inf \{x, y\}$ exists imply $\inf \{x, y\} \in D$.

We observe that P itself is always an ideal and a dual ideal. Also, since the empty set $\emptyset$ satisfies the conditions vacuously, it will also be considered to be an ideal and a dual ideal even if P is a lattice. For lattices this differs from the usual convention.

4. <u>Lemma</u>. If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of ideals (resp. dual ideals) of P then $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an ideal (resp. dual ideal) of P.

<u>Proof</u>: We prove the lemma for the case of ideals and invoke the principle of duality to establish the result for dual ideals.

(i) Let $x \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$ and let $y \leqslant x$. For each $\lambda \in \Lambda$ $x \in I_\lambda$ and thus $y \in I_\lambda$. Consequently $y \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

(ii) Let $x, y \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$ and let $\sup \{x, y\}$ exist. Then, for each $\lambda \in \Lambda$, $x, y \in I_\lambda$ and so $\sup \{x, y\} \in I_\lambda$. Thus $\sup \{x, y\} \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

Thus $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an ideal and the lemma is established.

Lemma 4 implies that the ideals and dual ideals of P form closure systems (Moore families, in the terminology of [1]). Thus ([1] p. 112):

5. Lemma. The ideals (resp. dual ideals) of a poset P form a complete lattice under set inclusion . If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of ideals (resp. dual ideals) then $\inf (I_\lambda \mid \lambda \in \Lambda) = \bigcap (I_\lambda \mid \lambda \in \Lambda)$ and $\sup (I_\lambda \mid \lambda \in \Lambda) = \bigcap (J \mid J$ is an ideal (dual ideal) and $\bigcup_\lambda I_\lambda \subseteq J)$.

6. Definition. a) For each $x \in P$ the set $(x] = \{y \mid y \leqslant x\}$ is said to be the principal ideal generated by x.

b) For each $x \in P$ the set $[x) = \{y \mid y \geqslant x\}$ is said to be the principal dual ideal generated by x.

To justify this definition it should be noted that:

7. Lemma. $(x]$ is an ideal of P and is the set intersection of all ideals containing x; $[x)$ is a dual ideal of P and is the set intersection of all dual ideals containing x.

Proof: By condition a)(i) of Definition 3 every ideal of P containing x includes $(x]$.

(i) Let $y \in (x]$, i.e. $y \leqslant x$, and let $z \leqslant y$. Then $z \leqslant x$, and so $z \in (x]$.

(ii) Let $y, z \in (x]$ and let $\sup\{y, z\}$ exist. Since $y, z \leqslant x$ then $\sup\{y, z\} \leqslant x$. Thus $\sup\{y, z\} \in (x]$.

Consequently the lemma is established for $(x]$ and, by duality, the lemma also holds for $[x)$.

8. Definition. An ideal (resp. dual ideal) I is said to be a pseudo-principal ideal (resp. dual ideal) of P if it can be obtained by taking a finite sequence of binary joins and meets of principal ideals (resp. principal dual ideals).

The pseudo-principal ideals (resp. pseudo-principal dual ideals) are a sublattice of the lattice of all ideals (resp. dual ideals) and could be described as the sublattice generated by the principal ideals (resp. principal dual ideals).

The concepts of ideal and dual ideal can be weakened by requiring only that order be preserved. This leads to the concept of hereditary sets.

9. Definition. A subset I of P is said to be a hereditary subset (resp. dual hereditary subset) of P if $x \in I$, $y \leqslant x$ (resp. $y \geqslant x$) imply $y \in I$ for all $x, y \in P$.

Every ideal of P is clearly a hereditary subset, and dually. The set $(x]$ is the smallest hereditary subset containing x and $[x)$ is the smallest dual hereditary subset containing x. The families of hereditary subsets and dual hereditary subsets of P are again lattices, but in this case

$$\sup (I_\lambda | \lambda \in \Lambda) = \bigcup (I_\lambda | \lambda \in \Lambda)$$

whenever $(I_\lambda | \lambda \in \Lambda)$ is a family of hereditary (resp. dual hereditary) subsets of P. Thus the sets of hereditary and dual hereditary subsets of P are distributive lattices, and, indeed, sublattices of the lattice of all subsets of P. We recall that a distributive lattice is one where one, and hence both, of the following properties hold:

1) for all $x, y, z \in L$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

2) for all $x, y, z \in L$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

As in the case of ideals, one can define _pseudo-principal_ hereditary and dual hereditary sets. A case of interest in the sequel is when $\sup \{x, y\}$ and $\inf \{x, y\}$ exist only when x and y are _comparable_, i.e. when either $x \leqslant y$ or $y \leqslant x$. In this case the concepts of ideal and hereditary set coincide, and dually.

10. _Definition_. A mapping $f : P \to L$, where L is a lattice, is said to be a _homomorphism_ if

(i) $x, y \in P$ and $\sup \{x, y\}$ exists imply

$$f(\sup \{x, y\}) = f(x) \vee f(y) ;$$

and

(ii) $x, y \in P$ and $\inf \{x, y\}$ exists imply

$$f(\inf \{x, y\}) = f(x) \wedge f(y) .$$

If P is also a lattice this concept agrees with that of "lattice homomorphism".

A weaker situation is:

11. *Definition*. A mapping $f : P \to L$, L a lattice, is said to be an *isotone map* if $x, y \in P$, $x \leqslant y$ imply $f(x) \leqslant f(y)$.

Of course, the concept of "isotone map" is meaningful even if L is a poset, not necessarily a lattice.

3. Free lattices.

Lattices of various degrees of freeness were discussed by Whitman [10] and [11], Dilworth [4], and Dean [3], [3a].

12. *Definition*. (Whitman [10]). The *free lattice on $\mathfrak{m}$ generators* consists of a set X of cardinality $\mathfrak{m}$, a lattice denoted $FL(\mathfrak{m})$, and a set injection $\varphi : X \to FL(\mathfrak{m})$ such that

(i) $\varphi(X)$ generates $FL(\mathfrak{m})$;

(ii) if L is a lattice and f_o is a set mapping $f_o : X \to L$ then there is a lattice homomorphism

$f : FL(\mathfrak{m}) \longrightarrow L$ such that $f\varphi = f_0$.

13. <u>Definition</u> (Dilworth [4]). The <u>free lattice generated by a poset</u> P consists of a lattice denoted $FL(P)$ and an injective homomorphism $\varphi : P \to FL(P)$ such that

(i) $\varphi(P)$ generates $FL(P)$;

(ii) given any lattice L and homomorphism $f_0 : P \to L$ there is a lattice homomorphism $f : FL(P) \longrightarrow L$ such that $f\varphi = f_0$.

14. <u>Definition</u> (Dilworth [4]). The <u>completely free lattice generated by a poset</u> P consists of a lattice denoted $CF(P)$ and an isotone injection $\varphi : P \to CF(P)$ such that

(i) $\varphi(P)$ generates $CF(P)$;

(ii) given a lattice L and an isotone map $f_0 : P \to L$ there is a lattice homomorphism $f : CF(P) \longrightarrow L$ such that $f\varphi = f_0$.

Certain special cases of $FL(P)$ wil be discussed in the sequel. Let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices indexed by a set Λ. Then $\bigcup(L_\lambda \mid \lambda \in \Lambda)$ can be regarded as a poset P where $\sup\{x, y\}$ and $\inf\{x, y\}$ exist if and only if $x, y \in L_\lambda$ for some $\lambda \in \Lambda$; thus $x \leqslant y$ if and only if $x, y \in L_\lambda$ for some $\lambda \in \Lambda$ and $x \leqslant y$ in that L_λ. In this case $FL(P)$ is said to be the free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$. An alternative definition is:

15. Definition. If $(L_\lambda \mid \lambda \in \Lambda)$ is an indexed family of lattices then the free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$ consists of a lattice L and an indexed family $(\varphi_\lambda \mid \lambda \in \Lambda)$ of lattice injections $\varphi_\lambda : L_\lambda \to L$ such that

(i) $\bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda)$ generates L ;

(ii) given any lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$, $\lambda \in \Lambda$, there is a lattice homomorphism $f : L \to L'$ such for each $\lambda \in \Lambda$ $f\varphi_\lambda = f_\lambda$.

The concept of free product of lattices can be extended in two directions. The first is:

16. Definition. Let $(L_\lambda \mid \lambda \in \Lambda)$ be an indexed family of lattices, let M be a lattice, and for each $\lambda \in \Lambda$ let $\psi_\lambda : M \to L_\lambda$ be a lattice injection. The amalgamated free product of the $(L_\lambda \mid \lambda \in \Lambda)$ over M consists of a lattice L and lattice injections $\varphi_\lambda : L_\lambda \to L$ such that for each $\lambda, \mu \in \Lambda$ $\varphi_\lambda \psi_\lambda = \varphi_\mu \psi_\mu$, satisfying:

(i) $\bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda)$ generates L ;

(ii) given any lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$ such that for $\lambda, \mu \in \Lambda$ $f_\lambda \psi_\lambda = f_\mu \psi_\mu$ then there is a lattice homomorphism $f : L \to L'$ such that $f\varphi_\lambda = f_\lambda$.

The concept of amalgamated free product, as well as that of free product, is quite general and properly belongs to the field of universal algebra. An alternative generalization of the concept of free product is peculiar to lattice theory. Let the indexing set Λ be a <u>poset</u>. Let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$. A partial order $\leqslant$ is defined on P by:

(i) if $x, y \in L_\lambda$, $\lambda \in \Lambda$, then $x \leqslant y$ if and only if $x \leqslant y$ in L_λ ;

(ii) if $\lambda \neq \mu$, $x \in L_\lambda$ and $y \in L_\mu$ then $x \leqslant y$ if and only if $\lambda < \mu$.

However, if x, y are incomparable $\sup\{x, y\}$ and $\inf\{x, y\}$ wil be considered only if x and y are in the same lattice. With this restriction on sup and inf $FL(P)$ is called the <u>partially ordered free product</u> of the $(L_\lambda \mid \lambda \in \Lambda)$.

<u>Example 1.</u> Let Λ be the poset depicted in Fig. 1. Let L_c have a smallest element w. Then if $x \in L_a$, $y \in L_b$ w is the least upper bound of x and y. However, by the above convention, $\sup\{x, y\}$ will not exist.

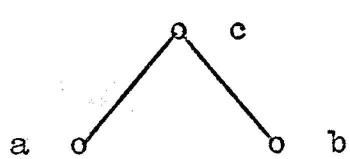

Fig. 1

An alternative approach to partially ordered free products is provided by:

17. Definition. Let Λ be a poset and $(L_\lambda \mid \lambda \in \Lambda)$ be a family of lattices indexed by Λ. The partially ordered free product of the $(L_\lambda \mid \lambda \in \Lambda)$ consists of a lattice L and lattice injections $\varphi_\lambda : L_\lambda \to L$ such that if $\lambda \leqslant \mu$ $\varphi_\lambda(x) \leqslant \varphi_\mu(y)$ for all $x \in L_\lambda$, $y \in L_\mu$, satisfying:

(i) $\bigcup(L_\lambda \mid \lambda \in \Lambda)$ generates L ;

(ii) given a lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$ such that if $\lambda < \mu$ $f_\lambda(x) \leqslant f_\mu(y)$ for all $x \in L_\lambda$, $y \in L_\mu$, then there is a lattice homomorphism $f : L \to L'$ such that $f\varphi_\lambda = f_\lambda$ for all $\lambda \in \Lambda$.

*4. $(\mathcal{M},\mathcal{N})$-structures.

The concepts of $\mathcal{M}$-ideals, $\mathcal{N}$-dual ideals, $(\mathcal{M},\mathcal{N})$-morphisms, and $FL(P; \mathcal{M},\mathcal{N})$ serve both as unifying concepts and generalizations of the ideas outlined in Sections 2 and 3. (See Dean [3a].)

18. Definition. Let $\langle P ; \leqslant \rangle$ be a poset. An $(\mathcal{M},\mathcal{N})$-structure on P consists of two families $\mathcal{M}$, $\mathcal{N}$ of finite non-empty subsets of P such that

(i) $A \in \mathcal{M}$ implies that $\sup A$ exists ;

and

(ii) $A \in \mathcal{N}$ implies that $\inf A$ exists.

A generalization of both ideal and hereditary subset is provided by:

19. Definition. a) A subset I of P is said to be an $\mathcal{M}$-ideal of P if

(i) $x \in I$ and $y \leqslant x$ imply $y \in I$;

(ii) $A \subseteq I$ and $A \in \mathcal{M}$ imply $\sup A \in I$.

b) A subset D of P is said to be an $\mathcal{N}$-dual ideal of P if

(i) $x \in D$ and $y \geqslant x$ imply $y \in D$;

(ii) $A \subseteq D$ and $A \in \mathcal{N}$ imply $\inf A \in D$.

If $\mathcal{M}$ consists of all pairs $x, y \in P$ such that $\sup \{x, y\}$ exists and $\mathcal{N}$ consists of all pairs $x, y \in P$ such that $\inf \{x, y\}$ exists then the concept of $\mathcal{M}$-ideal (resp. $\mathcal{N}$-dual ideal) agrees with that of ideal (resp. dual ideal). If $\mathcal{M} = \mathcal{N} = \emptyset$ then $\mathcal{M}$-ideals are hereditary subsets, and dually.

As for ideals we find that:

20. Lemma. If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of $\mathcal{M}$-ideals (resp. $\mathcal{N}$-dual ideals) of P then $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an $\mathcal{M}$-ideal (resp. $\mathcal{N}$-dual ideal).

Proof: The proof of a) is presented; that of b) follows by duality.

That $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is hereditary follows exactly as in Lemma 4. Let $A \in \mathcal{M}$ and $A \subseteq \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

Then for each $\lambda \in \Lambda$ $A \subseteq I_\lambda$, and so $\sup A \in I_\lambda$. Thus $\sup A \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

Thus $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an $\mathcal{M}$-ideal of P.

Thus the $\mathcal{M}$-ideals and $\mathcal{N}$-dual ideals form closure systems and the analogue of Lemma 5 holds.

Of interest in the sequel is:

21. Lemma. Let $\mathcal{M}, \mathcal{N}$ consist only of chains. If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of $\mathcal{M}$-ideals (resp. $\mathcal{N}$-dual ideals) then

$$\sup (I_\lambda \mid \lambda \in \Lambda) = \bigcup(I_\lambda \mid \lambda \in \Lambda).$$

Proof: This is clear since in this event $\mathcal{M}$-ideals are identical with hereditary subsets, and dually.

As in the case of ideals, $(x]$ is an $\mathcal{M}$-ideal, the smallest $\mathcal{M}$-ideal containing x, and dually for $[x)$. Pseudo-principal $\mathcal{M}$-ideals and pseudo-principal $\mathcal{N}$-dual ideals are defined in the obvious manner.

The concepts of isotone map and homomorphism can be integrated and generalized in the same manner. Let P be a poset and let L be a lattice.

22. Definition. A map $f : P \to L$ is said to be an $(\mathcal{M}, \mathcal{N})$-morphism if the following three properties are satisfied:

(i) f is isotone;

(ii) $A \in \mathcal{M}$ implies that $\bigvee f(A) = f(\sup A)$;

(iii) $A \in \mathcal{N}$ implies that $\bigwedge f(A) = f(\inf A)$.

If $\mathcal{M} = \mathcal{N} = \emptyset$ then an $(\mathcal{M}, \mathcal{N})$-morphism is an isotone map. If $\mathcal{M}$ consists of all pairs in P with a sup and $\mathcal{N}$ consists of all pairs in P with an inf then an $(\mathcal{M}, \mathcal{N})$-morphism is a homomorphism.

The various free lattices of Section 3 can be considered as special cases of $FL(P ; \mathcal{M}, \mathcal{N})$:

23. Definition. The $(\mathcal{M}, \mathcal{N})$-free lattice generated by a poset P with an $(\mathcal{M}, \mathcal{N})$-structure consists of a lattice $FL(P ; \mathcal{M}, \mathcal{N})$ and an $(\mathcal{M}, \mathcal{N})$-injection

$$\varphi : P \rightarrow FL(P ; \mathcal{M}, \mathcal{N})$$

such that

(i) $\varphi(P)$ generates $FL(P ; \mathcal{M}, \mathcal{N})$;

(ii) given a lattice L and an $(\mathcal{M}, \mathcal{N})$-morphism $f_0 : P \rightarrow L$, there is a lattice homomorphism $f : FL(P ; \mathcal{M}, \mathcal{N}) \rightarrow L$ such that $f\varphi = f_0$.

If $\mathcal{M} = \mathcal{N} = \emptyset$ then $FL(P ; \mathcal{M}, \mathcal{N}) = CF(P)$. If $\mathcal{M}$ consists of all pairs in P with a sup and $\mathcal{N}$ of all pairs in P with an inf then $FL(P ; \mathcal{M}, \mathcal{N}) = FL(P)$.

We should stress that $FL(P ; \mathcal{M}, \mathcal{N})$ is more than just a unification of the concepts of $FL(P)$ and $CF(P)$;

both partially ordered free products and amalgamated free products are forms of neither FL(P) nor CF(P) and yet, as will be evident in Chapter IV, they are special cases of $FL(P\,;\mathcal{M},\mathcal{N})$. Nor need $\mathcal{M}$, $\mathcal{N}$ consist solely of pairs:

Example 2. Let P be the lattice depicted in Fig. 2. Then $FL(P) = P$ and $CF(P) = FL(3)^b$. If $\mathcal{N} = \emptyset$ and $\mathcal{M}$ consists of exactly the set $\{a, b, c\}$ then $FL(P\,;\mathcal{M},\mathcal{N})$ is the quotient lattice of $FL(3)^b$ under the lattice congruence $\Theta(1, a \vee b \vee c)$ where a, b, c generate FL(3). $\Theta(x, y)$ denotes the smallest congruence relation identifying the elements x, y.

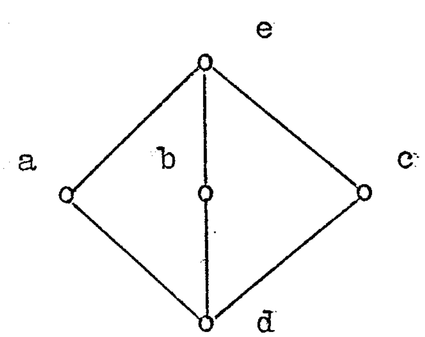

Fig. 2

We close this chapter by observing that all of the concepts presented here can also be defined if $\mathcal{M}$, $\mathcal{N}$ include infinite subsets of P. We exclude this case from our discussion because it would cause difficulties in the construction of $FL(P\,;\mathcal{M},\mathcal{N})$.

CHAPTER II

FL(P)

In this chapter the construction of FL(P) due to Dean [3a] is presented. FL(P) is characterized and a generalization of a result of Sorkin [9] on the extension of isotone maps is proved.

1. Lattice polynomials and covers.

Let X be a set. A lattice polynomial is an expression involving elements of X and two binary operation symbols $\smile$ and $\frown$. Each lattice polynomial A is assigned a length $\ell(A)$, the number of occurences of elements of X in the polynomial. A technical definition of a lattice polynomial is presented by mathematical induction on the length:

1. Definition. (i) If $x \in X$ then x is a polynomial of length 1, that is, $\ell(x) = 1$.

(ii) If A and B are polynomials of length ℓ_1, ℓ_2 then $A \smile B$ is a polynomial of length $\ell_1 + \ell_2$, i.e. $\ell(A \smile B) = \ell(A) + \ell(B)$.

(iii) If A and B are polynomials of length ℓ_1, ℓ_2 then $A \frown B$ is a polynomial of length $\ell_1 + \ell_2$, i.e. $\ell(A \frown B) = \ell(A) + \ell(B)$.

(Definition 1 continued)

The only lattice polynomials are those that can be obtained in a finite number of steps from (i), (ii), and (iii).

The set of lattice polynomials on X is denoted by $W(X)$.

2. <u>Lemma</u>. If $A \in W(X)$ then $\ell(A) \geqslant 1$ and $\ell(A) = 1$ if and only if $A \in X$.

Now let P be a poset. With each polynomial A is associated a pseudo-principal ideal of P, $\underline{A}$, the <u>lower cover</u> of A, and a pseudo-principal dual ideal $\overline{A}$, the <u>upper cover</u> of A. These are defined inductively:

3a) <u>Definition</u>. (i) If $\ell(A) = 1$, i.e. $A \in P$, then $\underline{A} = (A]$, the principal ideal of P generated by A.

(ii) If $A = A_0 \smile A_1$ then $\underline{A} = \underline{A_0} \vee \underline{A_1}$, the ideal join in P.

(iii) If $A = A_0 \frown A_1$ then $\underline{A} = \underline{A_0} \wedge \underline{A_1}$, the ideal meet in P.

The upper cover is defined in a dual manner:

3b) <u>Definition</u>. (i) If $\ell(A) = 1$ then $\overline{A} = [A)$, the principal dual ideal of P generated by A.

(ii) If $A = A_0 \smile A_1$ then $\overline{A} = \overline{A_0} \wedge \overline{A_1}$, the dual ideal meet.

(iii) If $A = A_0 \frown A_1$ then $\overline{A} = \overline{A}_0 \vee \overline{A}_1$, the dual ideal join.

Intuitively, elements of $W(P)$ represent elements of $FL(P)$. $\underline{A}$ is the set of all elements of P bounded above by A and, dually, $\overline{A}$ is the set of all elements of P bounded below by A.

4. <u>Lemma</u>. If $A \in W(P)$ and $x \in \underline{A}$, $y \in \overline{A}$ then $x \leqslant y$.

<u>Proof</u>: The proof proceeds by induction on $\ell(A)$.

(i) If $\ell(A) = 1$ then $A \in P$. Thus $\underline{A} = (A]$ and $\overline{A} = [A)$. Consequently $x \leqslant A$ and $A \leqslant y$. Thus $x \leqslant y$.

(ii) Now let $n > 1$ and let the conclusion of the theorem hold for all polynomials of length $< n$. Let $\ell(A) = n$. Thus either $A = A_0 \smile A_1$ or $A = A_0 \frown A_1$ where $\ell(A_0) < n$ and $\ell(A_1) < n$.

If $A = A_0 \smile A_1$ then $\underline{A} = \underline{A_0} \vee \underline{A_1}$ and $\overline{A} = \overline{A}_0 \wedge \overline{A}_1 = \overline{A}_0 \cap \overline{A}_1$. Let $y \in \overline{A}$; thus $y \in \overline{A}_0$ and $y \in \overline{A}_1$. Then, by induction, for all $z \in \underline{A_0}$, $z \leqslant y$. Thus $\underline{A_0} \subseteq (y]$ and, similarly, $\underline{A_1} \subseteq (y]$. By Lemma 1.7 $(y]$ is an ideal. Thus $\underline{A} = \underline{A_0} \vee \underline{A_1} \subseteq (y]$. Consequently if $x \in \underline{A}$ then $x \leqslant y$.

If $A = A_0 \frown A_1$ the result follows by the principle of duality.

Thus, by induction, the lemma follows.

5. <u>Coroll</u>. a) If $A \in W(P)$ and $x \in \overline{A}$ then $\underline{A} \subseteq (x]$.

b) If $A \in W(P)$ and $x \in \underline{A}$ then $\overline{A} \subseteq [x)$.

2. Construction of $FL(P)$.

An equivalence relation is defined on $W(P)$, where P is a poset. We show that the equivalence classes form a lattice and that this lattice is $FL(P)$. The first step is the construction of a quasi-order $\subseteq$ on $W(P)$, a generalization of the technique of Whitman [10], Dilworth [4], and Chen and Grätzer [2].

6. <u>Definition</u>. If $A, B \in W(P)$, set $A \subseteq B$ if it follows from the rules (1) to (6) below:

(1) $A = B$;

(2) $\overline{A} \cap \underline{B} \neq \emptyset$;

(3) $A = A_0 \smile A_1$ where $A_0 \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_0 \frown A_1$ where $A_0 \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_0 \smile B_1$ where $A \subseteq B_0$ or $A \subseteq B_1$;

(6) $B = B_0 \frown B_1$ where $A \subseteq B_0$ and $A \subseteq B_1$.

7. <u>Lemma</u>. If $A, B \in W(P)$ and $A \subseteq B$ then a) $\underline{A} \subseteq \underline{B}$ and b) $\overline{B} \subseteq \overline{A}$.

Proof: The proof of a) is presented; that of b) follows by duality.

The proof is by induction on $\ell(A) + \ell(B)$, which is $\geqslant 2$. If $A \subseteq B$ follows by rule (1) then the result is clear. Thus it may be assumed that $A \subseteq B$ does not follow by (1).

If $\ell(A) + \ell(B) = 2$ then $A \subseteq B$ must follow by rule (2). Thus $\bar{A} \cap \underline{B} \neq \emptyset$. Let $x \in \bar{A} \cap \underline{B}$. By Coroll. 5 $\underline{A} \subseteq (x]$ and, since $x \in \underline{B}$ and $\underline{B}$ is an ideal, $(x] \subseteq \underline{B}$. Thus $\underline{A} \subseteq \underline{B}$.

Now let $\ell(A) + \ell(B) = n > 2$, and assume that the conclusion of the lemma holds for all $A, B \in W(P)$ such that $\ell(A) + \ell(B) < n$.

If $A \subseteq B$ follows by rule (2) then the conclusion follows as above.

If $A \subseteq B$ follows by rule (3), that is, $A = A_0 \smile A_1$, $A_0 \subseteq B$ and $A_1 \subseteq B$, then $\underline{A_0} \subseteq \underline{B}$ since $\ell(A_0) + \ell(B) < \ell(A) + \ell(B)$. Similarly $\underline{A_1} \subseteq \underline{B}$ and, since $\underline{B}$ is an ideal, $\underline{A} = \underline{A_0} \vee \underline{A_1} \subseteq \underline{B}$.

If $A \subseteq B$ follows by rule (4) then $A = A_0 \frown A_1$ and, say, $A_0 \subseteq B$. Since $\ell(A_0) + \ell(B) < n$ then $\underline{A_0} \subseteq \underline{B}$. Thus $\underline{A} = \underline{A_0} \wedge \underline{A_1} \subseteq \underline{A_0} \subseteq \underline{B}$.

If $A \subseteq B$ follows by rule (5) then $B = B_0 \smile B_1$ and, say, $A \subseteq B_0$. Since $\ell(A) + \ell(B_0) < n$ then $\underline{A} \subseteq \underline{B_0}$.

Thus $\underline{A} \subseteq \underline{B_o} \subseteq \underline{B_o} \vee \underline{B_1} = \underline{B}$.

If $A \subseteq B$ follows by rule (6) then $B = B_o \frown B_1$, $A \subseteq B_o$, and $A \subseteq B_1$. Thus by the inductive hypothesis $\underline{A} \subseteq \underline{B_o}$ and $\underline{A} \subseteq \underline{B_1}$. Thus $\underline{A} \subseteq \underline{B_o} \cap \underline{B_1} = \underline{B_o} \wedge \underline{B_1} = \underline{B}$.

Thus, by induction, if $A \subseteq B$ then $\underline{A} \subseteq \underline{B}$ and, dually, $\overline{B} \subseteq \overline{A}$.

Using the result of this lemma we prove:

8. Lemma. The relation $\subseteq$ on $W(P)$ is a quasi-order; that is, $\subseteq$ is reflexive and transitive.

Proof: That $\subseteq$ is reflexive follows from rule (1).

The transitivity of $\subseteq$ is established by induction on $\ell(A) + \ell(B) + \ell(C) \geqslant 3$, where $A \subseteq B$ and $B \subseteq C$. If either $A \subseteq B$ or $B \subseteq C$ follows by rule (1) the result that $A \subseteq C$ is clear; thus in the proof it will be assumed that neither $A \subseteq B$ nor $B \subseteq C$ follows by rule (1).

Let $A \subseteq B$ and $B \subseteq C$, and let $\ell(A) + \ell(B) + \ell(C) = 3$. Then $\ell(A) = \ell(B) = \ell(C) = 1$. Thus $A \subseteq B$ follows by rule (2), i.e. $\overline{A} \cap \underline{B} \neq \emptyset$. Since $B \subseteq C$, $\underline{B} \subseteq \underline{C}$ by Lemma 7. Thus $\overline{A} \cap \underline{C} \neq \emptyset$, and so $A \subseteq C$ by rule (2).

Now we may assume that $A \subseteq C$ if $A \subseteq B$ and $B \subseteq C$ and $\ell(A) + \ell(B) + \ell(C) < n$.

Let $A \subseteq B$ and $B \subseteq C$, and let

$$\ell(A) + \ell(B) + \ell(C) = n .$$

If $A \subseteq B$ follows by rule (2) then, proceeding as above, $A \subseteq C$. If $B \subseteq C$ by rule (2) the dual argument establishes that $A \subseteq C$.

If $A \subseteq B$ follows by rule (3) then $A = A_0 \smile A_1$, $A_0 \subseteq B$, and $A_1 \subseteq B$. Since
$\ell(A_0) + \ell(B) + \ell(C) < \ell(A) + \ell(B) + \ell(C)$
then $A_0 \subseteq C$. Similarly, $A_1 \subseteq C$. Thus, by rule (3), $A \subseteq C$.

If $A \subseteq B$ follows by rule (4) then $A = A_0 \frown A_1$ and, say, $A_0 \subseteq B$. Thus, by the inductive hypothesis, $A_0 \subseteq C$ and consequently, by rule (4), $A \subseteq C$.

If $B \subseteq C$ follows by rule (5) or (6) the argument is the dual to the above. Thus only two cases remain: $B = B_0 \smile B_1$, $A \subseteq B$ by rule (5), and $B \subseteq C$ by rule (3); and, dually, $B = B_0 \frown B_1$, $A \subseteq B$ by rule (6), and $B \subseteq C$ by rule (4).

If $B = B_0 \smile B_1$, $A \subseteq B$ by rule (5), and $B \subseteq C$ by rule (3) then $A \subseteq B_0$, say, and $B_0 \subseteq C$, $B_1 \subseteq C$. Thus, since $\ell(A) + \ell(B_0) + \ell(C) < n$, $A \subseteq C$.

The other case is the dual.

Thus if $A, B, C \in W(P)$ where $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$. Thus $\subseteq$ is transitive also, and so $\subseteq$ is a quasi-order.

