I DEFLECTION OF MUONS BY THE EARTH'S MAGNETIC FIELD

II LIOUVILLE'S THEOREM APPLIED TO COSMIC RAYS

by

Allen Berkal

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Abstract

The charged muons detected by meson telescopes are bent as they pass through the earth's magnetic field. These particles have energies much reduced from the energies of the primary protons. Consequently, they might experience large deflections under the influence of the earth's magnetic field. One of the purposes of this thesis is to determine to what degree this process influences meson telescope observations.

The calculations take into account that muons are produced at different heights with different energies, that energy is dissipated as the charged particles traverse the atmosphere, and that muons decay in flight. Distribution curves are presented for the deflection spectrum for different viewing directions in an equatorial plane. It is found that the shape of the curve is not very sensitive to changes in the viewing direction in this plane. The maximum deflection for muons of like sign is about .30 radians and the average deflection is about .09 radians in a coordiante system in which it is possible to describe the deflection with one angle. It is concluded that corrections for this uncertainty in the viewing direction, although not negligible, would give little additional information in telescope observations with the present state of the art of cosmic ray telescopes.

Liouville's theorem in classical form is investigated with emphasis placed upon its application to cosmic ray problems. The theorem is presented in a pedagogic fashion.

Liouville's theorem is shown to be valid in a given electromagnetic field in a phase space in which the Newtonian momenta are used as coordinates. The relationship between the intensity of cosmic radiation and the density in phase space is derived. Liouville's theorem is shown to be the link between the intensities of radiation in different viewing directions at the same observation point. In conservative fields, isotropic radiation at infinity implies isotropic radiation everywhere. Conservative fields alone cannot produce the diurnal variation.

A slight modification of Axford's model for the diurnal variation is given as an example of the breakdown of Liouville's theorem. A frictional effect which causes the breakdown is introduced when the cosmic ray particles traverse regions of turbulence in the magnetic field. Since Liouville's theorem has been shown to be valid in a given electromagnetic field, that is, one which is a function of position and time only, it is concluded that the presence of the particles in these turbulent fields must influence the field. An analytical argument does not accompany this qualitative statement.

CONTENTS

PART I DEFLECTION OF MUONS BY THE EARTH'S MAGNETIC FIELD

PART ONE

DEFLECTION OF MUONS BY THE EARTH'S MAGNETIC FIELD

CHAPTER ONE

Introduction to Meson Deflection Calculations

Liouville's theorem is often applied to cosmic ray studies of isotropy. It is used to show that any isotropy that is observed cannot be due to interactions between particles and conservative fields. This is discussed in part II.

Experimentally, a detecting device which is able to discern directions can be used to investigate the existence of isotropies in the primary cosmic radiation. A commonly used device which supposedly has this property is the cosmic ray telescope. In practice, a cosmic ray telescope counts particles coming from a finite range of directions. There are four fundamental factors which affect its directional properties.

The primary cosmic ray particles are bent as they pass through the earth's magnetic field. To account for this it is necessary to calculate the asymptotic directions of the particles, the directions of motion that the primaries have when they first come under the influence of the earth's field. Often one speaks of the asymptotic cone of acceptance of the detector. This is the solid angle containing the asymptotic directions of approach that significantly contribute to the counting rate of the detector. It was shown by K. G. McCracken(1) that the variation in the asymptotic cones of acceptance from station to station would mean that different stations would see an anisotropy in the primary radiation in different ways. Because of the nature of the earth's magnetic field, particles will undergo larger deflections at low latitudes then at high latitudes. Therefore, in general low latitude stations smooth out the anisotropy because of their wide cones of acceptance. They also will shift the phase and reduce the amplitude of the anisotropy. On the other hand, the high latitude stations more faithfully reflect the actual anisotropy. The phase is more or less preserved and the amplitude is larger than for low latitude stations. In addition to having the observation station at a high latitude, the effects of deflections of the primaries can be minimized by selecting particles with high energies. High energy particles are bent to a lesser degree. By selecting high energy particles however, counting rates are reduced.

Secondly, the mesons detected by cosmic ray telescope are the result of decay of pions which in turn were produced in collisions by the primary protons. One would expect that in general the directions of the mesons and pions would be different than that of the parent particle, but from conservation of momentum considerations, that their directions would be collimated in some cone about the direction of the primary proton. The question remains as to how closely collimated are the mesons to the primary particles. High energy collisions produce closely collimated secondaries[2]. The decay of a pion into a muon and a neutrino is accompanied by small angle scattering and the muon beam is only slightly more divergent than the original beam. The half width of the muon distribution exceeds 1° only in case of very low pion energies. The pion rigidity has to be less than 1 Bev/c before the spread of muons is important[3]. A common procedure used in dealing with cosmic ray telescopes which detect high energy mesons only is to assume that the secondary particles are collimated in the direction of the primary particles.

Mesons are scattered through small angles as the result of elastic collisions in the atmosphere. The spreading caused by these collisions has been dealt with by Rossi[4]. The effect is usually neglected.

In practice, meson telescopes count particles coming from directions contained in a finite solid angle. Quite obviously then, there is a spread in the viewing directions one considers due to the geometry of the telescope. If only the solid angle of the telescope is reduced in size the counting rate is also reduced. To maintain counting rates and to decrease the solid angle requires large detectors separated by large distances.

The final aspect which affects the directional properties of the telescope is the deflection which mesons experience as they pass through the earth's magnetic field. Bonnevier and

Brunberg[5] investigated this aspect of the optics of cosmic ray telescopes. They considered tracing a particle of a given energy, 1 Bev, along its trajectory from the observation point up to a given height, 20 km. They did this for vertically orientated telescopes only. They found the deflection as a function of the magnitude of the earth's magnetic field and the dip angle. For a viewing station at Winnipeg (dip angle $77^{\circ}14'$ and magnetic intensity $B = 0.60544$ gauss) the value of the deflection calculated was approximately 1°. No attempt was made to account for the fact that mesons are produced at different levels in the atmosphere. No attempt was made to investigate how the deflections changed with energy.

It was the purpose of this study to make a more detailed analysis of the deflections of mesons in the earth's magnetic field in order to determine whether, and if necessary how, one would take this into account in determining the viewing directions of a cosmic ray telescope. The way this was done was to determine the theoretical distribution curve, the relative numbers plotted against the deflections of the mesons. This project was a continuation of work undertaken by E. Hung[6], which was motivated by experimental observations in the cosmic ray laboratory of Dr. S. Standil[7].

CHAPTER TWO

Approach Used to Determine Meson Deflections and Intensity

The investigation of the deflections of mesons in the earth's magnetic field required that three rather obvious, but nevertheless important, items be taken into account. First, mesons with opposite charges are bent in different directions. Secondly, the magnitude of a particle's deflection depends upon its height of production in the atmosphere, which may differ from particle to particle; and upon its energy which changes as the particle passes through the atmosphere. Thirdly, the meson are unstable particles and decay in flight. A procedure had been established by Harris and Escobar[8], and Hung[9] for finding differential intensities of mesons at ground level which considered these points. This procedure was modified in such a way as to enable the calculation of the distribution of the deflections undergone by mesons counted with a cosmic ray telescope. For the sake of completeness much of the approach used by Harris and Escobar, and Hung will be restated.

The following basic assumptions are necessary to the development of the fundamental calculations:

i) The primary particles undergo all their deflections outside the atmosphere. This is usually a good approximation as the depth of the atmosphere is much smaller than the radii of curvature of the primaries, which have large energies. This means that the primary particles undergo rectilinear motion in the atmosphere.

ii) Secondaries are degraded in energy; therefore, their radii of curvature are small compared to the depth of the earth's atmosphere and hence their deflection cannot be ignored.

iii) The surface of the atmosphere is considered to be a plane above the point of observation.

iv) The earth's magnetic field is taken to be constant throughout the depth of the atmosphere.

v) The primary radiation is independent of time and direction at the top of the atmosphere.

vi) The pions, which are produced in collisions by the primaries with particles in the atmosphere, decay immediately into muons.

vii) The pions and muons are collimated in the same direction as the primary proton at production. This assumption together with the previous one permits one to ignore pions as far as deflections are concerned.

viii) The production of mesons through pion decay occurs continuously throughout the atmosphere. Therefore the production occurs continuously along any path considered.

ix) The primary protons are merely absorbed exponentially in the atmosphere with a constant attenuation length L. That is, L is independent of energy.

x) The scattering of mesons in air is taken to be negligible.

There are two coordinate systems to which reference will be made. In both systems the origin is located at the point of observation, and both systems are chosen to be right handed orthogonal bases.

Recalling that the magnetic induction, $\underline{B}$, is taken to be constant with height, it is convenient to choose a coordinate system, to be called system K, (x, y, z), such that the z axis is anti-parallel to $\underline{B}$ (figure 1). The y axis is in a plane containing the z axis and the geomagnetic axis and is directed toward the geomagnetic north. Then the x axis points toward the geomagnetic east. This coordinate system is especially suited for considering deflections of charged particles in the earth's magnetic field since the deflections are the result of the moving charge's spiral motion about the lines of force. Therefore, in the K coordinate system the deflection can be given by the change in the azimuthal angle, $\Delta\phi$.

The coordinate system which has the vertical direction as the Z axis will be called system K', (X, Y, Z) (figure 1). Here the Y axis points toward the geographical north and the X axis toward the geographical east. This coordinate system is convenient for describing orientations in the laboratory.

The transformation between coordinate systems K' and K is given in appendix one.

In figure 1, the trajectory of a meson in coordinate system K is shown. The following symbols, given in

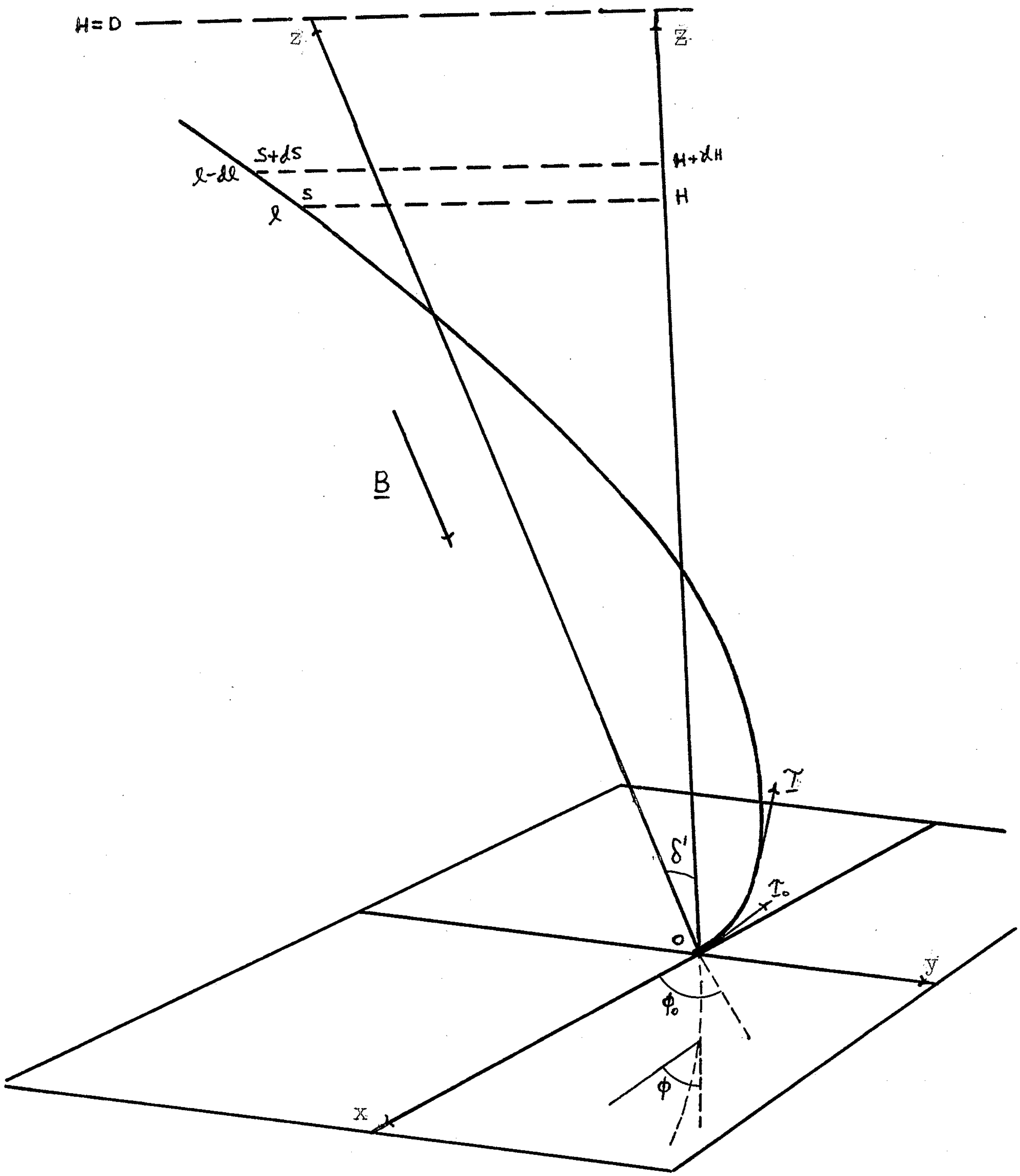


Figure1. Path of Positive Muon In Earth's Magnetic Field

alphabetical order, are to be used in the ensuing section:

$\mathbf{B}$, (0, 0 -B)	--	magnetic induction at O (gauss)
D	--	depth of atmosphere at observation point (gm cm^{-2})
δ'	--	angle between Oz and vertical
$\mathbf{F} = F\boldsymbol{\tau}$	--	resistive force of air upon meson in flight (gm sec^{-2}cm)
g	--	gravitational acceleration (cm sec^{-2})
H	--	the altitude measured from O upward along the vertical direction (gm cm^{-2})
l	--	the path length of the trajectory measured in the direction of motion (cm)
λ	--	the angle between $\boldsymbol{\tau}$ and the vertical
λ_0	--	the angle between $\boldsymbol{\tau}_0$ and the vertical
m	--	mass of air in a volume V (gm)
M	--	average gram molecular weight of air (gm)
μ	--	rest mass of muon (Mev/c^2)
$\mathbf{n}$	--	unit vector in the vertical direction, (0, sinδ', cosδ')
P	--	pressure of atmosphere at H (dyne cm^{-2})
$\mathbf{p}$	--	momentum of muon (gm cm sec^{-1})
ϕ	--	azimuthal angle in K coordinate system
ϕ_0	--	azimuthal angle of observation direction in K system
q	--	charge of muon (electronic charge in e. s. u.)
R	--	energy of the muon at point of observation in terms of its residual range in air (gm cm^{-2})

$\mathcal{R}$ -- universal gas constant (dyne cm (gm mole)$^{-1}$ ($^{\circ}$abs)$^{-1}$)

ρ -- density of air (gm cm^{-3})

S -- the path length of the trajectory measured from O upward (gm cm^{-2})

T -- temperature of atmosphere ($^{\circ}$abs)

$\underline{\tau}$ -- unit vector anti-parallel to the tangent vector of the particles trajectory

$\underline{\tau}_0$ -- $\underline{\tau}$ corresponding to direction of observation

θ -- the angle between Oz and $\underline{\tau}$

$\underline{v} = -v\underline{\tau}$ -- velocity of muon (cm sec^{-1})

Before deriving the equations which are necessary for the calculations perhaps a word should be said about the general procedure to be followed. A telescope with a given orientation is taken. (Here only directions in the equatorial plane were considered but the argument is not restricted to this case.) One then considers a meson of a given charge which arrives at the telescope with a given energy. The meson's trajectory is then extended back to the top of the atmosphere, allowing for energy loss by the particle during flight. Once this has been achieved, the number of mesons produced at each point along the trajectory, which have the proper energy at production to just reach the telescope with the correct amount of energy, is found. The next step requires the determination of the probability that a meson produced at a given point on the trajectory with the right amount of energy will not decay before reaching the telescope. The trajectory, the number produced at each point, and the probability of survival, will be sufficient to calculate the number of particles arriving at the telescope which satisfy given conditions.

For the purposes required, the determination of the trajectory of a muon is equivalent to finding the deflection of the particle together with the path length S and the angle λ at each point of the trajectory.

The deflection can be found by considering a muon in flight. The equation of motion is

$$\frac{d\underline{p}}{dt} = \frac{q\underline{v} \times \underline{B}}{c} + \underline{F} \tag{1}$$

but

$$\left.\begin{aligned} \underline{p} &= -p\,\underline{\Upsilon} \\ \underline{F} &= F\,\underline{\Upsilon} \\ \underline{v} &= -v\,\underline{\Upsilon} \end{aligned}\right\} \tag{2}$$

Therefore,

$$-\frac{d(p\underline{\Upsilon})}{dt} = -p\frac{d\underline{\Upsilon}}{dt} - \underline{\Upsilon}\frac{dp}{dt} = -\frac{qv\underline{\Upsilon}\times\underline{B}}{c} + F\underline{\Upsilon} \tag{3}$$

However, $\frac{dl}{dt} = v$. Applying the chain rule and dividing by v,

$$-p\frac{d\underline{\Upsilon}}{dl} - \underline{\Upsilon}\frac{dp}{dl} = -q\frac{\underline{\Upsilon}\times\underline{B}}{c} + \frac{F\underline{\Upsilon}}{v} \tag{4}$$

Take the dot product with $\underline{\Upsilon}$ of both sides, remembering that $\underline{\Upsilon}\cdot\underline{\Upsilon}=1$ implies that $\frac{d\underline{\Upsilon}}{dl}\cdot\underline{\Upsilon} = 0$

$$-\frac{dp}{dl} = \frac{F}{v} \tag{5}$$

Substitute this result into equation (4) to give

$$\frac{d\underline{\Upsilon}}{dl} = \frac{-q\,\underline{B}\times\underline{\Upsilon}}{c\,p} \tag{6}$$

or

$$d\underline{\Upsilon} = -\frac{q\,\underline{B}\times\underline{\Upsilon}}{c\,p}\,dl \tag{7}$$

From the definitions, $\frac{ds}{\rho} = -dl$, so that

$$d\underline{\gamma} = \frac{q\,\underline{B} \times \underline{\gamma}\, ds}{c\rho p} \tag{8}$$

Recalling that $\underline{B} \equiv (0, 0, -B)$, equation (8) yields

$$\left.\begin{aligned} d\gamma_x &= \frac{qB\gamma_y\, ds}{c\rho p} \\ d\gamma_y &= -\frac{qB\gamma_x\, ds}{c\rho p} \\ d\gamma_z &= 0 \end{aligned}\right\} \tag{9}$$

These equations give

$$d(\gamma_x + i\gamma_y) = \frac{qB}{c\rho p}\, ds(\gamma_y - i\gamma_x) = \frac{iqB}{c\rho p}\, ds(\gamma_x + i\gamma_y) \tag{10}$$

which can be integrated to give

$$\ln\left[\frac{\gamma_x + i\gamma_y}{\gamma_{x_0} + i\gamma_{y_0}}\right] = \int_S^0 \frac{iqB}{c\rho p}\, ds' = -i\int_0^S \frac{qB}{c\rho p}\, ds' \tag{11}$$

One can write $\gamma_{x_0} + i\gamma_{y_0} = \text{constant} \times e^{i\phi_0}$. Therefore,

$$\gamma_x + i\gamma_y = \text{CONSTANT}\; e^{i\left[\phi_0 - \int_0^S \frac{qB}{c\rho p} ds'\right]} \tag{12}$$

The third equation in (9) yields the result,

$$\gamma_z = \text{constant} \tag{13}$$

One can also write

$$\underline{\gamma} = \sin\theta(\underline{i}\cos\phi + \underline{j}\sin\phi) + \underline{k}\cos\theta \tag{14}$$

On comparing components of $\underline{\gamma}$, it is found that $\cos\theta = \text{constant}$. This implies that $\theta = \text{constant}$ and $\sin\theta = \text{constant}$. Hence,

$$\phi = \phi_0 - \int_0^S \frac{qB}{c\rho p}\, ds' \tag{15}$$

The following empirical formula which is accurate to within one per cent for $30 < R + S < 6000$ gm cm^{-2}, where $R + S$ is the residual range at any point along the trajectory, gives an approximate expression for p as a function of R and S [10,11].

$$\frac{1}{p} = \frac{1}{\mu c}\left(\frac{53.5}{56 + R + S} - 0.00207\right) \tag{16}$$

ρ is obtained as a function of H by using the ideal gas law.

$$pV = \frac{m\mathcal{R}T}{M} \tag{17}$$

On substituting

$$\left.\begin{array}{l} p = (D-H)g \\ \text{and} \quad \rho = \frac{m}{V} \end{array}\right\} \tag{18}$$

into (17)

$$\frac{1}{\rho} = \frac{\mathcal{R}T}{Mg(D-H)} \tag{19}$$

The value of $\frac{\mathcal{R}}{Mg}$ used was 2870 cm $(^{o}\text{abs})^{-1}$ (for dry air); so that

$$\frac{1}{\rho} = \frac{2870\ T}{(D-H)} \tag{20}$$

The substitution for p and ρ from equations (16) and (20) into equation (15) gives the result,

$$\phi(H) - \phi_0 = -\frac{2870\,g\,B}{\mu c^2}\int_0^S \left\{\frac{53.5}{56+R+S'} - 0.00207\right\}\frac{T(H)\,dS'}{(D-H)} \tag{21}$$

where the usual function notation has been used.

From the definitions of λ, S, and H it is seen that

$$dS = \frac{dH}{\cos\lambda(H)} \tag{22}$$

When this is substituted into (22)

$$\Delta\phi(H) = \phi(H) - \phi_0 = -\frac{2870\,g\,B}{\mu c^2}\int_0^H \left\{\frac{53.5}{56+R+S'} - 0.00207\right\}\frac{T(H')}{(D-H')}\frac{dH'}{\cos\lambda(H')} \tag{23}$$

$\Delta\phi$ is in radians.

Since λ is the angle between $\underline{r}$ and the vertical it is clear that $\cos\lambda = \underline{m}\cdot\underline{r} = (\underline{i}\,0 + \underline{j}\sin\delta' + \underline{k}\cos\delta')\cdot(\underline{i}\sin\theta\cos\phi + \underline{j}\sin\theta\sin\phi + \underline{k}\cos\theta)$

or

$$\cos\lambda = \sin\delta'\sin\theta\sin\phi + \cos\delta'\cos\theta \tag{24}$$

$\cos\lambda$ is then a function of ϕ as δ' and θ are constant angles. If $\sin\phi$ is expanded as a Taylor series about $\phi = \phi_0$, $\cos\lambda$ becomes

$$\cos\lambda = \sin\delta'\sin\theta\left[\sin\phi_0 + \Delta\phi\cos\phi_0 - \Delta\phi^2\frac{\sin\phi_0}{2} - \Delta\phi^3\frac{\cos\phi_0}{6} + \Delta\phi^4\frac{\sin\phi_0}{24} + \cdots\cdots\right] + \cos\delta'\cos\theta \tag{25}$$

By rearranging terms and taking the series only as far as necessary for the calculations[12] one gets

$$\cos\lambda(H) = \cos\lambda_0 + \sin\delta'\sin\theta\cos\phi_0\left[\Delta\phi(H) - \frac{\Delta\phi^3(H)}{6}\right] + \sin\delta'\sin\theta\sin\phi_0\left[-\frac{\Delta\phi^2(H)}{2} + \frac{\Delta\phi^4(H)}{24}\right] \tag{26}$$

where $\cos \lambda_0 = \sin \delta' \sin \theta \cos \phi_0 + \cos \delta' \cos \theta$

Equation (22) can be integrated to give

$$S(H) = \int_0^H \frac{dH'}{\cos \lambda(H')} \tag{27}$$

Equations (23), (26), and (27) constitute a set of coupled integral equations which are difficult to solve in closed form. The following series of successive approximations can be used to solve this set of equations:

i) Initially choose $\Delta \phi(H) = 0$

ii) Substitute into equation (26) to give initial values for $\cos \lambda(H)$

iii) Substitute into (27) to find S(H)

iv) Put the values of S(H) back into equation (23) to obtain new values for $\Delta \phi(H)$

v) Find refined values for $\cos \lambda(H)$ in equation (26) from the $\Delta \phi(H)$ above. In turn, find S(H) by using the $\cos \lambda(H)$ in equation (27). Repeat steps (iii) to (v).

If the series converge then repetition of the above process gives values which are closer and closer to the correct solutions. Some criterion can be established to determine where the series can be terminated.

Once the trajectory has been found, the next step is to get an expression for the number of mesons produced at each point along the trajectory. Sands[13], Olbert[14], and Harris and Escobar[15] developed such a relationship. Empirically it was determined that at the top of the atmosphere the number of mesons produced per gram, per second, per steradian, in a given direction with residual ranges at production in the region R' to $R'+dR'$; $G(R')dR'$, was such that the relationship

$$G(R')\,dR' = A(a+R')^{-m}\,dR' \tag{28}$$

was valid, where A, a, m are parameters. It was reasoned that only "a" could depend upon the observation point and direction, since the counting rate for high energy mesons would be independent of these factors. In fact it was shown that "a" is a function of the geomagnetic cutoff rigidity only. Therefore, the empirical differential range spectrum at production should be written as $G(R',a)dR'$. At a point below the top of the atmosphere the number of mesons produced will differ from the number produced at the top of the atmosphere by a factor which accounts for the exponential absorption of the primary protons. Thus at a point below the top of the atmosphere, the number of mesons produced per gram, per second, per steradian, in a given direction, such that the residual range at production is in the region R' to $R'+dR'$, is given by

$$G(R',a)\exp(-y/L) \tag{29}$$

where y is the atmosphere transversed by the primary proton before production and L is the constant attenuation length for the absorbtion of the primary protons. From the assumption of rectilinear motion $y = \frac{(D-H)}{\cos\lambda}$ where λ is the angle which the negative tangent to the protons trajectory makes with the vertical. This of course is the same λ associated with the meson's motion at production since the meson is collimated in the same direction as the proton. By putting in the experimentally determined values of A,

7.31×10^4 gm^{-2} cm^2 sec^{-1} sterad^{-1}; n, 3.58; and L, 120 gm cm^{-2},

$$G(R+S,a) = \frac{7.31\times 10^4}{(a+R+S)^{3.58}} \exp\left\{\frac{-(O-H)}{120\cos\lambda}\right\} \text{gm}^{-2}\text{cm}^{-2}\text{sec}^{-1}\text{sterad}^{-1} \quad (30)$$

for $100 < R + S < 6000$ gm cm^{-2} (16).

Since negative and positive particles experience deflections in opposite directions, they are treated separately. This means that it is necessary to have separate range production spectra which depend on the charge. To find these quantities G_+ and G_-, which will be indicated by G_σ where $\sigma = \pm 1$, it is useful to recall that because the primary particles are predominately protons, conservation of charge requires that more positive then negative secondaries be produced. To account for this one introduces the positive excess at production, δ. δ is defined by

$$\delta(R+S,a) = \frac{(G_+ - G_-)}{(G_+ + G_-)}.\frac{1}{2} \quad (31)$$

where $G_+ + G_- = G(R+S,a)$. Thus

$$\left.\begin{aligned} G_+ + G_- &= G(R+S,a) \\ \text{and} \quad G_+ - G_- &= \tfrac{1}{2}\delta G(R+S,a) \end{aligned}\right\} \quad (32)$$

The solution to these equations is

$$G_\sigma = \frac{1}{2}\left[1 + \frac{\sigma}{2}\delta(R+S,a)\right] G(R+S,a) \quad (33)$$

If the expression for G(R S,a) from (30) is substituted into (33)

$$G_\sigma(R+S,a)\,dR = \frac{1}{2}\left[1+\frac{\sigma}{2}\delta(R+S,a)\right]\frac{7.31\times 10^4}{(a+R+S)^{3.58}}\exp\left\{\frac{-(O-H)}{120\cos\lambda}\right\} dR \quad (34)$$

$$\text{gm}^{-1}\text{sec}^{-1}\text{sterad}^{-1}$$

In words, $G_\sigma(R+S,a)dR$ is the number of mesons with sign σ produced at a point S along a trajectory, in a given direction, per gram, per second, per steradian, and which have energies at production in the range $R+S$ to $R+S+dR$. If these particles survive the decay process and reach the telescope they will have residual ranges in the region R to R + dR.

Let $\omega(R+S)$ be the probability that a meson produced with energy R+S will survive a distance S. The law of decay in the laboratory frame is

$$\frac{d\omega'}{dt} = -\frac{1}{\tau'}\omega' \qquad (35)$$

where $\tau' = \tau/\sqrt{1-v^2/c^2}$ and τ is the mean life time of the mesons in a frame at rest. That is, the rate of change of probability at any time is proportional to the probability of survival up to that point. The element of time,

$$dt = \frac{dl}{v} = -\frac{dS}{\rho v} \qquad (36)$$

Then upon substitution into equation (35)

$$\frac{d\omega'}{\omega'} = \frac{dS}{\rho v \tau/\sqrt{1-v^2/c^2}} \qquad (37)$$

However,

$$\frac{p}{\mu} = \frac{v}{\sqrt{1-v^2/c^2}} \qquad (38)$$

Thus $$\frac{d\omega'}{\omega'} = \frac{\mu\, dS}{\rho p \tau} \qquad (39)$$

When this is integrated,

$$\ln\left[\frac{\omega(R+S)}{1}\right] = \int_S^0 \frac{\mu\, dS'}{\rho p \tau} = -\int_0^S \frac{\mu\, dS'}{\rho p \tau} \qquad (40)$$

Recalling equation (15), one sees that

$$\int_0^S \frac{dS'}{\rho p} = \frac{c\,\Delta\phi}{q B} \qquad (41)$$

Therefore, upon substitution equation (40) becomes

$$\omega(R+S) = \exp\left\{\frac{-c\mu\Delta\phi}{q B \tau}\right\} \qquad (42)$$

If $\Delta\phi$ is expressed as a function of R and H then ω is also a function of R and H.

Using the results contained in equations (23), (26), (27), (34) and (42), it is possible to find expressions for various types of intensity. The number of particles of charge σ which reach the telescope with residual range R to R + dR and which are produced at a point S, in dS, is given by

$$G_\sigma(R+S, a)\, dR\, dS\, \omega(R, S, \sigma) \tag{43}$$

$$\text{or} \quad \frac{1}{2}\left[1+\frac{1}{2}\sigma\,\delta(R+S, a)\right]\frac{7.31\times10^{4}}{(a+R+S)^{3.58}} \exp\left\{\frac{-(D-H)}{120\cos\lambda(H)}\right\}\omega\, dS\, dR \quad \text{cm}^{-2}\text{sec}^{-1}\text{sterad}^{-1} \tag{44}$$

If this expression is integrated over the length of the trajectory, o to S (D), the result is the number of particles of charge σ which reach the telescope with residual ranges R to R + dR. This is commonly called the differential intensity. Thus, the differential intensity spectrum for particles with charge σ, $i_\sigma(R, \Upsilon_o)$, is given by

$$i_\sigma(R, \Upsilon_o) = \int_0^D \frac{1}{2}\left[1+\frac{1}{2}\sigma\,\delta(R+S, a)\right]\frac{7.31\times10^{4}}{(a+R+S)^{3.58}} \exp\left\{\frac{-(D-H)}{120\cos\lambda(H)}\right\}\frac{\omega(R, H, \sigma)\, dH}{\cos\lambda(H)} \quad \text{gm}^{-1}\text{sec}^{-1}\text{sterad}^{-1} \tag{45}$$

where the integration over S from 0 to S (D) has been replaced by an integration over H by utilizing relationship (27). Notice that the intensity is a function of the telescope's viewing direction Υ_o. It is readily seen that the corresponding integral intensity, $I_\sigma(R, \Upsilon_o)$, can be found when the integration of the differential intensity is carried out over R. The lower limit of R is the minimum energy needed by a meson in order to be counted. This is determined by the telescope geometry. This limit must be consistant with the restrictions placed upon the calculation by equations (16) and (30). These restrictions place an upper limit upon the integration. It has been estimated that less than one per cent of the particles are ignored by imposing the constants of these limits[17].

Thus

$$I_\sigma(R_1, T_0) = \int_{R_1}^{R_2}\int_0^D \frac{1}{2}\left[1+\frac{1}{2}\sigma\delta(R+S, a)\right]\frac{7.31\times10^4}{(a+R+S)^{3.58}}\exp\left\{\frac{-(D-H)}{120\cos\lambda(H)}\right\}\frac{\omega(R,H,\sigma)\,dH\,dR}{\cos\lambda(H)}$$

$$\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1} \qquad (46)$$

where $\omega(R, H, \sigma) = \exp\left\{\frac{-c\mu\Delta\phi(R,H)}{qB\tau}\right\}$. Finally, the number of particles which reach the telescope with residual ranges R_1 to R_2 and which are produced in the range H_1 to H_2, $W(H_2, R_2)$, is given by

$$W(H_2, R_2) = \int_{R_1}^{R_2}\int_{H_1}^{H_2}\frac{1}{2}\left[1+\frac{1}{2}\sigma\delta(R+S, a)\right]\frac{7.31\times10^4}{(a+R+S)^{3.58}}\exp\left\{\frac{-(D-H)}{120\cos\lambda(H)} - \frac{c\mu\Delta\phi(R,H)}{qB\tau}\right\}\frac{dH\,dR}{\cos\lambda(H)}$$

$$\text{cm}^{-2}\,\text{sec}^{-1}\,\text{sterad}^{-1} \qquad (47)$$

The equations derived here were used to obtain a distribution curve of the deflections undergone by mesons reaching the telescope.

CHAPTER THREE

Numerical Computations

The deflection in the earth's magnetic field undergone by a meson of charge σ, which arrives at a telescope in a given direction, depends upon the residual range, R, at the observation point and upon the height, H, where it was produced. Because this is so, it is possible for particles with different ranges, R_1 and R_2 for example, to be bent the same amount, $\Delta\phi$, provided they are produced at different heights, H_1 and H_2. Given an R and a height of production, H, equations (23), (26), and (27) can be employed to give a unique value of $\Delta\phi$. When these values of R, H, and $\Delta\phi$ are substituted into the integrand of integral (47), the number of particles produced at height H, with residual range R at observation is given by the value of this integrand. These are not necessarily the only particles which undergo this deflection, $\Delta\phi$. To obtain the distribution curve one must add up all the particles which have been bent the same amount. The way this was done is the subject of this section.

Because an expression could not be found analytically which gave the distribution curve for a given viewing direction, it was necessary to construct a histogram which in the limit as the intervals became small would give the smooth curve. One way of doing this was to break up the R-H plane into rectangular boxes bounded by the lines of constant R and H. Corresponding to the ij^{TH} box there is the number of particles which have residual ranges in the interval R_{i-1} to R_i and which were produced at heights between H_{j-1} and H_j. This number was found for each box in the plane by using equation (47). In principle, the boxes in the R-H plane could have been chosen so small that all the particles associated with a box underwent approximately the same deflection. An average deflection was found using the values of the deflections corresponding to the R,H pairs at the corners of the box. The number of particles found for each box was then associated with the average deflection for that box. The spectrum of deflections was divided into intervals and when it was determined in which interval the average deflection for a given box fell the number of particles for that box was added to the existing sum of particles in that interval. When this was done for all the boxes in the R-H plane the histogram was obtained.

Integrations were approximated using the trapazoid method as this was deemed sufficient to obtain results with satisfactory accuracy. The top of the atmosphere was taken to be $H = 1000$ gm cm^{-2}. 200 gm cm^{-2} was taken to be the minimum residual range required by a meson in order that it be counted by the telescope. This lower energy cutoff corresponded approximately to the lower energy cutoff of the non-rotating cosmic ray telescopes of Dr. S. Standil and R. Briggs at the University of Manitoba.[18] The maximum value of the residual range at observation was taken to be 4400 gm cm^{-2}. The reason for choosing this value lies in the fact that R + S must satisfy at all points along the trajectory the relation $100 < R+S < 6000$ gm cm^{-2} in order that relations (15) and (30) be valid. For a telescope located on the parallel of latitude passing through Winnipeg, 64.4°N, and in the direction in the equatorial plane, $\gamma = 0$,* that makes the least angle with the vertical, the apparent depth of the atmosphere is $1000/\cos 64.4^{\circ}$ gm cm^{-2}, or 1600 gm cm^{-2}. For any other viewing direction in the equatorial plane, γ, the apparent depth of the atmosphere is given by $1000/\cos\gamma \cos 64.4^{\circ}$ gm cm^{-2}. The maximum value of $R+S$ for any viewing direction is approximately the sum of R and the apparent depth of the atmosphere. Strictly speaking therefore, the maximum residual range allowed in these calculations does depend on the viewing direction; however, since the calculations were performed for small γ's only, this dependence was ignored. Note that the case of large γ can be readily taken into account with only a small modification. In practice there are limits placed upon the number of boxes one can consider in the R-H plane and on the size of the intervals considered in the deflection spectrum. As the calculations were computerized, these limits were related to the finite capacity of the computer memory unit and by the finite speed with which the computer operates. It was decided that .01 radians would be a reasonable magnitude to take for the size of the intervals along the $\Delta\phi$ axis of the distribution histogram. The basis for this decision was the fact that this would give about forty points between zero deflection and what was estimated to be the maximum. The

* For definition of γ see page 72.

size of the boxes in the R-H plane were originally chosen to be $\Delta H = 40$ gm cm^{-2} and $\Delta R = 200$ gm cm^{-2}. This corresponded roughly to the limits of the capability of the 1620 IBM computer. It became evident after examining initial results that the size of the boxes would have to be changed. Work was transferred to the IBM 360 when it became available.

All equations and expressions were put into forms that were convenient for numerical computation. The following is a list of relations adapted for computer manipulation of the IBM 360:

i) The atmospheric height was divided into 160 intervals by writing

$$\left.\begin{aligned} H = H(N) &= 1040 - 40N \quad \text{gm cm}^{-2} \quad N = 1, 2, \cdots 21 \\ &= 200 - 4(N-21) \quad \text{gm cm}^{-2} \quad N = 22, 23, \cdots 61 \\ &= 40 - .4(N-61) \quad \text{gm cm}^{-2} \quad N = 62, 63, \cdots 161 \end{aligned}\right\} \tag{48}$$

ii) The intervals of residual range at observation, R were given by

$$R = R(L) = 4400 - L \quad \text{gm cm}^{-2} \quad L = 1, 2, \cdots\cdots 20 \tag{49}$$

The intervals of R and H defined here indicate how the R-H plane was divided into boxes in order to facilitate the numerical computations.

iii) Equation (23),

$$\Delta\phi(R, H) = \frac{-2870\, q\, B}{\mu c^{2}} \int_0^H \left\{ \frac{53.5}{56 + R + S(H)} - 0.00207 \right\} \frac{T(H')}{(D - H')} \frac{dH'}{\cos\lambda(H')}$$

was converted into a useful form by writing

$$\left.\begin{aligned} U(N) &= \frac{-2870\, q\, B}{\mu c^{2}} \frac{T(N)}{(D - H(N))} \left(\frac{\Delta H}{2}\right) \quad N = 1, 2, \cdots\cdots 160 \\ &= \frac{-0.098675430\, \sigma T(N)}{1000 - H(N)} \quad \text{where } \Delta H = 40 \text{ gm cm}^{-2} \\ U(161) &= \frac{-0.0987543\, \sigma T(161)}{1} \end{aligned}\right\} \tag{50}$$

and

$$V(N) = \left\{ \frac{53.5}{56+R+S(N)} - 0.00207 \right\} \frac{1}{\cos\lambda(N)} \tag{51}$$

Then

$$\left.\begin{aligned}
\Delta\phi(L,N) &= \sum_{N'=2}^{N} \left\{ U(N'-1)V(N'-1) + U(N')V(N') \right\} + \Delta\phi(L,N-1) \\
&\qquad N = 2, \cdots\cdots 21 \\
\Delta\phi(L,N) &= \sum_{N'=22}^{N} \left\{ \frac{U(N'-1)V(N'-1) + U(N')V(N')}{10} \right\} + \Delta\phi(L,N-1) \\
&\qquad N = 22, 23, \cdots\cdots 61 \\
\Delta\phi(L,N) &= \sum_{N'=62}^{N} \left\{ \frac{U(N'-1)V(N'-1) + U(N')V(N')}{100} \right\} + \Delta\phi(L,N-1) \\
&\qquad N = 62, 63, \cdots\cdots 161
\end{aligned}\right\} \tag{52}$$

Here the different forms of $\Delta\phi(L,N)$ correspond to different ranges of H. ΔH was taken to be 40 gm cm^{-2} in the definition of U(N). However, in fact, ΔH has the values 40, 4, and .4 gm cm^{-2} depending on the range of H one is integrating over. Notice also that in equation (50) a special situation exists for U(161). This arose because mathematically equation (23) diverges if H is allowed to take on the value D. The expression used for the density was found by applying the ideal gas law to the atmosphere. This implies that the atmosphere is of limitless extent because $h \propto \int_{H=0}^{D} \frac{dH}{(D-H)}$, where h is the height of the atmosphere in cm. This in turn means that B is taken as being constant to infinity and that mesons are produced everywhere. Although the factor accounting for meson decay, is an exponential with exponent proportional to $\Delta\phi$ and limits the number of mesons with large $\Delta\phi$ to small values; it is necessary to restrict the values of $\Delta\phi$ in the numerical calculations to finite values. This can be done by not integrating to the very top of the atmosphere or by introducing a new expression for the density in the last interval of height ΔH. The latter effectively constrains the atmosphere to a finite region in space. This was done when formally $D-H = 1$ gm cm^{-2} was substituted into the ideal gas law to find the density. It should be noted that if the interval of height, ΔH, which causes the difficulty is small, the number of mesons surviving to the telescope will be small, and hence it is not a matter of great importance in which of these two ways this interval is treated.

iv) Equation (26), which gives the cosine of the angle between the negative tangent vector and the vertical,

$$\cos\lambda(H) = \cos\lambda_0 + \sin\delta'\sin\theta\cos\phi_0\left[\Delta\phi(H) - \frac{\Delta\phi^3(H)}{6}\right] + \sin\delta'\sin\theta\sin\phi_0\left[-\frac{\Delta\phi^2(H)}{2} + \frac{\Delta\phi^4(H)}{24}\right]$$

was written

$$\cos\lambda(N) = \cos\lambda_0 + \sin\delta'\sin\theta\cos\phi_0\left[\Delta\phi(L,N) - \frac{\Delta\phi^3(L,N)}{6}\right] - \frac{1}{2}\sin\delta'\sin\theta\sin\phi_0\left[\Delta\phi^2(L,N) - \frac{\Delta\phi^4(L,N)}{12}\right] \quad (53)$$

and $\cos\lambda_0 = \sin\delta'\sin\theta\cos\phi_0 + \cos\delta'\cos\theta$

v) Equation (27), giving the path length

$$s(H) = \int_0^H \frac{dH'}{\cos\lambda(H')}$$

became

$$\left.\begin{aligned}
S(N) &= \sum_{N'=2}^{N} 20.0\left[\frac{1}{\cos\lambda(N'-1)}+\frac{1}{\cos\lambda(N')}\right]+S(N-1) \qquad N=2,3,\cdots 21\\
S(N) &= \sum_{N'=22}^{N} 2.0\left[\frac{1}{\cos\lambda(N'-1)}+\frac{1}{\cos\lambda(N')}\right]+S(N-1) \qquad N=22,23,\cdots\cdots 61\\
S(N) &= \sum_{N'=22}^{N} 0.2\left[\frac{1}{\cos\lambda(N'-1)}+\frac{1}{\cos\lambda(N')}\right]+S(N-1) \qquad N=62,63,\cdots\cdots 161
\end{aligned}\right\} \tag{54}$$

vi) Equation (47) which gives the number of particles associated with a box in the R-H plane,

$$W(H_2,R_2)=\int_{R_1}^{R_2}\int_{H_1}^{H_2}\frac{1}{2}\left[1+\frac{1}{2}\sigma\delta(R+S,a)\right]\frac{7.31\times10^4}{(a+R+S)^{3.58}}\exp\left\{\frac{-(D-H)}{120\cos\lambda(H)}-\frac{c\mu\Delta\phi(R,H)}{qB\gamma}\right\}\frac{dH\,dR}{\cos\lambda(H)}$$

was evaluated in steps.

$$Y(N)=\frac{1}{2}\left[1+\frac{1}{2}\sigma\delta(L)\right]\frac{7.31\times10^4}{(a+R(L)+S(L,N))^{3.58}}\frac{[H(N)-H(N-1)]}{2\cos\lambda(N)} \tag{55}$$

$$EPSLON(N)=\frac{-c\mu}{qB\gamma}\Delta\phi(N,L)=-8.765987\sigma\Delta\phi(N,L) \tag{56}$$

Then the number of particles arriving at the telescope with residual range R(L), having been produced in the interval of height H(N-1) to H(N) is

$$X(N,L) = Y(N)+Y(N-1)$$

except when N = 22 and N = 62 (57)

$$X(22,L) = Y(22)+\frac{Y(21)}{10}$$

and $$X(62,L) = Y(62)+\frac{Y(61)}{10}$$

Finally W(H,R) became

$$W(N,L)=X(N,L)+X(N,L-1) \tag{58}$$

Notice that $\Delta R/2$ was not included in W(N,L) because only the relative number was of importance and ΔR is the same for all intervals.

vii) The value of the deflection for the N-L box, AN, was found from

$$AN = \frac{1}{4}\left[\Delta\phi(N-1, L-1) + \Delta\phi(N, L) + \Delta\phi(N, L-1) + \Delta\phi(N-1, L)\right] \qquad (59)$$

viii) Once the deflection and relative number for the N-L box was found, the N-L box was broken into 4 small boxes. The relative number was taken to be W(N, L) for each of the smaller boxes. Four average deflections corresponding to the four boxes created were determined from the value AN assigned to the center of the N-L box and from the values of the deflection at the corners.

ix) The average deflection for all the particles counted given by the ratio of the weighted sum of the deflections and the sum of the relative numbers.

$$\overline{\Delta\phi} = SD/SP \qquad (60)$$

The computations were done in the following order:

i) The sign of the particle, σ, was chosen.

ii) For a given value of L and $R = R(L)$, $\Delta\phi(N)$ was set equal to zero for all N.

iii) Cos $\lambda(N)$ was determined for all N from equation (53).

iv) From equation (54), S(N) was found for all N.

v) New values of $\Delta\phi$ were determined from equation (52).

vi) New values of cos $\lambda(N)$ were found from (53).

vii) New values of S(N) were found from (52).

viii) If the new value of S(161) differed in absolute value from the previous value by more than 1 gm cm^{-2}

but less than 900 gm cm^{-2} the computations were repeated from step (v). If the critical difference was less than 1 gm cm^{-2}, the new values of S(N) were used in the ensuing steps.

ix) The steps from (ii) to (viii) were repeated for all the different values of L.

x) The relative number associated with each main box was found from equations (55), (56), (57), and (58).

xi) The deflection associated with each main box was found from (57).

xii) The deflections and the relative numbers associated with each of the smaller boxes was found.

xiii) It was determined in which interval each deflection belonged and then the appropriate relative number was added to the existing total for that interval.

xiv) The average deflection was found.

xv) The whole process was repeated for particles with the opposite charge to give the distribution curve for the viewing direction chosen, γ .