Thus, by Lemma 1.1, there is an equivalence relation $\sim$ on $W(P)$ where $A \sim B$ if and only if $A \subseteq B$ and

$B \subseteq A$. For each $A \in W(P)$ the equivalence class of A modulo $\sim$ is denoted $\langle A \rangle$. Thus $W(P)/\sim$ is a poset with a partial order $\leqslant$ defined by:

For all $A, B \in W(P)$ $\langle A \rangle \leqslant \langle B \rangle$ if and only if $A \subseteq B$.

9. Lemma. $W(P)/\sim$ is a lattice under $\leqslant$; for each $A, B \in W(P)$ $\langle A \rangle \vee \langle B \rangle = \langle A \smile B \rangle$ and $\langle A \rangle \wedge \langle B \rangle = \langle A \frown B \rangle$.

Proof: By rules (1) and (5) of Definition 6 $A \subseteq A \smile B$ and $B \subseteq A \smile B$. Thus $\langle A \rangle \leqslant \langle A \smile B \rangle$ and $\langle B \rangle \leqslant \langle A \smile B \rangle$; that is, $\langle A \smile B \rangle$ is an upper bound of $\{\langle A \rangle, \langle B \rangle\}$.

Now let $C \in W(P)$ and $\langle A \rangle \leqslant \langle C \rangle$, $\langle B \rangle \leqslant \langle C \rangle$. Thus $A \subseteq C$ and $B \subseteq C$; thus, by rule (3) of Definition 6, $A \smile B \subseteq C$; i.e. $\langle A \smile B \rangle \leqslant \langle C \rangle$.

Thus $\langle A \smile B \rangle$ is the least upper bound of $\{\langle A \rangle, \langle B \rangle\}$.

The second half of the conclusion follows by the principle of duality.

Now we show that P is embedded in $W(P)/\sim$.

10. Lemma. If $A, B \in P$ then $A \subseteq B$ in $W(P)$ if and only if $A \leqslant B$ in P .

Proof: Let $A \subseteq B$; then $\underline{A} \subseteq \underline{B}$. Thus $(A] \subseteq (B]$ and so $A \in (B]$, that is, $A \leqslant B$.

Conversely, if $A \leqslant B$ then $B \in [A) = \overline{A}$ and

and $B \in (B] = \underline{B}$; thus $B \in \bar{A} \cap \underline{B}$. Thus $\bar{A} \cap \underline{B} \neq \emptyset$ and so $A \subseteq B$.

11. <u>Lemma</u>. a) If $A, B \in P$ and $\sup \{A, B\}$ exists then $\sup \{A, B\} \sim A \smile B$.

b) If $A, B \in P$ and $\inf \{A, B\}$ exists then $\inf \{A, B\} \sim A \frown B$.

<u>Proof</u>: The proof of a) is presented; that of b) follows by duality.

In P , $A \leqslant \sup \{A, B\}$ and $B \leqslant \sup \{A, B\}$. Thus by Lemma 10 $A \subseteq \sup \{A, B\}$ and $B \subseteq \sup \{A, B\}$. Consequently, by rule (3) of Definition 6,

$$A \smile B \subseteq \sup \{A, B\}.$$

Now $\underline{A \smile B} = \underline{A} \vee \underline{B}$ and $A \in \underline{A}$, $B \in \underline{B}$; thus $A, B \in \underline{A} \vee \underline{B}$ and, by hypothesis, $\sup \{A, B\}$ exists. Since $\underline{A} \vee \underline{B}$ is an ideal of P $\sup \{A, B\} \in \underline{A \smile B} = \underline{A} \vee \underline{B}$. Also $\sup \{A, B\} \in \overline{\sup \{A, B\}}$; thus

$$\overline{\sup \{A, B\}} \cap \underline{A \smile B} \neq \emptyset .$$

Consequently, by rule (2), $\sup \{A, B\} \subseteq A \smile B$.

Thus $\sup \{A, B\} \sim A \smile B$.

12. <u>Definition</u>. A map $\varphi : P \rightarrow W(P)/\sim$ is defined; if $A \in P$ then $\varphi(A) = \langle A \rangle$.

13. <u>Lemma</u>. $\varphi : P \longrightarrow W(P)/\sim$ is an injective homomorphism; that is, φ is 1-1 and preserves existing

sup and inf.

Proof: (i) φ is injective:

Let $A, B \in P$ and $\varphi(A) = \varphi(B)$. Thus $\langle A \rangle = \langle B \rangle$ and so $A \subseteq B$ and $B \subseteq A$. Thus, by Lemma 10, $A \leqslant B$ and $B \leqslant A$. Thus $\varphi(A) = \varphi(B)$ implies that $A = B$.

(ii) Let $A, B \in P$ and let $\sup \{A, B\}$ exist. Then

$$\varphi(A) \vee \varphi(B) = \langle A \rangle \vee \langle B \rangle = \langle A \smile B \rangle = \langle \sup \{A, B\} \rangle = \varphi(\sup \{A, B\}).$$

(iii) If $A, B \in P$ and $\inf \{A, B\}$ exists then this case is the dual of (ii).

Thus we have established the lemma.

Now we need only show that the pair $(\varphi, W(P)/\sim)$ satisfies the universal mapping property for $FL(P)$. Let L be a lattice and let $f_o : P \to L$ be a homomorphism. We define a map $F : W(P) \to L$ by induction on the length of the elements of $W(P)$:

(i) if $\ell(A) = 1$, i.e. $A \in P$, then $F(A) = f_o(A)$;

(ii) if $A = A_o \smile A_1$ then $F(A) = F(A_o) \vee F(A_1)$;

(iii) if $A = A_o \frown A_1$ then $F(A) = F(A_o) \wedge F(A_1)$.

14. Lemma. Let $x \in P$ and $A \in W(P)$.

a) If $x \in \underline{A}$ then $f_o(x) \leqslant F(A)$.

b) If $x \in \overline{A}$ then $f_o(x) \geqslant F(A)$.

Proof: Since b) is the dual of a), only a) need be proved. The proof proceeds by induction on $\ell(A)$.

If $\ell(A) = 1$ then $A \in P$ and so $\underline{A} = (A]$. Thus $x \leqslant A$ and thus $f_o(x) \leqslant f_o(A) = F(A)$. (A homomorphism is clearly isotone.)

Now let $\ell(A) = n > 1$ and assume a) to be true for all polynomials of length $< n$. Since $\ell(A) > 1$, either $A = A_o \smile A_1$ or $A = A_o \frown A_1$ for some $A_o, A_1 \in W(P)$.

If $A = A_o \frown A_1$ then $\underline{A} = \underline{A_o} \cap \underline{A_1}$ and thus $x \in \underline{A_o}$ and $x \in \underline{A_1}$. Since $\ell(A_o), \ell(A_1) < n$, $f_o(x) \leqslant F(A_o)$ and $f_o(x) \leqslant F(A_1)$. Thus $f_o(x) \leqslant F(A_o) \wedge F(A_1) = F(A)$.

If $A = A_o \smile A_1$ let

$$I = \{y \in P \mid f_o(y) \leqslant F(A_o) \vee F(A_1)\} .$$

We first show that I is an ideal of P. If $y \in I$ and $z \leqslant y$ then $f_o(z) \leqslant f_o(y) \leqslant F(A_o) \vee F(A_1)$, and so $z \in I$. If $y_o, y_1 \in I$ and $\sup \{y_o, y_1\}$ exists then $f_o(\sup \{y_o, y_1\}) = f_o(y_o) \vee f_o(y_1)$ and $f_o(y_o) \leqslant F(A_o) \vee F(A_1)$, $f_o(y_1) \leqslant F(A_o) \vee F(A_1)$. Thus $\sup \{y_o, y_1\} \in I$, and so I is an ideal of P.

Since $\ell(A_o) < n$, whenever $x \in \underline{A_o}$

$$f_o(x) \leqslant F(A_o) \leqslant F(A_o) \vee F(A_1) .$$

Thus $\underline{A_o} \subseteq I$ and, similarly, $\underline{A_1} \subseteq I$. Since I is an ideal, $\underline{A} = \underline{A_o} \vee \underline{A_1} \subseteq I$. Thus if $x \in \underline{A}$ then

$$f_o(x) \leqslant F(A_o) \vee F(A_1) = F(A) .$$

Thus the lemma is established.

15. Lemma. If $A, B \in W(P)$ and $A \subseteq B$ then $F(A) \leqslant F(B)$.

Proof: The proof proceeds by induction on $\ell(A) + \ell(B) \geqslant 2$.

If $\ell(A) + \ell(B) = 2$ then $A, B \in P$. Thus $A \subseteq B$ implies that $A \leqslant B$ (Lemma 10). Thus $f_0(A) \leqslant f_0(B)$ and so, by the definition of F , $F(A) \leqslant F(B)$.

Now let $\ell(A) + \ell(B) = n > 2$ and let the conclusion hold for all pairs whose lengths add to an integer $< n$. $A \subseteq B$ must follow from one of the rules of Definition 6.

If $A \subseteq B$ follows from rule (1) the result is clear.

If $A \subseteq B$ follows from rule (2), i.e. if $\overline{A} \cap \underline{B} \neq \emptyset$, then there is an $x \in P$ such that $x \in \overline{A}$, $x \in \underline{B}$. Thus, by Lemma 14, $F(A) \leqslant f_0(x) \leqslant F(B)$. Thus $F(A) \leqslant F(B)$.

If $A \subseteq B$ follows from rule (3) then $A = A_0 \smile A_1$, $A_0 \subseteq B$, $A_1 \subseteq B$. Since $\ell(A_0) + \ell(B) < n$, $F(A_0) \leqslant F(B)$ and, similarly, $F(A_1) \leqslant F(B)$. Thus

$$F(A) = F(A_0) \vee F(A_1) \leqslant F(B) .$$

If $A \subseteq B$ follows from rule (4) then $A = A_0 \frown A_1$ and, say, $A_0 \subseteq B$. Thus, by the inductive hypothesis, $F(A_0) \leqslant F(B)$; so

$$F(A) = F(A_0) \wedge F(A_1) \leqslant F(A_0) \leqslant F(B) .$$

The dual arguments apply if the quasi-inequality follows by rule (5) or (6).

Thus if $A \subseteq B$ then $F(A) \leqslant F(B)$.

16. Coroll. If $A, B \in W(P)$ and $A \sim B$ then $F(A) = F(B)$.

Thus we can define a map

$$f : W(P)/\sim \ \longrightarrow L$$

by

$$f(\langle A\rangle) = F(A) \quad \text{for all} \quad A \in W(P) .$$

17. Lemma. $f : W(P)/\sim \ \longrightarrow L$ is a lattice homomorphism and $f\varphi = f_o$.

Proof: $W(P)/\sim$ is a lattice by Lemma 9. The map f is well-defined by Coroll. 16. Also

$$f(\langle A\rangle \vee \langle B\rangle) = f(\langle A \smile B\rangle) = F(A \smile B) = F(A) \vee F(B)$$
$$= f(\langle A\rangle) \vee f(\langle B\rangle) ,$$

and dually.

Thus f is a lattice homomorphism.

Now let $A \in P$. Then $\varphi(A) = \langle A\rangle$; thus $f\varphi(A) = f(\langle A\rangle) = F(A) = f_o(A)$. Thus $f\varphi = f_o$.

Thus the proof is complete.

Since P clearly generates $W(P)/\sim$ Lemma 17 shows that:

18. Theorem (Dean [3a]). The pair $(\varphi, W(P)/\sim)$ is the free lattice generated by the poset P , and so

$$W(P)/\sim \ = FL(P) .$$

3. Characterization of $FL(P)$.

Let L be a lattice generated by a poset P . A problem of some interest is that of finding necessary and sufficient conditions on L so that L be isomorphic to $FL(P)$. Dean solved the corresponding problem for $CF(P)$ ([3], Theorem 6, and Chapter III of this work). In Theorem 7 of the above reference Dean states a sufficient condition for L to be $FL(P)$, provided that L is a sublattice of a completely free lattice; however, his condition implies that $FL(P)$ is isomorphic to $CF(P)$ ([3] Theorem 4) and so does not contribute to the problem beyond his result for $CF(P)$. In this section the problem is solved, modulo the structure of the pseudo-principal ideals and pseudo-principal dual ideals of P .

19. Theorem. Let L be a lattice generated by a subset P . If P is regarded as a poset under the partial order induced by L then there is a lattice isomorphism from L onto $FL(P)$ extending $\varphi: P \to FL(P)$ if and only if the following two conditions hold:

(i) a) for all $x, y \in L$ such that $(x] \cap P$ and $(y] \cap P$ are ideals of P

$$(x \vee y] \cap P = ((x] \cap P) \overset{P}{\vee} ((y] \cap P) ,$$

the ideal join in P , and dually

b) for all $x, y \in L$ such that $[x) \cap P$ and

$[y) \cap P$ are dual ideals of P

$$[x \wedge y) \cap P = ([x) \cap P) \overset{P}{\vee} ([y) \cap P) \,,$$

the dual ideal join in P ;

(ii) given $x_o, x_1, y_o, y_1 \in L$ such that
$x_o \wedge x_1 \leqslant y_o \vee y_1$, $x_o \wedge x_1 \nleqslant y_i$, $x_i \nleqslant y_o \vee y_1$, $i \in \{0, 1\}$,
then there is a $p \in P$ such that

$$x_o \wedge x_1 \leqslant p \leqslant y_o \vee y_1 \,.$$

20. Coroll. Condition (i) implies that, for all $x \in L$, $(x] \cap P$ is a pseudo-principal ideal of P and $[x) \cap P$ is a pseudo-principal dual ideal of P .

Throughout the proof of this theorem $(x]$ (and dually $[x)$) denotes the set of all y in L such that $y \leqslant x$ (dually, $y \geqslant x$) and not just those elements in P .

First the necessity of the conditions is established. L may be taken as $W(P)/\sim$ if we identify P and $\varphi(P)$. Three lemmas are proved:

21. Lemma. If $A \in W(P)$ then
a) $(\langle A \rangle] \cap P = \underline{A}$, and b) $[\langle A \rangle) \cap P = \overline{A}$.

Proof: The proof of a) is presented, and b) follows by duality.

Let $p \in (\langle A \rangle] \cap P$. Thus $p \leqslant \langle A \rangle$ and so $p \subseteq A$. Thus, by Lemma 7, $\underline{p} \subseteq \underline{A}$, that is, $p \in \underline{A}$. Thus

$$(\langle A \rangle] \cap P \subseteq \underline{A} \,.$$

Now let $p \in \underline{A}$. Thus, since $p \in \bar{p}$, $\bar{p} \cap \underline{A} \neq \emptyset$. Consequently $p \subseteq A$, that is, $p \leqslant \langle A \rangle$. Thus $p \in (\langle A \rangle] \cap P$ and so $\underline{A} \subseteq (\langle A \rangle] \cap P$.

Thus $\underline{A} = (\langle A \rangle] \cap P$.

22. Lemma. If $A_0, A_1 \in W(P)$ then

a) $(\langle A_0 \rangle \vee \langle A_1 \rangle] \cap P = ((\langle A_0 \rangle] \cap P) \overset{P}{\vee} ((\langle A_1 \rangle] \cap P)$

and

b) $[\langle A_0 \rangle \wedge \langle A_1 \rangle) \cap P = ([\langle A_0 \rangle) \cap P) \overset{P}{\vee} ([\langle A_1 \rangle) \cap P)$.

Proof: Part a) is proved; part b) follows dually.

$$
\begin{aligned}
(\langle A_0 \rangle \vee \langle A_1 \rangle] \cap P &= (\langle A_0 \smile A_1 \rangle] \cap P \\
&= \underline{A_0 \smile A_1} \qquad \text{by Lemma 21,} \\
&= \underline{A_0} \overset{P}{\vee} \underline{A_1} \\
&= ((\langle A_0 \rangle] \cap P) \overset{P}{\vee} ((\langle A_1 \rangle] \cap P)
\end{aligned}
$$

again by Lemma 21.

Lemma 22 establishes the necessity of condition (i).

23. Lemma. Let $A_0, A_1, B_0, B_1 \in W(P)$ and let $\langle A_0 \rangle \wedge \langle A_1 \rangle \leqslant \langle B_0 \rangle \vee \langle B_1 \rangle$, $\langle A_0 \rangle \wedge \langle A_1 \rangle \nleqslant \langle B_i \rangle$, and $\langle A_i \rangle \nleqslant \langle B_0 \rangle \vee \langle B_1 \rangle$, $i \in \{0, 1\}$. Then there is a $p \in P$ such that $\langle A_0 \rangle \wedge \langle A_1 \rangle \leqslant p \leqslant \langle B_0 \rangle \vee \langle B_1 \rangle$.

Proof: Since $\langle A_0 \rangle \wedge \langle A_1 \rangle \leqslant \langle B_0 \rangle \vee \langle B_1 \rangle$ then $A_0 \frown A_1 \subseteq B_0 \smile B_1$ in $W(P)$. However, for $i \in \{0, 1\}$, $A_0 \frown A_1 \nsubseteq B_i$ and $A_i \nsubseteq B_0 \smile B_1$. Thus the above relation must be derived by an application of rule (2) of Definition 6.

Thus $\overline{A_0 \frown A_1} \cap \underline{B_0 \smile B_1} \neq \emptyset$. Let
$p \in \overline{A_0 \frown A_1} \cap \underline{B_0 \smile B_1}$; thus $\overline{A_0 \frown A_1} \cap \underline{p} \neq \emptyset$, and so
$A_0 \frown A_1 \subseteq p$.

Similarly $p \subseteq B_0 \smile B_1$.

Thus

$$\langle A_0 \rangle \wedge \langle A_1 \rangle \leqslant p \leqslant \langle B_0 \rangle \vee \langle B_1 \rangle .$$

Thus we have established the necessity of the conditions.

To establish the sufficiency several lemmas are stated.

Let $f_0 : P \to L$ be the embedding of P in L ; that is, for all $x \in P$, $f_0(x) = x$.

24. Lemma. For all $x \in P$

a) $(x] \cap P$ is an ideal of P ;

b) $[x) \cap P$ is a dual ideal of P .

Proof: This is clear since $(x] \cap P = \underline{x}$ and $[x) \cap P = \overline{x}$.

25. Lemma. Condition (i) implies that $f_0 : P \to L$ is a homomorphism.

Proof: Let $x, y \in P$ and let $\sup \{x, y\}$ exist. Then $x \vee y \leqslant \sup \{x, y\}$. Thus

$$f_0(\sup \{x, y\}) \geqslant f_0(x) \vee f_0(y) .$$

By Lemma 24 and condition (i) a)

$$(x \vee y] \cap P \quad = \quad ((x] \cap P) \overset{P}{\vee} ((y] \cap P) ,$$

and so $(x \vee y] \cap P$ is an ideal of P . Since

$x, y \in (x \vee y] \cap P$, $\sup\{x, y\} \in (x \vee y] \cap P$. Thus $\sup\{x, y\} \leqslant x \vee y$, and so $f_o(\sup\{x, y\}) = f_o(x) \vee f_o(y)$.

This argument and the dual imply the truth of this lemma.

Consequently we have a map $F : W(P) \to L$ extending f_o such that the corresponding map $f : FL(P) \to L$ is a lattice homomorphism. The map f is clearly onto since P generates L . The injectivity of f follows from:

26. <u>Lemma</u>. Condition (i) implies that for all $A \in W(P)$ a) $\underline{A} = (F(A)] \cap P$ and b) $\overline{A} = [F(A)) \cap P$.

<u>Proof</u>: Part a) is proved and part b) follows by duality.

The proof proceeds by induction on $\ell(A)$.

If $\ell(A) = 1$ then $A \in P$ and so $F(A) = f_o(A) = A$. Thus $\underline{A} = (A] \cap P = (F(A)] \cap P$.

Now let $n > 1$ and assume that the result is true for all lattice polynomials of length $< n$. Let $\ell(A) = n$.

If $A = A_o \frown A_1$ then

$$(F(A)] = (F(A_o) \wedge F(A_1)] = (F(A_o)] \cap (F(A_1)] \text{ . Thus}$$

$$\begin{aligned}(F(A)] \cap P &= (F(A_o)] \cap P \cap (F(A_1)] \cap P \\ &= \underline{A_o} \cap \underline{A_1} \\ &= \underline{A} \quad \text{since } \ell(A_o), \ell(A_1) < n .\end{aligned}$$

If $A = A_o \smile A_1$ then $(F(A)] = (F(A_o) \vee F(A_1)]$. Since $\ell(A_o)$, $\ell(A_1) < n$

$$(F(A_o)] \cap P = \underline{A_o} \quad \text{and} \quad (F(A_1)] \cap P = \underline{A_1} .$$

Thus $(F(A_o)] \cap P$ and $(F(A_1)] \cap P$ are ideals of P and so, by condition (i) a),

$$(F(A)] \cap P = ((F(A_o)] \cap P) \overset{P}{\vee} ((F(A_1)] \cap P) .$$

Thus

$$(F(A)] \cap P = \underline{A_o} \overset{P}{\vee} \underline{A_1} = \underline{A} .$$

Since F is surjective this lemma also establishes Coroll. 20.

27. Lemma. Let $A \in W(P)$ and $x \in P$.

a) If $x \leqslant F(A)$ then $x \subseteq A$ in $W(P)$;

b) if $x \geqslant F(A)$ then $A \subseteq x$ in $W(P)$.

Proof: Part b) is the dual of a) and thus we give only the proof of a).

If $x \leqslant F(A)$ then $x \in (F(A)] \cap P$ and thus $x \in \underline{A}$.
Thus $\overline{x} \cap \underline{A} \neq \emptyset$ and the result follows.

Now we state the crucial lemma:

28. Lemma. If $A, B \in W(P)$ and $F(A) \leqslant F(B)$ then $A \subseteq B$.

Proof: The proof proceeds by induction on $\ell(A) + \ell(B)$.

If $\ell(A) + \ell(B) = 2$ the result is clear.

Let $n > 2$ and let the conclusion hold for all polynomials whose lengths add to an integer $< n$. Let $\ell(A) + \ell(B) = n$.

If $A \in P$ or $B \in P$ then Lemma 27 establishes the result.

If $A = A_0 \smile A_1$ then $F(A) = F(A_0) \vee F(A_1) \leqslant F(B)$. Thus, for $i \in \{0, 1\}$, $F(A_i) \leqslant F(B)$ and $\ell(A_i) + \ell(B) < n$; $A_i \subseteq B$ and so $A \subseteq B$.

If $B = B_0 \frown B_1$ the dual argument applies.

The only remaining case is when $A = A_0 \frown A_1$ and $B = B_0 \smile B_1$. In this case

$$F(A) = F(A_0) \wedge F(A_1) \leqslant F(B_0) \vee F(B_1) = F(B).$$

If there is an $i \in \{0, 1\}$ such that $F(A_i) \leqslant F(B)$ then, by induction, $A_i \subseteq B$ and thus $A \subseteq B$.

The dual argument applies if there is an $i \in \{0, 1\}$ such that $F(A) \leqslant F(B_i)$.

On the other hand, if for all $i \in \{0, 1\}$ $F(A_0) \wedge F(A_1) \nleqslant F(B_i)$ and $F(A_i) \nleqslant F(B_0) \vee F(B_1)$ then condition (ii) implies that there is a $p \in P$ such that

$$F(A_0) \wedge F(A_1) \leqslant p \leqslant F(B_0) \vee F(B_1).$$

By Lemma 27 it follows that $A \subseteq p \subseteq B$. Thus $A \subseteq B$.

29. Coroll. $F(A) = F(B)$ implies that $A \sim B$.

Thus f is injective and so $f^{-1} : L \to FL(P)$ is an isomorphism. Clearly f^{-1} extends φ. Consequently the proof of Theorem 19 is complete.

Conditions (i) a), (i) b), and (ii) are independent;

we exhibit two examples to illustrate this. It may be noted that in both examples Coroll. 20 holds.

<u>Example 1.</u> Let L be the lattice depicted in Fig. 1. Let $P = \{a, b, c\}$. P generates L. Condition (i) a) fails in this case:
$(b] \cap P = \{b\}$ and $(c] \cap P = \{c\}$, both of which are ideals. Thus $((b] \cap P) \vee ((c] \cap P) = \{b, c\}$. However, since $b \vee c = z$, $(b \vee c] \cap P = \{a, b, c\}$. The other conditions hold.

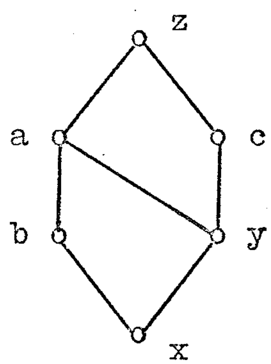

Fig. 1

The dual of this lattice is one where only (i) b) fails.

<u>Example 2.</u> Let L be the lattice depicted in Fig. 2. Let $P = \{a, b, c, d\}$. P generates L. Condition (ii) fails:
$b \vee d = z = a \wedge c$. Thus
$a \wedge c \leqslant b \vee d$, $a \wedge c \nleqslant b$,
$a \wedge c \nleqslant d$, $a \nleqslant b \vee d$, $c \nleqslant b \vee d$,
and $z \notin P$.
The other conditions hold for L.

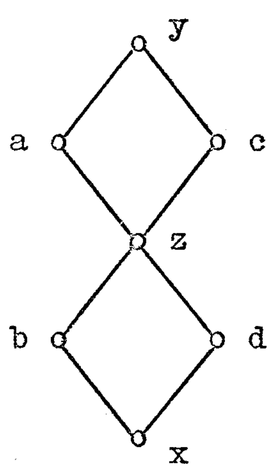

Fig. 2

Example 3. This is an example to show that (ii) may hold and Coroll. 20 a) may fail. Let $P = \{a, b, c\}$. Then $(y] \cap P = \{a, b\}$ and $\sup \{a, b\} = c$; thus $(y] \cap P$ is not an ideal. In the dual lattice Coroll. 20 b) fails.

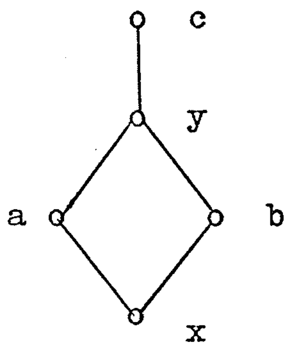

Fig. 3

A question related to the result in Theorem 19 is: given a poset P and a subset Q of P, what intrinsic conditions on P and Q are necessary and sufficient for the lattice generated by Q in $FL(P)$ to be isomorphic to $FL(Q)$? A partial answer to this question is provided.

Let P be a poset and let $Q \subseteq P$. We denote by $\mathcal{J}(Q)$ the set of pseudo-principal ideals of P which are obtained by taking joins and meets of a finite sequence of principal ideals of P generated by elements of Q. We denote by $\mathcal{D}(Q)$ the set of pseudo-principal dual ideals of P defined dually. As above, $\varphi(P)$ in $FL(P)$ is identified with P.

30. Theorem. Let the following conditions hold:

(i) a) For all $I, J \in \mathcal{J}(Q)$, if $I \cap Q$ and

$J \cap Q$ are ideals in Q then

$$(I \overset{P}{\vee} J) \cap Q = (I \cap Q) \overset{Q}{\vee} (J \cap Q) ,$$

and

b) for all $I, J \in \mathcal{D}(Q)$, if $I \cap Q$ and $J \cap Q$ are dual ideals in Q then

$$(I \overset{P}{\vee} J) \cap Q = (I \cap Q) \overset{Q}{\vee} (J \cap Q) ;$$

(ii) given $I_o, I_1 \in \mathcal{I}(Q)$, $D_o, D_1 \in \mathcal{D}(Q)$ such that, for each $i \in \{0, 1\}$,

$$(I_o \overset{P}{\vee} I_1) \cap D_i = \emptyset , \quad (D_o \overset{P}{\vee} D_1) \cap I_i = \emptyset , \text{ and}$$
$$(I_o \overset{P}{\vee} I_1) \cap (D_o \overset{P}{\vee} D_1) \neq \emptyset$$

then

$$(I_o \overset{P}{\vee} I_1) \cap (D_o \overset{P}{\vee} D_1) \cap Q \neq \emptyset .$$

Then there is an isomorphism from L , the sublattice of $FL(P)$ generated by Q , onto $FL(Q)$. The restriction of this isomorphism to Q is $\varphi_Q : Q \to FL(Q)$.

Proof: We show that the conditions of Theorem 19 obtain.

Let $x, y \in L$ and let $(x] \cap Q$ and $(y] \cap Q$ be ideals of Q . By applying Coroll. 20 to $FL(P)$, $(x] \cap P$, $(y] \cap P$ are pseudo-principal ideals of P and, since Q generates L ,

$$(x] \cap P, (y] \cap P \in \mathcal{I}(Q) .$$

Now $(x] \cap P \cap Q = (x] \cap Q$ and $(y] \cap P \cap Q = (y] \cap Q$, and so are ideals of Q . Thus, by condition (i) of our theorem

$$\left[((x] \cap P) \overset{P}{\vee} ((y] \cap P)\right] \cap Q = ((x] \cap Q) \overset{Q}{\vee} ((y] \cap Q) .$$

Since, by Theorem 19,

$$(x \vee y] \cap P = ((x] \cap P) \overset{P}{\vee} ((y] \cap P)$$

we find that

$$(x \vee y] \cap Q = ((x] \cap Q) \overset{Q}{\vee} ((y] \cap Q) .$$

Thus condition (i) a) of Theorem 19 holds for L .

The truth of (i) b) is established in a dual manner.

Now let $x_0, x_1, y_0, y_1 \in L$, $x_0 \wedge x_1 \leqslant y_0 \vee y_1$, $x_i \nleqslant y_0 \vee y_1$, $x_0 \wedge x_1 \nleqslant y_i$ for all $i \in \{0, 1\}$. Then there is a $p \in P$ such that

$$x_0 \wedge x_1 \leqslant p \leqslant y_0 \vee y_1 .$$

Thus $[x_0 \wedge x_1) \cap (y_i] = \emptyset$ and $(y_0 \vee y_1] \cap [x_i) = \emptyset$ for all $i \in \{0, 1\}$ and

$$[x_0 \wedge x_1) \cap (y_0 \vee y_1] \cap P \neq \emptyset .$$

Thus, applying Theorem 19 to $FL(P)$,

$$\left[([x_0) \cap P) \overset{P}{\vee} ([x_1) \cap P)\right] \cap ((y_i] \cap P) = \emptyset$$

and

$$\left[((y_0] \cap P) \overset{P}{\vee} ((y_1] \cap P)\right] \cap ([x_i) \cap P) = \emptyset$$

for all $i \in \{0, 1\}$, and

$$\left[((y_0] \cap P) \overset{P}{\vee} ((y_1] \cap P)\right] \cap \left[([x_0) \cap P) \overset{P}{\vee} ([x_1) \cap P)\right] \neq \emptyset .$$

Thus

$$\left[((y_0] \cap P) \overset{P}{\vee} ((y_1] \cap P)\right] \cap \left[([x_0) \cap P) \overset{P}{\vee} ([x_1) \cap P)\right] \cap Q \neq \emptyset,$$

that is,

$$[x_0 \wedge x_1) \cap P \cap (y_0 \vee y_1] \cap P \cap Q \neq \emptyset ,$$

that is, $[x_0 \wedge x_1) \cap (y_0 \vee y_1] \cap Q \neq \emptyset$.

Thus there is a $q \in Q$ such that

$$x_0 \wedge x_1 \leqslant q \leqslant y_0 \vee y_1 .$$

Applying Theorem 19 to L, our result follows.

4. Sorkin's theorem.

Sorkin [9] proved a rather surprising result: if L_λ, $\lambda \in \Lambda$, and L are lattices and, for each $\lambda \in \Lambda$, $f_\lambda : L_\lambda \to L$ is an isotone map--but not necessarily a lattice homomorphism--then the f_λ, $\lambda \in \Lambda$, can be extended to an isotone map from the free product of the L_λ to L. In this section a generalization of this result is proved. The proof is much simpler than Sorkin's, even though he considered only a special case of our result.

Let L be a lattice and let L^* be the completion of L. We consider L as a subset of L^*.

31. Theorem. Let P be a poset and $f_0 : P \to L$ be an isotone map. Then there is a map $f : FL(P) \to L^*$ such that $f\varphi = f_0$ satisfying:

(i) f is isotone;

(ii) if f_0 preserves all existing sup (resp. inf) of pairs in P then f preserves join (resp. meet) in $FL(P)$;

(iii) if each non-empty pseudo-principal ideal of P is a set union of finitely many principal ideals, and dually, then $\mathrm{Im}(f) \subseteq L$.

Proof: For each $A \in W(P)$ define $\varphi(A) = \bigvee f_0(\underline{A})$ and $\overline{\varphi}(A) = \bigwedge f_0(\overline{A})$. Define $F : W(P) \to L^*$ inductively on $\ell(A)$:

(1) if $\ell(A) = 1$, i.e. $A \in P$, then $F(A) = f_0(A)$;

(2) if $A = A_0 \smile A_1$ then
$F(A) = \varphi(A) \vee F(A_0) \vee F(A_1)$;

(3) if $A = A_0 \frown A_1$ then
$F(A) = \overline{\varphi}(A) \wedge F(A_0) \wedge F(A_1)$.

32. Lemma. a) If $x \in \underline{A}$ then $f_0(x) \leqslant F(A)$.
b) If $x \in \overline{A}$ then $f_0(x) \geqslant F(A)$.

Proof: Part a) is proved; b) follows dually.

We proceed by induction on $\ell(A)$:

If $\ell(A) = 1$ then $A \in P$; thus if $x \in \underline{A}$ then $x \leqslant A$ and so $f_0(x) \leqslant f_0(A) = F(A)$, since f_0 is isotone.

Now assume that $n > 1$ and that the result holds for all $A \in W(P)$ such that $\ell(A) < n$. Let $\ell(A) = n$; then either $A = A_0 \smile A_1$ or $A = A_0 \frown A_1$.

If $A = A_0 \smile A_1$ then $x \in \underline{A}$ implies $f_0(x) \in f_0(\underline{A})$. Thus $f_0(x) \leqslant \varphi(A) \leqslant \varphi(A) \vee F(A_0) \vee F(A_1) = F(A)$.

If $A = A_0 \frown A_1$ then $\underline{A} = \underline{A_0} \cap \underline{A_1}$. If $x \in \underline{A}$

then $x \in \underline{A_0}$ and $x \in \underline{A_1}$. Thus, by the inductive hypothesis,

$$f_o(x) \leqslant F(A_o) \quad \text{and} \quad f_o(x) \leqslant F(A_1) .$$

If $y \in \overline{A}$ then $x \leqslant y$ (Lemma 4). Thus $f_o(x) \leqslant f_o(y)$; thus $f_o(x) \leqslant z$ for all $z \in f_o(\overline{A})$. Consequently $f_o(x)$ is a lower bound of $f_o(\overline{A})$; that is,

$$f_o(x) \leqslant \bigwedge f_o(\overline{A}) = \overline{\varphi}(A) .$$

Consequently $f_o(x) \leqslant F(A_o)$, $f_o(x) \leqslant F(A_1)$, and $f_o(x) \leqslant \overline{\varphi}(A)$. Thus $f_o(x) \leqslant F(A)$.

33. <u>Coroll</u>. For all $A \in W(P)$

$$\underline{\varphi}(A) \leqslant F(A) \leqslant \overline{\varphi}(A) .$$

We now state the crucial lemma:

34. <u>Lemma</u>. If $A, B \in W(P)$ and $A \subseteq B$ then

$$F(A) \leqslant F(B) .$$

<u>Proof</u>: The proof is by induction on $\ell(A) + \ell(B)$. $A \subseteq B$ must be derived by rules (1) to (6) of Definition 6. If $A = B$ the result is clear; thus we need only consider rules (2) through (6), and we may assume that $A \neq B$.

If $\ell(A) + \ell(B) = 2$ then $A \subseteq B$ must be derived by rule (2); thus there is an $x \in P$ such that $x \in \overline{A} \cap \underline{B}$. Thus, by Lemma 32, $F(A) \leqslant f_o(x) \leqslant F(B)$, and the result follows.

Now assume that $\ell(A) + \ell(B) = n > 2$.

If $A \subseteq B$ is derived through rule (2) the proof is identical with that presented above.

If $A \subseteq B$ is derived by rule (3) then $A = A_0 \cup A_1$, $A_0 \subseteq B$, and $A_1 \subseteq B$. Thus, by the inductive hypothesis, $F(A_0) \leqslant F(B)$ and $F(A_1) \leqslant F(B)$. Since $A \subseteq B$ then, by Lemma 7, $\underline{A} \subseteq \underline{B}$; thus $f_0(\underline{A}) \subseteq f_0(\underline{B})$ and so $\bigvee f_0(\underline{A}) \leqslant \bigvee f_0(\underline{B})$. Thus $\underline{\varphi}(A) \leqslant \underline{\varphi}(B) \leqslant F(B)$, by Coroll. 33. Thus

$$F(A) = \underline{\varphi}(A) \vee F(A_0) \vee F(A_1) \leqslant F(B) .$$

If $A \subseteq B$ is derived by rule (4) then $A = A_0 \frown A_1$ and, say, $A_0 \subseteq B$. By the inductive hypothesis, $F(A_0) \leqslant F(B)$. Thus

$$F(A) = \overline{\varphi}(A) \wedge F(A_0) \wedge F(A_1) \leqslant F(A_0) \leqslant F(B) .$$

The dual arguments apply if $A \subseteq B$ follows from rule (5) or (6).

Thus the lemma is proved.

From the above lemma we conclude that $A \sim B$ implies that $F(A) = F(B)$. Thus we can define a map $f : FL(P) \to L^*$ by $f(\langle A\rangle) = F(A)$ for all $A \in W(P)$. If $\langle A\rangle, \langle B\rangle \in FL(P)$ and $\langle A\rangle \leqslant \langle B\rangle$ then $A \subseteq B$; thus $F(A) \leqslant F(B)$, that is, $f(\langle A\rangle) \leqslant f(\langle B\rangle)$. Thus f is isotone. From the definition of F on polynomials of length 1 it is clear that $f\varphi = f_0$. Thus (i) follows.

35. <u>Lemma</u>. If for all $x, y \in P$ such that $\sup \{x, y\}$

exists $f_o(\sup\{x, y\}) = f_o(x) \vee f_o(y)$ then for all $A, B \in W(P)$ $F(A \smile B) = F(A) \vee F(B)$.

<u>Proof</u>: We observe that $\{x \mid f_o(x) \leqslant F(A) \vee F(B)\}$ is an ideal of P . By Lemma 32 this ideal includes $\underline{A}$ and $\underline{B}$ and so includes $\underline{A \smile B}$. Thus if $z \in f_o(\underline{A \smile B})$ then $z \leqslant F(A) \vee F(B)$; that is, $\underline{\varphi}(A \smile B) \leqslant F(A) \vee F(B)$ and so

$$F(A \smile B) = F(A) \vee F(B) .$$

This lemma and the dual establish (ii).

To establish (iii) we consider the lattice L^b of Section 1.1. Under the convention that $\bigvee\emptyset = 0$ and $\bigwedge\emptyset = 1$, $\underline{\varphi}(A), \overline{\varphi}(A) \in L^b$ for all $A \in W(P)$. Now we need only observe that $0 < F(A) < 1$ for all $A \in W(P)$; thus 0 and 1 are never images under F and thus f maps to L .

It was observed by Professor Grätzer that part (i) of Theorem 31 holds in a more general situation:

Let K be any lattice of which P is a subset. Then there is an isotone $f : K \to L^*$ extending f_o . We can define f by $f(x) = \bigvee f_o((x] \cap P)$ for all $x \in K$.

* CHAPTER III

$FL(P ; \mathcal{M}, \mathcal{N})$

In this chapter the results of Chapter II are extended to $FL(P ; \mathcal{M}, \mathcal{N})$. As an application of this extension we derive the results of Dean [3] concerning $CF(P)$.

1. $FL(P ; \mathcal{M}, \mathcal{N})$.

Let P be a poset and let there be an $(\mathcal{M}, \mathcal{N})$-structure defined on P . With each $A \in W(P)$ we associate a pseudo-principal $\mathcal{M}$-ideal $\underline{A}$ and a pseudo-principal $\mathcal{N}$-dual ideal $\overline{A}$ as in Definition 2.3:

1. Definition. (i) If $A \in P$ then $\underline{A} = (A]$ and $\overline{A} = [A)$;

(ii) if $A = A_0 \smile A_1$ then $\underline{A} = \underline{A_0} \vee \underline{A_1}$, the $\mathcal{M}$-ideal join in P , and $\overline{A} = \overline{A}_0 \wedge \overline{A}_1$, the $\mathcal{N}$-dual ideal meet in P ;

(iii) if $A = A_0 \frown A_1$ then $\underline{A} = \underline{A_0} \wedge \underline{A_1}$, the $\mathcal{M}$-ideal meet in P , and $\overline{A} = \overline{A}_0 \vee \overline{A}_1$, the $\mathcal{N}$-dual ideal join in P .

Since, for any $x \in P$, $(x]$ is an $\mathcal{M}$-ideal of P and $[x)$ is an $\mathcal{N}$-dual ideal of P Lemma 2.4 and its corollary remain true in the $(\mathcal{M}, \mathcal{N})$-case.

The quasi-order $\subseteq$ on $W(P)$ is defined in the

$(\mathfrak{M}, \mathfrak{N})$-case exactly as in Definition 2.6, except that $\overline{A}$ and $\underline{B}$ refer to $(\mathfrak{M}, \mathfrak{N})$-covers.

2. Definition. If $A, B \in W(P)$ set $A \subseteq B$ if it follows from rules (1) to (6) below:

(1) $A = B$;

(2) $\overline{A} \cap \underline{B} \neq \emptyset$;

(3) $A = A_0 \smile A_1$ where $A_0 \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_0 \frown A_1$ where $A_0 \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_0 \smile B_1$ where $A \subseteq B_0$ or $A \subseteq B_1$;

(6) $B = B_0 \frown B_1$ where $A \subseteq B_0$ and $A \subseteq B_1$;

The fact that $\subseteq$ is a quasi-order follows exactly as in Chapter II and so the equivalence relation $\sim$ is defined on $W(P)$:

3. Definition. If $A, B \in W(P)$ then $A \sim B$ if and only if $A \subseteq B$ and $B \subseteq A$.

$W(P)/\sim$ is a lattice exactly as in Chapter II and we can define the map $\varphi: P \to W(P)/\sim$ where $\varphi(A) = \langle A \rangle$ for all $A \in P$. That φ is isotone is clear, and the proof of injectivity is identical with that in Chapter II. The proof that φ is an $(\mathfrak{M}, \mathfrak{N})$-morphism follows if we replace Lemma 2.11 by:

4. Lemma. a) If $\{x_0, \ldots, x_{n-1}\} \subseteq P$ and

$\{x_0, \ldots, x_{n-1}\} \in \mathcal{M}$ then

$\sup \{x_0, \ldots, x_{n-1}\} \sim (\ldots(x_0 \smile x_1) \smile \ldots \smile x_{n-2}) \smile x_{n-1}$.

b) If $\{x_0, \ldots, x_{n-1}\} \subseteq P$ and $\{x_0, \ldots, x_{n-1}\} \in \mathcal{N}$ then

$\inf \{x_0, \ldots, x_{n-1}\} \sim (\ldots(x_0 \frown x_1) \frown \ldots \frown x_{n-2}) \frown x_{n-1}$.

Proof: The proof of a) proceeds exactly as that presented in Lemma 2.11. Rule (3) must be applied several times and we must observe that

$\underline{(\ldots(x_0 \smile x_1) \smile \ldots \smile x_{n-2}) \smile x_{n-1}} = \underline{x_0} \vee \ldots \vee \underline{x_{n-1}}$;

since $\{x_0, \ldots, x_{n-1}\} \subseteq \underline{x_0} \vee \ldots \vee \underline{x_{n-1}}$ and

$\{x_0, \ldots, x_{n-1}\} \in \mathcal{M}$,

$\sup \{x_0, \ldots, x_{n-1}\} \in \underline{x_0} \vee \ldots \vee \underline{x_{n-1}}$.

The proof that the pair $(\varphi, W(P)/\sim)$ satisfies the universal mapping property for $FL(P ; \mathcal{M}, \mathcal{N})$ proceeds exactly as in Chapter II. We need only observe that if L is a lattice and $f_0 : P \to L$ is an $(\mathcal{M}, \mathcal{N})$-morphism then the set

$$I = \{y \in P \mid f_0(y) \leqslant F(A_0) \vee F(A_1)\} , \quad A_0, A_1 \in W(P) ,$$

is an $\mathcal{M}$-ideal of P , and dually.

5. Theorem (Dean [3a]). The pair $(\varphi, W(P)/\sim)$ is the $(\mathcal{M}, \mathcal{N})$-free lattice generated by P , and so

$$W(P)/\sim \; = FL(P ; \mathcal{M}, \mathcal{N}) .$$

The characterization of $FL(P ; \mathcal{M}, \mathcal{N})$ is provided by:

6. Theorem. Let L be a lattice generated by a subset P. If P is regarded as a poset under the partial order induced by L then there is a lattice isomorphism from L onto $FL(P ; \mathfrak{M}, \mathfrak{N})$ extending $\varphi : P \to FL(P ; \mathfrak{M}, \mathfrak{N})$ if and only if the following two conditions hold:

(i) a) for all $x, y \in L$ such that $(x] \cap P$ and $(y] \cap P$ are $\mathfrak{M}$-ideals of P

$$(x \vee y] \cap P = ((x] \cap P) \overset{P}{\vee} ((y] \cap P) ,$$

the $\mathfrak{M}$-ideal join in P, and dually

b) for all $x, y \in L$ such that $[x) \cap P$ and $[y) \cap P$ are $\mathfrak{N}$-dual ideals of P

$$[x \wedge y) \cap P = ([x) \cap P) \overset{P}{\vee} ([y) \cap P) ,$$

the $\mathfrak{N}$-dual ideal join in P;

(ii) given $x_0, x_1, y_0, y_1 \in L$ such that

$x_0 \wedge x_1 \leqslant y_0 \vee y_1$, $x_0 \wedge x_1 \nleqslant y_i$, $x_i \nleqslant y_0 \vee y_1$, $i \in \{0,1\}$, then there is a $p \in P$ such that

$$x_0 \wedge x_1 \leqslant p \leqslant y_0 \vee y_1 .$$

7. Coroll. Condition (i) implies that, for all $x \in L$, $(x] \cap P$ is a pseudo-principal $\mathfrak{M}$-ideal of P and $[x) \cap P$ is a pseudo-principal $\mathfrak{N}$-dual ideal of P.

The proof proceeds exactly as that of Theorem 2.19. We need only observe that condition (i) implies that the injection $f_0 : P \to L$ is an $(\mathfrak{M}, \mathfrak{N})$-morphism:

Let $\{x_0, \ldots, x_{n-1}\} \in \mathcal{M}$. For each $i \leqslant n-1$ $(x_i] \cap P$ is an $\mathcal{M}$-ideal. Applying condition (i) a) repeatedly,

$$(x_0 \vee \ldots \vee x_{n-1}] \cap P = ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P)$$

and

$$\{x_0, \ldots, x_{n-1}\} \subseteq ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P) ;$$

thus

$$\sup\{x_0, \ldots, x_{n-1}\} \in ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P)$$

and so

$$\sup \{x_0, \ldots, x_{n-1}\} \leqslant x_0 \vee \ldots \vee x_{n-1} .$$

The reverse inequality is obvious. The dual argument applies if $\{x_0, \ldots, x_{n-1}\} \in \mathcal{N}$.

A useful corollary to Theorem 6 is:

8. Theorem. a) If the lattice of $\mathcal{M}$-ideals of P is distributive then condition (i) a) of Theorem 6 can be replaced by

(i') a) for all $x_0, \ldots, x_{n-1} \in P$

$$(x_0 \vee \ldots \vee x_{n-1}] \cap P = ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P) .$$

b) If the lattice of $\mathcal{N}$-dual ideals of P is distributive then condition (i) b) of Theorem 6 can be replaced by

(i') b) for all $x_0, \ldots, x_{n-1} \in P$

$$[x_0 \wedge \ldots \wedge x_{n-1}) \cap P = ([x_0) \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ([x_{n-1}) \cap P) .$$

Proof: We prove a); the truth of b) follows dually.

We observe first that (i') a) implies that $(x] \cap P$ is an $\mathcal{M}$-ideal for all $x \in L$:

Let $\{x_0, \ldots, x_{n-1}\} \in \mathcal{M}$, and let

$x_0, \ldots, x_{n-1} \in (x] \cap P$;

then $(x_0 \vee \ldots \vee x_{n-1}] \subseteq (x]$. Thus

$$\begin{aligned}((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P) &= (x_0 \vee \ldots \vee x_{n-1}] \cap P \\ &\subseteq (x] \cap P\end{aligned}$$

and

$$\{x_0, \ldots, x_{n-1}\} \subseteq ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P),$$

an $\mathcal{M}$-ideal. Consequently

$$\sup\{x_0, \ldots, x_{n-1}\} \in ((x_0] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((x_{n-1}] \cap P)$$

and so $\sup\{x_0, \ldots, x_{n-1}\} \in (x] \cap P$.

Now we must prove that

$$(x \vee y] \cap P = ((x] \cap P) \overset{P}{\vee} ((y] \cap P)$$

for all $x, y \in L$. Since P generates L the elements of L can be represented by lattice polynomials over P. More formally, we define a map $F : W(P) \to L$ inductively on the length of elements of $W(P)$:

(1) if $A \in P$ then $F(A) = A$;

(2) if $A = A_0 \smile A_1$ then $F(A) = F(A_0) \vee F(A_1)$;

(3) if $A = A_0 \frown A_1$ then $F(A) = F(A_0) \wedge F(A_1)$.

Since P generates L F is surjective.

Thus we need only show that $A, B \in W(P)$ implies

$$(F(A) \vee F(B)] \cap P = ((F(A)] \cap P) \overset{P}{\vee} ((F(B)] \cap P) \ .$$

We proceed by induction on $\ell(A) + \ell(B)$.

If $\ell(A) + \ell(B) = 2$ then $A, B \in P$ and so the result follows directly from condition (i') a).

Let $n > 2$ and assume that the result holds for all polynomials whose lengths add to an integer less than n . Let $\ell(A) + \ell(B) = n$. Then, by the construction of $W(P)$, we can find polynomials $A_o, \ldots, A_{r-1}, B_o, \ldots, B_{s-1}$ in $W(P)$ such that

$$F(A) = F(A_o) \vee \ldots \vee F(A_{r-1}) \ ,$$
$$\ell(A_o) + \ldots + \ell(A_{r-1}) = \ell(A) \ ,$$

and

$$F(B) = F(B_o) \vee \ldots \vee F(B_{s-1}) \ ,$$
$$\ell(B_o) + \ldots + \ell(B_{s-1}) = \ell(B) \ .$$

For each meaningful index i either $A_i \in P$ or $A_i = X \frown Y$, $X, Y \in W(P)$, and $B_i \in P$ or $B_i = X \frown Y$, $X, Y \in W(P)$. (Note that r or s may very well be equal to 1 .)

If $A_i \in P$ for all $i \leqslant r-1$ and $B_j \in P$ for all $j \leqslant s-1$ then, by condition (i') a)

$$(F(A) \vee F(B)] \cap P = (F(A_o) \vee \ldots \vee F(A_{r-1}) \vee F(B_o) \vee \ldots \vee F(B_{s-1})] \cap P$$

$$= ((F(A_o)] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((F(A_{r-1})] \cap P) \overset{P}{\vee} ((F(B_o)] \cap P) \overset{P}{\vee} \ldots \overset{P}{\vee} ((F(B_{s-1})] \cap P)$$

$$= ((F(A_0) \vee \dots \vee F(A_{r-1})] \bigcap P) \stackrel{P}{\bigvee} ((F(B_0) \vee \dots \vee F(B_{s-1})] \bigcap P)$$
$$= ((F(A)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P) .$$

Otherwise assume for the sake of simplicity that $A_0 = D_0 \frown C_0$. Then

$$F(A) = \{F(C_0) \wedge F(D_0)\} \vee F(A_1) \vee \dots \vee F(A_{r-1}) ,$$

and, since $\ell(A_0) + \dots + \ell(A_{r-1}) < n$,

$$(F(A)] \bigcap P = \{((F(C_0)] \bigcap P) \underset{P}{\bigwedge} ((F(D_0)] \bigcap P)\} \stackrel{P}{\bigvee} ((F(A_1)] \bigcap P) \stackrel{P}{\bigvee} \dots \stackrel{P}{\bigvee} ((F(A_{r-1})] \bigcap P) .$$

Let $C = (\dots(C_0 \smile A_1) \smile \dots \smile A_{r-2}) \smile A_{r-1}$
and $D = (\dots(D_0 \smile A_1) \smile \dots \smile A_{r-2}) \smile A_{r-1}$.
Then, by the distributivity of the $\mathcal{M}$-ideals and the inductive hypothesis,

$$(F(A)] \bigcap P = ((F(C)] \bigcap P) \underset{P}{\bigwedge} ((F(D)] \bigcap P) .$$

Thus

$$((F(A)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P) = \{((F(C)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P)\} \underset{P}{\bigwedge} \{((F(D)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P)\}$$

by distributivity. Now $\ell(C) + \ell(B) < n$ and $\ell(D) + \ell(B) < n$. Thus, by the inductive hypothesis,

$$((F(A)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P)$$
$$= (F(C) \vee F(B)] \bigcap (F(D) \vee F(B)] \bigcap P ,$$

noting that $\underset{P}{\bigwedge} = \bigcap$. However,

$$F(A) \vee F(B) \leqslant (F(C) \vee F(B)) \wedge (F(D) \vee F(B)) .$$

Thus

$$(F(A) \vee F(B)] \bigcap P \subseteq ((F(A)] \bigcap P) \stackrel{P}{\bigvee} ((F(B)] \bigcap P) .$$

But, since $(F(A) \vee F(B)] \bigcap P$ is an $\mathcal{M}$-ideal

$$((F(A)] \cap P) \overset{P}{\vee} ((F(B)] \cap P) \subseteq (F(A) \vee F(B)] \cap P .$$

Thus the theorem is proved.

It may be noted that the proof of the theorem holds true under the seemingly weaker condition that the lattice, denoted temporarily by M, of pseudo-principal $\mathcal{M}$-ideals be distributive. However this is illusory; the lattice of all $\mathcal{M}$-ideals of P is isomorphic to the lattice of ideals of the lattice M and so is distributive if M is ([1], p. 129). A dual statement clearly holds for the $\mathcal{N}$-dual ideals.

Now let P be a poset and let $Q \subseteq P$. In attempting to generalize Theorem 2.30 to the $(\mathcal{M}, \mathcal{N})$-case we note that an $(\mathcal{M}, \mathcal{N})$-structure is tied to the specific poset considered. Thus if P has an $(\mathcal{M}, \mathcal{N})$-structure it is meaningless to consider $FL(Q ; \mathcal{M}, \mathcal{N})$ unless $Q = P$. However, if we let $\mathcal{M}'$ and $\mathcal{N}'$ be defined on Q, then the theorem can be stated. As in Chapter II we let $\mathcal{L}(Q)$ denote the set of those pseudo-principal $\mathcal{M}$-ideals of P obtained by taking a finite sequence of joins and meets of principal $\mathcal{M}$-ideals of P generated by elements of Q. The dual concept is denoted by $\mathcal{J}(Q)$. We denote the join of $\mathcal{M}$-ideals or $\mathcal{N}$-dual ideals of P by $\overset{P}{\vee}$, and the join of $\mathcal{M}'$-ideals or $\mathcal{N}'$-dual ideals of Q by $\overset{Q}{\vee}$.

9. Theorem. Let the following conditions hold:

(i) a) For all $I, J \in \mathcal{I}(Q)$, if $I \cap Q$ and $J \cap Q$ are $\mathcal{M}'$-ideals of Q then

$$(I \overset{P}{\vee} J) \cap Q = (I \cap Q) \overset{Q}{\vee} (J \cap Q) ,$$

and

b) for all $I, J \in \mathcal{D}(Q)$, if $I \cap Q$ and $J \cap Q$ are $\mathcal{N}'$-dual ideals of Q then

$$(I \overset{P}{\vee} J) \cap Q = (I \cap Q) \overset{Q}{\vee} (J \cap Q) ;$$

(ii) given $I_0, I_1 \in \mathcal{I}(Q)$, $D_0, D_1 \in \mathcal{D}(Q)$ such that, for each $i \in \{0, 1\}$,

$$(I_0 \overset{P}{\vee} I_1) \cap D_i = \emptyset , \quad (D_0 \overset{P}{\vee} D_1) \cap I_i = \emptyset , \text{ and}$$

$$(I_0 \overset{P}{\vee} I_1) \cap (D_0 \overset{P}{\vee} D_1) \neq \emptyset$$

then

$$(I_0 \overset{P}{\vee} I_1) \cap (D_0 \overset{P}{\vee} D_1) \cap Q \neq \emptyset .$$

Then there is an isomorphism from L , the sublattice of $FL(P ; \mathcal{M}, \mathcal{N})$ generated by Q , onto $FL(Q ; \mathcal{M}', \mathcal{N}')$; the restriction of this isomorphism to Q is

$$\varphi_Q : Q \to FL(Q ; \mathcal{M}', \mathcal{N}') .$$

The proof of this theorem is a word-for-word duplicate of that of Theorem 2.30.

It should be noted that the only relation between the $(\mathcal{M}, \mathcal{N})$-structure on P and the $(\mathcal{M}', \mathcal{N}')$-structure on Q is that postulated in the two conditions of the theorem. Thus, for example, let $\mathcal{M}' = \mathcal{N}' = \emptyset$, and let $\mathcal{M}$ be the set of all pairs in P that have a sup, and

$\mathcal{N}$ the set of all pairs in P that have an inf; the theorem provides a condition for the lattice in FL(P) generated by Q to be isomorphic to CF(Q) .

An interesting question is: given two structures $(\mathcal{M}_0, \mathcal{N}_0)$ and $(\mathcal{M}_1, \mathcal{N}_1)$ on a poset P , when will the $(\mathcal{M}_0, \mathcal{N}_0)$-free lattice be isomorphic to the $(\mathcal{M}_1, \mathcal{N}_1)$-free lattice? The answer is provided in the following theorem and its corollary.

10. Theorem. Consider $(\varphi_0 , FL(P ; \mathcal{M}_0, \mathcal{N}_0))$ and $(\varphi_1 , FL(P ; \mathcal{M}_1, \mathcal{N}_1))$. There is a lattice homomorphism

$$f : FL(P ; \mathcal{M}_0, \mathcal{N}_0) \to FL(P ; \mathcal{M}_1, \mathcal{N}_1)$$

such that $f \varphi_0 = \varphi_1$ if and only if every pseudo-principal $\mathcal{M}_1$-ideal is an $\mathcal{M}_0$-ideal and, dually, every pseudo-principal $\mathcal{N}_1$-dual ideal is an $\mathcal{N}_0$-dual ideal.

Proof: We first prove the "if" part. By the universal mapping property for $FL(P ; \mathcal{M}_0, \mathcal{N}_0)$ (Definition 1.23) we need only show that φ_1 is an $(\mathcal{M}_0, \mathcal{N}_0)$-morphism.

Since φ_1 is an $(\mathcal{M}_1, \mathcal{N}_1)$-morphism it is clearly isotone

Let $\{x_0, \ldots, x_{n-1}\} \in \mathcal{M}_0$ and let $x = \sup \{x_0, \ldots, x_{n-1}\}$. Since $\varphi_1(x_i) \leqslant \varphi_1(x)$ for each $i \leqslant n-1$, $\varphi_1(x_0) \vee \ldots \vee \varphi_1(x_{n-1}) \leqslant \varphi_1(x)$.

By Coroll. 7 applied to $FL(P ; \mathcal{M}_1, \mathcal{N}_1)$ the set

$$I = \varphi_1^{-1}((\varphi_1(x_0) \vee \ldots \vee \varphi_1(x_{n-1})] \cap \varphi_1(P))$$

is a pseudo-principal $\mathcal{M}_1$-ideal of P ; thus, by the condition of the theorem, I is an $\mathcal{M}_o$-ideal of P. Since $\{x_o, \ldots, x_{n-1}\} \in \mathcal{M}_o$ and $\{x_o, \ldots, x_{n-1}\} \subseteq I$

$$x = \sup \{x_o, \ldots, x_{n-1}\} \in I .$$

This implies that $\varphi_1(x) \leqslant \varphi_1(x_o) \vee \ldots \vee \varphi_1(x_{n-1})$. Thus $\varphi_1(x) = \varphi_1(x_o) \vee \ldots \vee \varphi_1(x_{n-1})$.