Three pieces of information had to be supplied before the numerical work could be started. First, the value of "a" in equation (55) had to be given. Because "a" changes slowly with viewing direction[19] one value was used for all directions. "a" was taken as 510 gm cm^{-2}[20]. Second, the positive excess at production, $\delta(R+S, a)$ was replaced by the positive excess at sea level, because the former was not known[21]. The twenty values for $\delta(R)$ were read as data in the program (appendix two). Finally the temperature was determined at various heights from data for The Pas, Manitoba as this was the only data available[22]. Incorporating these values in the equations described previously, a program was written for the IBM 360 which quickly gives the required histogram for different values of gamma (appendix three).

Because of the limitations of the IBM 1620 computer, satisfactory results were not obtained when attempts were made to calculate the distribution histogram on this machine, using the boxes in the R-H plane method. An alternative procedure which was tried is briefly presented here for the sake of completeness; also because it might be a convenient starting point for future calculations.

Again an attempt was made to construct a distribution histogram with .01 radian intervals along the $\Delta\phi$ axis. This method amounted to integrating equation (47) in the R-H plane between contours which corresponded to constant values of $\Delta\phi$, the deflection in the K coordinate system. As written, equation (47) could not be integrated analytically, but this difficulty was overcome by fitting surfaces for the relative number as a function of R and H, and for R as a function of $\Delta\phi$ and H. Polynomials of fifth order were fit by the least square method to a group of points generated in a previous program. For each point, four corresponding quantities were specified, the deflection, the relative number, the height, and the residual range at observation.

In the argument that follows the symbols defined below will be used.

ANUM -- number of particles with deflections between two specified values of $\Delta\phi$.

RN(R,H) -- relative number of particles corresponding to a height of production H and a residual range at observation R

P -- deflection (radians)

R -- residual range at observation gm cm^{-2}

P1, P2 -- values of deflection along contours which act as limits for integration in equation (47)

H1, H2 -- limits of integration for H in equation (47)

It was assumed that the relative number, RN(R, H), could be represented sufficiently well by a fifth order polynomial in R and H,

$$RN(R,H) = \sum_{\substack{i=1 \\ i+j\leq 7}}^{6} \sum_{j=1}^{6} C_{ij} R^{i-1} H^{j-1} \tag{61}$$

and that the residual range R could be approximated as a fifth order polynomial in P and H,

$$R(P,H) = \sum_{\substack{l=1 \\ l+k\leq 7}}^{6} \sum_{k=1}^{6} D_{lk} P^{l-1} H^{k-1} \tag{62}$$

If equation (47) is integrated between two contours in the R-H plane along which P has the values P1 and P2 respectively, then the number of particles with deflections between P1 and P2 is given by

$$ANUM = \int\int RN(R,H)\, dR\, dH \tag{63}$$

On substitution from (61)

$$\text{ANUM} = \int_{H_1}^{H_2} \int_{R(P1,H)}^{R(P2,H)} \sum_{\substack{i=1 \\ i+j \le 7}}^{6} \sum_{j=1}^{6} C_{ij} R^{i-1} H^{j-1} \, dR \, dH \tag{64}$$

By integrating over R

$$\text{ANUM} = \int_{H_1}^{H_2} \sum_{\substack{i=1 \\ i+j \le 7}}^{6} \sum_{j=1}^{6} \frac{C_{ij} R^{i}}{i} \Bigg|_{R(P1,H)}^{R(P2,H)} H^{j-1} \, dH \tag{65}$$

When P1 and P2 were substituted into equation (62), R(P1, H) and R(P2, H) were found as power series in H. From these series, the powers of R(P1,H) and R(P2,H) were obtained. They were written

$$R^{1}(P1, H) = \sum_{m=1}^{6} Q1_{m}(P1) H^{m-1} \tag{66}$$

where $$Q1_{m}(P1) = \sum_{l=1}^{l=7-m} D_{lm} P^{l-1} \tag{67}$$

and $$R^{i}(P1, H) = \sum_{t=1}^{t=5i+1} Qi_{t}(P1) H^{t-1} \tag{68}$$

Equations (65), (66), and (68) were combined to give

$$\text{ANUM} = \int_{H1}^{H2} \sum_{\substack{i=1 \\ i+j \le 7}}^{6} \sum_{j=1}^{6} C_{ij} \left[\sum_{t=1}^{5i+1} \left\{ \frac{Qi(P2,t) - Qi(P1,t)}{i} \right\} H^{t-1} \right] H^{j-1} \, dH \tag{69}$$

Finally, integration over H gave the required relationship,

$$\text{ANUM} = \sum_{\substack{i=1 \\ i+j \le 7}}^{6} \sum_{j=1}^{6} C_{ij} \left[\sum_{t=1}^{5i+1} \left\{ \frac{Qi(P2,t) - Qi(P1,t)}{i} \right\} \left\{ \frac{H2^{t+j-1} - H1^{t+j-1}}{t+j-1} \right\} \right] \tag{70}$$

This method of obtaining the distribution of deflection histogram was programmed for the IBM 360 computer. It did not give satisfactory results because the polynomials were poor approximations to the actual functions.

CHAPTER FOUR

Discussion of Results

An elementary consideration of the deflections of mesons in the earth's magnetic field would lead one to believe that the distribution curve for mesons of a given sign would be a function describing some kind of peak. That is, small deflections correspond to mesons which were produced near the observation point or to mesons produced with large energies. Not many exponentially attenuated primaries reach the lower layers of the atmosphere to produce mesons; and there are fewer mesons produced with large energies as the production spectrum employed indicates. On the other hand, one would expect few mesons to have large deflections. Large deflections correspond to mesons produced in the upper regions of the atmosphere. Since mesons decay in flight, few would survive the large distances necessary to reach the telescope. As the survival probability is controlled by an exponential factor it is likely that the distribution curve for deflections will behave in a somewhat similar manner for large deflections. For very large values of the deflection the relative number of particles should tend toward zero. Between the extremes in deflection one would expect the two factors, one pertaining to the number of particles produced at a given height and the other concerning the survival probability of the mesons, to produce a maximum.

The qualitative argument described above is indeed supported by the quantitative determination of the deflection distribution curve. Examples of the type of curves obtained using the program given in the appendix are given in figures 2 and 3. These curves correspond to viewing directions of gamma equal to 5 and +25 degrees respectively. There are two peaks, one for each type of charge. The positive particles suffer negative deflections while the negative mesons undergo positive deflections. Notice that the maximum deflection is approximately .30 radians or about 17 degrees for both positive and negative mesons. The average deflections have been calculated in the same program and work out to be roughly .095 radians or about 5.4 degrees in the K coordinate system. The results indicate that the distribution curve in the K system is not sensitive to changes in the viewing direction. It is perhaps useful to know that the time taken to calculate eleven distribution histograms was about three minutes.

It was mentioned previously that the K coordinate system in which all the calculations were performed is not the system usually used to describe events and orientations in the laboratory. A most convenient coordinate system in the cosmic ray laboratory is the K' system described earlier. A program was written for the computer which converted the deflections, $\Delta\phi$, in the K coordinate system to the K' system in terms of the zenith and azimuthal angles θ and ϕ. In addition, the angle between the negative tangent to the particle's motion and the direction of the telescope was calculated for given values of $\Delta\phi$. This angle was called α. For viewing directions given by γ equal to 5 and +25 degrees the value of α

Figure 2(a). Distribution Curve of Deflections of Positive Muons, Detected At $\gamma = 5^{\circ}$. Figure 2(a) And 2(b) Are Not Normalized to Unity.

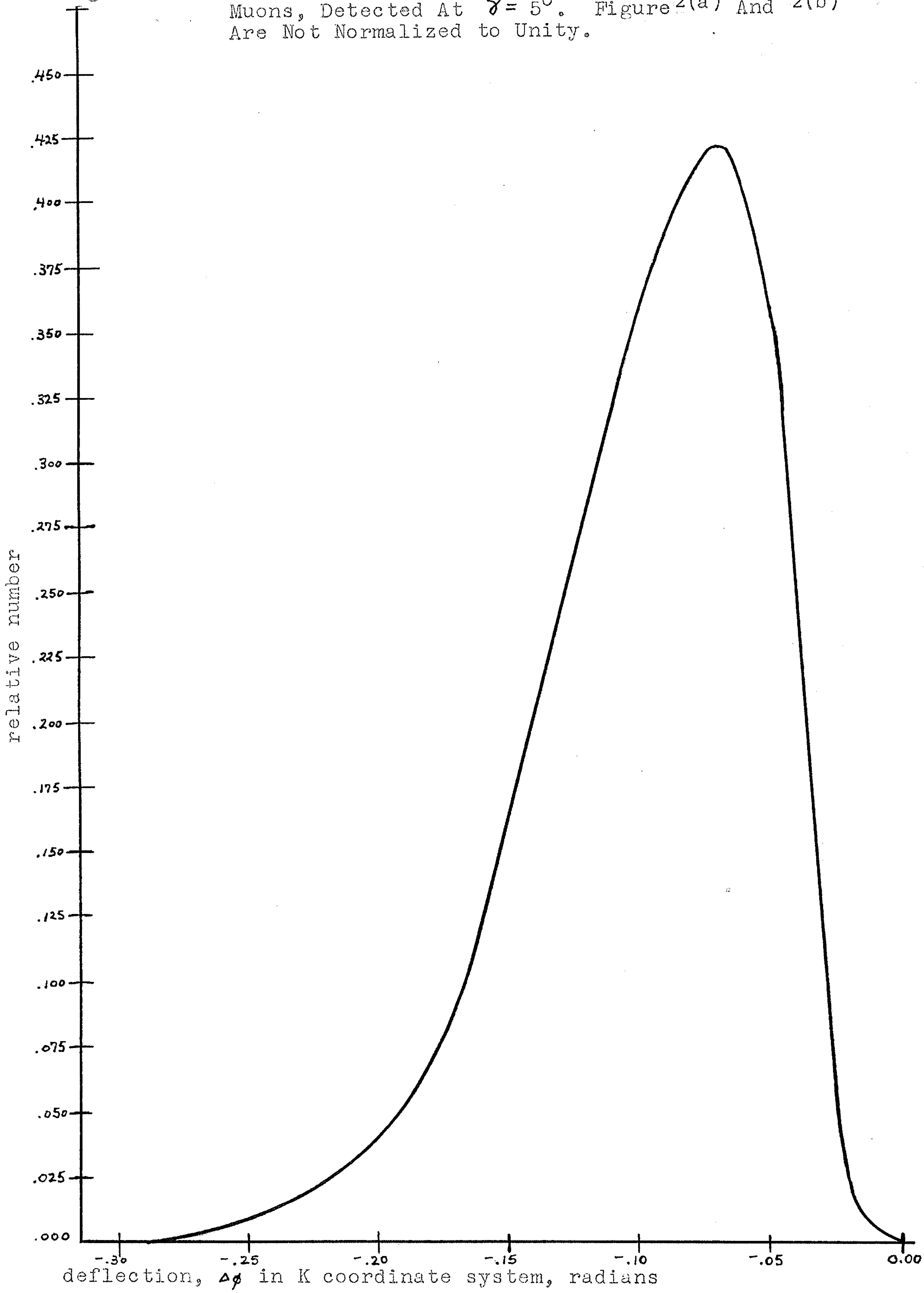


Figure 2(b). Distribution Curve of Deflections of Negative Muons, Detected At $\gamma = 5^{\circ}$. Figures 2(a) And 2(b) Are Not Normalized To Unity.

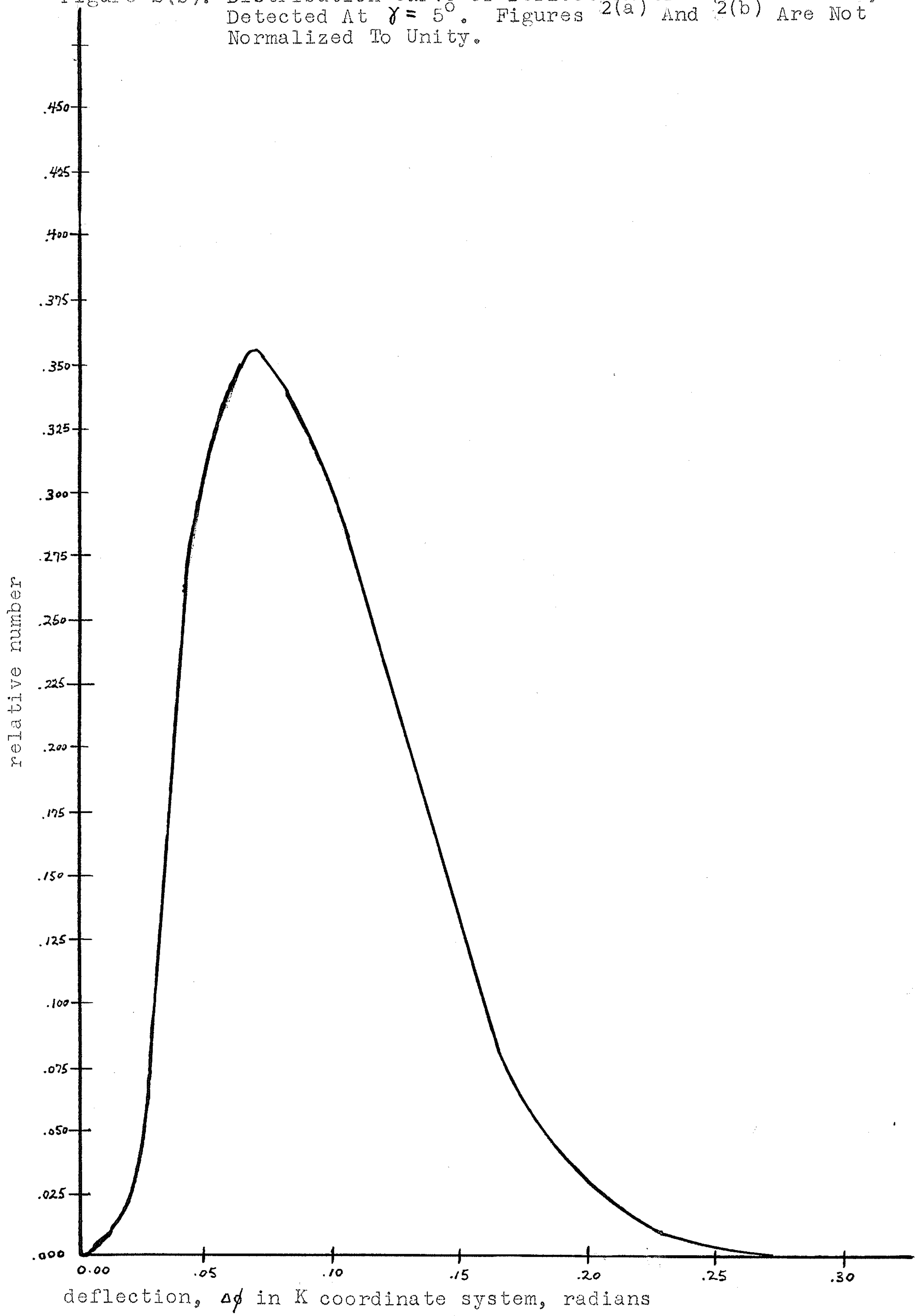


Figure 3(a). Distribution Curve of Deflections of Positive Muons, Detected at $\gamma = 25^{o}$. Area Under Curve Is Normalized To Unity.

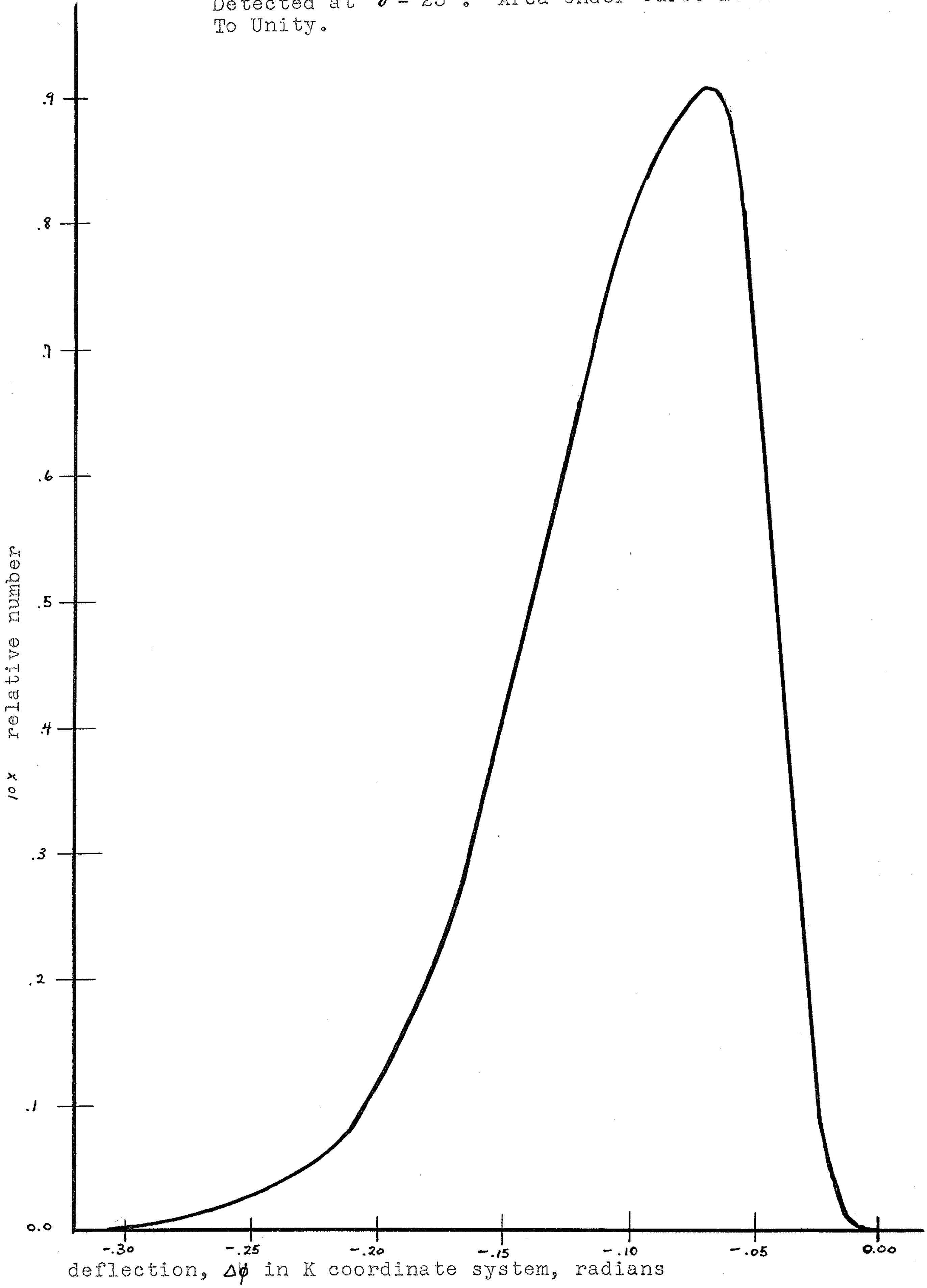
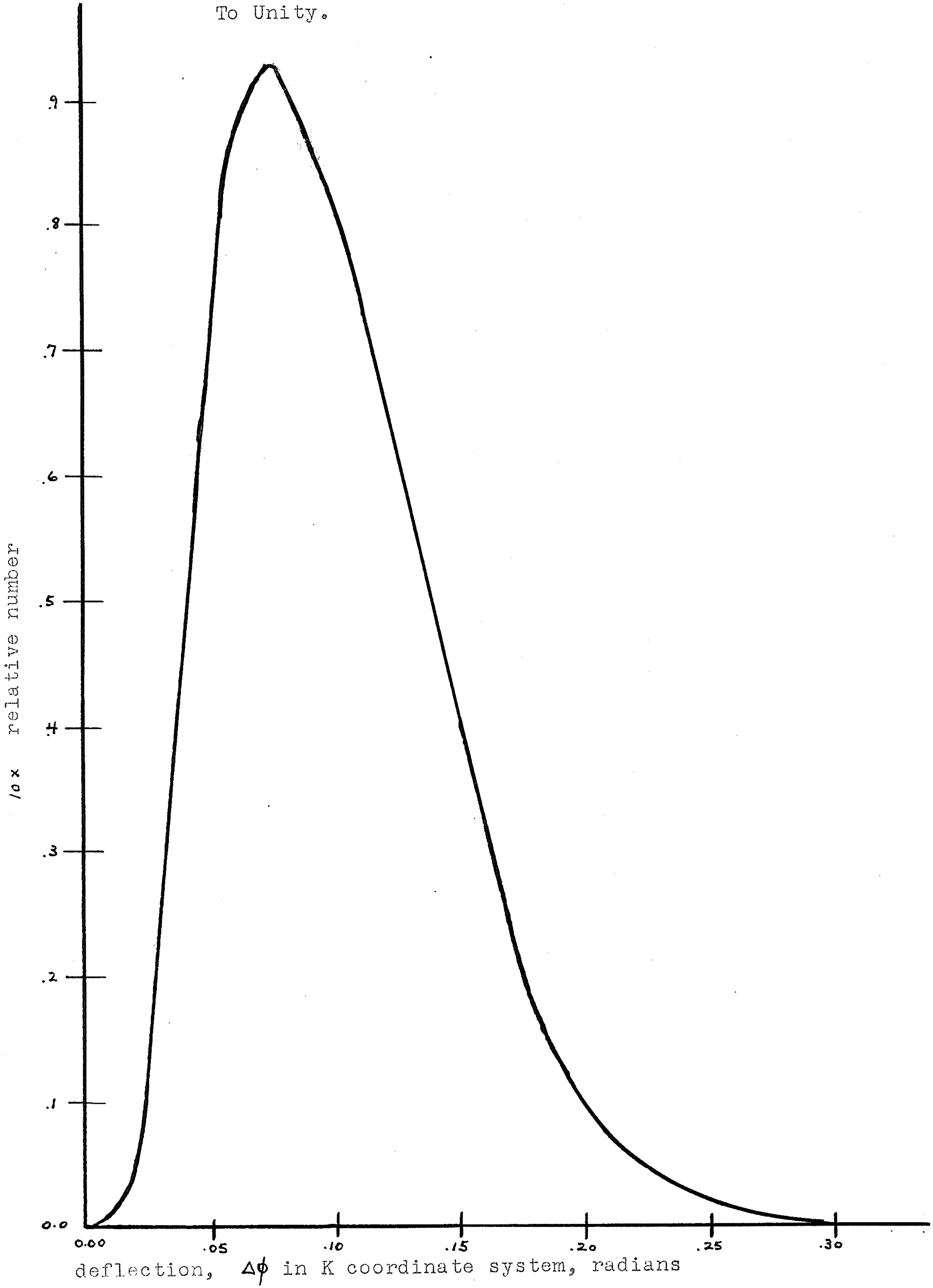


Figure 3(b). Distribution Curve of Deflections of Negative Muons, Detected at $\gamma = 25^{\circ}$. Area Under Curve Is Normalized To Unity.



corresponding to the average deflections was approximately 3.8 degrees.