This fact and the dual, along with the isotonicity of φ_1, imply that φ_1 is an $(\mathcal{M}_o, \mathcal{N}_o)$-morphism and so the existence of f follows.

Now we prove the "only if" part. Let

$$f : FL(P ; \mathcal{M}_o, \mathcal{N}_o) \rightarrow FL(P ; \mathcal{M}_1, \mathcal{N}_1)$$

be a lattice homomorphism such that $f\varphi_o = \varphi_1$; let I be a pseudo-principal $\mathcal{M}_1$-ideal of P. By the construction of $FL(P ; \mathcal{M}_1, \mathcal{N}_1)$ there is a $y \in FL(P ; \mathcal{M}_1, \mathcal{N}_1)$ such that $\varphi_1(I) = (y] \cap \varphi_1(P)$.

Let $\{x_o, \ldots, x_{n-1}\} \in \mathcal{M}_o$, $\{x_o, \ldots, x_{n-1}\} \subseteq I$, and let $x = \sup \{x_o, \ldots, x_{n-1}\}$. Then $\varphi_1(x_i) \leqslant y$ for each $i \leqslant n-1$; thus $\varphi_1(x_o) \vee \ldots \vee \varphi_1(x_{n-1}) \leqslant y$. Since $\varphi_1 = f\varphi_o$ and f is a lattice homomorphism

$$f(\varphi_o(x_o) \vee \ldots \vee \varphi_o(x_{n-1})) \leqslant y .$$

Since $\{x_o, \ldots, x_{n-1}\} \in \mathcal{M}_o$ and φ_o is an $(\mathcal{M}_o, \mathcal{N}_o)$-morphism, $\varphi_o(x_o) \vee \ldots \vee \varphi_o(x_{n-1}) = \varphi_o(x)$. Thus $\varphi_1(x) = f\varphi_o(x) \leqslant y$ and, by the definition of y, $x \in I$. Thus I is an $\mathcal{M}_o$-ideal.

This result and its dual establish the "only if" part.

Thus the proof of the theorem is complete.

We should like to remark that every pseudo-principal $\mathcal{M}_1$-ideal need not be a pseudo-principal $\mathcal{M}_0$-ideal. Indeed, if $\mathcal{M}_0 = \mathcal{N}_0 = \emptyset$, $\mathcal{M}_1$ consists of all pairs in P with a sup and $\mathcal{N}_1$ of all pairs with an inf, then $FL(P ; \mathcal{M}_0, \mathcal{N}_0) = CF(P)$ and $FL(P ; \mathcal{M}_1, \mathcal{N}_1) = FL(P)$. The theorem then states the well-known relation between CF(P) and FL(P). We note that an $\mathcal{M}_0$-ideal is a hereditary set and clearly every pseudo-principal ideal of P is a hereditary set. However, as the following example illustrates, it need not be a pseudo-principal hereditary set.

Example 1.

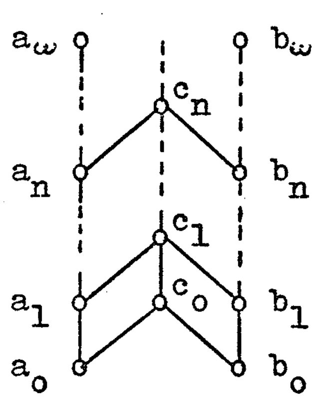

Fig. 1

Let the poset P, depicted in Fig. 1, consist of the elements a_ω, b_ω, a_i, b_i, c_i, $i = 0, 1, \ldots$. The partial order is as indicated in Fig. 1. For any non-negative integers i, j $\sup\{a_i, b_j\} = c_k$, where k is the greater of i, j. Thus P is the ideal join of $(a_\omega]$ and $(b_\omega]$ and so is a pseudo-principal ideal. However P is not a pseudo-principal hereditary set.

11. Coroll. There is a lattice isomorphism (onto) $f : FL(P\,;\mathcal{M}_0, \mathcal{N}_0) \to FL(P\,;\mathcal{M}_1, \mathcal{N}_1)$ such that $f\varphi_0 = \varphi_1$ if and only if every pseudo-principal $\mathcal{M}_0$-ideal is an $\mathcal{M}_1$-ideal and conversely for pseudo-principal $\mathcal{M}_1$-ideals, and, dually, every pseudo-principal $\mathcal{N}_0$-dual ideal is an $\mathcal{N}_1$-dual ideal and conversely for pseudo-principal $\mathcal{N}_1$-dual ideals.

Proof: We need only observe that if

$$f : FL(P\,;\mathcal{M}_0, \mathcal{N}_0) \to FL(P\,;\mathcal{M}_1, \mathcal{N}_1)$$

and

$$g : FL(P\,;\mathcal{M}_1, \mathcal{N}_1) \to FL(P\,;\mathcal{M}_0, \mathcal{N}_0)$$

are lattice homomorphisms such that $f\varphi_0 = \varphi_1$ and $g\varphi_1 = \varphi_0$ then fg is the identity restricted to $\varphi_1(P)$ and gf is the identity restricted to $\varphi_0(P)$; since $\varphi_0(P)$ and $\varphi_1(P)$ are generating sets of the respective lattices fg and gf are the identity maps. Thus they are inverses of each other, and the result follows.

In conclusion, we remark that the Sorkin theorem, Theorem 2.31, remains true for $FL(P\,;\mathcal{M},\mathcal{N})$ with the obvious proviso that ideals are replaced by $\mathcal{M}$-ideals and dually. Part (ii) should be replaced by

(ii) if f_0 preserves sup of all sets in $\mathcal{M}$ (resp. inf of all sets in $\mathcal{N}$) then f preserves join (resp. meet) in $FL(P\,;\mathcal{M},\mathcal{N})$.

The proof is an obvious analogy of that presented in Section 2.4.

2. Completely free lattices.

As a simple application of the results in Section 1 we derive the main results concerning CF(P) in the paper of Dean [3]. We note that CF(P) is just FL(P ; $\emptyset$, $\emptyset$) and that the concept of $\emptyset$-ideal (resp. $\emptyset$-dual ideal) coincides with the concept of hereditary subset (resp. dual hereditary subset). $\emptyset$-ideal and $\emptyset$-dual ideal join is set union (Lemma 1.21).

12. Theorem. In the construction of CF(P) from W(P) rule (2) of Definition 2 can be replaced by:

(2') $A, B \in P$ and $A \leqslant B$.

Proof: If $A, B \in P$ then $\overline{A} \cap \underline{B} \neq \emptyset$ is equivalent to $A \leqslant B$.

We need only show that if $\ell(A) + \ell(B) > 2$ and $\overline{A} \cap \underline{B} \neq \emptyset$ then the result $A \subseteq B$ can be obtained by applications of rules (1), (2'), (3), (4), (5), (6). This is done by induction on $\ell(A) + \ell(B)$.

If $A = A_o \smile A_1$ then $A \subseteq B$ implies that $A_o \subseteq B$, $A_1 \subseteq B$ and, by induction, $A_o \subseteq B$, $A_1 \subseteq B$ can be obtained by replacing rule (2) by rule (2'). Then $A \subseteq B$ follows by applying rule (3). The dual argument applies if $B = B_o \frown B_1$.

If $A = A_o \frown A_1$ then $\overline{A} = \overline{A}_o \cup \overline{A}_1$ and so $(\overline{A}_o \cup \overline{A}_1) \cap \underline{B} \neq \emptyset$, that is, one of $\overline{A}_o \cap \underline{B}$, $\overline{A}_1 \cap \underline{B}$

is non-empty, say $\overline{A}_o \cap \underline{B} \neq \emptyset$. Then $A_o \subseteq B$ and so $A \subseteq B$ can be derived by applying rule (4).

The dual argument applies if $B = B_o \smile B_1$.

Thus we have established Dean's construction of CF(P) ([3] , Definition 1).

13. Theorem ([3] , Theorem 4). FL(P) and CF(P) are identical if and only if P has the following two properties:

(i) given $x, y, z \in P$, $z = \sup\{x, y\}$ if and only if $z = x$ or $z = y$,

and dually

(ii) given $x, y, z \in P$, $z = \inf\{x, y\}$ if and only if $z = x$ or $z = y$.

Proof: It is clear that ideals of P are hereditary subsets and dually. Thus, in view of Coroll.11, we need only show that condition (i) is equivalent to the requirement that all pseudo-principal hereditary subsets be ideals. A dual requirement holds for condition (ii).

Assume that all pseudo-principal hereditary subsets are ideals and let $z = \sup\{x, y\}$. Since the join of hereditary subsets is set union, the pseudo-principal hereditary subset $(x] \cup (y]$ is an ideal, and $\{x, y\} \subseteq (x] \cup (y]$. Thus $z \in (x] \cup (y]$ and so $z \leqslant x$ or $z \leqslant y$. But, clearly, $x \leqslant z$ and $y \leqslant z$; thus

$z = x$ or $z = y$, and so condition (i) holds.

Now assume that condition (i) holds. Let I be a hereditary subset of P ; clearly $x \in I$ and $y \leqslant x$ imply that $y \in I$. Let $x, y \in I$ and let $\sup\{x, y\}$ exist. By condition (i) either $\sup\{x, y\} = x$ or $\sup\{x, y\} = y$; that is, $\sup\{x, y\} \in I$. Thus every hereditary subset is an ideal.

Consequently the theorem is proved.

14. Theorem. Let L be a lattice generated by a subset P . If P is regarded as a poset under the partial order induced by L there is a lattice isomorphism from L onto $CF(P)$ extending $\varphi : P \to CF(P)$ if and only if the following two conditions hold:

(i) a) for all $x, x_0, \ldots, x_{n-1} \in P$ $x \leqslant x_0 \vee \ldots \vee x_{n-1}$ implies that $x \leqslant x_i$ for some $i \leqslant n-1$, and dually,

b) for all $x, x_0, \ldots, x_{n-1} \in P$ $x \geqslant x_0 \wedge \ldots \wedge x_{n-1}$ implies that $x \geqslant x_i$ for some $i \leqslant n-1$;

(ii) given $x_0, x_1, y_0, y_1 \in L$, $x_0 \wedge x_1 \leqslant y_0 \vee y_1$ implies that either $x_0 \wedge x_1 \leqslant y_i$ or $x_i \leqslant y_0 \vee y_1$ for some $i \in \{0, 1\}$.

Proof: The lattice of hereditary subsets (resp. dual hereditary subsets) of P is a sublattice of the lattice of all subsets of P and so is distributive. Thus we need only show that our condition (i) is equivalent to condition (i') of Theorem 8, specialized to $FL(P\ ;\ \emptyset, \emptyset)$, and that our condition (ii) is equivalent to condition (ii) of Theorem 6.

Condition (i) a) of our theorem is equivalent to the requirement that

$$(x_0 \vee \dots \vee x_{n-1}] \cap P = ((x_0] \cap P) \cup \dots \cup ((x_{n-1}] \cap P)$$

for all $x_0, \dots, x_{n-1} \in P$; this is equivalent to condition (i') a) of Theorem 8 because $\overset{P}{\bigvee} = \bigcup$ for hereditary sets. The dual establishes the equivalence of conditions (i) b) and (i') b).

We note that our condition (ii) is stronger than condition (ii) of Theorem 6. Thus we need only show that our condition (ii) holds if L is CF(P) .

Since $CF(P) = FL(P\ ;\ \emptyset, \emptyset)$ condition (ii) of Theorem 6 holds. Thus if $x_0 \wedge x_1 \nleqslant y_i$ and $x_i \nleqslant y_0 \vee y_1$ for all $i \in \{0, 1\}$ and if $x_0 \wedge x_1 \leqslant y_0 \vee y_1$ then $[x_0 \wedge x_1) \cap (y_0 \vee y_1] \cap P \neq \emptyset$. Thus

$$[x_0 \wedge x_1) \cap \{((y_0] \cap P) \cup ((y_1] \cap P)\} \neq \emptyset$$

and so either $[x_0 \wedge x_1) \cap (y_0] \neq \emptyset$ or $[x_0 \wedge x_1) \cap (y_1] \neq \emptyset$, which contradicts the hypothesis that $x_0 \wedge x_1 \nleqslant y_i$. This contradiction establishes condition (ii) of our theorem.

Thus the proof is complete.

15. Coroll. ([3], Theorem 6). Let T be a subset of $CF(P)$ and let $L(T)$ be the sublattice of $CF(P)$ generated by T. A necessary and sufficient condition that $L(T)$ be isomorphic to $CF(Q)$, where Q is a poset isomorphic to T, is that, for any $t, t_o, \ldots, t_{n-1} \in T$, $t \leqslant t_o \vee \ldots \vee t_{n-1}$ implies that $t \leqslant t_i$ for some $i \leqslant n-1$, and dually.

Proof: We need only observe that condition (ii) of Theorem 14 holds in $L(T)$ because it holds in $CF(P)$ and $L(T)$ is a sublattice of $CF(P)$.

In the same vein we can answer a question left unanswered by Dean:

16. Coroll. Let Q be a subset of $CF(P)$. Then $L(Q)$, the sublattice of $CF(P)$ generated by Q, is isomorphic to $FL(Q)$ if and only if, for all $x, y \in L(Q)$,

$(x \vee y] \cap Q = ((x] \cap Q) \overset{Q}{\vee} ((y] \cap Q)$ and

$[x \wedge y) \cap Q = ([x) \cap Q) \overset{Q}{\vee} ([y) \cap Q)$.

Proof: We apply Theorem 6, noting that condition (ii) holds vacuously because $L(Q) \subseteq CF(P)$.

CHAPTER IV

FREE PRODUCTS

We apply the results of the previous two chapters to free products, partially ordered free products, and a special case of amalgamated free product. For a partially ordered set in general there is no direct way of finding joins of ideals and dual ideals nor of distinguishing which are pseudo-principal. For the special cases discussed in this chapter, however, there is an explicit algorithm, in terms of the lattice structure of the factors, that resolves these questions.

1. Free products.

Let Λ be an indexing set and let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices. As outlined in Section 1.3, the set $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$ can be considered to be a poset; if $x, y \in P$ we say that $x \leqslant y$ in P if and only if there is a $\lambda \in \Lambda$ such that $x, y \in L_\lambda$ and $x \leqslant y$ in L_λ. Consequently if $x \in L_\lambda$, $y \in L_\mu$, $\lambda, \mu \in \Lambda$, then $\sup\{x, y\}$ (resp. $\inf\{x, y\}$) exists if and only if $\lambda = \mu$, in which case $\sup\{x, y\} = x \vee y$ (resp. $\inf\{x, y\} = x \wedge y$) in L_λ. The free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$ is $(\varphi, FL(P))$. The restriction of φ to L_λ is denoted by φ_λ and is a lattice injection. Unless there is danger of confusion we denote lattice join and meet in $FL(P)$ and in each L_λ by the same symbols $\vee$ and $\wedge$. We first establish the structure of the pseudo-principal ideals and dual ideals of P. As throughout this work the empty set, even in a lattice, is treated as an ideal and a dual ideal.

1. Lemma. If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of sets such that for each $\lambda \in \Lambda$ I_λ is an ideal (resp. dual ideal) of L_λ then $\bigcup(I_\lambda \mid \lambda \in \Lambda)$ is an ideal (resp. dual ideal) of P.

Proof: Let each I_λ, $\lambda \in \Lambda$, be an ideal of L_λ.

(i) Let $x, y \in P$, $y \in \bigcup(I_\lambda \mid \lambda \in \Lambda)$, and $x \leqslant y$. Then there is a $\lambda \in \Lambda$ such that $y \in I_\lambda \subseteq L_\lambda$. Thus $x \in L_\lambda$ and, since I_λ is an ideal of L_λ, $x \in I_\lambda$. Consequently $x \in \bigcup(I_\lambda \mid \lambda \in \Lambda)$.

(ii) Let $x, y \in \bigcup(I_\lambda \mid \lambda \in \Lambda)$ and let $\sup\{x, y\}$ exist; then there is a $\lambda \in \Lambda$ such that $x, y \in L_\lambda$ and $\sup\{x, y\} = x \vee y$. Since the L_λ are mutually disjoint so are the I_λ; thus $x, y \in I_\lambda$ and so $x \vee y \in I_\lambda \subseteq I$.

Thus $\bigcup(I_\lambda \mid \lambda \in \Lambda)$ is an ideal of P. The principle of duality establishes the result for dual ideals.

2. Lemma. If I is an ideal (resp. dual ideal) of P and $\lambda \in \Lambda$ then $I \cap L_\lambda$ is an ideal (resp. dual ideal) of L_λ.

Proof: Let I be an ideal of P.

(i) Let $x \in I \cap L_\lambda$, $y \in L_\lambda$, and $y \leqslant x$. Then $y \in I$ and so $y \in I \cap L_\lambda$.

(ii) Let $x, y \in I \cap L_\lambda$. Then $x, y \in I$ and $x \vee y = \sup\{x, y\} \in I$, that is, $x \vee y \in I \cap L_\lambda$.

Thus $I \cap L_\lambda$ is an ideal of L_λ. The dual argument of course also applies.

3. Lemma. Let I, J be ideals (resp. dual ideals) of P. For each $\lambda \in \Lambda$

$$(I \vee J) \cap L_\lambda = (I \cap L_\lambda) \vee (J \cap L_\lambda) .$$

(The right-hand side refers to ideal join in L_λ.)

Proof: Let I, J be ideals of P. $I \vee J$ is an ideal of P and so $(I \vee J) \cap L_\lambda$ is an ideal of L_λ. Since both $I \cap L_\lambda$ and $J \cap L_\lambda$ are ideals of L_λ included in $(I \vee J) \cap L_\lambda$, $(I \cap L_\lambda) \vee (J \cap L_\lambda) \subseteq (I \vee J) \cap L_\lambda$.

To prove containment in the other direction we observe that the set L_λ is an ideal of L_λ for each $\lambda \in \Lambda$. Thus by Lemma 1 the set

$$I' = ((I \cap L_\lambda) \vee (J \cap L_\lambda)) \cup \bigcup (L_\mu \mid \mu \neq \lambda)$$

is an ideal of P. Since $I = \bigcup (I \cap L_\mu \mid \mu \in \Lambda)$, and similarly for J, $I, J \subseteq I'$. Thus $I \vee J \subseteq I'$ and so

$$(I \vee J) \cap L_\lambda \subseteq I' \cap L_\lambda = (I \cap L_\lambda) \vee (J \cap L_\lambda) .$$

The dual argument establishes the result for dual ideals.

4. Lemma. If I is a pseudo-principal ideal (resp. pseudo-principal dual ideal) of P then

(i) $I \cap L_\lambda = \emptyset$ for all but a finite number of $\lambda \in \Lambda$;

(ii) if $I \cap L_\lambda \neq \emptyset$ then $I \cap L_\lambda$ is a principal ideal (resp. principal dual ideal) of P.

Proof: For any ideals I, J of P and $\lambda \in \Lambda$ $(I \wedge J) \cap L_\lambda = (I \cap L_\lambda) \wedge (J \cap L_\lambda)$. This fact, along with the result of Lemma 3 establishes (ii); we need only observe that in a lattice pseudo-principal ideals and dual ideals are principal.

Part (i) follows from the observation that if $x \in L_\lambda$, $\lambda \in \Lambda$, then, in P, $(x] \cap L_\mu = \emptyset$ if $\mu \neq \lambda$ and from the fact that pseudo-principal ideals are obtained from a finite sequence of principal ideals.

As before, we consider the set $W(P)$ of polynomials over P. With each $A \in W(P)$ we assiciate a pair of elements of L_λ, for each $\lambda \in \Lambda$, provided they exist, which we call the λ-covers of A.

5. Definition. a) If $\lambda \in \Lambda$ and $\underline{A} \cap L_\lambda \neq \emptyset$ then the generator of the principal ideal $\underline{A} \cap L_\lambda$ of L_λ is denoted by $\underline{A}_\lambda$, and is said to be the lower λ-cover of A; if $\underline{A} \cap L_\lambda = \emptyset$ we say that the lower λ-cover of A does not exist.

b) If $\lambda \in \Lambda$ and $\overline{A} \cap L_\lambda \neq \emptyset$ then the generator of the principal dual ideal $\overline{A} \cap L_\lambda$ of L_λ is denoted by $\overline{A}^\lambda$ and is said to be the upper λ-cover of A; if

$\overline{A} \cap L_\lambda = \emptyset$ then we say that the upper λ-cover of A does not exist.

Intuitively, each $A \in W(P)$ represents an element $\langle A \rangle$ of $FL(P)$ and if $\{x \in L_\lambda \mid x \leqslant \langle A \rangle\} \neq \emptyset$ then $\underline{A}_\lambda$ exists and $\underline{A}_\lambda = \sup (x \in L_\lambda \mid x \leqslant \langle A \rangle)$. The dual statement holds for $\overline{A}^\lambda$.

6. <u>Coroll</u>. $\underline{A} = \bigcup((\underline{A}_\lambda] \mid \underline{A}_\lambda \text{ exists})$ and $\overline{A} = \bigcup([\overline{A}^\lambda) \mid \overline{A}^\lambda \text{ exists})$.

<u>Proof</u>: This follows from the observation that $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$ and thus if $I \subseteq P$ then

$$I = \bigcup(I \cap L_\lambda \mid \lambda \in \Lambda) .$$

We now describe an algorithm that can be used to find $\underline{A}_\lambda$ and $\overline{A}^\lambda$ for each $\lambda \in \Lambda$, $A \in W(P)$.

7. <u>Lemma</u>. (i) If $A \in P$ and $A \in L_\lambda$ then $\underline{A}_\lambda = \overline{A}^\lambda = A$ and $\underline{A}_\mu$, $\overline{A}^\mu$ are undefined for $\mu \neq \lambda$.

(ii) If $A, B, C \in W(P)$, $A = B \smile C$ and $\lambda \in \Lambda$ then $\overline{A}^\lambda$ is defined if and only if $\overline{B}^\lambda$ and $\overline{C}^\lambda$ exist, and $\overline{A}^\lambda = \overline{B}^\lambda \vee \overline{C}^\lambda$.

$\underline{A}_\lambda$ is defined if and only if at least one of $\underline{B}_\lambda$, $\underline{C}_\lambda$ exists; $\underline{A}_\lambda = \underline{B}_\lambda$ if only $\underline{B}_\lambda$ exists; $\underline{A}_\lambda = \underline{C}_\lambda$ if only $\underline{C}_\lambda$ exists; $\underline{A}_\lambda = \underline{B}_\lambda \vee \underline{C}_\lambda$ if both $\underline{B}_\lambda$ and $\underline{C}_\lambda$ exist.

(iii) If $A, B, C \in W(P)$, $A = B \frown C$ and $\lambda \in \Lambda$ then $\underline{A}_\lambda$ is defined if and only if $\underline{B}_\lambda$ and $\underline{C}_\lambda$ exist, and $\underline{A}_\lambda = \underline{B}_\lambda \wedge \underline{C}_\lambda$.

$\overline{A}^\lambda$ is defined if and only if at least one of $\overline{B}^\lambda$, $\overline{C}^\lambda$ exists; $\overline{A}^\lambda = \overline{B}^\lambda$ if only $\overline{B}^\lambda$ exists; $\overline{A}^\lambda = \overline{C}^\lambda$ if only $\overline{C}^\lambda$ exists; $\overline{A}^\lambda = \overline{B}^\lambda \wedge \overline{C}^\lambda$ if both $\overline{B}^\lambda$ and $\overline{C}^\lambda$ exist.

Proof: This follows from Lemma 3. We need only observe that if L is a lattice and $x, y \in L$ then $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. The dual facts also hold: $[x) \vee [y) = [x \wedge y)$ and $[x) \wedge [y) = [x \vee y)$.

8. Lemma. If $\lambda, \mu \in \Lambda$, $A \in W(P)$, and $\underline{A}_\lambda$, $\overline{A}^\mu$ exist then $\lambda = \mu$.

Proof: $\underline{A}_\lambda \in \underline{A}$, $\overline{A}^\mu \in \overline{A}$. Thus, by Lemma 2.4, $\underline{A}_\lambda \leqslant \overline{A}^\mu$. $\underline{A}_\lambda \in L_\lambda$ and $\overline{A}^\mu \in L_\mu$; thus, by the definition of the partial order on P , we conclude that $\lambda = \mu$.

A special case of some interest in the sequel occurs when the free peoduct of two lattices is considered.

9. Lemma. Let $|\Lambda| = 2$, $\Lambda = \{0, 1\}$. Then for each $A \in W(P)$ exactly one of the following four possibilities holds:

(i) $\underline{A}_0$, $\overline{A}^0$ exist and $\underline{A}_1$, $\overline{A}^1$ do not exist;

(ii) $\underline{A}_1$, $\overline{A}^1$ exist and $\underline{A}_0$, $\overline{A}^0$ do not exist;

(iii) $\underline{A}_0$, $\underline{A}_1$ exist and $\overline{A}^0$, $\overline{A}^1$ do not exist;

(iv) $\overline{A}^0$, $\overline{A}^1$ exist and $\underline{A}_0$, $\underline{A}_1$ do not exist.

<u>Proof</u>: These four conditions are clearly mutually exclusive. By Lemma 8 the first part of each condition implies the second. Thus we need only show that for each $A \in W(P)$ the first part of at least one of these four conditions holds. We proceed by induction on the length of the elements of $W(P)$.

If $A \in W(P)$ and $\ell(A) = 1$ then either $A \in L_0$ or $A \in L_1$ and so, by Lemma 7, either (i) or (ii) holds.

Let $n > 1$ and assume that the lemma is true for all lattice polynomials of length $< n$. Let $A \in W(P)$ and $\ell(A) = n$.

If $A = B \cup C$ and $\underline{A}_0$ does not exist then, by Lemma 7, neither $\underline{B}_0$ nor $\underline{C}_0$ exists. Since $\ell(B) < n$, $\ell(C) < n$, the conclusion of our lemma can be applied to B and C, and we find that $\overline{B}^1$ and $\overline{C}^1$ exist; thus $\overline{A}^1$ exists. A similar argument shows that if $\underline{A}_1$ does not exist then $\overline{A}^0$ exists. Thus A must satisfy at least one of the four conditions of our lemma.

If $A = B \cap C$ the dual argument establishes the result.

This result cannot be extended to those cases where $|\Lambda| > 2$:

<u>Example 1.</u> Let $\Lambda = \{0, 1, 2\}$ and for each $\lambda \in \Lambda$ let L_λ be the trivial lattice: $L_\lambda = \{a_\lambda\}$. Then

$$(a_0 \frown a_1) \smile (a_0 \frown a_2) \smile (a_1 \frown a_2)$$

has neither lower covers nor upper covers.

10. Lemma. If $A, B \in W(P)$ then $\overline{A} \cap \underline{B} \neq \emptyset$ if and only if there is a $\lambda \in \Lambda$ such that $\overline{A}^{\lambda}$, $\underline{B}_{\lambda}$ are defined and $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$.

Proof: If $\overline{A} \cap \underline{B} \neq \emptyset$ then there is a $\lambda \in \Lambda$ and an $x \in L_{\lambda}$ such that $x \in \overline{A}$, $x \in \underline{B}$. Thus, by Lemma 4, $\overline{A}^{\lambda} \leqslant x \leqslant \underline{B}_{\lambda}$; that is, $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$.

If $\overline{A}^{\lambda}$, $\underline{B}_{\lambda}$ are defined and $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$ then, since $\overline{A}^{\lambda} \in \overline{A}$, $\underline{B}_{\lambda} \in \overline{A}$ and so $\underline{B}_{\lambda} \in \overline{A} \cap \underline{B}$. Thus $\overline{A} \cap \underline{B} \neq \emptyset$.

11. Lemma. Let $A, B \in W(P)$ and $A \subseteq B$. Then for each $\lambda \in \Lambda$

a) if $\underline{A}_{\lambda}$ exists then $\underline{B}_{\lambda}$ exists and $\underline{A}_{\lambda} \leqslant \underline{B}_{\lambda}$;

b) if $\overline{B}^{\lambda}$ exists then $\overline{A}^{\lambda}$ exists and $\overline{A}^{\lambda} \leqslant \overline{B}^{\lambda}$.

Proof: By Lemma 2.7 $\underline{A} \subseteq \underline{B}$. Thus $\underline{A} \cap L_{\lambda} \subseteq \underline{B} \cap L_{\lambda}$. Since $\underline{A} \cap L_{\lambda} \neq \emptyset$, $\underline{B} \cap L_{\lambda} \neq \emptyset$ and part a) follows.

Part b) is dual to a) and so the lemma is established.

12. Theorem. The quasi-order $\subseteq$ on $W(P)$ satisfies:

If $A, B \in W(P)$ then $A \subseteq B$ if and only if it follows from one of the following conditions:

(1) $A = B$;

(2) there is a $\lambda \in \Lambda$ such that $\overline{A}^{\lambda}$, $\underline{B}_{\lambda}$ exist and $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$;

(3) $A = A_o \smile A_1$ where $A_o \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_o \frown A_1$ where $A_o \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_o \smile B_1$ where $A \subseteq B_o$ or $A \subseteq B_1$;

(6) $B = B_o \frown B_1$ where $A \subseteq B_o$ and $A \subseteq B_1$.

The relation $\sim$ on $W(P)$, where $A \sim B$ if and only if $A \subseteq B$ and $B \subseteq A$, is an equivalence relation and $W(P)/\sim$ is a lattice. For each $\lambda \in \Lambda$ the mapping $\varphi_\lambda : L_\lambda \to W(P)/\sim$, where if $x \in L_\lambda$ then $\varphi_\lambda(x) = \langle x \rangle$, is a lattice embedding. The system $(\varphi_\lambda, \lambda \in \Lambda ; W(P)/\sim)$ is the free product of the $(L_\lambda \mid \lambda \in \Lambda)$.

<u>Proof</u>: This follows directly from Theorem 2.18 and Lemma 10.

13. <u>Coroll</u>. Let Λ' be a subset of Λ such that $\lambda \in \Lambda - \Lambda'$ implies that L_λ is a chain. Then rule (2) of Theorem 12 may be replaced by

(2') a) there is a $\lambda \in \Lambda'$ such that $\overline{A}^\lambda$, $\underline{B}_\lambda$ exist and $\overline{A}^\lambda \leqslant \underline{B}_\lambda$; or

b) there is a $\lambda \in \Lambda$ such that $A, B \in L_\lambda$ and $A \leqslant B$.

<u>Proof</u>: This result is similar to Theorem 3.12 . We need only show that if $A, B \in W(P)$ and $A \subseteq B$ then this fact can be derived from successive applications of rules (1), (2'), (3), (4), (5), (6). We proceed by mathematical induction on $\ell(A) + \ell(B)$.

If $\ell(A) + \ell(B) = 2$ then $A \subseteq B$ derives by rule (2') b).

Let $\ell(A) + \ell(B) = n > 2$ and let the result be true for any two polynomials the sum of whose lengths is less than n. If $\ell(A) + \ell(B) = n$ and $A \subseteq B$ then, in view of the induction hypothesis, we need only show that this quasi-inequality can be derived by a single application of one of the rules (1), (2'), (3), &c. from polynomial inequalities of shorter length. Thus we need only consider the case where $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$ and $\lambda \in \Lambda - \Lambda'$.

If $A = C \smile D$, $C, D \in W(P)$, or $B = C \frown D$ then $A \subseteq B$ can be derived by an application of rule (3) or (6) respectively.

If $B = C \smile D$ then at least one of $\underline{C}_{\lambda}$, $\underline{D}_{\lambda}$, say $\underline{C}_{\lambda}$, exists or both $\underline{C}_{\lambda}$ and $\underline{D}_{\lambda}$ exist. In the former case $\underline{B}_{\lambda} = \underline{C}_{\lambda}$ and in the latter case $\underline{B}_{\lambda} = \underline{C}_{\lambda} \vee \underline{D}_{\lambda}$; since L_{λ} is a chain $\underline{C}_{\lambda} \vee \underline{D}_{\lambda} = \underline{C}_{\lambda}$ or $\underline{D}_{\lambda}$, say $\underline{C}_{\lambda}$. Thus in either case $\overline{A}^{\lambda} \leqslant \underline{C}_{\lambda}$ and, since $\ell(A) + \ell(C) < n$ and $A \subseteq B$ can be derived from $A \subseteq C$ by using rule (5), the result holds.

The case where $A = C \frown D$ is dual to the above.

Thus the corollary is established.

We now state the characterization of the free product:

14. Theorem. Let L be a lattice and let $(L_{\lambda} \mid \lambda \in \Lambda)$ be a family of sublattices of L whose set union generates L. There is a lattice isomorphism from L onto the free

product of $(L_\lambda \mid \lambda \in \Lambda)$ extending each φ_λ if and only if the following three conditions hold:

(i) given $\lambda, \mu \in \Lambda$, $x \in L_\lambda$, $y \in L_\mu$, then $x \leqslant y$ implies that $\lambda = \mu$;

(ii) a) given $x \in L_\lambda$, $y, z \in L$, then $x \leqslant y \vee z$, $x \nleqslant y$, $x \nleqslant z$ imply that there are $y', z' \in L_\lambda$ such that $y' \leqslant y$, $z' \leqslant z$ and $x \leqslant y' \vee z'$, and dually

b) given $x \in L_\lambda$, $y, z \in L$, then $x \geqslant y \wedge z$, $x \ngeqslant y$, $x \ngeqslant z$ imply that there are $y', z' \in L_\lambda$ such that $y' \geqslant y$, $z' \geqslant z$ and $x \geqslant y' \wedge z'$;

(iii) given $x_0, x_1, y_0, y_1 \in L$, then $x_0 \wedge x_1 \leqslant y_0 \vee y_1$, $x_0 \wedge x_1 \nleqslant y_i$, $x_i \nleqslant y_0 \vee y_1$, $i \in \{0, 1\}$, imply that there is a $\lambda \in \Lambda$ and a $z \in L_\lambda$ such that $x_0 \wedge x_1 \leqslant z \leqslant y_0 \vee y_1$.

Proof: This is an application of Theorem 2.19 to the case of the free product of lattices. Condition (i) is equivalent to the fact that the set union of the L_λ as a subset of L is isomorphic to the poset P defining the free product.

In view of Coroll. 2.20 condition (ii)is equivalent to condition (i) of Theorem 2.19. We need only observe that if I, J are ideals of a lattice and $x \notin I, J$ then $x \in I \vee J$ if and only if there are $x_0 \in I$, $x_1 \in J$ such that $x \leqslant x_0 \vee x_1$, and the dual fact for dual ideals.

Condition (iii) is just a restatement of condition (ii) of Theorem 2.19.

As an application of our methods we prove a special case of a theorem of Jónsson [8]. Jónsson's result, holding for certain algebras of which lattices are a special case, is proved in an entirely different manner.

15. Lemma. Let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices. For each $\lambda \in \Lambda$ let M_λ be a sublattice of L_λ. If L is the free product of $(L_\lambda \mid \lambda \in \Lambda)$ then the sublattice of L generated by $\bigcup(M_\lambda \mid \lambda \in \Lambda)$ is isomorphic to the free product of the family $(M_\lambda \mid \lambda \in \Lambda)$.

Proof: We apply Theorem 2.30. Denote the principal ideal of L_λ generated by $x \in L_\lambda$ by $(x]_\lambda$, and the dual concept by $[x)_\lambda$. Let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$ and $Q = \bigcup(M_\lambda \mid \lambda \in \Lambda)$. In view of Lemmas 1, 2, and 3 condition (i) a) of Theorem 2.30 is equivalent to:

(i') a) for all $I, J \in \mathcal{J}(Q)$, and $\lambda \in \Lambda$, if $I \cap M_\lambda$, $J \cap M_\lambda$ are ideals in M_λ then

$$\{(I \cap L_\lambda) \vee (J \cap L_\lambda)\} \cap M_\lambda = (I \cap M_\lambda) \vee (J \cap M_\lambda),$$

the join on the right being that of ideals of M_λ.

Using Lemmas 2, 3, and 4, if $I \in \mathcal{J}(Q)$ and $I \cap L_\lambda \neq \emptyset$ then there is an $x \in M_\lambda$ such that $I \cap L_\lambda = (x]_\lambda$. Condition (i') a) needs verification only if $I \cap L_\lambda \neq \emptyset$, $J \cap L_\lambda \neq \emptyset$. Thus $I \cap L_\lambda = (x]_\lambda$, $J \cap L_\lambda = (y]_\lambda$, $x, y \in M_\lambda$. Clearly $I \cap M_\lambda$, $J \cap M_\lambda$

are ideals of M_λ,

$$(I \cap L_\lambda) \vee (J \cap L_\lambda) = (x \vee y]_\lambda \text{ , and}$$

$$(I \cap M_\lambda) \vee (J \cap M_\lambda) = (x \vee y]_\lambda \cap M_\lambda$$

since $x, y, x \vee y \in M_\lambda$. Thus we have established condition (i') a).

The dual argument establishes condition (i) b) of Theorem 2.30.

To establish condition (ii) let I_0, I_1, D_0, D_1 be as in Theorem 2.30 (ii). Then there is a $\lambda \in \Lambda$ such that

$$(I_0 \overset{P}{\vee} I_1) \cap (D_0 \overset{P}{\vee} D_1) \cap L_\lambda \neq \emptyset \text{ .}$$

However, $I_0 \overset{P}{\vee} I_1 \in \mathcal{I}(Q)$ and $D_0 \overset{P}{\vee} D_1 \in \mathcal{D}(Q)$ and clearly neither is disjoint from L_λ. Thus, as above, there are $x, y \in M_\lambda$ such that

$$(I_0 \overset{P}{\vee} I_1) \cap L_\lambda = (x]_\lambda \text{ , } \quad (D_0 \overset{P}{\vee} D_1) \cap L_\lambda = [y)_\lambda \text{ .}$$

Thus $y \leqslant x$ and so

$$(I_0 \overset{P}{\vee} I_1) \cap (D_0 \overset{P}{\vee} D_1) \cap M_\lambda \neq \emptyset \text{ ,}$$

and condition (ii) is established.

16. <u>Theorem</u>. Let $|\Lambda| > 1$ and let $(\varphi_\lambda, \lambda \in \Lambda ; L)$ be the free product of $(L_\lambda \mid \lambda \in \Lambda)$. Let $\Lambda' \subseteq \Lambda$. $L - \bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda')$ is a sublattice of L if and only if L_λ is a chain for each $\lambda \in \Lambda'$.

<u>Proof</u>: Let L_λ be a chain for each $\lambda \in \Lambda'$. We need only show that given $x, y \in L - \bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda')$ then neither $x \vee y$ nor $x \wedge y$ can be elements of

$\varphi_\lambda(L_\lambda)$, $\lambda \in \Lambda'$. It is convenient to identify $\varphi_\lambda(L_\lambda)$ and L_λ .

Let $x \vee y = z \in L_\lambda$ where L_λ is a chain. Applying Theorem 14, $z \leqslant x$ or $z \leqslant y$; for otherwise there would be $x', y' \in L_\lambda$, $x' \leqslant x$, $y' \leqslant y$ and $z \leqslant x' \vee y'$, which, since L_λ is a chain, implies that $z \leqslant x'$ or $z \leqslant y'$. Since $x, y \leqslant z$, $z = x$ or $z = y$. Thus if $x \vee y \in L_\lambda$ either $x \in L_\lambda$ or $y \in L_\lambda$.

The dual argument holds for $x \wedge y$.

To prove the converse let L_λ , $\lambda \in \Lambda'$, not be a chain. Then there are distinct elements $x, y, z \in L_\lambda$ such that $x \vee y = z$. Since $|\Lambda| > 1$ there is a $\mu \in \Lambda$ distinct from λ . Let $d \in L_\mu$ and consider the polynomials $A = (x \smile d) \frown z$, $B = (y \smile d) \frown z$ in $W(P)$. By Lemma 7, $\overline{A}^\lambda = \overline{B}^\lambda = z$, $\underline{A}_\lambda = x$, $\underline{B}_\lambda = y$. Since $\overline{A}^\lambda \neq \underline{A}_\lambda$, $\overline{B}^\lambda \neq \underline{B}_\lambda$ then neither A nor B represent elements of L_λ . In view of Lemma 8 they cannot represent elements of L_ν , $\nu \neq \lambda$. Thus $\langle A\rangle, \langle B\rangle \in L - \bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda')$.

Since $A \subseteq z$, $B \subseteq z$ we conclude that $A \smile B \subseteq z$. Now $\overline{z}^\lambda = z$ and $\underline{A \smile B}_\lambda = x \vee y = z$; thus $z \subseteq A \smile B$ and so $z \sim A \smile B$. Thus $\langle A\rangle \vee \langle B\rangle \in \varphi_\lambda(L_\lambda)$. Thus the converse is established.

Finally we note that, by Lemmas 1 and 4, every non-empty pseudo-principal ideal (resp. pseudo-principal dual ideal) of P is a set union of finitely many principal ideals (resp. principal dual ideals) of P. Thus, by Theorem 2.31:

17. Theorem (Sorkin [9]). If $(L_\lambda \mid \lambda \in \Lambda)$ is a family of lattices, L_λ a lattice, and, for each $\lambda \in \Lambda$, $f_\lambda : L_\lambda \to L$ is an isotone map, then there is an isotone map f from the free product of $(L_\lambda \mid \lambda \in \Lambda)$ to L such that $f\varphi_\lambda = f_\lambda$ for each $\lambda \in \Lambda$.

*2. Partially ordered free products.

Let the indexing set Λ be a poset and let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of pairwise disjoint lattices. As outlined in Section 1.3, the set $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$ can be considered to be a poset; if $x, y \in L_\lambda$ then $x \leqslant y$ in P if and only if $x \leqslant y$ in L_λ, and if $x \in L_\lambda$, $y \in L_\mu$, $\lambda \neq \mu$, then $x \leqslant y$ in P if and only if $\lambda < \mu$. If we let the families $\mathcal{M}$ and $\mathcal{N}$ consist of all pairs $x, y \in P$ such that x, y are elements of the same lattice L_λ then $FL(P\,; \mathcal{M}, \mathcal{N})$ is the partially ordered free product of the $(L_\lambda \mid \lambda \in \Lambda)$. As in Section 1, we define λ-covers.

18. Lemma. Let $(I_\lambda \mid \lambda \in \Lambda)$ be a family of sets such that $I_\lambda \subseteq L_\lambda$ for each $\lambda \in \Lambda$. Then $\bigcup(I_\lambda \mid \lambda \in \Lambda)$ is an $\mathcal{M}$-ideal (resp. $\mathcal{N}$-dual ideal) of P if and only if:

(i) I_λ is an ideal (resp. dual ideal) of L_λ for each $\lambda \in \Lambda$;

(ii) if $I_\lambda \neq \emptyset$ and $\mu < \lambda$ (resp. $\mu > \lambda$) then $I_\mu = L_\mu$.

<u>Proof</u>: We consider the case of $\mathcal{M}$-ideals and invoke the principle of duality in the case of $\mathcal{N}$-dual ideals.

Let $I = \bigcup(I_\lambda \mid \lambda \in \Lambda)$ be an $\mathcal{M}$-ideal of P . Since $\emptyset$ is always an ideal we need only consider the case when $I_\lambda \neq \emptyset$. If $x \in I_\lambda$ and $y \in L_\lambda$, $y \leqslant x$, then $y \leqslant x$ in P ; thus $y \in I$ and so $y \in I_\lambda$. If $x, y \in I_\lambda$ then, since $x, y \in L_\lambda$, $\{x, y\} \in \mathcal{M}$. Thus $x \vee y = \sup \{x, y\} \in I$; consequently $x \vee y \in I_\lambda$. Thus I_λ is an ideal of L_λ and condition (i) is verified.

Let $\mu < \lambda$, $I_\lambda \neq \emptyset$, and $x \in L_\mu$. There is a $y \in I_\lambda$ and so $x \leqslant y$ in P ; thus $x \in I$ and so $I_\mu = L_\mu$. Consequently condition (ii) is verified.

To prove the converse assume that conditions (i) and (ii) hold; we show that I is an $\mathcal{M}$-ideal. Let $x \in I$, $x \in L_\lambda$, and let $y \leqslant x$. Thus $x \in I_\lambda$. If $y \in L_\lambda$ then, since I_λ is an ideal of L_λ , $y \in I_\lambda \subseteq I$. If $y \notin L_\lambda$ then $y \in L_\mu$ where $\mu < \lambda$ and, by condition (ii), $y \in I_\mu \subseteq I$. Thus $x \in I$ and $y \leqslant x$ imply $y \in I$.

Let $x, y \in I$, $\{x, y\} \in \mathcal{M}$. Then there is a $\lambda \in \Lambda$ such that $x, y \in L_\lambda$. Thus $x, y \in I_\lambda$ and, by condition (i), $\sup \{x, y\} = x \vee y \in I_\lambda \subseteq I$. Thus I is an

$\mathcal{M}$-ideal and the lemma is established.

19. Lemma. Let I, J be $\mathcal{M}$-ideals (resp. $\mathcal{N}$-dual ideals) of P. For each $\lambda \in \Lambda$

$$(I \vee J) \cap L_\lambda = (I \cap L_\lambda) \vee (J \cap L_\lambda) .$$

Proof: The proof duplicates that of Lemma 3. We need only observe that the only difficulty occurs when $(I \cap L_\lambda) \vee (J \cap L_\lambda) \neq L_\lambda$. In this case the set I' of Lemma 3 can be replaced by the $\mathcal{M}$-ideal

$$\{(I \cap L_\lambda) \vee (J \cap L_\lambda)\} \bigcup \{\bigcup (L_\mu \mid \mu \not\geq \lambda)\} .$$

20. Lemma. If I is a pseudo-principal $\mathcal{M}$-ideal (resp. pseudo-principal $\mathcal{N}$-dual ideal) of P then

(i) $I \cap L_\lambda = \emptyset$ or L for all but a finite number of $\lambda \in \Lambda$;

(ii) if $I \cap L_\lambda \neq \emptyset, L_\lambda$ then $I \cap L_\lambda$ is a principal ideal (resp. principal dual ideal) of L_λ.

The proof is an obvious generalization of that of Lemma 4.

In Lemma 18 we note that for an $A \in W(P)$, $\lambda \in \Lambda$, it is possible for $\underline{A} \cap L_\lambda \neq \emptyset$ and yet for $\underline{A} \cap L_\lambda$ not to be principal. Indeed, $\underline{A}$ need not be a set union of finitely many principal $\mathcal{M}$-ideals. The dual observations hold for $\overline{A}$.

Example 2. Let Λ be the lattice depicted in Fig. 1. Let $x \in L_a$, $y \in L_b$, and let L_c have no greatest element. Then $\underline{x \frown y} = L_c$. Consequently $\underline{x \frown y}$ is not a finite union of principal $\mathcal{M}$-ideals, nor is $\underline{x \frown y} \cap L_c$ a principal ideal of L_c.

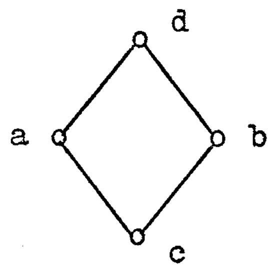

Fig. 1

In order to define λ-covers we introduce the lattices L^b_λ. For each $A \in W(P)$ and $\lambda \in \Lambda$, $\underline{A}_\lambda$ and $\overline{A}^\lambda$, if they exist, will be elements of L^b_λ.

21. Definition. (i) If $A \in P$ and $A \in L_\lambda$ then $\underline{A}_\lambda = \overline{A}^\lambda = A$; $\underline{A}_\mu$ exists if and only if $\mu \leqslant \lambda$ and $\underline{A}_\mu = 1$ if $\mu < \lambda$; $\overline{A}^\mu$ exists if and only if $\mu \geqslant \lambda$ and $\overline{A}^\mu = 0$ if $\mu > \lambda$.

(ii) If $A, B, C \in W(P)$, $A = B \smile C$, and $\lambda \in \Lambda$ then $\overline{A}^\lambda$ is defined if and only if $\overline{B}^\lambda$ and $\overline{C}^\lambda$ exist and $\overline{A}^\lambda = \overline{B}^\lambda \vee \overline{C}^\lambda$.

$\underline{A}_\lambda$ is defined if and only if at least one of $\underline{B}_\lambda$, $\underline{C}_\lambda$ exists; $\underline{A}_\lambda = \underline{B}_\lambda$ if only $\underline{B}_\lambda$ exists; $\underline{A}_\lambda = \underline{C}_\lambda$ if only $\underline{C}_\lambda$ exists; $\underline{A}_\lambda = \underline{B}_\lambda \vee \underline{C}_\lambda$ if both exist.

(iii) If $A, B, C \in W(P)$, $A = B \frown C$, and $\lambda \in \Lambda$ then $\underline{A}_\lambda$ is defined if and only if $\underline{B}_\lambda$ and $\underline{C}_\lambda$ exist and $\underline{A}_\lambda = \underline{B}_\lambda \wedge \underline{C}_\lambda$.

$\overline{A}^\lambda$ is defined if and only if at least one of $\overline{B}^\lambda$, $\overline{C}^\lambda$

exists; $\overline{A}^{\lambda} = \overline{B}^{\lambda}$ if only $\overline{B}^{\lambda}$ exists; $\overline{A}^{\lambda} = \overline{C}^{\lambda}$ if only $\overline{C}^{\lambda}$ exists; $\overline{A}^{\lambda} = \overline{B}^{\lambda} \wedge \overline{C}^{\lambda}$ if both exist.

22. Coroll. a) For each $A \in W(P)$, $\underline{A}_{\lambda} \in L_{\lambda}$ for only finitely many $\lambda \in \Lambda$.

b) For each $A \in W(P)$, $\overline{A}^{\lambda} \in L_{\lambda}$ for only finitely many $\lambda \in \Lambda$.

23. Lemma. Let $\lambda \in \Lambda$, $A \in W(P)$.

(i) If $\underline{A} \cap L_{\lambda} \neq \emptyset, L_{\lambda}$ then $\underline{A} \cap L_{\lambda}$ is the principal ideal of L_{λ} generated by $\underline{A}_{\lambda}$; if $\overline{A} \cap L_{\lambda} \neq \emptyset$, L then $\overline{A} \cap L_{\lambda}$ is the principal dual ideal of L_{λ} generated by $\overline{A}^{\lambda}$.

(ii) If $\underline{A} \cap L_{\lambda} = L_{\lambda}$ and L_{λ} is not a principal ideal of L_{λ} then $\underline{A}_{\lambda} = 1 \in L_{\lambda}^{b}$; if $\overline{A} \cap L_{\lambda} = L_{\lambda}$ and L_{λ} is not a principal dual ideal of L_{λ} then $\overline{A}^{\lambda} = 0 \in L_{\lambda}^{b}$.

(iii) If $\underline{A} \cap L_{\lambda} = \emptyset$ then $\underline{A}_{\lambda}$ is undefined; if $\overline{A} \cap L_{\lambda} = \emptyset$ then $\overline{A}^{\lambda}$ is undefined.

(iv) If $\underline{A} \cap L_{\lambda} = L_{\lambda}$ and L_{λ} is a principal ideal of L_{λ} then $\underline{A}_{\lambda} = 1 \in L_{\lambda}^{b}$ or $\underline{A}_{\lambda}$ is the greatest element of L_{λ}; if $\overline{A} \cap L_{\lambda} = L_{\lambda}$ and L_{λ} is a principal dual ideal of L_{λ} then $\overline{A}^{\lambda} = 0 \in L_{\lambda}^{b}$ or $\overline{A}^{\lambda}$ is the least element of L_{λ}.

Proof: This is a straight-forward calculation, using Lemma 19 and Definition 21.

As an illustration of part (iv):

<u>Example 3.</u> Let Λ be the poset depicted in Fig. 2. For each $\lambda \in \Lambda$ let $L_\lambda = \{a_\lambda\}$. Then $\underline{a_\omega} \cap L_\lambda = L_\lambda$ for all $\lambda \in \Lambda$; $\underline{a_\omega}_\omega = a_\omega$ and $\underline{a_\omega}_\lambda = 1$ if $\lambda < \omega$.

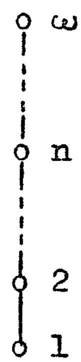

Fig. 2

24. <u>Lemma</u>. If $\lambda, \mu \in \Lambda$, $A \in W(P)$, and $\underline{A}_\lambda$, $\overline{A}^\mu$ exist then $\lambda \leqslant \mu$.

<u>Proof</u>: By Lemma 23 $\underline{A} \cap L_\lambda \neq \emptyset$ and $\overline{A} \cap L_\mu \neq \emptyset$. Let $x \in \underline{A} \cap L_\lambda$, $y \in \overline{A} \cap L_\mu$; by Lemma 2.4 in the $(\mathfrak{m}, \mathfrak{n})$ case $x \leqslant y$ and so, by definition of the partial order on P, $\lambda \leqslant \mu$.

25. <u>Lemma</u>. a) If $A \in L_\lambda$, $B, C \in W(P)$, $A \subseteq B \smile C$, $A \nsubseteq B$, and $A \nsubseteq C$ then $\underline{B \smile C}_\lambda \in L_\lambda$, and $A \leqslant \underline{B \smile C}_\lambda$.

b) If $B \frown C \subseteq A$, $B \nsubseteq A$, and $C \nsubseteq A$ then $\overline{B \frown C}^\lambda \in L_\lambda$, and $\overline{B \frown C}^\lambda \leqslant A$.

<u>Proof</u>: We prove a) and invoke duality in the case of b).

Because of the hypotheses $\overline{A} \cap \underline{B \smile C} \neq \emptyset$. Thus there is a $\mu \in \Lambda$ such that $\overline{A} \cap \underline{B \smile C} \cap L_\mu \neq \emptyset$. Since $\overline{A} \cap L_\mu \neq \emptyset$, $\mu \geqslant \lambda$. Thus, by Lemma 18, $\underline{B \smile C} \cap L_\lambda \neq \emptyset$. Thus $\underline{B \smile C}_\lambda$ exists. If $\underline{B \smile C}_\lambda = 1$ then either $\underline{B}_\lambda$ or $\underline{C}_\lambda = 1$, say $\underline{B}_\lambda = 1$. (The only case not immediately obvious is $\underline{B \smile C}_\lambda = \underline{B}_\lambda \vee \underline{C}_\lambda$; by the definition of L^b_λ either $\underline{B}_\lambda$ or $\underline{C}_\lambda = 1$.) Then $\underline{B} \cap L_\lambda = L_\lambda$ and so $\overline{A} \cap \underline{B} \neq \emptyset$, that is, $A \subseteq B$, contradicting the hypothesis of the lemma. Thus $\underline{B \smile C}_\lambda \in L_\lambda$ and, clearly, $A \leqslant \underline{B \smile C}_\lambda$.

This argument and the dual establish the lemma.

26. Lemma. If $A, B, C, D \in W(P)$, $A \frown B \subseteq C \smile D$, $A \frown B \nsubseteq C, D$, and $A, B \nsubseteq C \smile D$, then there is a $\lambda \in \Lambda$ such that $\overline{A \frown B}^\lambda \in L_\lambda$ and $\underline{C \smile D}_\lambda \in L_\lambda$, and

$$\overline{A \frown B}^\lambda \leqslant \underline{C \smile D}_\lambda .$$

Proof: By the definition of the quasi-order on $W(P)$

$$\overline{A \frown B} \cap \underline{C \smile D} \neq \emptyset .$$

Thus there is a $\lambda \in \Lambda$ such that

$$\overline{A \frown B} \cap \underline{C \smile D} \cap L_\lambda \neq \emptyset .$$

Thus $\overline{A \frown B}^\lambda$, $\underline{C \smile D}_\lambda$ exist and, as in the proof of Lemma 25, $\overline{A \frown B}^\lambda \neq 0$, $\underline{C \smile D}_\lambda \neq 1$. As in the proof of Lemma 10 it follows that $\overline{A \frown B}^\lambda \leqslant \underline{C \smile D}_\lambda$.

We observe also:

27. Lemma. If $A, B \in W(P)$ and $A \subseteq B$, $\lambda \in \Lambda$, then

a) if $\underline{A}_\lambda$ exists then $\underline{B}_\lambda$ exists, and $\underline{A}_\lambda \leqslant \underline{B}_\lambda$;

b) if $\overline{B}^\lambda$ exists then $\overline{A}^\lambda$ exists, and $\overline{A}^\lambda \leqslant \overline{B}^\lambda$.

In view of Lemmas 25 and 26, Theorem 12 still holds in the case of partially ordered free products; we replace condition (2) by:

(2) there is a $\lambda \in \Lambda$ such $\overline{A}^\lambda \in L_\lambda$, $\underline{B}_\lambda \in L_\lambda$, and $\overline{A}^\lambda \leqslant \underline{B}_\lambda$.

We should like to point out that one could very well dispence with the requirement that $\overline{A}^\lambda \neq 0$, $\underline{B}_\lambda \neq 1$; this, however, would entail investigating infinitely many upper and lower λ-covers in order to determine whether or not $A \subseteq B$. In the approach we chose to follow this determination is effective, in view of Coroll. 22.

We observe that, with the obvious modification, Coroll. 13 holds in the case of partially ordered free products.

Theorem 14 holds if we replace its condition (i) by:

(i) given $\lambda, \mu \in \Lambda$, $x \in L_\lambda$, $y \in L_\mu$ then $x \leqslant y$ implies that $\lambda \leqslant \mu$.

Lemma 15 holds with no change in the statement of the result. The proof in our present case proceeds exactly as the proof given, provided that one observes that if $I, J \in \mathcal{J}(Q)$ and $I \cap L_\lambda = L_\lambda$ then $I \cap M_\lambda = M_\lambda$ and so

$$((I \cap L_\lambda) \vee (J \cap L_\lambda)) \cap M_\lambda = (I \cap M_\lambda) \vee (J \cap M_\lambda) .$$

Parenthetically, it may be remarked that the method of Jónsson [8] alluded to in the discussion of Lemma 15 does not apply to this case.

Theorem 16 also holds in our case. We need only observe that, in the notation of Theorem 16, $\bar{d}^\lambda = 0$ if it exists and $\underline{d}_\lambda = 1$ if it exists. Thus the upper and lower λ-covers of A and B are as given in Theorem 16. Furthermore, since $\underline{A}_\lambda, \underline{B}_\lambda \neq 1$ and $\bar{A}^\lambda, \bar{B}^\lambda \neq 0$, neither A nor B can be equivalent to an element of L_ν, $\nu \neq \lambda$.

Since pseudo-principal $\mathfrak{M}$-ideals need not be finite set unions of principal $\mathfrak{M}$-ideals, and dually, it is somewhat surprising that the equivalent of Theorem 17 holds:

28. <u>Theorem</u>. Let Λ be a poset, $(L_\lambda \mid \lambda \in \Lambda)$ a family of lattices, and L a lattice. If for each $\lambda \in \Lambda$ there is an isotone map $f_\lambda : L_\lambda \to L$ such that $\lambda < \mu$ implies $f_\lambda(x) \ll f_\mu(y)$ for all $x \in L_\lambda$, $y \in L_\mu$ then there is an isotone map f from the partially ordered free product of $(L_\lambda \mid \lambda \in \Lambda)$ to L ; for each $\lambda \in \Lambda$ $f \varphi_\lambda = f_\lambda$.

<u>Proof</u>: Define $f_0 : P \to L$ by $f_0(x) = f_\lambda(x)$ if $x \in L_\lambda$. Then f_0 is isotone. Referring to the proof of Theorem 2.31 in the $(\mathfrak{M}, \mathfrak{N})$-case, we need only show that

$F(A) \in L$ for all $A \in W(P)$. We observe that if $A = B \smile C$ then

$$F(A) = \bigvee(f_o(\underline{A}_\lambda) \mid \underline{A}_\lambda \neq 1) \vee F(B) \vee F(C) .$$

This holds because $\underline{A}_\lambda = 1$ implies that either $\underline{B}_\lambda$ or $\underline{C}_\lambda = 1$, say $\underline{B}_\lambda = 1$. Thus $\underline{A} \cap L_\lambda = \underline{B} \cap L_\lambda$ and so if $x \in \underline{A} \cap L_\lambda$ then $f_o(x) \leqslant F(B) \leqslant F(B) \vee F(C)$. Since $\{ \lambda \in \Lambda \mid \underline{A}_\lambda$ exists and $\underline{A}_\lambda \neq 1\}$ is finite then $F(A) \in L$ provided that $F(B), F(C) \in L$.