The program used to determine the distribution of deflection curve could be easily modified to find a number of other quantities. It could be used to find intensities. The modification involved in finding differential and integral intensities would require the limits in equation (47) to be changed. To find the differential intensity equation (45) would be used and to find the integral intensity equation (46) would be used. The program for doing this is given in appendix four. Once the intensities have been found for a number of viewing directions, the opportunity arises to determine the exponent, n, in the experimentally used $\cos^n$ formula,

$$I = I_0 \cos^n \theta$$

where θ is the zenith angle in the K' system and I_0 is the vertical intensity.

By integrating equation (47) over the whole range of R but over a narrow range of H one can determine the number of particles which were produced in a given layer of the atmosphere and which are counted by a telescope in a given direction. In addition, the average height of production could be found by dividing the sum of the weighted heights of production by the sum of the weights, the number of particles counted. A further bit of information can be obtained relatively easily if one determines the average height of production for particles in a given narrow range of R. Then one can see how the average height of production changes with R.

The program for doing this is given in appendix five.

No attempt was made in the theory to account for π^- mesons being captured. Thus, the procedure employed in the preceding sections would not predict a positive temperature effect which is the result of the total cross section per unit volume for π^- meson capture decreasing with increasing temperature. However, the influence of the temperature of the atmosphere on counting rates can be investigated if one keeps in mind the limitations of the theory. The effect that would be calculated would involve the change in density of the atmosphere or equivalently the change in the average height of production in cm. with temperature. The data for the temperature corresponding to different heights, H, can be changed by simply changing the data cards. It should be noted that it is difficult to obtain reliable data about the temperature distribution in the atmosphere, particularly so if one wants the change in the temperature distribution with time.

Changes in counting rates with fluctuations in pressure can also be handled with the existing techniques. It would involve changing the value of the height of the top of the atmosphere in gm cm^{-2} in the program and adding or subtracting levels of H in the integration involved.

Finally, one should note that by changing values of α, β, δ', and "a" the program can be adopted to give results for different observation locations.

CHAPTER FIVE

Conclusions

The distribution curve for the deflections undergone in the earth's magnetic field by mesons counted by a telescope satisfying certain conditions was found. For a cosmic ray telescope, in an equatorial plane, looking south, $\gamma = 0$, the asymptotic directions at production of the mesons counted, were all contained in a cone about the axis of symmetry, the telescope's viewing direction, with a one half apex angle 11.5 degrees. The average deflection was given by a half apex angle of 3.8 degrees. That is, on the average the directions of the mesons at production make an angle of 3.8 degrees with the viewing direction of the telescope. The results obtained from other directions given by the angle γ were similar.

It must be said that the magnitudes of meson deflections are not so large that they in themselves completely destroy the directional properties of the cosmic ray telescope. In the present state of the art it is probably unnecessary to correct for the bending of mesons as the other uncertainties introduced by the finite solid angle of the telescope and the spread in the asymptotic directions of the primaries are predominant. However, it is conceivable that as the resolving power of directional detecting devices improves the spread in effective viewing directions caused by meson deflection might have to be taken into account. One possible advance would be the development of a cosmic ray telescope that would differentiate between positively and negatively charged mesons with relatively high energies. A telescope of this type, but detecting low energy mesons, has been used[23].

The distribution curve obtained could be used as a histogram with intervals of $\Delta\phi$ equal to .01 radians to give the number of particles counted which have suffered deflections between two values of $\Delta\phi$ in the K coordinate system. The two values must differ by more than .01 radians. A table has been constructed which can be used to convert deflections in the K coordinate system to deflections in the K' coordinate system.

PART TWO

LIOUVILLE'S THEOREM APPLIED TO COSMIC RAYS

CHAPTER ONE

Introduction to Liouville's Theorem Applied to Cosmic Rays

It is often enlightening in dealing with problems concerning cosmic ray intensities to employ a theorem from statistical mechanics, Liouville's theorem. Consider a phase space, the coordinates of which are the generalized coordinates and generalized momenta of the cosmic ray particles, and in which there are enough points so that one can talk about a density of points in phase space. Liouville's theorem says that under certain conditions the density in phase space is constant in time as one moves along the path determined by the equations of motion of the system. It is the purpose here to present the conditions necessary for Liouville's theorem to be valid and to ascertain when the theorem is applicable to conditions found in cosmic ray problems. In order to clarify the reasons for breakdown of the theorem the problem of the diurnal variation is presented as an example.

The counting rate of a counting device at the top of the atomosphere which detects particles above the cutoff rigidity can be related to the density in phase space. This relation of course is the connection between Liouville's theorem and cosmic ray physics. The purpose of having a directional detecting device is to be able to determine the directions of arrival of the primary cosmic ray particles. Does the existence of preferred directions imply that sources of cosmic rays lie along the associated asymptotic directions? The asymptotic directions take into account the bending of the charged particles through the terrestrial magnetic field. Assuming Liouville's theorem is valid for cosmic ray particles as they traverse space, that cosmic radiation is isotropic and independent of the observation point at large distances from the earth, and that only conservative electric fields act, then one can say that the radiation is isotropic near the earth. In other words, the counting rates for particles above the cutoff rigidity are the same in any direction. Anisotropies cannot be created by conservative forces acting on cosmic ray particles as they journey through space if these forces are consistent with the conditions of Liouville's theorem. Under these conditions an observed anisotropy is indicative of an anisotropic distribution of the sources of cosmic ray particles. It is shown later that a time independent electromagnetic field which is a function of position only (the presence of the cosmic ray particles does not influence this field to any significant extent) satisfies the conditions of Liouville's theorem. Such a field cannot therefore create an anisotropy. On the other hand, an irregular magnetic field which interacts with the particles in such a way that the field is influenced by the particles to a significant extent does not satisfy the constraints necessary to apply Liouville's theorem.

In the development of the theories concerning the origin of variations in the counting rate with direction, such as the diurnal variation, it is important to remember that any mechanism proposed must be such that it violates the conditions necessary for Liouville's theorem to be valid or it must involve non-conservative forces, or both. Thus, the argument of Ahluwalia and Dessler[24] that the diurnal variation was due to the drift velocity of particles in the perpendicular electric and magnetic fields associated with the solar wind was negated by D. Stern's implementation of Liouville's theorem[25]. This theorem can be used as a test to eliminate the possiblility that certain mechanisms are responsible for observed variations.

CHAPTER TWO

General Theory

Before a statement and proof of Liouville's theorem is given, a brief resumé is presented of some of the relevant fundamental notions of statistical mechanics.

A system is a physical object of interest. In cosmic ray problems a charged particle is a system. The collection of all systems of the same structure but distributed over a range of different possible states is called an ensemble. Here the ensemble shall mean the collection of all cosmic ray particles. Each system has n degrees of freedom and is characterized by the generalized coordinates $q_1, q_2 \ldots q_n$ and the generalized momenta $p_1, p_2, \ldots p_n$. The states of our systems are specified by the generalized coordinates and generalized momenta of the particles. There are three degrees of freedom per system. Phase space is a conceptual Euclidean space with the generalized coordinates and momenta serving as the coordinates. Phase space therefore has 2n axes. For each system of the ensemble there is one point in phase space.

For statistical purposes there is no need to distinguish between individual systems. It is sufficient to know the number of systems at a given time which correspond to different regions of phase space. The ensemble is considered to consist of such a large number of systems that the distribution in phase space is continuous. Then the density or distribution function, $\rho(q,p,t)$, is defined as the number of points per unit volume of phase space. When this is known the ensemble is regarded as completely specified. The number of systems, δN, in the range, $dq_1 dq_2 \cdots\cdots dq_m dp_1 dp_2 \cdots\cdots dp_m$ is given by

$$\delta N = \rho(q, p, t)\, dq_1 dq_2 \cdots\cdots dq_m\, dp_1 dp_2 \cdots\cdots dp_m$$

Statement of Liouville's Theorem (26, 27, 28).

Consider an ensemble of noninteracting systems, where initially $d\tau = dq_1 \cdots\cdots dq_m dp_1 \cdots\cdots dp_m$ is to be regarded as the element of volume in phase space and $\rho(q, p, t)$ is the density in phase space. The phase space is considered to have no sources or sinks. In other words, systems are not created or destroyed. Each system moves in accordance with Hamilton's equations of motion,

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

or equivalently with the following form of Lagrange's equations of motion,

$$\frac{d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}{dt} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Each point in phase space moves under the restrictions imposed by the equations of motion of the corresponding mechanical system. Consider at one instance of time the volume element $d\tau$, the boundary of which is formed by some surface of neighboring system points. The position of the system points defining the volume change with time hence it appears that the shape and volume of the region might change with time. Liouville's theorem says that the volume element $d\tau$ is constant with respect to time as it is followed along its dynamical path. An equivalent statement of Liouville's theorem would be that the density in phase space as one moves with the region under consideration is constant with respect to time.

Proof:

This proof will make use of the continuity equation in phase space. There are no sources or sinks in phase space. Consider the volume element in phase space at time t,

$$dV = dq_1 \cdots\cdots dq_m \, dp_1 \cdots\cdots dp_m$$

This volume element is regarded as being fixed in the space. It does not move with the system points it contains. The number of points in this volume at time t

$$dN = \rho(q, p, t) \, dq_1 \cdots\cdots dq_m \, dp_1 \cdots\cdots dp_m$$

As time progresses the system points in phase space move. In a time δt system points will flow across each "face" of the volume element. The number of points entering in time δt through the face located at q_i and perpendicular to the q_i axis is

$$\rho \dot{q}_i \, \delta t \, dq_1 \, dq_2 \cdots\cdots dq_{i-1} \, dq_{i+1} \cdots\cdots dq_m \, dp_1 \cdots\cdots dp_m$$

The number of points entering through the other "face" of the volume element perpendicular to the q_i axis in time δt at $q_i + dq_i$ is

$$-\left(\rho + \frac{\partial \rho}{\partial q_i} dq_i + \cdots\right)\left(\dot{q}_i + \frac{\partial \dot{q}_i}{\partial q_i} dq_i\right) dq_1 \cdots dq_{i-1} \, dq_{i+1} \, dp_1 \cdots dp_m \, \delta t$$

where only terms up to first order in dq are considered. The "faces" perpendicular to the p axes can be handled in a similar fashion. Therefore, the change in the number of particles in the volume element, dV, in δt is

$$\delta(dN) = -\sum_{i=1}^{m} \left[\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) + \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i\right)\right] dq_1 \cdots\cdots dq_m \, dp_1 \cdots\cdots dp_m \, \delta t$$

Dividing by dV

$$\delta \rho = \delta\left(\frac{dN}{dV}\right) = -\left[\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) + \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i\right)\right] \delta t$$

Einstein notation used, i = 1, 2, . . . n

Therefore

$$\frac{\partial \rho}{\partial t} = -\left[\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) + \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\right]$$

However, Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Therefore,

$$\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = \rho\left(\frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial q_i \partial p_i}\right) = 0$$

and
$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)$$

or
$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i = 0$$

ρ is a constant of the motion.

The fact that ρ is a constant of the motion implies that the volume element is constant along the path of the motion. Consider an infinitesimal volume element in phase space, $d\tau$. As time progresses the system points defining the volume will move and the volume may have a different shape and volume. The number of points contained in the volume will be constant with time. If some system were to cross the boundary it would at some time occupy the position of a boundary point. Since the motion of a system is uniquely determined by its location in phase space at a given time, the system would travel together with the boundary point thereafter. Therefore the number of system points within a given volume element is constant with time. Since $dN = \rho d\tau$ is the number of points in $d\tau$ at all times

$$0 = \frac{d(dN)}{dt} = \frac{d\rho}{dt}d\tau + \rho\frac{d(d\tau)}{dt}$$

However, $\frac{d\rho}{dt}=0$ and $\rho=$ constant, therfore $d\Upsilon$ is a constant of the motion.

It has been seen that Liouville's theorem requires that the equations of motion describing each sytem can be put into the form of Hamilton's equations. It is useful to enquire whether the motion of charged particles through an electromagnetic field can be described by the equations of the Hamiltonian form.

Consider a nonrelativistic particle in a given electromagnetic field which is completely determined by a given scalar potential ϕ and a given vector potential $\underline{A}$. The potentials are functions of position and time only. The electric field, $\underline{E}$, and the magnetic induction, $\underline{B}$, are given in terms of the potentials in c.g.s. units by

$$\left.\begin{aligned} \underline{B} &= \nabla\times\underline{A} \\ \underline{E} &= -\frac{1}{c}\frac{\partial\underline{A}}{\partial t}-\nabla\phi \end{aligned}\right\} \tag{1}$$

The Lorentz force on a charged particle in an electromagnetic field is given by

$$\underline{F} = q\left[\underline{E}+\frac{1}{c}(\underline{v}\times\underline{B})\right] \tag{2}$$

Substituting for $\underline{B}$ and $\underline{E}$ from (1)

$$\underline{F} = q\left[-\nabla\phi-\frac{1}{c}\frac{\partial\underline{A}}{\partial t}+\frac{1}{c}(\underline{v}\times\nabla\times\underline{A})\right] \tag{3}$$

but

$$\begin{aligned} \left[\underline{v}\times\nabla\times\underline{A}\right]_x &= v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right) \\ &= v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}+v_x\frac{\partial A_x}{\partial x}-v_y\frac{\partial A_x}{\partial y}-v_z\frac{\partial A_x}{\partial z}-v_x\frac{\partial A_x}{\partial x} \\ &= \frac{\partial(\underline{A}\cdot\underline{v})}{\partial x}-\underline{v}\cdot\nabla A_x \end{aligned} \tag{4}$$

However $\quad \underline{v} \cdot \nabla A_x = \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t}$ (5)

as can be seen from the definition of the convective derivative since $\underline{A}$ is independent of $\underline{v}$. Therefore

$$\left[\underline{v} \times (\nabla \times \underline{A})\right]_x = \frac{\partial(\underline{A}\cdot\underline{v})}{\partial x} - \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} \tag{6}$$

Substituting from equation (6) into (3)

$$F_x = q\left[-\frac{\partial\phi}{\partial x} + \frac{\partial(\frac{1}{c}\underline{v}\cdot\underline{A})}{\partial x} - \frac{1}{c}\frac{dA_x}{dt}\right] \tag{7}$$

Since $\underline{A}$ is independent of $\underline{v}$, $A_x = \frac{\partial(\underline{A}\cdot\underline{v})}{\partial v_x}$

When this is substituted into (7)

$$F_x = q\left\{-\frac{\partial(\phi - \frac{1}{c}\underline{v}\cdot\underline{A})}{\partial x} - \frac{d\frac{\partial}{\partial v_x}[1/c(\underline{A}\cdot\underline{v})]}{dt}\right\} \tag{8}$$

ϕ is independent of velocity therefore

$$F_x = q\left\{\frac{\partial(-\phi + \frac{1}{c}\underline{v}\cdot\underline{A})}{\partial x} - \frac{d\left(\frac{\partial[1/c(\underline{A}\cdot\underline{v}) - \phi]}{\partial v_x}\right)}{dt}\right\} \tag{9}$$

Define U

$$U = \frac{q}{c}\underline{A}\cdot\underline{v} - q\phi \tag{10}$$

Then
$$F_x = \frac{\partial U}{\partial x} - \frac{d\left(\frac{\partial U}{\partial v_x}\right)}{dt} \tag{11}$$

Lagrange's equations are of the form

$$\frac{d\left(\frac{\partial T}{\partial v_x}\right)}{dt} - \frac{\partial T}{\partial x} = Q_x \tag{12}$$

where $\quad Q_x = \underline{F}\cdot\frac{\partial \underline{r}}{\partial x} = F_x$ and $T = \frac{1}{2}mv^2$

Therefore the equation of motion becomes

$$\frac{d\left(\frac{\partial T}{\partial v_x}\right)}{dt} - \frac{\partial T}{\partial x} = \frac{\partial U}{\partial x} - \frac{d\left(\frac{\partial U}{\partial v_x}\right)}{dt} \tag{13}$$

Taking all terms to one side and defining the Lagrangian as $L = T - U$

$$\frac{d\left(\frac{\partial L}{\partial v_x}\right)}{dt} - \frac{\partial L}{\partial x} = 0 \tag{14}$$

The other Lagrangian equations are found in a similar fashion. A Hamiltonian can be defined in the usual way where one defines the generalized momenta by

$$p_i = \frac{\partial L}{\partial v_i} = m v_i + \frac{e}{c} A_i \tag{15}$$

where i = 1, 2, 3. Hamilton's equations,

$$\dot{x}_i = \frac{\partial H}{\partial p_{x_i}} \quad \text{and} \quad \dot{p}_{x_i} = -\frac{\partial H}{\partial x_i}$$

are then valid in the case of a charged particle in a given electromagnetic field. Note that it was necessary for ϕ and $\underline{A}$ to be independent of the particle velocity.

Lagrangian and Hamiltonian equations of motion can be found for the case of relativistic particles.(29) The relativistic Lagrangian is

$$L = -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} + \frac{e}{c}(\underline{A} \cdot \underline{v}) - e\phi \tag{16}$$

where v is the velocity of the particle and m_0 is the rest mass. The momenta, p_i, used in the derivation of Hamilton's equations are defined by

$$p_i = \frac{\partial L}{\partial v_i} \text{ or } p = \frac{m_0 \underline{v}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} + \frac{e}{c}\underline{A} \tag{17}$$

The Hamiltonian, $H(p, q, t) = \sum p_i \dot{q}_i - L$. Therefore

$$H = m_0 c^2 \left\{ 1 + \left(\frac{1}{m_0^2 c^2}\right)\left[\left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_y - \frac{e}{c}A_y\right)^2 + \left(p_z - \frac{e}{c}A_z\right)^2\right]\right\}^{\frac{1}{2}} + e\phi \tag{18}$$

Because the motion of a charged particle in a given electromagnetic field can be described by Hamilton's equations, Liouville's theorem holds in a phase space where the coordinates are the coordinates of the particles and the generalized momenta, $p = m\underline{v} + \frac{e}{c}\underline{A}$ as seen in (15) and (17).

Liouville's theorem gains importance in cosmic ray physics from the fact that the density in the phase space corresponding to the ensemble of cosmic ray particles can be related to the counting rate of a directional detecting device. Consider an ideal detecting instrument which looks at primary particles which come from directions contained within an infinitesimal solid angle $d\omega$. The cross sectional area of the device is dA. Suppose that dN particles with components of momenta, parallel to the axis of the cone, between p_1 and $p_1 + dp_1$ are counted in time dt. Momentum here is defined as mass times velocity. The two mutually perpendicular components of momentum of a given particle perpendicular to the axis of the cone are bounded by dp_2 and dp_3 respectively, if the particle is counted. That is, dp_2 and dp_3 define the limits of momenta specified by the cone. Together p_1, dp_2, and dp_3 specify the solid angle. It is seen that the solid angle, $d\omega$, is given by

$$d\omega = \frac{\left(\frac{dp_2 t}{m}\right)\left(\frac{dp_3 t}{m}\right)}{\left(\frac{pt}{m}\right)^2} = \frac{dp_2\, dp_3}{p^2}. \tag{19}$$

In time t, a beam of particles would travel a distance $\frac{pt}{m}$ along the direction of the infinitesimal solid angle from the apex of the cone defining the solid angle. The lateral spreading of the beam in t is confined to the area $\left(\frac{dp_2 t}{m}\right)\left(\frac{dp_3 t}{m}\right)$. Since the solid angle is infinitesimal the normal direction to this area is parallel to the radii of the defining cone. Thus from the definition of solid angle the above expression is obtained for $d\omega$.

Let σ be the number of particles per unit volume of configuration space with components of momentum between p_1 and $p_1 + dp_1$ parallel to the axis of the cone. Then, the dN particles counted in time dt at t will have come from a configuration volume element

$$dV = dA\, v\, dt \tag{20}$$

where dA is the cross sectional area of the detector and $v = \frac{p_1}{m}$. Then

$$dN = \sigma dV \tag{21}$$

Not only do the dN particles with components of momenta along the axis between p_1 and $p_1 + dp_1$ come from a volume element dV in configuration space but they must also come from a volume element $dV_p = dp_1\, dp_2\, dp_3$ in momentum space. This means that the number of particles counted in dt at t, at the position of dA, with parallel components of momentum between p_1 and $p_1 + dp_1$ come from a volume element $dV dV_p$ in phase space. Therefore,

$$dN = \rho\, dV dV_p \tag{22}$$

where ρ is the density in phase space of the cosmic ray particles at the position of the telescope and at momentum p_1. Equations (21) and (22) give the relationship between ρ and σ,

$$\sigma = \rho\, dV_p \tag{23}$$

From (19)

$$d\omega = \frac{dp_2\, dp_3}{p_1^2}\ \frac{dp_1}{dp_1} = \frac{dV_p}{p_1^2\, dp_1} \tag{24}$$

Substituting into (22) from (20) and (24)

$$dN = \rho\, dV_p dV = \rho\, d\omega\, p_1^2\, dp_1\, dA\, v\, dt \tag{25}$$

Now the number of particles with components of momentum parallel to the axis of the cone, counted per unit area, per unit time, per unit solid angle, N', is given by

$$N' = \frac{dN}{d\omega\, dA\, dt} = \rho\, v\, p_1^2\, dp_1 = \frac{\rho\, p_1^3\, dp_1}{m} \tag{26}$$

The connection between the intensity, N', and the density in phase space, ρ, has been derived for the case where the Newtonian momenta were used as coordinates in phase space. For the case of cosmic ray particles in an electromagnetic field Liouville's theorem was shown to hold in a phase space where the momenta were given by

$$p_i = p_i' + \frac{e}{c} \underline{A}(\underline{r}_i) \tag{27}$$

where $p_i' = m\underline{v}_i$ and i refers to the i^{th} particle. W. Swann[30] showed that Liouville's theorem is still valid in the phase space where the p_i' are the momenta. The volume element in this phase space is $dq_1 \, dq_2 \cdots dq_m \, dp_1' \cdots dp_m'$. The volume element in the phase space in which Liouville's theorem has been shown to be valid is $dq_1 \cdots dq_m \, dp_1 \cdots dp_m$.

The relationship between these two volume elements is

$$dq_1 \cdots dq_m \, dp_1' \cdots dp_m' = J \, dq_1 \cdots dq_m \, dp_1' \cdots dp_m'$$

where J is the Jacobian corresponding to this transformation.