A dual argument applies if $A = B \frown C$.

Consequently an inductive argument establishes the theorem.

*3. Amalgamated free products.

As in Definition 1.16 let L_λ , $\lambda \in \Lambda$, M be lattices and let $\psi_\lambda : M \to L_\lambda$ be a lattice injection for each $\lambda \in \Lambda$. Matters will be greatly simplified if M is thought of as a sublattice of each L , and thus $\lambda \neq \mu$ implies $L_\lambda \cap L_\mu = M$. With this convention in mind let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$. We define a partial order on P . Let $\leqslant_\lambda$ denote the partial order on L_λ . Since M is a sublattice of L_λ for each $\lambda \in \Lambda$, if $x, y \in M$ and $\lambda, \mu \in \Lambda$ then $x \leqslant_\lambda y$ if and only if $x \leqslant_\mu y$. Let $x, y \in P$; if there is a $\lambda \in \Lambda$ such $x, y \in L_\lambda$ then we define $x \leqslant y$ if and only if $x \leqslant_\lambda y$; if $x \in L_\lambda$,

$y \in L_\mu$, $\lambda \neq \mu$, then we define $x \leqslant y$ if and only if there is a $z \in M$ such that $x \leqslant_\lambda z$ and $z \leqslant_\mu y$.

29. Lemma. The relation $\leqslant$ is well-defined and is a partial order on P. For each $\lambda \in \Lambda$, $\leqslant_\lambda$ is the restriction of $\leqslant$ to L_λ.

Proof: To show that $\leqslant$ is well-defined let $\lambda \neq \mu$, $x \in L_\lambda$ and $x \in L_\mu$. Let $y \in L_\nu$. Since $x \in M$ $x \leqslant_\lambda z_1 \leqslant_\nu y$, $z_1 \in M$, if and only if $x \leqslant_\nu y$. This and the dual argument show that $\leqslant$ is well-defined.

Now we show that $\leqslant$ is a partial order.

1) $\leqslant$ is clearly reflexive.

2) Let $x \in L_\lambda$, $y \in L_\mu$, and $x \leqslant y$, $y \leqslant x$. If $\lambda = \mu$ then clearly $x = y$. If $\lambda \neq \mu$ then there are $z_1, z_2 \in M$ such that

$$x \leqslant_\lambda z_1 \leqslant_\mu y \leqslant_\mu z_2 \leqslant_\lambda x .$$

Thus $z_1 \leqslant_\mu z_2$ and so $z_1 \leqslant_\lambda z_2$. Thus, since $\leqslant_\lambda$ is antisymmetric, $x = z_1 = z_2$. Thus $x \leqslant_\mu y \leqslant_\mu x$ and so $x = y$. Consequently $\leqslant$ is antisymmetric.

3) Let $x \in L_\lambda$, $y \in L_\mu$, $z \in L_\nu$ and $x \leqslant y \leqslant z$. If $\lambda \neq \mu$, $\mu \neq \nu$ then there are $z_1, z_2 \in M$ such that

$$x \leqslant_\lambda z_1 \leqslant_\mu y \leqslant_\mu z_2 \leqslant_\nu z .$$

Thus $z_1 \leqslant_\mu z_2$ and so $z_1 \leqslant_\lambda z_2$. Thus $x \leqslant_\lambda z_2 \leqslant_\nu z$. Consequently $x \leqslant z$. If $\lambda = \mu$ or $\mu = \nu$ similar arguments hold. Thus $\leqslant$ is transitive.

Consequently $\leqslant$ is a partial order on P and, from the definition, $\leqslant$ restricted to L_λ is $\leqslant_\lambda$.

Thus we have proved the lemma.

We now define an $(\mathfrak{M}, \mathfrak{N})$-structure on P ; $\mathfrak{M}$, $\mathfrak{N}$ consist of all pairs $\{x, y\}$ such that there is a $\lambda \in \Lambda$, $x, y \in L_\lambda$. It is clear that $FL(P ; \mathfrak{M}, \mathfrak{N})$ is the free product of $(L_\lambda \mid \lambda \in \Lambda)$ amalgamated by M .

We first extend Lemma 15 to amalgamated free products.

30. <u>Theorem</u>. For each $\lambda \in \Lambda$ let M_λ be a lattice such that $M \subseteq M_\lambda \subseteq L_\lambda$. Then the sublattice of the amalgamated free product of the $(L_\lambda \mid \lambda \in \Lambda)$ generated by $\bigcup(M_\lambda \mid \lambda \in \Lambda)$ is isomorphic to the free product of $(M_\lambda \mid \lambda \in \Lambda)$ amalgamated by M .

We note that the free product of the $(M_\lambda \mid \lambda \in \Lambda)$ must be amalgamated by the same sublattice as the $(L_\lambda \mid \lambda \in \Lambda)$:

<u>Example 4.</u> Let P be the poset depicted in Fig. 3. Let

$$L_0 = \{a, b, c, d\} ;$$
$$L_1 = \{a, d, e\} ;$$
$$M = \{a, d\} .$$

In L , the free product of L_0 and

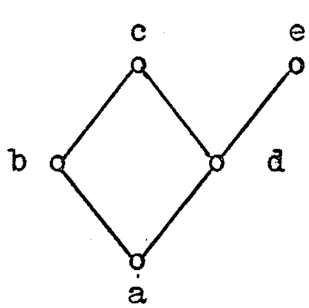

Fig. 3

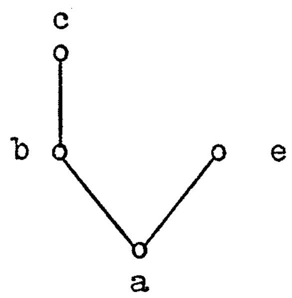

Fig. 4

L_1 amalgamated by M, $b \vee e = c \vee e$; $c = b \vee d$, $d \vee e = e$ and thus $c \vee e = b \vee d \vee e = b \vee e$.

Let $M_0 = \{a, b, c\}$;

$M_1 = \{a, e\}$;

$N = \{a\}$.

Then in L', the free product of M_0 and M_1 amalgamated by N, $b \vee e \neq c \vee e$. Thus the sublattice of L generated by $M_0 \bigcup M_1$ is not isomorphic to L'.

We prove Theorem 30 by applying Theorem 3.9. Let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$ as above and let $Q = \bigcup(M_\lambda \mid \lambda \in \Lambda)$. Let $\mathcal{M}'$, $\mathcal{N}'$ consist of all pairs $\{x, y\} \subseteq Q$ such that there is a $\lambda \in \Lambda$, $x, y \in L_\lambda$. Then $FL(Q; \mathcal{M}', \mathcal{N}')$ is the free product of $(M_\lambda \mid \lambda \in \Lambda)$ amalgamated by M.

We analyse the structure of $\mathcal{M}$-ideals in $\mathcal{I}(Q)$ and invoke the principle of duality in the case of $\mathcal{N}$-dual ideals of $\mathcal{D}(Q)$.

31. Lemma. If I is an $\mathcal{M}$-ideal of P then $I \bigcap Q$ is an $\mathcal{M}'$-ideal of Q.

Proof: $I \bigcap Q$ is clearly a hereditary subset of Q. Let $\{x, y\} \in \mathcal{M}'$, $x, y \in I \bigcap Q$. Thus $x, y \in M_\lambda$, $\lambda \in \Lambda$. Then $x, y \in L_\lambda$ and so $x \vee y \in I \bigcap M_\lambda$; thus $\sup_Q \{x, y\} \in I \bigcap Q$.

Given an $\mathcal{M}'$-ideal I of Q we define the set $(I] \subseteq P$; $(I] = \{x \in P \mid x \leqslant y$ for some $y \in I\}$.

32. Lemma. If I is an $\mathcal{M}'$-ideal of Q then $(I]$ is an $\mathcal{M}$-ideal of P .

Proof: $(I]$ is clearly a hereditary subset of P . Let $\{x, y\} \in \mathcal{M}$, $x, y \in (I]$. Thus there is a $\lambda \in \Lambda$, $x, y \in L_\lambda$ and there are $x', y' \in I$, $x \leqslant x'$, $y \leqslant y'$. If $x', y' \in M_\lambda$ then $\{x', y'\} \in \mathcal{M}'$ and so $x \vee y \leqslant x' \vee y' \in I$. If $x' \in M_\lambda$, $y' \in M_\mu$, $\lambda \neq \mu$, then $x \leqslant_\lambda x'$ and there is a $z \in M$ such that $y \leqslant_\lambda z \leqslant_\mu y'$; thus $z \in I$ and so $x \vee z \in I$ and $x \vee y \leqslant_\lambda x \vee z$. Similarly if $x' \in M_\nu$, $y' \in M_\mu$, λ, μ, ν distinct, then there are $z_1, z_2 \in M$ such that

$$x \leqslant_\lambda z_1 \leqslant_\nu x' ,$$
$$y \leqslant_\lambda z_2 \leqslant_\mu y' .$$

Thus $z_1, z_2 \in I$ and so $z_1 \vee z_2 \in I$ and $x \vee y \leqslant z_1 \vee z_2$. Thus in each of these cases $x \vee y \in (I]$.

Consequently $(I]$ is an $\mathcal{M}$-ideal of P .

33. Lemma. If I, J are $\mathcal{M}'$-ideals of Q then $(I \wedge_Q J] = (I] \wedge_P (J]$.

Proof: Since, for all $\mathcal{M}$, $\mathcal{M}$-ideal meet is set intersection we need only prove that $(I \cap J] = (I] \cap (J]$.

Clearly $(I \cap J] \subseteq (I] \cap (J]$.

Let $x \in (I] \cap (J]$; then there are $y \in I$, $z \in J$

such that $x \leqslant y, z$. If $x \in L_\lambda$, $y \in L_\mu$, $z \in L_\nu$ and $\lambda \neq \mu$, $\lambda \neq \nu$ then there are $y', z' \in M$ such that $x \leqslant_\lambda y' \leqslant_\mu y$ and $x \leqslant_\lambda z' \leqslant_\nu z$. Thus $y' \in I$, $z' \in J$ and so $y' \wedge z' \in I \cap J$; consequently $x \in (I \cap J]$.

Similar arguments, as presented in the proof of Lemma 32, apply to the other cases in which $x \leqslant y, z$.

Thus $(I] \cap (J] \subseteq (I \cap J]$, and so the lemma is established.

34. <u>Lemma</u>. If $I \in \mathscr{I}(Q)$ then $I = (I \cap Q]$.

<u>Proof</u>: Clearly $(I \cap Q] \subseteq I$; thus we need only show that $I \subseteq (I \cap Q]$. In view of the definition of $\mathscr{I}(Q)$ an inductive argument applies.

If I is a principal $\mathfrak{M}$-ideal of P generated by an element of Q then clearly $I \subseteq (I \cap Q]$ since $I \cap Q$ is the principal $\mathfrak{M}'$-ideal of Q generated by the same element.

Let $I \subseteq (I \cap Q]$ and $J \subseteq (J \cap Q]$. Then $I \underset{P}{\wedge} J \subseteq (I \cap Q] \underset{P}{\wedge} (J \cap Q] = ((I \cap Q) \underset{Q}{\wedge} (J \cap Q)]$ by Lemma 33. Thus $I \underset{P}{\wedge} J \subseteq ((I \cap J) \cap Q] = ((I \underset{P}{\wedge} J) \cap Q]$.

Also $I \subseteq (I \cap Q] \subseteq ((I \overset{P}{\vee} J) \cap Q]$ and similarly $J \subseteq ((I \overset{P}{\vee} J) \cap Q]$. Thus $I \overset{P}{\vee} J \subseteq ((I \overset{P}{\vee} J) \cap Q]$.

Consequently $I \subseteq (I \cap Q]$ for all $I \in \mathscr{I}(Q)$, and the lemma follows.

35. <u>Lemma</u>. If $I, J \in \mathscr{I}(Q)$ then

$$(I \overset{P}{\vee} J) \cap Q = (I \cap Q) \overset{Q}{\vee} (J \cap Q) .$$

<u>Proof</u>: Since $I \cap Q$, $J \cap Q \subseteq (I \overset{P}{\vee} J) \cap Q$ and since $(I \overset{P}{\vee} J) \cap Q$ is an $\mathcal{M}'$-ideal,

$$(I \cap Q) \overset{Q}{\vee} (J \cap Q) \subseteq (I \overset{P}{\vee} J) \cap Q .$$

To prove containment in the other direction we note that, since $I, J \in \mathcal{I}(Q)$,

$$I = (I \cap Q] \subseteq ((I \cap Q) \overset{Q}{\vee} (J \cap Q)] \text{, and similarly,}$$

$$J \subseteq ((I \cap Q) \overset{Q}{\vee} (J \cap Q)] ;$$

thus $I \overset{P}{\vee} J \subseteq ((I \cap Q) \overset{Q}{\vee} (J \cap Q)]$.

Clearly $I = (I] \cap Q$ for all $\mathcal{M}'$-ideals I of Q . Thus $(I \overset{P}{\vee} J) \cap Q \subseteq (I \cap Q) \overset{Q}{\vee} (J \cap Q)$.

Thus the lemma is established.

Lemma 35 and its dual establish condition (i) of Theorem 3.9.

36. <u>Lemma</u>. Let $I \in \mathcal{I}(Q)$, $D \in \mathcal{D}(Q)$ and let $I \cap D \neq \emptyset$. Then $I \cap D \cap Q \neq \emptyset$.

<u>Proof</u>: By Lemma 34 and its dual $I = (I \cap Q]$ and $D = [D \cap Q)$ where "[ )" is dual to "( ]" . Thus

$$(I \cap Q] \cap [D \cap Q) \neq \emptyset .$$

Let $x \in (I \cap Q] \cap [D \cap Q)$; then there are $y, z \in Q$ such that $z \in I \cap Q$, $y \in D \cap Q$, and $y \leqslant x \leqslant z$. Thus $y \leqslant z$, and thus $y \in I \cap Q$. Consequently

$$I \cap D \cap Q = (I \cap Q) \cap (D \cap Q) \neq \emptyset .$$

Lemma 36 implies condition (ii) of Theorem 3.9 and thus Theorem 30 is established.

Parenthetically it may be observed that the method of Jónsson [8] alluded to with respect to Lemma 15 is also applicable to Theorem 30.

For amalgamated free products in general upper and lower λ-covers cannot be defined. However they can be defined if the amalgamated sublattice M has finite <u>length</u> $n-1$. We recall that a poset has length $n-1$ if and only if there is a chain in the poset with n distinct elements and no chain of the poset has more than n distinct elements ([1] p. 5). In the remainder of this section M will be assumed to have length $n-1$.

35. <u>Lemma</u>. If I is an $\mathcal{M}$-ideal (resp. $\mathcal{N}$-dual ideal) of P and $I \cap M \neq \emptyset$ then $I \cap M$ is a principal ideal (resp. principal dual ideal) of M.

<u>Proof</u>: If I is an $\mathcal{M}$-ideal then, as in Lemma 31, $I \cap M$ is an ideal of M. Since M has finite length, every non-empty subset of M has a maximal element; thus $I \cap M$ is principal.

The dual argument applies if I is an $\mathcal{N}$-dual ideal.

36. <u>Definition</u>. a) If $x \in P$ and $(x] \cap M \neq \emptyset$ then the generator of $(x] \cap M$ is denoted by $c_0(x)$; if

$(x] \cap M = \emptyset$ then $c_o(x)$ is undefined.

b) If $x \in P$ and $[x) \cap M \neq \emptyset$ then the generator of $[x) \cap M$ is denoted by $C_o(x)$; if $[x) \cap M = \emptyset$ then $C_o(x)$ is undefined.

37. <u>Coroll</u>. If $x \in L_\lambda$, $y \in L_\mu$, $\lambda \neq \mu$ and $x \leqslant y$ then $C_o(x)$, $c_o(y)$ are defined and

$$x \leqslant C_o(x) \leqslant c_o(y) \leqslant y \text{ .}$$

We now proceed to analyse the stucture of pseudo-principal $\mathcal{M}$-ideals and $\mathcal{M}$-dual ideals. As above, if $x \in L_\lambda$ then $(x]$ and $(x]_\lambda$ are respectively the principal ideal generated by x in P and in L_λ , and dually.

38. <u>Lemma</u>. a) If $x \in L_\lambda$ then $(x] \cap L_\lambda = (x]_\lambda$ and $(x] \cap L_\mu = (c_o(x)]_\mu$, $\mu \neq \lambda$, if $c_o(x)$ is defined; $(x] \cap L_\mu = \emptyset$, $\mu \neq \lambda$, if $c_o(x)$ is undefined.

b) If $x \in L_\lambda$ then $[x) \cap L_\lambda = [x)_\lambda$ and $[x) \cap L_\mu = [C_o(x))_\mu$, $\mu \neq \lambda$, if $C_o(x)$ is defined; $[x) \cap L_\mu = \emptyset$, $\mu \neq \lambda$, if $C_o(x)$ is undefined.

<u>Proof</u>: Part b) is dual to a) and thus we need only prove a).

Clearly $(x] \cap L_\lambda = (x]_\lambda$. If $\mu \neq \lambda$ and $(x] \cap L_\mu \neq \emptyset$ then there is a $y \in L_\mu$ such that $y \leqslant x$. By the definition of the order $\leqslant$, $(x] \cap M \neq \emptyset$; thus

$c_o(x)$ exists and, by Coroll. 37, $(x] \cap L_\mu = (c_o(x)]_\mu$. Since $c_o(x) \in L_\mu$ for all $\mu \in \Lambda$ then $(x] \cap L_\mu = (c_o(x)]_\mu$ if $c_o(x)$ exists.

39. Lemma. Let $\Lambda' \subseteq \Lambda$ and let $(x_\lambda \mid \lambda \in \Lambda')$ be a family of mutually incomparable elements of P such that $x_\lambda \in L_\lambda$ for each $\lambda \in \Lambda'$. Then

a) $\bigcup((x_\lambda] \mid \lambda \in \Lambda')$ is an $\mathcal{M}$-ideal of P if and only if $\lambda, \mu \in \Lambda'$ and $c_o(x_\lambda)$ exists imply $c_o(x_\lambda) \leqslant x_\mu$;

b) $\bigcup([x_\lambda) \mid \lambda \in \Lambda')$ is an $\mathcal{N}$-dual ideal of P if and only if $\lambda, \mu \in \Lambda'$ and $C_o(x_\lambda)$ exists imply $C_o(x_\lambda) \geqslant x_\mu$.

Proof: Part b) follows from a) by the principle of duality.

Let $I = \bigcup((x_\lambda] \mid \lambda \in \Lambda')$ be an $\mathcal{M}$-ideal of P and let $\lambda, \mu \in \Lambda'$ and let $c_o(x_\lambda)$ exist. Then $c_o(x_\lambda), x_\mu \in L_\mu$ and $c_o(x_\lambda), x_\mu \in I$; thus $c_o(x_\lambda) \vee x_\mu \in I$ and so there is a $\nu \in \Lambda'$ such that $c_o(x_\lambda) \vee x_\mu \leqslant x_\nu$. Since $x_\mu \leqslant x_\nu$ then $\mu = \nu$; thus $c_o(x_\lambda) \vee x_\mu \leqslant x_\mu$ and so $c_o(x_\lambda) \leqslant x_\mu$.

To prove the converse let $x, y \in I$ and $\{x, y\} \in \mathcal{M}$; then there is a λ such that $x, y \in L_\lambda$ and $\mu, \nu \in \Lambda'$ such that $x \leqslant x_\mu$, $y \leqslant x_\nu$. If $\lambda \neq \mu$ then $c_o(x_\mu)$ exists and $x \leqslant c_o(x_\mu)$. Since $c_o(x_\mu) \leqslant x_\nu$ then $c_o(x_\mu) \vee y \leqslant x_\nu$. (This is true because if $\lambda \neq \nu$ then

$y \leqslant_\lambda c_o(x_\nu) \leqslant_\nu x_\nu$ and so

$$c_o(x_\mu) \vee y \leqslant_\lambda c_o(x_\mu) \vee c_o(x_\nu) \leqslant_\nu x_\nu \text{ .)}$$

Since $x \vee y \leqslant_\lambda c_o(x_\mu) \vee y$ then $x \vee y \in (x_\nu] \subseteq I$. If $\lambda = \mu = \nu$ then clearly $x \vee y \in (x_\nu] \subseteq I$.

Clearly I is a hereditary subset of P .

Thus I is an $\mathcal{M}$-ideal, and the lemma is proved.

40. Lemma. Let $x \in L_\lambda$, $y \in L_\mu$.

(i) If $\lambda = \mu$ then $(x] \cap (y] = (x \wedge y]$ and, dually, $[x) \cap [y) = [x \vee y)$.

(ii) If $\lambda \neq \mu$ then

a) $(x] \cap (y]$ is non-empty if and only if at least one of $c_o(x)$, $c_o(y)$ exists; if only $c_o(x)$ exists then $(x] \cap (y] = (c_o(x) \wedge y]$ and similarly if only $c_o(y)$ exists then $(x] \cap (y] = (x \wedge c_o(y)]$; if both $c_o(x)$, $c_o(y)$ exist then $(x] \cap (y] = (x \wedge c_o(y)] \cup (c_o(x) \wedge y]$;

b) $[x) \cap [y)$ is non-empty if and only if at least one of $C_o(x)$, $C_o(y)$ exists; if only $C_o(x)$ exists then $[x) \cap [y) = [y \vee C_o(x))$ and similarly if only $C_o(y)$ exists then $[x) \cap [y) = [x \vee C_o(y))$; if both $C_o(x)$, $C_o(y)$ exist then $[x) \cap [y) = [x \vee C_o(y)) \cup [C_o(x) \vee y)$.

Proof: (i) is clear.

We prove (ii) a). By the definition of the partial order on P , if $(x] \cap (y] \neq \emptyset$ then one of $c_o(x)$, $c_o(y)$ exists. If $c_o(y)$ exists then $c_o(y) \leqslant y$. Thus

$x \wedge c_o(y) \leqslant x, y$; consequently $(x \wedge c_o(y)] \subseteq (x] \cap (y]$. Let $c_o(x)$ not exist and let $z \in (x] \cap (y]$. Thus $z \leqslant x$ and, since $c_o(x)$ does not exist, $z \in L_\lambda$. Since $z \leqslant y$ then $z \leqslant c_o(y)$. Thus $z \leqslant x \wedge c_o(y)$. Thus

$$(x] \cap (y] = (x \wedge c_o(y)] .$$

A similar argument applies if only $c_o(x)$ exists.

If both exist then, in view of the above argument, we need only show that $(x] \cap (y] \subseteq (x \wedge c_o(y)] \cup (c_o(x) \wedge y]$. Let $z \in (x] \cap (y]$; thus $z \leqslant x, y$ and $z \in L_\nu$ where $\nu \neq \lambda$ or $\nu \neq \mu$, say $\nu \neq \lambda$. Then $z \leqslant c_o(x)$ and $z \leqslant y$; consequently $z \leqslant c_o(x) \wedge y$, as above. Thus

$$(x] \cap (y] \subseteq (x \wedge c_o(y)] \cup (c_o(x) \wedge y] .$$

Applying the principle of duality, (ii) b) is established, and so the lemma is proved.

We now proceed to describe the join of pseudo-principal $\mathcal{M}$-ideals.

41. <u>Definition</u>. If $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$ then we define elements $c_i^\lambda(x, y), C_i^\lambda(x, y) \in L_\lambda$, $c_i^\mu(y, x)$, $C_i^\mu(y, x) \in L_\mu$ for each $x \in L_\lambda$, $y \in L_\mu$ and each integer $i \geqslant 0$.

a) $c_o^\lambda(x, y) = x$ and $c_o^\mu(y, x) = y$; if $c_i^\lambda(x, y)$, $c_i^\mu(y, x)$ are defined and $c_o(c_i^\mu(y, x))$ does not exist then $c_{i+1}^\lambda(x, y) = c_i^\lambda(x, y)$; if $c_o(c_i^\mu(y, x))$ exists then $c_{i+1}^\lambda(x, y) = c_i^\lambda(x, y) \vee c_o(c_i^\mu(y, x))$.

$c^{\mu}_{i+1}(y, x)$ is defined similarly.

b) $C^{\lambda}_{o}(x, y) = x$ and $C^{\mu}_{o}(y, x) = y$; if $C^{\lambda}_{i}(x, y)$, $C^{\mu}_{i}(y, x)$ are defined and $C_o(C^{\mu}_{i}(y, x))$ does not exist then $C^{\lambda}_{i+1}(x, y) = C^{\lambda}_{i}(x, y)$; if $C_o(C^{\mu}_{i}(y, x))$ exists then $C^{\lambda}_{i+1}(x, y) = C^{\lambda}_{i}(x, y) \wedge C_o(C^{\mu}_{i}(y, x))$.

$C^{\mu}_{i+1}(y, x)$ is defined similarly.

We recall that the integer $n - 1$ is the length of the lattice M .

42. <u>Lemma</u>. Let $\lambda \neq \mu$ and let $x \in L_{\lambda}$, $y \in L_{\mu}$. Then

a) if $c_o(c^{\mu}_{n}(y, x))$ exists then

$$c_o(c^{\mu}_{n}(y, x)) \leqslant c^{\lambda}_{n}(x, y) ;$$

b) if $C_o(C^{\mu}_{n}(y, x))$ exists then

$$C_o(C^{\mu}_{n}(y, x)) \geqslant C^{\lambda}_{n}(x, y) .$$

<u>Proof</u>: We prove a) and use the principle of duality to establish b).

If neither $c_o(x)$ nor $c_o(y)$ exist then $c^{\mu}_{n}(y, x) = y$ and there is nothing to prove.

Thus we may assume that, say, $c_o(x)$ exists. Since $c_o(x) \leqslant c^{\lambda}_{i}(x, y)$ and $c_o(x) \leqslant c^{\mu}_{i}(y, x)$ for all $i > 0$, $c_o(c^{\lambda}_{i}(x, y))$ and $c_o(c^{\mu}_{i}(y, x))$ exist. Let $x_o = c_o(x), c_o(y)$, or $c_o(x) \vee c_o(y)$, whichever applies; if $i > 0$ let $x_i = c_o(c^{\lambda}_{i}(x, y)) \vee c_o(c^{\mu}_{i}(y, x))$. Then $x_o, \ldots, x_n \in M$ and $x_o \leqslant x_1 \leqslant \ldots \leqslant x_n$. Since the length

of M is $n-1$, there is a j, $0 \leqslant j < n$, such that $x_j = x_{j+1}$. If $j > 0$ then
$x_j = c_o(c_j^{\lambda}(x, y)) \vee c_o(c_j^{\mu}(y, x)) \leqslant c_{j+1}^{\lambda}(x, y),\ c_{j+1}^{\mu}(y, x)$.
Thus, since $x_j \in M$, $x_j \leqslant c_o(c_{j+1}^{\lambda}(x, y)) \leqslant x_{j+1}$ and $x_j \leqslant c_o(c_{j+1}^{\mu}(y, x)) \leqslant x_{j+1}$. Thus

$$c_o(c_{j+1}^{\lambda}(x, y)) = x_j \leqslant c_{j+1}^{\mu}(y, x)$$

and

$$c_o(c_{j+1}^{\mu}(y, x)) = x_j \leqslant c_{j+1}^{\lambda}(x, y) .$$

Thus, applying the inductive definition of c_i^{λ}, c_i^{μ}, $c_n^{\lambda}(x, y) = c_{j+1}^{\lambda}(x, y)$ and $c_n^{\mu}(y, x) = c_{j+1}^{\mu}(y, x)$; consequently $c_o(c_n^{\mu}(y, x)) \leqslant c_n^{\lambda}(x, y)$.

A similar argument applies if $j = 0$, and so the lemma is proved.

43. <u>Lemma</u>. Let $x \in L_{\lambda}$, $y \in L_{\mu}$. Then

a) if $\lambda = \mu$ then $(x] \vee (y] = (x \vee y]$, and if $\lambda \neq \mu$ then $(x] \vee (y] = (c_n^{\lambda}(x, y)] \bigcup (c_n^{\mu}(y, x)]$;

b) if $\lambda = \mu$ then $[x) \vee [y) = [x \wedge y)$, and if $\lambda \neq \mu$ then $[x) \vee [y) = [c_n^{\lambda}(x, y)) \bigcup [c_n^{\mu}(y, x))$.

<u>Proof</u>: Part b) is the dual of a), and thus we need prove only a).

If $\lambda = \mu$ the result is clear.

If $\lambda \neq \mu$ then, by Lemmas 39 and 42, $(c_n^{\lambda}(x, y)] \bigcup (c_n^{\mu}(y, x)]$ is an $\mathcal{M}$-ideal of P ; since $x \leqslant c_n^{\lambda}(x, y)$, $y \leqslant c_n^{\mu}(y, x)$ then

$$(x] \vee (y] \subseteq (c_n^{\lambda}(x, y)] \bigcup (c_n^{\mu}(y, x)] .$$

To prove containment in the other direction we observe that, for all $i \geqslant 0$, $c^{\lambda}_{i+1}(x, y) \leqslant c^{\lambda}_{i}(x, y) \vee c^{\mu}_{i}(y, x)$, and similarly for $c^{\mu}_{i+1}(y, x)$; since $c^{\lambda}_{o}(x, y) = x$, $c^{\mu}_{o}(y, x) = y$ we find that $c^{\lambda}_{n}(x, y) \in (x] \vee (y]$ and $c^{\mu}_{n}(y, x) \in (x] \vee (y]$. Thus

$$(c^{\lambda}_{n}(x, y)] \bigcup (c^{\mu}_{n}(y, x)] \subseteq (x] \vee (y],$$

and we have established the lemma.

Applying Lemmas 40 and 43:

44. Lemma. If M is of finite length then every non-empty pseudo-principal $\mathcal{M}$-ideal (resp. pseudo-principal $\mathcal{N}$-dual ideal) is a set union of a finite number of principal $\mathcal{M}$-ideals (resp. principal $\mathcal{N}$-dual ideals).