The Jacobian, $J = \dfrac{\partial(q_1 \cdots q_m \, p_1' \cdots p_m')}{\partial(q_1 \cdots q_m \, p_1 \cdots p_m)} = 1$

This is so because

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}, \quad \frac{\partial q_i}{\partial p_j} = 0, \quad \frac{\partial p_i}{\partial q_j} = \frac{e}{c}\frac{\partial A_i}{\partial q_j}, \quad \frac{\partial p_i}{\partial p_j} = \delta_{ij}$$

Therefore

$$J = \left|\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \frac{e}{c}\frac{\partial A_1}{\partial q_1} & \cdots & \frac{e}{c}\frac{\partial A_1}{\partial q_m} & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \frac{e}{c}\frac{\partial A_m}{\partial q_1} & \cdots & \frac{e}{c}\frac{\partial A_m}{\partial q_m} & 0 & & 1 \end{array}\right| = 1 \tag{28}$$

Then $dq_1 \cdots dq_n\, dp_1' \cdots dp_n' = dq_1 \cdots dq_n\, dp_1 \cdots dp_n =$ a constant along the dynamical path.

The results of Liouville's theorem still apply in (q, p') phase space. The density in phase space is constant as one follows a volume element along its path. If cosmic ray particles only interact with conservative electric and magnetic fields without significantly affecting the fields; that is, if the fields are always functions of position and time, then Liouville's theorem is valid as the particles traverse space. If this were so, and if one assumes that far away from earth the cosmic ray radiation is isotropic and independent of position then one could conclude that the radiation in the vicinity of earth is isotropic.

In a conservative field the change in the magnitude of the momentum of a particle in going from the observation point to infinity is independent of the path. Therefore if cosmic ray particles which arrive at a detector near the earth from different directions but with momenta equal in magnitude, are traced back along their trajectories to infinity they have equal momenta at infinity. Along the trajectories the density in phase space is constant because of Liouville's theorem. If one assumes that at infinity the density in phase space depends only upon the magnitude of the momentum, then the phase space densities are constant for the two trajectories. Referring to equation (26) one can conclude that the intensities in all directions are equal; the radiation is isotropic. If the fields are not steady and the electric field is no longer conservative then in general tracing particles with the same momentum at the observation point back along their respective trajectories will lead to different momenta at infinity. In this case, one can no longer say that radiation which is isotropic at infinity is isotropic at the observation point, even though Liouville's theorem is valid.

So far it has been shown that if a system can be described by Hamiltonian equations of motion, Liouville's theorem holds. This does not mean that the theorem is automatically negated if the system's equations of motion can not be put into Hamiltonian form. In order to determine some condition which will indicate the breakdown of Liouville's theorem consider the following. Suppose that the generalized form of Lagrange's equations is used.

$$\frac{d\left(\frac{\partial L}{\partial \dot{q}_i}\right)}{dt} - \frac{\partial L}{\partial q_i} = Q_i \tag{29}$$

where $Q_i = F_j \cdot \frac{\partial r_j}{\partial q_i}$. i indicates the generalized coordinate and j the particle. Einstein summation notation is used here and in what follows. By definition

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{30}$$

Therefore

$$\dot{p}_i - Q_i = \frac{\partial L}{\partial q_i} \tag{31}$$

Since $\quad H(p, q, t) = p_i \dot{q}_i - L(q, \dot{q}, t)$

then $\quad dH = p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \qquad (32)$

Substituting for $\frac{\partial L}{\partial q_i}$,

$$dH = \dot{q}_i \, dp_i - (\dot{p}_i - Q_i) dq_i - \frac{\partial L}{\partial t} dt$$

but $$dH(q,p,t) = \frac{\partial H}{\partial q_i}\, dq_i + \frac{\partial H}{\partial p_i}\, dp_i + \frac{\partial H}{\partial t}\, dt \tag{33}$$

The differentials in (32) and (33) are independent. Therefore equating coefficients in the two equations one gets the generalized Hamiltonian equations,

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + Q_i \end{aligned}\right\} \tag{34}$$

In Liouville's theorem a term appears on the right hand side. The continuity equation in phase space is found by analogy to $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0$ or $\frac{\partial \rho}{\partial t} + \underline{v}\cdot\nabla\rho + \rho\nabla\cdot\underline{v} = 0$

In phase space

$$\frac{\partial \rho}{\partial t} + \dot{q}_i \frac{\partial \rho}{\partial q_i} + \dot{p}_i \frac{\partial \rho}{\partial p_i} + \rho \frac{\partial \dot{q}_i}{\partial q_i} + \rho \frac{\partial \dot{p}_i}{\partial p_i} = 0 \tag{35}$$

Using the definition of the convective derivative

$$\frac{d\rho}{dt} + \rho\left[\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right] = 0 \tag{36}$$

Substituting for q_i and p_i from equation (34), the above becomes

$$\frac{d\rho}{dt} + \rho\left[\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} + \frac{\partial Q_i}{\partial p_i}\right] = 0$$

or $$\frac{d\rho}{dt} = -\rho \frac{\partial Q_i}{\partial p_i} \tag{37}$$

If the generalized forces are functions of the generalized momenta, Liouville's theorem breaks down.

CHAPTER THREE

Diurnal Variation and Liouville's Theorem

Observations indicate the presence of a diurnal variation in the cosmic ray intensity in the equatorial plane with an amplitude of about 0.4 per cent[31]. This corresponds to an anisotropy in the radiation reaching earth with the maximum occuring when the detecting device is looking backward along the earth's orbit and the minimum occuring when the direction of observation is forward along the earth's orbit.

If the radiation is assumed to be isotropic and independent of position at infinity, if one considers steady fields, and if Liouville's theorem is valid, then the radiation in directions which are accessible is isotropic in the vicinity of the earth. To explain the observed variation in intensity one can try to eliminate at least one of the three conditions which together predict isotropic radiation near earth.

Suppose that the first two conditions are met but a mechanism is proposed which explains the observed anisotropy as the result of the breakdown of Liouville's theorem in a steady electromagnetic field. It has been shown that Liouville's theorem holds valid in a given electromagnetic field, even if that field is time dependent. The theorem is valid both for the Canonical and the Newtonian momenta. Thus a contradiction arises as the third condition is met if the first two conditions hold. Any attempts to explain the diurnal variation which rely upon steady electromagnetic fields to destroy the isotropic nature of the radiation seem doomed to failure. One only needs to confront the hypothesis with the existence and ramifications of Liouville's theorem.

Stern[32] and Parker[33] have attempted to show the mechanism by which the isotropy is maintained in a conservative field. In effect, what happens is that a density gradient is set up along the direction of the electromagnetic drift velocity. This in turn, negates the anisotropy which would have resulted from the existence of the drift velocity above.

Parker[34] and Axford[35] have separately proposed essentially the same mechanism which they have successfully used to predict the diurnal variation. Their models give the observed directions for the anisotropy and also the observed amplitudes of the variation in intensity. They have attributed the variations in intensity to the existence of time dependent electromagnetic fields in the form of moving irregularities in the magnetic field beyond the orbit of the earth, and to a breakdown in Liouville's theorem. The point of interest here is that their mechanism involves the breakdown of Liouville's theorem in the presence of an electromagnetic field. The naïve viewpoint, which automatically relagated to oblivion any hypothesis, relying upon electromagnetic fields to destroy the validity of Liouville's theorem, must be scrutinized more closely. Liouville's theorem has been used to argue against the plausibility of proposals for the mechanism producing the diurnal variation, such as Ahluwalia's and Dessler's[36], which required nothing more than an appropriate steady electromagnetic field. The theorem has been shown to be valid in such a given field. On the other hand, Axford only requires a particular electromagnetic field in his model. His argument however implicitly contains the fact that Liouville's theorem breaks down. How is this seemingly paradoxial situation resolved?

In what follows a slightly modified version of Axford's model shall be presented. In essence, it contains the same features as the Parker model and produces the same results. The difference between the two presentations is one of approach only.

To begin with, it is necessary to state that the magnetic field is to be considered as being made up of two parts. There is a regular part and a fluctuating part superimposed. The regular part of the field is the "garden hose" model proposed by Parker[37]. Near the sun's equatorial plane a plasma escapes from the solar corona. The direction of this solar wind is approximately radial and its velocity, of the order of 300 km/sec, is indicated by V_s. Because of the high conductivity of the plasma, the lines of magnetic induction are locked in the plasma and are carried radially outward with a velocity V_s[38]. At any given time, consider the shape of a line of magnetic induction as shown in figure four. Suppose that the line of force is the result of plasma escaping from a region about point A at this time. The sections

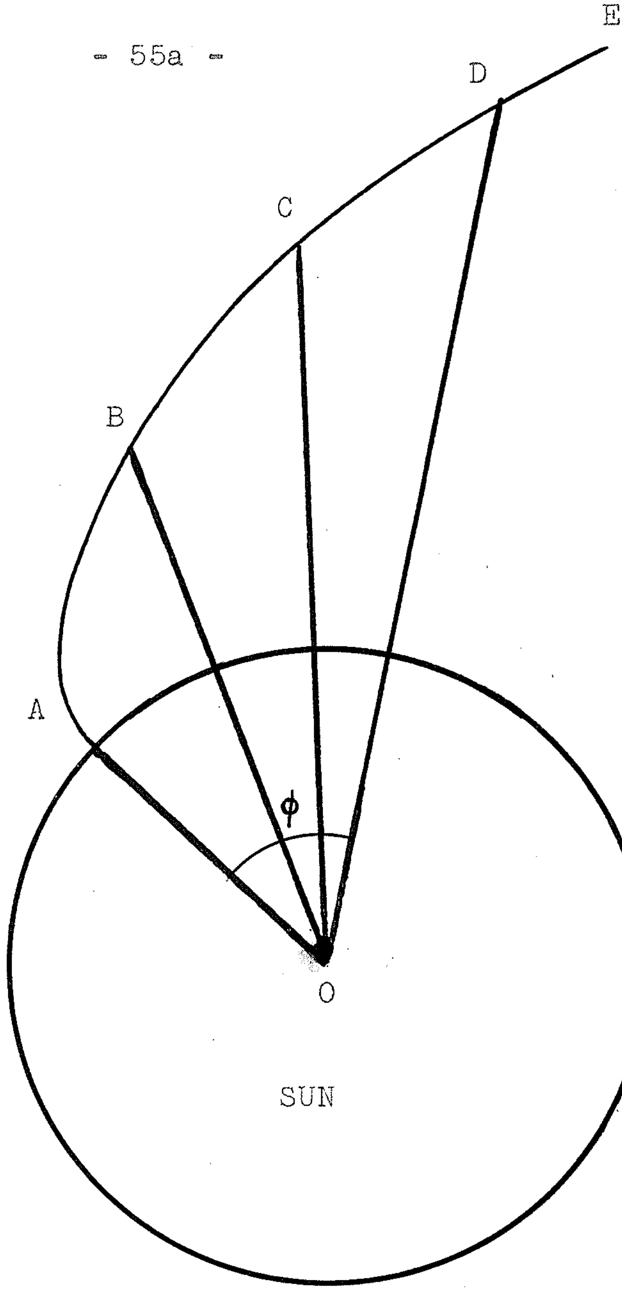


Figure 4. A Line Of Magnetic Induction Frozen In The Solar Wind

of the line of force at points B, C, D, and E must be the result of plasma, given off by this region at earlier times and because of the suns rotation at different locations in space. The plasma and hence the segments of the line of force have travelled distances from the surface of the sun in proportion to times elapsed since escape from the corona. The time period since escape is given by ϕ/Ω, where Ω is the angular velocity of the sun's equatorial region and ϕ is the polar coordinate describing the angular displacement of the line segment from the emitting region, A in figure four. The result therefore is that the lines of force form the Archimedes spirals given by

$$r = V_s\left(\frac{\phi}{\Omega}\right) \tag{1}$$

At a later time the situation is the same except that now the line of force has rotated through an angle. The "garden hose" field co-rotates with the sun.

As a prelude to further calculations consider figure five. Θ is the angle between the velocity of the solar wind at a point (r, ϕ) and the line of force. χ is the complement of Θ. Choose x, y axes as shown. Then the line of magnetic induction is described by $\underline{r} = r\cos\phi\,\underline{i} + r\sin\phi\,\underline{j}$

Substituting $r = \frac{V_s}{\Omega}\phi$,

$$\underline{r} = \frac{V_s}{\Omega}\left(\phi\cos\phi\,\underline{i} + \phi\sin\phi\,\underline{j}\right)$$

Then

$$\underline{n} = \frac{d\underline{r}}{d\phi} = \frac{V_s}{\Omega}\left[(-\phi\sin\phi + \cos\phi)\,\underline{i} + (\phi\cos\phi + \sin\phi)\,\underline{j}\right]$$

is a vector tangent to the line of force. Consider the vector

$$\underline{R} = \underline{V}_s - \underline{\Omega}\times\underline{r} = V_s\cos\phi\,\underline{i} + V_s\sin\phi\,\underline{j} - r\Omega\sin\phi\,\underline{i} + r\Omega\cos\phi\,\underline{j}$$

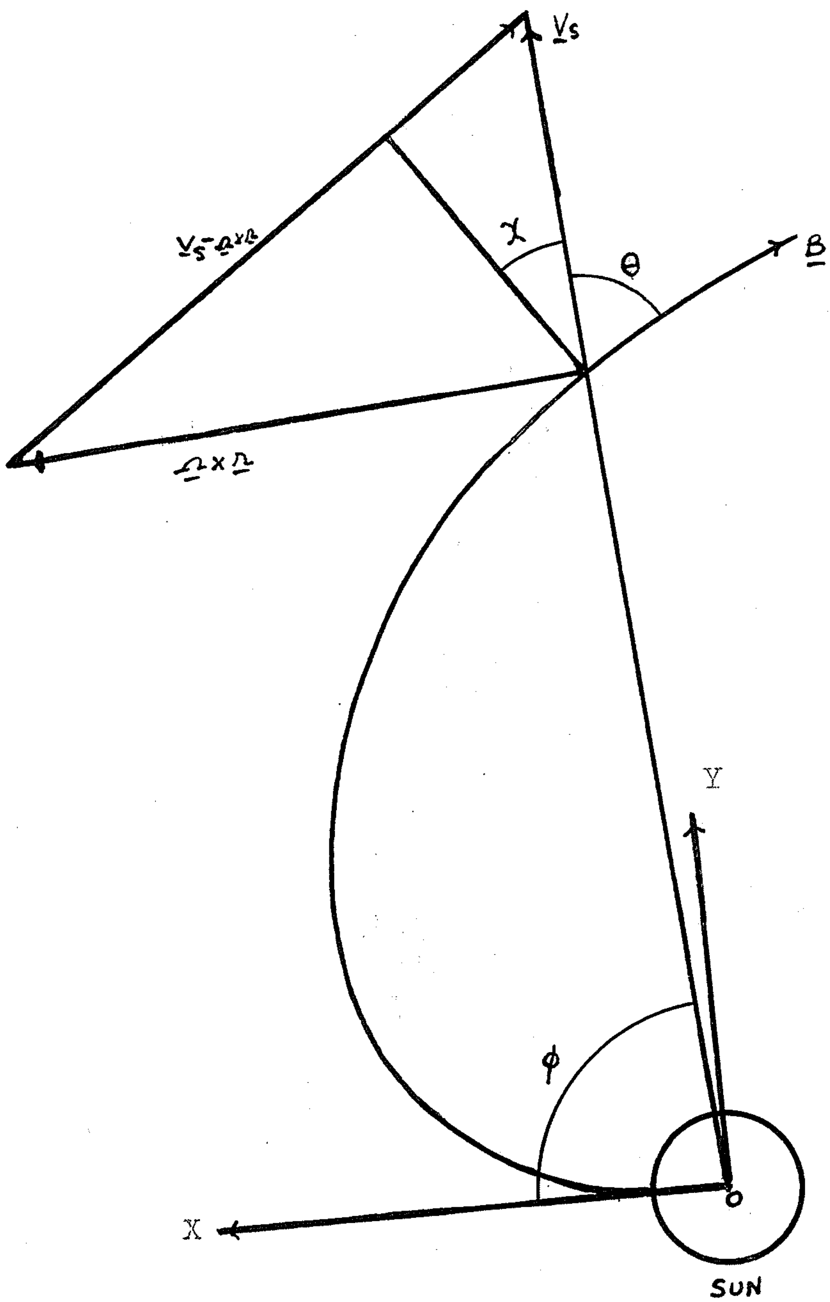


Figure 5. Relationship Between Vectors $\underline{\Omega} \times \underline{r}$ and $\underline{V}_s$

If one substitutes for $r\Omega$

$$\underset{\sim}{R} = V_s \left[(\cos\phi - \phi\sin\phi)\,\underset{\sim}{i} + (\phi\cos\phi + \sin\phi)\,\underset{\sim}{j} \right]$$

$\underset{\sim}{R}$ and $\underset{\sim}{N}$ are parallel. A line perpendicular to $\underset{\sim}{B}$ is then perpendicular to $\underset{\sim}{R}$. From the diagram this means that

$$\frac{V_s}{r\Omega} = \tan\chi \qquad (2)$$

In the coordinate system moving with the velocity of the solar wind there is only the magnetic field. From a fixed coordinate system in which the solar wind has a velocity V_s there is an induced electric field[39] given by

$$\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{V_s} \times \underset{\sim}{B} = 0 \qquad (3)$$

Therefore

$$E = \frac{1}{c} V_s B \sin\theta$$

but $\qquad \sin\theta = \cos\chi$

hence $\qquad c\,\frac{E}{B} = V_s \cos\chi$

V_s can be substituted for from (2) so that

$$\frac{cE}{B} = r\Omega \sin\chi \qquad (4)$$

This result will be useful later.

The fluctuating part of the magnetic field has the effect of scattering the cosmic ray particles. Turbulence in the solar wind causes irregularities in the magnetic field. These irregularities move with approximately the velocities of the solar wind. Axford uses the analogy between the motion of light ions in a magnetic field in the presence of heavier neutral atoms and the motion of cosmic ray particles through the magnetic field irregularities. No attempt has been successful in justifying the use of this analogy. If there is justification for employing it, one can show that Liouville's theorem cannot be applied in this situation.

A convenient place to start in the analysis of the motion of cosmic ray particles through an electromagnetic field, as described above, is the writing down of the appropriate Boltzmann's equation.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{v} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t}\right)_c \qquad (5)$$

where f ($\underline{x}$, $\underline{v}$, t) is the distribution function for the cosmic ray particles with mass m and charge e, $\underline{E}$ and $\underline{B}$ are the regular parts of the fields, and $\left(\frac{\partial f}{\partial t}\right)_c$ is the rate of change of the distribution function as a result of collisions with turbulent regions of the magnetic field. The results of taking the lowest order moments of the Boltzmann equation and integrating, (See appendix six.) are the continuity and conservation of momentum equations.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{u}) = \int \left(\frac{\partial f}{\partial t}\right)_c d^3v \qquad (6)$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = -\frac{1}{n}\nabla \cdot \underline{\underline{S}} + \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{u} \times \underline{B}\right) + \frac{1}{n}\int \left(\frac{\partial f}{\partial t}\right)_c \underline{v}\, d^3v \qquad (7)$$

where n is the density in configuration space, $\underline{U}$ is the streaming velocity of the cosmic ray particles and $\underline{\underline{S}}$ is a stress tensor which can be approximated by $\nabla \cdot \underline{\underline{S}} = \nabla\left(\frac{1}{3} n c^2\right)$ See appendix six. In order to evaluate the effect of the collisions with the turbulent regions of the magnetic field, Axford envokes the analogy of light ions colliding with heavier neutral atoms [40]. He uses a simple mean free path argument, where τ is the mean collision interval. The effect of collisions is to rearrange the velocities of the particles but not to change their density in configuration space.

Therefore

$$\int \left(\frac{\partial f}{\partial t}\right)_c d^3v = 0 \qquad (8)$$

From the analogy, the average change in momentum per collision of the particles is $m(\underline{V}_s - \underline{v})$, where $\underline{V}$ is the solar wind velocity and consequently the approximate velocity of the turbulent regions of the magnetic field, $\underline{v}$ is the velocity of the particle. See appendix seven. This change in the momentum on the average takes place in time τ. Therefore the average force per unit volume of configuration space due to collisions is given by $\frac{nm(\underline{V}_s - \underline{u})}{\tau}$. Since $\int \left(\frac{\partial f}{\partial t}\right)_c d^3v = 0$; that is, since there is no rate of change, because of collisions, in the number of particles per unit volume of configuration space, any change in the momentum per unit volume caused by collisions must be due to a change in the momentum of the particles already in the volume. Therefore the rate of change of momentum per unit volume due to collisions must be

$$m\int \left(\frac{\partial f}{\partial t}\right)_c \underline{v}\, d^3v = \frac{nm}{\tau}(\underline{V}_s - \underline{u})$$

This means that

$$\int \left(\frac{\partial f}{\partial t}\right)_c \underline{v}\, d^3v = \frac{nm}{\tau}(\underline{V}_s - \underline{u}) \qquad (9)$$

It should be noted that in the above discussion a friction-like force is introduced. This means that Liouville's theorem breaks down since frictional forces can only be treated as generalized forces in the Hamiltonian equations. It will now be shown that this type of force is consistent with the conditions $\left(\frac{\partial f}{\partial t}\right)_c$ was assumed to satisify. Consider the continuity equation for the distribution function $f(\underline{x}, \underline{v}, t)$

$$\frac{\partial f}{\partial t} + \frac{\partial (f v_i)}{\partial x_i} + \frac{\partial (f \dot{v}_i)}{\partial v_i} = 0$$

or $$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} + f \frac{\partial v_i}{\partial x_i} + f \frac{\partial \dot{v}_i}{\partial v_i} = 0 \tag{10}$$

where $i = 1, 2, 3$, and i refers to the vector components. Now suppose that $\dot{\underline{v}} = \frac{\underline{F}_e}{m} + \underline{F}_c$ where $\underline{F}_e$ is the Lorentz force and $m\underline{F}_c$ is the average force due to collisions. Then

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{\underline{F}_e}{m}\cdot\nabla_v f + \underline{F}_c\cdot\nabla_v f + f\nabla\cdot\underline{v} + f\nabla_v\cdot\dot{\underline{v}} = 0 \tag{11}$$

However $\nabla\cdot\underline{v} = 0$ since $\underline{v}$ is considered to be independent of $\underline{x}$. Therefore

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{\underline{F}_e}{m}\cdot\nabla_v f = -\underline{F}_c\cdot\nabla_v f - f\nabla_v\cdot\dot{\underline{v}} \tag{12}$$

but $\nabla_v\cdot\frac{\underline{F}_e}{m} = 0$, so that

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{\underline{F}_e}{m}\cdot\nabla_v f = -\underline{F}_c\cdot\nabla_v f - f\nabla_v\cdot\underline{F}_c \tag{13}$$

This is Boltzmann's equation. Equating the right hand sides of (5) and (13)

$$\left(\frac{\partial f}{\partial t}\right)_c = -\underline{F}_c\cdot\nabla_v f - f\nabla_v\cdot\underline{F}_c \tag{14}$$

Suppose $\underline{F}_c = k^2(\underline{V}_s - \underline{v})$. Then $\nabla_v\cdot\underline{F}_c = -3k^2$. Using these relations one can evaluate $\int\left(\frac{\partial f}{\partial t}\right)_c d^3v$ and $\int\left(\frac{\partial f}{\partial t}\right)\underline{v}\, d^3v$

$$\int\left(\frac{\partial f}{\partial t}\right)_c d^3v = 3k^2\int f\, d^3v - \int k^2(V_{si} - v_i)\frac{\partial f}{\partial v_i}\, d^3v \tag{15}$$

The second integral can be integrated by parts to give

$$\int\left(\frac{\partial f}{\partial t}\right)_c d^3v = 3k^2 n - 3k^2 n = 0 \tag{16}$$

if one takes $f = 0$ at infinity. Similarly, integrating by parts one find

$$\int\left(\frac{\partial f}{\partial t}\right)_c \underline{v}\, d^3v = 3k^2\int f\,\underline{v}\, d^3v - \int \underline{v}\, k^2(V_{si} - v_i)\frac{\partial f}{\partial v_i}\, d^3v$$
$$= 3k^2 n\underline{u} + nk^2(\underline{V}_s - 4\underline{u}) = nk^2(\underline{V}_s - \underline{u}) \tag{17}$$

So far it has been shown that the friction-like force introduced, is consistent with the assumptions made about the effect of collisions. To show in detail that the appropriate form of Liouville's theorem breaks down recall that

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i} + f \frac{\partial \dot{v}_i}{\partial v_i} = 0 \tag{18}$$

or
$$\frac{df}{dt} = - f \nabla_v \cdot \dot{\underline{v}} \tag{19}$$

However $\underline{F}_c = k^2(\underline{V}_s - \underline{v})$ and $\nabla_v \cdot \frac{F_e}{m} = 0$, therefore

$$\frac{df}{dt} = - f \nabla_v \cdot \left[k^2 (\underline{V}_s - \underline{v}) \right] = f 3 k^2 \tag{20}$$

Integrating this equation,
$$f = f_0 e^{3k^2 t} \tag{21}$$

Notice that $\frac{df}{dt} \neq 0$. This means that Liouville's theorem is not valid here. The fact that f is an exponential function of time is indicative of the notion that particles under a frictional force tend toward a limiting velocity. In this phase space this means that the density function tends toward infinity along a surface of constant velocity $\underline{V}_s$ and tends toward zero elsewhere.

The friction-like effect arises because of the presence of turbulent regions of magnetic field. However Liouville's theorem has been shown to be valid in any given electromagnetic field. This paradox can be can be resolved if one could show that the cosmic ray particles in passing through the turbulent regions of the magnetic field, modify the field in such a way that one can no longer consider the field to be a given function of position and time only. The field now would have to depend also on the velocities of the particles.

Returning to the development of the explanation of the diurnal variation, equations (6), (7), (8), and (9) can be combined to give

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{u}) = 0 \tag{22}$$

and
$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{n}\nabla\left(\frac{1}{3}nc^2\right) + \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{u}\times\underline{B}\right) + \frac{1}{\tau}(\underline{V}_s - \underline{u}) \tag{23}$$

Provided u changes slowly compared to the mean collision time, τ, and the gyrofrequency, $\omega = \frac{eB}{mc}$, the left hand side of (23), $\frac{d\underline{u}}{dt}$, can be neglected. Then

$$0 = -\frac{c^2}{3n}\nabla n + \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{u}\times\underline{B}\right) + \frac{1}{\tau}(\underline{V}_s - \underline{u}) \tag{24}$$

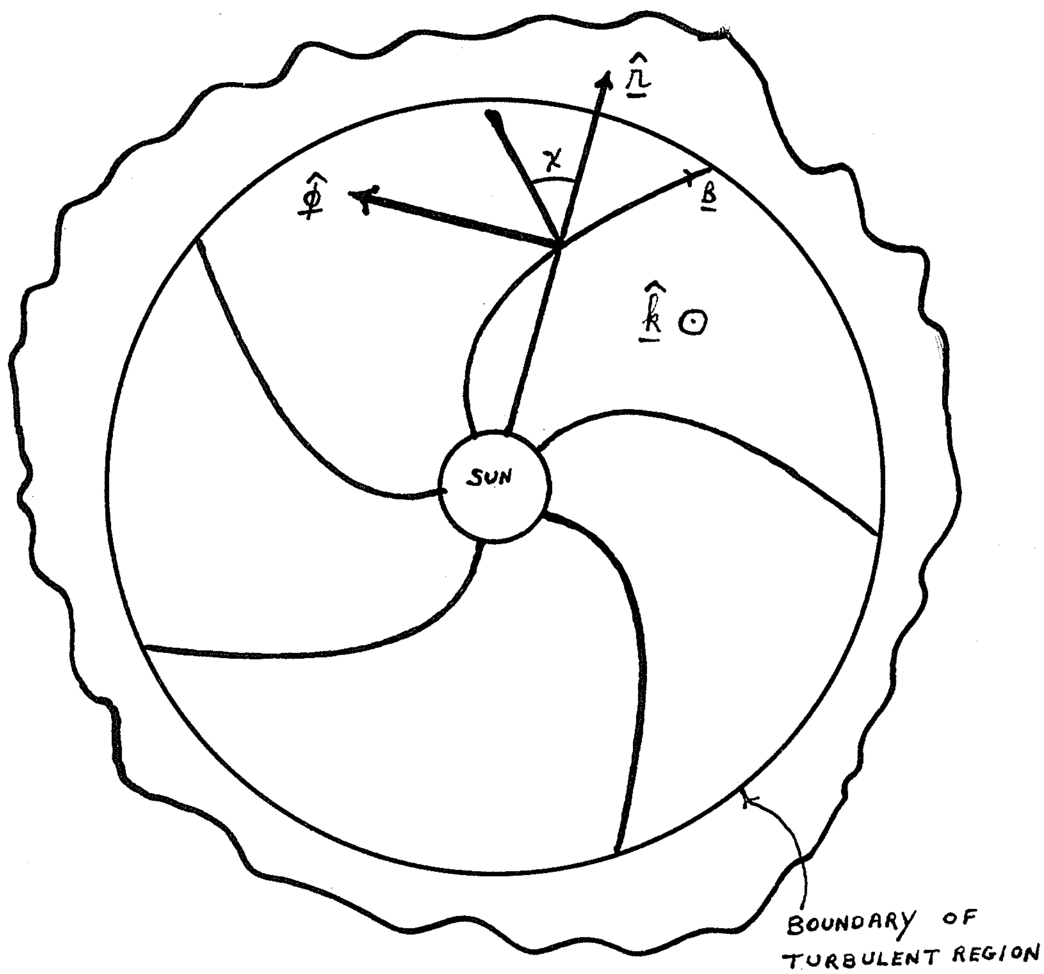
The model used here shall be limited to the ecliptic plane.* In this plane magnetic field lines of force form Archimedes spirals, and the electric field is perpendicular to the plane as indicated by equation (3). In a circular region about the sun which contains the earth's orbit the fields are considered to be very regular and τ is large. Outside the perimeter of this region the fields are considered to be very turbulent and τ is small. Cylindrical polar co-ordinates are chosen and the unit vectors in the principal directions are $\hat{r}$, $\hat{\phi}$, $\hat{k}$. See figure six. χ is the angle between the normal to the magnetic field and the radius vector. $\underline{V}_s$ is the velocity of the solar wind. Then

$$\begin{aligned}
\underline{B} &= B\sin\chi\,\hat{r} - B\cos\chi\,\hat{\phi} \\
\underline{E} &= \frac{1}{c}VB\cos\chi\,\hat{k} \\
\underline{V} &= V\hat{r} \\
\underline{u} &= u_r\hat{r} + u_\phi\hat{\phi} + u_z\hat{k}
\end{aligned} \tag{25}$$

From the symmetry of the model $\frac{\partial}{\partial\phi}$ and n=constant on the boundary with the turbulent region. It is assumed that

*Cylindrical modification of Axford's model

Figure 6. Turbulent Region of Magnetic Field



things change very slowly as one crosses the ecliptic plane in the $\hat{k}$ direction. Therefore $\frac{\partial}{\partial z}=0$. In deriving equation (24) it was assumed that

$$\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = 0$$

but $\frac{\partial}{\partial t} = \frac{\partial}{\partial z} = \frac{\partial}{\partial \phi} = 0$, therefore $\nabla = \hat{r}\frac{\partial}{\partial r}$ and

$$\frac{d\underline{u}}{dt} = u_r \frac{\partial \underline{u}}{\partial r} = 0$$

or

$$\frac{\partial (u_r{}^2)}{\partial r} = 0 \tag{26}$$

Because particles do not accumulate in the circular region and the region is not evacuated of cosmic ray particles U_r on the boundaries of the cavity. That is, $U_r = 0$ on the surface of the sun and on the boundary with the turbulent region. Because of (26) $U_r = 0$ everywhere.

It is now possible to substitute these conditions together with the expression for E from (4) into equation (24) written in cylindrical polar coordinates to give

$$0 = -\frac{c^2\tau}{3n}\frac{dn}{dr}\hat{r} + \frac{\tau e}{m}\Big[\frac{1}{c}V_s B\cos\chi\,\hat{k} + \frac{1}{c}(-u_\phi B\sin\chi\,\hat{k} + u_z B\sin\chi\,\hat{\phi} + u_z B\cos\chi\,\hat{r})\Big] + (V_s\hat{r} - u_\phi\hat{\phi} - u_z\hat{k}) \tag{27}$$

After substituting $\omega = \frac{eB}{mc}$, the component equations can be written

$$\left.\begin{aligned} 0\,u_\phi + \omega\tau\cos\chi\, u_z - \frac{c^2\tau}{3n}\frac{dn}{dr} &= -V_s \\ -\,u_\phi + \omega\tau\sin\chi\, u_z + 0\,\frac{dn}{dr} &= 0 \\ -\omega\tau\sin\chi\, u_\phi - u_z + 0\,\frac{dn}{dr} &= -V_s\,\omega\tau\cos\chi \end{aligned}\right\} \tag{28}$$

Equations (28) together with equation (2) can be solved to give

$$\frac{1}{3}c^2\tau\frac{dn}{dr} = \frac{[(\omega\tau)^2+1]\,V_s n}{[1+(\omega\tau\sin\chi)^2]} \tag{29}$$

$$u_z = \frac{\Omega r}{\omega\tau\sin\chi\left[1+\frac{1}{(\omega\tau\sin\chi)^2}\right]} \tag{30}$$

$$u_\phi = \frac{\Omega r}{\left[1 + \frac{1}{(\omega \tau \sin \chi)^2}\right]} \tag{31}$$

If one assumes that at the earth's orbit the field is regular and τ goes to infinity then

$$\left.\begin{array}{l} u_z = 0 \\ u_\phi = \Omega r \end{array}\right\} \tag{32}$$

The cosmic ray particles co-rotate with the sun. In addition it can be seen that

$$\frac{dn}{dr} = 0 \tag{33}$$

The density gradient which Stern showed balanced the effects of the drift velocity in static fields has been destroyed by the turbulent magnetic field outside this region. The choice of the boundary condition $\frac{\partial n}{\partial z} = 0$ eliminates the possibility of obtaining no cosmic ray streaming, $\underline{u} = 0$. By setting $\underline{u} = 0$ in equation (24) it is seen that

$$\frac{c^2}{3n} \frac{\partial n}{\partial z} = \frac{e}{m} E = \frac{e}{m} \frac{VB \cos \chi}{c} \tag{34}$$

Therefore for no streaming to take place a density gradient must be present. This gradient can be destroyed by having a turbulent region present.

It should be pointed out that the model assumed here is consistent with Stern's argument(41) if $\tau = \infty$. Setting $\underline{u} = 0$ and $\tau = \infty$, equation (24) becomes

$$\frac{c^2 \nabla n}{3n} = \frac{e}{m} \underline{E} \tag{35}$$

Equations (22) and (23) can be combined to give

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}_s) = (\nabla \cdot \underline{\underline{K}} \cdot \nabla)\, n \tag{36}$$

where $\underline{\underline{K}}$ is the diagonal diffusivity tensor with $K_\parallel = \frac{1}{3} c^2 \tau$ and $K_\perp = \frac{1}{3} \frac{c^2 \tau}{\left[1 + (\omega \tau)^2\right]}$ is a coordinate system with one axis parallel to $\underline{B}$. The streaming velocity parallel to $\underline{B}$ is

$$n u_{\parallel} = n V_{s\parallel} - K_{\parallel} \nabla_{\parallel} n$$

or
$$u_{\parallel} = V_{s\parallel} - \frac{c^2 \tau}{3} \nabla_{\parallel} (\ln n)$$

If $\tau = \infty$

$$\nabla_{\parallel} n = 0 \tag{37}$$

Equations (35) and (37) can be combined to give

$$\frac{1}{n} \nabla_{\perp} n = \frac{3 e \underline{E}}{\gamma m c^2}$$

This is the same expression obtained by Stern for $\frac{1}{\phi}\frac{d\phi}{dr}$, where the $\hat{\underline{r}}$ direction was the direction perpendicular to the magnetic field.

Finally, the model described above is only applicable to particles which are affected by the turbulent regions of the magnetic field. The upper limit here is 10^{11} ev.[42]

CHAPTER FOUR

Conclusions

Liouville's theorem was shown to be valid whenever the ensemble can be described by Hamilton's equations. The generalized forces must be such that they can be taken into the Hamiltonian. Liouville's theorem holds in any electromagnetic field which is a function of position and time only and which is not influenced an appreciable amount by the presence of the cosmic ray particles.

It appears that the existence of very turbulent magnetic fields causes the particles to be scattered in such a manner that the average force acting on the particles is a friction-like force. If one assumes the existence of such a force, the diurnal variation is correctly predicted. This, of course, causes the breakdown of Liouville's theorem. The fact that this force must be electromagnetic in nature prompts one to say that the particles influence the magnetic field in some way; otherwise one would be faced with a contradiction to Liouville's theorem.

The cylindrical modification of Axford's model for the diurnal variation predicts the observed results. It predicts that for particles above 10^{11} ev no variation will be observed. For low energy particles the idealized model of a perfectly regular field in the cavity, $\tau=\infty$, must be abandoned. The fluctuating part of the field is, in fact, present to some degree in the cavity and hence the very low energy particles will be scattered enough to minimize, if not wipe out, any anisotropy. The model pertains only to radiation in the ecliptic plane. What happens outside this plane is not discussed.

It is the existence of time dependent fields in the form of turbulent regions of magnetic field together with the breakdown of Liouville's theorem which no longer allows one to conclude that radiation is isotropic near the earth if one assumes isotropy far away.

A final point in regards to the friction-like effect is that the actual mechanism which produces this effect has not been explained. Only a qualitative idea of what might take place has been suggested.

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APPENDIX ONE

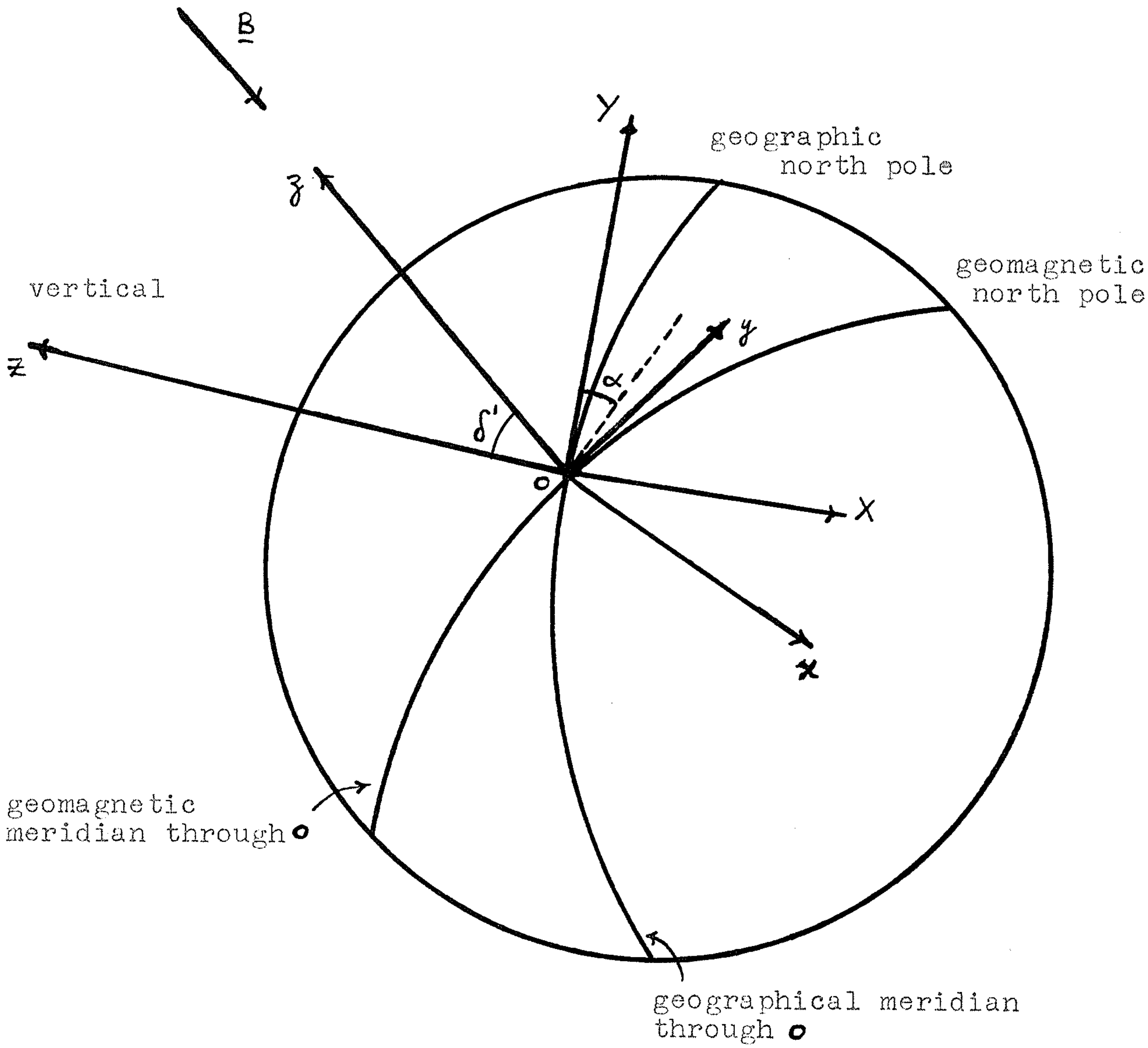
Transformation Between Coordinate Systems K' and K

The Z axis, the vertical direction, is in the yz plane. The yz plane, the plane through the observation point and containing the geomagnetic axis, intersects a spherical earth in a great circle. The vertical direction is just an extension of a radius of this circle and thus OZ is in the plane. (See figure 7.)

A compass needle confined to the XY plane would align itself in a direction corresponding to the projection of the y axis (which points toward the geomagnetic north) upon the XY plane. The angle between this projection and true north (the Y axes points toward true north) is the declination angle α. If the declination angle is east of true north, then a rotation about OZ of α in the clockwise sense bring OY into the plane containing Oy, Oz, OZ. That is, the yz and YZ planes coincide. OX and Ox are then in the same direction, as they are the normals to coincident planes. If the dip angle is north, (dip angle is the acute angle between the direction of $\underline{B}$ and the horizontal plane) a rotation about OX of δ', the angle between Oz and OZ, ($\delta' = 90^{\circ}$-dip angle) in the counter clockwise sense will bring the two sets of axes into coincidence. In matrix notation the transformation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta' & -\sin\delta' \\ 0 & \sin\delta' & \cos\delta' \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Figure 7. The K or (x,y,z) and The K' or (X,Y,Z) Coordinate Systems



dotted line is the projection of oy upon the XY plane

APPENDIX TWO

Values of $\delta(L)$, the Positive Excess, Read as Data for $L = 1, 2,$ 20

0.2265

0.2275

0.2285

0.2295

0.2305

0.2315

0.2325

0.2330

0.2335

0.2335

0.2335

0.2335

0.2330

0.2320

0.2300

0.2275

0.2235

0.2140

0.2080

0.2180

APPENDIX 3

```
C MESON DEFLECTION DISTRIBUTION    R-H PLANE DIVIDED INTO BOXES
C THE NUMBER OF PARTICLES FROM EACH BOX AND THE AVERAGE DEFLECTION FOUND
C
C
C                                 DEFINITIONS
C TEMPERATURE,T(N),READ IN FOR FIRST 61 LEVALS AND GENERATED IN PROGRAM
C FOR THE REMAINING 100 LEVELS
C EXCESS, THE POSITIVE EXCESS IS READ IN FOR EACH VALUE OF R
C GAMMA, THE ANGLE BETWEEN THE PROJECTION OF THE VERTICAL INTO THE
C EQUATORIAL PLANE AND THE VIEWING DIRECTION, IS GENERATED BUT CAN BE
C READ IN IF DESIRED
C ALPHA-ANGLE OF DECLINATION
C BETA-ANGLE OF LATITUDE
C DELTA-90 DEGREES MINUS DIP ANGLE
C THETA-ZENITH ANGLE OF TELESCOPE IN K' COORDINATE SYSTEM
C PHIC-AZIMUTH ANGLE OF TELESCOPE IN K' COORDINATE SYSTEM
C CLMDAO- ANGLE TELESCOPE MAKES WITH VERTICAL
C SP--SUM OF ALL PARTICLES COUNTED
C SD--SUM OF WEIGHTED DEFLECTIONS
C L--INDEX USED TO DISTINGUISH ENERGY LEVELS
C N--INDEX USED TO DISTINGUISH LEVELS OF HEIGHT
C S(1,N)-AN APPROXIMATION TO THE  PATH LENGTH
C S(2,N)-NEXT SUCCESSIVE APPROXIMATION TO THE PATH LENGTH FOLLOWING S(1,N)
C DLPHI(N,L)-DEFLECTION CORRESPONDING TO A PARTICLE AT A HEIGHT GIVEN
C BY N AND A RESIDUAL RANGE AT OBSERVATION GIVEN BY L
C CLMDA(N)-ANGLE BETWEEN NEGATIVE TANGENT AND VERTICAL AT HEIGHT N
C EPSLON(2,N)-EXPONENT IN FACTOR ACCOUNTING FOR MESON DECAY
C Y(N)-NUMBER OF PARTICLES PRODUCED AT A GIVEN HEIGHT WITH RESIDUAL
C RANGE AT OBSERVATION R
C X(N,L)-NUMBER PRODUCED BETWEEN LEVELS N-1 ANDN, WITH RESIDUAL RANGE
C GIVEN BY L
C W(N,L)-NUMBER PRODUCED BETWEEN LEVELS N-1 ANDN, WITH RESIDUAL RANGE
C BETWEEN L-1 AND L
C QA(I) OR QB(I)-NUMBER OF PARTICLES WITH DEFLECTIONS  IN I'TH INTERVAL
C
C NOTE THAT A NUMBER OF ARRAYS HAVE BEEN STORED IN THE SAME
C PLACE BY USING THE EQUIVALENCE STATEMENT
C
C
C
      DIMENSION QA(50),QB(50),T(161),U(161),S(2,161),V(2,161),DLPHI(161,
     120),CLMDA(161),EPSLON(2,161),Y(161),X(161,20),W(161,20),EXCESS(20)
      EQUIVALENCE (X(1,1),W(1,1)),(EPSLON(1,1),V(1,1),S(1,1)),(QA(1),QB(
     11)),(CLMDA(1),Y(1))
      READ(1,201)(T(I),I=1,61)
```

```
      READ(1,202)EXCESS
  202 FORMAT(10F6.4)
      SALPHA=SIN(.1678425)
      CALPHA=COS(.1678425)
      SBETA=SIN(.870919)
      CBETA=COS(.870919)
      SDELTA=SIN(.2228203)
      CDELTA=COS(.2228203)
  402 DO 90 I=1,11
      AN=I-1
      GAMMA=(5.-AN)*5.
      SGAMMA=SIN(.01*8.7266463*(5.-AN))
      CGAMMA=COS(.01*8.7266463*(5.-AN))
     0CTHETA=-SDELTA*SALPHA*SGAMMA+SDELTA*CALPHA*SBETA*CGAMMA+CDELTA*
     1CBETA*CGAMMA
      STHETA=(1.-CTHETA*CTHETA)**.5
     0SPHIO=(CDELTA*SALPHA*SGAMMA-CDELTA*CALPHA*SBETA*CGAMMA+SDELTA
     1*CBETA*CGAMMA)/STHETA
      CPHIO=(1.-SPHIO*SPHIO)**.5
      SDTSPO=SDELTA*STHETA*SPHIO
      SDTCPO=SDELTA*STHETA*CPHIO
      CLMDAO=SDELTA*STHETA*SPHIO+CDELTA*CTHETA
  406 DO 509 J=1,2
      DO 25 N=1,50
   25 QA(N)=0.0
      SP=0.0
      SD=0.0
      WRITE(3,100) GAMMA
  100 FORMAT('1','DEFLECTION',T25,'RELATIVE NUMBER',T60,'GAMMA=',F5.1,
     1'DEGREES')
      AN=J
      SIGMA=2.*(1.5-AN)
      DO 501 N=1,160
      IF(N-21)575,575,574
  575 TA=T(N)
      AN=(N-1)*1000.
      GO TO 588
  574 IF(N-61)586,586,587
  586 AN=AN+100.
      TA=T(N)
      GO TO 588
  587 TA=TA+.5
      AN=AN+10.
  588 U(N)=-9.867543*SIGMA*TA/(100000.-4.0*AN)
```