By Lemma 38:

45. Lemma. If M is of finite length and I is a pseudo-principal $\mathcal{M}$-ideal (resp. pseudo-principal $\mathcal{N}$-dual ideal) of P then

(i) $I \cap L_{\lambda}$ is either empty or principal in L_{λ} for each $\lambda \in \Lambda$;

(ii) for all but a finite number of $(\lambda, \mu) \in \Lambda \times \Lambda$, $I \cap L_{\lambda} = I \cap L_{\mu}$ and is generated by an element of M.

Thus, as in Definition 5, we can define upper and lower λ-covers for each $A \in W(P)$.

46. Definition. Let $A \in W(P)$.

a) If $\lambda \in \Lambda$ and $\underline{A} \cap L_\lambda \neq \emptyset$ then the generator of the principal ideal $\underline{A} \cap L_\lambda$ of L_λ , denoted $\underline{A}_\lambda$, is said to be the lower λ-cover of A ; if $\underline{A} \cap L_\lambda = \emptyset$ then we say that the lower λ-cover of A does not exist.

b) If $\lambda \in \Lambda$ and $\overline{A} \cap L_\lambda \neq \emptyset$ then the generator of the principal dual ideal $\overline{A} \cap L_\lambda$ of L_λ , denoted $\overline{A}^\lambda$, is said to be the upper λ-cover of A ; if $\overline{A} \cap L_\lambda = \emptyset$ then we say that the upper λ-cover of A does not exist.

47. Coroll. $\underline{A}_\lambda \in L_\lambda - M$ for only finitely many $\lambda \in \Lambda$, and dually; if $\underline{A}_\lambda, \underline{A}_\mu \in M$ then $\underline{A}_\lambda = \underline{A}_\mu$, and dually.

As in Lemma 7 we can describe an algorithm to determine $\underline{A}_\lambda$ and $\overline{A}^\lambda$ for each $\lambda \in \Lambda$ and $A \in W(P)$. We observe that if $\lambda, \mu \in \Lambda$ and $\underline{A}_\lambda$, $c_o(\underline{A}_\mu)$ exist then $c_o(\underline{A}_\mu) \leqslant \underline{A}_\lambda$. Thus, by Lemma 40, if $A, B \in W(P)$ and $C = A \frown B$ then $\underline{C}_\lambda$ exists if and only if $\underline{A}_\lambda$, $\underline{B}_\lambda$ exist and $\underline{C}_\lambda = \underline{A}_\lambda \wedge \underline{B}_\lambda$.

Similarly we observe that if $C = A \smile B$ then $\underline{C}_\lambda$ exists if and only if at least one of $\underline{A}_\lambda$, $\underline{B}_\lambda$ exists. To describe $\underline{C}_\lambda$ we observe that $\underline{A}_\mu, \underline{B}_\mu \in L_\mu - M$ for only finitely many $\mu_1, \ldots, \mu_r \in \Lambda$ distinct from λ ; thus we can define $x_1, \ldots, x_r$ such that for each i x_i exists if and only if at least one of $\underline{A}_{\mu_i}$, $\underline{B}_{\mu_i}$

exists and $x_i = \underline{A}_{\mu_i}$, $\underline{B}_{\mu_i}$, or $\underline{A}_{\mu_i} \vee \underline{B}_{\mu_i}$, whichever applies. Then

$$\underline{C}_\lambda = c_n^\lambda(\ldots c_n^\lambda(c_n^\lambda(x_o, x_1), x_2) \ldots, x_r)$$

where $x_o = \underline{A}_\lambda$, $\underline{B}_\lambda$, or $\underline{A}_\lambda \vee \underline{B}_\lambda$, whichever applies. The dual case also applies; the dual of x_i is $X_i = \overline{A}^{\mu_i}$, $\overline{B}^{\mu_i}$, or $\overline{A}^{\mu_i} \wedge \overline{B}^{\mu_i}$, whichever applies, and $X_o = \overline{A}^\lambda$, $\overline{B}^\lambda$, or $\overline{A}^\lambda \wedge \overline{B}^\lambda$. Thus:

48. Lemma. (i) If $A \in L_\lambda$ then $\underline{A}_\lambda = \overline{A}^\lambda = A$; $\underline{A}_\mu$, $\mu \neq \lambda$, exists if and only if $c_o(A)$ exists and in this event $\underline{A}_\mu = c_o(A)$; $\overline{A}^\mu$, $\mu \neq \lambda$, exists if and only if $C_o(A)$ exists and in this event $\overline{A}^\mu = C_o(A)$.

(ii) If $A, B, C \in W(P)$, $\lambda \in \Lambda$, and $A = B \smile C$ then $\overline{A}^\lambda$ exists if and only if $\overline{B}^\lambda$, $\overline{C}^\lambda$ exist and in this event $\overline{A}^\lambda = \overline{B}^\lambda \vee \overline{C}^\lambda$.

$\underline{A}_\lambda$ exists if and only if at least one of $\underline{B}_\lambda$, $\underline{C}_\lambda$ exists and in this event

$$\underline{A}_\lambda = c_n^\lambda(\ldots c_n^\lambda(c_n^\lambda(x_o, x_1), x_2) \ldots, x_r) .$$

(iii) If $A, B, C \in W(P)$, $\lambda \in \Lambda$, and $A = B \frown C$ then $\underline{A}_\lambda$ exists if and only if $\underline{B}_\lambda$, $\underline{C}_\lambda$ exist and in this event $\underline{A}_\lambda = \underline{B}_\lambda \wedge \underline{C}_\lambda$.

$\overline{A}^\lambda$ exists if and only if at least one of $\overline{B}^\lambda$, $\overline{C}^\lambda$ exists and in this event

$$\overline{A}^\lambda = C_n^\lambda(\ldots C_n^\lambda(C_n^\lambda(X_o, X_1), X_2) \ldots, X_r) .$$

49. Lemma. If $A, B \in W(P)$ then $\overline{A} \cap \underline{B} \neq \emptyset$ if and only if there is a $\lambda \in \Lambda$ such that $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$.

Proof: Since $\overline{A}^{\lambda} \in \overline{A}$ and $\underline{B}_{\lambda} \in \underline{B}$ the condition is sufficient.

If $\overline{A} \cap \underline{B} \neq \emptyset$ then there is a $\lambda \in \Lambda$ such that $\overline{A} \cap \underline{B} \cap L_{\lambda} \neq \emptyset$, that is, $[\overline{A}^{\lambda}) \cap (\underline{B}_{\lambda}] \neq \emptyset$; thus

$$\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda} .$$

Thus Theorem 12 holds in this case:

50. Theorem. Let M be of finite length. The quasi-order $\subseteq$ on $W(P)$ satisfies:

If $A, B \in W(P)$ then $A \subseteq B$ if and only if it follows from one of the following conditions:

(1) $A = B$;

(2) there is a $\lambda \in \Lambda$ such that $\overline{A}^{\lambda}$, $\underline{B}_{\lambda}$ exist and $\overline{A}^{\lambda} \leqslant \underline{B}_{\lambda}$;

(3) $A = A_0 \smile A_1$ where $A_0 \subseteq B$ and $A_1 \subseteq B$;

(4) $A = A_0 \frown A_1$ where $A_0 \subseteq B$ or $A_1 \subseteq B$;

(5) $B = B_0 \smile B_1$ where $A \subseteq B_0$ or $A \subseteq B_1$;

(6) $B = B_0 \frown B_1$ where $A \subseteq B_0$ and $A \subseteq B_1$.

The relation $\sim$ on $W(P)$, where $A \sim B$ if and only if $A \subseteq B$ and $B \subseteq A$, is an equivalence relation and $W(P)/\sim$ is a lattice. For each $\lambda \in \Lambda$ the mapping $\varphi_{\lambda}: L_{\lambda} \to W(P)/\sim$, defined as $\varphi_{\lambda}(x) = \langle x \rangle$ if $x \in L_{\lambda}$, is a lattice embedding. If $x \in M$ then $\varphi_{\lambda}(x) = \varphi_{\mu}(x)$

for all $\lambda, \mu \in \Lambda$. The system $(\varphi_\lambda, \lambda \in \Lambda ; W(P)/\sim)$ is the free product of $(L_\lambda \mid \lambda \in \Lambda)$ amalgamated by M .

Coroll. 13 also holds in the present case. We observe that if $A, B, C \in W(P)$, if L_λ is a chain, and if $\overline{A}^\lambda \leqslant \underline{B \smile C}_\lambda$ then $\overline{A}^\lambda \leqslant x_0$ or $\overline{A}^\lambda \leqslant c_0(x_i)$, $1 \leqslant i \leqslant r$. Now $x_0 = \underline{B}_\lambda$ or $\underline{C}_\lambda$, whichever is greater; thus if $\overline{A}^\lambda \leqslant x_0$ then $A \subseteq B$ or $A \subseteq C$. If $\overline{A}^\lambda \leqslant c_0(x_i)$ then $\overline{A}^\lambda \leqslant x_i$. If L_{μ_i} is not a chain then $\overline{A}^\lambda \leqslant x_i \leqslant \underline{B \smile C}_{\mu_i}$; thus $\overline{A}^{\mu_i} \leqslant c_0(\overline{A}^\lambda) \leqslant \underline{B \smile C}_{\mu_i}$, L_{μ_i} not a chain. If L_{μ_i} is a chain then $x_i \leqslant \underline{B}_{\mu_i}$ or $\underline{C}_{\mu_i}$ and so $\overline{A}^{\mu_i} \leqslant c_0(\overline{A}^\lambda) \leqslant \underline{B}_{\mu_i}$ or $\underline{C}_{\mu_i}$; thus $A \subseteq B$ or $A \subseteq C$. The dual argument applies if $\overline{B \frown C}^\lambda \leqslant \underline{A}_\lambda$.

Because the join of pseudo-principal $\mathcal{M}$-ideals and $\mathcal{N}$-dual ideals is rather complicated, Theorem 3.6 is the simplest characterization of amalgamated free products.

Theorem 16 generalizes as:

51. Theorem. Let $|\Lambda| > 1$ and let $(\varphi_\lambda, \lambda \in \Lambda ; L)$ be the free product of $(L_\lambda \mid \lambda \in \Lambda)$ amalgamated by the lattice M of finite length. Let $\Lambda' \subseteq \Lambda$ and let L_λ be a chain for each $\lambda \in \Lambda'$. Then

$$L - \bigcup(\varphi_\lambda(L_\lambda - M) \mid \lambda \in \Lambda')$$

is a sublattice of L .

Proof: Let $A, B \in W(P)$, $\lambda \in \Lambda'$, $C \in L_\lambda - M$, and let $A \smile B \sim C$. It suffices to show that $A \sim C$ or $B \sim C$.

Since $A \smile B \subseteq C$ then $A, B \subseteq C$. Also $C \subseteq A \smile B$. If, say, $C \subseteq A$ then $C \sim A$. Otherwise $C \subseteq A \smile B$ must follow by rule (2). By Coroll. 13 in our case there is a $\mu \in \Lambda - \Lambda'$ such that $\overline{C}^{\mu} \leqslant \underline{A \smile B}_{\mu}$. Since $A \smile B \subseteq C$, $\underline{C}_{\mu}$ exists and $\underline{A \smile B}_{\mu} \leqslant \underline{C}_{\mu}$. Thus $\overline{C}^{\mu} \leqslant \underline{C}_{\mu}$ and so $\langle C \rangle \in L_{\mu}$; since $C \in L_{\lambda}$, $\lambda \neq \mu$, this implies that $C \in M$, contradicting our assumption about C . Thus either $A \sim C$ or $B \sim C$.

The above argument and its dual establish the theorem.

We note that $L - \bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda')$ is not always a sublattice of L :

Example 5. Let $\Lambda = \{a, b\}$ and let x, y be unrelated elements of L_a such that $x \vee y = z$. Let L_b be a chain and let $L_a \cap L_b = \{z\}$. If L is the free product of L_a and L_b amalgamated by $\{z\}$ then $x, y \in L - L_b$, but $x \vee y \notin L - L_b$.

By Theorem 2.31 and Lemma 44 Sorkin's theorem applies to amalgamated free products:

52. Theorem. Let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of lattices and let M be a sublattice of finite length of

L_λ for each $\lambda \in \Lambda$. Let L be a lattice, and for each $\lambda \in \Lambda$ let $f_\lambda : L_\lambda \to L$ be an isotone map such that the restrictions of all f_λ, $\lambda \in \Lambda$, to M are equal. Then there is an isotone map f from the free product of the $(L_\lambda \mid \lambda \in \Lambda)$ amalgamated by M to L; for each $\lambda \in \Lambda$ f is an extension of f_λ.

CHAPTER V

CANONICAL REPRESENTATIONS

In this chapter the results of Chapter IV are applied to solve the problem of canonical representations for free products and partially ordered free products.

1. Introductory concepts.

Let L be a lattice generated by a poset P. As in Section 2.2 there is a unique mapping $F : W(P) \to L$ extending the embedding of P in L. If $A \in W(P)$, $x \in L$, and $F(A) = x$ then we say that A represents x in $W(P)$. Since every element of $W(P)$ has finite length we can say that $A \in W(P)$ is a minimal representation of $x \in L$ if

(i) $F(A) = x$;

(ii) if $F(B) = x$ then $\ell(B) \geqslant \ell(A)$.

To define the concept of canonical representation we must define the concept of two polynomials over P being equivalent up to commutativity and associativity. This concept is defined by induction on the length of polynomials.

1. Definition. If $A, B \in W(P)$ then A is said to be equivalent to B up to commutativity and associativity, denoted $A \equiv B$, if one of the following holds:

(i) $\ell(A) = \ell(B) = 1$ and $A = B$;

(ii) there are $A_o, A_1, A_2 \in W(P)$ such that $A = (A_o \smile A_1) \smile A_2$, $B = A_o \smile (A_1 \smile A_2)$; or, dually, $A = (A_o \frown A_1) \frown A_2$, $B = A_o \frown (A_1 \frown A_2)$;

(iii) there are $A_o, A_1, B_o, B_1 \in W(P)$ such that $A_o \equiv B_o$, $A_1 \equiv B_1$ and $A = A_o \smile A_1$ or $A_1 \smile A_o$, $B = B_o \smile B_1$ or $B_1 \smile B_o$; or, dually, $A = A_o \frown A_1$ or $A_1 \frown A_o$, $B = B_o \frown B_1$ or $B_1 \frown B_o$.

2. Coroll. (i) For all $A_o, A_1, B_o, B_1 \in W(P)$, $A_o \smile A_1 \not\equiv B_o \frown B_1$.

(ii) If $A, B, C \in W(P)$ and $A = B \smile C$ then A can be written as $A_o \smile A_1 \smile \dots \smile A_{r-1}$, $A_i \in W(P)$, where, for each i , either $A_i \in P$ or $A_i = B_i \frown C_i$. This representation is unique up to a permutation of the integers $0, 1, \dots, r-1$.

(iii) If $A, B, C \in W(P)$ and $A = B \frown C$ then A can be written as $A_o \frown A_1 \frown \dots \frown A_{r-1}$, $A_i \in W(P)$, where, for each i , either $A_i \in P$ or $A_i = B_i \smile C_i$. This representation is unique up to a permutation of the integers $0, 1, \dots, r-1$.

It is clear that $\equiv$ is an equivalence relation and is preserved under $\smile$ and $\frown$. It also is clear that if $A \equiv B$ then $F(A) = F(B)$.

3. Definition. If $x \in L$ then x is said to admit canonical representations in $W(P)$ if, for any two minimal representations A, B of x, $A \equiv B$. We say that L admits canonical representations in $W(P)$ if x admits canonical representations in $W(P)$ for all $x \in L$.

A problem of some interest is that of determining when $FL(P)$ or $FL(P\,;\mathcal{M},\mathcal{N})$ admits canonical representations in $W(P)$. In general this is a rather involved problem. A special case is:

4. Lemma (Dean [3], Theorem 5, Coroll. 1, p. 245). $CF(P)$ admits canonical representations in $W(P)$.

We now state several lemmas contributing to the problem of canonical representations in $FL(P)$. If $A \in W(P)$ is a minimal representation of $\langle A\rangle$ in $FL(P)$ then A is said to be a minimal polynomial.

5. Lemma. If $A, B \in W(P)$, $A \sim B$, and $\overline{A} \cap \underline{B} \neq \emptyset$ then there is an $x \in P$ such that $A \sim B \sim x$.

Proof: Since $B \subseteq A$ then, by Lemma 2.7, $\underline{B} \subseteq \underline{A}$. Thus $\overline{A} \cap \underline{A} \neq \emptyset$; let $x \in \overline{A} \cap \underline{A}$. Then $A \subseteq x \subseteq A$, and so $x \sim A$.

6. Lemma. Let $A_o, A_1, B_o, B_1 \in W(P)$ and let $A_o \smile A_1$, $B_o \frown B_1$ be minimal polynomials. Then

$$A_0 \smile A_1 \not\sim B_0 \frown B_1 .$$

Proof: Assume that $A_0 \smile A_1 \sim B_0 \frown B_1$. Since both are minimal $\ell(A_0) + \ell(A_1) = \ell(B_0) + \ell(B_1)$.

Now $A_0 \smile A_1 \subseteq B_0 \frown B_1$ and so, for each $i \in \{0, 1\}$, $A_i \subseteq B_0 \frown B_1$ and $A_0 \smile A_1 \subseteq B_i$.

Also $B_0 \frown B_1 \subseteq A_0 \smile A_1$. By Lemma 5, since $A_0 \smile A_1$, $B_0 \frown B_1$ are minimal representations, this quasi-inequality is not derived by rule (2) of Definition 2.6; thus either rule (4) or rule (5) applies. Rule (4) implies that there is an $i \in \{0, 1\}$ such that $B_i \subseteq A_0 \smile A_1$. Thus $A_0 \smile A_1 \sim B_i$ and, since $\ell(B_i) < \ell(A_0 \smile A_1)$, then $A_0 \smile A_1$ is not minimal. The dual argument applies if rule (5) is involved. Thus the proof is complete.

7. Lemma. Let $A, B \in W(P)$, $A \sim B$, both being minimal polynomials. Let

$$A \equiv A_0 \smile \ldots \smile A_{r-1} , \quad r > 1 ,$$

where each $A_i \in P$ or is of the form $C \frown D$; let

$$B \equiv B_0 \smile \ldots \smile B_{s-1} , \quad s > 1 ,$$

where each $B_j \in P$ or is of the form $E \frown F$. For each $i \leqslant r-1$, if $\ell(A_i) > 1$ then there is a $j \leqslant s-1$ such that $A_i \sim B_j$ and $\ell(A_i) = \ell(B_j)$, and conversely for each $j \leqslant s-1$.

Proof: $A_i \subseteq B \equiv B_0 \smile \ldots \smile B_{s-1}$. By the hypothesis on A_i and by Definition 2.6 one of the following three cases must hold:

(i) there is a $j \leqslant s-1$ such that $A_i \subseteq B_j$;

(ii) $A_i = C \frown D$ and, say, $C \subseteq B$;

(iii) $\overline{A}_i \cap \underline{B} \neq \emptyset$.

If case (ii) holds then, since $A_i \subseteq C$,
$A \subseteq A_o \smile \dots \smile A_{i-1} \smile C \smile A_{i+1} \smile \dots \smile A_{r-1} \subseteq B \subseteq A$.
Thus $A_o \smile \dots \smile A_{i-1} \smile C \smile A_{i+1} \smile \dots \smile A_{r-1} \sim A$, contradicting the minimality of A , since $\ell(C) < \ell(A_i)$.

If case (iii) holds then let $x \in \overline{A}_i \cap \underline{B}$; then
$A_i \subseteq x \subseteq B$. Thus
$A \subseteq A_o \smile \dots \smile A_{i-1} \smile x \smile A_{i+1} \smile \dots \smile A_{r-1} \subseteq B \subseteq A$, and so $A_o \smile \dots \smile A_{i-1} \smile x \smile A_{i+1} \smile \dots \smile A_{r-1} \sim A$. Since $\ell(A_i) > 1$ this also contradicts the minimality of A .

Thus case (i) applies. If $\ell(B_j) < \ell(A_i)$ then
$A \subseteq A_o \smile \dots \smile A_{i-1} \smile B_j \smile A_{i+1} \smile \dots \smile A_{r-1} \subseteq B \subseteq A$,
again contradicting the minimality of A . Thus
$\ell(B_j) \geqslant \ell(A_i)$. Consequently $\ell(B_j) > 1$ and, as for A_i , there is a $k \leqslant r-1$ such that $B_j \subseteq A_k$ and $\ell(A_k) \geqslant \ell(B_j)$.
If $k \neq i$ then, since $A_i \subseteq A_k$,

$$A \sim A_o \smile \dots \smile A_{i-1} \smile A_{i+1} \smile \dots \smile A_{r-1} ,$$

again contradicting the minimality of A . Thus $k = i$ and so $A_i \subseteq B_j \subseteq A_i$, that is , $A_i \sim B_j$. Since $\ell(B_j) \geqslant \ell(A_i)$ and $\ell(A_i) \geqslant \ell(B_j)$ then $\ell(A_i) = \ell(B_j)$.

Thus the proof is complete.

2. Canonical representations in free products.

Sorkin [9] presented an example illustrating that in general the free product of lattices need not admit canonical representations. We first extend this result.

Let $|\Lambda| > 1$ and let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices. As in Chapter IV let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$. Let L be the free product of the $(L_\lambda \mid \lambda \in \Lambda)$. We make continual use of the results of Section 4.1, especially Lemma 4.11. We remind the reader that $A \sim B$ implies that A and B have identical upper and lower covers, by this lemma.

8. Lemma. If there is a $\lambda \in \Lambda$ and $x, y, w \in L_\lambda$ such that x and y are incomparable and $x \vee y < w$ (or, dually, $x \wedge y > w$) then L does not admit canonical representations in $W(P)$.

Proof: Let $\mu \neq \lambda$ and let $d \in L_\mu$. Let $x \vee y = z \in L_\lambda$, $z < w$. We consider the polynomials $A = x \smile ((y \smile d) \frown w)$, $B = z \smile ((y \smile d) \frown w)$. Then $\ell(A) = \ell(B) = 4$ and $A \not\equiv B$.

Using the algorithm of Lemma 4.7: $\underline{A}_\lambda = \underline{B}_\lambda = z$; $\overline{A}^\lambda = \overline{B}^\lambda = w$; $\underline{A}_\nu, \underline{B}_\nu, \overline{A}^\nu, \overline{B}^\nu$ do not exist if $\nu \neq \lambda$.

We note first that $A \sim B$. Since $x \leqslant z$ then $A \subseteq B$; since $\underline{A}_\lambda = z$ then $z \subseteq A$ and so $B \subseteq A$. Thus $A \sim B$.

We now show that A is minimal.

Let $A \sim C$, $C \in W(P)$. Since $\underline{A}_\lambda \neq \overline{A}^\lambda$ then $\underline{C}_\lambda \neq \overline{C}^\lambda$ and thus $\ell(C) > 1$. Let $\ell(C) = 2$. If $C = a \smile b$, a, b in different lattices, then $\underline{C}_\lambda$ and $\overline{C}^\lambda$ cannot both exist. A similar situation is $C = a \frown b$. Thus $\ell(C) > 2$.

If $\ell(C) = 3$ then there are two cases; $C = C_o \smile C_1$ or $C = C_o \frown C_1$.

<u>Case 1</u>. $C = C_o \smile C_1$, $\ell(C_o) = 2$ and $C_1 \in P$: Since $\overline{C}^\lambda$ exists then $C_1 \in L_\lambda$ and $C_o = a \frown b$ where, say, $a \in L_\lambda$; thus $b \in L_\nu$, $\nu \neq \lambda$. Since $\underline{A}_\lambda = z$ then $C_1 = z$. Thus $C = (a \frown b) \smile z$, $a \in L_\lambda$, $b \in L_\nu$, $\nu \neq \lambda$.

Since $A \subseteq C$, $(y \smile d) \frown w \subseteq (a \frown b) \smile z$. $\overline{(y \smile d) \frown w}^\lambda = w \nsubseteq z = \underline{C}_\lambda$ and $\underline{C}_\nu$ is undefined for $\nu \neq \lambda$. Thus the above quasi-inequality cannot derive through rule (2) of Theorem 4.12. Thus there are four possibilities:

a) $(y \smile d) \frown w \subseteq z$;

b) $(y \smile d) \frown w \subseteq a \frown b$;

c) $y \smile d \subseteq (a \frown b) \smile z$;

d) $w \subseteq (a \frown b) \smile z$.

In a) $\overline{(y \smile d) \frown w}^\lambda = w$ and $\overline{z}^\lambda = z$; thus, by Lemma 4.11, case a) is impossible.

Similarly, in b) $\underline{(y \smile d) \frown w}_\lambda = y$ and $\underline{a \frown b}_\lambda$ is undefined; thus case b) is impossible.

In c) $d \subseteq (a \frown b) \smile z$. Since $\underline{d}_{\mu} = d$ and $\underline{(a \frown b) \smile z}_{\mu}$ is undefined, this case is also impossible.

Case d) also cannot hold since $\underline{w}_{\lambda} = w$, $\underline{(a \frown b) \smile z}_{\lambda} = z$ and $w \nleqslant z$.

Thus Case 1 cannot hold.

Case 2. $C = C_0 \frown C_1$, $\ell(C_0) = 2$ and $C_1 \in P$: Thus $C = (a \smile b) \frown C_1$, $a, b, C_1 \in P$. Since $\underline{C}_{\lambda}$ exists $C_1 \in L_{\lambda}$ and, say, $a \in L_{\lambda}$; thus $b \in L_{\nu}$, $\nu \neq \lambda$. $\overline{C}^{\lambda} = w$ implies that $C_1 = w$. Thus $C = (a \smile b) \frown w$.

Now $C \subseteq A$; since $\overline{C}^{\lambda} = w$ and $\underline{A}_{\lambda} = z$ the quasi-inequality cannot be obtained by rule (2). Thus one of the following four situations must hold:

a) $C \subseteq x$;

b) $C \subseteq (y \smile d) \frown w$;

c) $a \smile b \subseteq A$;

d) $w \subseteq A$.

Case a) implies that $\underline{C}_{\lambda} = z \leqslant x$.

Case b) implies that $z = \underline{C}_{\lambda} \leqslant \underline{(y \smile d) \frown w}_{\lambda} = y$.

Case c) implies, since $\underline{a \smile b}_{\nu} = b$, that $\underline{A}_{\nu}$ exists.

Case d) implies that $z = \underline{A}_{\lambda} \geqslant w$.

Thus Case 2 cannot hold.

Consequently $\ell(C) \geqslant 4$ and so A and B are minimal. Thus L does not admit canonical representations.

The dual argument applies if x and y are incomparable and $x \wedge y > w$. Thus the lemma is established.

9. <u>Lemma</u>. Let $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, and let L_λ, L_μ not be chains. Then L does not admit canonical representations in $W(P)$.

<u>Proof</u>: Let $x, y \in L_\lambda$ be incomparable and let $d_1, d_2 \in L_\mu$ be incomparable. Let $x \vee y = z$ and $d_1 \wedge d_2 = d$. We consider the polynomials

$A = x \smile ((y \smile d_1) \frown (y \smile d_2))$, $B = z \smile ((y \smile d_1) \frown (y \smile d_2))$.

Then $\ell(A) = \ell(B) = 5$ and $A \not\equiv B$.

Using the algorithm in Lemma 4.7

$$\underline{A}_\lambda = \underline{B}_\lambda = z ;$$
$$\underline{A}_\mu = \underline{B}_\mu = d ;$$
$\underline{A}_\nu$, $\underline{B}_\nu$ are undefined for $\nu \neq \lambda, \mu$;
$\overline{A}^\nu$, $\overline{B}^\nu$ are undefined for all $\nu \in \Lambda$.

We note first that $A \sim B$. Since $x \leqslant z$ then $A \subseteq B$; since $\overline{z}^\lambda = z$ and $\underline{A}_\lambda = z$ then $z \subseteq A$ and so $B \subseteq A$. Thus $A \sim B$.

We now show that A and B are minimal polynomials.

Let $A \sim C$, $C \in W(P)$. Since A has no upper covers then $\ell(C) \geqslant 2$.

If $\ell(C) = 2$ then $C \neq a \frown b$, $a, b \in P$, since C has no upper covers.

Otherwise, since $\underline{C}_\lambda = z$ and $\underline{C}_\mu = d$, $C = z \smile d$. Thus $(y \smile d_1) \frown (y \smile d_2) \subseteq z \smile d$. Since $(y \smile d_1) \frown (y \smile d_2)$ has no upper covers then either

a) $(y \smile d_i) \subseteq z \smile d$, $i = 1$ or 2, or

b) $(y \cup d_1) \cap (y \cup d_2) \subsetneq z$ or d .

Case a) implies that $d_i = \underline{y \cup d_i}_\mu \leqslant \underline{z \cup d}_\mu = d$, contradicting the incomparability of d_1 and d_2 .

Case b) is also impossible since $(y \cup d_1) \cap (y \cup d_2)$ has both a lower λ-cover and a lower μ-cover.

Thus $\ell(C) \geqslant 3$.

If $\ell(C) = 3$ then either $C = C_0 \cup C_1$ or $C = C_0 \cap C_1$. If $C = C_0 \cap C_1$ then either C_0 or $C_1 \in P$ and so C would have an upper cover. Consequently $C = C_0 \cup C_1$ and, say, $C_1 = c \in P$. If $C_0 = a \cup b$, $a, b \in P$, then, since C cannot be equivalent to a shorter polynomial by the above argument, then a, b, c are in three different lattices and so C would have three lower covers. Thus $C = (a \cap b) \cup c$, $a, b, c \in P$. Since $\underline{C}_\lambda = z$, $c = z$. However, a and b must lie in different lattices and thus $\underline{C}_\mu$ would not exist. This contradiction indicates that $\ell(C) > 3$.

If $\ell(C) = 4$ then there are two cases; $C = C_0 \cup C_1$ and $C = C_0 \cap C_1$.

<u>Case 1</u>. $C = C_0 \cup C_1$: Since $A \subsetneq C$ then $(y \cup d_1) \cap (y \cup d_2) \subsetneq C$. Since $(y \cup d_1) \cap (y \cup d_2)$ has no upper covers either

a) $y \cup d_i \subsetneq C$, $i = 1$ or 2 ,

or

b) $(y \cup d_1) \cap (y \cup d_2) \subseteq C_0$, say.

Case a) implies that $d_i \leqslant \underline{C}_\mu = d$, which contradicts $d < d_i$.