- 73 -

```
  6 IF(N-62)10,11,11
 10 U(N)=U(N)/10.
    GO TO 501
 11 U(N)=U(N)/100.
501 CONTINUE
    UA=U(21)/10.
    UB=U(61)/10.
    U(161)=-.9867543*SIGMA*.2726
    DO 517 L=1,20
    R=4600.-200.*L
    DO 507 N=1,161
    IF(N-21)576,576,577
576 AN=(N-1)*100.
    GO TO 600
577 IF(N-61)598,598,599
598 AN=AN+10.
    GO TO 600
599 AN=AN+1.
600 S(1,N)=.4*AN/CLMDAO
507 V(2,N)=(53.5/(56.+R+S(1,N))-0.00207)/CLMDAO
930 DLPHI(1,L)=0.0
    DO 703 N=2,161
    IF(N-22)503,8,506
  8 DLPHI(N,L)=DLPHI(N-1,L)+UA*V(2,N-1)+U(N)*V(2,N)
    GO TO 703
506 IF(N-62)503,9,503
  9 DLPHI(N,L)=DLPHI(N-1,L)+UB*V(2,N-1)+U(N)*V(2,N)
    GO TO 703
503 DLPHI(N,L)=DLPHI(N-1,L)+U(N-1)*V(2,N-1)+U(N)*V(2,N)
703 CONTINUE
    DO 504 N=1,161
504 CLMDA(N)=CLMDAO+SDTCPO*DLPHI(N,L)*(1.-.16666667*DLPHI(N,L)*
   1DLPHI(N,L))-.5*SDTSPO*DLPHI(N,L)*DLPHI(N,L)*(1.-1./12.*
   2DLPHI(N,L)*DLPHI(N,L))
    S(2,1)=0.0
    DO 505 N=2,161
    IF(N-21)580,580,581
581 IF(N-61)680,680,681
580 AN=20.
    GO TO 505
680 AN=2.
    GO TO 505
681 AN=.2
505 S(2,N)=S(2,N-1)+AN*(1./CLMDA(N-1)+1./CLMDA(N))
```

```
      IF(AN-1.)701,701,602
  602 IF(AN-900.)603,604,604
  603 DO 513 N=1,161
      S(1,N)=S(2,N)
  513 V(2,N)=(53.5 /(56.+R+S(1,N))-0.00207)/CLMDA(N)
      GO TO 980
  604 WRITE(3,801)AN
  801 FORMAT('1',E15.8)
  701 DO 516 N=1,161
      IF(N-21)590,590,591
  591 IF(N-61)592,592,593
  590 AN=(N-1)*10.
      Z=731000.
      GO TO 561
  592 AN=AN+1.
      Z=73100.
      GO TO 561
  593 AN=AN+.1
      Z=7310.
  561 EPSLON(2,N)=-8.765987*SIGMA*DLPHI(N,L)
  516 Y(N)=Z*(1.+.5*SIGMA*EXCESS(L))/CLMDA(N)
     1/((510.+R+S(1,N))**3.58)*EXP(-EPSLON(2,N)-
     2(1000.-4.*AN)/(120.*CLMDA(N)))
      YA=Y(21)/10.
      YB=Y(61)/10.
      DO 517 N=2,161
      IF(N-22) 525,524,526
  526 IF(N-62) 525,527,525
  524 X(N,L)=Y(N)+YA
      GO TO 517
  527 X(N,L)=Y(N)+YB
      GO TO 517
  525 X(N,L)=Y(N)+Y(N-1)
  517 CONTINUE
      DO 75 N=2,161
      DO 75 LI=2,20
      L=22-LI
   75 W(N,L)=X(N,L)+X(N,L-1)
      DO 5 N=2,161
      DO 5 L=2,20
      AN=.25*(DLPHI(N-1,L-1)+DLPHI(N,L)+DLPHI(N,L)+DLPHI(N-1,L))
      IF(AN)18,20,20
   20 DO 16 K=1,2
      DO 16 M=1,2
```

```
      II=N-2+K
      JJ=L-2+M
      D=.5*(AN+DLPHI(II,JJ))
      SP=SP+W(N,L)
      SD=SD+D*W(N,L)
      ID=D*100.
      ID=ID+1
   16 QA(ID)=QA(ID)+W(N,L)
      GO TO 5
   18 DO 17 K=1,2
      DO 17 M=1,2
      II=N-2+K
      JJ=L-2+M
      D=.5*(AN+DLPHI(II,JJ))
      SP=SP+W(N,L)
      SD=SD+D*W(N,L)
      ID=-D*100.
      ID=ID+1
   17 QB(ID)=QB(ID)+W(N,L)
    5 CONTINUE
C OUTPUT
      IF(J-1)778,777,778
  777 DO 511 N=1,50
      AN=-.01*N+.005
      QA(N)=QA(N)/SP
      WRITE(3,3)AN,QA(N)
    3 FORMAT(' 'E15.8,10X,E15.8)
  511 CONTINUE
      GO TO 512
  778 DO 510 N=1,50
      AN=.01*N-.005
      QA(N)=QA(N)/SP
      WRITE(3,3)AN,QB(N)
  510 CONTINUE
  512 AN=SD/SP
      WRITE(3,30)AN
   30 FORMAT('0','AVERAGE DEFLECTION=',E15.8)
  509 CONTINUE
   90 CONTINUE
      END
```

```
C   DIFFERENTIAL INTEGRAL INTENSITY CALCULATIONS
C
C
C                                    DEFINITIONS
C TEMPERATURE,T(N),READ IN FOR FIRST 61 LEVALS AND GENERATED IN PROGRAM
C FOR THE REMAINING 100 LEVELS
C EXCESS, THE POSITIVE EXCESS IS READ IN FOR EACH VALUE OF R
C GAMMA, THE ANGLE BETWEEN THE PROJECTION OF THE VERTICAL INTO THE
C EQUATORIAL PLANE AND THE VIEWING DIRECTION, IS GENERATED BUT CAN BE
C READ IN IF DESIRED
C ALPHA-ANGLE OF DECLINATION
C BETA-ANGLE OF LATITUDE
C DELTA-90 DEGREES MINUS DIP ANGLE
C THETA-ZENITH ANGLE OF TELESCOPE IN K' COORDINATE SYSTEM
C PHIO-AZIMUTH ANGLE OF TELESCOPE IN K' COORDINATE SYSTEM
C CLMDAO- ANGLE TELESCOPE MAKES WITH VERTICAL
C L--INDEX USED TO DISTINGUISH ENERGY LEVELS
C N--INDEX USED TO DISTINGUISH LEVELS OF HEIGHT
C S(1,N)-AN APPROXIMATION TO THE  PATH LENGTH
C S(2,N)-NEXT SUCCESSIVE APPROXIMATION TO THE PATH LENGTH FOLLOWING S(1,N)
C DLPHI(N,L)-DEFLECTION CORRESPONDING TO A PARTICLE AT A HEIGHT GIVEN
C BY N AND A RESIDUAL RANGE AT OBSERVATION GIVEN BY L
C CLMDA(N)-ANGLE BETWEEN NEGATIVE TANGENT AND VERTICAL AT HEIGHT N
C EPSLON(2,N)-EXPONENT IN FACTOR ACCOUNTING FOR MESON DECAY
C Y(N)-NUMBER OF PARTICLES PRODUCED AT A GIVEN HEIGHT WITH RESIDUAL
C RANGE AT OBSERVATION R
C  DI(J,L) -- DIFFERENTIAL INTENSITY FOR MESONS WITH CHARGE J
C  DINT(L) -- DIFFERENTIAL INTENSITY FOR MESONS WITH BOTH CHARGES
C  TI(J,L) -- INTEGRAL INTENSITY FOR MESONS WITH CHARGE J
C  TINT(L) -- INTEGRAL INTENSITY FOR MESONS BOTH CHARGES
C
C NOTE THAT A NUMBER OF ARRAYS HAVE BEEN STORED IN THE SAME
C PLACE BY USING THE EQUIVALENCE STATEMENT
C
C
C
      DIMENSION              T(161),U(161),S(2,161),V(2,161),DLPHI(161,
     120),CLMDA(161),EPSLON(2,161),Y(161),                EXCESS(20),
     3DI(2,20),DINT(20),TI(2,20),TINT(20)
      EQUIVALENCE                (EPSLON(1,1),V(1,1),S(1,1)),
     1   (CLMDA(1),Y(1))
      READ(1,201)(T(I),I=1,61)
  201 FORMAT(16F5.1)
      READ(1,202)EXCESS
```

```
      SALPHA=SIN(.1678425)
      CALPHA=COS(.1678425)
      SBETA=SIN(.870919)
      CBETA=COS(.870919)
      SDELTA=SIN(.2228203)
      CDELTA=COS(.2228203)
  402 DO 90 I=1,11
      AN=I-1
      GAMMA=(5.-AN)*5.
      SGAMMA=SIN(.01*8.7266463*(5.-AN))
      CGAMMA=COS(.01*8.7266463*(5.-AN))
     0CTHETA=-SDELTA*SALPHA*SGAMMA+SDELTA*CALPHA*SBETA*CGAMMA+CDELTA*
     1CBETA*CGAMMA
      STHETA=(1.-CTHETA*CTHETA)**.5
     0SPHIO=(CDELTA*SALPHA*SGAMMA-CDELTA*CALPHA*SBETA*CGAMMA+SDELTA
     1*CBETA*CGAMMA)/STHETA
      CPHIO=(1.-SPHIO*SPHIO)**.5
      SDTSPO=SDELTA*STHETA*SPHIO
      SDTCPO=SDELTA*STHETA*CPHIO
      CLMDAO=SDELTA*STHETA*SPHIO+CDELTA*CTHETA
  406 DO 509 J=1,2
      AN=J
      SIGMA=2.*(1.5-AN)
      DO 501 N=1,160
      IF(N-21)575,575,574
  575 TA=T(N)
      AN=(N-1)*1000.
      GO TO 588
  574 IF(N-61)586,586,587
  586 AN=AN+100.
      TA=T(N)
      GO TO 588
  587 TA=TA+.5
      AN=AN+10.
  588 U(N)=-9.867543*SIGMA*TA/(100000.-4.0*AN)
      IF(N-22)501,6,6
    6 IF(N-62)10,11,11
   10 U(N)=U(N)/10.
      GO TO 501
   11 U(N)=U(N)/100.
  501 CONTINUE
      UA=U(21)/10.
      UB=U(61)/10.
      U(161)=-.9867543*SIGMA*.2726
```

```
      R=4600.-200.*L
      DO 507 N=1,161
      IF(N-21)576,576,577
  576 AN=(N-1)*100.
      GO TO 600
  577 IF(N-61)598,598,599
  598 AN=AN+10.
      GO TO 600
  599 AN=AN+1.
  600 S(1,N)=.4*AN/CLMDAO
  507 V(2,N)=(53.5/(56.+R+S(1,N))-0.00207)/CLMDAO
  980 DLPHI(1,L)=0.0
      DO 703 N=2,161
      IF(N-22)503,8,506
    8 DLPHI(N,L)=DLPHI(N-1,L)+UA*V(2,N-1)+U(N)*V(2,N)
      GO TO 703
  506 IF(N-62)503,9,503
    9 DLPHI(N,L)=DLPHI(N-1,L)+UB*V(2,N-1)+U(N)*V(2,N)
      GO TO 703
  503 DLPHI(N,L)=DLPHI(N-1,L)+U(N-1)*V(2,N-1)+U(N)*V(2,N)
  703 CONTINUE
      DO 504 N=1,161
  504 CLMDA(N)=CLMDAO+SDTCPO*DLPHI(N,L)*(1.-.16666667*DLPHI(N,L)*
     1DLPHI(N,L))-.5*SDTSPO*DLPHI(N,L)*DLPHI(N,L)*(1.-1./12.*
     2DLPHI(N,L)*DLPHI(N,L))
      S(2,1)=0.0
      DO 505 N=2,161
      IF(N-21)580,580,581
  581 IF(N-61)680,680,681
  580 AN=20.
      GO TO 505
  680 AN=2.
      GO TO 505
  681 AN=.2
  505 S(2,N)=S(2,N-1)+AN*(1./CLMDA(N-1)+1./CLMDA(N))
      AN=ABS(S(1,161)-S(2,161))
      IF(AN-1.)701,701,602
  602 IF(AN-900.)603,604,604
  603 DO 513 N=1,161
      S(1,N)=S(2,N)
  513 V(2,N)=(53.5 /(56.+R+S(1,N))-0.00207)/CLMDA(N)
      GO TO 980
  604 WRITE(3,801)AN
  801 FORMAT('1',E15.8)
```

```
591 IF(N-61)592,592,593
590 AN=(N-1)*10.
    Z=731000.
    GO TO 561
592 AN=AN+1.
    Z=73100.
    GO TO 561
593 AN=AN+.1
    Z=7310.
561 EPSLON(2,N)=-8.765987*SIGMA*DLPHI(N,L)
516 Y(N)=Z*(1.+.5*SIGMA*EXCESS(L))/CLMDA(N)
   1/((510.+R+S(1,N))**3.58)*EXP(-EPSLON(2,N)-
   2(1000.-4.*AN)/(120.*CLMDA(N)))
    YA=Y(21)/10.
    YB=Y(61)/10.
    DI(J,L)=0.0
    DO 517 N=2,161
    IF(N-22) 525,524,526
526 IF(N-62) 525,527,525
524 DI(J,L)=DI(J,L)&Y(N)&YA
    GO TO 517
527 DI(J,L)=DI(J,L)&Y(N)&YB
    GO TO 517
525 DI(J,L)=DI(J,L)&Y(N)&Y(N-1)
517 CONTINUE
    TI(J,1)=0.0
    DO 75 L=2,20
 75 TI(J,L)=TI(J,L-1)&(DI(J,L)&DI(J,L-1))*100.
509 CONTINUE
    WRITE(3,100) GAMMA
100 FORMAT(1H1,T60,'GAMMA=',F5.1,' DEGREES')
    WRITE(3,3)
  3 FORMAT(1H ,'RESIDUAL RANGE',T20,'&VE DIFF. INT.',T39,'-VE DIFF. IN
   1T.',T58,'DIFF. INT.',T76,'&VE INTEGRAL INT.',T95,'-VE INTEGRAL INT
   2.',T115,'INTEGRAL INT.')
    DO 511 L=1,20
    TINT(L)=TI(1,L)&TI(2,L)
    DINT(L)=DI(1,L)&DI(2,L)
    R=(4600.-200.*L)
    WRITE(3,4) R,DI(1,L),DI(2,L),DINT(L),TI(1,L),TI(2,L),TINT(L)
  4 FORMAT(1H ,6(E15.8,4X),E15.8)
511 CONTINUE
 90 CONTINUE
    END
```

- 80 -

```
C   AVERAGE HEIGHTS OF PRODUCTION FOR &VE AND -VE MUONS AND FOR
C   DIFFERENT ENERGIES
C
C
C                              DEFINITIONS
C TEMPERATURE,T(N),READ IN FOR FIRST 61 LEVALS AND GENERATED IN PROGRAM
C FOR THE REMAINING 100 LEVELS
C EXCESS, THE POSITIVE EXCESS IS READ IN FOR EACH VALUE OF R
C GAMMA, THE ANGLE BETWEEN THE PROJECTION OF THE VERTICAL INTO THE
C EQUATORIAL PLANE AND THE VIEWING DIRECTION, IS GENERATED BUT CAN BE
C READ IN IF DESIRED
C ALPHA-ANGLE OF DECLINATION
C BETA-ANGLE OF LATITUDE
C DELTA-90 DEGREES MINUS DIP ANGLE
C THETA-ZENITH ANGLE OF TELESCOPE IN K  COORDINATE SYSTEM
C PHIO-AZIMUTH ANGLE OF TELESCOPE IN K  COORDINATE SYSTEM
C CLMDAO- ANGLE TELESCOPE MAKES WITH VERTICAL
C L--INDEX USED TO DISTINGUISH ENERGY LEVELS
C N--INDEX USED TO DISTINGUISH LEVELS OF HEIGHT
C S(1,N)-AN APPROXIMATION TO THE  PATH LENGTH
C S(2,N)-NEXT SUCCESSIVE APPROXIMATION TO THE PATH LENGTH FOLLOWING S(1,N)
C DLPHI(N,L)-DEFLECTION CORRESPONDING TO A PARTICLE AT A HEIGHT GIVEN
C BY N AND A RESIDUAL RANGE AT OBSERVATION GIVEN BY L
C CLMDA(N)-ANGLE BETWEEN NEGATIVE TANGENT AND VERTICAL AT HEIGHT N
C EPSLON(2,N)-EXPONENT IN FACTOR ACCOUNTING FOR MESON DECAY
C Y(N)-NUMBER OF PARTICLES PRODUCED AT A GIVEN HEIGHT WITH RESIDUAL
C RANGE AT OBSERVATION R
C X(N,L)-NUMBER PRODUCED BETWEEN LEVELS N-1 ANDN, WITH RESIDUAL RANGE
C GIVEN BY L
C  SH(J,L) -- WEIGHTED SUM OF HEIGHTS OF PRODUCTION FOR MUONS WITH
C             CHARGE J AND ENERGY GIVEN BY L
C  SW(J,L) -- SUM OF WEIGHTS USED ABOVE
C  TSH(J)  -- WEIGHTED SUM OF HEIGHTS OF PRODUCTION FOR MUONS WITH
C             CHARGE J
C  TSW(J)  -- SUM OF WEIGHTS USED ABOVE
C  AHP(J,L) -- AVERAGE HEIGHT OF PRODUCTION FOR MUONS WITH GHARGE J
C               AND ENERGY GIVEN BY L
C  AHPT(J) --- AVERAGE HEIGHT OF PRODUCTION FOR MUONS WITH GHARGE J
C  AHPB(L) --- AVERAGE HEIGHT OF PRODUCTION FOR MUONS WITH ENERGY
C               GIVEN BY L
C  TAHPB -- AVERAGE HEIGHT OF PRODUCTION FOR ALL MUONS
C  PATH(N) -- HEIGHT ABOVE OBSERVATION POINT ALONG VERTICAL (CM)
C  ROW(N) -- 1/DENSITY*DELTA H/2.
C
```

```
C NOTE THAT A NUMBER OF ARRAYS HAVE BEEN STORED IN THE SAME
C PLACE BY USING THE EQUIVALENCE STATEMENT
C
C
C
      DIMENSION                T(161),U(161),S(2,161),V(2,161),DLPHI(161,
     120),CLMDA(161),EPSLON(2,161),Y(161),X(161,20),            EXCESS(20)
     3,SH(2,20),SW(2,20),AHP(2,20),TSH(2),TSW(2),AHPB(20),PATH(161),
     4ROW(161),AHPT(2)
      EQUIVALENCE                 (EPSLON(1,1),V(1,1),S(1,1)),
     1     (CLMDA(1),Y(1))
      READ(1,201)(T(I),I=1,61)
  201 FORMAT(16F5.1)
      READ(1,202)EXCESS
  202 FORMAT(10F6.4)
      SALPHA=SIN(.1678425)
      CALPHA=COS(.1678425)
      SBETA=SIN(.870919)
      CBETA=COS(.870919)
      SDELTA=SIN(.2228203)
      CDELTA=COS(.2228203)
  402 DO 90 I=1,11
      AN=I-1
      GAMMA=(5.-AN)*5.
      SGAMMA=SIN(.01*8.7266463*(5.-AN))
      CGAMMA=COS(.01*8.7266463*(5.-AN))
     0CTHETA=-SDELTA*SALPHA*SGAMMA+SDELTA*CALPHA*SBETA*CGAMMA+CDELTA*
     1CBETA*CGAMMA
      STHETA=(1.-CTHETA*CTHETA)**.5
     0SPHIO=(CDELTA*SALPHA*SGAMMA-CDELTA*CALPHA*SBETA*CGAMMA+SDELTA
     1*CBETA*CGAMMA)/STHETA
      CPHIO=(1.-SPHIO*SPHIO)**.5
      SDTSPO=SDELTA*STHETA*SPHIO
      SDTCPO=SDELTA*STHETA*CPHIO
      CLMDAO=SDELTA*STHETA*SPHIO+CDELTA*CTHETA
  406 DO 509 J=1,2
      AN=J
      SIGMA=2.*(1.5-AN)
      DO 501 N=1,160
      IF(N-21)575,575,574
  575 TA=T(N)
      AN=(N-1)*1000.
      GO TO 588
  574 IF(N-61)586,586,587
  586 AN=AN+100.
```

```
      TA=T(N)
      GO TO 588
  587 TA=TA+.5
      AN=AN+10.
  588 U(N)=-9.867543*SIGMA*TA/(100000.-4.0*AN)
      ROW(N)=5740000.0*TA/(100000.-4.0*AN)
      IF(N-22)501,6,6
    6 IF(N-62)10,11,11
   10 U(N)=U(N)/10.
      ROW(N)=ROW(N)/10.
      GO TO 501
   11 U(N)=U(N)/100.
      ROW(N)=ROW(N)/100.
  501 CONTINUE
      UA=U(21)/10.
      UB=U(61)/10.
      U(161)=-.9867543*SIGMA*.2726
      ROWA=ROW(21)/10.
      ROWB=ROW(61)/10.
      ROW(161)=57400.0*272.6
      PATH(1)=0.0
      DO 35 N=2,161
      IF(N-22)31,32,33
   33 IF(N-62)31,34,31
   31 PATH(N)=PATH(N-1)&(ROW(N)&ROW(N-1))/2.0
      GO TO 35
   32 PATH(N)=PATH(N-1)&(ROW(N)&ROWA)/2.0
      GO TO 35
   34 PATH(N)=PATH(N-1)&(ROW(N)&ROWB)/2.0
   35 CONTINUE
      DO 517 L=1,20
      R=4600.-200.*L
      DO 507 N=1,161
      IF(N-21)576,576,577
  576 AN=(N-1)*100.
      GO TO 600
  577 IF(N-61)598,598,599
  598 AN=AN+10.
      GO TO 600
  599 AN=AN+1.
  600 S(1,N)=.4*AN/CLMDAO
  507 V(2,N)=(53.5/(56.+R+S(1,N))-0.00207)/CLMDAO
  980 DLPHI(1,L)=0.0
      DO 703 N=2,161
      IF(N-22)503,8,506
```

```
      GO TC 703
  506 IF(N-62)503,9,503
    9 DLPHI(N,L)=DLPHI(N-1,L)+UB*V(2,N-1)+U(N)*V(2,N)
      GO TO 703
  503 DLPHI(N,L)=DLPHI(N-1,L)+U(N-1)*V(2,N-1)+U(N)*V(2,N)
  703 CONTINUE
      DO 504 N=1,161
  504 CLMDA(N)=CLMDAO+SDTCPO*DLPHI(N,L)*(1.-.16666667*DLPHI(N,L)*
     1DLPHI(N,L))-.5*SDTSPO*DLPHI(N,L)*DLPHI(N,L)*(1.-1./12.*
     2DLPHI(N,L)*DLPHI(N,L))
      S(2,1)=0.0
      DO 505 N=2,161
      IF(N-21)580,580,581
  581 IF(N-61)680,680,681
  580 AN=20.
      GO TO 505
  680 AN=2.
      GO TO 505
  681 AN=.2
  505 S(2,N)=S(2,N-1)+AN*(1./CLMDA(N-1)+1./CLMDA(N))
      AN=ABS(S(1,161)-S(2,161))
      IF(AN-1.)701,701,602
  602 IF(AN-900.)603,604,604
  603 DO 513 N=1,161
      S(1,N)=S(2,N)
  513 V(2,N)=(53.5 /(56.+R+S(1,N))-0.00207)/CLMDA(N)
      GO TO 980
  604 WRITE(3,801)AN
  801 FORMAT('1',E15.8)
  701 DO 516 N=1,161
      IF(N-21)590,590,591
  591 IF(N-61)592,592,593
  590 AN=(N-1)*10.
      Z=731000.
      GO TO 561
  592 AN=AN+1.
      Z=73100.
      GO TO 561
  593 AN=AN+.1
      Z=7310.
  561 EPSLON(2,N)=-8.765987*SIGMA*DLPHI(N,L)
  516 Y(N)=Z*(1.+.5*SIGMA*EXCESS(L))/CLMDA(N)
     1/((510.+R+S(1,N))**3.58)*EXP(-EPSLON(2,N)-
     2(1000.-4.*AN)/(120.*CLMDA(N)))
```

- 84 -

```
      YB=Y(61)/10.
      DO 517 N=2,161
      IF(N-22) 525,524,526
  526 IF(N-62) 525,527,525
  524 X(N,L)=Y(N)+YA
      GO TO 517
  527 X(N,L)=Y(N)+YB
      GO TO 517
  525 X(N,L)=Y(N)+Y(N-1)
  517 CONTINUE
      DO 41 L=1,20
      SH(J,L)=0.0
      SW(J,L)=0.0
      DO 40 N=2,161
      SH(J,L)=SH(J,L)&X(N,L)*PATH(N)
   40 SW(J,L)=SW(J,L)&X(N,L)
   41 AHP(J,L)=SH(J,L)/SW(J,L)
      TSW(J)=0.0
      TSH(J)=0.0
      DO 42 L=2,20
      TSH(J)=(SH(J,L)&SH(J,L-1))*100.&TSH(J)
   42 TSW(J)=(SW(J,L)&SW(J,L-1))*100.&TSW(J)
      AHPT(J)=TSH(J)/TSW(J)
  509 CONTINUE
      TAHPB=(TSH(1)&TSH(2))/(TSW(1)&TSW(2))
      WRITE(3,100) GAMMA
  100 FORMAT(1H1,T60,'GAMMA=',F5.1,' DEGREES')
      WRITE(3,150)
  150 FORMAT(1H ,'RESIDUAL RANGE',T19,'AVE. HT. PROD. &VE MUONS',T46,'AV
     1E. HT. PROD. -VE MUONS',T75,'AVE. HT. PROD. ALL MUONS')
      DO 152 L=1,20
      AHPB(L)=(SH(1,L)&SH(2,L))/(SW(1,L)&SW(2,L))
      R=(4600.-200.*L)
      WRITE(3,151) R,AHP(1,L),AHP(2,L),AHPB(L)
  151 FORMAT(1H ,3(E15.8,10X),5X,E15.8)
  152 CONTINUE
      WRITE(3,153)
  153 FORMAT(1H ,/,(1H ,'AVE. HT. PROD. &VE MUONS',T31,'AVE. HT. PROD. -
     1VE MUONS',T60,'AVE. HT. PROD. ALL MUONS'))
      WRITE(3,154) AHPT(1),AHPT(2),TAHPB
  154 FORMAT(1H ,3(E15.8,15X))
      WRITE(3,160)
  160 FORMAT(1H )
      DO 158 N=1,54
```

```
      M2=3*N
      M3=3*N&1
      IF(N-54)156,157,157
  156 WRITE(3,155) M1,PATH(M1),M2,PATH(M2),M3,PATH(M3)
  155 FORMAT(        1H ,'HT.(',I3,')=',E15.8,' CM.',3X,'HT.(',I3,')=',E1
     15.8,' CM.'3X,'HT.('I3,')=',E15.8,' CM.')
      GO TO 158
  157 WRITE(3,159) M1,PATH(M1)
  159 FORMAT(1H ,'HT.(',I2,')=',E15.8,' CM')
  158 CONTINUE
   90 CONTINUE
      END
```

- 86 -

APPENDIX SIX

Moments of Boltzmann's Equation[43]

Suppose that $f(\underline{v},\underline{r},t)$ is the distribution function for the ensemble under consideration. It is assumed that f goes to zero at infinity. Let $Q(\underline{v})$ be some function of velocity, $\underline{v}$, only. All integrations in this section shall be over velocity space. Then

$$n(\underline{r},t) = \int f\, d^3v$$

is the density in configuration space at a point $\underline{r}$. In addition, a quantity $\bar{Q}(\underline{r},t)$ can be defined by

$$n\,\bar{Q}(\underline{r},t) = \int Q(\underline{v})\, f(\underline{v},\underline{r},t)\, d^3v$$

A bar over any quantity shall indicate a similar type of average value.

As a preliminary step to finding the required moments of the Boltzmann equation three useful integrals are investigated. The Einstein summation notation is used here with $i = 1, 2, 3$.

$$\int Q \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t}\int (Qf)\, d^3v = \frac{\partial[n\bar{Q}]}{\partial t} \tag{1}$$

$$\int Q v_i \frac{\partial f}{\partial x_i} d^3v = \frac{\partial}{\partial q_i}\int Q v_i f\, d^3v = \frac{\partial[n(\overline{v_i Q})]}{\partial x_i} \tag{2}$$

Integrating by parts,

$$\int Q \frac{dv_i}{dt}\frac{\partial f}{\partial v_i} d^3v = \sum_{j=1}^{3}\iint f\, Q \frac{dv_i}{dt}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f\, \frac{\partial[Q\frac{dv_i}{dt}]}{\partial v_i} d^3v$$

Because $f = 0$ at infinity

$$\int Q \frac{dv_i}{dt}\frac{\partial f}{\partial v_i} d^3v = -\int f\, \frac{\partial[Q\frac{dv_i}{dt}]}{\partial v_i} d^3v = -n\left[\overline{\frac{\partial(Q\frac{dv_i}{dt})}{\partial v_i}}\right] \tag{3}$$

Boltzmann's equation is

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{\underline{F}}{m}\cdot\frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t}\right)_c$$

or $$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial f}{\partial \underline{v}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{4}$$

since $\frac{\underline{F}}{m} = \frac{d\underline{v}}{dt}$, $\underline{F}$ being the none collision forces. $\left(\frac{\partial f}{\partial t}\right)_c$ is the rate of change of the distribution function due to collisions.

To find the lowest order moments of the Boltzmann's equation; that is, to integrate the equation as it is, set $Q = 1$. Equations (1), (2), (3) can then be used to integrate the terms on the left hand side of (4) to give

$$\frac{\partial n}{\partial t} + \frac{\partial (n\overline{v_i})}{\partial x_i} - n\left[\overline{\frac{\partial\left(\frac{dv_i}{dt}\right)}{\partial v_i}}\right] = \int \left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{5}$$

In the case of cosmic ray particles traversing an electromagnetic field

$$\frac{d\underline{v}}{dt} = \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{v} \times \underline{B}\right)$$

and $\frac{\partial\left(\frac{dv_i}{dt}\right)}{\partial v_i} = 0$. Therefore equation (5) becomes

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{u}) = \int \left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{6}$$

To find the first order moment of Boltzmann's equation, the integral of the equation after each term has been multiplied by $\underline{v}$, set $\underline{Q} = \underline{v}$. Again equations (1), (2), (3) can be used to evaluate the left hand side. Then

$$\frac{\partial (n\underline{u})}{\partial t} + \frac{\partial [n(\overline{\underline{v}v_i})]}{\partial x_i} - n\left[\overline{\frac{\partial\left\{\underline{v}\left(\frac{dv_i}{dt}\right)\right\}}{\partial v_i}}\right] = \int \underline{v}\left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{7}$$

Since $\frac{\partial}{\partial v_i}\left(\frac{dv_i}{dt}\right) = 0$

$$\frac{\partial (n\underline{u})}{\partial t} + \frac{\partial [n(\overline{\underline{v}v_i})]}{\partial x_i} - n\frac{d\underline{u}}{dt} = \int \underline{v}\left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{8}$$

The second term in this equation can be written as

$$\frac{\partial}{\partial x_1}\int \underline{v}\, v_1 f\, d^3v + \frac{\partial}{\partial x_2}\int \underline{v}\, v_2 f\, d^3v + \frac{\partial}{\partial x_3}\int \underline{v}\, v_3 f\, d^3v$$

$$= \underline{i}\left\{ \frac{\partial}{\partial x_1}\int v_1^2 f\, d^3v + \frac{\partial}{\partial x_2}\int v_1 v_2 f\, d^3v + \frac{\partial}{\partial x_3}\int v_1 v_3 f\, d^3v \right\} \tag{9}$$

$$+ \underline{j}\{\cdots\cdots\} + \underline{k}\{\cdots\cdots\}$$

Now define a random velocity $\underline{\omega} = \underline{v} - \underline{u}$, where $\underline{u}$ is the average velocity and $\underline{v}$ is the velocity of a particle. Then

$$\bar{\underline{\omega}} = \frac{1}{m}\int (\underline{v} - \underline{u}) f\, d^3v = 0 \tag{10}$$

Substituting for $\underline{v}$ in expression (9) and recalling that $\bar{\underline{\omega}} = 0$, the second term upon integration becomes

$$\underline{i}\left\{ \frac{\partial m(u_1^2 + \overline{\omega_1^2})}{\partial x_1} + \frac{\partial m(u_1 u_2 + \overline{\omega_1\omega_2})}{\partial x_2} + \frac{\partial m(u_1 u_3 + \overline{\omega_1\omega_3})}{\partial x_3} \right\} \tag{11}$$

$$+ \underline{j}\{\cdots\cdots\} + \underline{k}\{\cdots\cdots\}$$

In tensor notation this can be written as

$$\nabla\cdot(m\underline{u}\,\underline{u}) + \nabla\cdot(m\overline{\omega\omega})$$

When this is substituted for the second term, equation (8) becomes

$$\frac{\partial(m\underline{u})}{\partial t} + \nabla\cdot(m\underline{u}\,\underline{u}) + \nabla\cdot(m\overline{\underline{\omega}\underline{\omega}}) - m\frac{d\underline{u}}{dt} = \int \underline{v}\left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{12}$$

However, $\nabla\cdot(m\underline{u}\,\underline{u}) = m\underline{u}\cdot\nabla\underline{u} + \underline{u}\,\nabla\cdot(m\underline{u})$, and from equation (6)

$$\underline{u}\left[\nabla\cdot(m\underline{u})\right] = \underline{u}\int\left(\frac{\partial f}{\partial t}\right)_c d^3v - \underline{u}\frac{\partial m}{\partial t}$$

Therefore

$$\nabla\cdot(m\underline{u}\,\underline{u}) = m\underline{u}\cdot\nabla\underline{u} + \underline{u}\int\left(\frac{\partial f}{\partial t}\right)_c d^3v - \underline{u}\frac{\partial m}{\partial t}$$

Also $\quad m\dfrac{d\underline{u}}{dt} = \dfrac{m\bar{F}}{m}$, where $\bar{F} = e\underline{E} + \dfrac{\underline{u}\times\underline{B}}{c}$ (13)

and $\quad \dfrac{\partial m\underline{u}}{\partial t} = m\dfrac{\partial \underline{u}}{\partial t} + \underline{u}\dfrac{\partial m}{\partial t}$

On substitution of the results of (13) into (12)

$$m\frac{\partial\underline{u}}{\partial t} + m\underline{u}\cdot\nabla\underline{u} + \underline{u}\int\left(\frac{\partial f}{\partial t}\right)_c d^3v + \nabla\cdot\underline{\underline{S}} - \frac{m\bar{F}}{m} = \int\underline{v}\left(\frac{\partial f}{\partial t}\right)_c d^3v \tag{14}$$

where $\underline{\underline{S}} = m\overline{\underline{\omega}\underline{\omega}}$ is called the stress tensor.

If the distribution in the velocity space for $\underline{\omega}$ is spherically symmetric then the off diagonal elements of $\underline{\underline{S}}$ are zero. Then

$$\nabla \cdot \underline{\underline{S}} = \nabla(m\overline{\underline{\omega}\underline{\omega}}) = \underline{i}\,\frac{\partial(m\overline{\omega_1^2})}{\partial x_1} + \underline{j}\,\frac{\partial(m\overline{\omega_2^2})}{\partial x_2} + \underline{k}\,\frac{\partial(m\overline{\omega_3^2})}{\partial x_3}$$

but $\overline{\omega_1^2} = \overline{\omega_2^2} = \overline{\omega_3^2} = \frac{1}{3}\overline{\omega^2}$

Therefore

$$\nabla \cdot \underline{\underline{S}} = \underline{i}\,\frac{\partial\left(\frac{m\overline{\omega^2}}{3}\right)}{\partial x_1} + \underline{j}\,\frac{\partial\left(\frac{m\overline{\omega^2}}{3}\right)}{\partial x_2} + \underline{k}\,\frac{\partial(m\overline{\omega^2})}{\partial x_3} = \nabla\left(\frac{m\overline{\omega^2}}{3}\right) \qquad (15)$$

$\int\left(\frac{\partial f}{\partial t}\right)_c d^3v$ is the rate of change of density in configuration space which results from collisions of cosmic ray particles with irregularities in the magnetic field. The effect of these collisions is to redistribute the velocities of the particles such that in a coordinate system moving with the irregularities the particles appear to execute a random walk type of motion.(44) The density in configuration space is uneffected by collisions. Therefore $\int\left(\frac{\partial f}{\partial t}\right)_c d^3v = 0$

$m\int \underline{v}\left(\frac{\partial f}{\partial t}\right)_c d^3v$ is the rate of change of momentum per unit volume of configuration space. If the change in momentum in a collision on the average is $m(\underline{V} - \underline{u})$, where $\underline{V}$ is the velocity of the irregularities and $\underline{u}$ is the average velocity of the particles, and if is the average time between collisions, then the average force per unit volume equals $\frac{mn}{\tau}(\underline{V}-\underline{u})$

Therefore

$$\int \underline{v}\left(\frac{\partial f}{\partial t}\right) d^3v = \frac{n}{\tau}(\underline{V}-\underline{u}) \qquad (16)$$

If all those assumptions are made, the zero and first moments of Boltzmann's equation give the continuity and the conservation of momentum equations.

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{u}) = 0 \\ \text{and}\quad &\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{3n}\nabla(n\overline{w^2}) + \frac{e}{m}\left(\underline{E} + \frac{\underline{u}}{c}\times\underline{B}\right) + \frac{1}{\tau}(\underline{V}-\underline{u}) \end{aligned}\right\} \qquad (17)$$

In order to be able to work with equations (17) Axford approximates $\overline{w^2}$ by c^2.

APPENDIX SEVEN

Average Change in Momentum of a Light Particle in Collision with a Heavy Particle

Consider two particles of mass m and M respectively. The particles are to be treated as smooth elastic spheres. Suppose that in a coordinate system in which mass M is intially at rest, mass m has velocity $\underline{v}_o$ and collides with mass M at an angle θ as shown in figure 8. After the collision mass m has a velocity v with zenith angle Θ. The other particle has a velocity V at a zenith angle θ. Then the conservation of momentum and energy equations are

$$m v_0 = m v \cos \Theta + M V \cos \theta \qquad \text{momentum along z axis} \qquad (1)$$

$$0 = m v \sin \Theta + M V \sin \theta \qquad \text{momentum} \perp \text{ z axis} \qquad (2)$$

$$\frac{m v_0^2}{2} = \frac{m v^2}{2} + \frac{M V^2}{2} \qquad (3)$$

Eliminating Θ and v between (1), (2), and (3)

$$V = \frac{2 m v_0 \cos \theta}{(m+M)} \qquad (4)$$

Consider a beam of identical particles of mass m and velocity $\underline{v}_o$ uniformly distributed in the xy plane. If D is the distance between centers, the effective area presented for collisions is πD^2. The probability $p(\theta,\phi)d\theta\, d\phi$ of a collision occuring at a point (θ,ϕ) in θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ measured from the center of mass M is given by the ratio of the effective area presented for collisions by this region to the total effective area.

$$p(\theta,\phi)\, d\theta\, d\phi = \frac{D^2 \sin\theta \cos\theta\, d\theta\, d\phi}{\pi D^2} = \frac{\sin\theta\cos\theta\, d\theta\, d\phi}{\pi} \qquad (5)$$

The change in the momentum of mass m for a collision with point of impact at (θ, ϕ) is the negative of the change of momentum of the mass M. From (4)

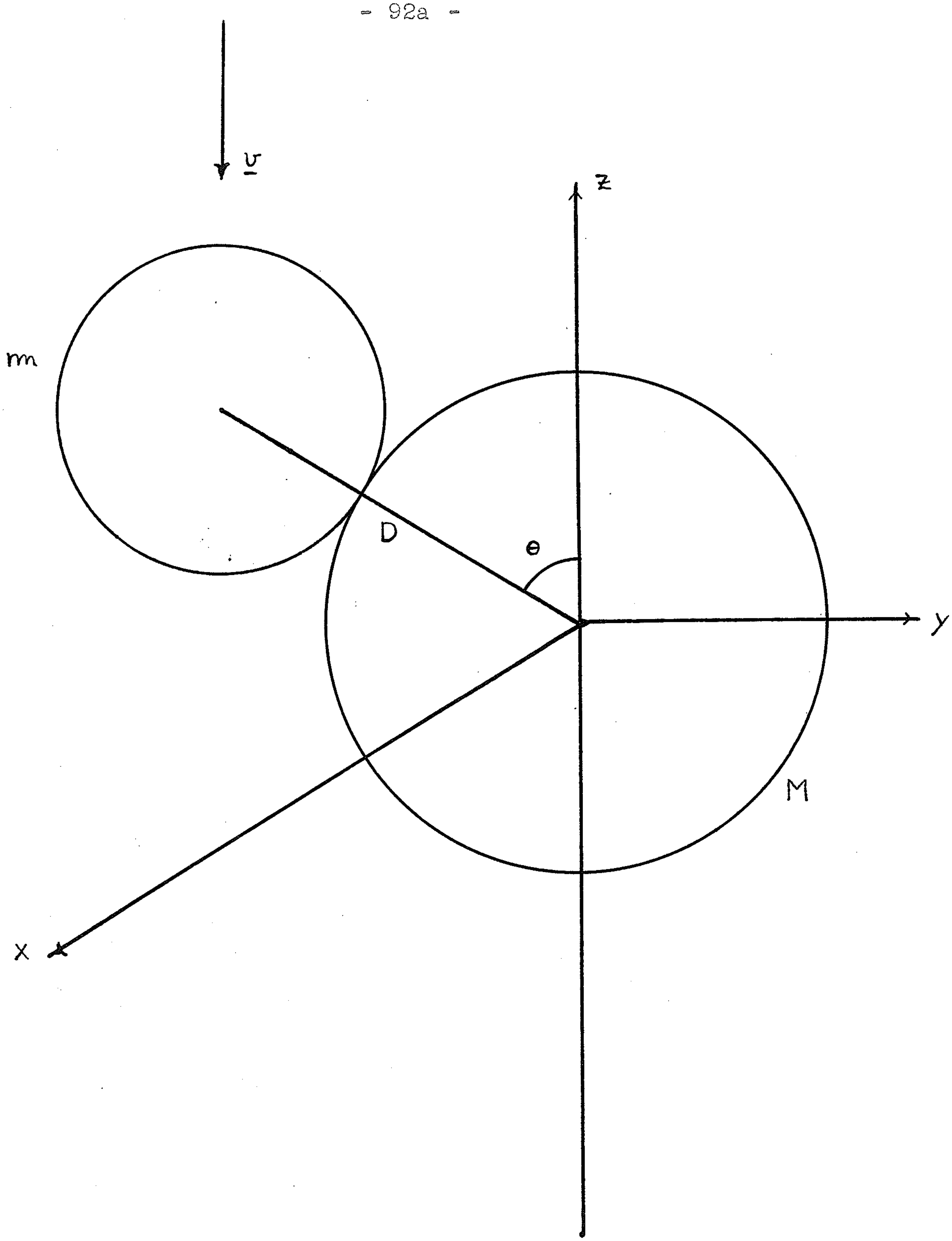


Figure Collision Between Light Particle and Heavy Particle

$$\Delta \not{p}_m(\theta,\phi) = \frac{2m v_0 \cos\theta\, M}{(m+M)} \left(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}\right) \qquad (6)$$

On the average the change in momentum of the particles of mass m that collide is

$$\overline{\Delta \not{p}_m} = \int \not{p}(\theta,\phi)\, \Delta \not{p}_m(\theta,\phi)\, d\theta\, d\phi \qquad (7)$$

Combining (5), (6), and (7)

$$\overline{\Delta \not{p}_m} = \frac{2m v_0 M}{(m+M)\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} \sin\theta\cos^2\theta \left(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}\right) d\theta\, d\phi$$

$$= \frac{2m v_0 M}{(m+M)\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\theta\cos^3\theta\, d\theta\, d\phi\, \underline{k}$$

$$= \frac{m v_0 M \underline{k}}{(m+M)}$$

if m<< M

$$\overline{\Delta \not{p}_m} = m v_0 \underline{k} \qquad (8)$$

The calculation was performed for the case where mass M was at rest. If mass M has an initial velocity $\underline{V}_0$ in the rest frame, then

$$\overline{\Delta \not{p}_m} = m\underline{V}_0 - m\underline{v}_0 \qquad (9)$$