In case b) $\underline{C_0}_\lambda$, $\underline{C_0}_\mu$ exists, and so $\ell(C_0) > 1$. Now $C_0 \neq E \cap F$; for $\ell(C_0) \leqslant 3$ and so either $\ell(E) = 1$ or $\ell(F) = 1$, and thus C_0 would have an upper cover, which is impossible since $(y \cup d_1) \cap (y \cup d_2)$ has none. Thus $C_0 = E \cup F$; thus $C \equiv E \cup F \cup C_1$. Since $\ell(C) = 4$ we may assume that $\ell(E) = 2$. The previous discussion has shown that C cannot be equivalent to a polynomial of shorter length; since C has only two lower covers $E = a \cap b$, $a, b \in P$. Since $\ell(C) = 4$, then $F, C_1 \in P$; since $\underline{C}_\lambda = z$, $\underline{C}_\mu = d$ it follows that

$$C \equiv (a \cap b) \cup (z \cup d) .$$

Thus, since $y \cup d_i \subseteq C$ is impossible as above, either

a) $(y \cup d_1) \cap (y \cup d_2) \subseteq a \cap b$,

or

b) $(y \cup d_1) \cap (y \cup d_2) \subseteq z \cup d$.

Case a) is impossible since $a \cap b$ has upper covers and $(y \cup d_1) \cap (y \cup d_2)$ does not.

Case b) implies that either

(i) $y \cup d_i \subseteq z \cup d$ for some i ; taking lower μ-covers we get the contradiction $d_i \leqslant d$;

or (ii) $(y \cup d_1) \cap (y \cup d_2) \subseteq z$ or d , which is impossible since both z and d have upper covers and

$(y \smile d_1) \frown (y \smile d_2)$ does not.

Thus Case 1 cannot hold.

Case 2. $C = C_o \frown C_1$: Since $C \subseteq A$ and C has no upper covers one of the following four must hold:

a) $C \subseteq x$;

b) $C \subseteq (y \smile d_1) \frown (y \smile d_2)$;

c) $C_o \subseteq A$;

d) $C_1 \subseteq A$.

Case a) implies that $\overline{C}^{\lambda}$ exists, and thus it cannot hold.

Case b) implies that $z = \underline{C}_{\lambda} \leqslant y$ and so cannot hold.

Since $A \subseteq C \subseteq C_o, C_1$ cases c) and d) imply that A is equivalent to a polynomial of length <4, of which the previous discussion has disposed. Thus Case 2 cannot hold.

Consequently $\ell(C) \geqslant 5$ and thus A, B are minimal.

Thus the proof is complete.

10. Lemma. Let λ, μ_1, μ_2 be distinct elements of Λ and let L_{λ} not be a chain. Then L does not admit canonical representations in $W(P)$.

Proof: Let $x, y \in L_{\lambda}$ be incomparable, and let $d_1 \in L_{\mu_1}$, $d_2 \in L_{\mu_2}$. Let $x \vee y = z$. As in the proof of Lemma 9, let

$$A = x \smile ((y \smile d_1) \frown (y \smile d_2)) , \quad B = z \smile ((y \smile d_1) \frown (y \smile d_2)).$$

As in Lemma 9 $A \sim B$. Also $\ell(A) = \ell(B)$ and $A \not\equiv B$.

We observe that

$$\underline{A}_\lambda = \underline{B}_\lambda = z ;$$

$\underline{A}_\nu$, $\underline{B}_\nu$ are undefined for all $\nu \neq \lambda$;

$\overline{A}^\nu$, $\overline{B}^\nu$ are undefined for all $\nu \in \Lambda$.

We show that A, B are minimal polynomials. Let $A \sim C$, $C \in W(P)$. Since A has no upper covers $\ell(C) \geqslant 2$.

If $\ell(C) = 2$ then $C \neq a \frown b$, $a, b \in P$, since C has no upper covers. Similarly, since C has only one lower cover, $C \neq a \smile b$, $a, b \in P$. Thus $\ell(C) \geqslant 3$.

If $\ell(C) = 3$ then either $C = C_0 \frown C_1$ or $C = C_0 \smile C_1$. The former situation is impossible since $\ell(C_0)$ or $\ell(C_1) = 1$ and so C would have an upper cover. Thus $C = C_0 \smile C_1$ and, say, $C_1 = c \in P$. Since C has only one lower cover then $C_0 = a \frown b$, $a, b \in P$, and $c \in L_\lambda$. In order that C have no upper λ-cover $a, b \notin L_\lambda$. Now $A \subseteq C$; thus

$$(y \smile d_1) \frown (y \smile d_2) \subseteq (a \frown b) \smile c .$$

Since $(y \smile d_1) \frown (y \smile d_2)$ has no upper covers the above quasi-inequality cannot follow by using rule (2) of Theorem 4.12. Thus one of the following three holds:

a) $(y \smile d_1) \frown (y \smile d_2) \subseteq c$;

b) $(y \smile d_1) \frown (y \smile d_2) \subseteq a \frown b$;

c) there is an $i \in \{0, 1\}$ such that $y \smile d_i \subseteq c$.

Both c and $a \frown b$ have upper covers and so neither a) nor b) can hold. Since $y \smile d_i$ has a lower μ_i-cover and C does not, c) cannot hold. Thus $\ell(C) \geqslant 4$.

If $\ell(C) = 4$ there are two cases; $C = C_o \smile C_1$ and $C = C_o \frown C_1$.

<u>Case 1</u>. $C = C_o \smile C_1$: Since $\underline{C}_\lambda$ exists we may assume that $\underline{C_o}_\lambda$ exists. Since C has no other lower covers either $\ell(C_o) = 1$ or $\ell(C_o) = 3$. If $\ell(C_o) = 3$ then, since C_o can have no other lower covers, $C_o \equiv a \smile (e \frown f)$ where $a \in L_\lambda$, $e, f \in P$; then $C_1 = b \in P$ and, since C has only a lower λ-cover, $b \in L_\lambda$. In this event, however, $C \sim (a \vee b) \smile (e \frown f)$ and so A would be equivalent to a polynomial of length 3, an eventuality that the previous discussion rejects.

Thus $\ell(C_o) \neq 3$ and so $C_o = a \in L_\lambda$. Thus $\ell(C_1) = 3$ and, in view of the above discussion, $C_1 \equiv b \frown C_2$, $b \in P$, and $\ell(C_2) = 2$. Thus $C \equiv a \smile (b \frown C_2)$. Consequently $(y \smile d_1) \frown (y \smile d_2) \subseteq a \smile (b \frown C_2)$, an eventuality that can be rejected exactly as the situation when $C = (a \frown b) \smile c$, $a, b, c \in P$, was rejected above.

Thus Case 1 cannot hold.

<u>Case 2</u>. $C = C_o \frown C_1$: This case is disposed of exactly as it was in Lemma 9.

Consequently $\ell(C) \geqslant 5$ and so, since $\ell(A) = 5$, A and B are minimal. Since $A \not\equiv B$ the lemma follows.

If L_λ is a chain for each $\lambda \in \Lambda$ then $L = CF(P)$; thus, by Lemma 4,

11. Lemma (Sorkin [9]). If L_λ is a chain for all $\lambda \in \Lambda$ then the free product of $(L_\lambda \mid \lambda \in \Lambda)$ admits canonical representations in $W(\bigcup(L_\lambda \mid \lambda \in \Lambda))$.

If $|\Lambda| = 1$,say $\Lambda = \{\lambda\}$, then the free product of $(L_\lambda \mid \lambda \in \Lambda)$ is L_λ and so every element can be represented by a unique polynomial of length 1 ; thus L_λ admits canonical representations in $W(L_\lambda)$.

There is only one case not considered above. This is the case where $|\Lambda| = 2$, say $\Lambda = \{\lambda, \mu\}$, L_μ is a chain, and L_λ is not a chain but the join of any two incomparable elements is maximal in L_λ and the meet of any two incomparable elements is minimal in L_λ . Thus L is Q^b where Q is the disjoint union of unrelated chains. We show that in this case the free product does admit canonical representations. We recall that 1 is the greatest element of Q^b and 0 is the smallest. As usual, $P = L_\lambda \bigcup L_\mu$.

12. Lemma. In the above notation, if $A \in W(P)$ and $\underline{A}_\mu$ does not exist then $A \subseteq 1$ and, dually, if $\overline{A}^\mu$ does not exist then $0 \subseteq A$. If $\underline{A}_\mu$ exists then $A \subseteq 1 \smile \underline{A}_\mu$ and if $\overline{A}^\mu$ exists then $0 \frown \overline{A}^\mu \subseteq A$.

Proof: We need only consider $\underline{A}_\mu$; the results for $\overline{A}^\mu$ follow by the principle of duality.

If $\underline{A}_\mu$ does not exist then, by Lemma 4.9, $\overline{A}^\lambda$ exists; then $A \subseteq \overline{A}^\lambda \leqslant 1$ and so $A \subseteq 1$.

Now assume that $\underline{A}_\mu$ exists. We prove that $A \subseteq 1 \smile \underline{A}_\mu$ by induction on $\ell(A)$.

If $\ell(A) = 1$ then $A = \underline{A}_\mu \subseteq 1 \smile \underline{A}_\mu$.

Let $\ell(A) = n > 1$ and let the result hold for any polynomial shorter than n.

If $A = B \frown C$ then $\underline{B}_\mu$ and $\underline{C}_\mu$ exist, by Lemma 4.7. Since B and C are shorter than A then $B \subseteq 1 \smile \underline{B}_\mu$, $C \subseteq 1 \smile \underline{C}_\mu$. Since L_μ is a chain we may assume that $\underline{B}_\mu \leqslant \underline{C}_\mu$. Thus $\underline{A}_\mu = \underline{B}_\mu$ by Lemma 4.7; thus $B \subseteq 1 \smile \underline{A}_\mu$ and so $A \subseteq B \subseteq 1 \smile \underline{A}_\mu$.

If $A = B \smile C$ then we may assume that either only $\underline{B}_\mu$ exists or both $\underline{B}_\mu$, $\underline{C}_\mu$ exist. If $\underline{C}_\mu$ does not exist then, as proved above, $C \subseteq 1$. By the inductive hypothesis $B \subseteq 1 \smile \underline{B}_\mu = 1 \smile \underline{A}_\mu$; thus $B \smile C \subseteq 1 \smile \underline{A}_\mu$. If both $\underline{B}_\mu$, $\underline{C}_\mu$ exist then, by the inductive hypothesis, $B \subseteq 1 \smile \underline{B}_\mu \subseteq 1 \smile \underline{A}_\mu$ and $C \subseteq 1 \smile \underline{C}_\mu \subseteq 1 \smile \underline{A}_\mu$; thus $A = B \smile C \subseteq 1 \smile \underline{A}_\mu$.

Thus the lemma is proved.

13. Lemma. The free product of L_λ and L_μ admits canonical representations.

Proof: Assume that the lemma is false. Then there is a smallest integer n such that there are $A, B \in W(P)$, $\ell(A) = \ell(B) = n$, $A \sim B$, $A \not\equiv B$, and A, B are minimal polynomials.

If $n = 1$ then $A, B \in P$; thus $A \sim B$ implies that $A = B$. Consequently $n > 1$.

By Lemma 6, $A = A_0 \smile A_1$ and $B = B_0 \frown B_1$, or vice versa, is impossible.

In view of the principle of duality we may assume that $A \equiv A_0 \smile \ldots \smile A_{r-1}$, $B \equiv B_0 \smile \ldots \smile B_{s-1}$, $r, s > 1$, where each A_i is an element of P or is of the form $C \frown D$, and similarly for each B_j.

If for each $i \leqslant r-1$ there is a $j \leqslant s-1$ such that $A_i \sim B_j$, $\ell(A_i) = \ell(B_j)$, and conversely for each $j \leqslant s-1$, then, by the minimality of A and B, j is uniquely determined by i, and conversely. Thus there are mappings
$f : \{0, \ldots, r-1\} \to \{0, \ldots, s-1\}$ and
$g : \{0, \ldots, s-1\} \to \{0, \ldots, r-1\}$
such that $A_i \sim B_{f(i)}$ for each $i \leqslant r-1$ and $B_j \sim A_{g(j)}$ for each $j \leqslant s-1$. If $g(f(i)) \neq i$ then, since $A_i \sim A_{g(f(i))}$, $A \sim A_0 \smile \ldots \smile A_{i-1} \smile A_{i+1} \smile \ldots \smile A_{r-1}$, contradicting the minimality of A. Thus for each $i \leqslant r-1$ $g(f(i)) = i$ and, similarly, for each $j \leqslant s-1$ $f(g(j)) = j$. Thus $r = s$ and f, g are permutations of $\{0, \ldots, r-1\}$. By the minimality of n,

$A_i \equiv B_{f(i)}$ since $\ell(A_i) = \ell(B_{f(i)})$. Thus $A \equiv B$, contradicting the assumption about A, B.

Consequently, in view of Lemma 7, we may assume without loss of generality that $\ell(A_o) = 1$.

If $A_o \in L_\mu$ then, since $A_o \subseteq B$ and L_μ is a chain, we apply Coroll. 4.13 and find a $j \leqslant s-1$ such that $A_o \subseteq B_j$. If $\ell(B_j) = 1$ then $A_o \leqslant B_j$; thus $B_j \in L_\mu$ and, using Coroll. 4.13 and proceeding as in Lemma 7, we find that $A_o = B_j$. Thus the above argument would apply. If $\ell(B_j) > 1$ then there is an $i \leqslant r-1$ such that $B_j \sim A_i$ and $\ell(A_i) = \ell(B_j) > 1$. Thus $i \neq 0$ and $A_o \subsetneqq A_i$; consequently $A \sim A_1 \cup \ldots \cup A_{r-1}$, again contradicting the minimality of A.

Thus we may assume that $A_o \in L_\lambda$ and that $A_o \nsubseteq B_j$ for all $j \leqslant s-1$. Since $A_o \subseteq B$, $\underline{B}_\lambda$ exists and $A_o \leqslant \underline{B}_\lambda$. Now $A_o \nleqslant \underline{B_j}_\lambda$ for all $j \leqslant s-1$ since $A_o \subseteq B_j$. Since, for any $x, y \in L_\lambda$, $x \vee y = x, y$ or 1 we may assume, without loss of generality, that $\underline{B_o}_\lambda \vee \underline{B_1}_\lambda = 1$. We may assume that $B_o \in L_\lambda$, for if $B_o, B_1 \notin L_\lambda$ then, by the above arguments, there are $i, k \neq 0$ such that $B_o \subseteq A_i$ and $B_1 \subseteq A_k$; since $A_o \leqslant \underline{B_o}_\lambda \vee \underline{B_1}_\lambda$ then $A_o \subseteq B_o \cup B_1 \subseteq A_i \cup A_k$ and so $A \sim A_1 \cup \ldots \cup A_{r-1}$, contradicting the minimality of A.

Consequently $B_0 \in L_\lambda$. Since $1 = \underline{B_0 \smile B_1}$, $1 \smile B_1 \subseteq B_0 \smile B_1$. Since $B_0 \leqslant 1$ then $B_0 \smile B_1 \subseteq 1 \smile B_1$. Thus

$$B_0 \smile B_1 \sim 1 \smile B_1 .$$

By Lemma 12, either $B_1 \subseteq 1$ or $\underline{B_1}_\mu$ exists and $B_1 \subseteq 1 \smile \underline{B_1}_\mu$. If $B_1 \subseteq 1$ then $B_0 \smile B_1 \sim 1$, contradicting the minimality of B. If $B_1 \subseteq 1 \smile \underline{B_1}_\mu$ then, since $B_0 \leqslant 1$,

$$B_0 \smile B_1 \subseteq 1 \smile \underline{B_1}_\mu \subseteq 1 \smile B_1 \sim B_0 \smile B_1 .$$

Thus $B_0 \smile B_1 \sim 1 \smile \underline{B_1}_\mu$; since $\underline{B_1}_\lambda$, $\underline{B_1}_\mu$ exist then $\ell(B_1) > 1$, again contradicting the minimality of B.

Thus our original assumptions about A and B cannot all hold.

Consequently the free product admits canonical representations.

The results of this section can be summarized as:

14. Theorem. The free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$ admits canonical representations if and only if at least one of the following holds:

(i) $|\Lambda| = 1$;

(ii) all L_λ are chains;

(iii) $|\Lambda| = 2$, say $\Lambda = \{\lambda, \mu\}$, L_μ is a chain, and $L = Q^b$ where Q is the disjoint union of unrelated chains.

*3. Canonical representations in partially ordered free products.

Let Λ be a poset and let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of lattices indexed by Λ. P denotes the poset introduced in Section 4.2. Let the partially ordered free product of the $(L_\lambda \mid \lambda \in \Lambda)$ be denoted by L. By the discussion in Section 4.2 the calculations in Lemmas 8, 9 and 10 apply to partially ordered free products.

If there are two incomparable $\lambda, \mu \in \Lambda$ and if L_λ contains two incomparable elements whose join is not maximal in L_λ, or whose meet is not minimal in L_λ, then the calculations of Lemma 8 apply; thus L does not admit canonical representations. A similar situation occurs if λ, μ are incomparable, L_λ is not a chain, and there is a $\nu \in \Lambda$ such that $\nu < \lambda$, $\nu \nleq \mu$ or $\nu > \lambda$, $\nu \ngeq \mu$; the w of Lemma 8 can be taken as an element of L.

If there are three mutually incomparable $\lambda, \mu, \nu \in \Lambda$ such that L_λ is not a chain then the calculations presented in Lemma 9 show that L does not admit canonical representations in $W(P)$.

If λ, μ are incomparable in Λ and L_λ, L_μ are not chains then, as in Lemma 10, L does not admit canonical representations.

If L is a chain for all $\lambda \in \Lambda$ then, by Lemma 4, L admits canonical representations.

There is only one case left to consider:

For each $\lambda \in \Lambda$, if L_λ is not a chain then either

(i) for all $\nu \in \Lambda$ either $\nu \geqslant \lambda$ or $\nu \leqslant \lambda$;

or

(ii) L_λ is of the form Q^b, Q a disjoint union of incomparable chains, $\Lambda' = \{\mu \in \Lambda \;\; \mu, \lambda$ are incomparable$\}$ is a chain, L_μ is a chain for all $\mu \in \Lambda'$, and $\nu > \lambda$ (resp. $\nu < \lambda$) and $\mu \in \Lambda'$ implies that $\nu > \mu$ (resp. $\nu < \mu$).

To dispose of this situation let I be a chain and let $(P_i \mid i \in I)$ be a family of posets such that, for each $i \in I$, P_i has an $(\mathcal{M}_i, \mathcal{N}_i)$-structure. Let $P = \bigcup(P_i \mid i \in I)$ be a poset such that

(i) if $x, y \in P_i$ then $x \leqslant y$ in P if and only if $x \leqslant y$ in P_i ;

(ii) if $x \in P_i$, $y \in P_j$, $i \neq j$, then $x \leqslant y$ if and only if $i < j$.

We define $\mathcal{M} = \bigcup(\mathcal{M}_i \mid i \in I)$, $\mathcal{N} = \bigcup(\mathcal{N}_i \mid i \in I)$ as the structure on P.

There is a natural embedding of $W(P_i)$ in $W(P)$ for each $i \in I$. By an abuse of notation $\underline{A}_i$ denotes the $(\mathcal{M}_i, \mathcal{N}_i)$-lower cover of A in P_i and $\underline{A}$ its $(\mathcal{M}, \mathcal{N})$-lower cover in P, and dually.

15. Lemma. If $A \in W(P_i)$, $i \in I$, then

a) $\underline{A} \cap P_i = \underline{A}_i$ and $\underline{A} \cap P_j = \emptyset$ if $j > i$;

b) $\overline{A} \cap P_i = \overline{A}^i$ and $\overline{A} \cap P_j = \emptyset$ if $j < i$.

Proof: A simple inductive argument will do. We need only observe that if J, K are $\mathcal{M}$-ideals of P such that $J \cap P_i$, $K \cap P_i$ are $\mathcal{M}_i$-ideals of P_i then

$$(J \vee K) \cap P_i = (J \cap P_i) \overset{i}{\vee} (K \cap P_i)$$

where $\overset{i}{\vee}$ denotes join of $\mathcal{M}_i$-ideals of P_i and $\vee$ denotes join of $\mathcal{M}$-ideals of P.

The dual situation, of course, holds for $\mathcal{N}$-dual ideals.

16. Coroll. The quasi-order on $W(P)$ defined by $(\mathcal{M}, \mathcal{N})$ restricted to $W(P_i)$ is the quasi-order on $W(P_i)$ defined by $(\mathcal{M}_i, \mathcal{N}_i)$.

Proof: In view of Definition 3.2 we need only show that if $A, B \in W(P_i)$, $i \in I$, then $\overline{A} \cap \underline{B} \neq \emptyset$ if and only if $\overline{A}^i \cap \underline{B}_i \neq \emptyset$.

Clearly $\overline{A}^i \cap \underline{B}_i \neq \emptyset$ implies that $\overline{A} \cap \underline{B} \neq \emptyset$.

Now let $\overline{A} \cap \underline{B} \neq \emptyset$; then there is a $j \in I$ such that $\overline{A} \cap \underline{B} \cap P_j \neq \emptyset$. Thus $\overline{A} \cap P_j \neq \emptyset$ and $\underline{B} \cap P_j \neq \emptyset$. By Lemma 15 $j = i$ and so $\overline{A}^i \cap \underline{B}_i \neq \emptyset$.

Thus the corollary follows.

17. Lemma. If $A \in W(P_i)$, $B \in W(P_j)$, and $i < j$ then $A \subseteq B$ in $W(P)$.

Proof: We establish this lemma by induction on $\ell(A) + \ell(B)$. If $\ell(A) + \ell(B) = 2$ then $A \in P_i$, $B \in P_j$ and so $A \leqslant B$; thus $A \subseteq B$.

If $B = C \smile D$, $C, D \in W(P_j)$, then, by induction, $A \subseteq C \subseteq B$.

If $B = C \frown D$, $C, D \in W(P_j)$, then, by induction, $A \subseteq C, D$; thus $A \subseteq B$.

The dual arguments work for A .

18. Lemma. If $A \in W(P)$ is minimal then there is an $i \in I$ such that $A \in W(P_i)$.

Proof: We proceed by induction on $\ell(A)$.

If $\ell(A) = 1$ the result is clear.

If $A = B \smile C$ then B, C are minimal in $W(P)$. By induction there are $i, j \in I$ such that $B \in W(P_i)$, $C \in W(P_j)$. If $i \neq j$ then, since I is a chain, we may assume that $i < j$. Thus, by Lemma 17, $B \subseteq C$ and so $A \sim C$, contradicting the minimality of A . Thus $i = j$ and so $A \in W(P_i)$.

The dual argument holds if $A = B \frown C$.

Thus the lemma follows.

19. Theorem. If $FL(P_i ; \mathcal{M}_i, \mathcal{N}_i)$ admits canonical representations in $W(P_i)$ for each $i \in I$, I a chain,

then $FL(P\ ;\ \mathfrak{M}, \mathfrak{N})$ admits canonical representations in $W(P)$.

<u>Proof</u>: Let $A, B \in W(P)$ be minimal and let $A \sim B$, $\ell(A) = \ell(B)$. By Lemma 18 there are $i, j \in I$ such that $A \in W(P_i)$, $B \in W(P_j)$.

If $i \neq j$ we may assume that $i < j$. Then $\ell(A) = = \ell(B) > 1$ and so, by Lemma 5, $\overline{B} \cap \underline{A} \neq \emptyset$. However $B \subseteq A$. Applying Definition 3.2, if $B = B_o \smile B_1$ then $B_o \subseteq A$; but $B_o \in W(P_j)$ and so, by Lemma 17, $A \subseteq B_o$. Thus $A \sim B_o$, contradicting the minimality of A .

If $B = B_o \frown B_1$ and $B_o \subseteq A$, say, then, as above, $A \sim B_o$. The dual arguments hold in the other two cases.

Thus $i = j$ and, by Coroll. 16, $A \sim B$ in $W(P_i)$. Thus $A \equiv B$.

Thus the theorem is proved.

Thus

20. <u>Theorem</u>. The partially ordered free product of the family of lattices $(L_\lambda \mid \lambda \in \Lambda)$ admits canonical representations if and only if either all L_λ are chains, or for each $\lambda \in \Lambda$ such that L_λ is not a chain either

(i) for each $\mu \in \Lambda$ either $\mu \leqslant \lambda$ or $\mu \geqslant \lambda$;

or

(ii) L_λ is of the form Q^b , Q a disjoint union of incomparable chains, $\Lambda' = \{\mu \in \Lambda \mid \lambda, \mu \text{ incomparable}\}$

is a chain, L_μ is a chain for all $\mu \in \Lambda'$, and $\nu > \lambda$ (resp. $\nu < \lambda$) and $\mu \in \Lambda'$ imply $\nu > \mu$ (resp. $\nu < \mu$).

Proof: We apply Theorem 19 and Theorem 14.

We define an equivalence relation $\equiv$ on Λ: $\lambda \equiv \mu$ if and only if there is no $\nu \in \Lambda$, L_ν not a chain, strictly between λ and μ. By conditions (i) and (ii) $\equiv$ is an equivalence relation and preserves the partial order on Λ. Let $I = \Lambda/\equiv$; I is a chain under the induced partial order. For each $i \in I$ let $P_i = \bigcup(L_\lambda \mid \lambda \in i)$. Let $\mathcal{M}_i$, $\mathcal{N}_i$ be the sets of all pairs in the same L_λ for all $\lambda \in i$.

If L_λ is a chain for all $\lambda \in i$ then $FL(P_i ; \mathcal{M}_i, \mathcal{N}_i) = CF(P_i)$ and so admits canonical representations.

If there is a $\lambda \in i$ such that L_λ is not a chain then, by conditions (i) and (ii), either $P_i = L_\lambda$ and so $FL(P_i ; \mathcal{M}_i, \mathcal{N}_i)$ admits canonical representations, or L_λ is of the form Q^b and $P_i = L_\lambda \bigcup M$ where $M = \bigcup(L_\mu \mid \mu, \lambda$ incomparable). In the latter case M is a chain and so $FL(P_i ; \mathcal{M}_i, \mathcal{N}_i)$ is identical with the free product of L_λ and M. Thus, by Theorem 14, $FL(P_i ; \mathcal{M}_i, \mathcal{N}_i)$ admits canonical representations.

Applying Theorem 19 the result follows.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The main results of this dissertation are best described in terms of the concept of the word problem. The word problem is applicable to any universal algebra, but here we shall restrict our discussion to lattices. The usual formulation of the word problem is (see Evans [5] and [6], and Grätzer [7], Chapter 4, in each of which the word problem is stated for universal algebras):

Let $A_0, B_0, \ldots, A_{r-1}, B_{r-1}$ be lattice polynomials on the finite set $S = \{x_0, \ldots, x_{n-1}\}$. Given any two lattice polynomials A, B on S, find an algorithm to decide, whenever L is a lattice and $a_0, \ldots, a_{n-1} \in L$ satisfy the relations

$A_i(a_0, \ldots, a_{n-1}) = B_i(a_0, \ldots, a_{n-1})$ for all $i \leqslant r-1$, whether or not

$A(a_0, \ldots, a_{n-1}) = B(a_0, \ldots, a_{n-1})$.

If $C \in W(S)$ then $C(a_0, \ldots, a_{n-1})$ denotes the element $F(C) \in L$ where F is defined as $F(x_i) = a_i$ for all $i \leqslant n-1$, $F(X \smile Y) = F(X) \vee F(Y)$, and $F(X \frown Y) = F(X) \wedge F(Y)$ for all $X, Y \in W(S)$.

For lattices the above problem was solved in the affirmative by Evans [5].

It is natural to attempt to extend the problem to either an infinite set S or an infinite number of relations. The major drawback to such an approach is that in general one would have no effective way to determine whether or not a specific infinite set of elements satisfies a specific infinite set of identities.

For the word problem, in view of the nature of freeness, one need only consider the "most free" lattice generated by S satisfying the relations $A_i = B_i$ for all $i \leqslant r-1$. Thus a satisfactory approach to these difficulties is to let S be a poset P and to let the set of relations be all relations compatible with some $(\mathcal{M}, \mathcal{N})$-structure on P. This leads to the problem of determining whether or not two polynomials $A, B \in W(P)$ represent the same element of $FL(P\,;\mathcal{M}, \mathcal{N})$. If P is infinite the aforementioned difficulties still remain. However one can discuss the <u>word problem modulo</u> a given mathematical structure on P:

Given $A, B \in W(P)$, is there an algorithm that reduces the problem of whether or not A, B represent the same element of $FL(P\,;\mathcal{M}, \mathcal{N})$ to a problem regarding the given mathematical structure on P?

With this point of view Section 2.2 provides a solution to the word problem for $FL(P)$ modulo the structure of the pseudo-principal ideals and pseudo-principal dual ideals as subsets of P. Similarly the discussion in Section 3.1 solves the word problem for $FL(P ; \mathcal{M}, \mathcal{N})$ modulo the structure of the pseudo-principal $\mathcal{M}$-ideals and the pseudo-principal $\mathcal{N}$-dual ideals as subsets of P.

The discussion in Chapter IV solves the word problem for the free product of lattices modulo their lattice structure, and that for partially ordered free products modulo the lattice structure of the factors and the partial order on the indexing set.

Chapter IV also shows that the word problem for the free product of lattices amalgamated by a lattice M of finite length is solvable modulo the lattice structure of the factors and the partial maps c_o and C_o. If the amalgamated lattice M is finite then the question of whether or not $c_o(x)$, $C_o(x)$ exist for some element x of a factor and, if either exists, what it is, is effectively computable. Thus if M is finite the word problem for the amalgamated free product is solvable modulo the lattice structure of the factors.

We conclude by mentioning two problems that are left open in this study.

Problem 1. Can our results be extended to infinitary lattice operations; that is, what can one say about $FL(P ; \mathcal{M}, \mathcal{N})$ if $\mathcal{M}$, $\mathcal{N}$ contain infinite subsets of P?

<u>Problem 2</u>. Can the results of Chapter V be extended to arbitrary posets? That is, find necessary and sufficient conditions on P so that FL(P) (or FL(P ; $\mathcal{M}, \mathcal{N}$)) admits canonical representations.

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